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Structural and Computational Aspects of Simple and Influence Games

Fabián Riquelme Csori

PhD. Thesis

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Advisors:

Dr. Xavier Molinero Albareda   Dr. María José Serna Iglesias

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Abbreviations and Symbols

$\mid$	"Such that" in set definitions.
$\mid \bullet \mid$	Cardinality of $\bullet$.
$\lceil \bullet \rceil$	Ceiling function.
$\lfloor \bullet \rfloor$	Floor function.
$\setminus$	Set subtraction.
$\neg \bullet$	Negation or complement of $\bullet$.
$\preceq, \prec$	Shift-order relationship.
$\alpha(X)$	Immediate successors of coalition X.
$\mathcal{B}(\Gamma)$	Bargaining set of game Γ.
$\beta_i(\Gamma)$	Banzhaf index of player i in game Γ.
$\beta'_i(\Gamma)$	Probabilistic Banzhaf index of player i in game Γ.
BDD	Binary decision diagram.
BDDF	Binary decision diagram form.
c	Collective decision vector; constant.
C	Collective decision function.
$\mathcal{C}(\Gamma)$	Core of game Γ.
$\mathcal{C}_\epsilon(\Gamma)$	ϵ-core of game Γ.
$\mathcal{C}_{\epsilon_1}(\Gamma)$	Least core of game Γ.
$d(\mathbf{p}, X)$	Deficit of coalition X under payoff $\mathbf{p}$.
$\mathbf{d}$	Deficit vector.
$\delta^-(i)$	Indegree of node i.
$\delta^+(i)$	Outdegree of node i.
$e(\mathbf{p}, X)$	Excess of coalition X under payoff $\mathbf{p}$.
$\mathbf{e}$	Excess vector.

E	Set of edges of a graph.
ϵ	Parameter small enough.
ELF	Explicit losing form.
EWF	Explicit winning form.
f	Labeling function;
	Generic measure on a collective decision model.
$F(X)$	Spread of influence from a team or coalition X.
$F_k(X)$	Spread of influence from X at step k.
$F^k(X)$	Family of teams whose influence spreads to k agents.
$\mathtt{F}$	Set of followers.
$\Phi_i(\Gamma)$	Shapley-Shubik index of player i in game Γ.
FCB	Fully condensed binary tree.
FCBF	Fully condensed binary tree form.
G	Graph.
$G[X]$	Graph induced by X.
$\mathcal{G}$	Family of simple games.
Γ	Simple game.
Γ^d	Dual of simple game Γ.
$\eta_i(\Gamma)$	Banzhaf value of player i; number of swings in i of a game.
i, j, h	Players of simple game; nodes of graph; actors of system.
$I(\Gamma)$	Set of imputations of game Γ.
$I^*(\Gamma)$	Set of preimputations of game Γ.
$\mathtt{I}$	Set of independent actors.
iff	"If and only if".
$\mathrm{Im}(f)$	Image of function f.
$K_{r,s}$	Bipartite graph.
$\mathcal{K}(\Gamma)$	Kernel of game Γ.
$\kappa_i(\Gamma)$	Shapley-Shubik value of player i.
$\mathtt{L}$	Set of opinion leaders.
$\mathcal{L}$	Set of losing coalitions.
$\mathcal{L}^M$	Set of maximal losing coalitions.
$\mathcal{L}^S$	Set of shift-maximal losing coalitions.
$\lambda(\mathcal{H})$	Minimal transversals or slices of hypergraph $\mathcal{H}$.
MLF	Maximal losing form.
MWF	Minimal winning form.
MWC	Minimal winning coalition.

$\mathtt{M}$	Set of mediators.
$\mathcal{M}$	Collective decision making model.
$\mu(\mathcal{H})$	Minimal kernel of hypergraph $\mathcal{H}$.
n	Number of players of a game; number of nodes of a graph; number of actors of a system.
N	Set of players of a simple game.
$\mathcal{N}(\Gamma)$	Nucleolus of game Γ.
$\mathbb{N}$	Set of natural numbers, including the number zero.
ν	Characteristic function of a game.
$\nu(\mathcal{H})$	Clutter of hypergraph $\mathcal{H}$.
$\mathbf{p}, \mathbf{q}, \mathbf{r}$	Payoff vectors.
p_i	Payoff of player i.
$p(X)$	Payoff of coalition X.
$P_G(i)$	Predecessors of node i in graph G.
$\mathcal{P}(X)$	Power set of X.
$\mathcal{P}_k(X)$	Subsets of $\mathcal{P}(X)$ with cardinality k.
$\mathcal{PK}(\Gamma)$	Prekernel of game Γ.
$\mathcal{PN}(\Gamma)$	Prenucleolus of game Γ.
PCB	Partially condensed binary tree.
PCBF	Partially condensed binary tree form.
q	Quota of either a weighted game or an influence game; fraction value of an OLF system.
$\mathbb{R}$	Set of real numbers.
S_i	Set of swings for a player i.
$S_G(i)$	Successors of node i in graph G.
$\mathcal{S}$	OLF system.
$\mathcal{S}(\Gamma)$	A stable set of game Γ.
$\mathcal{S}_X(\Gamma)$	A stable set of simple game Γ defined with MWC X.
SWF	Shift-minimal winning form.
$\tau(\mathcal{H})$	Transversals or blocker of hypergraph $\mathcal{H}$.
V	Set of nodes of a graph; set of actors of a system.
VWRF	Vector weighted representation form.
w	Weight function.
$\mathcal{W}$	Set of winning coalitions.
$\mathcal{W}^m$	Set of minimal winning coalitions.
$\mathcal{W}^s$	Set of shift-minimal winning coalitions.

ω, ω_i	Chow parameters.
WRF	Weighted representation form.
X, Y, Z	Coalitions of a simple game; subset of nodes of a graph; subset of actors of a system.

Abstract

Thesis title	: Structural and computational aspects of simple and influence games
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Simple games are a fundamental class of cooperative games. They have a huge relevance in several areas of computer science, social sciences and discrete applied mathematics. The algorithmic and computational complexity aspects of simple games have been gaining interest to the computer science community in the recent years.

In this thesis we consider different computational problems related to properties, parameters, and solution concepts of simple games. We analyze the computational complexity of these problems under different forms of representation of simple games, regular games and weighted games. We also analyze the complexity required to transform a game from one representation to another. In this scenario, we study the decisive problem, which is associated to the duality problem of hypergraphs and monotone Boolean functions. We prove that the problem of deciding whether a simple game in minimal winning form is decisive can be solved in quasi-polynomial time. We also show that this decisive problem can be polynomially reduced to the same problem but restricted to regular games in shift-minimal winning form.

Furthermore, we prove that the problem of deciding whether a regular game is strong in shift-minimal winning form is coNP-complete. Additionally, we prove that the width, one of the parameters of simple games, can be computed in polynomial time for games in minimal winning form. Regardless of the form of representation, we also analyze counting and enumeration problems for several subfamilies of simple games.

We introduce influence games, which constitute a new approach to study simple games based on a model of spread of influence in a social network, where influence spreads according to the linear threshold model. We show that influence games capture the whole class of simple games. Moreover, we study for influence games the complexity of the problems related to parameters, properties and solution concepts considered for simple games. We consider extremal cases with respect to demand of influence, and we show that, for these subfamilies, several problems become polynomial.

We finish with some applications inspired on influence games. The first set of results concerns the definition of collective choice models. For mediation systems, several of the problems of properties mentioned above are polynomial-time solvable. For influence systems, we prove that computing the satisfaction—a measure equivalent to the Rae index and similar to the Banzhaf value—is hard unless we consider some restrictions in the model. For OLFM systems, a generalization of OLF systems [255, 256], we provide an axiomatization of satisfaction. The second set of results establishes a connection with social network analysis. We apply power indices of cooperative games as centrality measures of social networks, and we define new centrality measures based on the spread of influence phenomenon. We compare all these measures on real networks with some classical centrality measures.

Keywords: Game theory, Simple games, Cooperative games, Weighted games, Regular games, Hypergraph, Computational complexity, Decisive problem, Duality problem, Power indices, Solution concepts, Spread of influence, Influence games, Multi-agent systems, Mediation systems, Collective choice models, Satisfaction, Social network analysis, Centrality.

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Part I

Preliminaries

Chapter 1

Introduction

Game theory arises in the first half of the 20th century from the need to study formally situations of conflict and cooperation between intelligent rational decision-makers [257, 258]. It is closely related with other disciplines such as decision theory, voting theory and social choice theory. Decision theory is devoted to identifying the values, uncertainties, rationality, optimality and other issues relevant in an individual decision making; voting theory studies the voting systems, i.e., methods by which voters make a choice between different options; and social choice theory studies how the individual preferences can be combined to reach a collective decision.

From the beginning, a relevant branch of game theory has been cooperative game theory [155], which addresses the study of *cooperative games*, also called *coalitional games* or *characteristic function games*. A cooperative game is a mathematical structure formed by a set of players that by forming coalitions can achieve a common benefit, enforcing a cooperative behavior. Unlike non-cooperative games, where individual players compete to obtain the highest payoffs for themselves, here the players form coalitions, so that the problem is how to divide the payoffs or the utility, trying to obtain together the highest benefits as posible. Cooperative games have been widely studied by the scientific community [258, 236, 65, 201, 212, 44].

A well known subclass of cooperative games is the class of *simples games*, also called *simple coalitional games*, *simple voting games* or *cooperative simple games*, in which the benefit that a coalition may have is always binary, i.e., a coalition may be *winning* or *losing*, depending on whether the players in the coalition are able to benefit themselves from the game by achieving together some goal. Simple games were introduced in the seminal work of game theory in 1944 [258], as an extensive class of cooperative games[1], whose study

> *"yields a body of information which is of value for a deeper understanding of the general theory".*

Furthermore, in the preface of [251] we can read:

> *"Few structures in mathematics arise in more contexts and lend themselves to more diverse interpretations than do hypergraphs or simple games."*

Simple games have a huge relevance in mathematics, computer science and social sciences, being used to solve and represent problems arising in voting theory, decision theory and social choice theory, logic and threshold logic, circuit complexity, computational complexity theory, artificial intelligence, geometry, linear programming, Sperner theory, order theory, etc. [251, 71, 75] They are closely related with other mathematical and computational structures, such as dual hypergraphs, Sperner families, antichains, monotone Boolean functions, free distributive lattices, monotone collective decision making systems and multi-agent systems, among others [251, 71, 75].

A lot of effort has been devoted to understand which conditions a simple game should have in order to meet some properties [251]. The development of computer science as a scientific discipline in the 1950s and early 1960s, helped to deal with these problems from a computational point of view. Recently computer scientists have begun to question what is the computational complexity of deciding properties in simple games [10, 92]. Several of these questions have been satisfactorily classified in complexity classes, and the others remain still open. The way in which the game is represented is crucial for this complexity analysis.

Nowadays, cooperation towards task execution when tasks cannot be performed by a single agent is one of the fundamental problems in both social

[1] However, they considered a more restricted class of games, which nowadays are known as *strong games*, i.e., simple games such that the complement of every losing coalition is a winning coalition. According to Isbell [124], the definition that nowadays is used was given by Gillies in 1953 [97] under the name of *pseudogames*.

and multi-agent systems. There has been a lot of research understanding collective tasks allocation under different models coming from cooperative game theory. Under such framework, in general, cooperation is achieved by splitting the agents into teams so that each team performs a particular task and the payoff of the team is split among the team members. Thus, cooperative game theory provides the fundamental tools to analyze this context [272, 273, 44, 184, 12, 54]. In this vein, it is relevant to mention the effort of some authors to make explicit the relationship or agreements among some players or actors in a game before the conformation of coalitions. This is the case of the *communication structures* of Myerson [191] and the *a priori unions* or *coalition structures* of Owen [205], that help to deal with additional real-world decision-making situations.

On the other hand, the ways in which people influence each other through their interactions in a social network has received a lot of attention in the last decade. Social networks have become a huge interdisciplinary research area with important links to sociology, economics, epidemiology, computer science, and mathematics [5, 127, 67, 116]. In the last decades the field has grown extensively with the development of Internet and the emergence of online social networks. A social network can be represented by a graph where each node is an agent and each edge represents the degree of influence of one agent over another one. Several "germs"—ideas, trends, fashions, ambitions, rules, etc.—can be initiated by one or more agents and eventually be adopted by the system. The mechanism defining how these motivations are propagated within the network, from the influence of a small set of initially "motivated" nodes, is called a model for *influence spread*.

Motivated by viral marketing and other applications, the problem that has been usually studied is the *influence maximization problem*, initially introduced by Domingos and Richardson [64, 226] and further developed in [136, 76]. This problem addresses the question of finding a set with at most k players having maximum influence, and it is NP-hard [64], unless additional restrictions are considered, in which case some generality of the problem is lost [226]. In social network analysis, the spread of influence is also related with other interesting concepts, like the homophily phenomenon [171, 247]. Two general models for spread of influence were defined in [136]: the *linear threshold model*, based in the first ideas of [108, 230], and the *independent cascade model*, created in the context of marketing by [100, 101]. Models for influence spread in the presence of multiple competing products has also been proposed and analyzed [23, 33, 5]. In such a setting there is also work done towards analyzing the problem from the point of view of non-cooperative game theory. Non-cooperative *influence games* were defined in 2011 by Irfan and Ortiz [122]. Those games, however, analyze the strategic aspects of two firms competing on the social network and differ from the context of this thesis.

Besides the spread of influence phenomenon, one of the most studied concepts in social network analysis is the notion of *centrality* and the development of mechanisms to measure how structurally important is an actor within a social network [85, 32, 247]. There are many centrality measures that provide different relevance criteria for the vertices within the network [261, 150]. However, one of the major challenges for a successful implementation of network management activities, such as viral marketing, is the identification of key persons with a central structural position within the network. For this purpose, social network analysis provides a lot of measures for quantifying a member's interconnectedness within social networks, providing strongly differing results with respect to the quality of the different centrality measures [148].

The main goals of this thesis are two. Firstly, increase the number of known complexity results for computational problems related to simple games and subclasses of simple games. Secondly, to establish a relationship between the spread of influence phenomenon on social networks and binary decisions in voting systems. This can be done through the definition of influence games, a new family of simple games based on the linear threshold model. We determine for this new construction the complexity of the above computational problems. Due to the interdisciplinary nature of these games, we also find for them some applications in other areas, such as multi-agent systems, decision theory, social choice and social network analysis.

1.1 Results Overview

Although this thesis is focused on simple games, the research topic is multidisciplinary. On one hand, we can mention game theory, voting theory, social choice theory and decision theory, that arise from economy and politics. On the other hand, we have the computational complexity theory, as part of theoretical computer science, that arises from computer sciences and discrete applied mathematics. Furthermore, we use several tools from other fields of discrete mathematics, such as graph theory, hypergraph theory and order theory, among others. We also need to mention social network analysis, that comes from an intersection between mathematics and sociology.

The approach of the thesis is theoretical and algorithmic. We study the computational aspects of problems of simple games. We show the computational complexity of well known properties of traditional subclasses of simple games. For some problems that can not be solved in polynomial time, we present enumeration algorithms with polynomial-delay, when the output size is exponential in terms of the size of the input.

We attempt to use standard notation, indicating in each case the references in which they are used. For any question about nonstandard mathematical symbols, see the chapter Abbreviations and Symbols at the beginning of the thesis.

In order to present our computational results, we classify the properties of simple games into three types: those that represent features of simple games, players, or coalitions. About the first ones, we focus on the *proper*, *strong* and *decisive* properties. Those properties have applications in several areas such as interactive decision making, distributed computing, logic and linear programming, category theory, social science, hypergraph theory and reliability theory [151, 129, 18, 251, 157, 71]. We consider several properties of players, such as the *dummy*, *passer*, *vetoer*, and *dictator* properties, among others, closely related to the computation of solution concepts. In the same vein, regarding the properties of coalitions, we consider the *blocking* and the *swing*, which are useful to compute solution concepts, as well as to represent simple games in a more succinct way [258, 16, 164, 251].

From the point of view of parameters, we study quantities that give interesting information about simple games. We study the *width* and the *length*, also used in decision making [221], among others. We also study solution concepts, that come from cooperative game theory as a way to measure profit allocations of players, by considering the profit of each coalition [10, 44].

Given a subclass of simple games, some problems of interest in simple game theory are: Given a simple game, does it belongs to that class? Could you give me one after another all the simple games in the class? How many simple games belong to that class?

The first of these problems is related to the *conversion problem*, which is the problem of computing a representation of a game given in another representation. It is relevant to ask which forms of representation are the more appropriate when we face a computational problem [129]. Both the feasibility of the representation and the computational complexity of the problem are aspects that must be considered; and the first of these aspects has direct implications about the second one. For instance, the *decisive* property can be decided in polynomial time when the game is given in *extensive winning form*; in quasi-polynomial time if the game is given in *minimal winning form* [84, 229], and the problem is coNP-complete if the game can be given in *weighted representation form* [92].

The second problem is the *enumeration problem*, that attempts to list without repetitions every game belonging to a class of simple games. Finally, the third one is the *counting problem*, which refers to find the number of elements of those classes. Note that every subclass of simple games is finite whether it is restricted to a given number of players. However, the subclasses of interest are typically huge, so the storage of their elements in memory is not feasible at all. In this scenario, the enumeration algorithms allow us to recover all the elements of a given subclass. The enumeration algorithms are especially useful in benchmarking, because through complex combinatorial operations, they may provide specific subfamilies of simple games for experiments, that would be difficult to achieve manually.

In general, we consider these problems for simple games, but also for the main subclasses of simple games, such as *regular games*, *weighted games* and *homogeneous games*, one of the most studied subclasses of weighted games [246].

In the context of both social and multi-agent systems, we propose to analyze cooperation based on a model for influence among the agents in their established network of trust and influence. Social influence is relevant to determine the global behavior of a social network and thus it can be used to enforce cooperation by targeting an adequate initial set of agents. From this point of view, we consider a simple and altruistic multi-agent system in which the agents are eager to perform a collective task but where their real engagement depends on the perception of the willingness to perform the task of other influential agents. We model this scenario by an *influence game*, a cooperative simple game in which a team of agents—or coalition—succeeds if it is able to convince enough number of agents to participate in the task. We take the deterministic linear threshold model [45, 5] as the mechanism for influence spread in the subjacent social network.

In the considered scenario we adopt the natural point of view of decision or voting systems, mathematically modeled as simple games [258]. This approach brings into the analysis all the parameters, properties and solution concepts for simple games previously mentioned.

1.2 Thesis Outline

This thesis is divided into three parts. The first one reviews the preliminary aspects of simple game theory; the second part presents the different forms of representation, and analyzes aspects of several problems on simple games; and the last one is devoted to influence games and their applications. In turn, each part is subdivided into chapters.

In Chapter 2 we establish basic definitions and notation related with graph theory and computational complexity. Here we also define simple games, their main properties, parameters, solution concepts and subfamilies.

In Chapter 3 we provide an easy way to access information about the different forms of representation of simple games, together with an analysis of their relationships from a computational point of view. We study the most classical forms of representation, such as the extended winning (losing) form and the minimal winning (maximal losing) form, but also representations derived from the binary trees and binary decision diagrams. For regular games we consider the shift-minimal winning form and the fully condensed binary trees. For weighted games we consider the usual weighted representation. In this sense, the chapter behaves like a survey. Moreover, we also analyze the complexity of the conversion problem in each case, i.e., the problem of transforming a game from one form of representation to another. Some of these results consider new algorithms which run with polynomial-delay. This chapter correspond to the following paper:

[181] X. Molinero, F. Riquelme, and M. Serna. Forms of representation for simple games: sizes, conversions and equivalences. Submitted to *Mathematical Social Sciences*, 2014.

In Chapter 4 we study the properties, parameters and solution concepts of simple games from a computational point of view. The known results are summarized on Tables 4.1 and 4.2. We solve some open computational problems, and we give ideas about how to deal with problems that remain open. The main results obtained in these issues are summarized on Tables 4.3 and 4.4. In particular, we show that the decisive problem for simple games is equivalent to the duality problem of hypergraphs, so it can be solved in quasi-polynomial time, instead of being coNP-complete, as was conjectured in [92]. We also prove that the decisive problem for weighted games in minimal winning form is polynomial time solvable. These results are presented in the following publication:

[229] F. Riquelme and A. Polyméris. On the complexity of the decisive problem in simple and weighted games. *Electronic Notes in Discrete Mathematics*, 37:21–26, 2011.

Moreover, we show that the decisive problem for regular games in shift-minimal winning form can also be solved in quasi-polynomial time, but it seems unlikely that there is a polynomial algorithm to solve it, as in the case of regular games in minimal winning form. This result can be found in the following document:

[216] A. Polyméris and F. Riquelme. On the complexity of the decisive problem in simple, regular and weighted games. CoRR, abs/1303.7122, 2013.

Another interesting result is that the width parameter—as well as the length—can be computed in polynomial time for simple games in either extended or minimal winning form. This problem was posted as open in [10], and its solution is presented in the following paper:

[177] X. Molinero, F. Riquelme, and M. Serna. Cooperation through social influence. Submitted to *European Journal of Operational Research*, 2013.

[175] X. Molinero, F. Riquelme, and M. Serna. Social influence as a voting system: A complexity analysis of parameters and properties. CoRR, abs/1208.3751v3, 2012.

The final part of the chapter is devoted to the enumeration and counting problems. We provide the main existing results from different analogous research areas, and we propose a novel procedure to enumerate subclasses of decisive games. These results remain unpublished, but a list of explicit enumerated decisive regular games is presented in Appendix A. The algorithm presented in this chapter was also used for counting decisive homogeneous games, whose numbers where introduced in the Online Encyclopedia of Integer Sequences (OEIS) [238], assuming the code $A189360$. The main ideas of the enumeration of decisive regular games were presented in the following conference:

[174] X. Molinero, A. Polyméris, F. Riquelme, and M. Serna. Efficient enumeration of complete simple games. In The Fifth International Conference on Game Theory and Management (GTM 2011). St. Petersburg, Russia, June 27-29, 2011.

In Chapter 5 we define the influence games based on both an influence graph and the linear-threshold model of influence spread. We show its expressiveness, by proving that they are equivalent to the whole family of simple games. We also analyze several of the previous properties, parameters and solution concepts for influence games. These results are summarized on Tables 5.1 and 5.2. For many cases in which the complexity of the problems is hard, we study some extremal cases for which the hard problems become polynomial time solvable. Most of these results form part of an article mentioned above:

[177] X. Molinero, F. Riquelme, and M. Serna. Cooperation through social influence. Submitted to *European Journal of Operational Research*, 2013.

[175] X. Molinero, F. Riquelme, and M. Serna. Social influence as a voting system: A complexity analysis of parameters and properties. CoRR, abs/1208.3751v3, 2012.

Additionally, some results regarding solution concepts for influence games were presented in the following conference:

[183] X. Molinero, F. Riquelme, and M. Serna. Solution concepts in influence games. In the 20th Conference of the International Federation

of Operational Research Societies (IFORS). Barcelona, Spain, July 13-18, 2014.

The last two chapters of the third part consider several applications of influence games in different topics of computer sciences.

In Chapter 6 we define some collective choice models, namely the opinion leader-follower systems or OLF systems, mediation systems, and influence systems. While the first one was defined in [255], the other two are new. Mediation systems are influence games in which we consider a new kind of actors called mediators, that are not part of the set of players. We prove that several problems considered above for influence games, can be solved for mediation systems in polynomial time. This model appears in the following proceedings:

[180] X. Molinero, F. Riquelme, and M. Serna. Star-shaped mediation in influence games. In K. Cornelissen, R. Hoeksma, J. Hurink, and B. Manthey, editors, *12th Cologne-Twente Workshop on Graphs and Combinatorial Optimization, Enschede, Netherlands, May 21-23, 2013*, volume WP 13-01 of *CTIT Workshop Proceedings*, pages 179–182, 2013.

For OLF systems and influence systems, we address the computational complexity of the *satisfaction* measure introduced in [255], that we show is equivalent to the *Rae index* of cooperative games [220]. We prove that computing the satisfaction measure is hard even for influence systems on bipartite digraphs. We also introduce subfamilies of these systems for which the satisfaction can be computed in polynomial time. These results are presented in the following paper:

[178] X. Molinero, F. Riquelme, and M. Serna. Measuring satisfaction in societies with opinion leaders and mediators. Submitted to *Discrete Applied Mathematics*, 2013.

We finish the chapter by proving the existence of an axiomatization of the satisfaction measure for opinion leader-follower through mediators systems or OLFM systems, a generalization of OLF systems that allows mediators in the set of actors. It is a generalization of the axiomatization given by [256] for the satisfaction measure in OLF systems. This results appear in the following document:

[228] F. Riquelme. Satisfaction in societies with opinion leaders and mediators: properties and an axiomatization. CoRR, abs/1405.3460, 2014.

Finally, in Chapter 7 we propose and study new centrality measures for social network analysis based on the spread of influence phenomenon. We provide evidence that power indices can be applied as centrality measures, and we also define the effort centrality and the satisfaction centrality. Additionally, we define a family of centrality measures that can be computed in polynomial time. We compare all these measures with some of the most traditional ones in real social networks. Most of these results are presented in the following publication:

[182] X. Molinero, F. Riquelme, and M. Serna. Power indices of influence games and new centrality measures for agent societies and social networks. In C. Ramos, P. Novais, C. E. Nihan, and J. M. Corchado, editors, *Ambient Intelligence - Software and Applications: 5th International Symposium on Ambient Intelligence, volume 291 of Advances in Intelligent Systems and Computing*, pages 23-30. Springer International Publishing, 2014.

[179] X. Molinero, F. Riquelme, and M. Serna. Power indices of influence games and new centrality measures for agent societies and social networks. CoRR, abs/1306.6929, 2013.

Additionally, the use of power indices as centrality measures and some preliminary validation were presented in the following conference:

[176] X. Molinero, F. Riquelme, and M. Serna. Centrality measures based on power indices for social networks. In The 26th European Conference on Operational Research (EURO2013). Rome, Italy, July 1-4, 2013.

Chapter 2

Mathematical Preliminaries

In this chapter we describe the preliminary definitions concerning graph theory, computational complexity and simple games. We assume basic knowledge in computer sciences and discrete mathematics.

As usual, $\mathcal{P}(N)$ denotes the power set of N, and n its cardinality, i.e., $n = |N|$.

2.1 Graphs and Hypergraphs

In this thesis we use standard notation for graph theory [26], mainly used in Chapters 3, 5 and 6. We also use hypergraphs, mainly in Section 4.2, for which we use notation from [18, 214].

Definition 2.1. Let $G = (V, E)$ be a *graph*, where $V(G)$ is the set of vertices or nodes, and $E(G)$ is the set of edges. Let $n = |V(G)|$ be the number of vertices. A *directed graph* is a graph whose edges have a direction associated with them. The directed edges are also known as *arcs*. An *undirected graph* is a graph without direction in the edges. An edge and a vertex on that edge are called *incident*. Two edges are called *adjacent* if they share a common vertex. Similarly, two vertices are called *adjacent* if they share a common edge.

For each $i \in V$, $S_G(i) = \{j \in V \mid (i, j) \in E\}$ is the set of *successors* of i, and $P_G(i) = \{j \in V \mid (j, i) \in E\}$ is the set of *predecessors* of i. We extend this notation to vertex subsets, given $X \subseteq V$, so that $S_G(X) = \{i \in V \mid \exists j \in X, i \in S_G(j)\}$ and $P_G(X) = \{i \in V \mid \exists j \in X, i \in P_G(j)\}$. Finally, let

$\delta^-(i) = |P_G(i)|$ and $\delta^+(i) = |S_G(i)|$ be the indegree and the outdegree of the node i, respectively. The *size* of a graph is its number of vertices, $|V|$.

We use simply V and E rather than $V(G)$ and $E(G)$, when there is no risk of confusion. Let $i, j \in V$ be a pair of nodes. If G is a directed graph, an edge from node i to node j is denoted by a pair $(i, j) \in E$; otherwise, if G is undirected, an edge between nodes i and j is denoted by $\{i, j\} \in E$.

All the graphs considered in this thesis are directed, unless otherwise stated, without loops and multiple edges. When there is no risk of confusion, an undirected edge between nodes i and j is also denoted by (i, j). For the remaining definitions, we consider interchangeably either directed or undirected graphs.

Usually nodes or edges need to be labeled.

Definition 2.2. A *weighted graph* is a graph (G, w) where $w : E(G) \to \mathbb{N}$ is a *weight function*. A *labeled graph* is a weighted graph (G, w, f) where $f : V(G) \to \mathbb{N}$ is a *labeling function*. An *unweighted labeled graph* is a graph (G, f) in which every edge has weight 1.

Sometimes, we just need to pay attention to some portion of an entirely graph. For this cases it is necessary the notion of subgraph.

Definition 2.3. Let $G = (V, E)$ be a graph. A *subgraph* $G' = (V', E')$ of G is a graph whose vertex set is a subset of that of G, i.e. $V' \subseteq V$, and whose adjacency relation is a subset of that of G restricted to this subset, i.e. $E' = \{(i, j) \in E \mid i \in V' \text{ and } j \in V'\}$. A subgraph G' of a graph G is *induced* if for all $i, j \in V'$, $(i, j) \in E'$ if and only if $(i, j) \in E$; i.e., G' is an induced subgraph of G if it has exactly the edges that appear in G over the same vertex set. If the vertex set of G' is the subset X of $V(G)$, then G' can be written as $G[X]$ and is said to be *induced by* X.

Another issue in graph theory is the connectivity in graphs.

Definition 2.4. A *path* in a graph is a sequence of edges which connect a sequence of vertices. Two vertices i and j are *connected* in a graph if it contains a path from i to j. A graph is *connected* if every pair of vertices in the graph is connected. A *connected component* is a maximal connected subgraph of a graph.

Now we define some kind of graphs with interesting properties that we use in the following chapters.

Definition 2.5. Let $G = (V, E)$ be a graph, then:

- G is *isolated* if it has no edges. It is denoted by I_n.
- G is a *cycle* if it is a path starting and ending at the same vertex, with no repetitions of vertices or edges allowed, other than the repetition of the starting and ending vertex.
- G is *complete* if every vertex is connected with the others.
- G is *bipartite* if V can be divided into two disjoint sets V_1 and V_2 such that every edge connects a vertex in V_1 to one in V_2, i.e., so that V_1 and V_2 are *independent sets*. Equivalently, a bipartite graph is a graph that does not contain odd-length cycles.
- G is a *complete bipartite graph* if it is complete and bipartite. It is denoted by $K_{r,s}$, where $r = |V_1|$ and $s = |V_2|$. We denote by $\vec{K}_{r,s}$ to the complete bipartite graphs in which the edges are all oriented from V_1 to V_2.
- G is a *star graph* if it is a bipartite graph $K_{1,k}$ formed by $k+1$ vertices.
- G is a *triangle* if it is the complete graph with three vertices that forms a cycle.
- G is a *tree* if it is connected and has no cycles.
- G is a *binary tree* if it is a tree such that each vertex has at most two child vertices. Vertices with children are called *inner nodes*. Vertices without children are called *terminal nodes*, *leaf nodes*, *outer nodes* or *external nodes*. The *root node* is the ancestor of all nodes. The *null tree* is the tree which does not have any vertex other than the root.

Observe that a graph where all edges are incident is either a triangle or a star.

In binary trees, child vertices are usually distinguished as "left" and "right" or by labels 0/1 on the corresponding arc. Any vertex in the data structure can be reached by starting at the root node and repeatedly following pointers to either the left or right child. In a binary tree, the outdegree of every vertex is at most two and it holds that $|E| = |V| - 1$. In some cases the terminal nodes are labeled.

Sometimes it is useful to consider a generalization of graphs where the edges can connect multiple vertices, in such a way that instead of a set of edges we obtain a family of sets. This generalization is known as hypergraph.

Definition 2.6. Let N be a finite set known as *ground set*, a *hypergraph* is a family $\mathcal{H} \subseteq \mathcal{P}(N)$ of *hyperedges* $X \subseteq N$.

We may represent a hypergraph $\mathcal{H}$ as an incidence matrix, whose rows represent the incidence vectors $x : N \rightarrow \{0,1\}$ of the hyperedges $X \in \mathcal{H}$. Thus, given $i \in N$, $x(i) = 1$ if and only if $i \in X$. The size of a hypergraph $\mathcal{H}$, i.e., the amount of bits needed in order to write down the family of incidence vectors that characterize the hypergraph, is $n \cdot |\mathcal{H}| = n \cdot |\mathcal{H}| \in \mathbb{N}$.

The following operators define several properties of hypergraphs. They come from [214].

Definition 2.7. Given a hypergraph $\mathcal{H}$ over a set N, then we define:

- $\neg(\mathcal{H}) = \{N \setminus X \mid X \in \mathcal{H}\}$ is the family of complementaries of $\mathcal{H}$.
- $\mu(\mathcal{H}) = \{X \in \mathcal{H} \mid \text{ for all } Z \in \mathcal{H}, Z \not\subset X\}$ is the *minimal* of $\mathcal{H}$, or the family of *irredundant* elements of $\mathcal{H}$.
- $\nu(\mathcal{H}) = \{Z \subseteq N \mid \text{ exists } X \in \mathcal{H}, X \subseteq Z\}$ is the *clutter* of $\mathcal{H}$, or the family of subsets of N that *respond* to $\mathcal{H}$.
- $\tau(\mathcal{H}) = \{Z \subseteq N \mid \text{ for all } X \in \mathcal{H}, X \cap Z \neq \emptyset\}$ is the *blocker* of $\mathcal{H}$, or the family of subsets of N that are *transversal* to $\mathcal{H}$.
- $\lambda(\mathcal{H}) = \mu(\tau(\mathcal{H}))$ are the *slices* of $\mathcal{H}$, or the family of *irredundant elements that are transversal* to $\mathcal{H}$.

Let $\mathcal{H}, \mathcal{K} \subseteq \mathcal{P}(N)$ be two hypergraphs. Since $\mathcal{H} \subseteq \mathcal{K}$ implies $\nu(\mathcal{H}) \subseteq \nu(\mathcal{K})$, the operator ν is monotone. On the other hand, since $\mathcal{H} \subseteq \mathcal{K}$ implies $\tau(\mathcal{K}) \subseteq \tau(\mathcal{H})$, the operator τ is antitone. The application of these operators can be seen in the following example.

Example 2.1. Let be $N = \{1,2,3\}$, consider the hypergraph given by $\mathcal{H} = \{\{1,2\},\{3\},\{1,2,3\}\}$. The families obtained from the operators defined above are the following.

N	123
$\mathcal{H}$	001
	110
	111

N	123
$\neg(\mathcal{H})$	000
	001
	110

N	123
$\mu(\mathcal{H})$	001
	110

N	123
$\nu(\mathcal{H})$	001
	011
	101
	110
	111

N	123
$\tau(\mathcal{H})$	011
	101
	111

N	123
$\lambda(\mathcal{H})$	011
	101

where each row in the binary matrices represents a hyperedge.

Note that the operators ν and τ are closely related:

Lemma 2.1. [214] Given a hypergraph $\mathcal{H} \subseteq \mathcal{P}(N)$, then it holds that:

- $\nu(\mu(\mathcal{H})) = \nu(\mathcal{H})$,
- $\tau(\mu(\mathcal{H})) = \tau(\mathcal{H})$,
- $\tau(\nu(\mathcal{H})) = \tau(\mathcal{H})$,
- $\tau(\tau(\mathcal{H})) = \nu(\mathcal{H})$, and
- $\mathcal{P}(N) \setminus \nu(\mathcal{H}) = \neg(\tau(\mathcal{H}))$.

Finally, we define three properties for hypergraphs that will be helpful in Section 4.2.

Definition 2.8. A pair of hypergraphs $(\mathcal{H}, \mathcal{K})$ over the same ground set is:

- *coherent*, if $\nu(\mathcal{H}) \subseteq \tau(\mathcal{K})$,
- *complete*, if $\nu(\mathcal{H}) \supseteq \tau(\mathcal{K})$, and
- *dual*, if it is both coherent and complete, i.e. $\nu(\mathcal{H}) = \tau(\mathcal{K})$.

2.2 Computational Complexity

In this section we recall the basic definitions and tools from [51, 207].

The *computational complexity theory* is a branch of theoretical computer science and mathematics focused on classifying computational problems according to their inherent difficulty. Thus, the problems that can be solved by algorithms within a given resource are collected together in a specific complexity class.

We usually distinguish between two kinds of complexity: *time complexity* and *space complexity*. More formally, a *complexity class* is a set of computational problems that can be solved by algorithms with times of execution—for the case of time complexity—or memory storage spaces—for the case of space complexity—upper bounded by a function family, usually in terms of the input size of the algorithm. From now on we call the computational problems just as "problems".

A standard way to describe computational complexity in terms of an upper bound is given by the big O notation.

Definition 2.9. Let g be a function defined on the natural numbers, we say that $O(g)$ is the set of functions $\{f(n) \mid \text{there exists positive constants } c \text{ and } n_0 \text{ such that } 0 \leq f(n) \leq cg(n) \text{ for all } n \geq n_0\}$.

Unless we say otherwise, we refer always to the time complexity. The most common time complexity classes refer to *decision problems*, i.e., problems that can be answered with a "yes" or "no".

Definition 2.10. The class P contains all the decision problems that can be solved in time which is polynomial in the size of the input. The class NP contains all the decision problems for which membership has a polynomial-time verifiable certificate. The class coNP contains all the decision problems whose complements are in NP. The class EXP contains all the decision problems that can be solved in time which is exponential in the size of the input. We denote sEXP as the *strict exponential class*, i.e., the class of all the decision problems which belong to EXP but cannot be solved in sub-exponential time in the size of the input. We denote QP as the class of all the decision problems that can be solved in quasi-polynomial—sub-exponential but super-polynomial—time, i.e., by functions like $n^{\log O(n)}$, where n is the size of the input.

To show that a decision problem belongs to the class P, it suffices to find a polynomial time algorithm that solves the problem. On the other hand, there exist other classes of decision problems known as “hard problems”, for which we need to define before what is a polynomial-time reduction. It is well known that a decision problem A can be encoded as a string of binary numbers, so that $A \subseteq \{0,1\}^*$.

Definition 2.11. A *polynomial-time reduction* from a decision problem $A \subseteq \{0,1\}^*$ to a decision problem $B \subseteq \{0,1\}^*$ is a polynomial-time computable function $f : \{0,1\}^* \to \{0,1\}^*$ such that, for every instance $x \in \{0,1\}^*$, $x \in A$ if and only if $f(x) \in B$.

Consider two decision problems A and B, a polynomial-time reduction f from A to B and an algorithm to solve B. If there exists a polynomial-time reduction from A to B, then an instance x of A can be solved with the algorithm that solves B, with the input given by the instance $f(x)$.

Now we can define the remaining most common time complexity classes.

Definition 2.12. The class NP-hard—in short, NPH—contains all the decision problems such that any problem in NP is polynomial time reducible to them. The class NP-complete—in short, NPC—contains all the decision problems that belong to NP and NP-hard.

Furthermore, the classes coNP, coNP-hard—in short, coNPH—and coNP-complete—in short, coNPC—contain respectively all the decision problems whose complements are in NP, NP-hard and NP-complete.

The NP-complete problem by antonomasia is the Satisfiability problem (SAT), for which no one knows if there is an algorithm able to solve it in polynomial time. The most accepted conjecture is that P $\neq$ NP, so the answer would be “no”. The membership of SAT in the class NP-complete was established by Cook in 1971 [50]. All other NP-complete problems can be polynomially reduced from SAT. Two lists of several NP-complete problems obtained through polynomial-time reductions from other NP-complete problems are presented in [133, 95].

An additional interesting concept is given by the problems that can be solved by an algorithm that runs in pseudo-polynomial time [95].

Definition 2.13. An algorithm runs in *pseudo-polynomial time* if its running time is polynomial in the numeric value of the input.

Observe that this numeric value can be exponential in the length of the input, that is given by the number of its digits. Further, there are several computational problems that require a set of values to solve them, instead of decision problems that just need a binary answer.

An *enumeration problem* is a problem which can be solved by an algorithm that in the output set determines successively each one of the possible solutions. The following definitions come from [129].

Definition 2.14. An enumeration problem can be solved with *incremental polynomial time* if there exists an algorithm so that given an input and a set of members of the output, find another member of the output, or determine that none exists, can be done in time polynomial in the combined sizes of the input and the given members of the output. Further, an enumeration problem can be solved with *polynomial-delay* if there exists an algorithm that returns the members of the output in some order, and such that the delay between any two consecutive members is bounded by a polynomial in the input size.

We denote Pd to the class of enumerating problems that can be solved with polynomial-delay. Note that an algorithm that runs with polynomial-delay implies that it runs in incremental polynomial time.

A *counting problem* refers to find the number of solutions of an enumerating problem. An algorithm that solves a counting problem returns a natural number. The most representative complexity classes for this kind of problems are the following.

Definition 2.15. The class #P or *sharp*-P contains all the counting problems such that the objects being counted can be verified in polynomial time. The class #P-hard—in short, #PH—contains all the counting problems such that any problem in #P is polynomial time reducible to them. The class #P-complete—in short, #PC—contains all the counting problems that belong to #P and #P-hard.

In Section 2.3.4 we introduce the enumeration and counting problems specifically related to simple games.

Regarding space complexity, there are complexity classes which are analogous than for time complexity.

Definition 2.16. The class PSPACE contains all the decision problems that can be solved by using a polynomial amount of space. The class EXPSPACE

Complexity class	**Description**
	Decision problems
P	Polynomial time solvable.
QP	Quasi-polynomial time solvable.
EXP	Exponential time solvable.
sEXP	Exponential but not sub-exponential time solvable.
NP	Non-deterministic polynomial time solvable.
NP-hard (NPH)	All NP problems are polynomial time reducible to it.
NP-complete (NPC)	In NP and NP-hard.
coNP	Its complement is in NP.
coNP-complete (coNPC)	Its complement is in NP-complete.
PSPACE	Polynomial amount of space solvable.
EXPSPACE	Exponential amount of space solvable.
	Enumeration problems
Pd	Polynomial-delay solvable.
	Counting problems
#P	Counting solutions of NP problems.
#P-hard (#PH)	All #P problems are polynomial time reducible to it.
#P-complete (#PC)	In #P and #P-hard.

Table 2.1: A list of complexity classes.

contains all decision problems that can be solved by using an exponential amount of space.

A succinct description of the previous complexity classes are summarize on Table 2.1.

We finish this section with a list of NP-hard problems that we use in the thesis for polynomial-time reductions [95].

Name: Vertex Cover
Input: Undirected graph $G = (V, E)$ and an integer k.
Question: Is there a set $X \subseteq V$ with $|X| \leq k$ such that each edge in G has at least one vertex in X?

Name: Set Cover
Input: Finite set S, a collection of subsets $C \subseteq S$, and an integer k.
Question: Is there a subset $C' \subseteq C$ with $|C'| \leq k$ such that

every element in S belongs to at least one member of C'?

Name: SET PACKING
Input: Collection C of finite sets, and an integer k.
Question: Is there a collection of disjoint sets $C' \subseteq C$ with $|C'| \geq k$?

Name: KNAPSACK
Input: Finite set S of items i with weights w_i, profits n_i; an integer k.
Question: Is there a set $C \subseteq S$ with $\sum_{i \in C} w_i \leq k$ such that the value $\sum_{i \in C} n_i$ is maximized?

It is known that using dynamic programming the KNAPSACK problem can be solved in pseudo-polynomial time [135].

2.3 Simple Games

Simple games can be defined by using the notion of hypergraph. We follow definitions and notations from [251].

Definition 2.17. Let N be a finite set, a *simple game* is a monotone hypergraph $\mathcal{W}$ over N, i.e., such that for all $X \in \mathcal{W}$, if $X \subseteq Z$, then $Z \in \mathcal{W}$.

The set N is known as the *grand coalition*. Let $\mathcal{W}$ be the set of all winning coalitions and $\mathcal{L} = \neg\mathcal{W} = \{X \subseteq N \mid X \notin \mathcal{W}\}$ be the set of all losing coalitions, we usually denote a simple game as a pair $\Gamma = (N, \mathcal{W})$.

As a subclass of cooperative games, the simple games satisfy another classical definition. For cooperative games, we use notation from [44, 10].

Definition 2.18. A *cooperative game* is a pair (N, ν), where N is a set of players and $\nu : \mathcal{P}(N) \to \mathbb{R}$ is the *characteristic/valuation function* of the game, that associates, for each *coalition* $X \subseteq N$, a payoff $\nu(X)$ which the coalition members may distribute among themselves.

Note that the characteristic function ν should not be confused with the operator $\nu(\mathcal{H})$ for a hypergraph $\mathcal{H}$. The clutter of a hypergraph is only used in this thesis in Section 4.2, and both concepts are clearly distinguished depending on the context.

By using cooperative games, a simple game can also be defined as follows.

Definition 2.19. A *simple game* is a cooperative game $\Gamma = (N, \nu)$ such that $\nu : \mathcal{P}(N) \to \{0, 1\}$, $\nu(\emptyset) = 0$, $\nu(N) = 1$ and ν is monotonic, i.e., so that $\nu(X) \leq \nu(Y)$ whenever $X \subseteq Y$. A coalition $X \subseteq N$ is *winning* whether $\nu(X) = 1$ and *losing* whether $\nu(X) = 0$.

We say that a simple game $(N, \mathcal{W})$ is given in extended winning representation form. However, there are many other forms of representations, as we shall see in Chapter 3. To study the complexity of a computational problem, we need to specify the form of representation on which the simple game is given. Nevertheless, for the most definitions regarding simple games, we use this form of representation, assuming that it is a generic form, and also that the form of representation is independent of what is being defined. We will use simple games as a pair (N, ν) only in the context of solution concepts, as in Section 2.3.2, because solution concepts comes from cooperative games.

It is clear that $\mathcal{P}(N)$ can be partitioned into $\mathcal{W}$ and $\mathcal{L}$, so that $X \in \mathcal{W}$ if and only if $X \notin \mathcal{L}$. Further, note that Definition 2.19 excludes the games (N, ν) with $\nu(\emptyset) = 1$ or $\nu(N) = 0$. Indeed, these games are considered trivial, because by monotonicity, all the coalitions of the first one are winning, and all the coalitions of the second one are losing. Several authors exclude these trivial games to avoid further difficulties in some proofs. We maintain this convention, although in most of the results there is no problem accepting the trivial games.

Note that simple games can also be defined through cooperative games by determining a threshold $q \in \mathbb{R}$, so that a coalition $X \subseteq N$ is winning whenever $\nu(X) \geq q$, and losing whenever $\nu(X) < q$. In this sense, the simple games are also known as *threshold games* [10].

To finish this section we define three operations on simple games: duality, union and intersection.

Like hypergraphs or Boolean functions, every simple game has a dual. The dual of a simple game is another simple game—or sometimes even the same—which is obtained by an involution operation. If we apply the same operation again over the new game, we obtain the original one.

Definition 2.20. Let $\Gamma = (N, \mathcal{W})$ be a simple game, its *dual* is the simple game $\Gamma^d = (N, \mathcal{W}^d)$, where for all $X \in \mathcal{W}^d$, $N \setminus X \in \mathcal{L}$.

Note that for every simple game Γ, it is clear that $(\Gamma^d)^d = \Gamma$.

For the remaining definitions, we need the following result.

Lemma 2.2. Let $\Gamma_1 = (N, \mathcal{W}_1)$ and $\Gamma_2 = (N, \mathcal{W}_2)$ be two simple games, both $(N, \mathcal{W}_1 \cup \mathcal{W}_2)$ and $(N, \mathcal{W}_1 \cap \mathcal{W}_2)$ are also simple games.

Proof. Let be $X \in \mathcal{W}_1 \cup \mathcal{W}_2$, it means that $X \in \mathcal{W}_1$ or $X \in \mathcal{W}_2$. If $X \subseteq Z$, then $Z \in \mathcal{W}_1$—by monotonicity of Γ_1—or $Z \in \mathcal{W}_2$—by monotonicity of Γ_2— so then $Z \in \mathcal{W}_1 \cup \mathcal{W}_2$ and $(N, \mathcal{W}_1 \cup \mathcal{W}_2)$ is a simple game.

Let be $X \in \mathcal{W}_1 \cap \mathcal{W}_2$, it means that $X \in \mathcal{W}_1$ and $X \in \mathcal{W}_2$. If $X \subseteq Z$, then $Z \in \mathcal{W}_1$ and $Z \in \mathcal{W}_2$—by monotonicity of Γ_1 and Γ_2, respectively—so then $Z \in \mathcal{W}_1 \cap \mathcal{W}_2$ and $(N, \mathcal{W}_1 \cap \mathcal{W}_2)$ is a simple game. □

From the above we can define the following.

Definition 2.21. The *intersection* of two simple games is the simple game where a coalition wins if and only if it wins in both games. In a similar way, the *union* of two simple games is the simple game where a coalition wins if and only if it wins in at least one of the two games.

2.3.1 Properties and Parameters

Now we discuss several relevant properties and parameters of simple games. We start with well known properties regarding to coalitions.

Definition 2.22. Let Γ be a simple game. A winning coalition is *minimal* if removing any of its players we obtain a losing coalition. A losing coalition is *maximal* if adding any player on it we obtain a winning coalition. A coalition $X \subseteq N$ is a *blocking coalition* if $N \setminus X$ is losing, and it is a *swing* if there exists a player $i \in X$ such that i is *critical*, i.e., $X \in \mathcal{W}$ and $X \setminus \{i\} \in \mathcal{L}$.

We denote as $\mathcal{W}^m = \{X \in \mathcal{W} \mid \text{ for all } Z \in \mathcal{W}, Z \not\subset X\}$ the set of *minimal winning coalitions* (MWCs), and as $\mathcal{L}^M = \{Y \in \mathcal{L} \mid \text{ for all } Z \in \mathcal{L}, Y \not\subset Z\}$ the set of *maximal losing coalitions*. We denote the set of swings for a player i as $S_i = \{X \in \mathcal{W} \mid X \setminus \{i\} \in \mathcal{L}\}$. Furthermore, we use $\mathcal{W}(\Gamma)$, $\mathcal{L}(\Gamma)$, $\mathcal{W}^m(\Gamma)$ and $\mathcal{L}^M(\Gamma)$ to denote the set of winning, losing, minimal winning and maximal losing coalitions of a simple game Γ, and simply $\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$ and $\mathcal{L}^M$ when there is no risk of ambiguity.

The sets $\mathcal{W}^m$ and $\mathcal{L}^M$ are useful to represent simple games in more succinct ways [258]. The blocking property was firstly defined in 1956 by Richardson [225], as a way to simplify the notation of simple games given in 1953 by Gillies [97]. The swing is a property which emerged from the definition of critical player, described at least since 1965 by Banzhaf [16].

Regarding simple games as a whole, the main properties are as follows.

Definition 2.23. A simple game $(N, \mathcal{W})$ is:

- *proper*, if for all $X \subseteq N$, $X \in \mathcal{W}$ implies $N \setminus X \in \mathcal{L}$, i.e., every winning coalition is a blocking;
- *strong*, if for all $X \subseteq N$, $X \in \mathcal{L}$ implies $N \setminus X \in \mathcal{W}$;
- *decisive*, if it is both strong and proper, i.e., $X \in \mathcal{W}$ iff $N \setminus X \in \mathcal{L}$.

Further, a simple game is *dual-comparable* if it is either proper or strong, *improper* if it is not proper, and *weak* if it is not strong.

Proper (simple) games are also known as *supperaditive* [204] or *coherent* [214]; strong games [258, 225, 123] as *complete* [214]; and decisive games as *constant-sum* [258], *zero-sum* [258] or *self-dual* [235]. The decisive property has been considered for simple games from its origins. However, in other topics such as Boolean functions and Boolean logic, the self-duality is even older, being defined in 1921 by Post [217]. Dual-comparability was firstly studied in 1961 by Muroga et al. [189].

These properties are closely related to many computational problems, such as the *dualization problem* of conjunctive normal forms (CNF) in logic and Boolean functions, the problem of computing the *minimal transversals* or *hitting sets* of a given hypergraph [18], the problem of computing the *maximal independent sets* of a hypergraph [151, 129], among others [71]. They are also of special interest for voting systems [251]. In particular, decisive games have applications in several areas, such as interactive decision making, distributed computing, logic and linear programming, category theory, social science, hypergraph theory and reliability theory [157]. Furthermore, in [251] we read:

> *"properness rules out the possibility of disjoint winning coalitions (...) while strongness rules out the possibility of two losing coalitions whose union is N (...) Some authors who view simple games as models of voting systems have little interest in simple games that are not proper. Their argument is that disjoint winning coalitions can allow contradictory decisions to be made by the voting body. (...) A less vigorous argument is sometimes*

> *raised against games that are not strong, and thus, the argument goes, leave some issues unresolved. Ramamurthy (1990) refers to 'the paralysis that may result from allowing a losing coalition to obstruct a decision.'"*

Note that Γ is proper (strong) if and only if Γ^d is strong (proper), and it is decisive if and only if $\Gamma = \Gamma^d$.

Following with the definitions, now we describe the main properties focused in players of simple games.

Definition 2.24. Let $(N, \mathcal{W})$ be a simple game and $i \in N$ a player:

- i is a *dummy* if $i \in X$ implies $X \notin \mathcal{W}^m$;
- i is a *passer* if $i \in X$ implies $X \in \mathcal{W}$, i.e. $\{i\} \in \mathcal{W}$;
- i is a *vetoer* if $i \notin X$ implies $X \notin \mathcal{W}$, i.e. $N \setminus \{i\} \in \mathcal{L}$;
- i is a *dictator* if $i \in X$ iff $X \in \mathcal{W}$, i.e. i is both passer and vetoer.
- Given a coalition $X \subseteq N$, i is *critical* in X if $X \in \mathcal{W}$ and $X \setminus \{i\} \in \mathcal{L}$.
- Given another player $j \in N$, i and j are *symmetric* if for all $X \subseteq N \setminus \{i, j\}$, $X \cup \{i\} \in \mathcal{W}$ if and only if $X \cup \{j\} \in \mathcal{W}$.

Dummies were defined in 1944 by von Neumann and Morgenstern [258] and the symmetry of players in 1966 by Maschler and Peleg [164]. Critical players, also known as *swing players* are used at least since 1965 by Banzhaf [16] to describe his popular power index, used to measure the voting power of players in a simple game. Passers, vetoers or *veto players* and dictators are usually used in problems related with solution concepts. To read about the Banzhaf index and other solution concepts, see Section 2.3.2.

It is easy to see that when there is a passer, then no other player can be a vetoer, and vice versa. Therefore, a simple game may have at most one dictator, and simple games with more than one passer or vetoer do not have a dictator. We can also relate these three kind of players with the properties of Definition 2.23.

Proposition 2.1. Let $\Gamma = (N, \mathcal{W})$ be a simple game. If a player $i \in N$ is:

- vetoer but not dictator, then Γ is proper and weak;

- passer but not dictator, then Γ is strong and improper;
- dictator, then Γ is decisive.

Proof. If i is a vetoer, then $X = N \setminus \{i\} \in \mathcal{L}$, so for all $Y \in \mathcal{W}$, it holds $i \in \mathcal{W}$, and the complement $N \setminus Y \subseteq X$. Therefore, by monotonicity, $N \setminus Y \in \mathcal{L}$, so the game is proper.

If i is a passer, then $X = \{i\} \in \mathcal{W}$, so for all $Y \in \mathcal{L}$, it holds $i \notin \mathcal{L}$, and $X \subseteq N \setminus Y$. Therefore, by monotonicity, $N \setminus Y \in \mathcal{W}$, so the game is strong.

If i is vetoer and passer, then for all $X \in \mathcal{W}$ it holds $i \in X$, and for all $Y \in \mathcal{L}$ it holds $i \notin Y$, which means that $|\mathcal{W}| = |\mathcal{L}|$, i.e., the game is decisive. □

Besides the properties, there are some parameters which give interesting information about simple games. The parameters can be useful, for instance, to measure the inputs of the algorithms and thus to deal with their computational complexities.

Definition 2.25. Let $\Gamma = (N, \mathcal{W})$ be a simple game:

- The *size* of Γ is $|\Gamma| = |N| \cdot |\mathcal{W}| = n \cdot |\mathcal{W}| \in \mathbb{N}$.
- The *length* of Γ is $length(\Gamma) = \min\{|X| \mid X \in \mathcal{W}\}$.
- The *width* of Γ is $width(\Gamma) = \min\{|X| \mid N \setminus X \in \mathcal{L}\}$.
- The *Nakamura number* of Γ is $\min\{|\mathcal{W}'| \mid \mathcal{W}' \subseteq \mathcal{W}, \bigcap \mathcal{W}' = \emptyset\}$.
- The *Coleman's power* of Γ is $|\mathcal{W}|/2^n$.
- The *Chow parameters* of Γ are the members of the vector $(\omega_1, \ldots, \omega_n, \omega)$, where for all $i \in N$, $\omega_i = |\mathcal{W}_i| = |\{X \in \mathcal{W} \mid i \in X\}|$ and $\omega = |\mathcal{W}|$.

The size of a simple game corresponds to a bound on the description of the game. Therefore, it depends of how the game is represented. This is discussed in detail in Chapter 3. The length and the width were firstly defined in 1990 by Ramamurthy [221] as indicators of efficiency for decision making. The Nakamura number was used in 1979 by Nakamura [192] to prove that the rationality of collective choice critically depends on the number of alternatives [131]. The Coleman's power, also known as the "power of collectivity to act", was introduced in 1971 by Coleman [49] as a way to represent the ease with which a decision can be made. At last, Chow parameters, also known as *degree sequence* [251], were introduced in 1961 by Chow [46] in the context of threshold functions, and then used by Dubey and Shapley [66] to study some properties of the Banzhaf index.

Given a simple game Γ, note that the width of Γ is equal to the length of Γ^d, and the length of Γ is equal to the width of Γ^d [10].

Now we define isomorphism and equivalence for simple games.

Definition 2.26. Let $\Gamma = (N, \mathcal{W})$ and $\Gamma' = (N', \mathcal{W}')$ be two simple games with the same number of players. Γ and Γ' are *isomorphic* if and only if there exists a bijective function $\varphi : N \to N'$, such that for every coalition $X \subseteq N$, $X \in \mathcal{W}$ if and only if $\varphi(X) \in \mathcal{W}'$. Moreover, when $N = N'$ and φ is the identity function, then we say that both simple games are *equivalent*.

Finally, we include an example that explains some of the concepts described in this section.

Example 2.2. Consider the simple game $\Gamma = (N, \mathcal{W})$ with $N = \{a, b, c, d, e\}$ and $\mathcal{W} = \{\{a, b, c, d, e\}, \{a, b, c, d\}, \{a, b, c, e\}, \{a, b, d, e\}, \{a, b, e\}, \{a, c, d, e\}, \{a, c, d\}, \{a, c, e\}, \{a, d, e\}, \{b, c, d, e\}, \{b, c, d\}, \{b, c, e\}, \{b, d, e\}, \{c, d, e\}, \{c, e\}, \{d, e\}\}$. Hence,

$$\mathcal{W}^m = \{\{a, b, e\}, \{a, c, d\}, \{b, c, d\}, \{c, e\}, \{d, e\}\}$$

and

$$\mathcal{L}^M = \{\{a, b, c\}, \{a, b, d\}, \{a, e\}, \{b, e\}, \{c, d\}\}.$$

This game is decisive, and it doesn't have any dummy, passer, vetoer or dictator. By definition, each player is critical in the MWCs to which it belongs, and therefore, every MWC is a swing. Players a and b are symmetric. The size of the game can be given by $n \cdot |\mathcal{W}| = 5 \cdot 16 = 80$, but it also could be defined—as we show in Chapter 3—by $n \cdot |\mathcal{W}^m| = 5 \cdot 5 = 25$. The length is $|\{c, e\}| = |\{d, e\}| = 2$. Since every coalition $X \subseteq N$ with $|X| = n - 1$ is winning, the width cannot be equals 1, so it is equal to $|\{d, e\}| = 2$. For the Nakamura number, note that the intersection of any pair of MWCs is nonempty; but $\{d, e\} \cap \{c, e\} \cap \{b, c, d\} = \emptyset$, so the Nakamura number is 3. The Coleman's power is equal to $16/32 = 0.5$, and the Chow parameters are given by the vector $(9, 9, 11, 11, 13, 16)$.

2.3.2 Solution Concepts

Solution concepts come from cooperative game theory as a way to measure profit allocations of players, by considering the profit of each coalition. A solution concept assigns for each game a set of *payoffs* or *allocations*. According to [10], the biggest attention in the development of cooperative game theory has been to devise solution concepts to explain equilibrium in different systems, in the sense of the profit allocations of players vs. the profit of each coalition in which they belong. Since simple games are a family of cooperative games, the solution concepts can also be studied when the profit of each coalition is binary. While there are some solution concepts whose definition is simplified in the context of simple games, there are others specifically created for simple games, like the Banzhaf index. In this thesis we concentrate in solution concepts in the context of simple games.

In this section we use notation mainly from [44, 10].

Definition 2.27. Let (N, ν) be a simple game, a *payoff* is a vector of non-negative numbers $\mathbf{p} = (p_1, \ldots, p_n)$ such that for all $i \in N$, p_i is the payoff of player i. The payoff of a coalition $X \subseteq N$ is given by $p(X) = \sum_{i \in X} p_i$.

The *excess* of X under $\mathbf{p}$ is $e(\mathbf{p}, X) = p(X) - \nu(X)$ and the *excess vector* of $\mathbf{p}$ is $\mathbf{e}(\mathbf{p}) = (e(\mathbf{p}, X_1), \ldots, e(\mathbf{p}, X_{2^n}))$, where $X_1, \ldots, X_{2^n}$ is the list of all coalitions over N ordered so that $e(\mathbf{p}, X_1) \leq \ldots \leq e(\mathbf{p}, X_{2^n})$.

The *deficit* of X under $\mathbf{p}$ is $d(\mathbf{p}, X) = \nu(X) - p(X)$ and the *deficit vector* of $\mathbf{p}$ is $\mathbf{d}(\mathbf{p}) = (d(\mathbf{p}, X_1), \ldots, d(\mathbf{p}, X_{2^n}))$ so that $d(\mathbf{p}, X_1) \geq \ldots \geq d(\mathbf{p}, X_{2^n})$.

A payoff is:

- *efficient* if $p(N) = \nu(N)$;
- *individual rational* if $p_i \geq \nu(\{i\})$, for all $i \in N$.
- *homogeneous* if each player receives either a payoff 0 or a fixed amount $\frac{1}{r}$, where r is the number of players with payoff greater than zero.

A *preimputation* is an efficient payoff vector. An *imputation* is an individual rational preimputation. The set of all preimputations of a game Γ is denoted by $I^*(\Gamma)$, and the set of all imputations is denoted by $I(\Gamma)$.

For other properties of payoffs for solution concepts, see [10, 44]. Recall that for simple games $\nu(N) = 1$, hence the efficiency property implies that for these games every preimputation has $p(N) = 1$. Further, in simple games the presence of imputations depends of the passer players [10].

Proposition 2.2. Let $\Gamma = (N, \nu)$ be a simple game, $I(\Gamma) \neq \emptyset$ if and only if there is at most one passer in the game. Further, if there is only one passer $i \in N$, then $I(\Gamma) = \{\mathbf{p}\}$ with $p_i = 1$ and $p_j = 0$, for all $j \in N \setminus \{i\}$.

Proof. If $I(\Gamma)$ is nonempty, then there exists a payoff $\mathbf{p}$ with $p(N) = 1$ and $p_i \geq \nu(\{i\})$ for all $i \in N$. This holds if either $\nu(\{i\}) = 0$ for all $i \in N$, or there exists a unique $j \in N$ such that $\nu(\{j\}) = 1$, i.e., j is a passer. If $\nu(\{j\}) = 1$, then an imputation $\mathbf{p}$ should have $p_i = 1$ and also $p_j = 0$ for all $j \in N \setminus \{i\}$, because $p(N) = 1$. So this is the unique possible imputation.

Finally, if there is another passer h, then $\nu(\{h\}) = 1$; but if $p_h = 1$, then $p(N) \geq p_i + p_h > 1$, so $I(\Gamma)$ is empty. □

According to [74, 44], two of the most relevant criteria to define a solution concept are the *fairness*, i.e., how well each player's payoff reflects its contribution, and the *stability*, i.e., what are the incentives for the players to stay in a coalition.

The most common solution concepts based on the fairness criterion are the *power indices*. The most classic and popular power indices are the *Banzhaf index* and the *Shapley-Shubik index*. The first one comes from 1946 and it was firstly introduced by Penrose [213], being rediscovered in 1965 by Banzhaf [16] and in 1971 by Coleman [49]. That is why the Banzhaf index is also known as *Penrose index*, *Penrose-Banzhaf index* or *Banzhaf-Coleman index*. Another name for the Banzhaf index is *Dahlingham index*, proposed in [66] as a combination of results from Dahl [53] and Allingham [3].

The second one comes from 1953, when Shapley proposed the *Shapley value* [232], a way to distribute the payoffs on the coalitions of a cooperative game according to what would be the payoff for the grand coalition, emphasizing the fairness criterion. The power index restricted to simple games was defined the following year by Shapley and Shubik [233].

Intuitively, the Banzhaf index is the proportion of coalitions in which a player plays a critical role.

Definition 2.28. Let $\Gamma = (N, \nu)$ be a simple game and $i \in N$ a player. The *Banzhaf value* $\eta_i(\Gamma)$ is the number of coalitions in which i is critical, i.e., $\eta_i(\Gamma) = |S_i|$. The *probabilistic Banzhaf index* of player i in Γ is the proportion of coalitions in which player i is critical, i.e.,

$$\beta'_i(\Gamma) = \frac{\eta_i(\Gamma)}{2^{n-1}}.$$

Some authors refer the probabilistic Banzhaf index just like Banzhaf index [167]. It also measures the probability that a player i turns a losing coalition into a winning coalition when each one of the other players decides independently with probability $\frac{1}{2}$ whether to join the coalition or not. However, note that the outcomes of the probabilistic Banzhaf index may not be efficient. That is why we define the *Banzhaf index*, sometimes called *normalized Banzhaf index*, starting from the probabilistic Banzhaf index:

$$\beta_i(\Gamma) = \frac{\beta'_i(\Gamma)}{\sum_{i \in N} \beta'_i(\Gamma)} = \frac{\eta_i(\Gamma)}{\sum_{i \in N} \eta_i(\Gamma)}.$$

With these considerations, the Banzhaf index is efficient, in the sense that $\sum_{i=1}^{n} \beta_i(\Gamma) = 1$. Note that if player i is dummy, then $\beta_i(\Gamma) = 0$.

To define the Shapley-Shubik index, first consider an ordered set of players N and a permutation $\pi : N \to N$. We said that the $\pi(i)$-th player is a *pivotal* if the coalition $\{\pi(1), \ldots, \pi(i-1)\} \in \mathcal{L}$ and further $\{\pi(1), \ldots, \pi(i-1), \pi(i)\} \in \mathcal{W}$.

Definition 2.29. Let $\Gamma = (N, \nu)$ be a simple game and $i \in N$ a player. The *Shapley-Shubik value* $\kappa_i(\Gamma)$ is the number of permutations for which the player i is a pivotal, i.e., $\kappa_i(\Gamma) = \sum_{X \in S_i} (n - |X|)!\ (|X| - 1)!$ The *Shapley-Shubik index* of player i in Γ, denoted by $\Phi_i(\Gamma)$, is

$$\Phi_i(\Gamma) = \frac{\kappa_i(\Gamma)}{n!}.$$

Intuitively, the Shapley-Shubik index is the proportion of permutations for which the player i is pivotal. Thus, if the players join the coalition in a random order, this power index measures the probability that a player i turns a losing coalition into a winning coalition.

This solution concept is also efficient, because $\sum_{i=1}^{n} \Phi_i(\Gamma) = 1$. Further, if player i is dummy, then $\Phi_i(\Gamma) = 0$, and if two players i and j are symmetric, then $\Phi_i(\Gamma) = \Phi_j(\Gamma)$.

There are many other power indices in the literature. For instance, both the *Deegan-Packel index* [59] and the *Holler index* [119] are based on the

MWCs. The second one is similar to Banzhaf index, but only considering the swings that are MWCs. For an overview about them, see [81, 88, 4, 22].

Now we concentrate on the most common solution concepts based on the stability criterion.

The first solution concept for cooperative games was defined considering this criterion. It was introduced in 1944 by von Neumann and Morgenstern [258] and it is called *stable set* or *von Neumann-Morgenstern solution.*

Definition 2.30. Let $\Gamma = (N, \nu)$ be a cooperative game. An imputation $\mathbf{p}$ *dominates* an imputation $\mathbf{q}$, which we denote $\mathbf{p} >_{\text{dom}} \mathbf{q}$, if there is a nonempty coalition $X \subseteq N$ such that $p_i > q_i$ for all $i \in X$ and $p(X) \leq \nu(X)$. Two imputations can dominate each other. A *stable set* $\mathcal{S}(\Gamma)$ of the game is a set of imputations which satisfies the following two properties:

1. Internal stability: No imputation in the stable set is dominated by another imputation in the set, i.e., for all $\mathbf{p}, \mathbf{q} \in \mathcal{S}(\Gamma)$, $\mathbf{p} \not>_{\text{dom}} \mathbf{q}$.

2. External stability: All imputation outside the stable set are dominated by at least one imputation in the set, i.e., for all $\mathbf{q} \in I(\Gamma) \setminus \mathcal{S}(\Gamma)$, there exists some $\mathbf{p} \in \mathcal{S}(\Gamma)$ with $\mathbf{p} >_{\text{dom}} \mathbf{q}$.

Note that a stable set is formed by imputations that are "maximal" with respect to the dominance relation. Thus, no stable set is a subset of another stable set [10]. A stable set may or may not exist for cooperative games [159, 160], and when exists it is typically neither a singleton nor unique [158], so for the same game we could define several stable sets $\mathcal{S}_1(\Gamma)$, $\mathcal{S}_2(\Gamma)$, etc.

In simple games there exist stable sets that can be characterized by using the set of MWCs. There are no stable sets if and only if the set of imputations is empty. Besides, if there is at least one imputation, the number of stable sets of the simple game is greater or equal to its number of MWCs.

Proposition 2.3. Let $\Gamma = (N, \nu)$ be a simple game and $X \in \mathcal{W}^m$. Then $\mathcal{S}_X(\Gamma) = \{\mathbf{p} \in I(\Gamma) \mid p(X) = 1\}$ is a stable set of the game.

Proof. Given $X \in \mathcal{W}^m$, we start by showing that $\mathcal{S}_X(\Gamma)$ is internal stable. Note that for all $\mathbf{p} \in \mathcal{S}_X(\Gamma)$ and $i \in N \setminus X$, it holds $p_i = 0$. Now consider

$Y \subseteq N$ and $\mathbf{p}, \mathbf{q} \in \mathcal{S}_X(\Gamma)$. If $X \subseteq Y$, there exist $i, j \in X$ such that $p_i > q_i$ and $p_j < q_j$, so for this kind of Y, $\mathbf{p}$ does not dominates $\mathbf{q}$. If $Y \subset X$, then $\nu(Y) = 0$, so to dominate $\mathbf{q}$ the imputation $\mathbf{p}$ must be such that $p(Y) = 0$; but this means that $p_i \not> q_i$ for some $i \in Y$, and hence $\mathbf{p} \not>_{\text{dom}} \mathbf{q}$. Therefore, $\mathcal{S}_X(\Gamma)$ is internal stable.

For external stability, note that every $\mathbf{q} \in I(\Gamma) \setminus \mathcal{S}(\Gamma)$ has $q(N) = 1$ and $q_i > 0$ for some $i \in N \setminus X$. Therefore, there exists some $\mathbf{p} \in \mathcal{S}_X(\Gamma)$ so that $p(X) > q(X)$, and moreover, with $p_i > q_i$ for all $i \in X$. Since $p(x) = \nu(X) = 1$, $\mathcal{S}_X(\Gamma)$ is external stable. Thus, $\mathcal{S}_X(\Gamma)$ is a stable set. □

Note that this result is not a double implication, because there can be losing coalitions that are not comparable with MWCs. For instance, in our Example 2.2 the maximal losing coalition $\{a, b, c\}$ is not comparable to any MWC.

In 1959, Gillies [98] formalized in modern terms the notion of *core* of a cooperative game. The concept was introduced in 1881 by Edgeworth [68], and it was initially known in economics as *contract curve* [132]. Although in 1944 von Neumann and Morgenstern [258] considered the core as an interesting concept, their work was focused on games with empty core.

Definition 2.31. Let $\Gamma = (N, \nu)$ be a cooperative game, the *core* is

$$\mathcal{C}(\Gamma) = \{\mathbf{p} \in I(\Gamma) \mid \text{ for all } X \subseteq N, e(\mathbf{p}, X) \geq 0\},$$

i.e., the set of all the imputations which have a non-negative excess over any coalition.

The core is an appealing solution concept, since its imputations guarantee that each coalition obtains at least what it could gain on its own. It is known that for any cooperative game, if the core is nonempty, then it is contained in all the stable sets of the game [10]. Moreover, if the core is a stable set then it is the unique stable set. [65]. However, the core is empty for many games of interest. In particular, for simple games we have a well known result [74, 44]:[1]

[1] In [44] the authors warn that the second part of this result is only valid for proper games, but this is exactly the case when there is a veto player, as we saw in Proposition 2.1.

Proposition 2.4. Let $\Gamma = (N, \nu)$ be a simple game, it has a nonempty core if and only if at least one player is a vetoer. Moreover, if the core is nonempty then $\mathcal{C}(\Gamma) = \{\mathbf{p} \in I(\Gamma) \mid p(X) = 1 \text{ and } p_i = 0, \text{ for all } i \in N \text{ non-vetoer}\}$.

From the above and Proposition 2.3, it is easy to deduce what follows.

Proposition 2.5. Let $\Gamma = (N, \nu)$ be a simple game. If $|\mathcal{W}^m| = 1$ and the core is nonempty, then the core is the only stable set.

The usual emptiness of the core leads to the definition of the ϵ-*core* in 1966 [234] and the *least core* in 1979 [166] as a way to ensure nonempty outcomes. The idea is to relax the notion of the core by allowing a small error in the inequalities.

Definition 2.32. Let $\Gamma = (N, \nu)$ be a cooperative game, the ϵ-*core* is

$$\mathcal{C}_\epsilon(\Gamma) = \{\mathbf{p} \in I^*(\Gamma) \mid \text{ for all } X \subseteq N, d(\mathbf{p}, X) \leq \epsilon\},$$

i.e., the set of all the preimputations which have a deficit at most equal to a parameter ϵ over any coalition. The *least core* of Γ, denoted by $\mathcal{C}_{\epsilon_1}(\Gamma)$, is the nonempty ϵ-core of the game with the smallest ϵ value.

Note that for a least core imputation $\mathbf{p}$, ϵ_1 represents the best deficit—or the worst excess—of $\mathbf{p}$. Note also that if ϵ is large enough, e.g., $\epsilon \geq 1$, then the ϵ-core is guaranteed to be nonempty. By definition, the least core is the intersection of all the ϵ-cores; it is always nonempty [10], it is not unique and it may contain many payoffs [74].

For monotone cooperative games, and hence also for simple games, it is known that given a least core imputation, every player belongs to a coalition which gets the worst excess for that imputation [10]. Further, for simple games with no vetoers, i.e., with empty core, there is no player that belongs to any coalition which gets the worst excess for the imputation [10]. The least core is also related to the length of a simple game [10]:

Proposition 2.6. Let (N, ν) be a simple game, $\mathbf{p} \in \mathcal{C}_{\epsilon_1}$ with $p_1 \geq \ldots \geq p_n$ and $\epsilon_1(\Gamma)$ the best deficit of $\mathbf{p}$. Then:

$$\mathit{length}(\Gamma) \geq \frac{1 - \epsilon_1(\Gamma)}{p_1}.$$

Proof. Let $X \subseteq N$ be such that $|X| = length(\Gamma)$. Since for all $Y \subseteq N$, $\nu(Y) - p(Y) \leq \epsilon_1(\Gamma)$, then $1 - p(X) \leq \epsilon_1(\Gamma)$. The minimum possible value of p_1 is $p(X)/|X|$, so $|X| \cdot p_1 \geq 1 \geq p(X) \geq 1 - \epsilon_1(\Gamma)$, and therefore $|X| \geq (1 - \epsilon_1(\Gamma))/p_1$. □

Note that for the simple game with $\mathcal{W} = \mathcal{P}(N) \setminus \emptyset$, it holds that $length(\Gamma) = \frac{1-\epsilon_1(\Gamma)}{p_1} = 1$. Similarly, for the simple game with $\mathcal{W} = \{N\}$, it holds that $length(\Gamma) = \frac{1-\epsilon_1(\Gamma)}{p_1} = n$.

Shortly before the ϵ-core was defined, in 1964 Aumann and Maschler introduced the *bargaining set* [6], a solution concept similar to the core. There are several slightly different definition of this concept [7]. Here we use a version based on [10].

Definition 2.33. Let $\Gamma = (N, \nu)$ be a cooperative game, $\mathbf{p}, \mathbf{q}, \mathbf{r}$ payoff vectors, $X, Y \subseteq N$ and $i, j \in N$. A pair $(\mathbf{p}, X)$ is an *objection* to $\mathbf{q}$ of i against j if $i \in X$, $j \notin X$, $\mathbf{p}(X) = \nu(X)$ and $p_h > q_h$ for all $h \in X$. A pair $(\mathbf{r}, Y)$ is a *counter-objection* to the objection $(\mathbf{p}, X)$ of j against i if $j \in Y$, $i \notin Y$, $\mathbf{r}(Y) = \nu(Y)$, $r_h \geq q_h$ for all $h \in Y \setminus X$ and $r_h \geq p_h$ for all $h \in X \cap Y$. The *bargaining set* $\mathcal{B}(\Gamma)$ is the set of all the imputations $\mathbf{r}$ such that for any objection $(\mathbf{p}, X)$ to $\mathbf{r}$ of any player i against player j, there is a counter-objection to $(\mathbf{p}, X)$ of j against i.

For simple games, the bargaining set is equivalent to the core if it is nonempty. However, in some cases the core may be empty but the bargaining set nonempty [70].

One year later, in 1965 Davis and Maschler [55] introduced the *kernel*, a solution concept that represents all the imputations where players cannot demand a part of the payoffs of the other players. The kernel can be defined by using similar concepts to objections and counter-objections of the bargaining set.

Definition 2.34. Let $\Gamma = (N, \nu)$ be a cooperative game, $\mathbf{p}$ a payoff vector and $i, j \in N$. A coalition $X \subseteq N$ is a *kernel-objection* to $\mathbf{p}$ of i against j if $i \in X$, $j \notin X$ and $p_j > \nu(\{j\})$. A coalition $Y \subseteq N$ is a *kernel-counter-objection* to the objection X of i against j if $j \in Y$, $i \notin Y$ and $e(\mathbf{p}, Y) \leq e(\mathbf{p}, X)$. Thus, the *kernel* $\mathcal{K}(\Gamma)$ is the set of all the imputations $\mathbf{p}$ such that for every kernel-objection X to $\mathbf{p}$ of any i against any other j, there is a kernel-counter-objection to X of j against i. Alternatively, the kernel can also be defined as follows.

The *maximum surplus* $s^{\nu}_{ij}(\mathbf{p})$ of player i over player j with respect to a payoff vector $\mathbf{p}$ is

$$s^{\nu}_{ij}(\mathbf{p}) = \max\{d(\mathbf{p}, X) \mid X \subseteq N \setminus \{j\}, i \in X\}.$$

Then the *kernel* of the game is the set of imputations $\mathbf{p} \in I(\Gamma)$ such that

$$(s^{\nu}_{ij}(\mathbf{p}) - s^{\nu}_{ji}(\mathbf{p}))(p_j - \nu(\{j\})) \leq 0$$

and

$$(s^{\nu}_{ji}(\mathbf{p}) - s^{\nu}_{ij}(\mathbf{p}))(p_i - \nu(\{i\})) \leq 0.$$

Intuitively, the surplus is a way to measure one player's bargaining power over another. It is known that the kernel is a subset of the bargaining set, and it is also nonempty [55].

An auxiliary solution concept related to the kernel and introduced in 1972 is the *prekernel* [165].

Definition 2.35. Let $\Gamma = (N, \nu)$ be a cooperative game, the *prekernel* $\mathcal{PK}(\Gamma)$ of the game is the set of preimputations $\mathbf{p} \in I^*(\Gamma)$ such that for all $i, j \in N$,

$$s^{\nu}_{ij}(\mathbf{p}) = s^{\nu}_{ji}(\mathbf{p}).$$

Note that for any cooperative game, the intersection among the prekernel and the set of imputations of the game is a subset of the kernel. However, the kernel is not a subset of the prekernel, and moreover, since the outcomes of the prekernel do not need to be individually rational, the prekernel is not a subset of the kernel [165]. Furthermore, both the prekernel preimputations and the kernel imputations inside any $\mathcal{C}_\epsilon$ coincide [165], i.e., those prekernel preimputations are imputations as well.

For simple games with no passers, symmetric players get both equal prekernel and kernel payoffs [10].

The last solution concept that we comment is the *nucleolus*, defined in 1969 by Schmeidler [231]—see also [166]—as a way to find fairest payoffs among the payoffs within the least core. In this sense, it can be thought as a refinement of the least core [44]. In some sense, it is the most stable payoff allocation scheme, and it is particularly desirable when the stability of the grand coalition is important [74, 231]. The nucleolus has also been used as an alternative of the power indices to measure power in voting systems [37].

Definition 2.36. Let (N, ν) be a cooperative game. The *nucleolus* $\mathcal{N}(\Gamma)$ is the imputation $\mathbf{p} \in I(\Gamma)$ with the lexicographically largest excess vector—or minimal deficit vector—on the game.

It is known that if the game has imputations, then the kernel always contains the nucleolus, and hence, it is guaranteed that is not empty. In fact, when $I(\Gamma) \neq \emptyset$ the nucleolus always exists and it returns a unique outcome for each game [231]. The nucleolus is in the least core, and if the core is nonempty, the nucleolus is also in the core [10].

For simple games, it is known that if a player i is a dummy, the nucleolus has a payoff $p_i = 0$, and moreover, $p_i = 0$ for all the kernel imputations [10]. Furthermore, if an imputation $\mathbf{p}$ is the nucleolus of a game and $e(\mathbf{p}, X)$ is the first element of the excess vector $\mathbf{e}(\mathbf{p})$, then for any player i there exists a coalition Y such that $i \in Y$ and $e(\mathbf{p}, X) = e(\mathbf{p}, Y)$ [10].

Like the prekernel for the kernel, we can also define the *prenucleolus*, which is also unique and it always exists [231].

Definition 2.37. Let (N, ν) be a cooperative game. The *prenucleolus* $\mathcal{PN}(\Gamma)$ is the preimputation $\mathbf{p} \in I^*(\Gamma)$ with the lexicographically largest excess vector on the game.

The prenucleolus always exists and it is unique while for every coalition $X \subseteq N$ with $|X| = 1$, $\nu(X) = 0$ [231]. Further, the prenucleolus is always contained in the least core [10].

To summarize, the inclusion relationships between the main solution concepts based on the stability criterion, are as follows, where $\mathcal{S}(\Gamma)$ denotes one of the many possible stable sets:

- $\mathcal{N}(\Gamma) \subseteq \mathcal{K}(\Gamma) \subseteq \mathcal{B}(\Gamma)$, $\mathcal{N}(\Gamma) \subseteq \mathcal{S}(\Gamma)$ and $\mathcal{N}(\Gamma) \subseteq \mathcal{C}_{\epsilon_1}(\Gamma) \subseteq \mathcal{C}_{\epsilon}(\Gamma)$.
- If $\mathcal{C}(\Gamma) \neq \emptyset$, then $\mathcal{N}(\Gamma) \subseteq \mathcal{C}(\Gamma) \subseteq \mathcal{S}(\Gamma)$.
- If $\mathcal{C}(\Gamma) \neq \emptyset$ and Γ is a simple game, then $\mathcal{C}(\Gamma) = \mathcal{B}(\Gamma)$.
- $\mathcal{PN}(\Gamma) \subseteq \mathcal{C}_{\epsilon}(\Gamma)$, $\mathcal{PK}(\Gamma) \cap I(\Gamma) \subseteq \mathcal{K}(\Gamma)$ and $\mathcal{PK}(\Gamma) \cap \mathcal{C}_{\epsilon}(\Gamma) = \mathcal{K}(\Gamma) \cap \mathcal{C}_{\epsilon}(\Gamma)$.

Therefore, if the core is nonempty, for simple games we have:

- $\mathcal{N}(\Gamma) \subseteq \mathcal{K}(\Gamma) \subseteq \mathcal{C}(\Gamma) = \mathcal{B}(\Gamma) \subseteq \mathcal{S}(\Gamma)$ and $\mathcal{N}(\Gamma) \subseteq \mathcal{C}_{\epsilon_1}(\Gamma) \subseteq \mathcal{C}_{\epsilon}(\Gamma)$.

For simple games with at least one imputation, $\mathcal{S}(\Gamma)$, $\mathcal{C}_{\epsilon_1}(\Gamma)$, $\mathcal{B}(\Gamma)$, $\mathcal{K}(\Gamma)$, $\mathcal{PK}(\Gamma)$, $\mathcal{N}(\Gamma)$ and $\mathcal{PN}(\Gamma)$ are always nonempty. In contrast, $\mathcal{C}(\Gamma)$ and $\mathcal{C}_{\epsilon}(\Gamma)$ can be empty.

We finish this section by applying the previous solution concepts in a specific simple game.

Example 2.3. Consider the same simple game of Example 2.2. The sets of swings for each player are $S_a = \{\{a,c,d\}\}$, $S_b = \{\{a,b,e\}, \{b,c,d\}\}$, $S_c = \{\{a,b,c,d\}, \{a,c,d\}, \{a,c,e\}, \{b,c,d\}, \{b,c,e\}, \{c,e\}\}$, $S_d = \{\{a,b,c,d\}, \{a,c,d\}, \{a,d,e\}, \{b,c,d\}, \{b,d,e\}, \{d,e\}\}$ and $S_e = \{\{a,b,c,e\}, \{a,b,d,e\}, \{a,b,e\}, \{a,c,e\}, \{a,d,e\}, \{b,c,e\}, \{b,d,e\}, \{c,d,e\}, \{c,e\}, \{d,e\}\}$. Thus, the Banzhaf indices are given by $\beta_a(\Gamma) = \frac{1}{25} = 0.04$, $\beta_b(\Gamma) = \frac{2}{25} = 0.08$, $\beta_c(\Gamma) = \beta_d(\Gamma) = \frac{6}{25} = 0.24$ and $\beta_e(\Gamma) = \frac{10}{25} = 0.4$; and the Shapley-Shubik indices are given by $\Phi_a(\Gamma) = \frac{4}{120} = 0.033$, $\Phi_b(\Gamma) = \frac{8}{120} = 0.067$, $\Phi_c(\Gamma) = \Phi_d(\Gamma) = \frac{28}{120} = 0.233$ and $\Phi_e(\Gamma) = \frac{48}{120} = 0.4$.

For the remaining solution concepts, first note that there are no vetoers in the game, so the core is empty. Furthermore, since there are no passers in the game, $I(\Gamma) \neq \emptyset$, so the nucleolus exists and it belongs to all the remaining solution concepts.

By using for instance the MWC $X_1 = \{a,b,e\}$, we can define a stable set $\mathcal{S}_{X_1}(\Gamma) = \{(p_a, p_b, 0, 0, p_e) \mid p_a + p_b + p_e = 1\}$, in such a way that some valid imputations of the stable set are $(0.3, 0.3, 0, 0, 0.4)$, $(0.1, 0.2, 0, 0, 0.7)$, $(0.5, 0, 0, 0, 0.5)$, etc. Finally, it is possible to verify that the imputation $(\frac{1}{9}, \frac{1}{9}, \frac{2}{9}, \frac{2}{9}, \frac{1}{3})$ is the lexicographically largest excess vector on the game, so this is the nucleolus of the game. This is not obvious because it requires to solve a linear program. In Section 4.1.2 we explain in detail how it is computed the nucleolus in a simple game.

2.3.3 Regular and Weighted Games

There exist two particular subfamilies of simple games which are very important because of its many applications [258, 123, 187, 251], as well as the possibility for them to be represented in succinct forms of representation. That is why we introduce regular games and weighted games at the end of this chapter. Their succinct representations and the relationships between them are stated in Chapter 3.

Definition 2.38. Let $\preceq$ be a *desirability relation* on a set of players N, that linearly order the players by increasing power. Given $X \subseteq N$, an *increasing-shift* on X, specified by a pair $(i, j) \in N \setminus X \times X$ such that $j \prec i$, is an operation which returns $Z = X \setminus \{j\} \cup \{i\}$. Given two winning coalitions $Y, Z \in \mathcal{W}$, we said that $Y \preceq Z$ if and only if there exists a finite sequence of increasing-shifts on Y which produces another coalition $Y' \subseteq Z$.

Note that the desirability relation on coalitions is not linear [251]. This relation was firstly introduced for simple games in 1958 [124], and it is also known as *desirability order* [251], *dominance relation* [82] or *Winder order* [268] in threshold logic. The notion of desirability relation was generalized to cooperative games in 1966 [164].

Depending on the order of the set players, the increasing-shift operation is also known as *right-shift* or *left-shift* [202, 144, 43] This operation has also been studied in other contexts such as non-cooperative games [35], fair division [36] and Boolean functions [113].

Now we can define the regular games [209, 163].

Definition 2.39. Let $\Gamma = (N, \mathcal{W})$ be a simple game where N is ordered by a desirability relation $\preceq$, then Γ is a *regular game* with respect to $\preceq$ if for all $X \in \mathcal{W}$, every increasing-shift returns an element of $\mathcal{W}$; i.e. if replacing a member of a winning coalition $X \in \mathcal{W}$ by a more powerful one, always yields a winning coalition.

Analogously to the sets $\mathcal{W}^m$ and $\mathcal{L}^M$, we denote $\mathcal{W}^s = \{X \in \mathcal{W} \mid \text{for all } Z \in \mathcal{W}, Z \not\preceq X\}$ as the set of *shift-minimal winning coalitions*, and $\mathcal{L}^S = \{Y \in \mathcal{L} \mid \text{ for all } Z \in \mathcal{L}, Y \not\preceq Z\}$ as the set of *shift-maximal losing coalitions*.

Regular games have been used at least since 1966 to study the kernel of a game [164]. In the context of monotone Boolean functions, they are known as *regular functions* at least since 1969 [235], and have been sometimes also called *directed games* [144]. Outside the game theory, they have been used to solve other problems like the regular set-covering problem [209] and the problem of separating hyperplanes [69]. In monotone Boolean functions, the shift-minimal winning coalitions are related with the *shelters* [210].

Moreover, we have the following [251].

Definition 2.40. A simple game $\Gamma = (N, \mathcal{W})$ is *linear* if there exists a linear re-ordering of N, for which Γ becomes regular.

Linear games are also known as *complete games* [41], *ordered games* [246], *swap-robust games* [251] or *2-monotone Boolean functions* [268].

Now we introduce weighted games.

Definition 2.41. Let $\Gamma = (N, \mathcal{W})$ be a simple game, then Γ is a *weighted game* if there exists a *weight function* $w : N \to \mathbb{R}$ and a real *quota* $q \in \mathbb{R}$ such that for all $X \subseteq N$, $X \in \mathcal{W}$ if and only if $w(X) \geq q$; where $w(X) = \Sigma\{w(a) \mid a \in X\}$.

Weighted games are also called *weighted voting games* or *weighted majority games*. They were defined in the origins of game theory in 1944 [258], but similar ideas were used one year before to define the Threshold Logic Unit (TLU), the first artificial neuron [169]. Some years later, they were deeply studied in 1956 [123] in the context of simple game theory, and since then weighted games have also been studied in many different contexts under different names: *Linearly separated truth functions*, to contact and to rectifier nets [170]; *linearly separable switching functions* or *threshold Boolean functions*, to separate circuits in switching circuit theory and analyze the threshold synthesis problem [121]; *trade robustness*, for voting theory and trade exchanges [249]; or *threshold hypergraphs*, to synchronizing parallel processes [103, 224]. Sometimes, the inherent concept of weight function is changed by the one of *threshold criteria*.

According to Hu [121]—see also [87, 90]—the weight function can be replaced by integer non-negative weights with $0 \leq q \leq w(N)$.

Weighted games clearly are simple (monotone) games, and one can always linearly re-order the elements of N such that $w : N \to \mathbb{R}$ becomes monotone; i.e. for all $i, j \in N$ with $i \prec j$, $w(i) \leq w(j)$. And then clearly the game becomes regular. So all weighted games are regular. The converse, however, does not hold [187, 251].

Despite of the fact that weighted games are a strict subclass of simple games, it is interesting to remark a well known result which says that every simple game can be expressed as the intersection or the union of a finite number of weighted games. This allow us to define the following [251].

Definition 2.42. A *vector-weighted game* is a simple game formed by the intersection of a finite number of weighted games. In a similar way, we say that a *co-vector-weighted game* is a simple game formed by the union of a finite number of weighted games.

Vector-weighted games are also called *vector-weighted systems* [250], *multiple weighted games* [10], *weighted multiple majority games* [2] or by some other combinations of these words; in the context of hypergraphs, they are known as *threshold intersection dimensions* [266, 162] and in switching functions as *canonical conjunctive forms* [187].

This leads us to define the following.

Definition 2.43. The *dimension* (*codimension*) of a simple game is the minimum number of weighted games whose intersection (union) is equivalent to that simple game.

The expressiveness of vector-weighted games was firstly shown in [128] for hypergraphs, and then expressed for simple games in [250]. The concept of codimension and the equivalence between simple games and the union of a finite number of weighted games was introduced in [89]. We highlight these observations in the following theorem.

Theorem 2.1. [250, 89] Every simple game can be represented as a vector-weighted game, an vice versa. Every simple game can be represented as a co-vector-weighted game, an vice versa.

In the same vein, we shall see other expressiveness results for influence games in Section 5.2.

Moreover, several subclasses of weighted games have been defined [258, 10, 251]. One of the most studied [246] is the class of *homogeneous games*, which was firstly defined together with the class of weighted games [258].

Definition 2.44. An *homogeneous game* is a weighted game with weight function w and quota q such that for all $X \subseteq N$, $w(X) = q$.

Due to the importance of the decisive problem of Definition 2.23, the decisive weighted games have its own name [92].

Definition 2.45. A *majority game* is a decisive weighted game. We say that a *sub-majority game* is a strong weighted game.

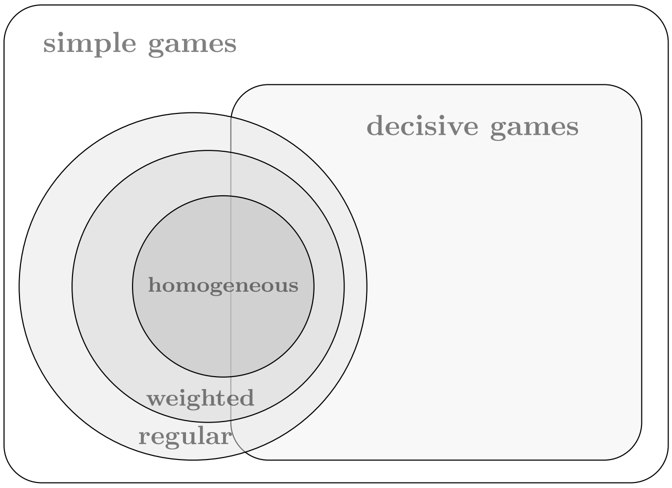


Figure 2.1: Inclusion relationship between subclasses of simple games.

Figure 2.1 illustrates the main subclasses of simple games considered in this chapter.

To finish with this section, note that every simple game that belongs to any of the subclasses defined in this section requires that the grand coalition be ordered. In most of this thesis we assume an increasing order given by $N = \{1, \ldots, n\}$; however, to simplify calculations we use in Sections 4.2.3 and 4.4.2 a decreasing order given by $N = \{n, \ldots, 1\}$. Nevertheless, the definitions and results apply to both orders.

2.3.4 Counting and Enumerating

In this section we introduce the problem of counting and enumerating subclasses of simple games.

In this thesis we concentrate especially on the subclasses presented in Section 2.3.3, considering also the properties of simple games given by Definition 2.23. We are not only interested in enumerating (and counting) simple, regular, weighted and homogeneous games, but we also pretend to enumerate (and count) these games that are decisive as well.

Both counting and enumeration algorithms of subclasses of simple games have usually a high complexity. The subclasses of our interest grow exponentially in function of n, so it is not possible to implement algorithms which run in polynomial time. However, we do not even know whether the considered counting problems are #P-hard, or if the enumeration problems are polynomial-delay solvable. Regardless the above, as we shall see in Section 4.4.1, some authors have designed algorithms that achieve a good performance for the cases that reach to solve.

The interest in counting the number of simple games begins long before the simple games were defined. According to [251], the first studies related to simple games have to do with the enumeration of free distributive lattices, started in 1897 by Dedekind [58]. It is well known that the number of simple games available for a given set of players coincides with the so called *Dedekind number*, and the problem of finding these numbers is called the *Dedekind's Problem* [274].

More than a half century later, in 1959 Isbell reactivated the interest on counting, but focusing in the family of majority games, i.e., decisive weighted games [125]. Since then, but especially with the rise of computer technology, many authors have worked in the enumeration of different subclasses of simple games, usually in the framework of monotone Boolean functions. In particular, Muroga et al. in 1970 provided several counting results for subclasses of monotone Boolean functions [190].

The known results for enumeration and counting are presented later in Section 4.4. In that Section we also present some approaches to the enumeration of decisive regular games.

Part II

Simple Games

Chapter 3

Representation and Conversion Problems

In this chapter we survey several standard forms of representation for simple games, regular games and weighted games. The forms of representation here considered are summarized in Table 3.1. Furthermore, an overview of the results of the conversion problem for these forms of representation is given in Tables 3.2–3.5. Results in bold face are new and question marks correspond to open problems. For subfamilies of simple games in Tables 3.4 and 3.5 the results are restricted to simple games in the considered subfamily. For these last two Tables, we shall see that the reversed conversion problems can be solved trivially in polynomial time.

In Section 3.1 we survey the main standard forms of representation for general simple games. Analogously, in Section 3.2 we present standard forms of representation for regular games, and in Section 3.3 the standard form of representation for weighted games. At the end of each section we show computational complexity results about the conversion problems among the different considered forms of representation.

3.1 Representations for Simple Games

First we turn our attention to the ways in which a simple game can be represented as the input to a problem.

Definition 3.1. Let $\mathcal{G}$ be a class of simple games, a *form of representation*

Simple games	
EWF	Extensive or explicit Winning Form
ELF	Extensive or explicit Losing Form
MWF	Extensive or explicit Minimal Winning Form
MLF	Extensive or explicit Maximal Losing Form
PCBF	Partially Condensed Binary Tree Form
BDDF	Binary Decision Diagram Form
VWRF	Vector Weighted Representation Form
coVWRF	co-vector Weighted Representation Form
Regular games	
FCBF	Fully Condensed Binary Tree Form
SWF	Shift-minimal Winning Form
Weighted games	
WRF	Weighted Representation Form

Table 3.1: Forms of representation considered in this chapter.

of $\mathcal{G}$ is a finite data structure which allows to represent each simple game $\Gamma \in \mathcal{G}$, as well as verifying in polynomial time if a given instance of the data structure represents a simple game Γ in $\mathcal{G}$ or not.

We use the notation $F(\Gamma)$ to denote a representation of simple game Γ in form F. As usual, $|F(\Gamma)|$ denotes the *size* of $F(\Gamma)$ that is the amount of bits needed to write down the data structure representing Γ.

Definition 3.2. Let F_1 and F_2 be two forms of representation for a class of simple games $\mathcal{G}$, the *conversion problem* from F_1 to F_2, denoted by $F_1 \rightsquigarrow F_2$, is the problem of computing $F_2(\Gamma)$ from $F_1(\Gamma)$, i.e.:

Name: $F_1 \rightsquigarrow F_2$
Input: Simple game Γ in F_1 form
Output: A representation of Γ in F_2 form.

It is well known that simple games can be described by *monotone Boolean functions* [251], also known as *positive Boolean function* [163], and therefore by several kinds of logical formulas [71].

Definition 3.3. A *monotone Boolean function* is a binary function $f : \{0,1\}^n \to \{0,1\}$ such that for all pair of vectors $v, w \in \{0,1\}^n$ with $v \le w$, it holds $f(v) \le f(w)$. A vector $x \in \{0,1\}^n$ is a *true vector* if $f(x) = 1$ and it is a *false vector* if $f(x) = 0$.

Output → Input ↓	EWF	MWF	ELF	MLF
EWF	–	P	EXP	P
MWF	sEXP **Pd**	–	EXP	EXP
ELF	EXP	EXP	–	P
MLF	sEXP	sEXP	sEXP **Pd**	–

Table 3.2: Computational complexity of the conversion problem from the row form to the column form, for representations of simple games based on explicit descriptions of set families.

Output → Input ↓	EWF	MWF	PCBF	BDDF
EWF	–	P	P	P
MWF	sEXP **Pd**	–	P	**P**
PCBF	sEXP **Pd**	P	–	**P**
BDDF	sEXP	sEXP	sEXP	–

Table 3.3: Computational complexity of the conversion problem from the row form to the column form, for representations of simple games based on variants of binary trees.

Observe that the true (false) vectors represent the winning (losing) coalitions of the corresponding simple game. We do not consider this succinct implicit representation and we refer to any of the references on Boolean formulas [262, 52].

Furthermore, there exist some forms of representation based on graphs that lead to the definition of subfamilies of simple games. These approaches are usually introduced in contexts of trading, management and flow interactions. Here we mention a few. *Vertex connectivity games* were motivated by the context of network reliability: Given a graph in which the set of vertices V is partitioned on three subsets (V_p, V_b, V_s), where V_s is the set of players of the game, a coalition $S \subseteq V_s$ is winning if and only if $S \cup V_b \cup V_p$ *fully connects* V_p, i.e., if for all vertices $u, v \in V_p$, there exists a path from u until v [13]. A simplification of *flow games*—a model of cooperative games based

Output → Input ↓	EWF	MWF	PCBF	BDDF	FCBF	SWF
FCBF	sEXP **Pd**	P	P	**P**	–	P
SWF	sEXP	sEXP **Pd**	sEXP	sEXP?	sEXP	–

Table 3.4: Computational complexity of the conversion problem from the row form to the column form for representations of regular games.

Output → Input ↓	EWF	MWF	PCBF	BDDF	FCBF	SWF
WRF	sEXP **Pd**	sEXP Pd	sEXP	sEXP	sEXP **Pd**	sEXP

Table 3.5: Computational complexity of the conversion problem from the row form to the column form for representations of weighted games.

on flow networks, which shortly are directed graphs with positive labels on the edges [130]—called *connectivity games* is a model where every edge has the same capacity 1, and the flow has a unitary value, in such a way that winning coalitions correspond to paths from the source to the sink, and losing coalitions are those subset of edges that do not induce a source-sink path [14]. In *spanning connectivity games* the players are the edges of an undirected weighted multigraph so that a coalition is winning if and only if the edges in the coalition constitute a connected spanning subgraph [11]. A *threshold variant* of *shortest path games*—given a flow network with various "sources" and "sinks", every shortest path $P = (v_1, \ldots, v_m)$ through the edges $(e_1, \ldots, e_{m-1})$ is a coalition with value $\sum_{i=1}^{m-1} w_i - m$ if v_1 is a source and v_m is a sink; or 0, otherwise [83, 259]—is a network with a threshold T such that a coalition is winning if and only if $\sum_{i=1}^{m-1} w_i - m \geq T$ [195]. Usually, these representations are related to cooperative games and do not consider the whole class of simple games. In Chapter 5 we introduce *influence games*, and we show that this class capture the whole class of simple games. It remains open to study the conversion problems related to these forms of representations based on graphs.

We finish this section with the notion of reasonable representation, that we use in the following chapters.

Definition 3.4. Let $\Gamma = (N, \mathcal{W})$ be a simple game, consider the game $\Gamma' = (N \cup \{x\}, \mathcal{W}')$, where x is a new player and $\mathcal{W}' = \{S \cup \{x\} \mid S \in \mathcal{W}\}$. A representation is *reasonable* if a representation of the game Γ' can be computed with only polynomial blow-up with respect to a given representation of the game Γ.

3.1.1 Explicit Set Families and Incidence Vectors

In this section, we describe several usual forms of representation for simple games corresponding to standard explicit forms of representation of set families [258]. We use matrix notation as an explicit representation of sets. This form of representation is mostly used in the context of hypergraphs and monotone Boolean functions [144, 71, 214]. Thus, each coalition $X \in \mathcal{P}(N)$ is represented by its incidence vector $x \in \{0,1\}^n$. For each player $i \in N$, x_i or $x(i)$ is a component of x. If $x_i = 1$ (resp. $x_i = 0$), then x_i is called a 1-component (resp. 0-component) of x. Given $x, y \in \{0,1\}^n$, recall that in lexicographic order $x \leq y$ if and only if either $x < y$ or $x = y$ if and only if $x_i \leq y_i$, for each $i \in N$ and considering as usual $0 < 1$.

We start with the earliest forms of representation defined by von Neumann and Morgenstern [258].

Definition 3.5. A simple game Γ is given in:

- *(Extensive or explicit) winning form (EWF)* as a pair $(N, \mathcal{W})$, where N is its set of players, and $\mathcal{W}$ its set of winning coalitions.
- *(Extensive or explicit) minimal winning form (MWF)* as a pair $(N, \mathcal{W}^m)$, where $\mathcal{W}^m$ is its set of MWCs.

Observe that both forms of representations are valid since, given a subset family, we can check in polynomial time whether it is monotonic and whether it is minimal. Indeed, to check minimality we can just test whether removing one of the elements of the coalition leads to a winning coalition.

Assuming that each set is represented by its incidence vector, we can represent a simple game by an incidence matrix with one row for each winning coalition (or each MWC) and a column for each player.

Example 3.1. Consider the simple game $\Gamma = (N, \mathcal{W})$ of Example 2.2. In matrix notation, $(N, \mathcal{W}^m)$ is the following:

N	$abcde$
	00011
	00101
$\mathcal{W}^m$	01110
	10110
	11001

Assuming a matrix representation, we have that $|\text{EWF}(\Gamma)| = |N| \cdot |\mathcal{W}| = n \cdot |\mathcal{W}| \in \mathbb{N}$ and $|\text{MWF}(\Gamma)| = n \cdot |\mathcal{W}^m|$. For any simple game Γ, $\mathcal{W}^m \subseteq \mathcal{W}$. Therefore $|\text{EWF}(\Gamma)| \leq |\text{MWF}(\Gamma)|$. Moreover, if $\mathcal{W} \neq \emptyset$ and $\mathcal{W} \neq \{N\}$, then we have a strict containment $\mathcal{W}^m \subset \mathcal{W}$. In the following example we provide a simple game whose representation in MWF is exponentially smaller than in EWF.

Example 3.2. Let Γ be a simple game which contains the empty coalition as winning coalition, i.e., such that all players are dummies. Hence, $\emptyset$ is the unique MWC, so that $|\mathcal{W}^m| = 1$, but the number of winning coalitions will be $|\mathcal{W}| = 2^n$, which is exponential in terms of n.

From the above we have the following known result.

Lemma 3.1 ([92])**.** The problem MWF⇝EWF can be solved in exponential time but it can not be solved in sub-exponential time. The problem EWF⇝MWF can be solved in polynomial time.

Besides EWF and MWF, there are other two classical forms of extensive representations based on losing coalitions. Those representation forms were also defined by von Neumann and Morgenstern [258].

Definition 3.6. A simple game Γ is given in:

- *(Extensive or explicit) losing form (ELF)* as a pair $(N, \mathcal{L})$, where N is its set of players, and $\mathcal{L}$ its set of losing coalitions.
- *(Extensive or explicit) maximal losing form (MLF)* as a pair $(N, \mathcal{L}^M)$, where $\mathcal{L}^M$ is its set of maximal losing coalitions.

Given a simple game in ELF and a subset $X \subseteq N$, we can check whether or not X belongs to $\mathcal{L}$ in polynomial time. Furthermore, when the game is given in MLF, checking whether a given subset family is maximal can also be done in polynomial time, since we can test whether adding one of the elements of the complement of the coalition leads to a losing coalition. These conditions characterize the fact that a set family is the set of (maximal) losing coalitions of a game. Thus, both forms of representations are valid.

The size of a simple game Γ in both representations is $|\text{ELF}(\Gamma)| = n \cdot |\mathcal{L}|$ and $|\text{MLF}(\Gamma)| = n \cdot |\mathcal{L}^M|$, respectively.

The conversion problem among representations based on losing coalitions and those based on winning coalitions was studied in [92]. While the polynomial results come from the monotonicity of simple games, the exponential ones are related with the size of the representations, in the same way than in Lemma 3.1.

Lemma 3.2 ([92])**.** The problems EWF⇝MLF, ELF⇝MLF, ELF⇝MWF can be solved in polynomial time. The problems EWF⇝ELF, MWF⇝ELF, MWF⇝MLF, MLF⇝EWF, MLF⇝EWF and ELF⇝ELF can be solved in exponential time but can not be solved in sub-exponential time.

Moreover, based on a result of reliability functions from [15], Aziz proved that the problem MWF⇝EWF is #P-complete [9]. Since $\mathcal{W} \cup \mathcal{L} = \mathcal{P}(N)$, note that there are three possibilities for a given simple game which belongs to a subclass:

1. Both $|\mathcal{W}|$ and $|\mathcal{L}|$ grow exponentially in terms of n.
2. $|\mathcal{W}|$ is polynomially bounded and $|\mathcal{L}|$ grows exponentially.
3. $|\mathcal{W}|$ grows exponentially and $|\mathcal{L}|$ is polynomially bounded.

When the first possibility holds, we need more accurate bounds to know the differences between using EWF or ELF. When the second one holds, then it is most useful to represent the game in EWF than ELF. Finally, when the third condition holds, the dual game is one of the second type and it can be represented in a most useful way in EWF. Remember by Definition 2.20 that we can always get the original game using the dual of the dual game with a potential increase in size.

Note that according to duality we have that $|\text{EWF}(\Gamma)| = |\text{ELF}(\Gamma^d)|$. This allows to obtain a representation for a game from the representation of its dual. Although this transformation might be useful to analyze the computational complexity of some problems defined on simple games, it is

Algorithm 1 GenerateEWFfromMWF

Input: A simple game Γ in MWF with $\mathcal{W}^m = \{X_1, \ldots, X_m\}$ sorted in lexicographic order.
Output: Γ in EWF.

Generate(X, R, i)
  for all $j \in R$ in increasing order
    $X = X \cup \{j\}$;
    if for all $k > i, X_k \not\subset X$
      print X;
      $R = N_j \setminus X$;
      Generate(X, R, i);
      $X = X \setminus \{j\}$;
$\{$main$\}$
for $i = 1, \ldots, m$ **do**
  print X_i;
  $R = N \setminus X_i$;
  Generate(X_i, R, i);

not part of our objectives as we are interested in comparing representations of the same game.

As we saw before, the sizes of the representations of simple games may be a strong argument to prove that there is no polynomial time algorithm for the conversion problem between some forms of representation. In these cases we can study the complexity of the enumeration problem.

Lemma 3.3. MWF⇝EWF can be solved with polynomial-delay.

Proof. First note that every MWC is a winning coalition, so they have to be printed. Let $(N, \mathcal{W}^m)$ be a simple game, as usual we assume that $N = \{1, \ldots, n\}$, $N_i = \{i, \ldots, n\}$ and $\mathcal{W}^m = \{X_1, \ldots, X_m\}$ according to increasing lexicographical order.

See Algorithm 1. In order to enumerate all the winning coalitions without repetitions, for each $X_i \in \mathcal{W}^m$ our algorithm enumerates only a subset of the $2^{n-|X|} - 1$ winning coalitions that contain X. The algorithm is a branch and cut algorithm that uses the usual backtrack tree providing an enumeration of the subsets of a set without repetitions. Recall that in such a tree a node has as children the superset obtained by adding one candidate element. The set of candidates is formed by those elements that are not in the current subset and that are posterior to all the elements in the current subset. An example of the backtracking tree for a set with four elements

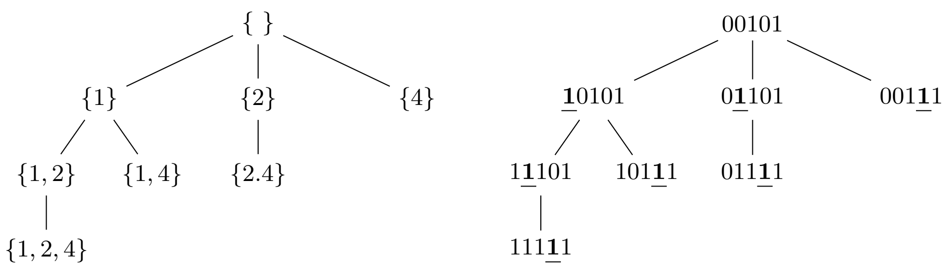


Figure 3.1: Backtrack tree for $\{1, 2, 4\}$ and backtrack tree for the enumeration without repetitions of all the winning coalitions that contain 00101.

is given in Figure 3.1. We consider such a tree for every given winning coalition.

We first sort the set of MWCs in increasing lexicographical order. Then, for each MWC X_i, we perform a traversal of the backtrack tree corresponding to all the subsets of $N \setminus X_i$ as described before. On the traversal our algorithm backtracks whenever it reaches a set X that is a superset of X_j, for some $j > i$. Thus our algorithm prints, for any MWC X_i, all the coalition that are supersets of X_i but not supersets of any MWC X_j with $i < j$. The first property guarantees that the winning coalitions are printed without repetitions. Furthermore, monotonicity guarantees that the algorithm prints all the winning coalitions.

Finally, take into account that in any of the backtracking trees constructed in an execution of Algorithm 1, the height is at most n, therefore any backtrack path has length bounded by n. In consequence, the number of steps between the printing of one winning coalition and the next one is polynomial and the claim follows. □

Example 3.3. Consider the simple game $(N, \mathcal{W}^m)$ of Example 3.1. Figure 3.2 shows the enumeration without repetition of all the winning coalitions of the game. They are printed in the following order: 00011, 10011, 01011, 00101, 10101, 01101, 00111, 01110, 01111, 10110, 11110, 10111, 11001, 11101, 11111, 11011.

For the case of forms of representations based in sets of losing coalitions we get an equivalent result.

Lemma 3.4. MLF⇝ELF can be solved with polynomial-delay.

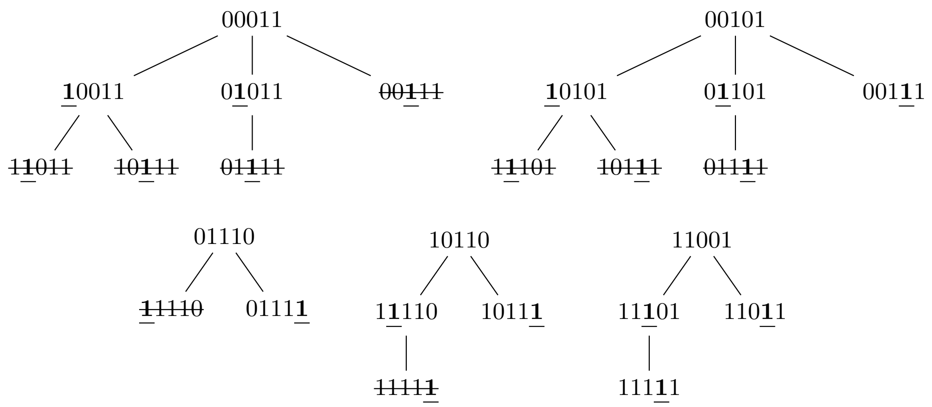


Figure 3.2: Computing MWF⇝EWF for the simple game of Example 3.1.

Proof. The proof is similar to that of Lemma 3.3 and it corresponds to Algorithm 2. Again our algorithm starts by sorting in lexicographic order the set of maximal losing coalitions. Algorithm 2 backtracks over the family of subsets associated to each maximal losing coalition. As before, to avoid repetitions when we deal with a maximal losing coalition X_i, we backtrack when the algorithm reaches a set that is also a subset of X_j, for some $j > i$. Again, all the backtracking trees have height at most n and thus Algorithm 2 works with polynomial-delay. □

It remains open to determine whether the conversions MWF⇝ELF, MWF⇝MLF, MLF⇝EWF and MLF⇝MWF can be solved with polynomial-delay.

3.1.2 Binary Tree Representations

In this section, we review several usual forms of representation for simple games using different families of (extended) binary trees [163]. There are several forms of representation based on directed binary trees [194].

As usual we assume a lexicographic order on the set N of players. Any family of subsets $\mathcal{F}$ can be represented by a binary tree in different ways. The simplest—and most costly—way of representation uses a complete binary tree with height n.

Definition 3.7. A *complete binary tree* B_c with height n is a binary tree such that for any node $t \in B_c$ at depth $j+1$, the left edge (respectively, right edge) from t represents that $x_{n-j} = 1$ (respectively, $x_{n-j} = 0$). Terminal

Algorithm 2 GenerateELFfromMLF

Input: A simple game Γ in MLF with $\mathcal{L}^M = \{X_1, \ldots, X_m\}$ sorted in lexicographic order.
Output: Γ in ELF.

GENERATE(X, R, i)
  for all $j \in R$
    $X = X \setminus \{j\}$;
    if for all $k > i, X_k \not\supset X$
      print X;
      $R = X \cap N_{j+1}$;
      GENERATE(X, R, i);
      $X = X \cup \{j\}$;

{main}
for $i = 1, \ldots, m$ **do**
  print X_i;
  $R = X_i$;
  GENERATE(X_i, R).

nodes are labeled with labels from the set $\{0, 1\}$. Thus, a terminal node $t \in B_c$ at depth n represents a vector with n components corresponding to the edge labels found in the path from the root to t. Those sets with vectors corresponding to paths ending in a terminal node with label 1 belong to the represented family. Sets whose vectors correspond to paths ending in a terminal node with label 0 do not belong to the family.

Let us explain this construction with an example.

Example 3.4. Figure 3.3 illustrates the complete binary tree for the set of winning coalitions of the simple game given in Example 3.1.

Note that every complete binary tree representing a subset family has 2^n terminal nodes and $2^n - 1$ inner nodes, including the root node. And thus its size depends on n and not on the size of the represented set. Therefore, complete binary trees are not the best form of tree representation. However, we include them here in order to introduce other variants of binary trees.

A first variation could be to avoid the vector components represented by the terminal nodes with label 0.

Example 3.5. Figure 3.4 represents a non-complete binary tree for the set of winning coalitions of the simple game of Example 3.1.

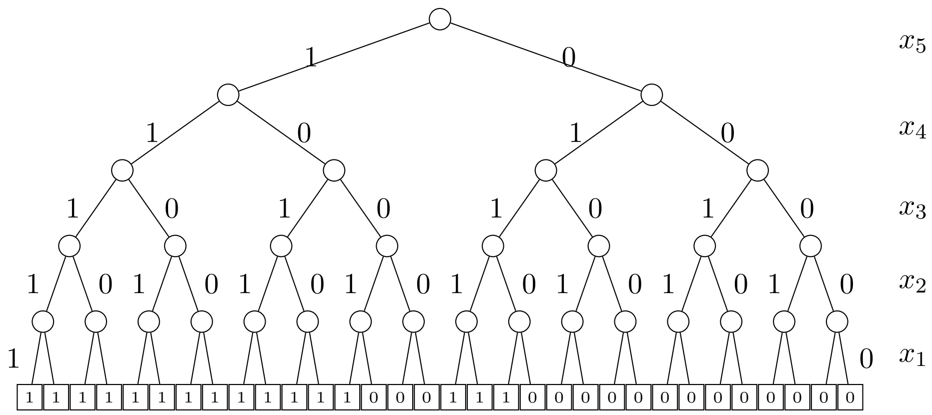


Figure 3.3: Complete binary tree $B_c(\mathcal{W})$ representing the winning coalitions for the simple game $\Gamma = (N, \mathcal{W})$ of Example 3.1. The labels in the last level of edges can be deduced from the other levels.

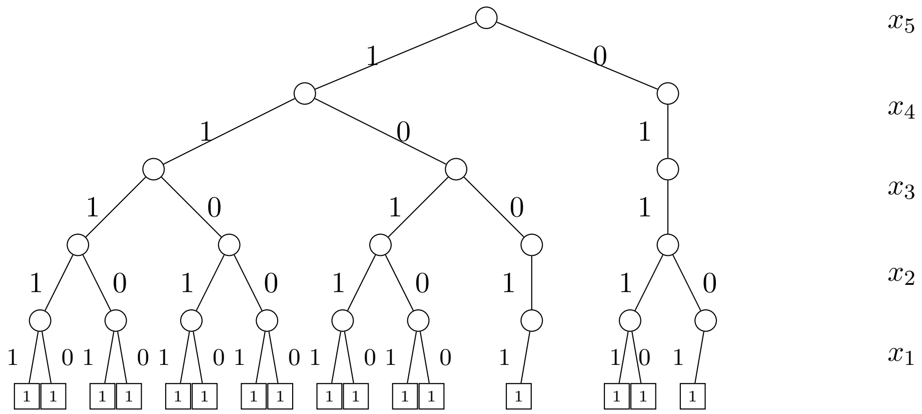


Figure 3.4: Binary tree $B(\mathcal{W})$ representing the winning coalitions for the simple game $\Gamma = (N, \mathcal{W})$ of Example 3.1.

From the computational point of view, any binary tree representation of one of the fundamental sets describing a simple game—$\mathcal{W}$, $\mathcal{L}$, $\mathcal{W}^m$ or $\mathcal{L}^M$—is related to the corresponding representation form in the same way. In what follows—as it has been done in the literature—we consider only binary tree representations of the subset of MWCs. We continue considering binary trees that are not necessarily complete.

We keep the edge labels, with the same meaning, i.e., an edge from a node at depth j to a node at depth $j+1$ labeled $l \in \{0, 1\}$ represents the case $x_{n-j} = l$. Terminal nodes—at depth n—do not have assigned labels. Furthermore, all terminal nodes are at depth n. As before, a terminal node

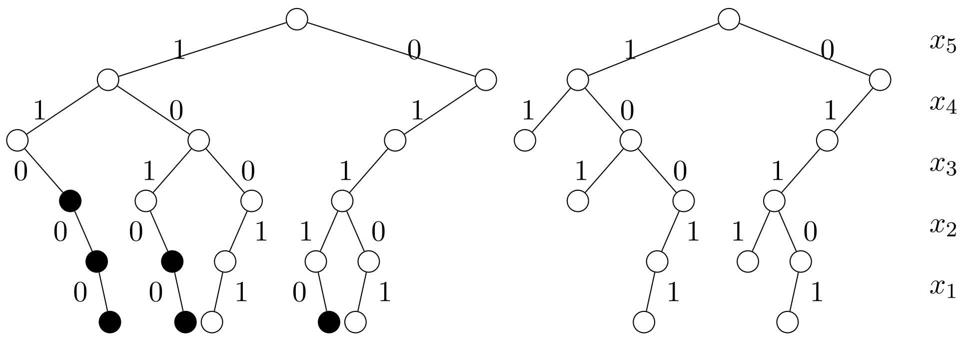


Figure 3.5: $B(\Gamma)$ (left) and $PCB(\Gamma)$ (right) representing the simple game Γ of Example 3.1. $PCB(\Gamma)$ is obtained from $B(\Gamma)$, by removing the marked nodes.

$t \in B$ at depth n represents a vector with components resulting by the edge labels in the path from the root to t. Those sets with vectors corresponding to paths ending in a terminal node belong to the subset family represented by the set. Observe that now the size of $B(\mathcal{F})$ is polynomially related to $|\mathcal{F}|$. This binary tree data structure has been used in relation with simple games in the context of monotone Boolean functions as a representation of the set of MWCs [163].

Definition 3.8. A simple game Γ is given in *binary tree form (BF)* when it is given by a binary tree representing the set $\mathcal{W}^m(\Gamma)$.

We use the notation $B(\Gamma)$ to denote a simple game Γ given in binary tree form. Observe that we can check in polynomial time whether a set belongs or not to the set represented by a binary tree. Therefore, we can check in polynomial time the minimality of the represented set. Thus, the binary tree form is a valid representation for simple games.

Example 3.6. Figure 3.5 at left depicts the binary tree $B(\Gamma)$ for the game $\Gamma = (N, \mathcal{W}^m)$ given in Example 3.1.

In the following we introduce two other forms of representation based on binary trees. Whereas the first one reduces the size of the binary trees, the second one, based on binary decision diagrams, reduces the size of a complete binary tree representation. In both cases the data structure allows to check in polynomial time if a given set belongs to the represented family. Therefore, both data structure are valid representation forms for simple games.

The first data structure is due to Makino [163].

Definition 3.9. A *partially condensed binary tree* (*PCB*) for a set family $\mathcal{F}$ is the subgraph of $B(\mathcal{F})$ which is obtained after removing recursively all the leaves whose parent has no edge labeled 1, i.e., whose parent has no left-child.

From the above definition we are able to define the following representation form.

Definition 3.10. A simple game Γ is given in *partially condensed binary tree form (PCBF)* if it is given by the PCB obtained from $B(\Gamma)$.

We denote by $PCB(\Gamma)$ the partially condensed binary tree representation of Γ. Recall that the tree provides a representation of $\mathcal{W}^m(\Gamma)$.

Example 3.7. The right tree in Figure 3.5 represents *PCB*(Γ) for the simple game given in Example 3.1. This tree is obtained from $B(\Gamma)$, given as the left tree in the same figure.

Our last representation form in this section is based on binary decision diagrams.

Definition 3.11. A *binary decision diagram* (*BDD*) is a directed, acyclic and labeled graph with decision nodes and two terminal nodes called 0-terminal and 1-terminal. A BDD is *ordered* (*OBDD*) if the players appear in the same order on all paths from the root; and it is *reduced* (*RBDD*) if all its isomorphic subgraphs are merged and it does not have any node with two isomorphic children. A binary decision diagram represents the set family formed by all the vectors extracted from paths ending in the 1-terminal.

BDDs are also called *branching programs* [263, 172]. They were introduced in the context of monotonic functions by Lee [153] and studied in depth by Akers [1] and Boute [34]. For each given ordering on N, there is a unique *reduced and ordered BDD* (*ROBDD*) representing the family [39, 40]. That is why usually by BDD it is meant the ROBDD corresponding to the lexicographic order.

Definition 3.12. A simple game Γ is given in *binary decision diagram form (BDDF)* by a BDD representing $\mathcal{W}^m(\Gamma)$.

Algorithm 3 GeneratingBDDfromBT

Input: A binary tree representing a subset set S.

Output: A BDD representing S.

1: Merging duplicate terminal nodes that share the same label (0 or 1).
2: For each level from the bottom to the top:
3: **loop**
4: Merging duplicate inner nodes whose left-child and right-child are connected with the same node.
5: Removing inner nodes with the same left-child and right-child.
6: **end loop**

The BDDF represents a set family in the same way as complete binary trees, but in a more succinct way. The MWCs are represented by paths leading to the 1-terminal.

Given a simple game Γ, we can compute $BDD(\Gamma)$ from $B(\Gamma)$. This can be done through Algorithm 3, which was described in [40]. After each step of the procedure, it is necessary to redirect all incoming arcs to the corresponding new nodes. Steps 4 and 5 must be repeated as many times as necessary. In general, starting with the level of the first component x_1, and ending with the level of the n-th component x_n. Usually left-children are denoted by a solid edge, whereas the right-children are denoted by a dashed edge. In what follows, we use a double edge to represent a node in which the left-child and the right-child are the same.

We can also consider the OBDD obtained by a procedure in which the rule implemented in step 5 of Algorithm 3 is not used. In such a BDD every path from the root to a terminal node has length $n + 1$. The so obtained BDD is called a *quasi-reduced binary decision diagram (QOBDD)* [17, 29]. Note that for a fixed variable ordering, both QOBDD and ROBDD are canonical representations, and hence, if we consider a QOBDD and its respective ROBDD—sharing inner nodes—then checking whether both BDDs are equivalent is trivial in the sense of computational complexity [115]. On the other hand, let Γ_1 and Γ_2 be two simple games, such that the sizes of their respective BDDs are $r, s \in \mathbb{N}$; therefore, according to [28], determine whether both BDDs are equal can be computed in $O(\min\{r, s\})$.

Example 3.8. Given the simple game of Example 3.1, Figure 3.6 illustrates the BDD obtained from the complete binary tree of Figure 3.3, after applying Algorithm 3.

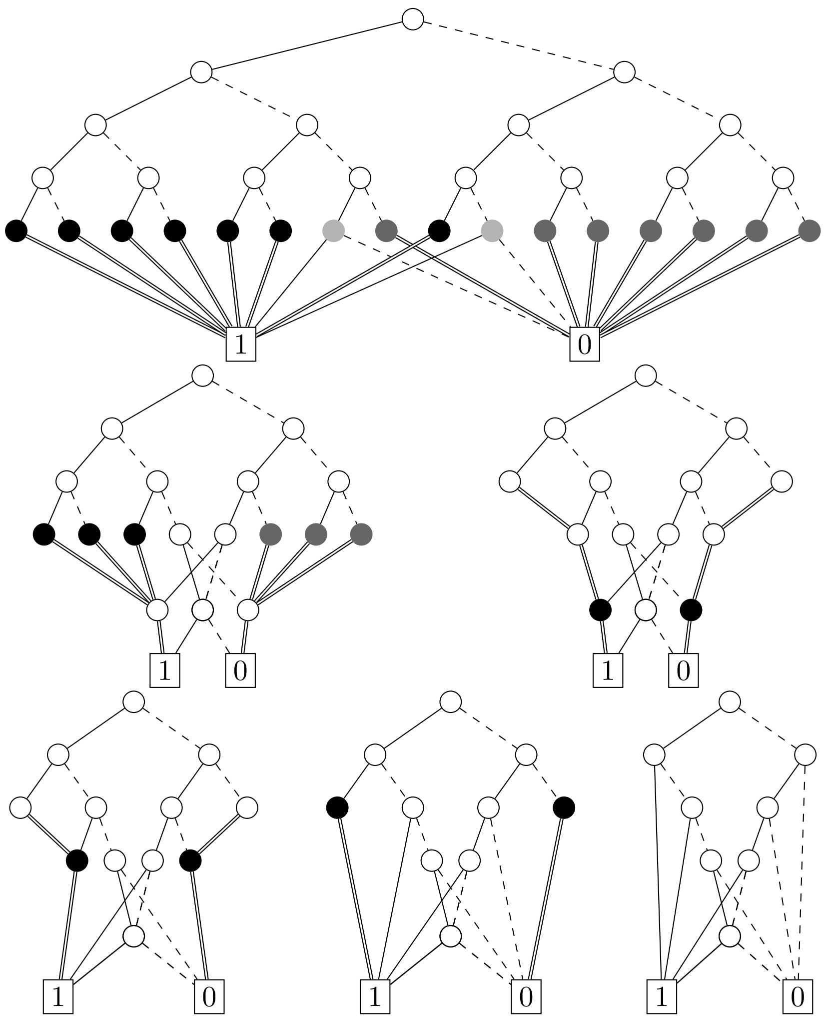


Figure 3.6: $BDD(\Gamma)$ obtained from the given binary tree $B(\Gamma)$ for the game Γ of Example 3.1. Marked nodes are merged or removed in the next step.

Let Γ be a simple game, we could construct, instead of $BDD(\mathcal{W}(\Gamma))$, any of the BDDs associated to any of the other fundamental set families, $BDD(\mathcal{L}(\Gamma))$, $BDD(\mathcal{W}^m(\Gamma))$ or $BDD(\mathcal{L}^M(\Gamma))$. However, the differences between them and their corresponding forms of representation are the same as for $BDD(\mathcal{W}(\Gamma))$ and MWF. In some cases these alternatives provide bigger representation sizes than the others. For the particular case of regular games, as we shall see in Section 3.2, some of those forms of representation are of equivalent size.

3.1.3 Representation Sizes and Conversion Problems

Now we summarize several known results related to the size of the forms of representation defined before. We introduce some new results and we use them to analyze the computational complexity of the corresponding conversion problems. As the representation based on binary trees and variants are based on the winning coalitions, we consider only the conversion problems with respect to EWF and MWF.

First of all, it is necessary to remark that both the size of the PCBs and BDDs (ROBDDs) depend on the ordering of players. It is interesting to note that while a specific ordering could provide a very small size, another one might mean an exponential size in terms of the number of players. Moreover, the problem of finding the best variable ordering for a BDD is NP-hard [25]. However, as in this work we use always an ordering of variables already given, we do not worry about this problem. Even so, there exist classes of simple games Γ for which $|BDD(\Gamma)|$ grows exponentially in terms of n, independently of the order of the variables. For instance, consider the Theorem 4 of [120], which shows a class of simple games—actually, a class of weighted games—for which $|BDD(\Gamma)| \in \Omega(2^{\sqrt{n}/2})$. Additionally, there exists a known result by Wegener which says that almost all QOBDDs for general Boolean functions—not only the monotone ones—have size $2^n/(2n)$ [264]. The same author proved that QOBDDs are at most a factor $n + 1$ larger than their corresponding ROBDDs [265].

The relevance of BDDs with polynomial size has led to Ishiura and Yajima to define the *PolyBDD* family [126]. This class, however, has no explicit characterization, so it will not be discussed here. An interesting open question is to characterize the simple games that can be represented by BDDs with polynomial size.

It is easy to see that a simple game in MWF, PCBF or BDDF may turn out to be very much smaller than in EWF. To see this, just consider the simple game Γ of Example 3.2, where $\mathcal{W}^m = \{\emptyset\}$. In this case, $|\mathcal{W}^m| = |PCB(\Gamma)| = |BDD(\Gamma)| = 1$, but $|\mathcal{W}| = 2^n$ is an exponential amount in terms of n.

There are simple games whose representations in BDDF turn out to be very much smaller than in MWF. For instance, the simple game given in Example 3.14 in Section 3.3. Moreover, for a simple game with $n > 1$, to represent a MWC in $PCB(\Gamma)$ it is always required to have at least two nodes.

Note that the game defined in Example 3.14 also shows that there exist simple games for which their representation in BDDF grows exponentially in terms of their representation in PCBF.

Hence, for every simple game Γ with n players, where n is sufficiently large, $|BDD(\Gamma)| \leq |PCB(\Gamma)| \leq n|\mathcal{W}^m| \leq n|\mathcal{W}|$. The above implies the following easy results.

Lemma 3.5. PCBF⇝EWF, BDDF⇝EWF, BDDF⇝MWF and also BDDF⇝PCBF can be solved in exponential time, and can not be solved in sub-exponential time.

The following result establishes those conversion problems that can be solved in polynomial time.

Lemma 3.6. EWF⇝PCBF, EWF⇝BDDF, MWF⇝PCBF, MWF⇝BDDF, PCBF⇝MWF and PCBF⇝BDDF can be solved in polynomial time.

Proof. Polynomiality of MWF⇝PCBF comes from [163]: Since the number of nodes of $PCB(\Gamma)$ is at most $O(n|\mathcal{W}^m|)$, then MWF⇝PCBF can be computed in $O(n|\mathcal{W}^m|)$ time. For PCBF⇝MWF, as the number of paths from the root to each terminal node is $|\mathcal{W}^m|$, we can explore totally $PCB(\Gamma)$ using breadth-first traversal in $O(|V| + |E|) = O(|PCB(\Gamma)|)$ time, considering at most n steps by each path for completing the coalitions that begins with zeros. Thus, we can compute PCBF⇝MWF in $O(n \cdot (|V| + |E|))$ time.

Polynomiality of EWF⇝PCBF and EWF⇝BDDF follow from the above. As $|BDD(\Gamma)| \leq |PCB(\Gamma)| \leq n|\mathcal{W}^m|$, for all $x \in \mathcal{W}^m$ we can build the corresponding path of $BDD(\Gamma)$ in $O(n|\mathcal{W}^m|)$ time. Then the partial $BDD(\Gamma)$ can be explored with breadth-first traversal in $O(|V| + |E|)$, joining each node without right-child to the 0-terminal. Hence, as $|V| \leq n|\mathcal{W}^m|$, then MWF⇝BDDF can be computed in $O(n|\mathcal{W}^m|)$ time.

For PCBF⇝BDDF, since all the leaves of $PCB(\Gamma)$ have label 1, they can be removed and replaced by a 1-terminal node, keeping the corresponding edges. Then connect each node without right-child to the 0-terminal, and finally merge duplicate inner nodes as in step 4 of Algorithm 3. All these steps can be computed in polynomial time. □

Note that, as MWF⇝PCBF and PCBF⇝MWF can be solved in polynomial time, the computational complexity of each problem raised about simple games is the same, regardless of which of these two forms of representation is chosen. On the other hand, if the simple game is given in BDDF, changing the form of representation to MWF, PCBF or EWF, may increase the complexity of the problem.

Let us now turn our attention to the cases in which the conversion problem requires exponential time. For PCBF⇝EWF it is enough to compute first an enumeration of MWF in polynomial time, according to Lemma 3.6, and then apply Algorithm 1.

Lemma 3.7. PCBF⇝EWF can be solved with polynomial-delay.

Note that Algorithm 1 cannot be applied for BDDs because we do not have explicitly all the MWCs, and by Lemma 3.5, BDDF⇝EWF cannot be computed in polynomial time. The existence of an algorithm with polynomial-delay for the conversion problems BDDF⇝EWF, BDDF⇝MWF and BDDF⇝PCBF remains open.

3.2 Representations for Regular Games

In what follows of this chapter, we assume that $N = \{1, \ldots, n\}$ is ordered according to the usual lexicographical order $\leq$.

It is known that a regular game is completely determined by the set of its shift-minimal winning coalitions—see [144], where authors are based on the unpublished result of [203]. Furthermore, once the desirability ordering is known, checking shift-minimality can be implemented in polynomial time. Thus, we can consider this set as a valid form of representation for regular games.

Definition 3.13. A regular game Γ is given in *shift-minimal winning form (SWF)* as a pair $(N, \mathcal{W}^s)$, where $\mathcal{W}^s$ is the set of shift-minimal winning coalitions.

As before, a matrix notation is useful to represent regular games in a computational context. Consider an injective function $\bar{f} : \{0,1\}^n \to \mathbb{N}^n$ such that each winning coalition x is associated with a unique integer vector $\bar{x}$, defined for all $a \in N$ as $\bar{f}(x)(a) = \bar{x}(a) = \Sigma\{x(b) \mid b \in N, a \preceq b\}$. Thus, $X \preceq Y$ if and only if $\bar{x} \leq \bar{y}$.

Example 3.9. It is easy to see that the simple game $\Gamma = (N, \mathcal{W}^m)$ of Example 3.1 is regular. Let be $\overline{\mathcal{W}}^m = \{\bar{x} \in \mathbb{N}^n \mid x \in \mathcal{W}^m\}$, then:

N	$abcde$
$\overline{\mathcal{W}}^m$	$22221 : \bar{x}_1$
	$22211 : \bar{x}_2$
	$33210 : \bar{x}_3$
	$32210 : \bar{x}_4$
	$32111 : \bar{x}_5$

Since $\bar{x}_2 \leq \bar{x}_1$ and $\bar{x}_4 \leq \bar{x}_3$, it holds that:

N	$abcde$
$\mathcal{W}^s$	00101
	10110
	11001

It is clear that for every regular game, its set of shift-minimal winning coalitions is a subset of all its MWCs. Furthermore, it is known that given a simple game in MWF, it can be decided whether the game is regular or linear in time polynomial in the size of the game. In fact, regularity can be decided in linear time and linearity in $O(n^2 + n|\mathcal{W}^m|)$-time [163].

In what follows we consider a more succinct version of the form of representation PCBF, restricted to regular games. However, it does not use the set of shift-minimal winning coalitions defined above, keeping the focus in the set of MWCs.

Definition 3.14. Given a simple game Γ and some total order over N, a *fully condensed binary tree* $FCB(\mathcal{W}^m(\Gamma))$ is a binary tree obtained from $PCB(\mathcal{W}^m(\Gamma))$ by recursively removing all edges (t_i, t_{i+1}) such that t_i has no right-child, and for each removed edge, merging both nodes t_i and t_{i+1}.

It is clear that $FCB(\mathcal{W}^m(\Gamma))$ can be carried out in polynomial time.

FCBs were defined by Makino [163] to decide in linear time the regularity of a monotone Boolean function. In a non-binary, but alphanumeric context, they were simultaneously defined in 1968 as *Patricia tries* [186] and without name [110], being nowadays also known as *radix trees*.

Like PCBs, each MWC is represented by a leaf and its path from the root. Analogously to previous data structures we have that determining whether a coalition belongs to the represented set can be done in polynomial time.

Definition 3.15. A regular game Γ is given in *fully condensed binary tree form (FCBF)* if $\mathcal{W}^m(\Gamma)$ is being represented by $FCB(\mathcal{W}^m(\Gamma))$.

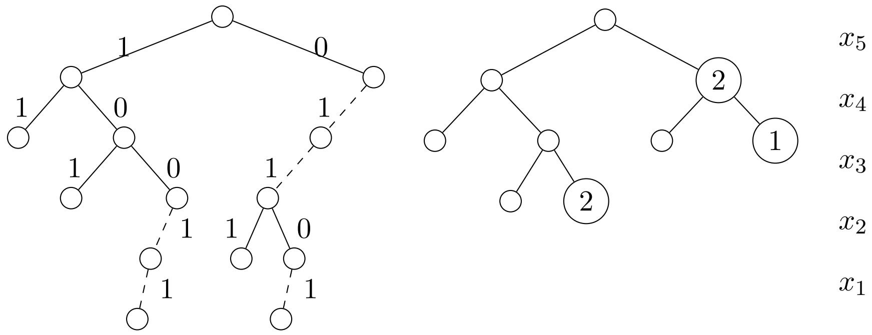


Figure 3.7: *FCB*(Γ) obtained from *PCB*(Γ) for the game Γ of Example 3.1. Nodes between dashed edges on the first tree are merged in the second one.

By abuse of notation, we use *FCB*(Γ) to denote *FCB*($\mathcal{W}^m(\Gamma)$).

For any regular game Γ, *FCB*(Γ) is always *complete*, in the sense that it has no inner nodes having only one child. For non-regular simple games this may not be true [163].

Example 3.10. Given the *PCB*(Γ) for the game of Example 3.1 (see Figure 3.5), we can construct the *FCB*(Γ) illustrated in Figure 3.7. The nodes have been labeled with integers representing the number of left-children merged with the current node. Each node labeled by the number m represents m components 1, and all the latest missing components are 0.

An additional succinct form of representation for regular games uses an invariant $(\vec{v}, \mathcal{M})$ where $\vec{v}$ is a set of players' classes and $\mathcal{M}$ is a matrix that considers some special type of winning coalitions [43]. This type of representation is less succinct than SWF and it is also less usual in computer science approaches, so we leave as an open problem to study the conversion problems for this case.

3.2.1 Conversion Problems for Regular Games

It is clear that $|FCB(\Gamma)| \leq |PCB(\Gamma)|$ for every simple game Γ. Moreover, since the number of root-nodes of both *FCB* and *PCB* is equal to $|\mathcal{W}^m|$, but their total number of nodes is lower than $n|\mathcal{W}^m|$, we conclude that the difference between $|FCB(\Gamma)|$ and $|PCB(\Gamma)|$ is always polynomial in terms of n.

Furthermore, as we showed before, it always holds $\mathcal{W}^s(\Gamma) \subseteq \mathcal{W}^m(\Gamma)$, so $|\mathcal{W}^s(\Gamma)| \leq |\mathcal{W}^m(\Gamma)|$. However, in this case there are subclasses of simple games in which the difference between $|\mathcal{W}^s|$ and $|\mathcal{W}^m|$ can grow exponentially in terms of n.

Example 3.11. Consider the simple game Γ with the biggest size $|\mathcal{W}^m(\Gamma)|$ as possible over N, i.e., the one given by Sperner [242]—see also [96]—with $\mathcal{W}^m(\Gamma) = \{X \subseteq N \mid |X| = \lfloor \frac{n}{2} \rfloor\}$. In this case, $|\mathcal{W}^m(\Gamma)| = \binom{n}{\lfloor n/2 \rfloor}$, but $|\mathcal{W}^s(\Gamma)| = 1$, because $\mathcal{W}^s(\Gamma) = \{1^{n/2}0^{n/2}\}$.

Note that in this example, $|PCB(\Gamma)| \geq \binom{n}{\lfloor n/2 \rfloor}$ and $|FCB(\Gamma)| \geq \binom{n}{\lfloor n/2 \rfloor}$, so the difference between $|\mathcal{W}^s(\Gamma)|$ and $|PCB(\Gamma)|$, as well as between $|\mathcal{W}^s(\Gamma)|$ and $|FCB(\Gamma)|$, can also grow exponentially in terms of n.

The above does not mean, however, that $|\mathcal{W}^s(\Gamma)|$ is always "small". Actually, as proved Krohn and Sudhölter [144]—see also [147]—there are subclasses of regular games for which the number of its shift-minimal winning coalitions also grows exponentially in terms of n—only that more slowly than the number of its MWCs.

Example 3.12. The following regular game Γ is the biggest as possible for $n = 8$, with $|\mathcal{W}^s(\Gamma)| = 14$. Note that the number of shift-minimal winning coalitions almost doubles the number of players. Obviously, the number of MWCs is even bigger.

N	$abcdefgh$
	00001110
	00010101
	00100011
	00111100
	01011010
	01100110
$\mathcal{W}^s$	01101001
	10010110
	10011001
	10100101
	11000011
	11011100
	11101010
	11110001

All the above, in addition to Lemma 3.5, implies the following results.

Lemma 3.8. For regular games, SWF⇝EWF, SWF⇝MWF, SWF⇝PCBF, SWF⇝FCBF, BDDF⇝FCBF and FCBF⇝EWF can be solved in exponential time, and can not be solved in sub-exponential time.

By the same considerations,we can prove the following result.

Lemma 3.9. For regular games, the conversion problems EWF⇝SWF, MWF⇝SWF, PCBF⇝SWF, EWF⇝FCBF, MWF⇝FCBF, PCBF⇝FCBF, FCBF⇝PCBF, FCBF⇝MWF, FCBF⇝BDDF and FCBF⇝SWF can be solved in polynomial time.

Proof. The two first claims follow from an algorithm similar to the one derived to solve the EWF⇝MWF problem. Just compare all (minimal) winning coalitions to each other, but deleting those whose respective vectors $\bar{x}$ are bigger in the lexicographic order. Since $\bar{f}(x)$ can be simultaneously computed when each (winning) coalition is compared, this procedure takes $O(n \cdot |\mathcal{W}|^2)$ time for the first case, and $O(n \cdot |\mathcal{W}^m|^2)$ for the second one.

Since PCBF⇝MWF and MWF⇝SWF can be solved in polynomial time, then PCBF⇝SWF can also be solved in polynomial time. The remaining results are deduced in the same way from PCBF⇝FCBF and FCBF⇝PCBF, which are clearly polynomial by what is said in Section 3.2. □

In addition, it is known that for regular games MWF⇝MLF can be solved in polynomial time [210], regardless the exponentiality for simple games in general. Unlike the previous cases, the size relationship between SWF and BDDF is less clear.

As stated in Section 3.1.3, in Theorem 4 of [120] is presented a subclass of regular games where for every game Γ of this subclass, $|BDD(\Gamma)|$ grows exponentially in function of n. However, it does not seems that in this example $|\mathcal{W}^s(\Gamma)|$ grows much lower than $|BDD(\Gamma)|$. Furthermore, although we saw in Example 3.11 a game Γ with $|\mathcal{W}^s(\Gamma)| = 1$, in this case $|BDD(\Gamma)|$ does not seems to increase too much in terms of n. For instance, with $n = 8$ and $n = 9$ we obtain respectively $|BDD(\Gamma)| = 27$ and $|BDD(\Gamma)| = 32$, in contrast to the sizes of each game in MWF, which are $n \cdot |\mathcal{W}^m| = 560$ and 1134, respectively. On the contrary, if in some example it were shown that both $|\mathcal{W}^s|$ and $|BDD(\Gamma)|$ have an exponential growth, it could not be deduced from here that SWF⇝BDDF can be solved in polynomial time.

To compute SWF⇝BDDF (or BDDF⇝SWF), we can always compute first SWF⇝MWF (resp. BDDF⇝MWF) and then MWF⇝BDDF (resp.

MWF⇝SWF). But if we do this, the whole process requires exponential time. The absence of known algorithms that are able to skip this intermediate step, leads us to state the following conjecture.

Conjecture 3.1. For regular games, BDDF⇝SWF and SWF⇝BDDF can be solved in exponential time, and can not be solved in sub-exponential time.

It is interesting to note that there are some efficient algorithms that benefit themselves from working with a smaller class of simple games such as regular games. For instance, in [19] is shown that computing either $BDD(\mathcal{W}^m(\Gamma))$ or $BDD(\mathcal{L}^M(\Gamma))$ from $BDD(\mathcal{W}(\Gamma))$ can be done in linear time in the input size, if the game Γ is regular. Further, they designed an algorithm to computing $BDD(\mathcal{W}^s(\Gamma))$ from $BDD(\mathcal{W}^m(\Gamma))$.

As at the end of Section 3.1.3, we finish this section with some enumeration results considering the same notation as the proof of Lemma 3.3.

Lemma 3.10. For regular games, SWF⇝MWF and FCBF⇝EWF can be solved with polynomial-delay.

Proof. Generation of $\mathcal{W}^m$ from $\mathcal{W}^s$ follows similar ideas than Algorithm 1 of Lemma 3.3, but considering as input the set $\mathcal{W}^s = \{x_1, \ldots, x_m\}$: See Algorithm 4.

Given $X \in \mathcal{W}^s$, let be $R = \{i \in X \mid i+1 \notin X\}$. It means that for any $j \in R$ we can do a 1-right-shift applied to j, i.e., to replace player j by player $j+1$. Note that given a winning coalition, to do a 1-right-shift implies that the new coalition is still winning. Steps 7-10 update the set R. Steps 7-8 consider the case where, being $j > 1$, $j-1 \in X$, $j \in X$ and $j+1 \notin X$, then a 1-right-shift applied to j implies that $j \notin R$ but $j-1 \in R$. Steps 9-10 consider the case where $j \in X$, $j+1 \notin X$ and $j+2 \notin X$, then a 1-right-shift applied to j implies that $j \notin R$ but $j+1 \in R$.

Again, the algorithm is a branch and cut algorithm that uses the usual backtrack tree providing an enumeration of all possible 1-right-shifts of a minimal winning set without repetitions. Now, for each new MWC X, we perform a traversal of the backtrack tree whenever it does not reach a set X_k such that $X_k \preceq X$, for any $k > i$—in this case it will be generated later—or $X_k \subset X$, for any $k < i$—in this case it is not minimal. These properties and the monotonicity guarantee that we generate all MWCs without repetitions.

Algorithm 4 GenerateMWFfromSWF

Input: A simple game Γ in SWF with $\mathcal{W}^s = \{X_1, \ldots, X_m\}$ sorted in card-lexicographic order.
Output: Γ in MWF.

Generate(X, R, i)
  for all $j \in R$ in increasing order
    $X = X \cup \{j+1\} \setminus \{j\}$;
    if (for all $k > i$, $X_k \not\supseteq X$) $\wedge$ (for all $k < i$, $X_k \not\subset X$)
      print X;
      $R = R \setminus \{j\}$;
      if $(j > 1) \wedge (j - 1 \notin X)$
        $R = R \cup \{j - 1\}$;
      if $j + 2 \notin X$
        $R = R \cup \{j + 1\}$;
      Generate(X, R, i);
      $X = X \cup \{j\} \setminus \{j+1\}$;
{main}
for $i = 1, \ldots, m$ **do**
  print X_i;
  $R = \{i \in X_i \mid i + 1 \notin X_i\}$;
  Generate(X_i, R, i);

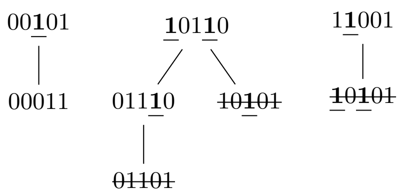


Figure 3.8: Computing SWF⇝MWF for the simple game of Example 3.1.

Following the same reasoning as for Algorithm 1 of Lemma 3.3, the number of steps between the printing of one minimal wining coalition and the next one is polynomial.

For FCBF⇝EWF, just compute FCBF⇝MWF in polynomial time—Lemma 3.9—and then apply Algorithm 1. □

We apply Algorithm 4 in the following example.

Example 3.13. Consider the simple game $(N, \mathcal{W}^s)$ of Example 3.9. Figure 3.8 shows the enumeration without repetition of all the MWCs of the game. They are printed in the following order: 00101, 00011, 10110, 01110, 11001.

We left as an open problem whether the conversion problem SWF⇝EWF can be solved with polynomial-delay. Similar to what happens for BDDs with Lemma 3.3—see Conjecture 3.1—Algorithm 4 cannot be used with BDDs. It also remains open to show whether for regular games BDDF⇝SWF can be solved with polynomial-delay.

3.3 Representations for Weighted Games

The following is the most natural way to represent weighted games.

Definition 3.16. A weighted game Γ is given in *weighted representation form (WRF)* as a tuple $[q; w_1, \ldots, w_n]$, where q is the quota and $w_1, \ldots, w_n$ are the weights of its players.

Observe that any tuple $[q; w_1, \ldots, w_n]$ represents a weighted game and thus the WRF is a valid representation for weighted games.

Given a simple game in MWF, it can be decided in polynomial time whether the game is weighted or not. This problem is known in the context of Boolean functions as *threshold synthesis problem* and it was solved by Peled and Simeone [209]. One common way to do this is solving the following system of linear inequalities:

$$w(X) > w(Y) \qquad \text{for all } X \in \mathcal{W}^m, Y \in \mathcal{L}^M \tag{3.1}$$

where $w = (w_1, \ldots, w_n)$ are the unknowns. The game is weighted if and only if the linear system has a solution, i.e., if it produces a weighted realization $[q; w_1, \ldots, w_n]$, where the quota can be derived from the weights by doing $q = \min\{w(X) \mid X \in \mathcal{W}^m\}$. Hence each solution of this linear system defines a weighted game.

Given an order over N, it is important to note that even though each simple (or regular) game can be univocally represented in EWF, MWF, PCBF or BDDF (or SWF or FCBF), a weighted game may be represented in infinite ways in WRF. This is true even for our restriction over natural numbers. Actually, given a weighted game with realization $[q; w_1, \ldots, w_n]$, the realizations $[cq; cw_1, \ldots, cw_n]$, with $c \in \mathbb{N}$, are all equivalent. But there may be further equivalent realizations, as for instance $[2; 2, 1, 1]$ and $[3; 3, 2, 1]$. In fact, given two realizations, determine whether both represent the same weighted game is NP-hard [167].

Regarding vector-weighted games, we know that any simple game $\Gamma = (N, \mathcal{W})$ can be represented by a certain number k of weighted games with realizations $[q^{(t)}; w_1^{(t)}, \ldots, w_n^{(t)}]$, such that for all $X \subseteq N$ and $1 \leq t \leq k$, $X \in \mathcal{W}$ if and only if $w^{(t)}(X) \geq q^{(t)}$, where $w^{(t)}(X) = \sum\{w_a^{(t)} \mid a \in X\}$. This expressions allow us to consider another valid form of representation for simple games.

Definition 3.17. A simple game Γ is given in *vector-weighted representation form (VWRF)* as a finite set of tuples $[q^{(t)}; w_1^{(t)}, \ldots, w_n^{(t)}]$, with $1 \leq t \leq k$ for some $k \in \mathbb{N}$.

The game represented by $[q^{(t)}; w_1^{(t)}, \ldots, w_n^{(t)}]$, for $1 \leq t \leq k$, is the game obtained as the intersection of the given family of weighted games.

In a similar way, regarding the codimension, we can consider the game Γ represented by $[q^{(t)}; w_1^{(t)}, \ldots, w_n^{(t)}]$, for $1 \leq t \leq k$, as the game obtained as the union of the given family of weighted games. We refer to such representation as *co-vector-weighted representation form (coVWRF)*. Since any simple game can be expressed as the union of a finite family of weighted games [89], coVWRF is a valid representation form for simple games.

Observe that, for any representation of weighted games that is closed under intersection or union, i.e., a representation of the intersection or union of two games that can be obtained in polynomial time, the conversion problem for simple games given in VWRF or coVWRF has the same complexity than for weighted games given in WRF.

A generalization of games constructed through binary operators is the family of *binary games* introduced in [80]. A binary game is defined by a propositional logic formula and a finite collection of weighted games. The boolean formula determines the requirements for a coalition to be winning in the described game. When considering only monotone formulas, binary games provide another representation of simple games.

3.3.1 Conversion Problems for Weighted Games

Ishiura and Yajima [126] proved that, for every weighted game in WRF, if each weight is bounded by a polynomial of n, then the sizes of their respective BDDs grow polynomially in terms of n. That is the case of the following example, in which however the number of MWCs grows exponentially in terms of n.

n	10	11	12	13	14	15	16	17	18	19	20
$\lvert BDD(\Gamma)\rvert$	74	91	117	150	184	223	274	331	388	459	545
$\lvert\mathcal{W}^m\rvert$	77	133	240	429	772	1414	2588	4742	8761	16273	30255

Table 3.6: Exponential growth of a class of weighted games in MWF, in terms of n.

Example 3.14. Consider, for every number of players n, the family of weighted games $\Gamma = [q; n, \ldots, 1]$ such that $q = \lceil (\sum_{i=1}^{n} i)/2 \rceil$. The number of MWCs for each one of these games is specified by the following sequence $s(q, n)$:

$$s(q,n) = \begin{cases} 0 & \text{if } q = 0 \text{ or } n = 0 \text{ or } q > \sum_{i=1}^{n} i \\ & \text{(weights are not enough to get } q\text{)} \\ n - q + s(q,q) & \text{if } q > n \\ & (\{\{n\}, \{n-1\}, \ldots, \{q+1\}\} \in \mathcal{W}^m) \\ 1 + s(q, n-1) & \text{if } q = n \quad \text{(take } n \text{ or not)} \\ s(q-n, n-1) + s(q, n-1) & \text{if } q < n \quad \text{(take } n \text{ or not)} \end{cases}$$

This sequence grows exponentially on n when $q = \lceil (\sum_{i=1}^{n} i)/2 \rceil$, as also we can see in Table 3.6. Such values can be checked with the application of Bolus [27].

Despite of this, in Section 3.2.1 was mentioned that there are subclasses of weighted games whose sizes in BDDF also grow exponentially in terms of n, independently of the ordering of the players and the weights considered in WRF [120]. Moreover, there exist exponential algorithms to solve WRF⇝BDDF [17, 29].

As the size of a weighted game in WRF is always $n + 1$, the first part of the following result is quite simple, and it is deduced from the previous Sections 3.1.3 and 3.2.1. For the second part, just note that every weighted game is a simple game with dimension 1.

Lemma 3.11. For weighted games, the conversion problems WRF⇝EWF, WRF⇝MWF, WRF⇝PCBF, WRF⇝BDDF, WRF⇝FCBF and also WRF⇝SWF can be solved in exponential time, and can not be solved in sub-exponential time. For simple games, the same occurs replacing WRF by VWRF.

Despite of this result, there exist some subclasses of weighted games for which WRF⇝BDDF turns out to be polynomial. Such is the case of homogeneous games.

Example 3.15. The regular game Γ of Example 3.11 is both weighted and homogeneous, and can be represented in WRF by the vector $[\frac{n}{2}; 1, 1, \ldots, 1]$.

It is known that every homogeneous game can be represented by a QOBDD with size $O(n^2)$, and that from its weighted representations, this QOBDD can be computed in $O(n^2 \cdot \log n)$ time [28]. Therefore, since $|BDD(\Gamma)| \leq |QOBDD(\Gamma)|$, then WRF⇝BDDF for homogeneous games can be solved in polynomial time.

Fredman and Khachiyan [84] showed that given a simple game Γ in MWF, $\mathcal{L}^M(\Gamma)$ can be computed in sub-exponential time; but until now, it remains open to show if this can be done in polynomial time. However, if Γ is regular or weighted, the problem turns out to be polynomial [210]. This plus the fact that the linear programming is polynomial [138] proves that the linear system (3.1) can be solved in polynomial time, implying together with the previous lemmas the following result.

Lemma 3.12. For weighted games, the conversion problems EWF⇝WRF, MWF⇝WRF, PCBF⇝WRF, BDDF⇝WRF, FCBF⇝WRF and also SWF⇝WRF can be solved in polynomial time.

Proof. We just prove the last statement. As mentioned in Section 3.2.1, according to Krohn and Sudhölter [144] each regular game Γ is completely determined by $\mathcal{W}^s(\Gamma)$. Therefore, the system of linear inequalities (3.1) can be replaced by the following:

$$w(X) > w(Y) \qquad \text{for all } X \in \mathcal{W}^s, Y \in \mathcal{L}^S \tag{3.2}$$

where $\mathcal{L}^S$ is the set of shift-maximal losing coalitions. Thus, as (3.2) has less inequalities than (3.1), then it is clear that (3.2) can also be solved in polynomial time.

For BDDF⇝WRF we refer to [30], where the author solves the problem for QOBDDs using linear programming and producing real weighted representations. □

Algorithm 5 GenerateMWFfromWRF

Input: $\Gamma = [q; w_1, \ldots, w_n]$ with $\sum_{i=1}^{n} w_i \geq q$ and $w_1 \geq \ldots \geq w_n$.
Output: Γ in MWF.

1: WRFtoMWF(X, i, q')
2:   **if** $i = n$
3:     **print** $X \cup \{n\}$; **return**;
4:   **else**
5:     **if** $(\sum_{j=i+1}^{n} w_j \geq q)$ WRFtoMWF$(X, i+1, q')$;
6:     **if** $(w_i \geq q')$ **print** $X \cup \{i\}$;
7:     **else** WRFtoMWF$(X \cup \{i\}, i+1, q' - w_i)$;
8:     **return**;
9: {main}
10: WRFtoMWF$(\emptyset, 1, q)$;

Algorithm 6 GenerateEWFfromWRF

Input: $\Gamma = [q; w_1, \ldots, w_n]$ with $\sum_{i=1}^{n} w_i \geq q$ and $w_1 \geq \ldots \geq w_n$.
Output: Γ in EWF.

1: WRFtoEWF(X, i)
2:   **if** $i \leq n$
3:     **if** $w(X \cup \{i\}) \geq q$
4:       **print** $X \cup \{i\}$;
5:     WRFtoEWF$(X \cup \{i\}, i+1)$;
6:     WRFtoEWF$(X, i+1)$;
7: {main}
8: WRFtoEWF$(\emptyset, 1)$;

We finish this section with some enumeration results. The proof of the following Lemma 3.13 is based on an algorithm of [167], that the authors used to compute the Deegan-Packel index [59] for weighted games in MWF.

Lemma 3.13. For weighted games, WRF⇝EWF and WRF⇝MWF can be solved with polynomial-delay.

Proof. WRF⇝MWF was shown by using Algorithm 5. Analogously to Algorithms 1 and 4, steps 1-8 form the recursive procedure started by the main routine of step 10. Each step has a time complexity bounded by $O(1)$ because of the updates of $\sum_{j=i+1}^{n} w_j$ at each iteration. The whole algorithm has time complexity $O(n|\mathcal{W}^m|)$ and memory complexity $O(n)$ [167].

For WRF⇝EWF, we show Algorithm 6 which is even simpler. Here also every step has a time complexity bounded by $O(1)$. Further, the algorithm has time complexity $O(n|\mathcal{W}|)$ and memory complexity $O(n)$. Both recursions of steps 5 and 6 allow to generate the winning coalitions in an orderly fashion, so that none is repeated. □

We think that there exists a similar procedure as Algorithms 5 and 6 to postulate the following.

Conjecture 3.2. For weighted games, WRF⇝SWF can be solved with polynomial-delay.

Regarding vector-weighted games, note that there exist simple games with exponential dimension in function of the number of players—see, for instance, Theorem 8 of [72]. Therefore, it is not clear the complexity of the conversion problem from other forms of representation to VWRF.

Chapter 4

Computational Problems

In this chapter we present several complexity results related to the computation of the properties, parameters and solution concepts for simple games defined in Sections 2.3.1 and 2.3.2.

In Section 4.1 we survey the known results presented in the literature. In Section 4.2 we prove new results related to the IsDecisive and IsStrong problem for simple games, regular games, and weighted games in different forms of representation. These results solve some open problems and refute some conjectures proposed in other works. In Section 4.3 we solve the Width open problem for simple games in MWF, we propose two new parameters related to the width and the length, and we introduce the IsDummy problem for regular games in SWF.

We finish this chapter with Section 4.4, where we focus on the problem of counting and enumerating subclasses of simple games. We survey the main known results on this topic, and we present an idea to deal with the enumeration of decisive regular games.

4.1 Known Complexity Results

4.1.1 Properties and Parameters

The known results of this section are summarized in tables. The absence of references means that the result is trivial, easily verifiable from its definition. Question marks "?" represent open problems.

Similarly to [92], we use the following general notation for the problems to be studied in this section.

Name: IsX
Input: Simple game Γ
Question: Does Γ satisfy property X?

Name: X
Input: Simple game Γ
Output: X(Γ), i.e. the value of the parameter X for Γ.

In general, we extend the notation IsX to the problem of deciding a property X for games, players or coalitions, considering an input formed by a simple game and players and/or coalitions. Additionally, we consider the following two problems.

Name: Iso
Input: Two games.
Question: Are both games isomorphic?

Name: Equiv
Input: Two games.
Question: Are both games equivalent?

Table 4.1 shows the known computational complexity results to decide some properties and parameters of simple games defined in Section 2.3.1, under different forms of representation. Some results of the table are updated in Sections 4.2 and 4.3. The inputs of the problems IsBlocking and IsSwing are a simple game in addition of a coalition. The inputs of the problems of "Properties of players" consider a simple game and a player, but the IsCritical problem also requires a coalition, and the IsSymmetric problem also requires another player.

We denote gIso as the class of problems reducible to graph isomorphism. Note that for simple games in MWF it is easy to see, using arguments from [161], that the Iso problem and the graph isomorphism problem are equivalent.

Regarding the dimension of a simple game, in [251] was shown that for $n \geq 1$ players, there is always a simple game of dimension n, and moreover, that the dimension of a simple game may be exponential in the number of

	Simple games			Regular	Weighted
Problem	EWF	MWF	VWRF	SWF	WRF
Properties of simple games					
IsProper	P	P	coNPC	?	coNPC [92]
IsStrong	P	coNPC [215]	coNPC	?	coNPC [92]
IsDecisive	P	?	coNPC	?	coNPC [10]
Properties of coalitions					
IsBlocking	P	P	P	P	P
IsSwing	P	P	P	P	P
Properties of players					
IsDummy	P	P	coNPC [167]	?	coNPC [167]
IsPasser	P	P	P	P	P
IsVetoer	P	P	P	P	P
IsDictator	P	P	P	P	P
IsCritical	P	P	P	P	P
IsSymmetric	P	P	coNPC [167]	P	coNPC [167]
Parameters					
Length	P	P	NPH [10]	P	P [10]
Width	?	?	P [10]	?	P [10]
Coleman's power	P	#PC [9]	#PC [9]	?	#PC [9]
Chow parameters	P	#PC [15, 9]	#PC [9]	?	#PC [99, 9]
Additional problems					
Equiv	P	P	coNPH [72]	?	coNPC [72]
Iso	gIso	gIso	?	?	?

Table 4.1: Known complexity results for properties and parameters of simple games.

players. Furthermore, it is known that given k weighted games, deciding whether the dimension of their intersection exactly equals k is NP-hard [63]. Given k weighted games, it remains open to show whether the problem of deciding if the codimension of their union exactly equals k is NP-hard.

For vector-weighted games in VWRF, these problems are at least as hard as for weighted games in WRF. The results without references for this form of representation are deduced from the column for WRF.

Properties of Simple Games

Given a simple game, to decide the IsProper problem it is just necessary to consider the MWCs of the game. As a matter of fact, if the complement $N \setminus X$ of a MWC X is losing, then it is clear that the complement $N \setminus Z$ of every coalition Z with $X \subseteq Z$ will be losing too.

Furthermore, given a simple game in EWF, to decide the IsStrong problem we can verifying if the complement $N \setminus X$ of every maximal losing coalition X is winning. Then it is clear that the complement $N \setminus Z$ of every coalition Z with $Z \subseteq X$ will be winning too. Since by Lemma 3.2 we know that EWF⇝MLF can be solved in polynomial time, the IsStrong problem can be solved in polynomial time when the simple game is given in EWF, ELF and MLF.

However, it is hard to decide the IsStrong problem when only the set of MWCs is given. Indeed, recall that MWF⇝MLF and MWF⇝ELF cannot be solved in sub-exponential time. This difficulty seems to be impossible to overcome even if the game is proper. Indeed, in [92] was conjectured that the IsDecisive problem is coNP-complete for simple games given in MWF. Fortunately, this assumption is wrong, as we shall see in Section 4.2.

Properties of Coalitions

The problems of this thesis regarding properties of coalitions are easy. In general, to decide whether a coalition $X \subseteq N$ is either blocking or a swing can always be done in polynomial time, whenever $\nu(X)$ can be determined in polynomial time.

Properties of Players

Most problems concerning properties of players are easy whenever for every coalition X, $\nu(X)$ can be computed in polynomial time. For instance, considering a player i, for the problems IsPasser and IsVetoer it is just necessary to determine whether $\nu(\{i\}) = 1$ and $\nu(N \setminus \{i\}) = 0$, respectively. However, there are some exceptions for succinct forms of representations.

For the IsDummy problem it is necessary to check for the given player i whether it is not contained in some MWC. Evidently this is easy when the set of MWCs is explicitly given, which is not the case for weighted games in WRF. As far as we know, the complexity of this problem for regular games in SWF has not been determined. As a matter of fact, it is clear that if there is some shift-minimal winning coalition that contains i, then i cannot be a dummy player; but it is not so clear what happens when this is not the case.

To solve the IsSymmetric problem in polynomial time it seems that at least the set of shift-minimal winning coalitions must be given explicitly. Let i, j two players in the grand coalition. For simple games in EWF, the problem is equivalent to deciding if for all $X \in \mathcal{W}$, $i \in X$ if and only if $j \in X$. For simple games in MWF, the problem is equivalent to deciding if for all $X \in \mathcal{W}^m$, when $i \in X$ and $j \notin X$, there exists another MWC $Y \in \mathcal{W}^m$ so that $i \notin Y$ and $j \in Y$. For regular games in SWF, the problem is equivalent to the previous one, considering $\mathcal{W}^s$ instead of $\mathcal{W}^m$, and the fact that for every $X \in \mathcal{W}^s$ with $i \notin X$ and $j \notin X$, it holds $X \preceq X \cup \{i\}$ and $X \preceq X \cup \{j\}$.

Parameters

Regarding the parameters, the Length problem can be computed in polynomial time even for weighted games in WRF: Given a weighted game $\Gamma = [q; w_1, \ldots, w_n]$, start with the player with less weight w_1, and keep adding more players with decreasing weights until $\sum_{i=1}^{k} w_i \geq q$; then $length(\Gamma) = k$. Since for any weighted game Γ its dual Γ^d can be obtained in polynomial time, $width(\Gamma)$ can also be computed in polynomial time [10]. However, the dualization of a simple game in less succinct forms of representation is not polynomial—see Section 4.2—so by the same reasoning we cannot deduce the complexity of the Width problem for simple games in other forms of representation. This open problem established in [10] is solved in Section 4.3.

About both the Coleman's power and Chow parameters problems, it is clear that the second one is at least as computationally hard as the first one. Further, if the Chow parameters problem can be computed in polynomial time, then the Coleman's power problem can also be computed in polynomial time.

4.1.2 Solution Concepts

There is a lot of work done about solution concepts, although most is focused in cooperative games rather than restricted to simple games. The computational complexity of solution concepts has been studied for many classes of cooperative games, such as assignment games [105, 240, 196], coalitional skill games [12], convex games [145], cyclic permutation games [241], flow games [105, 60, 218], induced subgraph games [61], matching games [137], min-cost spanning tree games [78], neighbor games [112], shortest path games [195], spanning connectivity games [11], standard tree games [107], threshold network flow games [14], vertex connectivity games [13], among several others.

There are various computational problems that can be defined over solution concepts. We consider four kind of problems, restricted to simple games, which were studied on [10, 73].

Name: Empty-X
Input: Simple game Γ
Question: Is the solution concept X empty in Γ?

Name: In-X
Input: Simple game Γ and payoff **p**
Question: Is the payoff **p** in the solution concept X of Γ?

Name: IsZero-X
Input: Simple game Γ and player i
Question: Is payoff of player i in game Γ zero according to solution X?

Name: Construct-X
Input: Simple game Γ
Output: A payoff in the solution concept X of Γ.

The known computational complexity results for these problems under different solution concepts are illustrated on Table 4.2. Note that the families of problems Empty-X and In-X only apply to sets, and the family of problems IsZero-X only applies to values. We omit from the table those problems that do not apply, like for instance Empty-Banzhaf-value or IsZero-Stable-set. The results without references are shown in the thesis of Aziz [10].

The rows labeled Empty-, In-, IsZero- and Construct- refer to the different kind of problems described above. The first column, below of each of those rows, refers to the solution concepts considered for that problem. The next four columns represent the complexity for each form of representation considered in the input of the problem. Thus, the fourth row, for instance, means that the problem Empty-Stable-Set is always nonempty for simple games in any representation form.

Henceforth, we denote the problems Construct-Banzhaf-value and Construct-Shapley-Shubik-value as Bval and SSval, respectively.

	Simple games			Weighted
Problem	EWF	MWF	VWRF	WRF
EMPTY-				
STABLE-SET	always nonempty			
CORE	P	P	P	P [73]
ϵ-CORE	P	P	NPH	NPH [73]
LEAST-CORE	always nonempty			
BARGAINING-SET	always nonempty if no passer			
KERNEL	always nonempty if no passer			
PREKERNEL	always nonempty			
NUCLEOLUS	always nonempty if no passer			
PRENUCLEOLUS	always nonempty			
IN-				
BANZHAF-VALUE	P	?	?	?
SHAPLEY-SHUBIK-VALUE	P	?	NPH	NPH
STABLE-SET	?	?	?	?
CORE	P	P	P	P [73]
ϵ-CORE	P	P	NPH	coNPH [73]
LEAST-CORE	P	P	NPH	NPH [73]
BARGAINING-SET	?	?	?	?
KERNEL	P	P	?	?
PREKERNEL	P	P	?	?
NUCLEOLUS	P	P	?	?
PRENUCLEOLUS	P	P	?	?
ISZERO-				
BANZHAF-VALUE	P	P	?	?
SHAPLEY-SHUBIK-VALUE	P	P	NPH	NPH
CORE	P	P	P	P [73]
NUCLEOLUS	P	P	NPH	coNPH [73]
PRENUCLEOLUS	P	P	NPH	NPH
CONSTRUCT-				
BANZHAF-VALUE	P [9]	#PC [9]	#PC [9]	#PC [219] NPC [168]
SHAPLEY-SHUBIK-VALUE	P [9]	#PC	#PC [9]	#PC [61] NPC [168]
STABLE-SET	P	P	P	P
CORE	P	P	P	P [73]
ϵ-CORE	P	P	NPH	NPH [73]
LEAST-CORE	P	P	NPH	NPH [73]
BARGAINING-SET	P	P	?	?
KERNEL	P	P	?	?
PREKERNEL	P	P	?	?
NUCLEOLUS	P	P	NPH	NPH [73]
PRENUCLEOLUS	P	P	NPH	NPH

Table 4.2: Known complexity results for solution concepts on simple games.

Power Indices

In Table 4.2 we consider the problems related to the Banzhaf value and the Shapley-Shubik value, instead of the Banzhaf index and the Shapley-Shubik index, respectively. It is clear by Definition 2.28 that since the denominator of the Shapley-Shubik index is fixed, the computation of the Shapley-Shubik index and the Shapley-Shubik value has the same complexity. This is also true for the probabilistic Banzhaf index and the Banzhaf value, but it is not necessarily true for the Banzhaf index and the Banzhaf value—see Definition 2.29. In this latter case we only can said that if the Banzhaf value can be computed, then it can be used to compute the Banzhaf index. Furthermore, it is known that for any form of representation, computing the Shapley-Shubik index is at least as hard as computing the Banzhaf index [10].

Note that both the BVAL and SSVAL problems correspond to compute $\eta_i(\Gamma)$ and $\kappa_i(\Gamma)$, respectively, for every player i on the simple game Γ. In despite of the hardness results for weighted games in WRF, in [167] was proved that there exist pseudo-polynomial time algorithms based on dynamic programming to solve both problems. Since then, several algorithms have been designed to solve the problem as quickly as possible [253, 140, 29].

Core, ϵ-Core and Least Core

The core imputations satisfy a system of weak linear inequalities. It is closed and convex, which means that it is a feasible set that can be solved by using linear programming. Moreover, for simple games the core is well characterized by Proposition 2.4. The computation of the core for simple games is easy even for succinct forms of representation.

To decide the EMPTY-CORE problem it is just enough to check for any player $i \in N$ whether $\nu(N \setminus \{i\}) = 0$. If this is true for at least one player, the answer of the problem is "yes"; otherwise is "no". The problems IN-CORE, ISZERO-CORE and CONSTRUCT-CORE can be solved by using the same characterization.

On the other hand, the least core can be computed for any simple game

(N,ν) by using the following linear program [74]:

$$\begin{array}{lll} \min & \epsilon & \\ \text{s.t.} & p_i \geq 0 & \text{for all } i \in N \\ & \sum_{i\in N} p_i = 1 & \\ & \sum_{i\in X} p_i \geq \nu(X) - \epsilon & \text{for all } X \in \mathcal{W}^m \end{array}$$

whose solution is a least core imputation and the smallest ϵ value of all possible ϵ-cores. Note that the second inequality considers only minimal winning coalitions. This is enough [10] because for any $X \in \mathcal{W}^m$ with $p(X) \geq \nu(X) - \epsilon = 1 - \epsilon$, if $X \subset Y$ then $p(Y) \geq 1 - \epsilon$; and if $Y \subset X$ then $p(Y) \geq 0 - \epsilon$. Therefore, as linear programming is solvable in polynomial time [138], the problems Construct-Least-core and In-Least-core can be solved in polynomial time whether the set of MWCs can be obtained in polynomial time in terms of the input size.

Despite the hardness results for weighted games in WRF, the problems Empty-ϵ-core, In-ϵ-core, In-LeastCore, Construct-ϵ-core and Construct-Least-core, can be solved by pseudo-polynomial time algorithms. Indeed, all these problems are polynomial-time solvable if the weights are at most polynomially large in n, or—equivalently—if they are represented in unary notation [73]. The same occurs for vector-weighted games in VWRF [73, 74].

In general, by using Proposition 2.6, if the problem Length is NP-hard then the problem In-ϵ-core is also NP-hard [10]. Furthermore, it is conjectured that if Length is NP-hard, then the problem Construct-Least-core is also NP-hard [10].

Nucleolus and Prenucleolus

For any reasonable form of representation of a simple game, verifying whether a payoff is an imputation or just a preimputation can be done in polynomial time. Therefore, in terms of the computational complexity, there is no difference between the problems concerning nucleolus or prenucleolus.

As the core, the nucleolus can also be computed with linear programming. In this case, it is necessary to solve a sequence of at most n successive linear programs [142]. The first linear program corresponds to the computation of the least core:

$$\begin{array}{lll} \min & \epsilon & \\ \text{s.t.} & p_i \geq 0 & \text{for all } i \in N \\ & \sum_{i \in N} p_i = 1 & \\ & \sum_{i \in X} p_i \geq \nu(X) - \epsilon & \text{for all } X \in \mathcal{W}' \end{array}$$

where, again, it is not necessary to consider all the set of coalitions, but only $\mathcal{W}' = \mathcal{W}^m \cup \{X \cup \{i\} \mid X \in \mathcal{W}^m, i \in N\}$ [223, 10]. Lets denote the output of this first linear program as $(\mathbf{p}^1, \epsilon^1)$. Let $\sum^1$ be the set of tight constraints for $(\mathbf{p}^1, \epsilon^1)$, which by a slight abuse of notation also represents all the coalitions $X \in \mathcal{W}'$ such that $p(X) = \nu(X) - \epsilon$. Thus, the second linear program is

$$\begin{array}{lll} \min & \epsilon & \\ \text{s.t.} & p_i \geq 0 & \text{for all } i \in N \\ & \sum_{i \in N} p_i = 1 & \\ & \sum_{i \in X} p_i = \nu(X) - \epsilon^1 & \text{for all } X \in \sum^1 \\ & \sum_{i \in X} p_i \geq \nu(X) - \epsilon & \text{for all } X \in \mathcal{W}' \setminus \sum^1 \end{array}$$

where the tight constraints now appear in an equality. The remaining coalitions appear in the same inequality that needs to be re-computed until the payoffs to all coalitions are determined, i.e., until the solution space of the current linear program consists of a single point [74]. Hence, the jth linear program is given by

$$\begin{array}{lll} \min & \epsilon & \\ \text{s.t.} & p_i \geq 0 & \text{for all } i \in N \\ & \sum_{i \in N} p_i = 1 & \\ & \sum_{i \in X} p_i = \nu(X) - \epsilon^1 & \text{for all } X \in \sum^1 \\ & \dots & \dots \\ & \sum_{i \in X} p_i = \nu(X) - \epsilon^{j-1} & \text{for all } X \in \sum^{j-1} \\ & \sum_{i \in X} p_i \geq \nu(X) - \epsilon & \text{for all } X \in \mathcal{W}' \setminus \bigcup_{k=1}^{j-1} \sum^k . \end{array}$$

The explanation of this computation can also be seen in [74].

Since the nucleolus is unique, it is clear that if Construct-Nucleolus can be solved in polynomial time, then the In-Nucleolus problem can be decided in polynomial time. Moreover, if the Construct-Least-core problem is NP-hard, then CONSTRUCT-NUCLEOLUS is also NP-hard [10].

Regarding weighted games in WRF, if the core is nonempty, then the nucleolus can be obtained in polynomial time.

Theorem 4.1 ([73]). Let Γ be a weighted game with nonempty core and k vetoers. The nucleolus $\mathcal{N}(\Gamma)$ is given by the homogeneous imputation $(p_1, \dots, p_n)$, such that for all $i \in N$ we have

$$p_i = \begin{cases} \frac{1}{k} & \text{if } i \text{ is vetoer} \\ 0 & \text{otherwise.} \end{cases}$$

When the core is empty, the CONSTRUCT-NUCLEOLUS problem is NP-hard, but as the problems CONSTRUCT-ϵ-CORE and CONSTRUCT-LEAST-CORE, it can also be solved by a pseudo-polynomial time algorithm, so the problem is polynomial-time solvable if the weights are at most polynomially large in n [73, 74]. The same occurs for vector-weighted games in VWRF [73, 74]. Moreover, it is also interesting to note that there are subclasses of weighted games for which the problems CONSTRUCT-NUCLEOLUS, IN-NUCLEOLUS and ISZERO-NUCLEOLUS are polynomial-time solvable, even when the core is empty.

Proposition 4.1 ([211]). Let $\Gamma = [q; w_1, \dots, w_n]$ be a decisive homogeneous game in which each dummy player gets zero weight. Then the nucleolus is $\mathcal{N}(\Gamma) = (w_1/w(N), \dots, w_n/w(N))$.

Stable Set

The stable set is a solution concept slightly studied in simple games. It is clear by Proposition 2.3 that the CONSTRUCT-STABLE-SET problem can be solved in polynomial time whenever a MWC can be obtained in polynomial time. However, the same proposition does not implies that the IN-STABLE-SET problem can be decided in polynomial time, as it is stated in [10]. Actually, given an imputation, by using this result we can only verifying whether it is contained in some of those stable sets provided by the MWCs. But if the answer is “no”, there may exist other stable sets in which the imputation is contained. In the absence of other explicit results, to our knowledge this problem remains open.

Kernel and Prekernel

The kernel has also been much studied in simple games, although unlike the core, for this solution concept there are not too many complexity results. Analogously to the core and the nucleolus, it is known that the kernel corresponds to a union of a finite number of closed convex polyhedra [55].

Given a simple game $\Gamma = (N, \nu)$ and a payoff $\mathbf{p} \in I(\Gamma)$, to decide both the IN-KERNEL and IN-PREKERNEL problems we can start by determining the $n(n-1)$ maximum surpluses

$$s_{ij}^{\nu}(\mathbf{p}) = \max\{\nu(X) - p(X) \mid X \in \mathcal{W}^m, i \in X, j \notin X\}.$$

Note that, as in the core, we can concentrate the computation at the MWCs. Indeed, for any $X \in \mathcal{W}^m$ and another coalition $X' \subseteq N$, if $X \subset X'$, then we have $\nu(X) - p(X) = 1 - p(X) \geq 1 - p(X') = \nu(X') - p(X')$, and if $X' \subset X$, then $\nu(X) - p(X) = 1 - p(X) \geq 0 - p(X') = \nu(X') - p(X')$. In the case that for all $X \in \mathcal{W}^m$, $j \in X$, then $s_{ij}^{\nu}(\mathbf{p})$ must be given by a losing coalition, because it does not contain j; therefore, the coalition that provides the maximum surplus containing i is the singleton $\{i\}$. In the case that for all $X \in \mathcal{W}^m$, when $j \notin X$ it also holds that $i \notin X$, then $s_{ij}^{\nu}(\mathbf{p})$ must be given by a non-MWC, provided by the coalition $X' \cup \{i\}$ with the minimum $p(X')$, so that $j \notin X'$. Finally, we verify whether the payoff in the maximum surpluses satisfy both the inequalities $(s_{ij}^{\nu}(\mathbf{p}) - s_{ji}^{\nu}(\mathbf{p}))(p_j - \nu(\{j\})) \leq 0$ and $(s_{ji}^{\nu}(\mathbf{p}) - s_{ij}^{\nu}(\mathbf{p}))(p_i - \nu(\{i\})) \leq 0$ for the kernel, and the equalities $s_{ij}^{\nu}(\mathbf{p}) = s_{ji}^{\nu}(\mathbf{p})$ for the prekernel.

We know that in simple games with a nonempty set of imputations, the intersection of the kernel and the least core coincides with the intersection of the prekernel and the least core. There exist two solution concepts, namely the *lexicographic prekernel* and the *lexicographic kernel*, that belong to these intersections [79]. It is also known that if the maximum surpluses and the MWCs can be obtained in polynomial time, then an outcome of both the lexicographic prekernel and the lexicographic kernel can be obtained in polynomial time [79]. Therefore, these solution concepts can be used to solve both the CONSTRUCT-KERNEL and CONSTRUCT-PREKERNEL problems in polynomial time, depending on the form of representation of the game.

For weighted games in WRF, the complexity of the IN-KERNEL, IN-

Prekernel, Construct-Kernel and Construct-Prekernel problems is still open. This does not mean that these problems have not been studied. We know at least the following.

Theorem 4.2 ([271, 10])**.** Let $\Gamma = [q; w_1, \ldots, w_n]$ be a weighted game without vetoers and such that $q \geq w_1 \geq \ldots \geq w_n$. Let $\mathbf{p} = (p_1, \ldots, p_n) \in I(\Gamma)$ be a homogeneous payoff, where $R = \{1, \ldots, r\} \subseteq N$ is the set of r players with payoff $\frac{1}{r}$ and $N \setminus R$ is the set of $n - r$ players with payoff equals zero. Let be $T(R) = \{1, \ldots, k\} \cup N \setminus R$, where $k = \max_{m \in [0,r]} \sum_{i \in T(R)} w_i < q$. This payoff $\mathbf{p}$ is in the kernel if and only if $w(T(R)) + w_r - w_{r+1} \geq q$ and $w(T(R)) - w_1 + w_{k+1} + w_r \geq q$.

Moreover, for some kind of weighted games, the kernel is equivalent to the nucleolus given by the homogeneous imputation of Theorem 4.1 [7].

Bargaining Set

Similarly to the stable set, the bargaining set has been little studied in simple games. The existence of several slightly different definitions [7] makes their study more difficult to address. However, every bargaining set can be obtained by a system of linear inequalities in the space of the payoffs [6]. Moreover, it is important to remember that in simple games, when the core is nonempty, it is equivalent to the bargaining set [70]. Therefore, all the open complexity problems related with the bargaining set are reduced to analyze simple games with empty core, i.e., with no vetoers.

Recall that the kernel, and hence the nucleolus, are contained in the bargaining set. Thus, given a subclass of simple games in certain representation form, if either the Construct-Kernel or the Construct-Nucleolus problem can be solved in polynomial time, then Construct-Bargaining-Set can also be solved in polynomial time.

4.2 Decisiveness and Strongness

In this section we solve several open complexity problems related to both the IsDecisive and the IsStrong problems for simple games. Recall by Definition 2.23 that a simple game is decisive if and only if it is proper and strong. The main results are illustrated in bold in Table 4.3, where the question mark remains as a conjecture.

	Simple games			Regular	Weighted
Problem	EWF	MWF	VWRF	SWF	WRF
Properties of simple games					
IsProper	P	P	coNPC	**P**	coNPC
IsStrong	P	coNPC	coNPC	**coNPC**	coNPC
IsDecisive	P	**QP**	coNPC	**QP?**	coNPC

Table 4.3: New complexity results for decisiveness of simple games.

The main result of the following Section 4.2.1 is Theorem 4.3, that solves a conjecture given by [92], which suggested that the IsDecisive problem for simple games in MWF, rather than QP, was coNP-complete.

In Section 4.2.2 we provide some results regarding weighted games. Given a simple game in MWF, we design an algorithm to decide in polynomial time whether the game is weighted or not. With some slight modifications, we can also decide in polynomial time whether the game is homogeneous or not. By using the first algorithm, we can also decide in polynomial time whether the game is majority—i.e., weighted and decisive—or sub-majority—i.e., weighted and strong. Thus, Theorem 4.5 solves another conjecture given by [92], which suggested that this last problem, rather than polynomial, was coNP-complete.

Finally, in Section 4.2.3 we show a polynomial-time reduction from the IsDecisive (resp. IsStrong) problem for simple games in MWF to the IsDecisive (resp. IsStrong) problem for regular games in SWF. Thus, we prove that the IsStrong problem for regular games in SWF is coNP-complete, and the IsDecisive problem is most probably not coNP-complete, but probably belongs to QP.

4.2.1 Decisiveness for Simple Games

In this subsection we establish the equivalence among the IsDecisive problem and the duality problem for hypergraphs. We show that decisiveness can naturally be represented in the context of hypergraph theory, in such a way that it can be decided for simple games in quasi-polynomial time.

Let $\Gamma = (N, \mathcal{W})$ be a simple game, note from Definition 2.7 that $\mathcal{W}$ is a hypergraph over N with $\nu(\mathcal{W}) = \mathcal{W}$ and $\mu(\mathcal{W}) = \mathcal{W}^m$. Thus, the amount of information needed to specify Γ is given by the size $n \cdot |\mu(\mathcal{W})|$ of $\mu(\mathcal{W})$. Note also that $\mathcal{L} = \neg(\tau(\mathcal{W}))$ is the set of losing coalitions, and the simple game $(N, \tau(\mathcal{W}))$ corresponds to the dual game of $(N, \mathcal{W})$.

Moreover, the properties of hypergraphs presented in Definition 2.8 are closely related to some properties of simple games.

Lemma 4.1. Let $\Gamma = (N, \mathcal{W})$ be a simple game, then:

- Γ is proper iff $(\mathcal{W}, \mathcal{W})$ is coherent iff $(\mu(\mathcal{W}), \mu(\mathcal{W}))$ is coherent,
- Γ is strong iff $(\mathcal{W}, \mathcal{W})$ is complete iff $(\mu(\mathcal{W}), \mu(\mathcal{W}))$ is complete,
- Γ is decisive iff $(\mathcal{W}, \mathcal{W})$ is dual iff $(\mu(\mathcal{W}), \mu(\mathcal{W}))$ is dual.

Proof. Just note that:
$(\mathcal{W}, \mathcal{W})$ is coherent iff $\nu(\mathcal{W}) \subseteq \tau(\mathcal{W})$ iff $\mathcal{W} \subseteq \tau(\mathcal{W})$, i.e. iff Γ is proper.
$(\mathcal{W}, \mathcal{W})$ is complete iff $\nu(\mathcal{W}) \supseteq \tau(\mathcal{W})$ iff $\mathcal{W} \supseteq \tau(\mathcal{W})$, i.e. iff Γ is strong.
$(\mathcal{W}, \mathcal{W})$ is dual iff $\nu(\mathcal{W}) = \tau(\mathcal{W})$ iff $\mathcal{W} = \tau(\mathcal{W})$, i.e. iff Γ is decisive. ☐

Example 4.1. In projective geometry, the *Fano Plane* is the smallest projective plane. It was introduced in simple game theory by [225] to define a subclass of simple games called *finite projective games*. Since then, it has been very much studied, due to it has special properties that make it a likely counterexample for different results—for instance, without going into details, it is the only non-partition game with the same number of minimal winning coalitions and players [246]—as well as a case in which some properties turn out to be the same—for instance, its reactive bargaining set coincides with its kernel [106].

Observe that the Fano Plane can be represented by a hypergraph $\mathcal{H}$ over $N = \{1, \dots, 7\}$, with seven evenly distributed hyperedges represented by the following incidence matrix.

N	1234567
$\mathcal{H}$	0000111
	0011010
	0101100
	0110001
	1001001
	1010100
	1100010

It is easy to check that $\mu(\mathcal{H}) = \mathcal{H}$ and prove that $(\mathcal{H}, \mathcal{H})$ is coherent. It is more difficult to prove the completeness of this pair—see the next Theorem 4.3—but in fact it is dual. So the game $(N, \nu(\mathcal{H}))$ is proper and strong, i.e., decisive. In addition, note that by its symmetry it is neither regular nor linear.

It is well known that the IsProper problem can be solved in polynomial time. In the context of hypergraphs, this is also clear because since ν is monotone, $(\mathcal{H}, \mathcal{H})$ is coherent if and only if for all $X, Z \in \mathcal{H}$, $X \cap Z \neq \emptyset$, a condition that can be verified in polynomial time.

Furthermore, it is known that the problem of deciding whether a pair of hypergraphs $(\mathcal{H}, \mathcal{K})$ is complete is coNP-complete [215]. Since there is a polynomial-time reduction from this problem to the same problem for the case when $\mathcal{H} = \mathcal{K}$ [84], then we conclude that the IsStrong problem for simple games in MWF is also coNP-complete. Another proof for this last result in simple games was shown in [92].

In turn, the IsDecisive problem for simple games in MWF belongs to QP, since the duality of a pair of hypergraphs $(\mathcal{H}, \mathcal{H})$ can be decided in quasi-polynomial time [84]. Thus, from all the above we conclude the following.

Theorem 4.3. For simple games in MWF, the IsProper problem belongs to P, the IsStrong problem is coNP-complete, and the IsDecisive problem belongs to QP.

Note that the IsDecisive problem is most probably not NP-hard, unless any NP-complete problem can be solved in quasi-polynomial time. But note also that the mentioned quasi-polynomial algorithm does not allow us to generate $\lambda(\mathcal{H}) = \mu(\tau(\mathcal{H}))$ in sub-exponential time, since $|\lambda(\mathcal{H})|$ can not be quasi-polynomially bounded by $|\mu(\mathcal{H})|$. To prove this statement, consider the following example.

Example 4.2. Let be $N' = \{1, \ldots, m\}$ with $m \in \mathbb{N}$, $n = 2m$ and further $\mathcal{H} = \{\{2i-1, 2i\} \mid i \in N'\}$. Therefore, the irredundant transversals of $\mathcal{H}$ are $\lambda(\mathcal{H}) = \{X \subseteq N \mid \text{ for all } i \in N', \text{ either } 2i-1 \in X \text{ or } 2i \in X\}$. So $|\mathcal{H}| = m$, but $|\lambda(\mathcal{H})| = 2^m$.

Algorithm 7 DecideWeighted

Input: A simple game in MWF represented by a hypergraph $\mathcal{H} = \mu(\mathcal{W})$.
Output: If the game is weighted, return "Yes"; otherwise "No".

1: **if** $\mathcal{H}$ is not linear, **return** "No";
2: Determine a linear ordering of N which makes $\mathcal{H}$ regular;
3: Generate $\mathcal{J} = \neg(\lambda(\mathcal{H}))$;
4: **if** there is not exists a weighted representation $[q; w_1, \ldots, w_n]$ s.t.:
   for all $X \in \mathcal{H},\ w(X) \geq q$;
   for all $Y \in \mathcal{J},\ w(Y) \leq q - 1$;
   for all $i \in N, \quad w_i \geq 0$;
  return "No";
5: **return** "Yes".

4.2.2 Decisiveness and Strongness for Weighted Games

The following result was firstly proved in the context of threshold functions. Now we provide a proof related to hypergraphs and simple games.

Theorem 4.4. [209] For simple games in MWF, deciding whether it is weighted or not, can be done in polynomial time.

Proof. Let us consider Algorithm 7. It is well known that steps 1 and 2 can be computed in polynomial time [163]. For step 3 we can use the algorithm provided in [210], that given $\mathcal{H} = \mu(\mathcal{W})$, it generates $\lambda(\mathcal{H})$ in linear time and at the same time proves that $|\lambda(\mathcal{H})| \leq n \cdot |\mathcal{H}| + 1$. Finally, step 4 only demands the solution of a system of linear inequalities, which can also be found in polynomial time [138]. Therefore, since each step can be accomplished in polynomial time, this algorithm runs in polynomial time. Finally, as the weighted games are the simple games that admit a representation in WRF, the algorithm is correct and it proves the theorem. □

Slightly modifying Algorithm 7, we obtain the following result.

Corollary 4.1. For simple games in MWF, deciding whether it is homogeneous or not, can be done in polynomial time.

Proof. Just replace in Algorithm 7 the first inequation of step 4 by the equation $w(X) = q$. □

Based on the same idea of Algorithm 7, we have also the following result.

Theorem 4.5. For simple games in MWF, deciding whether it is majority—i.e., weighted and decisive—or not, can be done in polynomial time.

Algorithm 8 DecideMajority

Input: A simple game in MWF represented by a hypergraph $\mathcal{H} = \mu(\mathcal{W})$.
Output: If the game is majority, return "Yes"; otherwise "No".
1: **if** $\mathcal{H}$ is not linear, **return** "No";
2: Generate $\mathcal{K} = \lambda(\mathcal{H})$;
3: **if** $\mathcal{K} \neq \mathcal{H}$ **return** "No";
4: **if** there is not exists a weighted representation $[q; w_1, \ldots, w_n]$ s.t.:
  for all $X \in \mathcal{H}$, $w(X) \geq q$;
  for all $Y \in \neg(\mathcal{K})$, $w(Y) \leq q - 1$;
  for all $i \in N$, $w_i \geq 0$;
 return "No";
5: **return** "Yes".

Proof. Let us consider Algorithm 8. The procedure to decide whether the game is weighted is the same as the one given by Algorithm 7. Therefore, we need to prove that the algorithm solves IsDecisive in polynomial time. Let $\Gamma = (N, \mathcal{W})$ be a simple game. By the proof of Lemma 4.1 we know that Γ is decisive if and only if $\tau(\mathcal{W}) = \mathcal{W}$. By Definition 2.7, applying the operator μ on the expression we obtain $\lambda(\mathcal{W}) = \mu(\mathcal{W})$. Moreover, it is easy to see that $\lambda(\mu(\mathcal{W})) = \lambda(\mathcal{W})$. Therefore, we can decide IsDecisive through the following question: Is $\lambda(\mathcal{H}) = \mathcal{H}$? which is equivalent to the two conditions given above. This question is addressed in the opposite way in step 3. Therefore, the algorithm is correct, and since the operators can be computed here in polynomial time [210], then it proves the theorem. □

Slightly modifying Algorithm 8, we also obtain the following result.

Theorem 4.6. For simple games in MWF, deciding whether it is sub-majority—i.e., weighted and strong—or not, can be done in polynomial time.

Proof. Note that a simple game is strong if and only if $\tau(\mathcal{W}) \subseteq \mathcal{W}$ if and only if $\lambda(\mathcal{W}) \subseteq \mu(\mathcal{W})$. Therefore, in the same vein than the proof of Theorem 4.5, we can decide IsStrong through the following question: Is $\lambda(\mathcal{H}) \subseteq \nu(\mathcal{H})$? This question is addressed in the opposite way by replacing in Algorithm 8 the inequality $\mathcal{K} \neq \mathcal{H}$ of step 3 by $\mathcal{K} \nsubseteq \nu(\mathcal{H})$. Thus, the algorithm is correct, and as in the previous theorem, it can be computed in polynomial time. □

It is known that up to eight players—see Table 4.5—all the simple games which are regular and decisive are also majority games. This implies that step 4 of Algorithm 8 is unnecessary in these cases, because the answer would always be "Yes". However, from nine players onwards, this is not always true. Indeed, it is known that for $n = 9$ there are 319124 regular and decisive games [144], but only 175428 majority games [190].

4.2.3 Decisiveness and Strongness for Regular Games

In this section we study both the IsStrong and IsDecisive problems for regular games given in SWF. We consider a decreasing lexicographical order of the players, so that $N = \{n, \ldots, 1\}$.

Lemma 4.2. Let $\Gamma = (N, \mathcal{W}^s)$ be a regular game in SWF and $Z \subseteq N$. Deciding whether $Z \in \mathcal{W}$ or $Z \in \mathcal{L}$ can be done in polynomial time in function of the size of Γ.

Proof. Given a coalition Z, consider its associated integer vector $\bar{z}$—see Section 3.2. If there is some $X \in \mathcal{W}^s$ such that $\bar{x} \leq \bar{z}$, then Z is winning; otherwise, it is losing. Note that this candidate can be computed in polynomial time when $\mathcal{W}^s$ is given. □

From the above result, we have the following.

Lemma 4.3. Given a regular game in SWF, the IsProper problem can be solved in polynomial time.

Proof. Let be $\Gamma = (N, \mathcal{W}^s)$, for all $X \in \mathcal{W}^s$, if $N \setminus X \in \mathcal{L}$—by Lemma 4.2 this can be determined in polynomial time—then by monotonicity the complement $N \setminus Z$ of every coalition Z with $X \subseteq Z$ will be losing too. □

Now we define some concepts related to the increasing-shift presented in Definition 2.38.

Definition 4.1. A *left-shift* on X is either an increasing-shift specified by a pair $(a, b) \in N \setminus X \times X$ such that $a = b + 1$; or, if $1 \in N \setminus X$, a replacement of X by $X \cup \{1\}$.

We use $X \subseteq' Z$ to denote the fact that there exists a sequence of left-shifts that can be applied on X to produce Z. Note that this relation $\subseteq'$ is a variation of the desirability relation, and it can also be decided in polynomial time.

Lemma 4.4. Given $X, Z \subseteq N$, we can decide whether $X \subseteq' Z$ in time polynomial in n.

Proof. Given $X, Z \subseteq N$, $X \subseteq' Z$ if and only if, for any $a \in N$, we have $|\{b \in X \mid a \preceq b\}| \leq |\{b \in Z \mid a \preceq b\}|$. If $X \subseteq' Z$, then the stated inequalities evidently hold. If the inequalities hold, then starting from X we can always perform a sequence of left-shifts that preserve the inequalities to finally produce Z. □

Note also that if $X \subseteq Z$ then $X \subseteq' Z$, i.e., $\subseteq'$ is a monotone variation of the relation $\subseteq$. Further, the complement of $\subseteq'$—as for $\subseteq$—is antitone, i.e., if $X \subseteq' Z$ then $N \setminus Z \subseteq' N \setminus X$.

From the above, we can define the operators $\nu', \tau', \mu', \lambda'$ like the already familiar ν, τ, μ, λ—see Definition 2.7—but with the relation $\subseteq'$ instead of $\subseteq$. For instance, given $\mathcal{H} \subseteq \mathcal{P}(N)$, $\tau'(\mathcal{H}) = \{Z \subseteq N \mid \text{ for all } X \in \mathcal{H}, X \not\subseteq' N \setminus Z\}$, and $\mu'(\mathcal{W}) = \{X \in \mathcal{W} \mid \text{ for all } Z \in \mathcal{W}, Z \not\subset' X\}$ yields the shift-minimal winning coalitions of the given game.

Observe that, given a hypergraph $\mathcal{W}$ over N, it holds that $\mathcal{W} \subseteq \nu(\mathcal{W}) \subseteq \nu'(\mathcal{W})$, so $(N, \mathcal{W})$ is a regular game if and only if $\nu'(\mathcal{W}) = \mathcal{W}$. Therefore, $\nu'(\mathcal{W}) = \nu(\mathcal{W}) = \mathcal{W}$ is equivalent to $\nu'(\mathcal{W}) \subseteq \mathcal{W}$, and hence regular games are $\subseteq'$-monotone games, as simple games are $\subseteq$-monotone games.

Note also that if Γ is regular, then $\tau(\mathcal{W}) = \tau'(\mathcal{W})$, so its dual game $(N, \tau(\mathcal{W}))$ is regular too.

Lemma 4.5. Let $\Gamma = (N, \mathcal{W})$ be a regular game:

- Γ is proper if and only if $\nu'(\mathcal{W}) \subseteq \tau'(\mathcal{W})$;
- Γ is strong if and only if $\nu'(\mathcal{W}) \supseteq \tau'(\mathcal{W})$; and
- Γ is decisive if and only if $\nu'(\mathcal{W}) = \tau'(\mathcal{W})$.

Proof. It follows directly from Lemma 4.1 and the variations of the operators based on the relation $\subseteq'$. □

For all the following results, we present a transformation T that from a simple game, it produces a regular game over an expanded grand coalition. We will show that this function reduces the IsDecisive problem for simple games to the same problem for regular games, and further, this reduction can be computed in polynomial time.

Definition 4.2. Let $N = \{n, \ldots, 1\}$ and $N' = \{2n, 2n-1, \ldots, 2, 1\}$. Let $\mathcal{P}(N)$ be ordered according to $\subseteq$ and $\mathcal{P}(N')$ be ordered according to $\subseteq'$. Let $T : \mathcal{P}(N) \to \mathcal{P}(N')$ be a function such that given $X \subseteq N$ and $a \in N$, $2a \in T(X)$ if and only if $a \in X$, and $2a - 1 \in T(X)$ if and only if $a \notin X$. For any $a \in N$, let $Z^a = \{2a\} \cup \{\{2b - 1 \mid b \in N, a \preceq b\} \subseteq N'$ and let $\mathcal{G}' = \{Z^a \mid a \in N\} \subseteq \mathcal{P}(N')$.

This definitions are applied later in Example 4.3. Note that T is an injective function, and for all $X, Z \subseteq N$, $X \subseteq Z$ if and only if $T(X) \subseteq' T(Z)$.

Furthermore, the image $T(\mathcal{P}(N)) = \{T(X) \mid X \subseteq N\}$ of $\mathcal{P}(N)$ contains only the coalitions $X' \in \mathcal{P}(N')$ such that, for all $a \in N$, either satisfy $2a \in X'$ or $2a - 1 \in X'$. Hence, if we restrict the codomain of the transformation to $T(\mathcal{P}(N))$, then T becomes bijective, i.e., what is called an *isomorphism*.

Lemma 4.6. For the transformation T and the set of coalitions $\mathcal{G}'$ defined above, it always holds that $\nu'(\mathcal{G}') \cup T(\mathcal{P}(N)) = \tau'(\mathcal{G}')$.

Proof. If $X' \in \nu'(\mathcal{G}') \cup T(\mathcal{P}(N))$ and $a \in N$, then $X' \not\subseteq' N' \setminus Z^a$, because $|\{b' \in X' \mid 2a - 1 \preceq b'\}| > n - a = |\{b' \in N' \setminus Z^a \mid 2a - 1 \preceq b'\}|$. Therefore, $\nu'(\mathcal{G}') \cup T(\mathcal{P}(N)) \subseteq \tau'(\mathcal{G}')$.

Now it remains to prove that $\nu'(\mathcal{G}') \cup T(\mathcal{P}(N)) \supseteq \tau'(\mathcal{G}')$. Note that this inclusion is equivalent to $\mathcal{P}(N') \setminus T(\mathcal{P}(N)) \subseteq \nu'(\mathcal{G}') \cup \nu'(\neg(\mathcal{G}'))$. Given $Z' \in \mathcal{P}(N') \setminus T(\mathcal{P}(N))$, let $a \in N$ be maximal such that either $\{2a, 2a-1\} \subseteq Z'$ or $\{2a, 2a-1\} \cap Z' = \emptyset$. If the first holds, then $Z^a \subseteq' Z'$; and if the second one holds, then $N' \setminus Z^a \subseteq' Z'$. □

Now we are able to present the following key result, that applies the transformation T on hypergraphs.

Lemma 4.7. Let be $\mathcal{H}, \mathcal{K} \subseteq \mathcal{P}(N)$ two hypergraphs, $\mathcal{H}' = \{T(X) \mid X \in \mathcal{H}\}$ and $\mathcal{K}' = \{T(Y) \mid Y \in \mathcal{K}\}$, then:

- $(\mathcal{H}, \mathcal{K})$ is coherent if and only if $(\mathcal{H}' \cup \mathcal{G}', \mathcal{K}' \cup \mathcal{G}')$ is coherent; and
- $(\mathcal{H}, \mathcal{K})$ is complete if and only if $(\mathcal{H}' \cup \mathcal{G}', \mathcal{K}' \cup \mathcal{G}')$ is complete.

Proof. According to Definition 2.8 and Lemma 4.5, for the first statement we need to prove that $\nu(\mathcal{H}) \subseteq \tau(\mathcal{K})$ if and only if $\nu'(\mathcal{H}' \cup \mathcal{G}') \subseteq \tau'(\mathcal{K}' \cup \mathcal{G}')$. Note that $\nu(\mathcal{H}) \subseteq \tau(\mathcal{K})$ if and only if $X \not\subseteq N \setminus Y$, for all $(X, Y) \in \mathcal{H} \times \mathcal{K}$; i.e., if and only if $X' \not\subseteq' N' \setminus Y'$, for all $(X', Y') \in \mathcal{H}' \times \mathcal{K}'$; i.e., if and only if $\nu'(\mathcal{H}') \subseteq \tau'(\mathcal{K}')$. Moreover, Lemma 4.6 implies $\nu'(\mathcal{G}') \subseteq \tau'(\mathcal{G}')$, $\nu'(\mathcal{H}') \subseteq \tau'(\mathcal{G}')$ and $\nu'(\mathcal{G}') \subseteq \tau'(\mathcal{K}')$. So we completed the proof of the first statement.

Analogously, for the second statement we prove that $\nu(\mathcal{H}) \supseteq \tau(\mathcal{K})$ if and only if $\nu'(\mathcal{H}' \cup \mathcal{G}') \supseteq \tau'(\mathcal{K}' \cup \mathcal{G}')$. Note that $\nu(\mathcal{H}) \supseteq \tau(\mathcal{K})$ if and only if $Z \in \nu(\mathcal{H})$, for all $Z \in \mathcal{P}(N)$ with $N \setminus Z \notin \nu(\mathcal{K})$; i.e., if and only if $Z \in \nu(\mathcal{H}')$, for all $Z' \in T(\mathcal{P}(N))$ with $N' \setminus Z' \notin \nu'(\mathcal{K}')$. And since $Z \in \nu(\mathcal{G}')$ for all $Z' \in \mathcal{P}(N') \setminus T(\mathcal{P}(N))$ with $N' \setminus Z' \notin \nu'(\mathcal{G}')$, we have proved the second statement. □

From Lemma 4.7, we obtain the main result of this section.

Theorem 4.7. For regular games given in SWF, the IsStrong problem is coNP-complete. If the IsDecisive problem can be solved in polynomial time for regular games in SWF, then the IsDecisive problem for simple games in MWF will be also polynomial-time solvable.

Proof. The IsDecisive (resp. IsStrong) problem for simple games $(N, \mathcal{W})$ can polynomially be reduced to the IsDecisive (resp. IsStrong) problem for regular games $(N', \mathcal{W}')$, specified by their shift-minimal winning coalitions $\mu'(\mathcal{W}')$: Just apply the reduction T by setting $\mathcal{H} = \mathcal{K} = \mathcal{W}$, and considering that $|\mathcal{G}'| \leq |N|$. □

This theorem translate the "bad news" of classical duality theory—i.e., the NP-completeness of the IsStrong problem—to corresponding "bad news" for the regular duality theory. However, it would also guaranty that "good news" for the regular case—i.e., a possible polynomial-time algorithm for the IsDecisive problem—translate to corresponding "good news" for the classical theory.

Observe that, considering regular games in SWF rather than MWF, the polynomial-time results of Theorems 4.4 and 4.5 do not hold any more. Moreover, we do not know if the IsDecisive problem for regular games in SWF can be solved in quasi-polynomial time. This problem remains open.

Recalling that $T(\mathcal{P}(N)) \cap \nu'(\mathcal{G}') = \emptyset$, we immediately get the following.

Corollary 4.2. Given $\mathcal{H} \subseteq \mathcal{P}(N)$, let $\mathcal{K} = \lambda(\mathcal{H})$, $\mathcal{H}' = \{T(X) \mid X \in \mathcal{H}\}$ and $\mathcal{K}' = \{T(Y) \mid Y \in \mathcal{K}\}$. Then $\mathcal{K}' \subseteq \lambda'(\mathcal{H}' \cup \mathcal{G}') = \mu'(\mathcal{K}' \cup \mathcal{G}')$. Furthermore, $\lambda(\mathcal{H})$ can be obtained from $\lambda'(\mathcal{H}' \cup \mathcal{G}')$.

Suppose that for any regular game in SWF, starting with $\mathcal{H}' = \mu'(\mathcal{W})$ we could determine $\mathcal{K}' = \lambda'(\mathcal{W})$ in time polynomially bounded by the input plus output size $n \cdot (|\mathcal{H}'| + |\mathcal{K}'|)$. This does not contradict Corollary 4.2; but according to Lemma 4.7 it would imply that all the so much investigated decision problems, that until now are only known to be quasi-polynomial, are in fact polynomial.

Let us now apply Corollary 4.2 in a pair of hypergraphs $(\mathcal{H}, \mathcal{K})$ that represents a simple game in SWF.

Example 4.3. Reconsider Example 4.2 at the end of Section 4.2.1, where $I = \{m, \ldots, 1\}$, $N = \{2m, 2m-1, \ldots, 2, 1\}$, $\mathcal{H} = \{\{2i, 2i-1\} \mid i \in I\}$ and $\mathcal{K} = \{Y \subseteq N \mid \text{ for all } i \in I, \text{ either } 2i \in Y \text{ or } 2i-1 \in Y\}$. Then $|\mathcal{G}'| = 2m$, so $|\mathcal{H}' \cup \mathcal{G}'| = 3m$. Therefore, with $k = 3m$, $|\mathcal{H}' \cup \mathcal{G}'| = k$ and $|\lambda'(\mathcal{H}' \cup \mathcal{G}')| \geq |\mathcal{K}'| = c^k$, where $c = 2^{1/3} > 1$; so the size of the hypergraph $\lambda'(\mathcal{H}' \cup \mathcal{G}')$ grows exponentially in the size of $\mu'(\mathcal{H}' \cup \mathcal{G}')$. The following table presents the case $m = 2$.

N	4321	N'	87654321
$\mathcal{H}$	1100	$\mathcal{H}'$	10100101
	0011		01011010
$\mathcal{K}$	1010	$\mathcal{K}'$	10011001
	1001		10010110
	0110		01101001
	0101		01100110
		Z^4	11000000
		Z^3	01110000
		Z^2	01011100
		Z^1	01010111

This also proves that for regular games in SWF, $|\lambda'(\mathcal{W})|$ can grow exponentially in function of $|\mu'(\mathcal{W})|$; although $|\lambda(\mathcal{W})|$ can be bounded linearly in function of $|\mu(\mathcal{W})|$ [209]. This result was already proven before, using different techniques, in Corollary 3 of [56].

4.3 Other Parameters and Properties

In this section we study the computational complexity of some parameters and properties for simple games. The main results are illustrated in bold in

Table 4.4.

	Simple games			Regular	Weighted
Problem	EWF	MWF	VWRF	SWF	WRF
Properties of players					
IsDummy	P	P	coNPC	**coNP**	coNPC
Parameters					
Width	**P**	**P**	P	**P**	P
Coleman's power	P	#PC	#PC	?	#PC
Chow parameters	P	#PC	#PC	?	#PC

Table 4.4: New complexity results for properties of players and parameters of simple games.

The Width problem has been considered explicitly open for simple games in EWF and MWF [10]. In Section 4.3.1 we show that both cases can be computed in polynomial time. In that section we also define and analyze two new parameters called *strict width* and *strict length*. They are closely related to the strict and the width, so for all the considered cases, they can also be computed in polynomial time.

Finally, in Section 4.3.2 we propose an approach to the study of the IsDummy problem for regular games in SWF.

4.3.1 The Width of a Simple Game

Let Γ be a simple game, recall from Definition 2.25 that the parameter $width(\Gamma) = \min\{|X| \mid N \setminus X \in \mathcal{L}\}$.

Note that whether a given simple game Γ is strong, then for all $Y \in \mathcal{L}$, $N \setminus Y \in \mathcal{W}$, so therefore, the width only can be the cardinality of a winning coalition, i.e., the width is the same that the length: $length(\Gamma) = \min\{|X| \mid X \in \mathcal{W}\}$.

Lemma 4.8. Let Γ be a simple game, $width(\Gamma) = n - \max\{|X| \mid X \in \mathcal{L}\}$.

Proof. Let be $Z \subseteq N$ such that $|Z| = \min\{|X| \mid N \setminus X \in \mathcal{L}\}$. If $Z \in \mathcal{L}$, then $|N \setminus Z| = \max\{|X| \mid X \in \mathcal{L}\}$, and due to $n = |Z| + |N \setminus Z|$, the lemma holds. If $Z \in \mathcal{W}$, then it also holds that $|N \setminus Z| = \max\{|X| \mid X \in \mathcal{L}\}$, because if there exists $Y \in \mathcal{L}$ with $|Y| > |N \setminus Z|$, this implies that $|N \setminus Y| < |Z|$, a contradiction. □

Now we define two new parameters.

Definition 4.3. Let $\Gamma = (N, \mathcal{W})$ be a simple game:

- The *strict length* of Γ is the minimum cardinality from which all the coalitions are winning, i.e.,
  $slength(\Gamma) = \min\{k \in \mathbb{N} \mid \mathcal{P}_k(N) \subseteq \mathcal{W}\}$.
- The *strict width* of Γ is the complement of the maximum cardinality to which all coalitions are losing, i.e.,
  $swidth(\Gamma) = n - \max\{k \in \mathbb{N} \mid \mathcal{P}_k(N) \subseteq \mathcal{L}\}$.

So we have the following.

Lemma 4.9. Let $\Gamma = (N, \mathcal{W})$ be a simple game, then:

- $width(\Gamma) = n + 1 - slength(\Gamma)$, and
- $length(\Gamma) = n + 1 - swidth(\Gamma)$.

Proof. For the first sentence, by Definition 4.3 and Lemma 4.8, we need to prove that $slength(\Gamma) = \max\{|X| \mid X \in \mathcal{L}\} + 1$. Let be $k \in \mathbb{N}$, suppose that $k = slength(\Gamma)$. Then for all $Y \in \mathcal{L}$, $|Y| < k$. Further, there exists at least one $Y \in \mathcal{L}$ with $|Y| = k - 1$, because otherwise $slength(\Gamma)$ would be $k - 1$, a contradiction. Therefore, $\max\{|X| \mid X \in \mathcal{L}\} + 1 = k - 1 + 1 = k$.

For the second sentence, we need to prove that $swidth(\Gamma) = n + 1 - \min\{|X| \mid X \in \mathcal{W}\}$. Now suppose that $k = n - swidth(\Gamma)$. Then for all $X \in \mathcal{W}$, $|X| > k$. Further, there exists at least one $X \in \mathcal{W}$ with $|X| = k+1$, because otherwise $swidth(\Gamma)$ would be $k + 1$, a contradiction. Therefore, $n + 1 - \min\{|X| \mid X \in \mathcal{W}\} = n + 1 - (k + 1) = n - k$. □

From the above we obtain the following result.

Theorem 4.8. Let Γ be a simple game in EWF or MWF, the problems Length, Width, sLength and sWidth can be computed in polynomial time.

Proof. Note that Length can trivially be computed in polynomial time. Therefore, as $length(\Gamma) = n+1-swidth(\Gamma)$, sWidth can also be computed in polynomial time. Let $k = slength(\Gamma)$. Observe that, by definition, all the coalitions with k players are winning in Γ but at least one coalition with cardinality $k-1$ is losing. Therefore, there is a MWC with cardinality k and there are no MWCs with cardinality $k+1$. Thus, computing k is equivalent to compute the maximum cardinality of a MWC. The last quantity can be obtained in polynomial time from a description of $\mathcal{W}^m$, so sLength can be computed in polynomial time. Finally, as $width(\Gamma) = n + 1 - slength(\Gamma)$, Width can be computed in polynomial time. □

Moreover, for regular games in SWF it holds an analogous result.

Theorem 4.9. Let Γ be a regular game in SWF, the problems Length, Width, sLength and sWidth can be computed in polynomial time.

Proof. If the game has a shift-minimal winning coalition $\{1, \ldots, k\} \in \mathcal{W}^s$, with $k \leq n$, then $slength(\Gamma) = k$, because for any other coalition $Z \in \mathcal{P}_k(N)$, it holds that $X \preceq Z$. Otherwise, $slength(\Gamma) = \max\{|X| \mid X \in \mathcal{W}^s\} + 1$, because for all $Z \in \mathcal{W}^s$, $\{1, \ldots, |Z|\} \preceq Z$, but $\{1, \ldots, |Z|\} \notin \mathcal{W}^s$. The remaining results are clear from Lemma 4.9 and the fact that Length here can also trivially be computed in polynomial time. □

4.3.2 Dummy Players in Regular Games

The following is an approach to the study of the IsDummy problem for regular games in SWF. Let Γ be a simple game, recall that a player $i \in N$ is dummy when it does not belong to any MWC.

Since $\mathcal{W}^s(\Gamma) \subseteq \mathcal{W}^m(\Gamma)$, it is clear that if for all $X \in \mathcal{W}^m$, $i \notin X$, then for all $X \in \mathcal{W}^s$, $i \notin X$. Hence, if there exists some $X \in \mathcal{W}^s$ such that $i \in X$, then i is not dummy. Furthermore, we have the following.

Lemma 4.10. Let $\Gamma = (N, \mathcal{W}^s)$ be a regular game in SWF. Given $X \in \mathcal{W}^s$, $i \in X$ and $j \notin X$ with $i \prec j$, deciding whether $X \setminus \{i\} \cup \{j\}$ is a MWC can be done in polynomial time.

Proof. We know by definition that $Z = X \setminus \{i\} \cup \{j\}$ is a winning coalition. Therefore, Z is a MWC if and only if for all $h \in N$, $Z \setminus \{h\} \in \mathcal{L}$; and by Lemma 4.2, we know that deciding $Z \setminus \{h\} \in \mathcal{L}$ can be done in polynomial time. □

From the above we have the following result.

Lemma 4.11. For regular games in SWF, the IsDummy problem belongs to coNP.

Proof. We can decide whether i is dummy by checking if for every shift-minimal winning coalition X and for every $Z \subseteq N$ obtained by increasing-shifts from X, Z is not a MWF. □

In despite of the above, the number of permutations required to decide whether a player is dummy can be exponential in the size of the input, so it is possible that given a regular game in SWF, the IsDummy problem could not be solved in polynomial time. In remains open to prove whether for regular games in SWF, IsDummy is coNP-hard.

4.4 Counting and Enumerating Results

In this section we focus on the study of counting and enumerating specific subclasses of simple games.

In Section 4.4.1 we survey the main known results related to the counting of the members of subclasses of simple games, and we present some known approaches to enumerate subclasses of simple games. In Section 4.4.2 we propose a new strategy to enumerate decisive regular games. We use an experimental approach. Although the correctness of the algorithm has not yet been formally proved, the correctness of the algorithm has been validated experimentally for the considered cases. By using these ideas, we can correctly enumerate all the decisive regular games up to eight players.

4.4.1 Known Results

The numbers of simple games in the subclasses defined in Chapter 3 are summarized in Table 4.5. The references on the second last column show the details of how the last value of the corresponding sequence was obtained. The last column shows the sequence numbers according to the Online Encyclopedia of Integer Sequences (OEIS) [238]. Further, most of the question marks refer to really difficult open combinatorial problems. In what follows we explain some general aspects about those results. It is important to remark that the counting of both the Dedekind numbers—i.e., the number of simple games— and the decisive games consider all the isomorphic games obtained by permutation of players. However, since in regular and weighted games the grand coalition is ordered, the remaining sequences of the table are up to isomorphism, i.e., the isomorphic games are counted only once.

	Number of players						
Subclass	1	2	3	4	5	6	7
Simple games	3	6	20	168	7,581	7,828,354	2,414,682,040,998
↪ Regular	3	5	10	27	119	1,173	44,315
↪ Weighted	3	5	10	27	119	1,113	29,375
↪ Homogeneous	1	3	8	23	76	293	1,307
↪ Decisive	1	2	4	12	81	2,646	1,422,564
↪ Regular	1	1	2	3	7	21	135
↪ Weighted	1	1	2	3	7	21	135

	Number of players	
Subclass	8	9
Simple games	56,130,437,228,687,557,907,788	?
↪ Regular	16,175,190	284,432,730,176
↪ Weighted	2,730,166	989,913,346
↪ Homogeneous	6,642	37,882
↪ Decisive	229,809,982,112	423,295,099,074,735,261,880
↪ Regular	2,470	319,124
↪ Weighted	2,470	175,428

	Number of players		
Subclass	10	Reference	OEIS
Simple games	?	[267]	*A*000372
↪ Regular	?	[91]	*A*132183
↪ Weighted	?	[248, 146]	*A*000617
↪ Homogeneous	239,490	[144] ($n = 9$)	*A*189359
↪ Decisive	?	[38]	*A*001206
↪ Regular	1,214,554,343	[141]	*A*109456
↪ Weighted	52,980,624	[190] ($n = 9$)	*A*001532

Table 4.5: Results of counting the number of members for subclasses of simple games.

Note that the enumeration problems require an additional computational effort than the counting problems, in order to represent explicitly the games that are enumerated. As a matter of fact, the last values of the sequences of Table 4.5 are obtained through counting algorithms, but usually the members of the respective subclasses have not been yet explicitly enumerated.

The number of simple games up to 4 players—including trivial games—were counted in 1897 by Dedekind [58]. The Dedekind number for $n = 5$ was discovered in 1940 [47], and for $n = 6$ in 1946 [260]. The next number was found in 1965 [48] and rediscovered again in 1976 [20]. The largest known Dedekind number, for $n = 8$, was found in 1991 by Wiedemann [267], using a similar method than in [20]. All these numbers were found in the context of order theory, using mathematical methods and enumerating the members of free distributive lattices defined over n generators.

It is well known that the number of members of a free distributive lattice is equivalent to the number of simple games. The definition of free distributive lattices generated by n elements, as we know it, was firstly introduced by Skolem in 1931 [237], being even older than the explicit definition of simple games of von Neumann and Morgenstern [258]. Furthermore, the Dedekind numbers also describe the number of monotone Boolean functions on n variables, the number of labeled Sperner families on n vertices, and the number of antichains in the power set of a grand coalition [274].

The decisive games up to 4 players were illustrated in 1897 by Dedekind, as the self-dual members of the free distributive lattices [58]. For 5 players they were counted in 1944 by von Neumann and Morgenstern [258]. For 6 players in 1959 by Gurk and Isbell [109]. For 7 players in 1992 by Loeb [156]—see also [24]—and for 8 players in 1995 by Loeb and Conway [157], requiring only twelve minutes of computation. This latter result takes advantages of symmetrical combinatorial structures called *maximal intersecting families* (MIFs), which allow us to skip the redundant and automorphic games. The number of decisive games for 9 players was published in 2013 by Brouwer et al. [38], and it was obtained through the counting of independent sets in a sparse graph; computation time is not mentioned in this case.

Regular games up to 8 players and decisive regular games up to 9 players were counted in 1995 by Krohn and Sudhölter [144]. The authors not only found the numbers but also enumerate the games. Their idea was to use linear programming in order to enumerate regular games represented as lattices formed by its shift-minimal winning coalitions. Taking advantage that these lattices are *rank-symmetric*—i.e. the complement of a shift-winning coalition on one level of the lattice is on their opposite side—the resulting algorithms are faster, but they require a lot of memory, and that is why they cannot enumerate games with larger number of players.

The number of regular games for 9 players was found in 2010 by Freixas and Molinero [91]—see also [90, 93]. As above, they also enumerate the regular games, allowing to verify the games that are weighted. Thus, they are also able to enumerate weighted games in WRF. Their idea is to characterize the regular games in a form of representation slightly less succinct than SWF. Thus, they generate the regular games, by using a brute-force mechanism, not avoiding repetitions of isomorphic games. From this point, they start a classification of weighted games. By using known upper bounds for the weights of the weighted games [187, 102] they minimize the number of inequalities required to define these games—in a similar way than expression (3.2) in Section 3.3.1—and then solve the obtained systems of linear inequalities by using linear programming. Unfortunately, these reduced systems of linear inequalities may also have an exponential number of restrictions, becoming in the main obstacle of this method.

The number of decisive regular games for 10 players was computed in 2008 by Knuth [141]. The first counting is based on regular games in a form of representation similar to SWF, and the second one on binary decision diagrams. In [146, 147] it is proposed an algorithm which counts *cliques*—subsets of an undirected graph such that every pair of nodes is connected by an edge—and establishes a bijection among these cliques and the regular games, achieving excellent results in computational speed to counting regular games.

The first serious approaches for counting weighted games are in the context of Boolean functions, i.e., have to do with counting threshold Boolean functions. Those functions were counted up to 6 players in 1959 by Muroga et al. [188], and for 7 players in 1964 by Winder [269]—see also [270, 62]. In 1967 Muroga et al. published several results counting threshold Boolean functions and subclasses of them by using linear programming. Among other results, the authors succeed to count the number of weighted games up to 8 players [190]. Several years later, in the context of simple game theory, first Tautenhahn in 2008 on his Master's thesis [248], and then Kurz [146] in 2011, count the number of weighted games for 9 players, again by using linear programming methods.

Regarding decisive weighted games, they were explicitly given up to 5 players in 1944 by von Neumann and Morgenstern [258], for 6 players in 1959 by Isbell and Gurk [109], and for 7 players in the same year by Isbell [125]. In the above mentioned result of Muroga et al. in 1967, they count the number of decisive weighted games up to 9 players [190]. The number of decisive weighted games for 10 players was given in OEIS by W. Lan—sequence A001532 [238]—but it is not documented how the author found this number.

Finally, regarding homogeneous games, they can be counted by using brute force mechanisms for a larger number of players. Besides the $239,490$ homogeneous games for 10 players showed on Table 4.5, we know that there are $1,661,564$ homogeneous games for $n = 11$, and $12,548,067$ for $n = 12$. The sequence could probably be extended. Moreover, Sudhölter provided in 1989 an explicit recursive formula to count them [244]. The author represents homogeneous games as *step functions*—i.e., a piecewise constant function with a finite number of pieces—in such a way that these games can be counted and enumerated. The number of homogeneous games up to 9 players was presented in 1995 by Krohn and Sudhölter [144]. In that paper the authors are not interested into count the homogeneous games for larger number of players. However, by using the mentioned recursive formula of Sudhölter, it could be possible to extend the counting results.

4.4.2 Enumerating Decisive Regular Games

In this section we propose a new strategy to enumerate decisive regular games, motivated by the Master thesis of Riquelme, which was supervised by Polyméris [227]. The main idea is to define a free distributive lattice on n generators, where each member is a decisive regular game in SWF. In what follows we consider a decreasing lexicographical order of the players, so that $N = \{n, \ldots, 1\}$.

The construction of the lattice attempts to ensure that all the decisive regular games with n players belong to it, and also that the games are never repeated. Furthermore, all the games are located into the lattice in such a way that each one of them can be recognized by its neighbors. The analogous lattice on $n + 1$ generators can be obtained constructively by expanding the domain of the existing games, and then obtaining from them all the missing games. The new games should be located into the new lattice, being connected to all the games from which they can be generated.

The main obstacle of this process is the exponential increase in the size of the lattices, in terms of n. However, this can be considered an intrinsic complexity of the problem, and the complexity should be analyzed in terms of the output size. Therefore, the efficiency of the algorithm must be based mainly on two aspects: first, that it does not generate replicated games; and second, that in order to generate the new lattice, the algorithm does not have to access to the elements of the original lattice more than once.

We start with the following definition.

Definition 4.4. Let $\Gamma = (N, \mathcal{W})$ be a simple game and let $Z \subseteq N$. We say that Γ *responds* to Z if either there exists $X \in \mathcal{W}$ with $X \subseteq Z$, or there exists $Y \in \mathcal{L}$ with $Z \subseteq Y$.

Note that a simple game is strong if and only if it responds to all $Z \subseteq N$.

For regular games in SWF, we know that the IsProper problem belongs to P but the IsStrong problem is coNP-complete—see Lemma 4.3 and Theorem 4.7, respectively. As we want to enumerate regular games which are decisive—i.e., proper and strong—it is important to avoid the computation of the IsStrong problem. For this purpose we introduce the following.

Definition 4.5. A regular game is *boring* if for all $X \in \mathcal{W}^m$, $1 \notin X$.

Note that every regular game with $1 \notin X$, for all $X \in \mathcal{W}^s$, is boring. The following result characterizes strong regular games from proper and boring regular games.

Lemma 4.12. Let Γ be a proper and boring regular game. If for all $Z \subseteq N$ with $1 \notin Z$, Γ responds to Z, then Γ is strong, and therefore decisive.

Proof. Let $Z \subseteq N$ with $1 \in Z$ and let $Z' = Z \setminus \{1\}$. If there exists $X \in \mathcal{W}$ with $X \subseteq Z'$, then it is clear that $X \subseteq Z$. If there exists $Y \in \mathcal{L}$ with $Z' \subseteq Y$, then $Z \preceq Y$, so therefore $Z \in \mathcal{L}$. Thus, Γ responds to Z. $\square$

For what follows we need an additional definition.

Definition 4.6. Let Γ be a regular game and $X \in \mathcal{W}^s(\Gamma)$. The set of *immediate successors* of X is the set $\alpha(X) = \{Z \subseteq N \mid X \prec Z$ and for all $Y \subseteq N$ with $X \prec Y \preceq Z$, then $Y = Z\}$.

Now we prove that from any decisive regular game we can obtain a boring decisive regular game. Recall that for decisive games, the complement of any winning coalition is losing, and vice versa.

Lemma 4.13. Every decisive regular game produces a boring decisive regular game.

Algorithm 9 BDRGfromDRG

Input: Decisive regular game $\Gamma = (N, \mathcal{W}^s)$ in SWF.
Output: Boring decisive regular game $\Gamma' = (N, \mathcal{W}^{s'})$ in SWF.

1: **repeat**
2: **if** for all $X \in \mathcal{W}^s$, $1 \notin X$, **return** $\Gamma' = \Gamma$;
3: Choose $X \in \mathcal{W}^s$ with $1 \in X$;
4: Define $X' = N \setminus X$;
5: $\mathcal{W}^s = \mathcal{W}^s \setminus \{X\} \cup \alpha(X)$;
6: $\mathcal{W}^s = \mathcal{W}^s \setminus \{Z \in \mathcal{W}^s \mid X' \preceq Z\} \cup \{X'\}$.

Proof. We prove that by exchanging winning coalitions by its complementary losing coalitions, we can provide a construction that from any decisive regular game produces a boring decisive regular game. See Algorithm 9. Step 5 converts a shift-minimal winning coalition that contains the player 1 into a losing coalition, in such a way that the game continues responding as many coalitions as possible. Step 6 brings the complementary losing coalition X' into the set of shift-minimal winning coalitions, and removes from $\mathcal{W}^s$ those coalitions Z so that $Z \succeq X'$.

Now we prove that the output is always a regular game in SWF. Let be $X \in \mathcal{W}^s$, $Z \in \alpha(X)$ and $Z' \in \mathcal{W}^s \setminus \{X\}$. If $Z \preceq Z'$, since $X \prec Z$ we have that $X \preceq Z'$, a contradiction. If $Z' \preceq Z$, since $X \prec Z$ and there is no $Y \neq Z$ with $X \prec Y \preceq Z$, then $Z' \preceq X$, a contradiction. Therefore, since step 6 maintains the regularity, then the output is a regular game in SWF.

Now we prove that the output is a decisive game. Note that Γ is proper, so $X \in \mathcal{W}(\Gamma)$ implies $N \setminus X \in \mathcal{L}(\Gamma)$. Therefore, in step 5 we also obtain proper games, since for all $Z \in \alpha(X)$, $Z \succ X$ and hence $N \setminus Z$ is losing. However, in this step the game is not strong, because both X and X' are losing. But this is fixed in step 6, where the coalition X' is added to the set of shift-minimal winning coalitions. Finally, since the coalitions $Z \in \mathcal{W}^s$ with $Z \succeq X'$ that are removed maintain the properness of the current game, the new game obtained in each repetition of the algorithm is decisive.

By step 2, it is clear that the output is always a boring game. Therefore, Algorithm 9 always returns a boring decisive regular game. □

In turn, note also that from boring decisive regular games we can obtain decisive regular games.

Lemma 4.14. Let $\Gamma = (N, \mathcal{W}^m)$ and $\Gamma' = (N \setminus \{1\}, \mathcal{W}^m)$ be two simple

Algorithm 10 DRGfromBDRG

Input: Boring decisive regular game $\Gamma = (N, \mathcal{W}^s)$ in SWF.
Output: Decisive regular game $\Gamma' = (N, \mathcal{W}^{s'})$ in SWF.

1: Choose $X \in \mathcal{W}^s$ with $1 \notin X$;
2: Define $X' = N \setminus X$;
3: $\mathcal{W}^s = \mathcal{W}^s \setminus \{X\} \cup \alpha(X)$;
4: $\mathcal{W}^s = \mathcal{W}^s \setminus \{Z \in \mathcal{W}^s \mid X' \preceq Z\} \cup \{X'\}$;
5: **return** $\Gamma' = \Gamma$.

games. Then Γ is a boring decisive regular game if and only if Γ' is a decisive regular game.

Proof. Let Γ be a boring decisive regular game. If $X \setminus \{1\} \in \mathcal{W}(\Gamma')$, then $N \setminus X \in \mathcal{L}(\Gamma)$, because Γ is proper; therefore, $N \setminus (X \setminus \{1\}) \in \mathcal{L}(\Gamma')$, so Γ' is proper. Note that Γ is boring, so for all $X \in \mathcal{L}(\Gamma)$, $1 \notin X$. If $X \setminus \{1\} \in \mathcal{L}(\Gamma')$, then $N \setminus (X \setminus \{1\}) \in \mathcal{W}(\Gamma')$, because $1 \in N \setminus (X \setminus \{1\})$; so Γ' is strong.

Let Γ' be a decisive regular game. Note that Γ is clearly boring, because we do not add new MWCs. If $X \in \mathcal{L}(\Gamma)$, then $N \setminus X \in \mathcal{W}(\Gamma)$, because 1 does not belong to any losing coalition; so Γ is strong. If $X \in \mathcal{W}(\Gamma)$, we have two cases: 1) if $1 \notin X$, it is clear that $X \setminus \{1\} \in \mathcal{W}(\Gamma')$, so then $N \setminus (X \setminus \{1\}) \in \mathcal{L}(\Gamma')$, because Γ' is proper; hence $N \setminus X \in \mathcal{L}(\Gamma)$, because $1 \notin X$, and thus Γ is proper. 2) if $1 \in X$, it is clear that $X \in \mathcal{W}(\Gamma) \setminus \mathcal{W}^m(\Gamma)$, so then $X \setminus \{1\} \in \mathcal{W}(\Gamma')$, and since Γ' is proper, $N \setminus (X \setminus \{1\}) \notin \mathcal{L}(\Gamma')$; therefore, $N \setminus X \in \mathcal{L}(\Gamma)$, because it is formed from the losing coalition $N \setminus (X \setminus \{1\})$ of Γ', without adding any additional player. Thus, Γ is proper.

Finally, since Γ' remains regular in both cases, then the lemma holds. □

Note that Lemma 4.14 implies that every decisive regular game on n players can be obtained from a boring decisive regular game on $n+1$ players. Moreover, based on Lemma 4.13 we can show the following.

Lemma 4.15. Every decisive regular game on n players can be obtained from a boring decisive regular game on n players.

Proof. We need to prove that by using the same exchange of coalitions used in the proof of Lemma 4.13, we can obtain decisive regular games from boring decisive regular games. See Algorithm 10, which is the same Algorithm 9, but without repetitions, and choosing in step 3 of the previous algorithm the coalitions $X \in \mathcal{W}^s$ with $1 \notin X$.

Just note that when some shift-minimal winning coalition $X \in \mathcal{W}^s$ becomes losing and $N \setminus X$ becomes winning, the new coalitions $Z \in \mathcal{W}^s$ with $1 \in Z$ are such that $X \prec Z$. □

In what follows we construct enumeration algorithms. To display these enumerations we use additional notation. To denote that we are enumerating or printing a decisive regular game, we draw the game within a box, like for instance:

10010
01110

When a decisive regular game Γ' is obtained from other decisive regular game Γ, by exchanging a coalition $X \in \mathcal{W}^s(\Gamma)$ by $N \setminus X$—see steps 3 and 4 of Algorithm 10—we include an arrow pointing from the coalition X to Γ', like this:

	10100
10010 →	01101
01110	10011

In turn, this notation derives some additional definitions.

Definition 4.7. Let $\mathcal{G}$ be a set of decisive regular games in SWF. We define a function $\delta : \mathcal{G} \times \mathcal{P}(N) \to \mathcal{G}$ such that for all $(\Gamma, X) \in \mathcal{G} \times \mathcal{P}(N)$, $\delta(\Gamma, X) = \Gamma'$, where $\mathcal{W}^s(\Gamma') = (\mathcal{W}^s(\Gamma) \setminus \{X\} \cup \alpha(X)) \setminus \{Z \in \mathcal{W}^s \mid N \setminus X \preceq Z\} \cup \{N \setminus X\}$.

We denote by a pair $(\mathcal{G}, \delta)$ a set of decisive regular games connected by arrows according to the relationship δ, i.e., such that for all $\Gamma, \Gamma' \in \mathcal{G}$, there exists a coalition X such that either $\delta(\Gamma, X) = \Gamma'$ or $\delta(\Gamma', X) = \Gamma$.

Note that $\mathcal{W}^s(\Gamma')$ corresponds to the set $\mathcal{W}^{s'}$ obtained from Γ by the exchange of X by $N \setminus X$ in steps 3 and 4 of Algorithm 10. Based on this algorithm, in the following result we construct an enumeration algorithm to list decisive regular games from a set of boring decisive regular games.

Lemma 4.16. There is an algorithm to enumerate all the decisive regular games up to 4 players.

Proof. Let be $N = \{n, \dots, 1\}$. We consider the enumerations for $n = 1$ and $n = 2$ as base cases, because by Table 4.5, we know that for both cases there is only one decisive regular game. For $n = 1$, the only decisive regular game in SWF is the one with $\mathcal{W}^s = \{1\}$. Let us print this game:

Algorithm 11 EnumeratingDRGs(WithRepetitions)

Input: A pair $(\mathcal{G}, \delta)$ of connected decisive regular games on $n-1$ players.
Output: A pair $(\mathcal{G}', \delta)$ of connected decisive regular games on n players.

1: $N = N \cup \{1\}$;
2: Define $(\mathcal{G}', \delta) = (\mathcal{G}, \delta)$;
3: **for all** $\Gamma \in \mathcal{G}$ **repeat**
4:     **print** Γ;
5:     **for all** $X \in \mathcal{W}^s$ with $1 \notin X$ **repeat**
6:         Define $X' = N \setminus X$;
7:         $\mathcal{W}^s = \mathcal{W}^s \setminus \{X\} \cup \alpha(X)$;
8:         $\mathcal{W}^{s'} = \mathcal{W}^s \setminus \{Z \in \mathcal{W}^s \mid X' \preceq Z\} \cup \{X'\}$;
9:         **print** $\Gamma' = (N, \mathcal{W}^{s'})$;
10:         $\mathcal{G}' = \mathcal{G}' \cup \Gamma'$;
11:         Define $\delta(\Gamma, X) = \Gamma'$;
12: **return** $(\mathcal{G}', \delta)$.

[1]

For $n = 2$, the only decisive regular game in SWF can be obtained by increasing the domain of the previous game, so we obtain the regular game with $\mathcal{W}^s = \{10\}$:

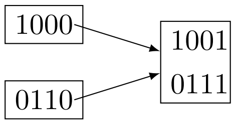
[10]

Now let us consider Algorithm 11, which repeats the procedure of Algorithm 10 for every shift-minimal winning coalition that does not contains the player 1. After each exchange of coalitions, this new algorithm enumerates the obtained decisive regular game, and further saves the connections created among the games due to this exchange.

For $n = 3$, note that $\mathcal{G}$ only contains the game with $\mathcal{W}^s = \{10\}$. In step 1 we increase again the domain of the previous game, obtaining a decisive regular game on three players with $\mathcal{W}^s = \{100\}$. Note that by Lemma 4.14, whenever we increase the domain of the games, we obtain a boring decisive regular game, so the game is enumerated in step 4. Since $\alpha(\{100\}) = \{101\}$, in step 7 we have $\mathcal{W}^s = \{101\}$; and then in step 8 we have $\mathcal{W}^s = \{011\}$, because $011 \prec 101$. Thus, we enumerate in step 9 the new decisive regular game with $\mathcal{W}^s = \{011\}$, and we save the changes in steps 10 and 11, obtaining the following output in step 12:

[100] → [011]

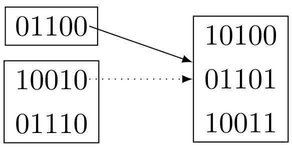
Analogously, for $n = 4$, we increase the domain of the previous games, obtaining two boring decisive regular games on four players. Let us consider first the game with $\mathcal{W}^s = \{1000\}$. Here $\alpha(\{1000\}) = \{1001\}$, so in step 8 we obtain $\mathcal{W}^s = \{1001, 0111\}$, because $\{0111\} \not\leq \{1001\}$. On the other hand, from the second game, with $\mathcal{W}^s = \{0110\}$, we have $\alpha(\{0110\}) = \{0111\}$, and moreover $\{1001\} \not\leq \{0111\}$. Therefore, from both games the algorithm produces the following output:



which is the third decisive regular game that exists on four players. □

Note that in the previous enumeration for $n = 4$, we obtain the same decisive regular game from two different boring decisive regular games. Such repetitions should be avoided if we eventually want to obtain algorithms with polynomial-delay. When we are enumerating decisive regular games on n players, the repetitions can be avoided: we recall the relation δ obtained from the created games on $n - 1$ players, which are represented in our notation by the arrows.

To denote a game that has been blocked, so it can not generate the same game that has already been enumerated, we use a dotted arrow, like this:



Our next result explains how we can avoid this kind of repetitions.

Lemma 4.17. There is an algorithm to enumerate all the decisive regular games up to 4 players, without repetition.

Proof. Let us consider Algorithm 12, which is the same Algorithm 11 but with the new step 6. Until 3 players, the procedure is the same than the explained in the proof of Lemma 4.16.

For $n = 4$, as usual we increase the domain of the games obtained on three players, obtaining two boring decisive regular games on four players: Γ_1, with $\mathcal{W}^s(\Gamma_1) = \{1000\}$; and Γ_2, with $\mathcal{W}^s(\Gamma_2) = \{0110\}$. For Γ_1, note that $\delta(\Gamma_1, 1000) = \Gamma_2$, so by step 6 we continue the procedure like in Lemma 4.16 and we obtain the game Γ_3 with $\mathcal{W}^s(\Gamma_3) = \{1001, 0111\}$. Note

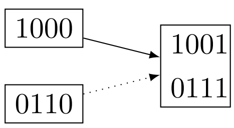
Algorithm 12 EnumeratingDRGsUpTo4

Input: A pair $(\mathcal{G}, \delta)$ of connected decisive regular games on $n-1$ players.
Output: A pair $(\mathcal{G}', \delta)$ of connected decisive regular games on n players.

1: $N = N \cup \{1\}$;
2: Define $(\mathcal{G}', \delta) = (\mathcal{G}, \delta)$;
3: **for all** $\Gamma \in \mathcal{G}$ **repeat**
4:   **print** Γ;
5:   **for all** $X \in \mathcal{W}^s$ with $1 \notin X$ **repeat**
6:     **if** $\delta(\Gamma, X) \neq \emptyset$ **then**
7:       Define $X' = N \setminus X$;
8:       $\mathcal{W}^s = \mathcal{W}^s \setminus \{X\} \cup \alpha(X)$;
9:       $\mathcal{W}^{s'} = \mathcal{W}^s \setminus \{Z \in \mathcal{W}^s \mid X' \preceq Z\} \cup \{X'\}$;
10:       **print** $\Gamma' = (N, \mathcal{W}^{s'})$;
11:       $\mathcal{G}' = \mathcal{G}' \cup \Gamma'$;
12:       Define $\delta(\Gamma, X) = \Gamma'$;
13: **return** $(\mathcal{G}', \delta)$.

that Γ_3 contains the coalition 0111, which is the complement of the coalition 1000 of Γ_1. Then, by step 12 the connection for Γ_1 is updated with $\delta(\Gamma_1, 1000) = \Gamma_3$.

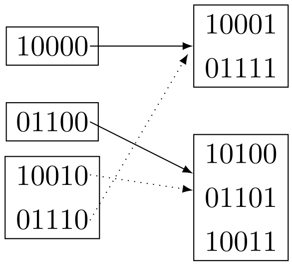
On the other hand, note that the remaining coalition of Γ_3, 1001, could also be obtained by the exchange of its complementary coalition, 0110, from other decisive regular game. But this game is Γ_2, which is the projection in $n = 3$ of the game obtained from Γ_1. Therefore, since $\delta(\Gamma_2, 0110) = \emptyset$, we avoid to repeat again the same game, and we obtain the following:



Therefore, we have enumerated all the decisive regular games up to 4 players, without repetition. □

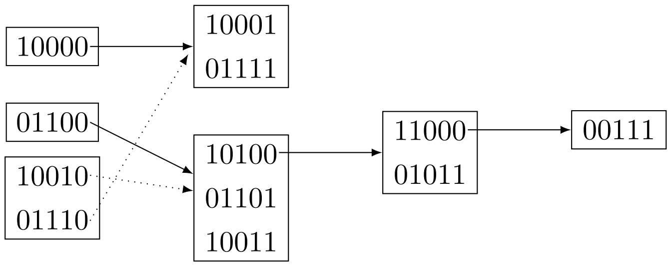
Let us continue applying Algorithm 12, from the pair $(\mathcal{G}, \delta)$ obtained for $n = 4$ at the end of the proof of Lemma 4.17.

For $n = 5$, we increase the domain of the previous games, obtaining three boring decisive regular games on five players. Applying the same procedure, we obtain:



Note that the third boring game does not generates new games, because it has only two coalitions, and its projection on $n = 4$ was pointed by exactly two boring games.

However, by Table 4.5 we know that there are seven regular decisive games on $n = 5$, instead of five. The remaining games must be obtained from the non-boring game with coalition 10100. Thus, considering the same Algorithm 12, but replacing step 3 by "**for all** $\Gamma \in \mathcal{G}'$ **repeat**", we obtain the seven decisive regular games for 7 players:



This leads us to propose the following conjecture.

Conjecture 4.1. There is an algorithm that, with enough memory resources, is able to enumerate all the decisive regular games, without repetition and with polynomial-delay.

Now we show an experimental validation of the previous procedure.

Experimental Validation

The procedure described before was implemented in C++. We checked that the algorithm enumerates all the $2,470$ decisive regular games on 8 players and thousands of decisive regular games for 9 players. We verified that all the enumerated games were decisive regular, and that they were not repeated. Furthermore, with an Intel Core 2 Duo processor clocked at 3.16 GHz, we obtained for less than 9 players very good runtimes, as we can see in Table 4.6.

n	1	2	3	4	5	6	7	8
time (sec.)	0.002	0.002	0.002	0.002	0.005	0.005	0.027	1.307

Table 4.6: Time results for enumerating decisive regular games.

In Appendix A we provide the list of all the 170 decisive regular games from $n = 1$ until $n = 7$.

Besides the correctness of the algorithm, the biggest problem of this strategy of enumeration is that to generate the set of all decisive regular games on n players, it is necessary to have in memory the set of all decisive regular games on $n-1$ players. As the size of these sets grows exponentially in terms of n, the enumeration is limited by memory resources. Indeed, it was not possible to enumerate explicitly the set of all decisive regular games on $n = 9$. After a long computation time, the computer ran out of memory space. In despite of the above, we have been able to enumerate a good set of decisive regular games over nine players. This set could be used for instance as benchmarking to study other properties of decisive regular games.

Example 4.4. The following is a decisive regular game in SWF. By using linear programming, we also know that it is not weighted.

N	987654321
$\mathcal{W}^s$	011011011
	011100100
	100011100
	100100011
	100101000
	101000000

Since every weighted game is regular—see Section 2.3.3— by Table 4.5 we can conclude that all the 2470 decisive regular games on eight players are also weighted. As far as we know, explicit examples of simple games with nine players, which are regular and decisive, but not weighted, have not been illustrated until now. However, another regular game that in fact meets these characteristics can be found in [245].

Part III

Influence games

Chapter 5

Influence Games

In this chapter we study *influence games*, a new subclass of simple games. Briefly, an influence game is described by an influence graph, modeling a social network, and a quota, indicating the required minimum number of agents that have to cooperate to perform successfully the task. A team will be successful or winning if it can influence at least as many individuals as the quota establishes. We take the spread of influence in the linear threshold model as the value that measures the power of a team. From this model we can study all the problems related to simple games, including properties, parameters and solution concepts, from the context of multi-agent systems, social networks, social choice, among other topics.

In Section 5.1 we define the model. In Section 5.2 we explain its expressiveness, by showing that influence games capture the whole class of simple games. In Section 5.3 we characterize the computational complexity of several problems defined in Chapter 2 for influence games. Finally, in Section 5.4 we analyze those problems for some particular extremal subclasses of influence games, with respect to the propagation of influence, showing tighter complexity characterizations.

5.1 Definitions and Preliminaries

Before introducing formally the family of influence games we need to define a family of labeled graphs and a process of spread of influence based on the *linear threshold model* [108, 230]. In this first analysis of influence games, we draw upon the deterministic version of the linear threshold model, in which node thresholds are fixed, as our model for influence spread following [45, 5]. We use standard graph notation following [26]. As in graph theory, here $n = |V|$ and $m = |E|$. For any $0 \le k \le n$, $\mathcal{P}_k(X)$ denotes the subsets of X with exactly k-elements.

Definition 5.1. An *influence graph* is a tuple (G, w, f), where $G = (V, E)$ is a weighted, labeled and directed graph without loops. As usual V is the set of vertices, agents or actors, E is the set of edges and $w : E \to \mathbb{N}$ is a *weight function*. Finally, $f : V \to \mathbb{N}$ is a labeling function that quantifies how influenceable each agent is. An agent $i \in V$ has *influence* over another $j \in V$ if and only if $(i, j) \in E$. We also consider the family of *unweighted influence graphs* (G, f) in which every edge has weight 1.

Given an influence graph (G, w, f) and an initial activation set $X \subseteq V$, the *spread of influence* of X is the set $F(X) \subseteq V$ which is formed by the agents activated through an iterative process. We use $F_k(X)$ to denote the set of nodes activated at step k. Initially, at step 0, only the vertices in X are activated, that is $F_0(X) = X$. At step $i > 0$, those vertices for which the sum of weights of the edges connecting nodes in $F_{i-1}(X)$ to them meets or exceeds their label functions are activated, i.e.,

$$F_i(X) = F_{i-1}(X) \cup \left\{ v \in V \mid \textstyle\sum_{u \in F_{i-1}(X), (u,v) \in E} w((u, v)) \ge f(v) \right\}.$$

The process stops when no additional activation occurs and the final set of activated nodes becomes $F(X)$.

Example 5.1. Figure 5.1 shows the spread of influence $F(X)$ in an unweighted influence graph $G = (V, f)$, with $V = \{a, b, c, d\}$, for the initial activation $X = \{a\}$. In the first step we obtain $F_1(X) = \{a, c\}$, and in the second step (the last one) we obtain $F(X) = F_2(x) = \{a, c, d\}$.

As the number of vertices is finite, for any $i > n$, $F_i(X) = F_{i-1}(X)$. Thus, $F(X) = F_n(X)$ and we have the following well known basic result.

Lemma 5.1. Given an influence graph (G, w, f) and a set of vertices X, the set $F(X)$ can be computed in polynomial time.

In what follows, unless otherwise stated, results and definitions will be stated for directed graphs. All of them can be restated for undirected graphs. Now we define influence games.

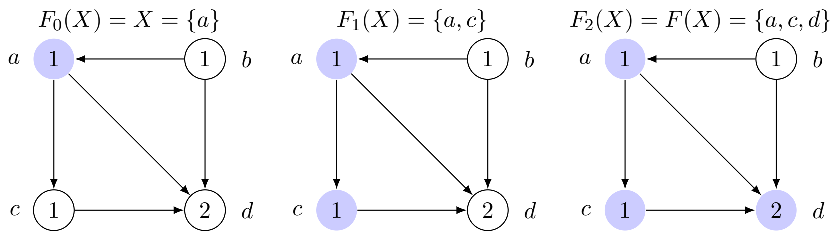


Figure 5.1: The spread of influence starting from the initial activation of $X = \{a\}$ on an unweighted influence graph.

Definition 5.2. An *influence game* is given by a tuple (G, w, f, q, N) where (G, w, f) is an influence graph, q is an integer *quota*, $0 \leq q \leq |V| + 1$, and $N \subseteq V$ is the *set of players*. $X \subseteq N$ is a *successful team* if and only if $|F(X)| \geq q$, otherwise X is an *unsuccessful team*.

As it was done for influence graphs, we also consider the family of *unweighted influence games* for the cases in which the graph G is unweighted. In such a case we use the notation (G, f, q, N).

Influence games adopt a correspondence with simple games.

Example 5.2. Let (G, f) be an influence graph and N any subset of agents. Two particular ranges of the quota lead to some trivial simple games. By setting $q = 0$, thus considering influence games of the form $(G, f, 0, N)$, we have that every team of agents is successful, therefore $(G, f, 0, N)$ is a representation of the simple game $(N, \mathcal{P}(N))$. When $q > |V(G)|$, the influence game (G, f, q, N) is a representation of the simple game $(N, \emptyset)$ as there are no successful teams in the game.

Let us provide an example of influence game based on the influence graph considered in Example 5.1.

Example 5.3. Consider the influence game $(G, f, 3, V(G))$, where (G, f) is the influence graph considered in Example 5.1. In this case, we have that $F(\{a\}) = \{a, c, d\}$, and thus $\{a\} \in \mathcal{W}$. The fundamental set families for Γ are $\mathcal{W}^m = \{\{a\}, \{b\}\}$, $\mathcal{L}^M = \{\{c, d\}\}$, $\mathcal{L} = \{\{c, d\}, \{c\}, \{d\}, \{\}\}$ and $\mathcal{W} = \mathcal{P}(V(G)) \setminus \mathcal{L}$.

5.2 Expressiveness

Influence games are monotonic as, for any $X \subseteq N$ and $i \in N$, if $|F(X)| \geq q$ then $|F(X \cup \{i\})| \geq q$, and if $|F(X)| < q$ then $|F(X \setminus \{i\})| < q$. Thus, every influence game is a simple game. Moreover, we will show that the opposite is also true.

Theorem 5.1. Every simple game can be represented by an unweighted influence game. Furthermore, when the simple game Γ is given in either EWF or MWF, an unweighted influence game representing Γ can be obtained in polynomial time.

Proof. Assume that a simple game Γ is given by $(N, \mathcal{W})$ or $(N, \mathcal{W}^m)$. It is already well known that given $(N, \mathcal{W})$, the family $\mathcal{W}^m$ can be obtained in polynomial time. Thus we assume in the following that the set of players and the set $\mathcal{W}^m$ are given.

In order to represent Γ as an influence game we first define an unweighted influence graph (G, f). The graph $G = (V, E)$ is the following. The set V of nodes is formed by a set with n nodes, $V_N = \{v_1, \ldots, v_n\}$, one for each player, and a set of nodes for each MWC. For any $X \in \mathcal{W}^m$, we add a new set V_X with $slength(\Gamma) - |X|$ nodes. We connect vertex v_i with all the vertices in V_X whenever $i \in X$. Finally, the label function is defined as follows, for any $1 \leq i \leq n$, $f(v_i) = 1$ and, for any $X \in \mathcal{W}^m$ and any $v \in V_X$, $f(v) = |X|$. Observe that in the influence game $(G, f, slength(\Gamma), V_N)$ a team is successful if and only if its players form a winning coalition in Γ. Therefore $(G, f, slength(\Gamma), V_N)$ is a representation of Γ as unweighted influence game. It remains to show that given $(N, \mathcal{W}^m)$ a description of $(G, f, slength(\Gamma), N)$ can be computed in polynomial time. For doing so it is enough to show that $slength(\Gamma)$ can be computed in polynomial time. Let $k = slength(\Gamma)$.

Observe that, by definition, all the coalitions with k players are winning in Γ but at least one coalition with size $k-1$ is losing. Therefore there is a MWC with size k and there are no MWCs with size $k+1$. Thus, computing k is equivalent to compute the maximum size of a MWC. The last quantity can be obtained in polynomial time from a description of $\mathcal{W}^m$. □

The following example provides an illustration of the construction.

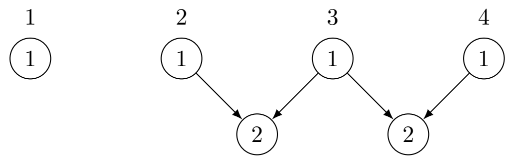


Figure 5.2: An unweighted influence graph associated to the simple game $(\{1,2,3,4\},\{\{1,2,4\},\{2,3\},\{3,4\}\})$.

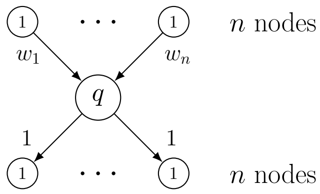


Figure 5.3: An influence graph (G, w, f) associated to the weighted game $[q; w_1, \ldots, w_n]$.

Example 5.4. Let $\Gamma = (\{1,2,3,4\},\{\{1,2,4\},\{2,3\},\{3,4\}\})$ be a simple game in MWF. We have that $slength(\Gamma) = 3$ because all subsets of N with cardinality 3 are winning, i.e., $\{1,2,3\},\{1,2,4\},\{1,3,4\},\{2,3,4\} \in \mathcal{W}$. For coalition $\{1,2,4\}$ we do not need to add nodes to the graph. For each of the teams $\{2,3\}$ and $\{3,4\}$, we need to add one node with label $3-2=1$. A drawing of the resulting unweighted influence graph is given in Figure 5.2.

The proof of Theorem 5.1 shows the expressiveness of the family of influence games with respect to the class of simple games. However, the construction cannot be implemented in polynomial time when the simple game is given in succinct ways like WRF. Observe also that the number of agents in the corresponding influence game is in general exponential in the number of players. For the particular case of weighted games in WRF, we can show that there exist representations by influence games having a polynomial number of agents.

Theorem 5.2. Every weighted game can be represented as an influence game. Furthermore, given a weighted representation of the game, a representation as an influence game can be obtained in polynomial time.

Proof. Let $[q; w_1, \ldots, w_n]$ be a weighted game, consider the influence game $(G, w, f, n+1, N)$, whose influence graph is shown in Figure 5.3. The n

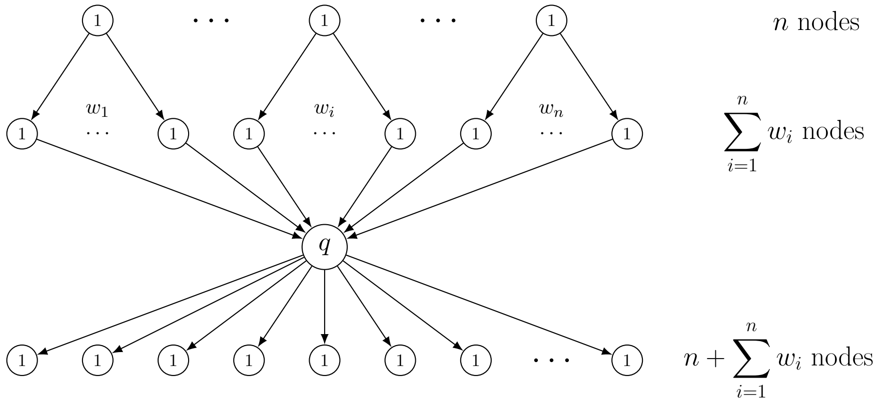


Figure 5.4: An unweighted influence graph (G, f) associated to the weighted game $[q; w_1, \dots, w_n]$.

nodes in the first level of G correspond to the set N, each of them has as associated label the value 1. Each node $i \in N$ is connected to a central node with label q and the corresponding edge has weight w_i. The n nodes in the last level are another set of n nodes with label 1. Observe that $X \subseteq N$ is a winning coalition in $[q; w_1, \dots, w_n]$ if and only if $\sum_{i \in X} w_i \geq q$. The last condition is equivalent to $|F(X)| \geq n + 1$. Thus we have that $X \subseteq N$ is a winning coalition in $[q; w_1, \dots, w_n]$ if and only if $X \subseteq N$ is a winning coalition in $(G, w, f, n + 1, N)$.

Finally, observe that the construction of $(G, w, f, n + 1, N)$ can be done in polynomial time with respect to the size of $[q; w_1, \dots, w_n]$. □

Observe that in the previous construction the size of the influence graph is polynomial in the number of agents but the overall construction is done in polynomial time in the size of the weighted representation. We can change slightly the construction and get a representation as unweighted influence game by increasing again the proportion of players.

Theorem 5.3. Every weighted game can be represented as an unweighted influence game. Furthermore, given a weighted representation of the game, a representation as unweighted influence game can be obtained in pseudo-polynomial time.

Proof. Let $[q; w_1, \dots, w_n]$ be a weighted game, consider the unweighted influence graph (G, f) sketched in Figure 5.4. The n nodes in the first level correspond to the set N. For any $i \in N$, node i is connected to a set of w_i different nodes in the second level representing its weight. Thus, $X \subseteq N$ is a winning coalition if and only if $\sum_{i \in X} w_i \geq q$, which is equivalent to $|F(X)| \geq n + \sum_{i=1}^{n} w_i$. Therefore, the influence game $(G, f, n + \sum_{i=1}^{n} w_i, N)$ is a representation of the given weighted game.

Observe that given $[q; w_1, \ldots, w_n]$, constructing the graph G requires time $O(n + w_1 + \cdots + w_n)$ and thus the construction can be done in pseudo-polynomial time. □

In the previous results we have assumed that a weighted representation of the game is given. It is known that there are weighted games whose weighted representation requires that $\max_{i \in N}\{w_i\}$ to be $(n+1)^{(n+1)/2}/2$ [208]. Therefore the construction of the previous lemma will require exponential space and time with respect to the number of players.

Our next result establishes the closure of influence games under intersection and union. Furthermore, we show that an influence game representing the resulting simple game can be obtained in polynomial time.

Theorem 5.4. Given two influence games, their intersection and union can be represented as an influence game. Furthermore, both constructions can be obtained in polynomial time.

Proof. Let $\Gamma = (G, w, f, q, N)$ be an influence game with $G = (V, E)$, recall that, for any $X \subseteq N$, $F_i(X) \subseteq V$ denotes the spread of influence of X in the i-th step of the activation process and that we can assume that $0 < i \leq n$. All the sets considered in our constructions are replications of either the set N or the set V. For sake of simplicity, we use the term *corresponding node* to refer to the same node in a different copy of N or V.

We start constructing an influence graph (G', w', f') as shown in Figure 5.5. G' has $2n+1$ columns of nodes. The first column F^0 represents V, and the remaining nodes are divided in pairs of sets (f^i, F^i), for $1 \leq i \leq n$. For any $1 \leq i \leq n$, the sets f^i and F^i have n nodes each, as a replication of the nodes in V. The edges are defined as follows, for any $1 \leq i \leq n$, a node $y \in F^{i-1}$ is connected to a node $z \in f^i$ if and only if $(y, z) \in E$. These edges have associated weight $w(y, z)$. Furthermore, every node in F^{i-1} is connected by an edge with weight 1 to its corresponding node in F^i. Every node in f^i is connected by an edge with weight 1 to its corresponding node

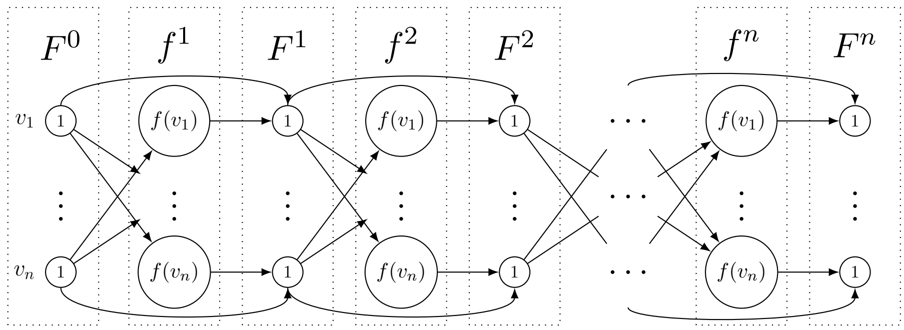


Figure 5.5: The influence graph (G', w', f') associated to the influence game (G, w, f, q, N).

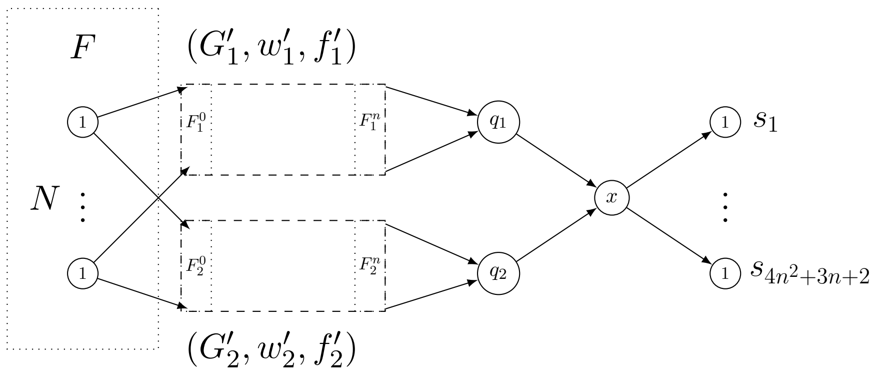


Figure 5.6: The influence graph associated to the intersection $(x = 2)$ or the union $(x = 1)$ of two influence games with influence graphs (G_1, w_1, f_1) and (G_2, w_2, f_2) and quotas q_1 and q_2 respectively.

in F^i. The labeling function assigns label 1 to all the nodes in sets F^i and maintains the original labeling for nodes in the sets f^i.

Note that after the activation of a team $X \subseteq F^0$ in (G', w', f'), for any $0 \leq i \leq n$, the set of nodes in F^i that are activated coincides with the set $F_i(X)$. Thus the subset of activated nodes in F^n coincides with $F(X)$. Observe also that (G', w', f') has $2n^2+n$ nodes and that it can be constructed in polynomial time in the size of a given influence game (G, w, f, q, N).

Now, given two influence games, namely $\Gamma_1 = (G_1, w_1, f_1, q_1, N)$ and $\Gamma_2 = (G_2, w_2, f_2, q_2, N)$, we construct the two influence graphs (G'_1, w'_1, f'_1) and (G'_2, w'_2, f'_2) as described before—see Figure 5.5. We use the construction depicted in Figure 5.6 to construct another influence graph. In this last

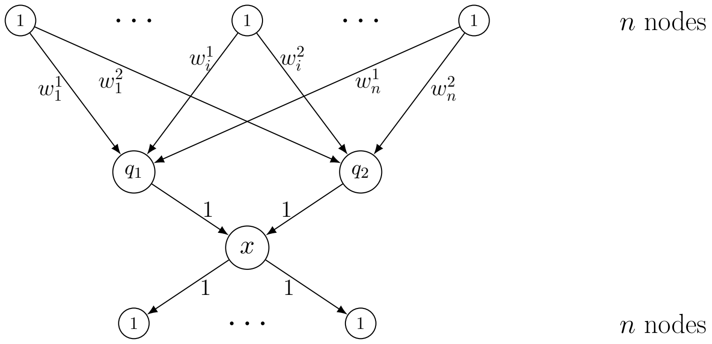


Figure 5.7: Influence graphs associated to $[q_1; w_1^1, \dots, w_n^1] \cap [q_2; w_1^2, \dots, w_n^2]$ ($x = 2$), and $[q_1; w_1^1, \dots, w_n^1] \cup [q_2; w_1^2, \dots, w_n^2]$ ($x = 1$).

construction we add a set F which is a copy of N. All the nodes in F have label 1. The nodes in F are connected to their corresponding nodes in F_1^0 and in F_2^0 through edges with weight 1. Furthermore, we add a node with label q_1, a node with label q_2, a node with label x, and a set with $4n^2+3n+2$ nodes. Those new nodes are connected according to the pattern given in Figure 5.6. The nodes in the last column, F_i^n, of (G_i', w_i', f_i') are all connected to the node with label q_i, for $i \in \{1, 2\}$. The nodes with labels q_1 and q_2 are connected to the node with label x which is connected to the last set of nodes. All those new connections have assigned weight 1. Observe that in total we have at most $2(2n^2 + n) + n + 3 + 4n^2 + 3n + 2$ nodes. Thus the overall construction can be computed in polynomial time.

Let $(G_\cup, w_\cup, f_\cup)$ be the influence graph obtained by setting $x = 1$ and $(G_\cap, w_\cap, f_\cap)$ be the influence graph obtained by setting $x = 2$. Consider the games $\Gamma_\cup = (G_\cup, w_\cup, f_\cup, 4n^2 + 3n + 2, F)$ and $\Gamma_\cap = (G_\cap, w_\cap, f_\cap, 4n^2 + 3n + 2, F)$. By construction a team X is successful in $\Gamma_\cup$ if and only if X is successful in either Γ_1 or Γ_2. Furthermore, a team is successful in $\Gamma_\cap$ if and only if X is successful in both Γ_1 and Γ_2. $\square$

It is interesting to note that it is possible to devise a construction representing the intersection or the union of weighted games as the influence games $(G_\cup', w', f_\cup', n{+}2, N)$ and $(G_\cap', w', f_\cap', n{+}3, N)$. The corresponding influence graphs $(G_\cup', w', f_\cup')$ and $(G_\cap', w', f_\cap')$ are shown in Figure 5.7—setting as before label x to be 1 or 2 depending on the considered operation. This new construction requires only a linear number of additional nodes, however the graph is weighted.

Thus, as any simple game can be represented as the intersection or union of a finite number of weighted games, we have an alternative way to show the completeness of the family of weighted influence games with respect to the class of simple games—Theorem 5.1. However, as the dimension and the codimension of a simple game might be exponential in the number of players—but bounded by the number of maximal losing and minimal winning coalitions, respectively [94, 89]—we cannot conclude that any simple game can be represented by a weighted influence game whose number of agents is polynomial in the number of players. For the particular case of unweighted influence game we know the following.[1]

Theorem 5.5. The family of unweighted influence games in which the number of agents in the corresponding influence graph is polynomial in the number of players is a proper subset of simple games.

Proof. We use a simple counting argument to show the result. Observe that, for any $n \geq 0$, there are more than $2^{(2^n/n)}$ simple games with n players [139]. Taking into account that a simple game has at most $n!$ isomorphic simple games we know that there are more that $2^{(2^n/n)}/n!$ different simple games on n players.

Consider an unweighted influence game with n players and $f(n)$ agents. The possibilities for the edge sets are less than $2^{(f(n)+1)^2}$. It suffices to consider label functions assigning values between 0 and $f(n)+1$. Thus there are at most $(f(n)+2)^{f(n)+2}$ possibilities for the labeling functions. Finally, for the quota, only $f(n)+2$ possibilities have to be considered. Thus, the number of unweighted influence games with n players and $f(n)$ agents is at most $2^{O(f(n)^2)}$.

Taking $f(n) = n^{\log n}$, the family includes all unweighted influence games with n players and polynomial number of agents. Taking the logarithm on both sides, one easily sees that $2^{O(f(n)^2)}$ is asymptotically smaller than $2^{(2^n/n)/n!}$. □

Even though we have shown in Theorem 5.2 that all games with polynomial dimension or polynomial codimension can be represented as weighted influence games in polynomial time—i.e., they admit weighted influence

[1]We thank Sascha Kurz for pointing out the proof of Theorem 5.5.

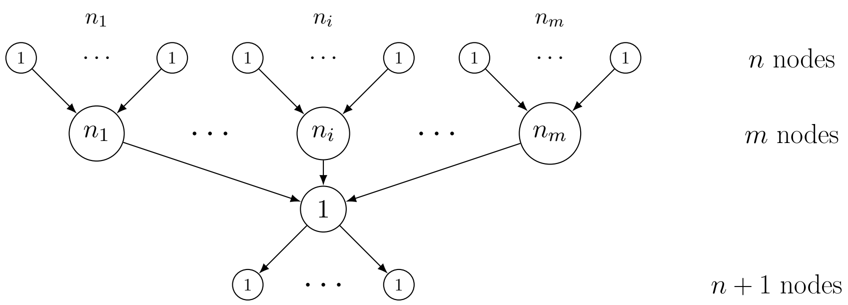


Figure 5.8: The simple game whose set of players $N = \{1, \ldots, n\}$ admits a partition $N_1, \ldots, N_m$ in such a way that $\mathcal{W} = \{S \subseteq N\,;\, \exists N_i \text{ with } N_i \subseteq S\}$ has exponential dimension, $n_1 \cdot \ldots \cdot n_{m-1}$ [94], but this game admits a polynomial unweighted influence graph (G, f) with respect to n for the corresponding unweighted influence game $(G, f, n+1, N)$.

graphs with polynomial number of agents—a fundamental open question is determining which simple games can be represented as an (unweighted) influence games with polynomial number of agents. In particular, it remains open to know whether there are games with exponential dimension that also require an exponential number of players in any representation as influence games.

In this line, we know that the simple game with exponential dimension with respect to the players of Section 2 in [94] can be represented by an unweighted influence game—see Figure 5.8—in polynomial time with respect to the number of players. Another candidate is the simple game with exponential dimension of Theorem 8 in [72] for which we have been unable to show whether it can be represented by a (unweighted) influence game with polynomial number of agents or not.

In Appendix B we show all the minimal unweighted influence games (G, f, q, N) with $N = V$, over three and four players, and their corresponding 10 and 30 representations as simple games in MWF.

5.3 Problems and Complexity

Our second set of results settles the complexity of the problems related to simple games presented in Section 2.3. Hardness results are obtained for unweighted influence games in which the number of agents in the network

	Influence games
Problem	(G, w, f, q, N)
Properties of simple games	
IsProper	coNPC
IsStrong	coNPC
IsDecisive	coNPC
Properties of coalitions	
IsBlocking	P
IsSwing	P
Properties of players	
IsDummy	coNPC
IsPasser	P
IsVetoer	P
IsDictator	P
IsCritical	P
IsSymmetric	P
Parameters	
Length	NPH
Width	NPH
Additional problems	
Equiv	coNPH
Iso	coNPH

Table 5.1: New results of complexity for properties and parameters of influence games.

is polynomial in the number of players, while polynomial time algorithms are devised for general influence games. The new results about properties and parameters are summarized in Table 5.1, complementing the results for simple games presented in Tables 4.1, 4.3 and 4.4.

The new results about solution concepts are summarized in Table 5.2, complementing the results for simple games presented in Table 4.2. Recall that problems Construct-Banzhaf-value and Construct-Shapley-Shubik-value are also denoted as Bval and SSval, respectively. We omit in this table the generic results of Table 4.2 regarding nonemptiness.

From Theorems 5.1 and 5.2 we know that all the computational problems related to properties and parameters that are computationally hard for simple games in EWF or MWF, as well as for weighted games in WRF, are also computationally hard for influence games. Nevertheless, the hardness results do not apply to unweighted influence games with polynomial, in the number of player, number of agents. In this section we address the computational complexity of problems for games with a polynomial number of agents. All the hardness proofs are given for the subclass formed by un-

	Influence games
Problem	(G, w, f, q, N)
EMPTY-	
CORE	P
IN-	
CORE	P
ϵ-CORE	NPH
ISZERO-	
CORE	P
CONSTRUCT-	
BANZHAF-VALUE	#PC
SHAPLEY-SHUBIK-VALUE	#PC
STABLE-SET	P
CORE	P

Table 5.2: New results of complexity for solution concepts of influence games.

weighted influence games on undirected influence graphs, which is a subset of all the other variations. The polynomial time algorithms are devised for the biggest class of general influence games, i.e., weighted influence games on directed graphs which includes all others.

Before starting to analyze problems we state here some basic results. From Lemma 5.1 we know that, for a given team X, we can compute in polynomial time the set $F(X)$. Therefore we have the following.

Lemma 5.2. For a given influence game (G, w, f, q, N), deciding whether a team $X \subseteq N$ is successful can be done in polynomial time.

Our next result concerns a particular type of influence games that we will use first as a basic construction, which associates an unweighted influence game to an undirected graph, and later as a representative of a particular subclass of influence games.

Definition 5.3. Given an undirected graph $G = (V, E)$, the unweighted influence game $\Gamma(G)$ is the game $(G, f, |V|, V)$ where, for any $v \in V$, the label $f(v)$ is the degree of v in G, i.e., $f(v) = d_G(v)$.

Recall that a set $S \subseteq V$ is a vertex cover of a graph G if and only if, for any edge $(u, v) \in E$, u or v (or both) belong to S. From the definitions we get the following result.

Lemma 5.3. Let G be an undirected graph. A team X is successful in $\Gamma(G)$ if and only if X is a vertex cover of G, Furthermore, the influence game $\Gamma(G)$ can be obtained in polynomial time, given a description of G.

Theorem 5.6. Computing LENGTH, WIDTH, sLENGTH and sWIDTH of an unweighted influence game is NP-hard.

Proof. For LENGTH, we provide a reduction from the minimum set cover problem. Let $C = \{C_1, \ldots, C_m\}$ be a collection of subsets of a universe with n elements. We associate to C the unweighted influence game (G, f, q, N) where $G = (V, E)$. The graph G has three disjoint sets of vertices: $Y = \{y_1, \ldots, y_m\}$, $T = \{t_1, \ldots, t_n\}$, and $Z = \{z_1, \ldots, z_{m+1}\}$, together with an additional vertex x. The components of the game are the following.

- $V = Y \cup T \cup \{x\} \cup Z$,
- $E = \{(y_j, t_i) \mid i \in C_j\} \cup \{(t_i, x) \mid 1 \leq i \leq n\} \cup \{(x, z_k) \mid 1 \leq k \leq m+1\}$,
- $f(y_j) = n + 1$, for any $1 \leq j \leq m$,
- $f(t_i) = 1$, for any $1 \leq i \leq n$,
- $f(z_k) = 1$, for any $1 \leq k \leq m + 1$,
- $f(x) = n$,
- $q = m + n + 1$ and
- $N = Y$.

Therefore, it is easy to see that a team $X \subseteq N$ succeeds if and only if it corresponds to a set cover, so the length of (G, f, q, N) coincides with the size of a minimum set cover.

For WIDTH we provide a reduction from the maximum set packing problem. Consider an influence game (G', f', q', N) where G' is constructed from G. We remove node $\{x\}$, add the connections $\{(t_i, z_k) \mid 1 \leq i \leq n, 1 \leq k \leq m + 1\}$, and set $f'(t_i) = 2$ for any $1 \leq i \leq n$. We keep $N = Y$ and set $q' = m + 1$. It is easy to see that a team $X \subseteq N$ is unsuccessful in (G', f', q', N) if and only if X corresponds to a set packing in C. Hence, the width of (G', f', q', N) is n minus the size of a maximum set packing of C.

The remaining results for sLENGTH and sWIDTH follow from the relationships of Lemma 4.9. □

The hardness result for LENGTH can also be obtained directly from Lemma 5.3 which provides a reduction from the minimum vertex cover problem. However, the reductions from the minimum set cover problem given in the previous theorem allow us to extract additional results about the complexity of approximation. In particular, the reductions in Theorem 5.6 imply that LENGTH is neither approximable within $(1-\epsilon)\cdot\log m$ nor within $c\cdot\log n$, for some $c>0$, and that WIDTH is not approximable within $m^{1/2-\epsilon}$, for any $\epsilon>0$, using the non-approximability results from [8] for the problems minimum set cover and minimum set packing.

Our next result settles the complexity of the computation of the Banzhaf and Shapley-Shubik values of a given player.

Theorem 5.7. Computing BVAL and SSVAL for a given influence game and a given player is #P-complete.

Proof. Both problems belong trivially to #P. To show hardness we construct a reduction for the problem of computing the number of vertex covers of a given graph which is known to be #P-complete [95]. Let G be a graph, we first construct the graph G' which is obtained from G adding a new vertex x and connecting x to all the vertices in G. The associated input to BVAL is formed by the influence game $\Gamma(G')$ and the player x. Observe that the reduction can be computed in polynomial time.

Let X be a successful team in $\Gamma(G')$ such that $x\in X$. When $X\neq V(G')$ we know that $X\setminus\{x\}$ must be a vertex cover of G. Furthermore $x\in C_x$ as $X\setminus\{x\}$ is not winning in $\Gamma(G')$. When $X=V(G')$, $X\setminus\{x\}$ is winning in $\Gamma(G')$ and thus $x\notin C_x$. As a consequence, we have that $\eta_x(\Gamma)$ coincides with the number of vertex covers of G minus one. As computing the number of vertex covers of a graph is #P-hard, we have that BVAL is #P-hard.

According to [10] (Theorem 3.29, page 50), to prove that SSVAL is #P-hard, it is enough to show that BVAL is #P-hard and that influence games verify the property of being a reasonable representation. In the remaining of this proof we show that influence games are a reasonable representation.

Let $\Gamma=(G,w,f,q,N)$ be an influence game, and assume that $G=(V,E)$ has n vertices and m edges. Consider the influence graph (G',w',f') where

- $G'=(V',E')$ and $V'=V\cup\{x,y\}\cup\{a_1,\dots,a_{2n}\}$,
- $E'=E\cup\{(x,y)\}\cup\{(v,y)\mid v\in V\}\cup\{(y,a_i)\mid 1\leq i\leq 2n\}$,
- $w'(e)=w(e)$, for any $e\in E$, and $w'(e)=1$, for any $e\in E'\setminus E$, and

- $f'(v) = f(v)$, for any $v \in V$, $f'(x) = 1$, $f'(y) = q + 1$, and $f'(a_i) = 1$, for any $1 \leq i \leq 2n$.

Finally, we consider the influence game $\Gamma^+ = (G', w', f', q', N')$ where $q' = 2n$ and $N' = N \cup \{x\}$.

From the previous construction, it follows that all the winning coalitions in Γ^+ must include x. Furthermore, $X \cup \{x\}$ is a winning coalition in Γ^+ if and only if X is a winning coalition in Γ. Therefore, Γ^+ is a representation of Γ' and has polynomial size with respect to the size of Γ. So, we conclude that influence games are a reasonable representation. □

For the following Theorem 5.8, we consider a new construction. Let $G = (V, E)$ be a graph where $V = \{v_1, \ldots, v_n\}$ and $E = \{e_1, \ldots, e_m\}$, and let k be an integer—which will be useful to consider a set cover of size k or less. Then the unweighted influence game $\Delta_1(G, k) = (G_1, f_1, q_1, N_1)$ is defined as follows, where Figure 5.9 shows the corresponding influence graph. $G_1 = (V_1, E_1)$ has $V_1 = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\}$ where $\alpha = m + n + 4$. The edges in E_1 are constructed as follows. We include the incidence graph of G: for any $e = (v_i, v_j) \in E$, we add to E_1 the edges (e, v_i), (e, v_j) and (e, y). For any $1 \leq i \leq n$, we add the edge (v_i, x). For any $1 \leq j \leq \alpha$, we add the edges (x, s_j) and (y, s_j). Finally, we add the edge (z, y). The labeling function f_1 is defined as: $f_1(v_i) = m + 2$, $1 \leq i \leq n$; $f_1(e_j) = 1$, $1 \leq j \leq m$; $f_1(s_\ell) = 1$, $1 \leq \ell \leq \alpha$; and $f_1(z) = 2$, $f_1(x) = k + 1$, $f_1(y) = m + 1$. The quota is $q_1 = \alpha$ and the set of players is $N_1 = \{v_1, \ldots, v_n, z\}$.

Observe that by construction the games $\Gamma(G)$ and $\Delta_1(G, k)$ can be obtained in polynomial time. As an immediate consequence of the definition, we have that X is a successful team in $\Delta_1(G, k)$ if and only if either $(|X \cap V| \geq k + 1)$ or $z \in X$ and $X \setminus z$ is a vertex cover in G.

Our next result settles the complexity of the problems which are coNP-complete.

Theorem 5.8. For unweighted influence games with polynomial number of vertices, the problems IsSymmetric, IsDummy, IsProper, IsStrong and IsDecisive are coNP-complete.

Proof. Membership in coNP follows from the definitions. To get the hardness results, we provide reductions from the complement of the Vertex Cover

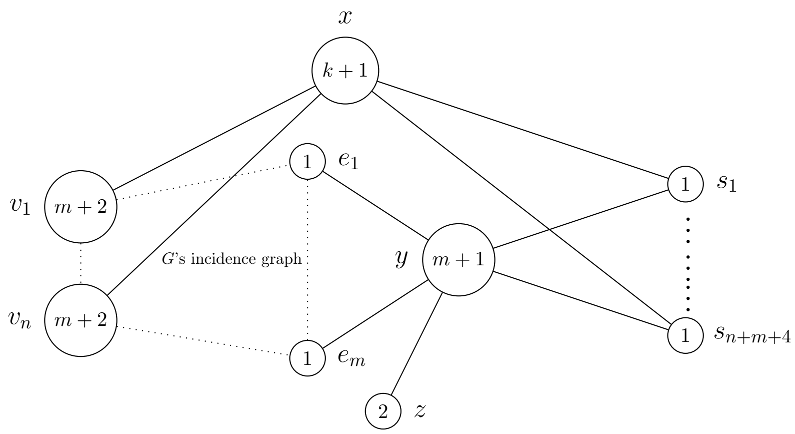


Figure 5.9: Influence graph (G_1, f_1) of the game $\Delta_1(G, k)$.

problem and some other problems derived from it. Let (G, k) be an input to Vertex Cover, as usual we assume that G has n vertices and m edges.

Let us start considering the IsDummy problem. Starting from $G = (V, E)$ and k, we construct the unweighted influence game $\Delta_1(G, k)$ and the pair $(\Delta_1(G, k), z)$ which is an instance of the IsDummy problem. If G has a vertex cover X with size k or less, by construction, we have that $X \cup \{z\}$ is a successful team of $\Delta_1(G, k)$. Furthermore, if X is a vertex cover of minimum size, we have that $X \cup \{z\}$ is a minimal successful team. Therefore, z is not a dummy player in $\Delta_1(G, k)$. If G does not have a vertex cover with size k or less and X is a successful team containing z, it must hold that $|X \setminus \{z\}| > k$, therefore $X \setminus \{z\}$ is a successful team. In consequence z is a dummy player in $\Delta_1(G, k)$. As the pair $(\Delta_1(G, k), z)$ is computable in polynomial time, we have the desired result.

Let us consider now the IsSymmetric problem. Starting from $G = (V, E)$ and k, we construct the unweighted influence game $\Delta_2(G, k) = (G_2, f_2, q_2, N_2)$ (see Figure 5.10). G_2 is obtained from the graph G_1 appearing in the construction of $\Delta_1(G, k)$ by adding two new vertices t and s and the edges (x, s), (y, s) and (t, s). Recall that the vertices of G_1 are $V(G_1) = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\}$. The label function is the following: $f_2(v) = f_1(v)$, for $v \in V(G_2) \cap V(G_1)$; $f_2(s) = 4$; $f_2(t) = 2$. Finally, $q_2 = \alpha + 1 = n + m + 5$ and $N_2 = \{v_1, \ldots, v_n, z, t\}$. Note that a description of G_2 can be obtained in polynomial time as well as a description of $\Delta_2(G, k)$ given a description of (G, k). Let us show that the construction

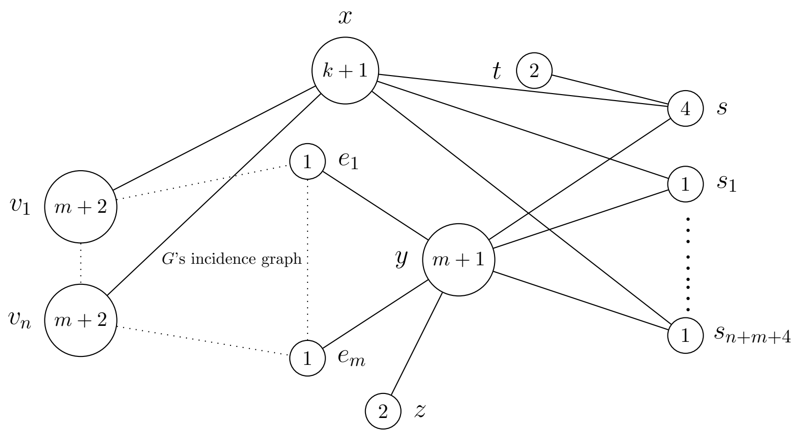


Figure 5.10: Influence graph (G_2, f_2) used in the definition of the game $\Delta_2(G, k)$.

is indeed a reduction.

When G has a vertex cover X of size k or less, by construction the team $X \cup \{z\}$ is successful in $\Delta_2(G, k)$ while the team $X \cup \{t\}$ is unsuccessful. Therefore z and t are not symmetric. When G does not have a vertex cover X of size k or less, by construction, any successful team Y must contain a subset with at least $k + 1$ vertices from $\{v_1, \ldots, v_n\}$. Therefore both $Y \cup \{z\}$ and $Y \cup \{t\}$ are successful teams in $\Delta_2(G, k)$, i.e., vertices z and t are symmetric.

To prove hardness for the next two problems, IsProper and IsDecisive, we provide a reduction from the following variation of the Vertex Cover problem:

Name:	Half vertex cover
Input:	Undirected graph with an odd number of vertices n.
Question:	Is there a vertex cover with size $(n - 1)/2$ or less?

We first show that the Half vertex cover problem is NP-complete. By definition the problem belongs to NP. To prove hardness we show a reduction from the Vertex Cover problem. Given a graph G with n vertices and an integer k, $0 \leq k \leq n$, we construct a graph $\hat{G}$—see Figure 5.11—as follows.

$\hat{G}$ has vertex set $\hat{V} = V(G) \cup X \cup Y \cup \{w\}$, where X has $n-k-1$ vertices,

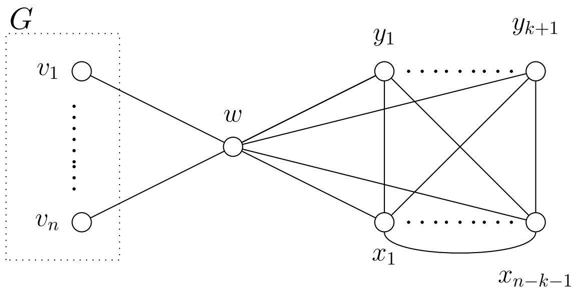


Figure 5.11: Graph $\hat{G}$ used to prove that HALF VERTEX COVER is NP-hard.

Y has $k+1$ vertices, and edge set

$$\begin{aligned}\hat{E} = E &\cup \{(x, x') \mid x \neq x' \wedge x, x' \in X\} \\ &\cup \{(x, y) \mid x \in X, y \in Y\} \\ &\cup \{(w, z) \mid z \in V \cup X \cup Y\}.\end{aligned}$$

By construction, $\hat{G}$ has $2n+1$ vertices, so it can be constructed in polynomial time. Note that any vertex cover S of $\hat{G}$ with minimum size has to contain w, all the vertices in X and no vertex from Y. The remaining of the cover, $S \cap V$ must be a minimum vertex cover of G. Therefore, G has a vertex cover of size k or less if and only if $\hat{G}$ has a vertex cover of size n or less.

Let us provide a reduction from the HALF VERTEX COVER to the ISPROPER and the ISDECISIVE problems. Let G be an instance of HALF VERTEX COVER with $2k+1$ vertices, for some value $k \geq 1$. Consider the unweighted influence game $\Delta_1(G, (n-1)/2) = (G_1, f_1, q_1, N_1)$. Recall that $V(G') = \{v_1, \ldots, v_n, e_1, \ldots, e_m, x, y, z, s_1, \ldots, s_\alpha\}$ where $\alpha = n + m + 4$, $q_1 = n + m + 5$, and $N_1 = \{v_1, \ldots, v_n, z\}$. Let $k = (n-1)/2$.

If G has a vertex cover X with $|X| \leq k$, the team $X \cup \{z\}$ is successful and, as $n + 1 - |X \cup \{z\}| > k$, we have that $N \setminus (X \cup \{z\})$ is also successful. Hence $\Delta_1(G, k)$ is not proper. When all the vertex covers of G have more than k vertices, any successful team Y of $\Delta_1(G, k)$ verifies $|Y \cap \{v_1, \ldots, v_n\}| > k$, i.e., $|Y \cap \{v_1, \ldots, v_n\}| \geq k + 1$. For a successful team Y, we have to consider two cases: $z \in Y$ and $z \notin Y$. When $z \in Y$, $N \setminus Y \subseteq \{v_1, \ldots, v_n\}$ and $|N \setminus Y| < n - k - 1 = k$. Thus, $N \setminus Y$ is an unsuccessful team. When $z \notin Y$, $|N \setminus (Y \cup \{z\})| \leq k$ and $N \setminus Y$ is again an unsuccessful team. So, we conclude that $\Delta_1(G,(n-1)/2)$ is proper. As $\Delta_1(G,(n-1)/2)$ can be obtained in polynomial time, the IsProper problem is coNP-hard.

Observe that when G is an instance of the Half vertex cover and all the vertex covers of G have more than $(n-1)/2$ vertices, the game $\Delta_1(G,(n-1)/2)$ is also decisive. When this condition is not met, the game $\Delta_1(G,(n-1)/2)$ is not proper and thus it is not decisive. Thus, we conclude that the IsDecisive problem is also coNP-hard.

To finish the proof we show hardness for the IsStrong problem. We provide a reduction from the complement of the following problem.

Name:	Half independent set
Input:	Undirected graph with an even number of vertices n.
Question:	Is there an independent set with size $n/2$ or higher?

The Half independent set trivially belongs to NP. Hardness follows from a simple reduction from the Half independent set. Starting from a graph G with an odd number of vertices we construct a new graph G' by adding one new vertex connected to all the vertices in G. This construction guarantees that G has a vertex cover of size $(n-1)/2$ or less if and only if G' has a vertex cover with size $n/2$ or less. As the complement of a vertex cover is an independent set, we have that G has a vertex cover of size $(n-1)/2$ if and only if G' has an independent set with size $n/2$ or higher.

Now we show that the complement of the Half independent set problem can be reduced to the IsStrong problem. We associate to an input to Half independent set the game $\Delta_3(G,n/2) = (G_3, f_3, n+m+5, N_3)$ where $N_3 = V \cup \{z\}$ and (G_3, f_3) is the influence graph described in Figure 5.12. Which is a variation of $\Delta_1(G,k)$ using ideas similar to those in the reduction from the Set Packing problem in Theorem 5.6.

When G has an independent set with size at least $n/2$, G also has an independent set X with $|X| = n/2$. It is easy to see that both the team $X \cup \{z\}$ and its complement are unsuccessful in $\Delta_3(G,n/2)$. Therefore, $\Delta_3(G,n/2)$ is not strong. Assume now that all the independent sets in G have less than $n/2$ vertices. Observe that, for a team X in $\Delta_3(G,n/2)$ with $|X \cap V| < n/2$, its complement has at least $n/2+1$ elements in V and thus it is successful. When $|X \cap V| > n/2$ the team is successful. Therefore we have to consider only those teams with $|X \cap V| = n/2$. In such a case, we

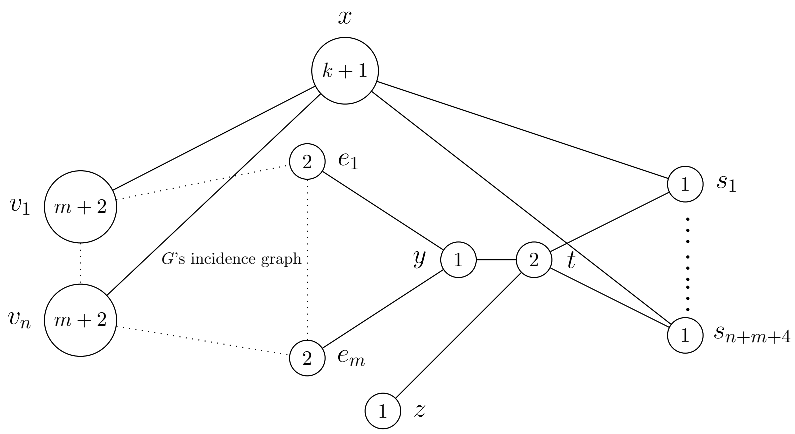


Figure 5.12: Influence graph (G_3, f_3) of the game $\Delta_3(G, k)$.

know that neither $X \cap V$ nor $V \setminus (X \cap V)$ are independent sets. Then, by construction, one of the sets X or $N \setminus X$ must contain z and is successful while its complement is unsuccessful. In consequence $\Delta_3(G, n/2)$ is strong. $\square$

The complexity of the remaining problems is summarized in the following theorem.

Theorem 5.9. For influence games, the problems IsPasser, IsVetoer, IsDictator, IsCritical, IsBlocking and IsSwing belong to P.

Proof. We provide characterizations of the properties in terms of the sizes of $F(X)$ for adequate sets X. Given an influence game $\Gamma = (G, w, f, q, N)$, $i \in N$ and $X \subseteq N$, we have.

- Player i is a passer if and only if $|F(\{i\})| \geq q$.
- Player i is a vetoer if and only if $|F(N \setminus \{i\})| < q$.
- Player i is a dictator if and only if $|F(N \setminus \{i\})| < q$ and $|F(\{i\})| \geq q$.
- Player i is critical for X if and only if $|F(X)| \geq q$ and $|F(X \setminus \{i\})| < q$.
- Team X is blocking if and only if $|F(N \setminus X)| < q$.
- Team X is a swing if and only if $|F(X)| \geq q$ and there is $i \in X$ for which $|F(X \setminus \{i\}) < q$.

Therefore, from Lemma 5.1, we get the claimed result. $\square$

Now we consider the complexity of the problems related to game isomorphism and equivalence. We state here the definitions for influence games.

Definition 5.4. Let $\Gamma = (G, w, f, q, N)$ and $\Gamma' = (G', w', f', q', N')$ be two influence games with the same number of players. Γ and Γ' are *isomorphic* if and only if there exists a bijective function $\varphi : N \to N'$, such that for every team $X \subseteq N$, $|F(X)| \geq q$ if and only if $|F(\varphi(X))| \geq q'$. Moreover, when $N = N'$ and φ is the identity function, then we say that the two influence games are *equivalent*.

Theorem 5.10. For unweighted influence games with polynomial number of vertices, the problem EQUIV is coNP-complete and the problem ISO is coNP-hard and belongs to Σ_2^p.

Proof. Membership to the corresponding complexity classes follows directly from the definition of the problems. For the hardness part we provide a reduction from the complement of the VERTEX COVER problem. Let G be a graph and consider the influence game $\Gamma_1 = \Delta_1(G, k)$ as defined before—see Figure 5.9. Recall that the set of players is $N_1 = \{v_1, \dots, v_n, z\}$. To define the second influence game Γ_2 we consider the weighted game with set of players N_1 and quota $q = k + 1$. The weights of the players are the following: $w(v_i) = 1$, for any $1 \leq i \leq n$, and $w(z) = 0$. A representation of Γ_2 as an unweighted influence game can be obtained in polynomial time using the construction of Theorem 5.3. Our reduction associates to an input to vertex cover (G, k) the pair of influence games (Γ_1, Γ_2). Observe that Γ_1 is equivalent (isomorphic) to Γ_2 if and only if G does not have a vertex cover of size k or less. □

We have been unable to provide a complete classification for the ISO problem. It remains open to show whether the problem is Σ_2^p-hard or not.

Regarding the remaining solution concepts, we have the following result.

Theorem 5.11. For influence games, the problems CONSTRUCT-STABLE-SET, EMPTY-CORE, IN-CORE, ISZERO-CORE and CONSTRUCT-CORE belong to P.

Proof. CONSTRUCT-STABLE-SET belongs to P by Proposition 2.3, because given an influence game, we can determine a minimal successful team in polynomial time: starting by the empty team, then continue adding vertices until obtain a team whose spread of influence meets the quota; this team is winning, but without the last vertex is unsuccessful, so the team is minimal.

The remaining problems belong to P by Proposition 2.4, since by Theorem 5.9 it is polynomial to decide whether a player is vetoer. □

We know by Theorem 5.6 that computing LENGTH in influence games is NP-hard, so according the conjecture of [10], for influence games CONSTRUCT-LEAST-CORE would be NP-hard, and hence CONSTRUCT-NUCLEOLUS would also be NP-hard. Furthermore, by Proposition 2.6 we have the following.

Corollary 5.1. For influence games, IN-ϵ-CORE is NP-hard.

5.4 Subclasses of Influence Games

In this section we focus in restricted subclasses of influence games. These restrictions could be useful to model particular systems, as well as to characterize subfamilies for which the computational complexity of some of the considered problems changes, becoming in general more tractable.

There are several ways to restrict an influence game (G, w, f, q, N). In the previous sections of this chapter, we have already considered unweighted influence games, as well as cases in which all the agents are players, i.e., $N = V$. Later, in Section 6.1.1, we restrict the labeling function so that it corresponds to the majority rule, when individuals are convinced when a majority of their neighbors are active. Further, in Chapter 6 we consider some restrictions in the topology of the influence graphs, considering bipartite graphs and star graphs. Bipartite graphs have been already used in the context of spread of influence through social networks [136], and also for collective choice model for societies [255, 256].

In what follows we consider two extreme cases of influence spread for undirected and unweighted influence games, with restricted levels of influence. In Section 5.4.1 we consider a maximum influence requirement, where agents adopt a behavior only when all its peers have already adopted it. In Section 5.4.2 we consider a minimum influence requirement, in which an agent gets convinced when at least one of its peers does. We show that, in both cases, the problems ISPROPER, ISSTRONG and ISDECISIVE, as well as computing WIDTH, have polynomial time algorithms. Computing LENGTH is NP-hard for maximum influence and polynomial time solvable for minimum influence.

5.4.1 Maximum Influence Requirement

Here we analyze first the case with maximum influence and maximum spread, that is games of the form $\Gamma = (G, f, |V|, V)$ where $f(v) = d_G(v)$, or, as we said in Definition 5.3, the game $\Gamma = \Gamma(G)$, for some graph G. When the graph G is disconnected with connected components $C_1, \dots, C_k$, the associated game $\Gamma(G)$ can be analyzed from the $\Gamma(C_1), \dots, \Gamma(C_k)$. Observe that, due to maximum spread, a successful team must influence all the vertices in the graph. Therefore, the members of a successful team in a connected component must influence all the vertices in their component. So, a team X is successful in $\Gamma(G)$ if and only if, for any $1 \leq i \leq k$, the team $X \cap V(C_i)$ is successful in $\Gamma(C_i)$. We analyze first the case in which G is connected.

Theorem 5.12. In an unweighted influence game Γ with maximum influence and maximum spread on a connected graph G it holds that:

- Γ is proper if and only if G is not bipartite.
- Γ is strong if and only if G is either a star or a triangle.
- Γ is decisive if and only if G is a triangle.

Proof. From Lemma 5.3 we know that the successful teams of $\Gamma = \Gamma(G)$ coincide with the vertex covers of G. We also recall that the complement of a vertex cover is an independent set.

If $G = (V, E)$ is bipartite, let (V_1, V_2) be a partition of V so that V_1 and V_2 are independent sets. In such a case, both V_1 and $V_2 = N \setminus V_1$ are successful teams in Γ. Therefore, Γ is not proper. For the opposite direction, if Γ is not proper, then the game admits two disjoint successful team, i.e, two disjoint vertex covers of G, and hence each of them must be an independent set. Thus the graph G is bipartite.

Now we prove that Γ is not strong if and only if G has at least two non-incident edges. Observe that a graph where all edges are incident is either a triangle or a star. If G has at least two non-incident edges $e_1 = (u_1, v_1)$ and $e_2 = (u_2, v_2)$, $\{u_1, v_1\}$ and $N \setminus \{u_1, v_1\}$ are both unsuccessful teams, therefore Γ is not strong. When the game is not strong, there is a team X

such that both X and $N \setminus X$ are unsuccessful. For this to happen it must be that there is an edge uncovered by X and another edge uncovered by $N \setminus X$. Thus G must have two non-incident edges.

Finally, it is well known that non-bipartite graphs has at least one odd cycle, so the only non-bipartite graph with all pair of edges incidents (proper and strong) is a triangle (decisive). □

When the graph is disconnected, a successful team X_i in the game $\Gamma(C_i)$ can be completed to a winning coalition in Γ. Observe that, if $\overline{X}_i = V(C_i) \setminus X_i$, the set $V \setminus \overline{X}_i$ is successful in Γ and contains X_i. For an unsuccessful team X_i in $\Gamma(C_i)$ both X_i and $V \setminus \overline{X}_i$ are unsuccessful in Γ. Therefore, the previous result can be extended to disconnected graphs by requesting the conditions to hold in all the connected components of the given graph.

Corollary 5.2. In an unweighted influence game Γ with maximum influence and maximum spread on a graph G the following properties hold.

- Γ is proper if and only if all the connected components of G are not bipartite.
- Γ is strong if and only if all the connected components of G are either a star or a triangle.
- Γ is decisive if and only if all the connected components of G are triangles.

Furthermore, the problems IsProper, IsStrong and IsDecisive belong to P for unweighted influence games with maximum influence and maximum spread.

In regard to the complexity of the two main parameters we have the following result.

Theorem 5.13. For unweighted influence games with maximum influence and maximum spread on a connected graph G, computing Length is NP-hard. Computing Width of $\Gamma(G)$ can be done in polynomial time even when G is disconnected.

Proof. As before we use the fact that $\Gamma(G)$ can be computed in polynomial time. Furthermore, from Lemma 5.3, $\mathit{length}(\Gamma(G))$ is the minimum size of a vertex cover of G. Therefore Length is NP-hard.

We prove that WIDTH can be computed in polynomial time by a case analysis. If G is just an isolated vertex or just one edge, the empty set is the unique unsuccessful team, thus $width(\Gamma) = 0$. Otherwise, either G has no edges or has at least one edge and an additional vertex. In the first case, the graph is an independent set with at least two vertices. Assume that $u \in V$, then $V \setminus \{u\}$ is unsuccessful and we conclude that $width(\Gamma) = n - 1$.

In the second case G has at least one edge $e = (u, v)$ and $V \setminus \{u, v\}$ is nonempty. We have again two cases, either G has an isolated vertex u or all the connected components of G have at least one edge. When u is an isolated vertex the team $V \setminus \{u\}$ is unsuccessful, therefore $width(\Gamma) = n-1$. When all the connected components of G have at least one edge, any team with $n-1$ nodes is a vertex cover, thus $width(\Gamma) < n-1$. Observe that the set $V \setminus \{u, v\}$ is not empty and, furthermore it does not cover the edge e, thus we have an unsuccessful team with $n - 2$ vertices. Thus, in this case $width(\Gamma) = n - 2$.

As the classification can be checked trivially in polynomial time we get the claimed result. □

For the case of maximum influence but not maximum spread, that is influence games of the form (G, f, q, V) where $f(v) = d_G(v)$ and $q < n$, the game cannot be directly analyzed from the games on the connected components, as the total quota can be fulfilled in different ways by the agents influenced in each component. Nevertheless, as the influence is maximum, any set of vertices X can influence another vertex u only when all the neighbors of u are included in X, alternatively when u becomes an isolated vertex after removing X. This leads to the following characterization of the successful teams.

Lemma 5.4. In an unweighted influence game with maximum influence $\Gamma = (G, d_G, q, V)$ where G has no isolated vertices, $X \subseteq V$ is a successful team if and only if removing X from G leaves at least $q - |X|$ isolated vertices.

This characterization gives rise to the following problem:

Name: AREISOLATED
Input: Graph $G = (V, E)$ and $q, k \in \mathbb{N}$.
Question: Is there $S \subseteq V$ such that $|S| \leq k$ and removing S from G

there are at least $q - k$ isolated vertices?

Observe that for $q = n$ we have that the solution S in the previous problem must be a vertex cover, and thus the AreIsolated problem is NP-hard.

Theorem 5.14. For influence games Γ with maximum influence, Length is NP-hard and Width belongs to P.

Proof. The hardness result follows from the previous observation. Observe that computing the minimum size of a solution to the AreIsolated problem is equivalent to compute the Length of the game $\Gamma = (G, d_G, q, V)$ and thus the later problem is NP-hard.

When computing the Width of $\Gamma = (G, d_G, q, V)$ we want to maximize the size of the unsuccessful teams. Therefore, we can restrict ourselves to analyze only unsuccessful teams X for which $F(X) = X$. We have that X is an unsuccessful team with $F(X) = X$ if and only if $|X| < q$ and every non isolated vertex in $V \setminus X$ remains non isolated in the subgraph induced by $N \setminus X$, i.e., $G[V \setminus X]$.

We consider first the case in which G has no isolated vertices. We first solve the problem of deciding whether, for a given α, it is possible to discard α nodes from G without leaving isolated vertices. For doing so we sort the sizes of the connected components of G in increasing order of size. As G has no isolated vertices all the connected components have at least two vertices. Assume that G has k connected components $C_1, \ldots, C_k$ with sizes $2 \leq w_1 \leq w_2 \leq \cdots \leq w_k$.

When $w_k = 2$, all the connected components have exactly two vertices. Therefore, if α is even and at most n, we can discard the α vertices in the first $\alpha/2$ components, without leaving isolated vertices. Otherwise, the removal of any set of size α will leave at least one isolated vertex.

When $w_k > 2$. We compute the first value j for which $\sum_{i=1}^{j} w_i \leq \alpha$ but $\sum_{i=1}^{j+1} w_i > \alpha$. Let $\beta = \sum_{i=1}^{j} w_i$. Let S_j be the set of vertices in the first j-components. If $\beta = \alpha$, S_j can be removed without leaving isolated vertices. When $\beta < \alpha$ we have two cases:

(1) $w_{j+1} > \alpha - \beta + 1$. Let $C \subset C_{j+1}$ be a set with $w_{j+1} - (\alpha - \beta)$ vertices such that $G[C]$ is connected. The vertices in S_j together with the $\alpha - \beta$ vertices of C_{j+1} not in C can be removed without leaving any isolated vertex.

(2) $w_{j+1} \leq \alpha - \beta + 1$. By construction, $\alpha < \beta + w_{j+1}$, thus $w_{j+1} = \alpha - \beta + 1$. If $j+1 < k$, removing the vertices in S_j together with $\alpha - \beta - 1$ vertices from the $j+1$-th component—as in case (1)—and one additional vertex from the k-th component leaves no isolated vertices. If $j + 1 = k$, the removal of any set of size α will leave at least one isolated vertex.

The previous characterization can be decided in polynomial time for any value of α. By performing the test for $\alpha = q-1, q-2, \ldots, 1$ we can compute in polynomial time the maximum value of α (α_{max}) for which α nodes can be discarded without leaving isolated vertices. As the Width of the game is just α_{max} we get the desired result for graphs without isolated vertices.

When G has n_0 isolated vertices, we consider the graph G' obtained from G by removing all the isolated vertices. Note that for a team X with $X = F(X)$ and any set Y of isolated vertices we have that $F(X \cup Y) = X \cup Y$, thus $width(\Gamma) = \min\{width(\Gamma') + n_0, q-1\}$. Therefore Width can be computed in polynomial time. □

5.4.2 Minimum Influence Requirement

Let be $\Gamma = (G, 1_V, q, N)$ where $1_V(v) = 1$ for any $v \in V$. Observe that if G is connected, the game has a trivial structure as any nonempty vertex subset of N is a successful team. For the disconnected case we can analyze the game considering an instance of the Knapsack problem. Assume that G has k connected components, $C_1, \ldots, C_k$. Without loss of generality, we assume that all the connected components of G have nonempty intersection with N. For $1 \leq i \leq k$, let $w_i = |V(C_i)|$ and $n_i = |V(C_i) \cap N|$.

Lemma 5.5. If a successful team X is minimal then it has at most one node in each connected component. Minimal successful teams are in a many-to-one correspondence with the MWCs of the weighted game $[q; w_1, \ldots, w_k]$.

Moreover, we have the following result.

Theorem 5.15. For unweighted influence games with minimum influence, the problems Length, Width, IsProper, IsStrong and IsDecisive belong to P.

Proof. Let $\Gamma = (G, 1_V, q, N)$ be an unweighted influence game with minimum influence.

To compute LENGTH, assume that the connected components of G are sorted in such a way that $w_1 \geq \cdots \geq w_k$. To minimize the size of a winning coalition we consider only those coalitions with at most one player in a connected component. Observe that, the $length(\Gamma)$ is the minimum j for which $\sum_{i=1}^{j} w_i \geq q$ but $\sum_{i=1}^{j-1} w_i < q$. Of course this value can be computed in polynomial time.

To compute WIDTH, observe that an unsuccessful team of maximum size can be obtained by computing a selection $S \subseteq \{1, \ldots, k\}$ of connected components in such a way that $\sum_{i \in S} w_i < q$ and $\sum_{i \in S} n_i$ is maximized. Computing such selection is equivalent to solving a KNAPSACK problem on a set of k items, item i having weight w_i and value n_i, and setting the knapsack capacity to q. As the KNAPSACK problem can be solved in pseudo-polynomial time and, in our case, all the weights and values are at most n, we conclude that WIDTH can be computed in polynomial time.

To compute ISSTRONG, observe that in order to minimize the influence of the complementary of a team X it is enough to consider only those teams X that contain all or none of the players in a connected component. Let $w_N = \sum_{i=1}^{k} w_i$, and let α_{max} be the maximum $\alpha \in \{0, \ldots, q-1\}$ for which there is a set $S \subseteq \{1, \ldots, k\}$ with $\sum_{i \in S} w_i = \alpha$. Note that α can be zero and thus S can be the empty set. Observe that Γ is strong if and only if $w_N - \alpha_{max} \geq q$. The value α_{max} can be computed by solving several instances of the KNAPSACK problem. As the weights are at most n, the value can be obtained in polynomial time.

To compute ISPROPER, note that to check whether the game is not proper it is enough to show that there is a winning coalition whose complement is also winning. For doing so we separate the connected components in two sets: those containing one player and those containing more that one player. Let $A = \{i \mid n_i = 1\}$ and $B = \{i \mid n_i > 1\}$. Let $N_A = \cup_{i \in A}(N \cap V(C_i))$ and $N_B = N \setminus N_A$. Let $w_A = \sum_{i \in A} w_i$ and $w_B = w_N - w_A$. As all the components in B have at least two vertices, we can find a set $X \subseteq N_B$ such that $|F(X)| = |F(N_B \setminus X)| = w_B$. Thus if $w_B \geq q$ the game is not proper. When $w_B < q$ the game is proper if and only if the influence game Γ' played on the graph formed by the connected components belonging to A and quota $q' = q - w_B$ is proper. Observe that Γ' is equivalent to the weighted game with a player for each component in $i \in A$ with associated weight w_i and quota q'. Let α_{min} be the minimum

$\alpha \in \{q', \ldots, w_A\}$ for which there is a set $S \subseteq A$ with $\sum_{i \in S} w_i = \alpha$. Observe that Γ' is proper if and only if $w_A - \alpha_{min} < q'$. The value α_{min} can be computed by solving several instances of the KNAPSACK problem having item weights polynomial in n. Therefore, α_{min} can be computed in polynomial time and the claim follows. □

Chapter 6

Collective Choice Models

In the previous chapter, through the definition of influence games, we studied the ways in which the actors of a multi-agent system influence each other through their interactions in a social network and, in particular, the social rules that can be used for the spread of influence. This chapter concerns to some applications of unweighted influence games in multi-agent systems, decision theory and social choice.

In Section 6.1 we define four collective choice models: *OLF systems*, *OLFM systems*, *mediation systems* and *influence systems*. All of them can be studied in the context of influence games.

OLF systems were defined in [255] as a kind of opinion leader-follower collective choice model. Opinion leadership is a well known and established model for communication policy in sociology and marketing. It comes from the *two-step flow of communication* theory proposed since the 1940s [152]. This theory recognizes the existence of collective decision making situations in societies formed by actors called *opinion leaders*, who exert influence over other kind of actors called the *followers*, becoming in a two-step decision process [152, 134]. In the first step of the process, all actors receive information from the environment, generating their own decisions; in the second step, a flow of influence from some actors over others is able to change the choices of some of them [252].

The other three models are new. OLFM systems are a generalization of OLF systems that supports inner nodes. In influence games, a set of agents have to take a decision among two possible alternatives with the help of the social environment or network of the system itself. However, sometimes not all individuals play the same role in the process of taking a decision. Thus, mediation systems are inspired in a multi-agent system with a very simple topology, but nevertheless it allows to study systems with a *mediator*, an external kind of actor that can exert influence, in different degrees, to the agents and thus help to reach a decision. In this sense, mediation systems are an extension of unweighted influence games, in order to incorporate an additional level of influence spread, exerted by the mediator. These systems allows to model a natural mediation schema occurring in society. Furthermore, we shall see that the main difference between influence games and OLF systems is that in the first ones only one of the two alternatives of the actors can be propagated, while in OLF systems it is propagated the alternative with majority. Two kind of influence systems are defined to deal with this difference.

In Section 6.2 we study some computational problems for mediation systems, showing that the presence of a single mediator facilitates the computation of several problems that are hard for influence games—see Table 5.1 of Section 5.3.

In Section 6.3 we study the computational complexity of computing the satisfaction or Rae index for influence systems. The satisfaction measure was defined in [255] for OLF systems, motivated by the theoretical study of the effects that different opinion leader-follower structures can exert in collective decision making systems. However, this measure is the same than the Rae index, a power index introduced by Douglas W. Rae [220] for anonymous games, that afterwards it was applied by Dubey and Shapley [66] for simple games. We show that the computation of this measure is hard, and then we present polynomial results for some particular cases. One of this particular cases is closely related to the mediation systems.

In Section 6.4 we generalize for OLFM systems some properties that the satisfaction measure meets for OLF systems. By using these properties, we provide an axiomatization of the satisfaction score for the case in which followers maintain their own initial decisions unless all their opinion leaders share an opposite inclination. This new axiomatization generalizes the one given by [256] for OLF systems under the same restrictions.

6.1 Models

In general, a *collective decision making model* or *collective choice model* $\mathcal{M}$ for a set of n actors is a decision system that defines a *collective (choice) decision function* $C_{\mathcal{M}}(x)$, where $x \in \{0,1\}^n$ is the initial decision vector or initial choice vector of the actors, assigning one of the values 1 or 0 as collective decision. Let V be a set of actors, abusing of notation we may consider $C_{\mathcal{M}}(X)$ instead of $C_{\mathcal{M}}(x)$ where $i \in X \subseteq 2^V$ if and only if $x_i = 1$. Note that the decision process may include many parameters in the model, but in all models considered in this chapter we assume that the collective decision function can be computed in polynomial time.

Definition 6.1. We said that $x \in \{0,1\}^n$ is an *initial decision vector*, where x_i represents the initial decision of the i-th actor of some decision system.

Observe that we can associate to any simple game a collective decision function in a natural way:

Definition 6.2. Let $\Gamma = (N, \mathcal{W})$ be a simple game. Let $x \in \{0,1\}^n$ be an initial decision vector of the players. The *collective decision function* associated to Γ is defined as follows:

$$C_\Gamma(x) = \begin{cases} 1 & \text{if } X(x) \in \mathcal{W}, \\ 0 & \text{otherwise} \end{cases}$$

where $X(x)$ is defined as $\{i \in N \mid x_i = 1\}$.

6.1.1 OLF Systems

OLF systems are structured on directed bipartite graphs. The decision process considers three kind of actors: *opinion leaders*, *followers* and *independent actors*. Opinion leaders cannot be influenced but they may exert their influence over its followers. Followers can change their initial decision when the influence from the leaders is high enough. Independent actor neither can influence nor can being influenced by others. At the end of the process all actors arrive to an stable solution and the collective decision function corresponds to the *simple majority voting system*.

Note that the influence games are able to represent a "more-than-two-step flow of communication", providing more complex influence relationships between the different actors than in this model. The model can be formalized as follows.

Definition 6.3. Let n be an odd number. An *opinion leader-follower system—OLF system*, in short—for a set of n actors is a pair $\mathcal{S} = (G, q)$ where $G = (V, E)$ is a bipartite digraph, representing the actors' relation, and q is a rational number with $1/2 \leq q < 1$, here called *fraction value*. The set V is partitioned into three subsets:

- The *opinion leaders*: $\mathtt{L}(G) = \{i \in V \mid P_G(i) = \emptyset \text{ and } S_G(i) \neq \emptyset\}$.
- The *followers*: $\mathtt{F}(G) = \{i \in V \mid P_G(i) \neq \emptyset \text{ and } S_G(i) = \emptyset\}$.
- The *independent actors*: $\mathtt{I}(G) = \{i \in V \mid P_G(i) = \emptyset \text{ and } S_G(i) = \emptyset\}$.

Moreover, the fraction value q represents the fraction of opinion leaders with the same inclination that is necessary to influence the decision of a follower.

When there is no risk of ambiguity, we simply use $\mathtt{I}$, $\mathtt{L}$ or $\mathtt{F}$ instead of $\mathtt{I}(G)$, $\mathtt{L}(G)$ or $\mathtt{F}(G)$. Note that in an OLF system $\mathcal{S} = ((V, E), q)$, if $(i, j) \in E$ then $i \in \mathtt{L}$ and $j \in \mathtt{F}$.

Now we define the collective decision process of an OLF system, according to [255].

Definition 6.4. Given an OLF system $\mathcal{S}$, the *collective decision vector* $c = c_{\mathcal{S}}(x)$ associated to an initial decision vector x is defined as

$$c_i = \begin{cases} b & \text{if } |\{j \in P_G(i) \mid x_j = b\}| > \lfloor q \cdot |P_G(i)| \rfloor, \\ x_i & \text{otherwise.} \end{cases} \tag{6.1}$$

where $b \in \{0, 1\}$ and $q \in [\frac{1}{2}, 1)$, and thus the *collective decision function* $C_{\mathcal{S}}(x)$ is defined as

$$C_{\mathcal{S}}(x) = \begin{cases} 1 & \text{if } |\{i \in V \mid c_i = 1\}| > |\{i \in V \mid c_i = 0\}|, \\ 0 & \text{if } |\{i \in V \mid c_i = 1\}| < |\{i \in V \mid c_i = 0\}|. \end{cases} \tag{6.2}$$

corresponding to the alternative with the greatest number of "votes" in the final choice vector.

Observe that an OLF system $\mathcal{S}$ requires the number of actors to be odd in order to ensure that decisions by the simple majority rule can be

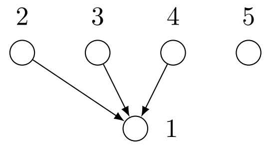


Figure 6.1: Example of an opinion leader-follower system.

reached [255, 256]. That is why inequalities in the last expression are strict. Furthermore, both leaders and independent actors always follow their own inclinations in the collective choice decision vector. A follower follows the majority decision among its predecessors or their own inclination. In particular, when a follower has an even number of predecessors and $q = \frac{1}{2}$, it is produced a tie, so the tiebreaker is given by the initial decision of the follower. Finally, note also that $S_G(\mathtt{L}) = \mathtt{F}$, $P_G(\mathtt{F}) = \mathtt{L}$ and $S_G(\mathtt{I}) = P_G(\mathtt{I}) = \emptyset$.

Example 6.1. Figure 6.1 illustrates a bipartite digraph $G = (V, E)$ corresponding to an OLF system $\mathcal{S} = (G, \frac{1}{2})$ over a set of five actors. For both initial decision vectors $x = (0, 1, 1, 0, 0)$ and $y = (1, 1, 1, 0, 0)$ we obtain the same collective decision vector $c_{\mathcal{S}}(x) = c_{\mathcal{S}}(y) = (1, 1, 1, 0, 0)$ and the same collective decision $C_{\mathcal{S}}(x) = C_{\mathcal{S}}(y) = 1$.

Note that the collective decision function of OLF systems is monotonic.

Lemma 6.1. Let $\mathcal{S} = (G, q)$ be an OLF system, its corresponding collective decision function is monotonic, with respect to inclusion, on $\mathcal{P}(V(G))$.

Proof. Let be $x \in \{0, 1\}^n$. If $i \in \mathtt{L} \cup \mathtt{I}$, as $c_i(x) = x_i$ and $x_i \in \{0, 1\}$, it is clear that $C(x - i) \leq C(x) \leq C(x + i)$. If $i \in \mathtt{F}$, we have three possibilities:
1) $x_i = 0$ and $c_i(x) = 1$, implying $c_i(x + i) = 1$, so $C(x) \leq C(x + i)$;
2) $x_i = 1$ and $c_i(x) = 0$, implying $c_i(x - i) = 0$, so $C(x - 1) \leq C(x)$; and
3) $x_i = c_i(x)$, which is the same case as for opinion leaders and independent actors. □

6.1.2 OLFM Systems

In this section we define *opinion leader-follower through mediators systems*—OLFM systems—as a generalization of the OLF systems of the previous section. OLFM systems allow us to model decision making situations with inner nodes that we called mediators, i.e., actors that behave as opinion leaders and followers, in the sense that they receive their influence from opinion leaders or other mediators, and can influence the followers or other mediators. Thus, while OLF systems are supported on directed bipartite graphs, OLFM systems are supported on layered digraphs.

Definition 6.5. A *layered digraph* is a digraph $G = (V, E)$ where V can be partitioned into k subsets $\mathcal{L}_1, \ldots, \mathcal{L}_k$ called *layers*, so that every edge connects a vertex from one layer to another vertex in a layer immediately below, i.e., for all $(a, b) \in E$, $a \in \mathcal{L}_i$ and $b \in \mathcal{L}_{i+1}$, for some $1 \leq i < k$.

This generalization allows to represent more complex social structures in which there are more than only two hierarchical levels.

Definition 6.6. An *opinion leader-follower through mediators system—OLFM system*, in short—for a set of n actors is a pair $\mathcal{S} = (G, q)$, where $G = (V, E)$ is a layered digraph and $q \in [\frac{1}{2}, 1)$ is the fraction value. The set V is partitioned into four subsets:

- The opinion leaders: $\mathtt{L}(G) = \{i \in V \mid P_G(i) = \emptyset \text{ and } S_G(i) \neq \emptyset\}$.
- The followers: $\mathtt{F}(G) = \{i \in V \mid P_G(i) \neq \emptyset \text{ and } S_G(i) = \emptyset\}$.
- The independent actors: $\mathtt{I}(G) = \{i \in V \mid P_G(i) = \emptyset \text{ and } S_G(i) = \emptyset\}$.
- The *mediators*: $\mathtt{M}(G) = \{i \in V \mid P_G(i) \neq \emptyset \text{ and } S_G(i) \neq \emptyset\}$.

As for OLF systems, for OLFM systems we also restrict our attention to an odd number of actors. Both the collective decision vector and the collective decision function of the system coincide with the ones for OLF systems—see expressions (6.1) and 6.2) in Definition 6.4. However, here the collective decision vector must be determined in order, starting from the actors in the first layer, then the ones in the second layer, and so on.

Observe that the opinion leaders and independent actors belong to the first layer of the graph, $\mathcal{L}_1$. The mediators are distributed into *layers of mediation*, whereas there are no mediators pointing to upper layers. The opinion leaders can only be connected with the mediators of the first layer of mediation, $\mathcal{L}_2$; the mediators of the last layer of mediation can only be connected with the followers, and the mediators of interlayers can only be connected with the mediators of the layer immediately below. Moreover, $\mathcal{L}_1 = \mathtt{L} \cup \mathtt{I}$ and for all $i \in \mathcal{L}_k$, $i \in \mathtt{F}$. Hence, the OLF systems can be seen as OLFM systems with only two layers, i.e., with $k = 2$. Note also that the

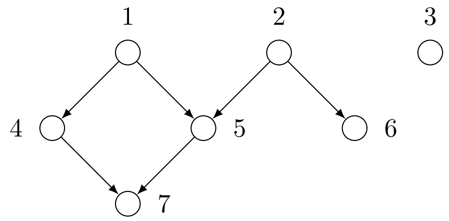


Figure 6.2: An OLFM system with one layer of mediation.

influence of actors in higher layers can affect the actors' decision in much lower layers.

Example 6.2. Figure 6.2 illustrates a graph G corresponding to an OLFM system $\mathcal{S} = (G, \frac{1}{2})$ over a set of seven actors. Here $\mathtt{L} = \{1, 2\}$, $\mathtt{I} = \{3\}$, $\mathtt{M} = \{4, 5\}$ and $\mathtt{F} = \{6, 7\}$. The computation of the collective decision function is shown in Table 6.1, where the initial decision vectors are ordered according to binary numeration. The vertical suspension points on the table indicate that both the collective decision vector and the collective decision function are the same than for the previous and the next decision vector.

It is clear by Lemma 6.1 that the collective decision function for OLFM systems is also monotonic.

6.1.3 Mediation Systems

In this section we consider a social network together with an external participant, namely the mediator. The mediator can exert influence on some nodes and accept advice from others, thus introducing a modification on the way that influence spreads through the network. On the bottom layer, the influence is exerted among the agents; on the second layer, the relationship of influence between the agents and an external mediator is kept.

We model the society by a set of nodes where the relation with the mediator can be expressed by three disjoint sets (A, B, C). The set A is formed by the agents that can influence the mediator but are not influenced by him. The set B contains those agents that influence and can be influenced by the mediator. Finally, the set C is formed by the agents that can be influenced by the mediator but cannot exert influence. This kind of relationship can be understood by means of the following star influence graph.

x	$c(x)$	$C(x)$	x	$c(x)$	$C(x)$	x	$c(x)$	$C(x)$
0000000			0110100	0110110	1	1001110	1001101	1
⋮	0000000	0	0110101	0110111	1	1001111	1001101	1
0001111			0110110	0110110	1	1010000	1011000	0
0010000			0110111	0110111	1	1010001	1011001	1
⋮	0010000	0	0111000	0110010	0	1010010	1011000	0
0011111			0111001	0110010	0	1010011	1011001	1
0100000	0100010	0	0111010	0110010	0	1010100	1011101	1
0100001	0100010	0	0111011	0110010	0	1010101	1011101	1
0100010	0100010	0	0111100	0110110	1	1010110	1011101	1
0100011	0100010	0	0111101	0110111	1	1010111	1011101	1
0100100	0100110	0	0111110	0110110	1	1011000	1011000	0
0100101	0100111	1	0111111	0110111	1	1011001	1011001	1
0100110	0100110	0	1000000	1001000	0	1011010	1011000	0
0100111	0100111	1	1000001	1001001	0	1011011	1011001	1
0101000	0100000	0	1000010	1001000	0	1011100	1011101	1
0101001	0100000	0	1000011	1001001	0	1011101	1011101	1
0101010	0100010	0	1000100	1001101	1	1011110	1011101	1
0101011	0100010	0	1000101	1001101	1	1011111	1011101	1
0101100	0100110	0	1000110	1001101	1	1100000		
0101101	0100111	1	1000111	1001101	1	⋮	1101111	1
0101110	0100110	0	1001000	1001000	0	1101111		
0101111	0100111	1	1001001	1001001	0	1110000		
0110000	0110010	0	1001010	1001000	0	⋮	1111111	1
0110001	0110010	0	1001011	1001001	0	1111111		
0110010	0110010	0	1001100	1001101	1			
0110011	0110010	0	1001101	1001101	1			

Table 6.1: Collective decision function for an OLFM system.

Definition 6.7. A *star influence graph* (A, B, C, k) is an unweighted influence graph $((V \cup \{x\}, E), f)$, where V can be partitioned into three sets $A, B, C \subseteq V$ and x is the *central node* or *mediator*, in such a way that $E = \{(u, x) \mid u \in A \cup B\} \cup \{(x, v) \mid v \in B \cup C\}$. The labeling function is given by $f(x) = k \in \mathbb{N}$ and $f(i) = 1$, for all $i \in V$.

Based on this definition, we can define the following.

Definition 6.8. A *star influence game* is a tuple (A, B, C, k, q), where (A, B, C, k) is a star influence graph, and a team $X \subseteq V$ is successful if and only if, for some $q \in \mathbb{N}$, either:

- $|X| \geq q$, or
- $|X \cap (A \cup B)| \geq k$ and $|X \cup B \cup C| \geq q$.

Thus, we can introduce our mediation systems as follows.

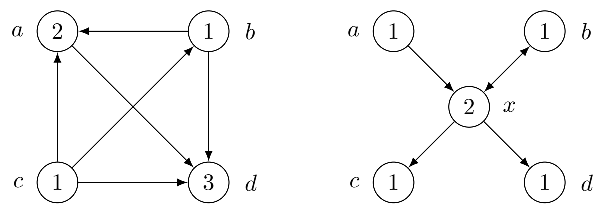


Figure 6.3: The spread of influence starting from the initial activation of $X = \{a\}$ on an unweighted influence graph.

Definition 6.9. A *star mediation influence game* is represented by a tuple (G, f, A, B, C, k, q), where (G, f) is an unweighted influence graph and (A, B, C, k) is a star influence graph, so that a team $X \subseteq V$ is successful if and only if, either:

- X is successful in the unweighted influence game (G, f, q, V), or
- $X \cup B \cup C$ is successful in (G, f, q, V) and also $|X \cap (A \cup B)| \geq k$.

Note that when a team is able to influence the mediator, then all the agents influenced by the mediator are activated and they propagate the alternative through the network.

Example 6.3. Let $\Gamma = (G, f, 3, V)$ be the influence game whose influence graph (G, f) is illustrated at the left of Figure 6.3. For Γ we have $\mathcal{W}^m = \{\{a, b, d\}, \{c\}\}$, $\mathcal{L}^M = \{\{a, b\}, \{a, d\}, \{b, d\}\}$, $\mathcal{L} = \mathcal{L}^M \cup \{\emptyset, \{a\}, \{b\}, \{d\}\}$ and $\mathcal{W} = \mathcal{P}(V) \setminus \mathcal{L}$. Now let $\Gamma' = (\{a\}, \{b\}, \{c, d\}, 2, 3)$ be the star influence game whose star influence graph—see Figure 6.3 at the right—is defined on the same vertices of Γ. Thus, $\mathcal{W}(\Gamma') = \{\{a, b\}, \{a, b, c\}, \{a, b, d\}, \{a, c, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. The team $\{a, b\}$ is obtained from the second condition of Definition 6.8, and the remaining teams from the first condition. Hence, for the star mediation influence game $\Gamma'' = (G, f, \{a\}, \{b\}, \{c, d\}, 2, 3)$ we obtain $\mathcal{W}(\Gamma'') = \mathcal{W}(\Gamma) \cup \{a, b\}$.

6.1.4 Influence Systems

In this section we consider two collective decision making models: the *oblivious influence system* and the *non-oblivious influence system*. In the first one, as in influence games—see Definition 5.2—the initial decision of the actors in $V \setminus N$ is not taken into account and a pessimistic point of view of their opinion is taken. In the second one, as in OLF systems—see Definition

6.4—the initial decision of actors in $V \setminus N$ is taken into account under some considerations.

Definition 6.10. An *oblivious influence system* is a collective decision making model defined on the set of vertices of an unweighted influence game $\Gamma = (G, f, q, N)$ whose collective decision function C_Γ is defined as follows:

$$C_\Gamma(x) = \begin{cases} 1 & \text{if } |F(X(x) \cap N)| \geq q, \\ 0 & \text{if } |F(X(x) \cap N)| < q \end{cases}$$

where $x \in \{0,1\}^{|N|}$ is the initial decision vector of the players.

Definition 6.11. A *non-oblivious influence system* is a collective decision making model defined on the set of vertices of an unweighted influence game $\Gamma = (G, f, q, N)$ whose collective decision function C_Γ is defined as follows:

$$C_\Gamma(x) = \begin{cases} 1 & \text{if } |\{i \in V \mid c_i = 1\}| \geq q, \\ 0 & \text{otherwise} \end{cases}$$

where $x \in \{0,1\}^{|N|}$ is the initial decision vector of the players, and the collective decision vector $c = c_\Gamma(x)$ is defined, for any $i \in N$, as

$$c_i = \begin{cases} 1 & \text{if } i \in F(X(x)), \\ 0 & \text{otherwise} \end{cases}$$

and for any $i \in V \setminus N$, as

$$c_i = \begin{cases} 1 & \text{if } p_i(x) \geq f(i) \text{ and } q_i(x) < f(i), \\ 0 & \text{if } q_i(x) \geq f(i) \text{ and } p_i(x) < f(i), \\ x_i & \text{otherwise,} \end{cases}$$

where $p_i(x) = |F(X(x) \cap N) \cap P(i)|$ and $q_i(x) = |P(i) \setminus F(X(x) \cap N)|$.

Note that for influence games in which $N = V$ the oblivious and non-oblivious model coincide because, in that case, for the non-oblivious model we have that $C_\Gamma(x) = 1$ if and only if $|\{i \in V \mid i \in F(X)\}| = |F(X)| \geq q$.

Now we analyze some properties of the collective decision functions.

Lemma 6.2. Let $\Gamma = (G, f, q, N)$ be an unweighted influence game. For both the oblivious and the non-oblivious system defined by Γ, the corresponding collective decision functions are monotonic, with respect to inclusion, on $\mathcal{P}(V(G))$.

Proof. For the oblivious case, let be $X, X' \subseteq V$, it is enough to observe that if $X \subseteq X' \subseteq V$ then $(X \cap N) \subseteq (X' \cap N) \subseteq N$, and by monotonicity of the spread of influence $C_\Gamma(X) \leq C_\Gamma(X')$.

For the non-oblivious case, let be $X \subseteq V$ and $i \notin X$. When $i \in N$ the monotonicity of the spread of influence gives that $C_\Gamma(X) \leq C_\Gamma(X \cup \{i\})$. When $i \notin N$ we have that $(F(X \cap N) \cap P(i)) \subseteq (F((X \cup \{i\}) \cap N) \cap P(i))$; therefore, $p_i(X) \leq p_i(X \cup \{i\})$ and $q_i(X) \geq q_i(X \cup \{i\})$, and by definition it follows that $C_\Gamma(X) \leq C_\Gamma(X \cup \{i\})$. □

Observe that the above property allow us to use the collective decision function of an influence system to define a simple game on the set of actors.

Note also that since OLFM systems are non-oblivious influence systems, then by Lemma 6.2 it is easy to see that the collective decision function for both OLFM and OLF systems are monotonic, as it was previously stated in Lemma 6.1.

In order to relate OLF systems with influence games we consider the following construction. Given an OLF system $\mathcal{S} = (G, q)$ we associate the unweighted influence game $\Gamma(\mathcal{S}) = (G, f, q', N)$ constructed as follows: $N = \mathtt{L} \cup \mathtt{I}$, $q' = \left\lfloor \frac{n}{2} \right\rfloor + 1$ and the labeling function f is defined as

$$f(i) = \begin{cases} \lceil q \cdot \delta^-(i) \rceil & \text{if } i \in \mathtt{F}, \\ 1 & \text{if } i \in \mathtt{L} \cup \mathtt{I}. \end{cases}$$

Furthermore, note that N omits the set of followers because, under the influence model, after their initial choice followers never can enforce their personal conviction and their final decision depends exclusively on whether the opinion leaders can influence them or not.

Lemma 6.3. The collective decision functions of both an OLF system $\mathcal{S} = (G, q)$ and the non-oblivious influence system defined by $\Gamma(\mathcal{S})$ coincide.

Proof. Let $\Gamma(\mathcal{S})$ be the oblivious influence system associated with $\mathcal{S}$. Let $X \subseteq V$ be the initial decision of the actors, $c = c_{\mathcal{S}}(X)$ and $c' = c_{\Gamma(\mathcal{S})}(X)$.

For any actor $i \in \mathtt{L} \cup \mathtt{I}$, by construction we have that $i \in F(X \cap N)$ if and only if $i \in X$. Observe that for $i \in \mathtt{L} \cup \mathtt{I}$, the nodes can not be influenced by any other node in $\mathcal{S}$. Therefore $c_i = c'_i$. For any actor $i \in \mathtt{F}$, we have that $\{j \in P(i) \mid x_j = 1\} = F(X \cap N) \cap P(i)$. Therefore $c_i = c'_i$ because the tie-breaking rule is the same in both systems. Thus, we have $C_{\mathcal{S}} = C_{\Gamma(\mathcal{S})}$ and the claim follows. □

As a consequence of the previous result we have a way to map OLF systems to a subfamily of the non-oblivious influence systems. In general, an OLF system cannot be cast as an oblivious influence system because the tie-breaking rules are different. Nevertheless, we can consider a subfamily, the *odd-OLF systems*, in which ties do not arise.

Definition 6.12. An *odd-OLF system* is an OLF system $(G, \frac{1}{2})$ in which, for any $i \in \mathtt{F}$, $\delta^-(i)$ is odd.

With no ties we can define a similar notion for influence graphs.

Definition 6.13. A *majority influence graph* is an influence graph (G, f) in which, for any $i \in V(G)$, $\delta^-(i)$ is either odd or zero. If $\delta^-(i)$ is odd, then $f(i) = (\delta^-(i) + 1)/2$ and if $\delta^-(i) = 0$, then $f(i) = 1$.

Similarly to Lemma 6.3, we have the following.

Lemma 6.4. The collective decision functions of both an odd-OLF system $\mathcal{S} = (G, \frac{1}{2})$ and the oblivious influence system defined by $\Gamma(\mathcal{S})$ coincide. The collective decision functions of both oblivious and non-oblivious influence systems on a majority influence graph coincide.

Proof. Let $\mathcal{S} = (G, \frac{1}{2})$ be and odd-OLF system and let $\Gamma(\mathcal{S})$ be the influence game associated to $\mathcal{S}$. Let $X \subseteq V$ be the initial decision of the actors. As in Lemma 6.3, for any actor $i \in \mathtt{L} \cup \mathtt{I}$ we have that $c_{\mathcal{S}}(i) = c_{\Gamma(\mathcal{S})(i)}$. For any actor $i \in \mathtt{F}$, we have that $\{j \in P(i) \mid x_j = 1\} = F(X \cap N) \cap P(i)$. Since $1/2 \leq q \leq 1$ and $\delta^-(i)$ is odd, we cannot have $|F(X \cap N) \cap P(i)| = |P(i)| - |F(X \cap N) \cap P(i)|$ and thus no tie in the size of the expanded predecessors arises. Therefore, in the oblivious model we have that $C_{\mathcal{S}} = C_{\Gamma(\mathcal{S})}$ and the claim follows. The proof for the second part of Lemma is analogous. □

Note that Lemma 6.4 is not true for OLF systems where some follower has even indegree, as it is shown in the following example.

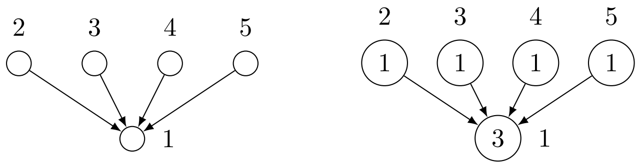


Figure 6.4: When followers have an even number of predecessors, the satisfaction in each model produces different results.

Example 6.4. Consider the OLF system $\mathcal{S} = (G, \frac{1}{2})$, whose graph $G = (E, V)$ is depicted on the left hand side of Figure 6.4, and its associated influence game $\Gamma(\mathcal{S}) = (G, f, 3, V \setminus \{1\})$ is shown on its right hand side. Note that in the influence game, $f(1) > \frac{1}{2}\delta^-(1)$, being $\delta^-(1)$ an even number, so that considering an initial activation containing exactly $\frac{1}{2}\delta^-(i) = 2$ predecessors of i, then the follower can not be influenced. Therefore, assuming an initial choice vector x with these characteristics, according to the oblivious influence system $C_\Gamma(x) = 0$, but for the non-oblivious influence system $C_\Gamma(x)$ depends of the initial choice x_1 of the follower 1.

6.2 Computational problems of mediation systems

In this section we study several problems of simple games and influence games for the mediation systems defined in Section 6.1.3.

For star influence games, the characterization of successful and unsuccessful teams given by Definition 6.8 allow us to decide in polynomial time whether a coalition of the associated simple game is either minimal winning or maximal losing.

Theorem 6.1. Let (A, B, C, k, q) be a star influence game. Given a team $X \subseteq A \cup B \cup C$, deciding whether X represents either a minimal winning coalition or a maximal losing coalition, can be done in polynomial time.

Proof. Note that for all $X \in \mathcal{W}^m$, $X \cap C \neq \emptyset$ if and only if $|X \cap A \cap B| < k$, so that every $X \subseteq (A \cup B \cup C)$ with $|X| = q$ and $X \cap C \neq \emptyset$ is a MWC. The remaining MWCs depend of the quota. If $q \leq k$, then every team $X \subseteq (A \cup B)$ with $|X| = q$ represents a MWC, and there are no other MWCs. On the other hand, if $q > k$, we need to distinguish two cases: If $q > k+|B|+|C|$, then every team $X \subseteq A$ with $|X| = q-|B|-|C|$ is a MWC; otherwise, every team $X \subseteq (A \cup B)$ with $|X| = k$ and $|X \cap A| \geq q-|B|-|C|$

Algorithm 13 EnumeratingMWCs

Input: Star influence game (A, B, C, k, q).
Output: Minimal winning coalitions of the associated game $(N, \mathcal{W}^m)$.

1: **print** $\{X \subseteq (A \cup B \cup C) \mid |X| = q, X \cap C \neq \emptyset, |X \cap A \cap B| < k\}$;
2: **if** $(q > k + |B| + |C|)$
3:     **print** $\{X \subseteq A \mid |X| = q - |B| - |C|\}$;
4: **else**
5:     **if** $(q > k)$
6:         **print** $\{X \subseteq (A \cup B) \mid |X| = k, |X \cap A| \geq q - |B| - |C|\}$;
7: **if** $(q \leq k)$
8:         **print** $\{X \subseteq (A \cup B) \mid |X| = q\}$;
9: **return**.

is a MWC. Since these conditions cover all possible cases, there is no other MWC. Further, since each condition can be verified in polynomial time, deciding whether X is a MWC can be done in polynomial time.

Furthermore, note that for all $X \in \mathcal{L}^M$, $|X| = q - 1$. Therefore, we need to consider two cases. If $|X \cap (A \cup B)| < k$, then $X \in \mathcal{L}^M$ if and only if $|X| = q - 1$; otherwise, $X \in \mathcal{L}^M$ if and only if $|X| = q - 1$ but also $(B \cup C) \subseteq X$. Since these conditions cover all possible cases, and both cases can be verified in polynomial time, then the theorem holds. □

Based in the previous result, we can construct algorithms with good performance to enumerate both the MWCs and the maximal losing coalitions of the associated simple game.

Theorem 6.2. Given a star influence game, both $\mathcal{W}^m$ and $\mathcal{L}^M$ can be enumerated with polynomial-delay.

Proof. Let us consider Algorithm 13, that follows the conditions given in the proof of Theorem 6.1 for MWCs. By using standard techniques in steps 1, 3, 6 and 8, we can list every MWC in polynomial time.

Analogously, Algorithm 14 follows the conditions given in the proof of Theorem 6.1 for maximal losing coalitions. Again, by using standard techniques in steps 3 and 5, we can list every maximal losing coalition in polynomial time. Note that in this case we only worry about the coalitions with size $q - 1$. □

From Theorem 6.1, we obtain that it is easy to decide several properties for star influence games.

Algorithm 14 EnumeratingMLCs

Input: Star influence game (A, B, C, k, q).
Output: Maximal losing coalitions of the associated game $(N, \mathcal{L}^M)$.

```
for all Y ⊆ N repeat
    if (|Y ∩ (A ∪ B)| < k)
        if (|Y| = q − 1) print Y;
    else
        if (|Y| = q − 1 and (B ∪ C) ⊆ Y) print Y;
return.
```

Theorem 6.3. For star influence games, the IsProper, IsStrong, IsDecisive, IsDummy, IsPasser, IsVetoer and IsDictator problems can be computed in polynomial time.

Proof. Let $X \subseteq N$ be a team, $\alpha = |A|$, $\beta = |B|$, $\gamma = |C|$; $n_\alpha = |X \cap A|$, $n_\beta = |X \cap B|$, $n_\gamma = |X \cap C|$; and $\bar{n}_\alpha = |A| - n_\alpha$, $\bar{n}_\beta = |B| - n_\beta$, $\bar{n}_\gamma = |C| - n_\gamma$. From the conditions of Definition 6.8, we can characterize every considered property through equations and linear systems. Recall that linear systems can be checked in polynomial time.

For IsProper, a star influence game is improper if at least one of the following four linear systems is satisfiable.

Case I:	**Case II:**	**Case III:**	**Case IV:**
$n_\alpha + n_\beta + n_\gamma \geq q$;	$n_\alpha + n_\beta + n_\gamma \geq q$;	$n_\alpha + n_\beta + n_\gamma < q$;	$n_\alpha + n_\beta + n_\gamma < q$;
$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma \geq q$.	$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma < q$;	$n_\alpha + n_\beta \geq k$;	$n_\alpha + n_\beta \geq k$;
	$\bar{n}_\alpha + \bar{n}_\beta \geq k$;	$n_\alpha + \beta + \gamma \geq q$;	$n_\alpha + \beta + \gamma \geq q$;
	$\bar{n}_\alpha + \beta + \gamma \geq q$.	$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma \geq q$.	$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma < q$;
			$\bar{n}_\alpha + \bar{n}_\beta \geq k$;
			$\bar{n}_\alpha + \beta + \gamma \geq q$.

For IsStrong, if at least one of the following linear systems is satisfiable, then the star influence game is weak.

Case I:	**Case II:**	**Case III:**	**Case IV:**
$n_\alpha + n_\beta + n_\gamma < q$;	$n_\alpha + n_\beta + n_\gamma < q$;	$n_\alpha + n_\beta + n_\gamma < q$;	$n_\alpha + n_\beta + n_\gamma < q$;
$n_\alpha + n_\beta < k$;	$n_\alpha + n_\beta < k$;	$n_\alpha + n_\beta \geq k$;	$n_\alpha + n_\beta \geq k$;
$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma < q$;	$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma < q$;	$n_\alpha + \beta + \gamma < q$;	$n_\alpha + \beta + \gamma < q$;
$\bar{n}_\alpha + \bar{n}_\beta < k$.	$\bar{n}_\alpha + \bar{n}_\beta \geq k$;	$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma \geq q$.	$\bar{n}_\alpha + \bar{n}_\beta + \bar{n}_\gamma < q$;
	$\bar{n}_\alpha + \beta + \gamma < q$.		$\bar{n}_\alpha + \bar{n}_\beta \geq k$;
			$\bar{n}_\alpha + \beta + \gamma < q$.

Let $i \in N$ be a player, the polynomial results for IsDummy, IsPasser, IsVetoer and IsDictator comes from the following characterizations:

	Conditions to fulfill the property		
	$i \in A$	$i \in B$	$i \in C$
Dummy	never	$\gamma = 0$ and $q > k + \beta + 1$	$q \geq k + \gamma$
Passer	either $q = 1$, or $k = 1$ and $q \leq \beta + \gamma + 2$	either $q = 1$, or $k = 1$ and $q \leq \gamma + 2$	$q = 1$
Vetoer	$q = \alpha+\beta+\gamma+1 = n$	$q = \alpha+\beta+\gamma+1 = n$	$q = \alpha+\beta+\gamma+1 = n$
Dictator	$q = \beta + \gamma + 2$ and $\alpha = k = 1$	$q = \gamma + 2$ and $\beta = k = 1$	never

Since every condition can be decided in polynomial time, the theorem holds. □

In despite of the above, since the successful teams of star mediation influence games depend of a general influence game, then from Theorems 5.8 and 5.9 we have the following.

Corollary 6.1. Given a star mediation influence game, the IsProper, IsStrong, IsDecisive and IsDummy problems are coNP-complete, while IsPasser, IsVetoer and IsDictator can be computed in polynomial time.

Observe that when we restrict the games to only certain types of actors, then we can get some interesting equivalences. The following result is related to the Equiv problem mentioned in Section 4.1.1.

Proposition 6.1. Let (A, B, C, k, q) be a star influence game, then:

- if $|A| = |C| = 0$, the games $(\emptyset, B, \emptyset, k, q)$, $(\emptyset, B, \emptyset, q, q)$ and $(\emptyset, B, \emptyset, k, k)$ are equivalent. Moreover, if $q \leq k$, $(\emptyset, B, \emptyset, k, q)$ and $(A, \emptyset, B, 1, q)$ are equivalent; otherwise, $(\emptyset, B, \emptyset, k, q)$ and $(\emptyset, \emptyset, B, 1, k)$ are equivalent.

- if $|B| = |C| = 0$, then if $q \leq k$, $(A, \emptyset, \emptyset, k, q)$, $(A, \emptyset, \emptyset, q, q)$ and $(A, \emptyset, \emptyset, k, k)$ are equivalent; otherwise, $(A, \emptyset, \emptyset, k, q)$ and $(A, \emptyset, \emptyset, 1, q)$ are equivalent.

- if $q \leq |B| + |C| + 1$ and $q \leq k$, then the games (A, B, C, k, q) and $(\emptyset, A \cup B \cup C, \emptyset, q, q)$ are equivalent.

We finish this section with some restrictions that help us to obtain characterizations of homogeneous games.

Proposition 6.2. Let $\Gamma = (A, B, C, k, q)$ be a star influence game, then:

1. if $|A| = |B| = 0$, Γ is always homogeneous with realization $[q; 1, \ldots, 1]$.
2. if $|A| = |C| = 0$, Γ is always homogeneous. If $q \leq k$, with realization $[q; 1, \ldots, 1]$, otherwise, with realization $[k; 1, \ldots, 1]$.
3. if $|B| = |C| = 0$, Γ is always homogeneous with realization $[q; 1, \ldots, 1]$.

6.3 Computing Satisfaction or Rae Index

According to [255], the *satisfaction* of an actor in a society refers to the number of possible decisions that all actors can take as a group, such that the collective decision coincides with the decision taken by the actor in the initial choice vector. This general formulation allow us to define the satisfaction measure for a generic collective decision making model.

Definition 6.14. Let $\mathcal{M}$ be a collective decision making model over a set of n actors. The *satisfaction measure* of the actor i is defined as follows:

$$\textsc{Sat}_{\mathcal{M}}(i) = |\{x \in \{0,1\}^n \mid C_{\mathcal{M}}(x) = x_i\}|.$$

It is relevant to note that when the collective decision making model $\mathcal{M}$ is monotonic, with respect to inclusion, the satisfaction measure coincides with the known *Rae index*.

Definition 6.15. Let $\mathcal{M}$ be a monotonic collective decision making model—such as a simple game—over a set of $n = |V|$ actors. The *Rae index* of the actor i is defined as follows:

$$\textsc{Rae}_{\mathcal{M}}(i) = |\{X \subseteq V \mid i \in X \in \mathcal{W} \text{ or } i \notin X \notin \mathcal{W}\}|.$$

In the context of simple games, Dubey and Shapley [66] established an affine-linear relation between the Rae index and the Banzhaf value [149]:

$$\textsc{Sat}(i) = \textsc{Rae}(i) = 2^{n-1} + \eta(i) \tag{6.3}$$

where $\eta(i) = \eta_i(\Gamma)$ denotes the Banzhaf value of player i in a game Γ—see Definition 2.28.

It is clear that this equality holds for any collective decision making model that is monotonic, such as OLF systems, OLFM systems, oblivious and non-oblivious influence systems.

In what follows we analyze the computational complexity of BVAL or CONSTRUCT-BANZHAF-VALUE, and that of the following problem, for influence systems, simple games and subfamilies of simple games.

Name: RAE
Input: A collective decision making model $\mathcal{M}$ and an actor i.
Output: $\text{RAE}_{\mathcal{M}}(i)$.

By equation (6.3), for simple games and influence games, the RAE problem and the BVAL problems are computationally equivalent. The second problem has received more attention and its complexity has been analyzed before—see Sections 2.3.2, 4.1.2 and 5.3. For instance, from Theorem 5.7 we know that for influences games the BVAL problem—and therefore, the RAE problem—is #P-complete. In the same vein, note that given an actor i and a value k, to determine whether $\text{RAE}(i) \leq k$ or $\text{RAE}(i) \geq k$ is NP-hard, because we can just to apply a dichotomic search taking into account that $0 \leq \text{RAE}(i) \leq 2^n$.

It is also interesting to note that the RAE problem is closely related to the CHOW PARAMETERS problem for simple games—see Section 4.1.1—but considering also losing coalitions. We continue with some additional results.

Lemma 6.5. Let $\mathcal{M}$ be any monotonic collective decision making model. For any player i we have $\text{RAE}(i) \geq 2^{n-1}$. Moreover, if i is dummy then $\text{RAE}(i) = 2^{n-1}$, and if i is a dictator then $\text{RAE}(i) = 2^n$.

Proof. The sentence $\text{RAE}(i) \geq 2^{n-1}$ is deduced from the equation (6.3). If i is dummy, we know from Section 2.3.2 that $\eta(i) = 0$, so then $\text{RAE}(i) = 2^{n-1}$. If i is dictator, then for any coalition $X \subseteq N$, if $X \in \mathcal{W}$ then $i \in X$, and if $X \notin \mathcal{W}$ then $i \notin X$, so hence $\text{RAE}(i) = 2^n$. □

The following example uses the reformulation of Lemma 6.4 to compute the satisfaction measure of an odd-OLF system.

Example 6.5. Let $\mathcal{S} = (G, \frac{1}{2})$ be an odd-OLF system whose graph is given in Figure 6.1. The equivalent oblivious influence system is $\Gamma(\mathcal{S}) = (G, f, 3, N)$, where $f(1) = 2$ and, for any $i \neq 1$, $f(i) = 1$. Therefore,

$\textsc{Rae}(1) = \textsc{Rae}(5) = 16$ and $\textsc{Rae}(2) = \textsc{Rae}(3) = \textsc{Rae}(4) = 24$. Note that for every actor $i \in V$, the number of winning coalitions that contain i coincides with the number of losing coalitions that not contain i, and it corresponds to $\textsc{Rae}(i)/2$.

In the following Section 6.3.1 we show that the Rae problem remains hard when the influence graph is restricted to be bipartite. In Section 6.3.2 we show two subfamilies of bipartite influence graphs on which Rae and Bval can be solved in polynomial time.

6.3.1 Hardness Results

Let (G, f) be an unweighted influence graph, for any $1 \le k \le n$ define $F^k(V, G, f) = \{X \subseteq V \mid |F(X \cap N)| = k\}$. When there is no risk of ambiguity, we say simply $F^k(V)$. Since $F^k(N) = \{X \subseteq N \mid |F(X)| = k\}$, we have $|F^k(V)| = 2^{|V|-|N|} \cdot |F^k(N)|$. Associated with this quantity we consider the following problem:

Name: Expansion
Input: An influence graph (G, f), a set of vertices N and an integer k.
Output: $|F^k(N)|$.

Our next result shows the relationship among the Rae and the Expansion problems for oblivious influence systems. Before stating it we introduce some notation. For an influence graph (G, f) and a vertex $i \in V(G)$, $\mathtt{F}_i = \{j \in S_G(i) \mid |P_G(j)| = 1\}$. We denote as $R(G, f, i)$ the influence graph (G', f') where $G' = G[V(G) \setminus (\mathtt{F}_i \cup \{i\})]$, $f'(j) = f(j)$ for $j \notin S_G(i)$, and $f'(j) = f(j) - 1$ for $j \in S_G(i)$.

Lemma 6.6. Let (G, f, q, N) be an oblivious influence system, the satisfaction, for the actors $i \in V(G) \setminus N$ that can not participate in the initial activation set and the ones without predecessors, is given by the following expression:

$$\textsc{Rae}(i) = \begin{cases} 2^{n-1} & \text{if } i \notin N \\ 2^{n-1} + 2^{n-|N|} \cdot |F^{q-1}(N \backslash \{i\}, R(G, f, i))| & \text{if } i \in N, P_G(i) = \emptyset, S_G(i) = \emptyset \\ 2^{n-1} + 2^{n-|N|} \cdot \sum_{j=1}^{r} |F^{q-j}(N \backslash \{i\}, R(G, f, i))| & \text{if } i \in N, P_G(i) = \emptyset, S_G(i) \neq \emptyset \end{cases}$$

where $r = 1 + |\mathtt{F}_i|$.

Proof. Let z be an initial decision vector, set $Z = \{i \in V \mid z_i = 1\}$ and $X = Z \cap N$.

For an actor $i \notin N$, we consider two cases. When $X \in \mathcal{W}$, $C_\Gamma(z) = 1$. Thus, actor i is satisfied only when $z_i = 1$. When $X \in \mathcal{L}$, $C_\Gamma(z) = 0$, hence actor i is satisfied only when $z_i = 0$. Therefore, since for any initial decision vector $V \setminus \{i\}$ there is only one way to complete the initial decision vector in such a way that the collective decision coincides with actor i's decision, then $\text{RAE}(i) = 2^{n-1}$.

For a player $i \in N$, we have three cases in which i is satisfied by the collective decision. The first two cases are the following: When $X \setminus \{i\} \in \mathcal{W}$ and $z_i = 1$, so $C_\Gamma(z) = 1 = z_i$; and when $X \setminus \{i\} \in \mathcal{L}$ and $z_i = 0$, so $C_\Gamma(z) = 0 = z_i$. These two first cases provide a total of 2^{n-1} initial decision vectors for which the collective decision coincides with the initial decision of player i. However, when $X \setminus \{i\} \in \mathcal{L}$ and $z_i = 1$, then $C_\Gamma(z) = 1$ if $X \in \mathcal{W}$. So we have another set of initial decision vectors for which player i is satisfied.

We have to count all the winning coalitions X so that $X \setminus \{i\}$ is losing. To remove the influence of actor i we have to take into account the influence graph $R(G, f, i)$.

When $S_G(i) = \emptyset$, actor i cannot influence any other actor. Observe that $Y \in F^{q-1}(N \setminus \{i\}, R(G, f, i))$ if and only if $Y \in \mathcal{L}$ but $Y \cup \{i\} \in \mathcal{W}$. Thus, as the system is oblivious, we have $2^{n-|N|} \cdot |F^{q-1}(N \setminus \{i\}, R(G, f, i))|$ additional initial decision vectors than can be completed, by adding $z_i = 1$, to an initial decision vector z with $C_\Gamma(z) = 1$.

When $S_G(i) \neq \emptyset$, player i can influence other actors. In this case we have to separate those vertices which can be influenced directly and only by i, those in the set $\mathtt{F}_i$, from the rest. Observe that all the vertices in $S_G(i) \setminus \mathtt{F}_i$ have degree at least 2. Now for a coalition Y, $Y \in \mathcal{L}$ but $Y \cup \{i\} \in \mathcal{W}$ if and only if $Y \in F^{q-j}(N \setminus \{i\}, R(G, f, i))$, for some $1 \leq j \leq r$. Taking into account that the system is oblivious, there are $2^{n-|N|} \cdot |F^{q-j}(N \setminus \{i\}, R(G, f, i))|$ additional initial vectors that can be completed, by adding $z_i = 1$, to an initial vector z giving expansion $q - j$, with $C_\Gamma(z) = 1$. □

Note that in the case of odd-OLF systems $N = \mathtt{L} \cup \mathtt{I}$. Furthermore, for $i \in \mathtt{I}$, $P_G(i) = S_G(i) = \emptyset$, and for $i \in \mathtt{L}$, $P_G(i) = \emptyset$ and $S_G(i) \neq \emptyset$. Therefore, the previous lemma provides a complete characterization of the

Rae measure. Note that it also shows that, as expected, opinion leaders have always a satisfaction greater or equal than the independent actors, and that both have always a satisfaction greater or equal than the followers.

For our hardness result we need to consider a variation of the counting vertex cover problem [95]:

Name: $\#\frac{2}{3}$-VC
Input: An undirected graph $G = (V, E)$.
Output: Number of vertex covers of size exactly $\frac{2}{3}|V|$ in G.

It is known that the problem of computing the number of independent sets with size exactly $\frac{2}{3}|V|$ in a graph is *hard*, in the sense that it cannot be computed by a sub-exponential time algorithm, unless the well known #P-complete #3-Sat problem—the counting version of the 3-satisfiability problem—could be computed in sub-exponential time [118]. Hence, as the complement of an independent set is a vertex cover, then the same result shows that $\#\frac{2}{3}$-VC is also hard. Now we are able to present the following result, by using the same notion of hardness.

Theorem 6.4. The Expansion problem is hard for directed bipartite influence graphs.

Proof. We provide a reduction from the $\#\frac{2}{3}$-VC problem. In our reduction we produce two influence graphs. Let $G = (V, E)$ be a graph, without loss of generality we assume that G is connected, $m = |E(G)|$, $n = |V(G)|$ is a multiple of three, and $n \geq 6$.

We construct a bipartite graph G_1 associated to G which is defined as follows. The set of vertices is given by $V(G_1) = V \cup \{E_1, \ldots, E_{n+2}\} \cup \{z\}$, where z is a new vertex, and for $1 \leq j \leq n+2$, E_j is a marked copy of E. Observe that $|V(G_1)| = n + (n+2)m + 1$. The set of arcs is the following:

$$E(G') = \{(u, e^j) \mid u \in V, e = \{u, v\} \in E \text{ and } e^j \in E_j \text{ is the marked copy of } e, 1 \leq j \leq n+2\} \cup \{(z, a) \mid a \in E_j, 1 \leq j \leq n+1\}.$$

G_1 is a directed bipartite graph and all the vertices have indegree either 0 or 3. Next we define the labeling function to define an associated influence graph (G_1, f_1). We set $f_1(u) = 1$, for $u \in V$, $f_1(z) = 1$, and $f_1(u) = 2$, for $u \notin V \cup \{z\}$.

Now we can define the reduction from $\#\frac{2}{3}$-VC to EXPANSION, which associates to G the input $h(G)$ to EXPANSION defined as follows

$$h(G) = \left((G_1, f_1), V \cup \{z\}, \frac{2}{3}n + (n+2)m + 1 \right)$$

Let be $X \subseteq V$ and $\alpha = |X|$. We analyze the expansion of the sets $X \cup \{z\}$ and X in (G_1, f_1).

When the initial activation set is $X \cup \{z\}$, we have two cases, either X is a vertex cover or not. When X is a vertex cover all vertices corresponding to edges get activated, therefore $|F(X \cup \{z\})| = \alpha + (n+2)m + 1$. This last quantity is equal to the required size only when $\alpha = \frac{2}{3}n$. When X is not a vertex cover, we know that at least one edge $e \in E$ is not covered, therefore $F(X \cup \{z\})$ misses—at least—all the marked copies of e. On the other hand, we assume $\alpha \leq n - 2$, because a set with either $n - 1$ or n vertices is indeed a vertex cover. Hence, we have $|F(X \cup \{z\})| \leq \alpha + (n+2)(m-1) + 2 \leq n - 2 + (n+2)m - (n+2) + 2 \leq (n+2)m - 2$ which is strictly smaller than the required size.

Now we consider the case when the initial activation set is X. Note that in G_1 only the copies of those edges with both endpoints in X are activated. In the case that, for every $e = (u, v) \in E$, $\{u, v\} \subseteq X$, since G is connected we have $|F(X)| = n + (n+2)m > k$. Otherwise, again, at least the copies of one edge and at least one vertex are not activated. Therefore we have $|F(X)| \leq \alpha + (n+2)(m-1) \leq n - 1 + (n+2)m - (n+2) = (n+2)m - 3$ which is strictly smaller than the required size.

From the previous case analysis, we have that the elements in $F^k(V \cup \{z\})$, for (G_1, f_1), are in a one-to-one correspondence with the vertex covers of size $\frac{2}{3}n$ in G. As the reduction can be computed trivially in polynomial time the claim holds. □

The hardness of the EXPANSION problem does not rule out the possibility of having some cases for which computing RAE or EXPANSION is easy. One easy case for EXPANSION is when the value of k is smaller that the minimum value of the labeling function over the actors not in N. It is easy to see that for an oblivious influence system, in that case $|F^k(N)| = \binom{n-|N|}{k}$. For an OLF system, we know by Lemma 6.6 that $\text{RAE}(i) = 2^{n-1}$, for any follower i, and thus it can be computed in polynomial time.

Observe that in the reduction provided in the proof of Theorem 6.4 we have constructed a majority influence graph. This leads us to the following result.

Theorem 6.5. Both the RAE and the BVAL problems are hard for directed bipartite majority influence systems.

Proof. We prove hardness by showing a polynomial time reduction from the EXPANSION problem to the RAE problem. Our construction starts with a directed bipartite influence graph (G, f), a set $N \subseteq V(G)$ and a value k verifying the conditions required to be an input to sc Expansion. Let be $n = |V(G)|$, we consider the influence graph (G', f') which is obtained from (G, f) by adding an isolated vertex z with label 1. We consider the influence system associated to the game $\Gamma(G, f) = (G', f', N \cup \{z\}, k+1)$ and the input to the RAE problem $(\Gamma(G, f), z)$.

In order to compute RAE(z) in $\Gamma(G, f)$, according to the second case in Lemma 6.6, we have to consider the reduced influence graph $R(G', f', z)$ and the parameter $q = k + 1$. By construction $R(G', f', z) = (G, f)$ and thus we have RAE$(z) = 2^n + 2^{n-|N|}|F_k(N, (G, f))|$.

Therefore, if we could solve RAE in polynomial time we are also able to solve EXPANSION in polynomial time, and the claim follows. □

As a consequence of the previous result and the Lemma 6.4 we have the following.

Corollary 6.2. Both the RAE and the BVAL problems are hard for OLF systems and oblivious and non-oblivious influence systems on bipartite graphs.

Observe that in our reduction, the quota may assume any value. Thus, the reduction does not show hardness for the RAE problem when it is being applied to odd-OLF systems, where the quota is different than the required majority. Therefore, it remains open to show the complexity of the RAE problem for odd-OLF systems.

6.3.2 Polynomially Results for Bipartite Influence Graphs

In this section we focus our attention on two classes of unweighted influence graphs, for which we can show that the EXPANSION and the RAE problems are polynomial time solvable under the oblivious influence system. Like in

Section 6.1.3, we consider an additional set of actors called mediators, that receive their influence from opinion leaders and may influence the followers, allowing several layers of influence, and hence establishing a more complex hierarchy among the different actors. Our families of bipartite graphs extend, in some sense, the OLF systems by allowing an intermediate set of actors that play the role of mediators between leaders and followers. The first family, the *strong influence graphs*, contains layered directed graphs in which the influence is exerted in an all to all fashion following a hierarchical structure. In this case the mediators are interposed between the leaders and the followers. The second family is based in the star influence games of Definition 6.8, and it contains only one mediator.

The mediators in the influence graph are not players. Thus, according to Lemma 6.6, for any mediator i, we have $\textsc{Rae}(i) = 2^{n-1}$.

Strongly mediated influence system

The first class is based on directed bipartite influence graphs. It verifies the property that any of the subgraphs constructed in Lemma 6.6 belong to the family. We show that the EXPANSION problem can also be solved in polynomial time for those subfamilies.

Our graphs family is defined recursively, by using isolated and complete bipartite graphs. Recall notation from Definitions 2.5 and 6.3.

Definition 6.16. The family of *strong hierarchical digraphs* is formed by directed bipartite graphs obtained by applying recursively the following rules:

- The graph I_a, for $a > 0$, and the graph $K_{a,b}$, for $a, b > 0$ are strong hierarchical digraphs.
- If H_1 and H_2 are strong hierarchical digraphs their disjoint union is a strong hierarchical digraph.
- If H_1 and H_2 are strong hierarchical digraphs and I_a is a set of a independent vertices, the graph H with

  $$V(H) = V(H_1) \cup V(H_2) \cup V(I_a) \text{ and}$$
  $$E(H) = E(H_1) \cup E(H_2) \cup E(I_a) \cup \{(u, v) \mid u \in \mathtt{F}, v \in V(I_a)\},$$

  where $\mathtt{F} = \mathtt{F}(H_1) \cup \mathtt{F}(H_2)$, is a strong hierarchical digraph.

We use the term *strong influence graph* to denote an influence graph (G, f) where G is a strong hierarchical digraph. Finally, a *strongly mediated influence system* is an influence system (G, f, q, N) where (G, f) is a strong influence graph and $N = \mathtt{L}(G) \cup \mathtt{I}(G)$.

Observe that, for a strong hierarchical digraph $G = (V, E)$, the set V can be partitioned into t subsets or *layers* $A_1, \ldots, A_t$, so that edges occur only between consecutive layers and are directed from layer i to layer $i+1$, with $i = \{1, \ldots, t-1\}$.

Furthermore, the graph G' which is obtained from G by removing a node $u \in \mathtt{I}(G) \cup \mathtt{L}(G)$ is also a strong hierarchical digraph. According to this, we use for the following result the constructions provided in Lemma 6.6, taking care of the fact that, by removing a vertex in an influence graph, some of the labels of its neighbors may become zero. Our algorithm will thus allow for a more general class of labeling functions in which some nodes might have associated a label zero and therefore form part of the expansion of any set.

Lemma 6.7. Let (G, f) be a strong influence graph in which for each $u \in V$, $0 \leq f(u) \leq n+1$. Let be $N = \mathtt{I}(G) \cup \mathtt{L}(G)$ and an integer k such that $0 \leq k \leq n$, then $|F^k(N)|$ can be computed in polynomial time.

Proof. For a given (G, f) our algorithm tabulates the function $T(a, b)$, with $0 \leq a \leq n$ and $0 \leq b \leq |\mathtt{F}(V)|$, defined as

$$T(a, b) = |\{X \subseteq N \mid |F(X)| = a \text{ and } |F(X) \cap \mathtt{F}(G)| = b\}|.$$

Observe that, if we can compute in polynomial time an array holding all the $T(a, b)$ values, we can obtain $|F^k(N)|$ by adding up the values in the row corresponding to k.

We show by induction how to construct an array storing the desired values inductively, following the structure of G. The base cases, according to Definition 6.16, are sets of isolated vertices and the complete bipartite digraphs.

When $G = I_\alpha$, all the actor are independent, therefore $F(X) = X$ for any set of actors X, and thus we have

$$T(a, 0) = \binom{\alpha}{a} \text{ for any } 0 \leq a \leq \alpha.$$

Observe that all those values can be computed in polynomial time.

When $G = K_{\alpha,\beta}$, for a set $X \subseteq \mathtt{L}(G)$ we have that $F(X) = X \cup \{u \in \mathtt{F}(G) \mid |X| \geq f(u)\}$. To express the function T we need an auxiliary function $R(c)$, $0 \leq c \leq n$, defined as $R(c) = |\{v \in \mathtt{F}(G) \mid f(v) \leq c\}|$. Observe that a vector storing the values of R can be computed by sorting the actors of $\mathtt{F}(G)$ in increasing order of labels and then counting the number of repeated values. Using this information, we know that a $X \subseteq \mathtt{L}(G)$ will expand its influence to all the followers u for which $f(u) \leq |X|$, so therefore we have

$$T(a,b) = \sum_{\{c \mid a=c+R(c), b=R(c)\}} \binom{\alpha}{c}.$$

The above values can be computed in polynomial time using a double scanning as follows:

1: Initialize $T(a,b)$ to 0;
2: for $c = 0$ to n
3: $\quad T(c+R(c), R(c)) = T(c+R(c), R(c)) + \binom{\alpha}{c}$.

We split the rest of the proof into two cases. The first case corresponds to connected strong influence graphs and the second one to disconnected strong influence graphs.

When G is connected assume that it is obtained from a set of connected strong influence graphs $\{G_1, \ldots, G_k\}$ and a set of isolated vertices I_α that have to be fully connected to a set of isolated vertices I_β in the next level. For the correctness of the proof it is relevant to consider as a unique set all the independent vertices that appear in the decomposition.

To get an expression for T we proceed inductively by considering the graphs $H_1, H_2, \ldots, H_k$, where H_i, $1 \leq i \leq k$ is the strong influence graph obtained by fully connecting the vertices with outdegree 0 in the graphs $G_1, \ldots, G_i$ to the vertices in I_β. Then we finalize by incorporating I_α and the connections until G_k to obtain G. We have to consider separately the first step, H_1, the intermediate steps, $H_2, \ldots, H_k$, and the last step, G. In order to avoid confusion we use T_i to denote the values of the function T when its definition is restricted to the graph H_i, and T'_i when it is restricted to G_i. As before, we consider the function R defined over the vertices in I_β, $R(c) = |\{v \in V(I_\beta) \mid f(v) \leq c\}|$.

For the first step assume that T' holds the values of the function T for the graph G_1. The equation for T_1 is quite similar to the one for $K_{\alpha,\beta}$

taking into account that the number of sets with the required expansion and number of followers is already precomputed in T'. Thus a set X activating $a_1 = |F^{G_1}(X)|$ nodes of which b_1 are followers will activate in addition $R(b_1)$ vertices in I_β. Hence we have

$$T_1(a,b) = \sum_{\{a_1,b_1 | a_1+R(b_1)=a, R(b_1)=b\}} T'_1(a_1,b_1)$$

which can be computed in polynomial time using a double scan.

For the intermediate step, $1 < i < k$, we want to obtain T_{i+1} from T_i and T'_{i+1}. The main difference now is that we have to compute the number of additional vertices in I_α that an expansion set in G_{i+1} will activate. For doing so we introduce some additional notation. For a value b, $b \in \mathrm{Im} f$, let $c(b) = c$ if and only if $R(c) = b$. For any d, $0 \leq d \leq |\mathtt{F}(G_{i+1})|$ and $0 \leq b \leq \beta$, define

$$\Delta(d,b) = \begin{cases} 0 & \text{if } d \leq c(b), \\ c(d) - c(b) & \text{otherwise.} \end{cases}$$

Observe that if b nodes in I_β are activated by a set in H_i, a set in $V(G_{i+1})$ activating d followers in G_{i+1} will activate a total of $\Delta(d,b)$ new followers in H_{i+1}. Therefore we have

$$T_{i+1}(a,b) = \sum_{\{a_1,b_1,a_2,b_2 | a=a_1+a_2, b=b_1+\Delta(b_2,b_1)\}} T_i(a_1,b_1) * T'_{i+1}(a_2,b_2).$$

As before, an array holding this set of values can be computed trivially in polynomial time.

For the last step, we have to join H_k with I_α. This case is similar to the previous one. We have only to take into account that I_α has a simpler structure and that $F^{I_\alpha}(X) = X$. Thus, from T_k we can define T restricted to G as follows:

$$T(a,b) = \sum_{\{a_1,b_1,a_2 | a=a_1+a_2, b=a_2+\Delta(a_2,b_1)\}} T_k(a_1,b_1) * \binom{\alpha}{a_2}.$$

Again, an array storing those values can be computed in polynomial time.

Finally, we have to consider the case in which the graph G is disconnected an thus it is the disjoint union of k directed and connected strong influence graphs $G_1, \ldots, G_k$, and possibly an independent set with size α. For the case of disjoint union, influence in different subgraphs cannot be aggregated. So, if G is the disjoint union of G_1 and G_2, for $X_1 \subseteq V(G_1)$ and $X_2 \subseteq V(G_2)$ we have that $F^G(X_1 \cup X_2) = F^{G_1}(X_1) \cup F^{G_2}(X_2)$. We proceed again by steps showing first how to define the T-values for the graphs $H'_2 \dots H'_k$, where H'_i is the disjoint union of $G_1, \dots, G_i$ and finally for G. Assume that T_i holds the T-values restricted to H_i and that T'_{i+1} correspond to those for G_{i+1}. As influence is just added we have that

$$T_{i+1}(a,b) = \sum_{\{a_1,b_1,a_2,b_2 | a=a_1+a_2, b=b_1+b_2\}} T_i(a_1,b_1) * T'(a_2,b_2).$$

Finally, observe that again an array holding the T-values can be computed from T_k as

$$T(a,b) = \sum_{\{a_1,b_1,a_2 | a=a_1+a_2, b=b_1+a_2\}} T_k(a_1,b_1) * \binom{\alpha}{a_2}.$$

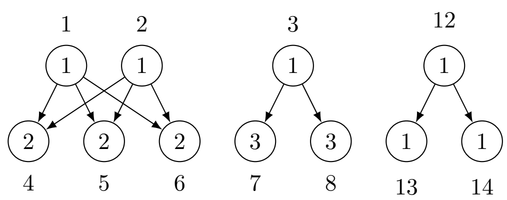
Since all those tables can be computed in polynomial time, the result follows. □

As we mention before the graphs constructed in Lemma 6.6, since (G, f) is a strong influence graph and $v \in \mathtt{L}(G) \cup \mathtt{F}(G)$ are also strong influence graphs, then the above results gives also the following:

Theorem 6.6. The Rae and Bval problems are polynomial time solvable in oblivious strongly mediated influence system.

Example 6.6. Consider the strongly mediated influence system given in Figure 6.5. According to Lemma 6.7, we have to consider first the disjoint union of the two connected components of G with at least one edge and finally incorporating to the graph the two independent actors, vertices 15 and 16, together.

For the second level in the graph we have three connected components $G_1 = G[1,2,4,5,6]$, $G_2 = G[3,7,8]$ and $G_3 = G[12,13,14]$:



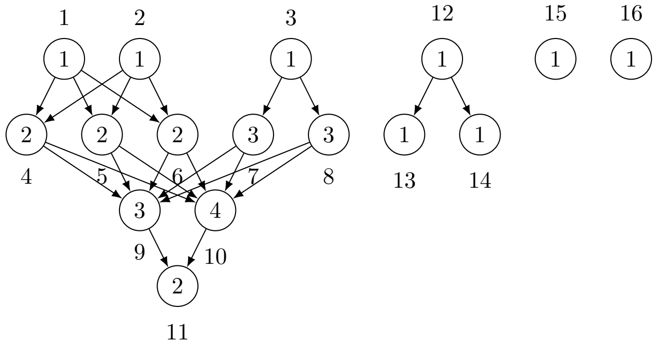
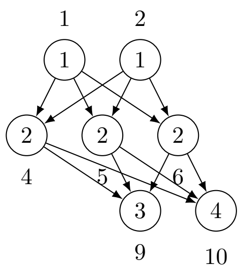


Figure 6.5: A strong influence graph with two layers of mediation.

In the next table we record the T-values for each subgraph.

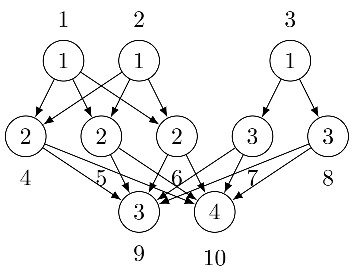
T_{G_1}	0	1	2	3	T_{G_2}	0	1	2	T_{G_3}	0	1	2
0	1	0	0	0	0	1	0	0	0	1	0	0
1	2	0	0	0	1	1	0	0	1	0	0	0
2	0	0	0	0	2	0	0	0	2	0	0	0
3	0	0	0	0	3	0	0	0	3	0	0	1
4	0	0	0	0	-	-	-	-	-	-	-	-
5	0	0	0	1	-	-	-	-	-	-	-	-

The following subgraph to be considered is $G_4 = G[\{x|1 \leq x \leq 11\}]$ which is created by combining G_1 and G_2 with an independent set with two vertices in the next layer. According to the algorithm, we have to compute in according to the first step rules the T-values for the graph H, so we have the following:



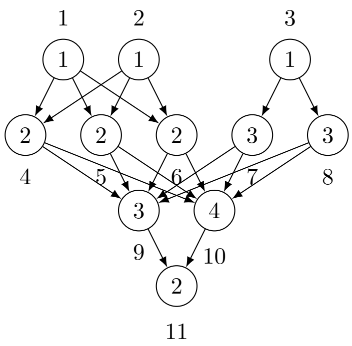
T_H	0	1	2
0	1	0	0
1	2	0	0
2	0	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	1	0
7	0	0	0

From H, following the equations of the second step, we can compute the T-values for the graph G_4, which are:



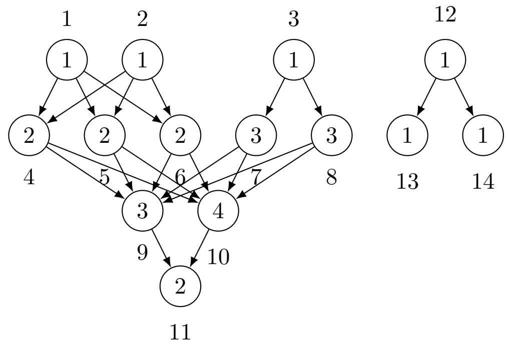
T_{G_4}	0	1	2
0	1	0	0
1	3	0	0
2	2	0	0
3	0	0	0
4	0	0	0
5	0	0	0
6	0	1	0
7	0	1	0
8	0	0	0
9	0	0	0
10	0	0	0

The next step is to incorporate the connections in the last layer to get G_1. For this we have to use the rule of the first step applied to G_4 and an independent set with one vertex. This gives the following:



T_{G_1}	0	1
0	1	0
1	3	0
2	2	0
3	0	0
4	0	0
5	0	0
6	1	0
7	1	0
8	0	0
9	0	0
10	0	0
11	0	0

Now the algorithm computes the T-values for the disjoint union of G_1 and G_2, obtaining:



$T_{G_1 \cup G_2}$	0	1	2	3
0	1	0	0	0
1	3	0	0	0
2	2	0	0	0
3	0	0	1	0
4	0	0	3	0
5	0	0	2	0
6	1	0	0	0
7	1	0	0	0
8	0	0	0	0
9	0	0	1	0
10	0	0	1	0
11	0	0	0	0
12	0	0	0	0
13	0	0	0	0
14	0	0	0	0

From this table we have to incorporate, as a disjoint union, an independent set of size 2, to get the T-values for G. The following table gives the T-values as well as the sum for each row, which corresponds to the value $|F^a(N)|$.

$T_{G_1 \cup G_2}$	0	1	2	3	4	5	$\lvert F^a(N) \rvert$
0	1	0	0	0	0	0	1
1	3	2	0	0	0	0	5
2	2	6	1	0	0	0	9
3	0	4	4	0	0	0	8
4	0	0	5	2	0	0	7
5	0	0	2	6	1	0	9
6	1	0	0	4	3	0	8
7	1	2	0	0	2	0	5
8	0	2	1	0	0	0	3
9	0	0	2	0	0	0	2
10	0	0	1	2	0	0	3
11	0	0	0	2	1	0	3
12	0	0	0	0	1	0	1
13	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0

Star influence systems

Another way to extend the odd-OLF systems is to allow the possibility that some actors at the same time could influence and be influenced by other actor. The next definition is based on the star influence games of Definition 6.8.

Definition 6.17. A *star influence system* is an influence game (G, f, q, N), where $V(G) = \mathtt{L} \cup \mathtt{F} \cup \mathtt{I} \cup \mathtt{R} \cup \{c\}$ is formed by a set of opinion leaders $\mathtt{L}$, a set of followers $\mathtt{F}$, a set of independent actors $\mathtt{I}$, a set of *reciprocal actors* $\mathtt{R}$, and a *central actor* $\{c\}$, who acts like a mediator between the followers and both the opinion leaders and the reciprocal actors. Thus, $E(G) = \{(u, c) \mid u \in \mathtt{L} \cup \mathtt{R}\} \cup \{(c, v) \mid v \in \mathtt{R} \cup \mathtt{F}\}$. For all $i \in V(G) \setminus \{c\}$, $f(i) = 1$, and $f(c) \in \{1, \ldots, |\mathtt{L}| + |\mathtt{R}|\}$. Further, $q \in (0, n]$ and $N = \mathtt{L} \cup \mathtt{R} \cup \mathtt{I}$.

Recall that by Theorem 6.4, the Expansion problem is hard for directed bipartite influence graphs. Now we prove that for oblivious star influence systems, the problem becomes polynomial time solvable.

Theorem 6.7. The Expansion problem can be solved in polynomial time on oblivious star influence systems.

Proof. Let (G, f, q, N) be a star influence system, we show how to compute $|F^k(N)|$ for any $0 \leq k \leq n$ in polynomial time. Let c be the central node of the star influence system. For $|F^k(N)|$, if $k < f(c)$ then there is no initial activation $X \in F^k(N)$ such that $c \in F(X)$; hence, $|F^k(N)|$ corresponds to the number of initial activations with k actors. On the contrary, if $k \geq f(c)$ then $|F^k(N)|$ corresponds to the number of initial activations that cannot influence the central actor, plus the initial activations that can do it; and for the latter case, it is necessary to take into account that $|F(X)|$ is at least $|\mathtt{R}| + |\mathtt{F}| + 1$—where 1 is for the central actor—plus the number of actors in $\mathtt{L} \cup \mathtt{I}$. Therefore, we have:

$$|F^k(N)| = \begin{cases} \dbinom{|\mathtt{L}| + |\mathtt{R}| + |\mathtt{I}|}{k} & \text{if } k < f(c), \\ \displaystyle\sum_{i=0}^{f(c)-1} \binom{|\mathtt{L}| + |\mathtt{R}|}{i} \binom{|\mathtt{I}|}{k-i} & \\ + \displaystyle\sum_{i+j=f(c)}^{|\mathtt{L}|+|\mathtt{R}|} \binom{|\mathtt{L}|}{i} \binom{|\mathtt{R}|}{j} \binom{|\mathtt{I}|}{k - i - (|\mathtt{R}| + |\mathtt{F}| + 1)} & \text{if } k \geq f(c). \end{cases}$$

Using the above the total amount can be computed in polynomial time. □

In Theorem 6.2 we proved that the enumeration of both the MWCs and the maximal losing coalitions can be done with polynomial-delay. Therefore, Theorem 6.7 allows to count the number of winning and losing coalitions for any star influence system in polynomial time, by computing $2^{|\mathtt{F}|+1} \cdot \sum_{i=q}^{n} |F^i(N)|$ and $2^{|\mathtt{F}|+1} \cdot \sum_{i=0}^{q-1} |F^i(N)|$, respectively.

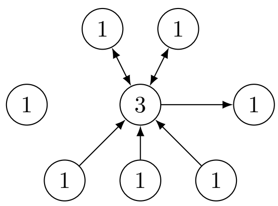


Figure 6.6: A star influence system.

Example 6.7. Consider the star influence system (G, f, q, N) of Figure 6.6 with $q = 4$. Here $|\mathtt{I}| = |\mathtt{F}| = 1$, $|\mathtt{R}| = 2$, $|\mathtt{L}| = 3$ and $f(c) = 3$. Hence, $|\mathcal{W}| = 2^2 \cdot \sum_{i=4}^{8} |F^i(N)| = 4 \cdot (0 + 3 + 12 + 13 + 4) = 128$, and $|\mathcal{L}| = 2^2 \cdot \sum_{i=0}^{3} |F^i(N)| = 4 \cdot (1 + 6 + 15 + 10) = 128$. Note that it holds that $|\mathcal{W}| + |\mathcal{L}| = 256 = 2^8$, as expected.

In order to transfer the previous result just observe that for a given star influence system (G, f, q, N) the graphs required in Lemma 6.6 are obtained from G by removing a vertex in N, and thus both are star influence systems. As a consequence we have the following.

Theorem 6.8. The problems Rae and Bval are polynomial solvable on oblivious star influence systems.

6.4 OLFM systems: Axiomatization of Rae index

For any collective decision making model $\mathcal{M}$, there exists an alternative definition for the satisfaction measure or Rae index, that facilitates the writing of some proofs of this section. The definition is as follows [255]:

$$\textsc{Rae}_{\mathcal{M}}(i) = \sum_{x \in \{0,1\}^n} \overline{\textsc{Rae}}_{\mathcal{M}}(i, x) \tag{6.4}$$

where

$$\overline{\textsc{Rae}}_{\mathcal{M}}(i, x) = \begin{cases} 1 & \text{if } C_{\mathcal{M}}(x) = x_i \\ 0 & \text{otherwise.} \end{cases}$$

In what follows, we use simply $\overline{\textsc{Rae}}(i, x)$ and $\textsc{Rae}(i)$ when there is no risk of confusion about $\mathcal{M}$.

Example 6.8. Let us consider the OLFM system given in Example 6.2. According to our computation, for this case we have $\textsc{Rae}(1) = 104$, $\textsc{Rae}(2) = \textsc{Rae}(5) = 88$, $\textsc{Rae}(3) = \textsc{Rae}(7) = 72$ and $\textsc{Rae}(4) = \textsc{Rae}(6) = 64$.

Note that since OLFM systems are non-oblivious influence systems—see Definition 6.11—then by Lemma 6.2 it is easy to see that the collective decision function under OLFM systems—like in OLF systems—is monotonic.

Consider the collective decision vector in expression (6.1), Definition 6.4. When the fraction value q is large enough, the expression becomes in

$$c_{i,\mathcal{S}}(x) = \begin{cases} b & \text{if } x_j = b \text{ for all } j \in P_G(i), \\ x_i & \text{otherwise} \end{cases} \tag{6.5}$$

where $b \in \{0, 1\}$, i.e., such that followers maintain their own initial decisions unless all their opinion leaders share an opposite inclination. This specific case was studied by [256], where the authors provided an axiomatization for the Rae index in OLF systems. For all what follows we shall consider OLFM systems also restricted to this case, so we dispense of the quota q.

Recall by Lemma 6.6 that in oblivious influence systems, the actors of the same type have always the same satisfaction score. However, this is not the case for OLFM systems. Observe that in Example 6.2, for instance, the satisfaction of a follower may be greater than the satisfaction of a mediator, and equal than the satisfaction of an independent actor.

For what follows, we denote a score as a function $f : V \to \mathbb{R}$ that assigns some real value to each actor of the system. The following properties were introduced by [255, 256] for OLF systems.

Definition 6.18. Let $\mathcal{S}$ and $\mathcal{S}'$ be OLF systems represented by the graphs G and G', respectively, with $V(G) = V(G')$. Let i, j, h be three different actors. A measure given by the function $f : V \to \mathbb{R}$ satisfies the properties:

1. **Symmetry**: if $S(i) = S(j)$ and $P(i) = P(j)$, then $f(i) = f(j)$.
2. **Dictator property**: if $S(i) = V \setminus \{i\}$, then $f(i) = 2^n$.
3. **Dictated independence**: if $|P_G(i)| = |P_{G'}(i)| = 1$, $f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(i)$.
4. **Equal gain property**: if $i \in \mathtt{L} \cup \mathtt{I}$, $j \in \mathtt{F}$ and $E(G') = E(G) \cup \{(i, j)\}$, then $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(j) - f_{\mathcal{S}}(j)$.
5. **Opposite gain property**: if $i \in \mathtt{L} \cup \mathtt{I}$, $j \in \mathtt{I}$ and $E(G') = E(G) \cup \{(i, j)\}$, then $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(j) - f_{\mathcal{S}'}(j)$.
6. **Horizontal neutrality**: if $i \in \mathtt{L} \cup \mathtt{I}$, $j \in \mathtt{F}$, $h \in \mathtt{L}$, $E(G') = E(G) \cup \{(i, j)\}$ and $h \in P_G(j)$, then $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(h) - f_{\mathcal{S}'}(h)$.

The above are desirable properties for scores. The symmetry property means that the score for actors with a symmetric position in the system is the same. A non-symmetrical measure could lead to unconventional results, e.g., two independent actors with different scores.

In this context, a *dictator* is an actor that points to all other actors of the system. Hence, in OLF systems there may be at most one dictator, and if $n > 1$, the dictator is always an opinion leader. Furthermore, if there is a dictator, then all other actors follow this actor, so they adopt as final decision the initial decision of the dictator. The dictator property states that the dictators have the highest score as possible. Observe that this notion corresponds to the dictator players of simple games. Furthermore, this property is closely related to Lemma 6.5.

The dictated independence states that all the followers with only one opinion leader have the same score. However, note that a follower who has only one opinion leader has always to follow this opinion leader. Therefore, since any actor with only one predecessor is a dummy, for Lemma 6.5 the dictated independence is equivalent to the following:

$$\text{if } |P(i)| = 1, \text{ then } f(i) = 2^{n-1}.$$

The remaining properties involve changes in the structure of the OLF systems, by assigning to an actor a new opinion leader. These properties were inspired by similar properties for solution concepts in cooperative game theory [255, 254]. In particular, the equal gain property is closely related with the fairness concept by [191].

In a reasonable score, the addition of an influence relationship—a directed edge—from one actor to another should increase the score of the first actor, because now it is exerting more influence in the system. In this scenario, we can consider two cases:

On the one hand, if the influenced actor was a follower before the addition of the edge, then the score of this follower should also increase, because now it is more difficult to change its initial decision. The equal gain property states that when a follower gets an additional opinion leader, the changes in scores of this follower and of its new opinion leader are the same. For a score that does not meet this property, the addition of a relationship between these kind of actors could be unfair for one of them.

On the other hand, if the influenced actor was an independent actor, then the score of this actor should decrease, because its final decision now depends of the initial decision of the opinion leader. The opposite gain property states that when an independent actor gets an opinion leader, the sum of the scores of these two actors does not change. For a score that does not meet this property, the addition of a relationship between two actors could be unfair for the opinion leader, because it is not getting a profit according to the effort it took to influence the independent actor.

Finally, horizontal neutrality is inspired by the properties considered for collusion of players in cooperative games with transferable utility [154, 111, 254]. This property states that, if a follower with at least one opinion leader gets an additional opinion leader, then the sum of scores of the old and new opinion leaders does not change. This means that the increase in the score for the new opinion leader comes fully from a decrease in the score for the other opinion leaders. For a score that does not meet this property, the new opinion leader could not get a profit according to the effort it took to influence an additional follower.

It is known that these properties hold for RAE in OLF systems.

Theorem 6.9 ([255, 256])**.** For OLF systems, the RAE score satisfies the six properties of Definition 6.18.

To show the axiomatization of satisfaction in OLF systems, in [256] the authors introduce an additional axiom, which corresponds to the total sum of the satisfaction scores over all actors, i.e., a satisfaction normalization.

Definition 6.19. Let $\mathcal{S}$ be an OLF system represented by a graph $G = (V, E)$. A score given by the function $f : V \to \mathbb{R}$ is normalized if it satisfies the following property:

7. **Satisfaction normalization**:
   $\sum_{i \in V} f(i) = \sum_{x \in \{0,1\}^n} |\{i \in V \mid C(x) = x_i\}|$.

It is not hard to see that these properties are independent, in the sense that there is no property that could be implied by other. In fact, note that the equation of property 4 considers a follower, the one of property 5 considers an independent actor that becomes in a follower, and the one property 6 does not consider any follower. Moreover, these seven properties provide an axiomatization of the satisfaction for OLF systems.

Theorem 6.10 ([256])**.** For OLF systems, the RAE score is the unique measure that satisfies the properties 1, 2, 3, 4, 5, 6 and 7 of Definitions 6.18 and 6.19.

In what follows we shall prove that all the properties for satisfaction in OLF systems also apply for OLFM systems. However, to establish an axiomatization in OLFM systems, we need to generalize the equal gain property and the opposite gain property, in order to consider the mediators in the layered graphs. Although it is not required for the axiomatization, we also introduce a generalization of the horizontal neutrality that is fulfilled for satisfaction in OLFM systems.

Definition 6.20. Let $\mathcal{S}$ and $\mathcal{S}'$ be two OLFM systems represented by the graphs G and G', respectively, such that $V(G) = V(G')$. Let i, j, h be three different actors, and $k \in \mathbb{N}$ such that $k \geq 0$. We say that a measure given by the function $f : V \to \mathbb{R}$ satisfies the properties:

4*b* **Equal absolute change property**: if $i \in \mathcal{L}_{k-1}$, $j \in \mathcal{L}_k$ and $E(G') = E(G) \cup \{(i, j)\}$, then either $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(j) - f_{\mathcal{S}}(j)$ or $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(j) - f_{\mathcal{S}'}(j)$.

5*b* **Opposite gain property**: if $i \in V$, $j \in \mathtt{I}$ and $E(G') = E(G) \cup \{(i, j)\}$, then either $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(j) - f_{\mathcal{S}'}(j)$ or $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(j) - f_{\mathcal{S}}(j)$.

6*b* **Power neutrality for two opinion leaders**: if $h \in \mathcal{L}_{k-1}$, $i \in \mathcal{L}_{k-1}$, $j \in \mathcal{L}_k$ with $P_G(j) = \{h\}$ and $E(G') = E(G) \cup \{(i, j)\}$, then either $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(h) - f_{\mathcal{S}'}(h)$ or $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(h) - f_{\mathcal{S}}(h)$.

Note that the opposite gain property is a generalization of the property 5 of Definition 6.18, because when $i \in \mathtt{L} \cup \mathtt{I}$, it only holds $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(j) - f_{\mathcal{S}'}(j)$. The equal absolute change property is a generalization of equal gain property, because when $i \in \mathtt{L} \cup \mathtt{I}$, it only holds $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(j) - f_{\mathcal{S}}(j)$. The power neutrality for two opinion leaders is a generalization of horizontal neutrality, because for $k = 2$, it only holds $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(h) - f_{\mathcal{S}'}(h)$. Moreover, the properties 4*b* and 6*b* were introduced by [255] for OLF systems—i.e., OLFM systems with two layers—not restricted to unanimity, i.e., so that followers can change their decisions based on a majority proportion of their opinion leaders.

The following result proves that all the previous properties are fulfilled by the satisfaction in OLFM systems.

Theorem 6.11. For OLFM systems, the RAE score satisfies the properties 1, 2, 3, 4*b*, 5*b* and 6*b* of Definitions 6.18 and 6.20.

Proof. For symmetry, for all $x \in \{0,1\}^n$, $P(i) = P(j)$ implies $c_i(x) = c_j(x)$. Further, as $S(i) = S(j)$, if $x_i \neq x_j$, then $c(x) = c(x - i + j)$; and if $x_i = x_j$, the satisfaction score does not change for the actors i and j.

For the dictator property, if $S(i) = V \setminus \{i\}$ we have that $i \in \mathtt{L}$ and $|\mathtt{I}| = |\mathtt{M}| = 0$, which is the same case proved for OLF systems in [255], i.e., as $C(x) = x_i$ for all $x \in \{0,1\}^n$, then $\textsc{Rae}(i) = 2^n$.

For the dictated independence, let be $P(i) = \{j\}$, then for all $x \in \{0,1\}^n$ it holds $c_i(x) = x_j$, so the collective choice $C(x)$ is independent of the decision of the actor i. Hence, let be $b = \{0,1\}$, if $C(x) = b$, there are exactly 2^{n-1} initial decision vectors with $x_i = b$, and 2^{n-1} with $x_i = 1 - b$.

For what follows, note that for every $x \in \{0,1\}^n$ such that $C_{\mathcal{S}}(x) = C_{\mathcal{S}'}(x)$, it holds $\overline{\textsc{Rae}}_{\mathcal{S}}(i,x) = \overline{\textsc{Rae}}_{\mathcal{S}'}(i,x)$, for all $i \in V$. Therefore, to determine $\textsc{Rae}_{\mathcal{S}}(i)$ and $\textsc{Rae}_{\mathcal{S}'}(i)$ we only need to consider the initial decision vectors $x \in \{0,1\}^n$ where $C_{\mathcal{S}}(x) \neq C_{\mathcal{S}'}(x)$.

For the equal absolute change property, first consider that $i \in \mathtt{L} \cup \mathtt{I}$ and $j \in \mathtt{M} \cup \mathtt{F}$. As $c_{j,\mathcal{S}}(x) \neq x_i$ and $c_{j,\mathcal{S}'}(x) = x_i$, then $C_{\mathcal{S}}(x) \neq x_i$ and $C_{\mathcal{S}'}(x) = x_i$; hence $\overline{\textsc{Rae}}_{\mathcal{S}'}(i,x) - \overline{\textsc{Rae}}_{\mathcal{S}}(i,x) = 1$. If $c_{j,\mathcal{S}}(x) = x_j$, then $x_j \neq x_i$, so $C_{\mathcal{S}}(x) = x_j$ and $C_{\mathcal{S}'}(x) \neq x_j$, which implies $\overline{\textsc{Rae}}_{\mathcal{S}}(j,x) - \overline{\textsc{Rae}}_{\mathcal{S}'}(j,x) = 1$; and if $c_{j,\mathcal{S}}(x) \neq x_j$, then $x_j = x_i$, so $C_{\mathcal{S}}(x) \neq x_j$ and $C_{\mathcal{S}'}(x) = x_j$, implying $\overline{\textsc{Rae}}_{\mathcal{S}'}(j,x) - \overline{\textsc{Rae}}_{\mathcal{S}}(j,x) = 1$. The possible change of inclinations or decisions of successors of j keeps that $C_{\mathcal{S}}(x) \neq C_{\mathcal{S}'}(x)$, and this does not contradicts the above. Thus, by expression (6.4) we have either $\textsc{Rae}_{\mathcal{S}'}(i) - \textsc{Rae}_{\mathcal{S}}(i) = \textsc{Rae}_{\mathcal{S}'}(j) - \textsc{Rae}_{\mathcal{S}}(j)$ or $\textsc{Rae}_{\mathcal{S}'}(i) - \textsc{Rae}_{\mathcal{S}}(i) = \textsc{Rae}_{\mathcal{S}}(j) - \textsc{Rae}_{\mathcal{S}'}(j)$.

Second, consider $i \in \mathtt{M}$. Note that in this case, the inclination of actor i also depends of their predecessors. To deal with this, just replace x_i in all the above equations by $c_i(x)$, and note that $c_i(x) = c_{i,\mathcal{S}}(x) = c_{i,\mathcal{S}'}(x)$, so if $x_i = c_i(x)$, we obtain the same equations, and if $x_i \neq c_i(x)$, we obtain that $\overline{\textsc{Rae}}_{\mathcal{S}}(i,x) - \overline{\textsc{Rae}}_{\mathcal{S}'}(i,x) = 1$, getting the same final equations that above.

For the opposite gain property, first consider $i \in \mathtt{L} \cup \mathtt{I}$ and $j \in \mathtt{I}$. As it must hold that $x_i \neq c_j$, then $C_{\mathcal{S}}(x) = x_j \neq x_i$ and $C_{\mathcal{S}'}(x) = x_i \neq x_j$; hence

$\overline{\text{RAE}}_{\mathcal{S}'}(i,x) - \overline{\text{RAE}}_{\mathcal{S}}(i,x) = 1$ and $\overline{\text{RAE}}_{\mathcal{S}}(j,x) - \overline{\text{RAE}}_{\mathcal{S}'}(j,x) = 1$. Second, consider that $i \in \texttt{M}\cup\texttt{F}$. For this case, just replace x_i in all the above equations by $c_i(x)$, and note that $c_i(x) = c_{i,\mathcal{S}}(x) = c_{i,\mathcal{S}'}(x)$, so it holds $c_i(x) \neq x_j$, $C_{\mathcal{S}}(x) = x_j$ and $C_{\mathcal{S}'}(x) = c_i(x)$; hence, $\overline{\text{RAE}}_{\mathcal{S}}(j,x) - \overline{\text{RAE}}_{\mathcal{S}'}(j,x) = 1$ and either $\overline{\text{RAE}}_{\mathcal{S}'}(i,x) - \overline{\text{RAE}}_{\mathcal{S}}(i,x) = 1$ or $\overline{\text{RAE}}_{\mathcal{S}}(i,x) - \overline{\text{RAE}}_{\mathcal{S}'}(i,x) = 1$. Thus, by expression (6.4) we have either $\text{RAE}_{\mathcal{S}'}(i) - \text{RAE}_{\mathcal{S}}(i) = \text{RAE}_{\mathcal{S}}(j) - \text{RAE}_{\mathcal{S}'}(j)$ or $\text{RAE}_{\mathcal{S}'}(i) - \text{RAE}_{\mathcal{S}}(i) = \text{RAE}_{\mathcal{S}'}(j) - \text{RAE}_{\mathcal{S}}(j)$.

For the power neutrality for two opinion leaders, first consider $i \in \texttt{L} \cup \texttt{I}$, $j \in \texttt{M} \cup \texttt{F}$ and $h \in \texttt{L}$. As $|P_G(j)| = 1$, $c_{j,\mathcal{S}}(x) = x_h$, and as $|P_{G'}(j)| = 2$, $c_{j,\mathcal{S}'}(x) \neq x_j$ iff $x_h = x_i \neq x_j$. Let $b \in \{0,1\}$, if $C_{\mathcal{S}}(x) = b$ and $C_{\mathcal{S}'}(x) = 1 - b$, then $c_{j,\mathcal{S}}(x) = b = x_h$ and $c_{j,\mathcal{S}'}(x) = 1 - b = x_i = x_j$, hence $\overline{\text{RAE}}_{\mathcal{S}'}(i,x) - \overline{\text{RAE}}_{\mathcal{S}}(i,x) = 1 = \overline{\text{RAE}}_{\mathcal{S}}(h,x) - \overline{\text{RAE}}_{\mathcal{S}'}(h,x)$. The possible change of inclinations of successors of j keeps that $C_{\mathcal{S}}(x) \neq C_{\mathcal{S}'}(x)$, and this does not contradicts the above. Thus, by expression (6.4) we have $\text{RAE}_{\mathcal{S}'}(i) - \text{RAE}_{\mathcal{S}}(i) = \text{RAE}_{\mathcal{S}}(h) - \text{RAE}_{\mathcal{S}'}(h)$.

Second, consider $h \in \texttt{M}$. For this case, just replace x_h in all the above equations by $c_h(x)$, and note that $c_h(x) = c_{h,\mathcal{S}}(x) = c_{h,\mathcal{S}'}(x)$, so either $\overline{\text{RAE}}_{\mathcal{S}}(h,x) - \overline{\text{RAE}}_{\mathcal{S}'}(h,x) = 1$ or $\overline{\text{RAE}}_{\mathcal{S}'}(h,x) - \overline{\text{RAE}}_{\mathcal{S}}(h,x) = 1$. Finally, consider $i \in \texttt{M}\cup\texttt{F}$. Replacing x_i by $c_i(x)$, where $c_i(x) = c_{i,\mathcal{S}}(x) = c_{i,\mathcal{S}'}(x)$, we obtain analogous equations. Therefore we have either $\text{RAE}_{\mathcal{S}'}(i) - \text{RAE}_{\mathcal{S}}(i) = \text{RAE}_{\mathcal{S}}(h) - \text{RAE}_{\mathcal{S}'}(h)$ or $\text{RAE}_{\mathcal{S}'}(i) - \text{RAE}_{\mathcal{S}}(i) = \text{RAE}_{\mathcal{S}'}(h) - \text{RAE}_{\mathcal{S}}(h)$. □

Note that the satisfaction normalization of Definition 6.19 remains the same for OLFM systems, because the collective decision function is the same. From the previous theorem, since for OLFM systems the satisfaction score satisfies power neutrality for two opinion leaders, then it also satisfies horizontal neutrality. Moreover, note that both properties 4*b* and 5*b* remain independent with the others. Indeed, in property 5*b* the actor $j \in V(G)$ is an independent actor, but in property 4*b* it is not true, because j belongs to level $k > 1$. Furthermore, there is no equation in these properties where both actors are in the same level, as it happens in property 6.

In what follows we prove an axiomatization of satisfaction for OLFM systems.

Theorem 6.12. For OLFM systems, the RAE score is the unique measure that satisfies the properties 1, 2, 3, 4*b*, 5*b*, 6 and 7 of Definitions 6.18, 6.19 and 6.20.

Proof. We know by Theorem 6.11 that in OLFM systems the RAE score satisfies properties 1, 2, 3, 4b, 5b, 6 and 7. To prove uniqueness it remains to show that, on the assumption that $f : V \to \mathbb{R}$ satisfies the seven axioms, then this score must be equal to RAE.

By Theorem 6.10, we know that if there are no mediators—i.e., we have an OLF system—then property 4b is replaced by property 4, so the score is equal to RAE. Now we proceed constructively.

First, given an OLFM system $\mathcal{S}$ without mediators, we can transform a follower i in a mediator by connecting it with an independent actor j, obtaining a new OLFM system $\mathcal{S}'$. Thus, by property 5b, it holds either $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}}(j) - f_{\mathcal{S}'}(j)$ or $f_{\mathcal{S}'}(i) - f_{\mathcal{S}}(i) = f_{\mathcal{S}'}(j) - f_{\mathcal{S}}(j)$. As $f_{\mathcal{S}}(i)$ and $f_{\mathcal{S}}(j)$ are uniquely determined by Theorem 6.10, both equations yield a system of linear equations easy to solve, so that the unknowns, $f_{\mathcal{S}'}(i)$ and $f_{\mathcal{S}'}(j)$, can be uniquely determined.

From the above, note that actor j in $\mathcal{S}'$ becomes in a follower. And also note that we can transform step by step other independent actors j in followers, such that $P_{G'}(j) = \{i\}$. In each step, satisfaction score can be uniquely determined by using the same property.

Secondly, suppose that we have an OLFM system $\mathcal{S}$ with only one layer of mediation, like the obtained above, with a follower $j \in \mathcal{L}_3$ so that $P_{G'}(j) = \{i\}$. Now we can transform a follower $h \in \mathcal{L}_2$ in a mediator, by connecting it with follower j, obtaining a new OLFM system $\mathcal{S}'$. Thus, by property 4b, it holds either $f_{\mathcal{S}'}(h) - f_{\mathcal{S}}(h) = f_{\mathcal{S}}(j) - f_{\mathcal{S}'}(j)$ or $f_{\mathcal{S}'}(h) - f_{\mathcal{S}}(h) = f_{\mathcal{S}'}(j) - f_{\mathcal{S}}(j)$. This is basically the same kind of system of linear equations obtained with property 5b), and as $f_{\mathcal{S}}(h)$ and $f_{\mathcal{S}}(j)$ are uniquely determined, then $f_{\mathcal{S}'}(h)$ and $f_{\mathcal{S}'}(j)$ can also be uniquely determined. We can also repeat this process by transforming new followers $h \in \mathcal{L}_2$ in mediators, obtaining in each step that satisfaction can be uniquely determined.

Of course, the same kind of transformations can be done to create lower layers, and therefore to produce any OLFM system.

Finally, note that property 1 implies that there is a constant $c \in \mathbb{R}$ such that for all $i \in \mathtt{I}$, $f(i) = c$. Hence, for every OLFM system, we can provide new independent actors and then using them as opinion leaders, followers or mediators, in such a way that f can always be uniquely determined. □

Chapter 7

Centrality in Social Networks

The aim of this chapter is to propose new centrality measures that can be used to analyze the relevance of the participants in a social network within a process related to spread of influence.

In Section 7.1 we define some traditional centrality measures, namely *degree*, *closeness*, *betweenness*, *flow closeness* and *flow betweenness*. After that, we provide new centrality measures. The first two correspond to the Banzhaf and the Shapley-Shubik power indices, which can be bring to this centrality context through the use of influence games. There is some previous work where the Shapley-Shubik index is used as centrality measure for specific game-theoretic networks [193, 173], but as far as we know, the Banzhaf index has not been used before for this purpose. Other two new centrality measures are the *effort* and the *satisfaction*, that take advantage directly from the notion of influence games. While effort's centrality measures the effort required to make the social network follow the opinion of an individual, satisfaction's centrality measures the level of satisfaction of each individual, so that it is influential if and only if it is taken into account. The last family of centrality measures consider only influence graphs, so they dispense of the quota of influence games. This help us to measure centrality in time polynomial in the size of the network.

In Section 7.2 we perform an experimental comparison between these new centrality measures and the classic ones. We compare them in some real social networks on which the computations can be performed in reasonable time.

7.1 Centrality Measures

A social network can be represented by a graph, where each vertex is an actor, individual, agent or player, and each edge connecting two vertices represents an interpersonal tie among the respective actors. These graphs may usually be directed, so that the interpersonal ties represent either a flow of communication or an influence relation from one actor to another. Furthermore, the graphs may be weighted, in such a way that the weight of every edge represents the strength of that interpersonal tie. As far we know, there are no traditional centrality measures defined for labeled graphs, despite of the fact that a labeling function in a social network may provide interesting additional information about the actors.

In this section we consider static networks, defined beforehand, so that the number of vertices remains unchanged and there is no creation, deletion nor strengthening of interpersonal ties. Recall that undirected graphs can be treated as symmetric directed graphs, considering that an undirected edge $\{i, j\}$ between two actors i and j is the same that two directed edges (i, j) and (j, i).

The *centrality* of a vertex refers to its relative importance inside of a network, and depends of structural aspects at a global level. Centrality is one of the most studied concepts in network analysis, and since the late 1970s in social network analysis [85, 86]. There are several centrality measures that provide different importance criteria to the vertices [143]. Three of the most well-known and widely applied are defined as follows [261, 150].

Definition 7.1. Let be a social network with a set of vertices V and $i \in V$.

- The *degree centrality* (C_D) corresponds to the indegree or outdegree of each actor, i.e.,

  $$C_D^-(i) = \delta^-(i) \quad \text{or} \quad C_D^+(i) = \delta^+(i).$$

  In normalized version:

  $$C_D'^-(i) = \frac{\delta^-(i)}{n-1} \quad \text{or} \quad C_D'^+(i) = \frac{\delta^+(i)}{n-1}.$$

  For undirected networks, $\delta(i) = \delta^-(i) = \delta^+(i)$, so C_D is without distinction C_D^- and C_D^+.

- The *closeness centrality* (C_C) is the inverse of the sum of shortest paths from i to the other actors, i.e., let D be the usual distance matrix of the network,
$$C_C(i) = \frac{1}{\sum_{i \neq j} (D)_{ij}}.$$
In normalized version:
$$C'_C(i) = \frac{n-1}{\sum_{i \neq j} (D)_{ij}}.$$
If there is no path from i to j, we assume that $(D)_{ij} = n$.

- Let b_{jk} be the number of shortest paths from the vertex j until k, and b_{jik} the number of these shortest paths that pass through i, with $i \neq j$, $i \neq k$ and $j \neq k$, then the *betweenness centrality* (C_B) is
$$C_B(i) = \sum_{j \neq k} \frac{b_{jik}}{b_{jk}}.$$
In normalized version:
$$C'_B(i) = \frac{1}{(n-1)(n-2)} \sum_{j \neq k} \frac{b_{jik}}{b_{jk}}.$$
If there is no path from j to k, then $\frac{b_{jik}}{b_{jk}} = 0$.

There are many other centrality measures based on the previous ones, such as the *Katz centrality*, *Bonacich centrality*, *Hubbell centrality*, *Newman betweenness*, among others [247]. The differences between these variations are few, and do not involve a change of paradigm. Additionally, there are other measures based on other ideas, like *Eigenvector* and *Alpha centrality* [247]. Some of these measures were initially defined only for undirected graphs. However, some of them can naturally be generalized to directed graphs and even to weighted graphs [243, 199].

Note that there are also traditional centrality measures that were defined exclusively for weighted graphs. That is the case of the flow betweenness [86] and the flow closeness [104], which are based on flow networks.

Definition 7.2. Let m_{jk} be the maximum flow from the node j to the node k, and m_{jik} the maximum flow from j to k that passes through the node i, with $i \neq j$, $i \neq k$ and $j \neq k$.

- The *flow betweenness centrality* (F_B) is given by

$$F_B(i) = \sum_{j \neq k} m_{jik}.$$

In normalized version:

$$F'_B(i) = \frac{\sum_{j \neq k} m_{jik}}{\sum_{j \neq k} m_{jk}}.$$

- The *flow closeness centrality* (F_C) is given by

$$F_C(i) = \sum_{i \neq k} m_{ik}.$$

Recently have been defined new centrality measures for flow networks, based on the previous ones [104].

Henceforth, we represent social networks as influence games (G, w, f, q), so we assume that $N = V$. The labeling of the vertices can be conditioned to the nature of the network. Note that the isolated vertices can not be convinced in any way. Since a power index is a measure of the importance of the players in a game, then we can use them as centrality measures, interpreting that an actor is more central in the network while it is more necessary to generate successful teams. In this scenario, observe that, like C_B, both the Banzhaf index β—from now on `Bz`—and the Shapley-Shubik index Φ—from now on `SS`—correspond to *medial measures*, in the sense that they take as reference the sets of actors which pass through a given vertex. Other measures like C_D or C_C are *radial measures* in the sense that they take as reference a given vertex which starts or ends some paths through the network [247].

Influence games can also provide new criteria to determine measures of centrality. The following centrality measure, for instance, takes advantage of the labeling function. Note that it does not consider explicitly the weight function, although it is implicitly considered in the spread of influence. Other measures that consider the weight function could be defined as well.

Definition 7.3. Let (G, w, f, q) be an influence game representing a social

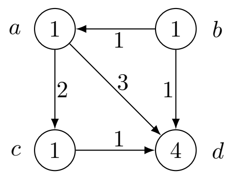


Figure 7.1: Influence game representing a small social network.

network, the *(minimum) effort* required by the network to choose a successful team that contains a required actor is given by

$$\texttt{Effort}(i) = \min\{f(S) \mid |F(S \cup \{i\})| \geq q\}.$$

A normalized version of the *effort centrality measure* (C_E) is

$$C_E(i) = \frac{f(N) - \texttt{Effort}(i)}{f(N)}.$$

Note that while greater is the required effort for a vertex, this vertex should be less central.

Example 7.1. Let us consider the influence game given by the influence graph of Figure 7.1 and a quota $q = 4$. Regarding the Banzhaf index, since actor b is the unique critical player, we have that $\eta(b) = 8$ and $\texttt{Bz}(b) = 1$, while $\eta(j) = 0$ and $\texttt{Bz}(j) = 0$ for $j \in \{a, c, d\}$. Regarding the Shapley-Shubik index, as $|\{b\}| = 1$, $|\{b, a\}| = |\{b, c\}| = |\{b, d\}| = 2$, $|\{b, a, c\}| = |\{b, a, d\}| = |\{b, c, d\}| = 3$ and $|\{b, a, c, d\}| = 4$, then $\kappa(b) = 24$ and $\texttt{SS}(b) = 1$, while $\texttt{SS}(j) = 0$ for $j \in \{a, c, d\}$. Furthermore, $\texttt{Effort}(b) = 1$, $\texttt{Effort}(a) = \texttt{Effort}(c) = 2$ and $\texttt{Effort}(d) = 5$, so $C_E(b) = 6/7 > C_E(a) = C_E(c) = 5/7 > C_E(d) = 2/7$.

The following measure is based on the satisfaction score defined in [255], which is equivalent to the Rae index as we saw in Section 6.3.

Definition 7.4. Let (G, w, f, q) be an influence game representing a social network, $\mathcal{W}_i = \{X \subseteq V(G) \mid i \in X, |F(X)| \geq q\}$ and $\mathcal{L}_{-i} = \{X \subseteq V(G) \mid i \notin X, |F(X)| < q\}$, the *satisfaction centrality measure* (C_S) is

$$C_S(i) = \frac{|\mathcal{W}_i| + |\mathcal{L}_{-i}|}{2^n}.$$

Note that we could define other measures based on C_E or C_S. For instance, we can define a simplified version of the effort, based on the parameter width:

$$\texttt{Width}(i) = \min\{|S| \mid |F(S \cup \{i\})| \geq q\}$$

and to consider

$$C_W(i) = \frac{n - \texttt{Width}(i)}{n}.$$

We have analyzed this measure C_W, but it does not provide significant differences on the results, so it is not considered in this thesis. Note that the normalizations chosen for both C_W and C_E require a subtraction in the numerator, like for C_C.

In the next section we shall see that these new centrality measures are useful for social networks with a relatively small number of actors. However, it is known by Theorem 5.7 that computing the Banzhaf value and the Shapley-Shubik value for influence games is #P-complete, as well as by Section 6.3.1 that computing the satisfaction score is usually a hard problem. Fortunately, we can define an additional family of centrality measures that dispense of the quota of influence games, so they only need an influence graph and thus can be computed in polynomial time in the size of the network.

Definition 7.5. Let (G, w, f) be an influence graph representing a social network, the *k-influence centrality* is given by

$$F_k(i) = \sum_{X \subseteq V, |X|=k, i \in X} |F(X)|.$$

In normalized version:

$$F'_k(i) = \frac{F_k(i)}{\sum_{j \in V} F_k(j)}.$$

Thus, for instance, when $k = 2$ the 2-influence centrality is equals to

$$F_2(i) = \sum_{j \in V} |F(\{i, j\})|.$$

Observe that flow measures consider networks where there exists information that is transported through the edges with an associated cost. In this case, it has influence exerted by the actors to other actors, and each

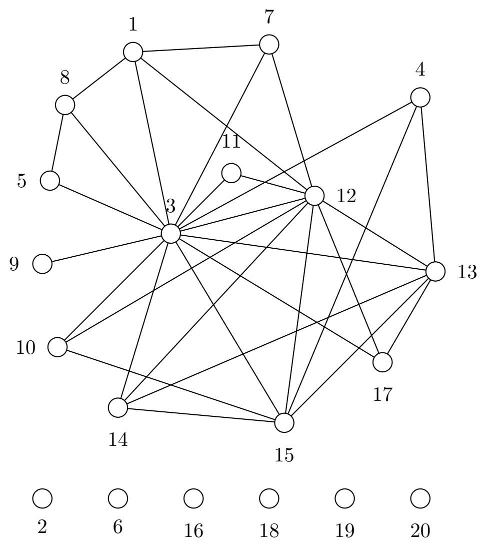


Figure 7.2: Social network of monkeys' interaction.

actor has an associated label that represents the difficulty to be influenced.

7.2 Cases of Study

In this section we consider three real social networks to compare the new centrality measures with the traditional ones. The first one, *monkeys' interaction*, corresponds to an unlabeled and undirected graph; the second one, *dining-table partners*, is a weighted directed graph; and the third one, *student Government discussion*, is a weighted and labeled directed graph.

We finish with an additional social network with a high number of actors. For this weighted directed graph we apply the 2-influence centrality measure, obtaining a good computational performance.

7.2.1 Monkeys' Interaction

Everett and Borgatti [77] provided a network that represents the real interactions amongst a group of 20 monkeys observed during three months alongside a river. It is an undirected graph where an edge $\{i, j\}$ exists when monkeys i and j were witnessed together in the river. The graph is formed by 6 isolated vertices and a connected component of 14 vertices, as shown in Figure 7.2. The authors considered the centrality measures C_D, C_C and

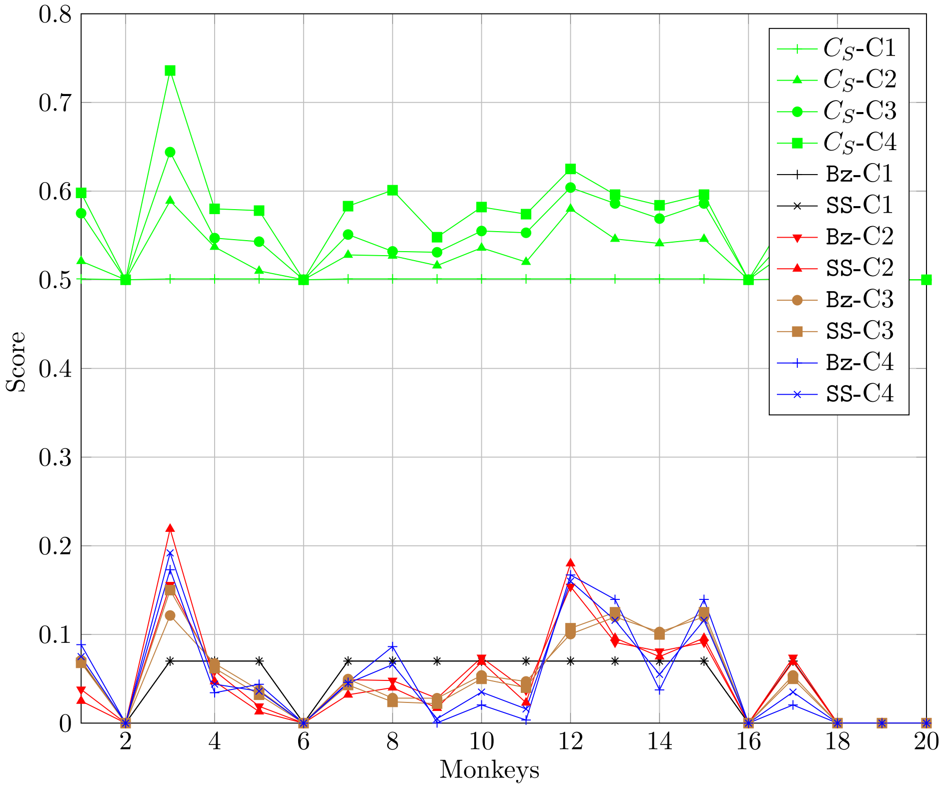


Figure 7.3: Comparisons between `Bz`, `SS` and C_S measures for every case in Monkeys' interaction network.

C_B, as well as generalized versions of these measures for groups instead of individuals. Years later, Latora and Marchiori [150] use the same network to compare the previous results with the measure called *information centrality*.

In order to analyze this network we assume that the graph is symmetric, and that the weight function is defined by $w(e) = 1$, for all edge $e \in E$. In the context of our work, this means that a monkey can influence and be influenced by other monkey if and only if they have interacted before. To deal with influence games we use a quota $q = 14$, which corresponds to the maximum spread of influence that can be obtained from a monkey. This helps to obtain lower measures in isolated vertices, as it is to be expected from a centrality measure. Now we consider the following natural labeling functions for every vertex $i \in V$:

				C1						C2						C3						C4					
Node	C'_D	C'_C	C'_B	Bz	SS	C'_E	C'_S	F'_2	F'_3	Bz	SS	C'_E	C'_S	F'_2	F'_3	Bz	SS	C'_E	C'_S	F'_2	F'_3	Bz	SS	C'_E	C'_S	F'_2	F'_3
1	5-6	5-6	5	1-14	1-14	1-14	1-14	1-14	7-20	11	11	1-14	11	9-10	11	6	6	1-10	5	**3**	5	5	5	1-20	4	4-19	4-6
2	15-20	15-20	7-20	15-20	15-20	15-20	15-20	15-20	1-6	15-20	15-20	15-20	15-20	11-17	12-17	15-20	15-20	15-20	15-20	8-19	11-18	15-20	15-20	1-20	15-20	4-19	7-18
3	**1**	**1**	**1**	1-14	1-14	1-14	1-14	1-14	7-20	**1**	**1**	1-14	**1**	**1**	**1**	**1**	**1**	11-14	**1**	**1**	**1**	**1**	**1**	1-20	**1**	**1**	**1**
4	7-11	7-11	7-20	1-14	1-14	1-14	1-14	1-14	7-20	8	8	1-14	6	11-17	9	7	7	1-10	11	8-19	11-18	10	9	1-20	11	4-19	7-18
5	12-13	12-13	7-20	1-14	1-14	1-14	1-14	1-14	7-20	14	14	1-14	14	18-19	19	12	12	1-10	12	6	6	8	10	1-20	12	4-19	7-18
6	15-20	15-20	7-20	15-20	15-20	15-20	15-20	15-20	1-6	15-20	15-20	15-20	15-20	11-17	12-17	15-20	15-20	15-20	15-20	8-19	11-18	15-20	15-20	1-20	15-20	4-19	7-18
7	7-11	7-11	7-20	1-14	1-14	1-14	1-14	1-14	7-20	9	10	1-14	9	9-10	10	10	10	1-10	10	8-19	11-18	7	8	1-20	8	4-19	7-18
8	7-11	7-11	6	1-14	1-14	1-14	1-14	1-14	7-20	10	9	1-14	10	5	5	13	13	11-14	13	7	8	6	6	1-20	**3**	**2-3**	**3**
9	14	14	7-20	1-14	1-14	1-14	1-14	1-14	7-20	13	13	1-14	13	18-19	18	14	14	1-10	14	20	20	14	14	1-20	14	20	20
10	7-11	7-11	7-20	1-14	1-14	1-14	1-14	1-14	7-20	6-7	6-7	1-14	7-8	6-8	7-8	8-9	8-9	1-10	7-8	8-19	9-10	11-12	11-12	1-20	9-10	4-19	7-18
11	12-13	12-13	7-20	1-14	1-14	1-14	1-14	1-14	7-20	12	12	1-14	12	20	20	11	11	1-10	9	8-19	19	13	13	1-20	13	4-19	19
12	**2**	**2**	**2**	1-14	1-14	1-14	1-14	1-14	7-20	**2**	**2**	1-14	**2**	**2**	**2**	5	4	1-10	**2**	**2**	**2**	**2**	**2**	1-20	**2**	**2-3**	**2**
13	**3-4**	**3-4**	**3-4**	1-14	1-14	1-14	1-14	1-14	7-20	**3-4**	**3-4**	1-14	**3-4**	**3-4**	**3-4**	**2-3**	**2-3**	11-14	**3-4**	4-5	**3-4**	**3-4**	**3-4**	1-20	5-6	4-19	4-6
14	5-6	5-6	7-20	1-14	1-14	1-14	1-14	1-14	7-20	5	5	1-14	5	6-8	6	4	5	1-10	6	8-19	7	9	7	1-20	7	4-19	7-18
15	**3-4**	**3-4**	**3-4**	1-14	1-14	1-14	1-14	1-14	7-20	**3-4**	**3-4**	1-14	**3-4**	**3-4**	**3-4**	**2-3**	**2-3**	11-14	**3-4**	4-5	**3-4**	**3-4**	**3-4**	1-20	5-6	4-19	4-6
16	15-20	15-20	7-20	15-20	15-20	15-20	15-20	15-20	1-6	15-20	15-20	15-20	15-20	11-17	12-17	15-20	15-20	15-20	15-20	8-19	11-18	15-20	15-20	1-20	15-20	4-19	7-18
17	7-11	7-11	7-20	1-14	1-14	1-14	1-14	1-14	7-20	6-7	6-7	1-14	7-8	6-8	7-8	8-9	8-9	1-10	7-8	8-19	9-10	11-12	11-12	1-20	9-10	4-19	7-18
18	15-20	15-20	7-20	15-20	15-20	15-20	15-20	15-20	1-6	15-20	15-20	15-20	15-20	11-17	12-17	15-20	15-20	15-20	15-20	8-19	11-18	15-20	15-20	1-20	15-20	4-19	7-18
19	15-20	15-20	7-20	15-20	15-20	15-20	15-20	15-20	1-6	15-20	15-20	15-20	15-20	11-17	12-17	15-20	15-20	15-20	15-20	8-19	11-18	15-20	15-20	1-20	15-20	4-19	7-18
20	15-20	15-20	7-20	15-20	15-20	15-20	15-20	15-20	1-6	15-20	15-20	15-20	15-20	11-17	12-17	15-20	15-20	15-20	15-20	8-19	11-18	15-20	15-20	1-20	15-20	4-19	7-18

Table 7.1: Comparison of centrality measures for the Monkeys' interaction network. The three more central values of some measures are highlighted in bold. For influence games we consider a quota $q = 14$.

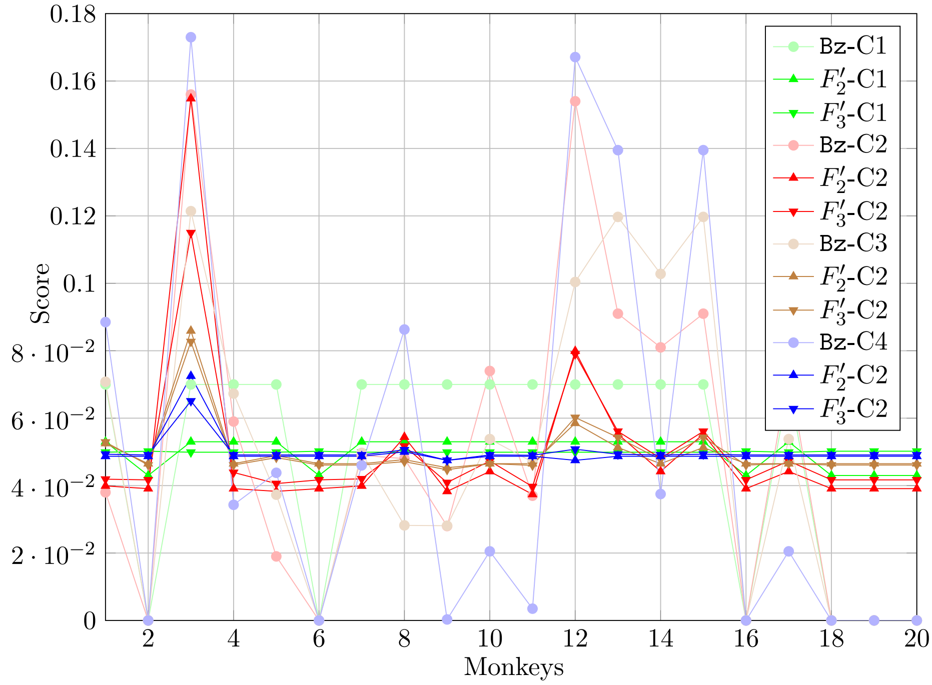


Figure 7.4: Comparisons between `Bz`, F_2 and F_3 measures for every case in Monkeys' interaction network.

- Case 1 (C1): Minimum influence required to convince, $f(i) = 1$.
- Case 2 (C2): Average influence required, $f(i) = \lceil deg(i)/2 \rceil$.
- Case 3 (C3): Majority influence required, $f(i) = \lfloor deg(i)/2 \rfloor + 1$.
- Case 4 (C4): Maximum influence required, $f(i) = deg(i)$.

The comparison between the rankings for traditional measures and the new ones are presented on Table 7.1. Note that for minimum influence required to convince (C1), we have that $|F(\{i\})| = 14$ for every non-isolated vertex i. Thus, for this case the new measures are not good representatives, because all the non-isolated vertices assume the same score. By the same reason, F_3-C1 produces an extreme case, when isolated actors are more central than the non-isolated ones, and C_E does not provide a good ranking for any case. However, for the remaining cases the labeling function is relevant.

For `Bz`, `SS`, C_S and sometimes for F_2 the only pairs of monkeys with the same ranking are $(10, 17)$ and $(13, 15)$. These measures allow a more

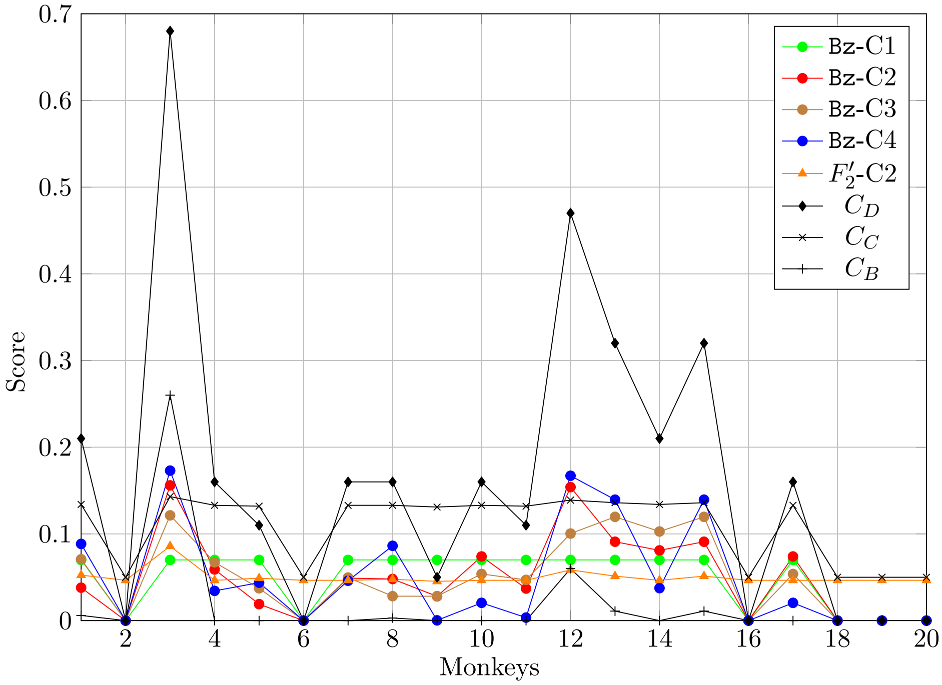


Figure 7.5: Comparisons between new and traditional centrality measures.

relevant hierarchization than the given by the others. Note that `Bz`, `SS` and C_S provide very similar rankings. In Figure 7.3 we can see that `Bz` and `SS` produce not only very similar rankings, but also similar scores. Moreover, subtracting a value equals 0.5 to the scores of C_S we obtain also similar values than for the power indices. This is due to the definition of the measures. Furthermore, remember by Lemma 6.5 that the score of satisfaction or Rae index is always greater or equal than 2^{n-1}, so it is predictable that the normalization produces scores greater or equal than 0.5. Additionally, note that for C_S the difference between the scores of the actors grows as the differences among the labels of the vertices—from case C1 until C4—are higher. Since `Bz`, `SS` and C_S produce relatively similar results, we compare the remaining new measures only with `Bz`.

See Figure 7.4. As we can see, F_2 and F_3 are very similar, and excluding case C1, they recognize—in the same way than the previous measures—the same actors as the most relevant. In contrast to C_S, the measures F_2 and F_3 are lower from one case to the next one, and the variability of the results is lower in general.

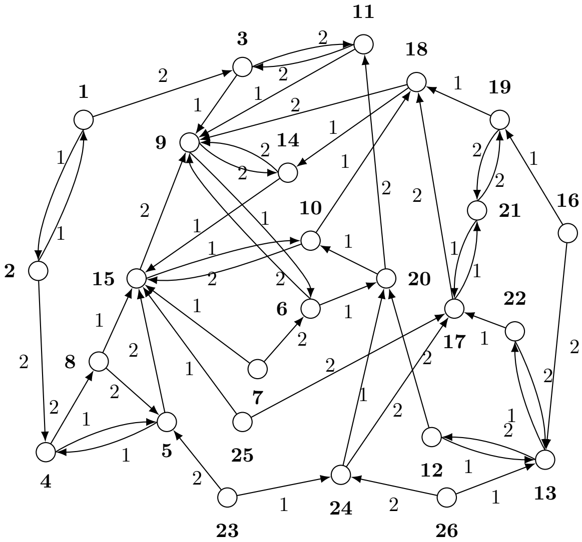


Figure 7.6: Original network for dining-table partners.

In Figure 7.5 we compare the traditional measures with Bz and F_2-C2. The results for the remaining measures are not so much different than for the latter two. Leaving aside the case 1 for the new measures, there are similarities between traditional measures and new ones. In the same way than traditional measures, the most central monkey for Bz, SS, C_S, F_2 and F_3 is 3. The second score is for monkey 12, except for Bz-C3 and SS-C3, where it is replaced by monkeys 13 and 15. Third score is for monkeys 13 and 15, except for C_S-C4, F_2-C4 and F_3-C4, where they are replaced by monkey 8. Finally, as expected, for almost every case the less central non-isolated monkey is monkey 9, except for Bz-C2, SS-C2 and C_S-C2, in which case is monkey 5.

7.2.2 Dining-Table Partners

A second real network is illustrated in Figure 7.6. It was firstly provided by a sociometric research [185] and, years later, it was also used to be handled and displayed by a computational application [57].

It represents the companion preferences of 26 girls living in one cottage at a New York state training school. Each girl was asked about who prefers as dining-table partner in first and second place. Thus, each girl is represented

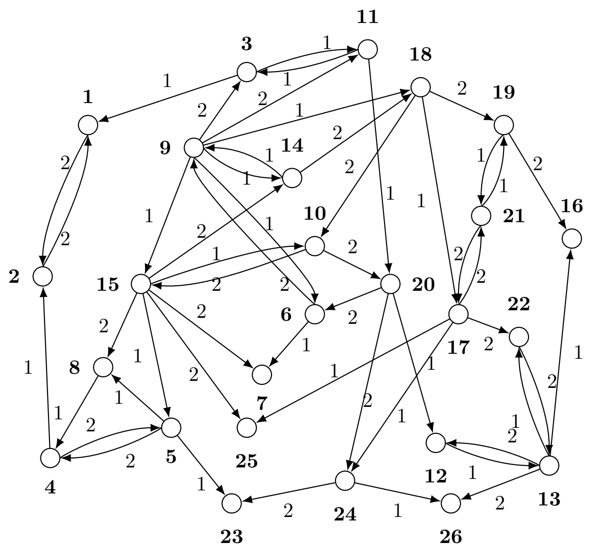


Figure 7.7: Influence graph from the original dining-table partners network.

by a vertex, and there is a directed edge (i, j) per each girl i preferring girl j as dining-table partner. Every vertex has an outdegree equals 2: edges with weight 1 denote the first option of the girl, and edges with weight 2 denote her second option.

We could assume that a girl has some ability to influence over another one which has chosen her as a partner. Figure 7.7 shows the corresponding network of this influence game, reversing each arc (i, j) by (j, i), so that a vertex points to another when the first one has some influence over the second one. Further, the weights of the edges must be exchanged, so that an original edge (i, j) with weight 1 now becomes in an edge (j, i) with weight 2, and vice versa. This is due to a girl has more influence over another one if that other has chosen her in first place rather than in second place. Of course, now every vertex has an indegree equals 2: one edge with weight 1 and the other with weight 2.

Instead of the Monkeys' interaction network, here there are no isolated vertices, but we can still obtaining scores for `Bz` and `SS` measures equals zero. For instance, see the scores for `Bz`-C1 and `SS`-C1 on Figure 7.8.

A common voting system is the one of *absolute majority*, in which an option wins whether it has more than the half of the votes. According to this idea, we consider for our experiments a quota $q = 14$, so that a team is

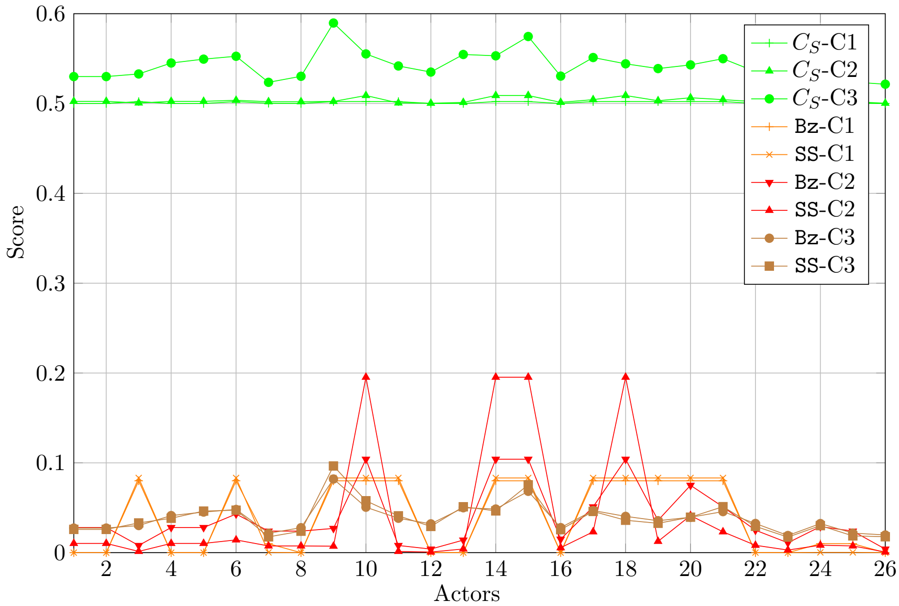


Figure 7.8: Comparisons between `Bz`, `SS` and C_S measures for every case in Dining-table partners network.

considered successful if and only if through its spread of influence, this team achieves to convince most of the girls. Moreover, for every vertex $i \in V$ we consider the following reasonable labeling functions:

- Case 1 (C1): Minimum influence required to convince, $f(i) = 1$.
- Case 2 (C2): Average influence required to convince, $f(i) = 2$.
- Case 3 (C3): Maximum influence required to convince, $f(i) = 3$.

The comparison between some traditional measures and the new ones are shown on Table 7.2. We avoid the indegree centrality C_D^- because since the indegree for each vertex is always 2, it does not provide any relevant information.

Analogously to the previous section, `Bz`-C1, `SS`-C1, C_S-C1 and C_E have several vertices with the same ranking, but when the required influence to convince increases, the values of the measures are more diverse for the power indices and satisfaction centrality. On the other hand, the measures `Bz`-C3, `SS`-C3 and C_S-C3 have the same values only for girls 1 and 2. Indeed, in

			C1						C2						C3					
Node	C'_{D^+}	C'_C	Bz	SS	C'_E	C'_S	F'_2	F'_3	Bz	SS	C'_E	C'_S	F'_2	F'_3	Bz	SS	C'_E	C'_S	F'_2	F'_3
1	17-21	20-21	16-26	19-26	13-26	19-26	23-25	24-25	11-14	10-13	5-26	11-14	12-15	11-14	21-22	19-20	7-26	21-22	16-22	17-21
2	17-21	20-21	16-26	19-26	13-26	19-26	23-25	24-25	11-14	10-13	5-26	11-14	12-15	11-14	21-22	19-20	7-26	21-22	16-22	17-21
3	8-16	9	1-12	1-12	1-12	1-12	1-9	1-9	23-24	23-24	5-26	23-24	25-26	25-26	18	14	**1-6**	18	16-22	16
4	8-16	13-14	16-26	19-26	13-26	19-26	17-19	20-22	11-14	10-13	5-26	11-14	12-15	11-14	10	12	7-26	10	7-11	10
5	5-7	13-14	16-26	19-26	13-26	19-26	17-19	20-22	11-14	10-13	5-26	11-14	12-15	11-14	9	8	7-26	9	**3-6**	4
6	8-16	7	1-12	1-12	1-12	1-12	1-9	1-9	8	8	5-26	8	9	9	6	6	7-26	6	7-11	7
7	22-26	22-26	13-15	13-15	13-26	13-15	21	16	17-19	16-18	5-26	17-19	18-22	18-22	24	25	7-26	24	23-26	23-26
8	17-21	15	16-26	19-26	13-26	19-26	17-19	20-22	17-19	16-18	5-26	17-19	18-22	18-22	20	22	7-26	20	16-22	22
9	**2-3**	**1**	1-12	1-12	1-12	1-12	1-9	1-9	15	19	5-26	15	11	15	**1**	**1**	**1-6**	**1**	**1**	**1**
10	8-16	5	1-12	1-12	1-12	1-12	1-9	1-9	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	**3**	**3**	**1-6**	**3**	**3-6**	**3**
11	8-16	8	1-12	1-12	1-12	1-12	1-9	1-9	23-24	23-24	5-26	23-24	25-26	25-26	13	10	**1-6**	13	12-15	13
12	17-21	17-18	16-26	16-18	13-26	16-18	13-15	13-15	25-26	25-26	5-26	25-26	18-22	18-22	17	18	7-26	17	16-22	17-21
13	4	16	16-26	16-18	13-26	16-18	13-15	13-15	21	21	5-26	21	10	10	4	5	7-26	4	**3-6**	5
14	8-16	**3**	1-12	1-12	1-12	1-12	1-9	1-9	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	5	7	7-26	5	7-11	8-9
15	**2-3**	**2**	1-12	1-12	1-12	1-12	1-9	1-9	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	**2**	**2**	**1-6**	**2**	**2**	**2**
16	22-26	22-26	16-26	19-26	13-26	19-26	16	18	20	20	5-26	20	23	23	19	21	7-26	19	16-22	17-21
17	**1**	10	1-12	1-12	1-12	1-12	10-12	10-12	6-7	6-7	5-26	6-7	5-6	5-6	7	9	7-26	7	**3-6**	6
18	5-7	4	1-12	1-12	1-12	1-12	1-9	1-9	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	**1-4**	11	13	7-26	11	7-11	11
19	8-16	12	1-12	1-12	1-12	1-12	10-12	10-12	9	9	5-26	9	16	16	14	15	7-26	14	12-15	14-15
20	5-7	6	1-12	1-12	1-12	1-12	1-9	1-9	5	5	5-26	5	7	7	12	11	7-26	12	7-11	8-9
21	8-16	11	1-12	1-12	1-12	1-12	10-12	10-12	6-7	6-7	5-26	6-7	5-6	5-6	8	4	**1-6**	8	12-15	12
22	17-21	17-18	16-26	16-18	13-26	16-18	13-15	13-15	16	15	5-26	16	8	8	15	17	7-26	15	16-22	17-21
23	22-26	22-26	16-26	19-26	13-26	19-26	26	26	22	22	5-26	22	24	24	26	26	7-26	26	23-26	23-26
24	8-16	19	13-15	13-15	13-26	13-15	20	17	10	14	5-26	10	17	17	16	16	7-26	16	12-15	14-15
25	22-26	22-26	13-15	13-15	13-26	13-15	22	19	17-19	16-18	5-26	17-19	18-22	18-22	23	23	7-26	23	23-26	23-26
26	22-26	22-26	16-26	19-26	13-26	19-26	23-25	23	25-26	25-26	5-26	25-26	18-22	18-22	25	24	7-26	25	23-26	23-26

Table 7.2: Comparison of centrality measures for the influence game version of the Dining-table partners network. The three more central values of some measures are highlighted in bold. For influence games we consider a quota $q = 14$.

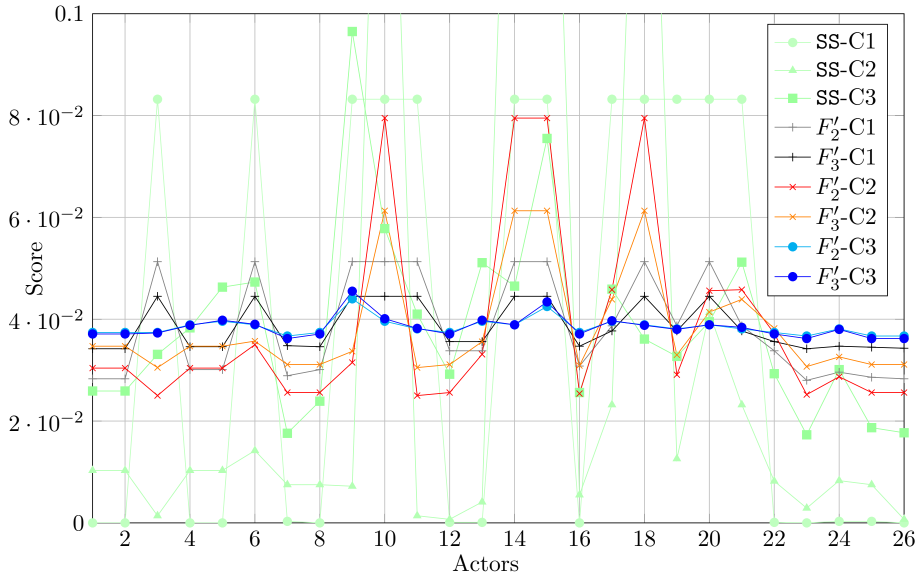


Figure 7.9: Comparisons between SS, F_2 and F_3 measures for every case in Dining-table partners network.

this sense girls 1 and 2 are equivalent for all the considered measures. Furthermore, as in the previous network the measures Bz, SS and C_S produce very similar rankings and scores. See Figure 7.8. For the remaining comparisons we use the measure SS, because for some of the three studied cases the results of the previous measures are similar to this one.

In Figure 7.9 we can see that the higher variability is obtained for the Case 2. Note that F_2 and F_3 are very similar too. Moreover, F_2-C1 and F_3-C1 have a similar behavior than SS-C1 and Bz-C1, although they present important differences on the actors 17, 19 and 21. To compare with the remaining measures, we consider in Figure 7.10 as representatives the measures SS and F_2. In that figure we can see that for this network different measures provide different centrality criteria.

The most central girls are highlighted in Table 7.2. Observe that girl 15 has a high centrality in all measures, but the high centrality of girl 9 depends of the considered case: for Case 1 and 3, girl 9 has a high centrality, but in Case 2 is far less central. Note that girl 13 is fairly central in C_D^+ and for the Case 3. For the other two cases, in despite of its high outdegree, only exist paths from this vertex to another four, which is a severe restriction for the new measures.

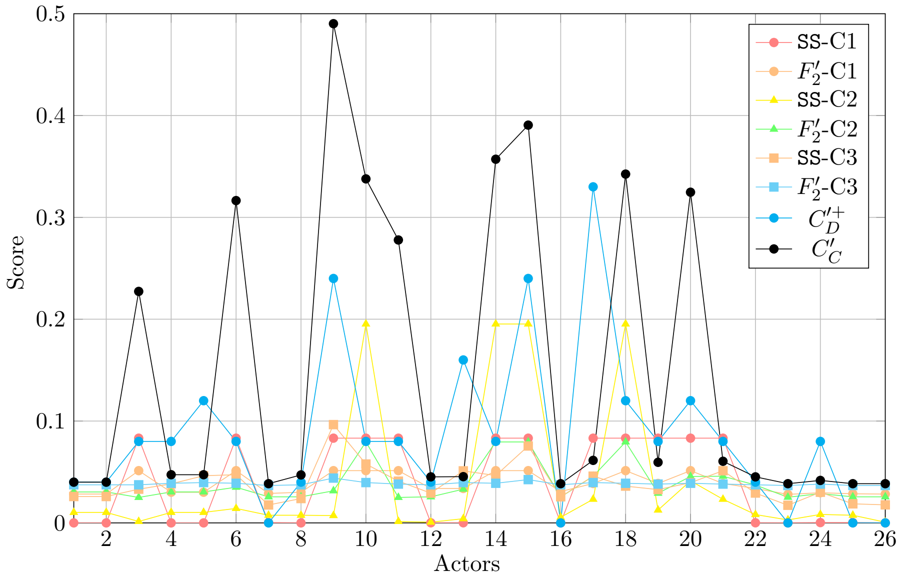


Figure 7.10: Comparisons between SS, F_2 and some traditional measures.

Additionally, unlike traditional measures, girl 10 plays an important role in our new measures. This is because in despite of neither having a high outdegree nor having too short paths to more distant vertices, she plays an essential role in the spread of influence to convince distant sets of girls, which in turn have no convincing power over her.

7.2.3 Student Government Discussion

Out third case of study considers the social network illustrated at the top of Figure 7.11. This network represents the communication interactions among different members of the Student Government at the University of Ljubljana in Slovenia. Data were collected through personal interviews in 1992 [117], being used later [57].

Every directed edge is a communication interaction and all of them have the same weight equals 1. Each vertex is a member of the Student Government, and unlike the previous cases, here vertices are labeled beforehand: There are three *advisors* labeled 1, seven *ministers* labeled 2, and one *prime minister* labeled 3. Note that these labels are not related with the spread of influence.

We slightly modified this network to obtain the influence graph at the bottom of Figure 7.11. We assume that every communication interaction is

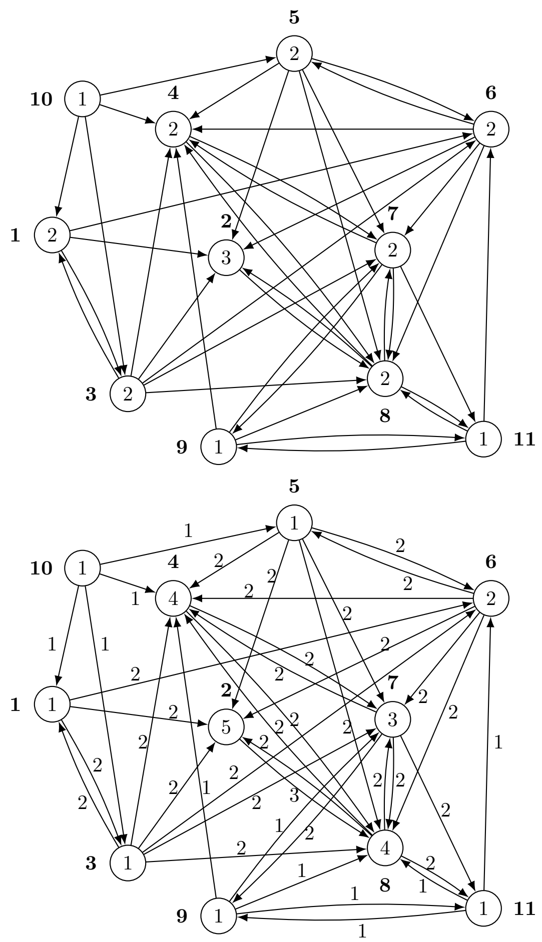


Figure 7.11: Student Government discussion network (up) and the adaptation to influence graph (down).

an attempt to influence over another student, and the capacity to influence depends on the student's position. For instance, the advise of a prime minister does not have the same effectiveness—marked with weight 3—than the advise of an advisor—marked with weight 1. Furthermore, since the labels of the vertices should represent the difficulty of each student $i \in N$ to be influenced according to their position in the Student Government, they have been changed by the following values:

- $f(i) = 1$ if i is an advisor.
- $f(i) = \lceil deg^{-}(i)/2 \rceil$ if i is a minister.

Scores

Node	C_D^-	C_D^+	C_C	C_B	Bz	SS	C_E	C_S	F_2	F_3
1	0.2	0.3	**0.357**	0.009	**0.164**	**0.176**	**0.91**	**0.516**	**0.110**	**0.100**
2	0.5	0.1	0.200	0.004	0.154	0.076	0.45	0.515	0.084	0.086
3	0.2	**0.6**	**0.435**	0.023	**0.164**	**0.176**	**0.91**	**0.516**	**0.110**	**0.100**
4	**0.7**	0.2	0.208	0.015	0.005	0.009	0.55	0.500	0.079	0.085
5	0.2	**0.5**	0.238	0.022	**0.164**	**0.176**	**0.91**	**0.516**	0.094	0.091
6	0.4	**0.5**	0.238	**0.127**	**0.164**	**0.176**	0.82	**0.516**	0.094	0.091
7	**0.6**	**0.4**	0.227	0.110	0.005	0.009	0.64	0.500	0.079	0.085
8	**0.8**	**0.4**	0.227	**0.159**	0.005	0.009	0.55	0.500	0.079	0.085
9	0.2	**0.4**	0.227	0.007	0.005	0.009	0.82	0.500	0.076	0.085
10	0.0	**0.4**	**0.556**	0.000	**0.164**	**0.176**	**0.91**	**0.516**	**0.119**	**0.108**
11	0.3	0.3	0.227	**0.122**	0.005	0.009	0.82	0.500	0.076	0.085

Rankings

Node	C_D^-	C_D^+	C_C	C_B	Bz	SS	C_E	C_S	F_2	F_3
1	7-10	8-9	**3**	8	6-10	7-11	**1-4**	**1-5**	**2-3**	**2-3**
2	4	11	11	10	11	6	11	6	6	6
3	7-10	**1**	**2**	5	6-10	7-11	**1-4**	**1-5**	**2-3**	**2-3**
4	**2**	10	10	7	**1-5**	**1-5**	9-10	7-11	7-9	7-11
5	7-10	**2-3**	4-5	6	6-10	7-11	**1-4**	**1-5**	4-5	4-5
6	5	**2-3**	4-5	**3**	6-10	7-11	5-7	**1-5**	4-5	4-5
7	**3**	4-7	6-9	4	**1-5**	**1-5**	8	7-11	7-9	7-11
8	**1**	4-7	6-9	**1**	**1-5**	**1-5**	9-10	7-11	7-9	7-11
9	7-10	4-7	6-9	9	**1-5**	**1-5**	5-7	7-11	10-11	7-11
10	11	4-7	**1**	11	6-10	7-11	**1-4**	**1-5**	**1**	**1**
11	6	8-9	6-9	**2**	**1-5**	**1-5**	5-7	7-11	10-11	7-11

Table 7.3: Comparison of centrality measures for the adapted version of the Student Government discussion network. The more central values of the measures are highlighted. We consider for influence games a quota $q = 6$.

- $f(i) = deg^-(i)$ if i is the prime minister.

Note that this labeling function provides a finer classification of the types of actors in the network. Moreover, for this network we consider a majority influence required to win, i.e., a quota $q = 6$.

Table 7.3 shows the results of the centrality measures corresponding to the adapted network of Figure 7.11.

Note that for this network, traditional measures provide different rankings. This can also be seen in Figure 7.12. In fact, none of the most central actors for C_C and C_B coincide, and while the most central vertex for C_C is the advisor 10, this is the less central according to C_B. This is because vertex 10 has a high accessibility to all other vertices, but it is not a good intermediary for connecting distant vertices through paths. In turn, regarding the new centrality measures, as usual `Bz` and `SS` are similar, as well as F_2 and F_3. Moreover, these last two measures are also similar to C_S. For this network, note also that C_B and F_2 produce the highest hierarchization

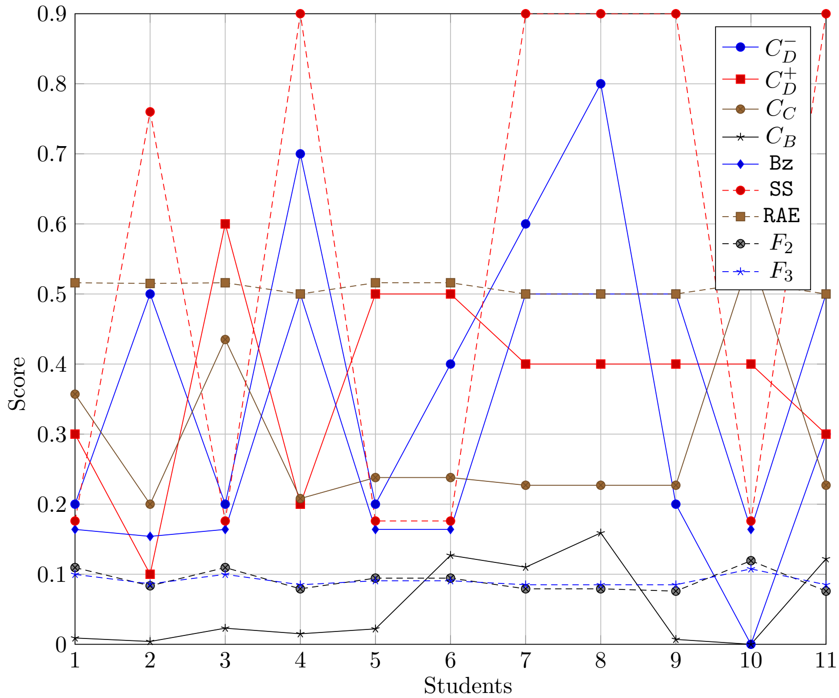


Figure 7.12: Comparisons between both new and traditional centrality measures for Student Government discussion network.

of the actors.

In general, note that the prime minister—node 2—does not have a high centrality. Regarding the power indices and the measures F_2 and F_3, this is because this actor has only been reported with minister 8, on which may exert some influence, but he has received many interactions—which we can understand as comments, advice, suggestions, etc.—from other ministers and advisors, exerting a strong influence on him. On the other hand, for C_E the low centrality of the prime minister is explained because he can not influence any other member by himself, and at the same time his activation requires the most highest effort.

7.2.4 Facebook interactions

We finish this chapter by considering a bigger social network, formed by $1,899$ students at University of California, Irvine. These students, interconnected by Facebook in the period between April to October 2004, shared $59,835$ online messages in total, represented by $20,296$ directed edges. The

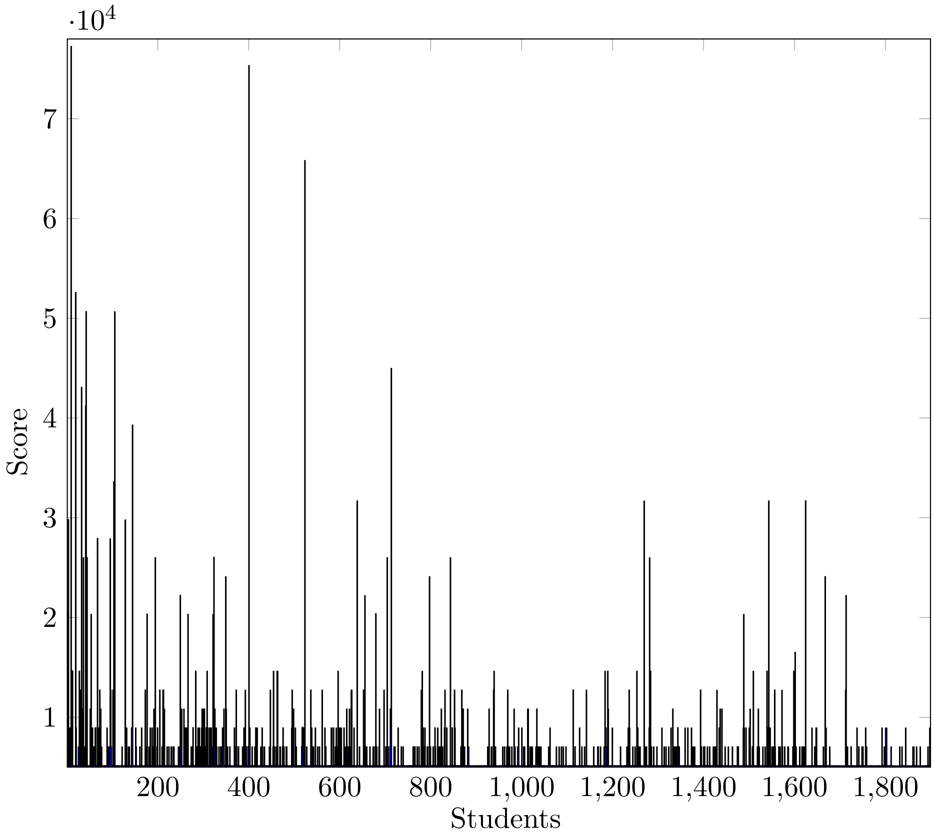


Figure 7.13: Centrality scores for the Facebook interactions network, by considering the F_2 measure with $p = 0.5$.

weight of each edge represents the number of messages sent from one student to another. This network can be downloaded from [197]. It was presented in [198] and also used in [200, 206] for different purposes than those discussed in this chapter.

In this case, for the labeling function we consider a parameter $p \in]0, 1]$ such that for each vertex $i \in V$,

$$f(i) = \left\lfloor p \cdot \sum_{j \in P(i)} w(j, i) \right\rfloor$$

where $w(j, i)$ is the weight of the edge (j, i) in the network.

The considered measures were implemented in C++. In Figure 7.13 we show the score results for the 2-influence centrality measure by considering

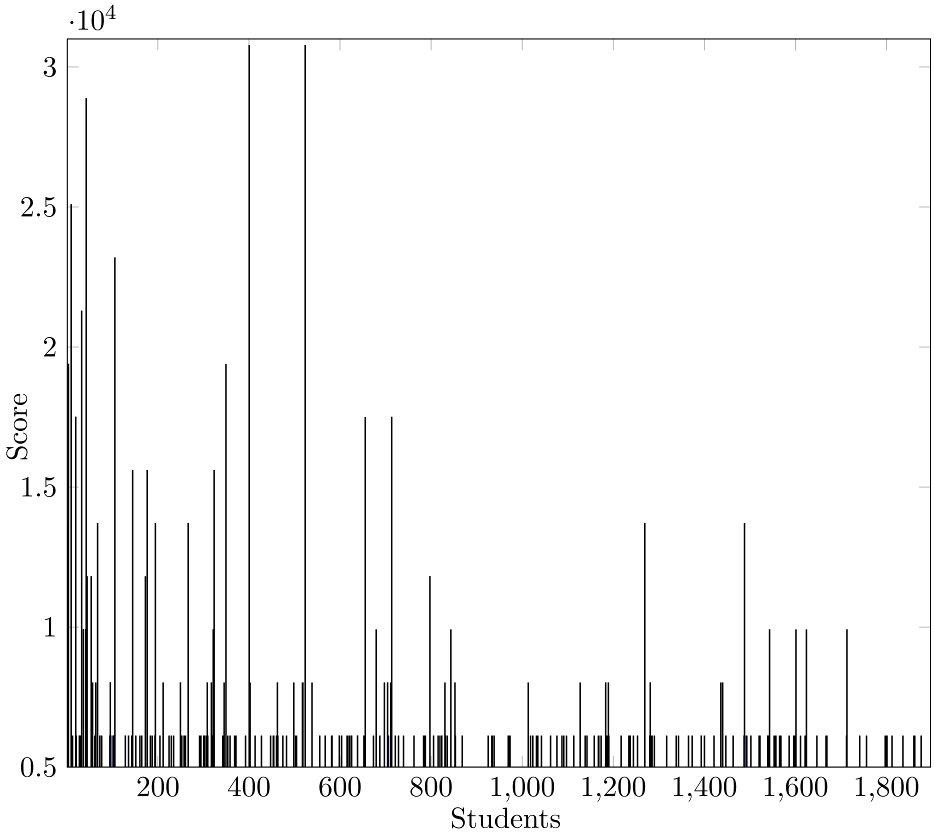


Figure 7.14: Centrality scores for the Facebook interactions network, by considering the F_2 measure with $p = 1$.

a parameter $p = 0.5$. For an Intel Core 2 Duo processor clocked at 3.16 GHz, it took 20 minutes and 23.834 seconds of computation. In this case, the most central vertex is 9 with a score $F_2(9) = 77,241$; the second one is vertex 400 with $F_2(400) = 75,330$, and the third one is vertex 523, with $F_2(523) = 65,810$.

Analogously, for $p = 1$ it took 14 minutes and 30.354 seconds of computation. The results are shown in Figure 7.14. In this case, the most central vertex is 523, with $F_2(523) = 30,769$; the second one is vertex 400 with $F_2(400) = 30,767$, and the third one is vertex 42 with $F_2(42) = 28,867$.

Note that measures based on the quota q for influence games—like `Bz`, `SS`, C_E or C_S—can not be computed in polynomial time. Indeed, none of these measures could obtain any result even in several weeks of computation.

Part IV

Conclusions

Concluding Remarks

In this thesis we have studied the computational aspects of many problems of simple games. By introducing the influence games, we have been able to apply several properties and problems from game theory, voting theory and decision theory into other fields like social network analysis, multi-agent systems and social choice.

Chapters 1 and 2 were devoted to explain the main known concepts related to this thesis.

In Chapter 3 we have summarized the most usual forms of representation for simple games, regular games and weighted games. We examined and obtained results about the complexity of the conversion problem related to these representation forms. In some cases where the conversion problem cannot be computed in polynomial time, we designed algorithms to solve this problem with polynomial-delay. Our results show that MWF, PCBF and FCBF representations are very compatible between them and with their relationship with the others. However, bear in mind that FCBF—introduced for regular games—will always be the most succinct of these forms of representation. On the other hand, BDDF seems to have a similar performance than SWF—recall that the second one can only be used to represent regular games. Indeed, both BDDF and SWF can be useful because of their succinctness, although the games in SWF and BDDF may also have an exponential size in terms of n, i.e., the number of players. In turn, weighted games have a more succinct form of representation, namely WRF, based on a $n+1$-vector of integers. However, it is known that there are weighted games whose weighted representation requires that $\max_{i\in N}\{w_i\}$ to be $(n+1)^{(n+1)/2}/2$ [208]. Furthermore, converting a weighted game from any of the considered forms of representation to WRF turns out to be polynomial, but converting from WRF to any other representation requires exponential time and can not be done in polynomial time. We conjecture that WRF⇝SWF can also be solved with polynomial-delay. Also recall that WRF is not univocal, in the sense that two weighted representations may represent the same game. In this vein, another interesting problem that is currently being studied by several researchers is to find the minimal integer representations of weighted games, i.e., the weighted representation given by the vector with the minimum integer numbers [22].

In Chapter 4 we started with a survey of the main computational complexity results regarding many properties, parameters and solution concepts for simple games. Although the list pretends to cover a wide spectrum of problems in simple game theory, it is not exhaustive. In particular, there exist several other power indices that have not been studied in this thesis and could be considered in the future, such as the Holler index, the Deegan-Packel index, the Johnston index, among many others. Several comparisons and open problems between power indices are presented in [21]. Another possible line of work is to find approximations of the values of power indices, by using probabilistic techniques [22].

In this chapter we have solved some relevant open problems, such as the decisive problem for simple games, regular games, and weighted games, which is equivalent to the duality problem of hypergraphs or monotone Boolean functions. For simple games in MWF and regular games in SWF, it remains open to show whether the IsDecisive problem is polynomial-time solvable or not. This problem remains open since 1996 [84] from the context of hypergraphs, Boolean functions and propositional logic. Furthermore, we solved the problem of computing the width parameter for a simple game in MWF, and we introduced the problem of deciding whether a player is dummy or not in a regular game in SWF. In general, most of the open problems are related to regular games in SWF, as for instance the IsDummy problem. These games have been studied enough in MWF, where many problems can be solved in polynomial time. However, as we have seen in this chapter and the previous one, it is also relevant to find out the computational complexity of the problems for regular games in their most succinct representation form.

These results, together with the ones in the previous chapter, can be useful for having at hand a comparative of the different usual forms of representation. When we want to solve some decision problem on simple games, if we know how the problem behaves for some particular form of representation, then we can know under which other forms of representation the problem has similar complexity. For instance, since the IsDecisive problem belongs to QP for simple games in MWF—see Theorem 4.3—now we deduce that it is also in QP for simple games in PCBF and for regular games in FCBF.

At the end of this chapter we proposed a way to explicitly enumerate decisive regular games. The ordering of the games within a lattice seems helpful to list without repetition the various elements of a class. The proposed enumeration algorithm is presented as a first line of research that could be explored in depth in future work. It remains open to prove the correctness of the algorithm, and attempting to parallelize the proposed algorithm, in order to improve its performance.

Chapter 5 is perhaps the most innovative contribution of the thesis. Inspired in the threshold model of spread of influence, the influence games represent winning coalitions as successful teams that can convince enough actors to perform a task. This approach reveals the importance of the influence between some players over others in order to form successful teams. Influence games are a new line of research that relates cooperative game theory with other topics like multi-agent systems or social networks.

Influence games have been introduced and extensively studied in the thesis. For these games we have determined the computational complexity of the main computational problems considered in Chapter 2. As a succinct way to represent simple games, several problems are hard in the context of influence games. However, there are also many properties of players and coalitions that are polynomial-time solvable. Moreover, there exist hard problems that for interesting subfamilies of influences games can be solved in polynomial time. It is interest to analyze influence games under other spreading models, in particular the linear threshold model with random thresholds. Another possible area of future research is to study both the counting problem and enumerating problem for influence games. There are many works related to count graphs that could help to this purpose. Furthermore, it would be of interest to determine the complexity of the conversion problems related to families of simple games defined through graphs, such as influence games or other families such as the ones studied in [44].

The last two chapters of this thesis emerge as applications of influence games. In Chapter 6 we have provided several collective choice models based on variations of influence games. For mediation systems, based on influence games restricted to star influence graphs, we proved that many problems become polynomial-time solvable, since they can be reduced to the problem of solving some systems of linear inequalities.

We also studied the satisfaction measure for the opinion leader-follower systems—OLF systems—and other influence systems inspired in this one. Interestingly enough, the satisfaction measure coincides with the well established Rae index, that is closely related to the Banzhaf value over the set of monotonic decision functions which can be casted as characteristic functions of simple games. We proved that even for influence systems on bipartite digraphs, the satisfaction measure is hard, in the sense that it cannot be computed by a sub-exponential time algorithm, unless the #P-complete #3-SAT problem could be computed in sub-exponential time. Besides this hardness result, we provided subfamilies of influence systems where the computation of the measure becomes polynomial-time solvable. We have shown that RAE and BVAL can be solved in polynomial time for oblivious strongly mediated influence system and oblivious star influence systems. It will be of interest to know if the problems can be solved in polynomial time for those families of graphs when we use the non-oblivious decision function. In this vein, it remains open to study the computational complexity of other measures for OLF systems and their generalizations through mediators. Due to the relationship between these models and simple games, all the power indices introduced in the context of simple games can also be studied here. As the satisfaction measure, power indices generally have a high complexity. Nevertheless, it is possible that under some restrictions, these measures may be computed in polynomial time. A first measure that could be studied is the *power* measure defined in [255, 256] because it is strongly related with satisfaction. However, the power measure has additional difficulties, since it concerns both the set of minimal winning coalitions $\mathcal{W}^m$ and the set of maximal losing coalitions $\mathcal{L}^M$ of simple games. All in all, for any actor i in a collective decision making model $\mathcal{M}$, *power* measure (POW) can be defined by the following:

$$\begin{aligned} \text{POW}(i) &= |\{x \in \{0,1\}^n \mid i \text{ has a swing in } C_{\mathcal{M}}(x)\}| \\ &= |\mathcal{W}_i^m| + |\mathcal{L}_{-i}^M| \end{aligned}$$

where $\mathcal{W}_i^m = \{X \subseteq V \mid i \in X, (X \cap N) \in \mathcal{W}^m, (X \setminus \{i\} \cap N) \in \mathcal{L}^M\}$ and $\mathcal{L}_{-i}^M = \{X \subseteq V \mid i \notin X, (X \cap N) \in \mathcal{L}^M, (X \cup \{i\} \cap N) \in \mathcal{W}^m\}$. While satisfaction is related to the Chow parameters, power measure is related with the *Holler index* [119], also studied in the context of simple game theory.

Additionally, in this chapter we provide an axiomatization for the satisfaction measure for OLFM systems, thus generalizing the corresponding results for OLF systems proposed in [256]. OLFM systems are OLF systems that incorporate mediators, allowing the presence of several layers of influence, and hence establishing a more general hierarchy among the different actors. Equation (6.3) suggests that through small modifications, such as a change in the normalization of Definition 6.19, it is also possible to define an axiomatization of the Banzhaf value for OLFM systems. In the same vein, note that there exist other axiomatizations of the Banzhaf value and other power indices for simple games [66, 21]. It also remains open to determine if the POW measure admits an axiomatization for OLF systems.

Our results show that there are many social models whose characteristics can be efficiently addressed from a computational point of view. One interesting direction for future work is to identify real social systems, whose behavior could be analyzed with the techniques developed in this thesis. For instance, by allowing connections among actors that belong to not immediately consecutive layers. A first approach could be dealing with the star mediation influence games proposed in Section 6.1.3. The interesting fact of this model is that despite it breaks the layered structure of the graphs, it is still simple, because the fraction value only affects the mediator.

Finally, in Chapter 7 we define new centrality measures for social networks represented by influence games. We have shown that weighted and labeled social networks can represent influence games, and thus we can define several measures over the set of actors in order to determine the relevance of these actors in the network. Our experimental results do not contradict the relevance criteria provided by traditional centrality measures. In some cases, the resulting hierarchization of the players between the new centrality measures and the traditional ones are almost equal. However, there are cases where the results are quite different. This is the case of the Student

Government discussion network studied in Section 7.2.3. Thus, the new proposed centrality measures provide new approaches and insights for social network analysis. Moreover, the new centrality measures defined here can be naturally used for edge-labeled and vertex-labeled directed graphs, a feature that is not supported by the most usual measures of centrality [31].

Influence games allow us to consider any power index as centrality measure. However, the biggest problem about some of these measures is their high computational complexity. To overcome this difficulty, we have proposed some alternative measures that consider influence graphs and they can be computed very efficiently for social networks with a huge number of actors. In this line, we proposed in Definition 7.5 the family of k-influence centrality measures. It could be interesting to study other possibilities. For instance, let $d(i, j)$ be the size of the shortest path from the vertex i to j, then a measure could be based on the influence of the set of neighbors at a distance k, i.e.,

$$F_k^N(i) = \sum_{\{j \in V \mid d(i,j)=k\}} |F(\{i, j\})|.$$

Another interesting line for future research is the comparison of the proposed centrality measures with others more related with flow networks, such as the flow betweenness and the flow closeness of Definition 7.2, or the recently defined flow-cost betweenness centrality measure [104]. An example of social network used to compare flow measurements is the Iranian Government's network [104]. In addition, there are several other repositories of social networks that could be used as cases of study [197, 239].

Part V

Appendices

Appendix A

List of decisive regular games

The following Table A.1 shows all the decisive regular games in SWF from $n = 1$ until $n = 7$. They were obtained by implementing the ideas of Section 4.4.2. Consider $N = \{n, \ldots, 1\}$.

n	decisive regular games in SWF						
1	1						
2	10						
3	011	100					
4	1001 0111	0110	1000				
5	00111	01011 11000	01101 10100 10011	01111 10001	10010 01110	01100	10000
6	001111 100011	001111 011001	001111 100101 011100	001111 101001 100110 011010	010110 001111 110001	001110	100011 010111 110000
	011001 010111 110000	010111 100101 011100 110000	100110 011010 010111 110000 101001	010110 110000	100011 011011 101000	101000 100101 011100 011011	100111 101000 011001
	011010 101000 100110	100001 011111	100100 100011 011101	011110 100010	100100 011100	011000	100000
7	1000000	0110000	1001000 0111000	1000100 0111100	1010000 0110100 1001100	1100000 0101100	0011100

1000010 0111110	1010000 0110010 1001110	0111000 1010000 1001010 0110110	1001000 0111010 1000110	0110100 1001100 1100000 1010010 0101110	0101100 1100010 0011110	0110010 1100000 0101110
1010000 0110110 1000110	0111000 1001010 1100000 0101110	0110100 1001100 1010010 0011110	0110010 0011110	1000110 1100000 0101110	0111000 1001010 0011110	1000110 0011110
1000001 0111111	1010000 0110001 1001111	0111000 1010000 1001001 0110111	1001000 0111001 1000111	0111100 1001000 1000101 0111011	1000100 0111101 1000011	0110100 1001100 1100000 1010001 0101111
1010000 1001100 0111000 0110101 1001011	1010000 0110100 1001101 0110011	0101100 1100001 0011111	1100000 1001100 0110100 0101101 1010011	0101100 0011101 1100011	0110010 1001110 1100000 0101111	1010000 1001010 0110110 0111001 1000111
0111000 1001010 0110110 1100000 1010001 0101111	0111010 1000110 1010000 1001001 0110111	1001100 1100000 1010010 0101110 0111000 0110101 1001011	0110100 1100000 1010010 0101110 1001101 0110011	0110100 1001100 1010010 0101110 1100001 0011111	1100010 0011110 1001100 0110100 0101101 1010011	0110010 0101110 1100001 0011111
0110110 1000110 1100000 1010001 0101111	1001010 1100000 0101110 0111001 1000111	0111000 1001010 0101110 1100001 0011111	1001100 1010010 0011110 0111000 0110101 1001011	0110100 1010010 0011110 1001101 0110011	1000110 0101110 1100001 0011111	1001010 0011110 0111001 1000111
0110001 1100000 0101111	1010000 1001001 0110111 0111001 1000111	0111000 1001001 1100000 0101111	1001000 0111011 1000011	0111100 1000101 1010000 0110111	1001100 1100000 1010001 0101111 0111000 0110101 1001011	0110100 1100000 1010001 0101111 1001101 0110011
0110100 1001100 1010001 0011111	1010000 0111000 1001011 0110011	1010000 1001100 0110101 1000111	1100001 0011111 1001100 0110100 0101101 1010011	1100000 0110100 0101101 0110011	1100000 1001100 0101101 0111000 1001011	0011101 1001100 0110100 1010011

0110010 1001110 1010001 0011111	1001010 0110110 0111001 1000111 1100000 1010001 0101111	0111000 1001010 0110110 1010001 0011111	0111010 1000110 1001001 1100000 0101111	1100000 1010010 0101110 0111000 1001011 0110011	1001100 1010010 0101110 0111000 0110101 1001011 1100001 0011111	1001100 1100000 1010010 0101110 0110101 1000111
0110100 1010010 0101110 1001101 0110011 1100001 0011111	1100010 0011110 0110100 0101101 0110011	1100010 0011110 1001100 0101101 0111000 1001011	0110110 1000110 1010001 0011111	1001010 0101110 0111001 1000111 1100001 0011111	1010010 0011110 0111000 1001011 0110011	1001100 1010010 0011110 0110101 1000111
0110001 0011111	1001001 0111001 1000111 1100000 0101111	0111000 1001001 0011111	1000011 1010000 0110111	0111100 1000101 1100000 0101111	1100000 1010001 0101111 0111000 1001011 0110011	1001100 1010001 0111000 0110101 1001011 0011111
1001100 1100000 1010001 0101111 0110101 1000111	0110100 1010001 1001101 0110011 0011111	1010000 0110011 1000111	1100001 0011111 0110100 0101101 0110011	1100001 0011111 1001100 0101101 0111000 1001011	1100000 0101101 0110011 1010100 0111000 1001011	1100000 1001100 0101101 1000111
0011101 0110100 0110011	0011101 1001100 0111000 1001011	1001010 0110110 0111001 1000111 1010001 0011111	0111010 1000110 1001001 0011111	1010010 0101110 0111000 1001011 0110011 1100001 0011111	1100000 1010010 0101110 0110011 1000111	1001100 1010010 0101110 0110101 1100001 0011111 1000111
1100010 0011110 0101101 0110011 1010100 0111000 1001011	1100010 0011110 1001100 0101101 1000111	1010010 0011110 0110011 1000111	1001001 0111001 1000111 0011111	1000011 1100000 0101111	0111100 1000101 0011111	1010001 0111000 1001011 0110011 0011111

	1100000 1010001 0101111 0110011 1000111	1001100 1010001 0110101 0011111 1000111	1100001 0011111 0101101 0110011 1010100 0111000 1001011	1100001 0011111 1001100 0101101 1000111	1100000 0111000 0101011	1100000 0101101 0110011 1010100 1000111	0011101 0110011 1010100 0111000 1001011
	0011101 1001100 1000111	1010010 0101110 0110011 1100001 0011111 1000111	1100010 0011110 0111000 0101011	1100010 0011110 0101101 0110011 1010100 1000111	1000011 0011111	1010001 0110011 0011111 1000111	1100001 0011111 0111000 0101011
	1100001 0011111 0101101 0110011 1010100 1000111	1100000 0101011 1011000 1000111	0011101 0111000 1100100 0101011	0011101 0110011 1010100 1000111	1100010 0011110 0101011 1011000 1000111	1100001 0011111 0101011 1011000 1000111	1100000 0100111
	0011101 1100100 0101011 1011000 1000111	0111000 0011011	1100010 0011110 0100111	1100001 0011111 0100111	1011000 1000111 0011011	0011101 1100100 0100111	0011011 1101000 0100111
	1110000 0010111	0001111					

Table A.1: List of decisive regular games in SWF from $n = 1$ until $n = 7$.

Appendix B

List of Influence Games

In this appendix we enumerate all the minimal unweighted influence games (G, f, q, N) with $N = V$, for 3 and 4 players. By "minimal" we refer to those influence games that are minimal with respect to their edges, labels and quota. We consider as restrictions that $0 \leq q \leq n$ and for all $i \in N$, $0 < f(i) \leq \delta^-(i)$; if $\delta^-(i) = 0$, we assume $f(i) = 1$.

Unweighted Influence Games for Three Players

Table B.1 shows the digraphs without loops for 3 nodes [222, 114]. Observe that there are 3 graphs disconnected and 13 connected. See the sequences $A000273$ and $A003085$ of [238].

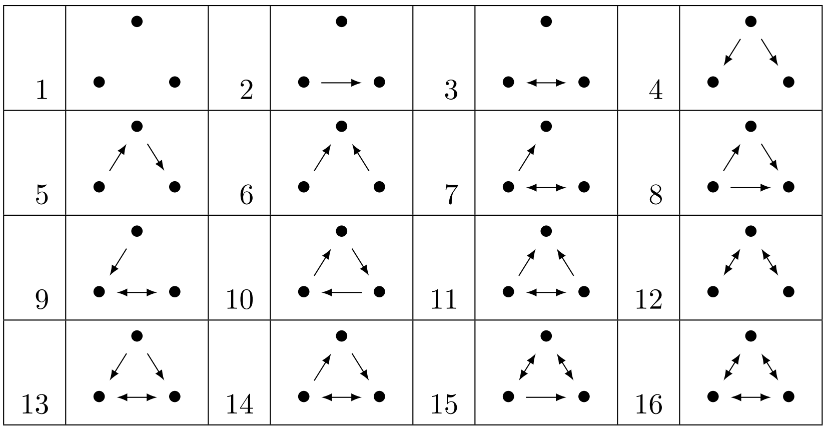


Table B.1: Digraphs without loops with 3 nodes.

The following Table B.2 shows all the minimal unweighted influence games corresponding to the 10 simple games up to isomorphism for $n = 3$. Observe that only the trivial games 1 and 5 cannot be represented by connected graphs.

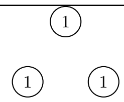
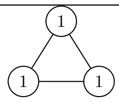
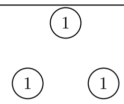
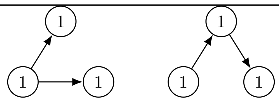
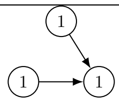
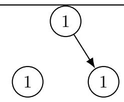
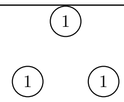
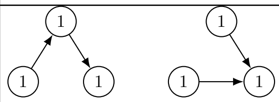
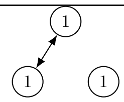
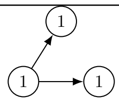
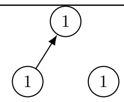
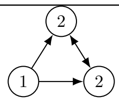
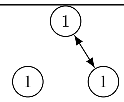
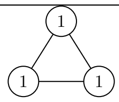
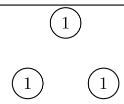
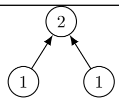
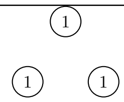
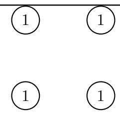
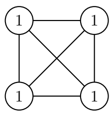
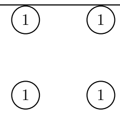
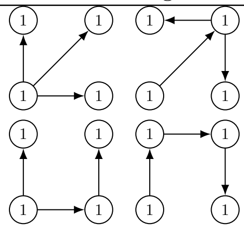
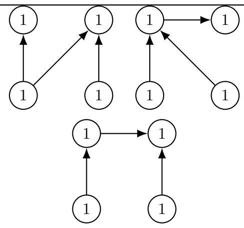
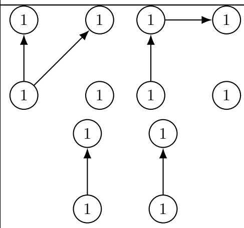
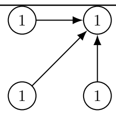
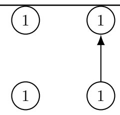
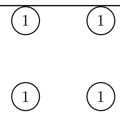
♯	$\mathcal{W}^m$	connected	q	♯ digraph	disconnected	q	♯ digraph
1	∅	–	–	–		4	1
2	000	 any connected graph with 2 edges	0	4, 5, 6		0	1
3	100		3	4, 5	–	–	–
4	110		3	6		3	2
5	111	–	–	–		3	1
6	010 100		2	5, 6		2	3
7	100 011		2	4		2	2
8	101 110		3	13		3	3
9	001 010 100	 any connected graph with 2 edges	1	4, 5, 6		1	1
10	110 101 011		2	6		2	1

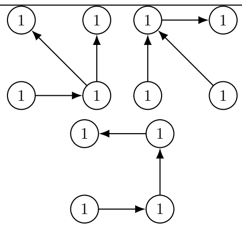
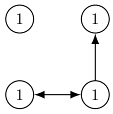
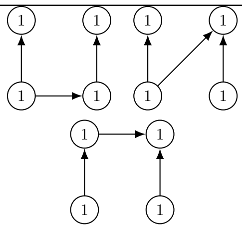
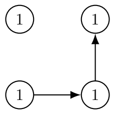
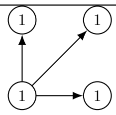
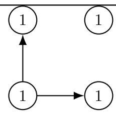
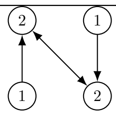
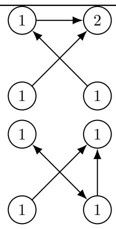
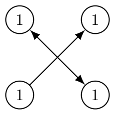
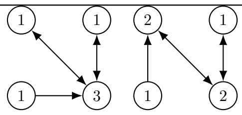
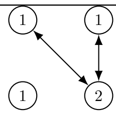
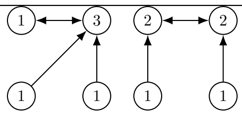
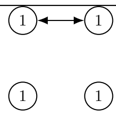
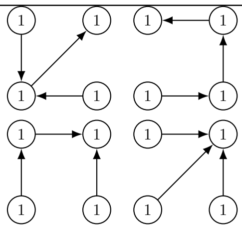
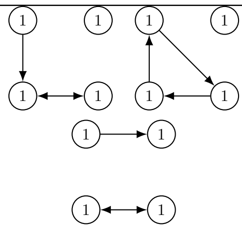
Table B.2: The minimal unweighted influence games corresponding to the 10 simple games up to isomorphism for $n = 3$.

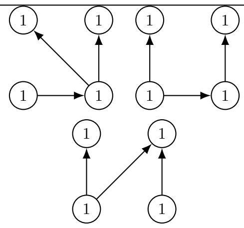
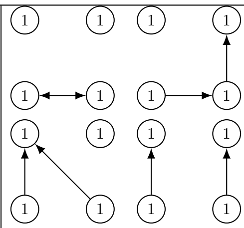
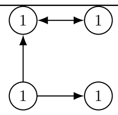
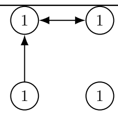
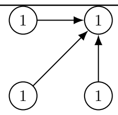
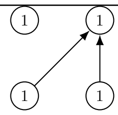
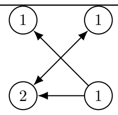
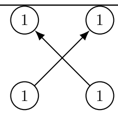
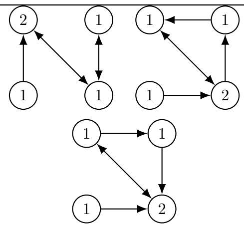
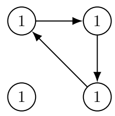
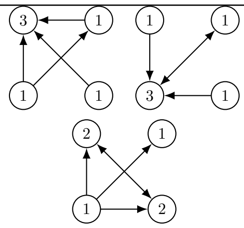
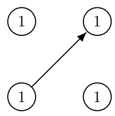
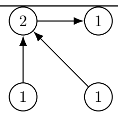
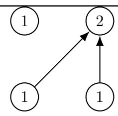
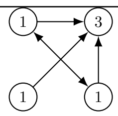
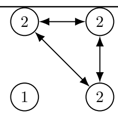
Unweighted Influence Games for Four Players

For $n = 4$ there are 218 directed graphs—including 19 disconnected graphs—and 30 simple games up to isomorphism; 29 of them are listed in [42], and here we add also $(N, \{\emptyset\})$. The number of the digraphs correspond to the codes given in [222].

The following Table B.3 shows all the minimal unweighted influence games corresponding to the 30 simple games up to isomorphism for $n = 4$.

♯	$\mathcal{W}^m$	connected	q	♯ digraph	disconnected	q	♯ digraph
1	$\emptyset$	–	–	–		5	$D21$
2	0000	 any connected graph with 3 edges	0	$D28, D31$ $D32, D33$ $D36, D38$ $D39, D40$		0	$D21$
3	1000		4	$D28, D31$ $D32, D38$	–	–	–
4	1100		4	$D33, D36$ $D39$		4	$D23, D25$ $D27$
5	1110		4	$D40$		4	$D22$
6	1111	–	–	–		4	$D21$

7	1000 0100		3	$D31, D36$ $D38$		3	$D29$
8	1000 0110		3	$D32, D33$ $D39$		3	$D25$
9	1000 0111		3	$D28$		3	$D23$
10	1100 0011		3	$D63$	–	–	–
11	1100 1010		3 4	$D39$ $D56$		4	$D37$
12	1100 1011		4	$D91, D92$		4	$D47$
13	1110 1101		4	$D62, D63$		4	$D24$
14	1000 0100 0010		2	$D36, D38$ $D39, D40$		2	$D34, D35$ $D37$

15	1000 0100 0011		2	$D31, D32$ $D33$		2	$D24, D25$ $D26, D27$
16	1000 0110 0101		3	$D54$		3	$D34$
17	1100 1010 0110		3	$D40$		3	$D26$
18	1100 1010 0101		3	$D54$		3	$D27$
19	1100 1010 1001		4	$D92, D93$ $D94$		4	$D35$
20	1100 1010 0111		3	$D61, D62$ $D72$		3	$D22$
21	1100 1011 0111		3	$D36$		3	$D26$
22	1110 1101 1011		4	$D88$		4	$D116$

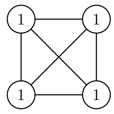
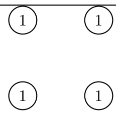
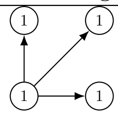
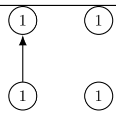
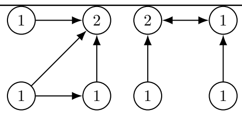
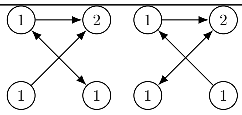
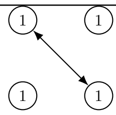
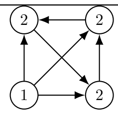
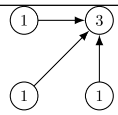
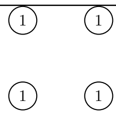
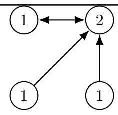
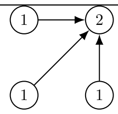
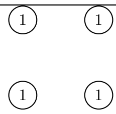
23	1000 0100 0010 0001	 any connected graph with 3 edges	1	$D28, D31$ $D32, D33$ $D36, D38$ $D39, D40$		1	$D21$
24	1000 0110 0101 0011		2	$D28$		2	$D22$
25	1100 1010 1001 0110		3	$D61, D63$	–	–	–
26	1100 1010 0101 0011		3	$D56, D64$		3	$D24$
27	1100 1010 1001 0111		3	$D129$	–	–	–
28	1110 1101 1011 0111		3	$D40$		3	$D21$
29	1100 1010 1001 0110 0101		3	$D62$	–	–	–
30	1100 1010 1001 0110 0101 0011		2	$D40$		2	$D21$

Table B.3: The minimal unweighted influence games corresponding to the 30 simple games up to isomorphism for $n = 4$.

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