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Introduction

Length functions. In mathematics, it is frequent to introduce real valued functions, which may attain infinity, to measure some finiteness properties of the objects we are dealing with (e.g., dimension of vector spaces, rank, composition length, logarithm of the cardinality). In 1968, Northcott and Reufel observed the underlying common properties of some particularly well-behaved invariants and axiomatized the abstract notion of *length function.* Indeed, given an Abelian category $\mathfrak{C}$, a function

$$L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$$

is a length function provided it satisfies the following properties:

(LF.1) L is *additive*, that is, $L(Y_2) = L(Y_1) + L(Y_3)$ for any short exact sequence $0 \to Y_1 \to Y_2 \to Y_3 \to 0$ in $\mathfrak{C}$;

(LF.2) L is *upper continuous*, that is, $L(Y) = \sup\{L(Y_\alpha) : \alpha \in \Lambda\}$ for any object Y in $\mathfrak{C}$ and any directed system $\mathcal{S} = \{Y_\alpha : \alpha \in \Lambda\}$ of sub-objects of Y such that $\sum_\Lambda Y_\alpha = Y$.

One of the goals of this thesis is to answer (at least partially) to the following question regarding the extension of length functions to modules over crossed products:

Question 0.1. *Let R be a ring, let G be a monoid and fix a crossed product $R{*}G$. Is it possible to find a map*

$$\begin{aligned}\{\textit{length functions on } R\text{-Mod}\} &\longrightarrow \{\textit{length functions on } R{*}G\text{-Mod}\}\\ L &\longmapsto L_{R*G}\end{aligned}$$

satisfying the formula $L(M) = L_{R*G}(R{*}G \otimes_R M)$ for any left R-module M?

Extension to polynomial rings. There is a classical way to answer Question 0.1 in the positive when $G = \mathbb{N}$ is the monoid of natural numbers and $R{*}G = R[\mathbb{N}] = R[X]$ is the ring of polynomials in one variable over R.

Indeed, let A be a left Noetherian ring with a distinguished central element $X \in A$. There is an important length function of the category of left A-modules called the *multiplicity* of X (see for example [79, Chapter 7]). Given a finitely generated left A-module ${}_AF$, the multiplicity of X in F is defined as

$$\mathrm{mult}_\ell(X, F) = \begin{cases} \ell(F/\phi_X(F)) - \ell(\mathrm{Ker}(\phi_X)) & \text{if } \ell(F/\phi_X(F)) < \infty;\\ \infty & \text{otherwise;}\end{cases} \tag{0.0.1}$$

where ℓ is the composition length and $\phi_X : F \to F$ is the endomorphism of F induced by left multiplication by X. Given an arbitrary left A-module ${}_AM$, one lets

$$\mathrm{mult}_\ell(X, M) = \sup\{\mathrm{mult}_\ell(X, F) : F \leqslant M \text{ fin. gen.}\}.$$

This classical notion of multiplicity was used by Vámos as a model to construct an L-*multplicity* of X based on a given length function L of A-Mod (see [98, Chapter 5]), just substituting ℓ in the above definition by an arbitrary length function L.

Let $A = R[X]$ be a polynomial ring over a left Noetherian ring R, and let $L : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. One can extend trivially L to A-Mod (just forgetting the action of X) and then take the L-multiplicity mult_L of the element $X \in A$, which is formally defined as in (0.0.1). This defines a map

$$\begin{aligned} \mathrm{mult} : \{\text{length functions on } R\text{-Mod}\} &\longrightarrow \{\text{length functions on } R[X]\text{-Mod}\} \\ L &\longmapsto \mathrm{mult}_L\,. \end{aligned}$$

The values of L can be recovered via the formula

$$L(M) = \mathrm{mult}_L(R[X] \otimes_R M)\,,$$

that holds for any left R-module M. This procedure of extending length functions from the category of modules over a given ring to the modules over its ring of polynomials is useful in many situations but it has the disadvantage that it just works in the Noetherian case.

In the recent paper [93], Salce, Vámos and the author studied the problem of the extension of a given length function on a category of modules R-Mod to a length function of (suitable subcategories of) $R[X]$-Mod, without any hypothesis on the base ring. The key idea is to see a left $R[X]$-module ${}_{R[X]}M$ as a pair $({}_RM, \phi_X)$ of a left R-module and a distinguished endomorphism, given by left multiplication by X. This allows us to see left $R[X]$-modules as discrete-time dynamical systems. Then, under suitable hypotheses, we can attach a dynamical invariant to $({}_RM, \phi_X)$, called *algebraic L-entropy* (see below for more details). Surprisingly enough, it turns out that the values of the algebraic L-entropy and of the L-multiplicity of a left $R[X]$-module coincide whenever these values are both defined.

A dynamical approach. Let us say something more about the dynamical aspects of this work. Indeed, given a set M and a self-map $\phi : M \to M$, one can consider the discrete-time dynamical system (M, ϕ), whose evolution law is

$$\mathbb{N} \times M \to M \quad \text{such that} \quad (n, x) \mapsto \phi^n(x)\,.$$

Depending on the possible structures on (M, ϕ) – for example when ϕ is a continuous self-map of a topological space M, or ϕ is an endomorphism of a module M – there exist various notions of entropy, which, roughly speaking, provide a tool to measure the “disorder”, “growth rate” or “mixing” of the evolution of the system.

In 1965, Adler, Konheim and McAndrew introduced the topological entropy, which is an invariant of dynamical systems (M, ϕ), where M is a compact space and ϕ is a continuous self-map. This concept was successively modified and generalized by Bowen, Hood and others.

Turning to the algebraic side, in the final part of the paper where the topological entropy was introduced, Adler et al. suggested a notion of entropy for a given endomorphism $\phi : G \to G$ of a discrete torsion Abelian group G:

$$\mathrm{ent}(\phi) = \sup\left\{\lim_{n\to\infty} \frac{\log|F + \phi(F) + \ldots + \phi^{n-1}(F)|}{n} : F \leqslant G,\ \log|F| < \infty\right\}\,.$$

In 1974, Weiss studied the basic properties of $\mathrm{ent}(-)$, also connecting it with the topological entropy of endomorphisms of profinite Abelian groups via the Pontryagin-Van Kampen duality. The turning point in the study of this notion of entropy came in 2009, when Dikranjan, Goldsmith, Salce and Zanardo proved the main properties of $\mathrm{ent}(-)$.

Another important step to make a notion of entropy available to module-theorists, is due to Salce and Zanardo [94]. Given a ring R and a suitable invariant $i : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, Salce and Zanardo defined a notion of entropy $\mathrm{ent}_i(\phi)$ for a given endomorphism ϕ of a left R-module, substituting by the invariant i the logarithm of the cardinality used to define $\mathrm{ent}(-)$ for endomorphisms of torsion Abelian groups. In the autor's master thesis [99], it was proved that if the invariant i is a length function satisfying suitable conditions then the resulting entropy is a length function. This result was published in the joint paper with Salce and Vámos [93], which also includes the connection with multiplicity we mentioned before.

At this point one should notice that the dynamical systems (M, ϕ) described above are just actions of the monoid $\mathbb{N}$ on the set M by iterations of ϕ, that is, morphisms from $\mathbb{N}$ to the monoid of self-maps of M. But, of course, there is nothing special about $\mathbb{N}$; in fact, given any monoid Γ, one can study dynamical systems (M, λ) where λ is a map that associates to any $\gamma \in \Gamma$ an endomorphism $\lambda_\gamma : M \to M$ (in the previous case $\lambda_n = \phi^n$ for all $n \in \mathbb{N}$). In this direction, it is worth noting that Ornstein and Weiss [81] extended the main results about the topological entropy of a self-map to the topological entropy of the action of an amenable group.

In this thesis we construct a general machinery to associate a notion of entropy to this kind of dynamical systems (M, λ). Most of the existing notions of entropy can be viewed as particular cases of this general framework.

A general scheme for entropies. To define our entropy function we need essentially four ingredients:

– a commutative semigroup $(S, +)$;

– a map $v : S \to \mathbb{R}_{\geqslant 0}$;

– a monoid Γ that acts on S via a homomorphism $\lambda : \Gamma \to \mathrm{Aut}(S, v)$;

– an averaging sequence $\mathfrak{s} = \{F_n : n \in \mathbb{N}\}$ of non-empty finite subsets of Γ.

We call the pair (S, v) a *pre-normed semigroup* and we define the $\mathfrak{s}$-entropy of the action λ as

$$\mathfrak{h}(\lambda, \mathfrak{s}) = \sup \left\{ \limsup_{n \in \mathbb{N}} \frac{v\left(\sum_{g \in F_n} \lambda_g(x)\right)}{|F_n|} : x \in S \right\} .$$

When $\Gamma = \mathbb{N}$ one usually takes $\mathfrak{s}$ to be the sequence of intervals $F_n = \{0, \dots, n-1\}$. When Γ is an amenable group (see Section 4.2), the most natural choice for the averaging sequence $\mathfrak{s}$ seems to be that of a Følner sequence.

We encode the above scheme in a category $\mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$ of left representations of Γ on pre-normed semigroups and this yields a general notion of entropy

$$\mathfrak{h}(-, \mathfrak{s}) : \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} .$$

Given a category $\mathfrak{C}$, an entropy function $\mathfrak{C} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is defined via a functor

$$F : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v) ,$$

letting the entropy of $Y \in \mathrm{Ob}(\mathfrak{C})$ be $\mathfrak{h}(F(Y), \mathfrak{s})$.

The philosophical point is that, whenever one has a notion of entropy in a given category, there is "usually" a functorial way to construct a "pre-normed semigroup", a monoid Γ and a suitable Γ-action such that the entropy in the category of representations of Γ on pre-normed semigroups equals the original entropy. It turns out that most of the usual notions of entropy can be defined this way. For example, the topological entropy con be defined via a semigroup of open covers, while the algebraic entropy $\mathrm{ent}(-)$ can be defined via a semigroup of finite subgroups.

The idea that all the notions of entropy in mathematics should be considered as different instances of a unique underlying concept is due to Dikran Dikranjan and he started working on it, at least, from 2006. In 2008, we started collaborating on a common project. During the years these ideas continued growing and we could include more and more examples in our general scheme. Some of this work, for actions of the monoid $\mathbb{N}$, will be included in the forthcoming paper [30] by Dikranjan and Giordano Bruno (see also the recent survey paper [29]).

The algebraic L-entropy. We can now come to our partial answer to Question 0.1. Given a crossed product $R{*}G$ of a ring R with a countable amenable group G, and given a length function $L : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, for any left $R{*}G$-modules ${}_{R*G}M$ we consider the semigroup

$$\mathrm{Fin}_L(M) = \{{}_RK \leqslant {}_RM : L(K) < \infty\}\,,$$

with the sum of submodules. We endow $\mathrm{Fin}_L(M)$ with the pre-norm

$$v_L : \mathrm{Fin}_L(M) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \ \text{ such that } \ v_L(K) = L(K)\,.$$

There is a natural left action $\lambda : G \to \mathrm{Aut}(\mathrm{Fin}_L(M))$ given by left multiplication. Given a Følner sequence $\mathfrak{s}$ of G, we can consider the $\mathfrak{s}$-entropy of the action λ in the category of pre-normed semigroups. With this procedure we construct an invariant

$$\mathrm{ent}_L : R{*}G\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}\,.$$

It turns out that ent_L is not well-behaved on the entire category $R{*}G$-Mod, so we define $\mathrm{lFin}_L(R{*}G)$ to be the subclass of $R{*}G$-Mod consisting of all left $R{*}G$-modules ${}_{R*G}M$ such that $L(K) < \infty$, for any finitely generated R-submodule K of M. For example, $\mathrm{lFin}_L(R{*}G)$ contains all the left $R{*}G$-modules ${}_{R*G}M$ such that $L(M) < \infty$. Furthermore, a consequence of the continuity of L on directed colimits of submodules is that, given a left R-module ${}_RK$ such that $L(K) < \infty$, the left $R{*}G$-module $R{*}G \otimes_R K$ is in $\mathrm{lFin}_L(R{*}G)$.
We prove the following

Theorem 8.18. *Given a ring R and a countable amenable group G, fix a crossed product $R{*}G$ and a discrete (i.e. the finite values of L form a subset of $\mathbb{R}_{\geqslant 0}$ which is order-isomorphic to $\mathbb{N}$) length function $L : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ which is compatible with $R{*}G$ (see Definition 8.2). Then, the invariant*

$$\mathrm{ent}_L : \mathrm{lFin}_L(R{*}G) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$$

satisfies the following properties:

(1) ent_L *is upper continuous;*

(2) $\mathrm{ent}_L(R{*}G \otimes_R K) = L(K)$ *for any L-finite left R-module K;*

(3) $\mathrm{ent}_L(N) > 0$ *for any non-trivial $R{*}G$-submodule $N \leqslant R{*}G \otimes_R K$;*

(4) ent_L *is additive.*

In particular, ent_L *is a length function of* $\mathrm{lFin}_L(R{*}G)$*.*

Thus, using entropy, we show that when G is an amenable group there are "many" length functions of R-Mod which can be extended to a "large" subcategory of $R{*}G$-Mod. Let us also remark that, even for $R = \mathbb{Z}$ and $L = \log_2 |-|$, Elek [36] proved that it is not always possible to find a length function of the category $R[G]$-Mod which takes a finite value on the module ${}_{R[G]}R[G]$, if G is not amenable. This shows that the answer to Question 0.1 is negative in general.

In order to understand how to apply Theorem 8.18, it is important to know when there are "enough" length functions in R-Mod that satisfy the hypotheses of the theorem. Starting with an easy example, if R is a field then all the length functions are multiples of the dimension of vector spaces. All such functions can be used to construct a well-behaved notion of entropy.

More generally, when R is a ring with left Gabriel dimension (e.g., a left Noetherian ring, see Chapter 2), we can prove a complete structure theorem for all the length functions of R-Mod, in terms of the Gabriel spectrum of R-Mod (i.e., the set of isomorphism classes of indecomposable injective modules). For this we re-prove, partially using different methods, and slightly generalize an old result of Vámos [98]. For the proof of this theorem we make use of some torsion-theoretic methods.

Using the structure of length functions, we can deduce the existence of many length functions of R-Mod that satisfy the hypotheses of Theorem 8.18 independently on the choice of the crossed product $R{*}G$, when R has left Gabriel dimension.

Three motivating problems. The interest in having a well-behaved invariant, like ent_L, for categories of modules over crossed products comes from some classical conjectures that we are now going to describe.

– *(Linear) Surjunctivity Conjecture.* A map is *surjunctive* if it is non-injective or surjective. Let A be a finite set and equip $A^G = \prod_{g\in G} A$ with the product of the discrete topologies on each copy of A. There is a canonical left action of G on A^G defined by

$$gx(h) = x(g^{-1}h) \text{ for all } g, h \in G \text{ and } x \in A^G\,.$$

A long standing open problem by Gottschalk [52] is that of determining whether or not any continuous and G-equivariant map $\phi : A^G \to A^G$ is surjunctive, we refer to this problem as the *Surjunctivity Conjecture.*

An analogous problem is as follows. Let $\mathbb{K}$ be a field, let V be a finite dimensional $\mathbb{K}$-vector space, endow V^G with the product of the discrete topologies and consider the canonical left G-action on V^G. It is asked whether any G-equivariant continuous and $\mathbb{K}$-*linear* map $V^G \to V^G$ is surjunctive, we refer to this problem as the *L-Surjunctivity Conjecture.*

– *Stable Finiteness Conjecture.* A ring R is *directly finite* if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. Furthermore, R is *stably finite* if the ring of $k \times k$ matrices $\mathrm{Mat}_k(R)$ is directly finite for all $k \in \mathbb{N}_+$. A long-standing open problem due to Kaplansky [64] is to determine whether the group ring $\mathbb{K}[G]$ is stably finite for any field $\mathbb{K}$, we refer to this problem as the *Stable Finiteness Conjecture.* Notice that, $\mathrm{Mat}_k(\mathbb{K}[G]) \cong \mathrm{End}_{\mathbb{K}[G]}(\mathbb{K}[G]^k)$, so an equivalent way to state the Stable Finiteness Conjecture is to say that any surjective endomorphism of a free left $\mathbb{K}[G]$-module of finite rank is injective.

– *Zero-Divisors Conjecture.* Another conjecture of Kaplasky (see [84] and [83]) affirms that $\mathbb{K}[G]$ is a domain for any torsion free group G and any field $\mathbb{K}$. We refer to this conjecture as the *Zero-Divisors Conjecture.*

All the above conjectures are open in general. In 1999, Gromov [53] defined sofic groups (see Definition 9.5) and proved a very general version of the Surjunctivity Conjecture (see also [105]) for this class of groups. Gromov's result also implies the L-Surjunctivity Conjecture for sofic groups (see also [13]).

The Stable Finiteness Conjecture was known in full generality for fields of characteristic 0, while there was no progress in the positive characteristic case until 2002, when Ara, O'Meara and Perera [5] proved that any crossed product $\mathbb{K}{*}G$ of a division ring $\mathbb{K}$ with an amenable group G is stably finite, and used this result to deduce the Stable Finiteness Conjecture for the class of residually amenable groups. Short after, Elek and Szabó [38] verified the conjecture for G a sofic group (see also [13] and [6] for alternative proofs).

Applications of entropy. Let us remark that the Surjuctivity Conjecture was classically known to hold for amenable groups: the usual proof in this particular case was an application of the topological entropy studied by Orstein and Weiss. This fact suggested that the notion of algebraic entropy should be applicable to the amenable case of the Stable Finiteness Conjecture. In fact, an application of Theorem 8.18 is the following:

Theorem 10.6. *Let R be a left Noetherian ring, let G be a countable amenable group and let $R{*}G$ be a crossed product. Let ${}_RK$ be a finitely generated left R-module, let ${}_{R*G}M = R{*}G \otimes_R K$, and let ${}_{R*G}N \leqslant {}_{R*G}M$. Then, any surjective endomorphism of left $R{*}G$-modules $\phi : N \to N$ is injective.*
In particular, $\mathrm{End}_{R*G}(N)$ is directly finite.

The hypothesis that G is amenable in the above theorem is essential. In fact, already for non-commutative free groups the above theorem fails (see Example 10.9).

A different application of the algebraic entropy is to problems related to the Zero-Divisors Conjecture. In fact, in [22] Chung and Thom used the topological entropy to study some cases of the Zero-Divisors Conjecture. We can generalize and complete their results as follows:

Theorem 10.10. *Let $\mathbb{K}$ be a division ring and let G be a countably infinite amenable group. For any fixed crossed product $\mathbb{K}{*}G$, the following are equivalent:*

(1) *$\mathbb{K}{*}G$ is a left and right Ore domain;*

(2) *$\mathbb{K}{*}G$ is a domain;*

(3) $\mathrm{ent}_{\dim}({}_{\mathbb{K}*G}M) = 0$*, for every proper quotient M of $\mathbb{K}{*}G$;*

(4) $\mathrm{Im}(\mathrm{ent}_{\dim}) = \mathbb{N} \cup \{\infty\}$.

In other words, the algebraic entropy detects the zero-divisors in $\mathbb{K}{*}G$. As an immediate consequence of the above theorem, we obtain that, in the above hypotheses, $\mathbb{K}{*}G$ is a domain if and only if it admits a flat embedding in a division ring.

A point-free approach in the sofic case. After viewing that in the amenable case we could generalize the known results about the Stable Finiteness Conjecture to crossed products, we tried to obtain a similar result for sofic groups. As we said, in this generality there is no hope to prove a result like Theorem 8.18 and so we need to find different tools to tackle the problem. Let us describe our strategy.

Let G be a group, let R be a ring and fix a crossed product $R{*}G$. Let K be a finitely generated left R-module, let ${}_{R*G}M = R{*}G \otimes_R K$ and consider an endomorphism of left $R{*}G$-modules $\phi : M \to M$. It is well-known that the poset $\mathcal{L}(M)$ of R-submodules of M (ordered by inclusion) is a lattice with very good properties. Consider the natural left action (by left multiplication) of G on $\mathcal{L}(M)$. The first important observation is that, at this level of lattices of R-submodules, there is essentially no difference between $R[G]$ and general crossed products (since the difference between the two construction is just in some units of R which leave invariant the R-submodules). Furthermore, ϕ induces a G-equivariant semi-lattice homomorphism

$$\Phi : \mathcal{L}(M) \to \mathcal{L}(M) \ \text{ such that } \ \Phi(H) = \phi(H)\,.$$

The second key-observation is that ϕ is injective (resp., surjective) if and only if Φ has the same property. Thus, using this construction we can translate our original problem in terms of some "well-behaved" lattices with a G-action and semi-lattice G-equivariant endomorphisms on them.

The surprising thing is to realize that, when proving the L-Surjunctivity Conjecture, one is actually working with the same kind of group actions on lattices as for the Stable Finiteness Conjecture. Let us be more precise: consider a left R-module N, take the product N^G endowed with the product of the discrete topologies and the usual left G-action, and consider a G-equivariant continuous endomorphism $\psi : N^G \to N^G$. If N is Artinian, one can show that the poset $\mathcal{N}(N^G)$ of closed submodules of N^G, ordered by *reverse* inclusion, is a lattice with many common features with a lattice of submodules of a discrete module. There is a natural right action of G on $\mathcal{N}(N^G)$, induced by the left G-action on N^G. Furthermore, ψ induces a G-equivariant semi-lattice homomorphism

$$\Psi : \mathcal{N}(N^G) \to \mathcal{N}(N^G) \ \text{ such that } \ \Psi(H) = \psi^{-1}(H)\,.$$

It turns out that ψ is injective (resp., surjective) if and only if Ψ is surjective (resp., injective). Thus, with this construction we can translate (a general form of) the L-Surjunctivity Conjecture in terms of lattices with a G-action and G-equivariant semi-lattice endomorphisms on them, exactly as we did for the Stable Finiteness Conjecture.

After these observations it was clear that the Stable Finiteness Conjecture and the L-Surjunctivity Conjecture should be treated as expressions in different languages of the same problem. In Chapter 2 we study the category of qframes, which are lattices with properties analogous to the lattices $\mathcal{L}({}_{R*G}M)$ and $\mathcal{N}(N^G)$ described above. Then, in Chapter 11, we prove a general theorem (see Theorem 11.5) for a G-equivariant endormorphism of left representations on qframes, where G is a sofic group. The proof is quite technical and uses the machinery of torsion and localization to reduce the problem to semi-Artinian qframes.

As a consequence, we obtain the following general version of the L-Surjunctivity Conjecture for sofic groups:

Theorem 11.8 *Let R be a ring, let G be a sofic group and let ${}_RN$ be an Artinian left R-module. Then any continuous and G-equivariant endomorphism $\phi : N^G \to N^G$ is surjunctive.*

Notice that the above theorem generalizes in different directions the main results of [15] and [14]. Furthermore, we prove a general version of the Stable Finiteness Conjecture in the sofic case, extending results of [38] and [5]:

Theorem 11.11 *Let R be a ring, let G be a sofic group, fix a crossed product $R{*}G$, let N_R be a finitely generated right R-module and let $M = R{*}G \otimes N$. Then,*

(1) *if N_R is Noetherian, then any surjective $R{*}G$-linear endomorphism of M is injective;*

(2) *if N_R has Krull dimension, then $\mathrm{End}_{R*G}(M)$ is stably finite.*

Another use of torsion theories: model approximations. In the last part of the thesis we study a problem of a different nature, the connection with the rest of the thesis comes from the methods we use. In fact, the formalism of torsion theories and localization of Grothendieck categories, that is used directly or indirectly in all our main results, is applied in Chapter 12 to clarify and generalize some recent results of Chachólski, Neeman, Pitsch, and Scherer about model approximations of the category of unbounded chain complexes over a Grothendieck category.

Let us start with some background for the problem. Model categories were introduced in the late sixties by Quillen [89]. A *model category* $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ is a bicomplete category $\mathbb{M}$ with three distinguished classes of morphisms, called respectively weak equivalences, fibrations and cofibrations, satisfying some axioms (see Definition 12.2). An important property of model categories is that one can invert weak equivalences, obtaining a new category, called the *homotopy category.*

The concept of model approximation was introduced by Chachólski and Scherer [21] with the aim of constructing homotopy limits and colimits in arbitrary model categories. The advantage of model approximations is that it is easier in general to prove that a given category has a model approximation than defining a model structure on it. On the other hand, model approximations allow to construct derived functors and to define the homotopy category. Consider a category $\mathfrak{C}$ with a distinguished class of morphisms $\mathcal{W}_{\mathfrak{C}}$ and a model category $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$. A *model approximation* of $(\mathfrak{C}, \mathcal{W}_{\mathfrak{C}})$ by $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ is a pair of adjoint functors

$$l : \mathfrak{C} \rightleftarrows \mathbb{M} : r$$

such that l sends the elements of $\mathcal{W}_{\mathfrak{C}}$ to elements of $\mathcal{W}$ and other technical conditions (for details see Definition 12.13).

Chachólski, Pitsch, and Scherer [19] introduced a useful model approximation for the category of unbounded complexes $\mathbf{Ch}(R)$ over a ring R, whose homotopy category is the usual derived category $\mathbf{D}(R)$. This construction encodes in a pair of adjoint functors the classical ideas to construct injective resolutions of unbounded complexes. The aim of the successive paper [18] of the same three authors together with Neeman, is to modify the construction of [19] in order to obtain a "model approximation for relative homological algebra".

Let us be more specific about the meaning of *relative homological algebra* in this context. Consider a Grothendieck category $\mathfrak{C}$, roughly speaking, an *injective class* is a suitable class $\mathcal{I}$ of objects of $\mathfrak{C}$ that is meant to represent a "different choice" for the injective objects in the category (see Definition 12.30). Chachólski, Pitsch, and Scherer [20] studied the injective classes of the category of modules Mod-R over a commutative ring R, classifying all the injective classes of injective objects.

Given a Grothendieck category $\mathfrak{C}$ and an injective class of injective objects $\mathcal{I}$, one says that a morphism of unbounded complexes $\phi^\bullet : M^\bullet \to N^\bullet$ is an *$\mathcal{I}$-quasi-isomorphism* provided $\mathrm{Hom}_{\mathfrak{C}}(\phi^\bullet, I)$ is a quasi-isomorphism of complexes of Abelian groups for all $I \in \mathcal{I}$. The following questions naturally arise:

Question 0.2. *In the above notation, denote by $\mathcal{W}_{\mathcal{I}}$ the class of $\mathcal{I}$-quasi-isomorphisms, then*

(1) *is it possible to find a model approximation for $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_{\mathcal{I}})$? If so, what does the homotopy category of such approximation look like?*

(2) *Is it possible (in analogy with [19]) to give an adjunction that encodes an inductive construction of the relative injective resolutions of unbounded complexes?*

Chachólski, Neeman, Pitsch, and Scherer [18] partially answer the above questions in case the category $\mathfrak{C}$ is the category of modules over a commutative Noetherian ring R. They showed that the above questions have a positive answer if R has finite Krull dimension. On the other hand, if the Krull dimension of R is not finite, there always exists an injective class of injectives $\mathcal{I}$ for which part (2) of the question has a negative answer.

In Chapter 12, we try to tackle the above questions in the general setting of Grothendieck categories. Our key observation is that there is a bijective correspondence between injective classes of injectives and hereditary torsion theories induced by the following correspondence:

$$\tau = (\mathcal{T}, \mathcal{F}) \longmapsto \mathcal{I}_\tau = \{\text{injective objects in } \mathcal{F}\}\,.$$

The bijection with hereditary torsion theories allows us to generalize the classification of the injective classes of injective object. We can now answer part (1) of Question 0.2 in full generality. First of all, recall that it is possible to associate to any hereditary torsion theory τ a localization of the category $\mathfrak{C}$, which is encoded in the following pair of adjoint functors:

$$\mathfrak{C} \underset{\mathbf{S}_\tau}{\overset{\mathbf{Q}_\tau}{\rightleftarrows}} \mathfrak{C}/\mathcal{T}\,,$$

where $\mathfrak{C}/\mathcal{T}$ is a Grothendieck category, which is called the *localization of* $\mathfrak{C}$ *at* τ. One extends the functors $\mathbf{Q}_\tau$ and $\mathbf{S}_\tau$ to the categories $\mathbf{Ch}(\mathfrak{C})$ and $\mathbf{Ch}(\mathfrak{C}/\mathcal{T})$ (just applying them degree-wise), this gives rise to an adjunction. Abusing notation, we use the same symbols for these new functors. Then, one proves that a morphism of complexes $\phi^\bullet$ in $\mathbf{Ch}(\mathfrak{C})$ is an $\mathcal{I}_\tau$-quasi-isomorphism if and only if $\mathbf{Q}_\tau(\phi^\bullet)$ is a quasi-isomorphism in $\mathbf{Ch}(\mathfrak{C}/\mathcal{T})$. Furthermore, if we endow $\mathbf{Ch}(\mathfrak{C}/\mathcal{T})$ with the canonical injective model structure, there is a model approximation

$$\mathbf{Q}_\tau :\ (\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_{\mathcal{I}_\tau}) \rightleftarrows \mathbf{Ch}(\mathfrak{C}/\mathcal{T})\ : \mathbf{S}_\tau\,.$$

The homotopy category associated with this model approximation has a very concrete form: it is precisely the derived category $\mathbf{D}(\mathfrak{C}/\mathcal{T})$, see Theorem 12.50.

The answer to part (2) of Question 0.2 is more delicate. First of all, one needs to understand what fails in the construction of [18]: the quotient category $\mathfrak{C}/\mathcal{T}$ may fail to be ($Ab.4^*$)-k for all $k \in \mathbb{N}$ (see Definition 12.26), even if $\mathfrak{C}$ is a very nice category (say a category of modules over a commutative Noetherian ring). This, a fortiori trivial, observation is sufficient to explain why one cannot always construct inductively the relative injective resolutions of unbounded complexes. In fact, there is no reason for an object $X^\bullet$ in the homotopy category $\mathbf{D}(\mathfrak{C}/\mathcal{T})$ of $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_{\mathcal{I}_\tau})$ for being isomorphic to the homotopy limit of its truncations if $\mathfrak{C}/\mathcal{T}$ is not ($Ab.4^*$)-k for any $k \in \mathbb{N}$. Going back to the original question, we can partially answer as follows: one can always find a model approximation of $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_{\mathcal{I}_\tau})$ by towers of model categories of half-bounded complexes over $\mathfrak{C}/\mathcal{T}$ provided $\mathfrak{C}/\mathcal{T}$ is ($Ab.4^*$)-k for some $k \in \mathbb{N}$.

Structure of the thesis

The thesis is organized in twelve chapters divided in five parts.

Part I encompasses the first three chapters and consists mainly of background material. Some sections contain basic results and are included with the intention to fix notations and to make the results of this thesis available to readers with diverse backgrounds. In general, we tend to omit the proofs for the most widely known results (giving instead references to the literature), while we include the proofs when we think that their arguments are particularly important for the comprehension or when the proofs available in the literature are not satisfactory for some reason.

In Chapter 1 we provide the necessary background in general category theory with emphasis on Abelian and Grothendieck categories. Furthermore, we recall the machinery of torsion theories and localization of Gabriel categories, stating the Gabriel-Popescu Theorem and some of its consequences.

In Chapter 2 we introduce the category of quasi-frame and we study the basic constructions in this category. Two useful tools in this context are the Krull and the Gabriel dimension of quasi-frames. Using the fact that the poset of sub-objects of a given object in a Grothendieck category is a quasi-frame, we re-obtain the classical notions of Krull and Gabriel dimension for such objects. We also introduce and study a relative version of the Gabriel dimension.

In Chapter 3 we provide the necessary background in topological groups and modules. In particular, in the first half of the chapter, after some preliminaries, we state the Pontryagin-Van Kampen Duality Theorem and the Fourier Inversion Theorem. In the second half of the chapter we give a complete proof of particular case of the Mülcer Duality Theorem between discrete and strictly linearly compact modules.

Part II is devoted to the study of entropy in a categorical setting, this part contains Chapters 4, 5 and 6.

In Chapter 4 we introduce the category of pre-normed semigroups and the category of left Γ-representations of a monoid Γ over a given category. Then, we introduce and study an entropy function in the category of left Γ-representations over the category of normed-semigroups. In the second part of the chapter we concentrate on the case when Γ is an amenable group.

Chapter 5 consist of a series of examples of classical invariants that can be obtained functorially using the entropy of pre-normed semigroups.

In Chapter 6 we prove a Bridge Theorem (generalizing a result of Peters) that connects the topological entropy on locally compact Abelian groups to the algebraic entropy on the dual, using the results of Chapter 3.

Part III is devoted to the study of length functions and to apply the machinery of entropy to extend length functions to crossed products. It consists of Chapters 7 and 8.

In Chapter 7 we prove a general structure theorem for length functions of Grothendieck categories with Gabriel dimension, this generalizes a result of Vámos. Given a ring R and a group G, we use the structure of length functions over R-Mod to give a precise criterion for a length function $L : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ to be "compatible" with a given crossed products $R{*}G$ (length functions compatible with $R{*}G$ are, roughly speaking, the functions that one may hope to extend to $R{*}G$-Mod).

In Chapter 8 we define the algebraic L-entropy of a left $R{*}G$-module M, where R is a general ring and G is a countable amenable group. The entire chapter is devoted to the proof of the main properties of entropy. In particular the proof of the additivity of the entropy function takes more than one third of the chapter.

In Part IV we apply the theory developed in the three previous parts of the thesis to some classical conjectures in group representations. This part encompasses Chapters 9, Chapter 10 and 11.

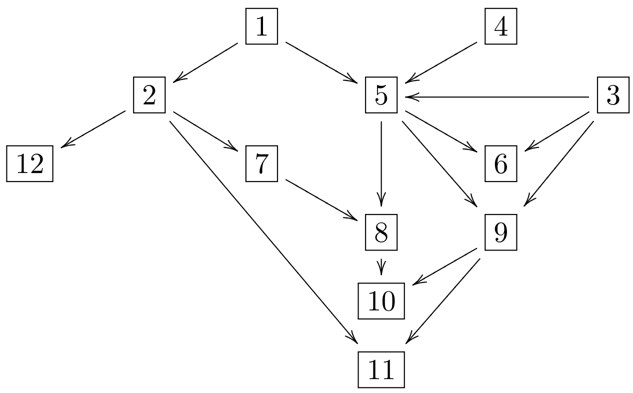
In Chapter 9 we state the conjectures we are interested in, that is, the Surjunctivity Conjecture, the L-Surjunctivity Conjecture, the Stable Finiteness Conjecture and the Zero-Divisors Conjecture. In order to state properly the (L-)Surjunctivity Conjecture we briefly recall some basics about cellular automata. We use the Müller Duality Theorem to prove some relations among the conjectures.

In Chapter 10 we concentrate on the amenable case of the above conjectures. In particular, we show how to use topological entropy to prove the surjunctivity conjecture for amenable groups and we use the algebraic L-entropy to study (general versions of) the Stable Finiteness and the Zero-Divisors Conjectures.

In Chapter 11 we concentrate on the sofic case of the L-Surjunctivity and of the Stable Finiteness Conjectures. In particular, we reduce both conjectures to a more general statement about endomorphisms of quasi-frames. This allows us to extend the known results on both conjectures.

Finally, Part V is devoted to the study of model approximations for relative homological algebra. In particular, we apply the machinery introduced in Chapters 1 and 2 to generalize and reinterpret some recent results of Chachólski, Neeman, Pitsch, and Scherer.

We conclude this introduction with the following "dependence graph" among the various chapters of the thesis:



Contents

Part I

Background material

Chapter 1

Categories and modules: an outline

In Chapter 1 we present the categorical concepts needed in the thesis. We include basic definitions and examples about general category theory and then we specialize to Abelian and Grothendieck categories. We also include some general facts about module theory and homological algebra. The chapter culminates with a discussion of the Gabriel-Popescu Theorem and some of its consequences.
The theory of localization of Abelian categories goes back to Gabriel's thesis [44] but we use here a slightly different approach, analogous to the treatment of localization in [96] for categories of modules: we first construct localizations in Grothendieck categories with enough injectives, we use these results to localize categories of modules (that clearly have enough injectives) and we deduce the Gabriel-Popescu Theorem which states that any Grothendieck category is a localization of a category of modules. As a byproduct, one obtains that any Grothendieck category has enough injectives and so we can localize any such category.

1.1 Categories and functors

1.1.1 Preliminar definition and basic examples

A *category* is an algebraic structure consisting of "*objects*" that are linked by "*arrows*" with two basic properties: the ability to compose the arrows associatively and the existence of an identity arrow for each object.

Definition 1.1. *A category $\mathfrak{C}$ consists of the following three data:*

– *a class of* objects $\mathrm{Ob}(\mathfrak{C})$*;*

– *a set of* morphisms $\mathrm{Hom}_{\mathfrak{C}}(A, B)$*, for every ordered pair of objects* (A, B) *of* $\mathfrak{C}$*;*

– *a* composition law

$$\begin{aligned} \mathrm{Hom}_{\mathfrak{C}}(A, B) \times \mathrm{Hom}(B, C) &\longrightarrow \mathrm{Hom}_{\mathfrak{C}}(A, C) \\ (\phi, \psi) &\longmapsto \psi \circ \phi \end{aligned}$$

for every ordered triple (A, B, C) *of objects of* $\mathfrak{C}$*.*

To underline the fact that a morphism ϕ *belongs to* $\mathrm{Hom}_{\mathfrak{C}}(A, B)$*, we also write* $\phi : A \to B$*. The morphisms from one object to itself are called* endomorphisms*, we let* $\mathrm{End}_{\mathfrak{C}}(A) = \mathrm{Hom}_{\mathfrak{C}}(A, A)$*. The above data are subject to the following axioms:*

(Cat.1) *given* $\phi_1 : A \to B$, $\phi_2 : B \to C$ *and* $\phi_3 : C \to D$, *we have* $(\phi_3 \circ \phi_2) \circ \phi_1 = \phi_3 \circ (\phi_2 \circ \phi_1)$;

(Cat.2) *for all* $A \in \mathrm{Ob}(\mathfrak{C})$, *there exists a morphism* $\mathrm{id}_A \in \mathrm{End}_{\mathfrak{C}}(A)$, *called* identity, *such that* $\mathrm{id}_A \circ \phi = \phi$ *and* $\psi \circ \mathrm{id}_A = \psi$ *for all* $B \in \mathrm{Ob}(\mathfrak{C})$, $\phi : B \to A$ *and* $\psi : A \to B$.

Sometimes we denote composition of morphisms in a category by juxtaposition, omitting the symbol "$\circ$".

Definition 1.2. *Let* $\mathfrak{C}$ *be a category and let* A, $B \in \mathrm{Ob}(\mathfrak{C})$. *A morphism* $\phi \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ *is an* isomorphism *if there exists* $\psi \in \mathrm{Hom}_{\mathfrak{C}}(B, A)$ *such that* $\psi\phi = \mathrm{id}_A$ *and* $\phi\psi = \mathrm{id}_B$. *An isomorphism* $\phi : X \to X$ *is said to be an* automorphism *of* X. *The set of automorphisms of* X *is denoted by* $\mathrm{Aut}_{\mathfrak{C}}(X)$.

If $\phi \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ is an isomorphism, there exists a unique $\psi \in \mathrm{Hom}_{\mathfrak{C}}(B, A)$ such that $\psi\phi = \mathrm{id}_A$ and $\phi\psi = \mathrm{id}_B$. We denote such ψ by ϕ^{-1} and we call it the *inverse* of ϕ.

Example 1.3. *Let* $\mathfrak{C}$ *be a category, we denote by* $\mathfrak{C}^{op}$ *the* opposite category *of* $\mathfrak{C}$, *that is, the category such that* $\mathrm{Ob}(\mathfrak{C}^{op}) = \mathrm{Ob}(\mathfrak{C})$ *and* $\mathrm{Hom}_{\mathfrak{C}^{op}}(A, B) = \mathrm{Hom}_{\mathfrak{C}}(B, A)$ *for every* $A, B \in \mathrm{Ob}(\mathfrak{C}^{op})$.

Example 1.4. *A* semi-group *is a pair* $(G, \cdot)$ *with* G *a set and where* $\cdot : G \times G \to G$ *is a binary associative operation (that is,* $(f \cdot g) \cdot h = f \cdot (g \cdot h)$ *for all* f, g *and* $h \in G$*). If there is a unit element* $e \in G$ *(that is,* $e \cdot g = g \cdot e = g$ *for all* $g \in G$*) then we say that the triple* $(G, \cdot, e)$ *is a* monoid. *In a monoid (or semi-group) we usually denote by* $\cdot$ *the operation and we denote by* e *the identity element. If the operation is commutative then it is denoted by* $+$ *and the identity element is denoted by* 0.
Any monoid G *can be considered as a category* $\mathfrak{C}_G$ *with a single object* $\bullet$ *and such that* $\mathrm{End}_{\mathfrak{C}_G}(\bullet) = G$. *Any category with one object is of this form.*
More generally, in a given category $\mathfrak{C}$, $\mathrm{End}_{\mathfrak{C}}(A)$, *with the operation induced by composition of morphisms and* id_A *as unit element, is a monoid for all* $A \in \mathrm{Ob}(\mathfrak{C})$.

Example 1.5. *We denote by* Set *the category of sets. The class of objects of* Set *is the class of all sets and the set of morphisms between two sets is the family of all functions between them. Composition and identity are as expected.*

Example 1.6. *We denote by* Top *the category of topological spaces. The class of objects of* Top *is the class of all topological spaces* (T, τ), *where* T *is a set and* τ *is a topology, that is a collection of subsets of* T *such that:*

- $\varnothing$ *and* $T \in \tau$;
- *arbitrary unions of elements of* τ *belong to* τ;
- *finite intersections of elements of* τ *belong to* τ.

The elements of τ *are called* open sets, *while the elements of the form* $T \backslash A$, *with* $A \in \tau$, *are called* closed sets. *A morphism* $\phi : (T_1, \tau_1) \to (T_2, \tau_2)$ *in* Top *is a* continuous map, *that is, a map* $\phi : T_1 \to T_2$ *such that* $\phi^{-1}(A) \in \tau_1$ *for all* $A \in \tau_2$. *Composition and identities are as expected.*

Example 1.7. *Let* I *be a set. A binary relation "*$\leqslant$*" on* I *is a* preorder *if it is transitive (i.e., if* $i \leqslant j$ *and* $j \leqslant k$, *then* $i \leqslant k$, *for all* i, j, $k \in I$*), and reflexive (i.e.,* $i \leqslant i$ *for all* $i \in I$*). If* $\leqslant$ *is a preorder on* I, *then the pair* $(I, \leqslant)$ *is said to be a* preordered set. *If* $\leqslant$ *is also antisymmetric (i.e., if* $i \leqslant j$ *and* $j \leqslant i$, *then* $i = j$, *for all* i, $j \in I$*) then it is a partial order and* $(I, \leqslant)$ *is a* partially ordered set *(or a* poset*). Furthermore,* $(I, \leqslant)$ *is a* totally ordered set *if it is a poset*

such that, for all $a, b \in I$, either $a \leqslant b$ or $b \leqslant a$.
Given a preordered set $(I, \leqslant)$, one can define a category $\mathrm{Cat}(I, \leqslant)$ *whose objects are the elements of I and, for all $i, j \in I$,*

$$\mathrm{Hom}_{\mathrm{Cat}(I,\leqslant)}(i,j) = \begin{cases} \{\bullet\} & \text{if } i \leqslant j; \\ \varnothing & \text{otherwise.} \end{cases}$$

In particular, given any set J, the discrete order *on J is a relation defined as follows: $i \leqslant j$ if and only if $i = j$. Of course this is a preorder. We call the category* $\mathrm{Cat}(J, \leqslant)$ *the* discrete category over J.

Example 1.8. *Let I be a set and let $\mathfrak{C}_i$ be a category, for all $i \in I$. The* product category $\prod_{i\in I} \mathfrak{C}_i$ *is defined as follows:*

- $\mathrm{Ob}(\prod_{i\in I} \mathfrak{C}_i) = \prod_{i\in I} \mathrm{Ob}(\mathfrak{C}_i)$, *where the $\prod_{i\in I}$ on the right hand side represents the cartesian product of classes;*
- $\mathrm{Hom}_{\prod_{i\in I} \mathfrak{C}_i}((C_i)_{i\in I}, (C'_i)_{i\in I}) = \prod_{i\in I} \mathrm{Hom}_{\mathfrak{C}_i}(C_i, C'_i)$, *for all $(C_i)_{i\in I}$, $(C'_i)_{i\in I} \in \mathrm{Ob}(\prod_{i\in I} \mathfrak{C}_i)$;*
- *composition is defined component-wise, using the composition laws in each $\mathfrak{C}_i$.*

Definition 1.9. *Given two categories $\mathfrak{C}_1$ and $\mathfrak{C}_2$, a* functor $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ *is a (generalized) function that*

(Func.1) *associates to any object A in $\mathfrak{C}_1$ an object $F(A)$ in $\mathfrak{C}_2$;*

(Func.2) *associates to each morphism $\phi : X \to Y \in \mathfrak{C}_1$ a morphism $F(f) : F(Y) \to F(X) \in \mathfrak{C}_2$. Furthermore, $F(\mathrm{id}_A) = \mathrm{id}_{F(A)}$, for all $A \in \mathrm{Ob}(\mathfrak{C}_1)$ and $F(\psi \circ \phi) = F(\psi) \circ F(\phi)$, for any pair of morphisms $\phi : X \to Y$ and $\psi : Y \to Z$.*

Let $\mathfrak{C}$ be a category, the obvious functor $\mathrm{id}_{\mathfrak{C}} : \mathfrak{C} \to \mathfrak{C}$ such that $F(X) = X$ and $F(\phi) = \phi$, for any object $X \in \mathrm{Ob}(\mathfrak{C})$ and any morphism $\phi : X \to Y$ in $\mathfrak{C}$, is said to be the *identity functor*. Notice also that, given three categories $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$, and functors $F : \mathfrak{C}_1 \to \mathfrak{C}_2$, $G : \mathfrak{C}_2 \to \mathfrak{C}_3$, there is a well defined composition $G \circ F : \mathfrak{C}_1 \to \mathfrak{C}_3$.

Example 1.10. *Let $\mathfrak{C}$ be a category. Any fixed object $A \in \mathrm{Ob}(\mathfrak{C})$ determines two functors*

$$\mathrm{Hom}_{\mathfrak{C}}(A, -) : \mathfrak{C} \to \underline{\mathrm{Set}} \quad \text{and} \quad \mathrm{Hom}_{\mathfrak{C}}(-, A) : \mathfrak{C}^{op} \to \underline{\mathrm{Set}}\,.$$

The functor $\mathrm{Hom}_{\mathfrak{C}}(A, -)$ *maps an object $B \in \mathrm{Ob}(\mathfrak{C})$ to the set* $\mathrm{Hom}_{\mathfrak{C}}(A, B)$ *and a morphism $\phi : B \to C$ to the map*

$$\mathrm{Hom}_{\mathfrak{C}}(A, \phi) : \mathrm{Hom}_{\mathfrak{C}}(A, B) \to \mathrm{Hom}_{\mathfrak{C}}(A, C) \quad \text{such that} \quad \psi \mapsto \phi \circ \psi\,.$$

The functor $\mathrm{Hom}_{\mathfrak{C}}(-, A)$ *is defined similarly.*

Example 1.11. *Given two semi-groups $(G_1, \cdot)$ and $(G_2, \cdot)$, a map $\phi : G_1 \to G_2$ is a* homomorphism *(resp., a* anti-homomorphism*) if $\phi(g \cdot h) = \phi(g) \cdot \phi(h)$ (resp., $\phi(g \cdot h) = \phi(h) \cdot \phi(g)$), for all g and $h \in G_1$. If G_1 and G_2 are monoids, then ϕ is a* homomorphism of monoids *(resp., anti-homomorphism of monoids) if it is a homomorphism of semi-groups (resp., anti-homomorphism of semi-groups) and $\phi(e) = e$. If $\phi : G_1 \to G_2$ is a homomorphism of monoids, one can see that it induces a functor $F_\phi : \mathfrak{C}_{G_1} \to \mathfrak{C}_{G_2}$ in the obvious way, similarly a anti-homomorphism $\psi : G_1 \to G_2$ corresponds to a functor $H_\psi : (\mathfrak{C}_{G_1})^{op} \to \mathfrak{C}_{G_2}$. All the functors among one-object categories have this form.*

Example 1.12. *The* category of semi-groups $\underline{\text{Semi}}$ *(resp., the* category of monoids $\underline{\text{Mon}}$*), is the category whose class of objects* $\text{Ob}(\underline{\text{Semi}})$ *(resp.,* $\text{Ob}(\underline{\text{Mon}})$*) is the class of all semi-groups (resp., monoids) and morphisms are semi-group (resp., monoid) homomorphisms with the usual composition of maps.*

Example 1.13. *A* group *is a monoid in which any element has an inverse. A group is* Abelian *if its operation is commutative. A* homomorphism *of (Abelian) groups is a semigroup homomorphism. We denote by* $\underline{\text{Group}}$ *(resp.,* $\underline{\text{Ab}}$*) the category of all (Abelian) groups and homomorphisms of groups among them.*
Notice that, given a category $\mathfrak{C}$ *and an object* $X \in \text{Ob}(\mathfrak{C})$*, the set* $\text{Aut}_{\mathfrak{C}}(X)$ *with the operation induced by composition is a group.*

Definition 1.14. *A* subcategory $\mathfrak{C}'$ *of a category* $\mathfrak{C}$ *is a category such that* $\text{Ob}(\mathfrak{C}')$ *is a subclass of* $\text{Ob}(\mathfrak{C})$*,* $\text{Hom}_{\mathfrak{C}'}(A, B)$ *is a subset of* $\text{Hom}_{\mathfrak{C}}(A, B)$ *for any pair of objects* $A, B \in \text{Ob}(\mathfrak{C}')$ *and such that composition and identity morphisms in* $\mathfrak{C}'$ *and in* $\mathfrak{C}$ *coincide.*

One can see that $\underline{\text{Mon}}$ is a subcategory of $\underline{\text{Semi}}$, which is a subcategory of $\underline{\text{Set}}$. In general, if $\mathfrak{C}'$ is a subcategory of $\mathfrak{C}$, then there is an *inclusion functor* $F : \mathfrak{C}' \to \mathfrak{C}$.

Definition 1.15. *Let* $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ *be a functor between two categories. For any pair of objects* A *and* $B \in \text{Ob}(\mathfrak{C}_1)$ *there is a map*

$$\text{Hom}_{\mathfrak{C}_1}(A, B) \longrightarrow \text{Hom}_{\mathfrak{C}_2}(F(A), F(B)) \quad \text{such that} \quad \phi \longmapsto F(\phi)\,.$$

If the above map is surjective for any pair of objects of $\mathfrak{C}_1$*, one says that the functor* F *is* full*, while if all such maps are injective one says that* F *is* faithful*. A functor which is both full and faithful is said to be* fully faithful*.*

An example of faithful functor is the inclusion of a subcategory $\mathfrak{C}'$ in a bigger category $\mathfrak{C}$. Notice that the inclusions of $\underline{\text{Ab}}$ in $\underline{\text{Group}}$ and of $\underline{\text{Group}}$ in $\underline{\text{Mon}}$ are full functors, while the inclusion of $\underline{\text{Mon}}$ in $\underline{\text{Semi}}$ is not full.

Definition 1.16. *Let* $\mathfrak{C}$ *be a category and let* $\mathfrak{C}'$ *be a subcategory. If the inclusion* $F : \mathfrak{C}' \to \mathfrak{C}$ *is full, we say that* $\mathfrak{C}'$ *is a* full subcategory *of* $\mathfrak{C}$*.*

Given a category $\mathfrak{C}$, in order to specify a full subcategory $\mathfrak{C}'$ of $\mathfrak{C}$, it is enough to specify $\text{Ob}(\mathfrak{C}')$.

Definition 1.17. *Let* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *be two categories and let* $F,\ F' : \mathfrak{C}_1 \to \mathfrak{C}_2$ *be two functors. A* natural transformation $\nu : F \Rightarrow F'$ *between* F *and* F' *is obtained taking for all* $C \in \mathfrak{C}_1$ *a morphism* $\nu_C : F(C) \to F'(C)$ *such that the following squares commute*

$$\begin{tikzcd}
F(C) \arrow[r, "F(\phi)"] \arrow[d, "\nu_C"'] & F(D) \arrow[d, "\nu_D"] \\
F'(C) \arrow[r, "F'(\phi)"'] & F'(D)
\end{tikzcd}$$

for all $D \in \text{Ob}(\mathfrak{C}_1)$ *and* $\phi \in \text{Hom}_{\mathfrak{C}_1}(C, D)$*. We say that* ν *is a* natural isomorphism *provided all the* ν_C *are isomorphisms.*

Example 1.18. *A category I is said to be* small *if its morphisms (and consequently its objects) form a set (not a proper class). Given a small category I and a category $\mathfrak{C}$ one can define the* functor category $\mathrm{Func}(I,\mathfrak{C})$ *as follows. The objects of* $\mathrm{Func}(I,\mathfrak{C})$ *are the functors from I to $\mathfrak{C}$, while the morphisms between two functors F, $F' : I \to \mathfrak{C}$ are the natural transformations $F \Rightarrow F'$. Composition and identities are as expected.*

Definition 1.19. *Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two categories.*

- *An* adjunction (F,G) *between $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is a pair of functors $F : \mathfrak{C}_2 \to \mathfrak{C}_1$ and $G : \mathfrak{C}_1 \to \mathfrak{C}_2$, such that the functor* $\mathrm{Hom}_{\mathfrak{C}_2}(-,G(-)) : \mathfrak{C}_2 \times \mathfrak{C}_1 \to \underline{\mathrm{Set}}$ *is naturally isomorphic to* $\mathrm{Hom}_{\mathfrak{C}_1}(F(-),-) : \mathfrak{C}_2 \times \mathfrak{C}_1 \to \underline{\mathrm{Set}}$. *In case (F,G) is an adjunction, we say that F is* left adjoint *to G and that G is* right adjoint *to F.*
- *A functor $G : \mathfrak{C}_1 \to \mathfrak{C}_2$ is an* equivalence *of categories if there exists a functor $F : \mathfrak{C}_2 \to \mathfrak{C}_1$ such that FG and GF are naturally isomorphic to the identity functors* $\mathrm{id}_{\mathfrak{C}_1}$ *and* $\mathrm{id}_{\mathfrak{C}_2}$ *respectively.*
- *An equivalence between $\mathfrak{C}_1^{op}$ and $\mathfrak{C}_2$ is said to be a* duality.

The proof of the following lemma is straightforward and so it is left to the reader.

Lemma 1.20. *Let $\mathfrak{C}$ be a category, let I be a set and let $\tilde{I}$ be the discrete category over I. Then, there is an equivalence of categories* $\mathrm{Func}(\tilde{I},\mathfrak{C}) \cong \prod_I \mathfrak{C}$.

Lemma 1.21. [96, Proposition 9.1, Ch. IV] *Let $\mathfrak{C}_1$ and $\mathfrak{C}_2$ be two categories and let $F : \mathfrak{C}_2 \to \mathfrak{C}_1$ be a functor. If G, $G' : \mathfrak{C}_1 \to \mathfrak{C}_2$ are both right adjoints to F, then there is a natural isomorphism $\nu : G \Rightarrow G'$.*

Thanks to the above lemma, adjoints are uniquely determined up to natural isomorphism, so we can speak about “the” right (or left) adjoint to a given functor

Lemma 1.22. [70, Theorem 1, Sec. 8, Ch. IV] *Let $\mathfrak{C}_1$, $\mathfrak{C}_2$ and $\mathfrak{C}_3$ be categories and let (F,G) and (H,K) be two adjunctions between $\mathfrak{C}_1$ and $\mathfrak{C}_2$, and $\mathfrak{C}_2$ and $\mathfrak{C}_3$ respectively. Then, the composition of the two adjunctions $(F \circ H, G \circ K)$ is an adjunction between $\mathfrak{C}_1$ and $\mathfrak{C}_3$.*

Let I, J be small categories and consider a functor $F : I \to J$. Given a category $\mathfrak{C}$, there is an induced functor

$$F_* : \mathrm{Func}(J,C) \to \mathrm{Func}(I,\mathfrak{C}),$$

defined by composition.

Definition 1.23. *Let I, J be small categories, let $\mathfrak{C}$ be a category and consider a functor $F : I \to J$. Then,*

- *the left adjoint $F^\dagger$:* $\mathrm{Func}(I,C) \to \mathrm{Func}(J,\mathfrak{C})$ *to F_* is the* left Kan extension *of F (if it exists);*
- *the right adjoint $F^\ddagger$:* $\mathrm{Func}(I,C) \to \mathrm{Func}(J,\mathfrak{C})$ *to F_* is the* right Kan extension *of F (if it exists).*

1.1.2 Universal constructions

Let $\mathfrak{C}$ be a category. In this subsection we briefly recall some constructions in $\mathfrak{C}$ that are defined by “universal properties”.

Definition 1.24. *Let $\mathfrak{C}$ be a category. An object $C \in \mathrm{Ob}(\mathfrak{C})$ is said to be* initial *(resp.,* terminal*) if, for all $A \in \mathrm{Ob}(\mathfrak{C})$ there exists a unique morphism $\phi : C \to A$ (resp., $\psi : A \to C$).*

Lemma 1.25. *Let $\mathfrak{C}$ be a category and let C and D be two initial (resp., terminal) objects in $\mathfrak{C}$. Then, there is a unique isomorphism $\phi : C \to D$.*

Proof. By definition of initial object, there is a unique morphism $\phi : C \to D$ and a unique morphism $\psi : D \to C$, furthermore the unique endomorphism of C is id_C and the unique endomorphism of D is id_D. It follows that $\phi \circ \psi = \mathrm{id}_D$ and $\psi \circ \phi = \mathrm{id}_C$, that is, $\psi = \phi^{-1}$ is the inverse of ϕ, which is the unique isomorphism going from C to D. Analogous considerations hold for terminal objects. □

A *universal property* is a condition imposing that a given object is the initial or final object in a suitable category. As we proved, this automatically ensures that an object satisfying such a property (if it exists) is unique up to unique isomorphism.

Definition 1.26. *Let I be a small category and let $F : I \to \mathfrak{C}$ be a functor; for all $i \in \mathrm{Ob}(I)$ we denote by C_i the object $F(i)$. The* colimit *of F is a pair $(\varinjlim F, (\epsilon_i)_{i\in \mathrm{Ob}(I)})$ with $\varinjlim F \in \mathrm{Ob}(\mathfrak{C})$ and $\epsilon_i \in \mathrm{Hom}_{\mathfrak{C}}(C_i, \varinjlim F)$, for all $i \in \mathrm{Ob}(I)$, such that $\epsilon_j \circ F(\phi) = \epsilon_i$, for any $\phi \in \mathrm{Hom}_I(i, j)$, and which satisfies the following universal property:*

($*$) *for any pair $(C, (\phi_i)_{i\in \mathrm{Ob}(I)})$ with $C \in \mathrm{Ob}(\mathfrak{C})$ and $\phi_i \in \mathrm{Hom}_{\mathfrak{C}}(C_i, C)$, for all $i \in \mathrm{Ob}(I)$, such that $\phi_j \circ F(\phi) = \phi_i$, for any $\phi \in \mathrm{Hom}_I(i, j)$, there exists a unique morphism $\Phi : \varinjlim F \to C$ such that $\phi_i = \Phi \circ \epsilon_i$ for all $i \in \mathrm{Ob}(I)$.*

Dually, consider a small category I and a functor $F : I^{op} \to \mathfrak{C}$. A pair $(\varprojlim F, (\pi_i)_{i\in \mathrm{Ob}(I)})$ with $\varprojlim F \in \mathrm{Ob}(\mathfrak{C})$ and $\pi_i \in \mathrm{Hom}_{\mathfrak{C}}(\varprojlim F, C_i)$ is a limit *of F if, when viewed in $\mathfrak{C}^{op}$, this pair is a colimit of the opposite functor $F^{op} : I \to \mathfrak{C}^{op}$.*

Notice that, given a category $\mathfrak{C}$, a small category I and a functor $F : I \to \mathfrak{C}$, one can define a category whose objects are pairs $(L, (\pi_i : L \to F(i))_{i\in \mathrm{Ob}(I)})$, with $L \in \mathrm{Ob}(\mathfrak{C})$, such that, given a morphism $\phi : i \to j$ in I, $F(\phi)\pi_i = \pi_j$. A morphism between two given objects $(L, (\phi_i)_{i\in \mathrm{Ob}(I)})$ and $(L', (\phi'_i)_{i\in \mathrm{Ob}(I)})$ is a morphism $\Phi \in \mathrm{Hom}_{\mathfrak{C}}(L, L')$ such that $\phi'_i\Phi = \phi_i$ for all $i \in \mathrm{Ob}(I)$. A colimit of F is an initial object of this category.

By the universal property, if a (co)limit exists, then it is uniquely determined up to a unique isomorphism, so there is no ambiguity in the notations $\varinjlim F$ and $\varprojlim F$.

Definition 1.27. *Let $(I, \leqslant)$ be a preordered set and let $\mathfrak{C}$ be a category. A* direct system *$\{C_i,\ \phi_{j,i} : i \leqslant j \in I\}$ consists of*

– *a family $\{C_i : i \in I\}$ of objects of $\mathfrak{C}$;*

– *a family $\{\phi_{j,i} : C_i \to C_j : i \leqslant j\}$ of morphisms such that $\phi_{k,j}\phi_{j,i} = \phi_{k,i}$, whenever $i \leqslant j \leqslant k$.*

An inverse system *is defined dually.*

Notice that, to specify a direct system $\{C_i,\ \phi_{j,i} : i \leqslant j \in I\}$ is equivalent to define a functor $F : \mathrm{Cat}(I, \leqslant) \to \mathfrak{C}$, dually, an inverse system $\{D_i,\ \psi_{i,j} : i \leqslant j \in I\}$ corresponds to a functor $G : \mathrm{Cat}(I, \leqslant)^{op} \to \mathfrak{C}$. In this case we also use the following notation

$$\varinjlim F = \varinjlim_{i\in I} C_i \quad \text{and} \quad \varprojlim G = \varprojlim_{i\in I} D_i\,.$$

Definition 1.28. *A category $\mathfrak{C}$ is* complete *(resp.,* cocomplete*) if for every small category I and every functor $F : I^{op} \to \mathfrak{C}$ (resp., $F : I \to \mathfrak{C}$), F has a limit (resp., a colimit).*

Lemma 1.29. [70, Corollary, Sec. 3, Ch. V] *Let $\mathfrak{C}$ be a complete (resp., cocomplete) category and let I be a small category. Then,* $\mathrm{Func}(I, \mathfrak{C})$ *is a complete (resp., cocomplete) category.*

Example 1.30. *Let I be a set and let $\mathcal{A} = (A_i)_{i\in I}$ be a family of objects of $\mathfrak{C}$ indexed by I. A* product *of $\mathcal{A}$ is a pair $(\prod \mathcal{A}, (\pi_i)_{i\in I})$, where $\prod \mathcal{A} \in \mathrm{Ob}(\mathfrak{C})$ and $\pi_i \in \mathrm{Hom}_{\mathfrak{C}}(\prod \mathcal{A}, A_i)$ for all $i \in I$, which satisfies the following universal property:*

(∗) for any pair $(P, (\phi_i)_{i\in I})$, with $P \in \mathrm{Ob}(\mathfrak{C})$ and $\phi_i \in \mathrm{Hom}_{\mathfrak{C}}(P, A_i)$ for all $i \in I$, there exists a unique morphism $\Phi : P \to \prod \mathcal{A}$ such that $\phi_i = \pi_i \circ \Phi$ for all $i \in I$.

Dually, the coproduct *of $\mathcal{A}$ is a pair $(\bigoplus \mathcal{A}, (\epsilon_i)_{i\in I})$, with $\bigoplus \mathcal{A} \in \mathrm{Ob}(\mathfrak{C})$ and $\epsilon_i \in \mathrm{Hom}_{\mathfrak{C}}(A_i, \bigoplus \mathcal{A})$ for all $i \in I$, which is a product of $\mathcal{A}$ in $\mathfrak{C}^{op}$.*
(Co)Products correspond to (co)limits of functors from the discrete category over the set I to the category $\mathfrak{C}$.

Example 1.31. *Let X, Y and $Z \in \mathrm{Ob}(\mathfrak{C})$, and consider two morphisms $\phi : Z \to X$ and $\phi' : Z \to Y$. A* push out *(also denoted by PO) of ϕ and ϕ' is a triple $(P, \alpha : X \to P, \alpha' : Y \to P)$, where $\alpha\phi = \alpha'\phi'$, that satisfies the following universal property:*

(∗) for any triple $(Q, f : X \to Q, f' : Y \to Q)$, where $f'\phi = \phi' f$, there exists a unique morphism $\Phi : P \to Q$ such that $\Phi\alpha' = f'$ and $\Phi\alpha = f$.

Dually, given two morphisms $\psi : X \to Z$ and $\psi' : Y \to Z$. A pull back *(also denoted by PB) of ψ and ψ' is a triple $(P, \beta : P \to X, \beta : P \to Y)$, where $\psi\beta = \psi'\beta'$ which is a PO in $\mathfrak{C}^{op}$.*
PBs and POs are respectively limits and colimit of functors from $\{\bullet \leftarrow \bullet \to \bullet\}$ to $\mathfrak{C}$.

Definition 1.32. *A non-empty category I is said to be* filtered *if*

- *given i and $j \in \mathrm{Ob}(I)$ there exists $k \in \mathrm{Ob}(I)$ such that $\mathrm{Hom}_I(i, k) \neq \varnothing \neq \mathrm{Hom}_I(j, k)$;*
- *given i, $j \in \mathrm{Ob}(I)$ and two arrows ϕ, $\psi \in \mathrm{Hom}_I(i, j)$ there exist $k \in \mathrm{Ob}(I)$ and $\xi \in \mathrm{Hom}_I(j, k)$ such that $\xi \circ \phi = \xi \circ \psi$.*

Example 1.33. *A preordered set $(I, \leqslant)$ is said to be* directed*(resp.,* downward directed*) if for all i, $j \in I$ there exists $k \in I$ such that $i \leqslant k$ and $j \leqslant k$ (resp., $i \geqslant k$ and $j \geqslant k$).*
Consider the category $\mathrm{Cat}(I, \leqslant)$ *defined in Example 1.7. Notice that* $\mathrm{Cat}(I, \leqslant)$ *is filtered if and only if $(I, \leqslant)$ is directed.*

Definition 1.34. *Let $\mathfrak{C}$ be a category, let I be a small category and let $F : I \to \mathfrak{C}$ (resp., $G : I^{op} \to \mathfrak{C}$) be a functor. If I is filtered, a colimit of F (resp., a limit of G) is said be a* filtered colimit *(resp.,* a filtered limit*).*

Definition 1.35. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories, let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor. We say that F* commutes with limits *(or* preserves limits*) if, for any small category I and any functor $G : I^{op} \to \mathfrak{C}$ that has a limit in $\mathfrak{C}$, the composition $FG : I^{op} \to \mathfrak{D}$ also has a limit and there is an isomorphism*

$$\varprojlim(FG) \to F(\varprojlim G)$$

that is compatible with the natural maps of limits. Functors that commute with colimits *(or* preserve colimits*) are defined dually.*
Similarly, if $F : \mathfrak{C}^{op} \to \mathfrak{D}$ is a functor, we say that F sends limits to colimits *if, for any small category I and any functor $G : I^{op} \to \mathfrak{C}$ that has a limit in $\mathfrak{C}$, the composition $FG : I \to \mathfrak{D}$ has a colimit and there is an isomorphism*

$$\varinjlim(FG) \to F(\varprojlim G)$$

that is compatible with the natural maps of limits and colimits. Functors that send colimits to limits *are defined dually.*
Restricting the class of possible small categories I, standard variations are possible. For example we say that $F : \mathfrak{C} \to \mathcal{D}$ commutes with (finite, countable) products*, if for any discrete category I over a (finite, countable) set and any functor $G : I^{op} \to \mathfrak{C}$ that has a product in $\mathfrak{C}$, the composition $FG : I^{op} \to \mathfrak{D}$ also has a product and there is an isomorphism $\varprojlim(FG) \to F(\varprojlim G)$ that is compatible with the natural maps of products.*

Lemma 1.36. [70, Theorem 1, Sec. 4, Ch. V] *Let $\mathfrak{C}$ be a category and let $C \in \mathrm{Ob}(\mathfrak{C})$. Then, the functor $\mathrm{Hom}_{\mathfrak{C}}(C, -) : \mathfrak{C} \to \underline{\mathrm{Set}}$ commutes with limits, while the functor $\mathrm{Hom}_{\mathfrak{C}}(-, C) : \mathfrak{C}^{op} \to \underline{\mathrm{Set}}$ sends colimits to limits.*

Lemma 1.37. [70, Theorem 1, Sec. 5, Ch. V] *Let $\mathfrak{C}$, $\mathfrak{D}$ be two categories and let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor. If F is a right adjoint then it preserves limits, while, if it is a left adjoint, it preserves colimits.*

Let I be a small category, let $\mathfrak{C}$ be a category and suppose that any functor $F : I \to \mathfrak{C}$ has a colimit. One can define a functor

$$\varinjlim : \mathrm{Func}(I, \mathfrak{C}) \to \mathfrak{C}$$

that associates to any functor $F \in \mathrm{Ob}(\mathrm{Func}(I, \mathfrak{C}))$ its colimit $\varinjlim F$. Indeed, let F and $F' \in \mathrm{Ob}(\mathrm{Func}(I, \mathfrak{C}))$, denote by $(\varinjlim F, (\epsilon_i)_{i \in \mathrm{Ob}(I)})$ and $(\varinjlim F', (\epsilon'_i)_{i \in \mathrm{Ob}(I)})$ the colimits of F and F' respectively, and take a natural transformation $\nu \in \mathrm{Hom}_{\mathrm{Func}(I,\mathfrak{C})}(F, F')$. Then, for all $i \in \mathrm{Ob}(I)$, there is a map $\phi_i = \epsilon'_i \circ \nu_i : F(i) \to \varinjlim F'$. By the universal property of the colimit, there is a unique morphism $\varinjlim \nu : \varinjlim F \to \varinjlim F'$, such that $\epsilon'_i \circ \nu_i = \varinjlim \nu \circ \epsilon_i$ for all $i \in \mathrm{Ob}(I)$.
Analogously, if any functor $F : \mathrm{Func}(I^{op}, \mathfrak{C}) \to \mathfrak{C}$ has a limit, one can show that there is a functor

$$\varprojlim : \mathrm{Func}(I^{op}, \mathfrak{C}) \to \mathfrak{C}\,.$$

With similar considerations (see also Lemma 1.20), given a set I and a category $\mathfrak{C}$ which has a coproduct (resp., product) for any I-indexed set of objects, one can define the coproduct functor $\bigoplus : \prod_I \mathfrak{C} \to \mathfrak{C}$ (the product functor $\prod : \prod_I \mathfrak{C} \to \mathfrak{C}$).

Lemma 1.38. [96, Proposition 8.8, Ch. IV] *Let $\mathfrak{C}$ be a complete category and let I, J be two small categories. Let $F : I \times J \to \mathfrak{C}$ be a functor and notice that it induces two functors*

$$\hat{F} : I \to \mathrm{Func}(J, \mathfrak{C}) \quad \textit{and} \quad \check{F} : J \to \mathrm{Func}(I, \mathfrak{C})\,.$$

Then, $\varinjlim(\varinjlim \hat{F}) = \varinjlim F = \varinjlim(\varinjlim \check{F})$.

Of course, the above lemma admits a dual formulation showing that "limits commute with limits".

Using the notions of limits and colimits one can give formulas to construct the left and right Kan extensions of a functor. Using these formulas one proves the following

Lemma 1.39. [65, Theorem 2.3.3] *Let I, J be small categories, let $F : I \to J$ be a functor and let $\mathfrak{C}$ be a category. Then,*

(1) *if $\mathfrak{C}$ has all colimits then $F^\dagger$ exists. Furthermore, if F is fully faithful, then $F^\dagger$ is fully faithful and there is a natural equivalence of functors $F_* F^\dagger \cong \mathrm{id}_{\mathrm{Func}(J,\mathfrak{C})}$;*

(2) *if $\mathfrak{C}$ has all limits then $F^\ddagger$ exists. Furthermore, if F is fully faithful, then $F^\ddagger$ is fully faithful and there is a natural equivalence of functors $F_* F^\ddagger \cong \mathrm{id}_{\mathrm{Func}(J,\mathfrak{C})}$;*

In the last part of this subsection we discuss the notions of kernel and cokernel.

Definition 1.40. *Let $\mathfrak{C}$ be a category. An object $C \in \mathrm{Ob}(\mathfrak{C})$ is a* zero-object *if it is both initial and terminal.*

If a zero-object exists, all the zero-objects in $\mathfrak{C}$ are isomorphic, so one can speak about the zero-object of $\mathfrak{C}$, which is usually denoted by 0. For any $A \in \mathrm{Ob}(\mathfrak{C})$ we denote by $\zeta_{0,A}$ (resp., $\zeta_{A,0}$) the unique element of $\mathrm{Hom}_{\mathfrak{C}}(A, 0)$ (resp., $\mathrm{Hom}_{\mathfrak{C}}(0, A)$). The morphisms of the form $\zeta_{B,A} = \zeta_{B,0} \circ \zeta_{0,A} \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ are called *zero-morphisms*. If we do not need to specify A and B we just write 0 for $\zeta_{B,A}$.

Definition 1.41. *Let $\mathfrak{C}$ be a category with a zero-object and let $\phi : A \to B$ be a morphism in $\mathfrak{C}$. A* kernel *of ϕ is a pair $(\mathrm{Ker}(\phi), k)$ with $\mathrm{Ker}(\phi) \in \mathrm{Ob}(\mathfrak{C})$ and $k \in \mathrm{Hom}_{\mathfrak{C}}(\mathrm{Ker}(\phi), A)$ such that $k\phi = 0$, which satisfies the following universal property*

(∗) *for any pair (K, k') with $K \in \mathrm{Ob}(\mathfrak{C})$ and $k' \in \mathrm{Hom}_{\mathfrak{C}}(K, A)$ such that $k'\phi = 0$, there exists a unique morphism $\psi : K \to \mathrm{Ker}(\phi)$ such that $k\psi = k'$.*

Dually, a pair $(\mathrm{CoKer}(\phi), c)$ with $\mathrm{CoKer}(\phi) \in \mathrm{Ob}(\mathfrak{C})$ and $c \in \mathrm{Hom}_{\mathfrak{C}}(B, \mathrm{CoKer}(\phi))$, is a cokernel *of ϕ if it defines a kernel of ϕ in $\mathfrak{C}^{op}$.*

1.1.3 (Pre)Additive and Abelian categories

Definition 1.42. *A category $\mathfrak{C}$ is* pre-additive *if it satisfies the following two axioms:*

(Add.1) *it has a zero-object;*

(Add.2) *given A, $B \in \mathrm{Ob}(\mathfrak{C})$, there is a map $+ : \mathrm{Hom}_{\mathfrak{C}}(A, B) \times \mathrm{Hom}_{\mathfrak{C}}(A, B) \to \mathrm{Hom}_{\mathfrak{C}}(A, B)$ such that $(\mathrm{Hom}_{\mathfrak{C}}(A, B), +)$ is an Abelian group and*

- *$(\psi_1 + \psi_2)\phi = \psi_1\phi + \psi_2\phi$, for all A, B, $C \in \mathrm{Ob}(\mathfrak{C})$, $\phi \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ and ψ_1, $\psi_2 \in \mathrm{Hom}_{\mathfrak{C}}(B, C)$;*
- *$\phi'(\psi_1 + \psi_2) = \phi'\psi_1 + \phi' + \psi_2$, for all B, C, $D \in \mathrm{Ob}(\mathfrak{C})$, ψ_1, $\psi_2 \in \mathrm{Hom}_{\mathfrak{C}}(B, C)$ and $\phi' \in \mathrm{Hom}_{\mathfrak{C}}(C, D)$.*

A pre-additive category is additive *if it satisfies the following axiom:*

(Add.3) *all finite products and coproducts exist.*

Example 1.43. *The category $\underline{\mathrm{Ab}}$ is additive. In fact, the zero-object in $\underline{\mathrm{Ab}}$ is the trivial Abelian group, while the additive structure on $\mathrm{Hom}_{\underline{\mathrm{Ab}}}(A, B)$, for any two Abelian groups A and B, is given by point-wise addition. Furthermore, given a set I and Abelian groups A_i, for all $i \in I$, one defines*

$$\prod_I A_i = \{(x_i)_{i\in I} : x_i \in A_i\} \quad \text{and} \quad \bigoplus_I A_i = \{(x_i)_{i\in I} \in \prod_I A_i : |i \in I : x_i \neq 0| < \infty\}\,.$$

For all i, $j \in I$, let $\delta_i^j : A_i \to A_j$ be a group homomorphism such that $\delta_i^j(x) = x$ if $i = j$ and $\delta_i^j(x) = 0$ otherwise. For all $j \in I$, there are canonical group homomorphisms

$$\pi_j : \prod_I A_i \to A_j \quad \text{and} \quad \epsilon_j : A_j \to \bigoplus_I A_i\,,$$

such that $\pi_j((x_i)_{i\in I}) = x_j$ *and* $\epsilon_j(x) = (\delta^i_j(x))_{i\in I}$. *One can show that* $(\prod_I A_i, (\pi_i)_{i\in I})$ *and* $(\bigoplus_I A_i, (\epsilon_i)_{i\in I})$ *are respectively the product and the coproduct of the family* $\{A_i : i \in I\}$. *So* <u>Ab</u> *has not only finite products and coproducts, but it has a product and a coproduct for any set of objects.*

Definition 1.44. *Given two pre-additive categories* $\mathfrak{C}$ *and* $\mathfrak{C}'$, *a functor* $F : \mathfrak{C} \to \mathfrak{C}'$ *is* additive *if, given* A *and* $B \in \mathrm{Ob}(\mathfrak{C})$, *the canonical map*

$$\mathrm{Hom}_{\mathfrak{C}}(A, B) \to \mathrm{Hom}_{\mathfrak{C}'}(F(A), F(B))$$

is a homomorphism of Abelian groups.

Proposition 1.45. [70, Proposition 4, Sec. 2, Ch. VIII] *Let* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *be additive categories and let* $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ *be a functor. Then, the following are equivalent:*

(1) *F is additive;*

(2) *F commutes with finite coproducts.*

Example 1.46. *Given a pre-additive category* $\mathfrak{C}$, *we usually consider the functors* $\mathrm{Hom}_{\mathfrak{C}}(X, -)$ *and* $\mathrm{Hom}_{\mathfrak{C}}(-, X)$, *as functors* $\mathfrak{C} \to$ <u>Ab</u> *and* $\mathfrak{C}^{op} \to$ <u>Ab</u>, *respectively. Considering these functors with target the category of Abelian groups, it is not difficult to show that they are both additive.*

Example 1.47. *A* ring *is a quintuple* $(R, \cdot, +, 1, 0)$ *such that* $(R, \cdot, 1)$ *is a monoid, that we call the* multiplicative structure *of* R, *and* $(R, +, 0)$ *is an Abelian group, that we call the* additive structure *of* R. *Furthermore, one supposes that the multiplicative and the additive structures are compatible, that is:*

$$(r + s) \cdot t = (r \cdot t) + (s \cdot t) \quad \textit{and} \quad t \cdot (r + s) = (t \cdot r) + (t \cdot s) \,,$$

for all r, s *and* $t \in R$. *A ring is* commutative *if* $\cdot$ *is a commutative operation.*
Given two rings $(R, \cdot, +, 1, 0)$ *and* $(R', \cdot', +', 1', 0')$, *a map* $\phi : R \to R'$ *is a* ring homomorphism *if it is a homomorphism of monoids with respect to the addivite and multiplicative structures of* R *and* R'. *We denote by* <u>Ring</u> *the category of all rings with ring homomorphisms. It is not difficult to show that* <u>Ring</u> *is an additive category.*
For a given ring R, *the one-object category* $\mathfrak{C}_R$ *described in Example 1.4 is naturally a pre-additive category with the addition induced by the operation "+" in* R. *Furthermore, given a ring homomorphism* $\phi : R \to R'$, *it naturally induces an additive functor between* $\mathfrak{C}_R$ *and* $\mathfrak{C}_{R'}$.

Lemma 1.48. [70, Theorem 3, Sec. 1, Ch. IV] *Let* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *be two pre-additive categories and let* (F, G) *be an adjunction between them. Then,* F *is an additive functor if and only if* G *is an additive functor. Furthermore, in this case the natural maps*

$$\nu_{A,B} : \mathrm{Hom}_{\mathfrak{C}_2}(B, G(A)) \to \mathrm{Hom}_{\mathfrak{C}_1}(F(B), A)$$

are isomorphisms of Abelian groups for all $A \in \mathfrak{C}_1$ *and* $B \in \mathfrak{C}_2$.

Corollary 1.49. *Let* $\mathfrak{C}_1$ *and* $\mathfrak{C}_2$ *be additive categories and let* $F : \mathfrak{C}_1 \to \mathfrak{C}_2$ *be a functor. If* F *has a right or a left adjoint, then* F *is additive.*

Proof. By Lemma 1.37, a left adjoint functor preserves colimits and, in particular, it commutes with binary coproducts. So, if F is a left adjoint, it is additive by Proposition 1.45. On the other hand, if F has a left adjoint G, then G is additive and thus F is additive by Lemma 1.48. $\square$

In a preadditive category one can show that finite products and finite coproducts coincide. The following lemma can be proved using [96, Proposition 3.2, Ch. IV].

Lemma 1.50. *Let $\mathfrak{C}$ be a preadditive category and let $\{C_i : i \in I\}$ be a finite family of objects. Given a product $(\prod_{i\in I} C_i, (\pi_i)_{i\in I})$ there exist morphisms $\epsilon'_j : C_j \to \prod_{i\in I} C_i$ such that $(\prod_{i\in I} C_i, (\epsilon'_i)_{i\in I})$ is a coproduct. Dually, given a coproduct $(\bigoplus_{i\in I} C_i, (\epsilon_i)_{i\in I})$ there exist morphisms $\pi'_j : \bigoplus_{i\in I} C_i \to C_j$ such that $(\bigoplus_{i\in I} C_i, (\pi'_i)_{i\in I})$ is a product.*
In particular, $\{C_i : i \in I\}$ has a product if and only if it has a coproduct and, in this case,

$$\prod_{i\in I} C_i \cong \bigoplus_{i\in I} C_i\,.$$

Definition 1.51. *Let $\mathfrak{C}$ be an additive category. Given a morphism $\phi : A \to B$ in $\mathfrak{C}$, the* image $\mathrm{Im}(\phi)$ *(resp., the* coimage $\mathrm{CoIm}(\phi)$*) of ϕ is $\mathrm{Im}(\phi) = \mathrm{Ker}(\mathrm{CoKer}(\phi))$ (resp., $\mathrm{CoIm}(\phi) = \mathrm{CoKer}(\mathrm{Ker}(\phi))$). We say that the additive category $\mathfrak{C}$ is* Abelian *if and only if*

(Ab.1) *every morphism of $\mathfrak{C}$ has a kernel and a cokernel;*

(Ab.2) *for every morphism $\phi : A \to B$ in $\mathfrak{C}$, the canonical morphism from $\mathrm{Im}(\phi)$ to $\mathrm{CoIm}(\phi)$ is an isomorphism, that is, the unique map $\bar\phi$ making the following diagram commute (which exists by the universal properties of kernels and cokernels) is an isomorphism:*

$$\begin{tikzcd} \mathrm{Ker}(\phi) \arrow[r] & A \arrow[r] \arrow[d, "\phi"] & \mathrm{CoIm}(\phi) \arrow[d, "\bar\phi"] \\ \mathrm{CoKer}(\phi) & B \arrow[l] & \mathrm{Im}(\phi)\,. \arrow[l] \end{tikzcd} \tag{1.1.1}$$

Example 1.52. *The category Ab is an Abelian category. We have seen in Example 1.43 that Ab is additive. Furthermore, given a morphism $\phi : A \to B$, $\mathrm{Ker}(\phi)$ is the subgroup $\{x \in A : \phi(x) = 0\}$ with the natural inclusion in A, while $\mathrm{CoKer}(\phi)$ is the quotient $B/\{\phi(x) : x \in A\}$, with the natural projection $B \to B/\{\phi(x) : x \in A\}$.*

Definition 1.53. *Let $\mathfrak{C}$ be a category, let A, B and $X \in \mathrm{Ob}(\mathfrak{C})$ and fix three morphisms $\phi, \psi \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ and $\chi \in \mathrm{Hom}_{\mathfrak{C}}(X, A)$. Then,*

– χ separates ϕ *and ψ if $\phi \neq \psi$ implies $\phi\chi \neq \psi\chi$;*

– χ *is an* epimorphism *if it separates any pair of morphisms in $\mathrm{Hom}_{\mathfrak{C}}(A, B)$, for all $B \in \mathrm{Ob}(\mathfrak{C})$;*

– χ *is a* monomorphism *if it is an epimorphism in $\mathfrak{C}^{op}$.*

The following lemma is an immediate consequence of the definitions:

Lemma 1.54. *Let $\mathfrak{C}$ be a category, let A, B, $C \in \mathrm{Ob}(C)$ and consider two morphisms $\phi : A \to B$ and $\psi : B \to C$. The following statements hold true:*

(1) *if $\psi\phi$ is a monomorphism, then ϕ is a monomorphism;*

(2) *if $\psi\phi$ is an epimorphism, then ψ is an epimorphism.*

Example 1.55. *In the category Set (resp., Semi, Mon), monomorphisms are precisely injective maps (resp., injective homomorphisms), epimorphisms are surjective maps (resp., surjective homomorphisms) and the isomorphisms are bijective maps (resp., bijective homomorphisms).*

The following lemma will be useful later on

Lemma 1.56. *Let $\mathfrak{C}$ be a category and let $\phi : X \to Y$ be a morphism in $\mathfrak{C}$. A morphism $k' : K \to X$ is a kernel of ϕ if and only if it is a monomorphism such that:*

(∗) *for any pair (H, h) with $H \in \mathrm{Ob}(\mathfrak{C})$ and $h \in \mathrm{Hom}_{\mathfrak{C}}(H, X)$ such that $h\phi = 0$, there exists a morphism $\psi : H \to K$ such that $k'\psi = h$.*

Proof. Let $k : \mathrm{Ker}(\phi) \to X$ be a kernel of ϕ. It is clear that k satisfies $(*)$, let us show that k is a monomorphism. Indeed, given an object $Z \in \mathrm{Ob}(\mathfrak{C})$ and morphisms $\psi_1, \psi_2 \in \mathrm{Hom}_{\mathfrak{C}}(Z, \mathrm{Ker}(\phi))$ such that $k\psi_1 = k\psi_2$, since $\phi(k\psi_1) = 0 = \phi(k\psi_2)$ and using the definition of kernel, there exists a unique morphism $\psi : Z \to \mathrm{Ker}(\phi)$ such that $k\psi = k\psi_1 = k\psi_2$, that is $\psi_1 = \psi = \psi_2$.
On the other hand, let $k' : K \to X$ be a monomorphism that satisfies $(*)$. Let $Z \in \mathrm{Ob}(\mathfrak{C})$ and let $\psi : Z \to X$ be such that $\phi\psi = 0$. By $(*)$, there exists a morphism $\psi' : Z \to K$ such that $k'\psi' = \psi$, while such ψ' is unique by the fact that k' is a monomorphism. □

Notice that an isomorphism is both an epimorphism and a monomorphism. One can find examples where the converse does not hold true. On the other hand, in Abelian categories we can prove the following:

Lemma 1.57. [96, Propositions 2.3 and 4.1] *Let $\mathfrak{C}$ be an Abelian category and let $\phi : A \to B$ be a morphism. Then,*

(1) *ϕ is a monormophism if and only if* $\mathrm{Ker}(\phi) = 0$*;*

(2) *ϕ is an epimorphism if and only if* $\mathrm{CoKer}(\phi) = 0$*;*

(3) *ϕ is an isomorphism if and only if it is both mono and epi.*

Definition 1.58. *Let $\mathfrak{C}$ be an Abelian category.*

– *A sequence $C^1 \xrightarrow{d^1} C^2 \xrightarrow{d^2} \dots \xrightarrow{d^{n-1}} C^n$ of morphisms in $\mathfrak{C}$ is* exact *in C^i (for $i = 2, 3, \dots, n-1$) if the canonical morphism $\mathrm{Im}(\phi^{i-1}) \to C^i$ is a kernel for ϕ^i; it is* exact *if it is exact in C^i for all $i = 2, \dots, n-1$;*

– *a* short exact sequence *is an exact sequence of the form $0 \to A \to B \to C \to 0$.*

Proposition 1.59. [70, Lemma 5, Sec. 4, Ch. V] *Let $\mathfrak{C}$ be an Abelian category and consider the following commutative diagram*

$$\begin{tikzcd} & A' \arrow[r] \arrow[d, "f"] & B' \arrow[r] \arrow[d, "g"] & C' \arrow[r] \arrow[d, "h"] & 0 \\ 0 \arrow[r] & A \arrow[r] & B \arrow[r] & C & \end{tikzcd}$$

If the above diagram has exact rows, then there is an exact sequence

$$\mathrm{Ker}(f) \to \mathrm{Ker}(g) \to \mathrm{Ker}(h) \to \mathrm{CoKer}(f) \to \mathrm{CoKer}(g) \to \mathrm{CoKer}(h)\,.$$

Moreover, if $A' \to B'$ is a monomorphism, then so is $\mathrm{Ker}(f) \to \mathrm{Ker}(g)$, and if $B \to C$ is an epimorphism, then so is $\mathrm{CoKer}(g) \to \mathrm{CoKer}(h)$.

Definition 1.60. *Let $F : \mathfrak{C} \to \mathfrak{C}'$ be an additive functor between two Abelian categories. We say that F is*

– left exact *if* $0 \to F(A) \to F(B) \to F(C)$ *is an exact sequence for any short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathfrak{C}$*;*

– right exact *if* $F(A) \to F(B) \to F(C) \to 0$ *is an exact sequence for any short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathfrak{C}$*;*

– exact *if* $0 \to F(A) \to F(B) \to F(C) \to 0$ *is a short exact sequence for any short exact sequence* $0 \to A \to B \to C \to 0$ *in* $\mathfrak{C}$.

Notice that F is exact if and only if it is both left and right exact.

Example 1.61. *Given an Abelian category* $\mathfrak{C}$ *and* $C \in \mathrm{Ob}(\mathfrak{C})$*, the* Hom-*functors* $\mathrm{Hom}_{\mathfrak{C}}(C, -) : \mathfrak{C} \to \underline{\mathrm{Ab}}$ *and* $\mathrm{Hom}_{\mathfrak{C}}(-, C) : \mathfrak{C}^{op} \to \underline{\mathrm{Ab}}$ *(see Example 1.46) are both left exact (use the universal properties of kernels and cokernels).*

The following proposition can be proved using that (co)kernels are particular (co)limits.

Proposition 1.62. [96, Proposition 8.6, Ch. IV] *Let* $\mathfrak{C}$, $\mathfrak{C}'$ *be Abelian categories and let* $F : \mathfrak{C} \to \mathfrak{C}'$ *be an additive functor. Then,*

(1) *F is left exact if and only if F commutes with finite limits if and only if F preserves kernels;*

(2) *F is right exact if and only if F commutes with finite colimits if and only if F preserves cokernels.*

The following corollary follows from Proposition 1.62 and Lemma 1.37.

Corollary 1.63. *Let* $\mathfrak{C}$ *and* $\mathfrak{C}'$ *be Abelian categories. If* $F : \mathfrak{C} \to \mathfrak{C}'$ *is a left (resp., right) adjoint functor, then F is right (resp., left) exact.*

1.1.4 Subobjects and quotients

Definition 1.64. *Let A, B and B' be objects in a category $\mathfrak{C}$. Two monomorphisms $\phi : B \to A$ and $\phi' : B' \to A$ are* equivalent *if there exists an isomorphism $\psi : B \to B'$ such that $\phi = \phi'\psi$. An equivalence class of monomorphisms with target A is, by definition, a* subobject *of A.* Quotient objects *are defined dually as equivalence classes of epimorphisms.*
A subobject of a quotient object (or, equivalently, a quotient of a subobject) is said to be a segment.

With a little abuse, we say that some representing monomorphism $\phi : B \to A$ is a subobject, if there is no need to specify the morphism ϕ we just write $B \leqslant A$. There is a partial order relation between subobjects, in fact, given two subobjects $\phi : B \to A$ and $\phi' : B' \to A$, $B \leqslant B'$ if there is a morphism $\psi : B \to B'$ such that $\phi = \phi'\psi$. It easily follows that, if $B \leqslant B'$ and $B' \leqslant B$, then they represent the same subobject.

Notice that if $\mathfrak{C}$ has a zero-object $0 \in \mathrm{Ob}(\mathfrak{C})$, then $0 \leqslant X$ for all $X \in \mathrm{Ob}(\mathfrak{C})$.

Example 1.65. *In the category* $\underline{\mathrm{Group}}$ *any monomorphism into a group G is equivalent to the inclusion of a subgroup in G, while any epimorphism is equivalent to the canonical projection from G onto the quotient over a normal subgroup.*
Similar descriptions hold in $\underline{\mathrm{Ab}}$.

Definition 1.66. *Let $\mathfrak{C}$ be an Abelian category and let $A \in \mathrm{Ob}(\mathfrak{C})$. A subobject $B \leqslant A$ is a* direct summand *of A if there exists a subobject $C \leqslant A$ such that the canonical morphism $B \oplus C \to A$ is an isomorphism.*
Furthermore, a non-zero object $X \in \mathrm{Ob}(\mathfrak{C})$ is indecomposable *if its only direct summands are 0 and X.*

By definition, a direct summand is a sub-object and, by Lemma 1.50, it is also a quotient object.

Definition 1.67. *A category $\mathfrak{C}$ is said to be* well-powered *if for any object X there is a set (as opposed to a proper class) of (equivalence classes of) subobjects. We denote by $\mathcal{L}(X)$ the poset of subobjects of X, with the partial order relation $\leqslant$.*

For an example of an Abelian category which is not well-powered see [78, Corollary C.3.3].

Let $\mathfrak{C}$ be an Abelian category and let ϕ be a morphism in $\mathfrak{C}$. Using Lemma 1.54 one can show that the kernel of ϕ is a monomorphism, while its cokernel is an epimorphism. By the universal property of kernels (resp., cokernels), any other kernel (resp., cokernel) of ϕ represents the same subobject (resp., quotient object) of A. Thus, there is no ambiguity in writing $\mathrm{Ker}(\phi) \leqslant A$. For any subobject $A' \leqslant A$, we denote by A/A' the quotient object represented by the cokernel of the monomorphism $A' \to A$.

A consequence of the axiom (Ab.2) is the following

Lemma 1.68. *Let $\mathfrak{C}$ be an Abelian category and let $\phi : A \to B$ be a morphism. Then, $A/\mathrm{Ker}(\phi) \cong \mathrm{Im}(\phi)$.*

Given an object A, a subobject $\epsilon : C \to A$ and a morphism $\phi : A \to B$, the *restriction* of ϕ to C is the composition $\phi\restriction_C = \phi \circ \epsilon : C \to B$. The image of $\phi\restriction_C$ is denoted by $\phi(C)$, in particular, $\mathrm{Im}(\phi) = \phi(A)$. By definition of the image, there is an induced morphism $C \to \phi(C)$, such that the composition $C \to \phi(C) \to B$ is exactly $\phi\restriction_C$. Abusing notation we denote also this second morphism by $\phi\restriction_C$. On the other hand, given a sub-object $\epsilon : D \to B$, there is a sub-object $\phi^{-1}(D) \to A$ defined by the following pullback diagram:

$$\begin{tikzcd} \phi^{-1}(D) \arrow[r] \arrow[d] \arrow[dr, phantom, "\text{PB}"] & D \arrow[d, "\epsilon"] \\ A \arrow[r, "\phi"'] & B \end{tikzcd}$$

Definition 1.69. *Let $\mathfrak{C}$ be an Abelian category. Given an endomorphism $\phi : A \to A$, a sub-object C of A is ϕ-*invariant *if $\phi(C) \leqslant C$.*

Definition 1.70. *Let $(L, \leqslant)$ be a poset and let $F \subseteq L$ be a subset. An* upper bound *(resp.,* lower bound*) for F is an element $x \in L$ such that $x \geqslant f$ (resp., $x \leqslant f$) for all $f \in F$. The* least upper bound *or* join *(resp.,* greatest lower bound *or* meet*) of F is the minimum (resp., maximum) of the set of all the upper (resp., lower) bounds of F. Least upper bounds and greatest lower bounds may not exist but, if they do, we denote them respectively by $\bigvee F$ and $\bigwedge F$ or, by $f_1 \vee \cdots \vee f_k$ and $f_1 \wedge \cdots \wedge f_k$ if $F = \{f_1, \ldots, f_k\}$ is finite.*
A poset is a lattice *if any of its finite subsets has a least upper bound and a greatest lower bound. Furthermore, a lattice is* complete *if it has joins and meets for any of its subsets (finite or infinite).*
Given two lattices $(L, \leqslant)$, $(L', \leqslant)$ and a map $\phi : L \to L'$,

– ϕ *is a* homomorphism of semi-lattices *if* $\phi(a \vee b) = \phi(a) \vee \phi(b)$*, for all* $a, b \in L$*;*

– ϕ *is a* lattice homomorphism *if it is a semi-lattice homomorphism and* $\phi(x \wedge y) = \phi(x) \wedge \phi(y)$*, for all* x *and* $y \in L$*;*

– ϕ commutes with arbitrary joins *if* $\phi(\bigvee F) = \bigvee\{\phi(f) : f \in F\}$*, whenever these joins exist.*

Lemma 1.71. [96, Proposition 1.2, Ch. III] *Let* $(L, \leqslant)$ *be a poset. If any subset of* L *has a least upper bound, then* L *is a complete lattice.*

Of course, the above lemma admits a dual statement: if any subset of a given poset $(L, \leqslant)$ has a greatest lower bound, then L is a complete lattice.

Proposition 1.72. *Let* $\mathfrak{C}$ *be a well-powered Abelian category, let* $X, Y \in \mathrm{Ob}(\mathfrak{C})$ *and let* $\phi \in \mathrm{Hom}_{\mathfrak{C}}(X, Y)$*. Then,*

(1) $\mathcal{L}(X)$ *is a lattice;*

(2) *if* $\mathfrak{C}$ *is complete or cocomplete, then* $\mathcal{L}(X)$ *is complete;*

(3) *the induced map* $\Phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ *such that* $\Phi(C) = \phi(C)$ *for all* $C \in \mathcal{L}(X)$*, is a homomorphism of semi-lattices;*

(4) *if* $\mathfrak{C}$ *is cocomplete, then* Φ *commutes with arbitrary joins.*

Proof. Let $F = \{C_i : i \in I\}$ be a set of subobjects of $\mathcal{L}(X)$. By [96, Proposition 4.2, Ch. IV] we can construct the least upper bound (resp., greatest lower bound) of F as the image of the natural map $\bigoplus_I C_i \to X$ (resp., kernel of the natural map $X \to \prod_I X/C_i$) if this coproduct (resp., product) exists in $\mathfrak{C}$. Thus, when F is finite we can always construct its join and meet. Furthermore, if $\mathfrak{C}$ is complete (resp., cocomplete) any subset of $\mathcal{L}(X)$ has a join (resp., meet), so $\mathcal{L}(X)$ is a complete lattice by Lemma 1.71. This proves (1) and (2). To prove (3) and (4) see the proof of [96, Corollary 8.5, Ch.IV]. ☐

Definition 1.73. *Let* $\mathfrak{C}$ *be a well-powered Abelian category, let* $F = \{X_i : i \in I\}$ *be a set of subobjects of an object* X *in* $\mathfrak{C}$*. We denote respectively by* $\sum_{i\in I} X_i$ *and* $\bigcap_{i\in I} X_i$ *the least upper bound and the greatest lower bound of* F*, whenever they exist. If* $I = \{1, \dots, k\}$ *is a finite set, then we also use the notations* $\sum_{i\in I} X_i = X_1 + \dots + X_k$ *and* $\bigcap_{i\in I} X_i = X_1 \cap \dots \cap X_k$*, respectively.*

Lemma 1.74. [96, Proposition 5.2, Ch. IV] *Let* $\mathfrak{C}$ *be a well-powered Abelian category, let* $C \in \mathrm{Ob}(\mathfrak{C})$ *and let* C_1 *and* $C_2 \in \mathcal{L}(C)$*. Then,*

$$(C_1 + C_2)/C_1 \cong C_2/(C_1 \cap C_2)$$

Definition 1.75. *Let* M *be an object in* $\mathfrak{C}$*. A sub-object* $N \leqslant M$ *is* essential *if* $N \cap K = 0$ *implies* $K = 0$ *for any sub-object* $K \leqslant M$*. A morphism* $\phi : M \to M'$ *in* $\mathfrak{C}$ *is* essential *if* $\mathrm{Im}(\phi)$ *is an essential sub-object of* M'*.*

1.1.5 Grothendieck categories

Let $\mathfrak{C}$ be an Abelian category. In this subsection we introduce three further axioms together with their duals, these axioms were first defined by Grothendieck in [55]. They are not required in the definition of Abelian category but it is very common to work with Abelian categories satisfying some of these further assumptions.

(Ab.3) For every set $\{A_i : i \in I\}$ of objects of $\mathfrak{C}$, the coproduct $\bigoplus_{i\in I} A_i$ exists in $\mathfrak{C}$.

(Ab.3*) For every set $\{A_i : i \in I\}$ of objects of $\mathfrak{C}$, the product $\prod A_i$ exists in $\mathfrak{C}$.

We have seen in Example 1.43 that $\underline{\mathrm{Ab}}$ satisfies both (Ab.3) and (Ab.3*).

Lemma 1.76. [96, Corollary 8.3, Ch. IV] *Let $\mathfrak{C}$ be an Abelian category. Then, $\mathfrak{C}$ satisfies (Ab.3) (resp., (Ab.3*)) if and only if $\mathfrak{C}$ is cocomplete (resp., complete).*

(Ab.4) $\mathfrak{C}$ satisfies (Ab.3), and the coproduct of a family of monomorphisms is a monomorphism.

(Ab.4*) $\mathfrak{C}$ satisfies (Ab.3*), and the product of a family of epimorphisms is an epimorphism.

Lemma 1.77. *Let $\mathfrak{C}$ be an (Ab.3) Abelian category, then $\mathfrak{C}$ is (Ab.4) if and only if, for any set I, the functor $\bigoplus_I : \prod_I \mathfrak{C} \to \mathfrak{C}$ is exact. Dually, an (Ab.3*) category $\mathfrak{C}$ is (Ab.4*) if and only if $\prod_I : \prod_I \mathfrak{C} \to \mathfrak{C}$ is exact, for any set I.*

Proof. Let I be a set, by the dual of [96, Proposition 3.1, Ch. IV], $\bigoplus_I : \prod_I \mathfrak{C} \to \mathfrak{C}$ preserves epimorphisms, that is, it sends cokernels to cokernels. By Proposition 1.62 this means that $\bigoplus_I$ is right exact, while (Ab.4) is equivalent to say that $\bigoplus_I$ sends kernels to kernels, that is, it is left exact. □

(Ab.5) $\mathfrak{C}$ satisfies (Ab.3), and filtered colimits of exact sequences are exact.

(Ab.5*) $\mathfrak{C}$ satisfies (Ab.3*), and filtered limits of exact sequences are exact.

Proposition 1.78. [96, Proposition 1.1, Ch. V] *Let $\mathfrak{C}$ be a cocomplete Abelian category. Then the following are equivalent:*

(1) *$\mathfrak{C}$ satisfies (Ab.5);*

(2) *given $M \in \mathrm{Ob}(\mathfrak{C})$, $K \leqslant M$ and a directed system $\{N_i : i \in I\} \subseteq \mathcal{L}(M)$,*

$$\left(\sum_I N_i\right) \cap K = \sum_I (N_i \cap K)\,;$$

(3) *given a morphism $\phi : M \to M'$ and a directed system $\{N_i : i \in I\} \subseteq \mathcal{L}(M)$,*

$$\phi^{-1}\left(\sum_I N_i\right) = \sum_I \phi^{-1}(N_i)\,.$$

Definition 1.79. *Let $\mathfrak{C}$ be a category. A subclass $\mathcal{G}$ of* $\mathrm{Ob}(\mathfrak{C})$ generates $\mathfrak{C}$ *if every pair of distinct morphisms $f, g \in \mathrm{Hom}_{\mathfrak{C}}(A, B)$ there exists $X \in \mathcal{G}$ and $\chi \in \mathrm{Hom}_{\mathfrak{C}}(X, A)$ that separates ϕ and ψ. If $\mathcal{G} = \{G\}$ consists of a single object, we say that G is a* generator *of $\mathfrak{C}$. The definitions of* cogenerating class *and* cogenerator *are dual.*

Lemma 1.80. [96, Proposition 6.6, Ch. IV] *Let $\mathfrak{C}$ be an Abelian category with a generator, then $\mathfrak{C}$ is well-powered.*

Definition 1.81. *A* Grothendieck category *is an (Ab.5) Abelian category with a generator.*

1.1.6 Injective and projective objects

Definition 1.82. *Let $\mathfrak{C}$ be an Abelian category. An object $X \in \mathrm{Ob}(\mathfrak{C})$ is* injective *(resp.,* projective*) if the functor* $\mathrm{Hom}_{\mathfrak{C}}(-,X) : \mathfrak{C}^{op} \to \underline{\mathrm{Ab}}$ *(resp.,* $\mathrm{Hom}_{\mathfrak{C}}(X,-) : \mathfrak{C} \to \underline{\mathrm{Ab}}$*) is exact.*

Notice that an object X is injective in $\mathfrak{C}$ if and only if it is projective in $\mathfrak{C}^{op}$. In the following lemma we give equivalent characterizations of injective and projective objects. The proof given in [96, Proposition 6.1, Ch. I] can be easily adapted to our general context.

Lemma 1.83. *Let $\mathfrak{C}$ be an Abelian category and let $X \in \mathrm{Ob}(\mathfrak{C})$. The following are equivalent:*

(1) *X is an injective object;*

(2) $\mathrm{Hom}_{\mathfrak{C}}(\phi, X) : \mathrm{Hom}_{\mathfrak{C}}(B,X) \to \mathrm{Hom}_{\mathfrak{C}}(A,X)$ *is surjective for any monomorphism $\phi : A \to B$ in $\mathfrak{C}$;*

(3) *given a monomorphism $\phi : A \to B$ in $\mathfrak{C}$ and a morphism $\psi : A \to X$, there exists a morphism $\bar{\psi} : B \to X$ making the following diagram commutative:*

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r, "\phi"] \arrow[d, "\psi"'] & B \arrow[dl, dotted, "\bar{\psi}"] \\ & X & \end{tikzcd}$$

Dual characterizations hold for projective objects.

The following corollary is a direct consequence of Lemma 1.36.

Corollary 1.84. *Let $\mathfrak{C}$ be an Abelian category and let $\{C_i : i \in I\}$ be a set of objects. Then,*

(1) *$\bigoplus_I C_i$ is injective if and only if C_i is injective for all $i \in I$;*

(2) *$\prod_I C_i$ is projective if and only if C_i is projective for all $i \in I$.*

Lemma 1.85. [96, Proposition 1.4, Ch. X] *Let $\mathfrak{C}$ be a category, let $\mathfrak{D}$ be an Abelian subcategory, denote by $F : \mathfrak{D} \to \mathfrak{C}$ the inclusion functor and suppose that F has an exact left adjoint $G : \mathfrak{C} \to \mathfrak{D}$. Then, an object $E \in \mathrm{Ob}(\mathfrak{D})$ is injective in $\mathfrak{D}$ if and only if $F(E)$ is injective in $\mathfrak{C}$.*

Definition 1.86. *Let $\mathfrak{C}$ be an Abelian category. We say that $\mathfrak{C}$ has* enough injectives *if, for any $X \in \mathrm{Ob}(\mathfrak{C})$, there exists an injective object E and a monomorphism $X \to E$.*

When a category has enough injectives, it is interesting to know if there exists a "minimal" injective object containing a given object as a subobject:

Definition 1.87. *Let $\mathfrak{C}$ be an Abelian category and let $X \in \mathrm{Ob}(\mathfrak{C})$. An* injective envelope *of X is an object $E \in \mathrm{Ob}(\mathfrak{C})$ together with an essential monomorphism $X \to E$.*

Lemma 1.88. [96, Proposition 2.3, Ch. V] *Let $\mathfrak{C}$ be an Abelian category, let $X \in \mathrm{Ob}(\mathfrak{C})$ and let $\alpha : X \to E$, $\alpha' : X \to E'$ be two injective envelopes of X. Then, there is an isomorphism $\phi : E \to E'$ such that $\alpha' = \phi\alpha$. In particular, an injective envelope is unique up to isomorphism.*

Given an Abelian category $\mathfrak{C}$ and an object $X \in \mathrm{Ob}(\mathfrak{C})$, we denote by $E(X)$ the injective envelope of X, whenever it exists.

Lemma 1.89. [96, Proposition 2.5, Ch. V] *Let $\mathfrak{C}$ be a Grothendieck category. The following are equivalent:*

(1) *$\mathfrak{C}$ has enough injectives;*

(2) *any object of $\mathfrak{C}$ has an injective envelope.*

1.1.7 Categories of modules

Definition 1.90. *Let I be a small pre-additive category and let $\mathfrak{C}$ be a pre-additive category. The* additive functor category $\mathrm{Add}(I, \mathfrak{C})$ *is the full sub-category of* $\mathrm{Func}(I, \mathfrak{C})$ *whose objects are all the additive functors $I \to \mathfrak{C}$.*

Proposition 1.91. *Let $\mathfrak{C}$ be an Abelian category and let I be a small category. Then,*

(1) $\mathrm{Func}(I, \mathfrak{C})$ *is Abelian;*

(2) *if I is preadditive, then* $\mathrm{Add}(I, \mathfrak{C})$ *is an Abelian subcategory of* $\mathrm{Func}(I, \mathfrak{C})$*;*

(3) *if $\mathfrak{C}$ is complete (resp., cocomplete), then* $\mathrm{Add}(I, \mathfrak{C})$ *is complete (resp., cocomplete).*

Proof. Part (1) is [96, Propositions 7.1, Ch. IV], while part (2) is [96, Proposition 7.2, Ch. IV]. Finally, the proof of part (3) is completely analogous to the proof of [96, Proposition 8.7, Ch. IV] (see the comment after Proposition 8.8 on p. 102 of [96]). □

The following proposition is known under the name of Yoneda Lemma, it is important as it gives a family of projective generators for $\mathrm{Add}(I, \mathfrak{C})$.

Proposition 1.92. [96, Proposition 7.3 and Corollary 7.5, Ch. IV] *Let $\mathfrak{C}$ be an Abelian category and let I be a small preadditive category. For all $A \in \mathrm{Ob}(I)$ let $h^A = \mathrm{Hom}_I(A, -) : I \to \underline{\mathrm{Ab}}$ and $h_A = \mathrm{Hom}_I(-, A) : I^{op} \to \underline{\mathrm{Ab}}$. Then, there are natural isomorphisms:*

$$\mathrm{Hom}_{\mathrm{Add}(I,\mathfrak{C})}(h^A, T) \cong T(A) \quad \textit{and} \quad \mathrm{Hom}_{\mathrm{Add}(I^{op},\mathfrak{C})}(h_B, S) \cong S(B)\,,$$

for all A, $B \in \mathrm{Ob}(I)$, $T \in \mathrm{Add}(I, \mathfrak{C})$ and $S \in \mathrm{Add}(I^{op}, \mathfrak{C})$. The category I is equivalent to the full subcategory of $\mathrm{Add}(I, \mathfrak{C})$ *(resp., of* $\mathrm{Add}(I^{op}, \mathfrak{C})$*) whose objects are of the form h^B (resp., h_B), with $B \in \mathrm{Ob}(\mathfrak{C})$. Finally, the sets $(h^B)_{B \in \mathrm{Ob}(I)}$ and $(h_B)_{B \in \mathrm{Ob}(I)}$ are families of projective generators respectively for* $\mathrm{Add}(I, \mathfrak{C})$ *and* $\mathrm{Add}(I^{op}, \mathfrak{C})$.

Let I, J be small preadditive categories, consider an additive functor $F : I \to J$ and let $\mathfrak{C}$ be a Grothendieck category. Then, there is an induced functor

$$\mathrm{res}_F : \mathrm{Add}(J, \mathfrak{C}) \to \mathrm{Add}(I, \mathfrak{C})\,,$$

defined by composition (which is the restriction of the functor $F_* : \mathrm{Func}(J, \mathfrak{C}) \to \mathrm{Func}(I, \mathfrak{C})$ used in Definition 1.23). In the following lemma we prove the existence of what is usually called the "additive Kan extension along F", anyway in this context we prefer a different terminology, see Definition 1.94; for a proof of Lemma 1.93 and more details on this construction we refer to [75, Section 6].

Lemma 1.93. *Let I, J be small preadditive categories, let $\mathfrak{C}$ be a Grothendieck category and consider an additive functor $F : I \to J$. Then, ??F : $\mathrm{Add}(J, \mathfrak{C}) \to \mathrm{Add}(I, \mathfrak{C})$ is exact and it has a left adjoint.*

Definition 1.94. *Let I, J be small preadditive categories, let $\mathfrak{C}$ be a Grothendieck category and consider an additive functor $F : I \to J$. The functor* $\mathrm{res}_F : \mathrm{Add}(J, \mathfrak{C}) \to \mathrm{Add}(I, \mathfrak{C})$ *defined by composition with F is called the* scalar restriction along F *while its left adjoint* $\mathrm{ext}_F :$ $\mathrm{Add}(J, \mathfrak{C}) \to \mathrm{Add}(I, \mathfrak{C})$ *is called the* scalar extension along F.

In what follows we specialize to categories of modules.

Definition 1.95. *Let R be a ring and let $\mathfrak{C}_R$ be the pre additive category defined in Example 1.47, an additive functor $F : \mathfrak{C}_R \to \underline{\mathrm{Ab}}$ (resp., $G : \mathfrak{C}_R^{op} \to \underline{\mathrm{Ab}}$) is said to be a* left R-module *(resp.,* right R-module*). The category of left (resp., right) R-modules* $\mathrm{Add}(\mathfrak{C}_R, \underline{\mathrm{Ab}})$ *(resp.,* $\mathrm{Add}(\mathfrak{C}_R^{op}, \underline{\mathrm{Ab}})$*) is usually denoted by* R-Mod *(resp.,* Mod-R*).*

The following corollary can be deduced from Proposition 1.91, anyway the usual way to prove it is to explicitly construct kernels, cokernels, products and coproducts in categories of modules.

Corollary 1.96. *Let R be a ring, then* Mod-R *and* R-Mod *are complete and cocomplete Abelian categories.*

Consider a left R-module $F : \mathfrak{C}_R \to \underline{\mathrm{Ab}}$ and let $M = F(\bullet)$. We generally adopt the compact notation ${}_RM$, instead of $F : \mathfrak{C}_R \to \underline{\mathrm{Ab}}$, where the action of F on the morphisms of $\mathfrak{C}_R$ is encoded in the following "scalar multiplication":

$$R \times M \to M \quad (r, m) \mapsto rm = (F(r))(m)\,,$$

that satisfies the following properties:

(Mod.1) $r(m + n) = rm + rn$;

(Mod.2) $(r + s)m = rm + sm$;

(Mod.3) $(rs)m = r(sm)$;

(Mod.4) $1m = m$;

for all m, $n \in M$ and r, $s \in R$. Notice that, given two left R-modules ${}_RM$ and ${}_RM'$, a homomorphism of Abelian groups $\phi : M \to M'$ is a morphism in the category of left R-modules if and only if $\phi(rm) = r\phi(m)$.
Given a left R-module ${}_RM$, a *submodule* (that is, a subobject in the category R-Mod) is a subgroup $N \leqslant M$ such that $rn \in N$ for all $r \in R$ and $n \in N$.
Given a left R-module ${}_RM$ and a submodule $N \leqslant M$, the *quotient module* (that is, a quotient object in the category R-Mod) ${}_R(M/N)$ is the quotient group M/N with the following scalar multiplication

$$R \times M/N \to M/N \quad \text{such that} \quad (r, mN) \mapsto (rm)N\,.$$

Analogous definitions hold in Mod-R.

We denote by ${}_RR$ (resp., R_R) the Abelian group underlying R with left (resp., right) R-module structure coming from the multiplication in R. This module corresponds to the functor $h^\bullet$ (resp., $h_\bullet$) described by the Yoneda Lemma and it is a projective generator for R-Mod (resp., Mod-R). The submodules of ${}_RR$ are the *left ideals* of R, while the submodules of R_R are the *right ideals* of R.

Theorem 1.97. *Let R be a ring, then both* R-Mod *and* Mod-R *are complete and cocomplete Grothendieck categories with enough injectives and a projective generator.*

Proof. By Corollary 1.96, both R-Mod and Mod-R are complete and cocomplete Abelian categories generated respectively by ${}_RR$ and R_R, furthermore one can verify the axiom (Ab.5) by hand. The fact that they have enough injectives is proved, for example, in [96, Proposition 9.3, Ch. I]. □

Let R and S be two rings and let $\phi : R \to S$ be a ring homomorphism. As we said, this is the same as taking an additive functor $F_\phi : \mathfrak{C}_R \to \mathfrak{C}_S$ between the two one-object categories $\mathfrak{C}_R$ and $\mathfrak{C}_S$. Abusing notation, we denote res_{F_ϕ} by res_ϕ

$$\begin{aligned} \mathrm{res}_\phi : S\text{-Mod} &\longrightarrow R\text{-Mod} \\ (M : \mathfrak{C}_S \to \underline{\mathrm{Ab}}) &\longmapsto (M \circ F_\phi : \mathfrak{C}_R \to \underline{\mathrm{Ab}}) \end{aligned}$$

The left adjoint ext_ϕ to res_ϕ, that is, the scalar extension along F_ϕ, is usually constructed as a tensor product in this context, that is, $\mathrm{ext}_\phi \cong - \otimes_R {}_R R_S$ (see [75, Section 6]).

Definition 1.98. *Let R and S be two rings. A ring homomorphism $\phi : R \to S$ is* left (resp., right) flat *if* $\mathrm{ext}_\phi : R\text{-Mod} \to S\text{-Mod}$ *(resp.,* $\mathrm{ext}_\phi : \text{Mod-}R \to \text{Mod-}S$*) is an exact functor.*

Example 1.99. *Let D be a ring and recall that an element $x \in D$ is a* zero-divisor *if there exists a non-zero element $y \in D$ such that $xy = 0$. An element which is not a zero-divisor is said to be* regular*, if any non-zero element of D is regular then D is said to be a* domain*.*
Given a domain D, we let $\Sigma = \{x \in D : x \text{ is regular}\} = D \setminus \{0\}$ and we say that D is a left Ore domain *if it is a domain and $Dx \cap Dy \neq \{0\}$ for all $x, y \in D \setminus \{0\}$ (notice for example that any commutative domain is left Ore).*
Given a left Ore domain we define the left field of fractions *of D as follows: first we say that two elements $(s, d), (s', d') \in \Sigma \times D$ are equivalent, in symbols $(s, d) \sim (s', d')$, if there exist $a, b \in D$ such that $as = bs' \in \Sigma$ and $ad = bd'$. We denote by $[s, d]$ the equivalence class of (s, d) and we let $\Sigma^{-1}D = \Sigma \times D/\sim$ be the set of these equivalence classes. The set $\Sigma^{-1}D$ is a skew field when endowed with the following operations, for all $(s_1, d_1), (s_2, d_2) \in \Sigma \times D$:*

$$[s_1, d_1] \cdot [s_2, d_2] = [as_1, bd_2] \quad \text{and} \quad [s_1, d_1] + [s_2, d_2] = [u, cd_1 + dd_2],$$

where $ad_1 = bs_2 \in \Sigma$ and $u = cs_1 = ds_2 \in \Sigma$. There is a canonical injective ring homomorphism $\varepsilon : D \to \Sigma^{-1}D$, this induces a scalar restriction $\mathrm{res}_\varepsilon : \Sigma^{-1}D\text{-Mod} \to D\text{-Mod}$, and a scalar extension $\mathrm{ext}_\varepsilon : D\text{-Mod} \to \Sigma^{-1}D\text{-Mod}$. One can show that the scalar extension is exact in this particular case so that $\varepsilon : D \to \Sigma^{-1}D$ is a flat endomorphism of rings (see for example [96, Proposition 3.5, Ch. II]).

1.2 Homological algebra

1.2.1 Cohomology

In this subsection we recall some basic definitions and constructions inside the category of cochain complexes over a given Abelian category.

Definition 1.100. *Let $\mathfrak{C}$ be a an Abelian category and consider a sequence of objects and morphisms*

$$X^\bullet : \quad \cdots \longrightarrow X^{i-1} \xrightarrow{d^{i-1}} X^i \xrightarrow{d^i} X^{i+1} \longrightarrow \cdots$$

We say that $X^\bullet$ is a cochain complex *if $d^i d^{i-1} = 0$, for all $i \in \mathbb{Z}$. A cochain complex $X^\bullet$ is* exact *in X^i if $\mathrm{Im}(d^{i-1}) = \mathrm{Ker}(d^i)$. The cochain complex $X^\bullet$ is* exact *(or* acyclic*) if it is exact in X^i for all $i \in \mathbb{Z}$.*

Definition 1.101. *Given an Abelian category $\mathfrak{C}$ and two cochain complexes $C^\bullet$ and $D^\bullet$, a* morphism of cochain complexes *$\phi^\bullet : C^\bullet \to D^\bullet$ is a sequence of morphisms $\phi^i : C^i \to D^i$ such* *that the following diagram commutes:*

$$\begin{tikzcd} C^\bullet \arrow[d, "\phi^\bullet"] & \cdots \arrow[r] & C^{i-1} \arrow[r, "d_C^{i-1}"] \arrow[d, "\phi^{i-1}"] & C^i \arrow[r, "d_C^i"] \arrow[d, "\phi^i"] & C^{i+1} \arrow[r, "d_C^{i+1}"] \arrow[d, "\phi^{i+1}"] & \cdots \\ D^\bullet & \cdots \arrow[r] & D^{i-1} \arrow[r, "d_D^{i-1}"] & D^i \arrow[r, "d_D^i"] & D^{i+1} \arrow[r, "d_D^{i+1}"] & \cdots \end{tikzcd}$$

that is, $d_D^{i-1}\phi^{i-1} = \phi^i d_C^{i-1}$, *for all* $i \in \mathbb{Z}$.

Notice that the degree-wise composition of two composable morphisms of cochain complexes is a morphism of cochain complexes.

Definition 1.102. *Given an Abelian category* $\mathfrak{C}$ *we denote by* $\mathbf{Ch}(\mathfrak{C})$ *the category of cochain complexes and morphisms of cochain complexes. For all* $n \in \mathbb{Z}$, *we denote by* $\mathbf{Ch}^{\geqslant n}(\mathfrak{C})$ *(resp.,* $\mathbf{Ch}^{>n}(\mathfrak{C})$, $\mathbf{Ch}^{\leqslant n}(\mathfrak{C})$, $\mathbf{Ch}^{<n}(\mathfrak{C})$*) the full subcategory of* $\mathbf{Ch}(\mathfrak{C})$ *whose objects are all the cochain complexes* $C^\bullet$ *such that* $C^i = 0$ *for all* $i < n$ *(resp.,* $i \leqslant n$, $i > n$, $i \geqslant n$*).*

The category $\mathbf{Ch}(\mathfrak{C})$ of cochain complexes is an Abelian category whose zero-object is the complex whose i-th component is the zero-object of $\mathfrak{C}$, for all $i \in \mathbb{Z}$. The sum on the homomorphism sets is defined degree-wise exploiting the addition in the homomorphism groups in $\mathfrak{C}$. Similarly, one can construct degree-wise products, coproducts, kernels and cokernels.
Furthermore, one can show that $\mathbf{Ch}^{\geqslant n}(\mathfrak{C})$, $\mathbf{Ch}^{>n}(\mathfrak{C})$, $\mathbf{Ch}^{\leqslant n}(\mathfrak{C})$ and $\mathbf{Ch}^{<n}(\mathfrak{C})$ are Abelian subcategories of $\mathbf{Ch}(\mathfrak{C})$ for all $n \in \mathbb{Z}$.

Definition 1.103. *Given an Abelian category* $\mathfrak{C}$ *and a cochain complex* $X^\bullet$, *we denote by* $X^{\geqslant n} \in \mathrm{Ob}(\mathbf{Ch}^{\geqslant n}(\mathfrak{C}))$ *the* n-th truncation *of* $X^\bullet$, *that is*

$$X^{\geqslant n}: \qquad \cdots \longrightarrow 0 \longrightarrow 0 \longrightarrow \mathrm{CoKer}(d^{n-1}) \longrightarrow X^{n+1} \longrightarrow X^{n+2} \longrightarrow \cdots .$$

Similarly, given a homomorphism of cochain complexes $\phi^\bullet : X^\bullet \to Y^\bullet$, *we denote by* $\phi^{\geqslant n} : X^{\geqslant n} \to Y^{\geqslant n}$ *the obvious induced morphism. This allows to define the* n-th truncation functor*:*

$$\mathbf{Ch}(\mathfrak{C}) \to \mathbf{Ch}^{\geqslant n}(\mathfrak{C}) .$$

One can prove that the n-th truncation functor is the right adjoint to the inclusion functor $\mathbf{Ch}^{\geqslant n}(\mathfrak{C}) \to \mathbf{Ch}(\mathfrak{C})$.

Lemma 1.104. *Let* $\mathfrak{C}$ *and* $\mathfrak{D}$ *be two Abelian categories and let* $F : \mathfrak{C} \to \mathfrak{D}$ *be a functor. Applying* F *degree-wise we obtain a functor* $F^\bullet : \mathbf{Ch}(\mathfrak{C}) \to \mathbf{Ch}(\mathfrak{D})$. *If we suppose also that* F *is exact, then* $F^\bullet$ *is exact and*

$$H^n(F^\bullet(X^\bullet)) \cong F(H^n(X^\bullet)) ,$$

for any cochain complex $X^\bullet$ *over* $\mathfrak{C}$. *Furthermore, if* (F, G) *is an adjoint pair between Abelian categories, then* $(F^\bullet, G^\bullet)$ *is an adjunction between the respective categories of complexes.*

Proof. The first part follows using that exact functors preserve kernels and cokernels, and defined using these constructions. The second part follows by [70, Theorem 1, Sec. 8, Ch. 1]. □

In the above settings we will usually abuse notation and denote $F^\bullet$ simply by F.

Definition 1.105. *Let $\mathfrak{C}$ be an Abelian category and let $n \in \mathbb{Z}$. The n-th* cohomology functor

$$H^n : \mathbf{Ch}(\mathfrak{C}) \to \mathfrak{C}$$

is defined as follows. Given a cochain comples $X^\bullet$, $H^n(X^\bullet) = \mathrm{Ker}(d^n)/\mathrm{Im}(d^{n-1})$, while, given a morphism $\phi^\bullet : X^\bullet \to Y^\bullet$ of cochain complexes, $H^n(\phi^\bullet) : H^n(X^\bullet) \to H^n(Y^\bullet)$ is the map induced by ϕ^n.
A morphism $\phi^\bullet : X^\bullet \to Y^\bullet$ in $\mathbf{Ch}(\mathfrak{C})$ is said to be a quasi-isomorphism *if $H^n(\phi^\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$.*

Notice that a cochain complex $X^\bullet$ is exact if and only if $H^n(X^\bullet) = 0$ for all $n \in \mathbb{Z}$. In particular, a complex is exact if and only if it is quasi-isomorphic to the zero-complex.

Definition 1.106. *Given an Abelian category $\mathfrak{C}$ and a morphism $\phi^\bullet : X^\bullet \to Y^\bullet$ in $\mathbf{Ch}(\mathfrak{C})$, the* mapping cone $\mathrm{cone}(\phi^\bullet)$ *is a cochain complex whose n-th component is $X^{n+1} \oplus Y^n$ and whose differentials are represented in matrix form as follows:*

$$\begin{pmatrix} d_X^{n+1} & 0 \\ \phi^n & d_Y^n \end{pmatrix} : X^{n+1} \oplus Y^n \longrightarrow X^{n+1} \oplus Y^n\,.$$

The following lemma shows that the construction of the mapping cone provides a different way to look at quasi-isomorphisms: these are exactly the homomorphisms whose cone is exact.

Lemma 1.107. [104, Corollary 1.5.4] *Let $\mathfrak{C}$ be an Abelian category and let $\phi^\bullet : X^\bullet \to Y^\bullet$ be a morphism in $\mathbf{Ch}(\mathfrak{C})$. Then,*

(1) *$\phi^\bullet$ is a quasi-isomorphism;*

(2) *$H^n(\phi^\bullet)$ is an isomorphism for all $n \in \mathbb{Z}$;*

(3) $\mathrm{cone}(\phi^\bullet)$ *is an exact complex.*

We conclude the section with the following fundamental lemma.

Lemma 1.108. [104, Corollary 1.3.1] *Let $\mathfrak{C}$ be an Abelian category and let $0 \to X^\bullet \to Y^\bullet \to Z^\bullet \to 0$ be a short exact sequence in $\mathbf{Ch}(\mathfrak{C})$. Then, there is an exact sequence:*

$$\ldots \to H^n(X^\bullet) \to H^n(Y^\bullet) \to H^n(Z^\bullet) \to H^{n+1}(X^\bullet) \to H^{n+1}(Y^\bullet) \to H^{n+1}(Z^\bullet) \to \ldots$$

1.2.2 Injective resolutions and classical derived functors

In this section we recall some basic results about injective resolutions and derived functors. Let us start defining injective resolutions of objects in Abelian categories:

Definition 1.109. *Let $\mathfrak{C}$ be an Abelian category and let $X \in \mathrm{Ob}(\mathfrak{C})$. Identify X with an object in $\mathbf{Ch}(\mathfrak{C})$ whose components are all 0 but its 0-th component which is X. Then, an* injective resolution *of X is a quasi-isomorphism $\lambda : X \to E^\bullet$ where $E^\bullet \in \mathrm{Ob}(\mathbf{Ch}^{\geqslant 0}(\mathfrak{C}))$, and E^n is injective for all $n \in \mathbb{Z}$.*

The following lemma establishes the existence of injective resolutions and the so-called "Comparison Theorem".

Lemma 1.110. [104, Lemma 2.3.6 and Theorem 2.3.7] *Let $\mathfrak{C}$ be an Abelian category with enough injectives and let $\phi : X \to Y$ be a morphism in $\mathfrak{C}$. Then,*

(1) *X has an injective resolution;*

(2) *given two injective resolutions $\lambda_1 : X \to E_1^\bullet$ and $\lambda_2 : Y \to E_2^\bullet$, there is a homomorphism of complexes $\Phi^\bullet : E_1^\bullet \to E_2^\bullet$ such that $\lambda_2\phi = \Phi^0\lambda_1$.*

Using injective resolutions we can define right derived functors:

Definition 1.111. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be Abelian categories with enough injectives, let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor and let $n \in \mathbb{N}$. The n-*th right derived functor *$\mathrm{R}^nF(\phi) : \mathfrak{C} \to \mathfrak{D}$ of F is defined as follows:*

– *choose for any object $X \in \mathrm{Ob}(\mathfrak{C})$ an injective resolution $X \to E^\bullet(X)$;*

– *given $X \in \mathrm{Ob}(\mathfrak{C})$, let $\mathrm{R}^nF(X) = H^n(F(E^\bullet(X)))$;*

– *given a morphism $\phi : X \to Y$ in $\mathfrak{C}$, take a morphism $\Phi^\bullet : E^\bullet(X) \to E^\bullet(Y)$ as in Lemma 1.110 and let $\mathrm{R}^nF(\phi) = H^n(\Phi^\bullet) : \mathrm{R}^nF(X) \to \mathrm{R}^nF(Y)$.*

In the following proposition we collect the main general properties of derived functors needed in the thesis. For a proof see [104, Sections 2.4 and 2.5]

Proposition 1.112. *Let $\mathfrak{C}$ and $\mathfrak{D}$ be Abelian categories with enough injectives and let $F : \mathfrak{C} \to \mathfrak{D}$ be a functor. Then, for all $n \in \mathbb{N}$,*

(1) *R^nF is a well-defined functor and it does not depend on the choice of resolutions;*

(2) *if F is left exact, then R^0F is naturally isomorphic to F;*

(3) *if F is left exact and if $E \in \mathrm{Ob}(\mathfrak{C})$ is injective, then $\mathrm{R}^nF(E) = 0$ for all $n > 0$;*

(4) *F is an exact functor if and only if R^nF is the 0-functor for all $n > 0$;*

(5) *given a short exact sequence $0 \to X \to Y \to Z \to 0$ in $\mathfrak{C}$, there is a long exact sequence*

$$\begin{aligned} 0 \to \mathrm{R}^0F(X) \to \mathrm{R}^0F(Y) \to \mathrm{R}^0F(Z) \to \mathrm{R}^1F(X) \to \dots& \\ \dots \to \mathrm{R}^nF(Z) \to \mathrm{R}^{n+1}F(X) \to \mathrm{R}^{n+1}F(Y) \to \mathrm{R}^{n+1}F(Z) \dots& \end{aligned}$$

Let $\mathfrak{C}$ be an Abelian category with enough injectives. After identifying a given object $X \in \mathrm{Ob}(\mathfrak{C})$ with a cochain complex concentrated in degree 0, by Lemma 1.110 we can find a bounded below complex of injectives which is quasi-isomorphic to X. We can generalize this fact as follows:

Lemma 1.113. [65, Lemma 13.2.1] *Let $\mathfrak{C}$ be an Abelian category with enough injectives and let $n \in \mathbb{Z}$. Then any complex $X^\bullet \in \mathrm{Ob}(\mathbf{Ch}^{\geqslant n}(\mathfrak{C}))$ is quasi isomorphic to a complex $Y^\bullet \in \mathrm{Ob}(\mathbf{Ch}^{\geqslant n}(\mathfrak{C}))$ such that Y^m is injective for all $m \in \mathbb{Z}$.*

Given two complexes $X^\bullet$ and $Y^\bullet$ with differentials denoted respectively by $d_X^\bullet$ and $d_Y^\bullet$, $\mathcal{H}om(X^\bullet, Y^\bullet)$ is a cochain complex of Abelian groups whose n-th component is

$$\mathcal{H}om(X^\bullet, Y^\bullet)^n = \prod_{i\in\mathbb{Z}} \mathrm{Hom}_{\mathfrak{C}}(X^i, Y^{n+i}),$$

and whose n-th differential is the following group homomorphism:

$$d^n : \mathcal{H}om(X^\bullet, Y^\bullet)^n \to \mathcal{H}om(X^\bullet, Y^\bullet)^{n+1} \quad (\phi_i)_{i\in\mathbb{Z}} \mapsto (\phi_{i+1}d_X^i + (-1)^{n+1}d_Y^{n+i}\phi_i)_{i\in\mathbb{Z}}.$$

Definition 1.114. *Given an Abelian category $\mathfrak{C}$, a cochain complex $E^\bullet$ is said to be* dg-injective *if E^n is an injective object for all $n \in \mathbb{Z}$ and if the complex $\mathcal{H}om(X^\bullet, E^\bullet)$ is exact, for any exact cochain complex $X^\bullet$.*

Example 1.115. *Bounded complexes of injectives are dg-injective, while unbounded complexes of injectives may be non-dg-injective (see for example [65, Lemma 13.2.4]).*

1.3 Torsion theories and localization

1.3.1 Torsion theories

Definition 1.116. *Let $\mathfrak{C}$ be a Grothendieck category and let $\mathcal{A} \subseteq \mathrm{Ob}(\mathfrak{C})$ be a subclass. $\mathcal{A}$ is a* Serre class *if, given a short exact sequence*

$$0 \to A \to B \to C \to 0\,,$$

B belongs to $\mathcal{A}$ if and only if both B and C belong to $\mathcal{A}$. Furthermore, $\mathcal{A}$ is a hereditary torsion class *(or* localizing class*) if it is a Serre class and it is closed under taking arbitrary coproducts.*

Let us recall also the following definition.

Definition 1.117. *Let $\mathfrak{C}$ be a Grothendieck category, let $X \in \mathrm{Ob}(\mathfrak{C})$, let κ be an ordinal and let X_α be a subobject of X for all $\alpha < \kappa$. The family $\{X_\alpha : \alpha < \kappa\}$ is a* continuous chain *provided:*

– *$X_0 = 0$ and $X_\alpha \leqslant X_\alpha + 1$ for all $\alpha < \kappa$;*

– *if $\lambda < \kappa$ is a limit ordinal, then $X_\lambda = \sum_{\alpha<\lambda} X_\alpha$.*

If $\{X_\alpha : \alpha < \sigma\}$ is a continuous chain of subobjects of a given $X \in \mathrm{Ob}(\mathfrak{C})$, we denote $\sum_{\alpha<\sigma} X_\alpha$ also by $\bigcup_{\alpha<\sigma} X_\alpha$.

Lemma 1.118. *Let $\mathfrak{C}$ be a Grothendieck category and let $\mathcal{D} \subseteq \mathrm{Ob}(\mathfrak{C})$ be closed under taking subobjects and quotients. Then, the class $\mathcal{T}$ of all the $\mathcal{D}$-filtrated objects, that is, all the $X \in \mathrm{Ob}(\mathfrak{C})$ admitting a continuous chain $\{X_\alpha : \alpha < \kappa\}$ of subobjects such that $X_{\alpha+1}/X_\alpha \in \mathcal{D}$ for all $\alpha < \kappa$ and $\bigcup_{\alpha<\kappa} X_\alpha = X$, is a hereditary torsion subclass of $\mathfrak{C}$.*

Proof. Let $Y \leqslant X \in \mathfrak{C}$.
If $X \in \mathcal{T}$, we can find a continuous chain $\{X_\alpha : \alpha < \lambda\}$ such that $\bigcup_{\alpha<\lambda} X_\alpha = X$ and $X_{\alpha+1}/X_\alpha \in \mathcal{D}$ for all $\alpha < \lambda$. This implies (by the closure properties of $\mathcal{D}$) that $\{X_\alpha \cap Y : \alpha < \lambda\}$ and $\{(X_\alpha + Y)/Y : \alpha < \lambda\}$ are continuous chains with successive quotients belonging to $\mathcal{D}$. Thus, Y and $X/Y \in \mathcal{T}$.
On the other hand, if Y and $X/Y \in \mathcal{T}$, that is, there exist continuous chains $\{Y_\alpha : \alpha < \lambda_1\}$ and $\{X_\alpha/Y : \alpha < \lambda_2\}$ such that $\bigcup_{\alpha<\lambda_1} Y_\alpha = Y$, $\bigcup_{\alpha<\lambda_2} X_\alpha/Y = X/Y$ and with successive quotients in $\mathcal{D}$, then $\{Y_\alpha : \alpha < \lambda_1\} \cup \{X_\alpha : \alpha < \lambda_2\}$ is a continuous chain with successive quotients in $\mathcal{D}$ and whose union is X.
So far, this proves that $\mathcal{T}$ is a Serre class; we need to verify that $\mathcal{T}$ is closed under taking coproducts. Let $\{X_i : i \in I\}$ be a family of objects in $\mathcal{T}$ and suppose that, for all $i \in I$, $\{X_{i,\alpha} : \alpha < \lambda_i\}$ is a continuous chain with successive quotients in $\mathcal{D}$ and such that $\bigcup_{\alpha<\lambda_i} X_{i,\alpha} = X_i$. Chose a total order $\leqslant$ on I, and fix the lexicographic order on $\mathfrak{I} = \{(i, \alpha) : i \in I,\ \alpha < \lambda_i\}$. For all $(i, \alpha) \in \mathfrak{I}$ we let $X^*_{i,\alpha} = \bigoplus_{j<i} X_j \oplus X_{i,\alpha}$. Clearly, $\{X^*_{i,\alpha} : (i, \alpha) \in \mathfrak{I}\}$ is a continuous chain with successive quotients in $\mathcal{D}$ and whose union equals $\bigoplus_I X_i$. $\square$

Lemma 1.119. *Let $\mathfrak{C}$ be a Grotheindieck category and let $\mathcal{T}$ be a hereditary torsion class. For all $X \in \mathrm{Ob}(\mathfrak{C})$, let*

$$\mathbf{T}(X) = \sum \{T \leqslant X : T \in \mathcal{T}\}\,.$$

Then,

(1) $\mathbf{T}(X) = X$ *if and only if* $X \in \mathcal{T}$;

(2) $\mathbf{T}(\mathbf{T}(X)) = \mathbf{T}(X)$ *and* $\mathbf{T}(X/\mathbf{T}(X)) = 0$, *for all* $X \in \mathrm{Ob}(\mathfrak{C})$.

Proof. (1) and the first part of (2) are easy consequences of the closure properties of $\mathcal{T}$. For the second half of part (2), let $X \in \mathrm{Ob}(\mathfrak{C})$ and consider a subobject $Y \leqslant X/\mathbf{T}(X)$ such that $Y \in \mathcal{T}$. Then, there is a short exact sequence $0 \to \mathbf{T}(X) \to \pi^{-1}(Y) \to Y \to 0$, where $\pi : X \to X/\mathbf{T}(X)$ is the canonical projection. Since $\mathcal{T}$ is a Serre class, $\pi^{-1}(Y) \in \mathcal{T}$ and so $\pi^{-1}(Y) \leqslant \mathbf{T}(X)$, showing that $Y = 0$. ☐

Given a subclass $\mathcal{A}$ of objects of $\mathfrak{C}$, we define the following two classes

$$\mathcal{A}^{\perp} = \{X \in \mathfrak{C} : \mathrm{Hom}_{\mathfrak{C}}(A, X) = 0\,,\ \forall A \in \mathcal{A}\} \quad \text{and} \quad {}^{\perp}\mathcal{A} = \{X \in \mathfrak{C} : \mathrm{Hom}_{\mathfrak{C}}(X, A) = 0\,,\ \forall A \in \mathcal{A}\}\,,$$

which are called respectively *right* and *left orthogonal class* to $\mathcal{A}$.

Corollary 1.120. *Let $\mathfrak{C}$ be a Grotheindieck category and consider a subclass $\mathcal{T} \subseteq \mathrm{Ob}(\mathfrak{C})$ that is closed under taking subobjects. The following are equivalent:*

(1) *$\mathcal{T}$ is a hereditary torsion class;*

(2) $\mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp})$.

Proof. (1)⇒(2). It is easy to see that $\mathcal{T} \subseteq {}^{\perp}(\mathcal{T}^{\perp})$. On the other hand, let $X \in {}^{\perp}(\mathcal{T}^{\perp})$ and define $\mathbf{T}(X)$ as in Lemma 1.119. Then, we have a short exact sequence

$$0 \to \mathbf{T}(X) \to X \to X/\mathbf{T}(X) \to 0\,.$$

Notice that $X/\mathbf{T}(X) \in \mathcal{T}^{\perp}$ (use the fact that the image of an object in $\mathcal{T}$ is again in $\mathcal{T}$), so the canonical morphism $X \to X/\mathbf{T}(X)$ is the zero-morphism, that is, $X/\mathbf{T}(X) = 0$, that is, $X = \mathbf{T}(X) \in \mathcal{T}$.

(2)⇒(1). Suppose that $\mathcal{T} = {}^{\perp}(\mathcal{T}^{\perp})$ and let $0 \to X \to Y \to Z \to 0$ be a short exact sequence. If $Y \in \mathcal{T}$, then $X \in \mathcal{T}$ by hypothesis. Furthermore, given $M \in \mathcal{T}^{\perp}$ and a morphism $\phi : Z \to M$, the composition $Y \to Z \to M$ has to be 0, thus $\phi = 0$ and so $Z \in {}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$. On the other hand, if X, $Z \in \mathcal{T}$, $M \in \mathcal{T}^{\perp}$ and $\psi : Y \to M$, then the composition $X \to Y \to M$ is trivial and so, by exactness, there exists a unique morphism $\psi' : Z \to M$ such that $\psi'\pi = \psi$, where $\pi : Y \to Z$ is the canonical epimorphism. Since the unique morphism $Z \to M$ is the zero-morphism, we obtain that $\psi = 0$, so $Y \in {}^{\perp}(\mathcal{T}^{\perp}) = \mathcal{T}$. Thus, $\mathcal{T}$ is a Serre class. The fact that $\mathcal{T}$ is closed under coproducts follows by Lemma 1.36. ☐

Definition 1.121. *A* hereditary torsion theory τ *in $\mathfrak{C}$ is a pair of classes $(\mathcal{T}, \mathcal{F})$ such that*

– *$\mathcal{T}$ is a hereditary torsion class;*

– $(\mathcal{T})^{\perp} = \mathcal{F}$ *and* ${}^{\perp}(\mathcal{F}) = \mathcal{T}$.

*We call $\mathcal{T}$ and $\mathcal{F}$ respectively the class of the τ-*torsion *and of the τ-*torsion free *objects. Since all the torsion theories in the sequel are hereditary, we just say "torsion theory", "torsion class" and "torsion free class" to mean respectively "hereditary torsion theory", "hereditary torsion class" and "hereditary torsion free class".*

Example 1.122. (1) *Let $\mathfrak{C}$ be a Grothendieck category. The pairs $(0, \mathfrak{C})$ and $(\mathfrak{C}, 0)$ are torsion theories. We call them respectively the* trivial *and the* improper *torsion theory.*

(2) *Let G be an Abelian group; an element $g \in G$ is torsion if there exists a positive integer n such that $ng = 0$. The group G is torsion if any of its elements is torsion, while it is torsion free if its unique torsion element is 0. The class $\mathcal{T}$ of all torsion Abelian groups is a torsion class in $\underline{\mathrm{Ab}}$, while the class $\mathcal{F}$ of the torsion free Abelian groups is a torsion free class. In particular, $(\mathcal{T}, \mathcal{F})$ is a torsion theory.*

(3) *A generalization of part (2) is as follows: let D be a left Ore domain and denote by Σ the set of the regular elements of D. Then $\mathcal{T} = \{M \in \mathrm{Ob}(D\text{-Mod}) : \forall m \in M, \exists s \in \Sigma,$ such that $sm = 0\}$ is a torsion class. The orthogonal of $\mathcal{T}$ is $\mathcal{F} = \mathcal{T}^{\perp} = \{M \in \mathrm{Ob}(D\text{-Mod}) : \forall m \in M,\ s \in \Sigma,\ sm \neq 0\}$.*
Generalizing further, if we have two rings R, S, a flat ring homomorphism $\phi : R \to S$ and a torsion theory $(\mathcal{T}, \mathcal{F}) \in S$-Mod, then we can define a torsion theory $(\mathcal{T}_\phi, \mathcal{F}_\phi)$ in R-Mod letting $\mathcal{T}_\phi = \{M \in \mathrm{Ob}(R\text{-Mod}) : \mathrm{ext}_\phi(M) \in \mathcal{T}\}$.

(4) *Let $\mathfrak{C}$ be a Grothendieck category. Given an injective object E in $\mathfrak{C}$, one can define a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, with $\mathcal{T} = {}^{\perp}\{E\}$ and $\mathcal{F} = \mathcal{T}^{\perp}$; such τ is said to be the* torsion theory cogenerated by E. *Similarly, given a class $\mathcal{E}$ of injective objects, and letting $\mathcal{T}' = {}^{\perp}\{\mathcal{E}\}$ and $\mathcal{F}' = (\mathcal{T}')^{\perp}$, $\tau' = (\mathcal{T}', \mathcal{F}')$ is said to be the torsion theory* cogenerated by $\mathcal{E}$.

Lemma 1.123. *Let $\mathfrak{C}$ be a Grotheindieck category, let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory and let $\phi : X_1 \to X_2$ be a morphism in $\mathfrak{C}$. Let $\mathbf{T}_\tau(X_i) = \sum\{T \leqslant X_i : T \in \mathcal{T}\}$ $(i = 1, 2)$, then, $\phi(\mathbf{T}_\tau(X_1)) \leqslant \mathbf{T}_\tau(X_2)$.*

Proof. Notice that $\phi(\mathbf{T}_\tau(X_1)) = \phi(\sum\{T \leqslant X_1 : T \in \mathcal{T}\}) = \sum\{\phi(T) \leqslant X_1 : T \in \mathcal{T}\}$ (see Proposition 1.72). Furthermore, since $\mathcal{T}$ is a Serre class, $\{\phi(T) \leqslant X_1 : T \in \mathcal{T}\} \subseteq \{T \leqslant X_2 : T \in \mathcal{T}\}$. $\square$

Thanks to Lemma 1.123, we can define a functor $\mathfrak{C} \to \mathcal{T}$, that turns out to be the right adjoint to the inclusion $\mathcal{T} \to \mathfrak{C}$:

Definition 1.124. *Let $\mathfrak{C}$ be a Grotheindieck category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. The τ-*torsion functor *$\mathbf{T}_\tau : \mathfrak{C} \to \mathcal{T}$ is defined as follows:*

– $\mathbf{T}_\tau(X) = \sum\{T \leqslant X : T \in \mathcal{T}\}$, *for any object $X \in \mathrm{Ob}(\mathfrak{C})$;*

– $\mathbf{T}_\tau(\phi) : \mathbf{T}_\tau(X) \to \mathbf{T}_\tau(Y)$ *is the restriction of $\phi : X \to Y$, for any morphism ϕ in $\mathfrak{C}$.*

Lemma 1.125. *Let $\mathfrak{C}$ be a Grotheindieck category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then,*

(1) $\mathbf{T}_\tau(X) = 0$ *if and only if $X \in \mathcal{F}$. In particular, there is a short exact sequence $0 \to T \to X \to F \to 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$, for all $X \in \mathrm{Ob}(\mathfrak{C})$;*

(2) $\mathbf{T}_\tau$ *is right adjoint to the inclusion $\mathcal{T} \to \mathfrak{C}$;*

Proof. For part (1) see the first half of the proof of Corollary 1.120. For part (2) notice that, for all $T \in \mathcal{T}$ and $X \in \mathrm{Ob}(\mathfrak{C})$, $\mathrm{Hom}_{\mathfrak{C}}(T, X) \cong \mathrm{Hom}_{\mathfrak{C}}(T, \mathbf{T}_\tau(X)) = \mathrm{Hom}_{\mathcal{T}}(T, \mathbf{T}_\tau(X))$, where the isomorphism comes from the fact that $\mathcal{T}$ is closed under taking quotients, so $\phi(T) \leqslant \mathbf{T}_\tau(X)$ for all $\phi \in \mathrm{Hom}_{\mathfrak{C}}(T, X)$. $\square$

1.3.2 Localization

In this subsection we show how the concept of torsion theory is related to that of localization. Most of the material in this subsection is adapted from Gabriel's thesis [44].

Definition 1.126. *Let $\mathfrak{C}$ be a Grothendieck category. A* localization *of $\mathfrak{C}$ is a pair of adjoint functors $(\mathbf{Q}:\mathfrak{C}\to\mathfrak{D},\mathbf{S}:\mathfrak{D}\to\mathfrak{C})$, where $\mathbf{S}$ is fully faithful, $\mathbf{Q}$ is exact and $\mathfrak{D}$ is an Abelian category. In this situation, $\mathfrak{D}$ is said to be a* quotient category*, $\mathbf{Q}$ is a* quotient functor *and $\mathbf{S}$ is a* section functor*. The composition $\mathbf{L}=\mathbf{S}\circ\mathbf{Q}:\mathfrak{C}\to\mathfrak{C}$ is said to be the* localization functor*.*

Let us remark that, in other contexts, one may find definitions of localization of an Abelian category that do not require that the quotient functor is exact. Anyway, some of the concepts of localization commonly used in algebra are particular cases of the above definition, as the following example shows.

Example 1.127. *Let D be an Ore domain and let Σ be the set of regular elements of D. As we said in Example 1.99, the canonical ring homomorphism $\varepsilon:D\to\Sigma^{-1}D$, induces an adjunction $(\mathrm{res}_\varepsilon,\mathrm{ext}_\varepsilon)$. Furthermore, res_ε is fully faithful and ext_ε is exact, thus we can think to ext_ε as a quotient functor, to res_ε as a section functor and to $\Sigma^{-1}D$-Mod as a quotient category of D-Mod.*

Lemma 1.128. [66, Lemma 2.2] *Let $\mathfrak{C}$ be a Grothendieck category, let $(\mathbf{Q},\mathbf{S})$ be a localization of $\mathfrak{C}$ and denote by $\mathbf{L}=\mathbf{S}\circ\mathbf{Q}:\mathfrak{C}\to\mathfrak{C}$ the localization functor. For all $X\in\mathrm{Ob}(\mathfrak{C})$ there is a natural isomorphism $\mathbf{L}(X)\cong\mathbf{L}(\mathbf{L}(X))$.*

One can encounter different definitions of localization in other contexts, see for example [90]. Let us explain the connection between localizations and torsion theories. Indeed, starting with a localization $(\mathbf{Q}:\mathfrak{C}\to\mathfrak{D},\mathbf{S}:\mathfrak{D}\to\mathfrak{C})$ and letting $\mathbf{L}=\mathbf{S}\circ\mathbf{Q}$,

$$\mathrm{Ker}(\mathbf{L})=\{X\in\mathfrak{C}:\mathbf{L}(X)=0\}=\{X\in\mathfrak{C}:\mathbf{Q}(X)=0\}=\mathrm{Ker}(\mathbf{Q})$$

is a torsion class (use the exactness of $\mathbf{Q}$ and the fact that it is a left adjoint). Hence, the localization $(\mathbf{Q},\mathbf{S})$ induces a torsion theory $(\mathrm{Ker}(\mathbf{Q}),\mathrm{Ker}(\mathbf{Q})^{\perp})$.

On the other hand, one can construct a localization out of a torsion theory. In this section we describe this localization in case our Grothendieck category $\mathfrak{C}$ has enough injectives. By Theorem 1.97, this allows us to localize any category of modules.

Definition 1.129. *Let $\mathfrak{C}$ be a Grothendieck category with enough injectives. An object $X\in\mathrm{Ob}(\mathfrak{C})$ is* τ-local *provided $E(X)$ and $E(X)/X\in\mathcal{F}$. The* localization of $\mathfrak{C}$ at τ *is the full subcategory $\mathfrak{C}/\mathcal{T}$ of $\mathfrak{C}$ of the τ-local objects. The inclusion $\mathbf{S}_\tau:\mathfrak{C}/\mathcal{T}\to\mathfrak{C}$ is called the* τ-section functor*.*

Lemma 1.130. *Let $\mathfrak{C}$ be a Grothendieck category with enough injectives, let $\tau=(\mathcal{T},\mathcal{F})$ be a torsion theory and let X, X', $X''\in\mathcal{F}$. Then,*

1. *the object $\mathbf{L}_\tau(X)=\pi^{-1}(\mathbf{T}_\tau(E(X)/X))$, where $\pi:E(X)\to E(X)/X$ is the canonical projection, is τ-local;*
2. *given a morphism $\phi:X\to X'$, there exists a unique morphism $\mathbf{L}_\tau(\phi):\mathbf{L}_\tau(X)\to\mathbf{L}_\tau(X')$ such that the following square commutes*

$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ \downarrow & & \downarrow \\ \mathbf{L}_\tau(X) & \xrightarrow{\mathbf{L}_\tau(\phi)} & \mathbf{L}_\tau(X')\,; \end{array}$$

(3) *given two morpshims $\phi : X \to X'$ and $\phi' : X' \to X''$, then $\mathbf{L}_\tau(\phi' \circ \phi) = \mathbf{L}_\tau(\phi') \circ \mathbf{L}_\tau(\phi)$.*

Proof. (1) Notice that $X \leqslant \mathbf{L}_\tau(X) \leqslant E(X)$, thus $E(\mathbf{L}_\tau(X)) = E(X) \in \mathcal{F}$. Furthermore,

$$\frac{E(\mathbf{L}_\tau(X))}{\mathbf{L}_\tau(X)} = \frac{E(X)}{\mathbf{L}_\tau(X)} \cong \frac{E(X)/X}{\mathbf{L}_\tau(X)/X} \cong \frac{E(X)/X}{\mathbf{T}_\tau(E(X)/X)} \in \mathcal{F}\,.$$

(2) Let $\phi' : X \to E(X')$ be the composition of ϕ and $X' \to E(X')$. Since $X \to E(X)$ is a monomorphism we can extend ϕ' to a morphism $\bar\phi : E(X) \to E(X')$. We claim that $\bar\phi(\mathbf{L}_\tau(X)) \leqslant \mathbf{L}_\tau(X')$. In fact, $\bar\phi(\mathbf{L}_\tau(X))/X'$ is a quotient of $\bar\phi(\mathbf{L}_\tau(X))/(X' \cap \phi(X')) \cong \mathbf{L}_\tau(X)/\phi^{-1}(X') = \mathbf{L}_\tau(X)/X \in \mathcal{T}$, and so $\bar\phi(\mathbf{L}_\tau(X))/X' \in \mathcal{T}$. Thus, we can define $\mathbf{L}_\tau(\phi) : \mathbf{L}_\tau(X) \to \mathbf{L}_\tau(X')$ as the restriction of $\bar\phi$.
It remains to show the uniqueness of $\mathbf{L}_\tau(\phi)$. Indeed, let $\psi : \mathbf{L}_\tau(X) \to \mathbf{L}_\tau(X')$ be a morphism such that $\psi\restriction_X = \phi'$. Then, $(\mathbf{L}_\tau(\phi) - \psi)(X) = 0$ and so $\mathbf{L}_\tau(\phi) - \psi$ can be decomposed as $\mathbf{L}_\tau(\phi) - \psi = gf$, with $f : \mathbf{L}_\tau(X) \to \mathbf{L}_\tau(X)/X$ the canonical projection and $g : \mathbf{L}_\tau(X)/X \to \mathbf{L}_\tau(X')$. By definition $\mathbf{L}_\tau(X)/X \in \mathcal{T}$, while $\mathbf{L}_\tau(X') \in \mathcal{F}$, thus $g = 0$ and $\mathbf{L}_\tau(\phi) - \psi = 0f = 0$, showing that $\psi = \mathbf{L}_\tau(\phi)$.
(3) follows by the uniqueness proved in part (2). □

The above lemma allows us to define a localization functor $\mathbf{L}_\tau : \mathfrak{C} \to \mathfrak{C}$:

Definition 1.131. *Let $\mathfrak{C}$ be a Grothendieck category with enough injectives and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. The τ-*localization functor *$\mathbf{L}_\tau : \mathfrak{C} \to \mathfrak{C}$ is defined as follows:*

- $\mathbf{L}_\tau(X) = \pi^{-1}(\mathbf{T}_\tau(E(X)/X))$, *where $\pi : E(X) \to E(X)/X$ is the canonical projection, for all $X \in \mathcal{F}$;*
- $\mathbf{L}_\tau(X) = \mathbf{L}_\tau(X/\mathbf{T}_\tau(X))$, *for all $X \in \mathrm{Ob}(\mathfrak{C})$;*
- *given a morphism $\phi : X \to X'$ in $\mathfrak{C}$, $\mathbf{L}_\tau(\phi) : \mathbf{L}_\tau(X) \to \mathbf{L}_\tau(X')$ is the unique morphism that makes the following square commute*

$$\begin{tikzcd} X/\mathbf{T}_\tau(X) \arrow[r, "\bar\phi"] \arrow[d] & X'/\mathbf{T}_\tau(X') \arrow[d] \\ \mathbf{L}_\tau(X) \arrow[r, "\mathbf{L}_\tau(\phi)"] & \mathbf{L}_\tau(X'). \end{tikzcd}$$

*Furthermore, we let $\mathbf{Q}_\tau : \mathfrak{C} \to \mathfrak{C}/\mathcal{T}$ be the unique functor such that $\mathbf{L}_\tau = \mathbf{S}_\tau\mathbf{Q}_\tau$. $\mathbf{Q}_\tau$ is called τ-*quotient functor*. We say that τ is* exact *if $\mathbf{L}_\tau$ is an exact functor.*

Theorem 1.132. *Let $\mathfrak{C}$ be a complete Grothendieck category with enough injectives and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory on $\mathfrak{C}$. Then,*

(1) *$\mathbf{Q}_\tau$ is left adjoint to $\mathbf{S}_\tau$;*

(2) *$\mathfrak{C}/\mathcal{T}$ is a complete Grothendieck category;*

(3) *$\mathbf{Q}_\tau$ is an exact functor that commutes with coproducts;*

(4) *an object $E \in \mathfrak{C}/\mathcal{T}$ is injective if and only if $\mathbf{S}_\tau(E)$ is injective in $\mathfrak{C}$.*

Proof. (1) We have to prove that there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{C}/\mathcal{T}}(\mathbf{Q}_\tau(X), Y) \cong \mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{S}_\tau(Y)),$$

for all $X \in \mathrm{Ob}(\mathfrak{C})$ and $Y \in \mathfrak{C}/\mathcal{T}$. Since $\mathbf{S}_\tau$ is fully faithfull, this is the same as proving a natural isomorphism $\mathrm{Hom}_{\mathfrak{C}}(\mathbf{L}_\tau(X), Y) \cong \mathrm{Hom}_{\mathfrak{C}}(X, Y)$ (with X, $Y \in \mathrm{Ob}(\mathfrak{C})$ and Y τ-local). Let $\alpha : X \to \mathbf{L}_\tau(X)$ be the canonical morphism and notice that α induces a map

$$\alpha^* : \mathrm{Hom}_{\mathfrak{C}}(\mathbf{L}_\tau(X), Y) \xrightarrow{\cong} \mathrm{Hom}_{\mathfrak{C}}(X, Y) \quad \alpha^*(\phi) = \phi \circ \alpha\,.$$

When $X \in \mathcal{F}$, α^* is an isomorphism by Lemma 1.130(2), while the general case follows noticing that there is a natural isomorphism $\mathrm{Hom}_{\mathfrak{C}}(X, Y) \cong \mathrm{Hom}_{\mathfrak{C}}(X/\mathbf{T}_\tau(X), Y)$.

(2) By [96, Propositions 1.2 and 1.3, Ch. X] and part (1), we obtain that kernels exist in $\mathfrak{C}/\mathcal{T}$ and that it is enough to show that $\mathbf{Q}_\tau$ sends kernels to kernels. We show this in two steps:

- First of all we prove that, given monomorphism $\phi : X \to Y$ in $\mathfrak{C}$, $\mathbf{Q}_\tau(\phi) : \mathbf{Q}_\tau(X) \to \mathbf{Q}_\tau(Y)$ is a monomorphism in $\mathfrak{C}/\mathcal{T}$. It is sufficient to show that $\mathbf{L}_\tau(\phi)$ is a monomorphism in $\mathfrak{C}$. We can assume without loss of generality that X, $Y \in \mathcal{F}$ (otherwise substitute them by $X/\mathbf{T}_\tau(X)$ and $Y/\mathbf{T}_\tau(Y)$ respectively, and notice that the induced morphism $X/\mathbf{T}_\tau(X) \to Y/\mathbf{T}_\tau(Y)$ is again a monomorphism). In this case, $0 = \mathrm{Ker}(\phi) = \mathrm{Ker}(\mathbf{L}_\tau(\phi)) \cap X$ and thus $\mathrm{Ker}(\mathbf{L}_\tau(\phi)) = 0$ since X is essential in $\mathbf{L}_\tau(X)$.
- Consider now a morphism $\phi : X \to Y$ in $\mathfrak{C}$, let $k : K \to X$ be a kernel of ϕ and let us show that $\mathbf{Q}_\tau(k) : \mathbf{Q}_\tau(K) \to \mathbf{Q}_\tau(X)$ is a kernel of $\mathbf{Q}_\tau(\phi)$ in $\mathfrak{C}/\mathcal{T}$. We suppose without any loss in generality that X, $Y \in \mathcal{F}$ (in fact, letting $\bar\phi : X/\mathbf{T}_\tau(X) \to Y/\mathbf{T}_\tau(Y)$ be the induced map, $\mathrm{Ker}(\bar\phi) = \mathrm{Ker}(\phi)/\mathbf{T}_\tau(\mathrm{Ker}(\phi))$ and so $\mathbf{Q}_\tau(\mathrm{Ker}(\bar\phi)) = \mathbf{Q}_\tau(\mathrm{Ker}(\phi))$). Let $\bar Z \in \mathfrak{C}/\mathcal{T}$, let $\bar\psi : Z \to \mathbf{Q}_\tau(X)$ be a morphism such that $\mathbf{Q}_\tau(\phi)\bar\psi = 0$ and let $Z = \mathbf{S}_\tau(\bar Z)$, $\psi = \mathbf{S}_\tau(\bar\psi) : Z \to \mathbf{L}_\tau(X)$, so that $\mathbf{L}_\tau(\phi)\psi = 0$. Consider the restriction $\psi' : \psi^{-1}(X) \to X$; since $\psi'\phi = 0$, there exists a morphism $f : \psi^{-1}(X) \to K$ such that $kf = \psi'$. Notice that $Z/\psi^{-1}(X) = \psi(Z)/(\psi(Z) \cap X) = (Z + X)/X \leqslant \mathbf{L}_\tau(X)/X \in \mathcal{T}$ and so, $\mathbf{L}_\tau(\psi^{-1}(X)) = \mathbf{L}_\tau(Z) = Z$. Hence, $\mathbf{Q}_\tau(f) : \bar Z \to \mathbf{Q}_\tau(K)$ is such that $\mathbf{Q}_\tau(k)\mathbf{Q}_\tau(f) = \mathbf{Q}_\tau(\psi) = \bar\psi$. We conclude by Lemma 1.56 that $\mathbf{Q}_\tau(K)$ is a kernel for $\mathbf{Q}_\tau(\phi)$.

(3) $\mathbf{Q}_\tau$ is exact since it is a left adjoint (thus right exact) and it preserves kernels. The fact that it preserves coproducts follows by Lemma 1.37.

(4) This follows by Lemma 1.85. □

1.3.3 The Gabriel-Popescu Theorem and its consequences

The following theorem, usually known as the “Gabriel-Popescu Theorem”, was first proved in [88].

Theorem 1.133. [96, Theorem 4.1, Ch. X] *Let $\mathfrak{C}$ be a Grothendieck category, let G be a generator of $\mathfrak{C}$, let $R = \mathrm{End}_{\mathfrak{C}}(G)$ and denote by $T : \mathfrak{C} \to \text{Mod-}R$ the functor $T = \mathrm{Hom}_{\mathfrak{C}}(G, -)$ where, for any object $X \in \mathfrak{C}$, the right R-module structure on $T(X)$ is given by the following map*

$$\mathrm{Hom}_{\mathfrak{C}}(G, X) \times \mathrm{End}_{\mathfrak{C}}(G) \to \mathrm{Hom}_{\mathfrak{C}}(G, X) \quad \textit{such that} \quad (\phi, \rho) \mapsto \phi \circ \rho\,.$$

Then,

(1) *T is fully faithful;*

(2) *if we denote by* $\tau = (\mathcal{T}, \mathcal{F})$ *the torsion theory cogenerated by*

$$\mathcal{E} = \{E(T(X) \oplus E(T(X))/T(X)) : X \in \mathrm{Ob}(\mathfrak{C})\},$$

(see Example 1.122 (4)) then the composition $\mathfrak{C} \xrightarrow{T} \text{Mod-}R \xrightarrow{\mathbf{Q}_\tau} (\text{Mod-}R)/\mathcal{T}$ *is a natural equivalence of categories.*

The following corollaries are direct consequences of the above theorem and Theorem 1.132.

Corollary 1.134. *Let* $\mathfrak{C}$ *be a Grothendieck category, then*

(1) $\mathfrak{C}$ *is complete;*

(2) $\mathfrak{C}$ *has enough injective objects.*

Corollary 1.135. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion theory on* $\mathfrak{C}$*. Consider a family* $\{X_i : i \in I\} \subseteq \mathfrak{C}/\mathcal{T}$*, then*

$$\prod_{i \in I} X_i \cong \mathbf{Q}_\tau \left(\prod_{i \in I} \mathbf{S}_\tau(X_i) \right).$$

Proof. Being a right adjoint, $\mathbf{S}_\tau$ commutes with limits, thus $\mathbf{S}_\tau \left(\prod_{i \in I} X_i\right) \cong \prod_{i \in I} \mathbf{S}_\tau(X_i)$. Apply $\mathbf{Q}_\tau$ to conclude. □

Corollary 1.136. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion theory in* $\mathfrak{C}$*. Then* τ *is cogenerated by an injective object* E*.*

Proof. Take a generator G of $\mathfrak{C}$ and let E be the product of all the injective envelopes of the τ-torsion free quotients of G. Then, $\mathcal{T}$ coincides with ${}^\perp\{E\}$ and $\mathcal{F} = \mathcal{T}^\perp$ is the class of all the objects that embed in some product of copies of E. □

The proof of Corollary 1.136 shows that the torsion theories in a Grothendieck category $\mathfrak{C}$ form a set, not a proper class (in fact, one can bound the cardinality of this set by the cardinality of the power set of the family of quotients of a chosen generator G of $\mathfrak{C}$).

Definition 1.137. *Let* $\mathfrak{C}$ *be a Grothendieck category. We denote by* $\mathrm{Tors}(\mathfrak{C})$ *the poset of all the torsion theories on* $\mathfrak{C}$*, ordered as follows: given* $\tau = (\mathcal{T}, \mathcal{F})$ *and* $\tau' = (\mathcal{T}', \mathcal{F}') \in \mathrm{Tors}(\mathfrak{C})$*,*

$$\tau' \preceq \tau \ \text{ if and only if } \ \mathcal{T}' \subseteq \mathcal{T} \ \text{ if and only if } \ \mathcal{F} \subseteq \mathcal{F}'.$$

When $\tau' \preceq \tau$*, we say that* τ *is a* generalization *of* τ'*, while* τ' *is a* specialization *of* τ*.*

Corollary 1.138. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion theory. Then,* $\mathcal{F}$ *is closed under taking injective envelopes.*

The notion of stable torsion theory was introduced by Gabriel in [48], see also [82] and [7].

Definition 1.139. *Let* $\mathfrak{C}$ *be a Grothendieck category. A torsion theory* $\tau = (\mathcal{T}, \mathcal{F})$ *on* $\mathfrak{C}$ *is* stable *if* $\mathcal{T}$ *is closed under taking injective envelopes. Furthermore,* $\mathfrak{C}$ *is* stable *if any* $\tau \in \mathrm{Tors}(\mathfrak{C})$ *is stable.*

Corollary 1.140. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\tau = (\mathcal{T}, \mathcal{F})$ *be a torsion theory on* $\mathfrak{C}$*. The following statements are equivalent*

(1) *τ is stable;*

(2) *given $E \in \mathrm{Ob}(\mathfrak{C})$ is an injective object, then $E \cong \mathbf{T}_\tau(E) \oplus E/\mathbf{T}_\tau(E)$.*

Proof. (1)⇒(2). Identify $\mathbf{T}_\tau(E)$ and its injective envelope with subobjects of E, so that $\mathbf{T}_\tau(E) \leqslant E(\mathbf{T}_\tau(E)) \leqslant E$. Since $E(\mathbf{T}_\tau(E))$ is τ-torsion, $E(\mathbf{T}_\tau(E)) \leqslant \mathbf{T}_\tau(E)$. Thus, $\mathbf{T}_\tau(E) = E(\mathbf{T}_\tau(E))$. Having proved that $\mathbf{T}_\tau(E)$ is injective, the desired decomposition follows.

(2)⇒(1). Let $X \in \mathcal{T}$ and let E be an injective envelope of X. Then, $E = \mathbf{T}_\tau(E) \oplus E/\mathbf{T}_\tau(E)$ and so we can identify $E/\mathbf{T}_\tau(E)$ with a sub-object of E. Since X is an essential sub-object of E and $X \cap E/\mathbf{T}_\tau(E) = 0$, $E/\mathbf{T}_\tau(E) = 0$. □

Chapter 2

Lattice Theory

In Chapter 2 we introduce the category QFrame of quasi frames. In QFrame we study some lattice-theoretic notions that are usually introduced in module theory, such as the composition length, the uniform dimension and the socle series.
In the second part of the chapter we study the basic properties of two cardinal invariants of qframes: Krull dimension and Gabriel dimension. Using these notions we can define torsion and localization in the category QFrame.
Given a Grothendieck category $\mathfrak{C}$ and a torsion theory τ in $\mathfrak{C}$, there is a notion of τ-Gabriel dimension for the objects of $\mathfrak{C}$. We show that this notion can be defined using the Gabriel dimension of quasi frames and we deduce its basic properties.

2.1 The category of Quasi-frames

2.1.1 Lattices

Let $(L, \leqslant)$ be a lattice. Given two elements x and $y \in L$, the *segment* between x and y is

$$[x, y] = \{s \in L : x \leqslant s \leqslant y\}\,.$$

We also let $(x, y] = [x, y]\backslash\{x\}$, $[x, y) = [x, y]\backslash\{y\}$ and $(x, y) = [x, y]\backslash\{x, y\}$. Notice that $[x, y]$ is a lattice with the order induced by L.

Definition 2.1. *Let $(L, \leqslant)$ be a lattice. Then,*

- $(L, \leqslant)$ *is* bounded *if it has a maximum (usually denoted by* 1*) and a minimum (usually denoted by* 0*);*
- $(L, \leqslant)$ *is* modular *if, for all a, b and $c \in L$ with $a \leqslant c$,*
$$a \vee (b \wedge c) = (a \vee b) \wedge c\,;$$
- $(L, \leqslant)$ *is* distributive *if, for all a, b and $c \in L$,*
$$a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c) \quad \textit{and} \quad a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)\,;$$
- $(L, \leqslant)$ *is* upper-continuous *if it is complete and, for any directed subset $\{x_i : i \in I\}$ of L (or, equivalently, for any chain in L) and any $x \in L$,*
$$x \wedge \bigvee_{i\in I} x_i = \bigvee_{i\in I}(x \wedge x_i)\,;$$

– *given $a \in L$, an element c is a* pseudo-complement for *a if it is maximal with respect to the property that $a \wedge c = 0$. $(L, \leqslant)$ is* pseudo-complemented *if, for any choice of $x \leqslant y$ in L, any element $a \in [x, y]$ has a pseudo-complement in $[x, y]$, that is, an element $c \in [x, y]$ maximal with respect to the property that $a \wedge c = x$;*

– *$(L, \leqslant)$ is a* frame *if it is complete and, for any subset $\{x_i : i \in I\}$ of L and any $x \in L$,*

$$x \wedge \bigvee_{i \in I} x_i = \bigvee_{i \in I} (x \wedge x_i) \, .$$

Notice that a complete lattice always has a minimum $\bigvee \varnothing = 0$ and a maximum $\bigwedge \varnothing = 1$.

Example 2.2. (1) *The family of closed sets* Closed(X, τ) *of a topological spaces (X, τ), ordered by reverse inclusion, is a frame. In fact, given a set $\{C_i : i \in I\}$ of closed sets, $\bigvee C_i = \bigcap C_i \in \mathrm{Closed}(X, \tau)$, while $\bigwedge C_i = \overline{\bigcup_{i \in I} C_i}$, where for any subset $E \subseteq X$ we denote by $\overline{E} = \bigcap \{C : E \subseteq C \in \mathrm{Closed}(X, \tau)\}$ the* closure *of E in X;*

(2) *a total order $(L, \leqslant)$ is a frame, in fact, given $x \in L$ and $\{y_i : i \in I\} \subseteq L$, we have two cases*

$$x \wedge \bigvee_{i \in I} y_i = \begin{cases} x & \text{if } x \leqslant \bigvee_{i \in I} y_i; \\ \bigvee_{i \in I} y_i & \text{if } x \geqslant \bigvee_{i \in I} y_i. \end{cases}$$

In both cases, there exists (at least) an element $j \in I$ such that $x \leqslant y_j$, so $\bigvee_{i \in I} (x \wedge y_i) = \bigvee_{i \neq j} (x \wedge y_i) \vee x \geqslant x = x \wedge \bigvee_{i \in I} y_i$;

(3) *given a Grothendieck category $\mathfrak{C}$ and an object $M \in \mathrm{Ob}(\mathfrak{C})$, $\mathcal{L}(M)$ is an upper-continuous modular lattice. Indeed, it is a complete lattice by Lemma 1.72, since any Grothendieck category is well-powered, and it is upper-continuous by Proposition 1.78, for modularity see for example* [96, Proposition 5.3, Ch. IV]*).*

We collect in the following lemma some observations on the notions introduced in Definition 2.1, for a proof see Sections 2, 3 and 4 of Chapter III in [96].

Lemma 2.3. *Let $(L, \leqslant)$ be a lattice. Then,*

(1) *if L is distributive, then it is also modular;*

(2) *if L is upper-continuous and modular, then it is pseudo-complemented;*

(3) *L is complete if and only if any subset F of L has a meet. Furthermore, if L is complete then it is bounded;*

(4) *if L is a frame, then it is distributive, upper continuous, bounded, complete and pseudo-complemented.*

Definition 2.4. *Let $(L_1, \leqslant)$ and $(L_2, \leqslant)$ be two lattices and consider a map $\phi : L_1 \to L_2$. Then, ϕ* preserves segments *if $[\phi(a), \phi(b)] = \phi([a, b])$, for all $a \leqslant b \in L_1$.*

Example 2.5. *Let $\mathfrak{C}$ be a Grothendieck category and let $\phi : X \to Y$ be a morphism in $\mathfrak{C}$. Then, the map*

$$\Phi : \mathcal{L}(X) \to \mathcal{L}(Y) \quad \text{such that} \quad \Phi(C) = \phi(C) \, ,$$

induced by ϕ is a semi-lattice homomorphism that commutes with arbitrary joins and preserves segments. Indeed, Φ is a semi-lattice homomorphism that commutes with arbitrary joins by

Proposition 1.72 and the fact that any Grothendieck category is complete. To show that Φ sends segments to segments, let $K_1 \leqslant K_2 \in \mathcal{L}(X)$ and consider $K \in [\Phi(K_1), \Phi(K_2)]$. Then,

$$\begin{aligned} K &= \Phi(\phi^{-1}(K)) = \Phi(\phi^{-1}(K) \cap \phi^{-1}(\phi(K_2))) \\ &= \Phi(\phi^{-1}(K) \cap (K_2 + \mathrm{Ker}(\phi))) = \Phi((\phi^{-1}(K) \cap K_2) + \mathrm{Ker}(\phi)) \\ &= \Phi(\phi^{-1}(K) \cap K_2) + \Phi(\mathrm{Ker}(\phi)) = \Phi(\phi^{-1}(K) \cap K_2)\,, \end{aligned}$$

where in the first line we used that $K \leqslant \mathrm{Im}(\phi)$, while in the second line we used the modularity of $\mathcal{L}(X)$. Since $\phi^{-1}(K) \cap K_2 \in [K_1, K_2]$ we proved that Φ sends segments to segments.

Lemma 2.6. *Let $(L_1, \leqslant)$ and $(L_2, \leqslant)$ be two lattices and consider a map $\phi : L_1 \to L_2$. Then,*

(1) *if L_1 has a minimum element 0 and ϕ commutes with arbitrary joins, then $\phi(0)$ is a minimum in L_2;*

(2) *if L_1 and L_2 are bounded and ϕ preserves segments, then ϕ is surjective if and only if its image contains 0 and 1.*

Proof. Remember that $0 = \bigvee \varnothing$ and so, since ϕ commutes with arbitrary joins, $\phi(0) = \bigvee \varnothing$ in L_2. This proves (1). Part (2) easily follows from the definitions. □

Definition 2.7. *A* quasi-frame *(or* qframe*) is an upper-continuous modular lattice. A map between two quasi-frames is a* homomorphism of quasi-frames *if it is a homomorphism of semi-lattices that preserves segments and commutes with arbitrary joins.*
We denote by QFrame *the category of quasi-frames and homomorphisms of quasi-frames.*

Notice that, given a qframe $(L, \leqslant)$ and two elements $a \leqslant b \in L$, the segment $[a, b]$ is again a qframe, even if the inclusion $[a, b] \to L$ is not a homomorphism of qframes.

2.1.2 Constructions in QFrame

Let us introduce some terminology and some useful constructions in the category of qframes.

Definition 2.8. *Let $\phi : L_1 \to L_2$ be a homomorphism of qframes. The element $\mathrm{Ker}(\phi) = \bigvee_{\phi(x)=0} x \in L_1$ is called the* kernel *of ϕ. We say that ϕ is* algebraic *provided the restriction $\phi : [\mathrm{Ker}(\phi), 1] \to L_2$ of ϕ to $[\mathrm{Ker}(\phi), 1]$ is injective.*

Notice that an algebraic homomorphism of qframes is injective if and only if its kernel is 0.

Example 2.9. *Let $\mathfrak{C}$ be a Grothendieck category and let $\phi : X \to Y$ be a morphism in $\mathfrak{C}$. Then, the morphism $\Phi : \mathcal{L}(X) \to \mathcal{L}(Y)$ such that $\Phi(K) = \phi(K)$, for all $K \in \mathcal{L}(X)$, is algebraic. Indeed, notice that $\mathrm{Ker}(\Phi) = \mathrm{Ker}(\phi)$ and that, given K_1, $K_2 \in [\mathrm{Ker}(\phi), 1]$ such that $\Phi(K_1) = \Phi(K_2)$, we get $K_1 = K_1 + \mathrm{Ker}(\phi) = \phi^{-1}(\phi(K_1)) = \phi^{-1}(\phi(K_2)) = K_2 + \mathrm{Ker}(\phi) = K_2$.*

Definition 2.10. *Let $(L, \leqslant)$ be a qframe, let I be a set and let $\mathcal{F} = \{x_i : i \in I\}$ be a subset of L such that $x_i \neq 0$ for all $i \in I$. We say that $\mathcal{F}$ is a* join-independent family *if, for any $i \in I$,*

$$\left(\bigvee_{j \in I \setminus \{i\}} x_j\right) \wedge x_i = 0\,.$$

Furthermore, we say that $\mathcal{F}$ is a basis *for L if it is join-independent and $\bigvee_{i \in I} x_i = 1$. The* uniform dimension $\mathrm{u.dim}(L)$ *of L is*

$$\mathrm{u.dim}(L) = \sup\{|\mathcal{F}| : \mathcal{F} \text{ a finite join independent family in } L\}\,.$$

If $\mathrm{u.dim}(L) = 1$, *L is said to be a* uniform qframe.

Lemma 2.11. *Let $(L, \leqslant)$ be a qframe, let $x \in L$ and let $\{y_i : i \in I\}$ be a basis of L. If $x \neq 0$, there exists a finite subset of I such that $x \wedge \bigvee_{i \in F} x_i \neq 0$.*

Proof. Notice that $0 \neq x = x \wedge \bigvee\{\bigvee_{i\in F} y_i : F \subseteq I \text{ finite}\} = \bigvee\{x \wedge \bigvee_{i\in F} y_i : F \subseteq I \text{ finite}\}$, so for at least one finite subset F of I, $x \wedge \bigvee_{i\in F} x_i \neq 0$ □

Definition 2.12. *Let I be a set and, for all $i \in I$, let $(L_i, \leqslant)$ be a qframe. We construct the* product *of this family as follows:*

$$\prod_{i\in I} L_i = \{\underline{x} = (x_i)_I : x_i \in L_i\,,\ \textit{for all } i \in I\}$$

with the partial order relation defined by

$$(\ \underline{x} \leqslant \underline{y}\ ) \iff (\ x_i \leqslant y_i\,,\ \textit{for all } i \in I\ )\,.$$

One can prove that $\prod_{i\in I} L_i$ is again a qframe. Furthermore, for any subset $J \subseteq I$ the canonical surjective map

$$\pi_J : \prod_{i\in I} L_i \to \prod_{j\in J} L_j\,,$$

defined by $\pi_J((x_i)_I) = (x_j)_J$, and the canonical injective map

$$\epsilon_J : \prod_{j\in J} L_j \to \prod_{i\in I} L_i\,,$$

defined by $\epsilon_J((x_j)_J) = (y_i)_{i\in I}$, with $y_i = x_i$ if $i \in J$, while $x_i = 0$ for all $i \in I\backslash J$, are homomorphisms of qframes.

Definition 2.13. *Let $(L, \leqslant)$ be a qframe. A* congruence *on $(L, \leqslant)$ is a subset $\mathcal{R} \subseteq L \times L$ which satisfies the following properties:*

(Cong.1) *$\mathcal{R}$ is an equivalence relation;*

(Cong.2) *for all a, b and $c \in L$, (a, b) implies $(a \vee c, b \vee c)$;*

(Cong.3) *for all a, b and $c \in L$, (a, b) implies $(a \wedge c, b \wedge c)$.*

When $\mathcal{R}$ is a congruence, we write $a \sim b$ to denote that $(a, b) \in \mathcal{R}$. Furthermore, if $\mathcal{R}$ satisfies the following condition (Cong.4), then $\mathcal{R}$ is said to be a strong congruence*:*

(Cong.4) *for all $a \in L$ the equivalence class $[a]$ has a maximum.*

Lemma 2.14. *Let $(L, \leqslant)$ be a qframe and let $\mathcal{R}$ be a strong congruence on L. Let $L/\mathcal{R}$ be the set of equivalence classes in L and endow it with the following binary relation:*

$$(\ [a] \preceq [b]\ ) \iff (\ \exists a' \in [a] \textit{ and } b' \in [b] \textit{ such that } a' \leqslant b'\ )\,.$$

Then $\preceq$ is a partial order, and $(L/\mathcal{R}, \preceq)$ is a qframe. Furthermore, the canonical map $\pi : L \to L/\mathcal{R}$ such that $x \mapsto [x]$ is a surjective homomorphism of qframes which commutes with finite meets.

Proof. *$\preceq$ is a partial order.* It is clear that $\preceq$ is well-defined and reflexive. Let $a, b \in L$ be such that $[a] \preceq [b]$ and $[b] \preceq [a]$, let us show that $[a] = [b]$. Indeed, by definition there exists $a', a'' \in [a]$ and $b', b'' \in [b]$ such that $a' \leqslant b'$ and $b'' \leqslant a''$. Thus, $a \sim a' = a' \wedge b' \sim a' \wedge b'' \sim a'' \wedge b'' = b'' \sim b$, that is, $[a] = [b]$, proving that $\mathcal{R}$ is symmetric. Let us now verify transitivity, that is, given $[a] \preceq [b] \preceq [c] \in L/\mathcal{R}$, $[a] \preceq [c]$. To do so, take $a' \in [a]$, b' and $b'' \in [b]$, and $c' \in [c]$ such that $a' \leqslant b'$ and $b'' \leqslant c'$. It is then clear that $a' \leqslant b' \vee b'' \leqslant b' \vee c'$, and also that $c' = c' \vee b'' \sim c' \vee b'$, thus $[a] \preceq [c]$ as desired.
Lattice structure. Let a and $b \in L$ and let us show that $[a \wedge b]$ is a greatest lower bound for $[a]$ and $[b]$ in $L/\mathcal{R}$. Indeed, it is clear that $[a \wedge b]$ is $\preceq$ of both $[a]$ and $[b]$. Furthermore, given $c \in L$ such that $[c] \preceq [a]$ and $[c] \preceq [b]$, there exist $c_a, c_b \in [c]$, such that $c_a \leqslant a$ and $c_b \leqslant b$. Thus, $[c] = [c_a \wedge c_b] \preceq [a \wedge b]$, showing that $[a \wedge b]$ is a greatest lower bound. One can show analogously that $[a \vee b]$ is a least upper bound for $[a]$ and $[b]$.
Modularity. Let a, b and $c \in L$ and suppose $[a] \preceq [c]$. Choose $a' \in [a]$ and $c' \in [c]$ such that $a' \leqslant c'$, then, by the modularity of L, $a' \vee (b \wedge c') = (a' \vee b) \wedge c'$ and so

$$[a] \vee ([b] \wedge [c]) = [a'] \vee ([b] \wedge [c']) = ([a'] \vee [b]) \wedge [c'] = ([a] \vee [b]) \wedge [c] .$$

Completeness. Consider a family $\mathcal{F} = \{[x_i] : i \in I\}$ in $L/\mathcal{R}$, we claim that $[\bigvee_{i\in I} x_i]$ is a least upper bound for $\mathcal{F}$. In fact, it is clear that $[\bigvee_{i\in I} x_i] \succeq [x_j]$ for all $j \in I$. Furthermore, given $c \in L$ such that $[c] \succeq [x_i]$ for all $i \in I$, we can choose $x_i' \in [x_i]$ such that $x_i' \leqslant \bar{c}$ for all $i \in I$, where $\bar{c} = \bigvee[c]$. Letting $\bar{x}_i = \bigvee[x_i]$, $\bar{c} = \bar{c} \wedge x_i' \sim \bar{c} \wedge \bar{x}_i$ and so $\bar{x}_i \leqslant \bar{c}$, for all $i \in I$. Thus, $[c] = [\bar{c}] \succeq [\bigvee_{i\in I} \bar{x}_i] = [\bigvee_{i\in I} x_i]$.
$(L/\mathcal{R}, \preceq)$ is a qframe. We have just to verify upper continuity. Let $\{[x_i] : i \in I\}$ be a directed family in $L/\mathcal{R}$ and let $\bar{x}_i = \bigvee[x_i]$, for all $i \in I$. The set $\{\bar{x}_i : i \in I\}$ is directed and so, for all $x \in L$, $x \wedge \bigvee_{i\in I} \bar{x}_i = \bigvee_{i\in I}(x \wedge \bar{x}_i)$. Thus, by our description of the lattice operations,

$$[x] \wedge \bigvee_{i\in I}[x_i] = [x] \wedge \bigvee_{i\in I}[\bar{x}_i] = \bigvee_{i\in I}([x] \wedge [\bar{x}_i]) = \bigvee_{i\in I}([x] \wedge [x_i]) .$$

π is a surjective homomorphism of qframes that commutes with finite meets. It is all clear from the description of the lattice operation in $L/\mathcal{R}$ a part the fact that π preserves segments. So take $x \leqslant y \in L$ and consider $[z] \in [[x], [y]]$. Let $x' \in [x]$ and $z' \in [z]$ be such that $x' \leqslant z'$. Clearly, $x \leqslant z' \vee x \in [z]$, in fact, $x \sim x'$ implies $z' \vee x \sim z' \vee x' = z'$. Furthermore, $y \geqslant (z' \vee x) \wedge y \in [z]$, in fact, given $z'' \in [z]$ and $y' \in [y]$ such that $z'' \leqslant y'$, we obtain $(z' \vee x) \wedge y \sim z'' \wedge y \sim z'' \wedge y' = z''$. Thus, $(z' \vee x) \wedge y \in [x, y]$ and $\pi((z' \vee x) \wedge y) = [z]$. $\square$

Definition 2.15. *A qframe $(L, \leqslant)$ is* compact *if, for any subset $S \subseteq L$ such that $\bigvee S = 1$, there exists a finite subset $F \subseteq S$ such that $\bigvee F = 1$.*

Lemma 2.16. *Let $(L, \leqslant)$ be a qframe, let $x \in L$ and let $\{y_i : i \in I\}$ be a family such that $\bigvee y_i = 1$. If $[0, x]$ is compact, there exists a finite subset F of I such that $x \leqslant \bigvee_{i\in F} y_i$.*

Proof. Notice that $x = x \wedge \bigvee_{i\in I}\{\bigvee_{i\in G} y_i : G \subseteq I \text{ finite}\} = \bigvee\{x \wedge \bigvee_{i\in G} y_i : G \subseteq I \text{ finite}\}$. By compactness, there exists a finite subset K of the set of finite subsets of I such that $x = \bigvee\{x \wedge \bigvee_{i\in G} y_i : G \in K\}$. Taking $F = \bigcup_{G\in K} G$ we get

$$x = \bigvee \left\{ x \wedge \bigvee_{i\in G} y_i : G \in K \right\} \leqslant x \wedge \bigvee_{i\in F} y_i \leqslant x .$$

Thus, $x = x \wedge \bigvee_{i\in F} y_i$, which means exactly that $x \leqslant \bigvee_{i\in F} y_i$. $\square$

2.1.3 Composition length

Definition 2.17. *Let $(L, \leqslant)$ be a qframe. Given a finite chain*

$$\sigma : x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n$$

of elements of L, we say that the length $\ell(\sigma)$ of σ is the number of strict inequalities in the chain. A second chain $\sigma' : y_0 \leqslant y_1 \leqslant \cdots \leqslant y_m$ is a refinement *of σ if $\{x_0, x_1, \ldots, x_n\} \subseteq \{y_0, y_1, \ldots, y_m\}$.*

The following lemma is known as Artin-Schreier's Refinement Theorem.

Lemma 2.18. [96, Proposition 3.1, Ch. III] *Let $(L, \leqslant)$ be a qframe, let $a \leqslant b \in L$ and let*

$$\sigma_1 : a = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = b \quad \text{and} \quad \sigma_2 : a = y_0 \leqslant y_1 \leqslant \cdots \leqslant y_m = b\,.$$

Then, there exists a series $\sigma : a = z_0 \leqslant z_1 \leqslant \cdots \leqslant z_t = b$ that refines both σ_1 and σ_2.

Definition 2.19. *Let $(L, \leqslant)$ be a qframe. The* length *of L is*

$$\ell(L) = \sup\{\ell(\sigma) : \ \sigma \text{ a finite chain of elements of } L\} \in \mathbb{N} \cup \{\infty\}\,.$$

If $\ell(L) < \infty$ we say that L is a qframe of finite length.

For any element $x \in L$ we use the notation $\ell(x)$ to denote the length of the segment $[0, x]$. In the following lemmas we describe some properties of this numerical invariant.

A qframe $(L, \leqslant)$ is said to be *trivial* if it has just one element. In what follows, by *non-trivial* qframe we mean a qframe which contains at least two elements. Furthermore, $(L, \leqslant)$ is said to be an *atom* (or to be *simple*) if it has two elements.

Remark 2.20. *A qframe $(L, \leqslant)$ is trivial if and only if $\ell(L) = 0$, while it is an atom if and only if $\ell(L) = 1$.*

Definition 2.21. *Let $(L, \leqslant)$ be a qframe and consider a finite chain*

$$\sigma : 0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = 1$$

If all the segments $[x_i, x_{i+1}]$, with $i = 0, \ldots, n-1$, are simple, then we say that σ is a composition series*.*

Using Lemma 2.18, it is not difficult to deduce the following lemma, which is usually known as Jordan-Hölder Theorem.

Lemma 2.22. *Let $(L, \leqslant)$ be a qframe of finite length. Then,*

(1) *any finite chain in L can be refined to a composition series;*

(2) *any two composition series in L have the same length;*

(3) *$\ell(L) = n$ if and only if there exists a composition series of length n in L.*

Definition 2.23. *A qframe $(L, \leqslant)$ is*

– Noetherian *if any ascending chain $x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n \leqslant \ldots$ stabilizes at some point;*

– Artinian *if any descending chain $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_n \geqslant \ldots$ stabilizes at some point.*

Using Lemma 2.22, one can prove that $\ell(L) < \infty$ if and only if L is both Noetherian and Artinian (see also Lemma 2.35).

Lemma 2.24. *Let $(L, \leqslant)$ be a qframe. Then, L is Noetherian if and only if $[0, x]$ is compact for all $x \in L$.*

Proof. Suppose that L is Noetherian, let $x \in L$ and consider a subset $S \subseteq [0, x]$ such that $\bigvee S = x$. Consider an ascending chain in L defined inductively as follows:

- let $F_0 = \varnothing$ and let $x_0 = 0$;
- given $n \in \mathbb{N}$ for which F_n has already been defined, we have two possibilities. If $\bigvee F_n = S$ then we let $F_{n+1} = F_n$, otherwise there exists a finite subset F_{n+1} of S that contains F_n and such that $\bigvee F_n < \bigvee F_{n+1}$. We let $x_{n+1} = \bigvee F_{n+1}$.

Now, the sequence $x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n \leqslant \ldots$ is an infinite ascending sequence so that there exists $\bar{n} \in \mathbb{N}$ such that $x_n = x_{\bar{n}}$ for all $n \geqslant \bar{n}$. In particular, $x = \bigvee F_{\bar{n}}$, where $F_{\bar{n}}$ is a finite subset of S, proving that $[0, x]$ is compact.
On the other hand, suppose that $[0, x]$ is compact for all $x \in L$ and consider an ascending chain

$$x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n \leqslant \ldots$$

Let $\bar{x} = \bigvee_{i \in \mathbb{N}} x_i$ and notice that, by compactness, there exists a finite subset $F \subseteq \mathbb{N}$ such that $\bar{x} = \bigvee_{f \in F} x_f$. Letting $\bar{n} = \max\{F\}$, $x_m = x_{\bar{n}}$ for all $m \geqslant \bar{n}$. $\square$

Lemma 2.25. [26, Lemma 3.2] *Let $(L, \leqslant)$ be a qframe of finite length and let $x, y \in L$. Then,*

$$\ell(x \vee y) + \ell(x \wedge y) = \ell(x) + \ell(y)\,.$$

Lemma 2.26. *Let $\phi : L_1 \to L_2$ be a homomorphism of qframes:*

(1) *if ϕ is injective, then $\ell(L_1) \leqslant \ell(L_2)$;*

(2) *if ϕ is surjective, then $\ell(L_2) \leqslant \ell(L_1)$.*

Proof. (1) Let $\sigma : \ x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be a chain in L_1, then $\phi(\sigma) : \ \phi(x_1) \leqslant \phi(x_2) \leqslant \cdots \leqslant \phi(x_n)$ is a chain in L_2. Furthermore, if $x_i \neq x_j$, then $\phi(x_i) \neq \phi(x_j)$ by injectivity. Thus, $\ell(\phi(\sigma)) = \ell(\sigma)$ and so $\ell(L_1) \leqslant \ell(L_2)$.
(2) Let $\sigma : \ x_1 \leqslant x_2 \leqslant \cdots \leqslant x_n$ be a chain in L_2. Since ϕ is surjective, there exist $y_1, \ldots, y_n \in L_1$ such that $\phi(y_i) = x_i$ for all $i = 1, \ldots, n$. Clearly, $\sigma' : \ y_1 \leqslant (y_1 \vee y_2) \leqslant \cdots \leqslant (y_1 \vee y_2 \vee \ldots \vee y_n)$ and, for all $i = 1, \ldots, n$, $\phi(y_1 \vee \ldots \vee y_i) = \phi(y_1) \vee \ldots \vee \phi(y_i) = x_1 \vee \ldots \vee x_i = x_i$. If $x_i \neq x_{i+1}$, then $y_1 \vee \ldots \vee y_i \neq y_1 \vee \ldots \vee y_i \vee y_{i+1}$ and so $\ell(\sigma) \leqslant \ell(\sigma')$. Thus, $\ell(L_2) \leqslant \ell(L_1)$. $\square$

Corollary 2.27. *Let I be a set. For all $i \in I$, let $(L_i, \leqslant)$ be a non-trivial qframe and let $L = \prod_I L_i$. Then,*

$$\ell(L) = \begin{cases} \sum_{i \in I} \ell(L_i) & \text{if } I \text{ is finite;} \\ \infty & \text{otherwise.} \end{cases}$$

Proof. If $\ell(L_i) = \infty$ for some $i \in I$ there is nothing to prove, so we suppose that $\ell(L_i)$ is finite for all $i \in I$. Let $\epsilon_i : L_i \to L$ be the canonical inclusion and let $1_i = \bigvee \epsilon(L_i)$, for all $i \in I$. Notice that $\epsilon_i(L_i) = [0, 1_i]$, so $\ell(L_i) = \ell(1_i)$, and $L = [0, \bigvee_{i \in I} 1_i]$, so $\ell(L) = \ell(\bigvee_{i \in I} 1_i)$.
When I is finite, the proof follows by Lemma 2.25 and the fact that, $1_i \wedge \bigvee_{j \neq i} 1_j = 0$.
If I is not finite, then for any finite subset $J \subseteq I$, we have $\ell(\prod_J L_j) = \sum_J \ell(L_j) \geqslant |J|$ by the first part of the proof. Furthermore, $\ell(\prod_I L_i) \geqslant \ell(\prod_J L_j)$, by Lemma 2.26 applied to the maps $\pi_J : \prod_I L_i \to \prod_J L_j$. Thus, $\ell\left(\prod_I L_i\right) \geqslant \sup\{|J| : J \subseteq I \text{ finite}\} = \infty$. $\square$

Lemma 2.28. *Let $(L, \leqslant)$ be a qframe of finite length, let $(L', \leqslant)$ be a qframe, and let $\phi : L \to L'$ be a homomorphism of qframes. Then ϕ is injective if and only if it is surjective, if and only if $\ell(L) = \ell(\phi(L))$.*

Proof. Let us start proving that ϕ is injective if and only if $\ell(L) = \ell(\phi(L))$. Indeed, suppose that $\ell(L) = \ell(\phi(L))$ and let $x, y \in L$ be such that $\phi(x) = \phi(y)$. If, looking for a contradiction $x \neq y$, then either $x < x \vee y$ or $y < x \vee y$. Without loss of generality, we suppose that $x < x \vee y$. Take the chain $0 \leqslant x < x \vee y \leqslant 1$ between 0 and 1 and refine it to a composition chain

$$\sigma : \ 0 \leqslant \cdots \leqslant x < \cdots < x \vee y < \cdots \leqslant 1 ,$$

thus $\ell(\sigma) = \ell(L)$ (see Lemma 2.22). The image via a homomorphism of qframes of a composition chain is a (eventually shorter) composition chain in the image. Thus, $\ell(\phi(\sigma)) = \ell(\phi(L)) = \ell(L) = \ell(\sigma)$, in particular, $\phi(x) \neq \phi(x \vee y) = \phi(x) \vee \phi(y)$, which contradicts the fact that $\phi(x) = \phi(y)$. For the converse it is enough to verify that the image of a composition chain via an injective homomorphism is a composition chain in the image with the same length.
Let us now verify that ϕ is surjective if and only if $\ell(L) = \ell(\phi(L))$. Indeed, if ϕ is not surjective, that is $\bigvee \phi(L) \neq 1$, consider a composition chain $\sigma : 0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n = \bigvee \phi(L)$ in $[0, \bigvee \phi(L)]$. We can define a longer chain $\sigma' : 0 = x_0 \leqslant x_1 \leqslant \ldots \leqslant x_n < 1$ in L. Hence, $\ell(\phi(L)) = \ell(\sigma) < \ell(\sigma') \leqslant \ell(L)$. The converse is trivial since $L = \phi(L)$ clearly implies that $\ell(L) = \ell(\phi(L))$. □

2.1.4 Socle series

Definition 2.29. *Let $(L, \leqslant)$ be a qframe. The* socle $s(L)$ *of L is the join of all the atoms in L. For all $x \in L$, we let $s(x) = s([0, x])$.*

Lemma 2.30. *Let $(L, \leqslant)$ be a qframe and let I be a set. Then,*

(1) *$s(x) \leqslant x$ and $s(x_1) \leqslant s(x_2)$, for all $x \in L$ and $x_1 \leqslant x_2 \in L$;*

(2) *$s(\bigvee_{i \in I} x_i) = \bigvee_{i \in I} s(x_i)$, where $x_i \in L$ for all $i \in I$;*

(3) *$s(\bigwedge_{i \in I} x_i) = \bigwedge_{i \in I} s(x_i)$, where $x_i \in L$ for all $i \in I$;*

(4) *if $\phi : L \to L'$ is a homomorphism of qframes, then $\phi(s(L)) \leqslant s(L')$.*

Proof. Parts (1), (2) and (3) follow by the properties described in [26, page 47]. For part (4), notice that $\phi(s(L)) = \phi(\bigvee\{x \in L : [0, x] \text{ is an atom}\} = \bigvee\{\phi(x) : [0, x] \text{ is an atom}\} \leqslant \bigvee\{y \in L' : [0, y] \text{ is an atom}\}$ (use the fact that ϕ takes intervals to intervals). □

Thanks to part (4) of Lemma 2.30, we can give the following

Definition 2.31. *Let $(L, \leqslant)$ be a qframe and let* $\mathrm{Soc}(L) = [0, s(L)]$. *Furthermore, given a homomorphism $\phi : L \to L'$ of qframes, we denote by* $\mathrm{Soc}(\phi) : \mathrm{Soc}(L) \to \mathrm{Soc}(L')$ *the restriction of ϕ. This defines a covariant functor* $\mathrm{Soc} : \mathrm{QFrame} \to \mathrm{QFrame}$.

Definition 2.32. *Let $(L, \leqslant)$ be a qframe and let $x \in L$. We say that x is* completely meet irreducible *if, given a family $\{y_i\}_{i \in I} \subseteq [x, 1]$ such that $\bigwedge y_i = x$, there exists $i \in I$ such that $y_i = x$.*

Lemma 2.33. *Let $(L, \leqslant)$ be a qframe and let $x \in L$. Then, x is completely meet irreducible if and only if the lattice $[x, 1]$ is uniform and its socle is an atom.*

Proof. Suppose first that x is completely meet irreducible. To show that $[x,1]$ is uniform consider any two elements y_1, $y_2 \in (x,1]$ such that $y_1 \wedge y_2 = x$. Then either $y_1 = x$ or $y_2 = x$, so the family $\{y_1, y_2\}$ is not join-independent. If $\mathrm{Soc}([x,1]) \neq \{x\}$, then $\mathrm{Soc}([x,1])$ is an atom by uniformity. On the other hand, if looking for a contradiction $\mathrm{Soc}([x,1]) = \{x\}$, then for all $y \in (x,1)$, (x,y) is not empty (otherwise $[x,y]$ is an atom) and so we can inductively construct a transfinite sequence as follows:

- $y_1 = 1$;
- if α is an ordinal for which y_α is already constructed, $y_{\alpha+1}$ is an element in (x, y_α);
- if λ is a limit ordinal for which y_α is constructed for all $\alpha < \lambda$, and $(x, \bigwedge_{\alpha<\lambda} y_\alpha) \neq \varnothing$, we choose $y_\lambda \in (x, \bigwedge_{\alpha<\lambda} y_\alpha)$, otherwise we stop.

Of course, there exists a limit ordinal $\bar\lambda$ for which $x = \bigwedge_{\alpha<\bar\lambda} y_\alpha$, so we obtain a sequence $\{y_\alpha\}_{\alpha<\bar\lambda}$ such that $\bigwedge_{\alpha<\bar\lambda} y_\alpha = x$ but $x \neq y_\alpha$ for all $\alpha < \bar\lambda$, which is a contradiction.

On the other hand, suppose that $[x,1]$ is uniform and that $\mathrm{Soc}([x,1]) = [x,s]$ is an atom, and choose a family $\{y_i\}_{i\in I} \subseteq [x,1]$ such that $\bigwedge_{i\in I} y_i = x$. If, looking for a contradiction, $y_i \neq x$ for all $i \in I$, then $y_i \wedge s \neq x$ by uniformity and, since $[x,s]$ is an atom, $s \leqslant y_i$. Thus, $x = \bigwedge_{i\in I} y_i \geqslant s \wedge \bigwedge_{i\in I} y_i = \bigwedge_{i\in I}(s \wedge y_i) = s$, which is a contradiction. □

We can iterate the procedure that defines the socle as follows:

Definition 2.34. *Let $(L, \leqslant)$ be qframe. Then,*

- $s_0(L) = s(L)$;
- *for any ordinal α, $s_{\alpha+1}(L) = s([s_\alpha(L), 1])$;*
- *for any limit ordinal α, $s_\lambda(L) = \bigvee_{\alpha<\lambda} s_\alpha(L)$.*

L is semi-Artinian *if $s_\tau(L) = 1$ for some ordinal τ.*

It is not difficult to show that the uniform dimension of a semi-Artinian qframe is the length of its socle.

Lemma 2.35. [26, Theorem 5.2 and Proposition 5.3] *Let $(L, \leqslant)$ be a qframe. Then,*

(1) *L is semi-Artinian if and only if $[0,x]$ and $[x,1]$ are semi-Artinian for all $x \in L$;*

(2) *L is semi-Artinian and Noetherian if and only if $\ell(L) < \infty$.*

2.2 Krull and Gabriel dimension

2.2.1 Krull and Gabriel dimension

Definition 2.36. *Let $(L, \leqslant)$ be a qframe. The* Krull dimension $\mathrm{K.dim}(L)$ *of L is defined as follows:*

- $\mathrm{K.dim}(L) = -1$ *if and only if L is trivial;*
- *if α is an ordinal and we already defined what it means to have Krull dimension β for any ordinal $\beta < \alpha$,* $\mathrm{K.dim}(L) = \alpha$ *if and only if* $\mathrm{K.dim}(L) \neq \beta$ *for all $\beta < \alpha$ and, for any descending chain*
$$x_1 \geqslant x_2 \geqslant x_3 \geqslant \ldots \geqslant x_n \geqslant \ldots$$
in L, there exists $\bar n \in \mathbb{N}_+$ such that $\mathrm{K.dim}([x_n, x_{n+1}]) = \beta_n$ *for all $n \geqslant \bar n$ and $\beta_n < \alpha$.*

If K.dim$(L) \neq \alpha$ *for any ordinal* α *we set* K.dim$(L) = \infty$.

Notice that the qframes with 0 Krull dimension are precisely the Artinian qframes.

Definition 2.37. *A subclass* $\mathcal{X} \subseteq \mathrm{Ob}(\mathrm{QFrame})$ *is a* Serre class *if it is closed under isomorphisms and, given* $L \in \mathrm{Ob}(\mathrm{QFrame})$ *and* $x \leqslant y \leqslant z \in L$, $[x, y]$, $[y, z] \in \mathcal{X}$ *if and only if* $[x, z] \in \mathcal{X}$.

The class of all lattices with Krull dimension $\leqslant \alpha$ for some ordinal α is a Serre class (see [26, Proposition 13.5]).

Lemma 2.38. *Let* $(L_1, \leqslant)$ *and* $(L_2, \leqslant)$ *be qframes. If* K.dim(L_1) *exists and if there exists a surjective homomorphism of qframes* $\phi : L_1 \to L_2$, *then* K.dim$(L_1) \geqslant$ K.dim(L_2).

Proof. Let us proceed by induction on K.dim$(L_1) = \alpha$. If $\alpha = -1$, then clearly also K.dim$(L_2) = -1$. Suppose now that $\alpha > -1$ and that we already proved our result for all $\beta < \alpha$. If K.dim$(L_2) <$ K.dim(L_1) there is nothing to prove, so suppose that K.dim$(L_2) \nless$ K.dim(L_1) and let us show that K.dim$(L_2) =$ K.dim(L_1). Indeed, consider a descending chain in L_2

$$x_0 \geqslant x_1 \geqslant \cdots \geqslant x_n \geqslant \ldots$$

By the surjectivity of ϕ, we can choose $y_i \in L_1$ so that $\phi(y_i) = x_i$, for all $i \in \mathbb{N}$, let also $y'_i = \bigvee_{j \geqslant i} y_j$. It is not difficult to see that

$$y'_0 \geqslant y'_1 \geqslant \cdots \geqslant y'_n \geqslant \ldots$$

and that $\phi(y'_i) = \bigvee_{j \geqslant i} \phi(y_j) = x_i$. By definition of Krull dimension, there exists $\bar{n} \in \mathbb{N}_+$ such that K.dim$([y'_n, y'_{n+1}]) = \beta_n$ for all $n \geqslant \bar{n}$ and $\beta_n < \alpha$. By inductive hypothesis, K.dim$([x_n, x_{n+1}]) \leqslant$ K.dim$([y'_n, y'_{n+1}]) = \beta_n$, showing that K.dim$(L_2) \leqslant \alpha$, and so, K.dim$(L_2) = \alpha$. □

Definition 2.39. *Let* $(L, \leqslant)$ *be a qframe. We define the* Gabriel dimension G.dim(L) *of* L *by transfinite induction:*

- G.dim$(L) = 0$ *if and only if* L *is trivial. A qframe* S *is* 0*-simple (or just simple) if it is an atom;*

- *let* α *be an ordinal for which we already know what it means to have Gabriel dimension* β, *for all* $\beta \leqslant \alpha$. *A qframe* S *is* α*-simple if, for all* $0 \neq a \in S$, G.dim$([0, a]) \nleqslant \alpha$ *and* G.dim$([a, 1]) \leqslant \alpha$;

- *let* σ *be an ordinal for which we already know what it means to have Gabriel dimension* β, *for all* $\beta < \sigma$. *Then,* G.dim$(L) = \sigma$ *if* G.dim$(L) \nless \sigma$ *and, for all* $1 \neq a \in L$, *there exists* $b > a$ *such that* $[a, b]$ *is* β*-simple for some ordinal* $\beta < \sigma$.

If G.dim$(L) \neq \alpha$ *for any ordinal* α *we set* G.dim$(L) = \infty$.

Notice that the qframes with Gabriel dimension equal to 1 are precisely the semi-Artinian qframes. Also the class of all qframes with Gabriel dimension $\leqslant \alpha$ for some ordinal α is a Serre class (see part (1) of Lemma 2.42). For any ordinal α, G.dim$(S) = \alpha + 1$, for any α-simple qframe S.

Lemma 2.40. *Let* α *be an ordinal and let* $(L, \leqslant)$ *be an* α*-simple qframe. Any non-trivial subqframe of* L *is* α*-simple.*

Proof. We proceed by transfinite induction on α. If $\alpha = 0$, then L is an atom and there is no non-trivial sub-qframe but L itself. Let $\alpha > 0$ and choose $0 \neq b \leqslant a \in L$. By definition, $\mathrm{G.dim}([0,b]) \nleqslant \alpha$ so, to prove that $[0,a]$ is α-simple, it is enough to show that $\mathrm{G.dim}([b,a]) \leqslant \alpha$. If $\mathrm{G.dim}([b,a]) < \alpha$ there is nothing to prove, so let us consider the case when $\mathrm{G.dim}([b,a]) \nless \alpha$. Let $a' \in (b,a]$, choose a pseudo-complement c of a in $[a',1]$ and let $d \in [c,1]$ be such that $[c,d]$ is β-simple for some $\beta < \alpha$. Let $b' = d \wedge a$, then $[a',b']$ is non-trivial by the maximality included in the definition of pseudo-complement, furthermore, by modularity, $[a',b']$ is isomorphic to $[c, b' \vee c]$, which is a sub-qframe of $[c,d]$. By inductive hypothesis, $[a',b']$ is β-simple. This proves that $\mathrm{G.dim}([b,a]) = \alpha$, as desired. □

Theorem 2.41. *Let $(L,\leqslant)$ be a qframe. The following statements hold true:*

(1) *L has Krull dimension if and only if any segment of L has finite uniform dimension and L has Gabriel dimension;*

(2) *if L has Krull dimension, then* $\mathrm{K.dim}(L) \leqslant \mathrm{G.dim}(L) \leqslant \mathrm{K.dim}(L) + 1$*;*

(3) *if L is Noetherian, then there exists a finite chain $0 = x_0 \leqslant x_1 \leqslant \cdots \leqslant x_n = 1$ such that $[x_{i-1}, x_i]$ is α_i-simple for some ordinal α_i, for all $i = 1,\dots,n$. Furthermore, L has Krull dimension and* $\mathrm{G.dim}(L) = \mathrm{K.dim}(L) + 1$.

Proof. For (1), see Exercise (116) in [80] (an argument to solve that exercise can be found in [51]). For parts (2) and (3) see respectively [26, Theorem 13.9] and (statement and proof of) [26, Theorem 13.10]. □

In the following lemmas we collect some properties of Gabriel dimension. Their proof is inspired by the treatment in [80] but we prefer to give complete proofs also here.

Lemma 2.42. *Let L be a qframe with Gabriel dimension. The following statements hold true:*

(1) *if $a \leqslant b \in L$, then* $\mathrm{G.dim}([a,b]) \leqslant \mathrm{G.dim}(L)$*;*

(2) *if $a \in L$, then* $\mathrm{G.dim}(L) = \max\{\mathrm{G.dim}([0,a]), \mathrm{G.dim}([a,1])\}$*;*

(3) *given a subset $\mathcal{F} \subseteq L$ such that $\bigvee \mathcal{F} = 1$,* $\mathrm{G.dim}(L) = \sup\{\mathrm{G.dim}([0,x]) : x \in \mathcal{F}\}$.

(4) *if L is not trivial, then* $\mathrm{G.dim}(L) = \sup\{\mathrm{G.dim}([a,b]) : [a,b]$ *β-simple for some β*$\}$*;*

(5) $\mathrm{G.dim}(L) \leqslant \beta + 1$, *where* $\beta = \sup\{\mathrm{G.dim}([x,1]) : x \neq 0\}$.

Proof. Let $\mathrm{G.dim}(L) = \alpha$.

(1) We proceed by transfinite induction on α. If $\alpha = 0$, there is nothing to prove, as well as when $\alpha > 0$ and $\mathrm{G.dim}([a,b]) < \alpha$. Consider the case when $\alpha > 0$ and $\mathrm{G.dim}([a,b]) \nless \alpha$. Let $a' \in [a,b)$ and let us find $b' \in (a',b]$ such that $[a',b']$ is β-simple for some $\beta < \alpha$. Indeed, we consider a pseudo-complement c of b in $[a',1]$ and we let $d \in [c,1]$ be such that $[c,d]$ is β-simple for some $\beta < \alpha$. Let $b' = d \wedge b$. By modularity, $[a',b'] \cong [c,(d \wedge b) \vee c]$, which is an initial segment in $[c,d]$. By Lemma 2.40, $[a',b']$ is β-simple.

(2) Let $\beta_1 = \mathrm{G.dim}([0,a])$ and $\beta_2 = \mathrm{G.dim}([a,1])$. By part (1), $\alpha \geqslant \max\{\beta_1,\beta_2\}$. Let us show that $\alpha \leqslant \max\{\beta_1,\beta_2\}$, that is, given $1 \neq b \in L$ we need to find $c \in (b,1]$ such that $[b,c]$ is γ-simple for some $\gamma < \max\{\beta_1,\beta_2\}$. Indeed, given $1 \neq b \in L$, we distinguish two cases. If $a \leqslant b$, then $b \in [a,1]$ and so there is $c \in (b,1]$ such that $[b,c]$ is γ-simple for some $\gamma < \beta_2$. If $a \nleqslant b$, then there is $c \in [a \wedge b, a]$ such that $[b \wedge a, c]$ is γ-simple for some $\gamma < \beta_1$ and, by modularity, $[b, b \vee c] \cong [a \wedge b, c]$.

(3) Let $\sup\{\mathrm{G.dim}([0,x]) : x \in \mathcal{F}\} = \beta$. Given $1 \neq a \in L$, we have to show that there exists $b \in [a, 1]$ such that $[a, b]$ is γ-simple for some $\gamma < \beta$. By hypothesis, there exists $x \in \mathcal{F}$ such that $x \nleqslant a$. Thus, $x \neq x \wedge a \in [0, x]$ and so there exists $b' \in [x \wedge a, x]$ such that $[x \wedge a, b']$ is γ-simple for some $\gamma < \mathrm{G.dim}([0,x]) \leqslant \beta$. Let $b = b' \vee a$; by modularity $[a, b] \cong [x \wedge a, b']$ is γ-simple as desired.

(4) Consider a continuous chain in L defined as follows:

- $x_0 = 0$;
- if $\sigma = \tau + 1$ is a successor ordinal, then $x_\sigma = 1$ if $x_\tau = 1$, while x_σ is an element $\geqslant x_\tau$ such that $[x_\sigma, x_\tau]$ is β-simple for some β;
- $x_\sigma = \bigvee_{\tau<\sigma} x_\tau$ if σ is a limit ordinal.

Since we supposed that L has Gabriel dimension, then the above definition is correct and there exists an ordinal $\bar{\sigma}$ such that $x_{\bar{\sigma}} = 1$. Let us prove our statement by induction on $\bar{\sigma}$. If $\bar{\sigma} = 1$, there is nothing to prove. Furthermore, if $\bar{\sigma} = \tau + 1$, then by part (2), $\mathrm{G.dim}(L) = \max\{\mathrm{G.dim}([0, x_\tau]), \mathrm{G.dim}([x_\tau, x_{\bar{\sigma}}])$ and we can conclude by inductive hypothesis. If $\bar{\sigma}$ is a limit ordinal, one concludes similarly using part (3).

(5) It is enough to prove the statement for γ-simple lattices for all ordinals γ and then apply part (4). So, let γ be an ordinal and let L be γ-simple lattice. Then, $\mathrm{G.dim}(L) = \gamma + 1$ and we should prove that $\sup\{\mathrm{G.dim}([x, 1]) : x \neq 0\} \geqslant \gamma$. If, looking for a contradiction, $\sup\{\mathrm{G.dim}([x, 1]) : x \neq 0\} = \beta < \gamma$, then just by definition, L is β-simple, that is a contradiction. □

Corollary 2.43. *Let $(L, \leqslant)$ be a qframe and let α be an ordinal. Then,* $\mathrm{G.dim}(L) \leqslant \alpha$ *if and only if, for any element $x \neq 1$, there exists $y > x$ such that* $\mathrm{G.dim}([x, y]) \leqslant \alpha$.

Proof. Let $x_0 = 0$, for any ordinal γ let $x_{\gamma+1} = 1$ if $x_\gamma = 1$, otherwise we let $x_{\gamma+1}$ be an element $> x_\gamma$ such that $\mathrm{G.dim}([x_\gamma, x_{\gamma+1}]) \leqslant \alpha$. Furthermore, for any limit ordinal λ we let $x_\lambda = \bigvee_{\gamma<\lambda} x_\gamma$. Let us prove by transfinite induction that $\mathrm{G.dim}([0, x_\gamma]) \leqslant \alpha$ for all γ, this will conclude the proof since there exists γ such that $x_\gamma = 1$. Our claim is clear when $\gamma = 0$. Furthermore, if $\gamma = \beta + 1$ and $\mathrm{G.dim}([0, x_\beta]) \leqslant \alpha$, then by Lemma 2.42 (2), $\mathrm{G.dim}([0, x_\gamma]) \leqslant \alpha$. If γ is a limit ordinal and $\mathrm{G.dim}([0, x_\beta]) \leqslant \alpha$ for all $\beta < \gamma$, one concludes by Lemma 2.42 (3). □

Lemma 2.44. *Let $(L, \leqslant)$ be a qframe with Gabriel dimension, let $(L', \leqslant)$ be a qframe and let $\phi : L \to L'$ be a surjective morphism of qframes. Then,* $\mathrm{G.dim}(L') \leqslant \mathrm{G.dim}(L)$.

Proof. Let us proceed by transfinite induction on $\mathrm{G.dim}(L)$.
If $\mathrm{G.dim}(L) = 0$, then L is a trivial as well as L', so there is nothing to prove.
Suppose now that $\mathrm{G.dim}(L) = \alpha > 0$ and that we have already verified our claim for all $\beta < \alpha$. Let first $\alpha = \gamma + 1$ be a successor ordinal and let L be γ-simple. Then, for all $0 \neq a \in L$, $\mathrm{G.dim}([a, 1]) \leqslant \gamma$ and so, by inductive hypothesis, $\mathrm{G.dim}(\phi([a, 1])) \leqslant \gamma$. By Lemma 2.42 (5), $\mathrm{G.dim}(\phi(L)) \leqslant \gamma + 1 = \alpha$.
Let now $x' \in L'$, consider the set $\mathcal{S} = \{x \in L : \phi(x) = x'\}$ and let $\bar{x} = \bigvee \mathcal{S}$, so that $\phi(\bar{x}) = \bigvee_{x\in\mathcal{S}} \phi(x) = x'$. Let also $\bar{y} \geqslant \bar{x}$ be such that $[\bar{x}, \bar{y}]$ is β-simple for some $\beta < \alpha$ and let $y' = \phi(\bar{y}) \in L'$. Then, $y' \geqslant x'$, furthermore $y' \neq x'$ (since $y' = x'$ would imply that $\bar{y} \in \mathcal{S}$, that is, $\bar{y} = \bar{x}$, which is a contradiction). By the first part of the proof, $\mathrm{G.dim}([x', y']) \leqslant \beta + 1 \leqslant \alpha$. To conclude apply Corollary 2.43. □

2.2.2 Torsion and localization

Definition 2.45. *Let $(L, \leqslant)$ be a qframe, and let α be an ordinal. We define the α-*torsion part* of L as*
$$t_\alpha(L) = \bigvee\{x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\}\,.$$

For any given $a \in L$, we let $t_\alpha(a) = t_\alpha([0,a])$.

Lemma 2.46. *Let $(L, \leqslant)$ be a qframe, let $a,\ b \in L$ and let α be an ordinal. Then,*

(1) $t_\alpha(a) = a \wedge t_\alpha(1)$*;*

(2) $t_\alpha(a \vee b) \leqslant t_\alpha(a) \vee b$*, provided* $a \wedge b = 0$*;*

(3) $t_\alpha(a \vee b) = t_\alpha(a) \vee t_\alpha(b)$*, provided* $a \wedge b = 0$*;*

In particular, $t_\alpha(\bigvee_{i\in I} x_i) = \bigvee_{i\in I} t_\alpha(x_i)$ *for any join-independent set* $\{x_i : i \in I\}$ *in* L*.*

Proof. (1) By definition, $t_\alpha(a) \leqslant a \wedge t_\alpha(1)$. On the other hand, by upper continuity,
$$\begin{aligned} a \wedge \bigvee\{x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\} &= \bigvee\{a \wedge x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\} \\ &= \bigvee\{x \in [0,a] : \mathrm{G.dim}([0,x]) \leqslant \alpha\} = t_\alpha(a)\,. \end{aligned}$$

This works since the family $\{x \in L : \mathrm{G.dim}([0,x]) \leqslant \alpha\}$ is directed by part (2) of Lemma 2.42.
(2) Let $x \in [0, a \vee b]$ be such that $\mathrm{G.dim}([0,x]) \leqslant \alpha$, then $x \vee b \in [0, a \vee b]$ and $\mathrm{G.dim}([b, x \vee b]) = \mathrm{G.dim}([b \wedge x, x]) \leqslant \mathrm{G.dim}([0,x]) \leqslant \alpha$. This shows $(*)$ below:
$$\begin{aligned} t_\alpha(a \vee b) &= \bigvee\{x \in [0, a \vee b] : \mathrm{G.dim}([0,x]) \leqslant \alpha\} \\ &\overset{(*)}{\leqslant} \bigvee\{x \vee b \in [0, a \vee b] : \mathrm{G.dim}([b, x \vee b]) \leqslant \alpha\} \\ &= \bigvee\{x \in [b, a \vee b] : \mathrm{G.dim}([b,x]) \leqslant \alpha\} \\ &\overset{(**)}{=} \bigvee\{x \vee b : x \in [0,a] \text{ and } \mathrm{G.dim}([0,x]) \leqslant \alpha\} \\ &\overset{(\substack{*\\**})}{=} b \vee \bigvee\{x : x \in [0,a] \text{ and } \mathrm{G.dim}([0,x]) \leqslant \alpha\} = b \vee t_\alpha(a)\,, \end{aligned}$$

where $(**)$ holds since te map $x \mapsto x \vee b$ is an isomorphism between $[0,a]$ and $[b, b \vee a]$ (use the fact that $a \wedge b = 0$), and in $(\substack{*\\**})$ we used upper-continuity.
(3) It is clear that $t_\alpha(b) \vee t_\alpha(a) \leqslant t_\alpha(a \vee b)$. Using twice part (2) and the modularity of L,
$$\begin{aligned} t_\alpha(b) \vee t_\alpha(a) \leqslant t_\alpha(a \vee b) \leqslant (t_\alpha(a) \vee b) \wedge (t_\alpha(b) \vee a) &= t_\alpha(a) \vee (b \wedge (t_\alpha(b) \vee a)) \\ &= t_\alpha(a) \vee ((b \wedge a) \vee t_\alpha(b)) = t_\alpha(a) \vee t_\alpha(b)\,. \end{aligned}$$

where the last equality holds since $a \wedge b = 0$.
For the last part of the statement, notice that
$$\bigvee_{i\in I} x_i = \bigvee_{F\subseteq I \text{ finite}} \left(\bigvee_{i\in F} x_i\right)\,.$$

Thus, using upper-continuity and part (3) of the lemma,

$$\begin{aligned} t_\alpha\left(\bigvee_{i\in I} x_i\right) &= t_\alpha(1) \wedge \bigvee_{F\subseteq I \text{ finite}} \left(\bigvee_{i\in F} x_i\right) \\ &= \bigvee_{F\subseteq I \text{ finite}} \left(t_\alpha(1) \wedge \bigvee_{i\in F} x_i\right) = \bigvee_{F\subseteq I \text{ finite}} \left(\bigvee_{i\in F} t_\alpha(x_i)\right) = \bigvee_{i\in I} t_\alpha(x_i)\,. \end{aligned}$$

□

Lemma 2.47. *Let L be a qframe, let $x \in L$ and let $\{y_s : s \in S\} \subseteq L$. Suppose that*

(1) $[0, y_s] \cong [0, y_t]$ *for all* s, $t \in S$*;*

(2) $[0, y_s]$ *is Noetherian for some (hence all)* $s \in S$*;*

(3) $\{y_s : s \in S\}$ *is a basis for* L*.*

Then, $\mathrm{G.dim}([0, x])$ *is a successor ordinal.*

Proof. A consequence of Theorem 2.41 (3) is that, for all $s \in S$, $t_{\alpha+1}(y_s) \neq t_\alpha(y_s)$ for just finitely many ordinals α (the same α's for all $s \in S$). Furthermore, $\bigvee_{s\in S} t_\alpha(y_s) = t_\alpha(1)$ for all α, by the above lemma. Thus, $t_{\alpha+1}(1) \neq t_\alpha(1)$ for finitely many ordinals α. Notice also that $t_\alpha(x) = t_\alpha(1) \wedge x$ for all α, thus $t_{\alpha+1}(x) \neq t_\alpha(x)$ implies $t_{\alpha+1}(1) \neq t_\alpha(1)$ and so, $t_{\alpha+1}(x) \neq t_\alpha(x)$ for finitely many ordinals α. Hence, $\mathrm{G.dim}([0, x]) = \sup\{\alpha + 1 : t_{\alpha+1}(x) \neq t_\alpha(x)\} = \max\{\alpha + 1 : t_{\alpha+1}(x) \neq t_\alpha(x)\}$ is a successor ordinal. □

Proposition 2.48. *Let $(L, \leqslant)$ be a qframe and let α be an ordinal. Then,*

(1) $x \in [0, t_\alpha(1)]$ *if and only if* $\mathrm{G.dim}([0, x]) \leqslant \alpha$*;*

(2) *given a qframe $(L', \leqslant)$ and a homomorphism of qframes $\phi : L \to L'$, $\phi(t_\alpha(L)) \leqslant t_\alpha(L')$.*

Proof. (1) By part (3) of Lemma 2.42, $\mathrm{G.dim}([0, t_\alpha(1)]) \leqslant \alpha$ and so, by part (1) of the same lemma, $\mathrm{G.dim}([0, x]) \leqslant \mathrm{G.dim}([0, t_\alpha(1)]) \leqslant \alpha$ for all $x \in [0, t_\alpha(1)]$. On the other hand, if $\mathrm{G.dim}([0, x]) \leqslant \alpha$, then $x \leqslant t_\alpha(1)$ by construction.

(2) is an application of part (1) and Lemma 2.44. □

Definition 2.49. *Let α be an ordinal. Given a qframe $(L, \leqslant)$, we let $T_\alpha(L) = [0, t_\alpha(1)]$, while, given a homomorphism of qframes $\phi : L \to L'$, we let $t_\alpha(\phi) : T_\alpha(L) \to T_\alpha(L')$ be the restriction of ϕ. This defines a covariant functor $T_\alpha : \mathrm{QFrame} \to \mathrm{QFrame}$ that we call* α-torsion functor*.*

Definition 2.50. *Let $(L, \leqslant)$ be a qframe, let α be an ordinal and define the following relation between two elements x and y in L:*

$$(x \sim_\alpha y) \quad \textit{if and only if} \quad (\mathrm{G.dim}([x \wedge y, x \vee y]) \leqslant \alpha)$$

Lemma 2.51. *Let $(L, \leqslant)$ be a qframe and let α be an ordinal, then $\sim_\alpha$ is a strong congruence on L.*

Proof. The fact that $\sim_\alpha$ is a congruence follows by Lemma 2.42 (2) and [4, Proposition 2.4]. Furthermore, given $x \in L$, let us show that $\bigvee[x] \in [x]$. In fact,

$$\mathrm{G.dim}\left(\left[x, \bigvee_{y\in[x]} y\right]\right) = \mathrm{G.dim}\left(\left[x, \bigvee_{y\in[x]} x \vee y\right]\right) = \sup\{\mathrm{G.dim}([x, x \vee y]) : y \in [x]\} \leqslant \alpha\,,$$

by Lemma 2.42 (3). Thus, $\sim_\alpha$ is a strong congruence. □

We denote by $Q_\alpha(L)$ the quotient of L over $\sim_\alpha$ and by $\pi_\alpha : L \to Q_\alpha(L)$ the canonical surjective homomorphism.

Proposition 2.52. *Let $(L, \leqslant)$ and $(L', \leqslant)$ be qframes, let $\phi : L \to L'$ be a homomorphism of qframes, and let α be an ordinal.*

(1) *If $x \sim_\alpha y$ in L, then $\phi(x) \sim_\alpha \phi(y)$ in L';*

(2) $\mathrm{G.dim}(Q_\alpha(T_{\alpha+1}(L))) \leqslant 1$*, that is, $Q_\alpha(T_{\alpha+1}(L))$ is semi-Artinian for any ordinal α.*

Proof. (1) By Lemma 2.44, $\mathrm{G.dim}([x \wedge y, x \vee y]) \geqslant \mathrm{G.dim}(\phi([x \wedge y, x \vee y])) = \mathrm{G.dim}([\phi(x \wedge y), \phi(x) \vee \phi(y)])$. Furthermore, $\phi(x) \wedge \phi(y) \geqslant \phi(x \wedge y)$ and so $\mathrm{G.dim}([\phi(x) \wedge \phi(y), \phi(x) \vee \phi(y)]) \leqslant \mathrm{G.dim}[\phi(x \wedge y), \phi(x) \vee \phi(y)] \leqslant \alpha$, by Lemma 2.42 (1).
(2) Let $S = [a, b]$ be an α-simple segment of $T_{\alpha+1}(L)$. Then, $\pi_\alpha(S)$ is an atom since a is not α-equivalent to b (as $\mathrm{G.dim}(S) = \alpha+1$) and b is α-equivalent to any $c \in (a, b]$ (as $\mathrm{G.dim}([c, b]) \leqslant \alpha$). If $Q_\alpha(T_{\alpha+1}(L)) = 0$ there is nothing to prove, otherwise choose an element $x \in T_{\alpha+1}(L)$ such that $\pi_\alpha(t_{\alpha+1}(1)) \neq \pi_\alpha(x) \in Q_\alpha(T_{\alpha+1}(L))$ and let $\bar{x} = \bigvee[x] \in T_{\alpha+1}(L)$. Notice that $\mathrm{G.dim}([\bar{x}, t_{\alpha+1}(1)]) = \alpha + 1$ (otherwise $[x] = [t_{\alpha+1}(1)]$). By definition of Gabriel dimension, there exists $\bar{y} \in T_{\alpha+1}(L)$ such that $[\bar{x}, \bar{y}]$ is β-simple for some $\beta < \alpha + 1$ and, since $\bar{y} \notin [\bar{x}]$, we have $\beta = \alpha$. By the previous discussion, $[\pi_\alpha(x), \pi_\alpha(\bar{y})]$ is 0-simple. One can conclude by Corollary 2.43. □

Definition 2.53. *Let α be an ordinal. Given a qframe $(L, \leqslant)$, we let $Q_\alpha(L) = L_{/\sim_\alpha}$, while, given a homomorphism of qframes $\phi : L \to L'$, we let $Q_\alpha(\phi) : Q_\alpha(L) \to Q_\alpha(L')$ be the induced homomorphism. This defines a functor Q_α :* QFrame $\to$ QFrame *that we call* α-localization functor.

It is not difficult to show that Q_α is compatible with the composition of morphisms, so that the above definition is correct.

2.2.3 Gabriel categories and Gabriel spectrum

Definition 2.54. *Let $\mathfrak{C}$ be a Grothendieck category. The* Gabriel filtration *of $\mathfrak{C}$ is a transfinite chain $\{0\} = \mathfrak{C}_0 \subseteq \mathfrak{C}_1 \subseteq \cdots \subseteq \mathfrak{C}_\alpha \subseteq \ldots$ of torsion classes defined as follows:*

- $\mathfrak{C}_0 = \{0\}$;
- *suppose that α is an ordinal for which $\mathfrak{C}_\alpha$ has already been defined, an object $C \in \mathfrak{C}$ is said to be* α-cocritical *if $C' \notin \mathfrak{C}_\alpha$ and $C/C' \in \mathfrak{C}_\alpha$, for any non-trivial sub-object $C' \leqslant C$;*
- *suppose that σ is an ordinal for which $\mathfrak{C}_\beta$ has already been defined, for all $\beta < \sigma$. Then, $\mathfrak{C}_\sigma$ is the smallest torsion class containing $\mathfrak{C}_\beta$ and all the β-cocritical objects, for all $\beta < \sigma$.*

A Grothendieck category $\mathfrak{C}$ is said to be a Gabriel category *if $\mathfrak{C} = \bigcup_\alpha \mathfrak{C}_\alpha$.*

As one may expect, there is a relation between Gabriel categories and the Gabriel dimension for qframes. See Proposition 2.68 for a connection between these two concepts.

Let $\mathfrak{C}$ be a Gabriel category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. One can show that $\mathfrak{C}/\mathcal{T}$ is a Gabriel category as well (showing by induction that $\mathbf{Q}_\tau(\mathfrak{C}_\alpha) \subseteq (\mathfrak{C}/\mathcal{T})_\alpha$ for all α).

For any ordinal α, we let $\tau_\alpha = (\mathfrak{C}_\alpha, \mathfrak{C}_\alpha^\perp)$; in what follows, we write α-torsion (resp., torsion free, local,...) instead of τ_α-torsion (resp., torsion free, local,...). Furthermore, we let $\mathbf{T}_\alpha : \mathfrak{C} \to \mathfrak{C}_\alpha$, $\mathbf{S}_\alpha : \mathfrak{C}/\mathfrak{C}_\alpha \to \mathfrak{C}$ and $\mathbf{Q}_\alpha : \mathfrak{C} \to \mathfrak{C}/\mathfrak{C}_\alpha$ be respectively the α-torsion, the α-section and the α-quotient functors. Abusing notation, we use the same symbols for the functors $\mathbf{T}_\alpha : \mathfrak{C}_{\alpha+1} \to \mathfrak{C}_\alpha$ and $\mathbf{Q}_\alpha : \mathfrak{C}_{\alpha+1} \to \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, induced by restriction.

Definition 2.55. *Let $\mathfrak{C}$ be a Grothendieck category, an object $X \in \mathrm{Ob}(\mathfrak{C})$ is* simple *if its qframe of subobjects $\mathcal{L}(X)$ is an atom. Furthermore, X is said to be* cocritical *if it is α-cocritical for some α.*

Lemma 2.56. *Let $\mathfrak{C}$ be a Gabriel category, let $X \in \mathrm{Ob}(\mathfrak{C})$ and consider an ordinal α. Then,*

(1) *$X \in \mathfrak{C}_{\alpha+1}$ if and only if there exists an ordinal σ and a continuous chain $0 = N_0 \leqslant N_1 \leqslant \cdots \leqslant N_\sigma = X$, such that N_{i+1}/N_i is either α-cocritical or α-torsion for every $i < \sigma$.*

(2) *if α is a limit ordinal, then $X \in \mathfrak{C}_\lambda$ if and only if $X = \sum_{\alpha<\lambda} \mathbf{T}_\alpha(X)$.*

Proof. (1) Let $\mathcal{A}$ be the class of all objects which are union of a chain as in the statement. Since every hereditary torsion class is closed under taking direct limits and extension, we obtain the inclusion $\mathcal{A} \subseteq \mathfrak{C}_{\alpha+1}$. On the other hand, $\mathfrak{C}_{\alpha+1}$ is minimal between the hereditary torsion classes which contain $\mathfrak{C}_\alpha$ and the α-cocritical objects, thus the converse inclusion follows by the fact that $\mathcal{A}$ is a torsion class (apply Lemma 1.118 to the class $\mathfrak{C}_\alpha \cup \{\alpha\text{-cocritical objects}\}$).

(2) Notice that, since α is a limit ordinal, $\mathfrak{C}_\alpha$ is the smallest torsion class that contains all the torsion classes $\mathfrak{C}_\beta$ with $\beta < \alpha$. Let $\mathcal{D} = \{X \in \mathfrak{C} : X = \sum_{\beta<\alpha} \mathbf{T}_\beta(X)\}$, we have to show that $\mathfrak{C}_\alpha = \mathcal{D}$. Indeed, given $X \in \mathcal{D}$, we have an epimorphism $\bigoplus_{\beta<\alpha} \mathbf{T}_\beta(X) \to X$, and so, since $\mathfrak{C}_\alpha$ is closed under quotients and coproducts, $X \in \mathfrak{C}_\alpha$. Thus, $\mathcal{D} \subseteq \mathfrak{C}_\alpha$. On the other hand, it is not difficult to show that $\mathcal{D}$ is a torsion class and that it contains $\mathfrak{C}_\alpha$, for all $\alpha < \lambda$. By minimality, $\mathfrak{C}_\lambda \subseteq \mathcal{D}$. ☐

Corollary 2.57. *Let $\mathfrak{C}$ be a Gabriel category and let $0 \neq X \in \mathrm{Ob}(\mathfrak{C})$. Then, X has a cocritical subobject.*

Proof. Let α be the smallest ordinal such that $X \in \mathfrak{C}_\alpha$, we proceed by induction on α. If $\alpha = 1$, then by Lemma 2.56 (1), X has a simple subobject (as being 0-torsion means being trivial, while being 0-cocritical means being simple). Similarly, if $\alpha = \beta + 1$, then there exists an ordinal σ and a continuous chain $0 = X_0 \leqslant X_1 \leqslant \cdots \leqslant X_\sigma = X$, such that X_{i+1}/X_i is either β-cocritical or β-torsion for every $i < \sigma$. Let i the smallest ordinal for which X_i is not trivial. If X_i is β-critical then we are done, while if X_i is β-torsion, then we can conclude by inductive hypothesis. Finally, if α is a limit ordinal, then $X = \bigcup_{\beta<\alpha} \mathbf{T}_\beta(X)$, by Lemma 2.56 (1). Thus there exists some $\beta < \alpha$ for which $0 \neq \mathbf{T}_\beta(X) \leqslant X$ and, by the inductive hypothesis, $\mathbf{T}_\beta(X)$ has a cocritical subobject. ☐

Lemma 2.58. *Let $\mathfrak{C}$ be a Gabriel category, let $C \in \mathfrak{C}$ be an object and let α be an ordinal. The following are equivalent:*

(1) *C is α-cocritical;*

(2) *C is α-torsion free and $\mathbf{Q}_\alpha(C)$ is simple;*

(3) *there exists a simple object $S \in \mathfrak{C}/\mathfrak{C}_\alpha$ such that C embeds in $\mathbf{S}_\alpha(S)$.*

Proof. (1)⇒(2). If C is α-cocritical, then it is α-torsion free by definition. Let $0 \neq X \leqslant \mathbf{Q}_\alpha(C)$, then $0 \neq \mathbf{S}_\alpha(X) \leqslant \mathbf{L}_\alpha(C)$. Since C is essential in $\mathbf{L}_\alpha(C)$, then $C \cap \mathbf{S}_\alpha(X) \neq 0$. This induces a short exact sequence

$$0 \to \mathbf{S}_\alpha(X) \cap C \to C \to C/(\mathbf{S}_\alpha(X) \cap C) \to 0\,,$$

with $C/(\mathbf{S}_\alpha(X) \cap C) \in \mathfrak{C}_\alpha$. Applying $\mathbf{Q}_\alpha$, we obtain a short exact sequence $0 \to X \to \mathbf{Q}_\alpha(C) \to 0 \to 0$, showing that $X = \mathbf{Q}_\alpha(C)$, that is therefore simple.

(2)⇒(3). Since C is α-torsion free, it embeds in $\mathbf{L}_\alpha(C) = \mathbf{S}_\alpha(\mathbf{Q}_\alpha(C))$.

(3)⇒(1). $\mathbf{S}_\alpha(S)$ is α-torsion free, so C is α-torsion free. Furthermore, let $0 \neq X \leqslant C$, then $0 \neq \mathbf{Q}_\alpha(X) \leqslant \mathbf{Q}_\alpha(C) \leqslant S$ and, since S is simple, $\mathbf{Q}_\alpha(X) = \mathbf{Q}_\alpha(C) = S$. Thus, $\mathbf{Q}_\alpha(X/C) = \mathbf{Q}_\alpha(X)/\mathbf{Q}_\alpha(C) = 0$, that is, $X/C \in \mathfrak{C}_\alpha$. □

Corollary 2.59. *Let $\mathfrak{C}$ be a Gabriel category. Then, $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ is semi-Artinian for all $\alpha <$ G.dim$(\mathfrak{C})$.*

Proof. Let $0 \neq X \in \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ and consider $\mathbf{S}_\alpha(X) \in \mathfrak{C}_{\alpha+1}$. By Corollary 2.57, there exists a cocritical subobject $C \leqslant \mathbf{S}_\alpha(X)$. Since $\mathbf{S}_\alpha(X)$ is α-torsion free by construction, C is α-cocritical and so $\mathbf{Q}_\alpha(C)$ is simple in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ by Lemma 2.58. By the exactness of $\mathbf{Q}_\alpha$, $\mathbf{Q}_\alpha(C) \leqslant X$. Thus, we proved that any object in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ has a simple subobject. □

Proposition 2.60. *Let $\mathfrak{C}$ be a Gabriel category and let E and $E' \in \mathfrak{C}$ be injective objects. The following statements hold true:*

(1) *E is indecomposable if and only if there exists a cocritical object C such that $E \cong E(C)$;*

(2) *if E and E' are indecomposables and cogenerate the same torsion theory, then $E \cong E'$.*

Proof. (1) Suppose $E = E(C)$, where C is α-cocritical for some ordinal α and let E_1, $E_2 \leqslant E$ be two subobjects such that $E = E_1 \oplus E_2$ with $E_1 \neq 0$. Then $E_1 \cap C \neq 0$ and $C/(C \cap E_1)$ embeds in E_2. Since E_2 is α-torsion free and $C/(C \cap E_1)$ is α-torsion, $C/(C \cap E_1) = 0$, that is, $C \leqslant E_1$. Thus, $E_2 \cap C = 0$, which implies that $E_2 = 0$.

On the other hand, suppose $0 \neq E$ is indecomposable. Since $\mathfrak{C}$ is a Gabriel category, there exists a cocritical subobject $C \leqslant E$ (see Corollary 2.57). Since E is indecomposable, $E = E(C)$.

(2) Let $\tau = (\mathcal{T}, \mathcal{F})$ and $\tau' = (\mathcal{T}', \mathcal{F}')$ be the torsion theories cogenerated by E and E' respectively and suppose $\tau = \tau'$. By part (1), there exists an ordinal α and an α-cocritical object C such that $E = E(C)$. Thus, E is α-torsion free and so $\mathfrak{C}_\alpha \subseteq \mathcal{T} = \mathcal{T}'$. Furthermore, $C \leqslant E \in \mathcal{F} = \mathcal{F}'$ and so $\mathrm{Hom}_{\mathfrak{C}}(C, E') \neq 0$. Let $\phi : C \to E'$ be a non-trivial morphism and notice that it is necessarily a monomorphism, in fact, if $0 \neq \mathrm{Ker}(\phi) \leqslant C$ then $\phi(C) \cong C/\mathrm{Ker}(\phi) \in \mathfrak{C}_\alpha \subseteq \mathcal{T}'$, that contradicts the fact that $\phi(C) \leqslant E' \in \mathcal{F}'$. Thus, $C \cong \phi(C)$ and so $E' \cong E(\phi(C)) \cong E(C) = E$. □

Remark 2.61. *Let $\mathfrak{C}$ be a stable Gabriel category and let E be an indecomposable injective object. By Proposition 2.60, there is an ordinal α and an α-cocritical object C such that $E \cong E(C)$. By construction $C \in \mathfrak{C}_{\alpha+1} \backslash \mathfrak{C}_\alpha$ and, by stability, $E(C) \in \mathfrak{C}_{\alpha+1}$. This shows that, in stable Gabriel categories, any indecomposable injective object belongs to $\mathfrak{C}_{\alpha+1} \backslash \mathfrak{C}_\alpha$ for some α.*

Definition 2.62. *Let $\mathfrak{C}$ be a Grothendieck category. A torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ that can be cogenerated by the injective envelope of a cocritical object is said to be* prime.
The α-Gabriel spectrum $\mathrm{Sp}^\alpha(\mathfrak{C})$ *is the family of isomorphism classes of injective envelopes of α-cocritical objects. The* Gabriel spectrum $\mathrm{Sp}(\mathfrak{C})$ *of $\mathfrak{C}$ is the family of isomorphism classes of indecomposable injective objects in $\mathfrak{C}$.*

With this terminology, Proposition 2.60 translates as follows.

Corollary 2.63. *Let $\mathfrak{C}$ be a Gabriel category. Then the following map*

$$\begin{aligned} \mathrm{Sp}(\mathfrak{C}) &\longrightarrow \{\textit{prime torsion theories}\} \\ E &\longmapsto ({}^{\perp}E, ({}^{\perp}E)^{\perp}) \end{aligned}$$

is a bijection. Furthermore, $\mathrm{Sp}(\mathfrak{C}) = \bigcup_\alpha \mathrm{Sp}^\alpha(\mathfrak{C})$.

Using the above corollary, we usually identify the set of prime torsion theories with the Gabriel spectrum. In particular, we write $\pi \in \mathrm{Sp}(\mathfrak{C})$ (or $\pi \in \mathrm{Sp}^\alpha(\mathfrak{C})$) to mean that π is a prime torsion theory. Furthermore, we let $E(\pi)$ be a representative of the isomorphism class of the indecomposable injectives which cogenerate π.

Theorem 2.64. *Let $\mathfrak{C}$ be a Gabriel category and let $E \in \mathrm{Ob}(\mathfrak{C})$ be an injective object. For all $\pi \in \mathrm{Sp}(\mathfrak{C})$ there exists a set I_π such that*

$$E \cong E\left(\bigoplus_{\pi \in \mathrm{Sp}(\mathfrak{C})} E(\pi)^{(I_\pi)}\right).$$

Furthermore, the set of pairs $\{(\pi, |I_\pi|) : \pi \in \mathrm{Sp}(\mathfrak{C})\}$ uniquely determines E up to isomorphism.

The proof of the above theorem uses a general machinery that we are not interested to treat here. Thus, we give just a sketch of the proof, pointing to the literature for details.

Proof. Let $\mathfrak{C}_{Spec}$ be the spectral category of $\mathfrak{C}$ defined as follows:

– the objects of $\mathfrak{C}_{Spec}$ are exactly the injective objects of $\mathfrak{C}$;

– given two objects E_1, E_2 of $\mathfrak{C}_{Spec}$, the morphisms are defined as follows:

$$\mathrm{Hom}_{\mathfrak{C}_{Spec}}(E_1, E_2) = \varinjlim \mathrm{Hom}_{\mathfrak{C}}(E, E_2)\,, \quad E \text{ essential subobject of } E_1.$$

There is a canonical left exact functor $P : \mathfrak{C} \to \mathfrak{C}_{Spec}$ taking an object to its injective envelope. It is useful to notice that, given two objects X and $Y \in \mathfrak{C}$ we have that

$$P(X) \cong P(Y) \quad (\text{in } \mathfrak{C}_{Spec}) \iff E(X) \cong E(Y) \quad (\text{in } \mathfrak{C})\,. \tag{2.2.1}$$

For more details on this construction we refer to [96, Sec. 6 and 7, Ch. V]. By Corollary 2.57, every non-trivial object in $\mathfrak{C}$ has a cocritical subobject. By [96, Proposition 7.3, Ch. V], this implies that $\mathfrak{C}_{Spec}$ is a discrete (i.e., any object of $\mathfrak{C}_{Spec}$ is a coproduct of simple objects) spectral category. In particular, given an injective object $E \in \mathfrak{C}$, $P(E)$ decomposes in $\mathfrak{C}_{Spec}$ as the direct sum of indecomposable (i.e., simple) objects. Furthermore, it is not difficult to see that an object $P(C)$ is simple in $\mathfrak{C}_{Spec}$ if and only if $E(C)$ is indecomposable in $\mathfrak{C}$. We obtain in $\mathfrak{C}_{Spec}$ the following decomposition

$$P(E) = \bigoplus_{\pi \in \mathrm{Sp}(\mathfrak{C})} P(E(\pi))^{(I_\pi)}\,.$$

Thus, we get the desired decomposition $E = E\left(\bigoplus_{\pi \in \mathrm{Sp}(\mathfrak{C})} E(\pi)^{(I_\pi)}\right)$. For the uniqueness statement it is enough to apply Theorem 1 of [102] in the category $\mathfrak{C}_{Spec}$. $\square$

Lemma 2.65. *Let $\mathfrak{C}$ be a Gabriel category, let π, $\pi' \in \mathrm{Sp}(\mathfrak{C})$ and consider the following conditions (see Definition 1.137):*

(1) $\pi \preceq \pi'$;

(2) $\mathrm{Hom}_{\mathfrak{C}}(E(\pi'), E(\pi)) \neq 0$.

Then, (1) implies (2). If π is stable, also the converse holds.

Proof. By definition, $\pi \preceq \pi'$ if and only if any π'-torsion free object is π-torsion free. In this case, $E(\pi')$ is π-torsion free, thus $\mathrm{Hom}_{\mathfrak{C}}(E(\pi'), E(\pi)) \neq 0$. On the other hand, if $\mathrm{Hom}_{\mathfrak{C}}(E(\pi'), E(\pi)) \neq 0$ and π is stable, then $E(\pi')$ is not π-torsion, thus it is π-torsion free (see Lemma 1.140) and so $\pi \preceq \pi'$. ☐

Definition 2.66. *Let $\mathfrak{C}$ be a Grothendieck category. Given a subset $S \subseteq \mathrm{Sp}(\mathfrak{C})$, we say that S is* generalization closed *(resp.,* specialization closed*) if it contains all the prime torsion theories that are generalizations (resp., specializations) of its members.*

Theorem 2.67. *Let $\mathfrak{C}$ be a Gabriel category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Define the following subsets of* $\mathrm{Sp}(\mathfrak{C})$*:*

– $S(\tau) = \{\pi \in \mathrm{Sp}(\mathfrak{C}) : \mathbf{T}_\tau(E(\pi)) \neq 0\}$;

– $G(\tau) = \{\pi \in \mathrm{Sp}(\mathfrak{C}) : \mathbf{T}_\tau(E(\pi)) = 0\}$.

Then, $S(\tau) \cup G(\tau) = \mathrm{Sp}(\mathfrak{C})$ and this is a disjoint union. Furthermore, given $\tau' \in \mathrm{Tors}(\mathfrak{C})$, $\tau = \tau'$ if and only if $G(\tau) = G(\tau')$ if and only if $S(\tau) = S(\tau')$.
If $\mathfrak{C}$ is stable, then $S(\tau)$ and $G(\tau)$ are respectively specialization and generalization closed. Furthermore, any specialization (resp., generalization) closed subset of $\mathrm{Sp}(\mathfrak{C})$ is of the form $S(\tau)$ (resp., $G(\tau)$) for some $\tau \in \mathrm{Tors}(\mathfrak{C})$ and $S(-)$ (resp., $G(-)$) induces a bijection between $\mathrm{Tors}(\mathfrak{C})$ and the set of specialization (resp., generalization) closed subsets of $\mathrm{Sp}(\mathfrak{C})$.

Proof. Let τ and τ' be two torsion theories such that $G(\tau) = G(\tau')$. Given $X \in \mathcal{F}$, there exist sets I_π, for all $\pi \in G(\tau)$ such that $E(X) \cong E(\bigoplus_{\pi \in G(\tau)} E(\pi)^{(I_\pi)})$ (see Theorem 2.64). Since $G(\tau) = G(\tau')$, $E(X)$ is τ'-torsion free, showing that $\mathcal{F} \subseteq \mathcal{F}'$. One proves similarly that $\mathcal{F}' \subseteq \mathcal{F}$, so $\mathcal{F} = \mathcal{F}'$, that is, $\tau = \tau'$. Analogously, notice that $S(\tau) = S(\tau')$ implies $G(\tau) = \mathrm{Sp}(\mathfrak{C}) \backslash S(\tau) = \mathrm{Sp}(\mathfrak{C}) \backslash S(\tau') = G(\tau')$ and so $\tau = \tau'$ for the first part of the proof.
Assume now that $\mathfrak{C}$ is stable and let $\pi \preceq \pi' \in \mathrm{Sp}(\mathfrak{C})$, that is, $\mathrm{Hom}_{\mathfrak{C}}(E(\pi'), E(\pi)) \neq 0$ (see Lemma 2.65). If $\mathbf{T}_\tau(E(\pi')) \neq 0$, then $E(\pi') \in \mathcal{T}$ (by stability), thus any proper quotient of $E(\pi')$ is τ-torsion. Hence, $\mathbf{T}_\tau(E(\pi)) \neq 0$. We proved that $S(\tau)$ is specialization closed, the fact that $G(\tau)$ is generalization closed follows from the fact that it is the complement of $S(\tau)$.
Finally, let G be a generalization closed subset of $\mathrm{Sp}(\mathfrak{C})$ and let

$$\underline{\mathcal{T}} = {}^{\perp}\{E(\pi) : \pi \in G\}\,, \ \underline{\mathcal{F}} = \underline{\mathcal{T}}^{\perp} \ \text{ and } \ \underline{\tau} = (\underline{\mathcal{T}}, \underline{\mathcal{F}})\,.$$

Then $(\mathrm{Sp}(\mathfrak{C}) \backslash G) \cup G = \mathrm{Sp}(\mathfrak{C}) = S(\underline{\tau}) \cup G(\underline{\tau})$ and it is easy to see that $G \subseteq G(\underline{\tau})$. Let $\pi' \in \mathrm{Sp}(\mathfrak{C}) \backslash G$. If, looking for a contradiction, $E(\pi') \notin \underline{\mathcal{T}}$, then there exists $\pi \in G$ such that $\mathrm{Hom}_{\mathfrak{C}}(E(\pi'), E(\pi)) \neq 0$. By Lemma 2.65 π' is a generalization of π and so $\pi' \in G$, which is a contradiction. Hence, $\mathrm{Sp}(\mathfrak{C}) \backslash G \subseteq S(\underline{\tau})$ and so $S(\underline{\tau}) = \mathrm{Sp}(\mathfrak{C}) \backslash G$ and $G(\underline{\tau}) = G$. ☐

2.2.4 Relative Gabriel dimension in Grothendieck categories

Proposition 2.68. *Let $\mathfrak{C}$ be a Grothendieck category and let α be an ordinal. Then, $\mathfrak{C}_\alpha = \{X \in \mathrm{Ob}(\mathfrak{C}) : \mathrm{G.dim}(\mathcal{L}(X)) \leqslant \alpha\}$.*

Proof. The case $\alpha = 0$ being clear, we prove the statement by induction on α. Suppose that, for all $\beta < \alpha$

$$\mathfrak{C}_\beta = \{X \in \mathrm{Ob}(\mathfrak{C}) : \mathrm{G.dim}(\mathcal{L}(X)) \leqslant \beta\}\,.$$

By injective hypothesis, an object X is β-cocritical for some $\beta < \alpha$ if and only if $\mathcal{L}(X)$ is a β-simple qframe. Let $\mathcal{D} = \{X \in \mathrm{Ob}(\mathfrak{C}) : \mathrm{G.dim}(\mathcal{L}(X)) \leqslant \alpha\}$ and let us show that $\mathfrak{C}_\alpha = \mathcal{D}$. Indeed, $\mathfrak{C}_\alpha \subseteq \mathcal{D}$ since $\mathcal{D}$ is a torsion class (see Lemma 2.42) that contains $\mathfrak{C}_\beta$ and the β-cocritical objects, for all $\beta < \alpha$. On the other hand, let $X \in \mathcal{D}$ and consider $\mathbf{T}_\alpha(X)$. If, looking for a contradiction, $\mathbf{T}_\alpha(X) \neq X$, then there exists $Z \in (\mathbf{T}_\alpha(X), X]$ such that $\mathrm{G.dim}([\mathbf{T}_\alpha(X), Z]) < \alpha$. We obtain a short exact sequence

$$0 \to \mathbf{T}_\alpha(X) \to Z \to Z/\mathbf{T}_\alpha(X) \to 0\,.$$

Since $\mathfrak{C}_\alpha$ is a Serre class, $Z \in \mathfrak{C}_\alpha$, so $Z \leqslant \mathbf{T}_\alpha(X)$, that is a contradiction. Thus, $\mathcal{D} \subseteq \mathfrak{C}_\alpha$. □

The concept of Gabriel dimension in Grothendieck categories was introduced in [44] (under the name of "Krull dimension") and systematically studied in [51] and in many other papers and books after that. We introduce here a relative version of this invariant.

Definition 2.69. *Let $\mathfrak{C}$ be a Gabriel category, let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory and let $X \in \mathfrak{C}$ be an object. We define respectively the* τ-Gabriel dimension *of $\mathfrak{C}$ and the* τ-Gabriel dimension *of X as follows*

$$\mathrm{G.dim}_\tau(\mathfrak{C}) = \min\{\alpha : \mathfrak{C}/\mathcal{T} = (\mathfrak{C}/\mathcal{T})_\alpha\} \quad \textit{and} \quad \mathrm{G.dim}_\tau(X) = \min\{\alpha : \mathbf{Q}_\tau(X) \in (\mathfrak{C}/\mathcal{T})_\alpha\}\,.$$

When $\tau = (0, \mathfrak{C})$ is the trivial torsion theory, the τ-Gabriel dimension is called Gabriel dimension *and we denote it respectively by* $\mathrm{G.dim}(\mathfrak{C})$ *and* $\mathrm{G.dim}(X)$.

Let $\mathfrak{C}$ be a Gabriel category, let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory and let $X \in \mathrm{Ob}(\mathfrak{C})$. Notice that, by Proposition 2.68, $\mathrm{G.dim}_\tau(X) = \mathrm{G.dim}(\mathcal{L}(\mathbf{Q}_\tau(X)))$.

Given an object X in a Grothendieck category $\mathfrak{C}$, we say that X is Noetherian (resp., Artinian, semi-Artinian), provided $\mathcal{L}(X)$ is a qframe with the same property. Recall that a Grothendieck category $\mathfrak{D}$ is said to be semi-Artinian if $\mathfrak{D} = \mathfrak{D}_1$, that is, any of its objects is semi-Artinian, equivalently, every object in $\mathfrak{D}$ has a simple sub-object.

Lemma 2.70. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. Then:*

(1) $\mathrm{G.dim}_\tau(\mathfrak{C}) = \sup\{\mathrm{G.dim}_\tau(X) : X \in \mathfrak{C}\}$;

(2) *if $Y \leqslant X \in \mathfrak{C}$, then* $\mathrm{G.dim}_\tau(X) = \max\{\mathrm{G.dim}_\tau(Y), \mathrm{G.dim}_\tau(X/Y)\}$;

(3) *if $\{X_i : i \in I\}$ is a family of objects in $\mathfrak{C}$, then* $\mathrm{G.dim}_\tau(\bigoplus_I X_i) = \sup_I \mathrm{G.dim}_\tau(X_i)$;

(4) *if $N \in \mathfrak{C}$ is a Noetherian object, then $\mathbf{Q}_\tau(N)$ is Noetherian and* $\mathrm{G.dim}_\tau(N)$ *is a successor ordinal. Furthermore, there exists a finite series $0 = Y_0 < Y_1 < \cdots < Y_k = N$ such that Y_i/Y_{i-1} is cocritical for all $i = 1, \ldots, k$.*

Proof. (1) is trivial.

(2) By the exactness of $\mathbf{Q}_\tau$, $\mathcal{L}(\mathbf{Q}_\tau(Y))$ equals the segment $[0, \mathbf{Q}_\tau(Y)]$ in $\mathcal{L}(\mathbf{Q}_\tau(X))$, while $\mathcal{L}(\mathbf{Q}_\tau(X/Y))$ equals the segment $[\mathbf{Q}_\tau(Y), 1]$ in $\mathcal{L}(\mathbf{Q}_\tau(X))$. Thus,

$$\begin{aligned}\mathrm{G.dim}_\tau(X) &= \mathrm{G.dim}(\mathcal{L}(\mathbf{Q}_\tau(X)))\\ &= \max\{\mathrm{G.dim}([0, \mathbf{Q}_\tau(Y)]), \mathrm{G.dim}([\mathbf{Q}_\tau(Y), 1])\} = \max\{\mathrm{G.dim}_\tau(Y), \mathrm{G.dim}_\tau(X/Y)\}\,,\end{aligned}$$

where the second equality follows by Lemma 2.42 (2).

(3) Since $\mathbf{Q}_\tau$ commutes with coproducts, $\mathbf{Q}_\tau(\bigoplus_I X_i) = \bigoplus_I \mathbf{Q}_\tau(X_i)$. Thus, we have a family $\mathcal{F} = \{\mathbf{Q}_\tau(X_i) : i \in I\} \subseteq \mathcal{L}(\mathbf{Q}_\tau(\bigoplus_I X_i))$ such that $\bigvee \mathcal{F} = 1$. By Lemma 2.42 (3),

$$\begin{aligned}\mathrm{G.dim}_\tau\left(\bigoplus_I X_i\right) &= \mathrm{G.dim}\left(\mathcal{L}\left(\mathbf{Q}_\tau\left(\bigoplus_I X_i\right)\right)\right)\\ &= \sup_I \mathrm{G.dim}([0, \mathbf{Q}_\tau(X_i)]) = \sup_I \mathrm{G.dim}_\tau(\mathbf{Q}_\tau(X_i))\,.\end{aligned}$$

(4) Use Proposition 2.41 (3).

□

Definition 2.71. *Let $\mathfrak{C}$ be a Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$, let $\mathfrak{C}' = \mathfrak{C}/\mathcal{T}$ and consider a torsion theory $\tau' = (\mathcal{T}', \mathcal{F}') \in \mathrm{Tors}(\mathfrak{C}')$. The following class of objects of $\mathfrak{C}$ is a torsion class:*

$$\mathcal{T}_{\tau\circ\tau'} = \{X \in \mathfrak{C} : \mathbf{Q}_\tau(X) \in \mathcal{T}'\}\,.$$

We denote by $\tau \circ \tau'$ the torsion theory whose torsion class is $\mathcal{T}_{\tau\circ\tau'}$.

Notice that, just by definition, the quotient functors relative to τ, τ' and $\tau \circ \tau'$ fit in the following commutative diagram:

$$\begin{tikzcd} \mathfrak{C} \arrow[r, "\mathbf{Q}_\tau"] \arrow[rr, bend right, "\mathbf{Q}_{\tau\circ\tau'}"'] & \mathfrak{C}/\mathcal{T} \arrow[r, "\mathbf{Q}_{\tau'}"] & \mathfrak{C}/\mathcal{T}_{\tau\circ\tau'} \cong \mathfrak{C}'/\mathcal{T}' \end{tikzcd}\,.$$

Lemma 2.72. *Let $\mathfrak{C}$ be a Gabriel category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$, let $\mathfrak{C}' = \mathfrak{C}/\mathcal{T}$, denote by τ_α the torsion theory in $\mathfrak{C}'$ whose torsion class is $(\mathfrak{C}')_\alpha$ (the α-th member of the Gabriel filtration of $\mathfrak{C}'$), and let $X \in \mathfrak{C}$. Then:*

(1) $\mathrm{G.dim}_\tau(X) = \alpha + 1$ *if and only if* $\mathrm{G.dim}_{\tau\circ\tau_\alpha}(X) = 1$;

(2) $\mathrm{G.dim}_{\tau\circ\tau_\alpha}(X) = 0$ *implies that* $\mathrm{G.dim}_\tau(X) \leqslant \alpha$.

Proof. (1) $\mathrm{G.dim}_\tau(X) = \mathrm{G.dim}(\mathbf{Q}_\tau(X)) = \alpha + 1$ if and only if $\mathbf{Q}_\tau(X) \in (\mathfrak{C}')_{\alpha+1} \setminus (\mathfrak{C}')_\alpha$, that is, $\mathrm{G.dim}_{\tau\circ\tau_\alpha}(X) = \mathrm{G.dim}(\mathbf{Q}_{\tau_\alpha}(\mathbf{Q}_\tau(X))) = 1$.

(2) $\mathrm{G.dim}_{\tau\circ\tau_\alpha}(X) = 0$ if and only if $\mathbf{Q}_{\tau\circ\tau_\alpha}(X) = 0$, that is, $\mathbf{Q}_\tau(X) \in \mathrm{Ker}(\mathbf{Q}_{\tau_\alpha}) = (\mathfrak{C}')_\alpha$. Equivalently, $\mathrm{G.dim}_\tau(X) = \mathrm{G.dim}(\mathbf{Q}_\tau(X)) \leqslant \alpha$. □

Lemma 2.73. *Let $\mathfrak{C}$ be a stable Gabriel category, let $\tau \in \mathrm{Tors}(\mathfrak{C})$ and let $\pi = (\mathcal{T}, \mathcal{F})$ and $\pi' = (\mathcal{T}', \mathcal{F}')$ be two distinct prime torsion theories. If $\mathrm{G.dim}_\tau(E(\pi)) = \mathrm{G.dim}_\tau(E(\pi')) > -1$, then $\mathrm{Hom}_{\mathfrak{C}}(E(\pi), E(\pi')) = 0$.*

Proof. By Remark 2.61, $\mathrm{G.dim}_\tau(E(\pi)) = \mathrm{G.dim}_\tau(E(\pi')) = \beta + 1$ for some ordinal $\beta \geqslant -1$. Denote by $\tau_\beta \in \mathrm{Tors}(\mathfrak{C}/\mathcal{T})$ the torsion theory whose torsion class is $(\mathfrak{C}/\mathcal{T})_\beta$. Then, by Lemma 2.72, $\mathrm{G.dim}_{\tau\circ\tau_\beta}(E(\pi)) = \mathrm{G.dim}_{\tau\circ\tau_\beta}(E(\pi')) = 1$ and, by stability, both $E(\pi)$ and $E(\pi')$ are $\tau \circ \tau_\beta$-torsion free, so $\tau \circ \tau_\beta$-local, thus

$$\mathrm{Hom}_{\mathfrak{C}}(E(\pi), E(\pi')) = \mathrm{Hom}_{\mathfrak{C}}(\mathbf{L}(_{\tau\circ\tau_\beta}E(\pi)), \mathbf{L}_{\tau\circ\tau_\beta}(E(\pi'))) \cong \mathrm{Hom}_{\mathfrak{C}/\mathcal{T}_{\tau\circ\tau_\beta}}(E(\pi), E(\pi')) \, ,$$

so there is no loss in generality if we assume that τ is the trivial torsion theory and $\beta = 0$. Suppose now that $\mathrm{Hom}_{\mathfrak{C}}(E(\pi), E(\pi')) \neq 0$, that is, $E(\pi)$ is not π'-torsion, thus, by stability, $E(\pi)$ is π'-torsion free. Now, since we are assuming $\mathrm{G.dim}(E(\pi)) = 1$, there exists a simple object $S \in \mathfrak{C}$ such that $E(\pi) \cong E(S)$ and $\mathrm{Hom}_{\mathfrak{C}}(S, E(\pi')) \neq 0$. By the simplicity of S and the fact that $E(\pi')$ is indecomposable we obtain that $E(\pi') \cong E(S) \cong E(\pi)$, which is a contradiction. □

Corollary 2.74. *Let $\mathfrak{C}$ be a stable Gabriel category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ and let $\pi \neq \pi' \in \mathrm{Sp}(\mathfrak{C})$. If $E(\pi), E(\pi') \notin \mathcal{T}$ and $\mathrm{Hom}_{\mathfrak{C}}(E(\pi), E(\pi')) \neq 0$, then $\mathrm{G.dim}_\tau(E(\pi)) > \mathrm{G.dim}_\tau(E(\pi'))$.*

Proof. Looking for a contradiction, suppose that $\mathrm{G.dim}_\tau(E(\pi)) \leqslant \mathrm{G.dim}_\tau(E(\pi'))$. By Theorem 1.132, $\mathbf{Q}_\tau(E(\pi))$ is an indecomposable injective object in $\mathfrak{C}/\mathcal{T}$, thus, by Remark 2.61 there exists an ordinal α such that $\mathrm{G.dim}_\tau(E(\pi)) = \alpha + 1$. Let

- $\mathfrak{C}_1 = \mathfrak{C}/\mathcal{T}$, $\bar\tau_1 = (\mathcal{T}_1, \mathcal{F}_1) \in \mathrm{Tors}(\mathfrak{C}_1)$, where $\mathcal{T}_1 = (\mathfrak{C}_1)_\alpha$, $\tau_1 = \tau \circ \bar\tau_1 \in \mathrm{Tors}(\mathfrak{C})$, $E_1 = \mathbf{Q}_{\tau_1}(E(\pi))$ and $E_1' = \mathbf{Q}_{\tau_1}(E(\pi'))$;
- $\mathfrak{C}_2 = \mathfrak{C}_1/\mathcal{T}_1$, $\bar\tau_2 = (\mathcal{T}_2, \mathcal{F}_2) \in \mathrm{Tors}(\mathfrak{C}_2)$, where $\mathcal{T}_2 = {}^\perp\{E_1'\}$ and $\tau_2 = \tau \circ \bar\tau_2 \in \mathrm{Tors}(\mathfrak{C})$;
- $\mathfrak{C}_3 = \mathfrak{C}_2/\mathcal{T}_2$.

Both $E(\pi)$ and $E(\pi')$ are τ_1-local and so $\mathrm{Hom}_{\mathfrak{C}_2}(E_1, E_1') \cong \mathrm{Hom}_{\mathfrak{C}}(E(\pi), E(\pi')) \neq 0$. This means that E_1 is not $\bar\tau_2$-torsion, thus both $\mathrm{G.dim}_{\tau_2}(E(\pi))$ and $\mathrm{G.dim}_{\tau_2}(E(\pi'))$ are strictly bigger than 0. On the other hand, $\mathrm{G.dim}_{\tau_2}(E(\pi)) \leqslant \mathrm{G.dim}_{\tau_1}(E(\pi)) = 1$ (see Lemma 2.72), while $\mathrm{G.dim}_{\tau_2}(E(\pi')) = 1$ (since, given a cocritical sub-object C of $E(\pi')$, $\mathbf{L}_{\tau_2}(C)$ is simple). This contradicts the conclusion of Lemma 2.73. □

2.2.5 Locally Noetherian Grothendieck categories

Definition 2.75. *A Grothendieck category $\mathfrak{C}$ is* locally Noetherian *if it has a set of Noetherian generators. Equivalently, any object in $\mathfrak{C}$ is the direct union of the directed family of its Noetherian subobjects.*

Corollary 2.76. *Any locally Noetherian Grothendieck category $\mathfrak{C}$ is a Gabriel category. Moreover, if $\mathcal{G}$ is a set of Noetherian generators for $\mathfrak{C}$, $\mathrm{G.dim}(\mathfrak{C}) = \mathrm{G.dim}(\bigoplus \mathcal{G})$.*

Proof. Let $\mathcal{G}$ be a set of Noetherian generators of $\mathfrak{C}$. By Lemma 2.70 (4), each $G \in \mathcal{G}$ has Gabriel dimension. By Lemma 2.70 (3), $\bigoplus \mathcal{G}$ has Gabriel dimension and so, by Lemma 2.70 (2) any quotient of $\bigoplus \mathcal{G}$ has a smaller Gabriel dimension. Thus, by Lemma 2.70 (1), $\mathrm{G.dim}(\mathfrak{C})$ exists and it coincides with $\mathrm{G.dim}(\bigoplus \mathcal{G})$. □

In the following proposition and corollary we collect some results about locally Noetherian categories:

Proposition 2.77. [96, Proposition 4.3 and Corollary 4.4, Ch. V] *Let $\mathfrak{C}$ be a Grothendieck category. Then, $\mathfrak{C}$ is locally Noetherian if and only if directed colimits of injective objects are injective.*

By the above proposition, Theorem 2.64 has the following form in locally Noetherian categories.

Corollary 2.78. *Let $\mathfrak{C}$ be a locally Noetherian Grothendieck category and let $E \in \mathrm{Ob}(\mathfrak{C})$ be an injective object. For all $\pi \in \mathrm{Sp}(\mathfrak{C})$ there exists a set I_π such that*

$$E \cong \bigoplus_{\pi \in \mathrm{Sp}(\mathfrak{C})} E(\pi)^{(I_\pi)}\,.$$

Furthermore, the set of pairs $\{(\pi, |I_\pi|) : \pi \in \mathrm{Sp}(\mathfrak{C})\}$ uniquely determines E up to isomorphism.

A ring R is *left* (resp., *right*) *Noetherian* if ${}_RR$ (resp., R_R) is a Noetherian object in R-Mod (resp., Mod-R). Furthermore, a ring is *Noetherian* if it is both left and right Noetherian. By Corollary 2.76, $\mathrm{G.dim}(R\text{-Mod}) = \mathrm{G.dim}({}_RR)$ for any left Noetherian ring R.

In the last part of this subsection we specify some of the above results in the case when $\mathfrak{C} = R$-Mod is the category of modules over a commutative Noetherian ring. This particular case will be used in Chapter 12 to re-obtain the main results of [20], [18] and [19], from our general theory.

Lemma 2.79. [96, Proposition 4.5, Ch. VII] *Let R be a commutative Noetherian ring. Then, R-Mod is a stable Gabriel category.*

Definition 2.80. *Let R be a ring. The product of two two-sided ideals $\mathfrak{a}$, $\mathfrak{b} \subseteq R$ is $\mathfrak{ab} = \{ab : a \in \mathfrak{a}, b \in \mathfrak{b}\}$. A two-sided ideal $\mathfrak{p} \subseteq R$ is* prime *if, given two two-sided ideals $\mathfrak{a}$, $\mathfrak{b}$ such that $\mathfrak{ab} \subseteq \mathfrak{p}$, either $\mathfrak{a} \subseteq \mathfrak{p}$ or $\mathfrak{b} \subseteq \mathfrak{p}$. The* spectrum $\mathrm{Spec}(R)$ *of R is the poset of all the prime ideals in R (ordered by inclusion). Given $\mathfrak{p}$, $\mathfrak{q} \in \mathrm{Spec}(R)$, if $\mathfrak{p} \subseteq \mathfrak{q}$ we say that $\mathfrak{p}$ is a* generalization *of $\mathfrak{q}$ and that $\mathfrak{q}$ is a* specialization *of $\mathfrak{p}$.*

Lemma 2.81. *Let R be a commutative Noetherian ring, then there is a bijection*

$$\mathrm{Spec}(R) \to \mathrm{Sp}(R\text{-Mod})\,, \quad \mathfrak{p} \mapsto E(R/\mathfrak{p})\,.$$

Furthermore, given $\mathfrak{p}$, $\mathfrak{q} \in \mathrm{Spec}(R)$ and denoting by $\pi(\mathfrak{p})$ and $\pi(\mathfrak{q})$ the prime torsion theories cogenerated by $E(R/\mathfrak{p})$ and $E(R/\mathfrak{q})$ respectively,

$$(\mathfrak{p} \subseteq \mathfrak{q}) \quad \Longleftrightarrow \quad (\pi(\mathfrak{q}) \preceq \pi(\mathfrak{p}))\,.$$

Proof. The fact that this map is well-defined and bijective is [72, Proposition 3.1]. Furthermore, if $\mathfrak{p} \leqslant \mathfrak{q}$, then there is an epimorphism $R/\mathfrak{p} \to R/\mathfrak{q}$ and this can be used to show that $\mathrm{Hom}_R(E(R/\mathfrak{p}), E(R/\mathfrak{q})) \neq 0$. By Lemma 2.65, this shows that $\pi(\mathfrak{q}) \preceq \pi(\mathfrak{p})$. On the other hand, if $\pi(\mathfrak{q}) \preceq \pi(\mathfrak{p})$, then $E(R/\mathfrak{p})$ is not $\pi(\mathfrak{q})$-torsion and so, by stability, it is $\pi(\mathfrak{q})$-torsion free. In particular, $\mathrm{Hom}_R(R/\mathfrak{p}, E(R/\mathfrak{q})) \neq 0$. Consider a non-trivial morphism $\phi : R/\mathfrak{p} \to E(R/\mathfrak{q})$ and consider the restriction $\phi' : R/\mathfrak{p} \to \phi(R/\mathfrak{p})$. Since ${}_RR$ is a projective generator, there is a non-trivial morphism $\psi : R \to \phi(R/\mathfrak{p})$ that extends to a morphism $\bar{\psi} : R \to R/\mathfrak{p}$ such that $\phi'\bar{\psi} = \psi$. Then, $\mathfrak{p} \subseteq \mathrm{Ker}(\phi)$ by construction and $\mathrm{Ker}(\phi) \subseteq \mathfrak{q}$ by [72, Lemma 3.2]. Thus, $\mathfrak{p} \subseteq \mathfrak{q}$. □

Chapter 3

Duality

The aim of Chapter 3 is to illustrate two classical duality theorems, taking also the occasion to recall all the background needed to state and apply these theorems. Indeed, in the first part of the chapter we recall some basics about topological groups and harmonic analysis in locally compact groups. After that, we recall the Pontryagin-Van Kampen Duality Theorem for locally compact Abelian groups and the Fourier Inversion Theorem.
In the second part of the chapter we recall some facts about (strictly) linearly compact modules and we give a complete proof of the Müller Duality Theorem between discrete and strictly linearly compact modules in a particular case.

3.1 Pontryagin-Van Kampen Duality

3.1.1 Topological spaces, measures and integration

In Example 1.6 we defined a topological space (X, τ) to be a set X with a distinguished family τ of open subsets that satisfies suitable closure properties. An easy example for a topological space is given by the *discrete spaces*, that are the topological spaces in which any subspaces is open.

Definition 3.1. *Let (X, τ) be a topological space. A set $\mathcal{B}$ of open subsets of X is a* base *of τ if any non-empty open set is a union of elements of $\mathcal{B}$.*

In a discrete topological space (X, τ) the set $\{\{x\} : x \in X\}$ is a base for the topology.
In general, given a topological space (X, τ), it is possible to give a "local characterization" of τ, that is, we can describe τ via the collection of "filters of neighborhoods" of the points.

Definition 3.2. *Let $(I, \leqslant)$ be a poset and let $\mathcal{F} \subseteq I$ be a subset. Then, $\mathcal{F}$ is a* filter *if the following conditions hold:*

– *$\mathcal{F}$ is downward directed;*

– *for all $x \in \mathcal{F}$ and $y \in I$, if $x \leqslant y$, then $y \in \mathcal{F}$.*

Any downward directed set is said to be base of filter. *Furthermore, given a base of filter $\mathcal{B}$ and a filter $\mathcal{F}$, we say that $\mathcal{B}$ is a* base *for $\mathcal{F}$ if $\mathcal{F} = \{x \in I : \exists b \in \mathcal{B} \text{ s.t. } b \leqslant x\}$.*

Definition 3.3. *Let (X, τ) be a topological space. Then,*

– *given a point* $x \in X$, *a* neighborhood *of* x *is a set* V *that contains an open* A *such that* $x \in A \subseteq V$. *We denote by* $\mathcal{V}_X(x)$, *or just* $\mathcal{V}(x)$ *if* X *is clear from the context, the set of neighborhoods of* x. *Given a subset* $S \subseteq X$, *a* neighborhood *of* S *is a set* V *that is a neighborhood of each element of* S.

– X *is a* Hausdorff *space if, given* $x \neq y \in X$, *there exist* $V \in \mathcal{V}(x)$ *and* $U \in \mathcal{V}(y)$ *such that* $U \cap V = \varnothing$.

Notice that a base completely determines the topology of a space. Furthermore, given a topological space (X, τ) and a point $x \in X$, the family of neighborhoods $\mathcal{V}(x)$ is a filter in the poset of subsets of X (ordered by inclusion). The knowledge of these filters, for any point of X, completely determines the topology, in fact, a subset is open if and only if it is a neighborhood of each of its points. These observations can be used to prove the following elementary lemma.

Lemma 3.4. *Let* (X, τ) *and* (Y, τ') *be topological spaces and let* $\phi : X \to Y$ *be a map. Then the following are equivalent*

(1) ϕ *is continuous;*

(2) *given a base* $\mathcal{B}'$ *of* τ', $\phi^{-1}(B')$ *is open for all* $B' \in \mathcal{B}'$;

(3) *for all* $x \in X$, *given a base* $\mathcal{C}$ *of* $\mathcal{V}_Y(\phi(x))$, $\phi^{-1}(U) \in \mathcal{V}(x)$ *for all* $U \in \mathcal{C}$.

Definition 3.5. *Let* (X, τ) *be a topological space. Then,*

– *a* pre-base *of* τ *is a set* $\mathcal{B}$ *of open sets such that the set of finite intersections of members of* $\mathcal{B}$ *is is a base of* τ;

– *given* $x \in X$, *a* pre-base of neighborhoods *of* x *is a family* $\mathcal{B}$ *of neighborhoods of* x *such that the set of finite intersections of members of* $\mathcal{B}$ *is a base of the filter* $\mathcal{V}(x)$.

The following corollary is a direct consequence of Lemma 3.4.

Corollary 3.6. *Let* (X, τ) *and* (Y, τ') *be topological spaces and let* $\phi : X \to Y$ *be a map. Then the following are equivalent*

(1) ϕ *is continuous;*

(2) *given a pre-base* $\mathcal{B}'$ *of* τ', $\phi^{-1}(B')$ *is open, for all* $B' \in \mathcal{B}'$;

(3) *given* $x \in X$ *and a pre-bae* $\mathcal{B}_{\phi(x)}$ *of* $\mathcal{V}(\phi(x))$, $\phi^{-1}(U) \in \mathcal{V}(x)$, *for all* $U \in \mathcal{B}_{\phi(x)}$.

Definition 3.7. *Let* (X, τ) *be a topological space and let* $Y \subseteq X$ *be a subset:*

– *the* induced topology *on* Y *is defined by declaring open all the subsets of the form* $A \cap Y$, *with* A *open in* X. *We sometimes denote this topology by* $\tau\restriction_Y$;

– Y *is* compact *if for any family of open sets* $\{A_i : i \in I\}$ *such that* $Y \subseteq \bigcup_{i \in I} A_i$ *there exists a finite subset* $F \subseteq I$ *such that* $Y \subseteq \bigcup_{i \in F} A_i$.

The space X *is locally compact if any of its elements has a compact neighborhood.*

Compare the definition of compact space with Definition 2.15. Notice also that, given a topological space (X, τ) and a subspace $(Y, \tau\restriction_Y)$, the inclusion $Y \to X$ is a monomorphism in the category of topological spaces. Not any subobject in this category is of this form.

Definition 3.8. *Let X be a set, and let $\mathcal{F} = \{C_i : i \in I\}$ be a family of subsets of X. Then $\mathcal{F}$ has the* finite intersection property *if, for any finite subset $F \subseteq I$, $\bigcap_{i \in F} C_i \neq \varnothing$.*

Lemma 3.9. [87, Theorem 4, Ch. 2] *Let (X, τ) be a topological space. Then the following are equivalent:*

(1) *X is compact;*

(2) *any family of open subsets of X with the finite intersection property has non-empty intersection;*

Definition 3.10. *Let (X, τ) and (Y, τ') be topological spaces and let $\phi : X \to Y$ be a map. We say that ϕ is* open *(resp.,* closed*,* proper*), provided for any open (resp., closed, compact) subset $A \subseteq X$, $\phi(A)$ has the same property. Furthermore, ϕ is a* homeomorphism *if it is bijective, continuous and open.*

Notice that homeomorphisms are the isomorphisms in the category of topological spaces.

Theorem 3.11. *Let (X, τ) and (Y, τ) be topological spaces and let $\phi : X \to Y$ be a continuous map. Then,*

(1) *if X is compact (resp., locally compact), then so is any of its closed subsets;*

(2) *if Y is Hausdorff, then any of its compact subsets is closed;*

(3) *if X is compact and Y is Hausdorff, then ϕ is proper.*

In particular, if X is compact, Y is Hausdorff and ϕ is surjective, then ϕ is open.

Proof. For parts (1), (2) and (3) see respectively B), C) and D) in [87, Section 13, Ch. 2]. For the last part, consider an open subset $A \subseteq X$, then $X \backslash A$ is closed and so, by part (1), compact. By part (3), $\phi(X \backslash A)$ is compact and so, by part (2), it is closed. By the surjectivity of ϕ, $\phi(X \backslash A) = Y \backslash \phi(A)$, showing that $\phi(A)$ is open. □

Definition 3.12. *Let I be a set and let (X_i, τ_i) be topological spaces, for all $i \in I$. The* product *$(\prod_{i \in I} X_i, \tau)$ of these topological spaces is a topological space defined as follows:*

- *as a set $\prod_{i \in I} X_i = \{(x_i)_{i \in I} : x_i \in X_i\}$. For all $j \in I$, there are surjections $\pi_j : \prod_{i \in I} X_i \to X_j$ such that $\pi_j((x_i)_{i \in I}) = x_j$;*
- *a pre-base of τ is given by $\{\pi_j^{-1}(A) : j \in I,\, A \in \tau_j\}$.*

The topology τ is called the product topology.

Notice that, by definition, the maps π_j in the above definition are continuous and open. Furthermore, $((\prod_{i \in I} X_i, \tau), (\pi_j)_{j \in I})$ is a product in the category of topological spaces.
The following classical result is usually known as Tychonoff's Theorem.

Theorem 3.13. [87, Theorem 5, Ch. 2] *Let I be a set and let (X_i, τ_i) be a topological space for all $i \in I$. Then, the product $\prod_{i \in I} X_i$ is compact if and only if X_i is compact for all $i \in I$.*

In the second part of this subsection we introduce some basic definition about measures and Lebesgue integration.

Definition 3.14. *Let X be a set. A family Σ of subsets of X is a* σ-algebra *if*

– Σ *is not empty;*

– Σ *is closed under complementation, that is,* $X\backslash A \in \Sigma$ *for all* $A \in \Sigma$*;*

– Σ *is closed under countable unions.*

If (X, τ) *is a topological space, the* Borel sets *are the sets belonging to the smallest* σ*-algebra containing all the open sets.*

Definition 3.15. *Let* (X, τ) *be a topological space and let* Σ *be the* σ*-algebra of the Borel sets. A* Borel measure *on* X *is a function* $m : \Sigma \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *such that*

– $m(\varnothing) = 0$*;*

– $m(\bigcup_{n\in\mathbb{N}} B_n) = \sum_{n\in\mathbb{N}} B_n$*, if* $B_n \in \Sigma$ *and* $B_n \cap B_m = \varnothing$*, for all* $n \neq m \in \mathbb{N}$.

Furthermore, a Borel measure $m : \Sigma \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is* regular *if*

– *it is* outer regular*, that is,* $m(E) = \inf\{m(U) : E \subseteq U,\ U \text{ open}\}$*, for all* $E \in \Sigma$*;*

– *it is* inner regular*, that is,* $m(A) = \sup\{m(K) : K \subseteq A,\ K \text{ compact}\}$*, for all* $A \in \tau$.

Definition 3.16. *Let* (X, τ) *be a topological space and let* $\phi : X \to \mathbb{R}$ *and* $\psi : X \to \mathbb{C}$ *be maps. Then,*

– ϕ *is* (Borel) measurable *if* $\{x \in X : \phi(x) > a\}$ *is a Borel set, for all* $a \in \mathbb{R}$*;*

– ψ *is (Borel) measurable if its real and imaginary parts are measurable;*

– ψ *is* positive *if it is real-valued and* $\phi(x) \geqslant 0$ *(in* $\mathbb{R}$*), for all* $x \in X$*;*

– *the* support $\mathrm{supp}(\psi)$ *of* ψ *is defined as the closure of the set of points on which* ψ *is* $\neq 0$*, that is*

$$\mathrm{supp}(\psi) = \overline{\{x \in X : \psi(x) \neq 0\}} \subseteq X\,.$$

Recall that a continuous complex-valued function is always measurable.

Let X be a set, let ϕ, $\psi : X \to \mathbb{C}$ be maps and let $\lambda \in \mathbb{C}$. We use the following notations

– $\lambda\phi : X \to \mathbb{C}$ is the map such that $x \mapsto \lambda \cdot \phi(x)$;

– $\phi \cdot \psi : X \to \mathbb{C}$ is the map such that $x \mapsto \phi(x) \cdot \psi(x)$;

– $\phi + \psi : X \to \mathbb{C}$ is the map such that $x \mapsto \phi(x) + \psi(x)$;

– $|\phi| : X \to \mathbb{C}$ is the map such that $x \mapsto |\phi(x)|$ (the norm of $\phi(x)$, see Example 3.25).

Let C be a subset of X and define the *characteristic function* $\chi_C : X \to \mathbb{C}$ of C as

$$\chi_C(x) = \begin{cases} 1 & \text{if } x \in C; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3.17. *Let* (X, τ) *be a topological space, let* m *be a Borel measure on* X *and let* $\phi : X \to \mathbb{C}$ *be a measurable function. Then,*

– *a* measurable partition *of* X *is a family* $\{A_1, \ldots, A_k\}$ *of Borel subsets of* X*, such that* $\bigcup_{i=1}^k A_i = X$ *and* $A_i \cap A_j = \varnothing$ *for all* $i \neq j \leqslant k$*;*

– *if ϕ is positive, the* Lebesgue integral $\int_X \phi(x)\,dm(x)$ *(or simply $\int_X \phi\,dm$) is*

$$\sup\left\{\sum_{i=1}^{k}(\inf\{\phi(x) : x \in A_i\}m(A_i)) : \{A_1, \dots, A_k\} \text{ is a measurable partition of } X\right\}.$$

We say that ϕ is integrable *provided $\int_X \phi\,dm < \infty$*

– *if $\phi(X) \subseteq \mathbb{R}$, then one defines two maps $\phi^+ : X \to \mathbb{R}_{\geqslant 0}$ and $\phi^- : X \to \mathbb{R}_{\geqslant 0}$ such that, for all $x \in X$:*

$$\phi^+(x) = \max\{0, \phi(x)\} \quad \text{and} \quad \phi^-(x) = -\min\{0, \phi(x)\}.$$

If both ϕ^+ and ϕ^- are integrable, then ϕ is said to be integrable *and one defines the* Lebesgue integral *as $\int_X \phi dm = \int_X \phi^+ dm - \int_X \phi^-\,dm$;*

– *in the general case, one can define two functions ϕ_1, $\phi_2 : X \to \mathbb{R}$ such that $\phi(x) = \phi_1(x) + i\phi_2(x)$, for all $x \in X$. If both ϕ_1 and ϕ_2 are integrable, then ϕ is said to be* integrable *and one defines the* Lebesgue integral *as $\int_X \phi\,dm = \int_X \phi_1\,dm + i\int_X \phi_2\,dm$.*

Let (X, τ) be a topological space, let m be a Borel measure and let $\phi : X \to \mathbb{C}$ be a map. Then, ϕ is integrable if and only if $\int_X |\phi|\,dm < \infty$. In this case, given a Borel subset E of X, the function $\phi \cdot \chi_E$ is still integrable and we let

$$\int_E \phi\,dm = \int_X \phi \cdot \chi_E\,dm.$$

The following bounds for the integral follow directly from the definitions

$$\inf\{\phi(x) : x \in E\}m(E) \leqslant \int_E \phi\,dm \leqslant \sup\{\phi(x) : x \in E\}m(E). \tag{3.1.1}$$

We conclude this subsection with some general properties of Lebesgue integration. A proof of parts (1), (2) and (3) of the following lemma can be found in [42, Propositions 2.21, 2.22 and 2.23], part (4) is a consequence of [42, Proposition 2.13], while an argument to prove part (5) is given in [92, Appendix E8].

Lemma 3.18. *Let (X, τ) be a topological space, and let m be a Borel measure. Let ϕ, $\psi : X \to \mathbb{C}$ be integrable maps and let a, $b \in \mathbb{C}$. Then*

(1) *$\int_X a\phi + b\psi\,dm = a\int_X \phi\,dm + b\int_X \psi\,dm$;*

(2) *$|\int_X \phi\,dm| \leqslant \int_X |\phi|\,dm$;*

(3) *$\int_E \phi\,dm = \int_E \psi\,dm$ for any Borel subset $E \subseteq X$, if and only if $\int_X |\phi - \psi|\,dm = 0$, if and only if the set $\{x \in X : \phi(x) \neq \psi(x)\}$ is contained in a set of measure zero;*

(4) *if both ϕ and ψ are positive and $\phi(x) \leqslant \psi(x)$ for all $x \in X$, then $\int_X \phi\,dm \leqslant \int_X \psi\,dm$;*

(5) *there exists a sequence of compact subsets $C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n \subseteq \ldots \subseteq X$ such that $\int_X |\phi - \phi \cdot \chi_{C_n}|\,dm < 1/n$, for all $n \in \mathbb{N}_+$.*

3.1.2 Topological groups and Haar measure

Definition 3.19. *Let G be a group. A topology τ on G is a* group topology *if the map*

$$f : G \times G \to G \quad \textit{such that} \quad f(x, y) = xy^{-1},$$

is continuous, where $G \times G$ carries the product topology. A topological group *is a pair (G, τ) of a group G and a group topology τ on G. We denote by $\underline{\mathrm{TopGr}}$ the category whose objects are the topological groups and the morphisms are the continuous group homomorphisms. Given (G, τ), $(G', \tau') \in \mathrm{Ob}(\underline{\mathrm{TopGr}})$, we let $\mathrm{Hom}_{\underline{\mathrm{TopGr}}}(G, H) = \mathrm{CHom}(G, H)$. An isomorphism in the category $\underline{\mathrm{TopGr}}$, that is, a map which is both an isomorphism of groups and a homeomorphism, is said to be a* topological isomorphism.
A topological group (G, τ) is Hausdorff (resp., compact, locally compact), if it has the same properties as a topological space. Analogously, (G, τ) is Abelian if G is Abelian as a group. By an LC group *(resp.,* LCA group*) we mean a locally compact Hausdorff (Abelian) group. We denote by $\underline{\mathrm{LcGr}}$ and $\underline{\mathrm{LcaGr}}$ the full subcategory of $\underline{\mathrm{TopGr}}$, whose objects are the LC and LCA groups respectively.*

Let $(G, \cdot)$ be a group, let U, $V \subseteq G$ and let $x \in G$. We use the following notation

– $xU = \{xu : u \in U\}$;

– $Ux = \{ux : u \in U\}$;

– $U^{-1} = \{u^{-1} : u \in U\}$;

– $UV = \{uv : u \in U \text{ and } v \in V\}$.

Let (G, τ) be a topological group and let $x \in G$. Notice that, just by definition, the morphisms

$$\begin{array}{ccc} G \longrightarrow G & G \longrightarrow G & G \longrightarrow G \\ g \longmapsto xg & g \longmapsto gx & g \longmapsto x^{-1}gx \end{array} \tag{3.1.2}$$

are homeomorphisms. In particular, if $\mathcal{V}(e)$ is the family of neighborhoods of e in G, $\mathcal{V}(x) = \{xV : V \in \mathcal{V}(e)\} = \{Vx : V \in \mathcal{V}(e)\}$.

Lemma 3.20. [34, Theorem 2.1.1] *Let G be a group and let $\mathcal{V}(e)$ be the filter of all neighborhoods of e in some group topology τ on G. Then:*

(1) *for every $U \in \mathcal{V}(e)$ there exists $V \in \mathcal{V}(e)$ with $VV \subseteq U$;*

(2) *for every $U \in \mathcal{V}(e)$ there exists $V \in \mathcal{V}(e)$ with $V^{-1} \subseteq U$;*

(3) *for every $U \in \mathcal{V}(e)$ and for every $x \in G$ there exists $V \in \mathcal{V}(e)$ with $xVx^{-1} \subseteq U$.*

Conversely, if $\mathcal{V}$ is a filter on G satisfying (1), (2) and (3), then there exists a unique group topology τ on G such that $\mathcal{V}$ coincides with the filter of all τ-neighborhoods of e in G.

A consequence of the above lemma is that, to specify a group topology on a given group, one has just to specify a pre-base of the neighborhoods of a point. Another consequence is the following:

Corollary 3.21. *Let (G, τ) and (G', τ') be topological groups and let $\phi : G \to G'$ be a map. Then, ϕ is continuous if and only if $\phi^{-1}(N) \in \mathcal{V}_G(e)$, for all N in a given pre-base $\mathcal{V}_{G'}(e)$ of the neighborhoods of e in G'.*

Lemma 3.22. *Let (G, τ) be a topological group. Then, G is Hausdorff (resp., discrete) if and only if $\{e\}$ is closed (resp., open).*

Proof. If $\{e\}$ is open, then any point is open in G by (3.1.2) and, since arbitrary unions of opens are open, any subset of G is open; the converse is trivial. On the other hand, $\{e\}$ is closed if and only if all the points are closed, if and only if G is Hausdorff by [58, Theorem 4.8]. □

Let (G, τ) be a group and let S be a subset. There is a useful way to describe the closure $\bar{S}$ of S in G:

Lemma 3.23. *Let (G, τ) be a topological group, let $\mathcal{B}$ be a base of neighborhoods of e in G and let S be a subset. Then,*

$$\bar{S} = \bigcap_{V \in \mathcal{B}} SV = \bigcap_{V \in \mathcal{B}} VS\,.$$

Proof. Given $x \in G$, $x \notin \bar{S} = \bigcap_{S \subseteq C \text{ closed}} C$ if and only if there exists a closed set containing S such that $x \notin C$, if and only if there exists a neighborhood N of x such that $N \cap S = \varnothing$ (take for example $N = G \backslash C$). Equivalently, there exists $U \in \mathcal{B}$ such that $Ux \cap S = \varnothing$, that is, $x \notin SU^{-1} \supseteq \bigcap_{V \in \mathcal{B}} SV$. □

Let (G, τ) be a group and let H be a subgroup. One can show that the closure $\bar{H}$ is still a subgroup of G, and it is normal if H is normal. When not otherwise specified, we will assume that H carries the topology induced by G. Furthermore, if H is normal, then there is a natural group topology induced on the quotient group G/H. Indeed, letting $\pi : G \to G/H$ be the natural projection, the open subsets of G/H are exactly the images of the open sets in G. Notice that, by definition, the projection π is open and continuous.

Proposition 3.24. *Let (G, τ) be a topological group and let $H \leqslant G$ be a subgroup. Then,*

(1) *if H is open, then it is closed;*

(2) *if H is normal, then it is closed if and only if G/H is Hausdorff;*

(3) *if H is normal, then it is open if and only if G/H is discrete;*

(4) *if H is normal and closed, and G is Hausdorff, then G is compact (resp., locally compact) if and only if both H and G/H have the same property.*

Proof. (1) If H is open, then $G \backslash H = \bigcup_{g \in G \backslash H} Hg$ and each Hg is open in G. Thus, $G \backslash H$ is open in G.
(2) $G \backslash H$ is open if and only if $\pi(G \backslash H) = (G/H) \backslash \{e\}$ is open, if and only if $\{e\}$ is closed in G/H. Conclude using Lemma 3.22.
(3) H is open if and only if $\pi(H) = \{e\}$ is open in G/H. Conclude using Lemma 3.22.
(4) See [58, Theorem 5.2].

□

Example 3.25. (1) *The additive group $(\mathbb{R}, +)$ is a topological LCA group when endowed with the usual euclidean topology, that is, the unique topology having $\{\{x \in \mathbb{R} : |x| < 1/n\} : n \in \mathbb{N}_+\}$ as a base of neighborhoods of 0;*

(2) *given $x = a + ib \in \mathbb{C}$ (with $a, b \in \mathbb{R}$) we let $|x| = \sqrt{a^2 + b^2}$ be the norm of x. The euclidean topology on $(\mathbb{C}, +)$ has a base of neighborhoods of 0 consisting of sets of the form $\{x \in \mathcal{C} : |x| < 1/n\}$, with n ranging in $\mathbb{N}_+$. With this topology, the additive group $\mathbb{C}$ is a topological LCA group;*

(3) *consider the subset* $\mathbb{S} = \{x \in \mathbb{C} : |x| = 1\}$. *Then,* $(\mathbb{S}, \cdot)$ *is a group, that is an LCA group with the topology induced by* $\mathbb{C}$. *Consider* $\mathbb{R}$ *with the topology described in (1); then* $\mathbb{Z}$ *is a closed subgroup and we can take the quotient* $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. *The map*

$$\mathbb{T} \to \mathbb{S} \quad \textit{such that} \quad x \mapsto e^{2\pi i x}$$

is a topological isomorphism.

Given a family $\{(G_i, \tau_i) : i \in I\}$ of topological groups, the product group $\prod_I G_i$ endowed with the product topology is again a topological group. This construction gives us products in the categories <u>TopGr</u>, <u>LcGr</u> and <u>LcaGr</u>.

Definition 3.26. *Let* (G, τ) *be a topological group and let* Σ *be the* σ*-algebra of Borel subsets of* G. *A* left Haar measure *on* G *is a regular Borel measure* $\mu : \Sigma \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *such that* $\mu(xE) = \mu(E)$ *for all* $E \in \Sigma$ *and* $x \in G$ *and such that* $\mu(C) < \infty$ *for any compact subset* $C \subseteq G$.

Of course, one can define similarly right Haar measures. The two concepts coincide on Abelian groups.

Example 3.27. *Let* G *be a group endowed with the discrete topology, so that any subset of* G *is a Borel subset, while the compact subsets are precisely the finite ones. Then, a left Haar measure on* G *is given by* $\mu(E) = |E|$ *if* E *is a finite subset,* $\mu(E) = \infty$ *otherwise. Given* $\phi : G \to \mathbb{C}$ *one can verify that* $\int_G \phi \, d\mu = \sup\{\sum_{g \in F} \phi(g) : F \subseteq G \text{ finite}\}$.

In general it is not possible to prove the existence of a left Haar measure on a given topological group G, but this is possible under suitable hypotheses. The following theorem is proved in [58, Section 15].

Theorem 3.28. *Let* (G, τ) *be an LC group. Then, there exists a left Haar measure* $\mu : \Sigma \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *on the Borel subsets of* G. *Furthermore, if* μ' *is another left Haar measure, then there exists* $\lambda \in \mathbb{R}$ *such that* $\lambda\mu(E) = \mu'(E)$ *for all* $E \in \Sigma$.

Given a topological group (G, τ) and a function $\phi : G \to \mathbb{C}$ we define

$$\phi_a : G \to \mathbb{C} \quad \text{such that } \phi_a(x) = \phi(ax)\,.$$

A consequence of the left invariance and of the "uniqueness" of the left Haar measure is the following

Corollary 3.29. *Let* (G, τ) *be an LC group, let* μ *be a fixed left Haar measure, let* $\phi : G \to \mathbb{C}$ *be an integrable function and let* $a \in G$. *Then,* ϕ_a *is integrable and* $\int_G \phi \, d\mu = \int_G \phi_a \, d\mu$.

In the following lemma we introduce the *modulus*, that is a group homomorphism

$$\Delta : \mathrm{Aut}(G) \to \mathbb{R}_+$$

that tells us how to compute the measure of the image of a Borel set under a topological automorphism.

Lemma 3.30. [58, (15.26) pag. 208] *Let* (G, τ) *be an LC group and let* μ *be a fixed Haar measure on* G. *Letting* $\mathbb{R}_+$ *denote the multiplicative group of positive reals, there exists a group homomorphism*

$$\Delta : \mathrm{Aut}(G) \to \mathbb{R}_+, \ \textit{such that} \quad \mu(\alpha E) = \Delta(\alpha)\mu(E)$$

for every topological automorphism α *of* G *and every Borel subset* E *of* G.

Let us recall the following useful relation which allows to compute the integral of the composition of an integrable function with an automorphism (see again [58, (15.26) p. 208] for a proof).

Corollary 3.31. *Let (G, τ) be a LC group and let μ be a fixed Haar measure on G. If $\phi : G \to \mathbb{C}$ is integrable and $\alpha \in \mathrm{Aut}(G)$ is a topological automorphism, then*

$$\int_G \phi \circ \alpha^{-1} \, d\mu = \Delta(\alpha) \int_G \phi \, d\mu \, .$$

Definition 3.32. *Let (G, τ) be an LC group and let $\mu : \Sigma \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a Haar measure on G. Given an integrable function $\phi : G \to \mathbb{C}$, the* L^1-norm *of ϕ is*

$$||\phi||_1 = \int_G |\phi| \, d\mu \, .$$

A canonical example for an absolutely integrable function is given by the functions with compact support.

Lemma 3.33. [58, Theorem 12.7] *Let (G, τ) be an LC group, let μ be a fixed Haar measure on G and let ϕ, $\psi : G \to \mathbb{C}$ be two positive, integrable functions. Then, $||\phi + \psi||_1 \leqslant ||\phi||_1 + ||\psi||_1$.*

The following lemma is an easy consequence of the definition, we state it just because we will need this precise statement.

Lemma 3.34. *Let (G, τ) be an LC group, let μ be a fixed Haar measure on G and let ϕ, $\psi : G \to \mathbb{C}$ be two positive, absolutely integrable functions. Then, $||\phi - \psi||_1 \geqslant ||\phi||_1 - ||\psi||_1$.*

Proof. Since ϕ and ψ are positive, then $|\phi - \psi|(x) \geqslant (\phi - \psi)(x) = (|\phi| - |\psi|)(x)$, for all $x \in G$. Now apply Lemma 3.18 (4). □

In the setting of Definition 3.32, given two integrable functions ϕ, $\psi : G \to \mathbb{C}$, we say that ϕ is equivalent to ψ if $||\phi - \psi||_1 = 0$. By part (3) of Lemma 3.18, this is an equivalence relation.

Definition 3.35. *Let (G, τ) be an LC group and let $\mu : \Sigma \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a Haar measure on G. We let $L^1(G)$ be the set of equivalence classes of integrable functions on G. Furthermore, we let $L^1(G)^+ \subseteq L^1(G)$ be the family of equivalence classes that contain at least a positive function.*

By Lemma 3.18 (3), given two equivalent integrable functions ϕ and ψ, $\int_E \phi \, d\mu = \int_E \psi \, d\mu$ for any Borel subset E and $||\phi||_1 = ||\psi||_1$, for any choice of a Haar measure μ on G. By writing $\phi \in L^1(G)$, we mean that ϕ is a representative of an equivalence class of integrable functions. Furthermore, when working with an equivalence class in $\phi \in L^1(G)^+$, we generally choose a positive representative. By the previous observations, the value of integrals and of the L^1-norm does not depend on the specific choice of the representative.

Until now we could work with general LC groups. From now on we will assume commutativity, so we will work on LCA groups. Therefore we pass to the additive notation.

Definition 3.36. *Let G be an LCA group, let μ be a fixed Haar measure on G and let ϕ, $\psi \in L^1(G)$. The* convolution *of ϕ and ψ is*

$$\phi * \psi : G \to \mathbb{C} \, , \quad \phi * \psi(x) = \int_G \phi(g)\psi(x - g) d\mu \, .$$

In the following lemma we recall some properties of convolutions; the proof of parts (1), (2) and (3) can be found in [92, Sections 1.1.6 and 1.1.7], while the proof of (4) follows as in [92, Section 1.1.6 (e)] using positivity.

Lemma 3.37. *Let G be an LCA group, let μ be a fixed Haar measure on G and let ϕ, ϕ', $\psi \in L^1(G)$. Then*

(1) $\phi * \psi \in L^1(G)$;

(2) $(\phi + \phi') * \psi = \phi * \psi + \phi' * \psi$ *and* $\phi * \psi = \psi * \phi$;

(3) *if ϕ and ψ have compact supports, then* $\mathrm{supp}(\phi * \psi) \subseteq \mathrm{supp}(\phi) + \mathrm{supp}(\psi)$;

(4) *if ϕ and $\psi \in L^1(G)^+$, then* $||\phi * \psi||_1 = ||\phi||_1 ||\psi||_1$.

In the following remark we show how convolution acts with respect to translation.

Remark 3.38. [58, Remark 20.11] *Let G be an LCA group, let ϕ, $\psi \in L^1(G)$ and let g, $x \in G$. Then*

$$(\phi * \psi)_g(x) = (\phi * \psi)(x + g) = (\phi_g * \psi)(x) = (\phi * \psi_g)(x)\,.$$

Definition 3.39. *Let G be an LCA group. A function $\phi : G \to \mathbb{C}$ is said to be* positive-definite *if, for any positive integer n and any choice of $x_1, \dots, x_n \in G$ and $c_1, \dots, c_n \in \mathbb{C}$, the following condition holds true:*

$$\sum_{i,j=1}^{n} c_i \overline{c_j} \phi(x_i - x_j) \in \mathbb{R}_{\geqslant 0}\,, \tag{3.1.3}$$

where $\bar{c}$ is the complex conjugate of $c \in \mathbb{C}$.

Let us start with some basic fact about positive-definite functions, whose proof follows by [92, page 18] and [41, Corollary 3.21 and Proposition 3.35].

Lemma 3.40. *Let G be an LCA group and let $\phi : G \to \mathbb{C}$ be a positive-definite function. Then,*

(1) $\phi(-x) = \overline{\phi(x)}$, *for all $x \in G$;*

(2) $|\phi(x)| \leqslant \phi(0)$, *for all $x \in G$. In particular, if ϕ is positive then $\phi(G) \subseteq [0, \phi(0)] (\subseteq \mathbb{R})$ and so $\phi(0)$ is a maximum;*

(3) *there exists a continuous function $\psi : G \to \mathbb{C}$ such that $\{x \in G : \phi(x) \neq \psi(x)\}$ is contained in a set of Haar measure* 0.

In principle it is not easy to figure out what the condition of being positive-definite means; the following lemma allows one to produce a positive-definite function from a square integrable function and so also to obtain some natural examples.

Lemma 3.41. [92, Section 1.4.2] *Let G be an LCA group, let μ be a fixed Haar measure, let $\phi : G \to \mathbb{C}$ be such that $\int |\phi|^2 \, d\mu < \infty$ and let $\widetilde{\phi} : G \to \mathbb{C}$ be the function such that $\widetilde{\phi}(x) = \overline{\phi(-x)}$. Then, $\phi * \widetilde{\phi}$ is positive-definite.*

As a consequence of the above lemma, we get the following

Example 3.42. *Let G be an LCA group. Whenever $C = -C$ is a symmetric neighborhood of* 0 *in G with compact closure, the above lemma shows that the convolution $\chi_C * \chi_C$ is positive-definite.*

In the following lemma we study some properties of functions of the form $\chi_C * \chi_C$.

Lemma 3.43. *Let G be an LCA group, let $\alpha : G \to G$ be a topological automorphism and let $C \subseteq G$ be a Borel subset. Then,*

(1) $\chi_{\alpha C} = \chi_C \circ \alpha^{-1}$;

(2) $\chi_{\alpha C} * \chi_{\alpha C} = \Delta(\alpha)(\chi_C * \chi_C) \circ \alpha^{-1}$;

(3) $||(\chi_C * \chi_C) \circ \alpha^{-1}||_1 = \Delta(\alpha)\mu(C)^2$.

Proof. Let μ be a fixed Haar measure.

(1) follows from the definition of characteristic function.

(2) Let $x \in G$ and define two maps ϕ, $\psi : G \to \mathbb{C}$ such that $\phi(g) = \chi_C(\alpha^{-1}(g))\chi_C(\alpha^{-1}(x - g))$ and $\psi(g) = \chi_C(g)\chi_C(\alpha^{-1}(x) - g)$, for all $g \in G$. Notice that $\phi = \psi \circ \alpha^{-1}$, so by Corollary 3.31,

$$((\chi_C \circ \alpha^{-1}) * (\chi_C \circ \alpha^{-1}))(x) = \int_G \phi \, d\mu = \Delta(\alpha) \int_G \psi \, d\mu = \Delta(\alpha)(\chi_C * \chi_C)(\alpha^{-1}(x)) .$$

(3) By parts (1) and (2), $||(\chi_C * \chi_C) \circ \alpha^{-1}||_1 = ||\chi_{\alpha C} * \chi_{\alpha C}||_1/\Delta(\alpha)$. Furthermore, $\chi_{\alpha C} \in L^1(G)^+$, thus by Lemmas 3.37 and 3.30, $||\chi_{\alpha C} * \chi_{\alpha C}||_1 = ||\chi_{\alpha C}||_1^2 = \mu(\alpha C)^2 = \Delta(\alpha)^2\mu(C)^2$. □

Definition 3.44. *Let G be an LCA group. We denote by $\mathcal{P}(G)$ the family of equivalence classes of positive-definite functions $G \to \mathbb{C}$ (where two positive-definite functions are equivalent if they differ on a set of measure 0). Furthermore, we let $\mathcal{P}^1(G) = \mathcal{P}(G) \cap L^1(G)$ and $\mathcal{P}^1(G)^+ = \mathcal{P}(G) \cap L^1(G)^+$.*

Lemma 3.41 can be used to construct positive-definite functions with prescribed support:

Lemma 3.45. *In the above notation, let U be a compact neighborhood of 0. Then, there exists a non-trivial $\phi \in \mathcal{P}^1(G)^+$ such that $\mathrm{supp}(\phi) \subseteq U$.*

Proof. Let $V \subseteq U$ be a compact neighborhood of 0 such that $V - V \subseteq U$ and let V' be an open neighborhood of 0 contained in V. Notice that $\{0\}$ and $G \backslash V'$ are two closed and disjoint subsets of G. Thus there exists a continuous function

$$f : G \to [0, 1] \text{ such that } f(0) = 1 \text{ and } f(x) = 0 \ \forall x \in G\backslash V' ,$$

by the Uryssohn Lemma (see for example [87, page 67]). In particular, $\mathrm{supp}(f) \subseteq \overline{V'} \subseteq V$ and $\mathrm{supp}(\widetilde{f}) \subseteq -V$ (where $\widetilde{f}(x) = f(-x)$ as in Lemma 3.41). We let $\phi = f * \widetilde{f}$. By Lemma 3.37, $\mathrm{supp}(\phi) \subseteq V - V \subseteq U$ and $\phi \in L^1(G)$, while it is easily verified that ϕ is non-trivial and positive. The conclusion follows by Lemma 3.41. □

3.1.3 The duality theorem

Definition 3.46. *Let G be an LCA group and let $G^* = \mathrm{CHom}(G, \mathbb{T})$ be the additive group of continuous homomorphisms from G to $\mathbb{T}$. The* compact-open topology *on G^* is defined by taking as a base of neighborhoods of 0 the sets of the form*

$$\mathcal{W}(C, U) = \{\gamma \in G^* : \gamma(C) \subseteq U\}$$

with C a compact neighborhood of 0 in G and U an neighborhood of 0 in $\mathbb{T}$. Furthermore, given a continuous homomorphism of LCA groups $\phi : G \to H$, we define $\phi^* : H^* \to G^*$ by the following formula

$$\phi^*(\gamma) = \gamma \circ \phi ,$$

for all $\gamma \in H^*$.

Lemma 3.47. [34, Lemma 3.1.1 and Exercise 2.10.2(c)] *Let G, H be LCA groups and let $\phi : G \to H$ be a continuous homomorphism. Then,*

(1) *G^* is an LCA group;*

(2) *ϕ^* is a continuous homomorphism;*

(3) *G is compact (resp., discrete) if and only if G^* is discrete (resp., compact).*

By the above lemma, the correspondence described in Definition 3.46 gives us a functor $(-)^* : \underline{\mathrm{LcaGr}}^{op} \to \underline{\mathrm{LcaGr}}$. The Pontryagin-Van Kampen Duality states that $(-)^*$ is adjoint to itself and, furthermore, that it induces a duality, in other words the functor $(-)^{**}$ (that is, the composition of the functor $(-)^*$ with itself) is naturally equivalent to the identity functor. Furthermore, by Lemma 3.47 (3), this restricts to a duality between the category of discrete and the category of compact Hausdorff Abelian groups.

Theorem 3.48. [34, Theorem 3.2.7] *Define, for all $G \in \mathrm{Ob}(\underline{\mathrm{LcaGr}})$ the evaluation map $\omega_G : G \to G^{**}$ by $x \mapsto \omega_G(x)$, where*

$$\omega_G(x) : G^* \to \mathbb{T} \quad \text{is such that} \quad \omega_G(x)(\gamma) = \gamma(x)\,.$$

Then, $\omega : \mathrm{id}_{\underline{\mathrm{LcaGr}}} \Rightarrow (-)^{**}$ is a natural isomorphism of functors.

Using the Prontryagin-Van Kampen duality one can give the following useful characterization of positive-definite functions, which can be found in [92, p. 19]. This result is usually called Bochner's Theorem.

Theorem 3.49. *Let G be an LCA group. A continuous function $\phi : G \to \mathbb{C}$ is positive-definite if and only if there exists a (necessarily unique) regular measure m on G^* such that $m(G^*) < \infty$ and*

$$\phi(x) = \int_{G^*} \omega_G(x)\, dm\,, \quad \text{for all } x \in G\,.$$

We conclude this section recalling the notion of Fourier transform.

Definition 3.50. *Let G be an LCA group and let μ be a fixed Haar measure. Given $\phi \in L^1(G)$, the* Fourier transform *of ϕ is defined as*

$$\widehat{\phi} : G^* \to \mathbb{C} \quad \text{such that} \quad \widehat{\phi}(\gamma) = \int_G \phi(x)\gamma(-x)\, d\mu\,, \quad \forall \gamma \in G^*\,.$$

In the following lemma we describe the behavior of the Fourier transform with respect to convolution and composition with an automorphism.

Lemma 3.51. *Let G be an LCA group and let μ be a fixed Haar measure. Let ϕ, $\psi \in \mathcal{P}^1(G)^+$ and let $\alpha \in \mathrm{Aut}(G)$ be a topological automorphism. Then,*

(1) $\widehat{\phi \circ \alpha^{-1}} = \Delta(\alpha)\widehat{\phi} \circ \alpha^*$;

(2) $\widehat{\phi * \psi} = \widehat{\phi} \cdot \widehat{\psi}$.

Proof. For the proof of (1) one can proceed as in the following computation:

$$\widehat{\phi \circ \alpha^{-1}}(\gamma) = \int_G \phi(\alpha^{-1}(x))\gamma(-x)d\mu = \Delta(\alpha)\int_G \phi(x)\gamma(\alpha(-x))d\mu$$
$$= \Delta(\alpha)\int_G \phi(x)\alpha^*(\gamma)(-x)d\mu = \Delta(\alpha)\widehat{\phi}(\alpha^*(\gamma))\,,$$

where the appearance of $\Delta(\alpha)$ at the end of the first line is due to Corollary 3.31. Part (2) follows by [92, Section 1.2.4, Part (b)]. □

The following theorem is known as Fourier Inversion Theorem.

Theorem 3.52. *Let G be an LCA group, let μ be a fixed Haar measure and let $\phi \in L^1(G)$ be a continuous function.*

(1) *If ϕ is positive-definite, then $\widehat{\phi} \in L^1(G^*)^+$.*

(2) *If $\widehat{\phi} \in L^1(G^*)$, then $\phi(x) = \widehat{\widehat{\phi}}(-x)$, for all $x \in G$.*

Proof. Part (1) follows by the first part (a) of [92, Theorem on p. 22] and [41, Corollary (4.23)]. For the proof of part (2) we refer to [41, Page 102]. □

3.2 Müller's Duality

3.2.1 Generalities on topological rings and modules

Definition 3.53. *A* topological ring *is a pair (R, τ), where R is a ring and τ is a group topology on the Abelian group $(R, +)$ and such that the function*

$$R \times R \to R \quad \textit{such that} \quad (r, s) \mapsto rs$$

is continuous when $R \times R$ is endowed with the product topology. Given a topological ring (R, τ), a topological right R-module *is a pair (M_R, σ) where M_R is a right R-module and σ is a group topology on the Abelian group M, such that the function*

$$M \times R \to M \quad \textit{such that} \quad (m, r) \mapsto mr$$

is continuous when $M \times R$ is endowed with the product topology of τ and σ.
Analogous definitions hold for left R-modules.

In what follows we generally work with discrete rings (that is, topological rings endowed with the discrete topology) and topological left or right modules over them. As for topological groups, given a topological ring R and topological right R-modules (M, τ) and (N, τ'), (M, τ) is Hausdorff if and only if $\{0\}$ is closed. Furthermore, a homomorphism of right R-modules $\phi : M \to N$ is continuous if and only if $\phi^{-1}(V)$ is a neighborhood of 0 in M for any neighborhood V of 0 in N. If $M'_R \leqslant M$ is a submodule, we consider on M'_R and on $(M/M')_R$ the group topologies induced by τ. When endowed with these topologies, M'_R and $(M/M')_R$ are topological modules.

Definition 3.54. *Let R be a discrete ring and let (M_R, τ) be a topological right R-module. Then (M, τ) is* linearly topologized *if there is a (pre-)base of neighborhoods of 0 consisting of open submodules.*

We denote by LT-R *the category of linearly topologized Hausdorff right R-modules and continuous homomorphisms of right R-modules. We denote by* $\mathrm{CHom}_R(M_1, M_2)$ *the group of all the continuous homomorphisms from M_1 to M_2.*

Let R be a discrete ring and let M_R be a right R-module. Notice that the discrete topology on M is linear because a pre-base for this topology is $\{0\}$. In the following example we describe two classes of modules whose only possible linear Hausdorff topology is the discrete one.

Example 3.55. *Let R be a discrete ring and let (M_R, τ) be a linearly topologized Hausdorff right R-module. If either M_R is Artinian or if it is uniform and it has simple socle, then τ is the discrete topology. Indeed, let $\mathcal{B}$ be a base of $\mathcal{V}(0)$ consisting of open submodules. Since M is Hausdorff, then $\overline{\{0\}} = \bigcap_{V \in \mathcal{B}} V = \{0\}$. If M_R is uniform and it has simple socle, the element 0 of the qframe $\mathcal{L}(M)$ is completely meet irreducible (see Lemma 2.33), and so $\{0\} \in \mathcal{B}$. On the other hand, if looking for a contradiction M is Artinian and $\{0\} \notin \mathcal{B}$, then choose arbitrarily $V_1 \in \mathcal{B}$ and, for all $n \in \mathbb{N}_+$, let $V_{n+1} \in \mathcal{B}$ be such that $V_n > V_{n+1}$ (it exists since $V_n \neq 0$, $\bigcap \mathcal{B} = \{0\}$ and $\mathcal{B}$ is closed under taking intersections). Then, $\{V_n\}_{n \in \mathbb{N}_+}$ is an infinite descending sequence of submodules, contradicting the Artinianity of M.*

3.2.2 (Strictly) Linearly compact modules

Definition 3.56. *Let R be a discrete ring and let (M_R, τ) be a linearly topologized Hausdorff right R-module. A* (open, closed) linear variety *is a subset of M of the form $x + N$ where $x \in M$ and N is a (open, closed) submodule. Then,*

- *(M, τ) is* linearly compact *if any family of open linear varieties with the finite intersection property has non-empty intersection;*
- *(M, τ) is* strictly linearly compact *if it is linearly compact and any surjective continuous homomorphism $\phi : M \to M'$, with $(M', \tau') \in \mathrm{Ob}(\mathrm{LT}\text{-}R)$, is open.*

We denote by SLC-R *the full subcategory of* LT-R *whose objects are the strictly linearly compact modules.*

Compare the above definition with the characterization of compactness given in Lemma 3.9. In particular, in a compact space any family of open neighborhoods of the points with the finite intersection property is supposed to have non-empty intersection. In a linearly compact module this is supposed to happen just for families of open linear varieties. The definition of strict linear compactness is justified by the parallel with the last part of Theorem 3.11.

Lemma 3.57. *Let R be a discrete ring, let (M_R, τ) be a linearly topologized Hausdorff right R-module, let n be a positive integer and let $x_i + V_i$ be an open variety for any $i = 1, \ldots, n$. If $\bigcap_{i=1}^n x_i + V_i \neq \varnothing$, then there exists $y \in M$ such that $\bigcap_{i=1}^n x_i + V_i = y + \bigcap_{i=1}^n V_i$.*

Proof. Let us prove the result for $n = 2$, the general case follows by induction. Choose arbitrarily $y = x_1 + v_1 = x_2 + v_2 \in (x_1 + V_1) \cap (x_2 + V_2)$ and let us show that $(x_1 + V_1) \cap (x_2 + V_2) = y + (V_1 \cap V_2)$. We show first that $(x_1 + V_1) \cap (x_2 + V_2) \subseteq y + (V_1 \cap V_2)$. Indeed, given $z = x_1 + w_1 = x_2 + w_2 \in (x_1 + V_1) \cap (x_2 + V_2)$, $w_1 - w_2 = x_2 - x_1 = v_1 - v_2$ and so, $v_1 - w_1 = v_2 - w_2 \in V_1 \cap V_2$. Thus $y = x_1 + v_1 = x_1 + w_1 - w_1 + v_1 = z + v_1 - w_1$, showing that $z = y - (v_1 - w_1) \in y + (V_1 \cap V_2)$. On the other hand, given $w \in V_1 \cap V_2$, $y + w = x_1 + (v_1 + w) = x_2 + (v_2 + w) \in (x_1 + V_1) \cap (x_2 + V_2)$. This show that $(x_1 + V_1) \cap (x_2 + V_2) \supseteq y + (V_1 \cap V_2)$. □

In the following lemma we work out the definition of strictly linearly compact module in the discrete case.

Lemma 3.58. [103, Theorem 28.14] *Let R be a discrete ring and let (M_R, τ) be a discrete right R-module. Then, M is Artinian if and only if it is strictly linearly compact.*

Recall that a topological Abelian group (M, τ) is *complete* if it is complete in the uniform structure on M, defined by saying that a subset of $M \times M$ is an entourage if and only if it contains the set $\{(x, y) : x - y \in U\}$ for some $U \in \mathcal{V}(0)$ (we refer to Section 5.3 for more details about uniform spaces and completeness in this context).
For a linearly topologized Hausdorff right R-module (M_R, τ) there is another characterization for completeness. Indeed, given a linear base $\mathcal{B}$ for τ, completeness of M is equivalent to affirm that there is a topological isomorphism (i.e., isomorphism of right R-modules which is also a homeomorphism)

$$M \to \varprojlim_{V \in \mathcal{B}} M/V$$

where the limit is endowed with the subset topology induced by the product of the discrete topologies in $\prod_{V \in \mathcal{B}} M/V$.

The proof of the following properties can be found in [103, Chapter VII].

Proposition 3.59. *Let R be a discrete ring and let (M_R, τ) be a linearly topologized Hausdorff right R-module. Then,*

(1) *M is (strictly) linearly compact if and only if both N and M/N are (strictly) linearly compact (with respect to the induced topologies), for any closed $N \leqslant M$.*

(2) *If M is the product of a family $\{(N_i, \tau_i) : i \in I\}$, then M is (strictly) linearly compact if and only if N_i is (strictly) linearly compact for all $i \in I$;*

(3) *M is (strictly) linearly compact if and only if M is complete and M/B_i is (strictly) linearly compact discrete, where $\mathcal{B} = \{B_i : i \in I\}$ is a linear base for M.*

If R is a field, by part (3) of the above proposition and Lemma 3.58, a linearly topologized Hausdorff R-vector space is linearly compact if and only if it is strictly linearly compact, if and only if it is complete and it has a base of neighborhoods consisting of vector subspaces of finite codimension.

We will need also the following fact, which can be found again in [103, Chapter VII]:

Lemma 3.60. *Let R be a discrete ring and let (M_1, τ_1), $(M_2, \tau_2) \in \mathrm{Ob}(\mathrm{LT}\text{-}R)$. If M_1 is (strictly) linearly compact and $\phi : M_1 \to M_2$ is a continuous morphism, then $\phi(M_1)$ is (strictly) linearly compact.*

3.2.3 The duality theorem

We start fixing the setting that we will keep all along this subsection.

(Dual.1) *R is a ring that is linearly compact as a right R-module endowed with the discrete topology;*

(Dual.2) *K_R is a minimal injective cogenerator, that is, K_R is the injective envelope of the coproduct of a family of representatives of the simple right R-modules. We assume K_R is Artinian;*

(Dual.3) *we denote by A the endomorphism ring of K_R.*

Example 3.61. *The above setting for duality occurs, for example, when R is a (skew) field or a commutative local complete Noetherian ring (see* [71]*).*

Notice that K has a natural left A-module structure induced by the following map:

$$A \times K \to K \quad \text{such that} \quad \alpha k = \alpha(k)\,,$$

for all $k \in K$ and $\alpha \in \mathrm{Hom}_R(K,K) = A$.

Lemma 3.62. [76, Lemma 4] *In the setting (Dual.1, 2, 3), the left A-module ${}_AK$ is an injective cogenerator of* A-Mod.

Let ${}_AN$ be a left A-module and define a linearly topologized right R-module N^* as follows. As an Abelian group $N^* = \mathrm{Hom}_A(N,K)$, the R-module structure is induced by

$$N^* \times R \to N^* \quad \text{such that} \quad (\phi r)(n) = (\phi(n))r\,,$$

for all $n \in N$, $\phi \in N^*$ and $r \in R$. Furthermore, we consider on N^* the so-called *finite topology*, that is, the unique linear topology that has a base of neighborhoods of 0 composed by the submodules of the form

$$\mathcal{W}(F) = \{f \in N^* : f(x) = 0\,,\ \forall x \in F\} \qquad \text{for a finite subset } F \subseteq N\,.$$

Lemma 3.63. *In the setting (Dual.1, 2, 3), let ${}_AN$ be a left A-module. Then, $(N^*)_R$ is a strictly linearly compact right R-module.*

Proof. First of all, notice that $(N^*)_R$ is Hausdorff since ${}_AK$ is a cogenerator by Lemma 3.62. Furthermore, let $F = \{f_1, \dots, f_n\}$ be a finite subset of N. It is not difficult to verify that the map

$$(N^*/\mathcal{W}(F))_R \to K^F \ \text{ such that } \ \nu + \mathcal{W}(F) \mapsto (\nu(f_1), \dots, \nu(f_n))\,,$$

is an injective homomorphism. Thus, $(N^*/\mathcal{W}(F))_R$ embeds in the Artinian module K^F, and it is therefore Artinian.

For any finite subset $F \subseteq N$, let $\Phi_F : N^* \to N^*/\mathcal{W}(F)$ be the natural projection. Let also $X_R = \prod_{F\subseteq N \text{ finite}} N^*/\mathcal{W}(F)$, let $\pi_F : X \to N^*/\mathcal{W}(F)$ be the natural projection and let $\Phi : N^* \to X$ be the unique morphism such that $\pi_F\Phi = \Phi_F$ for any finite subset $F \subseteq N$. Notice that $\mathrm{Ker}(\Phi) = \bigcap_F \mathrm{Ker}(\Phi_F) = \bigcap_F \mathcal{W}(F) = \{0\}$ since N^* is Hausdorff.

Endow X with the product topology, that is, a pre-base of neighborhoods of 0 is given by $\{\pi_F^{-1}(\{0\}) : F \subseteq N \text{ finite}\}$. By Proposition 3.59 (2), this topology makes X into a strictly linearly compact right R-module. Let us verify that

(a) Φ is continuous;

(b) $A \leqslant N^*$ is open if and only if $\Phi(A) = A' \cap \Phi(N^*)$ with $A' \leqslant X$ open;

(c) $\Phi(N^*)$ is closed in X;

then N^* is topologically isomorphic to the closed submodule $\Phi(N^*)$ of X and so it is strictly linearly compact by Proposition 3.59 (1).
(a) Use that $\Phi^{-1}(\pi_F^{-1}(\{0\})) = \Phi_F^{-1}(\{0\}) = \mathcal{W}(F)$ is open in N^*.
(b) Notice that $\Phi(\mathcal{W}(F)) = \Phi(\Phi^{-1}(\pi_F^{-1}(\{0\}))) = \Phi(N^*) \cap \pi_F^{-1}(\{0\})$. Given an open submodule $A \leqslant N^*$, there exists a finite subset $F \subseteq N$ such that $\mathcal{W}(F) \subseteq A$. It follows that $A' = \Phi(A) + \pi_F^{-1}(\{0\}) \leqslant X$ is open and that $\Phi(A) = \Phi(A) + \Phi(\mathcal{W}(F)) = \Phi(A) + (\pi_F^{-1}(\{0\}) \cap \Phi(N^*)) = A' \cap \Phi(N^*)$.
(c) Consider the closure

$$\overline{\Phi(N^*)} = \bigcap_{F\subseteq N \text{ finite}} \Phi(N^*) + \pi_F^{-1}(\{0\})\,,$$

so $f \in \overline{\Phi(N^*)}$ if and only if, for any finite $F \subseteq N$ there exists $g_F \in N^*$ such that $f - \Phi(g_F) \in \pi_F^{-1}(\{0\})$, that is, $\pi_F(f) = \Phi_F(g_F)$. Define an element $g \in N^*$ as follows

$$g : N \to K \quad \text{such that} \quad g(n) = g_{\{n\}}(n)\,.$$

One can show that g is a homomorphism of left A-modules and that $f = \Phi(g) \in \Phi(N^*)$. Thus, $\overline{\Phi(N^*)} = \Phi(N^*)$. □

Let ${}_AN$ and ${}_AN'$ be two left A-modules and let $\psi : N \to N'$ be a morphism of left A-modules. We define

$$\psi^* : (N')^* \to N^* \quad \text{such that} \quad \psi^*(g) = g \circ \psi\,,$$

for all $g \in (N')^*$.

Lemma 3.64. *In the setting (Dual.1, 2, 3), let ${}_AN$ and ${}_AN'$ be two left A-modules and let $\psi : N \to N'$ be a homomorphism of left A-modules. Then, $\psi^* : (N')^* \to N^*$ is continuous.*

Proof. Let F be a finite subset of N and denote by $\mathcal{W}_N(F)$ the basic neighborhood of 0 in N^* corresponding to F. Then,

$$(\psi^*)^{-1}(\mathcal{W}_N(F)) = \mathcal{W}_{N'}(\psi(F))\,,$$

where $\mathcal{W}_{N'}(\psi(F))$ denotes the basic neighborhood of 0 in $(N')^*$ corresponding to $\psi(F)$, and it is therefore open. □

Given a strictly linearly compact right R-module (M_R, τ), we let ${}_A(M^*)$ be a discrete left A-module such that $M^* = \mathrm{CHom}_R(M, K)$ as an Abelian group, and the action of A is defined by

$$A \times M^* \to M^* \quad \text{such that } (\alpha, \phi) \mapsto \alpha \circ \phi\,,$$

for all $\phi \in M^*$ and $\alpha \in A = \mathrm{End}_R(K)$. Furthermore, given another strictly linearly compact right R-module (M', τ') and a continuous homomorphism $\phi : M \to M'$ we define the following homomorphism of left A-modules:

$$\phi^* : (M')^* \to M^* \quad \text{such that} \quad (\phi^*(f))(x) = f(\phi(x))\,,$$

for all $x \in M$ and $f \in (M')^*$.

Notice that we have defined two functors

$$(-)^* : (\text{SLC-}R)^{op} \to A\text{-Mod} \quad \text{and} \quad (-)^* : (A\text{-Mod})^{op} \to \text{SLC-}R\,. \tag{3.2.1}$$

In the following theorem we verify that these functors are a duality.

Theorem 3.65. *Let R be a ring, let K_R be a minimal injective cogenerator and let $A = \mathrm{End}_R(K)$. Suppose that R is linearly compact discrete and that K_R is Artinian. Then, the above functors* (3.2.1) *define a duality between* A-Mod *and* SLC-R.

Proof. We define two natural isomorphisms

$$\omega : \mathrm{id}_{A\text{-Mod}} \Rightarrow (-)^{**} \quad \text{and} \quad \omega : \mathrm{id}_{\text{SLC-}R} \Rightarrow (-)^{**}\,.$$

Indeed, let $(M, \tau) \in \mathrm{Ob}(\text{SLC-}R)$, $N \in \mathrm{Ob}(A\text{-Mod})$ and define the evaluation maps

$$\begin{aligned} \omega_M : M &\to M^{**} \\ x &\mapsto \omega_M(x) \end{aligned} \qquad\qquad \begin{aligned} \omega_N : N &\to N^{**} \\ y &\mapsto \omega_N(y)\,, \end{aligned}$$

where $\omega_M(x)(f) = f(x)$ for all $f \in M^*$ and $\omega_N(y)(g) = g(y)$ for all $g \in N^*$. It is easily seen that these maps are homomorphisms of modules and that ω_M is continuous. Furthermore, it is not difficult to check that both ω_M and ω_N are injective, using the fact that K_R and ${}_AK$ are injective cogenerators in Mod-R and A-Mod respectively (see Lemma 3.62). Let us prove that

(a) ω_M is surjective;

(b) ω_N is surjective.

Notice that part (b) is sufficient to show that $\omega : \mathrm{id}_{A\text{-Mod}} \Rightarrow (-)^{**}$ is a natural isomorphism. Furthermore, if part (a) holds, then by definition of strictly linearly compact, ω_N is also open. Thus, ω_N is a topological isomorphism and so also $\omega : \mathrm{id}_{\text{SLC-}R} \Rightarrow (-)^{**}$ is a natural isomorphism. It remains to verify (a) and (b).

(b) Let $\nu : N^* \to K$ be a continuous homomorphism, that is, there exists a finite subset $F = \{f_1, \dots, f_n\} \subseteq N$ such that $\mathcal{W}(F) \subseteq \mathrm{Ker}(\nu)$. In particular, there is an induced morphism of discrete modules

$$\bar{\nu} : N^*/\mathcal{W}(F) \to K\,.$$

Notice that $\mathcal{W}(F) = \bigcap_{i=1}^n \mathcal{W}(\{f_i\})$ and that $N^*/\mathcal{W}(\{f_i\})$ embeds in K for all $i = 1, \dots, n$. Thus, there is an embedding $\epsilon : N^*/\mathcal{W}(F) \to K^n$ and $\bar{\nu}$ factors through it (by injectivity), that is, there exists a map $\tilde{\nu} : K^n \to K$ such that the following diagram commutes:

$$\begin{tikzcd} N^*/\mathcal{W}(F) \arrow[r, "\bar{\nu}"] \arrow[dr, "\epsilon"'] & K \\ & K^n \arrow[u, "\tilde{\nu}"'] \end{tikzcd}$$

By Lemma 1.36, $\mathrm{Hom}_R(K^n, K) = (\mathrm{End}_R(K))^n = A^n$. Thus, there exist $a_1, \dots, a_n \in A$ such that $\tilde{\nu}((k_i)_{i=1,\dots,n}) = \sum_{i=1}^n a_i k_i$, for all $(k_i)_{i=1,\dots,n} \in K^n$. This means that, given $x \in N^*$,

$$\begin{aligned} \nu(x) &= \bar{\nu}(x + \mathcal{W}(F)) = \tilde{\nu}\epsilon(x + \mathcal{W}(F)) \\ &= \sum_{i=1}^n a_i x(f_i) = x\left(\sum_{i=1}^n a_i f_i\right)\,. \end{aligned}$$

Thus, letting $y = \sum_{i=1}^n a_i f_i \in N$, $\nu = \omega_N(y)$.

(a) Suppose, looking for a contradiction, that $\omega_M : M \to M^{**}$ is not surjective. Then, there exists $\xi \in M^{**} \setminus \omega_M(M)$. Since K_R is a cogenerator, there exists a morphism $\phi : M^{**} \to K$ such that $\phi(\xi) \neq 0$ and $\phi(\omega_M(M)) = 0$. Now, $\phi \in M^{***}$ and by part (b) there is $f \in M^*$ such that $\omega_{M^*}(f) = \phi$. Notice that, for all $x \in M$

$$f(x) = \omega_M(x)(f) = \omega_{M^*}(f)(\omega_M(x)) = \phi(\omega_M(x)) = 0\,,$$

that is, $f = 0$. This implies that $\phi = 0$, which contradicts the fact that $\phi(\xi) \neq 0$. $\square$

The setting of the above theorem is a particular situation of a more general setting where a duality theorem can be proved, see [77]. Notice also that the results in [77] have been generalized in various directions (see for example the bibliography of [74]). On the other hand, the particular statement above is powerful enough for our needs and it has the advantage that it is not necessary to define “canonical choices” of topologies as in [77].

Remark 3.66. *Theorem 3.65 can be used to recover Sections 4 and 5 in* [37]. *In particular, the weak exactness of the duality functors described in* [37, Section 5] *can be improved to real exactness.*

Part II

A general scheme for entropies

Chapter 4

Entropy on semigroups

In this chapter we introduce the category of commutative pre-normed semigroups and its non-full subcategory of normed semigroups. After that, we define the categories of left and right representations of a given monoid Γ on a category. Finally, we introduce a notion of entropy for representations on commutative pre-normed semigroups and we study some of its basic properties, with particular emphasis on the case when Γ is an amenable group.

4.1 Entropy for pre-normed semigroups

4.1.1 Pre-normed semigroups and representations

Definition 4.1. *Let $S = (S, \cdot)$ be a semigroup. A* pre-norm *on S is a non-negative real-valued map $v : S \longrightarrow \mathbb{R}_{\geqslant 0}$. A* norm *on S is a sub-additive pre-norm v, that is, $v(x \cdot y) \leqslant v(x) + v(y)$ for all x, $y \in S$. If v is a pre-norm (resp., a norm) on S, we say that the pair (S, v) is a* pre-normed *(resp.,* normed*)* semigroup. *Furthermore, given two pre-normed semigroups (S, v) and (S', v'), a semigroup homomorphism $\phi : S \to S'$ is said to be* contractive *if $v'(\phi(x)) \leqslant v(x)$ for all $x \in S$.*

For example, consider a qframe $(L, \leqslant)$. Then, the operation $\vee : L \times L \to L$ makes $(L, \vee)$ into a semigroup. If we consider the function $\ell : L \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, such that $\ell(x) = \ell([0, x])$, then (L, ℓ) is a normed semigroup. Furthermore, any homomorphism of qframes $\phi : L \to L'$ induces a contractive homomorphism $(L, \ell) \to (L', \ell)$.

Definition 4.2. *Let $\underline{\mathrm{Semi}}_v$ be the* category of commutative pre-normed semigroups*, whose objects are all commutative pre-normed semigroups, and with morphisms all the semigroup homomorphisms. We denote by $\underline{\mathrm{Semi}}_v^*$ the non-full subcategory of $\underline{\mathrm{Semi}}_v$ whose objects are normed semigroups and where all the morphisms are supposed to be contractive.*

Definition 4.3. *Let $\mathfrak{C}$ be a category and let $(\Gamma, \cdot)$ be a monoid. The category $\mathrm{l.Rep}_\Gamma(\mathfrak{C})$ of* left Γ-representations *(resp.,* right Γ-representations*) on $\mathfrak{C}$ is the functor category $\mathrm{Func}(\mathfrak{C}_\Gamma, \mathfrak{C})$ (resp., $\mathrm{Func}(\mathfrak{C}_\Gamma^{op}, \mathfrak{C})$), where $\mathfrak{C}_\Gamma$ is the one-object category defined in Example 1.4.*

In the notation of the above definition,

– the objects of $\mathrm{l.Rep}_\Gamma(\mathfrak{C})$ are monoid homomorphisms $\alpha : \Gamma \to \mathrm{End}_\mathfrak{C}(M)$, for some $M \in \mathrm{Ob}(\mathfrak{C})$;

– a morphism $\phi : (\alpha : \Gamma \to \mathrm{End}_\mathfrak{C}(M)) \to (\alpha' : \Gamma \to \mathrm{End}_\mathfrak{C}(M'))$ is a morphism $\phi : M \to M'$ in $\mathfrak{C}$

such that the following squares commute for all $g \in \Gamma$:

$$\begin{tikzcd} M \arrow[r, "\phi"] \arrow[d, "\alpha(g)"'] & M' \arrow[d, "\alpha'(g)"] \\ M \arrow[r, "\phi"] & M' \end{tikzcd}$$

Thus, a left Γ-representation is essentially a dynamical system on the "space" M and "time" indexed by Γ. When we want to underline this dynamical point of view we use the notation $\alpha \circlearrowright M$ (or $(\alpha, \Gamma) \circlearrowright M$ if we want to specify Γ) for $\alpha : \Gamma \to \mathrm{End}_{\mathfrak{C}}(M)$. Similar observations hold for $\mathrm{r.Rep}_\Gamma(\mathfrak{C})$, using anti-homomorphisms of monoids.

If the image of a left Γ-representation α is contained in $\mathrm{Aut}_{\mathfrak{C}}(M)$, we say that α is *invertible*. Notice that if Γ is a group, any left Γ-representation is necessarily invertible.

Example 4.4. *Let* $(\mathbb{N}, +)$ *be the cyclic monoid of natural numbers, let* R *be a ring and consider the category* R-Mod *of left* R*-modules. A left* $\mathbb{N}$*-representation on* R-Mod *is exactly the same as a left* $R[X]$*-module. In fact, it is a classical point of view (see for example* [63, Section 12] *or* [9, Chapter 7]*) that of considering a left* $R[X]$*-module* ${}_{R[X]}M$ *as a left* R*-module* ${}_RM$ *with a distinguished* R*-linear endomorphism* $\phi : M \to M$*, which represents the action of* X*. So,* ${}_{R[X]}M$ *can be viewed as the left* $\mathbb{N}$*-representation* $\alpha_\phi \circlearrowright M$*, where* $\alpha_\phi(n) = \phi^n$ *for all* $n \in \mathbb{N}$*.*

The above example can be generalized as follows:

Example 4.5. *Let* $(\Gamma, \cdot)$ *be a monoid and let* R *be a ring. The* monoid ring $R[\Gamma]$ *is defined as follows. For all* $g \in \Gamma$ *we take a symbol* $\underline{g}$*, then the elements of* $R[\Gamma]$ *are formal sums of the form*

$$\sum_{g \in \Gamma} r_g \underline{g},$$

with $r_g = 0$ *for all but a finite number of indices. The sum of two elements is defined componentwise, that is,* $(\sum_{g\in\Gamma} r_g\underline{g}) + (\sum_{g\in G} s_g\underline{g}) = \sum_{g\in\Gamma}(r_g + s_g)\underline{g}$*, while multiplication is given by*

$$\left(\sum_{g\in\Gamma} r_g\underline{g}\right)\left(\sum_{g\in\Gamma} s_g\underline{g}\right) = \sum_{g\in\Gamma}\left(\sum_{hk=g} s_h r_k\right)\underline{g}\,.$$

Notice that in particular $\underline{g} \cdot \underline{h} = \underline{gh}$*. Clearly,* $R[\mathbb{N}] = R[X]$*. Analogously, the ring of* Laurent polynomials *is defined as* $R[\mathbb{Z}] = R[X^{\pm 1}]$*, one can find the notations* $R[X_1, \dots, X_k] = R[\mathbb{N}^k]$ *and* $R[X_1^{\pm 1}, \dots, X_k^{\pm 1}] = R[\mathbb{Z}^k]$*. As in the previous example one can show that the category* $\mathrm{l.Rep}_\Gamma(R\text{-Mod})$ *is equivalent to* $R[\Gamma]$-Mod*.*

4.1.2 Entropy of representations on pre-normed semigroups

Let Γ be a monoid, let (S, v) be a commutative pre-normed semigroup and let $(\alpha, \Gamma) \circlearrowright S$ be a left Γ-representation. Let also $\mathcal{F}(\Gamma)$ be the family of finite subsets of Γ. For all $F \in \mathcal{F}(\Gamma)$ and $x \in S$, the *F-th α-trajectory of x* is the following element of S:

$$T_F(\alpha, x) = \prod_{f\in F} \alpha(f)(x)\,. \tag{4.1.1}$$

Definition 4.6. *Let* $(I, \leqslant)$ *be a directed poset and let* X *be a set:*

– *a* net *with values in* X *is a function* $f : I \to X$*; we usually denote the net* f *by* $(x_i)_{i\in I}$*, where* $x_i = f(i) \in X$*;*

– *when* X *is endowed with a topology* τ*, we say that* $x \in X$ *is a* limit *for the net* $(x_i)_{i\in I}$*, in symbols* $\lim_{i\in I} x_i = x$ *if, for all* $V \in \mathcal{V}(x)$*, there exists* $i_V \in I$ *such that* $x_i \in V$ *for all* $i \geqslant i_V$*. If a net has a limit then we say that the net* converges*;*

– *if* $X = \mathbb{R}$ *is the real numbers, the* limit superior *(resp., the* limit inferior*) of the net* $(x_i)_{i\in I}$*, is*

$$\limsup_{i\in I} x_i = \inf\left\{\sup_{j\geqslant i} x_j : i \in I\right\} \quad \left(\textit{resp.}, \limsup_{i\in I} x_i = \sup\left\{\inf_{j\geqslant i} x_j : i \in I\right\}\right).$$

Notice that, if I is the set $\mathbb{N}$ of natural numbers with the usual order, then a net in X is just a sequence in X.
In what follows we will use nets of subsets of a given set S, this means that our nets take values in the collection of all subsets of S.

Let $(I, \leqslant)$ be a directed set and let $\underline{x} = (x_i)_{i\in I}$ and $\underline{y} = (y_i)_{i\in I}$ be two nets in a group $(G, \cdot)$. Then $\underline{x} \cdot \underline{y} = (x_i \cdot y_i)_{i\in I}$ is again a net in G.

In the following lemma we collect some well-known fact about nets, their proof is analogous to the usual proof for sequences and can be found in many standard texts. We recall that, given a poset $(I, \leqslant)$ a subset $S \subseteq I$ is *cofinal* if and only if for all $i \in I$, there exists $s \in S$ such that $i \leqslant s$.

Lemma 4.7. *Let* $(I, \leqslant)$ *be a directed set and let* $\underline{x} = (x_i)_{i\in I}$ *and* $\underline{y} = (y_i)_{i\in I}$ *be two nets in* $\mathbb{R}$*. Then,*

(1) $\limsup(\underline{x} + \underline{y}) \leqslant \limsup \underline{x} + \limsup \underline{y}$ *(resp.,* $\liminf(\underline{x} + \underline{y}) \geqslant \liminf \underline{x} + \liminf \underline{y}$*);*

(2) *if* $x_i \leqslant y_i$ *for all* $i \in I$*, then* $\limsup \underline{x} \leqslant \limsup \underline{y}$ *(resp.,* $\liminf \underline{x} \leqslant \liminf \underline{y}$*);*

(3) *if* $S \subseteq I$ *is a directed subset and* $\underline{x}_S = (x_i)_{i\in S}$*, then* $\limsup \underline{x}_S \leqslant \limsup \underline{x}$ *(resp.,* $\liminf \underline{x}_S \geqslant \liminf \underline{x}$*). Furthermore, if* S *is cofinal in* I*, then* $\limsup \underline{x}_S = \limsup \underline{x}$ *(resp.,* $\liminf \underline{x}_S = \liminf \underline{x}$*);*

(4) $\underline{x}$ *converges if and only if* $\limsup \underline{x} = \liminf \underline{x}$*. In this case,* $\limsup \underline{x} = \lim \underline{x}$*;*

(5) *if either* $\underline{x}$ *or* $\underline{y}$ *converges, then* $\limsup(\underline{x} + \underline{y}) = \limsup \underline{x} + \limsup \underline{y}$ *(resp.,* $\liminf(\underline{x} + \underline{y}) = \liminf \underline{x} + \liminf \underline{y}$*).*

With the notion of net, we can define the following notion of entropy.

Definition 4.8. *Let* Γ *be a monoid, let* (S, v) *be a commutative pre-normed semigroup and let* $(\alpha, \Gamma) \circlearrowleft S$ *be a left* Γ*-representation. Let* $(I, \leqslant)$ *be a directed set and let* $\mathfrak{s} = \{F_i\}_{i\in I}$ *be a net of non-empty finite subsets of* Γ*. The* $\mathfrak{s}$-entropy of α at x *is*

$$\mathfrak{h}(\alpha, \mathfrak{s}, x) = \limsup_{i\in I} \frac{v(T_{F_i}(\alpha, x))}{|F_i|},$$

while the $\mathfrak{s}$-entropy of α *is* $\mathfrak{h}(\alpha, \mathfrak{s}) = \sup\{\mathfrak{h}(\alpha, \mathfrak{s}, x) : x \in S\}$.

Let us fix a monoid Γ and a net $\mathfrak{s} = \{F_i\}_{i\in I}$ of non-empty finite subsets of Γ. We also add to the non-negative reals $\mathbb{R}_{\geqslant 0}$ the symbol ∞ in such a way that $x + \infty = \infty + x = \infty$, for all $x \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$. The entropy we have just defined can be seen as a numerical invariant on the category of representations, that is

$$\mathfrak{h}(-,\mathfrak{s}) : \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v) \longrightarrow \mathbb{R}_{\geqslant 0} \cup \{\infty\}\,, \quad (\alpha \circlearrowleft (S,v)) \mapsto \mathfrak{h}(\alpha,\mathfrak{s})\,. \tag{4.1.2}$$

Remark 4.9. *In Definition 4.8 we defined the $\mathfrak{s}$-entropy for a left Γ-representation α on a* commutative *pre-normed semigroups (S, v). The reason to assume the commutativity of S comes from the definition of the trajectories (see* (4.1.1)*): in fact, if S is not commutative, it is not clear how to interpret the product $T_F(\alpha, x) = \prod_{f\in F} \alpha(f)(x)$. On the other hand, if we have an order in the monoid Γ, then we can define the trajectories taking the products following that order. We will adopt this approach in Section 5.1.*

We are now going to discuss some basic properties of such invariant. In particular, we study monotonicity under taking certain subrepresentations and quotients (Lemma 4.10), and invariance under conjugation (Corollary 4.11).

Lemma 4.10. *Let Γ be a monoid, let $\mathfrak{s} = \{F_i\}_{i\in I}$ be a net of non-empty finite subsets of Γ and let $\phi : (\alpha \circlearrowleft (S, v)) \to (\alpha' \circlearrowleft (S', v'))$ be a morphism of left Γ-representations on commutative pre-normed semigroups. The following statements hold true:*

(1) *if $v(x) \leqslant v'(\phi(x))$ for all $x \in S$, then $\mathfrak{h}(\alpha,\mathfrak{s}) \leqslant \mathfrak{h}(\alpha',\mathfrak{s})$;*

(2) *if ϕ is surjective and $v(x) \geqslant v'(\phi(x))$ for all $x \in S$, then $\mathfrak{h}(\alpha',\mathfrak{s}) \leqslant \mathfrak{h}(\alpha',\mathfrak{s})$.*

Proof. (1) Let $x \in S$ and $\varnothing \neq F \in \mathcal{F}(\Gamma)$, then

$$v(T_F(\alpha, x)) \leqslant v'\left(\prod_{g\in F} \phi\alpha(g)(x)\right) = v'\left(\prod_{g\in F} \alpha'(g)\phi(x)\right) = v'(T_F(\alpha', \phi(x)))\,.$$

Using the above inequality for any $F_i \in \mathfrak{s}$, one obtains that $h_{\underline{\mathrm{Semi}}_v}(\alpha,\mathfrak{s}, x) \leqslant h_{\underline{\mathrm{Semi}}_v}(\alpha',\mathfrak{s}, \phi(x)) \leqslant \mathfrak{h}(\alpha',\mathfrak{s})$ for all $x \in S$. One concludes taking the supremum with respect to x.
(2) Let $y \in S'$ and let $x \in S$ be such that $\phi(x) = y$. Then,

$$v(T_F(\alpha, x)) = v'\left(\prod_{g\in F} \phi\alpha(g)(x)\right) = v'\left(\prod_{g\in F} \alpha'(g)(y)\right) = v'(T_F(\alpha', y))\,.$$

for all $\varnothing \neq F \in \mathcal{F}(\Gamma)$. Using the above equality for any $F_i \in \mathfrak{s}$, one obtains that $h_{\underline{\mathrm{Semi}}_v}(\alpha',\mathfrak{s}, y) = h_{\underline{\mathrm{Semi}}_v}(\alpha,\mathfrak{s}, x) \leqslant \mathfrak{h}(\alpha,\mathfrak{s})$ for all $y \in S'$. One concludes taking the supremum with respect to y. □

An easy consequence of the above lemma is the following invariance of entropy under conjugation.

Corollary 4.11. *Let Γ be a monoid, let $\mathfrak{s} = \{F_i\}_{i\in I}$ be a net of non-empty finite subsets of Γ and let $\phi : (\alpha \circlearrowleft (S, v)) \to (\alpha' \circlearrowleft (S', v'))$ be an isomorphism of left Γ-representations on commutative pre-normed semigroups such that $v(x) = v'(\phi(x))$ for all $x \in S$. Then, $\mathfrak{h}(\alpha,\mathfrak{s}) = \mathfrak{h}(\alpha',\mathfrak{s})$.*

In the following definition we isolate a technical condition that allows us to compare the entropies of different flows.

Definition 4.12. *Let* Γ *be a monoid, let* $\mathfrak{s} = \{F_i\}_{i\in I}$ *be a net of non-empty finite subsets of* Γ *and let* $\alpha \circlearrowleft (S, v)$, $\alpha' \circlearrowleft (S', v')$ *be two left* Γ*-representations on commutative pre-normed semigroups. We say that* α is $\mathfrak{s}$-dominated by α' *if for all* $x \in S$*, there exist a sequence* $(y_n)_{n\in\mathbb{N}}$ *of elements of* S' *and, for all* $i \in I$*, a sequence of non-negative reals* $\{k_i(n)\}_{n\in\mathbb{N}}$ *which verify the following conditions:*

$$\lim_{n\to\infty} k_i(n) = 0 \quad \textit{and} \quad v(T_{F_i}(\alpha, x)) \leqslant v'(T_{F_i}(\alpha', y_m)) + |F_i| \cdot k_i(m)\,,$$

for all $m \in \mathbb{N}$ *and* $i \in I$.

The following proposition shows that domination gives a criterion to verify that the entropy of a representation is less than or equal to the entropy of a second representation.

Proposition 4.13. *Let* Γ *be a monoid, let* $\mathfrak{s} = \{F_i\}_{i\in I}$ *be a net of non-empty finite subsets of* Γ *and let* $\alpha \circlearrowleft (S, v)$, $\alpha' \circlearrowleft (S', v')$ *be two left* Γ*-representations on commutative pre-normed semigroups. If* α *is* $\mathfrak{s}$*-dominated by* α'*, then* $\mathfrak{h}(\alpha, \mathfrak{s}) \leqslant \mathfrak{h}(\alpha', \mathfrak{s})$.

Proof. Let $x \in S$. Consider the sequence $(y_n)_{n\in\mathbb{N}}$ of elements of S' and, for all $i \in I$, the sequence of non-negative reals $\{k_i(n)\}_{n\in\mathbb{N}}$ given by Definition 4.12. Fix $\varepsilon > 0$ and, for all $i \in I$, choose $n_{(\varepsilon,i)} \in \mathbb{N}$ such that $k_i(n_{(\varepsilon,i)}) < \varepsilon$. We obtain that:

$$\begin{aligned}\mathfrak{h}(\alpha, \mathfrak{s}, x) &= \limsup_{i\in I} \frac{v(T_{F_i}(\alpha, x))}{|F_i|}\\ &\leqslant \limsup_{i\in I} \frac{v'(T_{F_i}(\alpha', y_{n_{(\varepsilon,i)}})) + |F_i| \cdot k_i(n_{(\varepsilon,i)})}{|F_i|} \leqslant \mathfrak{h}(\alpha', \mathfrak{s}) + \varepsilon\,.\end{aligned}$$

As this holds for all $x \in S$ and $\varepsilon \in \mathbb{R}_{>0}$, we obtain $\mathfrak{h}(\alpha, \mathfrak{s}) \leqslant \mathfrak{h}(\alpha', \mathfrak{s})$. □

We consider now the product of two commutative pre-normed semigroups (S_1, v_1) and (S_2, v_2). Indeed, let $S = S_1 \times S_2$ be their direct product in the category of semigroups, that is, the cartesian product with degree-wise operation. Then, S becomes a pre-normed semigroup with the max-pre-norm, given by

$$v(x_1, x_2) = \max\{v_1(x_1), v_2(x_2)\}\,.$$

When the product is endowed with such norm, one can prove the following formula for the $\mathfrak{s}$-entropy:

Lemma 4.14. *Let* Γ *be a monoid, let* $\mathfrak{s} = \{F_i\}_{i\in I}$ *be a net of non-empty finite subsets of* Γ *and let* $\alpha_1 \circlearrowleft (S_1, v_1)$, $\alpha_2 \circlearrowleft (S_2, v_2)$ *be two left* Γ*-representations on commutative pre-normed semigroups and let* (S, v) *be the product of* (S_1, v_1) *and* (S_2, v_2)*. Then, the left* Γ*-representation* $\alpha \circlearrowleft (S, v)$ *given by* $\alpha(g) = (\alpha_1(g), \alpha_2(g))$ *for all* $g \in \Gamma$ *has* $\mathfrak{s}$*-entropy*

$$\mathfrak{h}(\alpha, \mathfrak{s}) = \max\{\mathfrak{h}(\alpha_1, \mathfrak{s}), \mathfrak{h}(\alpha_2, \mathfrak{s})\}\,.$$

Proof. Given $x = (x_1, x_2) \in S$, by definition $v(T_F(\alpha, x)) = \max\{v_1(T_F(\alpha_1, x_1)), v_2(T_F(\alpha_2, x_2))\}$, for all $\varnothing \neq F \in \mathcal{F}(\Gamma)$ so,

$$\begin{aligned}\mathfrak{h}(\alpha, \mathfrak{s}, x) &= \inf\left\{\sup\left\{\max\left\{\frac{v_k(T_{F_j}(\alpha_k, x_k))}{|F_j|} : k = 1, 2\right\} j \geqslant i\right\} i \in I\right\}\\ &= \inf\left\{\max\left\{\sup\left\{\frac{v_k(T_{F_j}(\alpha_k, x_k))}{|F_j|} : j \geqslant i\right\} k = 1, 2\right\} i \in I\right\}\\ &= \max\left\{\inf\left\{\sup\left\{\frac{v_k(T_{F_j}(\alpha_k, x_k))}{|F_j|} : j \geqslant i\right\} i \in I\right\} k = 1, 2\right\}\\ &= \max\{\mathfrak{h}(\alpha_1, \mathfrak{s}, x_1), \mathfrak{h}(\alpha_2, \mathfrak{s}, x_2)\}\,.\end{aligned}$$

Taking suprema with $x_1 \in S_1$ and $x_2 \in S_2$ we obtain the result. $\square$

Let us consider also the coproduct of two commutative pre-normed semigroups (S_1, v_1) and (S_2, v_2). Indeed, we let $S = S_1 \oplus S_2$ be their coproduct in the category of semigroups, that coincides again with the cartesian product. S becomes a pre-normed semigroup with the following pre-norm

$$v_\oplus(x_1, x_2) = v_1(x_1) + v_2(x_2)\,.$$

When S is endowed with such norm, one can prove one inequality of a weak "addition formula" for the $\mathfrak{s}$-entropy. The converse inequality is also true in many concrete situations (see Lemma 4.40) but it cannot be proved in full generality (see for example Walters' book [100, p. 176]):

Lemma 4.15. *Let Γ be a monoid, let $\mathfrak{s} = \{F_i\}_{i\in I}$ be a net of non-empty finite subsets of Γ and let $\alpha_1 \circlearrowleft (S_1, v_1)$, $\alpha_2 \circlearrowleft (S_2, v_2)$ be two left Γ-representations on commutative pre-normed semigroups. Then, letting $S = S_1 \oplus S_2$, the left Γ-representation $\alpha \circlearrowleft (S, v_\oplus)$ given by $\alpha(g) = (\alpha_1(g), \alpha_2(g))$ for all $g \in \Gamma$ has $\mathfrak{s}$-entropy*

$$\mathfrak{h}(\alpha, \mathfrak{s}) \leqslant \mathfrak{h}(\alpha_1, \mathfrak{s}) + \mathfrak{h}(\alpha_2, \mathfrak{s})\,.$$

Proof. Let $x = (x_1, x_2) \in S$ and $i \in I$, then by definition $v(T_{F_i}(\alpha, x)) = v_1(T_{F_i}(\alpha_1, x_1)) + v_2(T_{F_i}(\alpha_2, x_2))$. Dividing by $|F_i|$ and taking the lim sup with i varying in I we obtain

$$\mathfrak{h}(\alpha, \mathfrak{s}, x) = \limsup_{i\in I}\left(\frac{v_1(T_{F_i}(\alpha_1, x_1)) + v_2(T_{F_i}(\alpha_2, x_2))}{|F_i|}\right) \leqslant \mathfrak{h}(\alpha_1, \mathfrak{s}, x_1) + \mathfrak{h}(\alpha_2, \mathfrak{s}, x_2)\,.$$

Taking the supremum with x_1 and x_2 varying in S_1 and S_2 respectively, the thesis follows. $\square$

4.1.3 Bernoulli representations

Let Γ be a group, let S be a commutative monoid and let v be a pre-norm on S such that $v(1) = 0$. For all $g \in \Gamma$ let $S_g = S$ and consider the monoid $M = \bigoplus_{g\in\Gamma} S_g$, which becomes a pre-normed monoid with the pre-norm

$$v_\oplus(x) = \sum_{g\in\Gamma} v(x_g) \text{ for any } x = (x_g)_{g\in\Gamma} \in M\,.$$

We can naturally define a map:

$$\mathfrak{B}_S : \Gamma \to \mathrm{Aut}(M)\,, \quad \mathfrak{B}_S(h)(x_g)_{g\in\Gamma} = (x_{h^{-1}g})_{g\in\Gamma}\,,$$

for all $h \in \Gamma$ and $x = (x_g)_{g\in\Gamma} \in M$. Notice that, the following relation holds

$$\mathfrak{B}_S(h_1)(\mathfrak{B}_S(h_2)(x)) = (x_{h_2^{-1}h_1^{-1}g})_{g\in\Gamma} = \mathfrak{B}_S(h_1h_2)(x) \quad\Rightarrow\quad \mathfrak{B}_S(h_1)\mathfrak{B}_S(h_2) = \mathfrak{B}_S(h_1h_2)\,,$$

so $\mathfrak{B}_S$ is a left Γ-representation. If one prefers to work with right representations, one can similarly define $\mathfrak{B}'_S(h)(x_g)_g = (x_{hg})_g$. We call $(\mathfrak{B}_S, \Gamma) \circlearrowleft (M, v_\oplus)$ the *Bernoulli left Γ-representation* over S.

Lemma 4.16. *Let (S, v) be a commutative pre-normed monoid, let Γ a group and let $\mathfrak{B}_S \circlearrowleft (M, v_\oplus)$ be the Bernoulli left Γ-representation. Then, for any net $\mathfrak{s} = \{F_i\}_{i\in I}$ of non-empty finite subsets of Γ:*

$$\mathfrak{h}(\mathfrak{B}_S, \mathfrak{s}) \geqslant \sup\{v(x) : x \in S\}\,.$$

Proof. Identify S with $S_1 \subseteq M$ and notice that

$$v(T_{F_i}(\mathfrak{B}_S, x)) = v\left(\prod_{f \in F_i} \mathfrak{B}_S(f)(x)\right) = |F_i| v(x)\,,$$

for all $x \in S$ and $i \in I$. Hence $\mathfrak{h}(\mathfrak{B}_S, \mathfrak{s}, x) = v(x)$, and so $\mathfrak{h}(\mathfrak{B}_S, \mathfrak{s}) \geqslant \sup\{\mathfrak{h}(\mathfrak{B}_S, \mathfrak{s}, x) : x \in S\} = \sup\{v(x) : x \in S\} = \sup\{v(x) : x \in S\}$. □

We are going to verify the converse inequality for a particular choice of the sequence $\mathfrak{s}$ in Lemma 4.41.

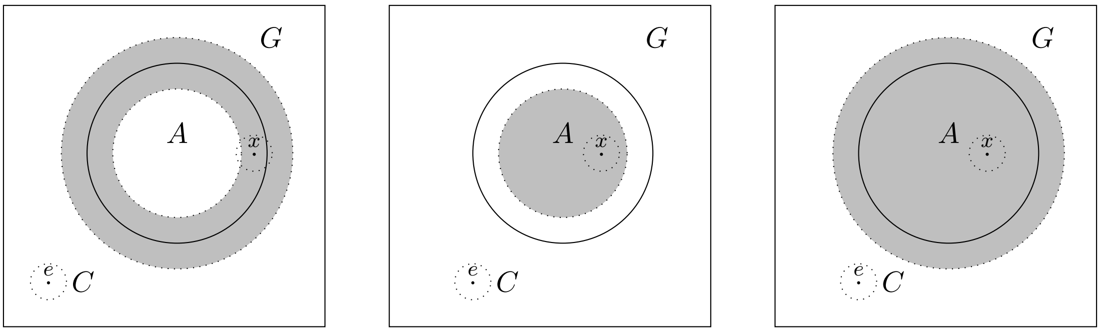
4.2 Representation of amenable groups

The original definition of amenability of a group G, in terms of a finitely additive invariant measure on the subsets of G, was introduced by von Neumann in 1929. We adopt here an equivalent definition of amenability (see Definition 4.19) introduced by Følner [43].

Definition 4.17. *Let G be a group and consider two subsets A, $C \subseteq G$, then*

- *the C-*interior *of A is $In_C(A) = \{x \in G : Cx \subseteq A\}$;*
- *the C-*exterior *of A is $Out_C(A) = \{x \in G : Cx \cap A \neq \emptyset\}$;*
- *the C-*boundary *of A is $\partial_C(A) = Out_C(A) \setminus In_C(A)$.*

If $e \in C$, one can imagine the above notions as in the following picture



$\partial_C(A)$ $In_C(A)$ $Out_C(A)$

The computations collected in the following lemma will be useful later on.

Lemma 4.18. *Let G be a group, let A, $C \subseteq G$ and $c \in G$. Then,*

(1) *$\partial_C(Ac) = \partial_C(A)c$ and $\partial_{Cc}(A) = c^{-1}\partial_C(A)$;*

(2) *if $A = \bigcup_{i \in I} A_i$ for some family $\{A_i : i \in I\}$ of subsets of G, then $\partial_C(A) \subseteq \bigcup_{i \in i} \partial_C(A_i)$;*

(3) *if $e \in C$, then $\partial_C(A) = (C^{-1}A \setminus A) \cup \bigcup_{c \in C} A \setminus c^{-1}A$.*

Proof. (1) Notice that

$$In_C(Ac) = \{x \in G : Cx \subseteq Ac\} = \{x \in G : Cxc^{-1} \subseteq A\} = \{xc \in G : Cx \subseteq A\} = In_C(A)c\,,$$

and that

$$\begin{aligned} Out_C(Ac) &= \{x \in G : Cx \cap Ac \neq \varnothing\} = \{x \in G : Cxc^{-1} \cap A \neq \varnothing\} \\ &= \{xc \in G : Cx \cap A \neq \varnothing\} = Out_C(A)c\,. \end{aligned}$$

Thus, $\partial_C(Ac) = Out_C(Ac) \backslash In_C(Ac) = Out_C(A)c \backslash In_C(A)c = (Out_C(A) \backslash In_C(A))c = \partial_C(A)c$. The proof of the second claim is analogous.

(2) Notice that

$$\begin{aligned} In_C(A) &= \{x \in G : Cx \subseteq A\} \supseteq \{x \in G : Cx \subseteq A_i \text{ for some } i \in I\} \\ &= \bigcup_{i \in I} \{x \in G : Cx \subseteq A_i\} = \bigcup_{i \in I} In_C(A_i)\,, \end{aligned}$$

and that

$$\begin{aligned} Out_C(A) &= \{x \in G : Cx \cap A \neq \varnothing\} = \{x \in G : Cx \cap A_i \neq \varnothing \text{ for some } i \in I\} \\ &= \bigcup_{i \in I} \{x \in G : Cx \cap A_i \neq \varnothing\} = \bigcup_{i \in I} Out_C(A_i)\,. \end{aligned}$$

Thus, $\partial_C(A) \subseteq \bigcup_{i \in I} Out_C(A_i) \backslash \bigcup_{i \in I} In_C(A) \subseteq \bigcup_{i \in I} \partial_C(A_i)$.

(3) Since $e \in C$, $In_C(A) \subseteq A \subseteq Out_C(A)$. Furthermore,

$$Out_C(A) \backslash A = \{x \in G \backslash A : \exists c \in C \text{ s.t. } cx \in A\} = \{x \in G \backslash A : \exists c \in C \text{ s.t. } x \in c^{-1}A\} = C^{-1}A \backslash A$$

and

$$A \backslash In_C(A) = \{x \in A : \exists c \in C \text{ s.t. } cx \notin A\} = \{x \in A : \exists c \in C \text{ s.t. } x \notin c^{-1}A\} = \bigcup_{c \in C} A \backslash c^{-1}A\,.$$

Thus, $\partial_C(A) = Out_C(A) \backslash In_C(A) = (Out_C(A) \backslash A) \cup (A \backslash In_C(A)) = (C^{-1}A \backslash A) \cup \bigcup_{c \in C} A \backslash c^{-1}A$. □

Definition 4.19. *A group G is* amenable *if and only if there exists a directed set $(I, \leqslant)$ and a net $\{F_i : i \in I\}$ of non-empty finite subsets of G such that, for any $C \in \mathcal{F}(G)$,*

$$\lim_I \frac{|\partial_C(F_i)|}{|F_i|} = 0\,. \tag{4.2.1}$$

Any such net is called a Følner net.

In the following lemma we collect some closure properties of the class of amenable groups.

Lemma 4.20. [16, Propositions 4.5.1, 4.5.4, 4.5.5 and 4.5.10] *Let G be a group and let $H \leqslant G$. Then,*

(1) *if G is amenable, then H is amenable;*

(2) *if H is normal, then G is amenable if and only if both H and G/H are amenable;*

(3) *if G is the directed colimit of a directed system of amenable groups, then G is amenable.*

Let X be a set and let A, $B \subseteq X$. The *symmetric difference* is the following subset of X:

$$A \Delta B = (A \backslash B) \cup (B \backslash A)$$

Lemma 4.21. *Let G be an amenable group and let $\mathfrak{s} = \{F_i\}_{i\in I}$ be a net of non-empty finite subsets of G. Then, the following are equivalent*

(1) $\lim_I |F_i \Delta gF_i|/|F_i| = 0$ *for all* $g \in G$*;*

(2) $\lim_I |F_i \Delta CF_i|/|F_i| = 0$ *for all non-empty* $C \in \mathcal{F}(G)$*;*

(3) $\mathfrak{s}$ *is a Følner net.*

Proof. (1)⇒(2). Let $C = \{g_1, \dots, g_n\} \in \mathcal{F}(G)$. Then, for all $i \in I$, $F_i \Delta CF_i \subseteq \bigcup_{k=1}^n (F_i \Delta g_k F_i)$. Thus, $\lim_I |F_i\Delta CF_i|/|F_i| \leqslant \sum_{k=1}^n \lim_I |F_i \Delta g_k F_i|/|F_i| = 0$.

(2)⇒(1) is trivial.

(2)⇒(3). First of all, notice that if $\lim_I |F_i \Delta KF_i|/|F_i| = 0$ for some $K \in \mathcal{F}(G)$, then in particular $\lim_I |F_i \backslash KF_i|/|F_i| = 0$ and $\lim_I |KF_i \backslash F_i|/|F_i| = 0$. Let $C \in \mathcal{F}(G)$, by Lemma 4.18 (1) we can suppose that $e \in C$. Furthermore, by Lemma 4.18 (3), $|\partial_C(F_i)| \leqslant |C^{-1}F_i \backslash F_i| \cup \sum_{c\in C} |F_i \backslash c^{-1}F_i|$ and so, $\lim_I |\partial_C(F_i)|/|F_i| \leqslant \lim_I |C^{-1}A\backslash A|/|F_i| \cup \sum_{c\in C} \lim_I |F_i \backslash c^{-1}F_i|/|F_i| = 0$.

(3)⇒(1). Let $g \in G$ and let $C = \{e, g^{-1}\}$. Then, $Out_C(F_i) = gF_i \cup F_i$ while $In_C(F_i) = gF_i \cap F_i$. Thus, $\partial_C(F_i) = F_i \Delta gF_i$ and so $\lim_i |F_i \Delta gF_i|/|F_i| = \lim_I |\partial_C(F_i)|/|F_i| = 0$. □

Lemma 4.22. *Let G be an amenable group, let $F \in \mathcal{F}(G)$ be non-empty and let $\{F_i\}_{i\in I}$ be a Følner net. Then, the following are Følner sequences:*

(1) $\{F_i F\}_{i\in I}$*;*

(2) $\{F_i \cup F\}_{i\in I}$*;*

(3) $\{F_i\}_{i\in J}$ *with $J \subseteq I$ a cofinal subset.*

Proof. (1) Let $C \in \mathcal{F}(G)$, then using Lemma 4.18

$$\lim_I \frac{|\partial_C(F_iF)|}{|F_iF|} \leqslant \lim_I \frac{|F||\partial_C(F_i)|}{|F_i|} = |F| \lim_I \frac{|\partial_C(F_i)|}{|F_i|} = 0\,,$$

proving (1).

(2) Let $C \in \mathcal{F}(G)$. Using again Lemma 4.18

$$\lim_I \frac{|\partial_C(F_i \cup F)|}{|F_i \cup F|} \leqslant \lim_I \frac{|\partial_C(F)| + |\partial_C(F_i)|}{|F_i|} = \lim_I \frac{|\partial_C(F)|}{|F_i|} + \lim_I \frac{|\partial_C(F_i)|}{|F_i|} = 0\,,$$

proving (2).

(3) is trivial. □

Corollary 4.23. *Let G be an infinite amenable group and let $\{F_i\}_{i\in I}$ be a Følner net. Then,* $\lim_{i\in I} |F_i| = \infty$.

Proof. We may suppose, without loss of generality, that $e \in F_i$ for all $i \in I$. Indeed, set $F_i' = F_i \cup \{e\}$, this gives a Følner net by Lemma 4.22 and clearly $\lim_{i\in I} |F_i| = \infty$ if and only if $\lim_{i\in I} |F_i'| = \infty$.

If there exists an element $g \in G$ and $i \in I$ such that $g \notin F_j$ for all $j \geqslant i$, then let $C = \{e, g^{-1}\}$ and notice that $g \in \partial_C(F_j)$ for all $j \geqslant i$. Thus,

$$0 = \lim_{i\in I} |\partial_C(F_j)|/|F_j| \geqslant \lim_{i\in I} 1/|F_j|\,.$$

This may happen only if $\lim_{i\in I}|F_i| = \infty$.

On the other hand, if for all $g \in G$ and for all $i \in I$ there exists $j \geqslant i$ such that $g \in F_j$, then choose an infinite family $\{g_n : n \in \mathbb{N}\}$ in G, let $S_n = \{i \in I : g_n \in F_i\} \subseteq I$ for all $n \in \mathbb{N}$ and let $\bar{S}_n = \bigcap_{k=0}^n S_k$. By construction, $\bar{S}_0$ is cofinal in I. If all the $\bar{S}_n$ are cofinal in I, in particular they are not empty and, given $s \in \bar{S}_n$, $|F_s| \geqslant |\{g_1, \dots, g_n\}| = n$ and so $\lim_{i\in I}|F_s| = \infty$. On the other hand, if there exists a minimum $n \in \mathbb{N}$ such that $\bar{S}_n$ is not cofinal in I, it means that $\bar{S}_{n-1}$ is cofinal in I and that there exists $s \in \bar{S}_{n-1}$ such that $g_n \notin F_t$ for all $t \geqslant s$. By the first part of the proof, $\lim_{i\in I}|F_i| = \lim_{i\in\bar{S}_{n-1}}|F_i| = \infty$. □

In the last part of the subsection we concentrate on countable amenable groups.

Remark 4.24. *If G is countable, then also $\mathcal{F}(G)$ is countable. Thus, given a net $\mathfrak{s} = \{F_i\}_{i\in I}$ in $\mathcal{F}(G)$, there is a countable subset J of I such that J is order-isomorphic to $\mathbb{N}$ and $\mathfrak{s}_J = \{F_i\}_{i\in J}$ is cofinal in $\mathfrak{s}$. By Lemma 4.22 (3), $\mathfrak{s}_J$ is Følner. This allows to always take $I = \mathbb{N}$ and just speak about* Følner sequences *in countable groups.*

Definition 4.25. *Given a group G and a Følner sequence $\mathfrak{s} = \{F_n : n \in \mathbb{N}\}$, we say that $\mathfrak{s}$ is a* Følner exhaustion *if*

- *$e \in F_0$ and $F_n \subseteq F_{n+1}$ for all $n \in \mathbb{N}$;*
- *$\bigcup_{n\in\mathbb{N}} F_n = G$.*

Example 4.26. *Every finite group is amenable. Furthermore, taking $G = \mathbb{Z}^k$, one can construct explicitly a Følner exhaustion $\{F_n : n \in \mathbb{N}\}$ as follows:*

$$F_n = \left\{\sum_{i=1}^k \lambda_i e_i : \lambda_i \in \mathbb{Z} \textit{ such that } \sum_{i=1}^k |\lambda_i| \leqslant n\right\}$$

where $\{e_i : i = 1, \dots, n\}$ are the canonical generators of G.

Lemma 4.27. *Let G be a countably infinite amenable group and let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence in G. Then there exists an increasing sequence $\{N(n)\}_{n\in\mathbb{N}}$ of natural numbers and a Følner exhaustion $\{S_n\}_{n\in\mathbb{N}}$ of G such that*

(1) *$F_{N(n)} \subseteq S_n$ for all $n \in \mathbb{N}$;*

(2) $\lim_{n\in\mathbb{N}} \frac{|F_{N(n)}|}{|S_n|} = 1$.

Proof. Since G is countable we can enumerate its elements, that is, $G = \{g_i : i \in \mathbb{N}\}$, we suppose that $g_0 = e$. For all $n \in \mathbb{N}$ let $A_n = \{g_0, \dots, g_n\}$, notice that $\{e\} \subseteq A_0 \subseteq \dots \subseteq A_n \subseteq \dots$ and $\bigcup_{n\in\mathbb{N}} A_n = G$. Let also $\{\varepsilon_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathbb{R}_{\geqslant 0}$ which converges to 0. Put $N(0) = 0$, $S_0 = F_0 \cup A_0$ and let $S_{n+1} = S_n \cup A_{n+1} \cup F_{N(n+1)}$, where $N(n+1) \geqslant N(n)$ is chosen inductively to satisfy

$$|\partial_{A_{n+1}}(S_{n+1})|/|S_{n+1}| < \varepsilon_{n+1} \quad \text{and} \quad \frac{|A_{n+1} \cup S_n|}{|F_{N(n+1)}|} < \varepsilon_{n+1}$$

this can be done since $\{F_i \cup (S_n \cup A_n)\}_{i\in\mathbb{N}}$ is a Følner sequence by Lemma 4.22 and $\lim_{n\to\infty}|F_n| = \infty$ by Corollary 4.23. Now, let $C \in \mathcal{F}(G)$, by construction there exists $\bar{n}$ such that $C \subseteq A_n$ for all $n \geqslant \bar{n}$, thus

$$\lim_{\mathbb{N}} \frac{|\partial_C(S_n)|}{|S_n|} \leqslant \lim_{n\geqslant\bar{n}} \frac{|\partial_{A_n}(S_n)|}{|S_n|} \leqslant \lim_{n\geqslant\bar{n}} \varepsilon_n = 0\,,$$

showing that $\{S_n\}_{n\in\mathbb{N}}$ is a Følner exhaustion. Furthermore, part (1) of the statement is clear, while part (2) can be shown as in the following computation:

$$\begin{aligned} 1 = \lim_{n\in\mathbb{N}} \frac{|F_{N(n)}|}{|F_{N(n)}|} &\geqslant \lim_{n\in\mathbb{N}} \frac{|F_{N(n)}|}{|S_n|} = \lim_{n\in\mathbb{N}} \frac{|F_{N(n)}|}{|S_n|} + \lim_{n\geqslant 1} \frac{|A_n \cup S_{n-1}|}{|F_{N(n)}|} \\ &= \lim_{n\geqslant 1} \frac{|F_{N(n)}| + |A_n \cup S_{n-1}|}{|S_n|} \geqslant \lim_{n\in\mathbb{N}} \frac{|S_n|}{|S_n|} = 1\,. \end{aligned}$$

□

4.2.1 Quasi-tilings

In what follows, we recall some terminology and results due to Ornstein and Weiss [81] (see also [62], [67] and [101]). All the proofs and statements of this subsection are adapted from the original papers with some slight changes to make them fit into our context.

Definition 4.28. *Let G be a group, let $A \in \mathcal{F}(G)$, let $\varepsilon \in (0,1)$, $\alpha \in (0,1]$ and $\delta \in [0,1)$, let I be a finite set and consider $\mathcal{A} = \{A_i\}_{i\in I} \subseteq \mathcal{F}(G)$. The family $\mathcal{A}$ is*

– ε-disjoint *if there is a family $\{B_i\}_{i\in I} \subseteq \mathcal{F}(G)$ such that*

- *$B_i \subseteq A_i$ and $|B_i| > (1-\varepsilon)|A_i|$, for all $i \in I$;*
- *$B_i \cap B_j = \varnothing$, whenever $i \neq j \in I$;*

– *a* α-cover *of A if $|A \cap (\bigcup_{i\in I} A_i)| \geqslant \alpha|A|$;*

– *a* δ-even cover *of A if*

- *$A_i \subseteq A$, for all $i \in I$;*
- *there exists $M > 0$ such that $\sum_{i\in I} \chi_{A_i}(x) \leqslant M$ for all $x \in A$, and $\sum_{i\in I} |A_i| \geqslant (1-\delta)|A|M$.*

Remark 4.29. *Let G be a group, let $A \in \mathcal{F}(G)$, let $\delta \in (0,1)$, let I be a finite set and consider $\mathcal{A} = \{A_i\}_{i\in I} \subseteq \mathcal{F}(G)$. If $\mathcal{A}$ is an δ-even cover, then $\mathcal{A}$ is an $(1-\delta)$-cover. In fact,*

$$\left| A \cap \left(\bigcup_{i\in I} A_i \right) \right| \overset{(*)}{=} \left| \bigcup_{i\in I} A_i \right| \overset{(**)}{\geqslant} \frac{1}{M} \sum_{i\in I} |A_i| \overset{\binom{*}{**}}{\geqslant} \frac{1}{M}(1-\delta)M|A| = (1-\delta)|A|\,,$$

where $(*)$ holds since $A_i \subseteq A$ for all $i \in I$, $(**)$ is a consequence of the fact that each $x \in A$ can belong to at most M elements of $\mathcal{A}$ and $\binom{*}{**}$ follows by the second condition in the definition of δ-even cover.

Definition 4.30. *Let G be a group, let $A \in \mathcal{F}(G)$, let $\varepsilon \in (0,1)$, let I be a finite set and consider $\mathcal{A} = \{A_i\}_{i\in I} \subseteq \mathcal{F}(G)$. The family $\mathcal{A}$ is an* ε-quasi-tiling *of A if there exists a family of* tiling centers *$\{C_i\}_{i\in I} \subseteq \mathcal{F}(G)$ such that*

– *$C_iA_i \subseteq A$ and $\{cA_i : c \in C_i\}$ forms an ε-disjoint family, for all $i \in I$;*

– *$C_iA_i \cap C_jA_j = \varnothing$, if $i \neq j \in I$;*

– *$\{C_iA_i : i \in I\}$ forms an $(1-\varepsilon)$-cover of A.*

It is a deep result, due to Ornstein and Weiss, that whenever G is a countable amenable group and $\{F_n\}_{n\in\mathbb{N}}$ is a Følner exhaustion, for any (small enough) $\varepsilon > 0$, one can find a nice family of subsets of G that ε-quasi-tiles F_n for all (big enough) $n \in \mathbb{N}$. In the rest of this subsection we are going to prove the following theorem.

Theorem 4.31. *Let G be a countably infinite amenable group, let $\{F_n\}_{n\in\mathbb{N}}$ and $\{F'_n\}_{n\in\mathbb{N}}$ be respectively a Følner sequence and a Følner exhaustion for G. Then, for all $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$, there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $\bar{n} \leqslant n_1 \leqslant \cdots \leqslant n_k$ and $\{F_{n_1}, \dots, F_{n_k}\}$ ε-quasi-tiles F'_m, for all big enough m.*

Before proceeding to the proof, we need a series of technical lemmas.

Lemma 4.32. *Let G be a group, let C and $A \in \mathcal{F}(G)$, let $\varepsilon \in (0,1)$ and let $c \in G$. If $|\partial_C(A)| < \varepsilon|A|$, then $|\partial_C(Ac)| < \varepsilon|Ac|$.*

Proof. By Lemma 4.18, $\partial_C(Ac) = \partial_C(A)c$ and so $|\partial_C(Ac)| = |\partial_C(A)|$. Similarly, $|A| = |Ac|$. □

Lemma 4.33. *Let G be a group, let I be a finite set, let $\mathcal{A} = \{A_i\}_{i\in I} \subseteq \mathcal{F}(G)$, let $A = \bigcup \mathcal{A}$, let $C \in \mathcal{F}(G)$ and let $\varepsilon, \delta \in (0,1)$.*

(1) *If $\mathcal{A}$ is $\varepsilon/2$-disjoint, then $\sum_{i\in I} |A_i| \leqslant (1+\varepsilon)|A|$;*

(2) *If $\mathcal{A}$ is $\varepsilon/2$-disjoint and $|\partial_C(A_i)| \leqslant \delta|A_i|$ for all $i \in I$, then $\partial_C(A) \leqslant \delta(1+\varepsilon)|A|$.*

Proof. (1) By definition of $\varepsilon/2$-disjointedness, there exists a family $\{B_i\}_{i\in I} \subseteq \mathcal{F}(G)$ such that $B_i \subseteq A_i$, $|B_i| > (1-\varepsilon/2)|A_i|$ and $B_i \cap B_j = \varnothing$, for all $i \neq j \in I$. Then,

$$|A| \geqslant \left|\bigcup_{i\in I} B_i\right| = \sum_{i\in I} |B_i| \geqslant (1-\varepsilon/2)\sum_{i\in I} |A_i|\,.$$

One concludes noticing that $(1-\varepsilon/2)^{-1} \leqslant 1+\varepsilon$.
(2) By Lemma 4.18, $\partial_C(A) \subseteq \bigcup_{i\in i} \partial_C(A_i)$, so

$$|\partial_C(A)| \leqslant \left|\bigcup_{i\in I} \partial_C(A_i)\right| \leqslant \sum_{i\in I} |\partial_C(A_i)| \overset{(*)}{\leqslant} \delta \sum_{i\in I} |A_i| \overset{(**)}{\leqslant} \delta(1+\varepsilon)|A|\,,$$

where $(*)$ holds since $|\partial_C(A_i)| \leqslant \delta|A_i|$ for all $i \in I$, while $(**)$ is true by part (1). □

Lemma 4.34. *Let G be a group, let A and S be two finite subsets of G containing e, let $H = In_{SS^{-1}}(A)$ and let $\varepsilon \in (0,1)$. If $|\partial_{SS^{-1}}(A)| \leqslant (1-\varepsilon)|A|$, then $\{Sg\}_{g\in H}$ is an ε-even cover of A.*

Proof. Let $M = |S|$, we have to prove that $\sum_{g\in H} \chi_{Sg}(x) \leqslant M$, for all $x \in A$ (which is true no element of A can belong to more that $|S|$ translates of S) and that $\sum_{g\in H} |Sg| \geqslant \varepsilon|A|M$. The fact that $|\partial_{SS^{-1}}(A)| \leqslant (1-\varepsilon)|A|$ implies that $|H| \geqslant \varepsilon|A|$ and so $\sum_{g\in H} |Sg| \geqslant \varepsilon|A|M$. □

Lemma 4.35. *Let G be a group, let $A \in \mathcal{F}(G)$ and let ε, $\delta \in (0,1)$. If $\mathcal{A}$ is a δ-even cover of A, then there is a subset $\mathcal{B} \subseteq \mathcal{A}$ such that*

(1) *$\mathcal{B}$ is ε-disjoint;*

(2) *it $\varepsilon(1-\delta)$-covers A;*

(3) *given* $B \in \mathcal{B}$, $|\bigcup \mathcal{B} \backslash B| < \varepsilon(1-\delta)|A|$.

Furthermore, if (1), (2) and (3) hold for some set $\mathcal{B}$*, then for all* $B \in \mathcal{B}$*,*

$$1 - \varepsilon(1-\delta) \geqslant \frac{|A \backslash \bigcup \mathcal{B}|}{|A|} > 1 - \varepsilon(1-\delta) - |B|/|A|\,.$$

Proof. Let $\mathcal{B}$ be a maximal ε-disjoint subfamily of $\mathcal{A}$. Suppose, looking for a contradiction, that $\mathcal{B}$ is not an $\varepsilon(1-\delta)$-cover of A (that is, $|\bigcup \mathcal{B}| < \varepsilon(1-\delta)|A|$) and consider the following claim:

$$(*) \quad \text{there exists } \bar{B} \in \mathcal{A} \text{ such that } |\bar{B} \cap \bigcup \mathcal{B}| < \varepsilon|\bar{B}|.$$

Let us verify $(*)$. Assume, looking for a contradiction, that $(*)$ does not hold and consider the following observations:

a) by our initial absurd hypothesis, $|\bigcup \mathcal{B}| < \varepsilon(1-\delta)|A|$;

b) by the negation of $(*)$, $\sum_{B \in \mathcal{A}} |B \cap \bigcup \mathcal{B}| \geqslant \varepsilon \sum_{B \in \mathcal{A}} |B|$;

c) by the definition of δ-even cover, there exists $M > 0$ such that $\sum_{B \in \mathcal{A}} |B| \geqslant (1-\delta)|A|M$;

d) by the definition of δ-even cover, given $x \in \bigcup \mathcal{B} (\subseteq A)$, there exist at most M different members $B \in \mathcal{A}$ such that $x \in B$. Thus, $\sum_{B \in \mathcal{A}} |B \cap \bigcup \mathcal{B}| \leqslant M|\bigcup \mathcal{B}|$.

Combining b) and c), $\sum_{B \in \mathcal{A}} |B \cap \bigcup \mathcal{B}| \geqslant \varepsilon(1-\delta)|A|M$, while, combining a) and d), $\sum_{B \in \mathcal{A}} |B \cap \bigcup \mathcal{B}| < \varepsilon(1-\delta)|A|M$, that is a contradiction. Thus, $(*)$ is verified. Now, consider $\bar{B} \in \mathcal{A}$ such that $|\bar{B} \cap \bigcup \mathcal{B}| < \varepsilon|\bar{B}|$ and notice that $\bar{B} \notin \mathcal{B}$ and that $\mathcal{B} \cup \{\bar{B}\}$ is still ε-disjoint. This contradicts the maximality of $\mathcal{B}$.
Thus, $\mathcal{B}$ satisfies properties (1) and (2) in the statement. Take now $B \in \mathcal{B}$ and notice that, if $|\bigcup \mathcal{B} \backslash B| \geqslant \varepsilon(1-\delta)|A|$, then $\mathcal{B}' = \mathcal{B} \backslash \{B\}$ is again a subset of $\mathcal{A}$ satisfying (1) and (2). Thus, removing a finite number of elements from $\mathcal{B}$, we can find a subset of $\mathcal{A}$ that satisfies (1), (2) and (3).
For the last part of the statement, notice that the bound $|A \backslash \bigcup \mathcal{B}|/|A| \leqslant 1 - \varepsilon(1-\delta)$ comes directly from condition (2), while the other bound can be computed as follows. Take $B \in \mathcal{B}$, then

$$\frac{|A \backslash \bigcup \mathcal{B}|}{|A|} = 1 - \frac{|\bigcup \mathcal{B} \backslash B| + |B|}{|A|} > 1 - \varepsilon(1-\delta) - \frac{|B|}{|A|}$$

□

We can finally prove the main result of this section:

Proof of Theorem 4.31. Fix $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$, and choose $k \in \mathbb{N}_+$ and $\delta > 0$ such that

$$\left(1 - \frac{\varepsilon}{2}\right)^k < \varepsilon \quad \text{and} \quad 6^k \delta < \frac{\varepsilon}{2}\,.$$

Notice that the second condition, together with the choice of ε, implies $\delta < 1/48$. Since $\{F_n\}_{n \in \mathbb{N}}$ is a Følner exhaustion, we can choose $\bar{n} \leqslant n_1 \leqslant n_2 \leqslant \ldots \leqslant n_k \in \mathbb{N}$ such that

$$\frac{|\partial_{F_{n_i} F_{n_i}^{-1}}(F_{n_{i+1}})|}{|F_{n_{i+1}}|} < \delta \quad \text{and} \quad \frac{|F_{n_i}|}{|F_{n_{i+1}}|} < \delta \tag{4.2.2}$$

Furthermore, since $\{F'_n\}_{n\in\mathbb{N}}$ is a Følner sequence, for any big enough $m \in \mathbb{N}$,

$$\frac{|\partial_{F_{n_k}F_{n_k}^{-1}}(F'_m)|}{|F'_m|} < \delta \quad \text{and} \quad \frac{|F_{n_k}|}{|F'_m|} < \delta\,. \tag{4.2.3}$$

Fix a positive integer m as above; we construct by downward induction a family $\{C_1, \dots, C_k\}$ with the following properties: letting $A_k = F'_m$ and $A_{j-1} = A_j \backslash F_{n_j}C_j$ for all $j = 2, \dots, k$, and letting $K_i = F_{n_i}F_{n_i}^{-1}$ for all $i = 1, \dots, k$,

(a_i) $\{F_{n_i}c : c \in C_i\}$ is ε-disjoint;

(b_i) $\{F_{n_i}c : c \in C_i\}$ is an $\varepsilon(1-6^i\delta)$-cover of A_i;

(c_i) $1-\varepsilon(1-\delta) \geqslant |A_i \backslash F_{n_i}C_i|/|A_i| > 1-\varepsilon(1-\delta)-\delta$, $|\partial_{K_i}(A_i)|/|A_i| < \delta$ and $|F_{n_i}|/|A_i| < \delta$.

Notice that, once we found such families, $F_{n_i}C_i = A_i \backslash A_{i-1}$ and so $F_{n_i}C_i \cap F_{n_j}C_j = \varnothing$ if $i \neq j$. Furthermore,

$$\begin{aligned}\frac{|A_k \backslash \bigcup F_{n_i}C_i|}{|A_k|} &= \frac{|A_1 \backslash F_{n_1}C_1|}{|A_k|} = \frac{|A_1 \backslash F_{n_1}C_1|}{|A_1|}\frac{|A_1|}{|A_2|}\cdots\frac{|A_{k-1}|}{|A_k|}\\ &\leqslant (1-\varepsilon(1-6^k\delta))(1-\varepsilon(1-6^{k-1}\delta))\dots(1-\varepsilon(1-\delta)) < (1-\varepsilon/2)^k < \varepsilon\,.\end{aligned}$$

Thus, $\{F_{n_1}, \dots, F_{n_k}\}$ ε-quasi-tiles $A_k = F'_m$ with tiling centers $\{C_1, \dots, C_k\}$ and the proof is concluded.

<u>*Case $i = k$*</u>. Let $A_k = F'_m$ and let $K_k = F_{n_k}F_{n_k}^{-1}$; with this notation, (4.2.3) reads as

$$\frac{|\partial_{K_k}(A_k)|}{|A_k|} < \delta \quad \text{and} \quad \frac{|F_{n_k}|}{|A_k|} < \delta\,,$$

notice that these are the two last conditions in (c_k). Consider $H_k = In_{K_k}(A_k)$, then $\{F_{n_k}h\}_{h\in H_k}$ is a δ-even cover of A_k, by Lemma 4.34. Thus, by Lemma 4.35, there exists $C_k \subseteq H_k$ that satisfies (a_k), (b_k) and (c_k).

<u>*Case $i < k$*</u>. Suppose we have already constructed $C_k, \dots, C_{i+1}$ with the desired properties. Then,

$$\begin{aligned}\frac{|\partial_{K_i}(A_i)|}{|A_i|} &\leqslant \frac{|\partial_{K_{i+1}}(A_{i+1})|}{|A_i|} + |C_{i+1}|\frac{\partial_{K_i}(F_{n_{i+1}})}{|A_i|}\\ &< \frac{\delta}{|A_{i+1}\backslash F_{n_{i+1}}C_{i+1}|}(|A_{i+1}| + |F_{n_{i+1}}||C_{i+1}|) && (\mathrm{c}_{i+1}) \text{ and } (4.2.2)\\ &< \frac{\delta|A_{i+1}|}{|A_{i+1}\backslash F_{n_{i+1}}C_{i+1}|}\left(1+\frac{1}{1-\varepsilon}\right) && (\mathrm{a}_{i+1})\\ &< \frac{\delta}{1-\varepsilon(1-\delta)-\delta}\left(1+\frac{1}{1-\varepsilon}\right) && (\mathrm{c}_{i+1})\\ &< 6\delta && \varepsilon < 1/4 \text{ and } \delta < 1/48,\end{aligned}$$

where the first inequality follows since $\partial_{K_i}(A_i) \subseteq \partial_{K_{i+1}}(A_{i+1}) \cup \bigcup_{c\in C_{i+1}} \partial_{K_i}(F_{n_{i+1}})c$ (use Lemma 4.18). Furthermore, using again (4.2.2), (c_{i+1}) and the initial bounds for ε and δ,

$$\frac{|F_{n_i}|}{|A_i|} = \frac{|F_{n_i}|}{|F_{n_{i+1}}|}\frac{|F_{n_{i+1}}|}{|A_{i+1}|}\frac{|A_{i+1}|}{|A_i|} < \frac{\delta^2}{1-\varepsilon(1-\delta)-\delta} < \delta\,.$$

This shows the second part of (c_i). Now, if we let $H_i = In_{K_i}(A_i)$, then $\{F_{n_i}h\}_{h\in H_i}$ is a 6δ-even cover of A_i, by Lemma 4.34. Thus, by Lemma 4.35, there exists $C_i \subseteq H_i$ that satisfies (a_i), (b_i) and (c_i). □

4.2.2 Non-negative real functions on finite subsets of an amenable group

In this subsection we recall some results and terminology about non-negative invariants for the finite subsets of G, that is, functions $f : \mathcal{F}(G) \to \mathbb{R}_{\geqslant 0}$.

Definition 4.36. *Let G be a group and let $f : \mathcal{F}(G) \to \mathbb{R}_{\geqslant 0}$. We say that f is*

– monotone *if $f(A) \leqslant f(A')$, for all $A \subseteq A' \in \mathcal{F}(G)$;*

– sub-additive *if $f(A \cup A') \leqslant f(A) + f(A')$, for all A, $A' \in \mathcal{F}(G)$;*

– (left) G-equivariant *if $f(gA) = f(A)$, for all $A \in \mathcal{F}(G)$ and $g \in G$.*

Notice that, given a group G and a G-equivariant function $f : \mathcal{F}(G) \to \mathbb{R}_{\geqslant 0}$, $f(\{g\}) = f(\{e\})$, for all $g \in G$. Thus, if f is also sub-additive, then $f(A) \leqslant \sum_{g\in A} f(\{g\}) = |A|f(\{e\})$, for all $A \in \mathcal{F}(G)$. Notice that a consequence of this is that, given a net $\mathfrak{s} = \{F_i\}_{i\in I}$ of non-empty finite subsets of G, then

$$\limsup_{i\in I} \frac{f(F_i)}{|F_i|} \leqslant \limsup_{i\in I} \frac{|F_i|f(\{e\})}{|F_i|} = f(\{e\}) .$$

In particular, any such limit superior is finite.

The following corollary is a consequence of Theorem 4.31. It is important to underline that the choice of the $n_1, \dots, n_k$ in the statement does not depend on the function f, but we can really find a family $\{n_1, \dots, n_k\}$, which works for all f at the same time.

Corollary 4.37. *Let G be a countably infinite amenable group and let $\{F_n\}_{n\in\mathbb{N}}$, $\{F'_n\}_{n\in\mathbb{N}}$ be respectively a Følner exhaustion and a Følner sequence in G. Then, for any $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$ there exist integers $n_1, \dots, n_k$ such that $\bar{n} \leqslant n_1 \leqslant \dots \leqslant n_k$ and, for any sub-additive and G-equivariant $f : \mathcal{F}(G) \to \mathbb{R}$*

$$\limsup_{n\to\infty} \frac{f(F'_n)}{|F'_n|} \leqslant M\varepsilon + \frac{1}{1-\varepsilon} \max_{1\leqslant i\leqslant k} \frac{f(F_{n_i})}{|F_{n_i}|} ,$$

where $M = f(\{e\})$.

Proof. Let $\varepsilon \in (0, 1/4)$ and $\bar{n} \in \mathbb{N}$. By Theorem 4.31, there exist positive integers $n_1, \dots, n_k$ such that $\bar{n} \leqslant n_1 \leqslant \dots \leqslant n_k$ and $\{F_{n_1}, \dots, F_{n_k}\}$ ε-quasi-tiles F'_n, for all big enough $n \in \mathbb{N}$. We let $C^n_1, \dots, C^n_k$ be the tiling centers for F'_n. Thus, when n is big enough,

$$F'_n \supseteq \bigcup_{i=1}^k C^n_i F_{n_i} \quad \text{and} \quad \left|\bigcup_{i=1}^k C^n_i F_{n_i}\right| \geqslant \max\left\{(1-\varepsilon)|F'_n| \ , \ (1-\varepsilon)\sum_{i=1}^k |C^n_i||F_{n_i}|\right\} .$$

Letting $f : \mathcal{F}(G) \to \mathbb{R}_{\geqslant 0}$ be a sub-additive and G-invariant function,

$$\begin{aligned}\frac{f(F'_n)}{|F'_n|} &\leqslant \frac{f\left(F'_n\backslash\bigcup_{i=1}^k C^n_i F_{n_i}\right)}{|F_n|} + \frac{f\left(\bigcup_{i=1}^k C^n_i F_{n_i}\right)}{|F'_n|} \leqslant M\frac{\left|F'_n\backslash\bigcup_{i=1}^k C^n_i F_{n_i}\right|}{|F'_n|} + \frac{f\left(\bigcup_{i=1}^k C^n_i F_{n_i}\right)}{\left|\bigcup_{i=1}^k C^n_i F_{n_i}\right|}\\ &\leqslant M\varepsilon + \frac{\sum_{i=1}^k |C^n_i| f(F_{n_i})}{(1-\varepsilon)\sum_{i=1}^k |C^n_i||F_{n_i}|} \leqslant M\varepsilon + \frac{1}{1-\varepsilon}\max_{1\leqslant i\leqslant k}\frac{f(F_{n_i})}{|F_{n_i}|} ,\end{aligned}$$

where $M = f(\{e\})$, as desired. □

The following result, generally known as "Ornstein-Weiss Lemma", is proved in [81] for a suitable class of locally compact amenable groups (a direct proof, along the same lines, in the discrete case can be found in [101], while a nice alternative argument, based on ideas of Gromov, is given in [67]).

Proposition 4.38. *Let G be a countably infinite amenable group and consider a monotone, sub-additive and G-equivariant function $f:\mathcal{F}(G)\to\mathbb{R}_{\geqslant 0}$. Then, for any Følner sequence $\{F_n\}_{n\in\mathbb{N}}$, the sequence $(f(F_n)/|F_n|)_{n\in\mathbb{N}}$ converges and the value of the limit $\lim_{n\in\mathbb{N}} f(F_n)/|F_n|$ is the same for any choice of the Følner sequence.*

Proof. Let $f:\mathcal{F}(G)\to\mathbb{R}_{\geqslant 0}$ be a monotone, sub-additive and G-equivariant function, and let $M=f(\{e\})$. Choose also a Følner exhaustion $\{F_n\}_{n\in\mathbb{N}}$ and a Følner sequence $\{F'_n\}_{n\in\mathbb{N}}$ in G. By Corollary 4.37, for all $\varepsilon\in(0,1/4)$ and $\bar n\in\mathbb{N}$ there exist $k\in\mathbb{N}$ and $\bar n\leqslant n_1<\cdots<n_k\in\mathbb{N}$ such that

$$\limsup_{n\in\mathbb{N}}\frac{f(F'_n)}{|F'_n|}\leqslant M\varepsilon+\frac{1}{1-\varepsilon}\max_{1\leqslant i\leqslant k}\frac{f(F_{n_i})}{|F_{n_i}|}\leqslant M\varepsilon+\frac{1}{1-\varepsilon}\sup_{\bar n\leqslant n}\frac{f(F_{n_i})}{|F_{n_i}|}$$

Since this holds for all $\varepsilon\in(0,1/4)$ and $\bar n\in\mathbb{N}$, we get $\limsup_{n\in\mathbb{N}}\frac{f(F'_n)}{|F'_n|}\leqslant\limsup_{n\in\mathbb{N}}\frac{f(F_n)}{|F_n|}$. Let now $\{N(n)\}$ be an increasing sequence of natural numbers such that $\lim_{n\in\mathbb{N}} f(F_{N(n)})/|F_{N(n)}|=\liminf_{n\in\mathbb{N}} f(F_n)/|F_n|$. Then, by the first part of the proof,

$$\limsup_{n\in\mathbb{N}} f(F_n)/|F_n|\leqslant\lim_{n\in\mathbb{N}} f(F_{N(n)})/|F_{N(n)}|=\liminf_{n\in\mathbb{N}} f(F_n)/|F_n|\,,$$

thus $\{f(F_n)/|F_n|\}$ converges to a limit λ. Using the same kind of argument one shows that, for any Følner exhaustion $\{S_n\}_{n\in\mathbb{N}}$, $\lim_{n\in\mathbb{N}} f(S_n)/|S_n|=\lambda$. Using Lemma 4.27, choose an increasing sequence $\{N(n)\}_{n\in\mathbb{N}}$ of natural numbers and a Følner exhaustion $\{S_n\}_{n\in\mathbb{N}}$ of G such that $F'_{N(n)}\subseteq S_n$ for all $n\in\mathbb{N}$ and $\lim_{n\in\mathbb{N}}\frac{|F'_{N(n)}|}{|S_n|}=1$. Notice that

$$\lambda\geqslant\limsup_{n\in\mathbb{N}}\frac{f(F'_{N(n)})}{|F'_{N(n)}|}\geqslant\liminf_{n\in\mathbb{N}}\frac{f(F'_{N(n)})}{|F'_{N(n)}|}=\liminf_{n\in\mathbb{N}}\frac{f(F'_{N(n)})+f(S_n\backslash F'_{N(n)})}{|S_n|}\geqslant\lim\frac{f(S_n)}{|S_n|}=\lambda\,.$$

□

The Ornstein-Weiss Lemma immediately implies the following convergence result for the entropy of left representations on normed semigroups. Notice that there exists a generalization of the above result to cancelable amenable semigroups (see [17]), using such stronger version of the Ornstein-Weiss Lemma one would obtain more general results about $\mathfrak{s}$-entropies. Furthermore, Krieger's proof (see [67]) of the Ornstein-Weiss Lemma holds for Følner nets in general, this would allow to extend the following corollary to amenable groups of any cardinality.

As usual, we are tacitly assuming Hypothesis $(*)$.

Corollary 4.39. *Let (M,v) be a normed semigroup, let Γ be a countably infinite amenable group and let $\alpha\circlearrowleft M$ be a left Γ-representation. For any given Følner sequence $\mathfrak{s}=\{F_n\}_{n\in\mathbb{N}}$ of Γ, the* lim sup *defining the $\mathfrak{s}$-entropy of α converges and its limit does not depend on the choice of the particular Følner sequence, provided the following conditions hold:*

(1) *$v(x),v(y)\leqslant v(xy)$ for all x and $y\in M$;*

(2) *$v(\alpha_g(x))=v(x)$ for all $x\in M$ and $g\in\Gamma$.*

Proof. One has just to show that, for all $x \in M$, the following function satisfies the hypothesis of the Ornstein-Weiss Lemma:

$$f_x : \mathcal{F}(\Gamma) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \quad \text{such that} \quad f_x(F) = v(T_F(\alpha, x)) .$$

Now, f_x is sub-additive by the sub-additivity of the norm, it is monotone by hypothesis (1) and it is left Γ-equivariant by hypothesis (2). □

4.2.3 Consequences of the convergence of defining limits

The convergence of defining limits has many nice consequences, in this subsection we show two of them: the additivity of $\mathfrak{s}$-entropy on the coproduct of two representations and a precise computation of the entropy of some Bernoulli flows.

Lemma 4.40. *Let Γ be a countably infinite amenable group and let $\mathfrak{s}$ be a Følner sequence. Consider two left Γ-representations $\alpha_1 \circlearrowleft (S_1, v_1)$ and $\alpha_2 \circlearrowleft (S_2, v_2)$ on normed semigroups and let $S = S_1 \oplus S_2$ and $\alpha \circlearrowleft (S, v_\oplus)$ be such that $\alpha(g) = (\alpha_1(g), \alpha_2(g))$ for all $g \in \Gamma$. If $v_i(s)$, $v_i(t) \leqslant v_i(st)$ for all s, $t \in S_i$ (with $i = 1, 2$) and $v_i(\alpha_i(g)(s)) = v_i(s)$ for all $s \in S_i$ (with $i = 1, 2$) and $g \in \Gamma$, then*

$$\mathfrak{h}(\alpha, \mathfrak{s}) = \mathfrak{h}(\alpha_1, \mathfrak{s}) + \mathfrak{h}(\alpha_2, \mathfrak{s}) .$$

Proof. The proof is the same of Lemma 4.15, using the fact that the limit of the sum of two converging sequences is the sum of the limits of the two sequences. □

Lemma 4.41. *Let (K, v) be a normed monoid such that*

(1) *$v(x)$, $v(y) \leqslant v(xy)$ for all $x, y \in K$;*

(2) *there exists an element $\bar{k} \in K$ such that* $\sup\{v(x) : x \in K\} = v(\bar{k})$.

*Given a countably infinite amenable group Γ and a Følner sequence $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$, the $\mathfrak{s}$-entropy of the Bernoulli Γ-flow is $\mathfrak{h}(\mathfrak{B}_K, \mathfrak{s}) =$
sup$\{v(x) : x \in K\}$.*

Proof. For all $g \in \Gamma$ let $(K_g, v_g) = (K, v)$ and let $M = \bigoplus_{g\in\Gamma} K_g$. Endow M with the coproduct norm $v_\oplus$ of the norms v_g. For any $F \in \mathcal{F}(\Gamma)$ let $\bar{k}_F \in M$ be such that $\bar{k}_F = (\bar{k}_{F,g})_{g\in\Gamma}$, where

$$\bar{k}_{F,g} = \begin{cases} \bar{k} & \text{if } g \in F; \\ e & \text{otherwise.} \end{cases}$$

These elements have the following useful properties:

(a) by definition, $v_\oplus(\bar{k}_F) = |F| \sup\{v(x) : x \in K\}$, for all $F \in \mathcal{F}(\Gamma)$. Furthermore, since $\bar{k}_F = \prod_{f\in F} \bar{k}_{\{f\}}$ and by our hypotheses (1) and (2), it follows that $v_\oplus(\alpha_g(\bar{k}_F)\alpha_h(\bar{k}_F)) = |gF \cup hF| \sup\{v(x) : x \in K\}$, for all $F \in \mathcal{F}(\Gamma)$ and $g, h \in \Gamma$. More generally $v_\oplus(T_{F'}(\alpha, \bar{k}_F)) = |F'F| \sup\{v(x) : x \in K\}$, for all F and $F' \in \mathcal{F}(\Gamma)$;

(b) given $x = (x_g)_{g\in\Gamma} \in M$, let $F = \{g \in \Gamma : x_g \neq e\}$ and notice that $v_\oplus(x) \leqslant v_\oplus(\bar{k}_F)$, by the choice of $\bar{k}$. Applying again our hypotheses, it follows that $v_\oplus(T_F(\alpha, x)) \leqslant v_\oplus(T_F(\alpha, \bar{k}_F))$.

The above property (b) shows that $\mathfrak{h}(\alpha,\mathfrak{s}) = \sup_{F\in\mathcal{F}(\Gamma)} \mathfrak{h}(\alpha,\mathfrak{s},\bar{k}_F)$. Now, using the Følner condition, we obtain

$$\lim_{n\to\infty} \frac{|F_nF|}{|F_n|} \leqslant \lim_{n\to\infty} \frac{|F_n \cup \bigcup_{f\in F} \partial_F(F_n)f|}{|F_n|} \leqslant 1 + \lim_{n\to\infty} \sum_{f\in F} \frac{|\partial_F(F_n)f|}{|F_n|} = 1\,. \tag{4.2.4}$$

Thus, also applying the above property (a),

$$\mathfrak{h}(\mathfrak{B}_K,\mathfrak{s},\bar{k}_F) = \limsup_{i\in I} \frac{|F_iF|\sup\{v(x) : x\in K\}}{|F_i|} \leqslant \lim_{i\in I} \frac{|F_iF|}{|F_i|} \sup\{v(x) : x\in K\} \overset{(*)}{=} \sup\{v(x) : x\in K\}\,,$$

where $(*)$ follows by (4.2.4). The converse inequality follows by Lemma 8.21. □

Chapter 5

Lifting entropy along functors

In this chapter we show how to lift along a functor the general notion of entropy defined in Chapter 4 for representations on normed semigroups. This allows us to define many classical invariants that we will describe.

Definition 5.1. *Let $\mathfrak{C}$ be a category, let Γ be a monoid and let $\mathfrak{s}$ be a net of non-empty finite subsets of Γ. Given a functor $F : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$ and an object $X \in \mathrm{Ob}(\mathfrak{C})$, we define*

$$\mathfrak{h}_F(X, \mathfrak{s}) = \mathfrak{h}(F(X), \mathfrak{s})\,.$$

The generality of the above definition will be needed in some concrete situations, nevertheless it is sometimes useful to lift entropies along a functor $F : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$ which "factors through" the category $\mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v)$. This means that, if $F(X) = (\alpha : \Gamma \to \mathrm{Aut}_{\underline{\mathrm{Semi}}_v}(S, v))$ for a given $X \in \mathrm{Ob}(\mathfrak{C})$, then v is a norm (not just a pre-norm) and α_g is contractive for all $g \in \Gamma$ (when G is a group, this implies that $\alpha_{g^{-1}}$ is contractive as well, so we have the stronger condition that $v(\alpha_g(x)) = v(x)$, for all $x \in S$). Furthermore, given a morphism $\phi : X \to Y$ in $\mathfrak{C}$, the image $F(\phi)$ is a morphism of representations which is induced by a contractive homomorphism of normed semigroups:

Definition 5.2. *Let $\mathfrak{C}$ be a category and let $F' : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$ be a functor. We say that F'* factors through $\mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v)$ *if there exists a functor $F : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v)$ such that $F' = E \circ F$, where $E : \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v) \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$ is the inclusion functor.*

Notice that in practice it is the same to give a functor $F : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v)$ or a functor $F' : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$ that factors through $\mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v)$. In what follows we will also write $\mathfrak{h}_F(X, \mathfrak{s})$ with $X \in \mathrm{Ob}(\mathfrak{C})$ to mean $\mathfrak{h}_{E\circ F}(X, \mathfrak{s})$.

Corollary 5.3. *Let $\mathfrak{C}$ be a category, let Γ be a group, let $\mathfrak{s}$ be a net of non-empty finite subsets of Γ and let $F : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}^*_v)$ be a functor. Given two objects X and Y in $\mathfrak{C}$,*

(1) *if $X \cong Y$, then $\mathfrak{h}_F(X, \mathfrak{s}) = \mathfrak{h}_F(X, \mathfrak{s})$;*

(2) *if F sends monomorphisms to monomorphisms and $X \leqslant Y$, then $\mathfrak{h}_F(X, \mathfrak{s}) \leqslant \mathfrak{h}_F(Y, \mathfrak{s})$;*

(3) *if F sends epimorphisms to epimorphisms and X is a quotient object of Y, then $\mathfrak{h}_F(X, \mathfrak{s}) \leqslant \mathfrak{h}_F(Y, \mathfrak{s})$;*

(4) *if F commutes with products, then $\mathfrak{h}_F(X \times Y, \mathfrak{s}) = \max\{\mathfrak{h}_F(X, \mathfrak{s}), \mathfrak{h}_F(Y, \mathfrak{s})\}$;*

(5) *if F commutes with coproducts, then $\mathfrak{h}_F(X \oplus Y, \mathfrak{s}) \leqslant \mathfrak{h}_F(X, \mathfrak{s}) + \mathfrak{h}_F(Y, \mathfrak{s})$. Furthermore, if G is a countably infinite amenable group and $\mathfrak{s}$ is a Følner sequence, then $\mathfrak{h}_F(X \times Y, \mathfrak{s}) = \mathfrak{h}_F(X, \mathfrak{s}) + \mathfrak{h}_F(Y, \mathfrak{s})$.*

We remark that, in the notation of the above corollary, F does not commute with coproducts in most of the concrete cases that we will study. Anyway, the strategy to prove the analog of part (5) above is to show that $F(X \oplus Y)$ is $\mathfrak{s}$-dominated by $F(X) \oplus F(Y)$, even if they are not isomorphic representations.

5.1 Statical and dynamical growth of groups

Definition 5.4. *A* digraph *$\mathcal{G}$ is a pair (V, E), where V is a set of* vertices *and $E \subseteq V \times V$ is a set of ordered pairs of vertices, called* edges. *Furthermore, $\mathcal{G}$ is a* labeled digraph *if we fix a map $E \to B$, from the set of edges to a set of* labelings *B. If we need to make explicit that the set of labelings is B we say that $\mathcal{G}$ is a B-labeled digraph.*

One can construct a category of B-labeled digraphs for a fixed set of labelings B. For doing so we need to introduce the notion of morphism of labeled digraphs:

Definition 5.5. *Let B be a set of labelings and consider two B-labeled digraphs $\mathcal{G}_1 = (V_1, E_1)$ and $\mathcal{G}_2 = (V_2, E_2)$. Let also $\phi_1 : E_1 \to B$ and $\phi_2 : E_2 \to B$ be the maps which define the labeling. A morphism of labeled digraphs $\alpha : \mathcal{G}_1 \to \mathcal{G}_2$ is a map $\alpha : V_1 \to V_2$ such that*

(1) *$(v, v') \in E_1$ implies that $(\alpha(v), \alpha(v')) \in E_2$, for all $v, v' \in V_1$;*

(2) *$\phi_1(v, v') = \phi_2(\alpha(v), \alpha(v'))$, for all $(v, v') \in E_1$.*

By definition, a digraph $\mathcal{G}$ consists of a set of vertices connected by directed edges. This simple structure is enough for introducing a concept of distance and a family of neighborhoods of any vertex of $\mathcal{G}$:

Definition 5.6. *Let $\mathcal{G} = (V, E)$ be a digraph. Given v, $v' \in V$, a (finite)* directed path *from v to v' is a sequence of edges $(v_0, v_1), (v_1, v_2), \ldots, (v_{n-1}, v_n) \in E$ such that $v_0 = v$ and $v_n = v'$. The* path distance *(or just distance) in $\mathcal{G}$ is a function $d : V \times V \to \mathbb{N} \cup \{\infty\}$ such that $d(v, v') = \{$number of edges in a minimal directed path between v and $v'\}$ if some directed path connecting v and v' exists; $d(v, v') = \infty$ otherwise. For all $v \in V$ and $n \in \mathbb{N}$ we let $N_n^{\mathcal{G}}(v) = \{v' \in V : d(v, v') \leqslant n\}$ be the n*-th neighborhood of v in $\mathcal{G}$. *If $\mathcal{G}$ is clear from the context we also denote $N_n^{\mathcal{G}}(v)$ by $N_n(v)$.*

Since the B-labeled digraphs for a fixed set B form a category, we can safely speak about isomorphic B-labeled digraphs or about labeled sub-digraphs. In particular, given a B-labeled digraph $\mathcal{G} = (V, E)$, with a little abuse of terminology we consider $N_n(v)$ (for all $v \in V$ and $n \in \mathbb{N}$) as a labeled sub-digraph of $\mathcal{G}$, where (v_1, v_2) is an edge in $N_n(v)$ (for v_1, $v_2 \in N_n(v)$) if and only if $(v_1, v_2) \in E$.

Our main example for a labeled digraph will be the following:

Definition 5.7. *Let G be a finitely generated group and let B be a finite symmetric set of generators. The* Cayley digraph *of G with respect to B is a B-labeled digraph $\mathcal{G}(G, B) = (V, E)$ such that the set of vertices V coincides with G and there is a directed edge (g, gb) for all $g \in G$ and $b \in B$; such edge is labeled by b.*
The distance between two vertices in $\mathcal{G}(G, B)$ is denoted by $d_B(-, -)$. For all $g \in G$ and $n \in \mathbb{N}$

we denote by $N_n(B, g)$ *the* n-th neighborhood of g in $\mathcal{G}(G, B)$. *In order to simplify notation, we usually denote* $N_n(B, e)$ *simply by* $N_n(B)$. *The* B-length *of an element* $x \in G$ *is* $\ell_B(x) = d_B(e, x)$*, so that* $N_n(B) = \{x \in G : \ell_B(x) \leqslant n\}$.

Example 5.8. *Let* $G = \mathbb{Z}^k = \mathbb{Z}e_1 \times \cdots \times \mathbb{Z}e_k$ *for some positive integer* k. *The canonical choice of generators is to take* $B = \{-e_1, \dots, -e_k, 0, e_1, \dots, e_k\}$. *Then,*

$$N_n(B) = \left\{ \sum_{i=1}^{k} \lambda_i e_i : \lambda_i \in \mathbb{Z} \text{ such that } \sum_{i=1}^{k} |\lambda_i| \leqslant n \right\} .$$

Definition 5.9. *Let* β, $\beta' : \mathbb{N} \to \mathbb{R}$ *be two functions, then* β' weakly dominates β *if there are constants* h *and* $k \in \mathbb{N}$ *such that* $\beta(n) \leqslant h\beta'(hn + k) + k$*, for all* $x \in \mathbb{N}$. *In symbols,* $\beta \preceq \beta'$. *We also say that* β *and* β' *are* weakly equivalent *if they weakly dominate one another; weak equivalence is an equivalence relation that we denote by* $\beta \sim \beta'$.
A given map $\beta : \mathbb{N} \to \mathbb{N}$ *has*

- polynomial growth *if* $\beta \sim f$*, where* $f : \mathbb{N} \to \mathbb{R}$ *is such that* $f(n) = n^k$ *for some* $k \in \mathbb{N}_+$*;*
- exponential growth *if* $\beta \sim g$*, where* $g : \mathbb{N} \to \mathbb{R}$ *is such that* $g(n) = h^n$ *for some real* $h > 1$*;*
- intermediate growth *if it is not of polynomial nor exponential growth and there exist* f, $g \colon \mathbb{N} \to \mathbb{N}$ *such that* $f \preceq \beta \preceq g$*, with* f *of polynomial growth and* g *of exponential growth.*

With all these definitions at hand, we can recall the notion of "growth rate of a group".

Definition 5.10. *Let* G *be a finitely generated group and denote by* $\mathcal{C}(G)$ *be the family of the finite subsets of* G *containing* e. *For all* $B \in \mathcal{C}(G)$*, the* growth function γ_B *of* G *relative to* B *is defined by*

$$\gamma_B : \mathbb{N} \to \mathbb{N} \qquad \gamma_B(n) = |N_n(B^*)| \,,$$

where $B^* = B \cup B^{-1}$ *and* $N_n(B)$ *is the* n*-th neighborhood of* e *in the Cayley digraph of the subgroup* $\langle B \rangle$ *of* G *generated by* B.

Lemma 5.11. *Let* G *be a group and let* B, $B' \in \mathcal{C}(G)$.

(1) *If* $\langle B' \rangle \subseteq \langle B \rangle$*, then* $\gamma_{B'} \preceq \gamma_B$. *In particular,* $\gamma_{B'} \sim \gamma_B$ *provided* $\langle B' \rangle = \langle B \rangle$*;*

(2) *If* $\langle B \rangle$ *is infinite, then* γ_B *has at least polynomial and at most exponential growth.*

Proof. (1) Since $\langle B \rangle \subseteq \langle B' \rangle = \bigcup_{n \in \mathbb{N}} N_n(B')$, there exists $k \in \mathbb{N}$ such that $B \subseteq N_k(B')$. Then, for all $n \in \mathbb{N}$, $N_n(B) \subseteq N_{n+k}(B')$, which implies that

$$\gamma_B(n) = |N_n(B)| \leqslant |N_{n+k}(B')| = \gamma_{B'}(n + k) \,,$$

showing that $\gamma_B \preceq \gamma_{B'}$. A similar argument gives $\gamma_{B'} \preceq \gamma_B$ in case $\langle B' \rangle = \langle B \rangle$.
(2) It is enough to prove that $n \leqslant \gamma_B(n) \leqslant |B|^n$, for all $n \in \mathbb{N}$. The upper bound comes by the definition of $N_n(B^*)$, which is the set of all words of length $\leqslant n$ in the alphabet B^* (recall that B is assumed to contain e). For the lower bound suppose, looking for a contradiction, that there exists a smallest $\bar{n} \in \mathbb{N}$ such that $\gamma_B(\bar{n}) < \bar{n}$ (such $\bar{n}$ is necessarily > 1 as γ_B takes values $\geqslant 1$). By the minimality of $\bar{n}$ and the monotonicity of γ_B, we get

$$\bar{n} - 1 \leqslant \gamma_B(\bar{n} - 1) \leqslant \gamma_B(\bar{n}) \leqslant \bar{n} - 1 \,.$$

This implies that $N_{\bar{n}}(B^*) = N_{\bar{n}-1}(B^*) = \langle B \rangle$, which contradicts our assumption of $\langle B \rangle$ being infinite. $\square$

Definition 5.12. *Let G be a group. Then,*

(1) *G has* polynomial growth *if γ_B has at most polynomial growth for all $B \in \mathcal{C}(G)$;*

(2) *G is of* exponential growth *if there exists $B \in \mathcal{C}(G)$ such that γ_B has exponential growth;*

(3) *G is of* intermediate growth *otherwise.*

Furthermore, we say that M has subexponential growth *if it has either polynomial or intermediate growth.*

The following lemma is a useful tool to find examples of amenable groups.

Lemma 5.13. *Let G be a finitely generated group of subexponential growth. Fixed a finite symmetric set $e \in B$ of generators, $\{N_n(B) : n \in \mathbb{N}\}$ is a Følner exhaustion for G.*

Proof. First of all, let us show that, for all $k \in \mathbb{N}$, $\lim_{\mathbb{N}} \gamma_B(n+k)/\gamma_B(n) = 1$. Indeed, suppose looking for a contradiction that there exists $\varepsilon > 0$ such that $\gamma_B(n+k) > (1+\varepsilon)\gamma_B(n)$. Then, $\gamma_B(nk) > (1+\varepsilon)^n|B|$ showing that γ_B weakly dominates the map $n \mapsto (1+\varepsilon)^n|B|$ which has exponential growth, a contradiction.

Let now $g \in G = \bigcup_{n\in\mathbb{N}} N_n(B)$ and let k be the minimal positive integer such that $g \in N_k(B)$. Then,

$$\lim_{\mathbb{N}} \frac{|N_n(B)g \backslash N_n(B)|}{|N_n(B)|} \leqslant \lim_{\mathbb{N}} \frac{|N_{n+k}(B) \backslash N_n(B)|}{|N_n(B)|} = \lim_{\mathbb{N}} \frac{|N_{n+k}(B)|}{|N_n(B)|} - \lim_{\mathbb{N}} \frac{|N_n(B)|}{|N_n(B)|} = 1 - 1 = 0\,,$$

by the first part of the proof. One should also verify that $\lim_{\mathbb{N}} |N_n(B) \backslash N_n(B)g|/|N_n(B)| = 0$. This follows by what we proved, observing that $|N_n(B) \backslash N_n(B)g| = |N_n(B)g^{-1} \backslash N_n(B)|$. $\square$

Definition 5.14. *The growth rate of the group G with respect to a set $B \in \mathcal{C}(G)$ is $\gamma(G,B) = \limsup_{n\to\infty} \log\gamma_B(n)/n$. We let also $\gamma(G) = \sup\{\gamma(G,B) : B \in \mathcal{C}(G)\}$.*

Observe that $\gamma(G) > 0$ if and only if G has exponential growth.

In the last part of the section we establish a connection between $\gamma(-)$ and the entropy of pre-normed semigroups. We omit the proof of the following easy lemma.

Lemma 5.15. *Let G be a group, endow $\mathcal{C}(G)$ with the following operation*

$$\cdot : \mathcal{C}(G) \times \mathcal{C}(G) \to \mathcal{C}(G) \quad \textit{such that} \quad (B_1, B_2) \mapsto B_1 B_2\,.$$

Let also $v(B) = \log|B|$ for all $B \in \mathcal{C}(G)$. Then, $(\mathcal{C}(G), \cdot, v)$ is a normed monoid. Furthermore, any group homomorphism $\phi : G_1 \to G_2$ induces a homomorphism of monoids

$$\Phi : \mathcal{C}(G_1) \to \mathcal{C}(G_2) \quad \textit{such that} \quad B \mapsto \phi(B)\,.$$

Let now G be a group and consider the trivial $\mathbb{N}$-representation

$$\alpha_G : \mathbb{N} \to \mathrm{End}(\mathcal{C}(G)) \quad \text{such that} \quad \alpha_G(n) = \mathrm{id}_{\mathcal{C}(G)}\,.$$

Let also $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$ be the sequence of finite subsets of $\mathbb{N}$ such that

$$F_n = \{0, 1, \dots, n\}\,.$$

Notice that $\mathcal{C}(G)$ is not a commutative semigroup in general. According to Remark 4.9, we can still define the $\mathfrak{s}$-entropy of α_G with respect to a fixed order in $\mathbb{N}$. We just choose the usual one (even if, for the trivial action any order would give the same result).

Notice that $T_{F_n}(\alpha_G, B) = N_n(B)$, and so $v(T_{F_n}(\alpha_G, B)) = \log(\gamma_B(n))$, for all $B \in \mathcal{C}(G)$ and for all $n \in \mathbb{N}$. Thus,

$$\gamma(G) = \sup\{\mathfrak{h}(\alpha_G, \mathfrak{s}, B) : B \in S\} = \mathfrak{h}(\alpha_G, \mathfrak{s}) .$$

One can interpret the above equation as saying that $\gamma(G)$ is the entropy of a dynamical system whose evolution law is given by the identity; it seems legit to call $\gamma(G)$ the *statical growth rate* of G. If one substitutes the trivial $\mathbb{N}$-representation on $\mathcal{C}(G)$ by some other, say the left $\mathbb{N}$-representation $\alpha_{(G,\phi)} : \mathbb{N} \to \mathrm{End}(\mathcal{C}(G))$ such that $\alpha_{(G,\phi)}(n)(B) = \phi^n(B)$ (for all $n \in \mathbb{N}$ and $B \in \mathcal{C}(G)$) for some endomorphism $\phi : G \to G$, one obtains what we can call the *dynamical growth rate* of G with respect to ϕ:

$$\gamma_\phi(G) = \sup\{\mathfrak{h}(\alpha_{(G,\phi)}, \mathfrak{s}, B) : B \in S\} = \mathfrak{h}(\alpha_{(G,\phi)}, \mathfrak{s}) .$$

We refer to [29] and its reference list for an account of classical and recent results on both the statical and the dynamical growth rates. We remark that in [29] one can also find a brief discussion of what happens reversing the order on $\mathbb{N}$.

5.2 Mean topological dimension

In this section we recall the definition of the mean topological dimension, given by Gromov [54] (see also [68] and [24]), showing that this invariant can be recovered using our general scheme for entropies.

Definition 5.16. *Let (X, τ) be a topological space. Then,*

- *an* open cover *of X is a family $\mathcal{U} = (U_i)_{i\in I}$ of open subsets of X, such that $\bigcup_{i\in I} U_i = X$. We denote by* $\mathrm{cov}(X)$ *the family of finite open covers of X;*
- *given $\mathcal{U}$, $\mathcal{V} \in \mathrm{cov}(X)$ and a continuous self-map $\phi : X \to X$, we let*
$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : U \in \mathcal{U}\,,\ V \in \mathcal{V}\} \quad \textit{and} \quad \phi^{-1}(\mathcal{U}) = \{\phi^{-1}U : U \in \mathcal{U}\}\,;$$
- *given $\mathcal{U}$, $\mathcal{V} \in \mathrm{cov}(X)$, $\mathcal{U}$ is a* refinement *of $\mathcal{V}$, in symbols $\mathcal{U} \preceq \mathcal{V}$ provided for all $U \in \mathcal{U}$ there exists $V \in \mathcal{V}$ such that $U \subseteq V$.*

Notice that $\mathcal{U} \vee \mathcal{V}$ is a refinement of both $\mathcal{U}$ and $\mathcal{V}$, furthermore a cover $\mathcal{W}$ refines both $\mathcal{U}$ and $\mathcal{V}$ if and only if it refines $\mathcal{U} \vee \mathcal{V}$.

Lemma 5.17. *Let (X, τ), (Y, τ') be topological spaces. Then,* $(\mathrm{cov}(X), \vee)$ *is a commutative monoid. Furthermore, given a continuous map $\phi : X \to Y$ the following map is a monoid homomorphism:*

$$\Phi : \mathrm{cov}(Y) \to \mathrm{cov}(X) \quad \textit{such that} \quad \Phi(\mathcal{U}) = \phi^{-1}(\mathcal{U})\,.$$

Proof. Let $\mathcal{U}$, $\mathcal{V} \in \mathrm{cov}(X)$, then $\mathcal{U} \vee \mathcal{V}$ is an open cover of X, in fact the intersection of two open sets is open and

$$X = X \cap X = \bigcup_{U\in\mathcal{U}} U \cap \bigcup_{V\in\mathcal{V}} V = \bigcup_{V\in\mathcal{V}} \left(\bigcup_{U\in\mathcal{U}} U \cap V\right) = \bigcup_{V\in\mathcal{V}} \bigcup_{U\in\mathcal{U}} (U \cap V) = \bigcup_{W\in\mathcal{U}\vee\mathcal{V}} W\,.$$

Furthermore, the associativity of $\vee$ follows by the associativity of the intersection and the unit in $\mathrm{cov}(X)$ is the open cover $\{X\}$. Finally, it is an easy exercise to show that Φ is a homomorphism of monoids. □

Definition 5.18. *Let (X,τ) be a topological space and let $\mathcal{U} = (U_i)_{i\in I} \in \mathrm{cov}(X)$. Then,*

– *given $x \in X$, the* local order *of $\mathcal{U}$ at x is* $\mathrm{ord}(\mathcal{U},x) = |\{j \in J : x \in U_j\}| + 1$;

– *the* global order *of $\mathcal{U}$ is* $\mathrm{ord}(\mathcal{U}) = \max\{\mathrm{ord}(\mathcal{U},x) : x \in X\}$;

– *the* topological dimension *of $\mathcal{U}$ is*

$$D(\mathcal{U}) = \min\{\mathrm{ord}(\mathcal{V}) : \mathcal{V} \textit{ is a finite open cover of } X \textit{ such that } \mathcal{V} \preceq \mathcal{U}\}\,.$$

Lemma 5.19. *Let (X,τ), (Y,τ') be topological spaces. Then, $(\mathrm{cov}(X), D)$ is a normed monoid. Furthermore, given a continuous map $\phi : X \to Y$, the induced map $\Phi : \mathrm{cov}(Y) \to \mathrm{cov}(X)$ is a contracting monoid homomorphism.*

Proof. The first part follows by [24, Lemma 2.5]. On the other hand, choose $\mathcal{V} \in \mathrm{ord}(X)$ such that $\mathcal{U} \preceq \mathcal{V}$ and $D(\mathcal{U}) = \mathrm{ord}(\mathcal{V})$. Given $x \in X$ such that $\mathrm{ord}(\Phi(\mathcal{V})) = \mathrm{ord}(\Phi(\mathcal{V}),x)$,

$$D(\Phi(\mathcal{U})) \leqslant \mathrm{ord}(\Phi(\mathcal{V})) = \mathrm{ord}(\Phi(\mathcal{V}),x) \leqslant \mathrm{ord}(\mathcal{V},\phi(x)) \leqslant \mathrm{ord}(\mathcal{V}) = D(\mathcal{U})\,.$$

□

Definition 5.20. *Let (X,τ) be a topological space, let Γ be a monoid and let $\alpha \circlearrowleft X$ be a left Γ-representation. For any finite open cover $\mathcal{U}$ of X, and $F \in \mathcal{F}(\Gamma)$ we let*

$$\mathcal{U}_{\alpha,F} = \bigvee_{f\in F} \alpha_f^{-1}(\mathcal{U})\,.$$

Given a net $\mathfrak{s} = \{F_i\}_{i\in I}$ of non-empty finite subsets of Γ, the topological $\mathfrak{s}$-mean dimension of α with respect to $\mathcal{U}$ *is*

$$\eta(\alpha,\mathfrak{s},\mathcal{U}) = \limsup_{i\in I} \frac{D(\mathcal{U}_{\alpha,F_i})}{|F_i|}\,.$$

The topological $\mathfrak{s}$-mean dimension of α *is $\eta(\alpha,\mathfrak{s}) = \sup\{\eta(\alpha,\mathfrak{s},\mathcal{U}) : \mathcal{U}$ a finite open cover$\}$.*

Proposition 5.21. *Define a functor $F : \mathrm{r.Rep}_\Gamma(\underline{\mathrm{Top}}) \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}_v^*})$ such that*

(1) *if $\alpha \circlearrowleft X \in \mathrm{Ob}(\mathrm{r.Rep}_\Gamma(\underline{\mathrm{Top}}))$ and $\alpha(g) = \phi : X \to X$, then $F(\alpha)(g) = \Phi : \mathrm{cov}(X) \to \mathrm{cov}(X)$ is such that $\Phi(\mathcal{V}) = \phi^{-1}(\mathcal{V})$;*

(2) *for any morphism $\phi : \alpha_1 \to \alpha_2$ of right Γ-representations, $F(\phi)$ is defined as in Lemma 5.19.*

Then, $\eta(\alpha,\mathfrak{s}) = \mathfrak{h}_F(\alpha,\mathfrak{s})$. Furthermore, if G is an amenable group and $\mathfrak{s}$ is a Følner sequence, then $\eta(\alpha,\mathfrak{s})$ converges, that is, $\eta(\alpha,\mathfrak{s}) = \lim_{n\in\mathbb{N}} D(\mathcal{U}_{\alpha,F_n})/|F_n|$.

5.3 Uniform spaces and their entropies

Let X be a set, given two subsets U and $V \subseteq X \times X$ we consider the following operations:

– $U^{-1} = \{(y,x) : (x,y) \in U\}$, the *inverse* of U;

– $U \circ V = \{(x,y) \in X \times X : \exists z \in X,\ (x,z) \in U \text{ and } (z,y) \in V\}$, the *composition* of U and V;

– $U(x) = \{y \in X : (x,y) \in U\}$, the *$U$-neighborhood of x*.

Definition 5.22. *Let X be a set. A* uniform structure *on X is a set $\mathfrak{U}$ of subsets of $X \times X$, whose elements are called* entourages *and satisfy the following axioms:*

(U1) *if $U \in \mathfrak{U}$, then U contains the diagonal $\Delta = \{(x,x) : x \in X\}$;*

(U2) *if $U \in \mathfrak{U}$, $V \subseteq X \times X$ and $U \subseteq V$, then $V \in \mathfrak{U}$;*

(U3) *if $U, V \in \mathfrak{U}$, then $U \cap V \in \mathfrak{U}$;*

(U4) *if $U \in \mathfrak{U}$, then there exists $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$;*

(U5) *if $U \in \mathfrak{U}$, then $U^{-1} \in \mathfrak{U}$.*

A pair $(X,\mathfrak{U})$ is said to be a uniform space. *Given two uniform spaces $(X,\mathfrak{U})$, $(Y,\mathfrak{V})$, a map $\phi : X \to Y$ is* uniformly continuous *if for every entourage $V \in \mathfrak{V}$ there exists an entourage $U \in \mathfrak{U}$ such that for every $(x_1, x_2) \in U$, $(\phi(x_1), \phi(x_2)) \in V$.*

The class of uniform spaces together with uniformly continuous maps form a category. There is a canonical functor from this category to the category of topological spaces that is described in the following lemma, whose proof follows from the definitions.

Lemma 5.23. *Let $(X,\mathfrak{U})$, $(Y,\mathfrak{V})$ be uniform spaces and let $\phi : X \to Y$ be a uniformly continuous map. The family $\{U(x) : U \in \mathfrak{U}, x \in X\}$ is a pre-base for a topology on X that we denote by $\tau_{\mathfrak{U}}$. Furthermore, ϕ is continuous when we endow X and Y with the topologies $\tau_{\mathfrak{U}}$ and $\tau_{\mathfrak{V}}$ respectively.*

Definition 5.24. *Let $(X,\mathfrak{U})$ be a uniform space, let $(I,\leqslant)$ be a directed set and let $(x_i)_{i\in I}$ be a net in X. Then,*

- *$(x_i)_{i\in I}$ is a* Cauchy net *if, for every entourage $V \in \mathfrak{U}$ there exists k such that for all i, $j \geqslant k$, $(x_i, x_j) \in V$;*
- *$(x_i)_{i\in I}$ is a* convergent net *if it converges with respect to the topology $\tau_{\mathfrak{U}}$ induced by $\mathfrak{U}$.*

Furthermore, $(X,\mathfrak{U})$ is complete *if any Cauchy net is a convergent net.*

Definition 5.25. *Let $(X,\mathfrak{U})$ be a uniform space. To any entourage $U \in \mathfrak{U}$, one associates a* basic uniform cover
$$\mathcal{C}(U) = \{U(x) : x \in X\}\,.$$
A cover $\mathcal{A}$ of X is said to be uniform *if there exists $U \in \mathfrak{U}$ such that $\mathcal{C}(U) \preceq \mathcal{A}$.*

The proof of the following lemma is an easy exercise.

Lemma 5.26. *Let $(X,\mathfrak{U})$ be a uniform space. If $\mathcal{A}$ and $\mathcal{B}$ are uniform covers and $\phi : X \to X$ is a uniformly continuous map, then both $\mathcal{A} \vee \mathcal{B}$ and $\phi^{-1}(\mathcal{A})$ are uniform covers.*

Our main example of uniform space are the topological groups:

Example 5.27. *Let (G,τ) be a topological group. There are two canonical uniform structures on G:*

- *an entourage for the* right uniformity *is a subset U of $G\times G$ that contains $\{(m,n) : mn^{-1} \in N\}$ for some $N \in \mathcal{V}_G(e)$;*
- *an entourage for the* left uniformity *is a subset U of $G\times G$ that contains $\{(m,n) : m^{-1}n \in N\}$ for some $N \in \mathcal{V}_G(e)$.*

Of course these two uniform structures coincide if M is Abelian. Furthermore, the topology induced by these two uniformities is the original topology τ.

Another important example of uniform space is described in the following proposition.

Proposition 5.28. [40, Section 8] *Let (X, τ) be a Hausdorff compact topological space. The set $\mathfrak{U}_\tau$ of all the neighborhoods of the diagonal $\Delta \subseteq X \times X$ is the unique uniform structure on X that induces the topology τ. Furthermore, a continuous map $X \to X$ is automatically uniformly continuous with respect to this uniform structure.*

5.3.1 Entropy in uniform spaces

Let Γ be a monoid, let $(X, \mathfrak{U})$ be a uniform space and consider a left Γ-representation $\alpha : \Gamma \to \mathrm{Aut}(X, \mathfrak{U})$. In this subsection we describe three different ways to define a notion of entropy for α; two of them, via separated and spanning sets, are classical and based on ideas of Bowen [10], the third is based on ideas described in [33] and generalizes the original definition of topological entropy introduced in [1].

Definition 5.29. *Let $(X, \mathfrak{U})$ be a uniform space, let $K \subseteq X$ be a compact subset (with respect to $\tau_{\mathfrak{U}}$), let $U \in \mathfrak{U}$, let Γ be a monoid and let $F \in \mathcal{F}(\Gamma)$. Given a left Γ-representation $\alpha : \Gamma \to$* $\mathrm{Aut}(X, \mathfrak{U})$,

- *a subset $S \subseteq X$ is said to (F, U)*-span *K* with respect to α, *if for every $k \in K$ there is $x \in S$ such that $(\alpha_g(k), \alpha_g(x)) \in U$ for all $g \in F$. We set*

$$r_F(U, K, \alpha) = \min\{|S| : S \ \ (F, U)\text{-spans } K \text{ with respect to } \alpha\}\,;$$

- *a subset $S \subseteq X$ is said to be (F, U)*-separated with respect to α, *if for each pair of distinct points $x, y \in S$ there exists $g \in F$ such that $(\alpha_g(x), \alpha_g(y)) \notin U$. We set*

$$s_F(U, K, \alpha) = \max\{|S| : S \subseteq K \text{ and } F \text{ is } (F, U)\text{-separated with respect to } \alpha\}\,;$$

- *if $\mathcal{A}$ is a uniform cover of X, let $N(K, \mathcal{A}) = \min\{|\mathcal{B}| : \mathcal{B} \preceq \mathcal{A}\}$. We set*

$$c_F(U, K, \alpha) = N\left(K, \bigvee_{g \in F} \alpha_g^{-1}(\mathcal{C}(U))\right).$$

The quantities $r_F(U, K, \alpha)$, $s_F(U, K, \alpha)$ and $c_F(U, K, \alpha)$ are well defined (and finite) as K is compact.

Lemma 5.30. *Let $(X, \mathfrak{U})$ be a uniform space, let Γ be a monoid and consider a left Γ-representation $\alpha : \Gamma \to \mathrm{Aut}(X, \mathfrak{U})$. If $U \in \mathfrak{U}$, K is a compact subset of X and $F \in \mathcal{F}(\Gamma)$, then*

(1) $\sigma_F(U, K, \alpha) \leqslant \sigma_F(W, K, \alpha)$, *with $\sigma = s, r, c$, provided $W \in \mathfrak{U}$ and $W \subseteq U$;*

(2) $s_F(U, K, \alpha) \leqslant c_F(W, K, \alpha)$, *for each $W \in \mathfrak{U}$ with $W^{-1} \circ W \subseteq U$;*

(3) $c_F(U, K, \alpha) \leqslant r_F(W, K, \alpha)$, *for each $W \in \mathfrak{U}$ with $W^{-1} \subseteq U$;*

(4) $r_F(U, K, \alpha) \leqslant s_F(U, K, \alpha)$.

Proof. (1) is a consequence of the definitions.
(2) By part (1), we only need to prove that K contains no $(F, W^{-1} \circ W)$-separated subsets with respect to α of size $> c_F(W, K, \alpha)$. Indeed, suppose there is a subset $S \subseteq K$ such that $|S| > c_F(W, K, \alpha)$. By definition, we can find two distinct elements $x_1, x_2 \in S$ and, for all $g \in F$, an element $y_g \in X$ such that

$$x_1, x_2 \in \bigcap_{g \in F} \alpha_g^{-1}(W(y_g)) \in \bigvee_{g \in F} \alpha_g^{-1}(\mathcal{C}(W)) .$$

Thus, $(\alpha_g(x_1), \alpha_g(x_2)) \in W^{-1} \circ W$, for all $g \in F$, proving that S is not $(F, W^{-1} \circ W)$-separated with respect to α.
(3) Let $E \subseteq K$ be a finite subset that (F, W)-spans K with respect to α. By definition, given $k \in K$, there exists $x \in E$ such that $(\alpha_g(k), \alpha_g(x)) \in W$ for all $g \in F$, that is,

$$k \in \alpha_g^{-1}(W^{-1}(\alpha_g(x))) \subseteq \alpha_g^{-1}(U(\alpha_g(x))) ,$$

for all $g \in F$. For all $x \in E$, let $B_x = \bigcap_{g \in F} \alpha_g^{-1}(U(\alpha_g(x)))$, so that $\mathcal{B} = \{B_x : x \in E\}$ covers K, $|\mathcal{B}| = |E|$ and $\mathcal{B} \preceq \bigvee_{g \in F} \alpha_g^{-1}(\mathcal{C}(U))$. Thus, by the definition of $c_F(U, K, \alpha)$, we have that $c_F(U, K, \alpha) \leqslant |E|$.
(4) follows by the fact that a maximal (F, U)-separated subset of K with respect to α, (F, U)-spans K with respect to α. □

Definition 5.31. *Let $(X, \mathfrak{U})$ be a uniform space, let $K \subseteq X$ be a compact subset, let $U \in \mathfrak{U}$ and let $\mathfrak{s} = \{F_i : i \in I\}$ be a net of non-empty finite subsets of a monoid Γ. Given a left Γ-representation $\alpha : \Gamma \to \mathrm{Aut}(X, \mathfrak{U})$, we define*

$$\sigma_{\mathfrak{s}}(U, K, \alpha) = \limsup_{i \in I} \frac{\log \sigma_{F_i}(U, K, \alpha)}{|F_i|} ,$$

where σ stands for r, s or c.

The following lemma follows easily by Lemma 5.30.

Lemma 5.32. *Let $(X, \mathfrak{U})$ be a uniform space, let Γ be a monoid, fix a net $\mathfrak{s}$ of non-empty finite subsets of Γ and consider a left Γ-representation $\alpha : \Gamma \to \mathrm{Aut}(X, \mathfrak{U})$. If $U \in \mathfrak{U}$ and K is a compact subset of X, then*

(1) *$s_{\mathfrak{s}}(U, K, \alpha) \leqslant c_{\mathfrak{s}}(W, K, \alpha)$, for each $W \in \mathfrak{U}$ with $W^{-1} \circ W \subseteq U$;*

(2) *$c_{\mathfrak{s}}(U, K, \alpha) \leqslant r_{\mathfrak{s}}(W, K, \alpha)$, for each $W \in \mathfrak{U}$ with $W^{-1} \subseteq U$;*

(3) *$r_{\mathfrak{s}}(U, K, \alpha) \leqslant s_{\mathfrak{s}}(U, K, \alpha)$.*

We conclude this subsection showing that the three approaches for defining the entropy of a representation on a uniform space are just different ways to introduce the same invariant:

Theorem 5.33. *Let $(X, \mathfrak{U})$ be a uniform space, let Γ be a monoid, fix a net $\mathfrak{s}$ of non-empty finite subsets of Γ and consider a left Γ-representation $\alpha : \Gamma \to \mathrm{Aut}(X, \mathfrak{U})$. Given a compact subset K of X and letting*

$$h_\sigma(K, \alpha, \mathfrak{s}) = \sup\{\sigma_{\mathfrak{s}}(U, K, \alpha) : U \in \mathcal{U}\}$$

where σ stands for r, s or c, $h_r(K, \alpha, \mathfrak{s}) = h_s(K, \alpha, \mathfrak{s}) = h_c(K, \alpha, \mathfrak{s})$.

Proof. We will prove the following inequalities:

$$h_s(K,\alpha,\mathfrak{s}) \overset{(*)}{\leqslant} h_c(K,\alpha,\mathfrak{s}) \overset{(**)}{\leqslant} h_r(K,\alpha,\mathfrak{s}) \overset{\binom{**}{*}}{\leqslant} h_s(K,\alpha,\mathfrak{s}).$$

Indeed, given $U \in \mathfrak{U}$, by the axiom (U4), there exists $V \in \mathfrak{U}$ such that $V \circ V \subseteq U$, furthermore, by the axioms (U3) and (U5), $W = V^{-1} \cap V \in \mathfrak{U}$. It follows that $W^{-1} \circ W \subseteq U$ and so, by Lemma 5.32 (1), $s_{\mathfrak{s}}(U,K,\alpha) \leqslant c_{\mathfrak{s}}(W,K,\alpha) \leqslant h_c(K,\alpha,\mathfrak{s})$. This proves $(*)$ by the arbitrariness of U. The proof of $(**)$ follows similarly using the axiom (U5) and Lemma 5.32 (2). Finally, $\binom{**}{*}$ is a direct application of Lemma 5.32 (3). □

By the above theorem one can give the following definition.

Definition 5.34. *Let $(X,\mathfrak{U})$ be a uniform space, let Γ be a monoid, let $\alpha : \Gamma \to \mathrm{Aut}(X,\mathfrak{U})$ be a left Γ-flow and let $\mathfrak{s}$ be a net of non-empty finite subsets of Γ. The* uniform $\mathfrak{s}$-entropy *of α is*

$$h_{\mathfrak{U}}(\alpha,\mathfrak{s}) := \sup_K h_r(K,\alpha,\mathfrak{s}) = \sup_K h_s(K,\alpha,\mathfrak{s}) = \sup_K h_c(K,\alpha,\mathfrak{s})$$

In general, the uniform $\mathfrak{s}$-entropy cannot be defined functorially via the semigroup $\mathfrak{s}$-entropy. One can generalize the notion of semigroup $\mathfrak{s}$-entropy considering the category of semigroups with suitable families of pre-norms and not just pre-normed semigroups, this is done in [32]. Anyway, we will see in the following subsections that the uniform $\mathfrak{s}$-entropy can be defined using the formalism of the semigroup entropy in case $(X,\mathfrak{U})$ is compact and Hausdorff or when X is an LC group and $\mathfrak{U}$ is its canonical (left or right) uniformity.

5.3.2 Topological entropy

The following definition generalizes the topological entropy defined in [1].

Definition 5.35. *Let (X,τ) be a compact Hausdorff topological space, let Γ be a monoid, let $\alpha \circlearrowleft (X,\tau)$ be a left Γ-representation and fix a net $\mathfrak{s} = \{F_i : i \in I\}$ of non-empty finite subsets of G. Given an open cover $\mathcal{A}$ of X one lets*

$$N(\mathcal{A}) = \min\{|\mathcal{B}| : \mathcal{B} \preceq \mathcal{A}\}.$$

The topological $\mathfrak{s}$-entropy of α *is defined as follows*

$$h_T(\alpha,\mathfrak{s}) = \sup\{h_T(\mathcal{A},\alpha,\mathfrak{s}) : \mathcal{A} \text{ open cover of } X\},$$

where

$$h_T(\mathcal{A},\alpha,\mathfrak{s}) = \limsup_{i\in I} \frac{\log\left(N\left(\bigvee_{g\in F_i}\alpha_g^{-1}\mathcal{A}\right)\right)}{|F_i|}.$$

Using Theorem 5.33 we obtain the following

Corollary 5.36. *Let (X,τ) be a compact Hausdorff space and denote by $\mathfrak{U}$ the unique uniform structure on X compatible with τ. Given a monoid Γ, a net $\mathfrak{s} = \{F_i : i \in I\}$ of non-empty finite subsets of Γ and a left Γ-representation $\alpha \circlearrowleft (X,\tau)$,*

$$h_T(\alpha,\mathfrak{s}) = h_{\mathfrak{U}}(\alpha,\mathfrak{s}).$$

Proof. For all $U \in \mathfrak{U}$, one can find an open refinement $\mathcal{A}_U$ of $\mathcal{C}(U)$ (just take, for all $x \in X$, an open neighborhood of x contained in $U(x)$). Furthermore, for any open cover $\mathcal{A}$ of X, there exists $U_{\mathcal{A}} \in \mathfrak{U}$ such that $\mathcal{C}(U_{\mathcal{A}}) \preceq \mathcal{A}$ (see, for instance, [40, Exercise 8.1.H]). Notice that, directly from the definitions,

$$h_T(\mathcal{A}, \alpha, \mathfrak{s}) \leqslant c_{\mathfrak{s}}(U_{\mathcal{A}}, X, \alpha) \quad \text{and} \quad c_{\mathfrak{s}}(U, X, \alpha) \leqslant h_T(\mathcal{A}_U, \alpha, \mathfrak{s})\,.$$

Also using the fact that $c_{\mathfrak{s}}(U, C, \alpha) \leqslant c_{\mathfrak{s}}(U, X, \alpha)$ for any compact subset $C \subseteq X$, one easily obtains that $h_T(\alpha, \mathfrak{s}) = h_{\mathfrak{U}}(\alpha, \mathfrak{s})$. □

Let CompTop be the category of compact Hausdorff topological spaces and continuous maps, and define a functor

$$F : \mathrm{CompTop} \to \underline{\mathrm{Semi}}_v$$

such that $F(X, \tau) = (\mathrm{cov}(X), N(-))$ for any compact Hausdorff space (X, τ) and that acts contravariantly on maps analogously to Lemma 5.17. Given a monoid Γ, F induces a functor $\widehat{F} : \mathrm{r.Rep}_\Gamma(\mathrm{CompTop}) \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$. It is now easy to see that $h_T(-, \mathfrak{s}) = \mathfrak{h}_{F^*}(-, \mathfrak{s})$ for any net $\mathfrak{s}$ of non-empty finite subsets of Γ.

5.3.3 Topological entropy on LC groups

Definition 5.37. *Let G be an LC group and denote by $\mathcal{C}(G)$ the set of compact neighborhoods of e in G (this notation is compatible with the one used in Section 5.1. In fact, the elements of $\mathcal{C}(G)$ are just finite subsets containing e, provided the group G is discrete). Let also Γ be a monoid and let $\alpha : \Gamma \to \mathrm{End}(G)$ be a left Γ- representation. Given $K \in \mathcal{C}(G)$ and $F \in \mathcal{F}(\Gamma)$, let*

$$C_F(\alpha, K) = \bigcap_{g \in F} \alpha_g^{-1} K\,.$$

Given a net $\mathfrak{s} = \{F_i : i \in I\}$ of non-empty elements of $\mathcal{F}(\Gamma)$, we let

$$k(K, \alpha, \mathfrak{s}) = \limsup_{i \in I} \frac{-\log(\mu(C_{F_i}(\alpha, K))}{|F_i|}\,.$$

Finally, let $k(\alpha, \mathfrak{s}) = \sup\{k(K, \alpha, \mathfrak{s}) : K \in \mathcal{C}(G)\}$.

Proposition 5.38. *In the notation of the above definition, $k(\alpha, \mathfrak{s})$ coincides with the uniform $\mathfrak{s}$-entropy $h_{\mathfrak{U}}(\alpha, \mathfrak{s})$ of α, where $\mathfrak{U}$ is the canonical right uniformity on G, provided $\sup_{i \in I} |F_i| = \infty$.*

Proof. We start proving that $h_{\mathfrak{U}}(\alpha, \mathfrak{s}) \leqslant k(\alpha, \mathfrak{s})$. Fix arbitrarily a compact subset $C \subseteq G$ and an entourage $U \in \mathfrak{U}$, we are going to show that

$$s_{\mathfrak{s}}(U, C, \alpha) \leqslant k(\alpha, \mathfrak{s})\,. \tag{5.3.1}$$

By the arbitrariness of C and U, this would imply $h_{\mathfrak{U}}(\alpha, \mathfrak{s}) \leqslant k(\alpha, \mathfrak{s})$. So, let us verify (5.3.1). Take two compact neighborhoods of e, K and $H \in \mathcal{C}(G)$, with the following properties:

(N1) $\{(x, y) : xy^{-1} \in K\} \subseteq U$;

(N2) $H^{-1}H \subseteq K$ and $HH \subseteq K$.

Now consider the cover $\{xH : x \in C\}$ of C and, using compactness, extract a finite sub-cover $\{xH : x \in C'\}$, for some finite subset $C' \subseteq C$. Then $N = \bigcup_{x\in C'} xK$ is a compact neighborhood of C and $\mu(N)$ is finite and not 0. Notice that, for any given $c \in C$ there exists $x_c \in C'$ such that $c \subseteq x_c H$ and so, a direct consequence of (N2) is that

$$cH \subseteq x_c HH \subseteq x_c K \subseteq N\,. \tag{5.3.2}$$

For all $i \in I$, let E_i be a maximal (F_i, U)-separated subset of C with respect to α (that is, $|E_i| = s_{F_i}(U, C, \alpha)$); by construction, the following conditions are verified:

$$\bigcup_{e\in E_i} eC_{F_i}(\alpha, H) \overset{(*)}{\subseteq} N \quad , \quad eC_{F_i}(\alpha, H) \cap fC_{F_i}(\alpha, H) \overset{(**)}{=} \varnothing\,,\ \forall e \neq f \in E_i\,,$$

where $(*)$ holds by (5.3.2). To verify $(**)$, notice that, since E_i is (F_i, U)-separated, there exists $g' \in F_i$ such that $(\alpha_{g'}(e), \alpha_{g'}(f)) \notin U$, which implies that $\alpha_{g'}(ef^{-1}) \notin K$, thus $ef^{-1} \notin \bigcap_{g\in F_i} \alpha_g^{-1}(K)$. By (N1), $ef^{-1} \notin \bigcap_{g\in F_i} \alpha_g^{-1}(H^{-1}H) = C_{F_i}(\alpha, H)^{-1}C_{F_i}(\alpha, H)$, which clearly implies $(**)$. We obtain that,

$$\mu(N) \geqslant \mu\left(\bigcup_{e\in E_i} eC_{F_i}(\alpha, H)\right) = \sum_{e\in E_i} \mu(eC_{F_i}(\alpha, H)) = s_{F_i}(U, C, \alpha)\mu(C_{F_i}(\alpha, H))\,.$$

Hence, $\log(s_{F_i}(U, C, \alpha)) \leqslant -\log(\mu(C_{F_i}(\alpha, H))) + \log(\mu(N))$ for all $i \in I$, which, dividing by $|F_i|$ and passing to the $\limsup$ implies (5.3.1), as desired.

On the other hand, let $K \in \mathcal{C}(G)$ be a compact neighborhood of 1 and for all $i \in I$, choose a minimal subset E_i of M that (F_i, U)-spans K, where $U = \{(x, y) : xy^{-1} \in K\} \in \mathfrak{U}$. It follows directly from the definitions that $K \subseteq \bigcup_{e\in E_i} eC_{F_i}(\alpha, K)$ for all $i \in I$, thus

$$r_{F_i}(U, K, \alpha)\mu(C_{F_i}(\alpha, K)) \geqslant \sum_{e\in E_i} \mu(eC_{F_i}(\alpha, K)) \geqslant \mu\left(\bigcup_{e\in E_i} eC_{F_i}(\alpha, K)\right) \geqslant \mu(K) > 0\,.$$

Hence, $-\log(\mu(C_{F_i}(\alpha, K))) \leqslant \log(r_{F_i}(U, K, \alpha)) - \log(\mu(K))$ for all $i \in I$, which, dividing by $|F_i|$ and passing to the $\limsup$ implies $h_{\mathfrak{U}}(\alpha, \mathfrak{s}) \geqslant r_{\mathfrak{s}}(U, K, \alpha) \geqslant k(K, \alpha, \mathfrak{s})$. By the arbitrariness of $K \in \mathcal{C}(G)$, one gets $h_{\mathfrak{U}}(\alpha, \mathfrak{s}) \geqslant k(\alpha, \mathfrak{s})$, concluding the proof. □

Definition 5.39. *Let G be an LC group, let Γ be a monoid, let $\alpha : \Gamma \to \mathrm{End}(G)$ be a left Γ-representation and let $\mathfrak{s}$ be a net of non-empty finite subsets of Γ. The common value $k(\alpha, \mathfrak{s}) = h_{\mathfrak{U}}(\alpha, \mathfrak{s})$ is called* topological entropy *and denoted by $h_T(\alpha, \mathfrak{s})$.*

One can take as a partial justification for the above terminology the fact that $h_T(\alpha, \mathfrak{s})$ really coincides with the topological entropy of α when G is compact (as shown in the previous subsection). We conclude this subsection showing how to recover $h_T(\alpha, \mathfrak{s})$ from our general scheme for defining entropies, in case α is an invertible representation.

Definition 5.40. *Let G be an LC group and fix an Haar measure μ on G. The* topological pre-normed semigroup *associated to G is*

$$\mathcal{C}_T(G) = (\mathcal{C}(G), \cap, v_T)\,,$$

where the operation is just intersection and, for all $K \in \mathcal{C}(G)$

$$v_T(K) = \begin{cases} -\log(\mu(K)) & \text{if } \mu(K) \leqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

Let G be an LC group, let Γ be a monoid and let $\alpha : \Gamma \to \mathrm{Aut}(G)$ be an invertible right representation. Fix an Haar measure μ on G such that $\mu(\bar{K}) = 1$ for a given $\bar{K} \in \mathcal{C}(M)$. Notice that $v_T(K) = 0$ if K contains $\bar{K}$, while $v_T(K) > 0$ if K is contained in $\bar{K}$. There is an induced left Γ-representation $\alpha_T : \Gamma \to \mathrm{Aut}_{\underline{\mathrm{Semi}}_v}(\mathcal{C}_T(G))$ such that $\alpha_T(g)(K) = \alpha_g^{-1}(K)$. With this notation, we have the following equalities:

$$C_F(\alpha, K) = T_F(\alpha_T, K)\,, \quad k(K, \alpha, \mathfrak{s}) = \mathfrak{h}(\alpha_T, \mathfrak{s}, K)$$

Thus, $h_T(\alpha, \mathfrak{s}) = k(\alpha, \mathfrak{s}) = \mathfrak{h}(\alpha_T, \mathfrak{s})$.

5.4 Algebraic entropies

5.4.1 Peters' entropy

In this subsection we introduce our candidate for being the "dual" of the topological $\mathfrak{s}$-entropy on LCA groups, that is, Peters' algebraic $\mathfrak{s}$-entropy. This is analogous to the entropies studied for $\mathbb{Z}$-representations in [85] and [86].

Definition 5.41. *Let G be an LCA group, let Γ be a monoid and let $\alpha : \Gamma \to \mathrm{End}(G)$ be a left Γ- representation. Given $K \in \mathcal{C}(G)$ and $F \in \mathcal{F}(\Gamma)$, let*

$$T_F(\alpha, K) = \sum_{g \in F} \alpha_g(K)\,.$$

Given a net $\mathfrak{s} = \{F_i : i \in I\}$ of non-empty elements of $\mathcal{F}(\Gamma)$, we let

$$h_A(K, \alpha, \mathfrak{s}) = \limsup_{i \in I} \frac{\log(\mu(T_{F_i}(\alpha, K))}{|F_i|}\,.$$

Peters' algebraic $\mathfrak{s}$-entropy *of α is $h_A(\alpha, \mathfrak{s}) = \sup\{h_A(K, \alpha, \mathfrak{s}) : K \in \mathcal{C}(G)\}$.*

Let G be an LCA group and fix an Haar measure μ on the Borel subsets of G; the *algebraic pre-normed semigroup* associated to G is $\mathcal{C}_A(G) = (\mathcal{C}(G), +, v_A)$, such that $K + K' = \{x + x' : x \in K \text{ and } x' \in K'\}$ for all K and $K' \in \mathcal{C}(M)$ and the pre-norm is defined by

$$v_A(K) = \begin{cases} \log(\mu(K)) & \text{if } \mu(K) \geqslant 1; \\ 0 & \text{otherwise.} \end{cases}$$

Given a left Γ-flow $\alpha \circlearrowleft G$, there is an induced left Γ-representation

$$\alpha_A : G \to \mathrm{End}(\mathcal{C}_A(M)) \quad \alpha_A(g)(K) = \alpha_g(K) \quad \text{for all } g \in G,\ k \in K\,.$$

Thus, $h_A(\alpha, \mathfrak{s}) = \mathfrak{h}(\alpha_A, \mathfrak{s})$.

5.4.2 Algebraic L-entropy

Definition 5.42. *Let $\mathfrak{C}$ be an Abelian category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant} \cup \{\infty\}$ be an invariant of $\mathfrak{C}$ (i.e., $L(0) = 0$ and $L(M) = L(N)$ whenever $M \cong N$). The invariant i is called sub-additive if the following conditions hold:*

(Inv.1) $L(N_1 + N_2) \leqslant L(N_1) + L(N_2)$ *for all subobjects N_1, N_2 of M;*

(Inv.2) *$L(M/N) \leqslant L(M)$ for every subobjects N of M.*

For all $M \in \mathrm{Ob}(\mathfrak{C})$, let $\mathrm{Fin}_L(M) = \{N \in \mathcal{L}(M) : i(N) < \infty\}$.

The following definition of algebraic entropy is a generalization of the entropy defined in [94]:

Definition 5.43. *Let $\mathfrak{C}$ be an Abelian category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant} \cup \{\infty\}$ be a sub-additive invariant. Let $M \in \mathrm{Ob}(\mathfrak{C})$, let Γ be a monoid and let $\alpha : \Gamma \to \mathrm{End}_{\mathfrak{C}}(M)$ be a left Γ-representation. For any $F \in \mathcal{F}(\Gamma)$ and $K \in \mathrm{Fin}_L(M)$ we let*

$$T_F(\alpha, K) = \sum_{f \in F} \alpha_f(K)\,.$$

Given a net $\mathfrak{s} = \{F_i\}_{i \in I}$ of non-empty finite subsets of Γ, the algebraic $\mathfrak{s}$-entropy of α *with respect to K is $\mathrm{ent}_L(\alpha, \mathfrak{s}, K) = \limsup_{i \in I} L(T_{F_i}(\alpha, K))/|F_i|$. Furthermore, the* algebraic $\mathfrak{s}$-entropy of α *is $\mathrm{ent}_L(\alpha, \mathfrak{s}) = \sup\{\mathrm{ent}_L(\alpha, \mathfrak{s}, K) : K \in \mathrm{Fin}_L(M)\}$.*

Let $\mathfrak{C}$ be an Abelian category and let L be a sub-additive invariant of $\mathfrak{C}$. Notice that, given $M \in \mathrm{Ob}(\mathfrak{C})$, $\mathrm{Fin}_L(M)$ is a sub-monoid of $(\mathcal{L}(M), +)$. Define a norm on $\mathrm{Fin}_L(M)$ setting

$$v_L(H) = L(H)$$

for any $H \in \mathrm{Fin}_L(M)$. For any morphism $\phi : M \to N$ in $\mathfrak{C}$, there is an induced morphism

$$\mathrm{Fin}_L(\phi) : \mathrm{Fin}_L(M) \to \mathrm{Fin}_L(N) \quad \text{such that} \quad \mathrm{Fin}_L(\phi)(H) = \phi(H)\,.$$

Moreover the norm v_L makes the morphism $\mathrm{Fin}_L(\phi)$ contractive by the property (Inv.2) of the invariant. Therefore, the assignments $M \to \mathrm{Fin}_L(M)$ and $\phi \to \mathrm{Fin}_L(\phi)$ define a functor $F : \mathfrak{C} \to \underline{\mathrm{Semi}}_v^*$. Given a monoid Γ, F induces a functor $\widehat{F} : \mathrm{l.Rep}_\Gamma(\mathfrak{C}) \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v^*)$. Let now $\mathfrak{s}$ be a net of non-empty finite subsets of Γ and let $\alpha \in \mathrm{Ob}(\mathrm{l.Rep}(\mathfrak{C}))$, one can show easily that $\mathrm{ent}_L(\alpha, \mathfrak{s}) = \mathfrak{h}_{\widehat{F}}(\alpha, \mathfrak{s})$.

Chapter 6

The Bridge Theorem

Let Γ be a monoid, let $\mathfrak{s}$ be a net of non-empty finite subsets of Γ, let $\mathfrak{C}$ and $\mathfrak{D}$ be two categories and let

$$F_1 : \mathfrak{C} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v) \quad \text{and} \quad F_2 : \mathfrak{D} \to \mathrm{l.Rep}_\Gamma(\underline{\mathrm{Semi}}_v)$$

be functors. If there exists a functor $G : \mathfrak{C} \to \mathcal{D}$ such that $F_2 \circ G = F_1$, then by definition

$$\mathfrak{h}_{F_1}(X, \mathfrak{s}) = \mathfrak{h}_{F_2}(G(X), \mathfrak{s}) \tag{6.0.1}$$

for all $X \in \mathrm{Ob}(\mathfrak{C})$. Of course, it may happen that we can find a functor $G : \mathfrak{C} \to \mathfrak{D}$ such that (6.0.1) holds but not necessarily $F_2 \circ G = F_1$. The expression "bridge theorem" was coined by Luigi Salce to indicate any theorem claiming the existence of a functor $G : \mathfrak{C} \to \mathfrak{D}$ (which is a bridge between $\mathfrak{h}_{F_1}(-, \mathfrak{s})$ and $\mathfrak{h}_{F_2}(-, \mathfrak{s})$) such that (6.0.1) holds for all $X \in \mathfrak{C}$. There are many examples of bridge theorems, this chapter is devoted to the proof of the Bridge Theorem between Peters' algebraic entropy and topological entropy on LCA groups. In this case the functor G is the Pontryagin-Van Kampen duality functor.

The statement of the following theorem (when $\Gamma = \mathbb{Z}$, $I = \mathbb{N}$ and $F_n = \{0, \dots, n-1\}$ for all $n \in \mathbb{N}$) is due to Justin Peters [85] and [86]. Anyway the proof given in [86] contains many misprints and, in some cases, some of the arguments are so obscure that we had to find different proofs for some lemmas. On the other hand, the machinery of semi-group entropies clarifies the ideas of Peters and allows for a clean proof of the following Bridge Theorem.

Theorem 6.1. *Let G be an LCA group, let Γ be a monoid, let $\mathfrak{s} = \{F_i\}_{i \in I}$ be a net of non-empty finite subsets of Γ and let $\alpha : \Gamma \to \mathrm{Aut}(G)$ be an invertible left representation. Then,*

$$h_A(\alpha, \mathfrak{s}) = h_T(\alpha^*, \mathfrak{s}) \,,$$

where $\alpha^* : \Gamma \to \mathrm{Aut}(G^*)$ is the right Γ-representation induced by the Pontryagin-Van Kampen duality, that is, $\alpha^*(g)(\gamma) = \gamma \circ \alpha(g) : G \to \mathbb{T}$, for all $\gamma \in G^*$ and $g \in G$.

In what follows, we fix the notation of the above theorem. Furthermore, as we explained in the previous sections, one can consider the induced G-flows $\alpha_A \circlearrowleft \mathcal{C}_A(G)$ and $\alpha_T^* \circlearrowleft \mathcal{C}_T(G^*)$ in order to define the algebraic $\mathfrak{s}$-entropy of α and the topological $\mathfrak{s}$-entropy of α^* respectively.
As the net $\mathfrak{s}$ remains unchanged throughout this chapter, we omit to specify it. In particular, we say that a Γ-representation dominates another Γ-representation to mean that it $\mathfrak{s}$-dominates it and we use the notations $h_A(\alpha)$ and $h_T(\alpha^*)$, instead of $h_A(\alpha, \mathfrak{s})$ and $h_T(\alpha^*, \mathfrak{s})$. In this notation, the Bridge Theorem consists in proving that

$$h_A(\alpha) = h_T(\alpha^*) \,.$$

Even if the definitions of algebraic and topological entropy seem to be "dual" to each other, the Bridge Theorem hides some difficulty. In fact, their definitions are based on measures of suitable *subsets* of G and G^*; thus, such definitions are not "categorical" –subobjects in the category of LCA groups are *closed subgroups*– and it is therefore difficult to translate their properties to the dual, as a duality is only useful for dualizing categorical statements.

The main idea in order to connect algebraic and topological entropy is to reformulate their definition in terms of positive positive-definite complex-valued and absolutely integrable functions on G and G^* respectively (see Propositions 6.5 and 6.6) and then use the Fourier Inversion Theorem to conclude (see Theorem 6.7).

6.1 Pre-normed semigroups of positive-definite functions

Let $\mathcal{P}^1(G)^+$ be the family of absolutely integrable, positive and positive definite functions on G. There are two canonical commutative and associative operations which can be defined in this family, namely convolution and pointwise product. In what follows we introduce two pre-normed semigroups based on these operations.

Definition 6.2. *Let $\mathcal{P}_A(G) = (\mathcal{P}^1(G)^+, *, w_A)$ be the semigroup $\mathcal{P}^1(G)^+$, where the operation is convolution and with the following pre-norm*

$$w_A(\phi) = \log\left(\frac{||\phi||_1}{\phi(0)}\right), \quad \textit{for all } \phi \in \mathcal{P}_A(G)\,.$$

Definition 6.3. *Let $\mathcal{P}_T(G^*) = (\mathcal{P}^1(G^*)^+, \cdot, w_T)$ be the semigroup $\mathcal{P}^1(G^*)^+$, where the operation is pointwise product and with the following pre-norm*

$$w_T(\psi) = \log\left(\frac{\psi(0)}{||\psi||_1}\right), \quad \textit{for all } \psi \in \mathcal{P}_T(G^*)\,.$$

As an example, consider the case when G is compact and $\mu(G) = 1$, then $||\phi||_1 \leqslant \phi(0)\mu(G) = \phi(0)$ (by Lemma 3.40 and (3.1.1)), thus $w_A(\phi) \leqslant 1$ for all $\phi \in \mathcal{P}_A(G)$. Similarly, G^* is discrete and we can let the Haar measure be the cardinality of subsets, then $||\psi||_1 = \sum_{\gamma\in G^*} \psi(\gamma) \geqslant \psi(0)$ and so $w_T(\psi) \geqslant 1$, for all $\psi \in \mathcal{P}_T(G^*)$.

Definition 6.4. *Let $\beta_A \circlearrowright \mathcal{P}_A(G)$ and $\beta_T \circlearrowright \mathcal{P}_T(G^*)$ be the invertible left Γ-representation defined as follows:*

$$\beta_A(g)(\phi) = \phi \circ \alpha_g^{-1} \quad \textit{and} \quad \beta_T(g)(\psi) = \Delta(\alpha_g)\psi \circ \alpha_g^*\,, \quad \textit{for all } g \in \Gamma\,.$$

The constant $\Delta(\alpha_g)$ in the definition of β_T is just a technicality and in fact the value of the entropy does not change if one removes it.

The hard part of the proof of Theorem 6.1 consist in proving the following propositions

Proposition 6.5. *In the above notation, $h_A(\alpha) = \mathfrak{h}(\beta_A)$.*

The proof of the above proposition is given in Subsection 6.2.1 and it consists in showing that α_A and β_A dominate each other. The topological analog of the above proposition is proved in Subsection 6.2.2 using a similar strategy. Here is the precise statement:

Proposition 6.6. *In the above notation, $h_T(\alpha^*) = \mathfrak{h}(\beta_T)$.*

When the above propositions are proved, one can conclude the proof of Theorem 6.1 using the following theorem.

Theorem 6.7. *In the above notation, consider the following map induced by the Fourier transform*

$$f : \mathcal{P}_A(G) \to \mathcal{P}_T(G^*)\,, \quad f(\phi) = \widehat{\phi}\,.$$

Then, f is an isomorphism in $\underline{\mathrm{Semi}}_v$ and it induces an isomorphism of left Γ-representations between β_A and β_T. In particular, $\mathfrak{h}(\beta_A) = \mathfrak{h}(\beta_T)$.

Proof. Let us start proving that the definition of f is correct, that is, $f(\phi)$ belongs to $\mathcal{P}^1(G^*)^+$, for all $\phi \in \mathcal{P}^1(G)^+$. Indeed, consider $\phi \in \mathcal{P}^1(G)^+$, then $\widehat{\phi} \in L^1(G^*)^+$ by the Fourier Inversion Theorem. Let now μ_ϕ be the non-negative and bounded (as $\phi \in L^1(G)^+$) regular measure defined on a generic Borel subset E of G by $\mu_\phi(E) = \int_E \phi(x)d\mu$. One can show that

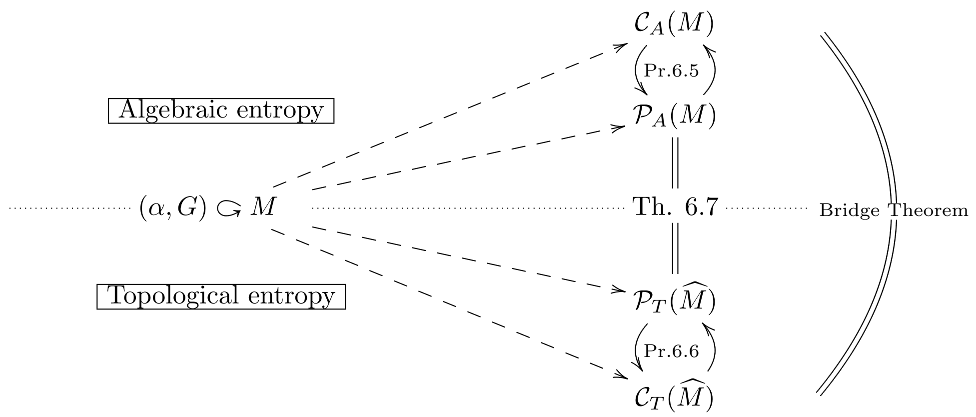
$$\widehat{\phi}(\gamma) = \int_G \phi(x)\gamma(-x)d\mu = \int_G \gamma(-x)d\mu_\phi = \int_G \gamma(x)d\mu_\phi$$

and so $\widehat{\phi} \in \mathcal{P}(G^*)$ by the Bochner Theorem. Furthermore, f is a bijective morphism of semigroups by Fourier Inversion Theorem and Lemma 3.51(2), while the fact that f induces a morphism of left Γ-representations follows by Lemma 3.51(1). Finally, f is norm preserving since

$$\widehat{\phi}(0) = \int_G \phi(x)\overline{0(x)}d\mu = \int_G \phi(x)d\mu = ||\phi||_1\,,$$

and so, also $||\widehat{\phi}||_1 = \widehat{\widehat{\phi}}(0) = \phi(-0) = \phi(0)$, by Fourier Inversion Theorem. The last statement about entropies follows by Corollary 4.11. $\square$

We summarize the scheme of the proof of Theorem 6.1 in the following picture:



6.2 Proofs

6.2.1 Proof of Proposition 6.5

Recall that we have an LCA group G and an invertible left Γ-representation $\alpha \circlearrowright G$, which induces two left Γ-representations on two different pre-normed semigroups that are functorially associated to G:

$$\begin{tikzcd}
 & \alpha \circlearrowright M \arrow[dl, dotted] \arrow[dr, dotted] & \\
\alpha_A \circlearrowright \mathcal{C}_A(M) & & \beta_A \circlearrowright \mathcal{P}_A(M)
\end{tikzcd}$$

We have to show that the entropy of the two induced representations is the same. We start proving that α_A is dominated by β_A. Roughly speaking this says that the way in which α acts on compact neighborhoods is controlled by the action on positive-definite complex functions. This, by Proposition 4.13, implies the inequality "$\leqslant$" in Proposition 6.5.

Lemma 6.8. *In the above notation, β_A dominates α_A.*

Proof. Let $C \in \mathcal{C}_A(G)$ and $F \in \mathcal{F}(\Gamma)$; for all $n \in \mathbb{N}$ we construct $\phi_n \in \mathcal{P}_A(G)$ and $\varepsilon_n \in \mathbb{R}_+$ such that

$$v_A(T_F(\alpha_A, C)) \leqslant w_A(T_F(\beta_A, \phi_n)) + 2|F| \log(\varepsilon_n) \quad \text{and} \quad \lim_{n\to\infty} \varepsilon_n = 1\,. \tag{6.2.1}$$

Indeed, let D' be an open neighborhood of 0 contained in C; by the compactness of C there exist $x_1, \dots, x_h \in G$ such that $C \subseteq (x_1 + D') \cup \dots \cup (x_h + D') = D$. One can verify that D is an open neighborhood of 0 in G with compact closure. For all $K \subseteq G$ and $n \in \mathbb{N}_+$, let

$$K^{(n)} = \{k_1 + \ldots + k_n : k_1, \dots, k_n \in K\}\,.$$

We let

$$\varepsilon_n = \frac{\mu\left(D^{(n+1)}\right)}{\mu\left(D^{(n)}\right)} \quad \text{and} \quad \phi_n = \chi_{D^{(n)}} * \chi_{D^{(n)}}\,;$$

thus, to conclude we have just to verify (6.2.1). In particular, the fact that $\lim_{n\to\infty} \varepsilon_n = 1$ follows by [39, Corollary 1.2] and so we have just to compute the value of $w_A(T_F(\beta_A, \phi_n))$.
For all $x \in T_F\left(\alpha_A, D^{(2)}\right)$ there exist x_f and $x'_f \in \alpha_A(f)(D^{(2)})$ for all $f \in F$ such that $x = \sum_{f\in F}(x_f + x'_f)$ and so, letting $\Delta(\alpha F) = \prod_{f\in F} \Delta(\alpha(f))$,

$$\begin{aligned} T_F\left(\beta_A, \phi_{n+1}\right)(x) &= T_F\left(\beta_A, \phi_{n+1}\right)_x(0) = \circledast_{f\in F} \left(\beta_A(f)\left(\phi_{n+1}\right)\right)_{x_f + x'_f}(0) \\ &= \Delta(\alpha F) \circledast_{f\in F} \left(\chi_{\alpha_A(f)\left(D^{(n+1)}\right)}\right)_{x_f} * \left(\chi_{\alpha_A(f)\left(D^{(n+1)}\right)}\right)_{x'_f}(0) \\ &\geqslant \Delta(\alpha F) \circledast_{f\in F} \left(\chi_{\alpha_A(f)\left(D^{(n)}\right)} * \chi_{\alpha_A(f)\left(D^{(n)}\right)}\right)(0) = T_F\left(\beta_A, \phi_n\right)(0)\,; \end{aligned} \tag{6.2.2}$$

where the first line follows by Remark 3.38, the equality in the second line comes from Lemma 3.43 and Remark 3.38, and the inequality in the third line follows since $(\chi_{\alpha_A(f)(D^{(n+1)})})_y \geqslant \chi_{\alpha_A(f)(D^{(n)})}$, for all $y \in \alpha_A(f)(C)$. Using (6.2.2) and (3.1.1) we get

$$||T_F(\beta_A, \phi_{n+1})||_1 \geqslant \int_{T_F(\alpha_A, D^{(2)})} T_F(\beta_A, \phi_{n+1})(x)d\mu \geqslant \mu(T_F(\alpha_A, D^{(2)})) \cdot T_F(\beta_A, \phi_n)(0)\,.$$

Furthermore, by Lemmas 3.37(4) and 3.43,

$$||T_F(\beta_A, \phi_n)||_1 = \prod_{f\in F} ||\beta_A(f)(\phi_n)||_1 = \Delta(\alpha F)\mu(C^{(n)})^{2|F|}\,.$$

Putting together all the above estimates, one can conclude as follows:

$$\begin{aligned} v_A(T_F(\alpha_A, C)) &\leqslant \log \mu(T_F(\alpha_A, D^{(2)})) \leqslant \log\left(\frac{||T_F(\beta_A, \phi_n)||_1}{T_F(\beta_A, \phi_n)(0)}\right) + \left(\log \frac{||T_F(\beta_A, \phi_{n+1})||_1}{||T_F(\beta_A, \phi_n)||_1}\right) \\ &= w_A(T_F(\beta_A, \phi_n)) + 2|F| \log\left(\frac{\Delta(\alpha F)\mu(D^{(n+1)})}{\Delta(\alpha F)\mu(D^{(n)})}\right) \\ &= w_A(T_F(\beta_A, \phi_n)) + 2|F| \log(\varepsilon_n)\,. \end{aligned}$$

□

We can now conclude the proof of Proposition 6.5 showing the converse inequality.

Lemma 6.9. *In the above notation, α_A dominates β_A.*

Proof. Let $\phi \in \mathcal{P}_A(G)$ and $F \in \mathcal{F}(\Gamma)$; for all $n \in \mathbb{N}_+$ we construct a $\psi_n \in \mathcal{P}_A(G)$ with compact support such that

$$w_A(T_F(\beta_A, \phi)) \leqslant w_A(T_F(\beta_A, \psi_n)) + 2|F| \log(1 + 1/n)\,. \tag{6.2.3}$$

After that we let $D_n = \mathrm{supp}(\psi_n) \in \mathcal{C}_A(G)$ and we verify that $w_A(T_F(\beta_A, \psi_n)) \leqslant v_A(T_F(\alpha_A, D_n))$, so that

$$w_A(T_F(\beta_A, \phi)) \leqslant v_A(T_F(\alpha_A, D_n)) + 2|F| \log(1 + 1/n)\,, \tag{6.2.4}$$

concluding the proof. Let us start with our program: we define $\phi' = \frac{1}{||\phi||_1}\phi$. Notice that $||\phi'||_1 = 1$ and

$$w_A(T_F(\alpha_A, \phi')) = \log\left(\frac{||T_F(\beta_A, \phi')||_1}{T_F(\beta_A, \phi')(0)}\right) = \log\left(\frac{||\phi||_1^{-|F|} \cdot ||T_F(\beta_A, \phi)||_1}{||\phi||_1^{-|F|} \cdot T_F(\beta_A, \phi)(0)}\right) = w_A(T_F(\beta_A, \phi))\,.$$

Now let $\psi = \phi' * \phi'$ and notice that $\psi \in \mathcal{P}_A(G)$ ($\phi' = \widetilde{\phi}'$ by Lemma 3.40, now apply Lemma 3.41) and $||\psi||_1 = 1$. Furthermore,

$$T_F(\beta_A, \psi)(0) = \int_G T_F(\beta_A, \phi')(x) \cdot T_F(\beta_A, \phi')(-x) d\mu \leqslant T_F(\beta_A, \phi')(0) \cdot ||T_F(\beta_A, \phi')||_1$$

and, by Lemma 3.37(4), $||T_F(\beta_A, \psi)||_1 = ||T_F(\beta_A, \phi')||_1^2$, thus

$$w_A(T_F(\beta_A, \psi)) \geqslant \log\left(\frac{||T_F(\beta_A, \phi')||_1^2}{T_F(\beta_A, \phi')(0)||T_F(\beta_A, \phi')||_1}\right) = w_A(T_F(\beta_A, \phi')) = w_A(T_F(\beta_A, \phi))\,.$$

For all $n \in \mathbb{N}$ there exists a compact symmetric neighborhood $C_n \in \mathcal{C}_A(G)$ such that $\phi_n = \phi' \cdot \chi_{C_n}$ satisfies $||\phi' - \phi_n||_1 < 1/(2n)$. We let $\psi_n = \phi_n * \phi_n \in \mathcal{P}_A(G)$ ($\phi_n = \widetilde{\phi_n}$ and so one can use Lemma 3.41). Furthermore, using Lemma 3.37 and the fact that $||\phi_n||_1 \leqslant ||\phi'||_1 = 1$, we get

$$\begin{aligned} 1 - ||\psi_n||_1 = ||\psi||_1 - ||\psi_n|| &\overset{3.34}{\leqslant} ||\psi - \psi_n||_1 \leqslant ||\psi - \phi' * \phi_n||_1 + ||\phi' * \phi_n - \psi_n||_1 \\ &\leqslant ||\phi'||_1||\phi' - \phi_n||_1 + ||\phi_n||_1||\phi' - \phi_n||_1 \leqslant 2||\phi' - \phi_n||_1 < 1/n\,. \end{aligned} \tag{6.2.5}$$

Notice also that $||T_F(\beta_A, \psi)||_1 = \prod_{f \in F} ||\beta_A(f)(\psi)||_1 = \Delta(\alpha F)||\phi'||_1^{2|F|} = \Delta(\alpha F)$ by Lemma 3.37 and (3.31). Similarly, $||T_F(\beta_A, \psi_n)||_1 = \Delta(\alpha F)||\psi_n||_1^{|F|}$. Furthermore, by construction $\phi_n \leqslant \phi'$, thus $\psi_n \leqslant \psi$ and, more generally, $\beta_A(g)(\psi_n) \leqslant \beta_A(g)(\psi)$ for all $g \in \Gamma$ and so $T_F(\beta_A, \psi_n)(0) \leqslant T_F(\beta_A, \psi)(0)$. Putting together all the above computations, we can verify (6.2.3):

$$\begin{aligned} w_A(T_F(\beta_A, \phi)) \leqslant w_A(T_F(\beta_A, \psi)) &= \log\left(\frac{||T_F(\beta_A, \psi_n)||_1}{T_F(\beta_A, \psi)(0)} \cdot \frac{||T_F(\beta_A, \psi)||_1}{||T_F(\beta_A, \psi_n)||_1}\right) \\ &\leqslant w_A(T_F(\beta_A, \psi_n)) + \log\left(\frac{\Delta(\alpha F)||\psi||_1^{|F|}}{\Delta(\alpha F)}\right) \\ &\overset{(6.2.5)}{\leqslant} w_A(T_F(\beta_A, \psi_n)) + |F| \log\left(1 + 1/n\right) \end{aligned}$$

As we said, to conclude we have just to verify that $w_A(T_F(\beta_A, \psi_n)) \leqslant v_A(T_F(\alpha_A, D_n))$, where $D_n = \mathrm{supp}(\psi_n) \in \mathcal{C}_A(G)$. Indeed, let $\chi = \chi_{T_F(\alpha_n, D_n)}$ and notice that, by Lemma 3.37, $\mathrm{supp}(T_F(\beta_A, \psi_n)) \subseteq T_F(\beta_A, D_n)$. Thus,

$$\begin{aligned}(\chi * T_F(\beta_A, \psi_n))(0) &= \int_G \chi(x) \cdot T_F(\beta_A, \psi_n)(-x) d\mu \\ &= \int_G T_F(\beta_A, \psi_n)(x) d\mu = ||T_F(\beta_A, \psi_n)||_1 \, .\end{aligned}$$

The conclusion now follows by Lemma 3.40 and (3.1.1) as in the following computation

$$\begin{aligned}0 &= \log((\chi * T_F(\beta_A, \psi_n))(0)) - \log(||T_F(\beta_A, \psi_n)||_1 \\ &= \log\left(\int_G \chi(x) \cdot T_F(\beta_A, \psi_n)(-x) d\mu\right) - \log\left(||T_F(\beta_A, \psi_n)||_1\right) \\ &= \log\left(\int_{T_F(\alpha_A, D_n)} T_F(\beta_A, \psi_n)(-x) d\mu\right) - \log\left(||T_F(\beta_A, \psi_n)||_1\right) \\ &\leqslant \log(\mu(T_F(\alpha_A, D_n)) \cdot T_F(\beta_A, \psi_n)(0)) - \log\left(||T_F(\beta_A, \psi_n)||_1\right) \\ &= v_A(T_F(\alpha_A, D_n)) - w_A(T_F(\beta_A, \psi_n)) \, .\end{aligned}$$

□

6.2.2 Proof of Proposition 6.6

Let us rapidly recall the situation. We have an LCA group G^* and invertible right Γ-representation $\alpha^* \circlearrowleft G^*$, which induces two left Γ-representations on two different pre-normed semigroups that are functorially associated to G^*:

$$\begin{tikzcd} & \alpha^* \circlearrowleft G^* \arrow[dl, dotted] \arrow[dr, dotted] & \\ \alpha_T \circlearrowleft \mathcal{C}_T(G^*) & & \beta_T \circlearrowleft \mathcal{P}_T(G^*) \end{tikzcd}$$

We have to show that the entropy of the two representations is the same. We divide the proof in two lemmas. We start proving that α_T dominates β_T. Roughly speaking this says that the way in which α^* acts on positive-definte complex functions on G^* is controlled by the action on compact neighborhoods. This implies the inequality "$\leqslant$" in Proposition 6.6, by Proposition 4.13.

Lemma 6.10. *In the above notation, α_T dominates β_T.*

Proof. Let us consider $F \in \mathfrak{s}$ and $\phi \in \mathcal{P}_T(G^*)$. We will show that, for all $n \in \mathbb{N}_+$, there exists a compact neighborhood V_n such that

$$w_T(T_F(\beta_T, \phi)) \leqslant v_T(T_F(\alpha_T, V_n)) + |F| \log((n+1)/n) \, . \tag{6.2.6}$$

In fact, letting $\varepsilon_n = n\phi(0)/(n+1)$, any compact neighborhood $V_n \in \mathcal{C}_T(G)$ contained in

$$V(\phi, n) = \left\{ f \in \widehat{M} : \phi(f) \geqslant \varepsilon_n \right\} \, ,$$

will work. To see this, one can verify that $\alpha_T(g)(V_n) \subseteq V(\beta_T(g)(\phi), n)$ and so

$$\beta_T(g)(\phi) \geqslant \varepsilon_n \cdot \chi_{\alpha_T(g)(V_n)} \, , \quad \forall \; g \in \Gamma \, .$$

In particular, $T_F(\beta_T, \phi) = \prod_{g \in F} \beta_T(g)(\phi) \geqslant \prod_{g \in F} \varepsilon_n \cdot \chi_{\alpha_T(g)(V_n)} = \varepsilon_n^{|F|} \cdot \chi_{T_F(\alpha_T, V_n)}$, where the last equality just follows noticing that the pointwise product of characteristic functions is the characteristic function of the intersection. We obtain that

$$\varepsilon_n^{-|F|} \cdot ||T_F(\beta_T, \phi)||_1 \geqslant ||T_F(\beta_T, \chi_{V_n})||_1 = \mu(T_F(\alpha_T, V_n))\,.$$

Taking logarithms one can derive (6.2.6). □

We can now conclude the proof of Proposition 6.6 showing the converse inequality.

Lemma 6.11. *In the above notation, β_T dominates α_T.*

Proof. Given $U \in \mathcal{C}_T(G^*)$, we can find a function $\phi \in \mathcal{P}_T(G^*)$ such that $\phi(0) = 1$ and $\mathrm{supp}(\phi) \subseteq U$, by Lemma 3.45. By Lemma 3.40, $\phi(0) = 1$ is a maximum for ϕ, thus $\phi \leqslant \chi_U$. Similarly, $\beta_T(g)(\phi) \leqslant \beta_T(g)(\chi_U)$ for all $g \in \Gamma$.
For all $F \in \mathfrak{s}$ we obtain the following inequality $||T_F(\beta_T, \phi)||_1 \leqslant ||T_F(\beta_T, \chi_U)||_1 = \mu(T_F(\alpha_T, U))$. To conclude notice that $T_F(\beta_T, \phi)(0) = \phi(0)^{|F|} = 1$ and so $w_T(T_F(\beta_T, \phi)) \geqslant v_T(T_F(\alpha_T, U))$. □

Part III

Length functions

Chapter 7

Length functions in Grothendieck categories

7.1 A Structure Theorem for length functions

7.1.1 Length functions

In any category $\mathfrak{C}$ it is possible to define real-valued invariants in order to measure various finiteness properties of the objects. In general, we call *invariant* of $\mathfrak{C}$, any map $i : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $i(X) = i(X')$ whenever X and X' are isomorphic objects in $\mathfrak{C}$.

If we make some stronger assumption on the structure of the category $\mathfrak{C}$, we can refine our definition of invariant in order to obtain a more treatable notion. Indeed, suppose that $\mathfrak{C}$ is an Abelian category. In this setting it seems natural to ask that, given a short exact sequence

$$0 \to X_1 \to X_2 \to X_3 \to 0 \tag{7.1.1}$$

in $\mathfrak{C}$, $i(X_2) = i(X_1) + i(X_3)$. In this case, we say that i is *additive* on the sequence (7.1.1). If i is additive on all the short exact sequences of $\mathfrak{C}$ and $i(0) = 0$, then we say that i is an *additive invariant* (or *additive function*).
In the following lemma we collect some useful properties of additive functions.

Lemma 7.1. *Let $\mathfrak{C}$ be an Abelian category and let $i : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be an additive function. Then,*

(1) $i(X) \geqslant i(Y)$ *for every segment Y of $X \in \mathrm{Ob}(\mathfrak{C})$;*

(2) $i(X_1 + X_2) + i(X_1 \cap X_2) = i(X_1) + i(X_2)$ *for every pair of sub-objects X_1, X_2 of $X \in \mathrm{Ob}(\mathfrak{C})$;*

(3) $\sum_{j\,odd} i(X_j) = \sum_{j\,even} i(X_j)$ *for every exact sequence $0 \to X_1 \to X_2 \to \cdots \to X_n \to 0$ in $\mathfrak{C}$.*

A natural assumption in the context of Grothendieck categories is the upper continuity of invariants: given an object $X \in \mathfrak{C}$ and a directed set $\mathcal{S} = \{X_\alpha : \alpha \in \Lambda\}$ of sub-objects of X such that $\sum_\Lambda X_\alpha = X$, we say that i is *continuous* on $\mathcal{S}$ if

$$i(X) = \sup\{i(X_\alpha) : \alpha \in \Lambda\}\,. \tag{7.1.2}$$

If i is continuous on all the directed systems of subobjects of the objects of $\mathfrak{C}$, we say that i is *upper continuous.* Obviously, upper continuity can be defined in arbitrary Abelian categories even if it seems more meaningful when all directed colimits exist and are exact.

Definition 7.2. *Let $\mathfrak{C}$ be an Abelian category. An additive and upper continuous invariant $i : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is said to be a* length function*.*

In what follows we generally denote length functions by the symbol L.

Definition 7.3. *Let $\mathfrak{C}$ be a Grothendieck category. An object $X \in \mathrm{Ob}(\mathfrak{C})$ is* finitely generated *(resp.,* Noetherian*) if and only if its qframe of subobjects $\mathcal{L}(X)$ is compact (resp.,* Noetherian*). Furthermore, a category $\mathfrak{C}$ is* locally finitely generated *(resp,.* locally Noetherian*) if there exists a set $\mathcal{F}$ of generators of $\mathfrak{C}$ such that each $G \in \mathcal{F}$ is finitely generated (resp,.* Noetherian*).*

Given a ring R, the category R-Mod is locally finitely generated (here ${}_RR$ is a finitely generated generator), while R-Mod is locally Noetherian if and only if R is a Noetherian ring.

The usual definition of length function is given in module categories, which are in particular locally finitely generated Grothendieck categories. In this special setting, the usual definition of upper continuity is different (see part (3) of the following proposition). We now show that we are not defining a new notion of upper continuity but just generalizing this concept to arbitrary Grothendieck categories (similar observations, with analogous proofs, were already present in [98] for module categories).

Proposition 7.4. *Let $\mathfrak{C}$ be a Grothendieck category and $L : \mathfrak{C} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be an additive function. Consider the following statements:*

(1) *L is a length function;*

(2) *given an object $M \in \mathrm{Ob}(\mathfrak{C})$, an ordinal κ and a continuous chain $\{M_\alpha : \alpha < \kappa\}$ of sub-objects of M such that $M = \sum_{\alpha<\kappa} M_\alpha$, $L(M) = \sup\{L(M_\alpha) : \alpha < \kappa\}$;*

(3) *for every object $M \in \mathrm{Ob}(\mathfrak{C})$, $L(M) = \sup\{L(F) : F$ finitely generated sub-object of $M\}$.*

Then (1)$\Leftrightarrow$(2) and (2)$\Leftarrow$(3). If $\mathfrak{C}$ is locally finitely generated, then the above statements are all equivalent.

Proof. (1)$\Rightarrow$(2) is trivial since continuous chains are directed posets. On the other hand, consider a directed poset $(I, \leqslant)$ and a direct system $\{M_i : i \in I\}$ of sub-objects of M. If I is finite then I has a maximum, so there is nothing to prove. On the other hand, if I is an infinite set, one shows as in the proof of [47, Lemma 1.2.10], that $(I, \leqslant)$ is the union of a continuous well-ordered chain of directed subsets, each of which has strictly smaller cardinality than I. One concludes by transfinite induction that (2)$\Rightarrow$(1).
Assume now (3) and consider a continuous chain as in part (2). For every finitely generated sub-object F of X, there exists $\alpha < \kappa$ such that $F \leqslant M_\alpha$ and so we obtain that

$$L(M) = \sup\{L(F) : \ F \text{ f.g. sub-object of } M\} \leqslant \sup\{L(M_\alpha) : \alpha < \kappa\} \leqslant L(M)\,.$$

To conclude, notice that if $\mathfrak{C}$ is locally finitely generated, then any object is the direct union of the direct system of its finitely generated sub-objects. In this situation, (1) implies (3). ☐

The notion of length function on a Grothendieck category is quite formal, this is why it seems useful to stop for a while and describe some concrete examples of length functions.

Example 7.5. *The logarithm of the cardinality $\log|-| : \mathbb{Z}\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, where $\log|G| = \infty$ whenever G is not finite, is a length function.*

Example 7.6. *Given a skew field* $\mathbb{K}$, *the* dimension $\dim : \mathrm{Ob}(\mathbb{K}\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is a length function. More generally, given a left Ore domain* D, *the* rank *of a left* D*-module* ${}_DM$ *is defined as* $\mathrm{rk}({}_DM) = \dim(\Sigma^{-1}D \otimes_D M)$. *This gives a length function* $\mathrm{rk} : \mathrm{Ob}(D\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$.

Example 7.7. *Let* $\mathfrak{C}$ *be a Grothendieck category. Then, the* composition length $\ell : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, *defined by* $\ell(M) = \ell(\mathcal{L}(M))$, *is a length function.*

Other examples can be obtained lifting a known length function along a localization functor as shown in Section 7.1.2. Another strategy to produce new examples is that of "linearly combining" some known length functions:

Definition 7.8. (1) *Given a length function* L *of* $\mathfrak{C}$ *and* $\lambda \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, *we consider the function*

$$\lambda L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \quad \textit{such that} \quad \lambda L(M) = \lambda \cdot (L(M))\,,\ \forall M \in \mathrm{Ob}(\mathfrak{C})\,,$$

with the convention that $\infty \cdot 0 = 0 \cdot \infty = 0$*;*

(2) *given a set* I *and additive functions* L_i *of* $\mathfrak{C}$ *for all* $i \in I$, *we consider the function*

$$\sum_{i\in I} L_i : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \quad \textit{such that} \quad \sum_{i\in I} L_i(M) = \sup\left\{\sum_{i\in F} L_i(M) : F \subseteq I \textit{ finite}\right\}\,,$$

for all $M \in \mathrm{Ob}(\mathfrak{C})$.

It is an exercise to prove that the sum of length functions and the multiplication of a length function by a constant are again length functions.

7.1.2 Operations on length functions

Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory on $\mathfrak{C}$. In this section we are going to show how the length functions of $\mathfrak{C}$ are related with the ones of $\mathcal{T}$ and $\mathfrak{C}/\mathcal{T}$.

Proposition 7.9. *In the above notation, there is a bijective correspondence*

$$f : \{\textit{length functions } L \textit{ of } \mathfrak{C} \textit{ with } \mathcal{T} \subseteq \mathrm{Ker}(L)\} \rightleftarrows \{\textit{length functions of } \mathfrak{C}/\mathcal{T}\} : g\ .$$

Proof. The maps f and g are defined in Lemmas 7.10 and 7.11 respectively. It follows by the definitions that they are inverse each other. $\square$

Lemma 7.10. *If* $L_\tau : \mathrm{Ob}(\mathfrak{C}/\mathcal{T}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is a length function, then there exists a unique length function* $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *such that* $L(M) = L_\tau(\mathbf{Q}_\tau(M))$ *for all* $M \in \mathrm{Ob}(\mathfrak{C})$. *Furthermore,* $\mathcal{T} \subseteq \mathrm{Ker}(L)$. *We set* $g(L_\tau) = L$.

Proof. Existence follows by the fact that $\mathbf{Q}_\tau$ is an exact functor that preserves colimits. Uniqueness is clear and the last statement comes from the fact that $\mathcal{T} = \mathrm{Ker}(\mathbf{Q}_\tau)$. $\square$

Lemma 7.11. *If* $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is a length function such that* $\mathcal{T} \subseteq \mathrm{Ker}(L)$, *then there exists a unique length function* $L_\tau : \mathrm{Ob}(\mathfrak{C}/\mathcal{T}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *such that* $L(M) = L_\tau(\mathbf{Q}_\tau(M))$ *for all* $M \in \mathrm{Ob}(\mathfrak{C})$. *We set* $f(L) = L_\tau$.

Proof. For all $M \in \mathrm{Ob}(\mathfrak{C})$, there is an exact sequence of the form $0 \to T_1 \to M \to \mathbf{S}_\tau\mathbf{Q}_\tau(M) \to T_2 \to 0$, with T_1, $T_2 \in \mathcal{T}$. As by hypothesis L is trivial on τ-torsion objects, we obtain by additivity that $L(M) = L(\mathbf{S}_\tau\mathbf{Q}_\tau(M))$. Using this simple observation, we can define

$$L_\tau(N) = L(\mathbf{S}_\tau(N)), \quad \text{for all } N \in \mathrm{Ob}(\mathfrak{C}/\mathcal{T}),$$

and verify that $L(M) = L(\mathbf{S}_\tau\mathbf{Q}_\tau(M)) = L_\tau(\mathbf{Q}_\tau(M))$ as desired. The uniqueness statement follows by the fact that the functor $\mathbf{Q}_\tau$ is essentially surjective. It remains to verify that L_τ is a length function. Indeed, let $0 \to N \to M \to M/N \to 0$ be a short exact sequence in $\mathfrak{C}/\mathcal{T}$. This induces an exact sequence $0 \to \mathbf{S}_\tau(N) \to \mathbf{S}_\tau(M) \to \mathbf{S}_\tau(M/N) \to T \to 0$, with $T \in \mathcal{T}$. Hence, $L_\tau(M) = L(\mathbf{S}_\tau(M)) = L(\mathbf{S}_\tau(N)) + L(\mathbf{S}_\tau(M/N)) + 0 = L_\tau(N) + L_\tau(M/N)$. The proof that L is upper continuous follows by a similar argument and transfinite induction. □

In the first part of this subsection we described how to transfer length functions along the adjoint pair $\mathbf{Q}_\tau : \mathfrak{C} \rightleftarrows \mathfrak{C}/\mathcal{T} : \mathbf{S}_\tau$; now we turn our attention to the adjunction $\mathbf{T}_\tau : \mathfrak{C} \rightleftarrows \mathcal{T} : inc$. In particular, whenever L is a length function on $\mathfrak{C}$, one can define its *restriction* to $\mathcal{T}$ as $L\restriction_\mathcal{T} : \mathcal{T} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, such that $L\restriction_\mathcal{T}(M) = L(M)$ for all $M \in \mathfrak{C}$, and prove that it is a length function. Notice that this can be applied to any full abelian subcategory $\mathcal{T}$ of $\mathfrak{C}$, not only to hereditary torsion classes.

On the other hand, if we start with a length function L on $\mathcal{T}$, we want to find a canonical way to extend it to the bigger category $\mathfrak{C}$. In [98] (see also [97]) Peter Vámos introduced a technique to extend length functions which works in a more general setting. Indeed, let $\mathcal{T}$ be a Serre subclass of $\mathfrak{C}$ and consider a length function L on $\mathcal{T}$. We start defining an invariant (which is not supposed to have any good property but that of being useful for our constructions) $L^* : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ as follows:

$$L^*(X) = \begin{cases} L(X) & \text{if } X \in \mathcal{T}; \\ 0 & \text{otherwise.} \end{cases} \tag{7.1.3}$$

Given an object $M \in \mathrm{Ob}(\mathfrak{C})$ and a series $\sigma : \ 0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = M$ of M, we let

$$\widehat{L}(\sigma) = \sum_{i \leqslant n} L^*(N_i/N_{i-1}).$$

Definition 7.12. *In the above notation, the* Vámos extension $\widehat{L} : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *of L to $\mathfrak{C}$ is the function defined by $\widehat{L}(X) = \sup\{\widehat{L}(\sigma) :$ σ ranging over all the finite series of $X\}$.*

In the next proposition we verify that $\widehat{L}$ satisfies the axioms of a length function.

Proposition 7.13. *Let $\mathcal{T}$ be a Serre subclass of a Grothendieck category $\mathfrak{C}$. If $L : \mathcal{T} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is a length function, then $\widehat{L} : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is a length function. Furthermore, if $L' : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is any additive function extending L, then $L'(M) \leqslant L(M)$ for all $M \in \mathfrak{C}$.*

Proof. We start by proving additivity. Let $0 \longrightarrow A \xrightarrow{\iota} B \xrightarrow{\pi} C \longrightarrow 0$ be a short exact sequence in $\mathfrak{C}$. Consider two series

$$\sigma_A : 0 \subseteq A_0 \subseteq \cdots \subseteq A_n = A\,, \quad \sigma_C : 0 \subseteq C_0 \subseteq \cdots \subseteq C_m = C\,.$$

For all $i = 0, \ldots, m$, let $B_i = \pi^{-1}(C_i)$, this defines a series of B

$$\sigma_B : 0 \subseteq \iota(A_0) \subseteq \cdots \subseteq \iota(A_n) \subseteq B_0 \subseteq \cdots \subseteq B_m = B\,.$$

Clearly, $\widehat{L}(\sigma_B) = \widehat{L}(\sigma_A) + \widehat{L}(\sigma_C)$, proving that $\widehat{L}(B) \geqslant \widehat{L}(A) + \widehat{L}(C)$. On the other hand, given a series $\sigma_B : 0 \subseteq B_0 \subseteq \cdots \subseteq B_n = B$, we let, for all $i = 1, \dots, n$, $A_i = \iota^{-1}(B_i)$ and $C_i = \pi(B_i)$. This defines two series

$$\sigma_A : 0 \subseteq A_0 \subseteq \cdots \subseteq A_n = A\,, \quad \sigma_C : 0 \subseteq C_0 \subseteq \cdots \subseteq C_n = C\,.$$

Furthermore, there are short exact sequences $0 \to A_i/A_{i-1} \to B_i/B_{i-1} \to C_i/C_{i-1} \to 0$. Using the additivity of L^* on $\mathcal{T}$ and the closure properties of $\mathcal{T}$, $\widehat{L}(\sigma_A) + \widehat{L}(\sigma_C) \geqslant \widehat{L}(\sigma_B)$, which implies that $\widehat{L}(B) \leqslant \widehat{L}(A) + \widehat{L}(C)$. It remains to prove upper continuity. Indeed, let $M \in \mathfrak{C}$ and consider a directed set $\{M_\alpha : \alpha \in \Lambda\}$ of sub-objects such that $\sum_\Lambda M_\alpha = M$. By additivity, $\widehat{L}(M) \geqslant \sup_\Lambda \widehat{L}(M_\alpha)$. On the other hand, given a series $\sigma :\ 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_n = M$, we prove by induction on $n \in \mathbb{N}_+$ that $\widehat{L}(\sigma) \leqslant \sup_\Lambda \widehat{L}(M_\alpha)$. We distinguish two cases:

(1) if $\widehat{L}(\sigma) = \infty$, then there exists a non-negative integer $m < n$ such that $N_{m+1}/N_m \in \mathcal{T}$ and $L(N_{m+1}/N_m) = \infty$. Notice also that $N_{m+1}/N_m = \sum_\Lambda((M_\alpha \cap N_{m+1}) + N_m)/N_m$ and so,

$$\begin{aligned}\sup_\Lambda \widehat{L}(M_\alpha) &\geqslant \sup_\Lambda \widehat{L}\left(\frac{(M_\alpha \cap N_{m+1}) + N_m}{N_m}\right)\\ &= \sup_\Lambda L\left(\frac{(M_\alpha \cap N_{m+1}) + N_m}{N_m}\right) = L(N_{m+1}/N_m) = \infty\,,\end{aligned}$$

where the first inequality comes by additivity of $\widehat{L}$ and the following equalities come by the fact that $\widehat{L}$ coincides with L on $\mathcal{T}$.

(2) Suppose now that $\widehat{L}(\sigma) < \infty$. If $n = 1$, then either $\widehat{L}(\sigma) = 0$ and there is nothing to prove, or $0 < \widehat{L}(\sigma) = L^*(M)$, but in this case $M \in \mathcal{T}$ and the thesis follows by the fact that L is a length function on $\mathcal{T}$. On the other hand, if $n > 1$, let

$$\sigma' :\ \ 0 = N_0 \subsetneq N_1 \subsetneq \cdots \subsetneq N_{n-1}\,, \text{ and } \quad \sigma'' :\ \ 0 \subsetneq N_n/N_{n-1}\,,$$

and notice that $\widehat{L}(\sigma) = \widehat{L}(\sigma') + \widehat{L}(\sigma'')$. Furthermore, $N_{n-1} = \sum_\Lambda(N_{n-1} \cap M_\alpha)$ and $N_n/N_{n-1} = \sum_\Lambda(M_\alpha + N_{n-1})/N_{n-1}$. By inductive hypothesis $\widehat{L}(\sigma') \leqslant \sup_\Lambda \widehat{L}(N_{n-1} \cap M_\alpha)$ and $\widehat{L}(\sigma'') \leqslant \sup_\Lambda \widehat{L}((M_\alpha + N_{n-1})/N_{n-1})$. Hence,

$$\widehat{L}(\sigma) = \widehat{L}(\sigma') + \widehat{L}(\sigma'') \leqslant \sup_\Lambda \widehat{L}(N_{n-1} \cap M_\alpha) + \sup_\Lambda \widehat{L}((M_\alpha + N_{n-1})/N_{n-1}) = \sup_\Lambda \widehat{L}(M_\alpha)\,,$$

where the last equality comes from the additivity of $\widehat{L}$ and the fact that the sum of suprema of two convergent nets is the supremum of the sum of the two nets.

□

We conclude this section proving that Vámos extension and lifting of length functions via a localization functor preserve linear combinations.

Lemma 7.14. *Let $\mathcal{T}$ be a Serre subclass of a Grothendieck category $\mathfrak{C}$, let Λ be a set and choose $\lambda(\alpha) \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ for all $\alpha \in \Lambda$.*

(1) *Given length functions $L_\alpha : \mathcal{T} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ for all $\alpha \in \Lambda$ and letting $L = \sum_\Lambda \lambda(\alpha) L_\alpha$, $\widehat{L} = \sum_\Lambda \lambda(\alpha)\widehat{L_\alpha}$.*

(2) *If $\mathcal{T}$ is closed under direct limits, $L_\alpha : \mathrm{Ob}(\mathfrak{C}/\mathcal{T}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ are length functions for all $\alpha \in \Lambda$ and $L = \sum_\Lambda \lambda(\alpha) L_\alpha$, then $g(L) = \sum_\Lambda \lambda(\alpha) g(L_\alpha)$, where g is defined in Lemma 7.10.*

Proof. (1) By the minimality of Vámos extension proved in Proposition 7.13, $\widehat{L} \leqslant \sum_\Lambda \lambda(\alpha)\widehat{L_\alpha}$. On the other hand, if $M \in \mathrm{Ob}(\mathfrak{C})$ and $\widehat{L}(M) = \infty$, then $\infty = \widehat{L}(M) \leqslant \sum_\Lambda \lambda(\alpha)\widehat{L_\alpha}(M)$ and there is nothing to prove. It remains to show that, if $\widehat{L}(M) < \infty$, then $\widehat{L}(M) \geqslant \sum_\Lambda \lambda(\alpha)\widehat{L_\alpha}(M)$. The case $|\Lambda| < \infty$ is essentially an application of Lemma 2.18. Suppose now that Λ is not finite and let $L_F = \sum_{\alpha\in F} \lambda(\alpha)L_\alpha$ for every non-empty finite subset $F \subseteq \Lambda$; by definition $L(M) = \sup\{L_F(M) : F \subseteq \Lambda \text{ finite}\}$. By the first part of the proof, $\widehat{L_F} = \sum_{\alpha\in F} \lambda(\alpha)\widehat{L_\alpha}$, so we have only to prove that $\widehat{L}(M) \geqslant \sup\{\widehat{L_F}(M) : F \subseteq \Lambda \text{ finite}\}$ for all $M \in \mathrm{Ob}(\mathfrak{C})$. This follows noticing that $\widehat{L_F}(\sigma) \leqslant \widehat{L}(\sigma)$ for any finite $F \subseteq \Lambda$ and any series σ of M.

(2) follows by definition of the map g. □

7.1.3 The classification in the semi-Artinian case

All along this subsection we denote by $\mathfrak{C}$ a *semi-Artinian* Grothendieck category, that is, a Gabriel category whose Gabriel dimension is 1. Notice that this is equivalent to say that any object in $\mathfrak{C}$ is the union of its socle series. The main result of this subsection is to give a structure theorem for all the length functions in $\mathfrak{C}$.

Lemma 7.15. *Let $\mathfrak{C}$ be a semi-Artinian Grothendieck category and let $L, L' : \mathfrak{C} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be two length functions. Then $L = L'$ if and only if their values on simple objects are the same.*

Proof. One implication is trivial, so suppose that L and L' coincide on simple objects. Consider an object $M \in \mathfrak{C}$ and write it as the union of a continuous chain

$$0 = N_0 \leqslant N_1 \leqslant \cdots \leqslant N_\alpha \leqslant \cdots \leqslant \bigcup_\alpha N_\alpha = M\,,$$

such that N_{i+1}/N_i is a simple object for all i (this can be done since M is the union of its socle series). By hypothesis $L(N_{i+1}/N_i) = L'(N_{i+1}/N_i)$ for all i. The conclusion follows by transfinite induction using additivity and upper continuity. □

Definition 7.16. *Let $\mathfrak{C}$ be a semi-Artinian Grothendieck category and let $\pi = (\mathcal{T}, \mathcal{F}) \in \mathrm{Sp}(\mathfrak{C})$. We let $\ell_\pi : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be the length function such that*

$$\ell_\pi(M) = \ell(\mathbf{Q}_\pi(M)) \quad \textit{such that} \quad \forall M \in \mathrm{Ob}(\mathfrak{C})\,.$$

That is, ℓ_π is the lifting of the composition length $\ell : \mathrm{Ob}(\mathfrak{C}/\mathcal{T}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ along the localization functor $\mathbf{Q}_\pi : \mathfrak{C} \to \mathfrak{C}/\mathcal{T}$. The functions of the form ℓ_π with $\pi \in \mathrm{Sp}(\mathfrak{C})$ are called atomic length function.

Lemma 7.17. *Let $\mathfrak{C}$ be a semi-Artinian Grothendieck category, let $\pi = (\mathcal{T}, \mathcal{F}) \in \mathrm{Sp}(\mathfrak{C})$, let $C(\pi)$ be the socle of $E(\pi)$ (which is a simple object) and let C be a simple object. Then,*

$$\ell_\pi(C) = \begin{cases} 1 & \textit{if } C \cong C(\pi); \\ \ell_\pi(C) = 0 & \textit{otherwise.} \end{cases}$$

Proof. By definition of ℓ_π, it is clear that $\ell_\pi(C) = 1$ if and only if $C \notin \mathcal{T}$ and that $\ell_\pi(C) = 0$ otherwise. So let us prove that $C \notin \mathcal{T}$ if and only if $C \cong C(\pi)$. Indeed, if there is an isomorphism $C \to C(\pi)$ one takes the composition with the inclusion $C(\pi) \to E(\pi)$ to get $\mathrm{Hom}_\mathfrak{C}(C, E(\pi)) \neq 0$, so $C \notin \mathcal{T}$. On the other hand, suppose $\mathrm{Hom}_\mathfrak{C}(C, E(\pi)) \neq 0$, then $\mathrm{Hom}_\mathfrak{C}(C, C(\pi)) = \mathrm{Hom}_\mathfrak{C}(C, \mathrm{Soc}(E(\pi))) \neq 0$ and, since any non-trivial morphism between two simple objects is an isomorphism, $C \cong C(\pi)$. □

In the following theorem we prove that any length function in $\mathfrak{C}$ is a linear combination of atomic length functions.

Theorem 7.18. *Let $\mathfrak{C}$ be a semi-Artinian Grothendieck category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. Then,*

$$L = \sum_{\pi \in \mathrm{Sp}(\mathfrak{C})} \lambda(\pi) \cdot \ell_\pi \,,$$

where $\lambda(\pi) = L(C(\pi))$, with $C(\pi) = \mathrm{Soc}(E(\pi))$ for all $\pi \in \mathrm{Sp}(\mathfrak{C})$. Furthermore, the constants $\lambda(\pi)$ are uniquely determined by L.

Proof. Let $L' = \sum_{\pi \in \mathrm{Sp}(\mathfrak{C})} \lambda(\pi) \cdot \ell_\pi$; we already mentioned that a linear combination of length functions is a length function, so L' is a length function. By Lemma 7.17, $\ell_\pi(C(\pi')) = 0$ for all $\pi' \neq \pi$, so $L'(C(\pi)) = \lambda(\pi)\ell_\pi(C(\pi)) = L(C(\pi))$ for all $\pi \in \mathrm{Sp}(\mathfrak{C})$, which shows that $L = L'$, by Lemma 7.15. The proof of the uniqueness statement is analogous. □

Using the uniqueness of the above decomposition, we can unambiguously give the following definition:

Definition 7.19. *Let $\mathfrak{C}$ be a semi-Artinian Grothendieck category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. The* support *of L is $\mathrm{Supp}(L) = \{\pi \in \mathrm{Sp}(\mathfrak{C}) : \lambda(\pi) \neq 0\}$.*

As an immediate consequences of the above theorem we obtain the following corollaries.

Corollary 7.20. *Let $\mathfrak{C}$ be a semi-Artinian Grothendieck category such that $|\mathrm{Supp}(\mathfrak{C})| = 1$. Then, any length function $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is a multiple of the composition length.*

Corollary 7.21. *Let D be a left Ore domain and denote by $\mathcal{T} \subseteq D\text{-Mod}$ the class of torsion left D-modules. The following are equivalent for a non-trivial length function $L : \mathrm{Ob}(D\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$:*

(1) *$\mathcal{T} \subseteq \mathrm{Ker}(L)$;*

(2) *there exists $\alpha \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $L = \alpha \cdot \mathrm{rk}$ (see Example 7.6).*

Furthermore, if $L(D) < \infty$ then the above equivalent conditions hold and $\alpha = L(D)$ in (2).

Proof. The implication (2)⇒(1) is trivial, in fact $\mathrm{Ker}(\mathrm{rk}) = \mathcal{T}$. Let us prove that (1)⇒(2). By Theorem 7.9, L is the lifting of a length function on $D\text{-Mod}/\mathcal{T} \cong Q\text{-Mod}$, where Q is the skew field of left fractions of D. Clearly, Q-Mod satisfies the hypotheses of the above corollary, thus there exists $\alpha \in \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $L = g(\alpha \cdot \dim_Q) = \alpha \cdot g(\dim_Q) = \alpha \cdot \mathrm{rk}$.
For the last statement, we suppose $L(D) < \infty$ and we verify (1). Let M be a torsion left D-module. Using upper continuity and additivity we can prove that $L(M) = 0$ if and only if $L(Dx) = 0$ for all $x \in M$. Thus, let $x \in M$ be a non-trivial element and consider the following exact sequences

$$0 \to \mathrm{Ann}_D(x) \to D \to Dx \to 0 \quad \text{and} \quad 0 \to D \to \mathrm{Ann}_D(x)\,,$$

where the second sequence exists as $\mathrm{Ann}_D(x)$ is a non-trivial left ideal of D (as M is torsion) and D is a domain. This shows that $L(D) = L(Dx) + L(\mathrm{Ann}_D(x)) \geqslant L(Dx) + L(D)$; thus $L(Dx) \leqslant L(D) - L(D) = 0$. Finally, $L(D) = \alpha \cdot \mathrm{rk}(D) = \alpha$. □

7.1.4 The main structure theorem

The main result of this subsection is to show that, analogously to the semi-Artinian case, any length function on a Gabriel category $\mathfrak{C}$ can be written as a linear combination of atomic length functions. We start defining atomic length functions in this context.

Definition 7.22. *Let $\mathfrak{C}$ be a Gabriel category, let $\alpha < \mathrm{G.dim}(\mathfrak{C})$ and let $\overline{\mathfrak{C}}_{\alpha+1} = \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$. Then, $\overline{C}_{\alpha+1}$ is semi-Artinian and so, given $\pi = (\mathcal{T},\mathcal{F}) \in \mathrm{Sp}(\overline{\mathfrak{C}}_{\alpha+1}) = \mathrm{Sp}^\alpha(\mathfrak{C})$, the length function $\ell_\pi : \mathrm{Ob}(\overline{\mathfrak{C}}_{\alpha+1}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is defined, as in the previous subsection, to be the lifting of the composition length in $\overline{\mathfrak{C}}_{\alpha+1}/\mathcal{T}$. Using Lemma 7.10 we can uniquely lift ℓ_π to a length function on $\mathfrak{C}_{\alpha+1}$ such that $\mathfrak{C}_\alpha \subseteq \mathrm{Ker}(\ell_\pi)$ and then, using Vámos extension, we extend it to a length function of $\mathfrak{C}$. Abusing notation, we denote this new function again by $\ell_\pi : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$. The functions of the form ℓ_π with $\pi \in \mathrm{Sp}(\mathfrak{C})$ are called* atomic length function*.*

Notice that, by definition, given $\pi \in \mathrm{Sp}^\alpha(\mathfrak{C})$, $\mathfrak{C}_\alpha \subseteq \mathrm{Ker}(\ell_\pi)$.

Definition 7.23. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. An object $M \in \mathrm{Ob}(\mathfrak{C})$ is L-finite if $L(M) < \infty$, we let $\mathrm{Fin}(L)$ be the Serre class of all the L-finite objects. Furthermore, we denote by $\overline{\mathrm{Fin}}(L)$ the minimal torsion class containing $\mathrm{Fin}(L)$.*

Definition 7.24. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. The* finite component *L^{fin} of L is the Vámos extension to $\mathfrak{C}$ of the restriction of L to $\mathrm{Fin}(L)$, that is $L^{fin} = \widehat{L\restriction_{\mathrm{Fin}(L)}} : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$.*
The infinite component *L^∞ of L is defined by*

$$L^\infty(M) = \begin{cases} 0 & \text{if } M \in \overline{\mathrm{Fin}}(L); \\ \infty & \text{otherwise.} \end{cases}$$

We remark that the finite component L^{fin} can very well assume infinite values (if L^{fin} is non-trivial, just take M such that $L^{fin}(M) \neq 0$, so $L^{fin}(\bigoplus_{\mathbb{N}} M) = \infty$); anyway its name is justified by the fact that L^{fin} is, by definition, determined by the finite values of L.

Lemma 7.25. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. Then, both the finite and the infinite component of L are length functions.*

Proof. The fact that L^{fin} is a legnth function follows by Proposition 7.13. On the other hand, given a short exact sequence $0 \to N \to M \to M/N \to 0$, $M \in \overline{\mathrm{Fin}}(L)$ if and only if N and $M/N \in \overline{\mathrm{Fin}}(L)$. Similarly, given an object $M \in \mathfrak{C}$ and a directed system of sub-objects $\{N_i : i \in I\}$ such that $\sum_{i\in I} N_i = M$, $M \in \overline{\mathrm{Fin}}(L)$ if and only if $N_i \in \overline{\mathrm{Fin}}(L)$ for all $i \in I$. Thus, also L^∞ is a length function. □

Notice that $L = L^{fin} + L^\infty$. This decomposition of L allows us to reduce the problem of finding a presentation of L as linear combination of atomic length functions to the same problem for L^∞ and L^{fin}.

Definition 7.26. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. For all $\alpha < \mathrm{G.dim}(\mathfrak{C})$ we let $\tau_\alpha^L = (\mathcal{T}_\alpha^L, \mathcal{F}_\alpha^L)$ be the torsion theory in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ such that $\mathcal{T}_\alpha^L = \{\mathbf{Q}_\alpha(M) : M \in \overline{\mathrm{Fin}}(L)\}$. The* infinite support *of L is the following subset of the spectrum*

$$\mathrm{Supp}^\infty(L) = \bigcup_{\alpha<\kappa} \mathrm{Supp}^\infty_\alpha(L)\,, \quad \textit{where} \quad \mathrm{Supp}^\infty_\alpha(L) = \{\pi \in \mathrm{Sp}^\alpha(\mathfrak{C}) : C(\pi) \in \mathcal{F}_\alpha^L\}\,.$$

Notice that in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, $\mathcal{T}_\alpha^L = \bigcap_{\pi\in\mathrm{Supp}_\alpha^\infty}\mathrm{Ker}(\ell_\pi)$, to show this use the correspondences described in Theorem 2.67 and the fact that torsion free classes are closed under taking injective envelopes.

Proposition 7.27. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. Then, $L^\infty = \sum_{\pi\in\mathrm{Supp}^\infty(L)} \infty\cdot\ell_\pi$.*

Proof. Let $L' = \sum_{\pi\in\mathrm{Supp}^\infty(L)} \infty\cdot\ell_\pi$. Both L^∞ and L' take values in $\{0,\infty\}$, thus they coincide if and only if $\mathrm{Ker}(L^\infty) = \mathrm{Ker}(L')$:

$$\mathrm{Ker}(L^\infty) = \overline{\mathrm{Fin}}(L) = \bigcap_{\alpha<\kappa}\{M\in\mathfrak{C} : \mathbf{Q}_\alpha\mathbf{T}_{\alpha+1}(M)\in\mathcal{T}_\alpha^L\} = \bigcap_{\pi\in\mathrm{Supp}^\infty(L)}\mathrm{Ker}(\ell_\pi) = \mathrm{Ker}(L')\,.$$

□

We can now turn our attention to the decomposition of L^{fin}.

Lemma 7.28. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. Then, there exists a unique family $\{L_{(\alpha)} : \mathrm{Ob}(\mathfrak{C}_{\alpha+1}) \to \mathbb{R}_{\geqslant 0}\cup\{\infty\} : \alpha < \mathrm{G.dim}(\mathfrak{C})\}$ of length functions such that*

(1) *$L^{fin} = \sum_{\alpha<\mathrm{G.dim}(\mathfrak{C})} L_\alpha$, where $L_\alpha : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0}\cup\{\infty\}$ is the Vámos extension of $L_{(\alpha)}$;*

(2) *$\mathfrak{C}_\alpha \subseteq \mathrm{Ker}(L_{(\alpha)})$, for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$.*

Proof. For all $\alpha \leqslant \mathrm{G.dim}(\mathfrak{C})$ we consider the Serre classes $\mathrm{Fin}^{(\alpha)}(L) = \mathrm{Fin}(L)\cap\mathfrak{C}_\alpha$ and $\mathrm{Fin}(L)$. We start defining inductively length functions $L^{(\alpha)} : \mathrm{Fin}^{(\alpha+1)}(L) \to \mathbb{R}_{\geqslant 0}\cup\{\infty\}$, and their Vámos extensions $L^\alpha : \mathrm{Fin}(L) \to \mathbb{R}_{\geqslant 0}\cup\{\infty\}$, for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$:

- $L^{(0)}(M) = L(M)$, for all $M \in \mathrm{Fin}^{(1)}(L)$;
- $L^{(\alpha)}(M) = L(M) - \sum_{\beta<\alpha} L^\beta(M)$, for all $M\in\mathrm{Fin}^{(\alpha+1)}(L)$ and $\alpha < \mathrm{G.dim}(\mathfrak{C})$.

It is not difficult to verify by transfinite induction that all the $L^{(\alpha)}$ and L^α are length functions. Let us verify the following claims by induction on $\mathrm{G.dim}(\mathfrak{C})$:

(1') $L(M) = \sum_{\alpha<\mathrm{G.dim}(\mathfrak{C})} L^\alpha(M)$, for all $M\in\mathrm{Fin}(L)$;

(2') $\mathrm{Fin}^{(\alpha)}(L) \subseteq \mathrm{Ker}(L^{(\alpha)})$, for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$.

If $\mathrm{G.dim}(\mathfrak{C}) = 1$ (i.e., $\mathfrak{C}$ is semi-Artinian), then $L(M) = L^{(0)}(M) = L^0(M)$ for all $M \in \mathrm{Fin}^{(1)}(L) = \mathrm{Fin}(L)$, proving (1'), while (2') is trivial since $\mathrm{Fin}^{(0)}(L) = \{0\}$.
Suppose now that $\mathrm{G.dim}(\mathfrak{C})$ is a limit ordinal. If $M\in\mathrm{Fin}^{(\alpha)}(L)$ for some $\alpha < \mathrm{G.dim}(\mathfrak{C})$, then by inductive hypothesis, $L(M) = \sum_{\beta<\alpha} L^\beta(M)$ and so $L^{(\alpha)}(M) = L(M) - \sum_{\beta<\alpha} L^\beta(M) = 0$, proving (2'). Furthermore, given $M\in\mathrm{Fin}(L)$, we can write $M = \bigcup_{\beta<\mathrm{G.dim}(\mathfrak{C})}(\mathbf{T}_\beta(M))$ (see Lemma 2.56). Then,

$$\begin{aligned} L(M) &= \sup_{\beta<\mathrm{G.dim}(\mathfrak{C})} L(\mathbf{T}_\beta(M)) = \sup_{\beta<\mathrm{G.dim}(\mathfrak{C})}\left\{\sum_{\alpha\leqslant\beta} L^\alpha(\mathbf{T}_\beta(M))\right\} \\ &= \sup_{\beta<\mathrm{G.dim}(\mathfrak{C})}\left\{\sum_{\alpha<\mathrm{G.dim}(\mathfrak{C})} L^\alpha(\mathbf{T}_\beta(M))\right\} = \sum_{\alpha<\mathrm{G.dim}(\mathfrak{C})} L^\alpha(M)\,, \end{aligned}$$

where the first equality follows by the upper continuity of L, the second one follows by part (1') of the inductive hypothesis ($\mathbf{T}_\beta(M) \in \mathfrak{C}_\beta$ and $\mathrm{G.dim}(\mathfrak{C}_\beta) = \beta < \mathrm{G.dim}(\mathfrak{C})$), the third one follows by the, already established, claim (2'), and the last equality uses the upper continuity of $\sum_{\alpha<\mathrm{G.dim}(\mathfrak{C})} L^\alpha$.

Finally, if $\mathrm{G.dim}(\mathfrak{C}) = \kappa+1$ is a successor ordinal, and $M \in \mathrm{Fin}^{(\kappa)}(L)$, then $L(M) = \sum_{\alpha<\kappa} L^\alpha(M)$ by inductive hypothesis and so $L^{(\kappa)}(M) = L(M) - \sum_{\alpha<\kappa} L^\alpha(M) = 0$, proving (2'). Furthermore, for any $M \in \mathrm{Fin}(L)$,

$$\sum_{\alpha\leqslant\kappa} L^\alpha(M) = \sum_{\alpha<\kappa} L^\alpha(M) + L^\kappa(M) \overset{(*)}{=} \sum_{\alpha<\kappa} L^\alpha(M) + L(M) - \sum_{\alpha<\kappa} L^\alpha(M) = L(M)\,,$$

where $(*)$ comes by the fact that $\mathfrak{C}_{\kappa+1} = \mathfrak{C}$ so $\mathrm{Fin}^{(\kappa+1)}(L) = \mathrm{Fin}(L)$ and $L^\kappa = L^{(\kappa)}$. Thus, also (1') is established.

For any $\alpha < \mathrm{G.dim}(\mathfrak{C})$ we define the length functions $L_{(\alpha)} : \mathfrak{C}_{\alpha+1} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ and $L_\alpha : \mathfrak{C} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$, as the Vámos extensions of $L^{(\alpha)}$ and L^α respectively. We can extend the above claims (1') and (2') to these new functions as follows. First of all, $L^{fin}(M) = \sum_{\alpha<\mathrm{G.dim}(\mathfrak{C})} L_\alpha(M)$ by (1') and the fact that Vámos extension preserves linear combinations. Furthermore, for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$ and $M \in \mathfrak{C}_\alpha$, $L_{(\alpha)}(M) = 0$ by the construction of Vámos extension and since, by (2'), $L^{(\alpha)}$ vanishes on all the factors belonging $\mathrm{Fin}^{(\alpha)}(\mathfrak{C})$ of any series of M. This shows that $\mathfrak{C}_\alpha \subseteq \mathrm{Ker}(L_{(\alpha)})$.

The proof of the uniqueness can be obtained by transfinite induction on $\mathrm{G.dim}(\mathfrak{C})$. □

By the above lemma, we have uniquely determined length functions $L_{(\alpha)} : \mathrm{Ob}(\mathfrak{C}_{\alpha+1}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$, such that $L_{(\alpha)}$ is trivial on $\mathfrak{C}_\alpha$. Thus, there exist unique length functions

$$\overline{L}_\alpha : \mathrm{Ob}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \tag{7.1.4}$$

such that $L_{(\alpha)}(M) = \overline{L}_\alpha(\mathbf{Q}_\alpha(M))$ for all $M \in \mathrm{Ob}(\mathfrak{C}_{\alpha+1})$, by Lemma 7.11. Notice also that all the categories of the form $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ are semi-artinian and so we have a well defined notion of support for the functions $\overline{L}_\alpha$.

Definition 7.29. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. The* finite support *of L is the following subset of $\mathrm{Sp}(\mathfrak{C})$:*

$$\mathrm{Supp}^{fin}(L) = \bigcup_{\alpha<\kappa} \mathrm{Supp}(\overline{L}_\alpha)\,.$$

With the above notion of support we can finally decompose L^{fin} as linear combination of atomic length functions.

Proposition 7.30. *Let $\mathfrak{C}$ be a Gabriel category and let $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. Then, there is a unique choice of constants $\lambda(\pi) \in \mathbb{R}_{>0}$ so that*

$$L^{fin} = \sum_{\pi\in\mathrm{Supp}^{fin}(L)} \lambda(\pi) \cdot \ell_\pi\,.$$

Proof. For all $\alpha < \mathrm{G.dim}(\mathfrak{C})$, we have a uniquely determined decomposition

$$\overline{L}_\alpha = \sum_{\pi\in\mathrm{Supp}(\overline{L}_\alpha)} \lambda(\pi)\ell_\pi$$

in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, by Theorem 7.18. Furthermore, by Lemma 7.14, this decomposition can be lifted to a decomposition in $\mathfrak{C}_{\alpha+1}$ and then extended to a decomposition in $\mathfrak{C}$, obtaining that

$$L_\alpha = \sum_{\pi \in \mathrm{Supp}(\overline{L}_\alpha)} \lambda(\pi)\ell_\pi\,, \quad \text{for all } \alpha < \kappa\,.$$

The desired decomposition now follows by Lemma 7.28. □

We conclude this section summarizing the main results on decomposition of length functions in Gabriel categories in the following theorem. We remark that this statement is analogous to the "Main Decomposition Theorem" in [98]

Theorem 7.31. *Let $\mathfrak{C}$ be a Gabriel category and $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ a length function. Then, there is a unique way to choose constants $\lambda(\pi) \in \mathbb{R}_{>0}$, for all $\pi \in \mathrm{Supp}^{fin}(L)$ such that*

$$L = L^{fin} + L^\infty = \sum_{\pi \in \mathrm{Supp}^{fin}(L)} \lambda(\pi) \cdot \ell_\pi + \sum_{\pi \in \mathrm{Supp}^\infty(L)} \infty \cdot \ell_\pi\,.$$

7.2 Length functions compatible with self-equivalences

Let $\mathfrak{C}$ be a Grothendieck category and recall that a functor $F : \mathfrak{C} \to \mathfrak{C}$ is an equivalence of categories if and only if

(Eq. 1) F is *essentially surjective*, i.e., for all $X \in \mathrm{Ob}(\mathfrak{C})$, there exists $Y \in \mathrm{Ob}(\mathfrak{C})$ such that $F(X) \cong Y$;

(Eq. 2) F is *fully faithful*.

A consequence of this characterization of self-equivalences is that any such functor preserves all the structures defined by universal properties, in particular, it commutes with direct and inverse limits and it preserves exactness of sequences. Furthermore, it commutes with injective envelopes and it preserves lattices of subobjects.

Let now $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function and fix a self-equivalence $F : \mathfrak{C} \to \mathfrak{C}$. It is easily seen that $L_F : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $L_F(M) = L(F(M))$ for all $M \in \mathrm{Ob}(\mathfrak{C})$ is a length function. In what follows we are going to study to what extent L_F can differ from L. The following example shows that L and L_F may be very different.

Example 7.32. *Consider a field K and consider the category $K \times K$-Mod $\cong K$-Mod $\times K$-Mod. This category is semi-Artinian and it has a self-equivalence $F : K \times K\text{-Mod} \to K \times K\text{-Mod}$ such that $(M, N) \mapsto (N, M)$ and $(\phi, \psi) \mapsto (\psi, \phi)$. If we take L to be the length function such that $L((M, N)) = \dim_K(M)$, then clearly $L_F((M, 0)) = 0 \neq \dim(M) = L((M, 0))$, provided $M \neq 0$.*

Definition 7.33. *Given a Grothendieck category $\mathfrak{C}$ and a self-equivalence $F : \mathfrak{C} \to \mathfrak{C}$, we say that a length function $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is* compatible *with F provided $L_F(M) = L(F(M)) = L(M)$ for all $M \in \mathrm{Ob}(\mathfrak{C})$.*

In this section we exploit the classification of length functions in Gabriel categories to find a necessary and sufficient condition on a length function to be compatible with a self-equivalence. Our motivation for studying compatibility of length functions with self-equivalences is the following. Given a ring R and a ring automorphism $\phi : R \to R$, we obtain a restriction of scalars

$$F_\phi : R\text{-Mod} \to R\text{-Mod}\,. \tag{7.2.1}$$

Notice that F_ϕ is a self-equivalence since clearly $F_\phi \circ F_{\phi^{-1}} \cong \mathrm{id}_{R\text{-Mod}} \cong F_{\phi^{-1}} \circ F_\phi$. We are interested in finding length functions L such that $L(F_\phi(M)) = L(M)$.

7.2.1 Orbit-decomposition of the Gabriel spectrum

We start with a technical result. Let $\mathcal{A}$ be a subclass of $\mathfrak{C}$, we denote by $\widetilde{\mathcal{A}}$ the class of all the objects of $\mathfrak{C}$ which are isomorphic to some object in $\mathcal{A}$.

Lemma 7.34. *Let $\mathfrak{C}$ be a Grothendieck category, let $F : \mathfrak{C} \to \mathfrak{C}$ a self-equivalence and let $\tau = (\mathcal{T}, \mathcal{F})$ be a torsion theory. The following are equivalent:*

(1) $\widetilde{F(\mathcal{T})} = \mathcal{T}$;

(2) $\widetilde{F(\mathcal{F})} = \mathcal{F}$;

(3) *given $X \in \mathrm{Ob}(\mathfrak{C})$, X is τ-local if and only if FX is τ-local;*

(4) $F\mathbf{L}_\tau = \mathbf{L}_\tau F$.

Proof. The equivalence between (1) and (2) follows since $\mathrm{Hom}_{\mathfrak{C}}(A, B) \cong \mathrm{Hom}_{\mathfrak{C}}(F(A), F(B))$, for all A, $B \in \mathrm{Ob}(\mathfrak{C})$ and by the fact that $\mathcal{F} = \mathcal{T}^{\perp}$ and $\mathcal{T} = {}^{\perp}\mathcal{F}$.
(2)⇒(3). Given a τ-local $X \in \mathrm{Ob}(\mathfrak{C})$, one can consider the following exact sequence

$$0 \to \mathbf{T}_\tau(FX) \to FX \to \mathbf{L}_\tau FX \to T \to 0\,,$$

where $T \cong \mathbf{T}_\tau(E(FX)/FX) \in \mathcal{T}$. Since F is an equivalence, it is exact and it commutes with injective envelopes, so $T \cong \mathbf{T}_\tau(F(E(X)/X))$ which is trivial by the fact that X is τ-local (implying that $E(X)/X \in \mathcal{F}$) and (2).
(3)⇒(1). It follows by the fact that the τ-torsion objects are exactly the objects not admitting non-trivial morphisms to a τ-local object.
(1)&(3)⇒(4). Let $X \in \mathrm{Ob}(\mathfrak{C})$ and consider the following exact sequence

$$0 \to \mathbf{T}_\tau(X) \to X \to \mathbf{L}_\tau(X) \to T \to 0\,,$$

where $T \in \mathcal{T}$. Applying $\mathbf{Q}_\tau F$ to the above sequence, using the exactness of such functor and (1), one gets $\mathbf{Q}_\tau F(X) \cong \mathbf{Q}_\tau F\mathbf{L}_\tau(X)$. Now, applying $\mathbf{S}_\tau$ and using (3) we obtain $\mathbf{L}_\tau F(X) \cong \mathbf{L}_\tau F\mathbf{L}_\tau(X) = F\mathbf{L}_\tau(X)$.
(4)⇒(3). Let $X \in \mathrm{Ob}(\mathfrak{C})$. Then, X is τ-local if and only if $X \cong \mathbf{L}_\tau(X)$, if and only if $FX \cong F\mathbf{L}_\tau(X)$. Thus, using (4), $FX \cong \mathbf{L}_\tau F(X)$, which is equivalent to say that FX is τ-local. □

In the following lemma we show that the equivalence $F : \mathfrak{C} \to \mathfrak{C}$ induces a bijection of $\mathrm{Sp}^0(\mathfrak{C})$ onto itself. This fact is then applied in Proposition 7.36 to show that F induces bijections of $\mathrm{Sp}^\alpha(\mathfrak{C})$ onto itself, for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$.

Lemma 7.35. *Let $\mathfrak{C}$ be a Gabriel category. For any simple object S, the object $F(S)$ is again simple. Furthermore, if we define a function*

$$f_0 : \mathrm{Sp}^0(\mathfrak{C}) \to \mathrm{Sp}^0(\mathfrak{C})$$

mapping $\pi \in \mathrm{Sp}^0(\mathfrak{C})$ to the isomorphism class of $F(E(\pi))$ in $\mathrm{Sp}^0(\mathfrak{C})$, then f_0 is well-defined and bijective.

Proof. The fact that F sends simples to simples follows by the fact that an equivalence preserves the lattice of sub-objects of a given object. For the second part of the statement, just notice that, given two simple objects S_1 and S_2, $S_1 \cong S_2$ is equivalent to $F(S_1) \cong F(S_2)$ and so, since any simple object is isomorphic to the socle of precisely one indecomposable injective, we are done. □

Proposition 7.36. *Let $\mathfrak{C}$ be a Gabriel category and $F : \mathfrak{C} \to \mathfrak{C}$ a self-equivalence. Then,*

(1_α) $\widetilde{F(\mathfrak{C}_\alpha)} = \mathfrak{C}_\alpha$;

(2_α) *the functor $F_\alpha : \mathfrak{C}/\mathfrak{C}_\alpha \to \mathfrak{C}/\mathfrak{C}_\alpha$ defined by the composition $F_\alpha = \mathbf{Q}_\alpha F \mathbf{S}_\alpha$ is an equivalence;*

for all $0 \leqslant \alpha < \mathrm{G.dim}(\mathfrak{C})$. In particular, via the identification $\mathrm{Sp}^\alpha(\mathfrak{C}) = \mathrm{Sp}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha) = \mathrm{Sp}^0(\mathfrak{C}/\mathfrak{C}_\alpha)$, each F_α induces a bijection

$$f_\alpha : \mathrm{Sp}^\alpha(\mathfrak{C}) \to \mathrm{Sp}^\alpha(\mathfrak{C}),$$

defined as in Lemma 7.35.

Proof. We prove simultaneously (1_α) and (2_α) by transfinite induction on α.
In case $\alpha = 0$, then (1_0) just says that $F(\{0\}) = \{0\}$, so it is trivially verified, while (2_0) is true as F_0 is just F.
Suppose now that (1_α) and (2_α) are verified for some $0 \leqslant \alpha < \mathrm{G.dim}(\mathfrak{C})$. Notice that $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ can be identified with $(\mathfrak{C}/\mathfrak{C}_\alpha)_1$, that is, the smallest hereditary torsion subclass of $\mathfrak{C}/\mathfrak{C}_\alpha$ containing all the simple objects. Since F_α is an equivalence, it sends hereditary torsion classes to hereditary torsion classes. Thus, the isomorphism closure of $F_\alpha(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha)$ is exactly $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$, as by Lemma 7.35 the image under an equivalence of a class containing all the simples, contains all the simples. Given $X \in \mathrm{Ob}(\mathfrak{C})$, we have the following equivalences:

$$\begin{aligned}(X \in \mathfrak{C}_{\alpha+1}) &\Leftrightarrow (\bar{X} := \mathbf{Q}_\alpha(X) \in \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha) \Leftrightarrow (\exists \bar{Y} \in \mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha : F_\alpha(\bar{Y}) \cong \bar{X}) \\ &\Leftrightarrow (\exists Y \in \mathfrak{C}_{\alpha+1} : F_\alpha \mathbf{Q}_\alpha(Y) \cong \bar{X}),\end{aligned}$$

where the second equivalence follows by (1_α). Now, to prove $(1_{\alpha+1})$ we have to show that the last of the above conditions is equivalent to $(\exists Y \in \mathfrak{C}_{\alpha+1} : F(Y) \cong X)$. The implication $(\exists Y \in \mathfrak{C}_{\alpha+1} : F(Y) \cong X) \Rightarrow (\exists Y \in \mathfrak{C}_{\alpha+1} : F_\alpha \mathbf{Q}_\alpha(Y) \cong \bar{X})$ is trivial, as $\mathbf{Q}_\alpha F(Y) \cong \mathbf{Q}_\alpha F \mathbf{S}_\alpha \mathbf{Q}_\alpha(Y) = F_\alpha \mathbf{Q}_\alpha(Y)$. For the converse implication, assume that there exists $Y \in \mathfrak{C}_{\alpha+1}$ such that $F_\alpha \mathbf{Q}_\alpha(Y) \cong \mathbf{Q}_\alpha(X)$. We have the following diagram with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & T_1 \arrow[r] & F(Y') \arrow[r] & \mathbf{S}_\alpha \mathbf{Q}_\alpha F(Y') \arrow[r] \arrow[d, "\cong"] & T_2 \arrow[r] & 0 \\ 0 \arrow[r] & T_3 \arrow[r] & X \arrow[r] & \mathbf{S}_\alpha \mathbf{Q}_\alpha X \arrow[r] & T_4 \arrow[r] & 0 \end{tikzcd}$$

where $Y' = \mathbf{S}_\alpha \mathbf{Q}_\alpha(Y)$ and $T_1, T_2, T_3, T_4 \in \mathfrak{C}_\alpha$. Using (2_α), one obtains $\mathfrak{C}_\alpha = \widetilde{F(\mathfrak{C}_\alpha)} \subseteq \widetilde{F(\mathfrak{C}_{\alpha+1})}$, so the first line of the diagram says that $\mathbf{S}_\alpha \mathbf{Q}_\alpha F(Y') \in \widetilde{F(\mathfrak{C}_{\alpha+1})}$, the isomorphism then shows that $\mathbf{S}_\alpha \mathbf{Q}_\alpha X \in \widetilde{F(\mathfrak{C}_{\alpha+1})}$. One concludes by the second line that $X \in \widetilde{F(\mathfrak{C}_{\alpha+1})}$, proving $(1_{\alpha+1})$.
In order to show $(2_{\alpha+1})$ one has to show that $F_{\alpha+1}$ is essentially surjective and fully faithful. The former is verified as follows: take $\bar{X} \in \mathfrak{C}/\mathfrak{C}_{\alpha+1}$, let $X = \mathbf{S}_{\alpha+1}\bar{X}$, choose $Y \in \mathfrak{C}$ such that $F(Y) \cong X$ and let $\bar{Y} = \mathbf{Q}_{\alpha+1} Y$, one concludes that

$$F_{\alpha+1}(\bar{Y}) = \mathbf{Q}_{\alpha+1} F \mathbf{S}_{\alpha+1} \mathbf{Q}_{\alpha+1}(Y) \overset{(*)}{\cong} \mathbf{Q}_{\alpha+1} \mathbf{S}_{\alpha+1} \mathbf{Q}_{\alpha+1} F(Y) \cong \mathbf{Q}_{\alpha+1} F(Y) \cong \bar{X},$$

where $(*)$ is given by Lemma 7.34 (4) (using that we already verified $(1_{\alpha+1})$, which corresponds to part (1) of that lemma). To verify full faithfulness take $X, Y \in \mathfrak{C}/\mathfrak{C}_{\alpha+1}$, then

$$\begin{aligned}\mathrm{Hom}_{\mathfrak{C}/\mathfrak{C}_{\alpha+1}}(X, Y) &\cong \mathrm{Hom}_{\mathfrak{C}}(\mathbf{S}_{\alpha+1}X, \mathbf{S}_{\alpha+1}Y) \cong \mathrm{Hom}_{\mathfrak{C}}(F\mathbf{S}_{\alpha+1}X, F\mathbf{S}_{\alpha+1}Y) \\ &\cong \mathrm{Hom}_{\mathfrak{C}/\mathfrak{C}_{\alpha+1}}(F_{\alpha+1}X, F_{\alpha+1}Y),\end{aligned}$$

where the last isomorphism follows as, by Lemma 7.34 (3), $F\mathbf{S}_{\alpha+1}X$ and $F\mathbf{S}_{\alpha+1}Y$ are both $(\alpha+1)$-local.
Finally, let $\lambda < \mathrm{G.dim}(\mathfrak{C})$ be a limit ordinal and assume (1_α), (2_α) for all $\alpha < \lambda$. Then, (1_λ) trivially follows recalling that $\mathfrak{C}_\lambda$ is the smallest hereditary torsion class containing $\mathfrak{C}_\alpha = \widetilde{F(\mathfrak{C}_\alpha)}$ for all $\alpha < \lambda$ and the same description can be given for $\widetilde{F(\mathfrak{C}_\lambda)}$. Furthermore, (2_λ) follows by (1_λ) and Lemma 7.34 exactly as in the successor case. $\square$

Motivated by the above proposition, we can give the following:

Definition 7.37. *Given a Gabriel category $\mathfrak{C}$, a self-equivalence $F : \mathfrak{C} \to \mathfrak{C}$ and a point $\pi \in \mathrm{Sp}^\alpha(\mathfrak{C}) \subseteq \mathrm{Sp}(\mathfrak{C})$ in the Gabriel spectrum, we let*

$$\mathcal{O}_F(\pi) = \{f_\alpha^n(\pi) : n \in \mathbb{Z}\}$$

*be the F-*orbit *of π, where $f_\alpha : \mathrm{Sp}^\alpha(\mathfrak{C}) \to \mathrm{Sp}^\alpha(\mathfrak{C})$ is the bijective map described in Proposition 7.36.*

It is clear that each point of the spectrum belongs to a unique F-orbit. In particular, the F-orbits induce a partition of the Gabriel spectrum.

7.2.2 A complete characterization in Gabriel categories

In the present subsection we use the orbit decomposition of the Gabriel spectrum and the classification of length functions in Gabriel categories in order to give a sufficient and necessary condition for a length function to be compatible with a given self-equivalence. The main result of this section is the following

Theorem 7.38. *Let $\mathfrak{C}$ be a Gabriel category, $F : \mathfrak{C} \to \mathfrak{C}$ be a self-equivalence and $L : \mathfrak{C} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. Then, F and L are compatible if and only if, for all $\pi \in \mathrm{Sp}(\mathfrak{C})$,*

(1) *if $\pi \in \mathrm{Supp}^\infty(L)$ then $\mathcal{O}_F(\pi) \subseteq \mathrm{Supp}^\infty(L)$;*

(2) *if $\pi \in \mathrm{Supp}^{fin}(L)$, then*

  (2.a) *$\mathcal{O}_F(\pi) \subseteq \mathrm{Supp}^{fin}(L)$;*

  (2.b) *$\lambda(\pi') = \lambda(\pi)$, for all $\pi' \in \mathcal{O}_F(\pi)$.*

Let us recall our decomposition of $L : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ obtained in Section 7.1.4. The first thing we did was to define a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$, where $\mathcal{T} = \overline{\mathrm{Fin}}(L)$ is the smallest hereditary torsion class containing $\mathrm{Fin}(L)$. This allowed us to write L a sum of its finite and infinite components:

$$L = L^{fin} + L^\infty ,$$

where L^∞ assumes the value 0 on $\mathcal{T}$ and ∞ elsewhere, while L^{fin} is the Vámos extension of the restriction of L to $\mathcal{T}$.
Similarly we can define a torsion theory $\tau_F = (\mathcal{T}^F, \mathcal{F}^F)$ with $\mathcal{T}^F = \overline{\mathrm{Fin}}(L_F)$. This induces a decomposition

$$L_F = L_F^{fin} + L_F^\infty .$$

Theorem 7.38 will follow showing that $\tau = \tau_F$ (or, equivalently, $L^\infty = L_F^\infty$) and $L^{fin} = L_F^{fin}$. In the setting of Theorem 7.38, we denote by f_α the self-bijection of $\mathrm{Sp}^\alpha(\mathfrak{C})$ induced by F.

Lemma 7.39. *In the above notation, the following are equivalent:*

(1) $\tau = \tau_F$, *that is,* $L^\infty = L^\infty_F$*;*

(2) $\pi \in \mathrm{Supp}^\infty(L)$ *implies* $\mathcal{O}_F(\pi) \subseteq \mathrm{Supp}^\infty(L)$*, for all* $\pi \in \mathrm{Sp}(\mathfrak{C})$*.*

Proof. Let $\alpha < \mathrm{G.dim}(\mathfrak{C})$ and choose $\pi \in \mathrm{Sp}^\alpha(\mathfrak{C})$. Then, $\pi \in \mathrm{Supp}^\infty_\alpha(L_F)$ if and only if $L^\infty_F(\mathbf{S}_\alpha(C(\pi))) = \infty$ (where $C(\pi) = \mathrm{Soc}(E(\pi))$ in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$), if and only if $L^\infty(F\mathbf{S}_\alpha(C(\pi))) = \infty$, if and only if $f_\alpha(\pi) \in \mathrm{Supp}^\infty_\alpha(L)$. Thus,

$$\mathrm{Supp}^\infty_\alpha(L_F) = f_\alpha(\mathrm{Supp}^\infty_\alpha(L)) .$$

By this equality it is clear that $\mathrm{Supp}^\infty_\alpha(L_F) = \mathrm{Supp}^\infty_\alpha(L)$ if and only if $\mathrm{Supp}^\infty_\alpha(L) = f_\alpha(\mathrm{Supp}^\infty_\alpha(L))$, which is equivalent to affirm that $\mathrm{Supp}^\infty_\alpha(L)$ is f_α and f_α^{-1}-invariant. This happens for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$ if and only if (2) is verified. □

We can now concentrate on showing that $L^{fin} = L^{fin}_F$ is equivalent to condition (2) in Theorem 7.38:

Lemma 7.40. *In the above notation, the following are equivalent:*

(1) $L^{fin} = L^{fin}_F$*;*

(2) *if* $\pi \in \mathrm{Supp}^{fin}(L)$*, then*

(2.a) $\mathcal{O}_F(\pi) \subseteq \mathrm{Supp}^{fin}(L)$*;*

(2.b) $\lambda(\pi') = \lambda(\pi)$*, for all* $\pi' \in \mathcal{O}_F(\pi)$*.*

Proof. For all $\alpha < \mathrm{G.dim}(\mathfrak{C})$, let

$$\overline{L}_\alpha,\ (\overline{L_F})_\alpha : \mathrm{Ob}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$$

be the functions described in (7.1.4) relative to L and L_F respectively. Using Lemma 7.34, one can follow the steps of the construction of L_α in the proof of Lemma 7.28 and show that $(L_F)_\alpha(M) = L_\alpha(F(M))$ for all $M \in \mathfrak{C}_{\alpha+1}$. Thus $(\overline{L_F})_\alpha(M) = \overline{L}_\alpha(F_\alpha(M))$ for all $M \in \mathrm{Ob}(\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha)$ (where as usual $F_\alpha = Q_\alpha F S_\alpha$). By the structure of length functions in Gabriel categories, $L^{fin} = L^{fin}_F$ if and only if $\overline{L}_\alpha = (\overline{L_F})_\alpha$ for all $\alpha < \mathrm{G.dim}(\mathfrak{C})$. Let $\alpha < \mathrm{G.dim}(\mathfrak{C})$, $\pi \in \mathrm{Sp}^\alpha(\mathfrak{C})$ and let $C(\pi) = \mathrm{Soc}(E(\pi))$ in $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$. Then,

$$(\overline{L_F})_\alpha(C(\pi)) = \overline{L}_\alpha(F_\alpha(C(\pi))) = \overline{L}_\alpha(C(f_\alpha(\pi))) .$$

Thus, $\mathrm{Supp}^{fin}_\alpha(L) = f_\alpha(\mathrm{Supp}^{fin}_\alpha(L_F))$ and so $\mathrm{Supp}^{fin}_\alpha(L) = \mathrm{Supp}^{fin}_\alpha(L_F)$ if and only if $\mathrm{Supp}^{fin}_\alpha(L)$ is f_α and f_α^{-1}-invariant, which is condition (2.a) in the statement. Furthermore, given $\pi \in \mathrm{Supp}^{fin}_\alpha(L)$, the constant associated to π in the decomposition of L is $\overline{L}_\alpha(C(\pi))$, while the constant associated to π in the decomposition of L_F is $\overline{L}_\alpha(C(f_\alpha(\pi)))$. Thus the two functions coincide if and only if $\overline{L}_\alpha$ is constant on the simple objects belonging to the same orbit under f_α, that is, condition (2.b) in the statement. □

7.2.3 Examples

Let K be a division ring and consider the category K-Mod of left K-modules. $\mathrm{Sp}(K\text{-Mod})$ consist of a single point, thus Theorem 7.38 says that any length function is compatible with any self-equivalence of K-Mod. On the other hand, we already proved in Corollary 7.20 that the length functions on K-Mod are just multiples of the composition length (which in this case is just the dimension over K) so this is not a very deep result.

The previous example can be generalized as follows. Let $\mathfrak{C}$ be a Gabriel category and consider the composition length $\ell : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$. Then, $\overline{\mathrm{Fin}}(\ell) = \mathfrak{C}_1$ and so $\mathrm{Supp}^{\infty}(\ell) = \mathrm{Sp}(\mathfrak{C}/\mathfrak{C}_1) = \bigcup_{\alpha \geqslant 1} \mathrm{Sp}^{\alpha}(\mathfrak{C})$, while $\mathrm{Supp}^{fin}(\ell) = \mathrm{Sp}(\mathfrak{C}_1) = \mathrm{Sp}^0(\mathfrak{C})$. Clearly both the finite and the infinite spectrum are invariant under any family of self-bijections $\{f_\alpha \colon \mathrm{Sp}^{\alpha}(\mathfrak{C}) \to \mathrm{Sp}^{\alpha}(\mathfrak{C}) : \alpha < \mathrm{G.dim}(\mathfrak{C})\}$. Furthermore, the constants associated to each $\pi \in \mathrm{Supp}^{fin}(\ell)$ in the decomposition of ℓ as linear combination of atomic functions are all 1. Thus, Theorem 7.38 can be applied to show that ℓ is compatible with any self-equivalence of $\mathfrak{C}$.

A further generalization of the above example can be achieved as follows. Let $\mathfrak{C}$ be a Gabriel category, for any $\alpha < \mathrm{G.dim}(\mathfrak{C})$ we define a length function

$$\ell_\alpha : \mathrm{Ob}(\mathfrak{C}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} \quad \text{such that} \quad \ell_\alpha(M) = \ell(\mathbf{Q}_\alpha(M))\,,$$

where ℓ is the composition length in $\mathfrak{C}/\mathfrak{C}_{\alpha+1}$. One can show that $\mathrm{Supp}^{fin}(\ell_\alpha) = \mathrm{Sp}^{\alpha}(\mathfrak{C})$ and $\mathrm{Supp}^{\infty}(\ell_\alpha) = \mathrm{Sp}(\mathfrak{C}/\mathfrak{C}_{\alpha+1}) = \bigcup_{\beta > \alpha} \mathrm{Sp}^{\beta}(\mathfrak{C})$, furthermore

$$\ell_\alpha = \sum_{\pi \in \mathrm{Sp}^{\alpha}(\mathfrak{C})} \ell_\pi + \sum_{\pi \in \mathrm{Sp}(\mathfrak{C}/\mathfrak{C}_{\alpha+1})} \infty \cdot \ell_\pi\,.$$

Theorem 7.38 implies that ℓ_α (and any of its multiples) is compatible with any self-equivalence of $\mathfrak{C}$.

Chapter 8

Algebraic L-entropy

8.1 Algebraic L-entropy for amenable group actions

8.1.1 Crossed products

Given a ring R and a group G, we constructed in Example 4.5 the group ring $R[G]$. In this subsection we introduce the concept of crossed product $R{*}G$, which is a generalization of $R[G]$. For more details on this kind of construction we refer to [84].

Definition 8.1. *Let R be a ring and let G be a group. A* crossed product *$R{*}G$ of R with G is a ring constructed as follows: as a set, $R{*}G$ is the collection of all the formal sums of the form*

$$\sum_{g\in G} r_g \underline{g}\,,$$

with $r_g \in R$ and $r_g = 0$ for all but finite $g \in G$, and where each $\underline{g}$ is a symbol uniquely assigned to a $g \in G$. Sum in $R{*}G$ is defined component-wise exploiting the addition in R:

$$\left(\sum_{g\in G} r_g\underline{g}\right) + \left(\sum_{g\in G} s_g\underline{g}\right) = \sum_{g\in G}(r_g + s_g)\underline{g}\,.$$

To define a product in $R{*}G$, one takes two maps $\sigma : G \to \mathrm{Aut}_{\underline{\mathrm{Ring}}}(R)$ and $\rho : G \times G \to U(R)$, (where $U(R)$ is the group of units of R); we denote the image of r via the automorphism $\sigma(g)$ by $r^{\sigma(g)}$, for all $g \in G$ and $r \in R$. We impose the following axioms on the maps σ and ρ, for all $r \in R$ and g_1, g_2, $g_3 \in G$:

(Cross.1) $\rho(g_1, g_2)\rho(g_1g_2, g_3) = \rho(g_2, g_3)^{\sigma(g_1)}\rho(g_1, g_2g_3)$;

(Cross.2) $r^{\sigma(g_2)\sigma(g_1)} = \rho(g_1, g_2)r^{\sigma(g_1g_2)}\rho(g_1, g_2)^{-1}$;

(Cross.3) $\rho(g_1, 1) = \rho(1, g_1) = 1$ *and* $\sigma(e) = 1$.

The product in $R{*}G$ is defined by the rule $(r\underline{g})(s\underline{h}) = rs^{\sigma(g)}\rho(g, h)\underline{gh}$, together with bilinearity, that is

$$\left(\sum_{g\in G} r_g\underline{g}\right)\left(\sum_{g\in G} s_g\underline{g}\right) = \sum_{g\in G}\left(\sum_{h_1h_2=g} r_{h_1}s_{h_2}^{\sigma(h_1)}\rho(h_1, h_2)\right)\underline{g}\,.$$

The axioms (Cross.1) and (Cross.2), are the exact conditions to make multiplication in $R{*}G$ associative and unitary, while (Cross.3) is telling us that $1_{R*G} = \underline{e}$. Of course the definition of $R{*}G$ does not depend only on R and G, as the choices of σ and ρ are fundamental for defining the product. Anyway one avoids a notation like $R[G, \rho, \sigma]$ and uses the more compact (though imprecise) $R{*}G$. Of course, the easiest example of crossed product is the group ring $R[G]$, which corresponds to trivial maps σ and ρ.

Notice that there is a canonical injective ring homomorphism

$$R \to R{*}G \quad r \mapsto r\underline{e}\,.$$

In view of this embedding we identify R with a subring of $R{*}G$. Notice also that the homomorphism $R \to R{*}G$ induces a scalar restriction functor

$$R{*}G\text{-Mod} \to R\text{-Mod}\,,\ \text{such that}\ {}_{R*G}M \mapsto {}_RM\,.$$

and a scalar extension functor

$$R\text{-Mod} \to R{*}G\text{-Mod}\,,\ \text{such that}\ {}_RM \mapsto R{*}G \otimes_R M\,.$$

On the other hand, in general there is no natural map $G \to R{*}G$ which is compatible with the operations. Anyway, the obvious assignment $g \mapsto \underline{g}$ respects the operations *modulo some units of* R. As described in (7.2.1), for all $g \in G$ there is a self-equivalence of the category R-Mod, induced by the ring automorphism $\sigma(g)$

$$F_{\sigma(g)} : R\text{-Mod} \to R\text{-Mod}\,.$$

Definition 8.2. *Let R be a ring, let G be a group and let $R{*}G$ be a given crossed product. A length function $L : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is said to be* compatible with $R{*}G$ *provided L is compatible with the self-equivalence $F_{\sigma(g)}$, for all $g \in G$.*

Notice that, given a ring R such that R-Mod has Gabriel dimension, the length functions ℓ_α of R-Mod described in Subsection 7.2.3 are compatible with any crossed product $R{*}G$ for any group G.

8.1.2 The action of G on monoids of submodules

Let R be a ring, let G be a group and fix a crossed product $R{*}G$. In this subsection we functorially associate to a left module over $R{*}G$ a left G-representation on a suitable submonoid of its monoid of submodules, this will allow us to define a notion of algebraic entropy in $R{*}G$-Mod, lifting the semigroup entropy along this functor.

Given a left $R{*}G$-module ${}_{R*G}M$, we denote by $(\mathcal{L}_R(M), +, 0)$ the monoid of left R-submodules of M. We let also

$$\lambda : G \to \mathrm{Aut}(\mathcal{L}_R(M)) \ \text{ such that } \ \lambda(g) = \lambda_g\,,$$

where $\lambda_g(K) = \underline{g}K$ for all $K \in \mathcal{L}_R(M)$. For any subset $F \subseteq G$ and any $K \in \mathcal{L}_R(M)$, the *F-th λ-trajectory of K* is

$$T_F(\lambda, K) = \sum_{g \in F} \lambda_g(K) = \sum_{g \in F} \underline{g}K\,.$$

Notice that $T_F(R{*}G, K) \in \mathcal{L}_R(M)$. The *(full) λ-trajectory of K* is the $R{*}G$-submodule of M generated by K, that is,

$$T_G(\lambda, K) = \sum_{g \in G} \lambda_g(K) = \sum_{g \in G} \underline{g}K = R{*}G \cdot K\,.$$

Lemma 8.3. *Let R be a ring, let G be a group, fix a crossed product $R{*}G$ and let ${}_{R*G}M$ be a left $R{*}G$-module. Then ${}_{R*G}M$ is finitely generated as a left $R{*}G$-module if and only if there exists a finitely generated R-submodule $K \in \mathcal{L}_R(M)$ such that $M = T_G(\lambda, K)$.*

Proof. If ${}_{R*G}M$ is finitely generated as a left $R{*}G$-module then choose a finite set of generators $x_1, \dots, x_n$, so that, $M = R{*}Gx_1 + \cdots + R{*}Gx_n = T_G(\lambda, x_1R + \cdots + x_nR)$. On the other hand, if $M = T_G(\lambda, K)$ with K finitely generated, than any finite set of generators of K generates M as $R{*}G$-module. □

Lemma 8.4. *Let R be a ring, let G be a group, fix a crossed product $R{*}G$ and let ${}_{R*G}M$ be a left $R{*}G$-module. Given a $R{*}G$-submodule ${}_{R*G}N \leqslant M$ and a subset $F \subseteq G$, there is a short exact sequence of left R-modules*

$$0 \to T_F(\lambda, K) \cap N \to T_F(\lambda, K) \to T_F(\lambda, (K+N)/N) \to 0\,.$$

Proof. The non-trivial maps in (8.4) are induced by the embedding $N \to M$ and by the projection $M \to M/N$. One can verify that the resulting sequence is exact noticing that $T_F(\lambda, (K+N)/N) = (T_F(\lambda, K) + N)/N$, in fact, $\underline{g}_1(k_1 + N) + \cdots + \underline{g}_n(k_n + N) = (\underline{g}_1 k_1 + \cdots + \underline{g}_n k_n) + N$ for all $k \in \mathbb{N}_+$, $k_1, \dots, k_n \in K$ and $g_1, \dots, g_n \in F$. □

Thanks to the following lemma, we can define the announced functor (see Definition 8.6)

$$R{*}G\text{-Mod} \to \mathrm{l.Rep}_G(\underline{\mathrm{Semi}}_v^*)\,.$$

Lemma 8.5. *Let R be a ring, let G be a group, let $R{*}G$ be a fixed crossed product and let $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function compatible with $R{*}G$. Let also ${}_{R*G}M$, ${}_{R*G}N$ be left $R{*}G$-modules and let $\phi : M \to N$ be a homomorphism of left $R{*}G$-modules. Then*

(1) $(\mathrm{Fin}_L(M), +, v_L)$ *is a normed monoid, where* $\mathrm{Fin}_L(M) = \{K \in \mathcal{L}_R(M) : L(K) < \infty\}$, *and* $v_L(K) = L(K)$ *for all* $K \in \mathrm{Fin}_L(M)$;

(2) *letting* $\lambda_g(K) = \underline{g}K = \{\underline{g} \cdot k : k \in K\}$ *for all* $g \in G$ *and* $K \in \mathrm{Fin}_L(M)$, *gives a homomorphism of groups*
$$\lambda : G \to \mathrm{Aut}(\mathrm{Fin}_L(M)), \quad \lambda(g) = \lambda_g : \mathrm{Fin}_L(M) \to \mathrm{Fin}_L(M)\,,$$
where $\mathrm{Aut}(\mathrm{Fin}_L(M))$ *denotes here the automorphism group of* $\mathrm{Fin}_L(M)$ *as a normed monoid;*

(3) *there is an induced contracting homomorphism of valued monoids* $\mathrm{Fin}_L(\phi) : \mathrm{Fin}_L(M) \to \mathrm{Fin}_L(N)$ *such that* $\mathrm{Fin}_L(\phi)(K) = \phi(K)$ *for all* $K \in \mathrm{Fin}_L(M)$. *Furthermore,* $\lambda_g \mathrm{Fin}_L(\phi) = \mathrm{Fin}_L(\phi)\lambda_g$, *for all* $g \in G$.

Proof. (1) By the additivity of L, $\mathrm{Fin}_L(M)$ is a sub-monoid of $\mathcal{L}_R(M)$. The fact that v_L is a norm can be proved as follows: let K_1, $K_2 \in \mathrm{Fin}_L(M)$, then $K_1 + K_2$ is a quotient of $K_1 \oplus K_2$, so $L(K_1) + L(K_2) = L(K_1 \oplus K_2) \geqslant L(K_1 + K_2)$.

(2) First of all one should verify that $\lambda_g(K)$ is an R-submodule of M. Indeed, given $r \in R$ and $k \in K$, $r(\underline{g}k) = (r\underline{g})k = (\underline{g}r^{\sigma(g^{-1})})k = \underline{g}(r^{\sigma(g^{-1})}k) \in \underline{g}K = \lambda_g(K)$. It is easy to see that each λ_g respects the operation and the unit of our monoid. Furthermore,

$$\lambda_g\lambda_h(K) = (\underline{g} \cdot \underline{h})K = (\rho(g,h)\underline{gh})K = \underline{gh}\rho(g,h)^{\sigma((gh)^{-1})}K = \underline{gh}K = \lambda_{gh}(K)\,.$$

To conclude, it remains to show that $v_L(\lambda_g(K)) \leqslant v_L(K)$ for any given $g \in G$ and $K \in \mathrm{Fin}_L(M)$. This follows from our assumption that L is compatible with $F_{\sigma(g)}$, in fact, $\lambda_g(K) \cong F_{\sigma(g)}(K)$ and so $v_L(\lambda_g(K)) = v_L(K)$.

(3) It is clear the $\mathrm{Fin}_L(\phi)$ is a monoid homomorphism. To show that is is contractive use the additivity of L and the fact that ϕ is in particular a homomorphism of left R-modules. Now, given $K \in \mathrm{Fin}_L(M)$ and $g \in G$,

$$\lambda_g(\mathrm{Fin}_L(\phi)(K)) = \{\underline{g}\phi(k) : k \in K\} = \{\phi(\underline{g}k) : k \in K\} = \mathrm{Fin}_L(\phi)(\lambda_g(K)),$$

where the second equality holds since ϕ is a homomorphism of left $R{*}G$-modules. □

Definition 8.6. *Let R be a ring, let G be a group, let $R{*}G$ be a crossed product and let $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function compatible with $R{*}G$. We define a functor*

$$\mathrm{Fin}_L : R{*}G\text{-Mod} \to \mathrm{l.Rep}_G(\underline{\mathrm{Semi}}_v^*)$$

that sends a left $R{*}G$-module ${}_{R*G}M$ to $\lambda : G \to \mathrm{Aut}(\mathrm{Fin}_L(M))$ and a homomorphism ϕ of left $R{*}G$-modules to $\mathrm{Fin}_L(\phi)$ (see Lemma 8.5).

In many cases, the monoid $\mathrm{Fin}_L(M)$ carries redundant information for our needs, for this reason it is usually useful to reduce to the smaller monoid consisting of the finitely generated modules in $\mathrm{Fin}_L(M)$. The following lemma, which is an immediate consequence of the upper continuity and of the discreteness of L, allows for such reduction. Before that, we need to recall the following definition:

Definition 8.7. *Let R be a ring and let $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function. We say that L is* discrete *if the set of finite values of L is isomorphic to $\mathbb{N}$ as an ordered set.*

Given a subset S of $\mathbb{R}_{\geqslant 0}$ that is order isomorphic (with the order induced by $\mathbb{R}$) to $\mathbb{N}$, then

– $\inf\{S'\} \in S'$ for all $S' \subseteq S$;

– $\sup\{S'\} \in S'$ for any bounded above $S' \subseteq S$.

Lemma 8.8. *Let R be a ring, let G be a group, fix a crossed product $R{*}G$ and let ${}_{R*G}M$ be a left $R{*}G$-module. Given a discrete length function $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ compatible with $R{*}G$ and given $K \in \mathrm{Fin}_L(M)$, there exists a finitely generated $K' \in \mathrm{Fin}_L(M)$ such that $L(K') = L(K)$.*

Proof. By the upper continuity of L, $L(K) = \sup\{L(H) : {}_RH \leqslant K \text{ fin. gen.}\}$. By the discreteness of L, the bounded set $\{L(H) : {}_RH \leqslant K \text{ fin. gen.}\}$ has a maximum and so one can take any finitely generated ${}_RK' \leqslant K$ that realizes this maximum. □

We need to introduce a last tool on the monoid of submodules $\mathcal{L}_R(M)$, that is, a closure operator. Indeed, we consider the torsion class $\mathrm{Ker}(L)$ of all left R-modules K such that $L(K) = 0$. The torsion functor relative to this class was denoted in [93] by $z_L : R\text{-Mod} \to \mathrm{Ker}(L)$, where, given $K \in R$-Mod,

$$z_L(K) = \{x \in K \mid L(Rx) = 0\};$$

$z_L(K)$ is called the *L-singular submodule* of K. If $z_L(K) = K$ (or, equivalently, $L(K) = 0$) we say that M is *L-singular.* There is a standard technique to associate a closure operator to any given torsion class (see [96]). In particular, given $K \in \mathcal{L}(M)$, we let $\pi\colon M \to M/K$ be the natural projection and we define

$$K_{L*} = \pi^{-1}(z_L(M/K))$$

to be the *L-purification* of K in M. An element $K \in \mathcal{L}(M)$ is said to be *L-pure* if $K_{L*} = K$, while, if $K \leqslant K' \in \mathcal{L}(M)$, we say that N is *L-essential in K'* if $L(K'/K) = 0$, that is, if $K \leqslant K' \leqslant K_{L*}$. With this terminology we can reformulate Lemma 8.8 as follows:

Corollary 8.9. *In the notation of Lemma 8.8, any $K \in \mathrm{Fin}_L(M)$ has an L-essential finitely generated submodule.*

We collect in the following lemma some useful properties of L-purifications, which follow by the fact that $(-)_{L*}$ is a closure operator associated to a torsion theory (see for example [96]). We give a complete proof for the sake of completeness.

Lemma 8.10. *Let R be a ring, let $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a length function, let M be a left R-module and consider two submodules K_1 and $K_2 \in \mathcal{L}(M)$. Then,*

(1) $((K_1)_{L*})_{L*} = (K_1)_{L*}$;

(2) $\left(\dfrac{K_1 + K_2}{K_1}\right)_{L*} = \dfrac{(K_1 + K_2)_{L*}}{K_1}$;

(3) $(K_1)_{L*} + (K_2)_{L*} \leqslant (K_1 + K_2)_{L*}$ *and* $((K_1)_{L*} + K_2)_{L*} = (K_1 + K_2)_{L*}$;

(4) $L((K_1 + K_2)/(K_1' + K_2')) = 0$ *whenever* $K_1' \leqslant K_1 \leqslant (K_1')_{L*}$ *and* $K_2' \leqslant K_2 \leqslant (K_2')_{L*}$.

Proof. (1) It is clear that $K_1 \leqslant (K_1)_{L*} \leqslant ((K_1)_{L*})_{L*}$, furthermore $(K_1)_{L*}/K_1$, $((K_1)_{L*})_{L*}/(K_1)_{L*}$ belong to $\mathrm{Ker}(L)$. The following short exact sequence

$$0 \to (K_1)_{L*}/K_1 \to ((K_1)_{L*})_{L*}/K_1 \to ((K_1)_{L*})_{L*}/(K_1)_{L*} \to 0$$

shows that $((K_1)_{L*})_{L*}/K_1 \in \mathrm{Ker}(L)$ and so $((K_1)_{L*})_{L*} \leqslant (K_1)_{L*}$.

(2) Consider the following commutative diagram

$$\begin{tikzcd} & M \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\ M/K_1 \arrow[rr, "\pi_3"] & & M/(K_1 + K_2) \end{tikzcd}$$

where π_1, π_2 and π_3 denote the natural projections. Then

$$\left(\frac{K_1 + K_2}{K_1}\right)_{L*} = \pi_3^{-1}\left(z_L\left(\frac{M}{K_1 + K_2}\right)\right) = \pi_1\left(\pi_2^{-1}\left(z_L\left(\frac{M}{K_1 + K_2}\right)\right)\right) = \frac{(K_1 + K_2)_{L*}}{K_1}$$

(3) Notice that K_1, $K_2 \leqslant K_1 + K_2$, hence $(K_1)_{L*}$, $(K_2)_{L*} \leqslant (K_1 + K_2)_{L*}$, showing that $(K_1)_{L*} + (K_2)_{L*} \leqslant (K_1 + K_2)_{L*}$. Furthermore, the inclusion $(K_1)_{L*} + K_2 \leqslant (K_1 + K_2)_{L*}$ proves that $((K_1)_{L*} + K_2)_{L*} \leqslant ((K_1 + K_2)_{L*})_{L*} = (K_1 + K_2)_{L*}$. For the converse inclusion one can use that $K_1 + K_2 \leqslant (K_1)_{L*} + K_2$ and so $(K_1 + K_2)_{L*} \leqslant ((K_1)_{L*} + K_2)_{L*}$.

(4) By hypothesis, $K_1' \leqslant K_1 \leqslant (K_1')_{L*}$ and $K_2' \leqslant K_2 \leqslant (K_2')_{L*}$. Thus,

$$\frac{K_1 + K_2}{K_1' + K_2'} \leqslant \frac{(K_1')_{L*} + (K_2')_{L*}}{K_1' + K_2'} \leqslant \frac{(K_1' + K_2')_{L*}}{K_1' + K_2'} \in \mathrm{Ker}(L)\,.$$

□

8.1.3 Definition of the L-entropy

Let us start this subsection with the definition of L-entropy.

Definition 8.11. *Let R be a ring, let G be a countably infinite amenable group, let $R{*}G$ be a crossed product, choose a Følner sequence $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$ and let $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a discrete length function compatible with $R{*}G$. Let also ${}_{R*G}M$ be a left $R{*}G$-module and denote by $\lambda : G \to \mathrm{Fin}_L(M)$ the left G-representation induced by the left $R{*}G$-structure of M. For all $K \in \mathrm{Fin}_L(M)$, the* algebraic L-entropy of ${}_{R*G}M$ at K *is*

$$\mathrm{ent}_L(\lambda, K) = \lim_{n\in\mathbb{N}} \frac{L(T_{F_n}(\lambda, K))}{|F_n|} .$$

The algebraic L-entropy of ${}_{R*G}M$ is $\mathrm{ent}_L(M) = \sup\{\mathrm{ent}_L(\lambda, K) : K \in \mathrm{Fin}_L(M)\}$.

Notice that the algebraic L-entropy is exactly the lifting of the semigroup entropy $\mathfrak{h}$ along the functor

$$\mathrm{Fin}_L : R{*}G\text{-Mod} \to \mathrm{l.Rep}_G(\underline{\mathrm{Semi}_v^*})$$

described in Subsection 8.1.2. In fact, the limit that defines ent_L exists by Corollary 4.39 (since the monoid $\mathrm{Fin}_L(M)$ satisfies conditions (1) and (2) in that corollary by Lemma 8.5). Similarly, the definition of ent_L does not depend on the choice of the Følner sequence $\mathfrak{s}$, as any such sequence gives rise to the same invariant.

Let us remark that, if R is a (skew) field and if we choose L to be the dimension of left vector spaces over R, then L is discrete and compatible with any crossed product $R{*}G$; more generally, this happens for all the functions ℓ_α described in Subsection 7.2.3. On the other hand, if $R{*}G = R[G]$, then the compatibility condition is trivially satisfied by any length function L and one just needs to assume discreteness.

It turns out that the L-entropy is not well-behaved on the whole category $R{*}G$-Mod but just on a suitable class of left $R{*}G$-modules with "enough" L-finite submodules:

Definition 8.12. *Let R be a ring and let M be a left R-module. We say that M is* locally L-finite *if $\mathrm{Fin}_L(M)$ contains all the finitely generated submodules of M. We denote by $\mathrm{lFin}(L)$ the class of all the locally L-finite left R-modules. Furthermore, given a group G and a crossed product $R{*}G$, we denote by $\mathrm{lFin}_L(R{*}G)$ the class of all the left $R{*}G$-modules ${}_{R*G}M$ such that ${}_RM \in \mathrm{lFin}(L)$.*

Notice that $\mathrm{lFin}(L)$ is closed under taking direct limits, quotients and submodules but not in general under extensions (see [93] for a counter-example).

In general, we will consider ent_L as an invariant on the class of locally L-finite $R{*}G$-modules and not on the whole class $\mathrm{Ob}(R{*}G\text{-Mod})$:

$$\mathrm{ent}_L : \mathrm{lFin}_L(R{*}G) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\} .$$

Remark 8.13. *We defined the algebraic L-entropy for left $R{*}G$-modules in case G is countable. Anyway, standard variations of the above arguments using Følner nets, allow one to define a similar invariant in case G is not countable.*

8.1.4 Basic properties

In this subsection we study the basic properties of the algebraic L-entropy. For simplicity we fix all along this subsection a ring R, a countably infinite amenable group G, a crossed product $R{*}G$ and a discrete length function $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ compatible with $R{*}G$. We also let ${}_{R*G}M$ be a locally L-finite left $R{*}G$-module and we denote by $\lambda : G \to \mathrm{Fin}_L(M)$ the left G-representation induced by the left $R{*}G$-structure of M.

Example 8.14. *If $L({}_RM) < \infty$, then $\mathrm{ent}_L({}_{R*G}M) = 0$. In fact, if ${}_RK \leqslant M$ is any L-finite R-submodule of M, then by definition $\mathrm{ent}_L(\lambda, K) \leqslant \lim_{n\to\infty} L(M)/|F_n| \leqslant \lim_{n\to\infty} L(M)/n = 0$ (for the second inequality use the fact that, as G is infinite we can take a Følner sequence such that $F_n \lneq F_{n+1}$ for all $n \in \mathbb{N}$, thus $|F_n| \geqslant n$).*

The following result allows us to redefine the algebraic entropy in terms of finitely generated submodules.

Proposition 8.15. *Let $K \in \mathrm{Fin}_L(M)$ and $H \leqslant K$ be an L-essential submodule. Then*

(1) $\mathrm{ent}_L(\lambda, H) = \mathrm{ent}_L(\lambda, K)$*;*

(2) $\mathrm{ent}_L({}_{R*G}M) = \sup\{\mathrm{ent}_L(\lambda, K) : K$ *finitely generated*$\}$.

Proof. (1) By definition of L-essential submodule, $K/H \in \mathrm{Ker}(L)$. Furthermore, for all $g \in G$, $\lambda_g(K)/\lambda_g(H) \cong F_{\sigma(g)}(K/H)$ so, as by hypothesis L is compatible with $F_{\sigma(g)}$, also $\lambda_g(K)/\lambda_g(H) \in \mathrm{Ker}(L)$. In particular, $\lambda_g(H)$ is L-essential in $\lambda_g(K)$ for all $g \in G$. By Lemma 8.10(4) and the additivity of L,

$$L(T_{F_n}(\lambda, K)) = L(T_{F_n}(\lambda, H))\,,$$

for all $n \in \mathbb{N}$, where $\{F_n\}_{n\in\mathbb{N}}$ is a Følner sequence. Therefore, $\mathrm{ent}_L(\lambda, K) = \mathrm{ent}_L(\lambda, H)$.
(2) The "$\leqslant$" inequality comes directly from the definition of entropy. On the other hand, by Lemma 8.8 any L-finite submodule K of M has an L-essential finitely generated submodule H and by part (1) $\mathrm{ent}_L(\lambda, K) = \mathrm{ent}_L(\lambda, H)$, which easily yields our claim. □

The definition of entropy in terms of finitely generated submodules given in Proposition 8.15 allows us to prove many important properties. In the following lemma we show that the entropy is monotone under taking submodules and quotients.

Lemma 8.16. *Let $N \leqslant M$ be an $R{*}G$-submodule. Then*

(1) $\mathrm{ent}_L({}_{R*G}M) \geqslant \mathrm{ent}_L({}_{R*G}N)$*;*

(2) $\mathrm{ent}_L({}_{R*G}M) \geqslant \mathrm{ent}_L({}_{R*G}(M/N))$.

Proof. Denote respectively by $\lambda' : G \to \mathrm{Fin}_L(N)$ and $\bar\lambda : G \to \mathrm{Fin}_L(M/N)$ the left G-representation induced by the left $R{*}G$-structure of N and M/N respectively.
(1) It is enough to notice that, whenever $K \leqslant N$ is an L-finite submodule of N, it is also an L-finite submodule of M and $\mathrm{ent}_L(\lambda', K) = \mathrm{ent}_L(\lambda, K) \leqslant \mathrm{ent}_L({}_{R*G}M)$.
(2) Given a finitely generated submodule $\bar K \leqslant M/N$, there exists a finitely generated (thus L-finite) submodule $K \leqslant M$ such that $(K + N)/N \cong \bar K$. Given a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$, by Lemma 8.4,

$$T_{F_n}(\bar\lambda, \bar K) = (T_{F_n}(\lambda, K) + N)/N \ \text{ and so } L(T_{F_n}(\bar\lambda, \bar K)) \leqslant L(T_{F_n}(\lambda, K)) \ \text{ for all } n \in \mathbb{N}\,.$$

Dividing by $|F_n|$ and taking the limit with $n \to \infty$ we get $\mathrm{ent}_L(\bar\lambda, \bar K) \leqslant \mathrm{ent}_L(\lambda, K) \leqslant \mathrm{ent}_L({}_{R*G}M)$. This ends the proof by Proposition 8.15. □

The above lemma has a converse in some particular situation:

Lemma 8.17. *Let $N \leqslant M$ be an $R{*}G$-submodule. Then,*

(1) $\mathrm{ent}_L({}_{R*G}M) = \mathrm{ent}_L({}_{R*G}N)$*, provided* $L(M/N) = 0$*;*

(2) $\mathrm{ent}_L({}_{R*G}M) = \mathrm{ent}_L({}_{R*G}(M/N))$*, provided* $L(N) = 0$*.*

Proof. One inequality of both statements follows by Lemma 8.16, thus we have just to verify the other one. We denote respectively by $\lambda' : G \to \mathrm{Fin}_L(N)$ and $\bar{\lambda} : G \to \mathrm{Fin}_L(M/N)$ the left G-representation induced by the left $R{*}G$-structure of N and M/N respectively.
Part (1) follows by Proposition 8.15, noticing that, whenever $K \leqslant M$ is an L-finite submodule, then $K \cap N$ is L-essential in K, thus, $\mathrm{ent}_L(\lambda, K) = \mathrm{ent}_L(\lambda, K \cap N) = \mathrm{ent}_L(\lambda', K \cap N) \leqslant \mathrm{ent}_L({}_{R*G}N)$
For part (2), let $K \in \mathrm{Fin}_L(M)$ and consider the short exact sequence described in Lemma 8.4, which shows that $L(T_{F_n}(\lambda, K)) = L(T_{F_n}(\bar{\lambda}, \bar{K})) + L(T_{F_n}(\lambda, K) \cap N) = L(T_{F_n}(\bar{\lambda}, \bar{K}))$. The conclusion follows. □

8.2 The algebraic entropy is a length function

In the present section we are going to prove the following

Theorem 8.18. *Let R be a ring, let G be a countably infinite amenable group, fix a crossed product $R{*}G$ and let $L : \mathrm{Ob}(R\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ be a discrete length function compatible with $R{*}G$. Then the invariant $\mathrm{ent}_L : \mathrm{lFin}_L(R{*}G) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ satisfies the following properties:*

(1) ent_L *is upper continuous;*

(2) $\mathrm{ent}_L(R{*}G \otimes_R K) = L(K)$ *for any L-finite left R-module K;*

(3) $\mathrm{ent}_L(N) > 0$ *for any non-trivial $R{*}G$-submodule $N \leqslant R{*}G \otimes_R K$;*

(4) ent_L *is additive.*

In particular, ent_L *is a length function on* $\mathrm{lFin}_L(R{*}G)$*.*

Part (1) will be verified in Subsection 8.2.1, parts (2) and (3) will be proved in Subsection 8.2.2 and part (4) will be the main result of Subsection 8.2.3. All along this section we will keep the notation of Theorem 8.18.

8.2.1 The algebraic entropy is upper continuous

In this subsection we are going to show that the algebraic L-entropy is an upper continuous invariant. We start with the following lemma that deals with the case when M is generated (as $R{*}G$-module) by an L-finite R-submodule K, that is, $M = T_G(\lambda, K)$. In such situation one does not need to take a supremum to compute entropy.

Lemma 8.19. *Let M be a left $R{*}G$-module such that $M = T_G(\lambda, K)$ for some $K \in \mathrm{Fin}_L(M)$, then*

$$\mathrm{ent}_L(M) = \mathrm{ent}_L(\lambda, K)\,.$$

Proof. Given a finitely generated R-submodule H of M, we can find a finite subset $e \in F \subseteq G$ such that $H \subseteq T_F(\lambda, K)$. This shows that $\mathrm{ent}_L(\lambda, H) \leqslant \mathrm{ent}_L(\lambda, T_F(\lambda, K))$. Now notice that, using the Følner condition,

$$\lim_{n\to\infty} \frac{|F_nF|}{|F_n|} \leqslant \lim_{n\to\infty} \frac{|F_n \cup \bigcup_{f\in F} \partial_F(F_n)f|}{|F_n|} \leqslant 1 + \lim_{n\to\infty} \sum_{f\in F} \frac{|\partial_F(F_n)f|}{|F_n|} = 1\,.$$

On the other hand, $|F_nF|/|F_n| \geqslant 1$ so $\lim_{n\to\infty} |F_nF|/|F_n| = 1$. We obtain that

$$\mathrm{ent}_L(\lambda, T_F(\lambda, K)) = \lim_{n\to\infty} \frac{T_{F_n}(\lambda, T_F(\lambda, K))}{|F_n|} = \lim_{n\to\infty} \frac{T_{F_nF}(\lambda, K)}{|F_n|} \cdot \frac{|F_n|}{|F_nF|} = \mathrm{ent}_L(\lambda, K)\,,$$

where the last equality comes from the fact that $\{F_nF\}_{n\in\mathbb{N}}$ is a Følner sequence by Lemma 4.22 and since the definition of ent_L does not depend on the choice of a particular Følner sequence. Thus, $\mathrm{ent}_L(\lambda, H) \leqslant \mathrm{ent}_L(\lambda, K)$ for any finitely generated $H \in \mathcal{L}(M)$; one concludes using Proposition 8.15. □

The upper continuity of ent_L can now be verified easily using the above lemma and Proposition 8.15:

Corollary 8.20. $\mathrm{ent}_L : \mathrm{lFin}_L(R{*}G) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ *is an upper continuous invariant.*

Proof. The fact that ent_L is an invariant can be derived by the definition and the fact that L is an invariant. Now, let $M \in \mathrm{lFin}_L(R{*}G)$, then by Proposition 8.15 and Lemmas 8.19 and 8.3 we get

$$\begin{aligned}\mathrm{ent}_L({}_{R*G}M) &= \sup\{\mathrm{ent}_L(\lambda, K) : K \text{ finitely generated } R\text{-submodule of } M\}\\ &= \sup\{\mathrm{ent}_L({}_{R*G}(T_G(\lambda, K)) : K \text{ finitely generated } R\text{-submodule of } M\}\\ &= \sup\{\mathrm{ent}_L({}_{R*G}N) : N \text{ finitely generated } R{*}G\text{-submodule of } M\}\,.\end{aligned}$$

□

8.2.2 Values on (sub)shifts

The present subsection is devoted to compute the values of the algebraic entropy on the $R{*}G$-modules of the form $M = R{*}G \otimes_R K$, for some left R-module K, and their $R{*}G$-submodules. Indeed, fix a left R-module K and let $M = R{*}G \otimes_R K$. As a left R-module there is a direct sum decomposition ${}_RM \cong \bigoplus_{g\in G} \underline{g}K$, so that one can uniquely represent a generic element $x \in M$ in the form $x = \sum_{g\in G} \underline{g}x_g$, where $x_g \in K$ for all $g \in G$ and $x_h = 0$ for almost all $h \in G$. Notice that

$$\underline{h}(\sum_{g\in G} \underline{g}x_g) = \sum_{g\in G} \underline{h}\,\underline{g}x_g = \sum_{g\in G} \underline{hg}\rho(h,g)^{\sigma(g)^{-1}} x_g = \sum_{g\in G} \underline{g}\,\rho(h, h^{-1}g)^{\sigma(h^{-1}g)^{-1}} x_{h^{-1}g}\,.$$

We denote the action of G on $\mathrm{Fin}_L(M)$ by

$$\beta : G \to \mathrm{Aut}(\mathrm{Fin}_L(M)), \quad \text{where } \beta(g) = \beta_g\,.$$

The choice of the greek letter β to represent this action comes from the *Bernulli actions* which are defined in ergodic theory and can be viewed as dual to the actions described here. We remark that the left action $\beta : G \to \mathrm{Fin}_L(M)$ is not isomorphic in general to the left Bernoulli G-representation $\mathfrak{B} : G \to \bigoplus_G \mathrm{Fin}_L(K)$ described in Subsection 4.1.3, even if we will show that these two representations have the same entropy.

Notice that β has the following properties, where F is a subset of G:

$$T_F(\beta, \underline{g}K) = \bigoplus_{h\in F} \underline{hg}K \quad \text{and} \quad T_G(\beta, \underline{g}K) = M\,. \tag{8.2.1}$$

In the following example we compute the algebraic entropy of Bernoulli shifts.

Example 8.21. *In the above notation, suppose $L(K) < \infty$. By Lemma 8.19 and* (8.2.1)*, we obtain that* $\mathrm{ent}_L({}_{R*G}M) = \mathrm{ent}_L(\beta, K)$*. Furthermore, again by* (8.2.1)*,* $L(T_F(\beta, K))/|F| = L(K)$*, for all $F \in \mathcal{F}(G)$. Therefore,* $\mathrm{ent}_L(M) = L(K)$*.*

The computation in the above example shows that the entropy of $M = R{*}G \otimes_R K$ is 0 if and only if $L(K) = 0$, if and only if $L(M) = 0$. Our next goal is to show that, if ${}_{R*G}N$ is a submodule of ${}_{R*G}M$, then $\mathrm{ent}_L({}_{R*G}N) = 0$ if and only if $L({}_RN) = 0$. This will be proved in Proposition 8.23 but first we need to recall some useful terminology and results from [12].

Let E and F be subsets of G. A subset $\mathcal{N} \subseteq G$ is an (E, F)*-net* if it satisfies the following conditions:

– the subsets $(gE)_{g\in\mathcal{N}}$ are pairwise disjoint, that is, $gE \cap g'E = \varnothing$ for all $g \neq g' \in \mathcal{N}$;

– $G = \bigcup_{g\in\mathcal{N}} gF$.

It is proved in [12, Lemma 2.2] that, for any subset E of a group G, one can always find an (E, EE^{-1})-net. The following lemma is a variation of [12, Lemma 4.3].

Lemma 8.22. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner sequence of G, let $E \subseteq F \subseteq G$ be finite subsets with $e \in F$ and let $\mathcal{N}$ be an (E, F)-net. Then there exist $0 < \alpha \leqslant 1$ and $n_0 \in \mathbb{N}$ such that $|F_n \cap \mathcal{N}| \geqslant \alpha \cdot |F_n|$, for all $n > n_0$.*

Proof. For each $n \in \mathbb{N}$, let $F_n^{+F} = Out_F(F_n) \cap \mathcal{N}$ and notice that $F_n^{+F} \setminus (F_n \cap \mathcal{N}) \subseteq \partial_F(F_n)$. Furthermore, since F_n is covered by the sets gF, $g \in F_n^{+F}$, we have $|F_n| \leqslant |F||F_n^{+F}|$. Let now $\alpha_1 = 1/|F|$, thus

$$\alpha_1|F_n| - |F_n \cap \mathcal{N}| \leqslant |F_n^{+F}| - |F_n \cap \mathcal{N}| \leqslant |F_n^{+F} \setminus (F_n \cap \mathcal{N})| \leqslant |\partial_F(F_n)|\,.$$

Let $0 < \alpha_2 < \alpha_1$; by the Følner condition, there exists $n_0 \in \mathbb{N}$ such that $|\partial_F(F_n)|/|F_n| \leqslant \alpha_2$ for all $n > n_0$. Thus, letting $\alpha = \alpha_1 - \alpha_2$, one has $0 < \alpha \leqslant 1$ and $|F_n \cap \mathcal{N}| \geqslant \alpha_1|F_n| - \partial_F(F_n) \geqslant \alpha|F_n|$, for all $n > n_0$. □

Proposition 8.23. *Let K be an L-finite left R-module and let ${}_{R*G}N$ be a submodule of ${}_{R*G}M$. Then,*

$$\mathrm{ent}_L({}_{R*G}N) = 0 \quad \textit{if and only if} \quad L({}_RN) = 0\,.$$

Proof. Suppose $L({}_RN) \neq 0$, then there exists $x \in N$ such that $L(Rx) \neq 0$. Let E be the set of all elements $h \in G$ such that, writing $x = \sum_{g\in G} \underline{g}x_g$, the component x_h is not 0. We fix an (E, EE^{-1})-net $\mathcal{N}$. Notice that, given $f_1 \neq f_2 \in \mathcal{N}$, then $\beta_{f_1}(Rx) \cap \beta_{f_2}(Rx) = 0$. Thus, by Lemma 8.22 we can find $n_0 \in \mathbb{N}$ and $0 < \alpha \leqslant 1$ such that

$$L(T_{F_n}(\beta, Rx)) \geqslant L(T_{F_n\cap\mathcal{N}}(\beta, Rx)) = |F_n \cap \mathcal{N}|L(Rx) \geqslant \alpha|F_n|L(Rx)$$

for all $n > n_0$. In particular, $\mathrm{ent}_L({}_{R*G}N) \geqslant \mathrm{ent}_L(\beta, Rx) \geqslant \alpha L(Rx) \neq 0$. □

8.2.3 The Addition Theorem

In the present subsection we complete the proof of Theorem 8.18 verifying a very strong property of the algebraic entropy, that is, its additivity on $\mathrm{lFin}_L(R{*}G)$. In particular, we have to verify that, given a locally L-finite left $R{*}G$-module M, and an $R{*}G$-submodule $N \leqslant M$,

$$\mathrm{ent}_L({}_{R*G}M) = \mathrm{ent}_L({}_{R*G}N) + \mathrm{ent}_L({}_{R*G}(M/N))\,. \tag{8.2.2}$$

We fix all along this subsection the following notations for the actions induced by the $R{*}G$-module structures:

$$\begin{array}{lll} \lambda : G \to \mathrm{Aut}(\mathrm{Fin}_L(M)) & \lambda' : G \to \mathrm{Aut}(\mathrm{Fin}_L(N)) & \bar\lambda : G \to \mathrm{Aut}(\mathrm{Fin}_L(M/N)) \\ \quad g \mapsto \lambda_g & \quad g \mapsto \lambda'_g = \lambda_g\restriction_{\mathrm{Fin}_L(N)} & \quad g \mapsto \bar\lambda_g \end{array}$$

We start proving the inequality "$\geqslant$" of (8.2.2).

Lemma 8.24. $\mathrm{ent}_L\left({}_{R*G}M\right) \geqslant \mathrm{ent}_L\left({}_{R*G}N\right) + \mathrm{ent}_L\left({}_{R*G}(M/N)\right).$

Proof. Let $K_1 \leqslant N$ and $K_2 \leqslant M/N$ be finitely generated R-submodules. Fix a finitely generated submodule $K \leqslant M$ such that $(K+N)/N = K_2$ and $K \cap N \supseteq K_1$. Given a finite subset $F \subseteq G$, by Lemma 8.4 there is a short exact sequence

$$0 \to T_F(\lambda, K) \cap N \to T_F(\lambda, K) \to T_F(\bar\lambda, K_2) \to 0\,.$$

Noticing that $T_F(\lambda', K_1) \subseteq T_F(\lambda, K) \cap N$, we get $L(T_F(\lambda, K)) \geqslant L(T_F(\lambda', K_1)) + L(T_F(\bar\lambda, K_2))$. Applying this inequality to the sets belonging to a Følner sequence $\{F_n\}_{n\in\mathbb{N}}$, yields

$$\mathrm{ent}_L({}_{R*G}M) \geqslant \mathrm{ent}_L(\lambda, K) \geqslant \mathrm{ent}_L(\lambda', K_1) + \mathrm{ent}_L(\bar\lambda, K_2)\,.$$

The result follows by the arbitrariness of the choice of K_1 and K_2. □

The first step in proving the converse inequality is to show that we can reduce the problem to the case when both M and N are finitely generated $R{*}G$-modules. This goal is obtained in the following corollary (which is just a reformulation of Corollary 4.37), and the subsequent two lemmas.

Corollary 8.25. *Let $\{F_n\}_{n\in\mathbb{N}}$ be a Følner exhaustion of G. Then, for any $\varepsilon \in (0, 1/4)$ and $n \in \mathbb{N}$ there exist $n_1, \dots, n_k \in \mathbb{N}$ with $n \leqslant n_1 \leqslant \dots \leqslant n_k$ such that, given an L-finite submodule $K \leqslant M$,*

$$\mathrm{ent}_L(\lambda, K) \leqslant \varepsilon \cdot L(K) + \frac{1}{1-\varepsilon} \cdot \max_{1\leqslant i\leqslant k} \frac{L(T_{F_{n_i}}(\lambda, K))}{|F_{n_i}|}\,.$$

Proof. This is a straightforward application of Corollary 4.37. In fact, the function $f_K : \mathcal{F}(G) \to \mathbb{R}_{\geqslant 0}$ such that $f_K(F) = L(T_F(\lambda, K))$ satisfies the hypotheses of such corollary for any L-finite R-submodule K of M, by Lemma 8.5. Furthermore, $\lim_{n\to\infty} f_K(F_n)/|F_n| = \mathrm{ent}_L(\lambda, K)$ by the definition of entropy. □

Lemma 8.26. *Consider a sequence $N_0 \leqslant N_1 \leqslant \dots \leqslant N_t \leqslant \dots \leqslant M$ of $R{*}G$-submodules of M such that $N = \bigcup_{t\in\mathbb{N}} N_t$ and let $\bar\lambda_t : G \to \mathrm{Aut}(\mathrm{Fin}_L(M/N_t))$ be the actions induced on the quotients. Then, given an L-finite submodule $K \leqslant M$ and letting $\bar K = (K+N)/N$ and $\bar K_t = (K+N_t)/N_t$ for all $t \in \mathbb{N}$,*

$$\mathrm{ent}_L(\bar\lambda, \bar K) = \inf_{t\in\mathbb{N}} \mathrm{ent}_L(\bar\lambda_t, \bar K_t)\,.$$

Proof. The inequality "$\leqslant$" follows by Lemma 8.16. On the other hand, for all $\varepsilon > 0$ there exists $n_\varepsilon \in \mathbb{N}$ such that

$$\frac{L(T_{F_n}(\bar\lambda, \bar K))}{|F_n|} \leqslant \mathrm{ent}_L(\bar\lambda, \bar K) + \varepsilon\,, \quad \text{for all } n \geqslant n_\varepsilon\,. \tag{8.2.3}$$

By Corollary 8.25, for any given $\varepsilon' \in (0, 1/4)$, there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $n_\varepsilon \leqslant n_1 \leqslant \dots \leqslant n_k$ and

$$\mathrm{ent}_L(\bar\lambda_t, \bar K_t) \leqslant \varepsilon' \cdot L(\bar K_t) + \frac{1}{1-\varepsilon'} \cdot \max_{1 \leqslant i \leqslant k} \frac{L(T_{F_{n_i}}(\bar\lambda_t, \bar K_t))}{|F_{n_i}|} \tag{8.2.4}$$

holds for all $t \in \mathbb{N}$. Now, for all $i \in \{1, \dots, k\}$, the set $\{L(T_{F_{n_i}}(\bar\lambda_t, \bar K_t)) : t \in \mathbb{N}\}$ is a set of values of L, all smaller than or equal to the finite value $L(T_{F_i}(\lambda, K))$. Since we supposed L to be discrete, this set has a minimum, say $L(T_{F_{n_i}}(\bar\lambda_{t_i}, \bar K_{t_i}))$. Let $s = \max_{1 \leqslant i \leqslant k} t_i$ and notice that

$$L(T_{F_{n_i}}(\bar\lambda_s, \bar K_s)) = L(T_{F_{n_i}}(\bar\lambda, \bar K))\,, \quad \text{for all } i = 1, \dots, k.$$

(This follows by the additivity of L and the fact that $L((N \cap (T_{F_{n_i}}(\lambda, K) + N_s))/N_s) = 0$, as this module is the union of modules of the form $(N_t \cap (T_{F_{n_i}}(\lambda, K) + N_s))/N_s$ with $t \geqslant s$, which are L-singular by the choice of s).
Using the above computations, we get

$$\begin{aligned}\inf_{t \in \mathbb{N}} \mathrm{ent}_L(\bar\lambda_t, \bar K_t) \leqslant \mathrm{ent}_L(\bar\lambda_s, \bar K_s) &\leqslant \varepsilon' \cdot L(\bar K_s) + \frac{1}{1-\varepsilon'} \cdot \max_{1 \leqslant i \leqslant k} \frac{L(T_{F_{n_i}}(\bar\lambda_s, \bar K_s))}{|F_{n_i}|} \\ &= \varepsilon' \cdot L(\bar K_s) + \frac{1}{1-\varepsilon'} \cdot \max_{1 \leqslant i \leqslant k} \frac{L(T_{F_{n_i}}(\bar\lambda, \bar K))}{|F_{n_i}|} \\ &\leqslant \varepsilon' \cdot L(K) + \frac{1}{1-\varepsilon'} \cdot (\mathrm{ent}_L(\bar\lambda, \bar K) + \varepsilon)\,.\end{aligned}$$

Letting ε' tend to 0 we obtain that $\inf_{t \in \mathbb{N}} \mathrm{ent}_L(\bar\lambda_t, K) \leqslant \mathrm{ent}_L(\bar\lambda, \bar K) + \varepsilon$. As this holds for all $\varepsilon > 0$, the conclusion follows. □

As we announced, we can now prove the following reduction to the case when ${}_{R*G}N$ and ${}_{R*G}M$ are finitely generated.

Lemma 8.27. *If* (8.2.2) *holds whenever* ${}_{R*G}N$ *and* ${}_{R*G}M$ *are finitely generated, then it holds in general.*

Proof. We already proved the inequality "$\geqslant$" in (8.2.2) always holds, thus we concentrate just on the converse inequality. Indeed, given a finitely generated submodule ${}_{R*G}K$ of ${}_{R*G}M$ we claim that

$$\mathrm{ent}_L({}_{R*G}K) = \mathrm{ent}_L({}_{R*G}(K \cap N)) + \mathrm{ent}_L({}_{R*G}((K + N)/N))\,. \tag{8.2.5}$$

Notice that, if we prove the above claim, we can easily conclude using upper continuity as follows:

$$\begin{aligned}\mathrm{ent}_L({}_{R*G}M) &= \sup\{\mathrm{ent}_L({}_{R*G}K) : K \leqslant M \text{ f.g.}\} \\ &= \sup\{\mathrm{ent}_L({}_{R*G}(K \cap N)) + \mathrm{ent}_L({}_{R*G}(K + N)/N) : K \leqslant M \text{ f.g.}\} \\ &\leqslant \sup\{\mathrm{ent}_L({}_{R*G}(K \cap N)) : K \leqslant M \text{ f.g.}\} + \sup\{\mathrm{ent}_L({}_{R*G}(K + N)/N) : K \leqslant M \text{ f.g.}\} \\ &= \mathrm{ent}_L({}_{R*G}N) + \mathrm{ent}_L({}_{R*G}(M/N))\,.\end{aligned}$$

It remains to verify claim (8.2.5). Indeed, choose a finitely generated R-submodule H of K, such that $K = T_G(\lambda, H)$. Notice also that $K \cap N = \bigcup_{n \in \mathbb{N}} T_G(\lambda, T_{N_n(B)}(\lambda, H) \cap N)$. For all $n \in \mathbb{N}$, we let H_n be an L-essential, finitely generated R-submodule of $T_{N_n(B)}(\lambda, H) \cap N$. By Proposition 8.15, we obtain that

$$\mathrm{ent}_L({}_{R*G}(N \cap K)) = \sup_{n\in\mathbb{N}} \mathrm{ent}_L(\lambda, T_{N_n(B)}(\lambda, H) \cap N) = \sup_{n\in\mathbb{N}} \mathrm{ent}_L(\lambda, H_n)\,.$$

We let $K' = \bigcup_{n\in\mathbb{N}} T_G(\lambda, H_n)$. Notice that K' is L-essential in $K \cap N$ (in fact, $(K \cap N)/K'$ is the union of modules of the form $((T_{N_n(B)}(\lambda, H) \cap N) + K')/K'$ and each of these modules is a quotient of an L-singular module of the form $(T_{N_n(B)}(\lambda, H) \cap N)/T_G(\lambda, H_n)$). By Lemma 8.17 and Proposition 8.15 we obtain that

$$\begin{aligned}\mathrm{ent}_L({}_{R*G}(K \cap N)) &= \mathrm{ent}_L({}_{R*G}K') \\ &= \lim_{n\to\infty} \mathrm{ent}_L({}_{R*G}T_G(\lambda, H_n)) \\ &= \sup_{n\in\mathbb{N}} \mathrm{ent}_L({}_{R*G}T_G(\lambda, H_n))\,.\end{aligned}$$

Similarly, one derives by Lemma 8.26 that

$$\begin{aligned}\mathrm{ent}_L({}_{R*G}(K/(K \cap N)) &= \mathrm{ent}_L({}_{R*G}(K/K')) \\ &= \lim_{n\to\infty} \mathrm{ent}_L({}_{R*G}(K/T_G(\lambda, H_n))) \\ &= \inf_{n\in\mathbb{N}} \mathrm{ent}_L({}_{R*G}(K/T_G(\lambda, H_n)))\,.\end{aligned}$$

By hypothesis, $\mathrm{ent}_L({}_{R*G}T_G(\lambda, H_n)) + \mathrm{ent}_L({}_{R*G}(K/T_G(\lambda, H_n))) = \mathrm{ent}_L({}_{R*G}K)$ for all $n \in \mathbb{N}$. Putting together all these computations we obtain:

$$\begin{aligned}\mathrm{ent}_L({}_{R*G}K) &= \lim_{n\to\infty} (\mathrm{ent}_L({}_{R*G}T_G(\lambda, H_n)) + \mathrm{ent}_L({}_{R*G}(K/T_G(\lambda, H_n)))) \\ &= \lim_{n\to\infty} \mathrm{ent}_L({}_{R*G}T_G(\lambda, H_n)) + \lim_{n\to\infty} \mathrm{ent}_L({}_{R*G}(K/T_G(\lambda, H_n))) \\ &= \mathrm{ent}_L({}_{R*G}(K \cap N)) + \mathrm{ent}_L({}_{R*G}(K/(K \cap N))\,,\end{aligned}$$

which verifies (8.2.5), concluding the proof. □

Finally, we have all the instruments to conclude the proof of the additivity of ent_L. The computations in the proof of the following lemma are freely inspired to the proof of the Abramov-Rokhlin Formula given in [101]. The context (and even the statements) in that paper is quite different but the ideas contained there can be perfectly adapted to our needs.

Lemma 8.28. $\mathrm{ent}_L\left({}_{R*G}M\right) \leqslant \mathrm{ent}_L\left({}_{R*G}N\right) + \mathrm{ent}_L\left({}_{R*G}(M/N)\right)$.

Proof. By Lemma 8.27, we can suppose that both M and N are finitely generated $R*G$-modules. In particular, there exists a finitely generated R-submodule $K' \leqslant N$ and a finitely generated R-submodule $\bar{K}_2 \leqslant M/N$ such that $N = T_G(\lambda', K')$ and $M/N = T_G(\bar{\lambda}, \bar{K}_2)$. Since $\bar{K}_2$ is finitely generated, there exists a finitely generated R-submodule K_2 of M, such that $(K_2 + N)/N = \bar{K}_2$. We let $K = K' + K_2$ and we notice that $M = T_G(\lambda, K)$. Finally, we let K_1 be an L-essential finitely generated submodule of $K \cap N$ containing K'. Notice that, by Lemma 8.19 we obtain that $\mathrm{ent}_L({}_{R*G}M) = \mathrm{ent}_L(\lambda, K)$, $\mathrm{ent}_L({}_{R*G}N) = \mathrm{ent}_L(\lambda', K \cap N) = \mathrm{ent}_L(\lambda', K_1)$ and $\mathrm{ent}_L({}_{R*G}(M/N)) = \mathrm{ent}_L(\bar{\lambda}, \bar{K}_2)$.

Let $\varepsilon \in (0, 1/4)$ and fix a Følner exhaustion $\{F_n\}_{n\in\mathbb{N}}$. By the existence of the limits defining the algebraic L-entropies, we can find $\bar{n} \in \mathbb{N}$ such that, for all $n > \bar{n}$

$$\left|\frac{L(T_{F_n}(\lambda, K))}{|F_n|} - \mathrm{ent}_L(M)\right| < \varepsilon\,, \quad \left|\frac{L(T_{F_n}(\lambda', K_1))}{|F_n|} - \mathrm{ent}_L(N)\right| < \varepsilon\,, \tag{8.2.6}$$

$$\left|\frac{L(T_{F_n}(\bar\lambda,\bar K_2))}{|F_n|}-\mathrm{ent}_L(M/N)\right|<\varepsilon\,.$$

For all $m\in\mathbb{N}$,

$$\frac{L(T_{F_m}(\lambda,K))}{|F_m|}=\frac{L(T_{F_m}(\lambda',K_1))}{|F_m|}+\frac{L(T_{F_m}(\lambda,K)/T_{F_m}(\lambda',K_1))}{|F_m|}$$

and so, for all $m\geqslant\bar n$,

$$\begin{aligned}\mathrm{ent}_L(M)<\frac{L(T_{F_m}(\lambda,K))}{|F_m|}+\varepsilon&=\frac{L(T_{F_m}(\lambda',K_1))}{|F_m|}+\frac{L(T_{F_m}(\lambda,K)/T_{F_m}(\lambda',K_1))}{|F_m|}+\varepsilon\\&<\mathrm{ent}_L(N)+\frac{L(T_{F_m}(\lambda,K)/T_{F_m}(\lambda',K_1))}{|F_m|}+2\varepsilon\,.\end{aligned}\tag{8.2.7}$$

In the remaining part of the proof we are going to show that there exists a positive integer k such that

$$\frac{L(T_{F_m}(\lambda,K)/T_{F_m}(\lambda',K_1))}{|F_m|}\leqslant\frac{1}{1-\varepsilon}\mathrm{ent}_L({}_{R*G}(M/N))+\varepsilon\left(L(K)(k+1)+\frac{1}{1-\varepsilon}\right),\tag{8.2.8}$$

for all big enough $m\in\mathbb{N}$. Applying (8.2.8) to (8.2.7), one gets

$$\mathrm{ent}_L(M)<\mathrm{ent}_L(N)+\frac{1}{1-\varepsilon}\mathrm{ent}_L(M/N)+\varepsilon\left(L(K)(k+1)+\frac{1}{1-\varepsilon}\right)+2\varepsilon$$

which, as it holds for all $\varepsilon\in(0,1/4)$, gives the desired inequality. Thus, to conclude we have to verify (8.2.8).

Since $\{F_n\}$ is a Følner exhaustion, $N=\bigcup_{n\in\mathbb{N}}T_{F_n}(\lambda',K_1)$ and so, for any L-finite submodule $H\leqslant M$, we can use the upper continuity of L to obtain that

$$L(H\cap N)=\lim_{n\to\infty}L(H\cap T_{F_n}(\lambda',K_1))\,.$$

By additivity, this implies that $L((H+N)/N)=\lim_{n\to\infty}L((H+T_{F_n}(\lambda',K_1))/T_{F_n}(\lambda',K_1))$ and, by the discreteness of L, this limit is the minimum of the values. By Theorem 4.31, there exist $\bar n<n_1<\dots<n_k\in\mathbb{N}$ such that $\{F_{n_1},\dots,F_{n_k}\}$ ε-quasi-tiles F_m for all $m\geqslant\bar n$. Applying the above argument with $H=T_{F_{n_i}}(\lambda,K)$ (for all $i=1,\dots,k$), we can find $\bar{\bar n}\in\mathbb{N}$ such that

$$L\left(\frac{T_{F_{n_i}}(\lambda,K)+T_{F_n}(\lambda',K_1)}{T_{F_n}(\lambda',K_1)}\right)=L\left(\frac{T_{F_{n_i}}(\lambda,K)+N}{N}\right)\tag{8.2.9}$$

for all $n\geqslant\bar{\bar n}$ and all $i=1,\dots,k$. From now on we suppose m to be a positive integer such that

$$m\geqslant\max\{\bar n,\bar{\bar n}\}\quad\text{ and }\quad|\partial_{F_{\bar{\bar n}}}(F_m)|/|F_m|\leqslant\varepsilon\,,\tag{8.2.10}$$

where the second condition can be assumed since $\{F_n\}_{n\in\mathbb{N}}$ is Følner. Since $\{F_{n_1},\dots,F_{n_k}\}$ ε-quasi-tiles F_m, we can choose tiling centers $C_1,\dots,C_k$, obtaining the following inequalities

$$|F_m|\geqslant\left|\bigcup_{i=1}^k C_iF_{n_i}\right|\geqslant\max\left\{(1-\varepsilon)|F_m|\ ,\ (1-\varepsilon)\sum_{i=1}^k|C_i||F_{n_i}|\right\},\tag{8.2.11}$$

which imply that

$$\frac{L(T_{F_m\setminus\bigcup_{i=1}^k C_iF_{n_i}}(\lambda,K))}{|F_m|}\leqslant\frac{|F_m\setminus\bigcup_{i=1}^k C_iF_{n_i}|}{|F_m|}L(K)=\left(1-\frac{\left|\bigcup_{i=1}^k C_iF_{n_i}\right|}{|F_m|}\right)L(K)\leqslant\varepsilon L(K)\,.$$

Applying this computation and using again (8.2.11), one gets:

$$\begin{aligned}\frac{L(T_{F_m}(\lambda,K)/T_{F_m}(\lambda',K_1))}{|F_m|}&\leqslant\frac{1}{|\bigcup_{i=1}^k C_iF_{n_i}|}L\left(\frac{T_{\bigcup_{i=1}^k C_iF_{n_i}}(\lambda,K)+T_{F_m}(\lambda',K_1)}{T_{F_m}(\lambda',K_1)}\right)+\varepsilon L(K)\\&\leqslant\frac{(1-\varepsilon)^{-1}}{\sum_{i=1}^k|C_i||F_{n_i}|}\sum_{i=1}^k L\left(\frac{T_{C_iF_{n_i}}(\lambda,K)+T_{F_m}(\lambda',K_1)}{T_{F_m}(\lambda',K_1)}\right)+\varepsilon L(K)\,.\end{aligned}\tag{8.2.12}$$

Now, let $t_i=(|C_i||F_{n_i}|)/\sum_{j=1}^k|C_j||F_{n_j}|$ and notice that $t_i\in(0,1)$ and $\sum_{i=1}^k t_i=1$. Then

$$\begin{aligned}&\frac{1}{\sum_{j=1}^k|C_j||F_{n_j}|}\sum_{i=1}^k L\left(\frac{T_{C_iF_{n_i}}(\lambda,K)+T_{F_m}(\lambda',K_1)}{T_{F_m}(\lambda',K_1)}\right)\\&\quad=\sum_{i=1}^k\frac{t_i}{|C_i||F_{n_i}|}L\left(\frac{T_{C_iF_{n_i}}(\lambda,K)+T_{F_m}(\lambda',K_1)}{T_{F_m}(\lambda',K_1)}\right)\,.\end{aligned}\tag{8.2.13}$$

Since $e\in F_{n_i}$, $C_i\subseteq F_m\subseteq In_{F_{\overline{\overline{n}}}}(F_m)\cup\partial_{F_{\overline{\overline{n}}}}(F_m)$ for all $i=1,\dots,k$ and so

$$\begin{aligned}&\frac{1}{|C_i||F_{n_i}|}L\left(\frac{T_{C_iF_{n_i}}(\lambda,K)+T_{F_m}(\lambda',K_1)}{T_{F_m}(\lambda',K_1)}\right)\leqslant\frac{1}{|C_i||F_{n_i}|}\sum_{c\in C_i}L\left(\frac{T_{cF_{n_i}}(\lambda,K)+T_{F_m}(\lambda',K_1)}{T_{F_m}(\lambda',K_1)}\right)\\&\leqslant\frac{1}{|C_i||F_{n_i}|}\left(\sum_{c\in C_i\cap In_{F_{\overline{\overline{n}}}}(F_m)}L\left(\frac{T_{F_{n_i}}(\lambda,K)+T_{c^{-1}F_m}(\lambda',K_1)}{T_{c^{-1}F_m}(\lambda',K_1)}\right)+|\partial_{F_{\overline{\overline{n}}}}(F_m)|L(K)\right)\\&\leqslant\frac{1}{|F_{n_i}|}L\left(\frac{T_{F_{n_i}}(\lambda,K)+T_{F_{\overline{\overline{n}}}}(\lambda',K_1)}{T_{F_{\overline{\overline{n}}}}(\lambda',K_1)}\right)+\frac{|\partial_{F_{\overline{\overline{n}}}}(F_m)|L(K)}{|C_i||F_{n_i}|}\\&=\frac{1}{|F_{n_i}|}L\left(\frac{T_{F_{n_i}}(\lambda,K)+N}{N}\right)+\frac{|\partial_{F_{\overline{\overline{n}}}}(F_m)|L(K)}{|C_i||F_{n_i}|}\end{aligned}\tag{8.2.14}$$

where the first inequality in the last line comes from the fact that (by definition of $In_{F_{\overline{\overline{n}}}}(F_m)$), $F_{\overline{\overline{n}}}\subseteq c^{-1}F_m$ for all $c\in C_i\cap In_{F_{\overline{\overline{n}}}}(F_m)$; the last equality is an application of (8.2.9)).

Let us assemble together the above computations:

$$
\begin{aligned}
\frac{L(T_{F_m}(\lambda, K)/T_{F_m}(\lambda', K_1))}{|F_m|} &\overset{(8.2.12)}{\leqslant} \frac{(1-\varepsilon)^{-1}}{\sum_{i=1}^{k}|C_i||F_{n_i}|}\sum_{i=1}^{k} L\left(\frac{T_{C_iF_{n_i}}(\lambda, K)) + T_{F_m}(\lambda', K_1))}{T_{F_m}(\lambda', K_1)}\right) + \varepsilon L(K) \\
&\overset{(8.2.13)}{\leqslant} \frac{1}{1-\varepsilon}\sum_{i=1}^{k}\frac{t_i}{|C_i||F_{n_i}|} L\left(\frac{T_{C_iF_{n_i}}(\lambda, K) + T_{F_m}(\lambda', K_1)}{T_{F_m}(\lambda', K_1)}\right) + \varepsilon L(K) \\
&\overset{(8.2.14)}{\leqslant} \frac{1}{1-\varepsilon}\sum_{i=1}^{k}\frac{t_i}{|F_{n_i}|} L\left(\frac{T_{F_{n_i}}(\lambda, K) + N}{N}\right) + \sum_{i=1}^{k}\frac{t_i}{1-\varepsilon}\frac{|\partial_{F_{\overline{\overline{n}}}}(F_m)|L(K)}{|C_i||F_{n_i}|} + \varepsilon L(K) \\
&\leqslant \sum_{i=1}^{k}\frac{t_i}{1-\varepsilon}\frac{T_{F_{n_i}}(\bar{\lambda}, \bar{K}_2)}{|F_{n_i}|} + \frac{1}{1-\varepsilon}\sum_{i=1}^{k}\frac{|\partial_{F_{\overline{\overline{n}}}}(F_m)|L(K)}{\sum_{i=1}^{k}|C_i||F_{n_i}|} + \varepsilon L(K) \\
&\overset{(8.2.11),(8.2.10),(8.2.6)}{\leqslant} \frac{1}{1-\varepsilon}(\mathrm{ent}_L({}_{R*G}(M/N)) + \varepsilon) + \varepsilon L(K)(k+1) \\
&= \frac{1}{1-\varepsilon}\mathrm{ent}_L({}_{R*G}(M/N)) + \varepsilon\left(L(K)(k+1) + \frac{1}{1-\varepsilon}\right).
\end{aligned}
$$

□

Part IV

Surjunctivity, Stable Finiteness and Zero-Divisors

Chapter 9

Description of the conjectures and known results

Recall that a *concrete category* is a pair ($\mathfrak{C}$, $F\colon \mathfrak{C} \to \underline{\mathrm{Set}}$), where $\mathfrak{C}$ is a category and F is a faithful functor. In a concrete category one says that a morphism ϕ is injective (resp., surjective, bijective) if so is the map $F(\phi)$. An object X of $\mathfrak{C}$ is said to be *hopfian* (resp., *cohopfian*) if any surjective (resp., injective) endomorphism of X is bijective. Such notion is usually introduced in categories of (Abelian) groups, rings, modules, or topological spaces.

In this chapter we describe some classical conjectures related with the concepts of (co)Hopficity, also underlying the relations among them and explaining some of the main known results.

9.1 Cellular Automata

In this section we introduce some basic definitions and facts about cellular automata. For more details we refer to [16].

Definition 9.1. *Let X be a set and let G be a group. We consider the product X^G of $|G|$-many copies of X, endowed with the product topology of the discrete topologies on each copy of G. We consider the elements of X^G as functions $x : G \to X$, so that for any subset F of G, there is a well-defined restriction $x\restriction_F : F \to X$. Consider the following left G-representation on X^G*

$$\lambda : G \to \mathrm{Aut}_{\underline{\mathrm{Top}}}(X^G) \quad \lambda(g) = \lambda_g$$

such that $(\lambda_g(x))(h) = x(g^{-1}h)$, for all $g, h \in G$. A cellular automaton *is a map $\phi : X^G \to X^G$ such that there exists a finite subset $W \subseteq G$ and a map $\tau : X^W \to X$ such that*

$$\phi(x)(g) = \tau(\lambda_g^{-1}(x)\restriction_W)\,.$$

The map τ is called a local defining map *and W is a* defying window *for ϕ. The set X is said to be the* alphabet *of our automaton.*

Let us give the following easier characterization of those maps $\phi : X^G \to X^G$ that are cellular automata. This characterization, due to Ceccherini-Silberstein and Coornaert, is a generalization of the classical Curtis-Hedlund Theorem.

Lemma 9.2. *Let X be a set and let G be a group. Consider the uniformity on X^G whose entourages of the diagonal are of the form:*

$$\mathcal{U}_F = \{(x, y) \in X^G \times X^G : x\restriction_F = y\restriction_F\}\,,$$

where F runs over all the finite subsets of G. Then, a map $\phi : X^G \to X^G$ is a cellular automaton if and only if ϕ is uniformly continuous and $\lambda_g\phi = \phi\lambda_g$, for all $g \in G$.

Proof. Suppose first that ϕ is a cellular automaton, let W be a defining window and let $\tau : X^W \to X$ be a local defining map. Then, for any finite subset F of G,

$$\begin{aligned}\phi^{-1}(\mathcal{U}_F) &= \{(x,y) \in X^G \times X^G : \phi(x)\restriction_F = \phi(y)\restriction_F\}\\ &= \{(x,y) \in X^G \times X^G : \phi(x)(f) = \phi(y)(f),\ \forall f \in F\}\\ &= \{(x,y) \in X^G \times X^G : \tau(\lambda_f^{-1}(x)\restriction_W) = \tau(\lambda_f^{-1}(y)\restriction_W),\ \forall f \in F\}\\ &\supseteq \{(x,y) \in X^G \times X^G : x\restriction_{fW} = y\restriction_{fW},\ \forall f \in F\} = \mathcal{U}_{FW}.\end{aligned}$$

Let $g \in G$ and $x \in X^G$, then

$$\lambda_g(\phi(x))(h) = \phi(x)(g^{-1}h) = \tau(\lambda_{g^{-1}h}^{-1}(x)\restriction_W) = \tau(\lambda_h^{-1}(\lambda_g(x))\restriction_W) = \phi(\lambda_g(x))(h)\,,$$

for all $h \in G$. This proves that $\lambda_g\phi = \phi\lambda_g$ for all $g \in G$.

On the other hand, suppose that ϕ is uniformly continuous and $\lambda_g\phi = \phi\lambda_g$, for all $g \in G$. Choose a finite subset $W \subseteq G$ such that $\phi^{-1}(\mathcal{U}_{\{e\}}) \supseteq \mathcal{U}_W$, notice that W exists by uniform continuity. By the choice of W, whenever we have two elements x, $y \in X^G$ such that $x\restriction_W = y\restriction_W$, $\phi(x)(e) = \phi(y)(e)$, thus we can unambiguously define a map $\tau : X^W \to X$, where $\tau(x : W \to X) = \phi(\tilde{x} : G \to W)(e)$, where $\tilde{x}$ is any element of X^G such that $\tilde{x}\restriction_W = x$. Now, given $g \in G$,

$$\phi(x)(g) = \lambda_{g^{-1}}\phi(x)(e) = \phi(\lambda_{g^{-1}}(x))(e) = \tau(\lambda_g^{-1}(x)\restriction_W)\,,$$

so τ is a defining map and W is a defining window for ϕ. $\square$

A map is *surjunctive* if it is non-injecitve or surjective. By the above lemma, one can see that, given a set X and a group G, all the automata $\phi : X^G \to X^G$ are surjunctive if and only if $(\lambda, G) \circlearrowleft X^G$ is co-Hopfian in the category of right G-representations on $\underline{\text{Top}}$.

The following long standing conjecture was stated by Gottschalk [52] in 1973:

Conjecture 9.3. *Let X be a finite set and let G be a group. Then, any cellular automaton $\phi : X^G \to X^G$ is surjunctive.*

We refer to Conjecture 9.3 as the *Surjunctivity Conjecture*. This classical problem, which is open in general, has been known for a long time to have a positive solution whenever G is an amenable group. It was just in 1999 when Gromov [53] came out with a general theorem solving the problem in the positive for the large class of sofic groups (see also [105]). In what follows we recall the definition of this class of groups.

Let V be a nonempty finite set and denote by S_V the symmetric group on V. Given two permutations σ_1 and $\sigma_2 \in S_V$ we let

$$d_V(\sigma_1,\sigma_2) = \frac{|\{v \in V : \sigma_1(v) \neq \sigma_2(v)\}|}{|V|}\,,$$

be the *normalized Hamming distance* between σ_1 and σ_2.

Definition 9.4. *Let G be a group, let $K \subseteq G$ be a subset and let $\varepsilon \geqslant 0$. Given a finite set V, a left* (K,ε)-quasi representation *of G on V is a map $\varphi : G \to S_V$ such that:*

(QA.1) $\varphi(e) = \mathrm{id}_V$;

(QA.2) *for all* k_1 *and* $k_2 \in K$, $d_V(\varphi(k_1k_2), \varphi(k_1)\varphi(k_2)) \leqslant \varepsilon$;

(QA.3) *for all* $k_1 \neq k_2 \in K$, $d_V(\varphi(k_1), \varphi(k_2)) \geqslant 1 - \varepsilon$.

Whenever we have a left quasi representation $\varphi : G \to S_V$ *we adopt the following notation. Given two subsets* $V' \subseteq V$ *and* $G' \subseteq G$, *we let* $G'V' = \{\varphi_g(v) : g \in G',\ v \in V'\}$. *In case* $V' = \{v\}$ *is a singleton set we let* $G'v = G'\{v\}$. *Similarly, if* $G' = \{g\}$ *is a singleton,* $gV' = \{g\}V'$. *Furthermore,* $gv = \varphi_g(v)$ *for all* $v \in V$ *and* $g \in G'$.

Notice that a left $(G, 0)$-quasi representation is just a left G-representation on Set.

For finitely generated groups, the following definition of sofic group is equivalent to the definition given in [105] and [53] (see [11]).

Definition 9.5. *A group* G *is* sofic *if, for any finite subset* $K \subseteq G$ *and for any positive constant* ε, *there exists a finite set* V *and a left* (K, ε)*-quasi representation of* G *on* V.

9.1.1 Linear cellular automata

One can define particular classes of cellular automata requiring the existence of a defining map with specific properties. In this subsection we consider automata defined by continuous linear maps:

Definition 9.6. *Let* G *be a group, let* R *be a topological ring, let* X *be a topological left* R*-module and consider a cellular automaton* $\phi : X^G \to X^G$. *We say that* ϕ *is a* linear cellular automaton *if there is a defining window* W *and a local defining map* $\tau : X^W \to X$ *that is a continuous homomorphism of left* R*-modules (where* X^W *is endowed with the product topology).*

One defines analogously linear cellular automata starting with topological right R-modules.

Lemma 9.7. *Let* G *be a group, let* R *be a topological ring, let* X *be a topological left* R*-module and consider a map* $\phi : X^G \to X^G$. *Endow* X^G *with the product topology and consider the following statements:*

(1) ϕ *is a linear cellular automaton;*

(2) ϕ *is a continuous and* $\lambda_g\phi = \phi\lambda_g$, *for all* $g \in G$.

Then, (1)⇒(2). If N *is discrete, then also (2)⇒(1).*

Proof. (1)⇒(2). Let $W \subseteq G$ be a defining window and let $\alpha : X^W \to X$ be the associated local defining map. For any subset $G' \subseteq G$ we let $\pi_{G'} : X^G \to X^{G'}$ be the canonical projection $\pi(x) = x\restriction_{G'}$. Recall that a typical basic neighborhood of 0 for the product topology on X^G is of the form $\pi^{-1}_{G'}(A)$ for some finite subset $G' \subseteq G$ and some open neighborhood A of 0 in $X^{G'}$.

For any open neighborhood A of 0 in X, $\phi^{-1}(\pi^{-1}_{\{g\}}(A)) = \pi^{-1}_{gW}(\tau^{-1}(A))$ is an open neighborhood in X^G. This is enough to show that ϕ is continuous since $\{\pi^{-1}_{\{g\}}(A) : g \in G\}$ is a prebase of the topology. The fact that $\lambda_g\phi = \phi\lambda_g$, for all $g \in G$ is true for any cellular automaton.

(2)⇒(1). When X is discrete this follows by Lemma 9.2. $\square$

A consequence of Lemma 9.7 is that, in the notation of the lemma, if $(\lambda, G) \circlearrowleft X^G$ is co-Hopfian in the category of left G-representations on topological left R-modules, then any linear cellular automaton $\phi : X^G \to X^G$ is surjunctive.

The following conjecture is analogous to the Surjunctivity Conjecture for a particular class of linear cellular automata.

Conjecture 9.8. *Let $\mathbb{K}$ be a field, let V be a finite dimensional $\mathbb{K}$-vector space and let G be a group. Then, any linear cellular automaton $\phi : V^G \to V^G$ is surjunctive.*

The above conjecture, to which we refer as the *Linear Surjunctivity Conjecture* was stated by Ceccherini-Silberstein and Coornaert (see [16]) and has a positive solution for fields of characteristic 0. Furthermore, Gromov's general surjunctivity theorem in [53] (see also [13]) shows that the L-Surjunctivity Conjecture holds for the class of sofic groups. Again, the general case is unknown.

9.2 Endormophisms of modules

Let us start with the following example:

Example 9.9. *Let R be a ring and let M be a Noetherian left R-module. Then, M is Hopfian. Indeed, given a surjective morphism $\phi : M \to M$, consider the following sequence of submodules:*

$$\mathrm{Ker}(\phi) \leqslant \mathrm{Ker}(\phi^2) \leqslant \cdots \leqslant \mathrm{Ker}(\phi^n) \leqslant \dots$$

By noetherianity there exists $\bar{n} \in \mathbb{N}_+$ such that $\mathrm{Ker}(\phi^n) = \mathrm{Ker}(\phi^{\bar{n}})$, for all $n \geqslant \bar{n}$. By the following sequence of isomorphisms induced by ϕ, we obtain that ϕ is a monomorphism and so it is injective:

$$\mathrm{Ker}(\phi) \cong \frac{\mathrm{Ker}(\phi^2)}{\mathrm{Ker}(\phi)} \cong \cdots \cong \frac{\mathrm{Ker}(\phi^{\bar{n}+1})}{\mathrm{Ker}(\phi^{\bar{n}})} = 0\,.$$

Let us consider also the following definition related to hopficity.

Definition 9.10. *A ring R is* directly finite *if $xy = 1$ implies $yx = 1$ for all $x, y \in R$. Furthermore, R is* stably finite *if the ring $\mathrm{Mat}_k(R)$ of $k \times k$ square matrices with coefficients in R, is directly finite for all $k \in \mathbb{N}_+$.*

The connection with hopficity is given in the following lemma.

Lemma 9.11. *Let R be a ring, let ${}_RM$ be a left R-module and consider the following statements:*

(1) *M is hopfian;*

(2) *$\mathrm{End}_R(M)$ is directly finite.*

In general, (1) implies (2). Furthermore, if M is projective, then (1) and (2) are equivalent.

Proof. (1)⇒(2). Let $\phi, \psi \in \mathrm{End}_R(M)$ be such that $\phi\psi = id$, which implies that ϕ is surjective. By the hopficity of M we obtain that ϕ is an automorphism, that is, $\phi\psi = \psi\phi = id$.
(2)⇒(1) assuming that M is projective. Let $\phi : M \to M$ be a surjective endomorphism. Consider the following diagram in which the dotted arrow is given by the projectivity of M

$$\begin{tikzcd} M \arrow[r, "\phi"] & M \arrow[r] & 0 \\ & M \arrow[u, "id"'] \arrow[ul, dotted, "\exists\psi"] & \end{tikzcd}$$

Thus, ϕ has a right inverse which, by (2), is also a left inverse. □

Lemma 9.12. *Let R be a ring, let ${}_RM$ and ${}_RN$ be left R-modules; suppose that ${}_RM$ is hopfian and ${}_RN$ is a direct summand of ${}_RM$. Then ${}_RN$ is hopfian as well.*
In particular, R is stably finite (if and) only if any finitely generated projective left R-module is hopfian.

Proof. Let ${}_RN' \leqslant M$ be a complement for N, that is $M \cong N \oplus N'$ and let $\phi : N \to N$ be a surjective endomorphism. Let $\Phi : M \to M$ be such that $\Phi(n, n') = (\phi(n), n')$, for all $n \in N$ and $n' \in N'$. Clearly Φ is surjective and $\mathrm{Ker}(\Phi) = \mathrm{Ker}(\phi) \oplus \{0\}$. Now, $\mathrm{Ker}(\Phi)$ is trivial by the hopficity of M and so also $\mathrm{Ker}(\phi) = 0$ concluding the proof. □

By the above lemma the class of Hopfian modules is closed by taking direct summands. On the other hand, in general the class of Hopfian modules is not closed under taking finite direct sums, not even over the ring $\mathbb{Z}$, for a classical (counter)example see [25, Example 3]. Similarly, the class of Hopfian modules is not closed under taking submodules. In order to obtain a class with better closure properties, Anna Giordano Bruno and the author used in [46] the concept of *hereditarily hopfian* Abelian group, that is, an Abelian group all of whose subgroups are Hopfian. We remark that prof. Brendan Goldsmith let us know that this concept is well-known to experts, even if it seems not to appear in the literature before [46].

Example 9.13. *The additive group of ring of p-adic integers $\mathbb{J}_p$ is Hopfian but not hereditarily Hopfian.*
In fact, any $\mathbb{Z}$-linear endomorphism of $(\mathbb{J}_p, +)$ is also $\mathbb{J}_p$-linear, so $\mathrm{End}_{\mathbb{Z}}(\mathbb{J}_p)$ is canonically isomorphic, as a ring, to the commutative ring $\mathrm{End}_{\mathbb{J}_p}(\mathbb{J}_p) \cong \mathbb{J}_p$. This shows that ${}_{\mathbb{J}_p}\mathbb{J}_p$ is directly finite (since $\mathrm{End}_{\mathbb{J}_p}(\mathbb{J}_p)$ is commutative) and so ${}_{\mathbb{J}_p}\mathbb{J}_p$ is Hopfian, being a projective $\mathbb{J}_p$-module (alternatively one can argue directly that ${}_{\mathbb{J}_p}\mathbb{J}_p$ is a Noetherian $\mathbb{J}_p$-module). Thus, also ${}_{\mathbb{Z}}\mathbb{J}_p$ is Hopfian.
On the other hand, ${}_{\mathbb{Z}}\mathbb{J}_p$ has infinite torsion-free rank, that is, there is a subgroup $G \leqslant \mathbb{J}_p$ of the form $G \cong \mathbb{Z}^{(\mathbb{N})}$ which is clearly not Hopfian. Hence, ${}_{\mathbb{Z}}\mathbb{J}_p$ is not hereditarily hopfian.

Generalizing from the case of Abelian groups we get the following definition:

Definition 9.14. *Given a ring R, we say that a left R-module M is* hereditarily Hopfian *if and only if all of its submodules are Hopfian.*

9.2.1 The Stable Finiteness Conjecture

A long-standing open question about directly finite rings is the following conjecture due to Kaplansky [64]

Conjecture 9.15. *For any field $\mathbb{K}$ and any group G, the group ring $\mathbb{K}[G]$ is stably finite.*

In case the field $\mathbb{K}$ is commutative and has characteristic 0, then the problem was solved in the positive by Kaplansky. There was no progress in the positive characteristic case until 2002, when Ara, O'Meara and Perera [5] proved that a group algebra $D[G]$ is stably finite whenever G is residually amenable and D is any division ring. This last result was generalized by Elek and Szabó [38] (see also [13] and [6] for alternative proofs), who proved the following

Theorem 9.16. *For any division ring $\mathbb{K}$ and any sofic group G, the group ring $\mathbb{K}[G]$ is stably finite.*

A straightforward consequence of the above theorem is that $\mathrm{Mat}_n(\mathbb{K}[G])$ is stably finite for any division ring $\mathbb{K}$ and any sofic group G. Now, by the Artin-Wedderburn Theorem, given a semisimple Artinian ring R, there exist positive integers k, $n_1, \ldots, n_k \in \mathbb{N}_+$ and division rings $\mathbb{K}_1, \ldots, \mathbb{K}_k$ such that $R \cong \mathrm{Mat}_{n_1}(\mathbb{K}_1) \times \cdots \times \mathrm{Mat}_{n_k}(\mathbb{K}_k)$. This implies that, $R[G] \cong \mathrm{Mat}_{n_1}(\mathbb{K}_1[G]) \times \cdots \times \mathrm{Mat}_{n_k}(\mathbb{K}_k[G])$, thus a consequence of the above theorem is that $R[G]$ is stably finite whenever R is semisimple Artinian and G sofic. This result can be further generalized as follows:

Remark 9.17. *[Ferran Cedó, private communication (2012)] If R is a ring with left Krull dimension, then $R[G]$ is stably finite. First of all, notice that, if I is a nilpotent ideal of R, then $I[G] = R[G]I$ is a nilpotent ideal of $R[G]$ and so one can reduce the problem modulo nilpotent ideals. Now, by [73, Corollary 6.3.8], the prime radical N of R is nilpotent and $N = P_1 \cap \cdots \cap P_m$, where $P_1, \ldots, P_m$ are minimal prime ideals. Thus, by [73, Proposition 6.3.5], R/N is a semiprime Goldie ring and so, by [73, Theorem 2.3.6] R/N has a classical semisimple Artinian ring of quotients S. In particular, $(R/N)[G]$ embeds in $S[G]$ and it is therefore stably finite.*

Both the proof of the residually amenable case due to Ara, O'Meara and Perera, and the proof of the sofic case due to Elek and Szabó, consist in finding a suitable embedding of $\mathbb{K}[G]$ in a ring which is known to be stably finite. Such methods are really effective but, as far as we know, cannot be used to obtain information on the modules over $\mathbb{K}[G]$. It seems natural to ask the following question related to Conjecture 9.15:

Question 9.1. *Let R be a ring, let G be a group, let $R{*}G$ be a crossed product and let M be a finitely generated left R-module. Under what conditions is $R{*}G \otimes_R M$ Hopfian (or hereditarily) Hopfian?*

In particular, Theorem 9.16 proves the Hopficity of $R{*}G \otimes_R R$ in the very particular case when R is a field, $R{*}G = R[G]$ and G is sofic. Furthermore, in [5] one can find a proof of the fact that any crossed product $D{*}G$ of a division ring D and an amenable group G, is stably finite.

We will generalize both these results in Chapters 10 and 11.

9.2.2 The Zero-Divisors Conjecture

Let us introduce another classical conjecture due to Kaplansky about group rings:

Conjecture 9.18. *Let $\mathbb{K}$ be a field and G be a torsion-free group. Then $\mathbb{K}[G]$ is a domain.*

Some cases of the above conjecture are known to be true but the conjecture is fairly open in general (for a classical reference on this conjecture see for example [83]). In most of the known cases, the strategy for the proof is to find an immersion of $\mathbb{K}[G]$ in some division ring. This is clearly sufficient but, in principle, it is a stronger property. To the best of the author's knowledge, the following question remains open:

Question 9.2. *Is it true that $\mathbb{K}[G]$ is a domain if and only if $\mathbb{K}[G]$ is a subring of a division ring?*

The above question is known to have positive answer if G is amenable (see [69, Example 8.16]).

9.3 Relations among the conjectures

9.3.1 Duality

In the introduction of [38], it is observed that the Surjunctivity Conjecture implies the Stable Finiteness Conjecture, in case $\mathbb{K}$ is a finite field. Roughly speaking, the idea is to consider $\mathbb{K}$ as an Abelian group, view $(\mathbb{K}[G])^k$ as a dense subgroup of the compact group $(\mathbb{K}^k)^{(G)}$ and to extend maps by continuity.
Let us give a different argument. In brief, consider the finite field $\mathbb{K}$ as a finite discrete Abelian group; then, applying Pontryagin-Van Kampen's duality to a G-equivariant endomorphism ϕ of the discrete group $(\mathbb{K}^k)^{(G)}$ (with the left G-action) we get a continuous G-equivariant endomorphism $\widehat{\phi}$ of the compact group $(\mathbb{K}^k)^G$ (with the right G-action) endowed with the product of the discrete topologies, and viceversa.
Thus, Pontryagin-Van Kampen's duality induces an anti-isomorphism

$$(\mathrm{End}_{\mathrm{l.Rep}_G(\underline{\mathrm{LcaGr}})}((\mathbb{K}^k)^{(G)}))^{op} \cong \mathrm{End}_{\mathrm{r.Rep}_G(\underline{\mathrm{LcaGr}})}((\mathbb{K}^k)^G)$$

between the ring of G-equivariant $\mathbb{K}$-endomorphisms of $(\mathbb{K}^k)^{(G)}$ and the ring of G-equivariant continuous $\mathbb{K}$-endomorphisms of $(\mathbb{K}^k)^G$. Ceccherini-Silberstein and Coornaert [13] give a different argument that shows that the same ring anti-isomorphism holds for arbitrary fields (they compose their map with the usual anti-involution on $\mathrm{Mat}_k(\mathbb{K}[G])$ to make it an actual ring isomorphism). This proves that the L-Surjunctivity Conjecture is equivalent to the Stable Finiteness Conjecture.

In this subsection we apply the Mülcer Duality Theorem, proved in Chapter 3, to relate questions regarding linear cellular automata to questions regarding modules. This process culminates in Corollary 9.22, which is a generalization of the above anti-isomorphism.

Given a ring R and a group G, by Lemmas 3.58 and 9.7, and Proposition 3.59, a linear cellular automaton whose alphabet is a discrete Artinian right R-module is exactly a morphism in $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$. In particular, the following corollary applies to show that such linear cellular automata have the so-called *closed image property.*

Corollary 9.19. *Let G be a group and let R be a ring. Let $\lambda_1 : G \to \mathrm{Aut}_{\mathrm{SLC}\text{-}R}(N_1)$ and $\lambda_2 : G \to \mathrm{Aut}_{\mathrm{SLC}\text{-}R}(N_2)$ be two right representations of G on strongly linearly compact right R-modules. Given a morphism of representations $\phi : N_1 \to N_2$, the image $\phi(N_1)$ is closed and invariant under the action of G on N_2.*

Proof. Apply Lemma 3.60. □

The following corollary of Theorem 3.65 provides a "bridge" between automata and homomorphisms of left $A[G]$-modules.

Corollary 9.20. *Let G be a group and consider the setting described in (Dual.1, 2, 3). The duality described in Theorem 3.65 induces a duality between $A[G]$-Mod and $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$.*

Proof. It is enough to notice that a left action $\rho : G \to \mathrm{Aut}_A(M)$ of G on a left A-module M corresponds to a right action $\rho^* : G \to \mathrm{Aut}_{\mathrm{SLC}\text{-}R}(M^*)$ of G on the dual module M^* (just letting $\rho^*(g) = (\rho(g))^*$ for all $g \in G$) and that a right action $\lambda : G \to \mathrm{Aut}_{\mathrm{SLC}\text{-}R}(N)$ on a strongly linearly compact right R-module N (notice that G acts via topological automorphisms) corresponds to a left action $\lambda^* : G \to \mathrm{Aut}_A(N^*)$ of G on N^*. Applying Theorem 3.65 one obtains a duality between $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$ and $\mathrm{l.Rep}_G(\mathrm{Mod}\text{-}A)$; to conclude just remember that $\mathrm{l.Rep}_G(A\text{-Mod})$ is equivalent to $A[G]$-Mod. □

The following corollary generalizes [16, Theorem 8.12.1]

Corollary 9.21. *Let G be a group, let R be a ring and let N be a strictly linearly compact right R-module. Let $X \leqslant N^G$ be a closed G-invariant submodule. Then, any bijective G-equivariant continuous homomorphism $\phi : X \to X$ has a continuous and G-equivariant inverse.*
Furthermore, in the setting described in (Dual.1, 2, 3) and letting $H = K^n$ for some positive integer n, any injective linear cellular automaton $\phi : H^G \to H^G$ has a left inverse that is a linear cellular automaton.

Proof. By Proposition 3.59, X is strictly linearly compact, so ϕ is a topological automorphism and thus its inverse $\psi : X \to X$ is automatically a topological automorphism. The fact that ψ is G-equivariant can be deduced from the fact that it is the inverse of a G-equivariant map.

For the second part, since H is strictly linearly compact discrete, an endomorphism of H^G is G-equivariant and continuous if and only if it is a linear cellular automaton. Furthermore, the dual of H^G is $A[G]^n$ that is a projective left $A[G]$-module, so H^G is an injective object in the category $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$. Now, an injective linear cellular automaton $\phi : H^G \to H^G$ is a monomorphism and so it has a left inverse $\psi : H^G \to H^G$ in $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$. By the previous discussion, ψ is a linear cellular automaton. $\square$

The following corollary improves [13, Theorem 1.3].

Corollary 9.22. *Let G be a group and consider the setting described in (Dual.1, 2, 3). Given a right G-representation $\lambda : G \to \mathrm{Aut}_{\mathrm{SLC}\text{-}R}(N)$ on a strictly linearly compact right R-module N, let M be the left $A[G]$-module which is dual to $\lambda \circlearrowleft N$. Then there is an anti-isomorphism of rings*

$$\begin{array}{ccc}(\mathrm{End}_{\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)}(N))^{op} & \longrightarrow & \mathrm{End}_{A[G]}(M) \\ \phi & \longmapsto & \phi^* .\end{array}$$

In particular, $\mathrm{End}_{\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)}((K^n)^G)$ *is anti-isomorphic to* $\mathrm{Mat}_n(A[G])$ *for any positive integer n. Hence, $A[G]$ is stably finite if and only if any linear cellular automaton $\phi : (K^n)^G \to (K^n)^G$ is surjunctive, for any positive integer n.*

Proof. The first statement is an easy consequence of duality. The fact that $\mathrm{End}_{\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)}(K^n)$ is anti-isomorphic to $\mathrm{Mat}_k(A[G])$ follows noticing that the dual of $(K^n)^G$ is exactly $A[G]^n$ and that $\mathrm{End}_{A[G]}(A[G]^n) \cong \mathrm{Mat}_n(A[G])$. The last statement follows by the previous one recalling that linear cellular automata $(K^n)^G \to (K^n)^G$ are exactly the continuous G-equivariant endomorphisms of $(K^n)^G$ and using the second part of Corollary 9.21. $\square$

9.3.2 Zero-divisors and pre-injectivity

The following lemma will allow us to translate the Zero-Divisors Conjecture in the language of linear cellular automata. Before that, consider a group G and the setting described (Dual.1, 2, 3), take a strictly linearly compact right R-module N and let $G \circlearrowleft N^G$ be the usual right action of G. Then, the dual of this right G-representation is the left $A[G]$-module $A[G] \otimes_A N^*$.

Lemma 9.23. *Let G be a group and consider the setting described in (Dual.1, 2, 3). Let N be a strictly linearly compact right R-module and let $M = N^*$ be its dual. The following are equivalent:*

(1) *any non-trivial endomorphism $A[G] \otimes_A M \to A[G] \otimes_A M$ of left $A[G]$-modules is injective;*

(2) *any non-trivial linear cellular automaton $N^G \to N^G$ is surjective.*

Proof. Let $\phi : N^G \to N^G$ be a linear cellular automaton. We have already noticed that ϕ is a morphism in $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$, let us show that ϕ is an epimorphism in this category if and only if it is surjective. Surjective implies epimorphism in any concrete category so let us suppose that ϕ is an epimorphism. By Lemma 3.60, $\mathrm{Im}(\phi)$ is closed in N^G and it is clearly G-invariant. We obtain the following sequence in $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$:

$$N^G \xrightarrow{\phi} N^G \xrightarrow{\pi} N^G/\mathrm{Im}(\phi)$$

where $\pi\phi = 0$ but, since we supposed that ϕ is an epimorphism, $\pi = 0$, that is, $\mathrm{Im}(\phi) = N^G$.

By the above discussion, condition (2) holds if and only if any linear cellular automaton $N^G \to N^G$ is an epimorphism in $\mathrm{r.Rep}_G(\mathrm{SLC}\text{-}R)$, thus, by duality, any endomorphism $A[G] \otimes_A M \to A[G] \otimes_A M$ in $A[G]$-Mod is a monomorphism, which is equivalent to condition (1) in the statement. □

We conclude this subsection combining the above result with a result of Ceccherini-Silberstein and Coornaert, for which we need the following

Definition 9.24. *Let R be a ring, let N be right R-module, let G be a group and let $\phi : N^G \to N^G$ be a linear cellular automaton. Then, ϕ is* pre-injective *if and only if the restriction $\phi : N^{(G)} \to N^{(G)}$ is injective.*

Proposition 9.25. *Let $\mathbb{K}$ be a field and let G be a group. The following are equivalent:*

(1) $\mathbb{K}[G]$ *is a domain;*

(2) *any non-trivial linear cellular automaton $\mathbb{K}^G \to \mathbb{K}^G$ is pre-injective;*

(3) *any non-trivial linear cellular automaton $\mathbb{K}^G \to \mathbb{K}^G$ is surjective.*

Proof. The equivalence between (1) and (3) is a consequence of Lemma 9.23, while the equivalence between condition (1) and condition (2) is [16, Corollary 8.16.12] □

The above proposition extends [16, Corollary 8.16.13] to non-amenable groups.

Chapter 10

The amenable case

10.1 Surjunctivity

Let X be a finite set and let G be an amenable group. The classical way to prove that any given cellular automaton $\phi : X^G \to X^G$ is surjunctive is via topological entropy. In this section we explain in detail this approach. First of all we need to give a formula to compute the entropy of left G-representations induced on the closed G-invariant subsets Y of X^G. Given an open cover $\mathcal{B}$ of X^G we let

$$\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}.$$

Notice that $\mathcal{B}_Y$ is an open cover of Y.

Lemma 10.1. *Let X be a finite set, let G be a group and let $Y \subseteq X^G$ be a G-invariant (with respect to the usual left G-representation $(\lambda, G) \circlearrowleft X^G$) closed subset. For any finite subset $F \subseteq G$, let $\pi_F : X^G \to X^F$ be the restriction map and let $\mathcal{A} = \{\pi_{\{e\}}^{-1}(x) : x \in X\}$. Then,*

(1) *let $\mathcal{C} = \{C_1, \dots, C_n\}$ be a finite open cover of Y, where $C_i = \pi_{F_i}^{-1}(x_i) \cap Y$ for some finite subset $F_i \subseteq G$ and $x_i \in X^{F_i}$, for all $i = 1, \dots, n$. Then, $\mathcal{C}$ has a refinement of the form $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ for some finite subset $F \subseteq G$.*

(2) *given an open cover $\mathcal{B}$ of Y, there is a finite subset $e \in F \subseteq G$ such that $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y \preceq \mathcal{B}$;*

(3) *$N((\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y) = |\pi_F(Y)|$ for any $F \subseteq G$.*

Proof. (1) Let $F = \bigcup_{i=1}^n F_i$ and notice that $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ is a partition. Furthermore, each C_i is a finite union of elements of $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ and any element of $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ is contained in some C_i (since $\bigcup_{i=1}^n C_i = Y = \bigcup(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ and since $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ is a partition). Thus, $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y \preceq \mathcal{C}$.
(2) First of all, we extract a finite sub-cover $\mathcal{B}'$ of $\mathcal{B}$. Let $\mathcal{B}' = \{B_1, \dots, B_n\}$, then each B_i is a union of sets of the form $\bigcap_{g\in F} \lambda_g^{-1}\pi_{F_i}^{-1}(x) \cap Y$ with $F \subseteq G$ finite and $x \in X^F$. Let $\mathcal{C}$ be the family of all such sets and let $\mathcal{C}'$ be a finite sub-cover of $\mathcal{C}$, which exists by compactness. Notice that $\mathcal{C}' \preceq \mathcal{C} \preceq \mathcal{B}' \preceq \mathcal{B}$ and that $\mathcal{C}'$ has a refinement of the desired form by part (1).
(3) Let $A_1, A_2 \in (\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$. By definition, there exist $x_i \in \pi_F(Y)$ such that $A_i = \pi_F^{-1}(x_i) \cap Y$ (for $i = 1, 2$) and so $A_1 = A_2$ if and only if $x_1 = x_2$. Thus, the sets in $(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A})_Y$ are in bijection with the elements of $\pi_F(Y)$. $\square$

The above technical lemma allows us to prove the following useful formula.

Proposition 10.2. *Let X be a finite set, let G be a countably infinite amenable group and let $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$ be Følner sequence for G. Given a closed G-invariant subset $Y \subseteq X^G$ with the induced G-action $(\lambda\restriction_Y, G) \circlearrowright Y$,*

$$h_T(\lambda\restriction_Y, \mathfrak{s}) = \lim_{n\in\mathbb{N}} \frac{\log|\pi_{F_n}(Y)|}{|F_n|} .$$

In particular, $h_T(\lambda\restriction_Y, \mathfrak{s}) \leqslant \log|X|$ and $h_T(\lambda, \mathfrak{s}) = \log|X|$.

Proof. Notice that, given two open covers $\mathcal{B}_1$, $\mathcal{B}_2$ of Y, if $\mathcal{B}_1 \preceq \mathcal{B}_2$, then $N(\mathcal{B}_1) \geqslant N(\mathcal{B}_2)$ and $\bigvee_{g\in F} \lambda_g^{-1}\mathcal{B}_1 \preceq \bigvee_{g\in F} \lambda_g^{-1}\mathcal{B}_1$ for any finite subset $F \subseteq G$. This shows that $h_T(\mathcal{B}_1, \lambda\restriction_Y, \mathfrak{s}) \geqslant h_T(\mathcal{B}_2, \lambda\restriction_Y, \mathfrak{s})$ whenever $\mathcal{B}_1 \preceq \mathcal{B}_2$.

By the above discussion and Lemma 10.1 (2)

$$h_T(\lambda\restriction_Y, \mathfrak{s}) = \sup\left\{ h_T\left(\left(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A}\right)_Y, \lambda\restriction_Y, \mathfrak{s}\right) : F \subseteq G \text{ finite}\right\} .$$

Consider now a finite subset $F \subseteq G$ and notice that

$$\begin{aligned} h_T\left(\left(\bigvee_{g\in F} \lambda_g^{-1}\mathcal{A}\right)_Y, \lambda\restriction_Y, \mathfrak{s}\right) &= \lim_{n\in\mathbb{N}} \frac{\log N\left(\left(\bigvee_{g\in F_nF} \lambda_g^{-1}\mathcal{A}\right)_Y\right)}{|F_nF|}\frac{|F_nF|}{|F_n|} \\ &\overset{(*)}{=} \lim_{n\in\mathbb{N}} \frac{\log N\left(\left(\bigvee_{g\in F_nF} \lambda_g^{-1}\mathcal{A}\right)_Y\right)}{|F_nF|} \\ &\overset{(**)}{=} h_T\left(\mathcal{A}_Y, \lambda\restriction_Y, \mathfrak{s}\right) \end{aligned}$$

where $(*)$ is true since $\lim_{n\in\mathbb{N}} |F_nF|/|F_n| = 1$ (this can be proved as in the proof of Lemma 8.19), while $(**)$ follows from the fact that F_nF is a Følner sequence (see Lemma 4.22) and the topological entropy does not depend on the choice of the Følner sequence.

The above computation proves that $h_T(\lambda\restriction_Y, \mathfrak{s}) = h_T\left(\mathcal{A}_Y, \lambda\restriction_Y, \mathfrak{s}\right)$, thus by Lemma 10.1 (3)

$$h_T(\lambda\restriction_Y, \mathfrak{s}) = \lim_{n\in\mathbb{N}} \frac{\log|\pi_{F_n}(Y)|}{|F_n|} .$$

The last statements follow since $\log|\pi_{F_n}(Y)| \leqslant \log|X^{F_n}| = |F_n|\log|X|$, for all $n \in \mathbb{N}$. □

For our application to surjunctivity we need also to show that the inequality $h_T(\lambda\restriction_Y, \mathfrak{s}) \leqslant \log|X|$ in the above proposition is strict whenever the inclusion $Y \subseteq X^G$ is proper. For that we need to apply the formalism of (E, F)-nets (see Subsection 8.2.2).

Lemma 10.3. *Let G be a group, let E, $K \subseteq G$ be finite subsets, let $F = EE^{-1}$ and let $\mathcal{N} \subseteq G$ be an (E, F)-net. Then,*

$$K \subseteq \bigcup_{g\in\mathcal{N}\cap In_E(K)} gF \cup \partial_F(K)F .$$

In particular, $|\mathcal{N} \cap In_E(K)|/|K| \geqslant 1/|F| - |\partial_F(K)|/|K|$.

Proof. By the definition of (E, F)-net, the translates of F cover G, so

$$K \subseteq \bigcup_{\mathcal{N}\cap Out_F(K)} gF .$$

Furthermore, $(\mathcal{N} \cap Out_F(K))\backslash(\mathcal{N} \cap In_E(K)) \subseteq Out_F(K)\backslash In_F(K) = \partial_F(K)$. Putting together these observations we obtain:

$$\begin{aligned} K\backslash \bigcup_{g\in\mathcal{N}\cap In_E(K)} gF &\subseteq \bigcup_{g\in\mathcal{N}\cap Out_F(K)} gF\backslash \bigcup_{g\in\mathcal{N}\cap In_E(K)} gF \\ &\subseteq \bigcup_{g\in(\mathcal{N}\cap Out_F(K))\backslash(\mathcal{N}\cap In_E(K))} gF \subseteq \partial_F(K)F\,. \end{aligned}$$

This shows that $|K| - |\bigcup_{g\in\mathcal{N}\cap In_E(K)} gF| = |K| - |\mathcal{N} \cap In_E(K)||F| \leqslant |\partial_F(K)||F|$. □

Using the above technical lemma we can give some concrete computation of topological entropy.

Proposition 10.4. *Let X be a finite set, let G be a countably infinite amenable group and let $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$ be Følner exhaustion for G. Given a closed G-invariant proper subset $Y \subsetneq X^G$ with the induced G-action $(\lambda\restriction_Y, G) \circlearrowright Y$, $h_T(\lambda\restriction_Y, \mathfrak{s}) \lneq \log|X|$.*

Proof. We verified in Proposition 10.2 that

$$h_T(\lambda\restriction_Y, \mathfrak{s}) = \lim_{n\in\mathbb{N}} \frac{\log|\pi_{F_n}(Y)|}{|F_n|} \leqslant \log|X|\,,$$

so we have to show that the above inequality is strict. Since $Y \subsetneq X^G$, there exists $\bar{n} \in \mathbb{N}$ such that $\pi_{F_n}(Y) \neq X^{F_n}$ for all $n \geqslant \bar{n}$, thus, $\log|\pi_{F_n}(Y)| \leqslant |F_n|\log|X| - 1$ for all $n \geqslant \bar{n}$. Let $E = F_{\bar{n}}$, let $F = EE^{-1}$ and let $\mathcal{N}$ be an (E, F)-net. Then, for all $n \geqslant \bar{n}$, let $\underline{F_n} = F_n\backslash\bigcup_{g\in\mathcal{N}\cap In_E(F_n)} gE$, so

$$\pi_{F_n}(Y) \subseteq \prod_{g\in\mathcal{N}\cap In_E(F_n)} \pi_{gE}(Y) \times X^{\underline{F_n}}\,.$$

Using Lemma 10.3, $|\mathcal{N} \cap In_E(F_n)|/|F_n| \geqslant 1/|F| - |\partial_F(F_n)|/|F_n|$ and so, for all $n \geqslant \bar{n}$,

$$\begin{aligned} \frac{\log|\pi_{F_n}(Y)|}{|F_n|} &\leqslant \frac{|\mathcal{N} \cap In_E(F_n)|\log|\pi_E(Y)| + (|F_n| - |\mathcal{N} \cap In_E(F_n)||E|)\log|X|}{|F_n|} \\ &\leqslant \frac{|\mathcal{N} \cap In_E(F_n)|\log(|X^E| - 1) + (|F_n| - |\mathcal{N} \cap In_E(F_n)||E|)\log|X|}{|F_n|} \\ &= \log|X| - \frac{|\mathcal{N} \cap In_E(F_n)|}{|F_n|}\log\left(\frac{|X^E|}{|X^E| - 1}\right) \\ &\leqslant \log|X| - \left(\frac{1}{|F|} - \frac{|\partial_F(F_n)|}{|F_n|}\right)\log\left(\frac{|X^E|}{|X^E| - 1}\right) \end{aligned}$$

By the above computation and the Følner condition,

$$\begin{aligned} \lim_{n\in\mathbb{N}} \frac{\log|\pi_{F_n}(Y)|}{|F_n|} = \lim_{n\geqslant\bar{n}} \frac{\log|\pi_{F_n}(Y)|}{|F_n|} &\leqslant \lim_{n\geqslant\bar{n}} \left(\log|X| - \left(\frac{1}{|F|} - \frac{|\partial_F(F_n)|}{|F_n|}\right)\log\left(\frac{|X^E|}{|X^E| - 1}\right)\right) \\ &= \log|X| - \frac{1}{|F|}\log\left(\frac{|X^E|}{|X^E| - 1}\right) < \log|X|\,, \end{aligned}$$

as desired. □

We are now ready for the announced result:

Theorem 10.5. *Let X be a finite set, let G be a countably infinite amenable group and let $\phi : X^G \to X^G$ be a cellular automaton. Then, ϕ is surjunctive.*

Proof. Suppose that ϕ is injective and let us prove that ϕ is surjective. Notice that $\phi(X^G) = Y$ is a closed and ϕ induces an homeomorphism between X^G and Y (see Theorem 3.11); furthermore, Y is a G-invariant subset of X^G. Let $\mathfrak{s} = \{F_n\}_{n\in\mathbb{N}}$ be Følner sequence for G, then $h_T(\lambda\restriction_Y, \mathfrak{s}) = h_T(\lambda, \mathfrak{s})$ since $\lambda\restriction_Y \circlearrowleft Y$ and $\lambda \circlearrowleft X^G$ are isomorphic representations. By Proposition 10.4 this implies that $Y = X^G$, that is, ϕ is surjective. □

10.2 Stable finiteness

In this section we use the theory of algebraic entropy to prove that a large class of left $R*G$-modules is hereditarily Hopfian, in case R is left Noetherian and G is amenable. We remark that this is a very strong version of Kaplasky's Stable Finiteness Conjecture in the amenable case, which can be re-obtained as a corollary.

Theorem 10.6. *Let R be a left Noetherian ring, G a countably infinite amenable group and let $R*G$ be a fixed crossed product. Then, for any finitely generated left R-module ${}_RK$, the left $R*G$-module $R*G \otimes_R K$ is hereditarily Hopfian.*
In particular, $\mathrm{End}_{R*G}(M)$ *is stably finite for any submodule ${}_{R*G}M \leqslant R*G \otimes_R K$.*

The proof of the above theorem makes use of the full force of the localization techniques introduced in Chapters 1 and 2. Such heavy machinery hides in some sense the idea behind the proof; this is the reason for which we prefer to give first the proof of the following more elementary statement, whose proof is far more transparent.

Lemma 10.7. *Let $\mathbb{K}$ be a division ring, let G be a countably infinite amenable group and fix a crossed product $\mathbb{K}*G$. For all $n \in \mathbb{N}_+$, $(\mathbb{K}*G)^n = \mathbb{K}*G \otimes \mathbb{K}^n$ is a hereditarily hopfian left $\mathbb{K}*G$-module.*

Proof. Let $n \in \mathbb{N}_+$ and choose $\mathbb{K}*G$-submodules $N \leqslant M \leqslant (\mathbb{K}*G)^n$ such that there exists a short exact sequence
$$0 \to N \to M \to M \to 0\,,$$
we have to show that $N = 0$. The length function $\dim : \mathbb{K}\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ is compatible with any crossed product, so we can consider the dim-entropy of left $\mathbb{K}*G$-modules. In particular, we have that $\mathrm{ent}_{\dim}(M) = \mathrm{ent}_{\dim}(M) + \mathrm{ent}_{\dim}(N)$ and
$$0 \leqslant \mathrm{ent}_{\dim}(N) \leqslant \mathrm{ent}_{\dim}(M) \leqslant \mathrm{ent}_{\dim}(\mathbb{K}*G \otimes \mathbb{K}^n) = n\,.$$
Thus, $\mathrm{ent}_{\dim}(N) = 0$. By Proposition 8.23, this implies that $\dim(N) = 0$, that is, $N = 0$. □

The same argument of the above proof can be used to prove Theorem 10.6, modulo the fundamental tool of Gabriel dimension:

Proof of Theorem 10.6. Consider a left $R*G$-submodule $M \leqslant R*G \otimes_R K$ and a short exact sequence of left $R*G$-modules
$$0 \to \mathrm{Ker}(\phi) \to M \xrightarrow{\phi} M \to 0\,.$$
In order to go further with the proof we need to show that, as a left R-module, the Gabriel dimension of $\mathrm{Ker}(\phi)$ is a successor ordinal whenever it is not -1 (i.e., whenever $\mathrm{Ker}(\phi) \neq 0$). This follows by the following

Lemma 10.8. *In the hypotheses of Theorem 10.6,* $\mathrm{G.dim}({}_RN)$ *is a successor ordinal for any non-trivial* R*-submodule* $N \leqslant R{*}G \otimes_R K$*.*

Proof. A consequence of Lemma 2.70 (4) is that $\mathbf{T}_{\alpha+1}(K)/\mathbf{T}_\alpha(K) \neq 0$ for just finitely many ordinals α. Notice that $\mathbf{T}_\alpha(R{*}G \otimes_R K) \cong R{*}G \otimes_R \mathbf{T}_\alpha(K)$, as left R-modules, for any ordinal α. Thus, $\mathbf{T}_{\alpha+1}(R{*}G \otimes_R K)/\mathbf{T}_\alpha(R{*}G \otimes_R K) \neq 0$ for finitely many ordinals. Notice also that $\mathbf{T}_\alpha(N) = \mathbf{T}_\alpha(R{*}G \otimes_R K) \cap N$ for all α, thus,

$$\frac{\mathbf{T}_{\alpha+1}(N)}{\mathbf{T}_\alpha(N)} = \frac{\mathbf{T}_{\alpha+1}(R{*}G \otimes_R K) \cap N}{\mathbf{T}_\alpha(R{*}G \otimes_R K) \cap N} \cong$$
$$\cong \frac{(\mathbf{T}_{\alpha+1}(R{*}G \otimes_R K) \cap N) + \mathbf{T}_\alpha(R{*}G \otimes_R K)}{\mathbf{T}_\alpha(R{*}G \otimes_R K)} \leqslant \frac{\mathbf{T}_{\alpha+1}(R{*}G \otimes_R K)}{\mathbf{T}_\alpha(R{*}G \otimes_R K)}$$

is different from zero for finitely many ordinals α. Thus,

$$\mathrm{G.dim}(N) = \sup\{\alpha + 1 : \mathbf{T}_{\alpha+1}(N)/\mathbf{T}_\alpha(N) \neq 0\} = \max\{\alpha + 1 : \mathbf{T}_{\alpha+1}(N)/\mathbf{T}_\alpha(N) \neq 0\}$$

is clearly a successor ordinal. □

Now, suppose that $\mathrm{Ker}(\phi) \neq 0$ and let $\mathrm{G.dim}(\mathrm{Ker}(\phi)) = \alpha + 1$. We want to show that

$$\phi\restriction_{\mathbf{T}_{\alpha+1}(M)} : \mathbf{T}_{\alpha+1}(M) \to \mathbf{T}_{\alpha+1}(M)$$

is surjective. Indeed, if there is $x \in \mathbf{T}_{\alpha+1}(M) \backslash \phi(\mathbf{T}_{\alpha+1}(M))$, it means that there exists $y \in M \backslash \mathbf{T}_{\alpha+1}(M)$ such that $\phi(y) = x$ (by the surjectivity of ϕ). This is to say that there is a short exact sequence

$$0 \to \mathrm{Ker}(\phi) \cap R{*}Gy \to R{*}Gy \to R{*}Gx \to 0\,,$$

with $\mathrm{G.dim}({}_R(R{*}Gy)) \gneq \alpha + 1 \geqslant \max\{\mathrm{G.dim}({}_R(\mathrm{Ker}(\phi) \cap R{*}Gy)), \mathrm{G.dim}({}_R(R{*}Gx))\}$, which contradicts Lemma 2.70 (2). Thus, we have a short exact sequence of left $R{*}G$-modules

$$0 \to \mathrm{Ker}(\phi) \to \mathbf{T}_{\alpha+1}(M) \to \mathbf{T}_{\alpha+1}(M) \to 0\,.$$

Consider the length function $\ell_\alpha : R\text{-Mod} \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ described in Subsection 7.2.3 and recall that $\mathrm{Ker}(\ell_\alpha)$ is exactly the class of all left R-modules with Gabriel dimension $\leqslant \alpha$. Furthermore, $\mathbf{T}_{\alpha+1}(K)$ is a Noetherian module, thus, $\mathbf{Q}_\alpha(\mathbf{T}_{\alpha+1}(K))$ is a Noetherian object in a semi-Artinian category, that is, an object with finite composition length, for this reason $\ell_\alpha(\mathbf{T}_{\alpha+1}(K)) = \ell(\mathbf{Q}_\alpha(\mathbf{T}_{\alpha+1}(K))) < \infty$. Using the computations of Example 8.21 and the Addition Theorem, we get

$$\mathrm{ent}_{\ell_\alpha}(\mathbf{T}_{\alpha+1}(R{*}G \otimes K)) = \mathrm{ent}_{\ell_\alpha}(R{*}G \otimes \mathbf{T}_{\alpha+1}(K)) = \ell_\alpha(\mathbf{T}_{\alpha+1}(K)) < \infty$$

and

$$\mathrm{ent}_{\ell_\alpha}(\mathbf{T}_{\alpha+1}(M)) = \mathrm{ent}_{\ell_\alpha}(\mathbf{T}_{\alpha+1}(M)) + \mathrm{ent}_{\ell_\alpha}(\mathrm{Ker}(\phi))\,.$$

Hence, $\mathrm{ent}_{\ell_\alpha}(\mathrm{Ker}(\phi)) = 0$ which, by Proposition 8.23, is equivalent to say that $\mathrm{Ker}(\phi) \subseteq \mathrm{Ker}(\ell_\alpha)$, contradicting the fact that $\mathrm{G.dim}(\mathrm{Ker}(\phi)) = \alpha + 1$. □

In the above proof we made use of the Addition Theorem for the algebraic entropy, which is quite a deep result. We want to underline that if one is only interested in the second part of the statement, that is, stable finiteness of endomorphism rings, then it is sufficient to use the weaker additivity of the algebraic entropy on direct sums, which can be verified as an easy exercise independently from the Addition Theorem.

Example 10.9. *Let G be a free group of rank $\geqslant 2$ and let $\mathbb{K}$ a field. It is well-known that $\mathbb{K}[G]$ is not left (nor right) Noetherian so we can find a left ideal ${}_{\mathbb{K}[G]}I \leqslant \mathbb{K}[G]$ which is not finitely generated. Furthermore, by [23, Corollary 7.11.8], $\mathbb{K}[G]$ is a free ideal ring, so I is free. This means that I is isomorphic to a coproduct of the form $\mathbb{K}[G]^{(\mathbb{N})}$ which is obviously not Hopfian.*

10.3 Zero-Divisors

In this section we provide an alternative argument to answer Question 9.2 for amenable groups (in the more general setting of crossed products) and we translate the amenable case of Conjecture 9.18 into an equivalent statement about algebraic entropy. This approach is inspired to the work of Nhan-Phu Chung and Andreas Thom [22]. Indeed, we can prove the following

Theorem 10.10. *Let $\mathbb{K}$ be a division ring and let G be a countably infinite amenable group. For any fixed crossed product $\mathbb{K}{*}G$, the following are equivalent:*

(1) *$\mathbb{K}{*}G$ is a left Ore domain;*

(2) *$\mathbb{K}{*}G$ is a domain;*

(3) $\mathrm{ent}_{\dim}({}_{\mathbb{K}*G}M) = 0$*, for every proper quotient M of $\mathbb{K}{*}G$;*

(4) $\mathrm{Im}(\mathrm{ent}_{\dim}) = \mathbb{N} \cup \{\infty\}$.

Before proving the above theorem we need to establish the following relation between the Ore property and the existence of a suitable length function.

Proposition 10.11. *A domain D is left Ore if and only if there exists a length function $L\colon \mathrm{Ob}(D\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $L(D) = 1$.*

Proof. If D is left Ore, then D is a flat subring of a division ring $\mathbb{K}$. Then there is an exact functor $\mathbb{K} \otimes_D - : D\text{-Mod} \to \mathbb{K}\text{-Mod}$ which commutes with direct limits. Thus, we can define the desired length function L simply letting $L({}_DM) = \dim_{\mathbb{K}}(\mathbb{K} \otimes_D M)$.
On the other hand, suppose that there is a length function $L\colon \mathrm{Ob}(D\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}$ such that $L(D) = 1$ and choose $x,\ y \in D\backslash\{0\}$. Since D is a domain, both Dx and Dy contain (and are contained in) a copy of D, thus $L(Dx) = L(Dy) = 1$. If, looking for a contradiction, $Dx \cap Dy = \{0\}$, then

$$1 = L(D) \geqslant L(Dx + Dy) = L(Dx \oplus Dy) = L(Dx) + L(Dy) = 2,$$

which is a contradiction. □

It is a classical result that any left Noetherian domain is left Ore (see for example [73, Theorem 1.15 in Chapter 2.1]). By the above proposition we can generalize this result as follows:

Corollary 10.12. *A domain with left Gabriel dimension is necessarily left Ore.*

Proof. Let D be a domain with left Gabriel dimension. First of all we verify that $\mathrm{G.dim}({}_DD)$ is not a limit ordinal. Indeed, if $\mathrm{G.dim}({}_DD) = \lambda$ is a limit ordinal, then $D = \bigcup_{\alpha<\lambda} \mathbf{T}_\alpha(D)$. This means that, for any non-zero $x \in D$, there exists $\alpha < \lambda$ such that $Dx \in \mathbf{T}_\alpha(D)$. Choose a non-zero $x \in D$, as D is a domain, there is a copy of D inside Dx. Thus, $\mathrm{G.dim}(D) \leqslant \mathrm{G.dim}(Dx) \leqslant \alpha$ for some $\alpha < \lambda$, a contradiction.
If $\mathrm{G.dim}({}_DD) = \alpha + 1$ for some ordinal α, then we can consider the length function

$$\ell_\alpha\colon \mathrm{Ob}(D\text{-Mod}) \to \mathbb{R}_{\geqslant 0} \cup \{\infty\}\ , \quad \ell_\alpha(M) = \ell(\mathbf{Q}_\alpha(M))\,.$$

To conclude one has to show that $\ell_\alpha(D) = 1$, that is, $\mathbf{Q}_\alpha(D)$ is a simple object. Since $\mathfrak{C}_{\alpha+1}/\mathfrak{C}_\alpha$ is semi-Artinian, there is a simple subobject S of $\mathbf{Q}_\alpha(D)$. Then $\mathbf{S}_\alpha(S)$ is a sub-module of $\mathbf{S}_\alpha\mathbf{Q}_\alpha(D)$. Identify $\mathbf{S}_\alpha(S)$, $\mathbf{S}_\alpha\mathbf{Q}_\alpha(D)$ and D with submodules of $E(D)$, since D is essential in $E(D)$, there is $0 \neq x$ such that $x \in \mathbf{S}_\alpha(S) \cap D$, but then $\mathbf{S}_\alpha(S)$ contains an isomorphic copy of D. Thus $\mathbf{Q}_\alpha\mathbf{S}_\alpha(S) = S$ contains an isomorphic copy of $\mathbf{Q}_\alpha(D)$, which is therefore simple. □

We can finally prove our result:

Proof of Theorem 10.10. (1)⇒(2) is trivial while (2)⇒(1) follows by Proposition 10.11 and the fact that the algebraic dim-entropy is a length function on $\mathbb{K}{*}G$-Mod such that $\mathrm{ent}_{\dim}({}_{\mathbb{K}*G}\mathbb{K}{*}G) = 1$.

(2)⇒(3). Consider a short exact sequence $0 \to {}_{\mathbb{K}*G}I \to {}_{\mathbb{K}*G}\mathbb{K}{*}G \to {}_{\mathbb{K}*G}M \to 0$, with $I \neq 0$. Choose $0 \neq x \in I$, then $\mathbb{K}{*}Gx \cong \mathbb{K}{*}G$, and so $\mathrm{ent}_{\dim}({}_{\mathbb{K}*G}M) = \mathrm{ent}_{\dim}({}_{\mathbb{K}*G}\mathbb{K}{*}G) - \mathrm{ent}_{\dim}({}_{\mathbb{K}*G}I) \leqslant 1 - 1 = 0$.

(3)⇒(4). Let us show first that for any finitely generate left $\mathbb{K}{*}G$-module ${}_{\mathbb{K}*G}F$, $\mathrm{ent}_{\dim}({}_{\mathbb{K}*G}F) \in \mathbb{N}$. In fact, choose a finite set of generators $x_1, \dots, x_n$ for F and, letting $F_0 = 0$ and $F_i = \mathbb{K}{*}Gx_1 + \cdots + \mathbb{K}{*}Gx_i$ for all $i = 1, \dots, n$, consider the filtration $0 \subseteq F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n = F$. By additivity,

$$\mathrm{ent}_{\dim}(F) = \sum_{i=1}^{n} \mathrm{ent}_{\dim}(F_i/F_{i-1}) \,.$$

All the modules $F_1/F_0, \dots, F_n/F_{n-1}$ are cyclics (i.e. quotients of $\mathbb{K}{*}G$), thus $\mathrm{ent}_{\dim}(F_i/F_{i-1}) \in \{0, 1\}$ by hypothesis. Hence, $\mathrm{ent}_{\dim}(F) \in \mathbb{N}$. To conclude one argues by upper continuity that the algebraic dim-entropy of an arbitrary left $\mathbb{K}{*}G$-module is the supremum of a subset of $\mathbb{N}$, thus it belongs to $\mathbb{N} \cup \{\infty\}$.

(4)⇒(2). Let $x \in \mathbb{K}{*}G$ and consider the short exact sequence

$$0 \to I \to \mathbb{K}{*}G \to \mathbb{K}{*}Gx \to 0$$

where $I = \{y \in \mathbb{K}{*}G : yx = 0\}$. Suppose that x is a zero-divisor, that is, $I \neq 0$ or, equivalently, $\dim({}_{\mathbb{K}}I) \neq 0$. By Proposition 8.23, $\mathrm{ent}_{\dim}(I) > 0$ and, by our assumption, $\mathrm{ent}_L(I) \geqslant 1$. Hence, using additivity, $\mathrm{ent}_{\dim}(\mathbb{K}{*}Gx) = 0$. Again by Proposition 8.23, this implies $\dim(\mathbb{K}{*}Gx) = 0$ and consequently $\mathbb{K}{*}Gx = 0$, that is, $x = 0$. Thus, the unique zero-divisor in $\mathbb{K}{*}G$ is 0. □

Chapter 11

The sofic case

In this chapter we describe a "point-free strategy" to solve the sofic case of both the L-Surjunctivity and of the Stabel Finiteness Conjectures. In particular, we will prove the following theorems.

Theorem 11.8 *Let R be a ring, let G be a sofic group and let ${}_RN$ be an Artinian left R-module. Then, any linear cellular automaton $\phi : N^G \to N^G$ is surjunctive.*

Notice that the above theorem generalizes in different directions the main results of [15] and [14]. Furthermore, the following general version of the Stable Finiteness Conjecture for sofic groups, generalizes results of [38] and [5]:

Theorem 11.11 *Let R be a ring, let G be a sofic group, fix a crossed product $R{*}G$, let N_R be a finitely generated right R-module and let $M_{R*G} = N \otimes_R R{*}G$.*

(1) *If N_R is Noetherian, then any surjective homomorphism $\phi : M \to M$ is injective;*

(2) *if N_R has Krull dimension, then $\mathrm{End}_{R*G}(M)$ is stably finite.*

The chapter is organized as follows: first we prove some general results for qframes in Section 11.1, then in Section 11.2 we deduce from these results the proof of the above theorems.

11.1 Main Theorems

11.1.1 The 1-dimensional case

I've learnt the arguments used in the proof of the following lemma while reading [38, proof of Proposition 4.4] and [105, proof of Lemma 3.1]. Also Lemma 11.2 is inspired by the argument used by Weiss to show surjunctivity of sofic groups.

Lemma 11.1. *Let G be a group, let K be a finite symmetric subset of G and let $H = KK$. Choose $n \in \mathbb{N}_{\geqslant 2}$, let ε be a positive constant such that $\varepsilon < \frac{1}{2n|H|^2}$, let V be a finite set, let $\varphi : G \to S_V$ be an (H, ε)-quasi-action of G on V and define the following set:*

$$\bar{V} = \{v \in V : hv \neq h'v \text{ and } (h_1h_2)v = h_1(h_2v), \text{ for all } h \neq h' \in H,\ h_1, h_2 \in H\}.$$

Then, the following statements hold true:

(1) $|\bar{V}| \geqslant (1 - 1/n)|V|$;

(2) *there is a subset $W \subseteq \bar{V}$ such that $Kv \cap Kw = \varnothing$ for all $v \neq w \in W$ and $|W| \geqslant |V|/2|H|$.*

Proof. (1) A given $v \in V$ belongs to $\bar{V}$ if and only if it satisfies the following two conditions:

(a) $\varphi_{h_1}(v) \neq \varphi_{h_2}(v)$ for all $h_1 \neq h_2 \in H$;

(b) $\varphi_{h_1 h_2}(v) = \varphi_{h_1}(\varphi_{h_2}(v))$ for all $h_1, h_2 \in H$.

There are less than $|H|^2$ equations in (a) and each of these equations can fail for at most $\varepsilon|V|$ elements v in V. Similarly, there are $|H|^2$ equations in (b) and each of these equations can fail for at most $\varepsilon|V|$ elements $v \in V$. Thus, the cardinality of $\bar{V}$ is at least

$$|V| - (|H|^2\varepsilon|V| + |H|^2\varepsilon|V|) \geqslant |V|(1 - 2|H|^2\varepsilon) \geqslant |V|(1 - 1/n)\,.$$

(2) Let W be a maximal subset of $\bar{V}$ with the property that $Kv \cap Kw = \varnothing$ for all $v \neq w \in W$. We claim that HW contains $\bar{V}$. In fact, if there is $v \in \bar{V}$ such that $v \notin HW$, this means that, for all $w \in W$, $Kv \cap Kw = \varnothing$, contradicting the maximality of W. Thus, $|\bar{V}| \leqslant |WH| \leqslant |W||H|$. To conclude, use that $2|\bar{V}| \geqslant |V|$ by part (1) and the choice of n. □

Lemma 11.2. *In the same setting of Lemma 11.1, let $(L_1, \leqslant)$ and $(L_2, \leqslant)$ be two qframes of finite length and consider a homomorphism of qframes $\Phi : L_1 \to L_2$. Let $l \in \mathbb{N}_{\geqslant 1}$ and suppose that*

(1) *there is distinguished family of elements $\{\bar{x}_v : v \in K\bar{V}\}$ such that*

  (1.1) $\bigvee_{K\bar{V}} \bar{x}_v = 1$ *;*

  (1.2) $\ell(\bar{x}_v) = l$*, for all $v \in K\bar{V}$;*

(2) $\ell\left(\bigvee_{v \in Kw} \Phi(\bar{x}_v)\right) \leqslant |K|l - 1$*, for all $w \in \bar{V}$.*

Then, $\ell(\mathrm{Im}(\Phi)) \leqslant \left(1 - \frac{1}{2|H|l}\right)|V|l$.

Proof. Choose a $W \subseteq \bar{V}$ as in part (2) of Lemma 11.1. By Lemma 2.25,

$$\ell(\Phi(L_1)) = \ell\left(\bigvee_{v \in K\bar{V}} \Phi(\bar{x}_v)\right) \leqslant \ell\left(\bigvee_{v \in K\bar{V} \setminus KW} \Phi(\bar{x}_v)\right) + \ell\left(\bigvee_{v \in KW} \Phi(\bar{x}_v)\right).$$

Furthermore,

$$\ell\left(\bigvee_{v \in KW} \Phi(\bar{x}_v)\right) \leqslant \sum_{w \in W} \ell\left(\bigvee_{v \in Kw} \Phi(\bar{x}_v)\right) \leqslant |W|(|K|l - 1)\,.$$

By the choice of W, $|K\bar{V} \setminus KW| = |K\bar{V}| - \sum_{w \in W}|Kw| = |K\bar{V}| - |W||K|$ and, by Lemma 2.26, $\ell\left(\bigvee_{v \in K\bar{V} \setminus KW} \Phi(\bar{x}_v)\right) \leqslant \ell\left(\bigvee_{v \in K\bar{V} \setminus KW} \bar{x}_v\right)$, thus

$$\ell\left(\bigvee_{v \in K\bar{V} \setminus KW} \bar{x}_v\right) \leqslant \sum_{v \in K\bar{V} \setminus KW} \ell(\bar{x}_v) = |K\bar{V} \setminus KW|l = (|K\bar{V}| - |W||K|)l \leqslant (|V| - |W||K|)l\,,$$

Putting together all these data, we get

$$\ell(\Phi(L_1)) \leqslant |W|(|K|l - 1) + (|V| - |W||K|)l = -|W| + |V|l \leqslant \left(1 - \frac{1}{2|H|l}\right)|V|l\,.$$

□

Theorem 11.3. *Let M be a qframe, let G be a sofic group, let $\rho : G \to \mathrm{Aut}(M)$ be a right action of G on M (we let $\rho(g) = \rho_g$ for all $g \in G$) and let $\phi : M \to M$ be a G-equivariant homomorphism of qframes, that is, $\rho_g\phi = \phi\rho_g$, for all $g \in G$. Choose an element $\bar{y} \in M$ such that*

(a) $\ell(\bar{y}) = l < \infty$;

(b) *the family $\{\bar{y}_g : g \in G\}$ is join-independent, where $\bar{y}_g = \rho_g(\bar{y})$ for all $g \in G$;*

(c) *there exists a finite symmetric subset $F \subseteq G$ such that $\phi(\bar{y}) \leqslant \bigvee_{g\in F} \bar{y}_g$ and $e \in F$.*

Fix an F as in (c) and let K be a finite symmetric subset of G containing F. Then, the following conditions are mutually exclusive:

(1) $\bar{y} \leqslant \bigvee_{g\in K} \phi(\bar{y}_g)$;

(2) $\ell\left(\bigvee_{g\in K} \phi(\bar{y}_g)\right) \leqslant |K|l - 1$.

Proof. Assume, looking for a contradiction, that both (1) and (2) are verified. We start by constructing some objects to which we want to apply Lemmas 11.1 and 11.2.

First we construct the objects mentioned in Lemma 11.1. Choose a positive integer $n \geqslant 2|H|l$, let $H = KK$, let ε be a positive constant such that $\varepsilon < \frac{1}{2n|H|^2}$, let V be a finite set, let $\varphi : G \to S_V$ be an (H, ε)-quasi-action of G on V and define

$$\bar{V} = \{v \in V : hv \neq h'v \text{ and } (h_1h_2)v = h_1(h_2v), \text{ for all } h \neq h' \in H,\ h_1, h_2 \in H\}.$$

Secondly, we construct the objects mentioned in Lemma 11.2. For a subset $G' \subseteq G$, we use the notation $\bar{y}_{G'} = \bigvee_{g\in G'} \bar{y}_g$ and, for all $v \in V$, we let $Q_v^{G'}$ be a qframe isomorphic to $[0, \bar{y}_{G'}]$. For all $v \in V$, we identify both $Q_v^e = Q_v^{\{e\}}$ and Q_v^K with sub-qframes of Q_v^H in such a way that there is an isomorphism of qframes

$$q_v : Q_v^H \longrightarrow [0, \bar{y}_H],$$

such that $q_v(Q_v^e) = [0, \bar{y}]$ and $q_v(Q_v^K) = [0, \bar{y}_K]$. For all $v \in \bar{V}$, the map $\sigma_v : Hv \to H$ such that $\sigma_v(hv) = h$ is well defined and bijective. So, given $v \in \bar{V}$ and $w \in Hv$ we let

$$q_v^w : Q_w^H \xrightarrow{\sim} [0, \bar{y}_{H\sigma_v(w)}]$$

be the composition $q_v^w = \rho_{\sigma_v(w)}q_w$. Let us introduce the following notation for all $G' \subseteq G$:

$$Q^{G'} = \prod_{v\in\bar{V}} Q_v^{G'}, \quad \forall G' \subseteq G.$$

For all $v \in \bar{V}$, we denote by $\iota_v^{G'} : Q_v^{G'} \to Q^{G'}$ the canonical inclusion in the product. Consider, for all $v \in \bar{V}$, the following homomorphism of qframes:

$$\Psi_v : Q^H \longrightarrow [0, \bar{y}_{HH}] \text{ such that } (a_w)_{w\in\bar{V}} \mapsto \bigvee_{w\in Hv\cap\bar{V}} q_v^w(a_w)$$

Given a, $b \in Q^H$, let

$$a \sim b \iff \Psi_v(a) = \Psi_v(b)\ \forall v \in \bar{V}.$$

This defines a strong congruence on Q^H and, by restriction, on Q^K. Let $L_1 = Q^K/_\sim$ and $L_2 = Q^H/_\sim$ and let $\pi_1 : Q^K \to L_1$ and $\pi_2 : Q^H \to L_2$ be the canonical projections. For all $v \in \bar{V}$, let $\Phi_v : Q_v^K \to Q_v^H$ be the unique map such that $q_v\Phi_v(x) = \phi(q_v(x))$, for all $x \in Q_v^K$. We let $\Phi : Q^K \to Q^H$ be the product of these maps, that is, $\Phi(x_v)_{v\in\bar{V}} = (\Phi_v(x_v))_{v\in\bar{V}}$. Given two elements $a \sim b \in Q^K$, $\Phi(a) \sim \Phi(b)$. In fact, for all $v \in \bar{V}$,

$$\Psi_v\Phi(a) = \bigvee_{w\in Hv\cap\bar{V}} \rho_{\sigma_v(w)} q_w \Phi_w(a_w) = \bigvee_{w\in Hv\cap\bar{V}} \rho_{\sigma_v(w)} \phi q_w(a_w) = \phi\left(\bigvee_{w\in Hv\cap\bar{V}} q_v^w(a_w)\right)$$
$$= \phi\left(\bigvee_{w\in Hv\cap\bar{V}} q_v^w(b_w)\right) = \ldots = \Psi_v\Phi(b)\,.$$

Let $\bar{\Phi} : L_1 \to L_2$ be the unique map such that $\bar{\Phi}\pi_1 = \pi_2\Phi$. One verifies that $\bar{\Phi}$ is a morphism of qframes.

Now that the setting is constructed we need to verify that the hypotheses (1) and (2) of Lemma 11.2 are satisfied. For all $v \in \bar{V}$ and $k \in K$ we let $x_k^v = \iota_v^K(q_v^{-1}(\bar{y}_k))$. Let us show that $x_k^v \sim x_{k'}^{v'}$ if and only if $kv = kv'$. Indeed, given $v,\ v' \in \bar{V}$ and $k,\ k' \in K$ such that $x_k^v \sim x_{k'}^{v'}$, notice that

$$\Psi_v(x_k^v) = \bar{y}_k \quad \text{and} \quad \Psi_v(x_{k'}^{v'}) = \rho_{\sigma_v(v')}\bar{y}_{k'}$$

if $v' \in Hv$, otherwise it is 0. Thus, $\sigma_v(v')k' = k$, that is, $v' = \sigma_v(v')v = (k')^{-1}kv$, so $k'v' = kv$ (here we are using that $v,\ v' \in \bar{V}$). Hence, given $w = kv \in K\bar{V}$, we can define $\bar{x}_w = \pi_1(x_k^v)$ without any ambiguity. Clearly $\bigvee_{v\in K\bar{V}} \bar{x}_v = 1$, let us show that the family $\{\bar{x}_v : v \in K\bar{V}\} \subseteq L_1$ is join-independent. Indeed, given $k'v' \in K\bar{V}$,

$$\bar{x}_{k'v'} \wedge \bigvee_{k'v'\neq kv\in K\bar{V}} \bar{x}_{kv} = \pi_1\left(x_{k'}^{v'} \wedge \bigvee_{k'v'\neq kw\in K\bar{V}} x_k^v\right) = \pi_1(0) = 0\,,$$

where the first equality comes from the definition of the $\bar{x}_w$ and the properties of π_1 (see Lemma 2.14), while the second equality holds since the family $\{x_k^v : kv \in K\bar{V}\} \subseteq Q^K$ is join-independent.

Furthermore, for all $w \in \bar{V}$:

$$\ell\left(\bigvee_{v\in Kw} \bar{\Phi}(\bar{x}_v)\right) = \ell\left(\bigvee_{v\in Kw} \bar{\Phi}\pi_1(Q_v^K)\right) = \ell\left(\bigvee_{v\in Kw} \pi_2\Phi_v(Q_v^K)\right) \leqslant \ell\left(\bigvee_{v\in Kw} \Phi(Q_v^K)\right)$$
$$\leqslant \ell(\phi([0,\bar{y}_K])) \leqslant |K|l - 1\,.$$

In the last part of the proof we obtain the contradiction we were looking for. Indeed, we claim that the restriction of π_2 to Q^e is injective and that $\pi_2(Q^e) \subseteq \bar{\Phi}(L_1)$. In fact, let $a = (a_v)_{v\in\bar{V}}$ and $b = (b_v)_{v\in\bar{V}} \in Q^e$ and suppose that $\pi_2(a) = \pi_2(b)$, that is, $a \sim b$. For all $v \in \bar{V}$ and $w \in Hv \cap \bar{V}$, by construction, $q_v^w(a_w),\ q_v^w(b_w) \leqslant \bar{y}_{\sigma_v(w)}$. So, using modularity and the independence of the family $\{\bar{y}_g : g \in G\}$,

$$q_v(a_v) = q_v(a_v) \vee 0 = h_v(a_v) \vee \left(\bigvee_{v\neq w\in Hv\cap\bar{V}} q_v^w(a_w) \wedge \bar{y}\right) = \bar{y} \wedge \left(q_v(a_v) \vee \bigvee_{v\neq w\in Hv\cap\bar{V}} q_v^w(a_w)\right)$$
$$= \bar{y} \wedge \Psi_v(a) = \bar{y} \wedge \Psi_v(b) = \ldots = q_v(b_v)\,,$$

that is, $a_v = b_v$, for all $v \in \bar{V}$. Our second claim follows by construction and the hypothesis (1). Also recalling the estimate for $|\bar{V}|$ given in Lemma 11.1, the two claims we just verified imply that

$$\ell(\mathrm{Im}(\bar{\Phi})) \geqslant \ell(\pi_2(Q^{(e)})) = \ell(Q^{(e)}) = |\bar{V}|l \geqslant \left(1 - \frac{1}{n}\right)|V|l\,.$$

Furthermore, by Lemma 11.2, $\ell(\mathrm{Im}(\bar{\Phi})) < \left(1 - \frac{1}{2|H|l}\right)|V|l$. Thus, $n < 2|H|l$, which is a contradiction. □

11.1.2 Higher dimensions

Lemma 11.4. *Let $(M, \leqslant)$ be a qframe, let G be a group, let $\rho : G \to \mathrm{Aut}(M)$ be a right action of G on M and consider an algebraic G-equivariant homomorphism of qframes $\phi : M \to M$. Suppose that there exists an element $y \in M$ such that $[0, y]$ is finitely generated and such that, letting $y_g = \rho_g(y)$ for all $g \in G$, the family $\{y_g : g \in G\}$ is a basis for M. Then,*

(1) *ϕ is surjective if and only if there exists a finite subset $K \subseteq G$ such that $y \leqslant \bigvee_{g\in K} \phi(y_g)$;*

(2) *ϕ is not injective if and only if there exist a finite subset $K \subseteq G$ and $0 \neq x \leqslant \bigvee_{g\in K} y_g$ such that $\phi(x) = 0$.*

Proof. (1) Suppose that ϕ is surjective, then $\bigvee_{g\in G} \phi(y_g) = \phi(1) = 1$. By Lemma 2.16, one can find a finite subset $K \subseteq G$ such that $y \leqslant \bigvee_{g\in G} \phi(y_g)$. On the other hand, if there exists $K \subseteq G$ such that $y \leqslant \bigvee_{g\in K} \phi(y_g)$, then $y_h \leqslant \bigvee_{g\in Kh^{-1}} \phi(y_g) \leqslant \phi(1)$ for all $h \in G$. Thus, $1 = \bigvee_{h\in G} y_h \leqslant \phi(1)$ and so ϕ is surjective.
(2) By the algebraicity of ϕ, if ϕ is not injective, there is a non-trivial element $x' \in \mathrm{Ker}(\phi)$. By Lemma 2.11, there exists a finite subset $K \subseteq G$ such that $x' \wedge \bigvee_{g\in K} y_g \neq 0$, so that $x = x' \wedge \bigvee_{g\in K} y_g$ is the element we were looking for. The converse is trivial. □

Theorem 11.5. *Let $(M, \leqslant)$ be a qframe, let G be a sofic group, let $\rho : G \to \mathrm{Aut}(M)$ be a right action of G on M and consider a surjective algebraic G-equivariant homomorphism of qframes $\phi : M \to M$. For a given element $y \in M$ such that $[0, y]$ is compact, consider the following conditions:*

(a_*) *$[0, y]$ is Noetherian;*

(a_*') *$\mathrm{K.dim}([0, y])$ exists and there is a homomorphism of qframes $\psi : M \to M$ such that $\phi\psi = \mathrm{id}$;*

(b_*) *letting $y_g = \rho_g(y)$ for all $g \in G$, the family $\{y_g : g \in G\}$ is a basis for M.*

If (b_*) and either (a_*) or (a_*') hold, then ϕ is injective.

Proof. Suppose, looking for a contradiction, that ϕ is not injective. Suppose that (b_*) is verified, so by Lemma 11.4, there exists a finite subset K of G such that

(1_*) $y \leqslant \bigvee_{g\in K} \phi(y_g)$;

(2_*) there exists $0 \neq x \leqslant \bigvee_{g\in K} y_g$ such that $\phi(x) = 0$.

Furthermore, since $[0, y]$ is compact, also $[0, \phi(y)]$ is compact and so there exists a finite subset $F \subseteq G$ such that

(3_*) $\phi(y) \leqslant \bigvee_{g\in F} y_g$.

In case (a_*) is verified, by Lemma 2.47 there exists an ordinal α such that $\mathrm{G.dim}([0, \mathrm{Ker}(\phi)]) = \alpha + 1$. On the other hand, if (a'_*) is verified, we let α be any ordinal such that $t_\alpha(x) \neq t_{\alpha+1}(x)$. In both cases, let $\bar{M} = Q_\alpha(T_{\alpha+1}(M))$ and denote by $\pi : T_{\alpha+1}(M) \to \bar{M}$ the canonical projection. We let $\bar{x} = \pi(t_{\alpha+1}(x))$ and $\bar{y} = \pi(t_{\alpha+1}(y))$. There is an induced right action of G on $\bar{M}$, $\bar{\rho} : G \to \mathrm{Aut}(\bar{M})$, where $\bar{\rho}_g = Q_\alpha(T_{\alpha+1}(\rho_g))$ for all $g \in G$. Of course, the map $\bar{\phi} = Q_\alpha(T_{\alpha+1}(\phi)) : \bar{M} \to \bar{M}$ is G-equivariant. One can prove that $\bar{\rho}_g(\bar{y}) = \pi(t_{\alpha+1}(y_g))$, for all $g \in G$, and so, whenever (b_*) is verified, the family $\{\bar{y}_g : g \in G\}$, where $\bar{y}_g = \bar{\rho}_g(\bar{y})$, is a basis of $\bar{M}$ (it is clear that $\bigvee \bar{y}_g = 1$, to see that this family is join-independent use that the canonical projection commutes with joins and finite meets by Lemma 2.14).

Suppose that (a_*) is verified. By Proposition 2.52 (2), $[0, \bar{y}]$ is semi-Artinian and, by (a_*), it is also Noetherian. Thus, $\ell(\bar{y}) = l < \infty$. Notice that, by ($3_*$), $\bar{\phi}(\bar{y}) \leqslant \bigvee_{g\in F} \bar{y}_g$ and, by (1_*), $t_{\alpha+1}(y) \in [0, \bigvee_{g\in K} \phi(y_g)]$, thus there exists $z \leqslant \bigvee_{g\in K} y_g$ such that $\phi(z) = t_{\alpha+1}(y)$. By the algebraicity of ϕ and Lemma 2.42 (2), $\mathrm{G.dim}([0, z]) = \max\{\mathrm{G.dim}([0, \mathrm{Ker}(\phi) \wedge z]), \mathrm{G.dim}([0, t_{\alpha+1}(y)])\} = \alpha + 1$, thus $z \in [0, \bigvee_{g\in K} t_{\alpha+1}(y_g)]$. Applying π, we obtain an element $\pi(z) \in [0, \bigvee_{g\in K} \bar{y}_g]$ such that $\bar{\phi}(\pi(z)) = \pi(\phi(z)) = \bar{y}$. Thus, $\bar{y} \leqslant \bigvee_{g\in K} \bar{\phi}(\bar{y}_g)$. By the choice of α, $\mathrm{Ker}(\bar{\phi}) \neq 0$ and so, by Lemma 2.11, there exists a finite subset $F' \subseteq G$ such that $\mathrm{Ker}(\bar{\phi}) \wedge \bigvee_{g\in F'} \bar{y}_g \neq 0$. Let K' be a finite symmetric subset of G which contains both F' and K, then

$$\bar{y} \leqslant \bigvee_{g\in K'} \bar{\phi}(\bar{y}_g) \quad \text{and} \quad \ell\left(\bigvee_{g\in K'} \bar{\phi}(\bar{y}_g)\right) \leqslant |K'|l - 1\,,$$

by the above discussion and Lemma 2.28. These two conditions cannot happen for the same K' by Theorem 11.3, so we get a contradiction.

Suppose now that (a'_*) is verified. We define $\bar{\psi} = Q_\alpha(T_{\alpha+1}(\psi)) : \bar{M} \to \bar{M}$, so that $\bar{\phi}\bar{\psi} = \mathrm{id}$. Consider the socle $\mathrm{Soc}(\bar{M}) = [0, s(\bar{M})]$ and notice that $s(\bar{M}) = \bigvee_{g\in G} s([0, \bar{y}_g])$. Since $[0, \bar{y}]$ is semi-Artinian and it has Krull dimension, then it is Artinian, thus, it has a socle of finite length: let $l = \ell(s(\bar{y}))$. By the choice of α, $\bar{x} \neq 0$ and $s(\bar{x}) = \bar{x} \wedge s(\bar{M}) \neq 0$, since, being $\bar{M}$ semi-Artinian, $s(\bar{M})$ is essential in $\bar{M}$. Since $\mathrm{Soc}(\bar{M})$ is fully invariant (see Lemma 2.30 (4)), $\bar{\phi}\restriction_{\mathrm{Soc}(\bar{M})}\bar{\psi}\restriction_{\mathrm{Soc}(\bar{M})} = id_{\mathrm{Soc}(\bar{M})}$. The family $\{s(\bar{y}_g) : g \in G\}$ is clearly join-independent. Furthermore, using the fact that $[0, s(\bar{y})]$ is compact (since it has finite length), also $[0, \bar{\phi}(s(\bar{y}))]$ and $[0, \bar{\psi}(s(\bar{y}))]$ are compact, so there exists a finite subset $F' \subseteq G$ such that $\bar{\phi}(s(\bar{y}))$, $\bar{\psi}(s(\bar{y})) \leqslant \bigvee_{g\in F'} s(\bar{y}_g)$. Let $K' \subseteq G$ be a finite symmetric subset that contains both F' and K, then

$$s(\bar{y}) = \bar{\phi}(\bar{\psi}(s(\bar{y}))) \leqslant \bar{\phi}\left(\bigvee_{g\in K'} s(\bar{y}_g)\right) \leqslant \bigvee_{g\in K'} \bar{\phi}(s(\bar{y}_g)) \quad \text{and} \quad \ell\left(\bigvee_{g\in K'} \bar{\phi}(s(\bar{y}_g))\right) \leqslant |K'|l - 1\,,$$

by Lemma 2.28 and the fact that $\bar{\phi}(s(\bar{x})) = 0$. This is a contradiction by Theorem 11.3. $\square$

11.2 Applications

11.2.1 L-Surjunctivity

In this subsection we use the general results we proved for qframes in the first half of the chapter to deduce a surjunctivity theorem for a suitable family of linear cellular automaton. Let us start defining a natural qframe associated with strictly linearly compact modules.

Definition 11.6. *Let R be a discrete ring and let M be a linearly topologized left R-module. We let $(\mathcal{N}(M), \leqslant)$ be the poset of submodules of M, ordered by* reverse *inclusion.*

Lemma 11.7. *Let R be a discrete ring, let M and N be strictly linearly compact left R-modules and let $\phi : M \to N$ be a continuous homomorphism of left R-modules. Then, $\mathcal{N}(M)$ and $\mathcal{N}(N)$ are qframes and the map*

$$\Phi : \mathcal{N}(N) \to \mathcal{N}(M) \quad \textit{such that} \quad \Phi(C) = \phi^{-1}(C)$$

is a homomorphism of qframes. Furthermore, if ϕ is injective then Φ is surjective and algebraic, and, under these hypotheses, Φ is injective if and only if ϕ is surjective.

Proof. It is easy to check that $\mathcal{N}(M)$ and $\mathcal{N}(N)$ are complete lattices (in fact, the maximum of $\mathcal{N}(M)$ is 0, while its minimum is M; furthermore the meet of two closed submodules is the closure of their sum, while the join of a family (finite or infinite) of closed submodules is their intersection). To show that $\mathcal{N}(M)$ is modular take A, B, $C \in \mathcal{N}(M)$ such that $A \leqslant C$ (that is, $C \subseteq A$). Using, the modularity of the lattice of all submodules $\mathcal{L}(M)$ of M with the usual order, one gets $C + (B \cap A) = (C + B) \cap A$, thus

$$\overline{C + (B \cap A)} = \overline{(C + B) \cap A} = \overline{(C + B)} \cap A\,,$$

which is the modular law in $\mathcal{N}(M)$. The fact that $\mathcal{N}(M)$ and $\mathcal{N}(N)$ are upper continuous is proved for example in [103, Theorem 28.20].

The map Φ is well-defined by the continuity of ϕ, that ensures that $\phi^{-1}(C) \in \mathcal{N}(M)$, for all $C \in \mathcal{N}(N)$. Since ϕ^{-1} commutes with arbitrary intersections, Φ commutes with arbitrary joins. Let now $C_1 \leqslant C_2 \in \mathcal{N}(N)$ and let us show that $\Phi([C_1, C_2]) = [\Phi(C_1), \Phi(C_2)]$. Indeed, given $C \in [\Phi(C_1), \Phi(C_2)]$, $\phi^{-1}(C_2) \subseteq C \subseteq \phi^{-1}(C_1)$, so that $C_2 \cap \phi(M) \subseteq \phi(C) \subseteq C_1 \cap \phi(M)$. Thus,

$$\begin{aligned} C &= \Phi(\phi(C)) = \Phi(\overline{\phi(C) + (C_2 \cap \phi(M))}) \\ &= \Phi((\overline{\phi(C) + C_2}) \cap \phi(M)) = \Phi(\overline{\phi(C) + C_2})\,, \end{aligned}$$

where in the first line we used that C contains the kernel of ϕ, while in the second line we applied the modular law. Since $\overline{\phi(C) + C_2} \in [C_1, C_2]$, Φ sends segments to segments and so it is a morphism of qframes.

Suppose now that ϕ is injective. To show that Φ is surjective notice that, by the injectivity of ϕ, $\Phi([0, 1]) = [0, \Phi(1)] = [0, \mathrm{Ker}(\phi)] = \mathcal{N}(M)$. It remains to show that Φ is algebraic: it is enough to notice that $\mathrm{Ker}(\Phi) = \phi(M)$ and that, given C_1, $C_2 \in [\phi(M), 1]$ such that $\Phi(C_1) = \Phi(C_2)$, then

$$C_1 = C_1 \cap \phi(M) = \phi(\phi^{-1}(C_1)) = \phi(\phi^{-1}(C_2)) = C_2 \cap \phi(M) = C_2\,.$$

Finally, since Φ is algebraic, Φ is injective if and only if $\mathrm{Ker}(\Phi) = 0$, that is, $\phi(M) = M$, which is equivalent to say that ϕ is surjective. □

Theorem 11.8. *Let R be a ring, let G be a sofic group and let ${}_RN$ be an Artinian left R-module. Then, any linear cellular automaton $\phi : N^G \to N^G$ is surjunctive.*

Proof. Suppose that $\phi : N^G \to N^G$ is an injective linear cellular automaton and let us prove that it is surjective.

By Lemmas 3.58 and 3.59 (2), N^G is strictly linearly compact so, by Lemma 11.7, $\mathcal{N}(N^G)$ is a qframe. Furthermore, the map

$$\rho : G \to \mathrm{Aut}(\mathcal{N}(N^G)) \quad \rho(g) = \rho_g\,,$$

such that $\rho_g(K) = \lambda_g^{-1}(K)$, for all $K \in \mathcal{N}(N^G)$ and $g \in G$, is a right action and the map

$$\Phi : \mathcal{N}(N^G) \to \mathcal{N}(N^G) \quad \Phi(K) = \phi^{-1}(K)\,,$$

for all $K \in \mathcal{N}(N^G)$, is a G-equivariant surjective algebraic homomorphism of qframes.
Let $y = \pi_e^{-1}(\{0\})$, where $\pi_e : N^G \to N^e$ is the usual projection, notice that $[0, y] \cong \mathcal{N}(N)$ is a Noetherian lattice and let $y_g = \rho_g(y)$, for all $g \in G$. It is clear that $\{y_g : g \in G\}$ is a basis for $\mathcal{N}(N^G)$.
By the above discussion, hypotheses (a_*) and (b_*) of Theorem 11.5 are satisfied and so Φ is injective. By Lemma 11.7, ϕ is surjective. □

11.2.2 Stable finiteness of crossed products

Lemma 11.9. *Let R be a ring, let M_R and N_R be right R-modules and let $\phi : M \to N$ be a homomorphism of right R-modules. Then, $(\mathcal{L}(M), \leqslant)$ and $(\mathcal{L}(N), \leqslant)$ are qframes and the map*

$$\Phi : \mathcal{L}(M) \to \mathcal{L}(N) \quad \text{such that} \quad \Phi(K) = \phi(K)$$

is a homomorphism of qframes. Furthermore, if ϕ is surjective, then Φ is surjective and algebraic, and, in this case, Φ is injective if and only if ϕ is injective.

Proof. In any given Grothendieck category, the posets of sub-objects are qframes (the maximum of $\mathcal{L}(M)$ is M, while its minimum is 0, furthermore, join and meet are given by sum and intersection respectively). By Proposition 1.72, Φ is a semi-lattice homomorphism which commutes with arbitrary joins. To show that Φ sends segments to segments, let $K_1 \leqslant K_2 \in \mathcal{L}(M)$ and consider $K \in [\Phi(K_1), \Phi(K_2)]$. Then,

$$\begin{aligned} K &= \Phi(\phi^{-1}K) = \Phi(\phi^{-1}K \cap \phi^{-1}\phi(K_2)) \\ &= \Phi(\phi^{-1}K \cap (K_2 + \mathrm{Ker}(\phi))) = \Phi((\phi^{-1}K \cap K_2) + \mathrm{Ker}(\phi)) \\ &= \Phi(\phi^{-1}K \cap K_2) + \Phi(\mathrm{Ker}(\phi)) = \Phi(\phi^{-1}K \cap K_2)\,, \end{aligned}$$

where in the first line we used that K is contained in the image of ϕ, while in the second line we used the modularity of $\mathcal{L}(M)$. Since $\phi^{-1}(K) \cap K_2 \in [K_1, K_2]$ we proved that Φ sends segments to segments, thus it is a morphism of qframes.
Suppose now that ϕ is surjective. Then, Φ is surjective as $\Phi(1) = \phi(M) = N$, which is the maximum of $\mathcal{L}(N)$. To show that Φ is algebraic, notice that $\mathrm{Ker}(\Phi) = \mathrm{Ker}(\phi)$ and that, given K_1, $K_2 \in [\mathrm{Ker}(\phi), 1]$ such that $\Phi(K_1) = \Phi(K_2)$, we get

$$K_1 = K_1 + \mathrm{Ker}(\phi) = \phi^{-1}(\phi(K_1)) = \phi^{-1}(\phi(K_2)) = K_2 + \mathrm{Ker}(\phi) = K_2\,.$$

Finally, notice that ϕ is injective if and only if $\mathrm{Ker}(\phi) = \mathrm{Ker}(\Phi) = 0$, which happen, by the algebraicity of Φ, if and only if Φ is injective. □

Lemma 11.10. *Let R be a ring, let G be a group, fix a crossed product $R{*}G$, let M_{R*G} be a right $R{*}G$-module and let $\phi : M_{R*G} \to M_{R*G}$ be an endomorphism of right $R{*}G$-modules. Letting $\mathcal{L}_R(M)$ denote the qframe of R-submodules of M, the following map*

$$\rho : G \to \mathrm{Aut}(\mathcal{L}_R(M)) \quad \rho \mapsto \rho_g : \mathcal{L}_R(M) \to \mathcal{L}_R(M)\,,$$

where $\rho_g(K) = K\underline{g}$, for all $g \in G$ and $K \in \mathcal{L}_R(M)$ is a right G-representation. Furthermore, the endomorphism of qframes

$$\Phi : \mathcal{L}_R(M) \to \mathcal{L}_R(M) \quad \text{such that} \quad \Phi(K) = \phi(K)$$

is G-equivariant.

Proof. Let $N \in \mathcal{L}_R(M)$, $r \in R$ and $g \in G$. Then, $\rho_g(N)r = N\underline{g}r = Nr^{\sigma(g)}\underline{g} \subseteq N\underline{g}$ and so $\rho_g(N) \in \mathcal{L}_R(M)$. Let now $\{N_i : i \in I\}$ a family of elements in $\mathcal{L}_R(M)$, then

$$\rho_g\left(\sum_{i\in I} N_i\right) = \left(\sum_{i\in I} N_i\right)\overline{g} = \sum_{i\in I}(N_i\overline{g}) = \sum_{i\in I}\rho_g(N_i)$$

so ρ_g is a semi-lattice homomorphism which commutes with arbitrary joins. Furthermore, given $g, h \in G$ and $N \in \mathcal{L}_R(M)$,

$$\rho_g(\rho_h(N)) = \rho_g(N\underline{h}) = N\underline{h}\underline{g} = N\tau(h,g)\underline{hg} = N\underline{hg} = \rho_{hg}(N)\,,$$

where the fourth equality holds since $\tau(h,g) \in U(R)$. In particular, $\rho_g\rho_{g^{-1}} = \rho_{g^{-1}}\rho_g = \mathrm{id}_{\mathcal{L}_R(M)}$ so, given a segment $[N_1, N_2]$ in $\mathcal{L}_R(M)$ and $N \in [\rho_g(N_1), \rho_g(N_2)]$, then $N = \rho_g(\rho_{g^{-1}}N)$ and $\rho_{g^{-1}}N \in [N_1, N_2]$. Thus we proved that each ρ_g is a homomorphism of qframes and that ρ is a right G-representation.

Finally, let us show that $\rho_g\Phi = \Phi\rho_g$. Indeed, given $N \in \mathcal{L}_R(M)$,

$$\rho_g\Phi(N) = \phi(N)\underline{g} = \phi(N\underline{g}) = \Phi(\rho_g(N))\,,$$

where the third equality holds since ϕ is a homomorphism of left $R{*}G$-modules. □

Theorem 11.11. *Let R be a ring, let G be a sofic group, fix a crossed product $R{*}G$, let N_R be a finitely generated right R-module and let $M = N \otimes R{*}G$. Then,*

(1) *if N_R is Noetherian, then any surjective $R{*}G$-linear endomorphism of M is injective;*

(2) *if N_R has Krull dimension, then* $\mathrm{End}_{R*G}(M)$ *is stably finite.*

Proof. The proof is an application of Theorem 11.5 and consists in translating the statement in a problem about qframes using the above lemmas.

Suppose first (1) and let $\phi : M \to M$ be a surjective endomorphism of right $R{*}G$-modules. Consider the qframe $\mathcal{L}_R(M)$ of all the right R-submodules of M (which is described in Lemma 11.9), with the right G-action described in Lemma 11.10. By the same lemma, ϕ induces a G-equivariant surjective algebraic homomorphism of qframes $\Phi : \mathcal{L}_R(M) \to \mathcal{L}_R(M)$.
Let $y = N \otimes \underline{e} \in \mathcal{L}_R(M)$, and notice that conditions (a_*) and (b_*) in Theorem 11.5 are verified for this choice of y. Thus, by the theorem, Φ is injective and this is equivalent to say that ϕ is injective by Lemma 11.9.

The proof of part (2) is analogous. □

Part V

Relative Homological Algebra

Chapter 12

Model approximations

12.1 Model categories and derived functors

In this section we recall some definitions and terminology about the general machinery of model categories.

Definition 12.1. *Let $\mathfrak{C}$ be a category and let $\mathcal{W}$ be a collection of morphisms in $\mathfrak{C}$. The pair $(\mathfrak{C}, \mathcal{W})$ is said to be a* category with weak equivalences *if, given two composable morphisms ϕ and ψ, whenever two elements of $\{\phi, \psi, \psi\phi\}$ belong to $\mathcal{W}$ so does the third. The elements of $\mathcal{W}$ are called* weak equivalences.

We now recall the definition of a model category. We will just give few concrete examples of model category (see Examples 12.3 and 12.12), we refer to [35] and [61] for further examples and properties.

Definition 12.2. *Let $\mathbb{M}$ be a complete and cocomplete category and let $\mathcal{W}$, $\mathcal{B}$ and $\mathcal{C}$ be three classes of morphisms; $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ is a* model category *provided the following conditions hold:*

(MC.1) *$(\mathfrak{C}, \mathcal{W})$ is a category with weak equivalences;*

(MC.2) *$\mathcal{W}$, $\mathcal{B}$ and $\mathcal{C}$ are closed under retracts (in the category of morphisms). That is, given a commutative diagram as follows:*

$$\begin{tikzcd}
X \arrow[r] \arrow[d, "\phi"'] \arrow[rr, bend left, dotted, "\mathrm{id}"] & X' \arrow[r] \arrow[d, "\phi'"] & X \arrow[d, "\phi"] \\
Y \arrow[r] \arrow[rr, bend right, dotted, "\mathrm{id}"'] & Y' \arrow[r] & Y
\end{tikzcd}$$

if ϕ' belongs to $\mathcal{W}$ (resp., $\mathcal{B}$ or $\mathcal{C}$), so does ϕ;

(MC.3) *consider the following diagram,*

$$\begin{tikzcd}
C \arrow[r] \arrow[d, "c"'] & B \arrow[d, "b"] \\
C' \arrow[r] \arrow[ur, dotted, "\psi"] & B'
\end{tikzcd}$$

where $b \in \mathcal{B}$ and $c \in \mathcal{C}$. If the external square commutes and either $b \in \mathcal{W}$ or $c \in \mathcal{W}$, then there exists ψ as above making the entire diagram commutative;

(MC.4) *given a morphism ϕ, there exist $b \in \mathcal{B} \cap \mathcal{W}$, $c \in \mathcal{C}$, $b' \in \mathcal{B}$ and $c' \in \mathcal{C} \cap \mathcal{W}$, such that*

$$\phi = bc \quad \text{and} \quad \phi = b'c' .$$

The elements of $\mathcal{W}$, $\mathcal{B}$, $\mathcal{B} \cap \mathcal{W}$, $\mathcal{C}$ and $\mathcal{C} \cap \mathcal{W}$ are called respectively weak equivalences, fibrations, acyclic fibrations, cofibrations and acyclic cofibrations*.*
Given an object $X \in \mathrm{Ob}(\mathbb{M})$, if the unique map from the initial object to X is a cofibration, then X is said to be cofibrant*. If the unique map from X to the terminal object is a fibration then X is said to be* fibrant*.*

The following example allows one to encode the machinery of classical homological algebra in the scheme of model categories.

Example 12.3. *Let $\mathfrak{C}$ be a Grothendieck category and recall that the category $\mathbf{Ch}(\mathfrak{C})$ of (unbounded) cochain complexes on $\mathfrak{C}$ is a complete and cocomplete category. Let $\mathcal{W}$ be the class of quasi-isomorphisms in $\mathbf{Ch}(\mathfrak{C})$, then $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W})$ is a category with weak equivalences.*
Let $\mathcal{B}$ be the class of all the epimorphisms with dg-injective kernels (see Definition 1.114) and let $\mathcal{C}$ be the class of monomorphisms, then $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}, \mathcal{B}, \mathcal{C})$ is a model category (see for example [61] or [45] for a proof).
Notice that, in this model category, the fibrant objects are exactly the dg-injective complexes, while any complex is cofibrant.

Consider now the following definition:

Definition 12.4. *Let $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ be a model category and let $X, Y \in \mathrm{Ob}(\mathbb{M})$.*
A morphism $\alpha : QX \to X$ is a cofibrant replacement *of X if QX is a cofibrant object and α is an acyclic fibration. Furthermore, given a morphism $\phi : X \to Y$ and two cofibrant replacements $\alpha : QX \to X$ and $\alpha' : QY \to Y$, a* cofibrant replacement *$Q\phi$ of ϕ is a morphism that makes the following square commute:*

$$\begin{tikzcd} QX \arrow[r, "\alpha"] \arrow[d, "Q\phi"'] & X \arrow[d, "\phi"] \\ QY \arrow[r, "\beta"] & Y. \end{tikzcd}$$

Dually, $\beta : X \to RX$ is a fibrant replacement *of X if RX is a fibrant object and β is an acyclic cofibration. Given a morphism $\phi : X \to Y$ and two fibrant replacements $\beta : X \to RX$ and $\beta' : Y \to RY$, a* cofibrant replacement *$R\phi$ of ϕ is a morphism that makes the following square commute:*

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "\phi"'] & RX \arrow[d, "R\phi"] \\ Y \arrow[r, "\beta"] & RY. \end{tikzcd}$$

The axioms that define a model category always allow one to find cofibrant an fibrant replacements for any object and morphism in a model category.

In the concrete situation of Example 12.3, a fibrant replacement of a cochain complex $X^\bullet \in \mathrm{Ob}(\mathbf{Ch}(\mathfrak{C}))$ is a quasi-isomorphism $X^\bullet \to E^\bullet$, where $E^\bullet$ is a dg-injective complex. By Lemma 1.113 we can always find a fibrant replacement for any left-bounded complex. A consequence of the existence of the model structure described in Example 12.3 is that any cochain complex in $\mathbf{Ch}(\mathfrak{C})$ is quasi-isomorphic to a dg-injective complex.

12.1.1 The homotopy category

Let $(\mathfrak{C}, \mathcal{W})$ be a category with weak equivalences. Ideally one would like to "invert" all the morphisms in the class $\mathcal{W}$ in order to obtain a new category $\mathfrak{C}[\mathcal{W}^{-1}]$ in which all the weak equivalences become isomorphisms. Furthermore, one also wants this process to be minimal in some sense. This can be formalized using the concept of universal localization:

Definition 12.5. *The* universal localization *of a category with weak equivalences* $(\mathfrak{C}, \mathcal{W})$ *is a pair* $(\mathfrak{C}[\mathcal{W}^{-1}], F)$ *of a category* $\mathfrak{C}[\mathcal{W}^{-1}]$ *and a canonical functor* $F : \mathfrak{C} \to \mathfrak{C}[\mathcal{W}^{-1}]$ *such that* $F(\phi)$ *is an isomorphism for all* $\phi \in \mathcal{W}$*. Furthermore, if* $G : \mathfrak{C} \to \mathfrak{D}$ *is a functor such that* $G(\phi)$ *is an isomorphism for all* $\phi \in \mathcal{W}$*, then there exists a unique functor* $G' : \mathfrak{C}[\mathcal{W}^{-1}] \to \mathfrak{D}$ *such that* $G'F = G$*.*

In the context of model categories, the category $\mathfrak{C}[\mathcal{W}^{-1}]$ is usually called *homotopy category* (see Proposition 12.7). In the following definition we give an explicit construction of the homotopy category of a category with weak equivalences. The drawback is that this construction may produce a proper class of morphism between two objects: in this case the result is not a category. For this reason we use the word "category" in quotes in the following definition.

Definition 12.6. *Let* $(\mathfrak{C}, \mathcal{W})$ *be a category with weak equivalences. We define the* homotopy "category" $Ho(\mathfrak{C})$ *of* $(\mathfrak{C}, \mathcal{W})$ *as follows. An object* $Ho(\mathfrak{C})$ *is just an object of* $\mathfrak{C}$*. For all* $X, Y \in \mathrm{Ob}(Ho(\mathfrak{C}))$*, we define* $\mathrm{Hom}_{Ho(\mathfrak{C})}(X, Y)$ *as the quotient of the class of all finite strings of composable morphisms* $(f_1, \ldots, f_n)$*, where* f_i *is either a morphism in* $\mathfrak{C}$ *or the formal inverse* $f_i = w_i^{-1}$ *of an arrow* $w_i \in \mathcal{W}$*, with respect to the equivalence relation* $\sim$ *generated by the following relations:*

(Ho.1) *given* $X \in \mathrm{Ob}(X)$*, consider the empty string* $()$ *at* X*. Then,* $() \sim (\mathrm{id}_X)$*;*

(Ho.2) $(f, g) \sim (g \circ f)$ *for all composable arrows* f*,* g *of* $\mathfrak{C}$*;*

(Ho.3) $(w, w^{-1}) \sim (\mathrm{id}_X) \sim (w^{-1}, w)$ *for all* $w : X \to Y \in \mathcal{W}$*.*

One says that the homotopy category of $(\mathfrak{C}, \mathcal{W})$ exists *if* $\mathrm{Hom}_{Ho(\mathfrak{C})}(X, Y)$ *is a set for any pair of objects in* $\mathfrak{C}$*. We denote by* $Ho(-) : \mathfrak{C} \to Ho(\mathfrak{C})$ *the obvious functor.*

The notions of model category, homotopy category and universal localization are interrelated as explained in the following proposition. For the proof see [60, Section 1.2] and [35, Proposition 5.11].

Proposition 12.7. *Let* $(\mathfrak{C}, \mathcal{W})$ *be a category with weak equivalences. The following statements hold true:*

(1) *if the homotopy category of* $(\mathfrak{C}, \mathcal{W})$ *exists, then* $(Ho(\mathfrak{C}), Ho(-))$ *is a universal localization of* $(\mathfrak{C}, \mathcal{W})$*;*

(2) *if we can choose two classes of morphisms* $\mathcal{B}$ *and* $\mathcal{C}$ *in* $\mathfrak{C}$ *such that* $(\mathfrak{C}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ *is a model category, then the homotopy category of* $(\mathfrak{C}, \mathcal{W})$ *exists. Furthermore, given two objects* X*,* $Y \in \mathrm{Ob}(\mathfrak{C})$*, there is a surjection*

$$\mathrm{Hom}_{\mathfrak{C}}(QX, RY) \to \mathrm{Hom}_{Ho(\mathfrak{C})}(X, Y) \quad \textit{such that} \quad \phi \mapsto (\alpha^{-1}, \phi, \beta^{-1}),$$

where $\alpha : QX \to X$ *and* $\beta : Y \to RY$ *are respectively a cofibrant replacement of* X *and a fibrant replacement of* Y*.*

Let us apply the above proposition to the setting of Example 12.3:

Example 12.8. *Let $\mathfrak{C}$ be a Grothendieck category and consider the injective model structure* $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}, \mathcal{B}, \mathcal{C})$ *on the cochain complexes over* $\mathfrak{C}$*. By Proposition 12.7, the universal localization* $Ho(\mathbf{Ch}(\mathfrak{C}))$ *of* $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W})$ *exists. The localized category* $Ho(\mathbf{Ch}(\mathfrak{C}))$ *is usually called the* derived category *of* $\mathfrak{C}$ *and it is denoted by* $\mathbf{D}(\mathfrak{C})$*.*
Consider now $X^\bullet$, $Y^\bullet \in \mathrm{Ob}(\mathbf{Ch}(\mathfrak{C}))$*, then there is a surjection*

$$\mathrm{Hom}_{\mathbf{Ch}(\mathfrak{C})}(X^\bullet, E^\bullet) \to \mathrm{Hom}_{\mathbf{D}(\mathfrak{C})}(X^\bullet, Y^\bullet)\,,$$

where $E^\bullet$ *is any dg-injective complex which is quasi-isomorphic to* $Y^\bullet$*.*
Choose now $n \in \mathbb{Z}$ *and notice that, just by definition, the* n*-th cohomology functor* $H^n : \mathbf{Ch}(\mathfrak{C}) \to \mathfrak{C}$ *sends quasi-isomorphisms to isomorphisms. By the properties of the universal localization, there is a unique functor* $\mathbf{D}(\mathfrak{C}) \to \mathfrak{C}$ *which makes the following diagram commutative:*

$$\begin{tikzcd} \mathbf{Ch}(\mathfrak{C}) \arrow[r, "H^n"] \arrow[d] & \mathfrak{C} \\ \mathbf{D}(\mathfrak{C}) \arrow[ur] & \end{tikzcd}$$

Abusing notation, we denote also this functor $\mathbf{D}(\mathfrak{C}) \to \mathfrak{C}$ *by* H^n*.*

12.1.2 Derived functors

Definition 12.9. *Let* $(\mathfrak{C}, \mathcal{W})$*,* $(\mathfrak{D}, \mathcal{W}')$ *be categories with weak equivalences and suppose that* $Ho(\mathfrak{C})$ *exists. Given a functor* $F : \mathfrak{D} \to \mathfrak{C}$*, a* total right derived functor *of* F *is a functor* $\mathrm{R}F : \mathfrak{D} \to Ho(\mathfrak{C})$ *together with a natural transformation* $Ho(-) \circ F \Rightarrow \mathrm{R}F$*.*

The existence of a model structure allows one to construct right derived functors explicitly:

Lemma 12.10. [21] *Let* $(\mathfrak{C}, \mathcal{W})$ *be a category with weak equivalences such that* $Ho(\mathfrak{C})$ *exists, let* $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ *be a model category and let* $F : \mathbb{M} \to \mathfrak{C}$ *be a functor that maps weak equivalences between fibrant objects to weak equivalences. Then the total right derived* $\mathrm{R}F$ *of* F *exists.*
Furthermore, $\mathrm{R}F$ *can be constructed as follows. Given* $X \in \mathrm{Ob}(\mathbb{M})$*, we take first a fibrant replacement* $X \to RX$ *and then we let* $\mathrm{R}F(X) = F(RX)$*;* $\mathrm{R}F$ *is defined similarly on morphisms. The natural transformation* $F \Rightarrow \mathrm{R}F$ *is induced by the morphisms* $F(X \to RX)$*.*

Let us return to the setting of Example 12.3. Consider two Grothendieck categories $\mathfrak{C}$, $\mathfrak{D}$ and an additive functor $F : \mathfrak{C} \to \mathfrak{D}$. We denote by $F : \mathbf{Ch}(\mathfrak{C}) \to \mathbf{Ch}(\mathfrak{D})$ the extension of F. If we endow $\mathbf{Ch}(\mathfrak{C})$ with its injective model structure, one can show that the composite $Ho(-) \circ F : \mathbf{Ch}(\mathfrak{C}) \to \mathbf{D}(\mathfrak{D})$ sends quasi-isomorphisms among dg-injective complexes to isomorphisms in $\mathbf{D}(\mathfrak{D})$. Thus there is a right derived functor $\mathrm{R}F : \mathbf{Ch}(\mathfrak{C}) \to \mathbf{D}(\mathfrak{D})$. It is now just an exercise to show that the classical derived functors $\mathrm{R}^nF : \mathfrak{C} \to \mathfrak{D}$ defined in Subsection 1.2.2, are a suitable restriction of the composition $H^n \circ \mathrm{R}F$, where $H^n : \mathbf{D}(\mathfrak{D}) \to \mathfrak{D}$ is the n-th cohomology functor.

Definition 12.11. *Let* I *be a small category and let* $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ *be a model category. The category* $\mathrm{Func}(I^{op}, \mathbb{M})$ *is naturally a category with weak equivalence with the following choice of weak equivalences:*

$$\mathcal{W}_I = \{\eta : F_1 \Rightarrow F_2 : (\eta(i) : F_1(i) \to F_2(i)) \in \mathcal{W},\ \forall i \in \mathrm{Ob}(I)\}$$

The total right derived functor of the limit functor $\varprojlim : \mathrm{Func}(I^{op}, \mathbb{M}) \to \mathbb{M}$ *is called* homotopy limit*. We denote the homotopy limit of a functor* $F : I^{op} \to \mathbb{M}$ *by* $\mathrm{holim}F$*.*

It is not known whether there exists a model structure on $\mathrm{Func}(I^{op}, \mathbb{M})$ with $\mathcal{W}_I$ as class of weak equivalences that allows one to construct homotopy limits. Anyway, there are positive answers for specific choices of the small category I. We consider just one example:

Example 12.12. *Let I be the category induced by the poset $\mathbb{N}$ with the inverse of its natural order, that is:*

$$I: \qquad 0 \longleftarrow 1 \longleftarrow 2 \longleftarrow 3 \longleftarrow \dots$$

Choose also a model category $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$. Given two functors $F, G : I \to \mathbb{M}$ and a natural transformation $\alpha : F \Rightarrow G$,

$$\begin{tikzcd}
F(0) \arrow[d, "\alpha_0"] & F(1) \arrow[l] \arrow[d, "\alpha_1"] & F(2) \arrow[l] \arrow[d, "\alpha_2"] & F(3) \arrow[l] \arrow[d, "\alpha_3"] & \dots \arrow[l] \\
G(0) & G(1) \arrow[l] & G(2) \arrow[l] & G(3) \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

we say that

(1) *α is a weak-equivalence if and only if $\alpha_i \in \mathcal{W}$ for all $i \in \mathbb{N}$;*

(2) *α is a cofibration if and only if $\alpha_i \in \mathcal{C}$ for all $i \in \mathbb{N}$;*

(3) *α is a fibration if and only if the following conditions hold true:*

   - *$\alpha_0 \in \mathcal{B}$;*
   - *for any given $i \in \mathbb{N}$, we consider the following diagram*

$$\begin{tikzcd}
F(i) \arrow[r] \arrow[d] & G(i) \arrow[d] \arrow[ddr] & \\
F(i+1) \arrow[r] \arrow[drr] & X \arrow[dr, dotted, "\phi_i"] & \\
& & G(i+1)
\end{tikzcd}$$

   where the small square is a push out diagram and ϕ_i is the unique map given by the universal property. Then, ϕ_i is a fibration for all $i \in \mathbb{N}$.

One can verify, that with the above choice of weak-equivalence, fibrations and cofibrations, $\mathrm{Func}(I, \mathbb{M})$ *is a model category.*

12.1.3 Model approximation

The concept of model approximation was introduced by Chachólski and Scherer in order to circumvent the difficulties in constructing homotopy limits (see [21]).

Definition 12.13. *Let $(\mathfrak{C}, \mathcal{W}_{\mathfrak{C}})$ be a category with weak equivalences. A* right model approximation *for $(\mathfrak{C}, \mathcal{W}_{\mathfrak{C}})$ is a model category $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ and a pair of functors*

$$l : \mathfrak{C} \rightleftarrows \mathbb{M} : r$$

satisfying the following conditions:

(MA.1) *l is left adjoint to r;*

(MA.2) *if $\phi \in \mathcal{W}_{\mathfrak{C}}$, then $l(\phi) \in \mathcal{W}$;*

(MA.3) *if ψ is a weak equivalence between fibrant objects in $\mathbb{M}$, then $r(\psi) \in \mathcal{W}_{\mathfrak{C}}$;*

(MA.4) *if $l(X) \to Y$ is a weak equivalence in $\mathbb{M}$ with X fibrant, the adjoint morphism $X \to r(Y)$ is in $\mathcal{W}_{\mathfrak{C}}$.*

Given two categories with weak equivalences $(\mathfrak{C}_1, \mathcal{W}_1)$ and $(\mathfrak{C}_2, \mathcal{W}_2)$, one can find conditions under which a model approximation of $(\mathfrak{C}_2, \mathcal{W}_2)$ gives automatically a model approximation of $(\mathfrak{C}_1, \mathcal{W}_1)$:

Lemma 12.14. *Let $(\mathfrak{C}_1, \mathcal{W}_1)$ and $(\mathfrak{C}_2, \mathcal{W}_2)$ be categories with weak equivalences and let*

$$l: \mathfrak{C}_1 \rightleftarrows \mathfrak{C}_2 : r$$

be an adjunction such that:

(1) *$\phi \in \mathcal{W}_1$ implies $l\phi \in \mathcal{W}_2$;*

(2) *$\psi \in \mathcal{W}_2$ implies $r\psi \in \mathcal{W}_1$;*

(3) *if a morphism $l(C) \to D$ is in $\mathcal{W}_2$, then the adjoint morphism $C \to r(D)$ is in $\mathcal{W}_1$.*

If $(\mathbb{M}, \mathcal{W}, \mathcal{B}, \mathcal{C})$ is a model category and $l' : \mathfrak{C}_2 \leftrightarrows \mathbb{M} : r'$ is a model approximation, then

$$L: \mathfrak{C}_1 \rightleftarrows \mathbb{M} : R$$

is a model approximation, where $L = l' \circ l$ and $R = r \circ r'$.

Proof. We have to verify conditions (MA.1)-(MA.4):
(MA.1) is [70, Theorem 1, Sec. 8, Ch. 1].
(MA.2) If $\phi \in \mathcal{W}_1$, then $l\phi \in \mathcal{W}_2$ by our hypothesis (1). Apply the definition of model approximation to obtain that $L(\phi) = l'(l(\phi))$ is a weak equivalence in $\mathbb{M}$.
(MA.3) If ψ is a weak equivalence between fibrant objects in $\mathbb{M}$, then $r'(\psi) \in \mathcal{W}_2$, by the definition of model approximation. Apply (2) to obtain that $R\psi = r(r'(\psi)) \in \mathcal{W}_1$.
(MA.4) Let $X \in \mathbb{M}$ be fibrant, let $Y \in \mathfrak{C}_1$ and let $X \to L(Y)$ be a weak equivalence in $\mathbb{M}$. The adjoint (under the adjunction (l', r')) morphism $r'(X) \to l(Y)$ belongs to $\mathcal{W}_2$, by the definition of model approximation. Using (3), the adjoint (under the adjunction (l, r)) morphism $r(r'(X)) \to Y$ belongs to $\mathcal{W}_1$. □

The above lemma provides a motivation for the following

Definition 12.15. *Let $(\mathfrak{C}_1, \mathcal{W}_1)$ and $(\mathfrak{C}_2, \mathcal{W}_2)$ be two categories with weak equivalences. An adjunction $l : \mathfrak{C}_1 \leftrightarrows \mathfrak{C}_2 : r$ is said to be* compatible with weak equivalences *if conditions (1), (2) and (3) in Lemma 12.14 are satisfied.*

12.1.4 Towers of models

In this subsection we recall the construction of the category of towers introduced in [19].

Definition 12.16. *Let $\mathbb{M}_\bullet = \{\mathbb{M}_n : n \in \mathbb{N}\}$ be a sequence of categories connected with adjunctions*

$$l_{n+1} : \mathbb{M}_{n+1} \rightleftarrows \mathbb{M}_n : r_n .$$

The category of towers on $\mathbb{M}_\bullet$, $\mathrm{Tow}(\mathbb{M}_\bullet)$ *is defined as follows:*

- *an* object *is a pair $(a_\bullet, \alpha_\bullet)$, where $a_\bullet = \{a_n \in \mathbb{M}_n : n \in \mathbb{N}\}$ is a sequence of objects one for each $\mathbb{M}_n$, and $\alpha_\bullet = \{\alpha_{n+1} : a_{n+1} \to r_n(a_n) : n \in \mathbb{N}\}$ is a sequence of morphisms;*
- *a* morphism *$f_\bullet : (a_\bullet, \alpha_\bullet) \to (b_\bullet, \beta_\bullet)$ is a sequence of morphisms $f_\bullet = \{f_n : a_n \to b_n : n \in \mathbb{N}\}$ such that $r_n(f_n) \circ \alpha_{n+1} = \beta_{n+1} \circ f_{n+1}$, for all $n \in \mathbb{N}$.*

If each $\mathbb{M}_n$ in the above definition is a bicomplete category, then one can construct limits and colimits component-wise in $\mathrm{Tow}(\mathbb{M}_\bullet)$, so, under these hypotheses, the category of towers is bicomplete.

Proposition 12.17. [19, Proposition 2.3] *Let $\mathbb{M}_\bullet = \{(\mathbb{M}_n, \mathcal{W}_n, \mathcal{B}_n, \mathcal{C}_n) : n \in \mathbb{N}\}$ be a sequence of model categories connected with adjunctions*

$$l_{n+1} : \mathbb{M}_{n+1} \rightleftarrows \mathbb{M}_n : r_n$$

and suppose that each r_n preserves fibrations and acyclic fibrations. Define the following classes of morphisms in $\mathrm{Tow}(\mathbb{M}_\bullet)$*:*

- *$\mathcal{W}_{\mathrm{Tow}} = \{f_\bullet : f_n \in \mathcal{W}_n\ ,\ \forall n \in \mathbb{N}\}$;*
- *$\mathcal{B}_{\mathrm{Tow}} = \{f_\bullet : f_n^* \in \mathcal{B}_n\ ,\ \forall n \in \mathbb{N}\}$, where f_n^* is constructed as follows. First we define an object $(p_\bullet, \pi_\bullet)$ in* $\mathrm{Tow}(\mathbb{M}_\bullet)$ *where each p_n comes from a pull-back diagram*

$$\begin{tikzcd} p_n \arrow[r, "\bar{f}_{n-1}"] \arrow[d, "\bar{\beta}_n"'] \arrow[dr, phantom, "\text{P.B.}"] & b_n \arrow[d, "\beta_n"] \\ r_{n-1}(a_{n-1}) \arrow[r, "r_{n-1}(f_{n-1})"'] & r_{n-1}(b_{n-1}) \end{tikzcd}$$

and $\pi_n = r_{n-1}(f_{n-1}) \circ \bar{\beta}_n$. Then, $f_n^* : a_n \to p_n$ is defined, using the universal property of the pull-back, as the unique morphism such that $\bar{\beta}_n f_n^* = \alpha_n$ and $\bar{f}_{n-1} f_n^* = f_n$.

- *$\mathcal{C}_{\mathrm{Tow}} = \{f_\bullet : f_n \in \mathcal{C}_n\ ,\ \forall n \in \mathbb{N}\}$*

Then, $(\mathrm{Tow}(\mathbb{M}_\bullet), \mathcal{W}_{\mathrm{Tow}}, \mathcal{B}_{\mathrm{Tow}}, \mathcal{C}_{\mathrm{Tow}})$ *is a model category.*

12.2 Local cohomology

In this subsection we introduce a general notion of local cohomology. The definitions and many arguments in the proofs are adapted directly from existing papers like [3], [2], [48], [49], [50] and many others. We give here complete proofs for completeness sake and because, to the best of the author's knowledge, there is no book or paper with a comprehensive exposition of these matters in a setting as general as we need in the sequel.

Definition 12.18. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$. The n-th* τ-local cohomology $\Gamma^n_\tau : \mathfrak{C} \to \mathcal{T}$ *is the n-th right derived functor of the τ-torsion functor $\mathbf{T}_\tau : \mathfrak{C} \to \mathcal{T}$.*

It is difficult to study the properties of local cohomology in full generality, so we need to impose one or more of the following hypotheses on the ambient category $\mathfrak{C}$ in almost all of our results:

(Hyp.1) $\mathfrak{C}$ is stable (see Definition 1.139).

(Hyp.2) $\mathfrak{C}$ is locally Noetherian (see Definition 2.75).

(Hyp.3) all the prime torsion theories on $\mathfrak{C}$ are exact (see Definition 1.131).

Example 12.19. *Let $\mathfrak{C} = R$-Mod be the category of left R-modules over a ring R. When $\mathfrak{C}$ satisfies (Hyp.1), (Hyp.2) and (Hyp.3), R is said to be* left effective. *Examples of left effective rings include:*

(1) *commutative Noetherian rings;*

(2) *left Noetherian Azumaya algebras, see [50, page 173];*

(3) *prime hereditary Noetherian quasi-local rings which are bounded orders in their classical rings of fractions, see [50, Example 2.3].*

Lemma 12.20. *Let $\mathfrak{C}$ be a Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ be stable and let $X \in \mathrm{Ob}(\mathfrak{C})$. Then,*

(1) $\Gamma^n_\tau(X) = 0$ *for all $n > 0$, provided X is τ-torsion;*

(2) *there is a natural isomorphism $\Gamma^n_\tau(X) \cong \Gamma^n_\tau(X/\mathbf{T}_\tau(X))$, for all $n > 0$;*

(3) *there is a natural isomorphism $\Gamma^{n+1}_\tau(X) \cong \Gamma^n_\tau(E(X)/X)$, for all $n > 0$;*

(4) *if X is τ-torsion free, then $\Gamma^1_\tau(X) \cong \mathbf{T}_\tau(E(X)/X)$.*

(5) *there is a natural isomorphism $\Gamma^{n+1}_\tau(X) \cong \mathrm{R}\mathbf{L}^n_\tau(X)$, for all $n > 0$;*

(6) *if X is τ-torsion free, then $\Gamma^1_\tau(X) \cong \mathbf{L}_\tau(X)/X$.*

Proof. (1) Let us consider an injective resolution $\lambda : X \to E^\bullet$ inductively as follows:

– $E^n = 0$ and $d^n = 0$ for all $n < 0$;

– $E^0 = E(X)$ and λ is the canonical embedding of X in its injective envelope;

– $E^1 = E(\mathrm{CoKer}(\lambda))$ and $d^0 = \varepsilon^0 \circ \pi^0$ where $\pi^0 : E^0 \to \mathrm{CoKer}(\lambda)$ is the canonical projection and ε^0 is the canonical embedding of $\mathrm{CoKer}(\lambda)$ in its injective envelope;

– for all $n \geqslant 1$ we let $E^{n+1} = E(\mathrm{CoKer}(d^{n-1}))$ and $d^n = \varepsilon^n \circ \pi^n$ where $\pi^n : E^n \to \mathrm{CoKer}(d^{n-1})$ is the canonical projection and ε^n is the canonical embedding of $\mathrm{CoKer}(d^{n-1})$ in its injective envelope.

It is an exercise to show that $\lambda : X \to E^\bullet$ is an injective resolution (this is well-known, for example one can use the dual argument of the proof of [104, Lemma 2.2.5]). Using the fact that τ is stable, we obtain that $E^n(X)$ is τ-torsion for all $n \in \mathbb{N}$ and so $E^\bullet(X) = \mathbf{T}_\tau(E^\bullet(X))$ is an exact complex in all degrees but, eventually, in the 0-th degree.

(2) Consider the short exact sequence $0 \to \mathbf{T}_\tau(X) \to X \to X/\mathbf{T}_\tau(X) \to 0$. This gives a long exact sequence in cohomology

$$\begin{aligned}0 \to \Gamma^0_\tau(\mathbf{T}_\tau(X)) \to \Gamma^0_\tau(X) \to \Gamma^0_\tau(X/\mathbf{T}_\tau(X)) \to& \\ \to \Gamma^1_\tau(\mathbf{T}_\tau(X)) \to \Gamma^1_\tau(X) \to \Gamma^1_\tau(X/\mathbf{T}_\tau(X)) \to \Gamma^2_\tau(\mathbf{T}_\tau(X)) \to \cdots ,&\end{aligned}$$

which implies the desired isomorphism as, by part (1), we have that $\Gamma^n_\tau(\mathbf{T}_\tau(X)) = 0$, for all $n > 0$.

(3)–(4) Consider the short exact sequence $0 \to X \to E(X) \to E(X)/X \to 0$. This gives a long exact sequence in cohomology

$$\begin{aligned}0 \to \mathbf{T}_\tau(X) \to \mathbf{T}_\tau(E(X)) \to \mathbf{T}_\tau(E(X)/X) \to \Gamma^1_\tau(X) \to \Gamma^1_\tau(E(X)) \to& \\ \Gamma^1_\tau(E(X)/X) \to \Gamma^2_\tau(X) \to \Gamma^2_\tau(E(X)) \to \Gamma^2_\tau(E(X)/X) \to \Gamma^3_\tau(X) \to \Gamma^3_\tau(E(X)) \to \dots&\end{aligned}$$

which implies the isomorphism in (3) as, $\Gamma^n_\tau(E(X)) = 0$ for all $n > 0$, being $E(X)$ is injective. If $X \in \mathcal{F}$, then $E(X) \in \mathcal{F}$ by stability and so $\mathbf{T}_\tau(E(X)) = 0$, proving (4).

(5)–(6) Fix an injective resolution $X \to E^\bullet$. By Lemma 1.140, $E^n \cong \mathbf{T}_\tau(E^n) \oplus E^n/\mathbf{T}_\tau(E^n)$ for all $n \in \mathbb{N}$. Consider the complex $\mathbf{T}_\tau(E^\bullet)$ and the quotient complex $E^\bullet/\mathbf{T}_\tau(E^\bullet)$, which are both complexes of injective objects. Notice that there is a short exact sequence in $\mathbf{Ch}(\mathfrak{C})$

$$0 \to \mathbf{T}_\tau(E^\bullet) \to E^\bullet \to E^\bullet/\mathbf{T}_\tau(E^\bullet) \to 0 .$$

The cohomologies of the complex $\mathbf{T}_\tau(E^\bullet)$ are exactly the τ-local cohomologies of M, while the cohomologies of $E^\bullet$ are all trivial but, eventually, the 0-th cohomology. Furthermore, $E^n/\mathbf{T}_\tau(E^n)$ is τ-torsion free and injective, so it is τ-local; in particular, $\mathbf{L}_\tau(E^n) \cong E^n/\mathbf{T}_\tau(E^n)$ for all $n \in \mathbb{N}$. We obtain an isomorphism of complexes $E^\bullet/\mathbf{T}_\tau(E^\bullet) \cong \mathbf{L}_\tau(E^\bullet)$, which shows that the cohomologies of the complex $E^\bullet/\mathbf{T}_\tau(E^\bullet)$ give exactly the right derived functors of the localization functor $\mathbf{L}$. Thus, we have a long exact sequence

$$\begin{aligned}0 \to \mathbf{T}_\tau(X) \to X \to \mathbf{L}_\tau(X) \to \Gamma^1_\tau(X) \to 0 \to \mathrm{R}^1\mathbf{L}_\tau(X) \to& \\ \to \Gamma^2_\tau(X) \to 0 \to \mathrm{R}^2\mathbf{L}_\tau(X) \to \Gamma^3_\tau(X) \to 0 \to \mathrm{R}^3\mathbf{L}_\tau(X) \to \cdots ,&\end{aligned}$$

which gives the desired isomorphisms. □

An immediate consequence of parts (5) and (6) of the above proposition is the following

Corollary 12.21. *Let $\mathfrak{C}$ be a Grothendieck category, let $\tau \in \mathrm{Tors}(\mathfrak{C})$ be stable and exact, and let $X \in \mathfrak{C}$. Then,*

$$\Gamma^n_\tau(X) = 0 \quad \forall n > 1 .$$

If X is τ-local, then also $\Gamma^0_\tau(X) = \Gamma^1_\tau(X) = 0$.

Corollary 12.22. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau_2 \preceq \tau_1 \in \mathrm{Tors}(\mathfrak{C})$ be stable. Then,*

$$(\Gamma^i_{\tau_1}(X) = 0 \, , \ \forall 0 \leqslant i \leqslant n) \ \Rightarrow \ (\Gamma^i_{\tau_2}(X) = 0 \, , \ \forall 0 \leqslant i \leqslant n)$$

for all $X \in \mathrm{Ob}(\mathfrak{C})$ and $n \in \mathbb{N}$.

Proof. We prove our statement by induction on $n \in \mathbb{N}$.
If $n = 0$, the result is clear as $0 = \Gamma^0_{\tau_1}(X) \cong \mathbf{T}_{\tau_1}(X) \supseteq \mathbf{T}_{\tau_2}(X) \cong \Gamma^0_{\tau_2}(X)$.
If $n \geqslant 1$ and suppose the result holds for any smaller integer. Consider the following isomorphisms:

(a) $\Gamma^0_{\tau_1}(X) \cong \Gamma^0_{\tau_2}(X) = 0$, as in the case $n = 0$, in particular X is both τ_1 and τ_2-torsion free;

(b) $\Gamma^1_{\tau}(X) \cong \Gamma^0_{\tau}(E(X)/X)$ for $\tau \in \{\tau_1, \tau_2\}$, by (a) and Lemma 12.20(4).

(c) $\Gamma^k_{\tau}(X) \cong \Gamma^{k-1}_{\tau}(E(X)/X)$ for all $1 < k \leqslant n$ and $\tau \in \{\tau_1, \tau_2\}$, by Lemma 12.20(3).

By (b) and (c), we have that $\Gamma^i_{\tau_1}(E(X)/X) = 0$ for all $0 \leqslant i \leqslant n-1$ and so, by inductive hypothesis, $\Gamma^i_{\tau_2}(E(X)/X) = 0$ for all $0 \leqslant i \leqslant n-1$. Applying again (b) and (c), $\Gamma^i_{\tau_2}(X) = 0$ for all $1 \leqslant i \leqslant n$. Hence, adding (a), $\Gamma^i_{\tau_2}(X) = 0$ for all $0 \leqslant i \leqslant n$ which is what we wanted to prove. ☐

Given a Noetherian object N in $\mathfrak{C}$, recall that N is automatically a *compact* object, that is, the functor $\mathrm{Hom}_{\mathfrak{C}}(N, -)$ commutes with direct limits (see for example [96, Proposition 3.4, Ch. V]). More explicitly, given a directed set Λ and a direct system $(X_\alpha, \phi_{\beta,\alpha})_\Lambda$ in $\mathfrak{C}$, we have a natural isomorphism

$$\varinjlim_{\alpha\in\Lambda} \mathrm{Hom}_{\mathfrak{C}}(N, X_\alpha) \cong \mathrm{Hom}_{\mathfrak{C}}\left(N, \varinjlim_{\alpha\in\Lambda} X_\alpha\right). \tag{12.2.1}$$

Proposition 12.23. *Let $\mathfrak{C}$ be a Grothendieck category satisfying (Hyp.2) and let $\tau \in \mathrm{Tors}(\mathfrak{C})$. Then,*

(1) *given two objects X and $M \in \mathrm{Ob}(\mathfrak{C})$, we have a natural isomorphism*

$$\mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau M) \cong \varinjlim_{Y} \mathrm{Hom}_{\mathfrak{C}}(X/Y, M),$$

with Y ranging in the family of sub-objects of X such that $X/Y \in \mathcal{T}$ (ordered by reverse inclusion);

(2) *all the τ-local cohomology functors commute with direct limits.*

Proof. (1) Let Y be a sub-object of X such that $X/Y \in \mathcal{T}$. For any morphism $\phi : X/Y \to M$, $\phi(X/Y) \leqslant \mathbf{T}_\tau(M) \leqslant M$ and so $\mathrm{Hom}_{\mathfrak{C}}(X/Y, \mathbf{T}_\tau(M)) \cong \mathrm{Hom}_{\mathfrak{C}}(X/Y, M)$. Furthermore, there is an injective map

$$-\circ p_Y : \mathrm{Hom}_{\mathfrak{C}}(X/Y, \mathbf{T}_\tau(M))(\cong \mathrm{Hom}_{\mathfrak{C}}(X/Y, M)) \longrightarrow \mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(M)) \tag{12.2.2}$$

where $p_Y : X \to X/Y$ is the canonical projection. By the universal property of the direct limit, there is a unique map Φ, making the following diagrams commutative, whenever $Y_1 \leqslant Y_2 \leqslant X$ and $X/Y_1 \in \mathcal{T}$:

$$\begin{tikzcd}
\mathrm{Hom}_{\mathfrak{C}}(X/Y_1, M) \arrow[dr, "-\circ p_{Y_1}"] \arrow[drr, bend left] & & \\
& \mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(M)) & \varinjlim_Y \mathrm{Hom}_{\mathfrak{C}}(X/Y, M). \arrow[l, "\exists!\,\Phi"'] \\
\mathrm{Hom}_{\mathfrak{C}}(X/Y_2, M) \arrow[uu] \arrow[ur, "-\circ p_{Y_2}"'] \arrow[urr, bend right] & &
\end{tikzcd}$$

Now, Φ is injective by the injectivity of the maps described in (12.2.2) and the commutativity of the above diagram; furthermore, one can show that Φ is surjective as follows: an element $\phi \in \mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(M))$ belongs to the image of Φ if and only if there exists $Y \leqslant X$ such that $X/Y \in \mathcal{T}$ and there is a morphism $\psi : X/Y \to M$ such that $\phi = \psi p_Y$. Given $\phi \in \mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(M))$, we easily get that $\phi(X) \in \mathcal{T}$ and so, letting $Y = \mathrm{Ker}(\phi)$, we have that $X/Y \in \mathcal{T}$. Furthermore, there is an induced (mono)morphism $\psi : X/Y \to \mathbf{T}_\tau(X)$ such that $\phi = \psi p_Y$, as desired.

(2) According to [56, Proposition 3.6.2], it suffices to verify that

(a) $\mathrm{R}^0\mathbf{T}_\tau \cong \mathbf{T}_\tau$ commutes with direct limits;

(b) $\varinjlim_\Lambda E_\alpha$ is $\mathbf{T}_\tau$-acyclic (i.e., $\Gamma_\tau^n(\varinjlim_\Lambda E_\alpha) = 0$ for all $n > 0$) for any directed system $(E_\alpha, \phi_{\beta,\alpha})_\Lambda$ of injective objects.

Let $(M_\alpha, \phi_{\beta,\alpha})_\Lambda$ be a directed system in $\mathfrak{C}$, over a directed set Λ. We have to verify that $\mathbf{T}_\tau(\varinjlim_\Lambda M_\alpha) \cong \varinjlim_\Lambda \mathbf{T}_\tau(M_\alpha)$. For this we show that there is a natural equivalence of functors $\mathrm{Hom}_{\mathfrak{C}}(-, \mathbf{T}_\tau(\varinjlim_\Lambda M_\alpha)) \cong \mathrm{Hom}_{\mathfrak{C}}(-, \varinjlim_\Lambda \mathbf{T}_\tau(M_\alpha))$, which implies (a) by the Yoneda Lemma. For all $X \in \mathfrak{C}$ there is a natural isomorphism

$$\mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(\varinjlim_{\Lambda} M_\alpha)) \cong \mathrm{Hom}(\varinjlim_{N} N, \mathbf{T}_\tau(\varinjlim_{\Lambda} M_\alpha))\,,$$

with N ranging in the family of Noetherian sub-objects of X, so

$$\mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(\varinjlim_{\Lambda} M_\alpha)) \cong \varprojlim_{N} \mathrm{Hom}(N, \mathbf{T}_\tau(\varinjlim_{\Lambda} M_\alpha))$$

This allows us to assume that X is itself Noetherian. In this case:

$$\begin{aligned}
\mathrm{Hom}_{\mathfrak{C}}(X, \mathbf{T}_\tau(\varinjlim_{\Lambda} M_\alpha)) &\cong \varinjlim_{Y} \mathrm{Hom}(X/Y, \varinjlim_{\Lambda} M_\alpha) && \text{with } X/Y \in \mathcal{T}, \text{ by (1)}\\
&\cong \varinjlim_{Y} \varinjlim_{\Lambda} \mathrm{Hom}(X/Y, M_\alpha) && \text{by (12.2.1)}\\
&\cong \varinjlim_{\Lambda} \varinjlim_{Y} \mathrm{Hom}(X/Y, M_\alpha)\\
&\cong \mathrm{Hom}(X, \varinjlim_{\Lambda} \mathbf{T}_\tau(M_\alpha))\,.
\end{aligned}$$

Part (b) follows by Proposition 2.78 (1), and the fact that injective objects are F-acyclic for any left exact functor F. $\square$

Lemma 12.24. *Let $\mathfrak{C}$ be a Grothendieck category satisfying (Hyp.1) and (Hyp.2), let $\bar{\pi} \in \mathrm{Sp}(\mathfrak{C})$, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ and let C be a cocritical object such that $E(C) \cong E(\bar{\pi})$.*
If $-1 < \mathrm{G.dim}_\tau(C) < \infty$, then $\mathrm{G.dim}_\tau(\mathbf{L}_{\bar{\pi}}(C)/C) < \mathrm{G.dim}_\tau(C)$.

Proof. By hypothesis $\mathrm{G.dim}_\tau(C) > -1$. Furthermore, if $\mathrm{G.dim}_\tau(C) = n+1$ for some $n \in \mathbb{N}$, we can denote by $\tau_n \in \mathrm{Tors}(\mathfrak{C}/\mathcal{T})$ the torsion theory whose torsion class is $(\mathfrak{C}/\mathcal{T})_n$. Then, $\mathrm{G.dim}_{\tau\circ\tau_n}(C) = 0$. Thus, there is no loss of generality in assuming that $\mathrm{G.dim}_\tau(C) = 0$ (otherwise substitute τ by $\tau \circ \tau_n$ and then use part (2) of Lemma 2.72).
Assuming that $\mathrm{G.dim}_\tau(C) = 0$, we have to show that $\mathbf{L}_{\bar{\pi}}(C)/C \in \mathcal{T}$. Let E be an injective object that cogenerates τ. By Proposition 2.78, there exist a set I, a family of prime torsion theories $\{\pi_i = (\mathcal{T}_i, \mathcal{F}_i) : i \in I\} \subseteq \mathrm{Sp}(\mathfrak{C})$ and a family of non-trivial cardinals $\{\alpha_i : i \in I\}$ such that $E \cong \bigoplus_{i\in I} E(\pi_i)^{(\alpha_i)}$, thus $\mathcal{T} = \bigcap_{i\in I} \mathcal{T}_i$ and so we reduced to prove that

$$\mathrm{Hom}_{\mathfrak{C}}(\mathbf{L}_{\bar{\pi}}(C)/C, E(\pi_i)) = 0\,, \tag{12.2.3}$$

for all $i \in I$. Let $i \in I$, if $\pi_i = \bar{\pi}$, then $\mathbf{L}_{\bar{\pi}}(C)/C \cong \mathbf{T}_{\bar{\pi}}(E(\bar{\pi})/C)$ is π_i-torsion by construction, so (12.2.3) follows. On the other hand, if $\pi_i \neq \bar{\pi}$, suppose looking for a contradiction, that $\mathrm{Hom}_{\mathfrak{C}}(\mathbf{L}_{\bar{\pi}}(C)/C, E(\pi_i)) \neq 0$ which implies $\mathrm{Hom}_{\mathfrak{C}}(\mathbf{L}_{\bar{\pi}}(C), E(\pi_i)) \neq 0$ which, by the injectivity of $E(\pi_i)$, implies $\mathrm{Hom}_{\mathfrak{C}}(E(\bar{\pi}), E(\pi_i)) \neq 0$. By Corollary 2.74, $0 = \mathrm{G.dim}_\tau(E(\bar{\pi})) > \mathrm{G.dim}_\tau(E(\pi_i))$, equivalently, $E(\pi_i) \in \mathcal{T}$, that is $\mathrm{Hom}_{\mathfrak{C}}(E(\pi_i), E) = 0$, which is clearly a contradiction. □

The following theorem is an improved version of [50, Proposition 2.4].

Theorem 12.25. *Let $\mathfrak{C}$ be a Grothendieck category satisfying (Hyp.1), (Hyp.2) and (Hyp.3), let $\tau \in \mathrm{Tors}(\mathfrak{C})$ and let $X \in \mathrm{Ob}(\mathfrak{C})$. Then, $\Gamma^n_\tau(X) \neq 0$ implies $\mathrm{G.dim}_\tau(X) + 2 \geqslant n$.*

Proof. By (Hyp.3), X is the direct union of its Noetherian sub-objects. By Proposition 12.23, the vanishing of τ-local cohomologies on Noetherian objects implies their vanishing on X. Thus we can suppose X to be Noetherian. By Lemma 2.70 (4), there exist sub-objects $0 = Y_0 \leqslant Y_1 \leqslant \cdots \leqslant Y_k = X$ such that Y_i/Y_{i-1} is cocritical for all $i = 1, \dots, k$. One can verify by induction on k that the vanishing of the τ-local cohomology functors on all the factors of the form Y_i/Y_{i-1} implies their vanishing on X. Thus we may suppose X to be cocritical, in particular $E(X) \cong E(\pi)$ for some $\pi \in \mathrm{Sp}(\mathfrak{C})$.

If $\mathrm{G.dim}_\tau(X)$ is not finite, then there is nothing to prove, therefore we suppose $\mathrm{G.dim}_\tau(X) = d < \infty$ and we proceed by induction on d. If $d = -1$, then $\Gamma^n_\tau(X) = 0$ for all $n > 0$, by Lemma 12.20 (1). Thus, $\Gamma^n_\tau(X) \neq 0$ implies $n = 0 \leqslant \mathrm{G.dim}_\tau(X) + 2 = -1 + 2 = 1$. If $d > -1$, consider the following long exact sequence:

$$\begin{aligned} 0 \to \Gamma^0_\tau(X) \to \Gamma^0_\tau(\mathbf{L}_\pi(X)) \to \Gamma^0_\tau(\mathbf{L}_\pi(X)/X) \to \Gamma^1_\tau(X) \to \Gamma^1_\tau(\mathbf{L}_\pi(X)) \to \\ \to \Gamma^1_\tau(\mathbf{L}_\pi(X)/X) \to \cdots \to \Gamma^n_\tau(X) \to \Gamma^n_\tau(\mathbf{L}_\pi(X)) \to \Gamma^n_\tau(\mathbf{L}_\pi(X)/X) \to \Gamma^{n+1}_\tau(X) \to \dots \end{aligned}$$

Notice that $\Gamma^0_\tau(X) = \Gamma^0_\tau(\mathbf{L}_\pi(X)) = 0$, since we supposed that $d > -1$ and so, by stability, X is τ-torsion free. Furthermore, using (Hyp.3) and Corollary 12.21, one can show that $\Gamma^n_\pi(\mathbf{L}_\pi(X)) = 0$ for all $n \in \mathbb{N}$. Using again that X (and so $E(\pi)$) is τ-torsion free, we get $\tau \preceq \pi$ and so we can apply Corollary 12.22 to show that $\Gamma^n_\tau(\mathbf{L}_\pi(X)) = 0$ for all $n > 0$. One obtains the following isomorphisms:

$$\Gamma^n_\tau(X) = \Gamma^{n-1}_\tau(\mathbf{L}_\pi(X)/X)\,, \quad \forall n \geqslant 1\,.$$

By Lemma 12.24, $\mathrm{G.dim}_\tau(\mathbf{L}_\pi(X)/X) < d$ and so we can apply our inductive hypothesis to show that $\Gamma^n_\tau(\mathbf{L}_\pi(X)/X) = 0$ for all $n > \mathrm{G.dim}_\tau(\mathbf{L}_\pi(X)/X) + 2$. Thus, if $\Gamma^n_\tau(X) \neq 0$ for some $n > 0$, then $\Gamma^{n-1}_\tau(\mathbf{L}_\pi(X)/X) \neq 0$ and so $n - 1 \leqslant \mathrm{G.dim}_\tau(\mathbf{L}_\pi(X)/X) + 2$, that is, $n \leqslant \mathrm{G.dim}_\tau(\mathbf{L}_\pi(X)/X) + 3 \leqslant \mathrm{G.dim}_\tau(X) + 2$. □

12.2.1 Exactness of products in the localization

In the definition of Grothendieck category one assumes direct limits to be exact but no assumption is required on the exactness of products. We will see in the last part of this paper that knowing that a Grothendieck category has "almost exact products" has very nice consequences on its derived category. In the following definition we precise the meaning of "almost exact products":

Definition 12.26. *Let $\mathfrak{C}$ be a Grothendieck category. For a non-negative integer n, $\mathfrak{C}$ is said to satisfy the axiom (Ab.4*)-k if, for any set I and any collection of objects $\{X_i\}_{i\in I}$,*

$$\prod_{i\in I}{}^{(n)} X_i = 0 \quad \forall n > k\,,$$

where $\prod^{(n)}_{i\in I}(-)$ is the n-th derived functor of the product $\prod : \mathfrak{C}^I \to \mathfrak{C}$.

Notice that condition (Ab.4*)-0 is exactly (Ab.4*) (see Subsection 1.1.5).

In general, there is no reason for a Grothendieck category to be (Ab.4*)-k for any k. Anyway, categories of modules are (Ab.4*) and, by the Gabriel-Popescu Theorem, any Grothendieck category is a quotient category of a category of modules. Thus, it seems natural to ask for sufficient conditions on a torsion theory $\tau = (\mathcal{T}, \mathcal{F})$ in an (Ab.4*) Grothendieck category $\mathfrak{C}$ that ensure that the quotient category $\mathfrak{C}/\mathcal{T}$ is (Ab.4*)-k for some $k \in \mathbb{N}$. We give two such conditions in the following lemma, a deeper criterion is given in Theorem 12.28.

Lemma 12.27. *Let $\mathfrak{C}$ be an (Ab.4*) Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ and suppose one of the following conditions holds:*

(1) *$\mathcal{T}$ is closed under taking products (in this case τ is said to be a TTF);*

(2) *τ is exact.*

Then, the quotient category $\mathfrak{C}/\mathcal{T}$ is still (Ab.4).*

Proof. Let I be a set and for each $i \in I$ consider objects A_i, B_i and $C_i \in \mathfrak{C}/\mathcal{T}$ such that

$$0 \to A_i \to B_i \to C_i \to 0 \quad \text{is exact in } \ \mathfrak{C}/\mathcal{T}\,.$$

Hence, for all $i \in I$, one obtains exact sequences $0 \to \mathbf{S}_\tau(A_i) \to \mathbf{S}_\tau(B_i) \to \mathbf{S}_\tau(C_i) \to T_i \to 0$ in $\mathfrak{C}$, where $T_i \in \mathcal{T}$. Using the (Ab.4*) property in $\mathfrak{C}$ we get an exact sequence

$$0 \to \prod_{i\in I} \mathbf{S}_\tau(A_i) \to \prod_{i\in I} \mathbf{S}_\tau(B_i) \to \prod_{i\in I} \mathbf{S}_\tau(C_i) \to \prod_{i\in I} T_i \to 0\,.$$

If τ is a TTF, then $\prod_{i\in I} T_i$ is τ-torsion and so we can apply $\mathbf{Q}_\tau$ to the above exact sequence obtaining the following short exact sequence

$$0 \to \prod_{i\in I} A_i \to \prod_{i\in I} B_i \to \prod_{i\in I} C_i \to \mathbf{Q}_\tau(\prod_{i\in I} T_i) = 0\,,$$

by Corollary 1.135 and the exactness of $\mathbf{Q}_\tau$. On the other hand, if $\mathbf{S}_\tau$ is exact, we get $T_i = 0$ for all $i \in I$ above and so again one can easily conclude. □

Theorem 12.28. *Let $\mathfrak{C}$ be an (Ab.4*) Grothendieck category which satisfies hypotheses (Hyp.1), (Hyp.2) and (Hyp.3), and let $\tau \in \mathrm{Tors}(\mathfrak{C})$. If $\mathrm{G.dim}(\mathfrak{C}/\mathcal{T}) = k < \infty$, then $\mathfrak{C}/\mathcal{T}$ is (Ab.4*)-$k+1$.*

Proof. Let $\{X_i\}_{i\in I}$ be a family of objects in $\mathfrak{C}/\mathcal{T}$. For all $i \in I$, choose an injective resolution $0 \to \mathbf{S}_\tau(X_i) \to E_i^\bullet$ of $\mathbf{S}_\tau(X_i)$ in $\mathfrak{C}$. Since $\mathbf{Q}_\tau$ is exact and sends injective objects to injective objects, the complex $\mathbf{Q}_\tau(E_i^\bullet)$ provides an injective resolution for X_i. Thus, for all $n > k+1$,

$$\begin{aligned}
\prod_{i\in I}{}^{(n)} X_i &= H^n\left(\prod_{i\in I} \mathbf{Q}_\tau\left(E_i^\bullet\right)\right) = H^n\left(\mathbf{Q}_\tau\left(\prod_{i\in I} \mathbf{L}_\tau\left(E_i^\bullet\right)\right)\right) && \text{by Corollary 1.135}\\
&= \mathbf{Q}_\tau\left(H^n\left(\prod_{i\in I} \mathbf{L}_\tau\left(E_i^\bullet\right)\right)\right) && \text{exact funct. commute with cohom.}\\
&= \mathbf{Q}_\tau\left(\prod_{i\in I} H^n\left(\mathbf{L}_\tau\left(E_i^\bullet\right)\right)\right) = \mathbf{Q}_\tau\left(\prod_{i\in I} \mathrm{R}\mathbf{L}_\tau^n\left(E_i^\bullet\right)\right) && \text{exact funct. commute with cohom.}\\
&= \mathbf{Q}_\tau\left(\prod_{i\in I} \Gamma_\tau^{n+1}\left(E_i^\bullet\right)\right) = 0 && \text{Lemma 12.20 and Theorem 12.25.}
\end{aligned}$$

□

Corollary 12.29. *Let $\mathfrak{C}$ be an (Ab.4*) Grothendieck category which satisfies hypotheses (Hyp.1), (Hyp.2) and (Hyp.3). If* $\mathrm{G.dim}(\mathfrak{C}) = k < \infty$*, then* $\mathfrak{C}/\mathcal{T}$ *is (Ab.4*)-*$k+1$ *for all* $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$*.*

12.3 Injective classes

Let $\mathfrak{C}$ be a Grothendieck category and let $\mathcal{I}$ be a class of objects of $\mathfrak{C}$. Slightly generalizing the setting of [20], we say that a morphism $\phi : X \to Y$ in $\mathfrak{C}$ is an *$\mathcal{I}$-monomorphism* if

$$\mathrm{Hom}_{\mathfrak{C}}(\phi, K) : \mathrm{Hom}_{\mathfrak{C}}(Y, K) \to \mathrm{Hom}_{\mathfrak{C}}(X, K)$$

is an epimorphism of Abelian groups for every $K \in \mathcal{I}$. We say that $\mathfrak{C}$ has *enough $\mathcal{I}$-injectives* if every object X admits an $\mathcal{I}$-monomorphism $X \to K$ for some $K \in \mathcal{I}$.

Definition 12.30. [20] *A subclass $\mathcal{I}$ of a Grothendieck category $\mathfrak{C}$ is an* injective class *(of $\mathfrak{C}$) if it is closed under products and direct summands, and $\mathfrak{C}$ has enough $\mathcal{I}$-injectives.*

Lemma 12.31. *Let $\mathcal{I}$ be an injective class of a Grothendieck category $\mathfrak{C}$. An object X of $\mathfrak{C}$ belongs to $\mathcal{I}$ if and only if* $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ *sends all $\mathcal{I}$-monomorphisms to epimorphisms of Abelian groups.*

Proof. Suppose that $\mathrm{Hom}_{\mathfrak{C}}(-, X)$ sends $\mathcal{I}$-monomorphisms to surjective morphisms. By definition of injective class, there is an $\mathcal{I}$-monomorphism $\varphi : X \to E$ for some $E \in \mathcal{I}$. Hence, $\mathrm{Hom}_{\mathfrak{C}}(\varphi, X) : \mathrm{Hom}_{\mathfrak{C}}(E, X) \to \mathrm{Hom}_{\mathfrak{C}}(X, X)$ is surjective and so X is a direct summand of E. Thus, X belongs to $\mathcal{I}$. The converse is trivial. □

Definition 12.32. *Let $\mathfrak{C}$ be a Grothendieck category and let $\mathcal{I}$ be an injective class. $\mathcal{I}$ is an* injective class of injectives *provided any object in $\mathcal{I}$ is an injective object. We denote by* $\mathrm{Inj}(\mathfrak{C})$ *the poset of all the injective classes of injectives in $\mathfrak{C}$, where, given $\mathcal{I}$ and $\mathcal{I}' \in \mathrm{Inj}(\mathfrak{C})$*

$$\mathcal{I} \preceq \mathcal{I}' \quad \textit{if and only if} \quad \mathcal{I} \subseteq \mathcal{I}' .$$

12.3.1 Examples

Before proceeding further we give some examples of injective classes (not necessarily of injectives).

Example 12.33. *Let $\mathfrak{C}$ be a Grothendieck category. Then*

(1) *$\mathcal{I} = 0$ is an injective class. In this case every morphism is an $\mathcal{I}$-monomorphism;*

(2) *$\mathcal{I} = \mathfrak{C}$ is an injective class. In this case, a morphism is an $\mathcal{I}$-monomorphism if and only if it has a left inverse (it is a splitting monomorphism);*

(3) *the class $\mathcal{I}$ of all injective objects is an injective class. With this choice, $\mathcal{I}$-monomorphisms are the usual monomorphisms.*

Part (3) of the above example can be generalized as follows:

Lemma 12.34. *Let $\mathfrak{C}$ be a Grothendieck category and let $\phi : X \to Y$ be a morphism in $\mathfrak{C}$. Then, ϕ is a monomorphism if and only if* $\mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ *is an epimorphism of Abelian groups for any injective object E of $\mathfrak{C}$.*
In particular, given an injective class $\mathcal{I}$ of $\mathfrak{C}$, every $\mathcal{I}$-monomorphism is in particular a monomorphism if and only if $\mathcal{I}$ contains the class of injectives.

Proof. If ϕ is a monomorphism, it is clear that $\mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ is an epimorphism for any injective object E. On the other hand, suppose that $\mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ is an epimorphism for any injective object E of $\mathfrak{C}$. Let $\psi : Z \to X$ be a morphism such that $\phi\psi = 0$, and let us verify that $\psi = 0$. We can suppose that ψ is a monomorphism, in fact, otherwise we can substitute ψ with the induced morphism $\bar{\psi} : Z/\mathrm{Ker}(\psi) \to X$. Thus, let $E = E(Z)$ and notice that $\mathrm{Hom}_{\mathfrak{C}}(\psi, E)$ and $\mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ are both epimorphisms and so $\mathrm{Hom}_{\mathfrak{C}}(\psi, E) \circ \mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ is an epimorphism, but $\mathrm{Hom}_{\mathfrak{C}}(\psi, E) \circ \mathrm{Hom}_{\mathfrak{C}}(\phi, E) = \mathrm{Hom}_{\mathfrak{C}}(\phi \circ \psi, E) = 0$. Thus, $\mathrm{Hom}_{\mathfrak{C}}(Z, E) = 0$ which is to say that $Z = 0$, as desired.
For the last statement, notice that if $\mathcal{I}$ contains all the injective objects then, by the first part, any $\mathcal{I}$-monomorphism has to be a monomorphism. On the other hand, suppose that any $\mathcal{I}$-monomorphism is a monomorphism. This means in particular that any given injective object E is such that $\mathrm{Hom}_{\mathfrak{C}}(-, E)$ sends $\mathcal{I}$-monomorphism to epimorphisms. By Lemma 12.31, this means that $E \in \mathcal{I}$. □

Example 12.35. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$, then $\mathcal{F}$ is an injective class. In fact, $\mathcal{F}$ is closed under taking products and direct summands. Furthermore, given an object X and $F \in \mathcal{F}$, apply the functor $\mathrm{Hom}_{\mathfrak{C}}(-, F)$ to the exact sequence $0 \to \mathbf{T}_\tau(X) \to X \to X/\mathbf{T}_\tau(X) \to 0$, to obtain the following exact sequence of Abelian groups:*

$$\mathrm{Hom}_{\mathfrak{C}}(X/\mathbf{T}_\tau(X), F) \to \mathrm{Hom}_{\mathfrak{C}}(X, F) \to \mathrm{Hom}_{\mathfrak{C}}(\mathbf{T}_\tau(X), F)\,.$$

Since $\mathrm{Hom}_{\mathfrak{C}}(\mathbf{T}_\tau(X), F) = 0$, the canonical projection $X \to X/\mathbf{T}_\tau(X)$ is an $\mathcal{F}$-monomorphism of X into an element of $\mathcal{F}$.

Example 12.36. *Let R be a ring and let $I \leqslant R$ be a two-sided ideal. We claim that the class*

$$\mathcal{I} = \{M \in R\text{-Mod} : IM = 0\}$$

is an injective class in R-Mod. In fact, $\mathcal{I}$ is closed under products and direct summands. Furthermore, given a left R-module M, we can always consider the canonical projection $p : M \to M/IM$, where $M/IM \in \mathcal{I}$. We have to show that p is an $\mathcal{I}$-monomorphism. Let $N \in \mathcal{I}$ and let $\phi : M \to N$ be a morphism. Given $x \in IM$, there exists $y \in M$ and $i \in I$ such that $iy = x$ and so, $\phi(x) = \phi(iy) = i\phi(y) = 0$ as $IN = 0$. Thus, ϕ factors through p as desired.

We remark that an injective class does not need to satisfy any reasonable closure property but closure under products and direct summands (that are assumed in the definition). In fact, an example of an injective class that is closed nor under subobjects or infinite direct sums is given by the class of all injective objects (at least in the non-locally Noetherian case). An injective class that is not closed under extensions is described in Example 12.36. For further examples we refer to [20].

12.3.2 Injective classes vs torsion theories

Definition 12.37. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$. We define the following subclass of $\mathfrak{C}$:*

$$\mathcal{I}_\tau = \{\textit{injective objects in } \mathcal{F}\}\,.$$

It is possible to give quite an explicit characterization of those morphisms that are $\mathcal{I}_\tau$-monomorphism:

Lemma 12.38. *Let $\mathfrak{C}$ be a Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ and let $\phi : X \to Y$ be a morphism in $\mathfrak{C}$. The following are equivalent:*

(1) ϕ *is an* $\mathcal{I}_\tau$*-monomorphism;*

(2) $\mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ *is an epimorphism for any injective object* E *which cogenerates* τ*;*

(3) $\mathrm{Ker}(\phi)$ *is* τ*-torsion;*

(4) $\mathbf{Q}_\tau(\phi)$ *is a monomorphism.*

Proof. The implication (1)⇒(2) is trivial since an injective object which cogenerates τ necessarily belongs to $\mathcal{I}_\tau$. Let us prove the implication (2)⇒(3). Choose an injective cogenerator E for τ and apply the functor $\mathrm{Hom}_{\mathfrak{C}}(-, E)$ to the exact sequence $0 \to \mathrm{Ker}(\phi) \to X \to Y$ obtaining the following exact sequence of Abelian groups

$$\mathrm{Hom}_{\mathfrak{C}}(Y, E) \to \mathrm{Hom}_{\mathfrak{C}}(X, E) \to \mathrm{Hom}_{\mathfrak{C}}(\mathrm{Ker}(\phi), E) \to 0\,.$$

If $\mathrm{Hom}_{\mathfrak{C}}(\phi, E)$ is an epimorphism then $\mathrm{Hom}_{\mathfrak{C}}(\mathrm{Ker}(\phi), E) = 0$ that is, $\mathrm{Ker}(\phi)$ is τ-torsion. The equivalence (3)⇔(4) follows by the exactness of $\mathbf{Q}_\tau$ and the fact that $\mathrm{Ker}(\mathbf{Q}_\tau) = \mathcal{T}$. It remains only to prove that (3)⇒(1). Indeed, given $K \in \mathcal{I}_\tau$ we can obtain as before an exact sequence $\mathrm{Hom}_{\mathfrak{C}}(Y, K) \to \mathrm{Hom}_{\mathfrak{C}}(X, K) \to \mathrm{Hom}_{\mathfrak{C}}(\mathrm{Ker}(\phi), K)$. Since K is τ-torsion free and $\mathrm{Ker}(\phi)$ is τ-torsion, $\mathrm{Hom}_{\mathfrak{C}}(\mathrm{Ker}(\phi), K) = 0$, as desired. □

Lemma 12.39. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$*. Then,* $\mathcal{I}_\tau \in \mathrm{Inj}(\mathfrak{C})$*.*

Proof. $\mathcal{I}_\tau$ is the intersection of $\mathcal{F}$ with the class $\mathcal{E}_{\mathfrak{C}}$ of all the injective objects in $\mathfrak{C}$. The closure properties of $\mathcal{F}$ and $\mathcal{E}_{\mathfrak{C}}$ imply that $\mathcal{I}_\tau$ is closed under products and direct summands. It remains to show that $\mathfrak{C}$ has enough $\mathcal{I}_\tau$-injectives. Indeed, let X be an object of $\mathfrak{C}$ and let $\phi : X \to E(X/\mathbf{T}_\tau(X))$ be the composition of the canonical morphisms $X \to X/\mathbf{T}_\tau(X)$ and $X/\mathbf{T}_\tau(X) \to E(X/\mathbf{T}_\tau(X))$. Clearly $E(X/\mathbf{T}_\tau(X)) \in \mathcal{I}_\tau$ and, furthermore, the kernel of ϕ is precisely $\mathbf{T}_\tau(X)$. By Lemma 12.38, ϕ is an $\mathcal{I}_\tau$-monomorphism. □

Let $\mathfrak{C}$ be a Grothendieck category. By the above lemma, we can associate an injective class of injectives to any given torsion theory. Let now $\mathcal{I} \in \mathrm{Inj}(\mathfrak{C})$ and define

$$\mathcal{F}_{\mathcal{I}} = \{\text{sub-objects of the elements of } \mathcal{I}\}\,.$$

Lemma 12.40. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\mathcal{I} \in \mathrm{Inj}(\mathfrak{C})$*. Then,* $\mathcal{F}_{\mathcal{I}}$ *is a torsion free class.*

Proof. Closure under taking sub-objects, products and injective envelopes easily follow by construction and the closure hypotheses on $\mathcal{I}$. It remains to prove that $\mathcal{F}_{\mathcal{I}}$ is closed under taking extensions. Let X be an object in $\mathfrak{C}$ and let $Y \leqslant X$ be a sub-object such that both Y and $X/Y \in \mathcal{F}_{\mathcal{I}}$. By construction, there exist I_1 and $I_2 \in \mathcal{I}$ such that $Y \leqslant I_1$ and $X/Y \leqslant I_2$. Let $\phi_1 : X \to I_1$ be a morphism extending the canonical inclusion $Y \to I_1$ (whose existence is ensured by the injectivity of I_1) and let $\phi_2 : X \to I_2$ be the composition of the canonical projection $X \to X/Y$ with the inclusion $X/Y \to I_2$. Define $\phi = \phi_1 \oplus \phi_2 : X \to I_1 \oplus I_2$. Then, $\mathrm{Ker}(\phi) = \mathrm{Ker}(\phi_1) \cap \mathrm{Ker}(\phi_2) = 0$ and so X is a sub-object of $I_1 \oplus I_2 \in \mathcal{I}$. □

Definition 12.41. *Let* $\mathfrak{C}$ *be a Grothendieck category and let* $\mathcal{I} \in \mathrm{Inj}(\mathfrak{C})$*. We define* $\tau_{\mathcal{I}}$ *to be the unique torsion theory on* $\mathfrak{C}$ *whose torsion free class is* $\mathcal{F}_{\mathcal{I}}$*.*

We are now ready to prove the main result of this subsection:

Theorem 12.42. *Let $\mathfrak{C}$ be a Grothendieck category. Then the map $\tau \mapsto \mathcal{I}_\tau$ is an order-reversing bijection between* Tors$(\mathfrak{C})$ *and* Inj$(\mathfrak{C})$. *The inverse bijection is given by the correspondence $\mathcal{I} \mapsto \tau_{\mathcal{I}}$.*

Proof. Let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ we want to prove that $\mathcal{F}_{\mathcal{I}_\tau} = \mathcal{F}$. The inclusion $\mathcal{F}_{\mathcal{I}_\tau} \subseteq \mathcal{F}$ is trivial, while, given $F \in \mathcal{F}$ and an injective cogenerator E for τ, there exists a set S such that $F \leqslant E^S \in \mathcal{I}_\tau$ and so $F \in \mathcal{F}_{\mathcal{I}_\tau}$.

On the other hand, let $\mathcal{I} \in \mathrm{Inj}(\mathfrak{C})$ and $\tau_{\mathcal{I}} = (\mathcal{F}, \mathcal{T})$. We want to prove that $\mathcal{I}_{\tau_{\mathcal{I}}} = \mathcal{I}$. The inclusion $\mathcal{I} \subseteq \mathcal{I}_{\tau_{\mathcal{I}}}$ is trivial, while, given $I \in \mathcal{I}_{\tau_{\mathcal{I}}}$, by definition $I \in \mathcal{F} = \mathcal{F}_{\mathcal{I}}$ and so I is an injective sub-object, and so a summand, of an element of $\mathcal{I}$, thus $I \in \mathcal{I}$. □

The above theorem together with Theorem 2.67 gives the following

Corollary 12.43. *Let $\mathfrak{C}$ be a Grothendieck category satisfying (Hyp.1). There are bijections*

$$\{\textit{Gen. closed subsets of } \mathrm{Sp}(\mathfrak{C})\} \rightleftarrows \mathrm{Inj}(\mathfrak{C}) \rightleftarrows \{\textit{Spec. closed subsets of } \mathrm{Sp}(\mathfrak{C})\}$$

$$G(\tau_{\mathcal{I}}) \longleftarrow\!\shortmid\; \mathcal{I} \longmapsto S(\tau_{\mathcal{I}})$$

12.3.3 Module categories

In this subsection we specialize our results about injective classes to categories of modules, re-obtaining as corollaries the main results of [20].

Definition 12.44. [27, 20] *Let R be a ring. A non-empty set $\mathcal{A}$ of left ideal of R is said to be a* torsion free set *(or,* saturated set*) if the following conditions hold:*

(NS.1) *$\mathcal{A}$ is closed under arbitrary intersections;*

(NS.2) *for all $x \in R$ and $I \in \mathcal{A}$, $(I : x) = \{r \in R : rx \in I\} \in \mathcal{A}$;*

(NS.3) *if a proper left ideal $J < R$ has the property that, for all $x \in R \setminus J$, there is $I \in \mathcal{A}$, such that $(J : x) \subseteq I$, then $J \in \mathcal{A}$.*

Let us recall the following fact from [27].

Lemma 12.45. [27, Corollary 2.3.14] *Let R be a ring. There is a bijective correspondence between* Tors(R-Mod) *and the family of torsion free sets of ideals of R.*

The following corollary, which is a consequence of Lemma 12.45 and Theorem 12.42, is a generalization of [20, Theorem 3.7].

Corollary 12.46. *Let R be a ring. There is a bijective correspondence between* Inj(R-Mod) *and the family of torsion free sets of ideals of R.*

The following corollary is a consequence of the above lemma and Corollary 12.43.

Corollary 12.47. [20, Corollary 3.9] *Let R be a commutative Noetherian ring. There is a bijective correspondence between* Inj(R-Mod) *and the family generalization closed sets of prime ideals of R.*

12.4 Model approximations for relative homological algebra

12.4.1 A relative injective model approximation

Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$. We extend the τ-quotient and the τ-section functors to categories of complexes applying them compont-wise:

$$\mathbf{Q}_\tau :\ \mathbf{Ch}(\mathfrak{C}) \rightleftarrows \mathbf{Ch}(\mathfrak{C}/\mathcal{T})\ : \mathbf{S}_\tau\,. \tag{12.4.1}$$

We use the same symbols for these new functors, it is an exercise to show that they are adjoint.

Definition 12.48. *Let $\mathfrak{C}$ be a Grothendieck category and let $\mathcal{I} \in \mathrm{Inj}(\mathfrak{C})$. A morphism $\phi^\bullet$ of cochain complexes is an* $\mathcal{I}$-quasi-isomorphism *provided* $\mathrm{Hom}_{\mathfrak{C}}(\phi^\bullet, K)$ *is a quasi-isomorphism of complexes of Abelian groups for all $K \in \mathcal{I}$.*
Given $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$, we define the following class of morphisms in $\mathbf{Ch}(\mathfrak{C})$

$$\mathcal{W}_\tau = \{\phi^\bullet : H^n(\mathrm{cone}(\phi^\bullet)) \in \mathcal{T}\,,\ \forall n \in \mathbb{Z}\}\,.$$

Recall that the mapping cone construction commutes with any additive functor (this can be easily verified by hand). This is used repeatedly in the following lemma.

Lemma 12.49. *Let $\mathfrak{C}$ be a Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ and denote by $\overline{\mathcal{W}}$ be the class of quasi-isomorphisms in $\mathbf{Ch}(\mathfrak{C}/\mathcal{T})$. The following are equivalent for a morphism $\phi^\bullet$ in $\mathbf{Ch}(\mathfrak{C})$:*

(1) *$\phi^\bullet \in \mathcal{W}_\tau$;*

(2) *$\mathbf{Q}_\tau(\phi^\bullet) \in \overline{\mathcal{W}}$;*

(3) *$\phi^\bullet$ is an $\mathcal{I}_\tau$-quasi-isomorphism.*

Furthermore, $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau)$ is a category with weak equivalences.

Proof. Since $\mathbf{Q}_\tau$ is exact, H^n and $\mathbf{Q}_\tau$ commute. Thus, for all $n \in \mathbb{N}$:

$$\mathbf{Q}_\tau(H^n(\mathrm{cone}(\phi^\bullet))) \cong H^n(\mathrm{cone}(\mathbf{Q}_\tau(\phi^\bullet)))\,.$$

This proves the equivalence between (1) and (2).
For the equivalence between (1) and (3), notice that $\mathrm{Hom}_{\mathfrak{C}}(\phi^\bullet, K)$ is a quasi-isomorphism for all $K \in \mathcal{I}_\tau$ if and only if, for all $n \in \mathbb{Z}$ and $K \in \mathcal{I}_\tau$,

$$0 = H^n(\mathrm{cone}(\mathrm{Hom}_{\mathfrak{C}}(\phi^\bullet, K))) \cong H^n(\mathrm{Hom}_{\mathfrak{C}}(\mathrm{cone}(\phi^\bullet), K)) \cong \mathrm{Hom}_{\mathfrak{C}}(H^n(\mathrm{cone}(\phi^\bullet)), K)\,.$$

Thus, $\mathrm{Hom}_{\mathfrak{C}}(\phi^\bullet, K)$ is a quasi-isomorphism for all $K \in \mathcal{I}_\tau$ if and only if $H^n(\mathrm{cone}(\phi^\bullet)) \in {}^\perp(\mathcal{I}_\tau) = \mathcal{T}$, for all $n \in \mathbb{Z}$. □

The following theorem answers part (1) of Question 0.2 in full generality:

Theorem 12.50. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$. Consider the category with weak equivalences $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau)$ and the injective model category $(\mathbf{Ch}(\mathfrak{C}/\mathcal{T}), \overline{\mathcal{W}}, \mathcal{B}, \mathcal{C})$ defined as in Example 12.3. Then, the adjunction*

$$\mathbf{Q}_\tau :\ (\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau) \rightleftarrows (\mathbf{Ch}(\mathfrak{C}/\mathcal{T}), \overline{\mathcal{W}}, \mathcal{B}, \mathcal{C})\ : \mathbf{S}_\tau$$

is a model approximation. Furthermore, the homotopy category relative to this model approximation is naturally isomorphic to the unbounded derived category $\mathbf{D}(\mathfrak{C}/\mathcal{T})$.

Proof. We have to verify conditions (MA.1)–(MA.4). Condition (MA.1) just states that $(\mathbf{Q}_\tau, \mathbf{S}_\tau)$ is an adjunction, while (MA.2), (MA.3) and (MA.4) are consequences of Lemma 12.49. The statement about the homotopy category follows by the explicit construction given in Proposition 5.5 of [21] and the fact that $\mathbf{Q}_\tau$ is essentially surjective. □

12.4.2 Approximations via towers of models

In this last subsection we try to approximate $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau)$ by a category of towers of models. Let us introduce the specific sequence of model categories we are interested in:

Lemma 12.51. *Let $\mathfrak{C}$ be a Grothendieck category, let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ and let $n \in \mathbb{N}_+$. There is a model category $(\mathbf{Ch}^{\geqslant -n}(\mathfrak{C}), \mathcal{W}_\tau^{\geqslant -n}, \mathcal{B}_\tau^{\geqslant -n}, \mathcal{C}_\tau^{\geqslant -n})$, where:*

- $\mathcal{W}_\tau^{\geqslant -n} = \{\phi^\bullet : \phi^\bullet$ *is a τ-quasi-isomorphism*$\}$;
- $\mathcal{B}_\tau^{\geqslant -n} = \{\phi^\bullet : \phi^i$ *is an epimorphism and* $\mathrm{Ker}(\phi^i) \in \mathcal{I}_\tau$ *for all* $i \geqslant -n\}$;
- $\mathcal{C}_\tau^{\geqslant -n} = \{\phi^\bullet : \phi^i$ *is a τ-monomorphism for all* $i \geqslant -n\}$.

For all $n \in \mathbb{Z}$, the above choice of weak equivalences, fibrations and cofibrations makes $(\mathbf{Ch}^{\geqslant -n}(\mathfrak{C}), \mathcal{W}_\tau^{\geqslant -n}, \mathcal{B}_\tau^{\geqslant -n}, \mathcal{C}_\tau^{\geqslant -n})$ into a model category.
Furthermore, there is an adjunction

$$l_{n+1} : \mathbf{Ch}^{\geqslant -n-1}(\mathfrak{C}) \rightleftarrows \mathbf{Ch}^{\geqslant -n}(\mathfrak{C}) : r_n \,, \tag{12.4.2}$$

where l_{n+1} is the obvious inclusion while r_n is the truncation functor. In this situation, r_n preserves fibrations and acyclic fibrations.

Proof. The proof can be obtained exactly as in the case when $\mathfrak{C}$ is a category of modules, see [18, Theorem 1.9]. □

Definition 12.52. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$. Consider the sequence $\mathbb{M}_\bullet = \{(\mathbf{Ch}^{\geqslant -n}(\mathfrak{C}), \mathcal{W}_\tau^{\geqslant -n}, \mathcal{B}_\tau^{\geqslant -n}, \mathcal{C}_\tau^{\geqslant -n}) : n \in \mathbb{N}\}$ defined in Lemma 12.51. We denote the category $(\mathrm{Tow}(\mathbb{M}_\bullet), \mathcal{W}_{\mathrm{Tow}}, \mathcal{B}_{\mathrm{Tow}}, \mathcal{C}_{\mathrm{Tow}})$ constructed as in Proposition 12.17 by $(\mathrm{Tow}_\tau(\mathfrak{C}), \mathcal{W}_{\mathrm{Tow}}, \mathcal{B}_{\mathrm{Tow}}, \mathcal{C}_{\mathrm{Tow}})$. Furthermore, we denote by $\mathrm{Tow} : \mathbf{Ch}(\mathfrak{C}) \to \mathrm{Tow}_\tau(\mathfrak{C})$ the so-called* tower functor*, which sends a complex $X^\bullet$ to the sequence of its successive truncations $\cdots \to X^{\geqslant -n} \to X^{\geqslant -n+1} \to \cdots \to X^{\geqslant 0}$ and acts on morphisms in the obvious way (see [19]). If $\tau = (0, \mathfrak{C})$ is the trivial torsion theory we denote $\mathrm{Tow}_\tau(\mathfrak{C})$ by $\mathrm{Tow}(\mathfrak{C})$.*

A typical object $X_\bullet^\bullet$ of $\mathrm{Tow}_\tau(\mathfrak{C})$ is a commutative diagram of the form

$$\begin{tikzcd}
\vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & X_2^{-2} \arrow[r, "d_2^{-2}"] \arrow[d] & X_2^{-1} \arrow[r, "d_2^{-1}"] \arrow[d, "t_2^{-1}"] & X_2^0 \arrow[r, "d_2^0"] \arrow[d, "t_2^0"] & X_2^1 \arrow[r, "d_2^1"] \arrow[d, "t_2^1"] & X_2^2 \arrow[r, "d_2^2"] \arrow[d, "t_2^2"] & \dots \\
& 0 \arrow[r] & X_1^{-1} \arrow[r, "d_1^{-1}"] \arrow[d] & X_1^0 \arrow[r, "d_1^0"] \arrow[d, "t_1^0"] & X_1^1 \arrow[r, "d_1^1"] \arrow[d, "t_1^1"] & X_1^2 \arrow[r, "d_1^2"] \arrow[d, "t_1^2"] & \dots \\
& & 0 \arrow[r] & X_0^0 \arrow[r, "d_0^0"] & X_0^1 \arrow[r, "d_0^1"] & X_0^2 \arrow[r, "d_0^2"] & \dots
\end{tikzcd}$$

where $(X_n^\bullet, d_n^\bullet)$ is a cochain complex for all $n \in \mathbb{N}$, and $X_n^m = 0$ for all $m < -n$.

12.4.3 Approximation of (Ab.4*)-k categories

Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$. The category $\mathrm{Tow}_\tau(\mathfrak{C})$ can be seen as a full subcategory of the category $\mathrm{Func}(\mathbb{N}, \mathbf{Ch}(\mathfrak{C}))$ of functors $\mathbb{N} \to \mathbf{Ch}(\mathfrak{C})$ and so we can restrict the usual limit functor to obtain a functor $\lim : \mathrm{Tow}_\tau(\mathfrak{C}) \to \mathbf{Ch}(\mathfrak{C})$.

In [19] and [18] the authors show that when $\mathfrak{C}$ is a category of modules over a commutative Noetherian ring of finite Krull dimension,

$$\mathrm{Tow} : (\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau) \rightleftarrows (\mathrm{Tow}_\tau(\mathfrak{C}), \mathcal{W}_{\mathrm{Tow}}, \mathcal{B}_{\mathrm{Tow}}, \mathcal{C}_{\mathrm{Tow}}) : \lim , \tag{12.4.3}$$

is a model approximation for all $\tau \in \mathrm{Tors}(\mathfrak{C})$. On the other hand, if the Krull dimension is not finite, one can always find counterexamples. In what follows we try to better understand this kind of construction when $\mathfrak{C}$ is a general Grothendieck category. First of all, notice that when we construct the homotopy category $\mathbf{D}(\mathfrak{C}/\mathcal{T})$ inverting the weak equivalences in $(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau)$ we are really doing two things at the same time:

(1) localize complexes over $\mathfrak{C}$ to complexes over $\mathfrak{C}/\mathcal{T}$;

(2) pass from a category of complexes over $\mathfrak{C}/\mathcal{T}$ to its derived category.

Our strategy is to separate the two operations in two different "steps", where each "step" corresponds to a pair of adjoint functors. The composition of these adjunctions is our candidate for a model approximation, as we will see in Theorem 12.54.

When $\mathfrak{C}$ is an (Ab.4*) Grothendieck category, let $X^\bullet \in \mathrm{Ob}(\mathbf{Ch}(\mathfrak{C}))$ and consider the sequence of truncations $\cdots \to X^{\geqslant -2} \to X^{\geqslant -1} \to X^{\geqslant 0}$. By Example 12.12 we can construct $\mathrm{holim}X^{\geqslant -i}$. One can prove as in [8, Application 2.4] that there is a quasi-isomorphism

$$X^\bullet \xrightarrow{\sim} \mathrm{holim}X^{\geqslant -i} . \tag{12.4.4}$$

This formula is useful as it allows to reduce many questions to half-bounded complexes. On the other hand, for (12.4.4) to hold, it is sufficient that the ambient category is (Ab.4*)-k for some finite k:

Theorem 12.53. [59, Theorem 1.3] *Let $\mathfrak{C}$ be a Grothendieck and assume that $\mathfrak{C}$ satisfies (Ab.4*)-k for some positive integer k. Then, for every $X^\bullet \in \mathrm{Ob}(\mathbf{Ch}(\mathfrak{C}))$, there is a quasi-isomorphism $X^\bullet \xrightarrow{\sim} \mathrm{holim}X^{\geqslant -i}$.*

We are now ready to prove our main result.

Theorem 12.54. *Let $\mathfrak{C}$ be a Grothendieck category and let $\tau = (\mathcal{T}, \mathcal{F}) \in \mathrm{Tors}(\mathfrak{C})$ be a torsion theory such that $\mathfrak{C}/\mathcal{T}$ is (Ab.4*)-k for some positive integer k. Then, the composition of the following adjunctions*

$$(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau) \underset{\mathbf{S}_\tau}{\overset{\mathbf{Q}_\tau}{\rightleftarrows}} (\mathbf{Ch}(\mathfrak{C}/\mathcal{T}), \overline{\mathcal{W}}) \underset{\lim}{\overset{\mathrm{Tow}}{\rightleftarrows}} (\mathrm{Tow}(\mathfrak{C}/\mathcal{T}), \mathcal{W}_{\mathrm{Tow}}, \mathcal{B}_{\mathrm{Tow}}, \mathcal{C}_{\mathrm{Tow}})$$

is a model approximation.

Proof. By Lemma 12.14, it is enough to show that $(\mathbf{Q}_\tau, \mathbf{S}_\tau)$ is compatible with weak equivalences (and this follows by Lemma 12.49) and that $(\mathrm{Tow}, \lim)$ is a model approximation. The proof that $(\mathrm{Tow}, \lim)$ is a model approximation is given in [19] for categories of modules. One can follow that proof almost without changes, using Theorem 12.53 (that applies here since $\mathfrak{C}/\mathcal{T}$ is (Ab.4*)-k) instead of Application 2.4 in [8]. □

Combining the above theorem with Theorem 12.28 we obtain the following corollary, which extends the main results of [18]:

Corollary 12.55. *Let $\mathfrak{C}$ be a Grothendieck category satisfying (Hyp.1), (Hyp.2) and (Hyp.3), and let $\tau \in \mathrm{Tors}(\mathfrak{C})$. If $\mathrm{G.dim}_\tau(\mathfrak{C}) < \infty$, then the composition*

$$(\mathbf{Ch}(\mathfrak{C}), \mathcal{W}_\tau) \underset{\mathbf{S}_\tau}{\overset{\mathbf{Q}_\tau}{\rightleftarrows}} (\mathbf{Ch}(\mathfrak{C}/\mathcal{T}), \overline{\mathcal{W}}) \underset{\lim}{\overset{\mathrm{Tow}}{\rightleftarrows}} (\mathrm{Tow}(\mathfrak{C}/\mathcal{T}), \mathcal{W}_{\mathrm{Tow}}, \mathcal{B}_{\mathrm{Tow}}, \mathcal{C}_{\mathrm{Tow}})$$

is a model approximation.

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List of Symbols

Index