Applications of Supersymmetry: Exact Results, Gauge/Gravity Duality and Condensed Matter

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A dissertation submitted to the University of Barcelona for the degree of Doctor of Philosophy

June 26, 2014

Acknowledgements

If there is something I have learned during these four years of doctorate, it is that research is very similar to a roller coaster, with some ups and downs. Sometimes everything works in a fluid and natural way and one obtains results, and other times you get stuck and difficulties are unavoidable. About these more difficult times, I would like to say that I have been tremendously fortunate to count on my advisor, Jorge G. Russo, I am very grateful to him for having placed his thorough knowledge and broad experience at my disposal. I have worked with you with mixed feelings of admiration, delight and gratitude.

I would also like to thank all the people I have collaborated with. It has been a pleasure to make my first contribution to science with Elisabetta Pallante. I want to thank Jérôme Gaillard, Niall T. Macpherson, Carlos Núñez and Daniel Thompson for a collaboration in which I learned so much. Specially, I am very grateful to Carlos Núñez for giving me the opportunity to make an stay at Swansea. I would also like to extend these thanks to all the people I met in Swansea for making such a pleasant stay.

I have to thank Diego Rodriguez, whose help with the subject to which the sixth chapter of this thesis is devoted has been invaluable. I have great memories from our discussions in Benasque, as well as those with Alfonso Ramallo, to whom I am grateful too.

I would like to express my gratitude to my professors during the Physics degree as well as during the master. Between these I have to mention Antonio Segui, Manuel Asorey, Luis J. Boya and Bartomeu Fiol, for introducing me in the world of research.

I owe much of my knowledge in Physics to these people, but I would not do any justice If I do not mention my mates during the doctorate: Adriana, Albert, Aldo, Blai, Dani, Francesco, Genís, Iván, Jacobo, Juan, Luis, Marina, Markus, Sergio and Xumeu.

I also have to acknowledge financial support from Ministry of Education, Culture and Sports FPU fellowship AP2009-3511, from the Department d'Estructura i Constituents de la Matèria and the Institut de Ciències del Cosmos, to whom all its staff I am also grateful.

Behind this thesis not only there is Physics, but also the dedication and the unconditional support of my loved ones, my parents, my brother and Leyre. Thank you for being with me even in the distance, this thesis is dedicated to you.

Agradecimientos

Si hay algo que he aprendido durante estos cuatro años de doctorado es que la invesitgación se parece mucho a una montaña rusa, con subidas y bajadas. Algunas veces todo sale de manera fluida y natural y se encuentran resultados, pero en otras ocasiones los atascos y las dificulatades son inevitables. Sobre estas últimas, me gustaría decir que he sido tremendamente afortunado por contar con mi director de tesis, Jorge G. Russo, le estoy muy agradecido por haber puesto sus amplios conocimientos y experiencia a mi disposición. Contigo he trabajado con una mezcla de admiración, placer y gratitud.

Me gustaría agradecer también a todas las personas con las que he colaborado. Ha sido un placer hacer mi primera contribución a la ciencia junto con Elisabetta Pallante. Quiero agradecer a Jérôme Gaillard, Niall T. Macpherson, Carlos Núñez y Daniel Thompson por una colaboración de la que he aprendido tanto. Especialmente, estoy muy agradecido a Carlos Núñez por darme la oportunidad de hacer una estancia en Swansea. También me gustaría hacer extensible estos agradecimientos a todas las personas que conocí en Swansea por hacer posible una estancia tan agradable.

Tengo que agradecer a Diego Rodriguez, cuya ayuda con el tema al cual el sexto capítulo de esta tesis está dedicado es inestimable. Tengo buenos recuerdos de las discusiones en Benasque, también con Alfonso Ramallo, a quién también estoy agradecido.

Me gustaría expresar mi gratitud a mis profesores de carrera y del master. Entre estos tengo que mencionar a Antonio Segui, Manuel Asorey, Luis J. Boya y Bartomeu Fiol, por introducirme en el mundo de la investigación.

Debo mucho de mi conocimiento en Física a estas personas, pero no haría justicia si no mencionara a mis compañeros de doctorado: Adriana, Albert, Aldo, Blai, Dani, Francesco, Genís, Iván, Jacobo, Juan, Luis, Marina, Markus, Sergio y Xumeu.

También tengo que agradecer la financiación por parte de la beca FPU AP2009-3511 del Ministerio de Educación, Cultura y Deportes, del Departament d'Estructura i Constituents de la Matèria y del Institut de Ciències del Cosmos, a cuya plantilla también estoy agradecido.

Detrás de esta tesis no sólo hay Física, también se encuentra la dedicación y el apoyo incondicional de mis seres queridos, mis padres, mi hermano y Leyre. Gracias por haber estado conmigo incluso en la distancia, esta tesis está dedicada a vosotros.

Contents

Chapter 1

Introduction

Historically, supersymmetry appeared from a way to circumvent the no-go theorem of Coleman and Mandula [1]. This theorem states that, under some assumptions, the most general Lie algebra of the S-matrix is that of the direct product of Poincaré and internal symmetry groups. One can relax the assumptions in which the Coleman-Mandula theorem is based, for example, we find conformal field theories whose symmetry group is an extension of the Poincaré group, but these theories are not described in terms of S-matrices and, hence, the Coleman-Mandula theorem does not apply as it relies in an S-matrix description. Haag, Sohnius and Lopuszanski [2] had already pointed out this, but more importantly, they generalized the Coleman-Mandula theorem by relaxing the statement of "Lie algebra", they considered the generalization of the notion of a Lie algebra to include commutators and anti-commutators to what it is known as a superalgebra or graded Lie algebra.

Then, apart from the usual Poincaré generators satisfying the usual commutation relations we also have additional complex Weyl spinor generators, Q^I_α ($\alpha = 1, 2$ and $I = 1, \ldots, \mathcal{N}$), and its conjugates, $(Q^I_\alpha)^\dagger = \bar{Q}^{I\dot{\alpha}}$, obeying the anticommutation relations

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta} Z^{IJ} \ , \qquad \{\bar{Q}^I_{\dot{\alpha}}, \bar{Q}^J_{\dot{\beta}}\} = \epsilon_{\dot{\alpha}\dot{\beta}} (Z^{IJ})^* \ . \tag{1.1}$$

The non-trivial extension of the Poincarè group comes from the anticommutator

$$\{Q^I_\alpha, \bar{Q}^J_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu \delta^{IJ} \ , \tag{1.2}$$

which generates translations. We are using the notation

$$\sigma^\mu_{\alpha\dot{\alpha}} = (\mathbb{I}_{\alpha\dot{\alpha}}, \sigma^i_{\alpha\dot{\alpha}}) \ , \qquad \bar{\sigma}^{\mu\dot{\alpha}\alpha} = (\mathbb{I}^{\dot{\alpha}\alpha}, -\sigma^{i\dot{\alpha}\alpha}) \tag{1.3}$$

where σ^i are the usual Pauli matrices. The supersymmetry algebra closes with the following commutators:

$$\begin{gathered} [M_{\mu\nu}, Q^I_\alpha] = i(\sigma_{\mu\nu})_\alpha{}^\beta Q^I_\beta \ , \qquad [M_{\mu\nu}, \bar{Q}^{I\dot{\alpha}}] = -i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}^{I\dot{\beta}} \ , \\ [P_\mu, Q^I_\alpha] = [P_\mu, \bar{Q}^J_{\dot{\alpha}}] = 0 \ . \end{gathered} \tag{1.4}$$

The $Z^{IJ} = -Z^{JI}$ in (1.1) are the central charges of the superalgebra, which appear in the case of extended supersymmetry $\mathcal{N} > 1$. $\sigma_{\mu\nu}$ are the spin 1/2 Lorentz generators given by

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu \bar{\sigma}^\nu - \sigma^\nu \bar{\sigma}^\mu\right) \ . \tag{1.5}$$

The upshot of the Haag-Sohnius-Lopuszanski theorem is that supersymmetry is the unique non-trivial way to extend the Poincaré group as a symmetry of the S-matrix. Therefore, supersymmetry has to be seriously considered as a possible new fundamental symmetry of nature. In addition, supersymmetry has been postulated as a candidate to address the following issues:

Naturalness: Let us compute the Standard Model leading contribution to the Higgs mass, which comes from its Yukawa coupling to the top quark,

$$\mathcal{L} = -\frac{y_t}{\sqrt{2}} H \bar{t}_L t_R + h.c. \tag{1.6}$$

the rest of the contributions does not alter the subsequent discussion. Then, the one-loop mass diagram, with top quarks running inside the loop, contributes to the Higgs mass by

$$\begin{aligned} \delta m^2\big|_{\text{top}} &= -iN_c \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \operatorname{Tr}\left[\frac{-iy_t}{\sqrt{2}} \frac{i}{\not{k} - m_t} \frac{-iy_t^*}{\sqrt{2}} \frac{i}{\not{k} - m_t}\right] \\ &= -N_c \frac{|y_t|^2}{8\pi^2} \int_0^{\Lambda^2} k_E^2 \mathrm{d}k_E^2 \frac{k_E^2 - m_t^2}{(k_E^2 + m_t^2)^2} \\ &= -N_c \frac{|y_t|^2}{8\pi^2} \left[\Lambda^2 - 3m_t^2 \log\left(\frac{\Lambda^2 + m_t^2}{m_t^2}\right) + \frac{2m_t^2 \Lambda^2}{\Lambda^2 + m_t^2}\right] \ , \end{aligned} \tag{1.7}$$

where, after going to Euclidean space, we have introduced a cut-off Λ that represents the energy scale where new physics is supposed to emerge, let us say the Planck scale $\sim 10^{19}$ GeV. Thus we see that the physical Higgs mass, $m_H^2 = m_0^2 + \delta m^2$, requires, due to the quadratic dependence in the cut-off, a severe fine-tuning between the bare term and one-loop corrections to get its actual value around 126 GeV. This is the naturalness problem. Then the only two possibilities are that the Standard Model is unnatural, or some new physics must enter at the order of the TeV scale in a particular way that renders the theory natural. This new physics cannot be an arbitrary particle that cancels out the one-loop quadratic divergence, since the cancellation does not hold all orders in perturbation theory. This new physics could be supersymmetry, since superpartner contributions enter in the precise way to cancel out these divergences.

Dark Matter: Supersymmetry provides some candidates for Dark Matter. Between many supersymmetric models, the most studied Dark Matter candidate is the lightest neutralino [3]. For example, let us consider the R-parity preserving MSSM (Minimal Supersymmetric Standard Model), whose field content is just that of the Standard Model supplemented with an extra Higgs doublet and the corresponding superpartners. Four neutralinos appear as a linear combination of the superpartners of the hypercharge gauge boson, B, the neutral component of the $SU(2)_L$ gauge triplet, W^3, and the neutral components of the two Higgs doublets. Depending on the particular model, the lightest of these neutralinos can be the lightest supersymmetric particle and hence, a candidate for Dark Matter, since unbroken R-parity forbids it to decay into Standard Model particles. Of course, there are other supersymmetric candidates for Dark Matter [4].

Gauge coupling unification: The combination of grand unified theories with supersymmetry works very nicely, since supersymmetry can achieve an accurate unification of the Standard Model gauge couplings, while grand unified theories based on Standard Model particle content do not manage a precise gauge coupling unification at any scale. For more details see, for instance, reference [5].

Supersymmetry might or might not be found at the LHC, even worse, it might not be realized as fundamental symmetry of nature at all. Nevertheless, supersymmetry still provides many valuable tools and this is the main topic of this thesis, to provide some novel examples of applications of supersymmetry beyond the construction of phenomenological models where it plays the role of a fundamental symmetry of nature. Along this thesis we will try to give examples of the following applications:

Dualities: Among various supersymmetric theories, several dualities arise that relate two of these theories. These dualities typically are of the form weak/strong coupling, which is a very remarkable property because it allows the study of a strongly coupled theory by doing perturbative computations in its weakly coupled dual theory. Although these theories might not be realized in nature, these dualities are a very remarkable feature that allow us to improve our knowledge of strongly coupled gauge theories in general.

Among these dualities we find, for example, the celebrated AdS/CFT or gauge/gravity duality [6]. Beyond the shadow of a doubt, this duality has revealed itself as a major source for the improvement of our understanding of quantum gravity and strongly coupled gauge theories. As originally stated, AdS/CFT conjectures an equivalence between type IIB superstring theory in $\mathrm{AdS}_5 \times S^5$ space-time and the conformal field theory $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills. This conjecture has been generalized to accommodate other gauge theories and other gravity theories in asymptotically anti de Sitter spaces.

Strictly speaking, the gauge/gravity duality is not a duality provided by supersymmetry in itself, but the theories involved in the duality heavily rely on supersymmetry and for sure it plays an important role as we will see.

Another remarkable duality is Seiberg duality [7]. As originally proposed, this is an infrared duality between super QCD-like theories with different gauge group ranks. This is again a duality between weakly and strongly coupled gauge theories and thus, it allows us to extract information about the strongly coupled regime.

Exact results: Supersymmetry plays an important role in this strong/weak coupling dualities, which are an extraordinary tool to understand the challenging strongly coupled regime of gauge theories. But, in some cases, supersymmetry also allows to compute some exact results in the gauge coupling! For this purpose we have localization techniques [8]. To apply localization to a certain theory it is essential the presence of a Grassmann-odd symmetry, for this reason, supersymmetric field theories seem to be a good candidate for the application of localization. Indeed, supersymmetric localization has turned out to be a very fruitful partnership and it has allowed to perform important test of the AdS/CFT conjecture [9].

Condensed Matter: Between the areas of condensed matter physics to which supersymmetry has something to contribute, we find random magnetic fields in Ising-like models, branched and linear polymers, electron localization in disordered media and so on [10]. Recently, supersymmetry has appeared in the context of superconductivity through the AdS/CFT correspondence [11]. It is in the field of superconductivity where we will give new applications of supersymmetry.

Although we will provide examples in these three areas in which supersymmetry appears as a relevant actor, we can find more applications to other areas like nuclear physics [12,13], optics [14] and others.

Organization of the thesis

This thesis is based on the results presented at:

[15] A. Barranco, E. Pallante and J. G. Russo,
$\mathcal{N}=1$ SQCD-like theories with N_f massive flavors from AdS/CFT and beta functions,
JHEP **1109** (2011) 086, [arXiv:1107.4002].

[16] A. Barranco and J. G. Russo,
Supersymmetric BCS,
JHEP **1206** (2012) 104, [arXiv:1204.4157].

[17] A. Barranco,
Supersymmetric BCS: Effects of an external magnetic field and spatial fluctuations of the gap,
JHEP **1307** (2013) 172, [arXiv:1301.0691].

[18] A. Barranco, J. Gaillard, N. T. Macpherson, C. Núñez and D. C. Thompson,
G-structures and Flavouring non-Abelian T-duality,
JHEP **1308** (2013) 018, [arXiv:1305.7229].

[19] A. Barranco and J. G. Russo,
Large N phase transitions in supersymmetric Chern-Simons theory with massive matter,
JHEP **1403** (2014) 012, [arXiv:1401.3672].

This thesis can be divided in three parts according to the three areas where we will use supersymmetry as mentioned above.

The first of these parts would correspond to chapter 2, devoted to the illustration of how we can obtain certain exact results in a supersymmetric theory by means of localization techniques. In particular, we will study the $\mathcal{N}=2$ super Chern-Simons theory with N_f flavors on a three-sphere, we will see how localization allows us to compute exactly its partition function in terms of a matrix integral and using this result, we will find that this theory presents two phase transitions in a certain decompactification limit [19].

The next three chapters, 3, 4 and 5, would make the second part of this thesis, where we will work with the gauge/gravity duality, with emphasis in the role played by supersymmetry.

Chapter 3 must be understood as a brief presentation of the AdS/CFT duality as originally proposed by Maldacena. We will also present the conditions that a supergravity solution must fulfill in order to preserve some supersymmetry. These conditions can be nicely expressed in terms of G-structures. This chapter is necessary to understand the following two chapters.

In chapter 4 we will try to extract some physics from a supergravity background claimed to be dual to $\mathcal{N}=1$ super QCD with N_f massive flavors and a quartic superpotential, according to the gauge/gravity correspondence. Departing from the Maldacena-Núñez supergravity background which, letting aside some subtleties, it is designed to be dual to pure $\mathcal{N}=1$ super Yang-Mills, we will give detailed information on how the addition of massless flavors in first place and massive flavors in second place is realized. We will pay special attention to how to extract the dual gauge theory physics provided by these backgrounds, in particular we will compute the β-functions of the dual field theories with focus on the emergence of fixed points in the RG flow [15]. We will find that Seiberg duality is realized in these backgrounds as well.

In chapter 5 we present the results obtained in [18]. We will compute the non-Abelian T-dual of the Klebanov-Witten supergravity background with and without flavors. The ideal objective would be to understand what is the effect of the application of a non-Abelian T-dual transformation in the dual gauge theory. However, we will pursue much modest goals, on the one hand, we will study the effects of non-Abelian T-duality on the G-structures of these supersymmetry preserving backgrounds. On the other hand, we will consider the application of non-Abelian T-duality for the construction of new flavoured solutions of supergravity. Contrary to chapter 4, in this chapter we will focus on the supersymmetry conditions satisfied by the generated backgrounds, rather than paying much attention to the physical properties.

Chapter 6 is the last application we are going to show and would correspond to the third part of this thesis. There we will review the construction of the first supersymmetric model of BCS superconductivity, as presented in [16], as well as some of its phenomenological consequences [17].

Finally, to comply with the rules of the University of Barcelona we end up with a summary also available in Spanish.

Chapter 2

Exact results in Chern-Simons theory

The study of strongly coupled gauge theories is a challenge for which we have very few tools. Among these tools we find localization techniques, which not only do they allow the exploration of strongly coupled regimes of field theories, but they also provide with exact results for interacting field theories.

For observables with a sufficient amount of supersymmetry, localization provides final expressions given in terms of a matrix integral. This integral is in general complicated, although much simpler and much more under control than the original functional integral. In the multicolor limit, the integral is dominated by a saddle-point and in some cases the saddle-point equations can be solved exactly.

Supersymmetric localization has led to the exact computation of the Euclidean partition function and vacuum expectation values of Wilson loop operators in many supersymmetric gauge theories in various dimensions. In the pioneering work by Pestun [8], the method of localization was used to obtain exact formulas for $\mathcal{N} = 2$ super Yang-Mills (SYM) theories on a four-sphere with arbitrary gauge group and matter content. Soon after the method was applied to the calculation of Euclidean path integrals in three-dimensional supersymmetric Chern-Simons theories on a three-sphere [20] and, since then, many other interesting examples have been worked out.

As we said, from supersymmetric localization we can obtain exact results in terms of the gauge coupling. Thus, it is very interesting to apply localization to gauge theories with known gravity dual. In this way, we can compare the results obtained from both the localization and the holographic approach in the strong coupling regime to test the gauge/gravity duality.

For example, using the previous ideas, the large N behavior of the free energy and Wilson loops in ABJM theory were determined [9]. ABJM [21, 22] is a three dimensional conformal field theory, built out of two copies of $U(N)$ Chern-Simons theory with opposite levels (couplings), k, and with bifundamental matter. The gravity dual of this theory is $\mathrm{AdS}_4 \times S^7/\mathbb{Z}_k$. Applying holographic methods to this $\mathrm{AdS}_4/\mathrm{CFT}_3$ correspondence, one finds that the free energy scales with $N^{3/2}$ in the strong 't Hooft coupling regime, whereas perturbation theory at weak coupling tells us that the free energy behaves as N^2. The localization computation in [9] perfectly interpolates between these two different behaviors, leading to a striking test of the AdS/CFT correspondence.

The large-N master field (i.e. the eigenvalue distribution that solves the corresponding matrix model) of several four-dimensional $\mathcal{N} = 2$ $U(N)$ super Yang-Mills theories has been also determined (for a recent review and references, see [23]). Among the different results that arise from this study, perhaps the most intriguing one is the emergence of large N quantum phase transitions [24, 25], which seem to be generic features of massive $\mathcal{N} = 2$ theories in the decompactification limit. This phenomenon was shown explicitly for $\mathcal{N} = 2^*$ SYM –obtained by the unique mass deformation of $\mathcal{N} = 4$ SYM preserving two supersymmetries– and $\mathcal{N} = 2$ SQCD with $2N_f$ flavors, with $N_f < N$. Large N phase transitions are familiar in gauge theories and they are due to singularities associated with the finite radius of convergence of planar perturbation theory [26, 27]. However, for the supersymmetric observables computed in [24,25], the physical origin of the phase transition appears to be different. When the coupling crosses a critical value, field configurations with extra massless multiplets contribute to the saddle-point, leading to discontinuities in vacuum expectation values of supersymmetric observables.

Similarly, one may expect that massive three-dimensional $\mathcal{N} = 2$ supersymmetric gauge theories on S^3 also exhibit interesting large N physics. In particular, one would like to know if Chern-Simons (CS) theories coupled to massive matter undergo quantum weak/strong coupling phase transitions. In this chapter, we will illustrate the power of supersymmetry to provide exact results for any value of the coupling. We will use localization results [20] and matrix model techniques [9, 28, 29] to address the previous question and study the large N limit of $U(N)$ $\mathcal{N} = 2$ super Chern-Simons theory with $2N_f$ massive flavors on the three-sphere (other studies of Chern-Simons matter theories at large N can be found in e.g. [30–32]).

We will start by reviewing the particular Chern-Simons matter theory in which we are interested. Next, we will review the localization technique and we will compute, as an example, how to obtain the matrix model from the path integral partition function of pure super Chern-Simons theory without matter. We will briefly outline how to build the matrix model in the more complicated case in which massive matter is included. Finally, we will solve the matrix model by saddle-point at large N and we will show that this theory exhibits phase transitions in a specific decompactification limit of the theory. For the first two sections we will mainly follow [20, 28]. We will use the conventions in [28] through all over this chapter and we collect them in the appendix.

2.1 Chern-Simons theory

Physics in $2+1$ dimensions has many interesting results different from those in the usual $3+1$ case. An example of these differences is the Chern-Simons theory. Consider the familiar Maxwell gauge theory in $3+1$ dimensions (or its non-Abelian version, Yang-Mills theory),

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - A_\mu J^\mu \ , \tag{2.1}$$

where $F_{\mu\nu}$ is the field strength and J is a conserved matter current. The immediate generalization of this theory to any space-time dimension, let us say d, is to allow the indices to run over $0, \ldots, d-1$. However, for $d = 3$ we find another gauge theory, namely

Chern-Simons theory. This theory complies with all the sensible requirements of being Lorentz invariant, gauge invariant and local. The Abelian CS Lagrangian is given by

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}A_\mu\partial_\nu A_\rho - A_\mu J^\mu \ . \tag{2.2}$$

Although this Lagrangian does not seem at first sight gauge invariant, because it is expressed in terms of the gauge field instead of the manifestly gauge invariant field strength, when computing how the previous Lagrangian changes under a gauge transformation ($A_\mu \to A_\mu + \partial_\mu\Lambda$) we find

$$\delta\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\partial_\mu\left(\Lambda(\epsilon^{\mu\nu\rho}\partial_\nu A_\rho - J^\mu)\right) \ , \tag{2.3}$$

i.e. the Lagrangian is gauge invariant up to a total derivative. Hence, if we can discard boundary terms, the action remains gauge invariant. However, this might not always be possible, but in the case we will be considering these boundary terms will not play any role.

One feature of CS theory worth remarking is that it is first-order in derivatives, this makes its canonical structure very different from that of Maxwell or Yang-Mills theory. Extensions of this theory to $3+1$ dimensions are not possible, because indices do not match up, although one can extend this theory to any odd space-time dimension.

It is possible to generalize the Lagrangian (2.2) to non-Abelian gauge symmetries. The non-Abelian CS theory is described by the Lagrangian

$$\mathcal{L}_{\mathrm{CS}} = \frac{k}{4\pi}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(A_\mu\partial_\nu A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right) \ . \tag{2.4}$$

Under a gauge transformation $A_\mu \to U^{-1}A_\mu U - iU^{-1}\partial_\mu U$, described by an element U of the gauge group, the variation of the previous Lagrangian is

$$\delta\mathcal{L}_{\mathrm{CS}} = i\frac{k}{4\pi}\epsilon^{\mu\nu\rho}\partial_\mu\,\mathrm{Tr}\left(\partial_\nu U U^{-1}A_\rho\right) + \frac{k}{12\pi}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right) \ , \tag{2.5}$$

where we identify a total derivative, as in the Abelian case, and an extra term. This new term is known as the winding number density of the group element U,

$$w(U) = \frac{1}{24\pi^2}\epsilon^{\mu\nu\rho}\,\mathrm{Tr}\left(U^{-1}\partial_\mu U U^{-1}\partial_\nu U U^{-1}\partial_\rho U\right) \ . \tag{2.6}$$

The integral of this expression, with appropriate boundary conditions, is an integer. Therefore, under a large gauge transformation, i.e. one with non trivial winding number n, the action changes by

$$\delta S_{\mathrm{CS}} = 2\pi k n \tag{2.7}$$

and then, to have a gauge invariant theory, we must require k to be quantized, $k \in \mathbb{Z}$.

2.1.1 $\mathcal{N} = 2$ supersymmetric theories in three dimensions

We are interested in coupling Chern-Simons theory to matter fields. To this purpose, it is convenient to have a description of super Chern-Simons theory in terms of off-shell supermultiplets. This was done in [33] for $\mathcal{N} = 1, 2, 4$, but we will only consider the $\mathcal{N} = 2$ case and how this theory couples to matter fields [34]. We will also show the $\mathcal{N} = 2$ super Yang-Mills action in three dimensions, since it will be necessary for the application of localization.

When considering the supersymmetric extension of Chern-Simons theory we have to deal with spinors in 3 dimensions. Eventually we will be interested in considering these theories on S^3, and so it is convenient to work on Euclidean space. The relevant rotation group is then $SO(3)$, or more precisely its universal cover $SU(2)$. As opposed to the standard Lorentz algebra in four dimensions, $su(2) \times su(2)$, in 3-dimensional Euclidean space the algebra is just $su(2)$ and then all spinors transform under the same representation. Thus, in three dimensions we do not distinguish between dotted and undotted spinor indices, as opposed to what we would do in four dimensions in order to refer to one $su(2)$ or the other.

As opposed to Minkowski space in three dimensions, where we can build Majorana spinors, in the Euclidean counterpart we cannot. Then, in Euclidean space, the spinorial generators will be Dirac spinors. For $\mathcal{N} = 2$ supersymmetry we have two independent spinor generators ϵ and $\bar{\epsilon}$. If we relate them by the usual hermitian conjugation, this would take us to the $\mathcal{N} = 1$ case, where we have the minimum amount of supersymmetry given by just a single complex spinor. In this sense, the $\mathcal{N} = 1$ supersymmetric algebra in 3-dimensional Euclidean space is more similar to the $\mathcal{N} = 2$ supersymmetric algebra in 3-dimensional Minkowski space than its $\mathcal{N} = 1$ counterpart.

We summarize the spinor conventions we use in appendix 2.A.

Before considering the aforementioned actions we first need to describe the basic building blocks to construct them, i.e. the $\mathcal{N} = 2$ supermultiplets in three dimensions. These are familiar from the point of view of the 4-dimensional $\mathcal{N} = 1$ supersymmetric algebra, since they can be obtained from dimensional reduction to three dimensions.

Vector hypermultiplet: The usual 4-dimensional $\mathcal{N} = 1$ vector multiplet contains a gauge field A_μ, a four-component Majorana spinor, χ, and a real auxiliary scalar field, D. When we dimensional reduce to three dimensions, the gauge field decomposes into a three-vector gague field, A_μ, and a real scalar field, σ, corresponding to the A_3 component in four dimensions. The 4-dimensional spinor splits into complex two-component spinors, λ and $\bar{\lambda}$, in three dimensions. In three dimensions we still keep the auxiliary field D. All these fields are valued in the Lie algebra of the gauge group, which we will take to be $U(N)$.

Under a supersymmetric transformation generated by the spinors ϵ and $\bar{\epsilon}$, these component fields transform in the following way:

$$
\begin{aligned}
\delta A_\mu &= \frac{i}{2}(\bar{\epsilon}\gamma_\mu\lambda - \bar{\lambda}\gamma_\mu\epsilon) \ , \\
\delta\sigma &= \frac{1}{2}(\bar{\epsilon}\lambda - \bar{\lambda}\epsilon) \ , \\
\delta\lambda &= \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^\mu D_\mu\sigma\right)\epsilon \ , \\
\delta\bar{\lambda} &= \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + D - i\gamma^\mu D_\mu\sigma\right)\bar{\epsilon} \ , \\
\delta D &= -\frac{i}{2}(\bar{\epsilon}\gamma^\mu D_\mu\lambda + D_\mu\bar{\lambda}\gamma^\mu\epsilon) + \frac{i}{2}(\bar{\epsilon}[\lambda,\sigma] + [\bar{\lambda},\sigma]\epsilon) \ .
\end{aligned}
\tag{2.8}
$$

where $D_\mu = \partial_\mu + i[A_\mu, \cdot]$ is the usual gauge covariant derivative and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i[A_\mu, A_\nu]$ is the field strength.

With the vector supermultiplet we can write the supersymmetric Chern-Simons action

$$
S_{\mathrm{CS}}^{\mathcal{N}=2} = -\frac{k}{4\pi}\int \mathrm{d}^3x \,\mathrm{Tr}\left[\epsilon^{\mu\nu\rho}\left(A_\mu\partial_\nu A_\rho + i\frac{2}{3}A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda + 2D\sigma\right] \ , \tag{2.9}
$$

where Tr is the trace in the fundamental representation. Of course, there is another supersymmetric, gauge invariant action we can write, the super Yang-Mills action,

$$
S_{\mathrm{YM}}^{\mathcal{N}=2} = \int \mathrm{d}^3x \,\mathrm{Tr}\left(\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}D^2 + \frac{i}{2}\bar{\lambda}\gamma^\mu D_\mu\lambda + \frac{i}{2}\bar{\lambda}[\sigma,\lambda]\right) \ . \tag{2.10}
$$

One can check that these actions are invariant under (2.8).

If we also want to couple vector superfields to matter we have to consider chiral multiplets.

Chiral hypermultiplets: The three dimensional $\mathcal{N} = 2$ chiral multiplet is made of a complex scalar field, ϕ, a two component complex spinor, ψ, and an auxiliary complex scalar, F. The supersymmetric transformations for these component fields are, when coupled to the vector supermultiplet:

$$
\begin{aligned}
\delta\phi &= \bar{\epsilon}\psi \ , & \delta\bar{\phi} &= \epsilon\bar{\psi} \ , \\
\delta\psi &= (i\gamma^\mu D_\mu\phi + i\sigma\phi)\epsilon + F\bar{\epsilon} \ , & \delta\bar{\psi} &= (i\gamma^\mu D_\mu\bar{\phi} + i\bar{\phi}\sigma)\bar{\epsilon} + \bar{F}\epsilon \ , \\
\delta F &= \epsilon(i\gamma^\mu D_\mu\psi - i\lambda\phi - i\sigma\psi) \ , & \delta\bar{F} &= \bar{\epsilon}(i\gamma^\mu D_\mu\bar{\psi} + i\bar{\phi}\bar{\lambda} - i\bar{\psi}\sigma) \ .
\end{aligned}
\tag{2.11}
$$

Hence, out of this multiplet, we can build the following supersymmetric action,

$$
\begin{aligned}
S_{\mathrm{matter}}^{\mathcal{N}=2} = \int \mathrm{d}^3x \Big(& D_\mu\bar{\phi}D^\mu\phi - i\bar{\psi}\gamma^\mu D_\mu\psi + \bar{F}F \\
& + \bar{\phi}\sigma^2\phi + i\bar{\phi}D\phi + i\bar{\psi}\sigma\psi + i\bar{\psi}\lambda\phi - i\bar{\phi}\bar{\lambda}\psi + W_F + \bar{W}_F\Big) \ ,
\end{aligned}
\tag{2.12}
$$

where, for example, by $\bar{\psi}\sigma\psi$ we mean $\bar{\psi}^a\sigma^\alpha(T_\alpha)_a^b\psi_b$ and α is an index over the Lie algebra, T_α are its generators, and the indices a and b are indices of the representation R in which the chiral multiplet is.

If we want to preserve scale invariance, the superpotential has to be quartic in the superfield [34], however, we will not consider any superpotential and thus we will take $W = 0$ in the previous action. In any case, conformal symmetry we will be broken when adding mass terms.

In terms of $\mathcal{N} = 2$ superspace, the matter action (2.12) with $W = 0$ can be written as

$$S_{\text{matter}} = \int \mathrm{d}^3x \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta}\, \bar{\Phi} e^{2V} \Phi \ , \tag{2.13}$$

where Φ is the chiral hypermultiplet and V is the vector hypermultiplet of the form described above. If we want to work with massive chiral multiplets [35], we can add mass terms by considering the coupling of the chiral multiplet to a background vector multiplet, V_m, just in the form shown in action (2.13), with V replaced by V_m. The conditions for the vanishing of the fermion variations of the background vector multiplet set $\sigma_m = m$, and the remaining background fields vanish. Then we just have to add the terms

$$+ \bar{\phi} m^2 \phi + i\bar{\psi} m \psi \tag{2.14}$$

to equation (2.12) to consider massive chiral superfields.

2.1.2 $\mathcal{N} = 2$ supersymmetric theories on the 3-sphere

To carry out localization we will work on a compact manifold $\mathcal{M}$, in particular a three sphere, S^3, with radius R. Working on a compact manifold provides an IR regulator for the theory, in this way some observables, like the free energy, are well defined.

The generalization of Chern-Simons theory to S^3 is straightforward,

$$S_{\text{CS}}^{S^3} = -\frac{k}{4\pi} \int \mathrm{d}^3x \epsilon^{\mu\nu\rho}\, \mathrm{Tr} \left[A_\mu \partial_\nu A_\rho + i\frac{2}{3} A_\mu A_\nu A_\rho \right] - \frac{k}{4\pi} \int \mathrm{d}^3x \sqrt{-\det g}\, \mathrm{Tr} \left[-\bar{\lambda}\lambda + 2D\sigma \right] \ . \tag{2.15}$$

The super Yang-Mills action and the matter action acquire new terms due to couplings of the scalars to the curvature. They are given by

$$\begin{aligned} S_{\text{YM}}^{S^3} = \int \mathrm{d}^3x \sqrt{-\det g}\, \mathrm{Tr} \Big[&\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{i}{2} \bar{\lambda} \gamma^\mu D_\mu \lambda \\ &+ \frac{i}{2} \bar{\lambda} [\sigma, \lambda] + \frac{1}{2} \left(D + \frac{\sigma}{R} \right)^2 - \frac{1}{4R} \bar{\lambda}\lambda \Big] \ , \end{aligned} \tag{2.16}$$

$$\begin{aligned} S_{\text{matter}}^{S^3} = \int \mathrm{d}^3x \sqrt{-\det g} \Big[&D_\mu \bar{\phi} D^\mu \phi - i\bar{\psi} \gamma^\mu D_\mu \psi + i\bar{\psi} \sigma \psi + i\bar{\psi} \lambda \phi \\ &- i\bar{\phi} \bar{\lambda} \psi + i\bar{\phi} D \phi + \bar{\phi} \sigma^2 \phi + \bar{F} F + \frac{3}{4} \frac{1}{R^2} \bar{\phi} \phi \Big] \ , \end{aligned} \tag{2.17}$$

where now the D_μ derivatives are covariant with respect to both the gauge field and the metric on S^3. We will use ∇_μ for covariant derivatives only with respect to the metric. We have also supposed canonical dimensions for fields in the matter supermultiplet. It can be checked that these actions are invariant under the supersymmetric transformations (2.8) and (2.11) adapted to the three sphere:

$$\begin{aligned}
\delta\lambda &= \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} - D + i\gamma^\mu D_\mu\sigma\right)\epsilon + i\frac{2}{3}\sigma\gamma^\mu\nabla_\mu\epsilon \ , \\
\delta\bar{\lambda} &= \left(-\frac{1}{2}\gamma^{\mu\nu}F_{\mu\nu} + D - i\gamma^\mu D_\mu\sigma\right)\bar{\epsilon} - i\frac{2}{3}\sigma\gamma^\mu\nabla_\mu\bar{\epsilon} \ , \\
\delta D &= -\frac{i}{2}(\bar{\epsilon}\gamma^\mu D_\mu\lambda + D_\mu\bar{\lambda}\gamma^\mu\epsilon) + \frac{i}{2}(\bar{\epsilon}[\lambda,\sigma] + [\bar{\lambda},\sigma]\epsilon) \\
&\quad - \frac{i}{6}\left(\nabla_\mu\bar{\epsilon}\gamma^\mu\lambda + \bar{\lambda}\gamma^\mu\nabla_\mu\epsilon\right) \ ,
\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}
\delta\psi &= (i\gamma^\mu D_\mu\phi + i\sigma\phi)\epsilon + F\bar{\epsilon} + \frac{i}{3}\gamma^\mu\nabla_\mu\epsilon\phi \ , \\
\delta\bar{\psi} &= (i\gamma^\mu D_\mu\bar{\phi} + i\bar{\phi}\sigma)\bar{\epsilon} + \bar{F}\epsilon + \frac{i}{3}\bar{\phi}\gamma^\mu\nabla_\mu\bar{\epsilon} \ ,
\end{aligned} \tag{2.19}$$

where we have only shown the transformations that receive corrections for being on the three sphere. We must stress again that these transformations are for fields with canonical dimensions, since for non-canonical ones, these transformations change. We also have to require ϵ and $\bar{\epsilon}$ to be Killing spinors to preserve supersymmetry, i.e. they must satisfy the condition

$$\nabla_\mu\epsilon = \gamma_\mu\epsilon' \tag{2.20}$$

for an arbitrary spinor ϵ' (we will see similar conditions in section 3.2). In our case, these are three equations, one of which determines ϵ' and the remaining two equations impose conditions on the Killing spinors. On the sphere the conformal Killing spinor equation can be written as

$$\nabla_\mu\epsilon = \pm\frac{i}{2}\gamma_\mu\epsilon \ , \tag{2.21}$$

each of which admits two solutions. In our case, we will take ϵ and $\bar{\epsilon}$ to be the Killing spinors solving the previous equation with positive sign.

If we split the δ-transformations (2.8) and (2.11) or their S^3 analog, (2.18) and (2.19), as $\delta = \delta_\epsilon + \delta_{\bar{\epsilon}}$ we can write the actions (2.16) and (2.17) in a δ-exact form,

$$\bar{\epsilon}\epsilon\mathcal{L}^{S^3}_{\text{YM}} = \delta_{\bar{\epsilon}}\delta_\epsilon \operatorname{Tr}\left(\frac{1}{2}\bar{\lambda} - 2D\sigma\right) \ , \tag{2.22}$$

$$\bar{\epsilon}\epsilon\mathcal{L}^{S^3}_{\text{matter}} = \delta_{\bar{\epsilon}}\delta_\epsilon\left(\bar{\psi}\psi - 2i\bar{\phi}\sigma\phi - \frac{1}{R}\bar{\phi}\phi\right) . \tag{2.23}$$

This fact will be important for localization.

Another possible way to obtain these theories on S^3 is the following [36]:

In first place, we have to find the 4-dimensional $\mathcal{N} = 1$ supersymmetric theory on $S^3 \times \mathbb{R}$. This is done starting with the corresponding Lagrangian in flat space and coupling it to off-shell supergravity in the old minimal formalism, where the gravity multiplet consists of: the metric; the gravitino, Ψ; and two auxiliary fields, M and b_μ, a complex scalar and a real vector respectively. The gravity multiplet enters as a background, so we do not consider its own action and all fermionic fields in the supergravity multiplet are turned to zero from the beginning.

Besides, the bosonic fields, either auxiliary or not, are set to arbitrary background values and do not need to satisfy the equations of motion. This method is used to obtain rigid supersymmetric theories on curved manifolds, even non-Lorentzian theories, in that case the auxiliary fields do not need to satisfy the reality conditions that one would impose in a Lorentzian theory and b_μ, M and $\bar{M}$ are considered to be independent complex functions (although at the end of the day, for the Lorentzian manifold $S^3 \times \mathbb{R}$, the auxiliary fields will satisfy the reality conditions). The only requirement is that the resulting Lagrangian respects some amount of supersymmetry of the original flat space theory.

The resulting Lagrangian depends on the original fields plus the metric and the auxiliary bosonic fields M, $\bar{M}$ and b_μ. For this Lagrangian to be supersymmetric, we must impose the supersymmetric variations of the gravitino to vanish, (more on this in section 3.2)

$$\delta\Psi(b_\mu, M, \zeta, \bar{\zeta}) = 0 \ , \qquad \delta\bar{\Psi}(b_\mu, \bar{M}, \zeta, \bar{\zeta}) = 0 \ . \tag{2.24}$$

where ζ and $\bar{\zeta}$ are Killing spinors. Since b_μ, M and $\bar{M}$ can be considered independent complex functions, we must impose both $\delta\Psi$ and $\delta\bar{\Psi}$ to vanish. Then, any solution with non-zero ζ, $\bar{\zeta}$ to equation (2.24) guarantees that the resulting Lagrangian preserves some supersymmetry.

In this way, one can build the $\mathcal{N} = 1$ supersymmetric matter action on $S^3 \times \mathbb{R}$. To obtain the theory on the three-sphere, one first rotates to Euclidean time the $\mathbb{R}$ direction and compactifies it to $S^3 \times S^1$ (now $b_4 = 3i/R$, with R the S^3 radius, and as anticipated the reality condition is not satisfied). The final step is to take the radius of S^1 to zero to end up with a three dimensional $\mathcal{N} = 2$ supersymmetric theory on S^3, as we were looking for.

2.2 Localization

Consider an action $S(\Phi)$, depending on a set of fields, Φ, invariant under the Grassmann-odd symmetry δ, i.e.

$$\delta S(\Phi) = 0 \ . \tag{2.25}$$

Then $\delta^2 = \delta_B$ is a Grassmann-even symmetry made of other possible symmetries of S. Since we are dealing with Lorentz and gauge invariant theories, this δ_B symmetry has to be made of combinations of gauge and Lorentz transformations.

Now consider the partition function corresponding to the previous action perturbed by a δ-exact term,

$$Z = \int \mathcal{D}\Phi \, e^{-S-t\delta V} \ , \tag{2.26}$$

where V is a fermionic functional, invariant under the symmetry δ_B. Then, it is easy to prove that the partition function is independent of the parameter t:

$$\frac{\mathrm{d}Z}{\mathrm{d}t} = -\int \mathcal{D}\Phi \, \delta V e^{-S-t\delta V} = -\delta \left(\int \mathcal{D}\Phi \, V e^{-S-t\delta V} \right) = 0 \ , \tag{2.27}$$

in the second equality we have integrated by parts and the missing term vanishes because of the properties $\delta S = \delta^2 V = 0$. We have also supposed that the δ-symmetry leaves the path integral measure invariant, that is, it must not suffer from anomalies. In the last equality we have used the fact that δ is a symmetry of the path integral, however, this result may not hold if the boundary term does not decay sufficiently fast in field configurations, but in most cases this does not happen.

The same derivation applies for the expectation value of an operator preserving the δ-symmetry, $\delta\mathcal{O} = 0$. Perturbing it with the term $-t\delta V$ we again find

$$\frac{d}{dt}\langle\mathcal{O}\rangle_t = \frac{d}{dt}\int \mathcal{D}\Phi\, \mathcal{O}e^{-S-t\delta V} = -\delta\left(\int \mathcal{D}\Phi\, \mathcal{O}V e^{-S-t\delta V}\right) = 0 \ . \tag{2.28}$$

If the partition function or the expectation value of the operator $\mathcal{O}$ do not depend on the parameter t, we can compute them for several values of t and all of them coincide with the original $t = 0$ integrals. Typically, one takes the limit $t \to \infty$, where simplifications usually occur. For instance, if δV has a positive definite bosonic part, $(\delta V)_B$, the partition function path integral "localizes" to a submanifold of field configurations with

$$(\delta V)_B = 0 \ . \tag{2.29}$$

And it turns out that, in many cases, this localized set of field configurations leads to a finite-dimensional integral.

As we see, the fact that S has to be invariant under a Grassman-odd symmetry makes supersymmetric theories the ideal context in which to apply these techniques. To illustrate the use of localization, we are going to compute in the next section the localized partition function of pure $\mathcal{N} = 2$ Chern-Simons theory on S^3.

2.2.1 Localization of Chern-Simons

Now we want to compute the partition function of pure $\mathcal{N} = 2$ super Chern-Simons theory with $U(N)$ gauge group on S^3 applying the localization methods described in the previous section.

We start by considering the CS action $S_{\text{CS}}^{S^3}$ given in (2.15) and we perturb it with the term $-tS_{\text{YM}}^{S^3}$ (2.16). We saw in (2.22) that the super Yang-Mills action is δ-exact, then it is appropriate as the perturbation term in localization. It also has positive definite bosonic part, therefore, in the $t \to \infty$ limit, bosonic fields must take classical values that ensure the vanishing of the bosonic part. Then, our theory localizes to

$$F_{\mu\nu} = 0 \ , \qquad D_\mu\sigma = 0 \ , \qquad D + \frac{\sigma}{R} = 0 \ . \tag{2.30}$$

The first of these equations implies that the gauge connection A_μ is flat and, since we are on a simply connected manifold, the only flat connection is $A_\mu = 0$. In consequence, we are left with the classical background

$$A_\mu = 0 \ , \qquad \sigma = \sigma_0 \ , \qquad D = -\frac{\sigma_0}{R} \ , \tag{2.31}$$

where σ_0 is constant. In the $t \to \infty$ limit, the partition function becomes, up to an overall constant,

$$Z = \int \mathrm{d}\sigma_0\, Z_{\text{1-loop}}[\sigma_0] e^{iS_{\text{cl}}[\sigma_0]} \ , \tag{2.32}$$

with a classical contribution coming from the CS action evaluated at (2.31),

$$S_{\text{cl}} = \frac{k}{2\pi R}\text{vol}(S^3)\,\text{Tr}(\sigma_0^2) \tag{2.33}$$

and a one-loop contribution coming from fluctuations of the fields that appears in the Yang-Mills action taking values around the locus (2.31),

$$\begin{aligned}\sigma &\to \sigma_0 + \frac{1}{\sqrt{t}}\sigma\ , \\ D &\to -\frac{\sigma_0}{R} + \frac{1}{\sqrt{t}}D\ , \\ \Phi &\to \frac{1}{\sqrt{t}}\Phi\ ,\end{aligned} \tag{2.34}$$

here Φ stands for any other field different than σ and D. The $1/\sqrt{t}$ factors have been introduced for convenience, these remove the overall t factor of the perturbation term and the $t \to \infty$ limit only allows us to maintain quadratic fluctuations of the fields.

To compute the one-loop contribution one has to perform a gauge-fixing. To be precise, we should have carried out the localization procedure on the gauge-fixed action. However, as shown in [8, 20], one can redefine the Grassmann odd symmetry employed in the localization computation by considering the BRST symmetry

$$\delta \to \delta_{\text{SUSY}} + \delta_{\text{BRST}}\ . \tag{2.35}$$

Then, instead of using the Yang-Mills action alone, one perturb the gauge-fixed CS action with an additional term $-t\delta(\bar{c}(\frac{1}{2}\xi b - \nabla^\mu A_\mu))$, which produces the usual gauge fixing terms

$$\partial_\mu \bar{c} D^\mu c - \frac{1}{2}\xi b^2\ + b\nabla^\mu A_\mu\ , \tag{2.36}$$

where t has been absorbed with the appropriate rescaling, just as we did in (2.34). b is the Lautrup-Nakanishi auxiliary field. Integrating out the b-field leads to the usual R_ξ-gauge fixing terms. We will work in the Landau gauge $\xi = 0$, then we can consider b as a Lagrange multiplier imposing Lorenz gauge condition.

After performing the expansion to quadratic order in the fields, the Yang-Mills action with gauge fixing terms becomes

$$\begin{aligned}S_{\text{YM}} = \frac{1}{2}\int \mathrm{d}^3x\sqrt{-\det g}\,\text{Tr}\Big(&- A^\mu \Delta A_\mu - [A_\mu, \sigma_0]^2 + \partial_\mu\sigma\partial^\mu\sigma + \Big(D + \frac{\sigma}{R}\Big)^2 \\ &+ i\bar{\lambda}\gamma^\mu\nabla_\mu\lambda + i\bar{\lambda}[\sigma_0, \lambda] - \frac{1}{2R}\bar{\lambda}\lambda + \partial_\mu\bar{c}\partial^\mu c + b\nabla^\mu A_\mu\Big)\ ,\end{aligned} \tag{2.37}$$

where Δ is the Laplacian and we have replaced the covariant derivative of c by an ordinary one once we neglect suppressed terms in t.

Now, we want to perform the path integral of the partition function corresponding to this action. First consider the term $(D + \sigma/r)^2$, σ can be absorbed after a redefinition of D and integration of the later just eliminates the $(D + \sigma/r)^2$ term.

Next we split the gauge field into a divergence and a divergenceless part,

$$A_\mu = \partial_\mu \phi + B_\mu \ , \qquad \text{with} \qquad \nabla^\mu B_\mu = 0 \ . \tag{2.38}$$

Integration over b imposes the constrain $\nabla^2\phi = 0$ and the integration of the fields ϕ, c and σ generate some determinants of ∇^2 on the sphere, each one to the appropriate power according to the field statistics. At the end of the day, these determinants cancel each other out. In any case, the integration over these fields produces contributions to the partition function independent of σ_0. At this point we are left with

$$Z = \int \mathrm{d}\sigma_0 \, e^{i\pi k R^2 \, \mathrm{Tr}\, \sigma_0^2} \int \mathcal{D}B \mathcal{D}\lambda D\bar{\lambda} \, e^{\int \mathrm{d}^3x \sqrt{-\det g}\, \mathcal{L}} \ , \tag{2.39}$$

where $\mathcal{L}$ is an abreviation for

$$\mathcal{L} = \frac{1}{2} \mathrm{Tr} \left(- B^\mu \Delta B_\mu - [B_\mu, \sigma_0]^2 + i\bar{\lambda}\gamma^\mu \nabla_\mu \lambda + i\bar{\lambda}[\sigma_0, \lambda] - \frac{1}{2R}\bar{\lambda}\lambda \right) \ . \tag{2.40}$$

The integrand in equation (2.32) is gauge invariant, then we can use this to choose σ_0 to be in the Cartan subalgebra of the Lie group. This simplifies the integration, since we end up with ordinary fields as opposed to matrix valued fields. However, we have to introduce the Vandermonde determinant that accounts for this change. It is computed in appendix 2.B, it gives:

$$\prod_{i<j}^{N} (\lambda_i - \lambda_j)^2 \ , \tag{2.41}$$

where λ_i are are the eigenvalues of σ_0. In general, for a gauge group G, the Vandermonde determinant becomes a product over the roots of G,

$$\prod_\alpha \alpha(\sigma_0) \ . \tag{2.42}$$

For the gauge group $U(N)$ the roots are given by $\lambda_i - \lambda_j$ and we recover expression (2.41). There is also a residual gauge symmetry given by the Weyl group, so we should divide the partition function by the order of the gauge group, but as it is a factor independent of σ_0 we will ignore it.

If now we apply the Cartan decomposition to B_μ (and in the same way for λ),

$$B_\mu = \sum_\alpha B_\mu^\alpha E_\alpha + H_\mu \ , \tag{2.43}$$

i.e. we express it as a sum of elements in the Cartan subalgebra, H_μ, and "ladder operators", E_α. These elements satisfy

$$[H, H'] = 0 \ , \qquad [H, E_\alpha] = \alpha(H) E_\alpha \ , \qquad [E_\alpha, E_\beta] = \begin{cases} N_{\alpha\beta} E_{\alpha+\beta} & \text{if } \alpha+\beta \text{ is a root} \\ H_\alpha & \text{if } \alpha+\beta = 0 \\ 0 & \text{otherwise} \end{cases}$$
$$\mathrm{Tr}[E_\alpha E_\beta] = \delta_{\alpha+\beta,0} \ , \tag{2.44}$$

where we have chosen the standard normalization for the last identity. Then as σ_0 is in the Cartan subalgebra and using these relations we are left with (2.39), where now $\mathcal{L}$ is

$$\mathcal{L} = \frac{1}{2}\sum_{\alpha}(B^{\mu}_{-\alpha}(-\Delta + \alpha(\sigma_0)^2)B_{\alpha} + \bar{\lambda}_{-\alpha}(i\gamma^{\mu}\nabla_{\mu} + i\alpha(\sigma_0) - \frac{1}{2R})\lambda_{\alpha}) \ . \tag{2.45}$$

After carrying out the remaining path integrals we are left with the determinant of the vector Laplacian on S^3. The eigenvalues and degeneracy of the vector Laplacian acting on the divergenceless part of a vector field are:

$$\text{eigenvalues:} \quad -\frac{1}{R^2}(l+1)^2 \ , \qquad \text{degeneracy: } 2l(l+2) \ ; \tag{2.46}$$

while those of the Dirac operator $i\gamma^{\mu}\nabla_{\mu}$ acting on fermionic modes are:

$$\text{eigenvalues:} \quad \pm\frac{1}{R}\left(l+\frac{1}{2}\right) \ , \qquad \text{degeneracy: } l(l+1) \ , \tag{2.47}$$

with $l = 1, 2, \ldots$ in both cases. Then we can write the one loop contribution as

$$Z_{\text{1-loop}} = \prod_{\alpha}\prod_{l=1}^{\infty}\frac{\det(\text{fermions})}{\sqrt{\det(\text{bosons})}} = \prod_{\alpha}\prod_{l=1}^{\infty}\frac{((l+i\alpha(\sigma_0))(-l-1+i\alpha(\sigma_0)))^{l(l+1)}}{((l+1)^2+\alpha(\sigma_0)^2)^{l(l+2)}} \ , \tag{2.48}$$

where we are setting $R = 1$, which can be easily recovered. After cancelations between factors in the numerator and in the denominator and using the fact that roots come in positive-negative pairs we get

$$Z_{\text{1-loop}} = \left(\prod_{l=1}^{\infty} l^4\right)\prod_{\alpha>0}\prod_{l=1}^{\infty}\left(1+\frac{\alpha(\sigma_0)^2}{l^2}\right)^2 \ . \tag{2.49}$$

The infinite product appearing in first place can be regularized by means of zeta function regularization, which gives 2 as a result. The remaining product in l is the infinite product representation of the sinh function. Therefore, we have

$$Z_{\text{1-loop}} = \prod_{\alpha>0}\left(\frac{2\sinh(\pi\alpha(\sigma_0))}{\pi\alpha(\sigma_0)}\right)^2 \ , \tag{2.50}$$

up to a σ_0-independent factor. Putting the classical and 1-loop contributions together, we find that the Vandermonde determinant cancels out the denominator of the previous formula, thus we are left with

$$Z = \int \mathrm{d}\mu \prod_{i<j} 4\sinh^2\frac{\mu_i-\mu_j}{2} e^{-\frac{1}{2g}\sum_i \mu_i^2} \ , \tag{2.51}$$

where we have already used that the Lie algebra of $U(N)$ has roots $\lambda_i - \lambda_j$. We have also rescaled the eigenvalues of σ_0,

$$\lambda = \frac{\mu}{2\pi} \tag{2.52}$$

and redefined the coupling constant,

$$g \equiv \frac{2\pi i}{k} \ . \tag{2.53}$$

At this point we have seen how to obtain the matrix model partition function of pure Chern-Simons theory, however, we are interested in Chern-Simons theories coupled to matter. To obtain the corresponding matrix model one has to proceed in an analogous way to the one we have followed in this section. When considering the addition of matter, we can take advantage of the fact that the matter Lagrangian can be written as a δ-exact term, (2.23), so we can perturb the action in the partition function with both $-tS_{\mathrm{YM}}$ and the additional term $-tS_{\mathrm{matter}}$ and proceed in a similar way to what we did in the absence of matter. Finally one obtains the following recipe to build the matrix model:

- The classical values of the fields are those shown in (2.31), while any other is set to zero.
- For each vector multiplet there is a factor

$$Z_{1\text{-loop}}^{\text{vector}} = \prod_{\alpha>0} \left(\frac{2\sinh(\pi\alpha(\sigma_0))}{\pi\alpha(\sigma_0)} \right)^2 . \tag{2.54}$$

  This is the contribution we have just computed.

- For every chiral hypermultiplet in a representation R, with mass given by a background vector multiplet as explained in equation (2.13), we have the contribution

$$Z_{1\text{-loop}}^{\text{chiral}} = \prod_{\rho} \frac{1}{2\cosh(\pi\rho(\sigma_0+m))} , \tag{2.55}$$

  where now the product is over the weights of the representation R. For example, in the fundamental representation of $U(N)$ we have

$$Z_{1\text{-loop}}^{\text{chiral}} = \prod_{i=1}^{N} \frac{1}{2\cosh(\pi(\lambda_i - m))} . \tag{2.56}$$

- Add the Vandermonde determinant,

$$\prod_{\alpha>0} \alpha(\sigma_0)^2 , \tag{2.57}$$

  to account for the gauge fixing of the matrix model to the Cartan subalgebra.

- Divide by the order of the Weyl group, this accounts for the remaining residual gauge symmetry after the gauge fixing to the Cartan subalgebra.

2.2.2 Wilson-Loop

Another observable that can be computed with localization is the vacuum expectation value (vev) of the 1/2 supersymmetric circular Wilson loop of a big circle of S^3 [20, 37].

Let us consider the following operator

$$W(C) = \left\langle \frac{1}{N} \mathrm{Tr}\, \mathcal{P} \exp\left(\oint_C \mathrm{d}\tau \, (iA_\mu \dot{x}^\mu + \sigma|\dot{x}|) \right) \right\rangle , \tag{2.58}$$

where the contour C is, for the moment, an arbitrary closed path on S^3. As we have seen, to apply localization we need some supersymmetry. Therefore, the variation of the operator (2.58) under the supersymmetric transformation (2.8),

$$\delta W(C) \propto \bar{\epsilon}(-\gamma_\mu \dot{x}^\mu + |\dot{x}|)\lambda + \bar{\lambda}(\gamma_\mu \dot{x}^\mu - |\dot{x}|)\epsilon \ , \tag{2.59}$$

must vanish, i.e.

$$\bar{\epsilon}(-\gamma_\mu \dot{x}^\mu + |\dot{x}|) = 0 \ , \tag{2.60}$$

$$(\gamma_\mu \dot{x}^\mu - |\dot{x}|)\epsilon = 0 \ , \tag{2.61}$$

for at least one non-trivial Killing spinor. We saw in (2.21) that the conformal Killing spinor equation on the three sphere is

$$\nabla_\mu \epsilon = +\frac{i}{2}\gamma_\mu \epsilon \ , \tag{2.62}$$

which admits two solutions. There is another conformal Killing spinor equation with the opposite sign which admits two more solutions but it is sufficient to consider the equation (2.62). If the Killing spinor ϵ satisfies equation (2.61), or what it is the same

$$\gamma_\mu \dot{x}^\mu \epsilon = \epsilon \ , \tag{2.63}$$

if we take τ to be the arc length. This condition can be only fulfilled for constant $\gamma_\mu \dot{x}^\mu$. This means that $\dot{x}^\mu$ can be expressed as a linear combination of the orthonormal frame on the three-sphere, which we can take, for example, the $SU(2)$ left invariant forms. Then we can choose our loop to be parallel to one vector of the frame, this means that the loop C is a big circle of S^3. Then we are left with the condition

$$\gamma_\mu \epsilon = \epsilon \ , \tag{2.64}$$

for a given μ. This condition is fulfilled by one of the solutions to the Killing equation (2.62), while we must take the remaining spinor equal to zero. As only half of the original Killing spinors take non-trivial values, this Wilson loop is 1/2 BPS.

Then the computation we did for the partition function is minimally changed for the computation of the expectation value of this Wilson loop. It does not change the localizing locus (2.31), neither it changes the computation of the one-loop contribution. Its difference comes from the classical contribution.

Then the vev of the Wilson loop localizes to a matrix integral obtained by replacing the fields by their classical values. For a $U(N)$ fundamental Wilson loop this amounts to insert the piece

$$\sum_i e^{\mu_i} \tag{2.65}$$

in the partition function.

2.3 $U(N)$ Chern-Simons with $2N_f$ massive flavors

Let us consider the $\mathcal{N}=2$ supersymmetric Chern-Simons theory with gauge group $U(N)$ on S^3 and level k, with a matter content given by $2N_f$ chiral multiplets of mass $m/2\pi$ (N_f fundamentals and N_f antifundamentals). For $m=0$, the theory is superconformal for any N_f [34,37]. The mass deformation for the chiral multiplets explicitly breaks classical scale invariance and hence conformal invariance. Applying the localization recipe given at the end of section 2.2.1, one finds that the partition function localizes to

$$Z_{N_f}^{U(N)} = \int \frac{\mathrm{d}^N\mu}{(2\pi)^N} \frac{\prod_{i<j} 4\sinh^2(\frac{1}{2}(\mu_i-\mu_j))\; e^{-\frac{1}{2g}\sum_i \mu_i^2}}{\prod_i \Big(4\cosh(\frac{1}{2}(\mu_i+m))\cosh(\frac{1}{2}(\mu_i-m))\Big)^{N_f}} \,, \tag{2.66}$$

where remember that

$$g = \frac{2\pi i}{k} \,. \tag{2.67}$$

The scalar field σ has mass dimensions, therefore, in (2.66) both μ and m scale with the radius of the three-sphere, R. The radius has been set to one for notational convenience. The dependence on the radius will be restored when considering the decompactification limit. Calculations will be performed for a real parameter $g>0$, which ensures the convergence of the integral. The dependence on k can be recovered in the final expressions for the supersymmetric observables by analytic continuation.

In the infinite N limit, the partition function can be determined by a saddle-point calculation. Here we will consider the Veneziano limit, where the 't Hooft coupling,

$$t \equiv gN \,, \tag{2.68}$$

and the Veneziano parameter,

$$\zeta \equiv \frac{N_f}{N} \,, \tag{2.69}$$

are kept fixed as $N\to\infty$. It is useful to define the potential as

$$V(\mu_i) = \sum_{i=1}^{N}\left(\frac{\mu_i^2}{2} + gN_f \log\left[2\cosh\frac{\mu_i+m}{2}\right] + gN_f\log\left[2\cosh\frac{\mu_i-m}{2}\right]\right) \,. \tag{2.70}$$

The saddle-point equations are then

$$\frac{1}{N}\sum_{j\neq i}\coth\frac{\mu_i-\mu_j}{2} = \frac{1}{t}V'(\mu_i) = \frac{\mu_i}{t} + \frac{\zeta}{2}\tanh\frac{\mu_i+m}{2} + \frac{\zeta}{2}\tanh\frac{\mu_i-m}{2} \,. \tag{2.71}$$

Introducing as usual the eigenvalue density

$$\rho(\mu) = \frac{1}{N}\sum_{i=1}^{N}\delta(\mu-\mu_i) \,, \tag{2.72}$$

the saddle-point equation (2.71) is converted into a singular integral equation:

$$⨍ \mathrm{d}\nu\, \rho(\nu)\coth\frac{\mu-\nu}{2} = \frac{\mu}{t} + \frac{\zeta}{2}\tanh\frac{\mu+m}{2} + \frac{\zeta}{2}\tanh\frac{\mu-m}{2} \,, \tag{2.73}$$

where the integral is defined by the principal value prescription. This matrix model can be solved exactly. The solution is explicitly constructed in section 2.5. For clarity, we will first discuss the solution directly in the decompactification limit, where, as we will see, the model exhibits the presence of quantum phase transitions.

Another observable in which we are interested is the vacuum expectation value of the Wilson loop explained in section 2.2.2, which localizes to the matrix integral,

$$W(C) = \left\langle \frac{1}{N} \sum_i e^{\mu_i} \right\rangle \ . \tag{2.74}$$

In the large N limit, this vacuum expectation value is just given by the average computed with the density function (2.72),

$$W(C) = \int \mathrm{d}\mu \ \rho(\mu) \ e^{\mu} \ . \tag{2.75}$$

2.4 Large N solution in the decompactification limit

Consider the integral equation (2.73). The term $\coth(\frac{1}{2}(\mu-\nu))$ represents a repulsive force among eigenvalues. For $t > 0$, the term μ/t is an harmonic force pushing the eigenvalues towards the origin. The last two terms, proportional to $\tanh(\frac{1}{2}(\mu \pm m))$, are forces pushing the eigenvalues towards $\mp m$, respectively.

If $t \gg 1$, the harmonic force is negligible. If, in addition, $m \gg 1$, then the potential is flat until $\mu = \mathcal{O}(m)$. As a result, the eigenvalues scale with m. Restoring the dependence on the radius R of S^3, we can make this limit precise introducing the coupling $\lambda \equiv t/mR$ and taking the decompactification limit at fixed λ, i.e.

$$\begin{aligned} &m \to mR \ , \qquad \mu \to \mu R \ , \qquad \text{with} \qquad R \to \infty \\ &t \equiv gN \to \infty \ , \qquad \lambda \equiv \frac{t}{mR} = \text{fixed} \ . \end{aligned} \tag{2.76}$$

It is worth stressing that t is dimensionless and a priori there is no reason why it should be scaled with mR. However, if the decompactification limit is taken at fixed $t \ll mR$, then its only effect is to decouple the matter fields, as this is equivalent to sending the masses to infinity. This may be compared with four-dimensional $\mathcal{N} = 2$ SYM theory coupled to massive matter, e.g. $\mathcal{N} = 2^*$ SYM or $\mathcal{N} = 2$ SCFT* which can be viewed as a UV regularization of pure $\mathcal{N} = 2$ SYM theory [25]. In that case, the limit of masses $M \to \infty$ at fixed 't Hooft coupling λ does not decouple the massive fields. In order to decouple the massive fields one needs to take at the same time $\lambda \to 0$ with fixed $MR\, e^{\frac{1}{\beta_0 \lambda}}$, where $\beta_0 < 0$ is the one-loop β function coefficient in $\beta_\lambda = \beta_0 \lambda^2$. In other words, $\lambda \to 0$ is required to renormalize a one-loop divergence, viewing M as UV cutoff. In Chern-Simons-matter theory, the 't Hooft coupling does not renormalize because it is proportional to a rational number, N/k. Thus, in the limit $mR \to \infty$ with fixed t, matter fields are decoupled and the theory just flows to pure $\mathcal{N} = 2$ Chern-Simons theory. In what follows we will refer to "decompactification limit" to the specific limit (2.76) where the most interesting physics arises. We will shortly see that this limit defines a regular limit of the theory.

We shall assume a one-cut solution where $\rho(\mu)$ is supported in an interval $\mu \in [-A, A]$, with unit normalization,

$$\int_{-A}^{A} \rho(\mu)\mathrm{d}\mu = 1 \ . \tag{2.77}$$

In the limit (2.76), the large N saddle-point equation simplifies to

$$\int_{-A}^{A} \mathrm{d}\nu\, \rho(\nu)\mathrm{sign}(\mu-\nu) = \frac{\mu}{m\lambda} + \frac{\zeta}{2}\left(\mathrm{sign}(\mu+m) + \mathrm{sign}(\mu-m)\right) \ , \tag{2.78}$$

where the dependence on R has completely canceled out and μ, m and λ can now take arbitrary values.

The solutions to (2.78) are different according to the value of the coupling λ. Consider first the case $0 < \zeta < 1$. This gives rise to three phases.

Phase I: $\lambda < 1$

This phase arises when $A < m$, implying that $|\mu| < m$ for any μ. Under these conditions, the sign functions on the right hand side of equation (2.78) cancel out. Flavors do not play any role and we find a uniform eigenvalue density:

$$\rho_{\mathrm{I}}(\mu) = \frac{1}{2m\lambda} \ , \tag{2.79}$$

supported in the interval $\mu \in [-m\lambda, m\lambda]$.

Phase II: $1 < \lambda < (1-\zeta)^{-1}$

In this interval of the coupling the eigenvalue density takes the form

$$\rho_{\mathrm{II}}(\mu) = \frac{1}{2m\lambda} + \frac{1}{2\lambda}(\lambda-1)\left(\delta(\mu+m) + \delta(\mu-m)\right) \ , \qquad \mu \in [-m, m] \ , \tag{2.80}$$

with $A = m$. The coefficients of the Dirac-δ functions are implied by the normalization condition (2.77), once $A = m$ is assumed. A further justification of this solution requires a regularization, which is provided automatically by the finite R exact solution presented below. We shall return to this solution in section 2.5.

Phase III: $\lambda > (1-\zeta)^{-1}$

In this case the saddle-point equation is solved by the eigenvalue density

$$\rho_{\mathrm{III}}(\mu) = \frac{1}{2m\lambda} + \frac{\zeta}{2}\left(\delta(\mu+m) + \delta(\mu-m)\right) \ , \qquad \mu \in [-m\lambda(1-\zeta), m\lambda(1-\zeta)] \ . \tag{2.81}$$

This is the solution that one would obtain by formal differentiation of (2.78) with respect to μ. In order for the δ functions to contribute to the integral in (2.78), we must require $A > m$, i.e. $\lambda > (1-\zeta)^{-1}$.

The above three solutions ρ_{I}, ρ_{II} and ρ_{III} will be reproduced in the next section by taking the decompactification limit in the general solution. They apply in three different intervals of the coupling λ and represent three different phases of the theory.

Thus, the picture is as follows. When $\lambda < 1$, the eigenvalues are uniformly distributed in the interval $[-m\lambda, m\lambda]$. The width of the eigenvalue distribution therefore increases with λ, until $\lambda = 1$, where the eigenvalue distribution is extended in the interval $[-m, m]$. Beyond $\lambda = 1$, there is still a uniform distribution in the interval $[-m, m]$, now with fixed width and a density that decreases as $1/\lambda$. At the same time, some eigenvalues begin to accumulate at $\mu = \pm m$. The width of the distribution stays fixed until λ overcomes $(1-\zeta)^{-1}$. Beyond this point, eigenvalues are uniformly distributed in an interval $[-m\lambda(1-\zeta), m\lambda(1-\zeta)]$, which expands as λ increases, but now with a fixed number N_f of eigenvalues accumulated at $\pm m$.

In the case $\zeta \geq 1$, i.e. $N_f \geq N$, the third phase disappears. The system has two phases I and II, represented by the solutions (2.79), (2.80), where now phase II holds in the interval $\lambda \in (1, \infty)$.

2.4.1 Free energy and critical behavior

The order of the phase transition is defined as usual by the analytic properties of the free energy:

$$F = -\frac{1}{N^2} \log Z \ . \tag{2.82}$$

We first consider $0 < \zeta < 1$ and compute its derivative with respect to the coupling, which is related to the second moment of the eigenvalue density,

$$\partial_\lambda F = -\frac{R}{2m\lambda^2}\langle \mu^2 \rangle = \begin{cases} -\frac{mR}{6} & \text{Phase I} \\ -\frac{mR}{6\lambda^3}(3\lambda - 2) & \text{Phase II} \\ -\frac{mR}{6\lambda^2}\left(\lambda^2(1-\zeta)^3 + 3\zeta\right) & \text{Phase III} \end{cases} \tag{2.83}$$

This implies a discontinuity in the third derivative at both critical points, $\lambda = 1$ and $\lambda = (1-\zeta)^{-1}$:

$$\partial_\lambda^3 (F_{\text{I}} - F_{\text{II}})\Big|_{\lambda=1} = -mR \ , \qquad \partial_\lambda^3 (F_{\text{II}} - F_{\text{III}})\Big|_{\lambda=(1-\zeta)^{-1}} = mR(1-\zeta)^5 \ . \tag{2.84}$$

Therefore, both phase transitions are third order. The free energy in the three phases is given by:

$$F_{\text{I}} = \frac{mR}{6}(6\zeta - \lambda) \ , \tag{2.85}$$

$$F_{\text{II}} = \frac{mR}{6\lambda^2}\Big(3(2\zeta - 1)\lambda^2 + 3\lambda - 1\Big) \ , \tag{2.86}$$

$$F_{\text{III}} = \frac{mR}{6\lambda}\Big((\zeta - 1)^3\lambda^2 + 3\zeta^2\lambda + 3\zeta\Big) \ , \tag{2.87}$$

up to a common numerical constant. Note that the free energy is complex upon analytic continuation to imaginary g. This is expected as the partition function (2.66) with imaginary g is complex.

In the case $\zeta \geq 1$, the expressions for the free energies F_{I} and F_{II} are the same, but, as explained, phase III disappears and phase II extends up to $\lambda = \infty$.

2.4.2 Wilson loop analyticity

We now compute (2.58) in the large R limit using the density functions (2.79), (2.80) and (2.81). We obtain $(0 < \zeta < 1)$

$$W(C) = \langle e^{\mu R} \rangle \sim \begin{cases} e^{mR\lambda} & \text{Phase I} \\ e^{mR} & \text{Phase II} \\ e^{mR\lambda(1-\zeta)} & \text{Phase III} \end{cases} \tag{2.88}$$

It follows a perimeter law, just like in massive (or asymptotically free) four-dimensional $\mathcal{N} = 2$ SYM theories [24, 25, 38, 39]. At the two critical points,

$$\partial_\lambda \log W(C) \sim \begin{cases} mR & \text{Phase I} \\ 0 & \text{Phase II} \\ mR(1-\zeta) & \text{Phase III} \end{cases} \tag{2.89}$$

Thus there is a discontinuity in the first derivative.[1]

2.5 Large N solution at finite R

2.5.1 General solution

The integral equation (2.73) can be solved in general for finite R using standard methods [28, 29]. It is convenient to make the following change of integration variables:

$$z_i = c e^{\mu_i} \ , \qquad c \equiv e^{t(1-\zeta)} \ . \tag{2.90}$$

Now we use the relations:

$$\mathrm{d}^N \mu \prod_{i<j} 4 \sinh^2 \frac{\mu_i - \mu_j}{2} = \mathrm{d}^N z \frac{\prod_{i<j}(z_i - z_j)^2}{\prod_i z_i^N} \ , \tag{2.91}$$

$$\prod_i \left(4 \cosh \frac{\mu_i + m}{2} \cosh \frac{\mu_i - m}{2} \right) = c^N \prod_i z_i^{-1} \left(1 + z_i \frac{e^{+m}}{c} \right) \left(1 + z_i \frac{e^{-m}}{c} \right) \ , \tag{2.92}$$

The partition function becomes

$$Z_{N_f}^{U(N)} = e^{-\frac{t}{2} N^2 (1-\zeta^2)} \int \mathrm{d}^N z \prod_{i<j} (z_i - z_j)^2 \ e^{-\frac{1}{g} \sum_i V(z_i)} \ , \tag{2.93}$$

which now exhibits a factor representing the Vandermonde determinant. The potential is given by

$$V(z) = \frac{1}{2} (\log z)^2 + t\zeta \log \left[\left(1 + z \frac{e^{+m}}{c} \right) \left(1 + z \frac{e^{-m}}{c} \right) \right] \ . \tag{2.94}$$

[1] Power-like factors in W (2.88) are not meaningful, since they are affected by subleading corrections which were discarded in the saddle-point equation (2.78). A formal calculation using the densities (2.79)-(2.81) including the power factors gives a W with discontinuities in the second derivatives. The discontinuity in the first derivative then appears in the infinite R limit.

Therefore, we have a usual matrix model with logarithmic terms in the potential. In these new variables, the saddle-point equation becomes

$$\fint_a^b \mathrm{d}z\, \hat{\rho}(z) \frac{1}{p-z} = \frac{1}{2t} V'(p) \ , \tag{2.95}$$

where $\hat{\rho}(z)\mathrm{d}z = \rho(\mu)\mathrm{d}\mu$. To compute the eigenvalue density one defines the auxiliary "resolvent" function as

$$\omega(p) = \frac{1}{N} \left\langle \sum_{i=1}^{N} \frac{1}{p - z_i} \right\rangle \ , \tag{2.96}$$

whose expression in the large N limit is

$$\omega(p) = \int \mathrm{d}z \frac{\hat{\rho}(z)}{p-z} \ . \tag{2.97}$$

For a generic potential $V(z)$, the resolvent is then given by [28, 29]

$$\omega(p) = \frac{1}{2t} \oint_{\mathcal{C}} \frac{\mathrm{d}z}{2\pi i} \frac{V'(z)}{p-z} \left(\frac{(p-a)(p-b)}{(z-a)(z-b)} \right)^{1/2} \ , \tag{2.98}$$

where $\mathcal{C}$ is a path enclosing the branch cut defined by the branch points a and b.[2] Then the eigenvalue density is obtained from the discontinuity of the resolvent across the branch cut,

$$\hat{\rho}(p) = -\frac{1}{2\pi i} \left(\omega(p + i\epsilon) - \omega(p - i\epsilon) \right) \ . \tag{2.99}$$

This can be easily seen since, according to (2.97),

$$\begin{aligned}
\omega(p \pm i\varepsilon) &= \int_{\mathbb{R}} \mathrm{d}z \frac{\hat{\rho}(z)}{p \pm i\epsilon - z} = \mathrm{P} \int \mathrm{d}z \frac{\hat{\rho}(z)}{p-z} + \int_{C_\varepsilon^{\mp}} \mathrm{d}z \frac{\hat{\rho}(z)}{p-z} \\
&= \mathrm{P} \int \mathrm{d}z \frac{\hat{\rho}(z)}{p-z} \mp i\pi \hat{\rho}(p) \ ,
\end{aligned} \tag{2.100}$$

where C_ε^+ (C_ε^-) is the contour around the pole $z = p$ in the upper (lower) half plane with (anti) clockwise orientation. By similar arguments one can see that the resolvent can be written as

$$\omega(p) = \frac{1}{2t} V'(p) - \frac{1}{2t} M(p) \sqrt{(p-a)(p-b)} \ , \tag{2.101}$$

with

$$M(p) = \oint_{\infty} \frac{\mathrm{d}z}{2\pi i} \frac{V'(z)}{z-p} \frac{1}{\sqrt{(z-a)(z-b)}} \ , \tag{2.102}$$

where the integral is done over the same path $\mathcal{C}$, but enclosing the point at infinity. These expressions immediately follow from equation (2.98).

The integral defining $M(p)$ contains two contributions, $M = M_1 + M_2$: M_1 coming from the potential term $(\log z)^2$, which is the one that appears in the pure Chern-Simons matrix model. This integral is computed in [29], we repeat here its computation for completeness. First of all, we cannot apply equation (2.102) directly, we need to deform the contour of integration as sketched in figure 2.1, surrounding the logarithm branch cut

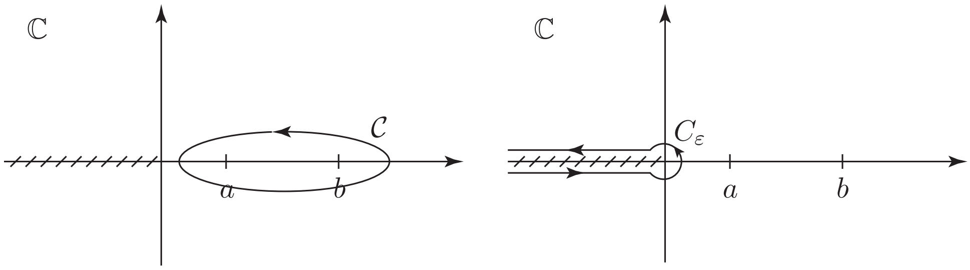

Figure 2.1: Integration contour for M_1.

in the negative real axis and the singularity at $z = 0$. Then we are left with the integral along the small circle C_ε and that coming from the logarithmic jump when crossing the negative real axis,

$$M_1(p) = -\oint_{C_\varepsilon} \frac{\mathrm{d}z}{2\pi i} \frac{\log z}{z(z-p)} \frac{1}{\sqrt{(z-a)(z-b)}} - \int_{-\infty}^{-\varepsilon} \mathrm{d}z \frac{1}{z(z-p)} \frac{1}{\sqrt{(z-a)(z-b)}} \ . \tag{2.103}$$

Both integrals in M_1 are divergent in the limit $\varepsilon \to 0$, however, singularities cancel out between both integrals and we are left with

$$M_1(p) = \frac{1}{p\sqrt{(p-a)(p-b)}} \log \frac{\left(\sqrt{a}\sqrt{p-b} - \sqrt{b}\sqrt{p-a}\right)^2}{p\left(\sqrt{p-a} - \sqrt{p-b}\right)^2} + \frac{2}{p\sqrt{ab}} \log \frac{\sqrt{a}+\sqrt{b}}{2\sqrt{ab}} \ . \tag{2.104}$$

The second piece M_2 is

$$M_2(p) = t\zeta \oint_\infty \frac{\mathrm{d}z}{2\pi i} \frac{1}{z-p} \frac{1}{\sqrt{(z-a)(z-b)}} \left(\frac{1}{ce^m + z} + \frac{1}{ce^{-m} + z} \right) \ . \tag{2.105}$$

There is no contribution from the residue at $z = \infty$, the only contributions come from the simple poles at $z = -ce^{\pm m}$. We find

$$M_2(p) = -t\zeta \left(\frac{1}{p + ce^m} \frac{1}{\sqrt{(a+ce^m)(b+ce^m)}} + (m \leftrightarrow -m) \right) \ . \tag{2.106}$$

Let us combine this with the contribution coming from the $(\log z)^2$ term. We write $\omega = \omega^{(1)} + \omega^{(2)}$, where

$$\omega^{(1)}(p) = -\frac{1}{2tp} \log \frac{\left(\sqrt{a}\sqrt{p-b} - \sqrt{b}\sqrt{p-a}\right)^2}{p^2\left(\sqrt{p-a} - \sqrt{p-b}\right)^2} - \frac{\sqrt{(p-a)(p-b)}}{tp\sqrt{ab}} \log \frac{\sqrt{a}+\sqrt{b}}{2\sqrt{ab}} \ , \tag{2.107}$$

$$\omega^{(2)}(p) = \frac{\zeta}{2} \left(\frac{1}{ce^m + p} + \frac{1}{ce^{-m} + p} \right) - \frac{1}{2t} M_2(p) \sqrt{(p-a)(p-b)} \ . \tag{2.108}$$

According to (2.97), the resolvent obeys the following boundary condition:

$$\omega(p) \sim \frac{1}{p} \ , \qquad \text{for} \qquad p \to \infty \ . \tag{2.109}$$

[2]Multi-cut solutions are not supported by the numerical results.

Imposing this asymptotic condition to the solution (2.107) and (2.108), we obtain two equations that determine the branch points a and b,

$$0 = \frac{\zeta}{2}\left(\frac{1}{\sqrt{(a+ce^m)(b+ce^m)}} + (m \leftrightarrow -m)\right) - \frac{1}{t\sqrt{ab}}\log\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{ab}} , \tag{2.110}$$

$$\begin{aligned} 1 = \zeta - \frac{\zeta}{2}\left(\frac{ce^m + \frac{1}{2}(a+b)}{\sqrt{(a+ce^m)(b+ce^m)}} + (m \leftrightarrow -m)\right) \\ + \frac{(\sqrt{a}+\sqrt{b})^2}{2t\sqrt{ab}}\log\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{ab}} + \frac{1}{t}\log\sqrt{ab} . \end{aligned} \tag{2.111}$$

Now, using the reflection symmetry of the original potential (2.70) prior to the change of variable (2.90), we find that a and b obey the relation,

$$ab = c^2 \equiv e^{2t(1-\zeta)} . \tag{2.112}$$

As a result, one of the two equations (2.110) or (2.111) becomes redundant. The solution for the eigenvalue density takes the form

$$\begin{aligned} \hat{\rho}(z) = \frac{1}{\pi t z}\frac{\sqrt{z-a}\sqrt{b-z}}{\sqrt{ab}}\log\left(\frac{\sqrt{a}+\sqrt{b}}{2\sqrt{ab}}\right) + \frac{1}{\pi t z}\tan^{-1}\left(\frac{\sqrt{z-a}\sqrt{b-z}}{z+\sqrt{ab}}\right) \\ - \frac{\zeta}{2\pi}\left(\frac{\sqrt{z-a}\sqrt{b-z}}{(ce^m+z)\sqrt{a+ce^m}\sqrt{b+ce^m}} + (m \to -m)\right) , \end{aligned} \tag{2.113}$$

with $z \in (a,b)$, $b = c^2 a^{-1}$ and a defined by one of the conditions (2.110) or (2.111).

The expression for the eigenvalue density takes a simpler form in terms of the original μ variable:

$$\rho(\mu) = \frac{1}{\pi t}\tan^{-1}\left(\frac{\sqrt{\cosh A - \cosh\mu}}{\sqrt{2}\cosh\frac{\mu}{2}}\right) + \frac{\zeta}{\pi}\frac{\cosh\frac{\mu}{2}\cosh\frac{m}{2}}{\cosh\mu + \cosh m}\frac{\sqrt{\cosh A - \cosh\mu}}{\sqrt{\cosh A + \cosh m}} \tag{2.114}$$

supported on the interval $\mu \in (-A, A)$, where A is given by the condition

$$\log\left(\cosh\frac{A}{2}\right) = \frac{1}{2}t(1-\zeta) + \frac{t\zeta\cosh\frac{m}{2}}{\sqrt{2}\sqrt{\cosh A + \cosh m}} , \tag{2.115}$$

for any $\zeta \geq 0$.

In the massless $m = 0$ case, the eigenvalue density becomes

$$\rho(\mu) = \frac{1}{\pi t}\tan^{-1}\left(\frac{\sqrt{\cosh A - \cosh\mu}}{\sqrt{2}\cosh\frac{\mu}{2}}\right) + \frac{\zeta}{2\pi}\sqrt{\operatorname{sech}^2\frac{\mu}{2} - \operatorname{sech}^2\frac{A}{2}} \tag{2.116}$$

$$\log X = -\frac{t}{2}(1-\zeta+\zeta X) , \qquad X \equiv \operatorname{sech}\frac{A}{2} . \tag{2.117}$$

In particular, if $\zeta = 0$, i.e. pure $\mathcal{N} = 2$ CS theory without matter, this reproduces the result of [28, 29]. This provides a check of our assumption that, for real g, eigenvalues lie on one cut in the real axes. For imaginary g, the cut lies in the complex plane.[3]

[3] It is simpler to perform the continuation to imaginary g after computing observables.

As the coupling t is gradually increased from zero, the eigenvalue density behaves as follows. At weak coupling, the classical force term μ/t in the saddle-point equation (2.73) is dominant, squeezing the eigenvalue distribution towards the origin. All eigenvalues are small and the kernel in the integral of equation (2.73) approaches the Hilbert kernel, leading to the Wigner semicircular distribution,

$$\rho(\mu) \approx \frac{1}{2\pi t}\sqrt{4t-\mu^2} \qquad \mu \in \left[-2\sqrt{t}, 2\sqrt{t}\right] \ , \qquad t \ll 1 \ . \tag{2.118}$$

Indeed, this expression can be obtained directly from (2.114). In fig. 2.2 we show this distribution as compared to the finite N eigenvalue density obtained numerically from eq. (2.71).

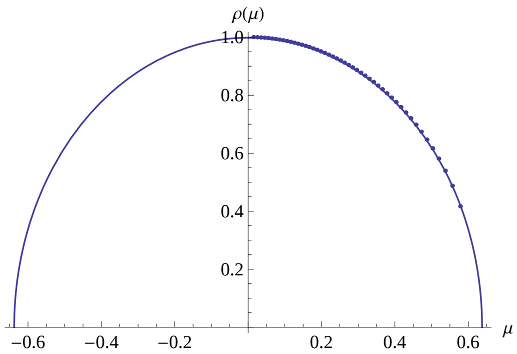

Figure 2.2: At $t \ll 1$ the eigenvalue density approaches the Wigner distribution ($t = 0.1$, $m = 50$, $\zeta = 0.25$). Solid line: eigenvalue distribution obtained analytically. Dots: numerical solution to (2.71) with $N = 100$.

As t is further increased, the eigenvalue distribution expands and gets flattened forming a plateau, until t gets close to $t \lesssim m$, when two peaks around $\mu \approx \pm m$ begin to form (fig. 2.3). For finite R, small peaks begin to show up already at $t \lesssim m$.

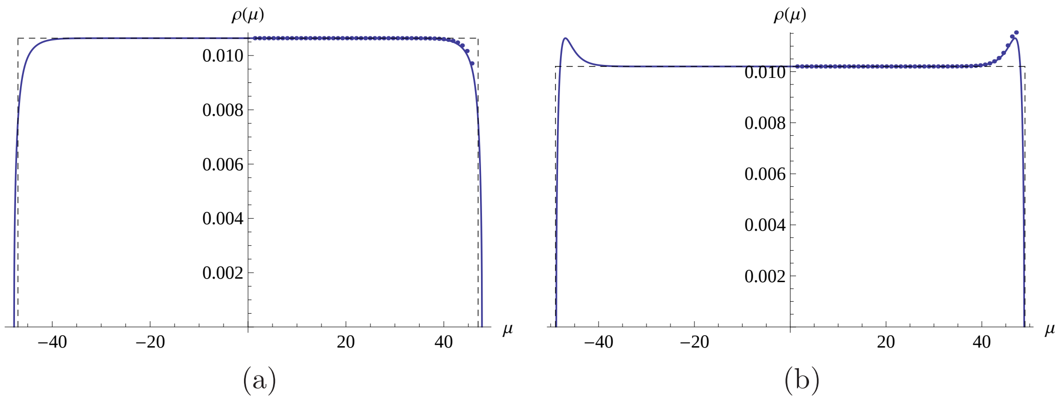

Figure 2.3: Eigenvalue density in phase I for $m = 50$, $\zeta = 0.25$ and (a) $t = 47$, (b) $t = 49$. Solid line: analytic solution. Dashed line: solution in the decompactification limit. Dots: numerical solution to (2.71) with $N = 100$.

As the coupling is increased beyond $t = m$, eigenvalues begin to accumulate around $\mu = \pm m$, enhancing the peaks and maintaining the plateau between them (this is shown in fig. 2.4). This would correspond to phase II in the decompactification limit, where peaks turn into Dirac delta functions. For $\zeta \geq 1$ this phase holds up to $t = \infty$: the eigenvalue distribution is uniform with support in a fixed interval $(-m, m)$, with a density decreasing as $1/t$, and with two peaks at $\mu = \pm m$, whose amplitudes increase until all eigenvalues get on the top of $\mu = \pm m$ as $t \to \infty$.

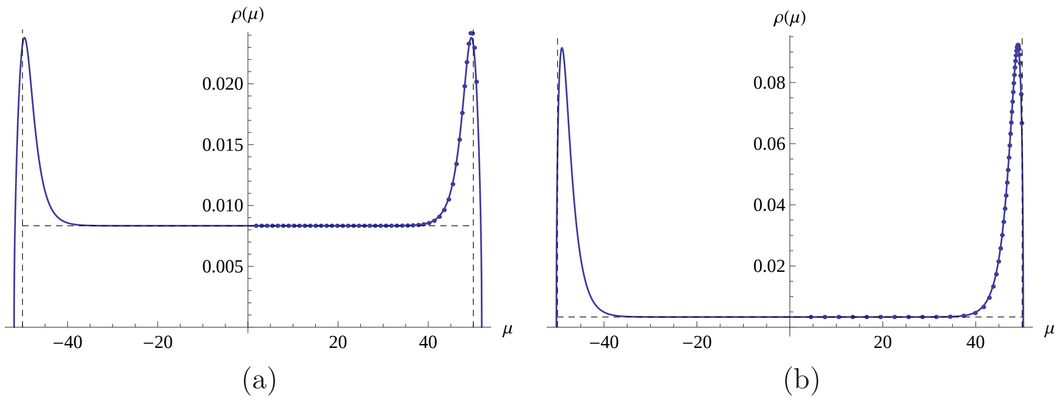

Figure 2.4: Eigenvalue density in phase II for $m = 50$ and (a) $t = 60$, $\zeta = 0.25$, (b) $t = 150$, $\zeta = 2$ (same conventions as in fig. 2.3).

When $0 < \zeta < 1$, phase II holds only in the interval $m < t < m/(1-\zeta)$. For $t > m/(1-\zeta)$, the plateau begins to extend beyond the peaks at $\mu = \pm m$, as shown in fig. 2.5. Each peak now contains $N_f/2$ eigenvalues. This reproduces the behavior found in section 2.4 for phase III.

Note that fig.2.4b and 2.5 display the eigenvalue density for the same value of $t = 150$ but different ζ. They illustrate the fact that when $\zeta \geq 1$ eigenvalues lie on the interval $[-m, m]$ for all $t > m$, whereas when $\zeta < 1$ the eigenvalue distribution extends beyond $\mu = \pm m$ as soon as t overcomes $m/(1-\zeta)$.

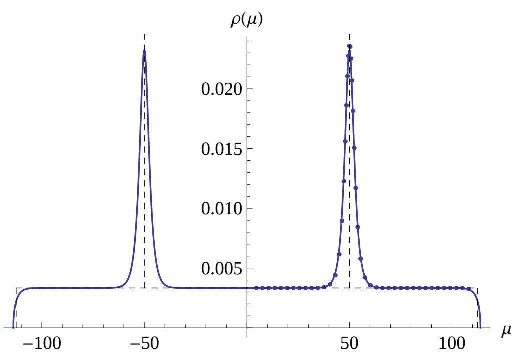

Figure 2.5: Eigenvalue density in phase III for $m = 50$, $t = 150$, $\zeta = 0.25$ (same conventions as in fig. 2.3).

Using the eigenvalue density (2.114), we can obtain the expression for the Wilson loop at finite R,

$$W(C) = \frac{1}{t}\sinh^2\frac{A}{2} + \frac{\zeta}{2}\frac{\sqrt{1+\cosh m}}{\sqrt{\cosh A + \cosh m}} \times \left(\cosh A - 1 + 2\cosh m\left(1 - \frac{\sqrt{\cosh A + \cosh m}}{\sqrt{1+\cosh m}}\right)\right) . \tag{2.119}$$

2.5.2 Decompactification limit

Let us examine the general formula for the eigenvalue density (2.114), (2.115) in the large R limit. It is convenient to restore the R dependence by the scaling $m \to mR$, $A \to AR$, $\mu \to \mu R$. For large R, (2.115) simplifies to the following form

$$A - \frac{1}{R}\log 4 = m\lambda(1-\zeta) + \frac{m\lambda\zeta}{\sqrt{e^{(A-m)R}+1}} , \tag{2.120}$$

where, again, we have introduced the parameter $\lambda \equiv t/mR$. We now solve this equation in the three different phases:

- $\lambda < 1$: Let us assume that $A < m$. In this case we can neglect the exponential inside the square root of (2.120). This gives $A \approx m\lambda$. Thus the $A < m$ phase appears when $\lambda < 1$.
- $1 < \lambda < (1-\zeta)^{-1}$: In this interval the solution is of the form:

$$A = m + \frac{1}{R}\log\left[\frac{\lambda^2\zeta^2}{(1-\lambda(1-\zeta))^2} - 1\right] + \mathcal{O}(R^{-2}) . \tag{2.121}$$

  As we will shortly see, the $\mathcal{O}(R^{-1})$ term is important in determining the density at $R \to \infty$. When $\zeta \geq 1$, this solution for A is real for any $\lambda > 1$, and in this case this phase extends up to $\lambda = \infty$. When $0 < \zeta < 1$, (2.121) solves (2.120) with real A provided $1 < \lambda < (1-\zeta)^{-1}$.
- $\lambda > (1-\zeta)^{-1}$: Let us now assume that $A > m$. In this case the last term of eq. (2.120) can be neglected and we end up with

$$A \approx m\lambda(1-\zeta) . \tag{2.122}$$

  Thus the solution arises only when $\zeta < 1$ and $A > m$ requires $\lambda > (1-\zeta)^{-1}$, in concordance with the analysis of section 2.4.

Consider now the eigenvalue density (2.114). The first term gives

$$\frac{1}{\pi m\lambda}\tan^{-1}\left(\frac{\sqrt{\cosh AR - \cosh\mu R}}{\sqrt{2}\cosh\frac{\mu R}{2}}\right) \underset{R\to\infty}{\longrightarrow} \begin{cases} 0 & , \quad |\mu| = A , \\ \frac{1}{2m\lambda} & , \quad |\mu| < A . \end{cases} \tag{2.123}$$

Therefore this is the term which gives the plateau, reproducing the same result of section 2.4.

Consider now the second term in (2.114). When $A < m$, this term vanishes at large R. If, instead, $A > m$, then this term generates two Dirac delta functions centered on $\pm m$ with normalization $\zeta/2$. For a trial function $f(\mu)$, one numerically finds that

$$R \int_{-A}^{A} \mathrm{d}\mu \, \frac{2}{\pi} \frac{\cosh \frac{\mu R}{2} \cosh \frac{mR}{2}}{\cosh \mu R + \cosh mR} \frac{\sqrt{\cosh AR - \cosh \mu R}}{\sqrt{\cosh AR + \cosh mR}} \, f(\mu) \;\longrightarrow\; f(m) + f(-m) \, , \tag{2.124}$$

at large R.

Finally, consider the intermediate case, phase II, where A is given by (2.121). We find a similar result as (2.124), but with an extra overall coefficient $(\lambda - 1)/(\zeta\lambda)$. This coefficient is produced by the correction of order $\mathcal{O}(R^{-1})$ in A. Thus the resulting ρ exactly matches the solution (2.80).

2.6 Comments

In summary, we have seen how supersymmetry can provide with interesting exact results, by making use of the localization methods. These methods allow us to reduce a complicated path integral to a simpler matrix model integral, which in some cases, as the one presented here, can be solved exactly.

As an example, we have studied $\mathcal{N} = 2$ Chern-Simons theory with massive matter on a three sphere. We have seen that in a particular decompactification limit mass deformations lead to new physics involving large N quantum phase transitions, like in $\mathcal{N} = 2$ massive four-dimensional SYM theories. These phase transitions produce non-analytic behavior in supersymmetric observables, like discontinuities in the first derivatives of the vev of the circular Wilson loop, which can be computed explicitly.

It is worth mentioning here that the partition function for this system can be computed exactly at finite N. This is done in [40], where different behaviors for the partition function are found depending on the relative values of the mass and the coupling parameters. This already gives evidence at finite N of the large N phase transitions studied here.

We have not included Fayet-Iliopoulos (FI) parameters. We can introduce a FI parameter, η, for each $U(1)$ factor of the gauge group. The addition of each of these FI terms translates into the addition of the classical term

$$S_{\mathrm{cl}}^{\mathrm{FI}} = 2\pi i \eta \, \mathrm{Tr}\, \sigma_0 \tag{2.125}$$

in the matrix model [35]. Including both FI and mass parameters may shed new light on the properties of these phase transitions. In particular, the exchange of mass and FI parameters exchanges mirror pairs of three-dimensional supersymmetric field theories [35, 41]. This indicates that certain massless theories deformed by FI parameters may also exhibit large N phase transitions in some limit. It would be interesting to study the consequences of this interplay in more detail.

It would also be interesting to perform similar studies to the one presented here for other mass deformations of three dimensional theories. For instance, in [42] they study the mass-deformed ABJM model on the three sphere, whose partition function is given in [35] and its mirror theory is the low energy limit of $U(N)$ $\mathcal{N} = 8$ super Yang-Mills.

In [42] they solve the saddle-point equation of this theory in a similar decompactification limit to the one presented here. As we do, they rescale the 't Hooft coupling with the radius R and study the theory at $R \to \infty$ with the rescaled 't Hooft coupling fixed. As in many localization computations like the one we have shown in this chapter, they solve the saddle-point equations of this model by analytically continuing the couplings to the complex plane. They do it in two possible ways, the analytic continuation is done for the rank of the gauge groups or in the Chern-Simons level. At large N, the density functions that solve the saddle point equations take a multiple step form an they undergo an infinite series of third order phase transitions that accumulate at strong coupling. However, the connection to the original ABJM theory that undoes the analytical continuation poses some problems and it still is an issue to be considered.

2.A Spinor conventions

In three dimensional Euclidean space, the Clifford algebra is defined by

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} , \tag{2.126}$$

where in our case $g_{\mu\nu}$ is just the Euclidean metric $\delta_{\mu\nu}$. If the Pauli matrices are given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} , \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} , \tag{2.127}$$

a possible choice of the Dirac matrices is then

$$\gamma^\mu = \gamma_\mu = \sigma_\mu . \tag{2.128}$$

and now we have

$$\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] = i\epsilon^{\mu\nu\rho}\gamma_\rho . \tag{2.129}$$

We will work with complex, two-component spinors, which are in the fundamental representation of the spin group $SU(2)$,

$$\psi_\alpha = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} . \tag{2.130}$$

The $SU(2)$ indices are raised or lowered with

$$\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha} = -\epsilon^{\alpha\beta} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} ; \qquad \psi^\alpha = \epsilon^{\alpha\beta}\psi_\beta , \qquad \psi_\alpha = \epsilon_{\alpha\beta}\psi^\beta . \tag{2.131}$$

This allows to define the $SU(2)$ inner product

$$\psi\chi \equiv \psi^\alpha\chi_\alpha = \psi^\alpha\epsilon_{\alpha\beta}\chi^\beta \tag{2.132}$$

and then, for example, we have

$$\psi\chi = \chi\psi , \qquad \psi\gamma^\mu\chi = \psi^\alpha(\gamma^\mu)_\alpha{}^\beta\chi_\beta = -\chi\gamma^\mu\psi . \tag{2.133}$$

We also define

$$\gamma^{\mu\nu} \equiv \frac{1}{2}[\gamma^\mu, \gamma^\nu] = i\epsilon^{\mu\nu\rho}\gamma_\rho . \tag{2.134}$$

2.B The Vandermonde determinant

Here we are going to show how the Vandermonde determinant appears in the partition function of a $U(N)$ gauge invariant matrix model. We will follow [29, 43].

Let us consider the matrix model partition function

$$Z = \frac{1}{\mathrm{vol}(U(N))} \int \mathrm{d}M e^{-S(M)} , \tag{2.135}$$

where M is an $N \times N$ hermitian matrix and the action is invariant under the adjoint action of gauge group $U(N)$, i.e.

$$S(M^U) = S(M) , \qquad \text{with} \qquad M^U = UMU^{-1} . \tag{2.136}$$

The measure $\mathrm{d}M$ is the Haar measure, given by

$$\mathrm{d}M = \prod_{i=1}^{N} \mathrm{d}M_{ii} \prod_{i<j} \mathrm{d}(\mathrm{Re}\, M_{ij})\mathrm{d}(\mathrm{Im}\, M_{ij}) . \tag{2.137}$$

In a matrix model it is usually convenient to work with diagonal matrices, so that the partition function becomes

$$Z = \int \prod_{i=1}^{N} \mathrm{d}\lambda_i J(\Lambda) e^{-S(\Lambda)} , \tag{2.138}$$

where Λ is the diagonalized M matrix,

$$M = V \Lambda V^\dagger , \qquad \text{with} \qquad \Lambda_{ij} = \lambda_i \delta_{ij} \tag{2.139}$$

and V is the element of $U(N)$ that diagonalizes M. We have to determine $J(\Lambda)$, this can be done by considering the choice of a diagonal M, namely Λ, as a gauge fixing condition and applying standard Faddev-Popov techniques. So let us introduce the quantity

$$\Delta^{-1}(M) = \int \mathrm{d}U \prod_{i<j} \delta[\mathrm{Re}(M^U)_{ij}]\delta[\mathrm{Im}(M^U)_{ij}] , \tag{2.140}$$

where $\mathrm{d}U$ is the Haar integration measure over the gauge group, which, of course, is gauge invariant. Then we can write the original partition function as

$$Z = \int \mathrm{d}M e^{-S(M)} \Delta(M) \int \mathrm{d}U \prod_{i<j} \delta[\mathrm{Re}(M^U)_{ij}]\delta[\mathrm{Im}(M^U)_{ij}] . \tag{2.141}$$

Both $\mathrm{d}M$ and $\Delta(M)$ are gauge invariant, hence, if we change $M \to M^{U^{-1}}$, we end up with

$$Z = \Omega_N \int \mathrm{d}M e^{-S(M)} \Delta(M) \prod_{i<j} \delta[\mathrm{Re}(M)_{ij}]\delta[\mathrm{Im}(M)_{ij}] , \tag{2.142}$$

where

$$\Omega_N = \int \mathrm{d}U \tag{2.143}$$

is the integral over the unitary group. As we say in the main text, we are not interested in the overall constant in the partition function, however, for completeness, we show here the result for the previous integral,

$$\frac{\Omega_N}{\mathrm{vol}(U(N))} = \frac{1}{N!}\frac{1}{(2\pi)^N} \ . \tag{2.144}$$

Therefore, it only remains to compute the quantity $\Delta(M)$. If we describe the gauge fixing condition by

$$F(M) = 0 \ , \qquad \text{i.e.} \qquad M_{ij} = 0 \qquad \text{for} \qquad i \neq j \ , \tag{2.145}$$

then $\Delta(M)$ is the standard Faddeev-Popov determinant,

$$\Delta(M) = \det \frac{\delta F(M^U)}{\delta A} \ , \tag{2.146}$$

for an infinitesimal gauge transformation, where A is the infinitesimal anti-hermitean matrix appearing in $U = e^A$. Then

$$F_{ij}(\Lambda^U) = (U\Lambda\, U^\dagger)_{ij} = [A, \Lambda]_{ij} + \ldots = A_{ij}(\lambda_j - \lambda_i) + \ldots \tag{2.147}$$

and the determinant (2.146) gives the square of the Vandermonde determinant:

$$\Delta(\Lambda) = \prod_{i<j}(\lambda_i - \lambda_j)^2 \ . \tag{2.148}$$

Chapter 3

Supersymmetry and Gauge/Gravity duality

In this chapter we move away from the applications presented in the previous chapter. In the next two chapters we will work in the context of the gauge/gravity duality. For this reason, we are going to review in first place the original AdS/CFT conjecture of Maldacena. Here and in the subsequent two chapters we will be interested in generalizations where the supergravity solutions look like a product of four uncompactified external dimensions and six compact internal dimensions. These solutions have to preserve supersymmetry, therefore, in the second part of this chapter we will review the concept of G-structures, in terms of which we can express the supersymmetry conditions that the backgrounds have to satisfy. It turns out that solving these supersymmetry conditions is simpler than solving the supergravity equations of motion and it can be proved that solutions to the supersymmetry conditions supplemented with Bianchi identities automatically satisfy the full set of equations of motion. We will also comment on how to find supersymmetric embeddings of D-branes by introducing the concept of calibrations.

3.1 AdS/CFT in a nutshell

In this section we are going to sketch the original AdS/CFT duality proposed by Maldacena [6]. This duality establishes a complete equivalence between $\mathcal{N} = 4$ $SU(N)$ super Yang-Mills theory and type IIB string theory on $\mathrm{AdS}_5 \times S^5$.[1] When making use of this duality one usually considers a regime leading to a weaker but more practical version of the duality, where it is enough to consider the low energy limit of the string theory, type IIB supergravity. Therefore, it is appropriate to present these two theories first.

[1]In the seminal paper of Maldacena [6] he also conjectures other dualities between certain conformal field theories of different dimensionality, d, and type IIB or M-theory gravity theories on $\mathrm{AdS}_{d+1} \times M_D$, with M_D a certain manifold of $D = 9 - d$ or $10 - d$ dimensions depending on whether we are in type IIB string theory or M-theory respectively.

3.1.1 $\mathcal{N}=4$ super Yang-Mills

$\mathcal{N}=4$ super Yang-Mills theory is made of a single $\mathcal{N}=4$ vector hypermultiplet in the adjoint representation of the gauge group $SU(N)$. The $\mathcal{N}=4$ hypermultiplet can be considered from the point of view of $\mathcal{N}=1$ supermultiplets as being made of one vector supermultiplet and three chiral supermultiplets. In consequence, the field content of this hypermultiplet is a vector field A_μ, six real scalars Φ^i $(i=1,\ldots,6)$ and four Weyl fermions λ^a_α, $\lambda^{\dagger\bar{a}}_{\dot\alpha}$. α and $\dot\alpha$ are chiral indices and $a=1,\ldots,4$ and $\bar{a}$ are in the $\mathbf{4}$ and $\bar{\mathbf{4}}$ representation of the global R-symmetry group $SU(4)_R \simeq SO(6)_R$. The Lagrangian for this theory is given by [44],

$$\begin{aligned}\mathcal{L}_{\mathcal{N}=4\ \mathrm{SYM}} = \mathrm{Tr}\Bigg[&-\frac{1}{2g^2_{\mathrm{YM}}}F_{\mu\nu}F^{\mu\nu} + \frac{\theta_{\mathrm{YM}}}{16\pi^2}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} \\ &- i\sum_a \bar\lambda^a\bar\sigma^\mu D_\mu\lambda_a - \frac{1}{g^2_{\mathrm{YM}}}\sum_i D_\mu\Phi^i D^\mu\Phi^i \\ &+ \left(\sum_{a,b,i} C^{ab}_i\lambda_a[\Phi^i,\lambda_b] + h.c.\right) + \frac{1}{2g^2_{\mathrm{YM}}}\sum_{i,j}[\Phi^i,\Phi^j]^2\Bigg]\,, \end{aligned} \tag{3.1}$$

in terms of two parameters: the coupling constant, g_{YM}, and the theta angle, θ_{YM}. The constants C^{ab}_i are related to the Dirac matrices associated to the Clifford representation of the R-symmetry group.

Apart from the R-symmetry and the supersymmetries, this theory also posesses the additional global symmetries:

- The β-function for this theory is exactly zero to all orders and, thus, the theory has conformal symmetry $SO(2,4) \simeq SU(2,2)$. This symmetry is generated by translations, Lorentz rotations, dilatations and special conformal transformations.
- Supersymmetry transformations and special conformal transformations do not commute, therefore, their commutator generates a new symmetry called "special conformal supersymmetry".

3.1.2 Type II supergravity

In this section we will briefly review both type IIA and type IIB supergravity, which are the low energy limit of type IIA/IIB string theory. Although the gravity dual of $\mathcal{N}=4$ super Yang-Mills involves type IIB supergravity or string theory, the gauge/gravity duality is not something specific of these two theories, it also applies to other field theories and gravity theories. For example, one can find dualities between certain field theories and type IIA supergravity. This possibility will appear in chapter 5, for this reason we review both IIA and IIB supergravity theories here.

Type II supergravity is a ten dimensional theory with 32 supersymmetries generated by two 16 component Majorana-Weyl spinors, ϵ^1 and ϵ^2. Whether these spinors have the same or opposite chirality corresponds to type IIB or type IIA supergravity respectively.

For the bosonic content of both type IIA and type IIB we distinguish two sectors inherited from string theory, depending on fields satisfying NS-NS (from Neveu-Schwarz) or R-R (from Ramond) boundary conditions. On the one hand, the bosonic NS content of both type IIA and type IIB is given by the metric g, the dilaton scalar field Φ and a three form H. On the other hand, in the RR sector we find the n-form fields, F_n, with $n = 0, 2, 4$ for type IIA and $n = 1, 3, 5$ for type IIB. The bosonic parts of the actions for both type IIB and type IIA supergravity are

$$\begin{aligned} S_{\text{IIB}} = \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_{10}} \sqrt{-\det g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left(F_1^2 + \frac{F_3^2}{3!} + \frac{1}{2}\frac{F_5^2}{5!} \right) \right] \\ - \frac{1}{2} \left(C_4 \wedge H \wedge \mathrm{d}C_2 \right) \ , \end{aligned} \tag{3.2}$$

$$\begin{aligned} S_{\text{Massive IIA}} = \frac{1}{2\kappa_{10}^2} \int_{\mathcal{M}_{10}} \sqrt{-\det g} \left[e^{-2\Phi} \left(R + 4(\partial\Phi)^2 - \frac{H^2}{12} \right) - \frac{1}{2} \left(F_0^2 + \frac{F_2^2}{2} + \frac{F_4^2}{4!} \right) \right] \\ - \frac{1}{2} \left(\mathrm{d}C_3 \wedge \mathrm{d}C_3 \wedge B + \frac{1}{3} F_0 \mathrm{d}C_3 \wedge B^3 + \frac{1}{20} F_0^2 B^5 \right) \ , \end{aligned} \tag{3.3}$$

where $\kappa_{10} = 8\pi^{7/2} g_s l_s^4$ is given in terms of the string coupling, g_s, and the string length, $l_s = \sqrt{\alpha'}$.

In type IIA supergravity, F_0 does not have any propagating degrees of freedom, the equations of motion force it to take a constant value, $F_0 = m$, called the Roman mass [45] (and hence we use the name "Massive type IIA" supergravity when it is included). For type IIB supergravity, F_5 must fulfill the self duality condition

$$\star F_5 = F_5 \ , \tag{3.4}$$

where the Hodge star operator is defined such that

$$\star \star F_n = (-1)^{n+1} F_n \ . \tag{3.5}$$

Problems in writing the type IIB supergravity action including this self duality condition are well known, here we must impose the self duality condition by hand.

The fluxes are defined in terms of potentials in the following way

$$H = \mathrm{d}B \ . \tag{3.6}$$

$$\text{Type IIB:} \quad F_1 = \mathrm{d}C_0 \ , \quad F_3 = \mathrm{d}C_2 - H \wedge C_0 \ , \quad F_5 = \mathrm{d}C_4 - H \wedge C_2 \ . \tag{3.7}$$

$$\text{Type IIA:} \quad F_0 = m \ , \quad F_2 = \mathrm{d}C_1 + F_0 B \ , \quad F_4 = \mathrm{d}C_3 - H \wedge C_1 + \frac{1}{2} F_0 B \wedge B \ . \tag{3.8}$$

In the absence of sources this leads to the following Bianchi identities

$$\mathrm{d}H = 0 \ . \tag{3.9}$$

$$\text{Type IIB:} \quad \mathrm{d}F_1 = 0 \ , \quad \mathrm{d}F_3 - H \wedge F_1 = 0 \ , \quad \mathrm{d}F_5 - H \wedge F_3 = 0 \ . \tag{3.10}$$

$$\text{Type IIA:} \quad \mathrm{d}F_0 = 0 \ , \quad \mathrm{d}F_2 - F_0 H = 0 \ , \quad \mathrm{d}F_4 - H \wedge F_2 = 0 \ . \tag{3.11}$$

The dual fluxes, are related to the previous ones by the expression

$$F_n = (-1)^{(n-1)(n-2)/2} \star F_{10-n} \tag{3.12}$$

and they are defined in terms of potentials as:

$$\text{Type IIB:} \qquad \star F_5 = F_5 \ , \qquad F_7 = \mathrm{d}C_6 - H \wedge C_4 \ , \qquad F_9 = \mathrm{d}C_8 - H \wedge C_6 \ . \tag{3.13}$$

$$\begin{aligned} \text{Type IIA:} \qquad & F_6 = \mathrm{d}C_5 - H \wedge C_3 + \frac{1}{3!} F_0 B^3 \ , \qquad F_8 = \mathrm{d}C_7 - H \wedge C_5 + \frac{1}{4!} F_0 B^4 \ , \\ & F_{10} = \mathrm{d}C_9 - H \wedge C_7 + \frac{1}{5!} F_0 B^5 \ . \end{aligned} \tag{3.14}$$

with the following equations of motion

$$\text{Type IIB:} \qquad \mathrm{d} \star F_1 + H \wedge \star F_3 = 0 \ , \qquad \mathrm{d} \star F_3 + H \wedge F_5 = 0 \ . \tag{3.15}$$

$$\text{Type IIA:} \qquad \mathrm{d} \star F_2 + H \wedge \star F_4 = 0 \ , \qquad \mathrm{d} \star F_4 + H \wedge F_4 = 0 \ . \tag{3.16}$$

All this can be expressed in terms of polyforms (sum of forms of different degree) in the following compact way

$$F_{\mathrm{IIB}} = \mathrm{d}_{-H} C_{\mathrm{IIB}} \ , \qquad F_{\mathrm{IIA}} = \mathrm{d}_{-H} C_{\mathrm{IIA}} + F_0 e^B \ , \tag{3.17}$$

with the combined Bianchi identities

$$\mathrm{d}_{-H} F_{\mathrm{IIB}} = 0 \ , \qquad \mathrm{d}_{-H} F_{\mathrm{IIA}} = 0 \ , \tag{3.18}$$

where $\mathrm{d}_H = \mathrm{d} + H\wedge$ is the nilpotent ($\mathrm{d}_H^2 = 0$) H-twisted exterior derivative acting on the polyforms

$$F_{\mathrm{IIB}} = F_1 + F_3 + F_5 + F_7 + F_9 \ , \qquad C_{\mathrm{IIB}} = C_0 + C_2 + C_4 + C_6 + C_8 \ , \tag{3.19}$$

$$F_{\mathrm{IIA}} = F_0 + F_2 + F_4 + F_6 + F_8 + F_{10} \ , \qquad C_{\mathrm{IIA}} = C_1 + C_3 + C_5 + C_7 + C_9 \ . \tag{3.20}$$

D-branes

In II supergravity there are also non-perturbative $(p+1)$-dimensional extended objects called Dp-branes [46]. These are solitonic solutions of the supergravity equations of motion. These solitons are also present as sources for closed strings in the high energy completion of supergravity provided by string theory. In addition, D-branes can be also considered from the point of view of perturbative open string theory. These objects can be found quantizing the open string with Dirichlet boundary conditions (and hence the name "D-brane") so that they appear as hypersurfaces where open strings can end. The AdS/CFT duality heavily relies on this doubled description of D-branes.

The D-brane action is given by the sum of a Dirac-Born-Infeld (DBI) and a Wess-Zumino (WZ) term. The bosonic part of this action for a single Dp-brane wrapping a submanifold Σ is given by

$$S_{Dp} = -T_{Dp} \int_\Sigma \mathrm{d}^{p+1}\xi \, e^{-\Phi} \sqrt{|\det(g|_\Sigma + \mathcal{F})|} + T_{Dp} \int_\Sigma C|_\Sigma \wedge e^{\mathcal{F}} \ , \tag{3.21}$$

where $|_\Sigma$ denotes the pull-back to the Dp-brane world volume Σ, for instance

$$g_{\mu\nu}|_\Sigma = \frac{\partial X^M}{\partial \xi^\mu} \frac{\partial X^N}{\partial \xi^\nu} g_{MN} \ , \tag{3.22}$$

where ξ are coordinates on the world-volume of the brane, whereas X are coordinates on the target space. Hence, g_{MN} is the target space metric. Moreover, we also find the quantity

$$\mathcal{F} = B\big|_{\Sigma} + 2\pi l_s^2 F \tag{3.23}$$

made of of the sum of the pull-back of the NS two form potential, B, plus the field strength of a $U(1)$ world-volume gauge field, $F = \mathrm{d}A$. Under a gauge transformation, these fields transform as

$$B \to B + \mathrm{d}\Lambda\ , \qquad A \to A - \frac{1}{2\pi l_s^2}\Lambda\big|_{\Sigma}\ , \tag{3.24}$$

leaving the gauge invariant combination $\mathcal{F}$ unchanged. Finally, T_{Dp} is the tension of the Dp-brane

$$T_{Dp} = \frac{1}{(2\pi)^p g_s l_s^{p+1}}\ . \tag{3.25}$$

Eventually, we will consider type II supergravity sourced by D-branes. Therefore, we will have to consider the previous supergravity actions (3.2) or (3.3) plus the DBI and WZ actions (3.21) of D-branes. Then, in the presence of sources, the Bianchi identities for RR-fields are modified to something of the form

$$\mathrm{d}_H F = -2\kappa_{10}^2\, j_{\text{sources}}\ , \tag{3.26}$$

and the equation of motion for H

$$\mathrm{d}(e^{-2\Phi} \star H) = \frac{1}{2}\sum_n \star F_n \wedge F_{n-2} + 2\kappa_{10}^2 \frac{\delta S_{Dp}}{\delta B}\ . \tag{3.27}$$

Finally, the Einstein equations and the dilaton equations of motion are

$$\text{Dilaton:} \qquad \nabla^2\Phi - \mathrm{d}\Phi\cdot\mathrm{d}\Phi + \frac{1}{4}R - \frac{1}{8}H\cdot H - \frac{1}{4}\frac{\kappa_{10}^2 e^{2\Phi}}{\sqrt{-\det g}}\frac{\delta S_{Dp}}{\delta\Phi} = 0 \tag{3.28}$$

$$\begin{aligned}\text{Einstein:} \qquad & R_{MN} + 2\nabla_M\nabla_N\Phi - \frac{1}{2}H_M H_N - \frac{1}{4}e^{2\Phi}F_M F_N \\ & -\kappa_{10}^2 e^{2\Phi}\left(T_{MN} + \frac{g_{MN}}{2\sqrt{-\det g}}\frac{\delta S_{Dp}}{\delta\Phi}\right) = 0\end{aligned} \tag{3.29}$$

where T_{MN} is the energy-momentum tensor associated to the sources

$$T_{MN} = -\frac{2}{\sqrt{-\det g}}\frac{\delta S_{Dp}}{\delta g^{MN}}\ . \tag{3.30}$$

Actions (3.2) and (3.3) do not have the standard Einstein-Hilbert term, since we find the scalar curvature multiplied by the factor $\exp(-2\Phi)$. Nevertheless, through a redefinition of the metric,

$$g^s_{MN} = e^{\Phi/2} g^E_{MN}\ , \tag{3.31}$$

we can recover the standard Einstein-Hilbert term. We have labelled the metric to distinguish the metric g^s_{MN} in the string frame, as given in (3.2) or (3.3), or in the Einstein frame, g^E_{MN}, after the previous redefinition. We will choose to work in the string frame, because later on, in chapter 5, we will consider T-dual transformations of supergravity backgrounds and these transformations are more conveniently expressed in the string frame. In chapter 5 we will also find explicit expressions for the equations of motion in presence of sources.

3.1.3 The Maldacena conjecture

To motivate the Maldacena conjecture let us start by considering a Dp-brane. Roughly speaking, one can view the Dp-brane from two points of view, one can consider its world volume action (3.21), or one can consider the geometry generated when this Dp-brane is placed in flat space. In the last case, the Dp-brane corresponds to a solution of type IIB supergravity action with the RR supergravity field F_{p+2} turned on. The equations of motion following from this action admit the supersymmetry preserving solution[2]

$$\mathrm{d}s^2 = H_p^{-1/2}\mathrm{d}x_{1,p}^2 + H_p^{1/2}\left(\mathrm{d}r^2 + r^2\mathrm{d}\Omega_{8-p}^2\right) \ , \tag{3.32}$$

with

$$H_p(r) = 1 + \frac{L^{7-p}}{r^{7-p}} \qquad \text{and} \qquad L^{7-p} = \frac{2\kappa_{10}^2 T_{Dp}}{(7-p)\mathrm{vol}(S^{8-p})} \ , \tag{3.33}$$

where $\mathrm{d}x_{1,p}^2$ is the $(p+1)$-dimensional Minkowski space and $\mathrm{d}\Omega_q^2$ is the metric of a q-sphere of radius one. This is the metric for an extremal black p-brane with horizon at $r = 0$ and represents the geometry generated by the Dp-brane.

The remaining pieces of the solution, the RR-form and the dilaton, are given by

$$F_{p+2} = \mathrm{d}H_p^{-1} \wedge \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^p \ , \qquad e^{\Phi} = g_s H_p^{(3-p)/4} \ . \tag{3.34}$$

At $r \to \infty$ the warping factor disappears, $H_p \to 1$, and we identify g_s with the string coupling at infinity in the last formula. There we can appreciate the importance of the $p = 3$ case, where the dilaton is constant and the previous identification holds for all r.

The RR charge of the D-brane can be ascertained from

$$Q_p = \frac{1}{\sqrt{2}\kappa_{10}} \int_{S^{8-p}} \star F_{p+2} = \sqrt{2}\kappa_{10} T_{Dp} \ , \tag{3.35}$$

Eventually, we will be interested in the geometry generated by a stack of N coincident Dp-branes, this can be obtained by the replacement

$$Q_p \to N Q_p \ . \tag{3.36}$$

and therefore, $L^{7-p} \to N L^{7-p}$.

The solution (3.32)-(3.34) describing the geometry of a Dp-brane is a classical gravitational solution requiring a quantum gravity completion at high energies. This completion is provided by string theory and therefore, this solution describes the dynamics at low energy of excitations of massless closed strings.

Consider now a stack of N coincident D3-branes (in this case we must impose the self duality condition on the RR 5-form, F_5), the system interpolates between ten dimensional Minkowski space at $r \gg L$ and the near horizon geometry at $r \ll L$, given by

$$\mathrm{d}s^2 \sim \frac{r^2}{L^2}\mathrm{d}x_{1,3}^2 + \frac{L^2}{r^2}\mathrm{d}r^2 + L^2\mathrm{d}\Omega_5^2 \ , \tag{3.37}$$

[2] As we will see in section 3.2, it is enough to solve the equations resulting from setting the gravitino and the dilatino supersymmetric transformations to zero to obtain supersymmetric solutions to the equations of motion.

where we identify an $\mathrm{AdS}_5 \times S^5$ geometry, with both the AdS-space and the 5-sphere with the same radius

$$L = (4\pi g_s N)^{1/4}\, l_s \ . \tag{3.38}$$

We can consider this system from a different point of view. In the presence of the stack of N D3-branes on flat space, type IIB string theory contains closed strings as excitations of empty space (bulk) or open strings, with both ends on the brane, as excitations of the D-branes. In the low energy limit only massless excitations remain and we can write an effective action describing the brane modes, the bulk modes and the interaction between them,

$$S = S_{\text{brane}} + S_{\text{bulk}} + S_{\text{int.}} \ . \tag{3.39}$$

Consider then the action S_{brane} that would correspond to a single Dp-brane. At low energies the effective action describing the dynamics of massless open strings is given by (3.21). Let us consider the simple case in which we place it in flat space without any RR field nor the NS B-field turned on. Then we can write the action (3.21) as

$$S = -T_{Dp} \int \mathrm{d}^{p+1}\xi \sqrt{|\det(\eta_{\mu\nu} + 2\pi l_s^2 F_{\mu\nu})|} \ . \tag{3.40}$$

Now, if the energy is below the string length scale $E \ll 1/l_s$, we can perform an l_s expansion and we get the following action

$$S = -\left(2\pi l_s^2\right)^2 T_{Dp} \int \mathrm{d}^{p+1}\xi \left(\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \ldots\right) \ , \tag{3.41}$$

where we identify the Maxwell action. In particular, we identify the gauge coupling with

$$g^2_{\text{Maxwell}} = (2\pi)^{p-2} g_s l_s^{p-3} \ . \tag{3.42}$$

Therefore, we find that D-branes support a gauge theory on their world-volume.

Actually, we are interested in a stack of N coincident Dp-branes, however, the complete expression of this multiple brane generalization of the DBI action is problematic. Problems arise when expanding the action and trying to include higher powers of fields due to the fact that the DBI action is supposed to be in the regime of slow varying background and world-volume fields, while strong fields are allowed. In this regime of validity one can drop terms involving derivatives of world-volume fields, but we encounter terms like

$$[D_\mu, D_\nu] \sim F_{\mu\nu} \ , \tag{3.43}$$

and whether discarding this type of terms or not becomes ambiguous. There are prescriptions like [47], that try to avoid this ambiguity. However, we will just consider the straightforward generalization of the one-brane action result (3.41) to a non-abelian $SU(N)$ gauge theory. Actually, the gauge group is $U(N)$, however, the extra gauge group factor, $U(1) = U(N)/SU(N)$, accounts for the overall position of the branes and can be ignored. This generalization is not so straightforward, since we can consider engineering the stack of N-branes by taking them from infinity to their actual position. Suppose then, that we already have N branes, if we add another brane and we suppose it does not backreact, it would probe the geometry (3.32)-(3.34), therefore we have to consider the action (3.21) in the background (3.32)-(3.34) and after performing the l_s expansion promote fields to those of a non-Abelian $SU(N)$ gauge theory.

It is convenient to work in the static gauge, according to which one uses diffeomorphism symmetry to set the first $p+1$ coordinates of the target space, X^M, equal to world-volume coordinates ξ^μ, and the remaining $9-p$ coordinates are identified with world-volume scalar fields $2\pi\alpha'\Phi^i$. After the l_s expansion and promoting fields to the non-abelian gauge symmetry (for which we must introduce a $1/2$ factor due to the normalization of the generators of the gauge group), the DBI-WZ action takes the form

$$S = -\frac{1}{2}\left(2\pi l_s^2\right)^2 T_{Dp}\int_\Sigma \mathrm{d}^{p+1}\xi\,\mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \partial_\mu\Phi^i\partial^\mu\Phi^i + \dots\right)\,, \tag{3.44}$$

plus an unimportant constant term that we have already neglected. We identify the Yang-Mills coupling as

$$g_{\mathrm{YM}}^2 = 2(2\pi)^{p-2}g_s l_s^{p-3}\,. \tag{3.45}$$

The theta term of Yang-Mills theory would be identified with the Wess-Zumino term

$$T_{Dp}\int C_{p-3}\wedge F\wedge F\,. \tag{3.46}$$

Given that we do not have the field C_{p-3} turned on, we infer that $\theta_{\mathrm{YM}}=0$.

The case of $D3$-branes is specially relevant, because, in addition to a $SU(N)$ gauge field, we find 6 scalar fields living in a $3+1$-dimensional world-volume. This is exactly the bosonic content of $\mathcal{N}=4$ super Yang-Mills. After completing the D3-brane action (3.21) with the fermionic content to have a supersymmetric theory, we end up with the full $\mathcal{N}=4$ super Yang-Mills theory.

In addition to S_{brane}, we also have the actions S_{bulk} and $S_{\mathrm{int.}}$. S_{bulk} describes the dynamics of closed string modes in the bulk and $S_{\mathrm{int.}}$ describes the interaction between brane and bulk modes. Then, in the so called decoupling limit, $l_s\to 0$, these interaction terms vanish, as well as higher derivative corrections of the bulk and brane actions. We are left out with two decoupled theories, on the one hand, the previously obtained $\mathcal{N}=4$ super Yang-Mills theory and on the other hand free supergravity in ten dimensional flat space.

What is the precise counterpart of the low energy or decoupling limit in the geometric view of the D-branes? If we just take the limit $l_s\to 0$ in the geometry (3.32) (with $p=3$) we may naively conclude that we just get free gravity in flat space, however this is not what happens. Due to the redshifted relation between the proper energy of an object at position r and its energy as seen from an asymptotic observer

$$E_\infty = E_p\Big/H_3^{1/4}\,, \tag{3.47}$$

we can have very energetic excitations in the near horizon region although seen as low energy excitations from the asymptotic observer. The right way to take the decoupling limit in the near horizon region consists of taking the $l_s\to 0$ limit while keeping $u\equiv r/l_s^2$ fixed. Therefore, the near horizon metric scales with α' and the proper energy in string units l_sE_p is kept fixed so that string excitations survive this limit. Then we are left with type IIB strings in a $\mathrm{AdS}_5\times S^5$ and, again, free gravity in flat space.

Therefore, we find that the D3-branes are described by two sets of two decoupled theories. In both sets of theories we find free supergravity in flat space, therefore it seems natural to identify the remaining theories. These and other considerations suggest the so called AdS/CFT or Maldacena conjecture, which states a duality between a theory of gravity,

- Type IIB superstring theory on $\mathrm{AdS}_5 \times S^5$. The parameters entering in this theory are the string coupling g_s and string length $l_s = \sqrt{\alpha'}$, the integer flux of the F_5 form, $N = \int_{S^5} F_5$, and the radius for both the anti de Sitter space and the sphere, L.

And a gauge theory in absence of gravity,

- $\mathcal{N} = 4$ super Yang-Mills in four dimensions with gauge group $SU(N)$ and Yang-Mills coupling g_{YM} in the superconformal phase, i.e. $\langle \Phi \rangle = 0$ without spontaneous symmetry breaking.

Apart from the identification of the rank of the gauge group and the flux of the F_5 form, the Yang-Mills and the string coupling are also identified according to (3.45),

$$g_{\mathrm{YM}}^2 = 4\pi g_s \tag{3.48}$$

and therefore, according to (3.38), we also encounter the following relation between the AdS radius and the 't Hooft coupling $\lambda \equiv g_{\mathrm{YM}}^2 N$,

$$\frac{L}{l_s} = \lambda^{1/4} \ . \tag{3.49}$$

As presented here, this is the strongest version of the duality, supposed to hold for generic values of these parameters. However, it is very hard to use this duality in its full generality and, usually, one takes limits where it becomes more tractable. For example, we do not know how to treat string theory at strong coupling, then it is logic to restrict ourselves to the weak coupling regime $g_s \to 0$ with l_s/L fixed. According to the identification of the parameters, this amounts to consider the 't Hooft limit $N \to \infty$ with 't Hooft coupling λ fixed. Even in this limit we do not know how to study string theory in curved backgrounds with RR-fluxes, hence, the next limit to take is the low energy limit, where string theory is replaced by supergravity. This amounts to neglect stringy corrections by taking the $l_s/L \to 0$ limit, whose gauge theory counterpart is the strong 't Hooft coupling limit, $\lambda \to \infty$. All these regimes are summarized in table 3.1.

Although the duality in the last regime is the weakest one, it still is extremely useful, since it provides a weak-strong coupling relation between two different theories, in such a way that it allows to obtain information of a gauge theory in the strong coupling regime from supergravity computations. Moreover, this weaker version of the duality is the most worked out and where one finds a vast amount of supporting evidence. In the next chapters we will work with some extensions of this AdS/CFT duality, always working in this low energy regime of validity.

Exact equivalence	
Type IIB string theory on $AdS_5 \times S^5 \; \forall \, g_s, l_s/L$	$\mathcal{N}=4$ $SU(N)$ SYM in 4D $\forall \, g_{\rm YM}, N$
$4\pi g_s = g_{\rm YM}^2 \qquad (L/l_s)^4 = \lambda \equiv g_{\rm YM}^2 N$	

Classical limit	
Classical string theory $g_s \to 0$ with l_s/L fixed	t' Hooft limit $g_{\rm YM} \to 0$ with λ fixed

Low energy limit	
Type IIB supergravity $g_s \to 0 \quad l_s/L \to 0$	Large 't Hooft coupling $g_{\rm YM} \to 0 \quad \lambda \to \infty$

Table 3.1: The different regimes of the AdS/CFT conjecture.

3.1.4 Matching of the symmetries

One of the first checks of this duality is the matching of global symmetries of both theories.

For example, we saw that $\mathcal{N}=4$ super Yang-Mills, being a conformal theory, must remain invariant under dilatations

$$D : x^\mu \to \Lambda x^\mu \; , \tag{3.50}$$

for a given constant Λ and x^μ are the gauge theory coordinates. Now consider the metric (3.37). The coordinates x^μ can be considered as the world-volume coordinates of the D3-branes, and hence they are identify with the gauge theory coordinates, while r and the coordinates characterizing the five sphere are those transverse to the brane.

As expected, the metric (3.37) remains invariant under the transformation (3.50), supplemented with the additional transformation

$$r \to r/\Lambda \tag{3.51}$$

and consequently, we see that short distances or the UV regime of the gauge theory correspond to physics near the boundary ($r \to \infty$) of AdS and the way round, the IR regime of the gauge theory corresponds to physics near the horizon ($r \to 0$) of AdS. This leads to the identification of the RG flow of the gauge theory with the r coordinate of the AdS space-time. Although the RG flow is trivial in the present case of $\mathcal{N}=4$ super Yang-Mills because it is a conformal theory, the Maldacena conjecture can be generalized for non-conformal theories with non-trivial RG flow and a UV fixed point, but the identification of the RG flow with the radial coordinate still holds. In that case, in the gravity side we no longer have an AdS space-time but an asymptotically AdS one, in such a way that the dilatation (3.50)-(3.51) (and the whole transformations of the conformal group) remains an isometry only at the boundary of the asymptotically AdS space-time. This case will be studied in the next chapter.

Not only is $\mathcal{N}=4$ super Yang-Mills invariant under dilatations but it also is invariant under the conformal group $SO(2,4)$. This is precisely the isometry group of the AdS$_5$

space, which can be easily seen when writing the AdS_5 space as an hyperboloid,

$$X_0^2 + X_5^2 - \sum_{i=1}^{4} X_i^2 = L^2 \ , \tag{3.52}$$

embedded in the $p+1$-dimensional flat space with metric

$$\mathrm{d}s^2 = -\mathrm{d}X_0^2 - \mathrm{d}X_5^2 + \sum_{i=1}^{4} \mathrm{d}X_i^2 \ . \tag{3.53}$$

The $SO(6)$ R-symmetry group of $\mathcal{N}=4$ super Yang-Mills also matches the isometry of the S^5 part of the metric (3.37).

An analogous matching occurs for fermionic symmetries. $\mathrm{AdS}_5 \times S^5$ is a maximally supersymmetric solution of type IIB supergravity and thus it has 32 supercharges, the same as $\mathcal{N}=4$ super Yang-Mills, where one finds 16 Poincaré supercharges supplemented up to 32 superconformal charges by the whole superconformal group.

3.1.5 Field/Operator correspondence

The Maldacena conjecture as stated so far lacks of a prescription to relate in a precise way the two theories involved in the duality. That prescription was proposed in [48, 49] and relates correlation functions of gauge invariant operators of $\mathcal{N}=4$ super Yang-Mills, $\mathcal{O}(x)$, sourced by a function $\phi_0(x)$, to those of a dual field propagating in $\mathrm{AdS}_5 \times S^5$ space, $\phi(x,r)$, whose boundary value is precisely $\phi_0(x)$. This prescription is given by the following formula:

$$\left\langle e^{\int \mathrm{d}^4x \phi_0(x)\mathcal{O}(x)} \right\rangle_{\mathrm{CFT}} = Z_{\mathrm{string}}\Big[\phi(x,r)\big|_{\partial\mathrm{AdS}} = \phi_0(x) \Big] \ . \tag{3.54}$$

Correlation functions in the gauge theory can be obtained from the previous formula by differentiation with respect to ϕ_0. However, the right-hand side of this equation is not easy to compute and, as explained above, one usually takes the more tractable large N and large λ limit. In that case, the right hand side of (3.54) can be substituted by

$$Z_{\mathrm{string}} \approx e^{-S_{\mathrm{sugra}}} \ , \tag{3.55}$$

where S_{sugra} is the on-shell supergravity action.

As an example of this duality between operators in the gauge theory and AdS bulk fields, consider the Yang-Mills term $g_{\mathrm{YM}}^{-2} \operatorname{Tr} F^2$. Thinking of the coupling constant as a spurious field, we identify, according to the formula (3.54), the operator $\mathcal{O} = \operatorname{Tr} F^2$, sourced by $\phi_0 = g_{\mathrm{YM}}^{-2}$. On the other hand we have seen that the Yang-Mills coupling is equivalent to the string coupling and the latter is identified with the dilaton value at the boundary of AdS. Therefore, the field that would appear in the right-hand side of (3.54), dual to $\operatorname{Tr} F^2$, is the dilaton.

3.2 Supersymmetry conditions

Beyond the shadow of a doubt, extending the previous duality between $\mathcal{N} = 4$ super Yang-Mills and supergravity in $\mathrm{AdS}_5 \times S^5$ to other theories is a desirable objective. For example, it would be interesting to reduce the amount of supersymmetry, in particular, we will be interested in dualities for $\mathcal{N} = 1$ four dimensional gauge theories. Another interesting feature would be to break conformal invariance allowing a non-trivial RG flow, in such a way that the duality would allow to study this RG flow, these will be the purpose of next chapter.

In the present section we will describe the conditions that the ten-dimensional supergravity background needs to satisfy to host a $\mathcal{N} = 1$ supersymmetric gauge theory in four dimensions.

In section 3.1.2 we have presented just the bosonic content of type II supergravity. This is enough because we will be interested in bosonic solutions, where just the bosonic fields are turned on, while the fermion fields are set to zero. Schematically, the form of supersymmetric transformations is

$$\delta_\epsilon \text{Bosons} = \text{Fermions}\, \epsilon \ , \tag{3.56}$$

$$\delta_\epsilon \text{Fermions} = \text{Bosons}\, \epsilon \ , \tag{3.57}$$

where ϵ is the parameter of the supersymmetric transformation. To have some supersymmetry, there must exist some ϵ's that satisfy the system (3.56)-(3.57). For the setup in which just bosonic fields are turned on the first set of equations (3.56) is automatically fulfilled, while we must impose the second set of equations (3.57).

The fermionic content of type II supergravity consists of a doublet of gravitini, ψ_M, and a doublet of dilatini, λ, where each member of the doublet has opposed or the same chirality depending on whether we are in type IIA or type IIB supergravity, respectively. The supersymmetric transformations of these fields, which must be set to zero, are

$$\delta\psi^1_M = \left(\nabla_M - \frac{1}{4}\not{H}_M\right)\epsilon^1 + \frac{1}{16}e^{\Phi}\not{F}\Gamma_M\Gamma_{(10)}\epsilon^2 \ , \tag{3.58}$$

$$\delta\psi^2_M = \left(\nabla_M + \frac{1}{4}\not{H}_M\right)\epsilon^2 - \frac{1}{16}e^{\Phi}\not{\sigma(F)}\Gamma_M\Gamma_{(10)}\epsilon^1 \ , \tag{3.59}$$

$$\delta\lambda^1 = \left(\not{\partial}\Phi - \frac{1}{2}\not{H}\right)\epsilon^1 + \frac{1}{16}e^{\Phi}\Gamma^M\not{F}\Gamma_M\Gamma_{(10)}\epsilon^2 \ , \tag{3.60}$$

$$\delta\lambda^2 = \left(\not{\partial}\Phi + \frac{1}{2}\not{H}\right)\epsilon^2 - \frac{1}{16}e^{\Phi}\Gamma^M\not{\sigma(F)}\Gamma_M\Gamma_{(10)}\epsilon^1 \ , \tag{3.61}$$

where $\Gamma_{(10)}$ is the 10-dimensional chirality operator, F is the F_{IIA} or F_{IIB} polyform and σ is an operator that when acting on a p-form reverts its indices, explicitly

$$C_p = \frac{1}{p!}C_{i_1\ldots i_p}\mathrm{d}x^{i_1}\wedge\ldots\wedge\mathrm{d}x^{i_p} \Rightarrow \sigma(C_p) = \frac{1}{p!}C_{i_1\ldots i_p}\mathrm{d}x^{i_p}\wedge\ldots\wedge\mathrm{d}x^{i_1} = (-1)^{p(p-1)/2}C_p \ . \tag{3.62}$$

Finally, the slash notation means contraction with antisymmetrized product of gamma matrices, for example,

$$\not{C}_p = \frac{1}{p!}C_{M_1\ldots M_p}\Gamma^{M_1\ldots M_p} \ . \tag{3.63}$$

Suppose now that we want to build a solution to type II supergravity of the form of a warped product of four-dimensional Minkowski space and a six-dimensional internal manifold M,

$$\mathrm{d}s_{10}^2 = e^{2A(y)}\mathrm{d}s_{1,3}^2 + g_{mn}(y)\mathrm{d}y^m\mathrm{d}y^n \ , \tag{3.64}$$

with all background fluxes preserving the Poincaré symmetry of $\mathbb{R}^{1,3}$. This forces them to depend only on the internal coordinates y^m, moreover, the NS H-field can only have indices in the internal space and the RR polyform is decomposed as follows:

$$F = \mathrm{vol}_4 \wedge e^{4A}\tilde{F} + \hat{F} \ , \tag{3.65}$$

where vol_4 is the volume form of the unwarped Minkowski part of the space-time. $\hat{F}$ and $\tilde{F}$ have only internal indices and they are related by $\tilde{F} = \star_6\sigma(\hat{F})$.

According to the ansatz for the metric and the fluxes, the Lorentz group is broken to $SO(1,3)\times SO(6)$ and the ten dimensional spinors of supersymmetric transformations are split into $Spin(1,3)$ Weyl spinors, ζ^i, and $Spin(6)$ Weyl spinors, η_a^i,

$$\epsilon^1 = \zeta^1 \otimes \sum_a \eta_a^1 + c.c. \ , \tag{3.66}$$

$$\epsilon^2 = \zeta^2 \otimes \sum_a \eta_a^2 + c.c. \ . \tag{3.67}$$

To have four-dimensional $\mathcal{N}=1$ theory there should exist a single four-dimensional conserved spinor and two internal spinors. Then, the ten-dimensional Majorana-Weyl spinors of type II supergravity can be decomposed in the following way:

$$\begin{aligned} \epsilon^1 &= \zeta_+ \otimes \eta_+^1 + \zeta_- \otimes \eta_-^1 \ , \\ \epsilon^2 &= \zeta_+ \otimes \eta_\mp^2 + \zeta_- \otimes \eta_\pm^2 \ , \end{aligned} \tag{3.68}$$

for type IIA/IIB supergravity[3]. The signs $\pm$ refer to the chirality of the spinors and we choose a basis in which $(\eta_+)^* = \eta_-$. One further requires the internal spinors to be globally defined, this requirement can be translated into the language of G-structures.

3.2.1 G-structures

The two internal spinors η_+^1 and η_+^2 define for the ansatz (3.64) an $SU(3)\times SU(3)$ structure. We will be interested in the two extreme cases in which these spinors are always parallel, corresponding to an $SU(3)$ structure, and the case when they are nowhere parallel, this corresponds to an $SU(2)$ structure. Let us clarify what this means.

Definition of G-structure

To this purpose let us define what a G-structure is. Consider a compact manifold M of dimension d, which in our case will be the internal manifold in (3.64) (hence $d=6$), with some patches U_α and U_β. Consider the tangent bundle TM, with fiber in each point $p\in M$ the space of tangent vectors T_pM (later we will consider the dual cotangent bundle

[3] Here and from now on, upper signs will refer to type IIA and lower signs to IIB supergravity.

T^*M). Then the tangent frame bundle FM is the bundle over M with fiber in each point $p \in M$ the set of ordered bases of the tangent space T_pM. This just means that in each point of a patch U_α we have the set of d independent vectors

$$e_a = e^i{}_a \frac{\partial}{\partial x_i} , \tag{3.69}$$

forming a basis of T_pM. If we consider a different patch U_β the new frames e'_a, defined over points $p \in U_\beta$, must be related with the previous ones on the overlap of the two patches $U_\alpha \cap U_\beta$ through the expression

$$e'^i{}_a = \frac{\partial x'^i}{\partial x^j} e^j{}_a . \tag{3.70}$$

The relation between both frames can also be expressed in terms of the transition functions $t_{\beta\alpha}(p) \in GL(d, \mathbb{R})$,

$$e'^i{}_a = e^i{}_b (t_{\beta\alpha})^b{}_a , \tag{3.71}$$

which satisfy the consistency relations

$$t_{\alpha\beta} t_{\beta\alpha} = 1 , \qquad t_{\alpha\beta} t_{\beta\gamma} = t_{\alpha\gamma} , \tag{3.72}$$

the last one for a triple overlap of patches. Therefore, the transition functions form a group called the structure group, for the moment $GL(d, \mathbb{R})$. However, if we can choose the local frame in the different patches in such a way that it is possible to reduce the structure group to a subgroup $G \subset GL(d, \mathbb{R})$, it is said that the manifold M has a G-structure.

It is appropriate to characterize these G-structures by means of non-degenerated, globally defined, G-invariant tensors or spinors. Since these tensors or spinors are globally defined on the manifold M, it is possible to choose frames with some components not changing along the different patches of M. Therefore the transition functions must leave these objects invariant and hence reducing the structure group to G. This is illustrated in figure 3.1.

The fluxless case: $SU(3)$-structure

In the presence of a metric of the form (3.64), the structure group reduces to $Spin(3,1) \times Spin(6)$. Now let us consider the supersymmetry conditions (3.58)-(3.61) in the absence of fluxes. These conditions simplify to

$$\delta\psi_M^{1,2} = \nabla_M \epsilon^{1,2} = 0 , \qquad \delta\lambda^{1,2} = \partial\!\!\!/ \Phi \epsilon^{1,2} = 0 . \tag{3.73}$$

This requires a topological condition, that is the existence of globally defined non-vanishing spinors. Consider the internal part of the spinors $\epsilon^{1,2}$, and let us suppose that it is the same Weyl spinor η_+ with positive chirality. This spinor transforms in the fundamental **4** of $SU(4) \simeq Spin(6)$, so we can choose a basis in which this spinor takes the form

$$\eta_+ = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \eta_0 \end{pmatrix} . \tag{3.74}$$

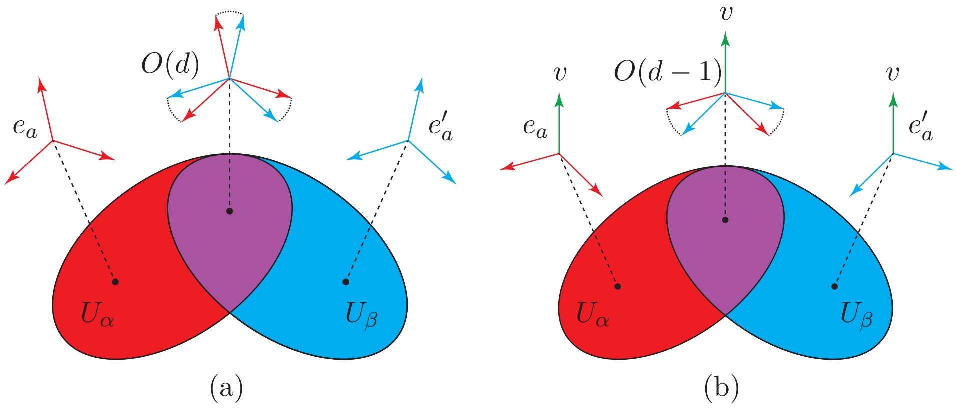

Figure 3.1: Suppose that the structure group in (a) is $O(d)$. If like in (b) we find a globally defined vector v, we can choose the frames with one component along this vector, then the transition functions must leave this vector invariant and the structure group is further reduced to $O(d-1)$. Figure reproduced from [50].

Therefore, as the spinor is globally defined and it must be invariant under the transition functions between patches, the transformations leaving η_+ invariant are reduced to the $SU(3)$ subgroup of $SU(4)$,

$$\begin{pmatrix} U & 0_{3\times 1} \\ 0_{1\times 3} & 1 \end{pmatrix}, \qquad \text{with } U \in SU(3). \tag{3.75}$$

Then the structure group is reduced to $SU(3)$, and hence, the internal manifold of (3.64) has a $SU(3)$ structure.

The supersymmetry conditions (3.73) also impose differential conditions. The first equation in (3.73) sets that ϵ^1 and ϵ^2, and thus η_+, must be covariantly constant, which implies that the internal space also has $SU(3)$ holonomy and therefore M is a Calabi-Yau three-fold. It can be shown that compactifying on a Calabi-Yau three fold preserves 1/4 of the supersymmetry. Eventually we will be interested in extensions of the AdS/CFT duality to less supersymmetric theories, in particular $\mathcal{N} = 1$ supersymmetric gauge theories, therefore this is the type of manifold we will compactify on to reduce the supersymmetry of the field theory from $\mathcal{N} = 4$ to $\mathcal{N} = 1$.

The second condition from (3.73) implies a constant dilaton. From the Minkowski part of (3.64), supersymmetry conditions only impose the four dimensional part of $\epsilon^{1,2}$ to be constant. In principle, one can choose the four dimensional part of $\epsilon^{1,2}$ to be different, leading to $\mathcal{N} = 2$ supersymmetry, but the supersymmetry conditions (3.58)-(3.61) in the presence of fluxes forces them to be equal, implying $\mathcal{N} = 1$ supersymmetry.

We can characterize the $SU(3)$-structure in a different way. Since we have at our disposal a globally defined nowhere vanishing spinor η_+, for the splitting (3.68) let us take $\eta^1_+ = a\eta_+$ and $\eta^2_+ = b\eta_+$ with $||\eta_+|| = \eta^\dagger_+\eta_+ = 1$, we can build out of it a real two-form and a complex three-form. In components, they are

$$J_{mn} = -\frac{i}{|a|^2}\eta^{1\dagger}_+\gamma_{mn}\eta^1_+ , \qquad \Omega_{mnp} = -\frac{i}{a^2}\eta^{1\dagger}_-\gamma_{mnp}\eta^1_+ , \tag{3.76}$$

where η_- is the complex conjugate, i.e. $\eta_- = \eta_+^*$ and $\gamma_{i_1 \dots i_n}$ is the antisymmetrized product of n gamma matrices. These forms have to satisfy the compatibility conditions,

$$J \wedge \Omega = 0 \ , \qquad \Omega \wedge \bar{\Omega} = -\frac{8i}{3!} J^3 \tag{3.77}$$

and the normalization condition,

$$\Omega \wedge \bar{\Omega} = 8i \mathrm{vol}(M) \ . \tag{3.78}$$

Then the topological requirement that characterizes an $SU(3)$-structure, the existence of a globally defined non-vanishing spinor, can be rephrased as the requirement of the existence of the globally defined forms (3.76) satisfying the conditions (3.77) and (3.78). Moreover, the differential condition of $SU(3)$ holonomy implies

$$\mathrm{d}J = \mathrm{d}\Omega = 0 \ . \tag{3.79}$$

J and Ω are called the Kähler form and the holomorphic $(3,0)$-form respectively.

$SU(3) \times SU(3)$ structure

The presence of fluxes changes the properties of supersymmetric solutions. In that case, it is convenient to work in the context of Generalized Complex Geometry. Very roughly speaking, this consists of taking the concepts presented in the definition of G-structures, which are based on the tangent bundle, and generalize them based on the concept of generalized tangent bundle, which is the sum of the tangent and the cotangent bundle $TM \oplus T^*M$.

In the absence of fluxes we have seen that the topological requirement of the existence of a globally defined spinor translates into the requirement that the structure group of TM is $SU(3)$. Now we are going to see a more general case in which the ansatz (3.64) admits two globally defined spinors, η^1 and η^2, (which do not need to be everywhere independent) and this topological requirement can also be expressed as the requirement that the generalized tangent bundle $TM \oplus T^*M$ has structure group $SU(3) \times SU(3)$. Depending on the relation between the two spinors, in some particular cases this is reduced to a $SU(3)$-structure or $SU(2)$-structure on TM.

From these spinors, η^1 and η^2, we can build two pure spinors on $TM \oplus T^*M$,

$$\Psi_1 = \eta_+^1 \otimes \eta_+^{2\dagger} \ , \qquad \Psi_2 = \eta_+^1 \otimes \eta_-^{2\dagger} \ , \tag{3.80}$$

recall that $\eta_+^* = \eta_-$. The definition of pure spinors is given in appendix 3.A, but essentially, these pure spinors can be seen as polyforms using the Clifford map (3.130) and the Fierz identity

$$\eta^1 \otimes \eta^{2\dagger} = \frac{1}{8} \sum_k \frac{1}{k!} (\eta^{2\dagger} \gamma_{i_k \dots i_1} \eta^1) \gamma^{i_1 \dots i_k} \ . \tag{3.81}$$

Therefore, the generalized tangent bundle has a $SU(d/2) \times SU(d/2)$-structure if there exist two globally defined pure spinors Ψ_1 and Ψ_2, which satisfy the compatibility conditions (for more details look at [50]),

$$\langle \Psi_1, \bar{\Psi}_1 \rangle = \langle \Psi_2, \bar{\Psi}_2 \rangle \neq 0 \ , \tag{3.82}$$

$$\langle \Psi_1, \mathbb{X} \cdot \Psi_2 \rangle = \langle \bar{\Psi}_1, \mathbb{X} \cdot \Psi_2 \rangle = 0 \ , \qquad \forall \mathbb{X} \in TM \oplus T^*M \ , \tag{3.83}$$

defined through the Mukai pairing of two polyforms

$$\langle \Psi_1, \Psi_2 \rangle = (\Psi_1 \wedge \sigma(\Psi_2))|_{(\mathrm{top})} \ , \tag{3.84}$$

where $|_{(\mathrm{top})}$ means the projection of the polyform on the form of higher order.

The most general relation between the two spinors η^1 and η^2 is given by

$$\eta^2_+ = c\eta^1_+ + \frac{1}{2} V^i \gamma_i \eta^1_- \ , \tag{3.85}$$

for some complex constant c and complex vector V^i. It is suitable to characterize both spinors in terms of orthogonal, normalized ones, η_+ and χ_+. If $||\eta^1_+|| = a$ and $||\eta^2_+|| = b$, we can take the following parameterization

$$\eta^1_+ = a\eta_+ \ , \qquad \eta^2_+ = b(\cos\varphi \eta_+ + \sin\varphi \chi_+) \ , \tag{3.86}$$

with $\eta^\dagger_+ \eta_+ = \chi^\dagger_+ \chi_+ = 1$, $\eta^\dagger_+ \chi_+ = 0$ and the parameter $\varphi \in [0, \pi/2]$ describes the angle between η^1 and η^2. Hence, at the points with $\varphi = 0$, these spinors are parallel and it is not necessary to introduce χ_+. At the points where $\varphi \neq 0$ the orthogonal spinors η_+ and χ_+ define, in general, a "local" $SU(2)$-structure which can also be defined in terms of the forms

$$z_i = \eta^\dagger_- \gamma_i \chi_+ \ , \qquad j_{ij} = -\frac{i}{2} \eta^\dagger_+ \gamma_{ij} \eta_+ + \frac{i}{2} \chi^\dagger_+ \gamma_{ij} \chi_+ \ , \qquad \omega_{ij} = i\chi^\dagger_+ \gamma_{ij} \eta_+ \ , \tag{3.87}$$

which, in principle, can change along the manifold and hence the name local. The first relation in (3.87) is just a redefinition of the complex vector appearing in (3.85),

$$V^i \equiv \frac{b}{a^*} z^i \sin\varphi \ , \qquad \chi_+ \equiv \frac{1}{2} z^i \gamma_i \eta_- \ , \tag{3.88}$$

once the following compatibility conditions are taken into account:

$$j \wedge \omega = \omega \wedge \omega = 0 \ , \qquad z \lrcorner j = z \lrcorner \omega = z \lrcorner \bar{\omega} = 0 \ , \qquad \omega \wedge \bar{\omega} = 2j^2 \ , \tag{3.89}$$

which must be fulfilled by the forms defining the local $SU(2)$-structure. Eventually, we will also use the notation $z \equiv v + iw$, with v and w real.

Making use of (3.80) and (3.81), we can write the pure spinors in terms of the above forms:

$$\Psi_1 = \frac{ab^*}{8} e^{\frac{1}{2} z \wedge \bar{z}} \wedge \left(\cos\varphi e^{-ij} + i \sin\varphi \omega\right) \ , \qquad \Psi_2 = \frac{ab}{8} z \wedge \left(\sin\varphi e^{-ij} - i\cos\varphi \omega\right) \ . \tag{3.90}$$

Then the following structures are defined [51, 52]:

$SU(3)$-structure: If $\varphi = 0$ everywhere. The pure spinors (Ψ_1, Ψ_2) are type $(0, 3)$. According to (3.90) they are given by

$$\Psi_1 = \frac{ab^*}{8} e^{\frac{1}{2} z \wedge \bar{z}} \wedge e^{-ij} \equiv \frac{ab^*}{8} e^{-iJ} \ , \qquad \Psi_2 = -i\frac{ab}{8} z \wedge \omega \equiv -i\frac{ab}{8} \Omega \ . \tag{3.91}$$

We can express the J and Ω forms in terms of spinors:

$$J = j - \frac{i}{2}\bar{z}\wedge z \qquad \Rightarrow \qquad J_{ij} = -i\eta_+^\dagger \gamma_{ij}\eta_+ \ , \tag{3.92}$$

$$\Omega = z\wedge\omega \qquad \Rightarrow \qquad \Omega_{ijk} = -i\eta_-^\dagger \gamma_{ijk}\eta_+ \ , \tag{3.93}$$

and we obtain essentially the same expressions appearing in (3.76). Then we identify J with the Kähler form and Ω with the holomorphic (3,0)-form, expressed in terms of the local $SU(2)$-structure forms. They must satisfy the compatibility conditions (3.77), which can be obtained from (3.89).

Static $SU(2)$-structure: If $\varphi = \pi/2$ everywhere. The pure spinors are type $(2,1)$ and are given by

$$\Psi_1 = i\frac{ab^*}{8}e^{\frac{1}{2}z\wedge\bar{z}}\wedge\omega \ , \qquad \Psi_2 = \frac{ab}{8}z\wedge e^{-ij} \ , \tag{3.94}$$

with z, j and ω given in terms of spinors in (3.87) and satisfying the compatibility conditions (3.89). Note that an $SU(2)$-structure can be identified as the intersection of the two $SU(3)$-structures defined by the spinors η_+ and χ_+,

$$J_1 = j - \frac{i}{2}\bar{z}\wedge z \ , \qquad \Omega_1 = z\wedge\omega \ , \tag{3.95}$$

$$J_2 = j - \frac{i}{2}z\wedge\bar{z} \ , \qquad \Omega_2 = \bar{z}\wedge\omega \ , \tag{3.96}$$

$$j = \frac{1}{2}(J_1 + J_2) \ , \qquad \omega = \frac{1}{2}\bar{z}\lrcorner\Omega_1 = \frac{1}{2}z\lrcorner\Omega_2 \ . \tag{3.97}$$

Intermediate $SU(2)$-structure: Neither of the previous cases. In a generic point where $\varphi \neq 0, \pi/2$, the pure spinors are type $(0,1)$. If the angle φ changes along the manifold, we have a dynamical $SU(2)$-structure.

To sum up, we have seen that the topological requirement of the existence of two globally defined spinors can be rephrased in terms of two other descriptions: the existence of globally defined forms or the existence of two pure spinors in $Spin(d,d)$, both of them subject to the fulfillment of some compatibility conditions.

3.2.2 Supersymmetric conditions in terms of pure spinors

As we have seen, from the two internal spinors η^1 and η^2 we can define two $Spin(6,6)$ pure spinors or polyforms, Ψ_1 and Ψ_2,

$$\Psi_1 = \Psi_\mp = \eta_+^1 \otimes (\eta_\mp^2)^\dagger \ , \qquad \Psi_2 = \Psi_\pm = \eta_+^1 \otimes (\eta_\pm^2)^\dagger \ , \tag{3.98}$$

where upper/lower indices correspond to type IIA/IIB supergravity, thus, chirality of pure spinors is interchanged when going from type IIA to type IIB supergravity. We have seen that the topological condition of the existence of globally defined spinors can be recast in terms of the pure spinors (3.98). To ensure supersymmetry we must also impose the differential conditions. In [51] (see also [53, 54]) it was found that the supersymmetry differential conditions (3.58)-(3.61) can be extremely simplified if they are written in terms of these pure spinors. The BPS conditions are now

$$\begin{aligned}
&\mathrm{d}_H\left(e^{2A-\Phi}\Psi_\mp\right) = e^{2A-\Phi}\mathrm{d}A\wedge\bar{\Psi}_\mp + \frac{e^{2A}}{16}\left[\left(|a|^2-|b|^2\right)F_- + i\left(|a|^2+|b|^2\right)\star_6 F_+\right] \ ,\\
&\mathrm{d}_H\left(e^{2A-\Phi}\Psi_\pm\right) = 0 \ ,
\end{aligned} \tag{3.99}$$

for type IIA/IIB. The RR fluxes entering on the right-hand side of this equation are defined for the type IIA case as

$$F_-^{\mathrm{IIA}} = F_0 - F_2 + F_4 - F_6 \ , \qquad F_+^{\mathrm{IIA}} = F_0 + F_2 + F_4 + F_6 \ , \tag{3.100}$$

and for the type IIB case

$$F_-^{\mathrm{IIB}} = F_1 - F_3 + F_5 \ , \qquad F_+^{\mathrm{IIB}} = F_1 + F_3 + F_5 \ . \tag{3.101}$$

Indeed it turns out, as we will see in a moment (3.118), that the norm of the spinors η^1 and η^2 has to be the same, $||\eta^1||^2 = |a|^2 = ||\eta^2||^2 = |b|^2$, to allow for supersymmetric D-branes. This further simplifies equation (3.99).

Moreover, it can be shown [55–57] that equations (3.99) together with Bianchi identities imply the equations of motion for all the type II supergravity fields, i.e: Einstein's equations, the dilaton equation and the equations of motion for the NS field, H, and the RR fields, F.

This is major simplifications due to the fact that BPS equations are first order differential equations and it is easier to solve first order rather than second order differential equations, as it is the case for the supergravity equations of motion.

This also holds in the presence of backreacting sources [57], if the sources are introduced in a supersymmetric way, which can be done with the help of generalized calibrations.

3.2.3 Calibrations

One is often interested, particularly in the context of the AdS/CFT correspondence, in the possibility that D-branes may wrap certain submanifolds of the geometry in a supersymmetric way. It turns out that supersymmetric branes, in the case where no flux or D-brane world-volume fields are turned on, minimize their world-volume [58] and they are said to be calibrated in the sense of [59]. Minimizing the world-volume of a submanifold is a difficult task involving second order differential equations. However, if the submanifold is calibrated and thus, it is endowed with a calibration form, the problem reduces to solving first-order differential equations.

A calibration form ϖ is a closed l-form on a manifold M that bounds the volume of any l-dimensional oriented submanifold Σ,

$$\mathrm{d}^l\sigma\sqrt{|\det g|_\Sigma|} \geq \varpi|_\Sigma \ . \tag{3.102}$$

In every point there must exist subspaces saturating the above bound, then the submanifold Σ is said to be calibrated if in every point $p\in\Sigma$ the bound (3.102) is saturated,

$$\mathrm{d}^l\sigma\sqrt{|\det g|_\Sigma|} = \varpi|_\Sigma \ . \tag{3.103}$$

It is easy to see that a calibrated submanifold Σ is that of minimal volume within its homology class. To see this, consider another submanifold Σ' in the same homology class, then there is a third submanifold $\mathcal{B}$ whose border corresponds to $\partial\mathcal{B} = \Sigma' - \Sigma$. Then we find

$$\begin{aligned}\mathrm{vol}(\Sigma') = \int_{\Sigma'} \mathrm{d}^l\sigma \sqrt{|\det g|_{\Sigma'}|} \geq \int_{\Sigma'} \varpi|_{\Sigma'} = \int_{\Sigma} \varpi|_{\Sigma} + \int_{\mathcal{B}} \mathrm{d}\varpi|_{\mathcal{B}} \\ = \int_{\Sigma} \varpi|_{\Sigma} = \int_{\Sigma} \mathrm{d}^l\sigma \sqrt{|\det g|_{\Sigma}|} = \mathrm{vol}(\Sigma) \ ,\end{aligned} \quad (3.104)$$

where we have used the calibration bound (3.102) for Σ', then we have decomposed the integral over Σ' as a sum over Σ and $\partial\mathcal{B}$ and we have applied the Stoke's theorem to the latter, which vanishes due to the fact that ϖ is closed. Finally, we have used that Σ is calibrated (3.103).

Of course, in the geometries we are interested in we will have both NS and RR fluxes and this simple calibration condition is not enough to establish supersymmetric D-brane configurations. In this case one has to consider the concept of "generalized calibrations" to include the possibility of fluxes [53, 60–62].

For a static space-filling D-brane with worldvolume field strength $\mathcal{F} = B|_\Sigma + 2\pi l_s^2 \mathrm{d}A$ wrapping an internal l-cycle Σ, we can define its energy density as

$$\mathcal{E}(\Sigma, \mathcal{F}) = e^{4A}\left(e^{-\Phi}\sqrt{|\det(g|_\Sigma + \mathcal{F})|}\mathrm{d}\sigma^1 \wedge \ldots \wedge \mathrm{d}\sigma^l - \left(\tilde{C}|_\Sigma \wedge e^{\mathcal{F}}\right)\Big|_{(l)}\right) \ , \quad (3.105)$$

where $\tilde{C}$ is the potential of the field strength $\tilde{F}$ in the $4+6$ decomposition (3.65). The energy per unit volume of the external space is

$$E(\Sigma, \mathcal{F}) = -\frac{S_{D(3+l)}}{T_{(3+l)}\mathrm{vol}_4} = \int_\Sigma \mathcal{E}(\Sigma, \mathcal{F}) \ . \quad (3.106)$$

Then a generalized calibration form ϖ is a d_H-closed polyform on M of definite parity which satisfies the algebraic condition given by the bound

$$\mathcal{E}(\Sigma, \mathcal{F}) \geq \left(\varpi|_\Sigma \wedge e^{\mathcal{F}}\right)\Big|_{(l)} \ . \quad (3.107)$$

Then a generalized submanifold, made of the pair $(\Sigma, \mathcal{F})$, is calibrated if in every point $p \in \Sigma$ the bound is saturated.

Due to the fact that ϖ is d_H closed, by similar arguments to those presented in the absence of fluxes or world-volume fields, one can see that a generalized calibrated submanifold minimizes its energy (within its generalized homology class).

The generalized calibration ϖ is not globally defined since it depends on the gauge choice for the RR potentials. For example, under the gauge transformation

$$\delta\tilde{C} = e^{-4A}\mathrm{d}_H\lambda \ , \quad (3.108)$$

in terms a given polyform λ made of even/odd forms for type IIA/IIB supergravity, the generalized calibration transforms as

$$\delta\varpi = -\mathrm{d}_H\lambda \ . \quad (3.109)$$

Therefore, an alternative definition of the generalized calibration is often used, where the energy density in (3.107) is replaced by just its DBI part,

$$e^{4A-\Phi}\sqrt{|\det(g|_\Sigma+\mathcal{F})|}\,\mathrm{d}\sigma^1\wedge\ldots\wedge\mathrm{d}\sigma^l \geq \left(\tilde{\varpi}|_\Sigma\wedge e^{\mathcal{F}}\right)\Big|_{(l)} . \tag{3.110}$$

With this alternative definition the generalized calibration is gauge invariant, but it is no longer d_H-closed,

$$\mathrm{d}_H\tilde{\varpi} = e^{4A}\tilde{F} . \tag{3.111}$$

Both $\tilde{\varpi}$ and ϖ are related by the formula

$$\tilde{\varpi} = \varpi + e^{4A}\tilde{C} . \tag{3.112}$$

Relation with κ-symmetry

Another approach to check if a D-brane embedding is supersymmetric, previous to the use of calibrations, was κ-symmetry, [63–67]. The consideration of κ-symmetric embeddings will provide us with natural calibration forms, as we will see.

Consider a Dp-brane extended along Σ_p, which consists of the time direction plus q flat directions and a $p-q$ cycle of the internal space. The D-brane preserves the supersymmetry of the background generated by

$$\epsilon = \begin{pmatrix} \epsilon^1 \\ \epsilon^2 \end{pmatrix} , \tag{3.113}$$

if it satisfies the relation [68]

$$\epsilon = \Gamma\epsilon , \tag{3.114}$$

where Γ is the κ-symmetry matrix of the Green-Schwarz formulation for D-branes. This operator is given by

$$\Gamma = \begin{pmatrix} 0 & \hat{\Gamma} \\ \hat{\Gamma}^{-1} & 0 \end{pmatrix} , \tag{3.115}$$

$$\hat{\Gamma} = \frac{1}{\sqrt{|\det(g|_\Sigma+\mathcal{F})|}}\sum_{2n+l=p+1}\frac{1}{n!l!2^n}\epsilon^{a_1\ldots a_{2n}b_1\ldots b_l}\mathcal{F}_{a_1a_2}\ldots\mathcal{F}_{a_{2n-1}a_{2n}}\Gamma_{b_1\ldots b_n} , \tag{3.116}$$

$$\hat{\Gamma}^{-1}(\mathcal{F}) = \hat{\Gamma}^\dagger(\mathcal{F}) = (-1)^{\mathrm{Int}[(p+3)/2]}\hat{\Gamma}_{Dp}(-\mathcal{F}) , \tag{3.117}$$

where a_i and b_i are indices on Σ, hence, Γ_a are the ten dimensional gamma matrices pulled back to the D-brane.

From the κ-symmetry condition (3.114) it follows that ϵ^1 and ϵ^2 must have equal norm, and therefore, after the splitting (3.68)

$$||\eta^1_+||^2 = ||\eta^2_+||^2 . \tag{3.118}$$

Furthermore, from the definition of Γ, it is easy to see that we can build the projector $P=\frac{1}{2}(\mathbb{I}-\Gamma)$, from which we obtain the bound

$$\epsilon^\dagger P\epsilon = \epsilon^\dagger P^\dagger P\epsilon \geq 0 \qquad \Rightarrow \qquad \epsilon^\dagger\epsilon \geq \epsilon^\dagger\Gamma\epsilon , \tag{3.119}$$

which, after using the definition of Γ, can be stated in the form

$$\sqrt{|\det(g|_\Sigma + \mathcal{F})|} \geq \frac{1}{||\epsilon^1||^2} \sum_{2n+l=p+1} \frac{1}{n!l!2^n} \epsilon^{a_1\dots a_{2n}b_1\dots b_l} \mathcal{F}_{a_1a_2}\dots\mathcal{F}_{a_{2n-1}a_{2n}} \epsilon^{2\dagger}\Gamma_{b_1\dots b_n}\epsilon^1 \ . \tag{3.120}$$

The equality in the bound holds when (3.114) is fulfilled, therefore, the previous equation is analogous to the calibration condition given in (3.110) and provides a way to find the calibration forms in terms of spinor bilinears. Indeed, if one further works out the splitting (3.68) on the operator $\hat{\Gamma}$ and one considers how the κ-symmetry conditions realize in the four and six-dimensional parts, one arrives exactly at the condition (3.110) with calibrations given by (see [53])

$$\text{Spacetime filling brane } (q=3): \qquad \tilde{\varpi} = -\frac{8e^{4A-\Phi}}{|a|^2}\mathrm{Re}(i\Psi_1) \tag{3.121}$$

$$\text{Domain wall } (q=2): \qquad \tilde{\varpi} = \frac{8e^{3A-\Phi}}{|a|^2}\mathrm{Re}(e^{i\theta}\Psi_2) \tag{3.122}$$

$$\text{Effective string } (q=1): \qquad \tilde{\varpi} = \frac{8e^{2A-\Phi}}{|a|^2}\mathrm{Re}(\Psi_1) \tag{3.123}$$

$$\text{Effective particle } (q=0): \qquad \text{Not supersymmetric}$$

where θ is an arbitrary constant phase. After some work, (details can be found in [53]) one can see that the calibration condition for the calibrations given by (3.121) and (3.122) reduces to the imaginary and real parts of the first equation in (3.99), respectively, while the calibration (3.123) reproduces the remaining supersymmetric condition in (3.99). Therefore, there is a nice interpretation of the supersymmetric conditions that the supergravity background must fulfill in terms of allowed D-brane configurations (when both internal spinors have the same norm).

3.A Pure spinors and polyforms

A polyform is a sum of forms of different degree. Then, a section of the generalized tangent bundle, call it a generalized vector, $\mathbb{X} = (x, X)$ where $x \in TM$ and $X \in T^*M$, acts on a polyform Ψ as

$$\mathbb{X}\cdot\Psi = x \lrcorner \Psi + X \wedge \Psi \ , \tag{3.124}$$

where the contraction between two p-forms $A_p \lrcorner B_p$ is defined as

$$A_p \lrcorner B_p = \frac{1}{p!} A^{\mu_1\dots\mu_p} B_{\mu_1\dots\mu_p} \ . \tag{3.125}$$

The generalized tangent bundle is also equipped with a metric, defined by the coupling of vectors and one-forms

$$\eta(\mathbb{X},\mathbb{Y}) = \frac{1}{2}(x\lrcorner Y + y \lrcorner X) \ . \tag{3.126}$$

Therefore, elements of the generalized tangent bundle act as a Clifford algebra

$$\{\mathbb{X},\mathbb{Y}\}\cdot\Psi = (\mathbb{X}\cdot\mathbb{Y} + \mathbb{Y}\cdot\mathbb{X})\cdot\Psi = 2\eta(\mathbb{X},\mathbb{Y})\Psi \ , \tag{3.127}$$

where, in comparison to the usual Clifford algebra of gamma matrices, $\mathbb{X}$ and $\mathbb{Y}$ play the role of the matrices and η is a $SO(d,d)$ metric. In this way, we see that there should be an isomorphism between polyforms and spinors, since we can also consider Ψ as a $Spin(d,d)$ spinor. Irreducible representations of $Spin(d,d)$ are Majorana-Weyl. The Majorana condition corresponds to consider real forms, while the Weyl condition restricts the polyforms to a sum of forms of even order, for positive chirality, or a sum of forms of odd order, for negative chirality.

In addition, we will be interested in pure spinors. Ψ is a pure spinor if the space

$$L_\Psi = \{\mathbb{X} \in TM \oplus T^*M | \mathbb{X} \cdot \Psi = 0\} \tag{3.128}$$

has dimension d, half that of $TM \oplus T^*M$. For example, pure spinors of the more familiar $Spin(d)$ (with d even) are those spinors, ψ, for which the space

$$L_\psi = \left\{ z \in \mathbb{C}^d | z_a \gamma^a \psi = 0 \right\} \tag{3.129}$$

has dimension $d/2$.

$Spin(d,d)$ pure spinors on $TM \oplus T^*M$ can be written as a bi-product of $Spin(d)$ spinors, which are pure in six dimensions, since any $Spin(6)$ Weyl spinor is pure. This can be done through the Clifford map

$$C = \sum_k \frac{1}{k!} C^{(k)}_{i_1 \dots i_k} \mathrm{d}x^{i_1} \wedge \dots \wedge \mathrm{d}x^{i_k} \quad \longleftrightarrow \quad C = \sum_k \frac{1}{k!} C^{(k)}_{i_1 \dots i_k} \gamma^{i_1 \dots i_k} \ , \tag{3.130}$$

which establishes an isomorphism between polyforms and bi-spinors.

Finally, any non-degenerate complex pure spinor can be decomposed in the form [69]

$$\Psi = \Omega_k \wedge e^{i\omega + b} \ , \tag{3.131}$$

where ω, b are real two forms and Ω_k is a complex decomposable k-form and it is said that the pure spinor is of type k, i.e. the type of a pure spinor is the smallest degree of the component forms when it is seen as a polyform.

Chapter 4

Gravity dual to $\mathcal{N} = 1$ SQCD-like theories with massive flavors

In this chapter we want to explore some extensions of the AdS/CFT duality to non-maximally supersymmetric backgrounds where conformal symmetry is lost and it is only recovered asymptotically.

An important type of these backgrounds corresponds to gravity duals to $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills theory with an arbitrary number N_f of fundamental flavors. The $N_f = 0$ case was constructed by Maldacena and Núñez (MN) [70], building up on a geometry previously found in [71]. Massless flavors in the fundamental representation of the $SU(N)$ gauge theory can be incorporated following the idea of [72] by adding N_f spacetime filling branes. The resulting holographic models [73–75] have led to many interesting physical insights, including, for instance, aspects of Seiberg duality (see also [76–78]). However, the presence of a singularity in the IR region limits the applicability of this geometry. Recently, a new $\mathcal{N} = 1$ supersymmetric geometry has been found by Conde, Gaillard and Ramallo [79] which includes the previous ones as particular cases, but more generally can circumvent the IR singularity. The aim of this chapter is to use this framework to construct new solutions and investigate new physical properties, with the aim of understanding the extent to which these geometries can describe aspects of $\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills theory with N_f fundamental massive flavors.

$\mathcal{N} = 1$ supersymmetric $SU(N)$ Yang-Mills theories with N_f massless flavors, analogously to their non-supersymmetric counterpart, are likely to abandon the QCD-like confined phase for sufficiently large number of flavors and develop a conformal phase before the loss of asymptotic freedom. The restoration of conformal symmetry and the presence of a so called conformal window in the number of flavors would thus identify a new family of non-Abelian gauge theories which is worth to explore.

Until now, the emergence of conformal symmetry in theories without supersymmetry has been discussed in the context of Schwinger-Dyson equations for chiral symmetry in the ladder approximation [80, 81], truncated non-perturbative RG flows [82, 83], supersymmetry inspired conjectures [84] and deformation theory [85]. The proof of existence of a conformal window however depends on our ability to describe these theories in a non-perturbative manner, following the evolution of parameters all the way from strong coupling to weak coupling. Lattice studies are currently the only ones to provide a fully non perturbative analysis, and in $N=3$ QCD they have recently produced evidence that $N_f=12$ is plausibly close to the end-point of a conformal window [86–91]. Similar results have been found in [92] using the world-line formalism.

A further insight comes from supersymmetric gauge field theories. The renormalization group of $\mathcal{N}=1$ supersymmetric QCD (SQCD) has been extensively studied, and the perturbative β function for the gauge coupling is given by the well known NSVZ formula [93]. However, any rigorous prediction for the existence and width of a conformal window would require to derive the β functions and anomalous dimensions of the theory in a non-perturbative way. Holographic techniques may allow to study the renormalization group flow beyond perturbation theory as we are going to see. In particular, for $N_f=2N$, a prediction arising from our study is the existence of a non-trivial UV fixed point at some strong coupling g_*. Consistency with the RG evolution at weak coupling requires the existence of an IR fixed point at $g'_*<g_*$, as we shall discuss. Notice that the presence of a UV fixed point at strong coupling in addition to an IR fixed point has been already conjectured in the pioneering work by Banks-Zaks [94], and might lead to a mechanism of disappearance of the conformal window via the annihilation of a pair of fixed points as suggested in [95].

This chapter is organized as follows. After briefly comment on the twisting mechanism that allows to preserve some supersymmetry when branes wrap curved manifolds, we will review the Maldacena-Núñez supergravity solution and how the physical properties of the field theory dual are obtained from this background. Later, we will review how massless flavors are added and the field theory physics these new backgrounds account for, for example, the realization of Seiberg duality. Finally, we will review the backgrounds recently constructed in [79], where flavors are massive and we will give some physical criteria to uniquely select a single background. Then we will compute the gauge coupling β-functions for the three cases: i) $N_f<2N$, ii) $N_f=2N$ and iii) $N_f>2N$; and we will make some comments about Seiberg duality.

4.1 Wrapped D-branes and the topological twist

To reduce the amount of supersymmetry to get an $\mathcal{N}=1$ gauge theory we can consider D-branes wrapping cycles of Calabi-Yau manifolds in a supersymmetric way. In principle, a D-brane with world-volume along a curved manifold cannot preserve supersymmetry because we should find a covariantly constant spinor and, since we are on a curved manifold, we have to take into account the spin connection, ω_μ. Then we have to consider the equation

$$(\partial_\mu+\omega_\mu)\epsilon=0\ , \tag{4.1}$$

which does not admit a solution at first sight. However, supersymmetry can be preserved if the field theory living on the world-volume of the brane is topologically twisted [96]. Let us explain what this means.

If the theory possesses some global R-symmetry, we can consider an external gauge field that couples to this R-symmetry current, so that equation (4.1) is modified to

$$(\partial_\mu + \omega_\mu - A_\mu)\epsilon = 0 \ . \tag{4.2}$$

Then, after the identification $\omega_\mu = A_\mu$, it is easy to find supersymmetry preserving solutions, because finding covariantly constant spinors reduces in this case to find just a constant spinor,

$$\partial_\mu \epsilon = 0 \ . \tag{4.3}$$

If the Dp-brane is wrapping a q-dimensional cycle, Σ, inside the Calabi-Yau fold, A_μ is identified with the connection of the normal bundle $N\Sigma$ and the fact that $N\Sigma$ is not trivially fibered over Σ allows for the identification between A_μ and the spin connection on Σ, ω_μ. In this way, the theory on the $(p+1)$-dimensional world-volume is twisted, this means that the behavior of the different fields under a Lorentz transformation, namely their spin, is changed, but the important thing is that the field theory on the flat $(p+1-q)$-dimensional part remains untwisted.

For example consider a D5-brane in flat space. The mere presence of the brane breaks the Lorentz group of the ten dimensional space into

$$SO(1,9) \to SO(1,5) \times SO(4)_R \tag{4.4}$$

and $SO(4)$ is identified with the R-symmetry group of the brane world-volume gauge theory. If the D5-brane wraps a two-sphere, we have the further breaking

$$SO(1,5) \times SO(4)_R \to SO(1,3) \times SO(2) \times SO(4)_R \ . \tag{4.5}$$

$SO(2) \simeq U(1)$ is the tangent bundle of the sphere and the twist is performed identifying this $U(1)$ with some $U(1) \subset SO(4)_R \simeq SU(2)_l \times SU(2)_r$, for example, $U(1)_l \subset SU(2)_l$ or equivalently $U(1)_r \subset SU(2)_r$. Working out how the field content of the brane world-volume transforms after this symmetry breaking pattern and the group identification, one arrives at the field content of an $\mathcal{N} = 1$ vector multiplet in four dimensions.

4.2 Maldacena-Núñez background

Here we elaborate on the model presented in [70], based on a four-dimensional supergravity solution previously found in [71]. In the setup of [70], Maldacena and Núñez considered a stack of N D5-branes wrapping a compact supersymmetric 2-cycle inside a CY three-fold.

By the twisting procedure explained above the background preserves four supercharges and it is claimed to be dual to $\mathcal{N} = 1$ super Yang-Mills in four dimensions plus some Kaluza-Klein adjoint matter. In the string frame, the ten dimensional metric is given by

$$\mathrm{d}s^2 = g_s \alpha' N e^{\Phi(r)} \left[\frac{1}{g_s \alpha' N} \mathrm{d}x_{1,3}^2 + \mathrm{d}r^2 + e^{2h(r)}(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2) + \frac{1}{4}(\tilde{\omega}^i - A^i)^2 \right] \ . \tag{4.6}$$

The angles $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$ parametrize a two-sphere. The coordinate r is a dimensionless quantity related to the actual radial coordinate by a factor $\sqrt{g_s \alpha' N}$ to recover the right dimensions. $\tilde{\omega}$ are the $SU(2)$ left-invariant one-forms parametrizing the three sphere:

$$
\begin{aligned}
\tilde{\omega}^1 &= \cos\psi \mathrm{d}\tilde{\theta} + \sin\psi \sin\tilde{\theta}\mathrm{d}\tilde{\phi} \ , \\
\tilde{\omega}^2 &= -\sin\psi \mathrm{d}\tilde{\theta} + \cos\psi \sin\tilde{\theta}\mathrm{d}\tilde{\phi} \ , \\
\tilde{\omega}^3 &= \mathrm{d}\psi + \cos\tilde{\theta}\mathrm{d}\tilde{\phi} \ ,
\end{aligned}
\tag{4.7}
$$

where the angles take values in $\tilde{\theta} \in [0,\pi]$, $\tilde{\phi} = [0, 2\pi)$ and $\psi \in [0, 4\pi)$. The dilaton $\Phi(r)$, the function $h(r)$ and the $SU(2)_L$ gauge field A, turned on to implement the twist, are given by

$$
e^{-2\Phi} = e^{-2\Phi_0}\frac{2e^h}{\sinh(2r)} \ , \qquad e^{2h} = r\coth(2r) - \frac{r^2}{\sinh^2(2r)} - \frac{1}{4} \ , \tag{4.8}
$$

$$
\begin{aligned}
A^1 &= -a(r)\mathrm{d}\theta \ , \\
A^2 &= a(r)\sin\theta\, \mathrm{d}\phi \ , \qquad \text{with} \quad a(r) = \frac{2r}{\sinh(2r)} \ . \\
A^3 &= -\cos\theta\, \mathrm{d}\phi \ ,
\end{aligned}
\tag{4.9}
$$

There is also a RR three-form F_3, satisfying the Bianchi identity $\mathrm{d}F_3 = 0$,

$$
F_3 = \mathrm{d}C_2 = -\frac{g_s\alpha' N}{4}\bigwedge_{i=1}^{3}(\tilde{\omega}^i - A^i) + \frac{g_s\alpha' N}{4}\sum_{i=1}^{3} F^i \wedge (\tilde{\omega}^i - A^i) \ , \tag{4.10}
$$

$$
\begin{aligned}
C_2 = \frac{g_s\alpha' N}{4}\Big(& a(r)(\sin\theta\, \mathrm{d}\phi \wedge \tilde{\omega}_2 - \mathrm{d}\theta \wedge \tilde{\omega}_1) \\
&+ (\psi + \psi_0)(\sin\theta\, \mathrm{d}\theta \wedge \mathrm{d}\phi - \sin\tilde{\theta}\, \mathrm{d}\tilde{\theta} \wedge \mathrm{d}\tilde{\phi}) + \cos\theta\cos\tilde{\theta}\mathrm{d}\phi \wedge \mathrm{d}\tilde{\phi}\Big) \ ,
\end{aligned}
\tag{4.11}
$$

where we also show the corresponding potential $C_{(2)}$ and $F^i = \mathrm{d}A^i + \frac{1}{2}\epsilon^{ijk}A^j \wedge A^k$ is the field strength corresponding to the gauge field A. Moreover, the color D5-branes have been encoded into a flux, having N of them translates into the flux quantization condition for the RR three-form,

$$
-\frac{1}{2\kappa_{10}^2 T_5}\int_{S^3(\tilde{\theta},\tilde{\phi},\psi)} F_3\big|_{S^3} = N \ , \tag{4.12}
$$

where $2\kappa_{10}^2 T_5 = 4\pi^2 g_s \alpha'$. Notice that the factor $a(r)$ in the gauge field makes this background regular, as oppose to the type IIB supergravity solution with $a = 0$.

To identify the relevant cycle the D5-branes are wrapping we should consider the five dimensional internal part of the metric (4.6) in the $r \to \infty$ limit,

$$
\begin{aligned}
\mathrm{d}s_5^2 = g_s\alpha' N e^{\Phi(r)}\Big[& r(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) \\
&+ \frac{1}{4}(\mathrm{d}\tilde{\theta}^2 + \sin^2\tilde{\theta}\mathrm{d}\tilde{\phi}^2) + \frac{1}{4}(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi + \cos\tilde{\theta}\mathrm{d}\tilde{\phi})^2\Big] \ ,
\end{aligned}
\tag{4.13}
$$

which is the metric of a $T^{1,1}$ manifold, we will comment more on this metric in section 5.2. Together with the r coordinate, this metric defines de CY_3 manifold of the target space, characterized by the topologically non-trivial three and two cycles

$$S^2: \quad \theta = \pm\tilde{\theta} \ , \qquad \phi = -\tilde{\phi} \ , \quad \psi = \text{any} \ . \tag{4.14}$$

$$S^3: \quad \theta = \phi = 0 \ . \tag{4.15}$$

The two cycle the D5-branes are wrapping can be either $\theta = \tilde{\theta}$ or $\theta = -\tilde{\theta}$ in (4.14), since both have the same volume and are equivalent from the field theory point of view. The value of ψ is determined by requiring that the cycle is that of minimal volume. One gets

$$S^2: \quad \theta = \tilde{\theta} \ , \qquad \phi = -\tilde{\phi} \ , \quad \psi = 0 \text{ mod } 2\pi \ . \tag{4.16}$$

$$S^2: \quad \theta = -\tilde{\theta} \ , \quad \phi = -\tilde{\phi} \ , \quad \psi = \pi \text{ mod } 2\pi \ . \tag{4.17}$$

4.2.1 $\mathcal{N} = 1$ super Yang-Mills field theory

Now we are going to collect some basic facts about $SU(N)$ $\mathcal{N} = 1$ super Yang-Mills field theory here, since we will want to see later if the Maldacena-Núñez solution presented in the previous section can reproduce them.

The $\mathcal{N} = 1$ SYM action can be written in terms of the holomorphic gauge coupling,

$$\tau \equiv \frac{\theta_{\rm YM}}{2\pi} + i\frac{4\pi}{g_{\rm YM}^2} \ , \tag{4.18}$$

in the simple form

$$\begin{aligned} S_{\mathcal{N}=1 \ \rm SYM} &= \frac{\tau}{16\pi i}\int \mathrm{d}^4x\, \mathrm{d}^2\theta\, \mathcal{W}^{a\alpha}\mathcal{W}^a_\alpha + h.c. \\ &= \int \mathrm{d}^4x \left[-\frac{1}{4g_{\rm YM}^2}F^{a\mu\nu}F^a_{\mu\nu} - \frac{\theta_{\rm YM}}{32\pi^2}F^{a\mu\nu}\tilde{F}^a_{\mu\nu} \right. \\ &\qquad \left. + \frac{i}{g_{\rm YM}^2}\bar{\lambda}^a\bar{\sigma}^\mu D_\mu\lambda^a + \frac{1}{2g_{\rm YM}^2}D^aD^a \right] \ , \end{aligned} \tag{4.19}$$

where $\tilde{F}^{a\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\alpha\beta}F^a_{\alpha\beta}$ and

$$\mathcal{W}^a_\alpha = -i\lambda^a_\alpha(y) + \theta_\alpha D^a(y) - (\sigma^{\mu\nu}\theta)_\alpha F^a_{\mu\nu}(y) - (\theta\theta)\sigma^\mu D_\mu\bar{\lambda}^a(y) \tag{4.20}$$

is the field strength chiral superfield, written in terms of the superspace coordinate $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$. $F^a_{\mu\nu}$ is the usual field strength of the gauge field A^a_μ, which, together with the gaugino, λ^a_α, and the auxiliary scalar field, D^a, form the vector supermultiplet $V = (A^a_\mu, \lambda^a, D^a)$.

β-function and the holomorphic coupling:

By holomorphicity arguments, we know that the (holomorphic) gauge coupling runs only at one loop and its β-function is given by [97–99]

$$\beta(g_{\rm YM}) = \mu\frac{\partial g_{\rm YM}}{\partial\mu} = -\frac{b}{16\pi^2}g_{\rm YM}^3 \ , \tag{4.21}$$

where μ is an arbitrary renormalization scale and the coefficient b is given by

$$b = \frac{11}{3}T(\text{Ad}) - \frac{2}{3}T(F) - \frac{1}{3}T(S) \ . \tag{4.22}$$

The index $T(R)$ is the one-half Dynkin index of the representation R. In this case, $T(F)$ and $T(S)$ mean the sum of indices over all the fermions and all the complex scalars in their respective representations. For the case at hand, $\mathcal{N}=1$ pure super Yang-Mills with gauge group $SU(N)$, $b=3N$.

The Renormalization Group equation (4.21) has the running coupling solution

$$\frac{1}{g^2_{\text{YM}}(\mu)} = -\frac{b}{8\pi^2}\log\frac{|\Lambda|}{\mu} \ , \tag{4.23}$$

where $|\Lambda|$ is the dynamically generated scale. This allows to define the holomorphic intrinsic scale

$$\Lambda \equiv |\Lambda| e^{i\theta_{\text{YM}}/b} \ , \tag{4.24}$$

in terms of which the running of the holomorphic gauge coupling is

$$\tau = \frac{b}{2\pi i}\log\frac{\Lambda}{\mu} + f(\Lambda,\mu) \ , \tag{4.25}$$

where the first term is the one-loop contribution and the second term accounts for non-perturbative instantonic contributions. The second term has to be an holomorphic function of Λ, invariant under $\Lambda \to e^{2\pi i/b}\Lambda$, because, according to (4.24), this only changes the θ_{YM} angle by 2π. It can be also expanded in Taylor series of positive powers of Λ, because the limit $\Lambda \to 0$ must coincide with the perturbative result. Therefore we can write the holomorphic coupling as

$$\tau = \frac{b}{2\pi i}\log\frac{\Lambda}{\mu} + \sum_{n=1}^{\infty} a_n \left(\frac{\Lambda}{\mu}\right)^{nb} \ . \tag{4.26}$$

The β-function (4.21) is also referred as the β-function computed in the Wilsonian renormalization scheme. Usually, one is interested in the β-function of the canonical gauge coupling for canonically normalized fields. This needs of a redefinition of the vector superfield from which the action (4.19) is built, $V \to g_{\text{YM}}V$. Both β-functions, the Wilsonian and the canonical one, are not related by just a simple change in the renormalization scheme, because of a rescaling anomaly. This anomaly sets the following relation between both types of coupling:

$$\frac{1}{g_W^2} = \frac{1}{g_c^2} + \frac{N}{4\pi^2}\log g_c \ , \tag{4.27}$$

where the subscript c refers to the canonically normalized fields coupling and the subscript W refers to the Wilsonian scheme. Inserting this relation in (4.21), one obtains the so called NSVZ β-function for canonically normalized fields,

$$\beta_{NSVZ} = -3\frac{g^3_{\text{YM}}N}{16\pi^2}\left(1 - \frac{g^2_{\text{YM}}N}{8\pi^2}\right)^{-1} \ . \tag{4.28}$$

and we have already renamed the subscript of the gauge coupling $c \to \mathrm{YM}$. This was computed for the first time using instanton methods by Novikov, Shifman, Vainshtein and Zakharov [93] in the Pauli-Villars scheme.

The β-function, either (4.28) or (4.21), is renormalization scheme independent at one and two loops.

$U(1)_R$ symmetry and the gaugino condensate:

Given that the gauge field in the vector multiplet has R-charge $R = 0$, and the gaugino has $R = 1$, this theory has a $U(1)_R$ symmetry, which turns out to be anomalous. This can be seen calculating the triangle anomaly between one global $U(1)_R$ current and two gluons (fig. 4.1).

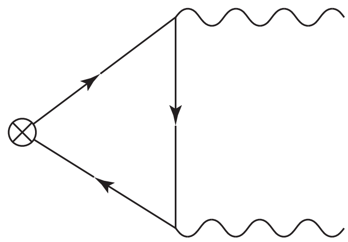

Figure 4.1: Triangle diagram with one $U(1)_R$ global current and two gluons at the vertices and fermions running on the loop. This diagram produces the anomaly that breaks the $U(1)_R$ symmetry.

Under an $U(1)_R$ shift, the gaugino and the SYM action transform as

$$\lambda \to e^{i\epsilon}\lambda\ , \qquad S \to S - \epsilon \int \mathrm{d}^4x\, \partial_\mu(\bar{\lambda}\bar{\sigma}^\mu\lambda) \tag{4.29}$$

and using, for example, Fujikawa's path integral derivation [100, 101] of the mentioned triangle anomaly, we obtain a non conserved current, $j^\mu = \bar{\lambda}\bar{\sigma}^\mu\lambda$, where the anomaly is given by

$$\partial_\mu j^\mu = \frac{N}{16\pi^2} F^{a\mu\nu}\tilde{F}^a_{\mu\nu}\ . \tag{4.30}$$

This result is not changed by higher-loops contributions [102,103]. In general, each fermion of the theory will contribute to this anomaly by a factor proportional to its R-charge times the index of the representation of $SU(N)$ under which it transforms,

$$\partial_\mu j^\mu = \frac{1}{16\pi^2}\left[\sum_{\text{fermions}} R\, T(\text{rep.})\right] F^{a\mu\nu}\tilde{F}^a_{\mu\nu}\ . \tag{4.31}$$

Then, the $U(1)_R$ transformation (4.29), is equivalent to the following shift in the θ_{YM} angle

$$\theta_{\mathrm{YM}} \to \theta_{\mathrm{YM}} - 2N\epsilon \tag{4.32}$$

In consequence, this transformation is a symmetry only when $\epsilon = k\pi/N$, with k being an integer. Then we see that the $U(1)_R$ classical symmetry is explicitly broken to $\mathbb{Z}_{2N}$.

Let us promote the holomorphic gauge coupling τ to a background spurious chiral superfield, so that the shifts (4.29) and (4.32) define a spurious $U(1)_R$ symmetry

$$\lambda \to e^{i\epsilon}\lambda\ , \qquad \tau \to \tau + \epsilon\frac{N}{\pi}\ . \tag{4.33}$$

Consider now the effective superpotential that is generated when modes above a certain energy scale μ are integrated out. Under the spurious symmetry the superpotential transforms as $W_{\text{eff}} \to e^{2i\epsilon}W_{\text{eff}}$, since it has R-charge $R=2$. Therefore, if the theory has no massless degrees of freedom, holomorphicity and symmetry fix the effective superpotential potential to be

$$W_{\text{eff}} \propto \mu^3 e^{2\pi i\tau/N}\ . \tag{4.34}$$

Then the F-term of the background spurious field τ acts as a source for the $\lambda^a\lambda^a$ operator. We can obtain the gaugino condensate computing

$$\langle\lambda^a\lambda^a\rangle = 16\pi i\frac{\partial\log Z}{\partial F_\tau}\ . \tag{4.35}$$

In the IR, if there are no massless particles, the effective action is just the effective superpotential. Discarding non-perturbative corrections, the previous expression reduces to

$$\langle\lambda^a\lambda^a\rangle \propto 16\pi i\frac{\partial}{\partial F_\tau}\int \mathrm{d}^2\theta\, W_{\text{eff}} = 16\pi i\partial_\tau W_{\text{eff}} = -\frac{32\pi^2}{N}\mu^3 e^{2\pi i\tau}\ . \tag{4.36}$$

Substituting here the expression of the running holomorphic coupling (4.26),

$$\langle\lambda^a\lambda^a\rangle \propto -\frac{32\pi^2}{N}\Lambda^3\ , \tag{4.37}$$

where we have not written the non-perturbative contributions to the running holomorphic coupling, since they only contribute like a phase. This gaugino condensate transforms as

$$\langle\lambda^a\lambda^a\rangle \to e^{2i\epsilon}\langle\lambda^a\lambda^a\rangle \tag{4.38}$$

under the shift (4.29). Then it is not invariant for any value of $\epsilon = k\pi/N$. To keep it invariant the allowed values of ϵ are given by $k=0,N$. In consequence, the presence of this gaugino condensate implies that in the IR the symmetry $\mathbb{Z}_{2N}$ spontaneously breaks to

$$\mathbb{Z}_{2N} \to \mathbb{Z}_2\ . \tag{4.39}$$

There are then N different vacua, each of which with its own $\mathbb{Z}_2$ symmetry, characterized by the change in the θ_{YM}-angle, $\theta_{\text{YM}} \to \theta_{\text{YM}} + 2\pi k/N$, in which the gaugino condensate takes different values.

4.2.2 Field theory results from the Maldacena-Núñez background

Consider the DBI-WZ action (3.21) for a D5-brane in the Maldacena-Núñez background explained at the beginning of the section,

$$\begin{aligned} S_{\text{DBI}} &= -T_{D5}\int \mathrm{d}\xi^6 e^{-\Phi}\sqrt{\left|\det\left(g\big|_\xi + 2\pi l_s^2 F\right)\right|}\ , \\ S_{\text{WZ}} &= T_{D5}\int\left((2\pi l_s^2)^2 C_2\big|_\xi \wedge F\wedge F + C_6\big|_\xi\right)\ , \end{aligned} \tag{4.40}$$

We now proceed in analogous way to section 3.1.3, where we found that $SU(N)$ $\mathcal{N}=4$ super Yang-Mills is the gauge theory that describes the low energy dynamics of N coincident D3-branes in flat space. By expanding now the action (4.40) in l_s up to quadratic terms in the world-volume fields, one arrives at

$$\begin{aligned} S = &-\frac{1}{2}T_{D5}\int \mathrm{d}^4x \int_{S^2} e^{-\Phi}\sqrt{-\det g}\left(1+(2\pi l_s^2)^2\frac{1}{4}F^a_{\mu\nu}F^{a\mu\nu}\right) \\ &+\frac{1}{4}(2\pi l_s^2)^2 T_{D5}\int \mathrm{d}^4x F^a_{\mu\nu}\tilde{F}^{a\mu\nu}\int_{S^2} C_2 \\ &+\ldots \end{aligned} \tag{4.41}$$

once the fields are promoted to those of a $SU(N)$ non-Abelian gauge theory, in order to consider the effect of having a stack of N coincident D5-branes. In the expansion (4.41) we find the bosonic part of the $SU(N)$ $\mathcal{N}=1$ SYM action, plus a constant energy-density term. In this expansion we can identify the gauge coupling, g_{YM}, and the θ_{YM}-angle in terms of the supergravity quantities:

$$\frac{1}{g^2_{YM}} = \frac{1}{2(2\pi)^3 g_s\alpha'}\int_{S^2} e^{-\Phi}\sqrt{-\det g} = \frac{N}{16\pi^2}\left(4e^{2h}+(a-1)^2\right)\ , \tag{4.42}$$

$$\theta_{YM} = -\frac{1}{2\pi g_s\alpha'}\int_{S^2} C_2 = -N\psi_0\ , \tag{4.43}$$

once we have replaced the value of the D5-brane tension (3.25) and considered the cycle (4.14) for definiteness.

From the relation (4.42) it is manifest the AdS/CFT prescription that relates the radial variable with the energy scale of the field theory, since large values of r correspond to small values of the coupling, attained at the UV of $\mathcal{N}=1$ SYM theory. Conversely, small values of r should correspond to the IR of the field theory, where it becomes strongly coupled.

According to (4.32), under a $U(1)_R$ transformation characterized by the parameter ϵ, the θ_{YM}-angle receives a shift $\theta_{YM}\to\theta_{YM}-2N\epsilon$, the identification (4.43) shows that this is realized in the gravity side through shifts in the angular variable $\psi\to\psi+2\epsilon$. However, these shifts do not correspond to isometries of the Maldacena-Núñez metric (4.6), the terms $(\tilde{\omega}^i - A^i)^2$ for $i=1,2$ are not invariant under shifts in ψ. This is not a problem, because we know that these shifts actually are not a symmetry of the quantum gauge theory. The true symmetry is $\mathbb{Z}_{2N}$, which spontaneously breaks to $\mathbb{Z}_2$ in the IR.

Let us consider the UV $r\to\infty$ limit of the MN background. In this limit, the a function vanishes and the explicit ψ dependence disappears from the metric (see (4.13)) and shifts in this angle become an isometry of the metric, although the dependence still is present in the RR potential. Since the UV limit has removed the dependence of the metric on ψ, the condition that fixed the value of ψ in (4.16) or (4.17) no longer does it and the 2-cycle is given by (4.14). Hence, integrating the RR potential over the 2-cycle in the θ_{YM}-angle expression (4.43) gives

$$\frac{1}{4\pi^2 g_s\alpha'}\int C_2 = \frac{N}{2\pi}(\psi+\psi_0)\ . \tag{4.44}$$

This integral is allowed to change by integer values. This implies that the shifts

$$\psi\to\psi+\frac{2\pi}{N}k \qquad k\in\mathbb{Z} \tag{4.45}$$

are a symmetry of the MN solution at large r, which is the supergravity counterpart of the $\mathbb{Z}_{2N}$ symmetry of the gauge theory.

On the other hand, in the IR the function a is turned on and the only remaining symmetry of the supergravity solution are shifts

$$\psi \to \psi + 2\pi k \ , \tag{4.46}$$

since the dependence of the metric on ψ is through trigonometric functions. This corresponds to the $\mathbb{Z}_2$ symmetry of the gauge theory in the IR. In this way, the MN solution is able to reproduce the symmetry breaking pattern of the gauge theory.

From this analysis it becomes clear that the a function is responsible for this symmetry breaking pattern. On the gauge theory side, this symmetry breaking is accompanied by gaugino condensation, then it is conjectured that both quantities are duals of each other [104, 105],

$$\langle\lambda^2\rangle \leftrightarrow a(r) \ . \tag{4.47}$$

This allows us to establish the precise relation between the radial coordinate and the energy scale of the gauge theory, which was already established in a qualitative way in the discussion of section 3.1.4 or after equations (4.42) and (4.43). Taking into account that the gaugino condensate has dimension three, the relation is

$$\frac{\Lambda^3}{\mu^3} \sim a(r) \ , \tag{4.48}$$

where, again, μ is an arbitrary renormalization scale at which the gaugino condensate is defined and Λ is the intrinsic scale of the gauge theory generated through dimensional transmutation by quantum corrections. Notice that we have identified a supergravity field with a protected operator of the gauge theory, i.e. an operator whose dimension does not change by quantum corrections. This must be in this way, since the dimension of supergravity fields do not change with the radial coordinate.

The precise energy-radius relation (4.48) allows us to compute the β-function of the gauge coupling from the supergravity side. At large r, discarding exponentially suppressed terms,

$$a(r) = 4re^{-2r} + \mathcal{O}(e^{-6r}) \ , \qquad \frac{1}{g_{YM}^2} = \frac{N}{16\pi^2}\left(4r + \mathcal{O}(e^{-2r})\right) \ , \tag{4.49}$$

we get the β-function:

$$\beta = \frac{\partial g_{YM}}{\partial \log\frac{\mu}{\Lambda}} = -3\frac{g_{YM}^3 N}{16\pi^2}\left(1 - \frac{g_{YM}^2 N}{8\pi^2}\right)^{-1} \ , \tag{4.50}$$

which is the complete perturbative NSVZ β-function of $\mathcal{N}=1$ SYM. This is a surprising result for various reasons:

- First of all, we are able to reproduce the correct β-function in the perturbative regime of the gauge theory, where, in principle, the supergravity approximation does not hold.

- Not only this, but if we carry out the computation that led to (4.50) without throwing away exponential suppressed terms,

$$\beta = -3\frac{g_{YM}^3 N}{16\pi^2}\left(1 - \frac{g_{YM}^2 N}{8\pi^2} + \frac{2\exp\left(-\frac{16\pi^2}{g_{YM}^2 N}\right)}{1-\exp\left(-\frac{16\pi^2}{g_{YM}^2 N}\right)}\right)^{-1} , \tag{4.51}$$

  the study through the gravity dual seems to indicate there are some non-perturbative contributions. However, the interpretation of these new contributions is not very clear as it might be contaminated from Kaluza-Klein modes, not belonging to the gauge theory, which cannot be disentangled from the gauge theory degrees of freedom, as we will see in a moment.

- Another fact worth mentioning is that it seems surprising that the simple relation (4.48) leads to the β-function (4.50) in the particular Pauli-Villars renormalization scheme. We could have modified the relation (4.48) by an analytic function of the gauge coupling,

$$\frac{\Lambda^3}{\mu^3} = f(g_{YM})a(r) , \tag{4.52}$$

  changing the β-function. Nevertheless, this redefinition enters in the β-function beyond two loops. The same happens for gauge theory computations of the β-function using different renormalization schemes. Thus, the universality of the two-loop coefficient of the NSVZ β-function is maintained. Therefore, the above redefinition (4.52) of the energy-radius relation should account for different renormalization schemes in the gauge theory.

The Maldacena-Núñez solution describes the $SU(N)$ $\mathcal{N}=1$ SYM in a particular vacuum of the N different ones. One can run over the different vacua by considering a gauge transformation of the $SU(2)_L$ gauge field A,

$$A \to A' = U^{-1}AU + iU^{-1}\mathrm{d}U , \tag{4.53}$$

where $U \in SU(2)_L$. Under such a gauge transformation the twisted part of the metric can be written as

$$\sum_{i=1}^{3}(\omega^i - A'^i)^2 = \frac{1}{2}\,\mathrm{Tr}(\omega - A')^2 = \frac{1}{2}\,\mathrm{Tr}(U\omega U^{-1} - i\mathrm{d}UU^{-1} - A)^2 . \tag{4.54}$$

Then, a global transformation corresponds to a rotation of the three-sphere parametrized by ω, while a local transformation will also contribute to the twist, in addition to the rotation of the three-sphere.

In particular, we can choose $U = \exp i\epsilon\sigma_3$ with ϵ independent of the space-time coordinates. Then the two-cycle (4.16) now becomes

$$S^2 : \qquad \theta = -\tilde{\theta} , \qquad \phi = -\tilde{\phi} , \qquad \psi = 2\epsilon \bmod 2\pi . \tag{4.55}$$

The integral of the two form potential implies that this corresponds to a different θ-vacuum, $\theta = -N(\psi_0 + 2\epsilon)$, so that ϵ is allowed to change by $2\epsilon = 2\pi k/N$ for integer k. Then, changing the gauge field A with $k = 0, \ldots, N-1$, we run over the N vacua of the gauge theory.

However, not everything works so well with the duality between the Maldacena-Núñez solution and $\mathcal{N}=1$ SYM. Within the supergravity regime, in which we have computational control, there is not a complete decoupling between four dimensional gauge degrees of freedom and Kaluza-Klein modes. Basically, when the energy scale of the four dimensional gauge theory becomes of the order of the energy scale characterized by the radius of the 2-cycle the branes are wrapping, the Kaluza-Klein modes, which in principle are not present in the gauge theory considered, start to appear in the spectrum and we start to explore a six dimensional field theory.

To lay in the supergravity regime we must require a small curvature. The curvature for the Maldacena-Núñez background is of the order

$$R \sim \frac{1}{\alpha' g_s N} \ , \tag{4.56}$$

and hence, we must take $\alpha' g_s N \to \infty$. On the other hand, at small distances, Kaluza-Klein modes have a mass roughly given by

$$m_{KK}^2 \sim \frac{1}{\mathrm{vol}(S^2)} \sim \frac{1}{\alpha' g_s N} \ . \tag{4.57}$$

Therefore, within the supergravity regime we cannot decouple the Kaluza-Klein modes at strong coupling. In consequence, one has to consider the results obtained in this way with some care.

Indeed, in [75] a proposal for the action describing the weakly coupled field theory was made taking into account these Kaluza-Klein modes. This proposal is based on some previous results [106, 107], where the spectrum of these Kaluza-Klein modes was studied. In the field theory we find a massless $\mathcal{N}=1$ vector multiplet V and a tower of massive chiral and vector Kaluza-Klein multiplets, Φ_k and V_k. We will denote the corresponding field strengths of the vector multiplets by $\mathcal{W}$ and $\mathcal{W}_k$. The proposal of [75] is

$$\begin{aligned} S = &\int \mathrm{d}^4x\, \mathrm{d}^4\theta \sum_k \left(\Phi_k^\dagger e^V \Phi_k + \mu_k |V_k|^2 \right) \\ &+ \int \mathrm{d}^4x\, \mathrm{d}^2\theta\ \mathcal{W}\mathcal{W} + \sum_k \Big(\mathcal{W}_k \mathcal{W}_k + \mu_k |\Phi_k|^2 + W(\Phi_k, V_k) \Big) + h.c. \end{aligned} \tag{4.58}$$

with a superpotential cubic in the chiral superfields and some interaction between chiral and vector superfields,

$$W = \sum_{ijk} z_{ijk} \Phi_i \Phi_j \Phi_k + \sum_k f(\Phi_k) \mathcal{W}_k \mathcal{W}_k \ . \tag{4.59}$$

4.3 Flavoring the Maldacena-Núñez background

In order to make contact with supersymmetric QCD-like theories one would like to add flavors to the Maldacena-Núñez background previously presented. More precisely we will consider the addition of chiral multiplets transforming in the fundamental representation.

As well as in the large N limit [108] of pure $SU(N)$ Yang-Mills, where Feynman diagrams can be associated with Riemann surfaces that organize themselves into a genus expansion similar to the one appearing in closed string theory. If one considers flavors, these Riemann surfaces now admit boundaries and, thus, one identifies an expansion of string theory with both open and closed strings.

Therefore, the supergravity counterpart of the addition of flavors consists of endorsing the original closed string background with an open string sector. This is done by considering the presence of N_f "flavor" branes (this approach was initiated by [72]) in the type IIB supergravity background previously presented, which already encodes the N "color" branes.

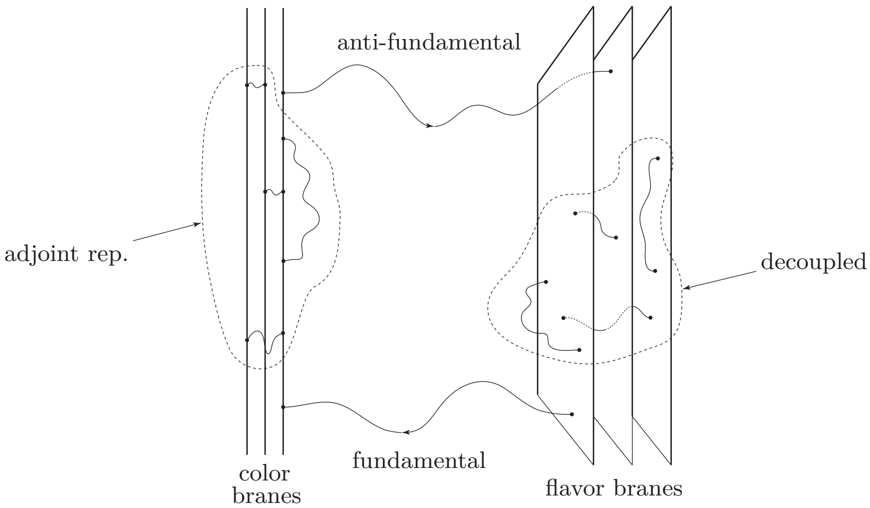

Figure 4.2: Picture of the color-flavor branes scheme.

Another motivation for the addition of this extra set of flavor branes comes from the open string picture as depicted in figure 4.2. Consider first the flavorless background generated by just the stack of N color branes. We have fields transforming in the adjoint representation of the gauge group $SU(N)$ and they can be considered as $N \times N$ matrix objects. Adding now N_f fundamental fields, corresponds to consider $N_f \times N$ matrix objects. Then, if we label each end of an open string depending on the brane it is attached to, we will find:

- Strings with both ends on the color branes, whose low energy excitations will correspond to these $N \times N$ fields.

- Strings starting on a flavor brane and ending on a color brane, representing a $N \times N_f$ object, i.e. a fundamental field (or anti-fundamental field if it starts and ends the way round).

- Strings with both ends on the flavor branes. This would correspond to fields transforming under the adjoint representation of a gauge $SU(N_f)$ group. We can get rid of these undesired fields by taking the quotient between the volume of flavor branes and that of color branes to be infinite [72]. For example, as we saw in (4.42), the gauge coupling is related to the volume of the internal space along which the brane is extended. Then, if we consider that the flavor branes are extended along a non-compact cycle of the internal space, the corresponding gauge coupling would vanish. In this way, we manage to decouple the undesired fields and the $SU(N_f)$ symmetry group can be considered now as a global symmetry.

For a number of flavor branes not very large, $N_f \ll N$, the addition of these branes can be done in the probe approximation, i.e. we can neglect the backreaction of the flavor branes, considering them in the background without deforming the geometry. This is equivalent in the field theory to the quenched approximation, where fundamental fields do not run inside loops. This can be easily seen from the point of view of the 't Hooft $1/N$ expansion. For example, consider the scattering between n mesons, whose Feynman diagrams may have an associated Riemann surface with w windows (corresponding to internal fundamental loops), h handles and b boundaries. Then the mesonic n point correlation function has the dependence [109]

$$\langle M(x_1)\dots M(x_n)\rangle \sim \left(\frac{N_f}{N}\right)^w N^{(2-2h-b-n/2)} \,. \tag{4.60}$$

Then in the 't Hooft limit

$$g_{YM}\to 0\,,\qquad N\to\infty\,,\qquad \text{with } \lambda=g_{YM}^2 N \text{ and } N_f \text{ fixed,} \tag{4.61}$$

the dominant contribution comes from surfaces with no windows, $w=0$, that is the quenched approximation.

However, we find another interesting limit, the Veneziano limit [110], given by

$$g_{YM}\to 0\,,\qquad N\to\infty\,,\qquad N_f\to\infty\,,\qquad \text{with } \lambda=g_{YM}^2 N \text{ and } \zeta\equiv\frac{N_f}{N} \text{ fixed.} \tag{4.62}$$

This is also referred as topological expansion, since only non-planar diagrams are suppressed, unlike the 't Hooft expansion (4.61), where both non-planar and planar diagrams with fundamental loops are suppressed. Although the Veneziano limit may capture more physics than the 't Hooft limit, since it takes into account more Feynman diagrams, this approach requires more effort in the supergravity analogue, because the number of flavor branes is large and we can no longer neglect their backreaction. We will work in this latter setup.

Then we will consider N_f backreacting D5-branes extended along a non-compact, calibrated two-cycle together with the 3+1 dimensions of the gauge theory. As already mentioned, we require this two-cycle to be non-compact to introduce a $SU(N_f)$ global symmetry. We also require the two-cycle to be calibrated to preserve supersymmetry.

To obtain this supersymmetric cycle is not an easy task because when we depart from the probe approximation and we have to consider backreacting branes, the dynamics of the system is governed by the combination of the supergravity action, encoding the color branes as a flux, plus the DBI-WZ action of the flavor branes,

$$S = S_{\text{sugra}} + S_{\text{flavor}} \ . \tag{4.63}$$

Then, to find where the flavor branes can be placed to preserve supersymmetry, we need to know the geometry; but the geometry is obtained by solving the combined action (4.63), where we need to know how the flavor branes are placed to specify S_{flavor}. The only way to deal with this problem is to propose an ansatz general enough to account for the backreaction of the flavor branes, but not so general in order to allow us to find the supersymmetric embeddings. This ansatz might be guided by the results obtained in the probe approximation.

The supersymmetric cycles that we can consider in the background [70] were studied in [111], where they addressed the problem of adding flavors in the probe approximation (this is also studied in [112]) by means of κ-symmetry, which we have seen it is equivalent to the use of calibrations (section 3.2.3). The cycle we are going to consider corresponds to the "cylinder solutions" of [111], since the flavor branes extend along the coordinates r, ψ (as well as the coordinates of the gauge theory) at any value of $\theta, \phi, \tilde{\theta}, \tilde{\phi}$. Then, in the Maldacena-Núñez background the metric induced on the world-volume takes the form (writing just the transverse directions to the gauge theory)

$$\mathrm{d}s_2^2 = g_s \alpha' N e^{\Phi} \left(\mathrm{d}r^2 + \frac{1}{4} \mathrm{d}\psi^2 \right) \ , \tag{4.64}$$

which is conformally equivalent to a cylinder. As the embedding reaches the $r = 0$ position, we have strings of zero length stretching between the flavor and color branes. Hence, this setup corresponds to massless flavors. On the contrary, if the embedding of the flavor branes did not reach the origin, we would have strings of minimal non-zero length, l, given by the minimum distance between both types of branes. The mass of these strings would be given by this length times the string tension, $m = Tl$, and they we would represent massive flavors.

Although we have discussed this embedding for the Maldacena-Núñez background, since in [111] they work in the probe approximation, the same conclusions hold for the backreacted geometry, which we come to discuss now.

The dynamics of the system is described by the action (4.63), where

$$S_{\text{IIB sugra}} = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \, e^{-2\Phi} \sqrt{-\det g} \left[R + 4(\partial^\mu \Phi)(\partial_\mu \Phi) - \frac{1}{12} e^{2\Phi} F_3^2 \right] \ , \tag{4.65}$$

$$S_{\text{flavor}} = T_{D5} \sum^{N_f} \left(- \int_{\mathcal{M}_6} \mathrm{d}^6\xi \, e^{-\Phi} \sqrt{-\det g}\Big|_{\mathcal{M}_6} + \int_{\mathcal{M}_6} C_6\Big|_{\mathcal{M}_6} \right) \ , \tag{4.66}$$

and $\mathcal{M}_6$ is the world-volume of the flavor branes and pullbacks are understood in S_{flavor}. F_3 must satisfy the flux quantization condition (4.12).

As we have already said, flavor branes extend along the directions $x_0, x_1, x_2, x_3, r, \psi$, each one of them at a fixed value of $\theta, \phi, \tilde{\theta}, \tilde{\phi}$. Then, if we derive the equations of motion following from the sum of these two actions, we would obtain Einstein and flux equations involving Dirac δ-functions depending on the latter set of coordinates, where the branes are localized. Something of the form

$$\mathrm{d}F_3 = 2\kappa_{10}^2 T_{D5} \sum_{i=1}^{N_f} \delta^{(4)}(\vec{r}-\vec{r}_i) \tag{4.67}$$

and similarly for Einstein's equations, where more Dirac δ-functions would appear in the strength energy tensor. This is a very hard problem, as we have to solve second order, nonlinear, partial differential equations with localized sources.

However, there is a huge simplification we can perform. As proposed in [113], since we are taking the Veneziano limit $N_f \sim N \to \infty$, we can consider the branes as a "fluid" homogeneously smeared over the transverse dimensions $\theta, \phi, \tilde{\theta}, \tilde{\phi}$. This introduces two major simplifications. First of all, the angular dependence is erased, so that we only have to consider ansatzes depending just on the radial coordinate. The second simplification consists of replacing the sum in (4.66) by an integral over these transverse coordinates, obtaining in this way a full ten dimensional integral,

$$S_{\text{flavor}} = T_{D5}\left(-\frac{N_f}{4\pi^2}\int \mathrm{d}^{10}x \sin\theta \sin\tilde{\theta} e^{-\Phi}\sqrt{-\det g} + \int \Xi_4 \wedge C_6\right) , \tag{4.68}$$

where we will take $\Xi_4 = \frac{N_f}{16\pi^2}\sin\theta\sin\tilde{\theta}\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\mathrm{d}\tilde{\theta}\wedge\mathrm{d}\tilde{\phi}$, which is known as the smearing form. This amounts to replace the Dirac δ-functions by a constant density. When the smearing is performed, the violation of the Bianchi identity is given in terms of a continuous distribution of charge,

$$\mathrm{d}F_3 = 2\kappa_{10}^2 T_{D5}\Xi_4 \ . \tag{4.69}$$

The smearing form is the brane charge density and we see it must be a closed form.

Then, modifying the Maldacena-Núñez background, Casero, Núñez and Paredes [73] proposed the following ansatz for the metric and RR-form:

$$\begin{aligned} \mathrm{d}s^2 = e^{\Phi(r)}\Big[&\mathrm{d}x_{1,3}^2 + e^{2k(r)}\mathrm{d}r^2 + e^{2h(r)}(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\phi^2) \\ &+ \frac{e^{2g(r)}}{4}\Big((\tilde{\omega}_1-A_1)^2+(\tilde{\omega}_2-A_2)^2\Big) + \frac{e^{2k(r)}}{4}(\tilde{\omega}_3-A_3)^2\Big] , \end{aligned} \tag{4.70}$$

$$F_{(3)} = \frac{g_s\alpha' N}{4}\left(\sum_{i=1}^{3}F_i\wedge(\tilde{\omega}_i-B_i) - \bigwedge_{i=1}^{3}(\tilde{\omega}_i-B_i) - \zeta\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge(\mathrm{d}\psi+\cos\tilde{\theta}\mathrm{d}\tilde{\phi})\right) , \tag{4.71}$$

where $\tilde{\omega}$ are the $SU(2)$ left-invariant one-forms given in (4.7). A is the $SU(2)_L$ gauge field that appeared in (4.9), although the dependence of a with the radial coordinate r can be different now, determined by the new BPS equations. The RR three form is the sum of a closed form with the same structure as in the flavorless case (4.10), plus a term generating the violation of the Bianchi identity depending on the Veneziano parameter ζ. B is an $SU(2)$ gauge connection following the same pattern as the gauge field A,

$$B_1 = -b(r)\mathrm{d}\theta \ , \qquad B_2 = b(r)\sin\theta\, d\phi \ , \qquad B_3 = -\cos\theta\,\mathrm{d}\phi \ , \tag{4.72}$$

and F_i is its corresponding field strength.

Until now we have been careful showing the $g_s\alpha'$ units, however it is more comfortable to work in units in which $g_s\alpha' = 1$. In this case, the $g_s\alpha'$ factors can be thought to be absorbed into N and N_f (see (4.69)-(4.71)), hence they are easily recovered by $N \to g_s\alpha' N$ and $N_f \to g_s\alpha' N_f$. Notice that in comparison to the Maldacena-Núñez metric (4.6), the factors N of the internal part of the metric have been absorbed into the factors e^{2g}, e^{2h} and e^{2k}.

It is convenient to introduce the following set of vielbeins

$$e^{x^i} = e^{\Phi/2}\mathrm{d}x^i \ , \qquad e^r = e^{\Phi/2+k}\mathrm{d}r \ , \tag{4.73}$$

$$e^{\theta} = -e^{\Phi/2+h}\mathrm{d}\theta \ , \qquad e^{\phi} = e^{\Phi/2+h}\sin\theta\mathrm{d}\phi \ , \tag{4.74}$$

$$e^1 = \frac{1}{2}e^{\Phi/2+g}(\tilde{\omega}_1 - A_1) \ , \qquad e^2 = \frac{1}{2}e^{\Phi/2+g}(\tilde{\omega}_2 - A_2) \ , \qquad e^3 = \frac{1}{2}e^{\Phi/2+k}(\omega_3 - A_3) \ . \tag{4.75}$$

To obtain the undetermined functions $a(r)$, $b(r)$, $g(r)$, $h(r)$, $k(r)$ and the dilaton, $\Phi(r)$, we must solve the set of BPS equations. The BPS equations impose the following projections on the ten-dimensional Killing spinor

$$\Gamma_{12}\epsilon = \Gamma_{\theta\phi}\epsilon \ , \qquad \Gamma_{r123}\epsilon = (\cos\alpha + \sin\alpha\Gamma_{\phi 2})\epsilon \ , \qquad i\epsilon^* = \epsilon \ , \tag{4.76}$$

for $\alpha(r)$ an arbitrary function to be determined. After the 4+6 split, $\epsilon_1 = \epsilon_2 = (\zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_-)$, where we are taking $i\gamma_{r\theta\phi 123}\eta_+ = \eta_+$, the ten dimensional Γ-matrices decompose into

$$\Gamma_\mu = \hat{\gamma}_\mu \otimes \mathbb{I} \ , \qquad \Gamma_a = \mathbb{I} \otimes \gamma_a \ . \tag{4.77}$$

We see that the internal manifold of this system possesses a $SU(3)$-structure described by the spinor η. This allows to write the BPS equations in an extremely simple way. As we saw in section 3.2, this $SU(3)$-structure, can be described in terms of two pure spinors

$$\Psi_+ = \frac{1}{8}e^{-iJ} \ , \qquad \Psi_- = -\frac{i}{8}\Omega, \tag{4.78}$$

where we consider that the spinor η_+ is already normalized. According to the projections (4.76) after the splitting (4.77) and using the formulas (3.76), the $SU(3)$-structure forms can be expressed as

$$J = e^{r3} + (\cos\alpha e^{\phi} + \sin\alpha e^2) \wedge e^{\theta} + (\cos\alpha e^2 - \sin\alpha e^{\phi}) \wedge e^1 \ , \tag{4.79}$$

$$\Omega = (e^r + ie^3) \wedge ((\cos\alpha e^{\phi} + \sin\alpha e^2) + ie^{\theta}) \wedge ((\cos\alpha e^2 - \sin\alpha e^{\phi}) + ie^1) \tag{4.80}$$

or performing the frame rotation in e^{ϕ}, e^2:

$$\tilde{e}^{\phi} = \cos\alpha e^{\phi} + \sin\alpha e^2 \ , \qquad \tilde{e}^2 = -\sin\alpha e^{\phi} + \cos\alpha e^2, \tag{4.81}$$

where J and Ω take the canonical form.

We find that the supersymmetry conditions (3.99) can be written in the simple form

$$\mathrm{d}\left(e^{\Phi/2}J\right) = -e^{3\Phi/2}\star_6 F_3 \ , \qquad \mathrm{d}\Omega = 0 \ , \qquad \mathrm{d}(e^{-\Phi/2}J\wedge J) = 0 \ . \tag{4.82}$$

It turns out that this set of equations can be partially integrated and they can be recast in a simple way in terms of a "master" differential equation [75]. After the integration the supersymmetric equations are

$$e^{2(\Phi-\Phi_0)} = \frac{1}{2}\frac{\sinh(2r)}{e^{h+g+k}} \ , \qquad b = \frac{2r}{\sinh(2r)} \ , \tag{4.83}$$

$$\cos\alpha = \frac{P\sinh(2r) - Q\cosh(2r)}{P\cosh(2r) - Q\sinh(2r)} \ , \qquad a e^{h+g}\cot\alpha = \frac{a^2-1}{4}e^{2g} - e^{2h} \ , \tag{4.84}$$

$$e^{2g} = P\coth(2r) - Q \ , \qquad a = \frac{P}{P\cosh(2r) - Q\sinh(2r)} \ , \tag{4.85}$$

$$e^{2h} = \frac{1}{4}\frac{P^2-Q^2}{P\coth(2r)-Q} \ , \qquad e^{2k} = \frac{1}{2}(P'+N_f) \ , \tag{4.86}$$

where we have introduced the functions $P(r)$ and $Q(r)$, which are defined in terms of the functions $g(r)$ and $a(r)$ from the inverse of the relation shown in (4.85),

$$Q \equiv (a\cosh(2r) - 1)e^{2g} \ , \qquad P \equiv a e^{2g}\sinh(2r) \ . \tag{4.87}$$

The remaining BPS conditions impose over these functions the following constraints:

$$Q = \coth(2r)\left((2N - N_f)\left(r - \frac{1}{2}\tanh(2r)\right) + q_0\right) \ , \tag{4.88}$$

$$P'' + (P'+N_f)\left(\frac{P'-Q'+2N_f}{P+Q} + \frac{P'+Q'+2N_f}{P-Q} - 4\coth(2r)\right) = 0 \ . \tag{4.89}$$

where Φ_0 and q_0 are integration constants. Hence, solving the BPS equations reduces to finding solutions to the "master equation" (4.89), which determines the full background.

The solutions to these BPS equations were studied in [73–75] and it is beyond the scope of this work to review them. Furthermore, all of them seem to suffer from IR singularities. This singularity has a simple explanation, as we are dealing with massless flavors, all flavor branes are forced to pass through the origin, thus the brane density blows up at that point.

It is worth mentioning here that despite of the presence of an IR singularity, some IR physics can be extracted from these backgrounds. Several criteria, for example [114, 115], have been proposed to determine if these type of singular backgrounds should be considered to extract some physical results without spoiling the application of the gauge/gravity techniques. Following the criterion [114], the solutions considered in [73–75] are of the good type and they should be considered. This criterion states that if the temporal component of the metric (in Einstein frame) g_{tt} is bounded, one should consider the singularity as a good physical one. Indeed, when computing IR observables usually different contributions of metric singularities cancel each other out, leaving a sensible physical result, however this does not always happen and a resolution of the singularity would be desirable. This will be consider in section 4.4 by introducing massive flavors.

The BPS solutions corresponding to backgrounds including massive flavors must coincide in the UV regime with the solutions of the BPS equations shown above. For this reason, we will review the asymptotic behavior of these solutions, discussed in full detail in [75].

For large radius, i.e. in the ultraviolet of the dual gauge theory, a generic solution behaves exponentially

$$P = k\; e^{4r/3} + \mathcal{O}(1)\;, \tag{4.90}$$

where k is an integration constant. There are also special solutions with the following linearly rising large r asymptotic:

$$P = |2N - N_f|\; r + \mathcal{O}(1)\;, \qquad N_f \neq 2N\;. \tag{4.91}$$

A further analysis is required to see if the geometry can actually be extended to $r \to \infty$ or if it meets a singularity before. This will be discussed below. When $N_f = 2N$, there are special solutions with the following asymptotic behavior:

$$P = P_0 + \mathcal{O}(e^{-cr})\;, \qquad P_0 = \frac{8N}{\xi(4-\xi)}\;, \tag{4.92}$$

with

$$q_0 = \frac{4N(\xi-2)}{\xi(4-\xi)}\;, \qquad c = 1 + \sqrt{9 - 4\xi + \xi^2}\;, \tag{4.93}$$

where ξ is a parameter allowed to change in the range $0 < \xi < 4$. [1]

4.3.1 Flavorless limit: Deformed Maldacena-Núñez solution

If the presence of (massless) flavor branes originates an IR singularity, one should expect that setting $N_f = 0$ in the setup explained in the previous section should produce regular solutions and, indeed, it is what happens. One also may expect that taking $N_f = 0$ should render the background to the Maldecena-Núñez solution, however, this is not the case. Since the ansatz (4.70) is more general than the ansatz (4.6), in addition to the regular solution of the Maldacena-Núñez model, there is a one-parameter deformation, first found in [73], that leads to solutions with regular behavior at $r = 0$. The infrared asymptotic of this unflavored one-parameter family of solutions has been explicitly written for P in [75],

$$P = h_1 r + \frac{4h_1}{15}\left(1 - \frac{4N^2}{h_1^2}\right) r^3 + \frac{16h_1}{525}\left(1 - \frac{4N^2}{3h_1^2} - \frac{32N^4}{3h_1^4}\right) r^5 + \mathcal{O}(r^7)\;, \tag{4.94}$$

where h_1 is the parameter that labels each solution of the family. When $h_1 = 2N$, one recovers the Maldacena–Núñez solution. It is worth noting that the resulting function Q is the same for any value of h_1,

$$Q = N(2r\coth(2r) - 1) \tag{4.95}$$

and the integration constant q_0 has to be chosen in order to avoid a pole in Q, which would spoil the regular behavior at $r = 0$.

In [73] they claim that different members of this family of solutions are related to changes in the dynamics and masses of the KK modes.

---

[1] In [74,75] they also find a set of solutions in which the origin of space is not at $r = 0$, and r is allowed to extend to $-\infty$, then the BPS equations and the master equation are slightly modified [75]. These solutions correspond to take $a = b = 0$, which in the flavorless case of the Maldacena-Núñez background leads to singular solutions. However, if the number of flavors is greater or equal to the number of colors, $N_f \geq N$, the IR singularity is of the "good" type and these solutions are claimed to describe strongly coupled physics of $\mathcal{N} = 1$ gauge theory with vanishing gaugino condensate. Nevertheless, the UV behavior of these solutions coincides at leading order with that of the solutions presented in the main text.

4.3.2 Field theory comparison

To propose an action describing the field theory dual to the background presented along section 4.3, we start with the proposed flavorless field theory action (4.58). To introduce flavors we must consider a pair of chiral multiplets Q and $\tilde{Q}$ transforming in the fundamental and antifundamental representation of the gauge group $SU(N)$ and flavor group $SU(N_f)$. Then we should add to (4.58) the canonical kinetic terms

$$\int \mathrm{d}^4x\,\mathrm{d}^4\theta\,\mathrm{Tr}\left[Q^\dagger e^V Q + \tilde{Q}^\dagger e^V \tilde{Q}\right] \ , \tag{4.96}$$

plus some superpotential term that couples these flavors to the Kaluza-Klein adjoint chiral multiplets. In [73] they propose a superpotential term of the form

$$W = \sum_k \kappa_k \tilde{Q}\Phi_k Q \ , \tag{4.97}$$

motivated by the fact that this is the only allowed way to couple fundamentals to an adjoint in $\mathcal{N}=2$ SQCD. κ is an undetermined coupling constant.

Let us consider the total action, given by the addition of (4.58) to (4.96) with the additional superpotential term (4.97). In the IR we can integrate out the massive Kaluza-Klein modes to end up with the folloeing action

$$S \sim \int \mathrm{d}^4x\,\mathrm{d}^4\theta\left[Q^\dagger e^V Q + \tilde{Q}^\dagger e^V \tilde{Q}\right] + \int \mathrm{d}^4x\,\mathrm{d}^2\theta\,\mathrm{Tr}\left[\mathcal{W}_\alpha^2 + W'\right] + h.c. \ , \tag{4.98}$$

where the new superpotential is quartic in the chiral multiplets.

$$W' \sim \sum_k \frac{\kappa_k^2}{2\mu_k^2}(\tilde{Q}Q)^2 \ . \tag{4.99}$$

We have omitted IR irrelevant superpotential terms of the order $\mathcal{O}((\tilde{Q}Q)^3)$. To integrate out the Kaluza-Klein degrees of freedom we must lay in an energy scale below m_{KK}, where the supergravity approximation may fail. Hence, this is not a very clean computation so that, expression (4.98) with the superpotential (4.99) has to be taken with a grain of salt.

Symmetries

- The term (4.97) explicitly breaks the flavor symmetry $SU(N_f)\times SU(N_f)$ to its diagonal group $SU(N_f)$. This agrees with the flavor brane set-up we have presented. We do not have a system of D5 and anti-D5 branes, which would reproduce the $SU(N_f)\times SU(N_f)$ flavor symmetry. Since we only have flavor D5-branes, the flavor symmetry group is $SU(N_f)$ (without taking into account the smearing).

- In [75] they argue that $SU(N_f)$ is not the actual symmetry. Because of the smearing, flavor branes are separated, breaking the symmetry to the group $U(1)^{N_f}$. Therefore, the superpotential (4.97) should be replaced by

$$W \sim \kappa^{ij}\tilde{Q}_{ai}\Phi^{ab}Q_{bj} \ , \tag{4.100}$$

with $i, j = 1, \ldots, N_f$ in the fundamental or antifundamental of $SU(N_f)$ and $a, b = 1, \ldots, N$ in the fundamental or antifundamental of $SU(N)$ and the rest of the discussion remains unchanged. Nevertheless, the $SU(N_f)$ symmetry is recovered in the IR.

- The $U(1)_R$ symmetry breaking pattern is now

$$U(1)_R \to \mathbb{Z}_{2N-N_f} \to \mathbb{Z}_2 \;, \tag{4.101}$$

where the first breaking is explicit, due to quantum corrections, and the second one is spontaneous, due to the formation of the gaugino condensate. By analogous arguments to those shown in section 4.2.2, the flavored supergravity background can reproduce this pattern. Although now one can identify the gaugino condensate with the a or b functions.

Notice that the solution $N_f = 2N$ is special because it preserves the $U(1)_R$ symmetry, there is no anomaly. Of course, this is what happens in the field theory. According to the formula (4.31) for the R-symmetry anomaly, the theory described by the action (4.98) has an anomaly proportional to

$$T(\mathrm{Ad})R_{\mathrm{gaugino}} + 2N_f T(\mathrm{F})(R_Q - 1) \;; \tag{4.102}$$

the indices are $T(\mathrm{Ad}) = N$, $T(\mathrm{F}) = 1/2$ and the gaugino has R-charge $R_{\mathrm{gaugino}} = 1$ and for the chiral supermultiplets we have $R_Q = 1/2$, which can be seen, for example, from the superpotential. Then the previous expression is exactly zero for $N_f = 2N$.

β-function

The NSVZ β-function for pure $\mathcal{N} = 1$ super Yang-Mills was explained in section 4.2.1. Its generalization to the presence of massless flavors is given by

$$\beta_{g_{\mathrm{YM}}} = -\frac{g_{\mathrm{YM}}^3}{16\pi^2} \frac{3N - N_f(1-\gamma_0)}{1 - \frac{g_{\mathrm{YM}}^2 N}{8\pi^2}} \;, \tag{4.103}$$

where γ_0 is the anomalous dimension of the fundamental superfields.

As we have already mentioned, the master equation (4.89) admits many solutions whose UV asymptotics is shown in (4.90), (4.91) and (4.92). There are also several IR behaviors [75] and, hence, a plethora of solutions interpolating between these two regimes giving rise to different backgrounds. Here, we lack of a criterion to determine whether a solution is relevant or not, indeed, all of them produce singular backgrounds at the IR. This situation will ameliorate in section 4.4, where we will consider the generalization to backgrounds realizing massive flavors. As in the massless case, in section 4.4 we will find several solutions giving rise to different backgrounds, but we will have at our disposal a physical criterion to determine uniquely the relevant solution. Moreover, contrary to the massless case, these solutions do not suffer from IR singularities.

Therefore, we will postpone the discussion of the β-function obtained with gravity computations until section 4.7. Since the comparisons with the NSVZ β-function only makes sense in the UV, the conclusions in section 4.4 are valid for the massless flavor case presented along this section as well.

Seiberg duality

Let us consider the following two theories:

- SQCD or electric theory:
  This is a similar field theory to the one we have been considering along the section. It is $SU(N)$ $\mathcal{N}=1$ SYM with $N_f > N$ flavors, i.e. N_f chiral supermultiplets in the fundamental representation of the gauge group and N_f in the anti-fundamental, but no superpotential. This theory has the global symmetry $SU(N_f)\times SU(N_f)\times U(1)\times U(1)_R$, under which the fields have the following quantum numbers:

	$SU(N)$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_R$
Q	$\mathbf{N}$	$\mathbf{N_f}$	$\mathbf{1}$	1	$1-\frac{N}{N_f}$
$\overline{Q}$	$\overline{\mathbf{N}}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	-1	$1-\frac{N}{N_f}$

Table 4.1: Quantum numbers for SQCD. The R-charge shown is that of the scalar in the supermultiplet.

- SQCD+M or magnetic theory:
  This is the same theory as the previous one, except now the gauge group is $SU(\tilde{N})$ with $\tilde{N}=N_f-N$ and we have an extra fundamental chiral superfield, $\tilde{M}$, coupled to the flavors q and $\bar{q}$ through the superpotential

$$W=\lambda\tilde{M}_i^j q_j\bar{q}^i \ , \tag{4.104}$$

  where λ is a dimensionless constant. Now the quantum numbers are:

	$SU(\tilde{N})$	$SU(N_f)$	$SU(N_f)$	$U(1)$	$U(1)_R$
q	$\mathbf{N}$	$\mathbf{N_f}$	$\mathbf{1}$	1	$\frac{N}{N_f}$
$\overline{q}$	$\overline{\mathbf{N}}$	$\mathbf{1}$	$\overline{\mathbf{N_f}}$	-1	$\frac{N}{N_f}$
$\tilde{M}$	$\mathbf{1}$	$\mathbf{N_f}$	$\overline{\mathbf{N_f}}$	0	$2\left(1-\frac{N}{N_f}\right)$

Table 4.2: Quantum numbers for SQCD+M. The R-charge shown is that of the scalar in the supermultiplet.

As originally proposed by Seiberg [7], these two theories, with different behaviors at weak coupling, are conjectured to describe the same IR physics. More precisely, their Green functions become exactly the same in the limit in which all external momenta are taken to zero after the identification of some gauge invariant operators between both theories. For example, we can define the baryonic operators

$$SQCD: \quad B_{i_1\dots i_N}=\epsilon^{n_1\dots n_N}Q_{n_1 i_1}\dots Q_{n_N i_N} \ , \quad \bar{B}^{i_1\dots i_N}=\epsilon_{n_1\dots n_N}\bar{Q}^{n_1 i_1}\dots Q^{n_N i_N} \ , \tag{4.105}$$

$$SQCD+M: \quad b_{i_1\dots i_{\tilde{N}}}=\epsilon^{n_1\dots n_{\tilde{N}}}q_{n_1 i_1}\dots q_{n_{\tilde{N}} i_{\tilde{N}}} \ , \qquad \bar{b}^{i_1\dots i_{\tilde{N}}}=\epsilon_{n_1\dots n_{\tilde{N}}}\bar{q}^{n_1 i_1}\dots q^{n_{\tilde{N}} i_{\tilde{N}}} \ , \tag{4.106}$$

where n is a gauge index and i, a flavor index. Then we identify

$$B_{i_1\ldots i_N} \leftrightarrow \epsilon_{i_1\ldots i_N j_1\ldots j_{\tilde{N}}} b^{j_1\ldots j_{\tilde{N}}} \ , \tag{4.107}$$
$$\bar{B}^{i_1\ldots i_N} \leftrightarrow \epsilon^{i_1\ldots i_N j_1\ldots j_{\tilde{N}}} \bar{b}_{j_1\ldots j_{\tilde{N}}} \ . \tag{4.108}$$

In the same way the mesonic operator of SQCD, $M_i^j = \bar{Q}^{jn} Q_{ni}$, is identify with the fundamental field of SQCD+M

$$M \leftrightarrow \tilde{M} \ , \tag{4.109}$$

not with the magnetic meson $m_i^j = \bar{q}^{jn} q_{ni}$, since quantum numbers simply do not match.

This duality is extremely useful since it is a weak-strong coupling duality and, therefore, a valuable tool to extract non-perturbative information in one theory by doing perturbative computations in its dual. Some consistency checks of this duality are:

- Global symmetries match.
- Seiberg duality is an involution.
- The dimension of moduli spaces for both theories and gauge invariant operators match.
- Anomalies for both theories match.
- It is consistent under mass or vev deformations. For example, we can give mass to a flavor and integrate it out, then the electric theory is a $SU(N)$ theory with $N_f - 1$ flavors, while the dual theory should be a $SU(N_f - N - 1)$ gauge theory with $N_f - 1$ flavors and it actually is. The reason is that a mass term for the electric flavors generates a linear term for the magnetic field $\tilde{M}$, which forces the magnetic squarks to take a vev. This generates a Higgs mechanism that renders the gauge group of the dual magnetic theory to $SU(N_f - N - 1)$ and the number of flavors to $N_f - 1$. This will be relevant for section 4.7.

However, neither of the two theories involved in the duality as originally proposed by Seiberg corresponds to the field theory dual of the supergravity background we have been studying along this section. Rather we have SQCD supplemented with a quartic superpotential. The effects of the introduction of a quartic superpotential are very nicely discussed in [116] and this modification of the original Seiberg duality turns out to be even more interesting since the duality holds along a RG flow, not just at the IR.

Due to the importance of Seiberg duality, it is then natural to ask what the supergravity counterpart of Seiberg duality is. In [73–75] they proposed that Seiberg duality is realized in the background (4.70)-(4.71) through the interchange of the spheres $(\theta, \phi) \leftrightarrow (\tilde{\theta}, \tilde{\phi})$. According to (4.12) the number of colors of the electric theory is given by the integral of the RR three-form over the three-sphere $S^3 \sim (\tilde{\theta}, \tilde{\phi}, \psi)$. Then, the number of colors of the magnetic theory would be given by integrating over the dual three-sphere $\tilde{S}^3 \sim (\theta, \phi, \psi)$:

$$- \frac{1}{2\kappa_{10}^2 T_{D5}} \int_{\tilde{S}^3(\theta,\phi,\psi)} F_3\big|_{\tilde{S}^3} = N_f - N \ , \tag{4.110}$$

which is the right number of colors for an interpretation in terms of a Seiberg duality. Moreover, the interchange of the two spheres does not play any role in the violation of the Bianchi identity, from which we can obtain the number of flavors, and thus, the number of flavors remains the same.

This duality is reflected at the level of the BPS system of equations by keeping it invariant under the interchange

$$(N, N_f) \leftrightarrow (N_f - N, N_f) \ , \qquad Q(r) \leftrightarrow -Q(r) \ . \tag{4.111}$$

This also leaves the master equation invariant. Then, two different field theories correspond to the same gravity solution and, therefore, they describe the same physics.

Nevertheless, this supergravity implementation of Seiberg duality should be taken carefully and one should try to support it with checks like those of the original Seiberg duality. For example, in [74] they verify the R-symmetry anomaly matching.

4.4 Considering massive flavors

The solutions to the BPS equations of the previous system or, what it is the same, the solutions to the master equation (4.89) show an IR singularity. This singularity is a common feature of supergravity solutions introducing massless flavors engineered through the smearing procedure previously explained. This singularity can be understood from the fact that the brane density blows up at $r=0$, because all flavor branes pass through the origin, since the flavors are massless. Although this singularity is of the good type, it makes the interpretation of the IR field theory not so clear.

One possibility to alleviate this consists of hiding the singularity behind an horizon [117], which corresponds to having a thermal bath in the field theory. Another possibility, which is the one we will be considering, consists of taking massive quarks, so that the branes do not reach the origin and, hence, the singularity is avoided.

We will take the same metric ansatz as in (4.70), respecting the $SU(3)$-structure of the internal manifold, but we will consider a slightly different RR three-form. The most general ansatz for a RR three-form just depending on the radial coordinate and compatible with supersymmetry is [79]

$$F_3 = \frac{N}{4}\left(-\prod_{i=1}^{3}(\tilde{\omega}^i - B^i) + \sum_{i=1}^{3}(F^i + f^i)\wedge(\tilde{\omega}^i - B^i)\right) \ , \tag{4.112}$$

where the change with respect to (4.71) comes from the f^i functions. They follow a similar structure to that of F^i,

$$f^1 = -L_1(r)\mathrm{d}r\wedge\mathrm{d}\theta \ , \qquad f^2 = L_1(r)\sin\theta\,\mathrm{d}r\wedge\mathrm{d}\phi \ , \qquad f^3 = L_2(r)\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi \ , \tag{4.113}$$

in terms of two functions $L_1(r)$ and $L_2(r)$ to be determined. Hence, the smearing form describing the density of brane charge, appearing in the Bianchi identity (4.69), changes to

$$\begin{aligned}\Xi_3 = &-\frac{N}{16\pi^2}\sin\theta\,\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge\left(L_2\tilde{\omega}^1\wedge\tilde{\omega}^2 - L_2'\mathrm{d}r\wedge\tilde{\omega}^3\right)\\ &+\frac{NL_1}{16\pi^2}\mathrm{d}r\wedge\left(\mathrm{d}\theta\wedge\tilde{\omega}^2\wedge\tilde{\omega}^3 + \mathrm{d}\phi\wedge\left(\sin\theta\,\tilde{\omega}^1\wedge\tilde{\omega}^3 + \cos\theta\,\mathrm{d}\theta\wedge\tilde{\omega}^2\right)\right) \ .\end{aligned} \tag{4.114}$$

The new BPS equations set a relation between the functions L_1 and L_2,

$$L_1 = -\frac{L_2'}{2\cosh(2r)} \ . \tag{4.115}$$

Then, defining the function $S(r)$ as

$$S(r) \equiv -\frac{N}{N_f} L_2(r) \ , \tag{4.116}$$

we can write the functions that parametrize the RR three-form and the smearing form in terms of the $S(r)$ function.

The remaining BPS conditions allow us to write a master equation analogous to (4.89), generalized for massive flavors,

$$Q = \coth(2r)\left[q_0 + \int_0^r \frac{2N - N_f S(\rho)}{\coth^2(2\rho)}\right] \ , \tag{4.117}$$

$$P'' + N_f S' + (P' + N_f S)\left(\frac{P' - Q' + 2N_f S}{P+Q} + \frac{P' + Q' + 2N_f S}{P-Q} - 4\coth(2r)\right) = 0 \tag{4.118}$$

and the rest of the relations imposed by the BPS equations in the massless case (4.83)-(4.86) have the same form, except for

$$b = \frac{2r}{\sinh(2r)} - \frac{N_f}{2N}\left[\frac{S(r)}{\cosh(2r)} + \frac{2}{\sinh(2r)}\int_0^r \mathrm{d}\rho\, \tanh^2(2\rho) S(\rho)\right] \tag{4.119}$$

and

$$e^{2k} = \frac{1}{2}(P' + N_f S(r)) \ . \tag{4.120}$$

Setups where the function $S(r)$ vanishes below a certain value, $r \le r_q$, describe the physics of $\mathcal{N} = 1$ SYM with massive flavors, whose mass is related to the separation of the flavor and color branes, r_q, and the quartic coupling we already had in the massless case.

Notice that in the limit $S \to 1$ we recover the massless situation of the previous section and the limit $S \to 0$ is equivalent to set $N_f = 0$, i.e. we should get the solution presented in 4.3.1. This situation is similar to the approach used in [118–120] where massive flavors are reproduced from the massless case by the substitution $N_f \to N_f S(r)$, for the function $S(r)$ interpolating between zero in the IR ($r \to 0$) and one in the UV ($r \to \infty$). Nevertheless, performing this naive substitution in the flavored background of section 4.3 does not give the whole RR three-form (4.112) of the "massive" background.

Once a profile $S(r)$ is determined, a solution is obtained by first computing $Q(r)$ in (4.117) and then solving equation (4.118) for $P(r)$. It should be noted that regularity of the geometry (see the expressions for e^{2h} (4.86) and e^{2k} (4.120)) requires

$$P > |Q| \ , \qquad P' > -N_f S \ . \tag{4.121}$$

In the particular cases $S = 0$ and $S = 1$ one finds solutions which already appeared in the literature.

4.5 Supersymmetric embedding

In section 4.3, flavor branes were introduced following [111], where they make use of κ-symmetry to find supersymmetric embeddings for the D-branes. However, as we saw in section 3.2.3 we have a powerful way of finding supersymmetric embeddings by means of the use of calibrations.

The fact that the background we are considering has an $SU(3)$-structure results very useful, as it provides a natural calibration form (see equations (3.91) and (3.121)):

$$\omega = -e^{\Phi} J \ . \tag{4.122}$$

Having an $SU(3)$-structure also implies that the internal manifold is complex and so, we can introduce a set of complex coordinates $z_1, \ldots, z_4$ parametrizing a deformed conifold [121], thus satisfying

$$z_1 z_2 - z_3 z_4 = 1 \tag{4.123}$$

and related to the radial coordinate through

$$\sum_{i=1}^{4} |z_i|^2 = 2\cosh(2r) \ . \tag{4.124}$$

In this way we can write the metric, the Kähler $(1,1)$-form J, and the holomorphic $(3,0)$-form Ω as

$$\begin{aligned} ds_6^2 &= \frac{1}{2} h_{\alpha\bar\beta} (dz^\alpha \otimes d\bar z^{\bar\beta} + d\bar z^{\bar\beta} \otimes dz^\alpha) \ , &(4.125)\\ J &= \frac{i}{2} h_{\alpha\bar\beta} dz^\alpha \wedge d\bar z^{\bar\beta} \ , &(4.126)\\ \Omega &= -\frac{e^{2\Phi_0}}{2z_3} dz_1 \wedge dz_2 \wedge dz_3 \ . &(4.127)\end{aligned}$$

The explicit dependence with the original coordinates is given by [79]

$$\begin{aligned} z_1 &= -e^{-\frac{i}{2}(\phi+\tilde\phi)} \left(e^{r+\frac{i}{2}\psi} \sin\frac{\theta}{2} \sin\frac{\tilde\theta}{2} - e^{-r-\frac{i}{2}\psi} \cos\frac{\theta}{2}\cos\frac{\tilde\theta}{2} \right) \ , \\ z_2 &= e^{\frac{i}{2}(\phi+\tilde\phi)} \left(e^{r+\frac{i}{2}\psi} \cos\frac{\theta}{2} \cos\frac{\tilde\theta}{2} - e^{-r-\frac{i}{2}\psi} \sin\frac{\theta}{2}\sin\frac{\tilde\theta}{2} \right) \ , \\ z_3 &= e^{\frac{i}{2}(\phi-\tilde\phi)} \left(e^{r+\frac{i}{2}\psi} \cos\frac{\theta}{2} \sin\frac{\tilde\theta}{2} + e^{-r-\frac{i}{2}\psi} \sin\frac{\theta}{2}\cos\frac{\tilde\theta}{2} \right) \ , \\ z_4 &= -e^{-\frac{i}{2}(\phi-\tilde\phi)} \left(e^{r+\frac{i}{2}\psi} \sin\frac{\theta}{2} \cos\frac{\tilde\theta}{2} + e^{-r-\frac{i}{2}\psi} \cos\frac{\theta}{2}\sin\frac{\tilde\theta}{2} \right) \ . \end{aligned} \tag{4.128}$$

One can characterize the supersymmetric D5-brane embeddings by two algebraic equations in terms of the complex coordinates:

$$F_1(z_i) = 0 \ , \qquad F_2(z_i) = 0 \ . \tag{4.129}$$

In [79], Conde, Gaillard and Ramallo proved, for the calibration given in (4.122), that the embedding described by the previous holomorphic equations satisfies the calibration condition (3.103) and, therefore, the embedding is supersymmetric.

However, this is not the whole story. It may happen that embeddings of the form (4.129) produce a backreaction of the flavor branes not compatible with the ansatz shown around (4.112).

In particular, the choice made in [79] is given by the following embedding parametrized by two complex constants A and B:

$$z_3 = A z_1 \ , \qquad z_4 = B z_2 \ , \tag{4.130}$$

for which flavor branes backreact in a compatible way with the ansatz.

This equation, together with (4.123) and (4.124) determines the minimum distance r_q that this embedding reaches

$$\cosh(2r_q) = \frac{\sqrt{1+|A|^2}\sqrt{1+|B|^2}}{|1-AB|} \ . \tag{4.131}$$

It depends on the moduli of A and B, as well as their phase. By demanding that the WZ term of the action of the full set of D5 branes in the ten-dimensional theory coincides with the action obtained from the embeddings one arrives at [79]

$$S(r) = \frac{\sqrt{\cosh 4r - \cosh 4r_q}}{\sqrt{2}\sinh(2r)}\Theta(r-r_q) \ . \tag{4.132}$$

Notice that $S(r)$ is continuous at $r = r_q$, while $S'(r)$ diverges as $S'(r) \sim (r-r_q)^{-1/2}$ near r_q, and it is thus singular. To avoid this singularity in [79] they have proposed a brane setup for which the tip of the branes, r_q, is "smeared", so that an average should be made over brane distributions with different tip positions, weighted with a density function $\rho(r_q)$. After performing the change of variables $y = \cosh(4r)$ and $y_q = \cosh(4r_q)$ with $y \geq 1$, and assuming that the branes are distributed over the whole space $0 < r < \infty$, then the profile function will be given by

$$S(y) = \int_1^y dy_q \ \rho(y_q) \frac{\sqrt{y-y_q}}{\sqrt{y-1}} \ , \tag{4.133}$$

where the measure function $\rho(y_q)$ satisfies the normalization condition

$$\int_1^\infty \ dy_q \rho(y_q) = 1 \ . \tag{4.134}$$

4.6 Simple solutions for massive flavors

In this section we will present some requirements for the profile function $S(r)$ and we will propose a simple function $S(r)$ that fulfills them. Once we have a good profile function we can find the solutions to the master equation (4.118). However, not all the solutions to the master equation will be interesting for us, we will take some plausible physical assumptions in such a way that the relevant solution will be uniquely determined for each value of the parameter $\zeta \equiv N_f/N$.

On the gauge field theory side, one expects that the asymptotic physics for N_f massive flavors at high and low energies should be as follows:

a) At energies lower than the flavor mass (infrared limit) it should converge to the unflavored case, $S=0$, since in the deep IR we can integrate out all massive flavors.

b) At high energies (ultraviolet limit) flavors seem to be massless, so it should converge to the N_f massless flavor case, $S=1$.

This picture can be realized by the gravity dual background when the function $S(r)$ interpolates between the infrared/small radius limit $S(r)\to 0$ for $r \ll r_q$ – with r_q being a measure of the common quark mass – and the ultraviolet/large radius limit $S(r)\to 1$ for $r \gg r_q$.

Thus we are interested in solutions for massive flavors that approach the deformed MN solution (4.94) in the infrared, i.e. in the small radius limit $r\to 0$. In [79], to describe flavors with a given mass $\mathcal{O}(y_q)$ with some spread, a measure function $\rho(y_q)$ with a finite support around y_q was chosen. Here, we slightly depart from this approach. Given the freedom in the choice of distribution of branes, we will conveniently adopt a smooth distribution $\rho(y_q)$ of branes, chosen to meet the following requirements:

- $S(r)$ is assumed to be a monotonous increasing function of r, in agreement with the idea that degrees of freedom are integrated out when we move along the RG flow from the UV to the IR, varying between $S=0$ and $S=1$, approaching $S=1$ at infinity.

- We demand $S(r)\sim r^4$ (or greater powers) for $r\sim 0$, so that the curvature invariants of the geometry near $r=0$ are the same as in the deformed MN solution. In this way we ensure that the metric is regular at the origin (and that there are no massless flavors, which would generate the awkward IR singularity).

- In order to have a more tractable differential equation (4.118), we demand that S is such that the integral (4.117) defining Q can be explicitly performed with a simple result for Q.

- Finally, we demand that $\rho(y_q)$ is positive definite and satisfies the normalization condition (4.134).

We found an extremely simple choice that meets all these requirements:

$$S(r) = (\tanh(2r))^{2n}\ , \qquad\qquad n = 2,\, 3,\, \dots \tag{4.135}$$

This corresponds to a distribution of branes with masses concentrated around the maximum of $S'(r)$, at

$$r_{\max} = \operatorname{arccoth}\left(\sqrt{\frac{3+2n+2\sqrt{4n+2}}{2n-1}}\right)\ , \tag{4.136}$$

which increases with n (for large n, $r_{\max} \sim 1/4 \log n$). The spread Δr decreases with n. In order to determine $\rho(y_q)$, we note that the integral defining S is related to an Abel Transform as follows

$$2\partial_y(\sqrt{y-1}\; S(y)) = \mathcal{A}[\rho(y)] = \int_1^y dy_q\, \frac{\rho(y_q)}{\sqrt{y-y_q}}\;. \tag{4.137}$$

The inverse Abel Transform formula is

$$\rho(y_q) = \frac{2}{\pi}\partial_{y_q}\int_1^{y_q}\frac{\partial_y\left(\sqrt{y-1}\; S(y)\right)}{(y_q-y)^{1/2}}dy\;. \tag{4.138}$$

It is easy to verify that the normalization condition (4.134) is satisfied for this measure function. For the choice (4.135), we find

$$\rho_{(n)}(y_q) = \frac{4\sqrt{2}\,\Gamma(n+\frac{3}{2})}{\sqrt{\pi}(n-1)!}\,\frac{(y_q-1)^{n-1}}{(y_q+1)^{n+\frac{3}{2}}}\;. \tag{4.139}$$

In particular, if we take $n=2$,

$$\rho_{(n=2)}(y_q) = \frac{15(y_q-1)}{\sqrt{2}(1+y_q)^{\frac{7}{2}}}\;. \tag{4.140}$$

Next, we compute $Q(r)$ in (4.117). The basic integral we need is

$$\int_0^r dr\;\tanh^m(2r) = \frac{\tanh^{m+1}(2r)}{2(m+1)}\,{}_2F_1\left[1,\tfrac{1}{2}(1+m),\tfrac{1}{2}(3+m),\tanh^2(2r)\right]\;. \tag{4.141}$$

For integer m, this reduces to simple expressions. Thus we find

$$\begin{aligned} Q_{(n)}(r) &= \frac{1}{2}(2N-N_f)(2r\coth(2r)-1) \\ &\quad -\frac{N_f}{2}\left(1+\sum_{k=1}^{n+1}\left(\frac{\tanh^{2k-1}(2r)}{2k}-\frac{\tanh^{k-1}(2r)}{k}-\frac{\tanh^{k+n}(2r)}{k+n+1}\right)\right)\;, \end{aligned} \tag{4.142}$$

$$Q_{(n=2)}(r) = \frac{1}{2}(2N-N_f)(2r\coth(2r)-1)+\frac{N_f}{6}\tanh^2(2r)+\frac{N_f}{10}\tanh^4(2r)\;. \tag{4.143}$$

Notice that we have set $q_0=0$. The reason is that the term $q_0\coth(2r)$ produces a singular behavior at $r=0$, thus violating our condition that the solution reduces to the deformed MN solution at $r=0$.

In the following section we will proceed to the analysis of solutions $P(r)$ of the master equation (4.118) as a function of the Veneziano parameter $\zeta = N_f/N$. In all cases we will use the $S(r)$ given by (4.135) with $n=2$ and hence Q given by (4.143).

In general, the resulting differential equation (4.118) admits the following boundary conditions:

$$P \approx \begin{cases} p_0+\mathcal{O}(r^3) \\ h_1 r+\mathcal{O}(r^3) \end{cases} \qquad\qquad r\sim 0\;, \tag{4.144}$$

$$P \approx \begin{cases} |2N-N_f|\,r & \zeta\neq 2 \\ P_0+e^{-cr} & \zeta=2 \\ k\,e^{4r/3} & \text{any }\zeta \end{cases} \qquad\qquad r\gg 1\;. \tag{4.145}$$

Notice that the solutions with boundary conditions at $r \gg 1$ already appeared in section 4.3, these are (4.90), (4.91) and (4.92). For each $\zeta \neq 2$, the solution to (4.118), $P(r)$, is uniquely determined if we demand the following asymptotic conditions:

a) At $r \sim 0$, $P \sim h_1 r$, i.e. the solution reduces to the deformed Maldacena-Núñez solution with the asymptotic behavior given by (4.94).

b) At large r, the solution has the linear behavior (4.91), $P \sim |2N - N_f|\, r$.

For a generic integration constant h_1 above some critical value, the large r asymptotic behavior is $P \sim e^{4r/3}$, as discussed earlier. At a critical value of h_1 the solution has the linear behavior $P \sim (2N - N_f)r$, or constant for $\zeta = 2$, and at any lower h_1 it meets a singularity before reaching $r = \infty$. Hence, the condition of linear behavior at infinity specifies the solution uniquely.[2]

In order to solve the differential equation (4.118) numerically, as mentioned above we take the brane distribution (4.135) with $n = 2$, and Q given in (4.143). This describes massive flavors with a mass around $r \approx 0.5$ (see (4.136)), determined by the maximum of $S'(r)$, shown in fig. 4.3 together with $S(r)$.

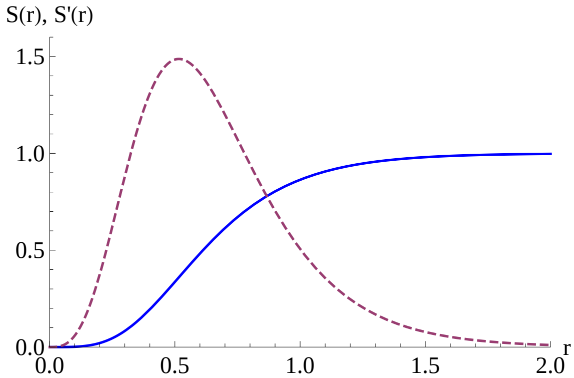

Figure 4.3: $S(r)$ (solid line) and $S'(r)$ (dashed line). The maximum of $S'(r)$ at $r \approx 0.5$ indicates the characteristic mass scale of the massive flavors.

Since we have to meet boundary conditions at zero and infinity, we employ a shooting method. This determines the critical h_1. Figures 4.4 (a), (b), (c), (d) illustrate the solutions in the three cases $N_f < 2N$, $N_f = 2N$ and $N_f > 2N$.

- In the first case we take $\zeta = 7/4$, for which we find

$$\frac{h_1}{N_f} \cong 1.53218706 \ , \qquad \zeta = \frac{7}{4} \ , \tag{4.146}$$

and the solution is reported in fig. 4.4 (a).

[2] The solutions with exponential behavior at infinity have a constant dilaton and become Ricci flat, which is not the expected asymptotic behavior for holographic applications. Some interesting applications of these solutions as describing properties of 6d field theories have nevertheless been found in [75].

- In the special case $\zeta = 2$ the solution that starts with $P \cong h_1 r$ near $r = 0$ and asymptotes to a constant at infinity has

$$\frac{h_1}{N_f} \cong 1.42475837 \ , \qquad \zeta = 2 \ . \tag{4.147}$$

The large radius behavior is given by

$$P = P_0 - e^{-c(r-r_1)} + \mathcal{O}(e^{-4r}) \ , \tag{4.148}$$

with

$$P_0 = \frac{32N}{15} \ , \qquad c = 1 + \frac{\sqrt{21}}{2} \ , \qquad Q \to \frac{8N}{15} \ , \tag{4.149}$$

where r_1 is a numerical constant. The solution is shown in fig. 4.4 (b).

- Finally, fig. 4.4 (c) shows a case with $\zeta > 2$, taking in particular $\zeta = 7/3$, for which we find

$$\frac{h_1}{N_f} \cong 1.35890843 \ , \qquad \zeta = \frac{7}{3} \ . \tag{4.150}$$

Note that $\zeta = 7/3$ is related to $\zeta = 7/4$ (used in fig. 4.4 (a)) by $\zeta \to \zeta/(\zeta - 1)$, which is produced by the change $N \to N_f - N$. We have made this choice for later comparison between theories related by a naive Seiberg duality transformation. We will comment on this below.

More generally, one can determine h_1 as a function of ζ, with $0 < \zeta < \infty$, as shown in fig. 4.5. For $\zeta \to 0$ we obtain $h_1/N_f \to \infty$. Indeed one can verify that $h_1 \to 2N_f/\zeta = 2N$ as $N_f \to 0$, recovering the MN boundary condition at $r = 0$ for P discussed above. Furthermore, we note that for large ζ the critical h_1 approaches a finite asymptotic value,

$$\frac{h_1}{N_f} \cong 1.72102763 \ , \qquad \zeta \to \infty \ . \tag{4.151}$$

The reason is that for $\zeta \gg 1$, one can scale $P \to N_f P$ so that the master equation (4.118) becomes independent of ζ, as Q becomes proportional to N_f, see (4.143). This scaling solution is shown in fig. 4.4 (d).

4.7 Gauge coupling β-function

To compute the β function of the gauge coupling in the dual field theory we first need to identify the gauge coupling constant in terms of geometrical quantities. For the metric (4.70), which is the same for both the massless and massive flavor background, this has been done in [73] and the computation is completely analogous to that performed in (4.42). The gauge coupling turns out to be directly related to the P function as follows

$$\frac{8\pi^2}{g^2_{\mathrm{YM}}} = 2\left(e^{2h} + \frac{e^{2g}}{4}(a-1)^2\right) = \tanh(r)\ P(r) \ . \tag{4.152}$$

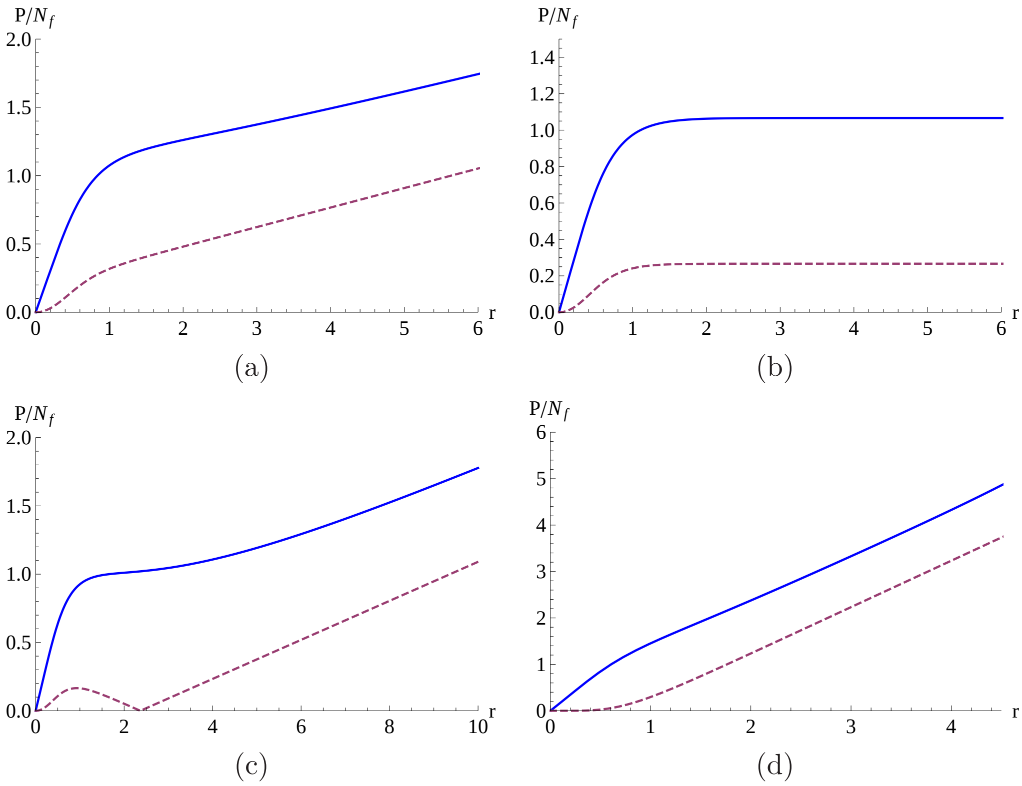

Figure 4.4: The function $P(r)/N_f$ solution to the master equation (4.118) that matches between the deformed Maldacena-Núñez solution (4.94) in the infrared ($r \to 0$) and the linear behavior in the ultraviolet ($r \to \infty$). The dashed line corresponds to $Q(r)/N_f$ ($|Q(r)|/N_f$ in fig. c). (a) $\zeta = 7/4$ (b) $\zeta = 2$. (c) $\zeta = 7/3$. (d) $\zeta = \infty$.

Since it is clear when we refer to the Yang-Mills coupling we will remove from now on the YM subscript to clear the notation.

The second crucial ingredient necessary to obtain any β function in the dual field theory is the precise relation between the radial coordinate r of the supergravity background and the energy scale of the gauge theory. This was already discussed in section 4.2.2 for the Maldacena-Núñez model, where the following relation was obtained

$$\left(\frac{\Lambda}{\mu}\right)^3 \sim a(r) \;, \tag{4.153}$$

which gives rise to the UV behavior

$$\frac{\mu}{\Lambda} \sim e^{\frac{2r}{3}} \;, \qquad r \gg 1 \;. \tag{4.154}$$

In extending the relation between μ and r to models with $N_f \neq 0$ massless flavors, one needs to consider a number of issues. In particular, interesting solutions exist with $a = b = 0$ in (4.70)-(4.71) and its massive generalization, so one should seek for other possible definitions of the energy scale than (4.153). As emphasized in [73, 75], for a class of

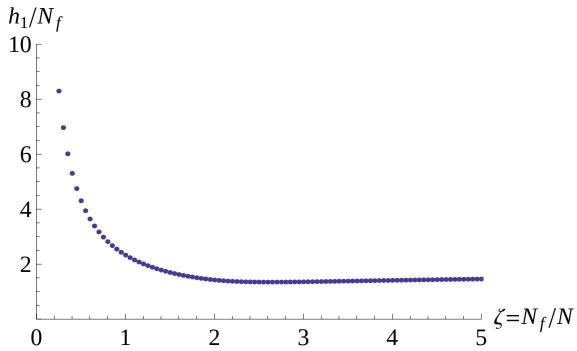

Figure 4.5: Near $r=0$, the solutions are required to behave as $P \approx h_1 r$ to approach the deformed MN solution. The figure shows the critical values of the parameter h_1 which are required for P to have linear behavior at infinity.

flavored $\mathcal{N}=1$ supersymmetric models, the same UV relation (4.154) arises from any of the following identifications

$$\left(\frac{\Lambda}{\mu}\right)^3 \sim a(r) \ , \qquad \left(\frac{\Lambda}{\mu}\right)^3 \sim b(r) \ , \qquad \left(\frac{\Lambda}{\mu}\right)^3 \sim e^{-2\Phi(r)} \ . \tag{4.155}$$

The relations (4.154) and (4.155) can be generically written in the form

$$\left(\frac{\Lambda}{\mu}\right)^3 = F(r) \ , \qquad F(r) \to e^{-2r} \quad \text{for} \quad r \to \infty \ . \tag{4.156}$$

Different choices of F are analogous to the ambiguity that appears on the field theory side in the choice of renormalization scheme. Using (4.152) and (4.156), we obtain the following expression for the β function:

$$\beta_{\frac{8\pi^2}{g^2}} = -\frac{3F}{F'}\partial_r(\tanh r \ P) = -\frac{3F}{F' \cosh^2 r}(\sinh r \cosh r \ P' + P) \tag{4.157}$$

and by knowing F and the solution P, we can now compute $\beta_{8\pi^2/g^2}$ and hence β_g. Differences between the possible radius/energy relations in (4.155) eventually arise in the IR. However, we have verified that all relations in (4.155) lead to qualitatively similar results. For the calculations that follow, we will adopt the prescription (4.153). In this way, when $N_f=0$, we recover the β function of the MN model (specifically, the β function obtained in [122]).

It is convenient to rescale away the parameter N_f in the master equation (4.118) by the change $P = N_f \tilde{P}$ and $Q = N_f \tilde{Q}$. This leads to the following scaling for the β function,

$$\beta_g = \frac{1}{\sqrt{N_f}}\beta_{\tilde{g}}(x,\tilde{g}) \ , \qquad\qquad g = \frac{1}{\sqrt{N_f}}\tilde{g} \ , \tag{4.158}$$

where

$$\frac{8\pi^2}{\tilde{g}^2} = \tanh(r)\,\tilde{P}(r) \ . \tag{4.159}$$

In what follows we will thus compute $\beta_{\tilde{g}}$. Note that, in terms of the 't Hooft coupling $\lambda \equiv g^2 N$, one has $\tilde{g}^2 = \zeta\lambda$, $\zeta = N_f/N$, and

$$\beta_\lambda = f(x,\lambda)\ . \tag{4.160}$$

This can be compared with the NSVZ β function (4.103), which in terms of λ reads

$$\beta_\lambda = -\frac{\lambda^2}{8\pi^2(1-\frac{\lambda}{8\pi^2})}(3 - x(1-\gamma_0))\ . \tag{4.161}$$

This agrees with the structure of the holographic β function (4.160), i.e. in the large N limit at fixed N_f/N it only depends on λ and ζ.

At this point it is useful to recall some basic facts of the NSVZ β function. It was suggested by Seiberg [7] that a conformal window for SQCD should exist for $\frac{3}{2}N < N_f < 3N$, where a family of massless SQCD theories with N_f massless flavors develop an IR fixed point at finite coupling. All flavored gauge theories in the conformal window would be deconfined and chiral symmetry restored. The lower end-point should be considered a lower-bound on the actual value. A non-trivial IR fixed point can be found if $\zeta \approx 3$ [94]. Indeed, using the explicit form of the one-loop anomalous dimension the vanishing of the β function requires

$$\frac{3}{\zeta} - 1 = -\gamma_0 = \frac{1}{8\pi^2}\, g^2 N + \mathcal{O}(g^4 N^2)\,. \tag{4.162}$$

It is clear that this fixed point moves towards the strongly coupled region as ζ decreases from 3 to lower values. This assumes a small value of the anomalous dimension. As we will see below, the present holographic system, like the one of [74, 75] seems to involve large values of the anomalous dimension γ_0, in fact $\gamma_0 = -1/2$ in the UV.

The calculations that follow use our specific choice for the embedding function $S(r) = \tanh^4(2r)$. However, the structure of the fixed points seems to be a generic property of the solutions of the master equation (4.118) with linear dilaton asymptotic and any embedding function S with $S(r) \to 1$ at infinity. This asymptotic includes previously known solutions with massless flavors.

The linear dilaton asymptotics of these types of backgrounds preclude the emergence of an anti de Sitter geometry at infinity, which should be a more appropriate description near the UV fixed points. Despite this fact and despite the above mentioned ambiguities in the definition of the holographic beta function, we will find some remarkable coincidences with the expected behavior in flavored SQCD.

4.7.1 $N_f < 2N$

The β function for the gauge theory with massive fundamental flavors is obtained by taking the solution $P(r)$ found in the previous section (see fig. 4.4a) and applying the formula (4.157). The result is shown in fig. 4.6a.

The β function has a UV fixed point at $g=0$, where it has the following behavior

$$\beta_g \approx -\frac{3}{32\pi^2}(2N - N_f)g^3\,. \tag{4.163}$$

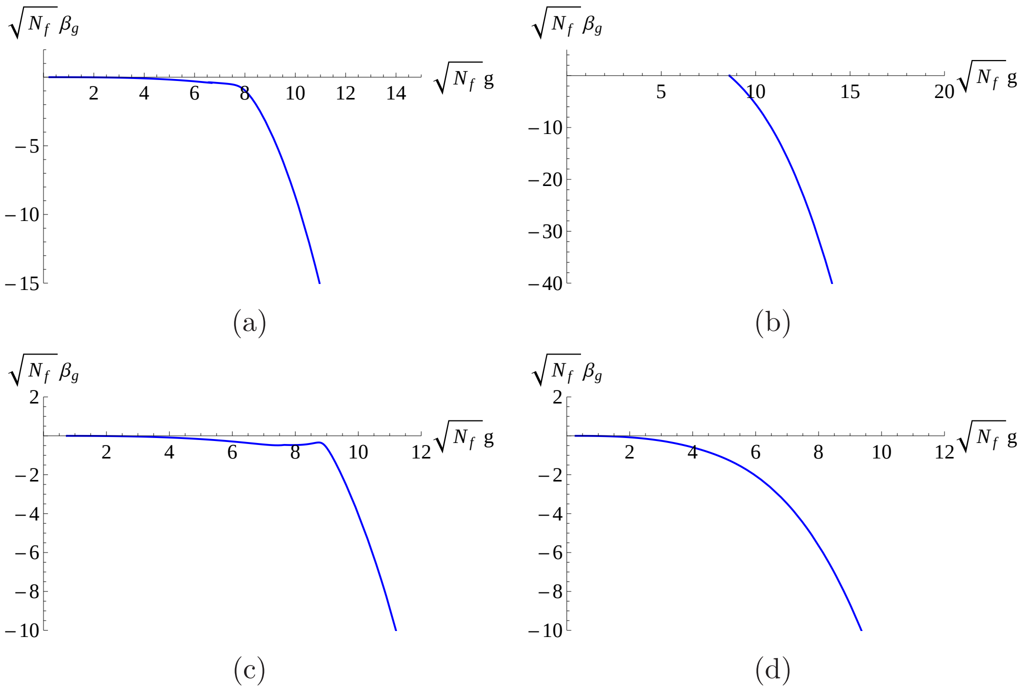

Figure 4.6: $\sqrt{N_f}\beta_g$ as a function of $\sqrt{N_f}g$, corresponding to the supergravity solutions in fig. 4.4a,b,c,d. (a) $\zeta = 7/4$. (b) $\zeta = 2$. (c) $\zeta = 7/3$. (d) $\zeta = \infty$.

Remarkably, this exactly agrees with the NSVZ β function (4.103) near $g = 0$, if $\gamma_0 = -1/2$ in the UV – where mass terms can be neglected. A similar conclusion was reached in the case of the backgrounds with $S = 1$ [74, 75]. This is not surprising, since in the UV our S differs from $S = 1$ by exponentially suppressed terms, which do not affect the leading behavior in (4.163). It would be interesting to have an independent derivation of the anomalous dimension γ_0 by holographic methods, but presently it is not clear to us what the correct prescription would be.[3]

The β function of fig. 4.6a is zero at $g = 0$, negative and monotonically decreasing for $g > 0$, thus implying asymptotic freedom and ordinary confinement in the IR, where $g \to \infty$. In particular, we find no additional IR or UV fixed points at finite coupling.

4.7.2 $N_f = 2N$

Using the solution $P(r)$ found in the previous section (see fig. 4.4b) we determine the β function, shown in fig. 4.6b. We can see that a non-trivial UV fixed point $g = g_*$ appears. Although we do not have the gravity solution that describes the missing branch $g < g_*$, some interesting features can be inferred by comparing with the NSVZ β function (4.103) in this UV region where mass terms can be neglected. For $N_f = 2N$, the NSVZ β function

[3] In [74] an attempt was made to compute γ_0 by proposing that the quartic coupling of the gauge theory should be identified with some quotient of the volumes of different cycles of the manifold.

becomes

$$\beta_g = -\frac{g^3 N}{16\pi^2(1 - \frac{g^2 N}{8\pi^2})}(1 + 2\gamma_0)\ . \tag{4.164}$$

Again, it is consistent with our results if $\gamma_0 \to -1/2$ in the UV and g flows to g_*. Moreover, since the perturbative NSVZ β function is negative near the UV fixed point at $g = 0$, by continuity there must be at least another point g'_*, with $g'_* < g_*$, where the β function vanishes. In the simplest assumption that there is only one such point, this would be an IR fixed point. The resulting picture is in fact similar to the one proposed by Seiberg (for a discussion on the effect of mass terms see e.g. [116]). Obviously, a description using massive flavors like the present one cannot describe the emergence of a conformal fixed point in the infrared. However, given the presence of the UV fixed point at $g = g_*$, the IR fixed point seems to be the simplest possibility that permits a negative beta function near $g = 0$. The combined presence of a pair of IR and UV fixed points is also a prerequisite for the existence of a mechanism in which the disappearance of the conformal window is due to the annihilation of a pair of fixed points [95]. Notice that if for $N_f = 2N$ the IR fixed point appears at 't Hooft coupling $\lambda = \mathcal{O}(1)$ (as suggested by a naive extrapolation of (4.162)), it would be very difficult to see it by means of perturbative and holographic techniques.

4.7.3 $N_f > 2N$

Using now the solution $P(r)$ of fig. 4.4c we determine the β function for the case $\zeta = 7/3$. This is shown in fig. 4.6c. The β function has, like in the $N_f < 2N$ case, a UV fixed point at $g = 0$, where it has the behavior

$$\beta_g \approx -\frac{3}{32\pi^3}(N_f - 2N)g^3\,. \tag{4.165}$$

This exactly agrees with the NSVZ β function (4.103) of the Seiberg dual gauge theory with $\tilde{N} = N_f - N$ near $g = 0$, if again we set $\gamma_0 = -1/2$ in the UV. This strongly suggests that in the UV region the background obtained with our boundary conditions describes, when $N_f > 2N$, the Seiberg dual system.

It must be stressed that in the present case Seiberg duality is only an approximate relation that depends on the scale of energy (see [79]). The idea is that at a given scale μ one can integrate out massive flavors which have mass greater than μ and remain with a reduced number of light flavors. In the present framework, this reduced number of flavors at an energy scale r is effectively described by $N_f(r) \equiv N_f\, S(r)$. With our choice of $S(r)$, massive flavors are accumulated near $r \approx 0.5$ (see fig. 4.3). In the infrared region, where $r \sim 0$, one has $S \sim r^4$ so $N_f(r) \to 0$, as expected since in this region the energy scale is much smaller than the characteristic mass of the flavors. On the other hand, in the UV region, $S \to 1$ and $N_f(r) \to N_f$, which is consistent with the fact that at this scale of energies all flavors look massless. As observed in [79], the master equation (4.118) remains invariant under $N \to N_f(r) - N$ and $N_f(r) \to N_f(r)$. This transformation changes $Q(r) \to -Q(r)$. This is the only sense in which Seiberg duality can be applied to the present system (in particular, $N \to N_f - N$ and $N_f \to N_f$ is not a symmetry of the master equation) and it is consistent with our proposal that the solution $P(r)$ of fig. 4.4c describes the Seiberg dual system at an energy scale much larger than the characteristic mass of the flavors, where $N_f(r) \to N_f$.

Having obtained a gravity solution for the "Seiberg dual" system, the question is how to identify a background dual to the original gauge theory. When $N_f > 2N$, we expect that the gauge theory will develop a Landau pole. This means that the theory cannot be extended beyond a certain UV scale. On the gravity side, it means that the geometry should terminate at a maximum value of r, where it probably has a singularity. Indeed, there is a one-parameter family of solutions with parameter h_1 that at $r = 0$ approach the deformed MN solution, but at some finite r they meet a singularity where $P = |Q|$. These are the solutions which have an h_1 whose value is anything lower than the critical h_1 of fig. 4.5. In this case we lack a clear criterion to pick a unique solution in this family that is dual to the original gauge theory. It should also be noted that the application of holography is difficult to justify for singular backgrounds that do not get to infinity.

The β function in fig. 4.6c exhibits a local maximum precisely near the g_* where a fixed point appears in the $N_f = 2N$ case. Indeed, as N_f approaches $2N$, the local maximum approaches the line $\beta_g = 0$ and occurs at large values of r. In the strict $N_f = 2N$ limit, the branch $g < g_*$ disappears from the figure, because the solution gets to $r = \infty$ already at $g = g_*$.

Finally, fig. 4.6d shows the gauge coupling β function computed in the infinite flavor limit, that is, for the solution shown in fig. 4.4d. It shares similar features with the case $\zeta = 7/3$, except that the local maximum has disappeared. The disappearance of the local maximum can be understood as follows: for $\zeta = \infty$, one has $\tilde{\zeta} = N_f/\tilde{N} \to 1$, where $\tilde{N} = N_f - N$. Thus one is computing the β function of the "Seiberg dual" system with $\zeta = 1$. For $\zeta = 1$, the β_g indeed looks very similar to fig. 4.6d.

4.8 Comments

We have investigated the new gravity backgrounds found in [79] dual to $\mathcal{N} = 1$ supersymmetric gauge theories with massive fundamental flavors. This is a step forward in the endeavor to describe QCD-like theories through the AdS/CFT duality which started with [70].

These backgrounds are characterized by a profile function $S(r)$ which encodes the flavor brane distribution. In the specific backgrounds we have studied we have chosen a continuous $S(r) = \tanh^4(2r)$, with support in the whole space $0 < r < \infty$, which leads to a simple analytic expression for the function $Q(r)$, and thus permits a more straightforward integration of the master equation (4.118) that determines $P(r)$, hence the complete geometry. The solutions –parametrized by $\zeta \equiv N_f/N$– were uniquely determined by imposing boundary conditions that ensure regularity at $r = 0$ and acceptable asymptotic behavior at infinity. In this way, solutions are free from the IR singularity that affects the massless flavor $S = 1$ case of [74, 75].

An interesting open problem would be to find regular backgrounds that can describe the massless flavor limit in a controllable manner. The current approach uses an $S(r)$ function that determined the flavor mass scale $r \sim M_f$. One can attempt to study the limit $M_f \to 0$ within this context. Although this approach seems to be affected by singularity problems similar to those of the massless $S = 1$ case, it is possible that some universal properties can be learned by studying this limit in detail.

Other possible direction would be the exploration of the physical consequences of different profile functions $S(r)$. Some examples in this direction can be found in [123,124].

For the particular profile function studied here, we have investigated properties of the gauge coupling beta function and the possible emergence of fixed points. As explained in [74,75], the main feature that seems to determine the properties of the dual gauge theory is the presence of quartic operators in the superpotential that arise upon integration of the Kaluza-Klein modes of the original string theory. These operators are of the form $W \sim (Q^r \tilde{Q}_u)(Q^u \tilde{Q}_r)$, with gauge indices contracted inside the parentheses and lead to a sextic potential in the scalar fields. They become marginal when the anomalous dimension γ_0 is $-1/2$. For this value of the anomalous dimension the NSVZ β function (4.103) becomes

$$\beta_g = -\frac{3g^3}{32\pi^2}\,\frac{2N - N_f}{1 - \frac{g^2 N}{8\pi^2}}\,. \tag{4.1}$$

We have seen that this expression agrees with the holographic β function in the UV region for $N_f \leq 2N$, and also for $N_f > 2N$ if we replace $N \to N_f - N$. We argued that for $N_f > 2N$ our backgrounds should therefore describe the Seiberg dual system, in the generalized sense discussed in sect. 4.3. After this replacement, the β function stays negative for all $N_f > 2N$. As discussed, in the $N_f > 2N$ case, finding a gravity description of the original system before Seiberg duality is difficult because, as the expression (4.1) indicates, asymptotic freedom is lost and thus a Landau pole is expected, presumably meaning that the gravity solution must encounter a singularity at some radius r.

For $N_f = 2N$ we found that the theory has a UV fixed point, which hints at the presence of an IR fixed point at some lower coupling, if one is to match continuously with standard perturbative results. In this context, we stress that our theory only converges to the massless case in the UV, and it can only asymptotically recover the presence of conformal fixed points. For $N_f < 2N$ we have not found any evidence of an IR fixed point, perhaps suggesting that in the presence of quartic operators the "conformal window" opens and closes at $N_f = 2N$.

The effect of higher dimensional operators such as $(Q^r \tilde{Q}_u)(Q^u \tilde{Q}_r)$ –which in the component Lagrangian leads to terms (scalars)^6 and (scalar)^2 (fermion)^2–has a counterpart in non-supersymmetric QCD. It produces effects that are similar to well known non-perturbative effects related to chiral dynamics in the low energy effective field theory. For example, a quartic fermion operator has a key role in the emergence of chiral symmetry breaking and must have a role in the disappearance of conformality: Schwinger-Dyson gap equation for the fermion propagator implies a direct relation between the onset of chiral symmetry breaking, thus the presence of a non vanishing chiral condensate, and the point where the four-fermion operator becomes relevant in the RG flow [80,81,125]. This happens by lowering the number of flavors in QCD-like theories, starting from the point where asymptotic freedom sets in. At sufficiently low N_f chiral symmetry will always be broken and conformality is lost.

Chapter 5

Non-Abelian T-dual of Klebanov-Witten with flavors and G-structures

Although the idea of generalising T-duality to non-Abelian isometry groups has rather old roots [126], it is only recently that it has been studied as full solution-generating symmetry of supergravity [127–133]. The recent work of Itsios et al. [132, 133] considered the application of this duality transformation in IIB supergravity backgrounds preserving $\mathcal{N} = 1$ supersymmetry. For instance applying an $SU(2)$ non-Abelian T-duality to the internal space of the Klebanov–Witten background ($\mathrm{AdS}_5 \times T^{1,1}$) results in a solution of type IIA which retains the AdS_5 factor and has a lift to M-theory which corresponds to the geometries obtained in [134] from wrapping M5 branes on an S^2. In [133] similar dualizations were applied to non-conformal geometries (Klebanov–Tsetylin, Klebanov–Strassler and wrapped D5 models like the ones shown in the previous chapter) resulting in a new class of smooth solutions of massive type IIA supergravity. The field theory interpretation of these massive IIA solutions is, as yet, undetermined. However an analysis of the gravity solution indicates they retain rich RG dynamics displaying signatures of Seiberg duality, domain walls and confinement in the IR.

A common feature of the geometries obtained in [133] is that they retain four dimensional Poincaré invariance and it was argued that they should also retain $\mathcal{N} = 1$ supersymmetry. The conditions for a solution of type II supergravity to possess these symmetries can be very elegantly stated using the language of G-structures (see section 3.2).

The first purpose of this chapter is to study the effects of non-Abelian T-duality on these G-structures and thereby to give credence to the conjecture made in [133] that in general the result of the dualization will be to take an $SU(3)$-structure background to one with $SU(2)$-structure.

A second purpose of this chapter concerns the application of non-Abelian T-duality in the construction of new flavored solutions of supergravity. The string dual view on the addition of fundamental matter to the field theories has already studied in chapter 4.

For the background we will consider in this chapter, namely Klebanov-Witten [135], the addition of flavors works in a similar way as in chapter 4.

A generic feature about these solutions encoding the dynamics of N_f fields transforming in the fundamental representation of the $SU(N)$ gauge group is that the string backgrounds should represent sources localised on those SUSY-preserving submanifolds. In chapter 4 we have already seen that the complications associated with the non-linear and coupled partial differential equations this problem requires, lead to the consideration of smeared sources. The SUSY-preserving way of implementing this smearing is also described by the G-structures classifying the original (unflavored) background, see [136], [137] for details.

Hence, there is a rich interplay between G-structures and the dynamics of SUSY sources in supergravity. This is one of the themes of this work.

The main goal is to geometrize part of the information of the works [132, 133]. The idea being that once in a geometric context, a future physical analysis will become more clear and systematic. On the other hand, we emphasize the underlying motivation: the utility of non-Abelian T-duality is to produce backgrounds (hard to obtain by an educated guess) that being smooth, they define a dual QFT. So, understanding the geometric side of the non-Abelian T-duality will help to characterize a set of new strongly coupled field theories.

In this chapter we will start by reviewing non-Abelian T-duality and the Roček-Verlinde recipe to obtain the T-dual of a given background. It will be useful to show how G-structures transform under T-duality. Then we will work with the Klebanov-Witten background. We will show its $SU(3)$-structure and we will explicitly construct the $SU(2)$-structure of its T-dual. Finally, we will present the flavored Klebanov-Witten model and its T-dual.

5.1 Non-Abelian T-duality

In this section we present some useful overview of non-Abelian T-duality, a comprehensive treatment may be found in [133].

T-duality first appeared in the context of closed strings for toroidal compactifications [138–140]. In the simple case of one single direction compactified on a circle of radius R, the target space coordinate corresponding to this direction must satisfy the boundary condition

$$X(\sigma+\pi,\tau) = X(\sigma,\tau) + 2\pi R m\ , \qquad m \in \mathbb{Z}\ , \tag{5.1}$$

where (σ,τ) are world-sheet coordinates and m is the winding number, that is the number of times the string winds around the circle. The mode expansion for a closed string with winding number m must incorporate the previous boundary condition and it becomes

$$X(\sigma,\tau) = x + 2\alpha' p\tau + 2Rm\sigma + \ldots \tag{5.2}$$

and the dots refer to string excitation modes, not modified by the compactification. x and p are the center of mass and the total string momentum components in the direction of the circle along which we compactify. Since this direction is compact, the momentum component is quantized

$$p = \frac{n}{R}\ , \qquad n \in \mathbb{Z}\ , \tag{5.3}$$

where n is the Kaluza-Klein excitation number. Then we see that there is no change if we consider compactifications on circles of radius $R \leftrightarrow \alpha'/R$, if, besides, we interchange the winding and the Kaluza-Klein excitation numbers $m \leftrightarrow n$ (with the additional change in the dilaton $\Phi \to \Phi - \log R/\sqrt{\alpha'}$ [141]). This is illustrated in figure 5.1.

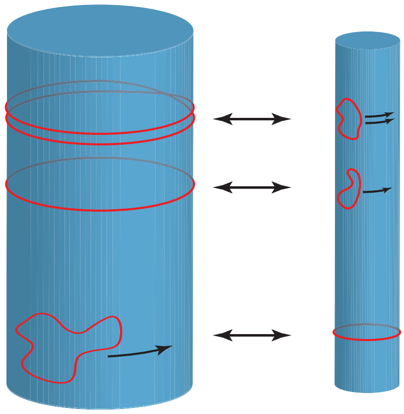

Figure 5.1: Diagram of T-duality acting on closed strings with a direction compactified on a circle.

This was extended by Buscher [142,143] to non-flat conformal backgrounds. One starts by considering the σ-model built out of a B-field, the dilaton, Φ, and a metric, g, enjoying an abelian isometry, which in adapted coordinates (θ, x^α) acts as translations in θ. Then the dual σ-model is built out of the tilded quantities

$$\tilde{g}_{00} = \frac{1}{g_{00}} \,, \qquad \tilde{g}_{0\alpha} = \frac{B_{0\alpha}}{g_{00}} \,, \qquad \tilde{g}_{\alpha\beta} = g_{\alpha\beta} - \frac{1}{g_{00}}(g_{0\alpha}g_{0\beta} - B_{0\alpha}B_{0\beta}) \,, \tag{5.4}$$

$$\tilde{B}_{0\alpha} = \frac{g_{0\alpha}}{g_{00}} \,, \qquad \tilde{B}_{\alpha\beta} = B_{\alpha\beta} - \frac{1}{g_{00}}(g_{0\alpha}B_{0\beta} - g_{0\beta}B_{0\alpha}) \,, \tag{5.5}$$

$$\tilde{\Phi} = \Phi - \frac{1}{2}\log g_{00} \,. \tag{5.6}$$

The transformation above is known as abelian T-duality. An alternative construction of this transformation was given by Roček and Verlinde [144], more useful to generalize these concepts to non-Abelian T-duality. The three-step Roček-Verlinde procedure consists of:

1. One starts by gauging the abelian $U(1)$ isometry. To this purpose, one must introduce an auxiliary gauge field A.

2. Then the gauge field is demanded to be flat by the introduction of the Lagrange multiplier term $v\mathrm{d}A$. Integrating out the Lagrange multiplier leads to the original σ-model.

3. But we do not integrate out the Lagrange multiplier. Now we integrate out the gauge field considering the Lagrange multiplier as a new dynamical variable. Finally, fixing the gauge, one obtains the T-dual σ model.

This approach can be readily generalized to the case of non-Abelian isometries and provides a putative non-Abelian T-duality transformation [126, 145–147]. Unlike its Abelian counter part, this non-Abelian T-duality typically destroys the isometries dualized (though they can be recovered as non-local symmetries of the string σ-model [148]). Due to global complications, it is thought that this non-Abelian dualization is not a full symmetry of string (genus) perturbation theory however it remains valid as a solution-generating symmetry of supergravity. In this regard, it is still very useful in the context of the AdS/CFT correspondence.

Let us see in more detail how this non-Abelian T-duality works. First consider a bosonic string σ-model in a NS background, we will assume that this background admits some isometry group G and that background fields can be expressed in terms of left-invariant Maurer–Cartan forms, $L^i = -i\,\mathrm{Tr}(g^{-1}\mathrm{d}g)$, for this group. That is to say the target space metric has a decomposition

$$\mathrm{d}s^2 = G_{\mu\nu}(x)\mathrm{d}x^\mu \mathrm{d}x^\nu + 2G_{\mu i}(x)L^i + g_{ij}(x)L^i L^j \ , \tag{5.7}$$

with corresponding expressions for the NS two-form B and dilaton Φ. The non-linear σ-model is

$$S = \int \mathrm{d}^2\sigma \Big(Q_{\mu\nu}\partial_+ x^\mu \partial_- x^\nu + Q_{\mu i}\partial_+ x^\mu L^i_- + Q_{i\mu} L^i_+ \partial_- x^\mu + E_{ij} L^i_+ L^j_- \Big) \ , \tag{5.8}$$

where

$$Q_{\mu\nu} = G_{\mu\nu} + B_{\mu\nu} \ , \quad Q_{\mu i} = G_{\mu i} + B_{\mu i} \ , \quad Q_{i\mu} = G_{i\mu} + B_{i\mu} \ , \quad E_{ij} = g_{ij} + b_{ij} \ , \tag{5.9}$$

and $L^i_\pm$ are the left-invariant forms pulled back to the world sheet. To obtain the dual σ-model one first gauges the isometry by making the replacement

$$\partial_\pm g \to D_\pm g = \partial_\pm g - A_\pm g \ , \tag{5.10}$$

in the Maurer–Cartan forms. Also, the addition of a Lagrange multiplier term $-i\mathrm{Tr}(vF_{+-})$ enforces a flat connection.

After integrating this Lagrange multiplier term by parts, one can solve for the gauge fields to obtain the T-dual model. Finally, we must gauge fix the redundancy by, for example, setting $g = \mathbb{I}$ [1].

We obtain the Lagrangian,

$$\tilde{S} = \int \mathrm{d}^2\sigma \Big(Q_{\mu\nu}\partial_+ x^\mu \partial_- x^\nu + (\partial_+ v_i + \partial_+ x^\mu Q_{\mu i})(E_{ij} + f_{ij}{}^k v_k)^{-1}(\partial_- v_j - Q_{j\mu}\partial_- x^\mu) \Big) \ , \tag{5.11}$$

from which the T-dual metric and B-field can be ascertained. As with Abelian T-duality [142, 143] the dilaton receives a shift from performing the above manipulations in a path integral given by

$$\hat{\Phi}(x, v) = \Phi(x) - \frac{1}{2}\log(\det M) \ , \tag{5.12}$$

[1] More general gauge fixing choices are allowed and will in fact be exploited in this paper. For details of these we refer the reader to [133]. In this section we assume the gauge fixing choice of $g = \mathbb{I}$.

where we have defined $M_{ij} = E_{ij} + f_{ij}{}^k v_k$, which will play a prominent role in what follows. This shift in the dilaton ensures conformal invariance of the dual model, at least at first order in α'.

Using the equations of motion, one can ascertain the following transformation rules for the world-sheet derivatives

$$\begin{aligned} L^i_+ &= -(M^{-1})_{ji}\,(\partial_+ v_j + Q_{\mu j}\partial_+ x^\mu) \ , \\ L^i_- &= M^{-1}_{ij}(\partial_- v_j - Q_{j\mu}\partial_- x^\mu) \ , \\ \partial_\pm x^\mu &= \text{invariant} \ . \end{aligned} \tag{5.13}$$

These relations provide a classical canonical equivalence between the two T-dual σ-models [148, 149].

The consequence of this is that left and right movers couple to different sets of vielbeins for the T-dual geometry. Suppose that we define frame fields for the initial metric (5.7) by

$$\mathrm{d}s^2 = \eta_{AB} e^A e^B + \sum_{i=1}^{\dim G} \delta_{ab} e^a e^b \ , \quad e^A = e^A_\mu \mathrm{d}x^\mu \ , \quad e^a = \kappa^a_i L^i + \lambda^a_\mu \mathrm{d}x^\mu \ . \tag{5.14}$$

Then by making use of the transformation rules (5.13) one finds that after T-dualization left and right movers couple to the vielbeins

$$\begin{aligned} \hat{e}^a_+ &= -\kappa M^{-T}(\mathrm{d}v + Q^T \mathrm{d}x) + \lambda \mathrm{d}x \ , \qquad & \hat{e}^A_+ &= e^A \ , \\ \hat{e}^a_- &= \kappa M^{-1}(\mathrm{d}v - Q\,\mathrm{d}x) + \lambda \mathrm{d}x \ , & \hat{e}^A_- &= e^A \ , \end{aligned} \tag{5.15}$$

in which M^{-T} is the inverse transpose of the matrix M defined above. Both these frame fields define the T-dual target space metric obtained from (5.11) given by

$$\mathrm{d}\hat{s}^2 = \eta_{AB} e^A e^B + \sum_{i=1}^{\dim G} \delta_{ab} \hat{e}^a_+ \hat{e}^b_+ = \eta_{AB} e^A e^B + \sum_{i=1}^{\dim G} \delta_{ab} \hat{e}^a_- \hat{e}^b_- \ . \tag{5.16}$$

Since these frame fields define the same metric they must be related by a Lorentz transformation and indeed

$$\hat{e}_+ = \Lambda \hat{e}_- \ , \quad \Lambda = -\kappa M^{-T} M \kappa^{-1} \ . \tag{5.17}$$

We note that $\det \Lambda = (-1)^{\dim G}$, this will have the consequence that the dualization of an odd-dimensional isometry group maps between type IIA and IIB theories whereas that of an even-dimensional group preserves the chirality. This Lorentz transformation induces an action on spinors defined by the invariance property of gamma matrices [2];

$$\Omega^{-1}\Gamma^a \Omega = \Lambda^a{}_b \Gamma^b \ . \tag{5.18}$$

We are particularly interested in performing this duality in supergravity backgrounds of relevance to the AdS/CFT correspondence which are typically supported by RR fluxes.

[2] Unfortunately, the existing notation in the literature means we have the same symbol Ω for the spinorial transformation matrix and for the $SU(3)$-structure three-form. We trust the reader will infer from the context which is meant.

Then one ought to, in principle, reconsider the above derivation in a formalism suitable for including RR fluxes. In the case of Abelian and Fermionic T-duality [150] (T-duality when the full supersymmetric action, including fermions, is considered and duality is performed along fermionic directions) this has explicitly been done in the pure spinor approach [151,152] and a simple extrapolation of these results to this non-Abelian context leads to the following conclusion which can also be motivated from the considerations of [153]. The dual RR fluxes are obtained by right multiplication by the above matrix Ω on the RR bispinor (this can be viewed equivalently as a Clifford multiplication on the RR polyform/pure spinor). Explicitly, the T-dual fluxes are given by [127]:

$$e^{\hat{\Phi}}\hat{\not{F}} = e^{\Phi}\not{F}\cdot\Omega^{-1}\ , \tag{5.19}$$

where the RR polyforms are defined by equations (3.19)-(3.20) and the slashed notation in equation (3.63).

For many applications knowledge of the transformation laws for the gauge-invariant field strengths is sufficient. However, in some applications we will also be interested on how the RR potentials themselves transform. The potentials are given in equations (3.7), (3.8) or (3.17), related to the field strengths. Actually we will need to be a bit more general than those expressions when we consider the addition of sources, see appendix 5.B.

We propose that the potentials so defined have a straightforward transformation rule:

$$e^{\hat{\Phi}}\hat{\not{C}} = e^{\Phi}\not{C}\cdot\Omega^{-1}\ . \tag{5.20}$$

We should comment briefly about a subtlety; the potentials in the equation above have to be chosen in such a way that the T-duality can be readily performed. In other words, for the transformation rule to be as above, the potentials C_p should have a vanishing Lie derivative along the Killing vectors of the isometry dualized. A less judicious choice of potentials would require composing the above transformation law with an appropriate gauge transformation that first brings the potential into the desired form (this is well explained in [154] for the NS two-form potential which does not need to have a vanishing Lie derivative under the isometry dualized but instead it obeys $\mathcal{L}_k B = \mathrm{d}\xi$).

Although we have not shown that (5.20) implies (5.19) in all generality, we find that it does indeed generate the correct transformation in the case at hand. The essential step in a general proof would be to show that the Clifford multiplication implied by the spinor contraction in (5.20) commutes with the action of the twisted differential d_H. One may be confident that this is true in all generality since this is indeed the case with Abelian T-duality [154] and we shall see that in a certain basis the transformation rules do become very similar to the Abelian case.

We end this section by remarking the status of supersymmetry under non-Abelian T-duality. Supersymmetry does not need to be preserved by T-duality (Abelian or not).[3] Whether (and how much) supersymmetry is preserved depends on how the Killing vectors about which we dualize act on the supersymmetry. The action of a vector on a spinor, which is only well defined when the vector is Killing, is given by [155–157]

$$\mathcal{L}_k\epsilon = k^\mu D_\mu\epsilon + \frac{1}{4}\nabla_\mu k_\nu\gamma^{\mu\nu}\epsilon\ . \tag{5.21}$$

[3]In principle, supersymmetry can even be enhanced by T-duality but given that non-Abelian T-duality destroys isometry this seems rather unlikely in this case.

If, when acting on the Killing spinor of the initial geometry, this vanishes automatically for all the Killing vectors that generate the action of G then we anticipate supersymmetry to be preserved in its entirety. If, on the other hand, this vanishes only for some projected subset of Killing spinors then we expect only a corresponding projected amount of supersymmetry to be preserved in the T-dual.[4] In this paper we consider the case of $\mathcal{N}=1$ supersymmetry which is invariant under the above action of G so that the non-Abelian duality should preserve supersymmetry. Suppose we start with ten-dimensional Majorana-Weyl Killing spinors ϵ^1 and ϵ^2, then the Killing spinors in the T-dual will be given by

$$\hat{\epsilon}^1 = \epsilon^1 \ , \quad \hat{\epsilon}^2 = \Omega \cdot \epsilon^2 \ . \tag{5.22}$$

5.1.1 Transformation of G-structures under T-duality

As already seen in section 3.2, the supersymmetric conditions can be recast in terms of G-structures. Therefore, it is very useful to know the non-Abelian T-dual of these structures. To obtain the transformation rules one can work explicitly with the T-dual Killing spinors defined in equation (5.22) and construct from first principles the pure-spinors $\Psi_\pm$ describing the G-structures explained in section 3.2, namely $SU(3)$ and $SU(2)$-structures. Alternatively, for the spinor-phobic one can circumvent this by using the following transformation rules on the polyforms

$$\not{\Psi}_+^{SU(2)} = i\not{\Psi}_-^{SU(3)}\Omega^{-1} \ , \qquad \not{\Psi}_-^{SU(2)} = \not{\Psi}_+^{SU(3)}\Omega^{-1} \ . \tag{5.23}$$

The D-brane generalized calibrations follows from this as shown in appendix 5.A.

Let us just remark at this stage that the condition of supersymmetry being preserved as detailed in equation (5.21) simply translates (using the Liebniz derivation property obeyed the Lorentz-Lie derivative [155–157]) into the invariance of the pure-spinors under the regular Lie derivative acting on forms:

$$\mathcal{L}_k \epsilon = 0 \Rightarrow \mathcal{L}_k \Psi_\pm = 0 \ . \tag{5.24}$$

For the case of the Abelian T-duality one can show that this criteria does indeed ensure that supersymmetry is preserved after T-duality [154]. The essence of the proof is that up to terms proportional to this Lie derivative, the twisted differential d_H commutes with the Clifford multiplication rule (c.f. equation (5.23)) used to extract the T-dual pure spinors. Using this, one can infer that supersymmetry is preserved by the dualization. Although we have not verified the details, the situation here appears to be exactly analogous, indeed as we shall shortly see one can find a basis in which the non-Abelian T-duality essentially mimics the Abelian case.

In the following sections, we will consider two examples that will make clear various points discussed above. The first case-study will be the non-Abelian T-dual of the Klebanov-Witten system as presented in [132, 133]. We will explicitly show the $SU(2)$-structure of the solution (and hence its SUSY preservation). We will then consider the background obtained by adding fundamental fields (quarks) to the Klebanov-Witten field theory [158] (conversely, we will consider the addition of source-branes to the Klebanov-Witten background). With the essential help of the $SU(2)$-structure formalism.

[4] In [129] this was confirmed to be true in general for a large class of backgrounds.

5.2 Example 1: Unflavored Klebanov-Witten and its T-dual

In this section we shall examine the Klebanov-Witten geometry, its T-dual and explicitly demonstrate the $SU(2)$-structure of the T-dual.

5.2.1 Klebanov-Witten model

In the same way we saw that $\mathcal{N}=4$ super Yang-Mills can be obtained as the low-energy dynamics of D3-branes in flat space, the Klebanov-Witten model follows from considering the low-energy dynamics of D3-branes on the conifold.

The conifold is a six dimensional singular space, which can be considered as the three complex dimensional space characterized by the condition

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 = 0 \ , \tag{5.25}$$

embedded in $\mathbb{C}^4$ with a singularity at the origin of $\mathbb{C}^4$. One can also use the alternative coordinates

$$Z_{\alpha\beta} = z_i(\sigma^i)_{\alpha\beta} + iz_4\mathbb{I}_{\alpha\beta} \tag{5.26}$$

in such a way that the condition (5.25) becomes in terms of the new variables

$$\det Z = 0 \ . \tag{5.27}$$

This space has the symmetry $SO(4)$ acting on the index $i=1,\ldots,4$, or alternatively $SO(4)\simeq SU(2)\times SU(2)$, each $SU(2)$ acting on the α and β indices, respectively. There is also an additional $U(1)$ rotating each z_i by a common phase.

The metric of this space can be written in the form [121]

$$\mathrm{d}s_6^2 = \mathrm{d}r^2 + r^2\mathrm{d}s^2_{T^{1,1}} \ , \tag{5.28}$$

where $T^{1,1}$, the base of the cone, is defined as the coset $\frac{SU(2)\times SU(2)}{U(1)}$, and its metric is

$$\mathrm{d}s^2_{T^{1,1}} = \frac{1}{9}\left(\mathrm{d}\psi + \cos\theta\mathrm{d}\phi + \cos\tilde\theta\mathrm{d}\tilde\phi\right)^2 + \frac{1}{6}(\mathrm{d}\theta^2 + \sin^2\theta\mathrm{d}\phi^2) + \frac{1}{6}(\mathrm{d}\tilde\theta^2 + \sin^2\tilde\theta\mathrm{d}\tilde\phi^2) \ , \tag{5.29}$$

where (θ,ϕ) and $(\tilde\theta,\tilde\phi)$ parametrize two spheres in the usual way and $\psi\in[0,4\pi)$. The relation between these coordinates and the holomorphic ones is giben by

$$Z = r^{3/2}\begin{pmatrix} e^{\frac{i}{2}(\psi-\phi-\tilde\phi)}\sin\frac{\theta}{2}\sin\frac{\tilde\theta}{2} & e^{\frac{i}{2}(\psi-\phi+\tilde\phi)}\sin\frac{\theta}{2}\cos\frac{\tilde\theta}{2} \\ e^{\frac{i}{2}(\psi+\phi-\tilde\phi)}\cos\frac{\theta}{2}\sin\frac{\tilde\theta}{2} & e^{\frac{i}{2}(\psi+\phi+\tilde\phi)}\cos\frac{\theta}{2}\cos\frac{\tilde\theta}{2}\end{pmatrix} \ . \tag{5.30}$$

In analogy to the duality obtained in section 3.1.3 by placing N D3-branes in flat space, placing them now at the tip of the conifold, generates the following metric

$$\mathrm{d}s^2 = \left(1+\frac{L^4}{r^4}\right)^{-1/2}\mathrm{d}x^2_{1,3} + \left(1+\frac{L^4}{r^4}\right)^{1/2}\left(\mathrm{d}r^2 + r^2\mathrm{d}s^2_{T^{1,1}}\right) \ , \tag{5.31}$$

with

$$L = \left(\frac{27}{4}\pi g_s N\right)^{1/4} l_s \ , \tag{5.32}$$

whose near horizon limit is $\mathrm{AdS}_5 \times T^{1,1}$,

$$\mathrm{d}s^2 = \frac{r^2}{L^2}\mathrm{d}x_{1,3}^2 + \frac{L^2}{r^2}\mathrm{d}r^2 + L^2\mathrm{d}s^2_{T^{1,1}} \ , \tag{5.33}$$

together with N units of RR flux supporting the geometry

$$F_5 = \frac{4}{g_s L}\Big(\mathrm{vol}(\mathrm{AdS}_5) - L^5\mathrm{vol}(T^{1,1})\Big) \ . \tag{5.34}$$

We will work with the following frame fields for this geometry

$$\begin{gathered} e^{y^\mu} = \frac{r}{L}\mathrm{d}y^\mu \quad (\mu = 0\ldots 3) \ , \quad e^r = \frac{L}{r}\mathrm{d}r \ , \quad e^\phi = \lambda_1 \sin\theta \mathrm{d}\phi \ , \quad e^\theta = \lambda_1 \mathrm{d}\theta \ , \\ e^1 = \lambda_1\sigma_1 \ , \quad e^2 = \lambda_1\sigma_2 \ , \quad e^3 = \lambda\left(\sigma_3 + \cos\theta\mathrm{d}\phi\right) \ , \end{gathered} \tag{5.35}$$

in which $\lambda_1^2 = \frac{L^2}{6}$ and $\lambda^2 = \frac{L^2}{9}$ and we have renamed the $SU(2)$ left-invariant one-forms of equation (4.7):

$$\begin{aligned} \sigma_1 &= \tilde{\omega}_2 = (-\sin\psi\mathrm{d}\tilde{\theta} + \cos\psi\sin\tilde{\theta}\mathrm{d}\tilde{\phi}) \ , \\ \sigma_2 &= \tilde{\omega}_1 = (\cos\psi\mathrm{d}\tilde{\theta} + \sin\psi\sin\tilde{\theta}\mathrm{d}\tilde{\phi}) \ , \\ \sigma_3 &= \tilde{\omega}_3 = (\cos\tilde{\theta}\mathrm{d}\tilde{\phi} + \mathrm{d}\psi) \ . \end{aligned} \tag{5.36}$$

Since the conifold is a Calabi-Yau three-fold, it preserves 1/4 of the original supersymmetries and thus, the dual gauge theory describing the low-energy dynamics of the branes must be $\mathcal{N} = 1$ supersymmetric, indeed superconformal as we have the AdS_5 factor in the metric. The precise dual gauge theory was obtained by Klebanov and Witten in [135], it is an $\mathcal{N} = 1$ superconformal $SU(N) \times SU(N)$ gauge theory which can be described by a two-node quiver and has two sets of bi-fundamental matter fields A_α in the $(\mathbf{N}, \bar{\mathbf{N}})$ representation of the gauge group and B_β in the $(\bar{\mathbf{N}}, \mathbf{N})$. The indices α and β correspond to the two sets of $SU(2)$ global symmetries. These fields appear as the identification with the coordinates

$$Z_{\alpha\beta} = A_\alpha B_\beta \ . \tag{5.37}$$

Besides the $SU(2) \times SU(2)$ global symmetry, the field theory has a $U(1)_R$ symmetry identified with the remaining $U(1)$ symmetry of the conifold, which shifts the coordinate ψ. For this $U(1)_R$ to be non-anomalous we must require both the R-charges of A and B to be 1/2. In a superconformal theory we can relate the dimension of chiral superfields with their superconformal R-charge [159]

$$d = \frac{3}{2}R \tag{5.38}$$

and hence A and B have conformal dimension 3/4.

Furthermore, to give mass to some undesired massless multiplets we must introduce some superpotential. Superpotentials must have R-charge $R = 2$ and, in this case, it must be an exactly marginal operator so as not to break conformal symmetry, therefore it must be a quartic operator made of A and B. It should also respect the $SU(2)\times SU(2)\times U(1)_R$ global symmetry, then the superpotential is fixed up to a constant, λ,

$$W=\frac{\lambda}{2}\epsilon^{\alpha\alpha'}\epsilon^{\beta\beta'}\,\mathrm{Tr}\left(A_\alpha B_\beta A_{\alpha'}B_{\beta'}\right)\ . \tag{5.39}$$

For reference we state the ten-dimensional spinors of KW in the basis (5.35) given by

$$\epsilon_1=\sqrt{\frac{r}{L}}\Big(\zeta_+\otimes\eta_++\zeta_-\otimes\eta_-\Big)\ ,\quad \epsilon_2=\sqrt{\frac{r}{L}}\Big(i\,\zeta_+\otimes\eta_+-i\,\zeta_-\otimes\eta_-\Big)\ . \tag{5.40}$$

The chiralities in these expressions are defined with respect to four and six-dimensional chirality matrices

$$\gamma_{(4)}=i\,\gamma^{y^0y^1y^2y^3}\ ,\quad \gamma_{(6)}=-i\,\gamma^{\phi\theta 123r}\ , \tag{5.41}$$

such that under the ten-dimensional chirality operator $\Gamma_{(10)}=\gamma_{(4)}\otimes\gamma_{(6)}$ both ϵ_1 and ϵ_2 are positive. In addition the spinor η_+ is constant and normalised such that $\eta_+^\dagger\eta_+=1$. Supersymmetry imposes the following projections on the spinor (as above $\eta_+=(\eta_-)^*$),

$$\gamma^{r3}\eta_+=\gamma^{12}\eta_+=\gamma^{\phi\theta}\eta_+=-i\,\eta_+\ . \tag{5.42}$$

Using these expressions, we can determine the $SU(3)$-structure of KW in the basis (5.35) to be

$$\begin{aligned} J&=e^{\theta\phi}-e^{12}+e^{3r}\ ,\\ \Omega&=(e^2+i\,e^1)\wedge(e^\theta+i\,e^\phi)\wedge(e^3+i\,e^r)\ . \end{aligned} \tag{5.43}$$

5.2.2 T-dual of the Klebanov-Witten model

The non-Abelian T-dual of this geometry with respect to the $SU(2)$ global symmetry defined by the σ_i was constructed in [132, 133]. The result is an $\mathcal{N}=1$ supersymmetric solution of type IIA whose NS sector is given by[5]

$$\begin{aligned} \mathrm{d}\hat{s}^2=&\mathrm{d}s^2_{\mathrm{AdS}_5}+\lambda_1^2(\mathrm{d}\theta^2+\sin^2\theta\mathrm{d}\varphi^2)+\frac{\lambda_1^2\lambda^2}{\Delta}x_1^2\hat{\sigma}_3^2\\ &+\frac{1}{\Delta}\Big((x_1^2+\lambda^2\lambda_1^2)\mathrm{d}x_1^2+(x_2^2+\lambda_1^4)\mathrm{d}x_2^2+2x_1x_2\mathrm{d}x_1\mathrm{d}x_2\Big)\ ,\\ \hat{B}=&-\frac{\lambda^2}{\Delta}\Big[x_1x_2\mathrm{d}x_1+(x_2^2+\lambda_1^4)\mathrm{d}x_2\Big]\wedge\hat{\sigma}_3\ ,\\ e^{-2\hat{\Phi}}=&\frac{8}{g_s^2}\,\Delta\ , \end{aligned} \tag{5.44}$$

where $\hat{\sigma}_3=\mathrm{d}\psi+\cos\theta\mathrm{d}\varphi$ and

$$\Delta\equiv\lambda_1^2x_1^2+\lambda^2(x_2^2+\lambda_1^4)\ . \tag{5.45}$$

The gauge fixing used in this case is given by fixing the Lagrange multiplier v_2 to zero,

$$v_2=0\ ,\qquad v_1=2x_1\ ,\qquad v_3=2x_2\ , \tag{5.46}$$

[5] We have set $L=1$ which may be restored by appropriate rescalings.

and we have renamed v_1 and v_2. For different gauge choices the dual backgrounds are locally diffeomorphic and one has to be careful with global properties of the dual coordinates, for example, a systematic way to find the periodicity of the dual coordinates is an open problem in non-Abelian T-duality. More details may be found in [133], where the range of ψ is restricted to 2π and that of x_1 to the positive real line to remove a singularity of the dual background.

The metric evidently has an $SU(2) \times U(1)_\psi$ isometry and for a fixed value of (x_1, x_2) the remaining directions give a squashed three-sphere. This geometry is supported by two and four-form RR fluxes which may be computed using equation (5.19) and whose explicit form can be found in [133]. We remark in passing that the lift of this geometry to eleven dimensions has an interpretation in terms of recently discovered $\mathcal{N} = 1$ SCFT's obtained from wrapping M5 branes on a Riemann surface (of genus zero in this case) [134].

One can establish the left and right-moving T-dual frames for this geometry along the lines of equation (5.15). The frames in the AdS direction are unaltered as are e^θ and e^ϕ. In the directions dualized we find new frame fields $\hat{e}^i_\pm$ for $i = 1 \dots 3$. The plus and minus T-dual frames are related by a Lorentz transformation which, as described in section 5.1, induces a transformation on spinors given by [6],

$$\Omega = \frac{\Gamma_{(10)}}{\sqrt{\Delta}} \Big(-\lambda_1^2 \lambda \Gamma^{123} + \lambda_1 x_1 \cos\psi \, \Gamma^1 + \lambda_1 x_1 \sin\psi \, \Gamma^2 + \lambda x_2 \Gamma^3 \Big) \; . \tag{5.47}$$

This defines the Killing spinors of the T-dual to be

$$\hat{\epsilon}_1 = \epsilon_1 \; , \qquad \hat{\epsilon}_2 = \Omega \cdot \epsilon_2 \; . \tag{5.48}$$

Implementing the four-six decomposition one finds from (5.40) using (5.42) that

$$\begin{aligned} \hat{\epsilon}_1 &= \sqrt{\frac{r}{L}} \Big(\zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_- \Big) \; , \\ \hat{\epsilon}_2 &= \sqrt{\frac{r}{L}} \Big(\zeta_+ \otimes \hat{\eta}_-^2 + \zeta_- \otimes \hat{\eta}_+^2 \Big) \; , \end{aligned} \tag{5.49}$$

where

$$\hat{\eta}_-^2 = -\frac{i}{\sqrt{\Delta}} \Big(\lambda_1^2 \lambda \gamma^r + \lambda_1 x_1 \cos\psi \, \gamma^1 + \lambda_1 x_1 \sin\psi \, \gamma^2 + \lambda x_2 \gamma^3 \Big) \eta_+ \; , \qquad \hat{\eta}_+^2 = (\hat{\eta}_-^2)^* . \tag{5.50}$$

It is clear that in this basis, the T-dual Killing spinors depend not only on the radial coordinate but also on the T-dual coordinates x_1, x_2. It is helpful to work in a different basis in which this new spinor can be expressed as simply as possible. In addition, we would like the new vielbein basis to preserve the geometric structure defined by η_+, because ϵ_1 is invariant under the non-Abelian T-duality. To do so we perform a rotation to a new

[6] The careful reader will not confuse this matrix Ω and its inverse Ω^{-1} with the complex three-form defining an $SU(3)$-structure, that appears for example in equation (5.43).

basis $\tilde{e} = R\hat{e}$ (ordered as $r, \phi, \theta, 1, 2, 3$) with the rotation matrix

$$R = \frac{1}{\sqrt{1+\zeta\cdot\zeta}} \begin{pmatrix} 1 & 0 & 0 & \zeta^1 & \zeta^2 & \zeta^3 \\ 0 & \sqrt{1+\zeta\cdot\zeta} & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{1+\zeta\cdot\zeta} & 0 & 0 & 0 \\ -\zeta^1 & 0 & 0 & 1 & -\zeta^3 & \zeta^2 \\ -\zeta^2 & 0 & 0 & \zeta^3 & 1 & -\zeta^1 \\ -\zeta^3 & 0 & 0 & -\zeta^2 & \zeta^1 & 1 \end{pmatrix} \tag{5.51}$$

with

$$\zeta^1 = \frac{x_1 \cos\psi}{\lambda\lambda_1} , \qquad \zeta^2 = \frac{x_1 \sin\psi}{\lambda\lambda_1} , \qquad \zeta^3 = \frac{x_2}{\lambda_1^2} . \tag{5.52}$$

Notice that these parameters are reflecting the structure of the spinor transformation matrix Ω. The rotated vielbeins are given, in coordinate frame, by:

$$\begin{aligned} \tilde{e}^r &= \frac{\lambda\lambda_1^2 \mathrm{d}r - r(x_1 \mathrm{d}x_1 + x_2 \mathrm{d}x_2)}{r\sqrt{\Delta}} , & \tilde{e}^\phi &= \lambda_1 \sin\theta\, \mathrm{d}\phi , \\ \tilde{e}^1 &= \lambda_1 \frac{r\lambda(x_1 \sin\psi\, \hat{\sigma}_3 - \cos\psi\, \mathrm{d}x_1) - x_1 \cos\psi\, \mathrm{d}r}{r\sqrt{\Delta}} , & \tilde{e}^\theta &= \lambda_1 \mathrm{d}\theta , \\ \tilde{e}^2 &= -\lambda_1 \frac{r\lambda(x_1 \cos\psi\, \hat{\sigma}_3 + \sin\psi\, \mathrm{d}x_1) + x_1 \sin\psi \mathrm{d}r}{r\sqrt{\Delta}} , & \tilde{e}^3 &= -\frac{\lambda x_2 \mathrm{d}r + \lambda_1^2 r\, \mathrm{d}x_2}{r\sqrt{\Delta}} . \end{aligned} \tag{5.53}$$

Then in this new basis (in which the gamma matrices are of course also rotated $\tilde{\gamma} = R\gamma$), we can easily show that

$$\begin{aligned} \tilde{\epsilon}_1 &= \sqrt{\frac{r}{L}} \Big(\zeta_+ \otimes \eta_+ + \zeta_- \otimes \eta_- \Big) , \\ \tilde{\epsilon}_2 &= \sqrt{\frac{r}{L}} \Big(\zeta_+ \otimes \tilde{\eta}_-^2 + \zeta_- \otimes \tilde{\eta}_+^2 \Big) , \end{aligned} \tag{5.54}$$

with $\tilde{\eta}_+^2 = (\tilde{\eta}_-^2)^*$ and,

$$\tilde{\eta}_-^2 = -i\, \tilde{\gamma}^r \eta_+ . \tag{5.55}$$

Note that, as required for type IIA supergravity, the new spinors have opposite chirality. With this simple relation between $\tilde{\eta}_-^2$ and η_+, we clearly see that they are never parallel, hence we have an $SU(2)$-structure. Because we were careful about the definition of our new vielbein basis, the projections on η_+ are not modified,

$$\tilde{\gamma}^{r3}\eta_+ = \tilde{\gamma}^{12}\eta_+ = \tilde{\gamma}^{\phi\theta}\eta_+ = -i\, \eta_+ , \tag{5.56}$$

but the projections obeyed by $\tilde{\eta}_-^2$ are different

$$-\tilde{\gamma}^{r3}\tilde{\eta}_-^2 = \tilde{\gamma}^{12}\tilde{\eta}_-^2 = \tilde{\gamma}^{\phi\theta}\tilde{\eta}_-^2 = -i\, \tilde{\eta}_-^2 . \tag{5.57}$$

The Killing spinors define two different $SU(3)$-structures

$$\begin{aligned} J^1 &= \tilde{e}^{\theta\phi} + \tilde{e}^{21} - \tilde{e}^{3r} , \\ \Omega^1 &= (\tilde{e}^2 + i\, \tilde{e}^1) \wedge (\tilde{e}^\theta + i\, \tilde{e}^\phi) \wedge (-\tilde{e}^3 + i\, \tilde{e}^r) , \\ J^2 &= \tilde{e}^{\theta\phi} + \tilde{e}^{21} + \tilde{e}^{3r} , \\ \Omega^2 &= (\tilde{e}^2 + i\, \tilde{e}^1) \wedge (\tilde{e}^\theta + i\, \tilde{e}^\phi) \wedge (-\tilde{e}^3 - i\, \tilde{e}^r) , \end{aligned} \tag{5.58}$$

whose intersection is the $SU(2)$-structure given by (see (3.97))

$$\begin{aligned} v + iw &= -\tilde{e}^3 + i\tilde{e}^r \ , \\ j &= \tilde{e}^{\theta\phi} + \tilde{e}^{21} \ , \\ \omega &= (\tilde{e}^2 + i\,\tilde{e}^1) \wedge (\tilde{e}^\theta + i\,\tilde{e}^\phi) \ . \end{aligned} \tag{5.59}$$

An explicit check shows that these do indeed satisfy the dilatino and gravitino equations that follow from equation (3.99).

Note that it makes sense to mix e^r with e^1, e^2 and e^3 when performing the rotation (5.51) because the geometric structure links e^r and e^3 in the projection $\gamma^{r3}\eta_+ = -i\,\eta_+$. Actually the choice of this rotation appears clearer when considering that, because of the geometric structure, the transformation of the spinor ϵ_2 can be written very easily as $\Omega\,\epsilon_2 = -\tilde{\Gamma}^r\epsilon_2$. It is in this new basis that the transformation closely resembles the T-duality of the Abelian case.

5.3 Example 2: Flavored Klebanov-Witten and its T-dual

5.3.1 Flavoring the Klebanov-Witten model

An important step if one is to try and use the AdS/CFT paradigm to understand QCD-like dynamics is to incorporate fundamental flavors into the gauge theory and corresponding gravity descriptions. A first step in this direction is to add a finite number N_f of fundamental flavors which in the IIB set-up is typically achieved by the inclusion of extra flavor D-branes as we explained in the previous chapter when adding flavors to the Maldacena-Núñez background.

In the case at hand we will consider adding $2N_f$ D7 branes to the KW geometry in such a way that supersymmetry is preserved. We will work beyond the probe approximation in the Veneziano limit. We first describe the gauge theory engineered from the D3-D7 system in the conifold. We consider D7 branes parallel to the D3 stack in the Minkowski directions with the remaining four directions embedded holomorphically and non-compactly in the conifold. The strings that run between the D7 and the D3 give rise to massless flavors. To avoid gauge anomalies on the field-theory side of the description and supergravity tadpoles on the string side of it, one must include two branches of D7 branes giving rise to fundamental chiral superfields for each gauge group ($q, \tilde{q}$ in the $(\mathbf{N}, \mathbf{1})$ and $(\bar{\mathbf{N}}, \mathbf{1})$ and $Q, \tilde{Q}$ in the $(\mathbf{1}, \mathbf{N})$ and $(\mathbf{1}, \bar{\mathbf{N}})$). The superpotential for this theory is given by [158],

$$W = \frac{\lambda}{2}\epsilon^{ij}\epsilon_{mn}Tr\left(A_iB^mA_jB^n\right) + h_1\tilde{q}^aA_1Q_a + h_2\tilde{Q}^aB_1q_a \ . \tag{5.60}$$

Notice that the $SU(2)$ global symmetries are explicitly broken by the embedding of the D7 branes - this symmetry will be recovered by smearing the sources. The addition of flavors implies that the theory loses conformality; a positive beta function is generated and *a priori* one expects a Landau pole in the UV.

We now turn to the gravity description. By considering the κ-symmetry projectors one can determine that the supersymmetric embeddings of D7 branes in the KW background to lie along two branches (the y^μ denote the Minkowski directions) [73, 76, 158],

$$\xi = (y^\mu, r, \psi, S^2), \qquad \tilde{\xi} = (y^\mu, r, \psi, \tilde{S}^2), \tag{5.61}$$

where S^2 and $\tilde{S}^2$ are the two-spheres parametrised by θ, ϕ and $\tilde{\theta}$, $\tilde{\phi}$ respectively. To avoid the D7 charge tadpole we must include N_f branes on both branches. One can write an action for the whole system consisting of supergravity together with DBI and WZ terms of the D7 branes (in string frame)

$$\begin{aligned} S_{\mathrm{DBI}} &= -\sum^{N_f} \int_{\xi} \mathrm{d}^8\xi e^{-\Phi} \sqrt{\left|\det g|_{\xi}\right|} - \sum^{N_f} \int_{\tilde{\xi}} \mathrm{d}^8\tilde{\xi} e^{-\Phi} \sqrt{\left|\det g|_{\tilde{\xi}}\right|} \ , \\ S_{\mathrm{WZ}} &= \sum^{N_f} \int \left(C_8|_{\xi} + C_8|_{\tilde{\xi}} \right) \ . \end{aligned} \tag{5.62}$$

We do not activate the gauge field on the brane itself and since there is no NS two-form in this geometry the WZ term is simple. Note that the two stacks of flavor branes introduce a $U(N_f) \times U(N_f)$ symmetry (although the diagonal axial $U(1)$ is anomalous).

Now, as we are considering the case where the number of flavor branes goes to infinity, we can smear them and consider that each stack is distributed homogeneously across the two-sphere it does not wrap.[7] In a field-theory perspective the $U(N_f)$ flavor symmetries are broken to $U(1)^{N_f}$. The supergravity effect can be encoded by introducing a smearing form:

$$\Xi_2 = -\frac{N_f}{4\pi} \left(\sin\theta \mathrm{d}\theta \wedge \mathrm{d}\varphi + \sin\tilde{\theta} \mathrm{d}\tilde{\theta} \wedge \mathrm{d}\tilde{\phi} \right) \ . \tag{5.63}$$

The smearing procedure essentially boils down to replacing the DBI and WZ contributions of equation (5.62) with

$$\begin{aligned} S_{\mathrm{DBI}} &\to -\sum_{Nf} \int \mathrm{d}^{10}x e^{-\Phi} \left(\sin\tilde{\theta} \sqrt{\left|\det g|_{\xi}\right|} + \sin\theta \sqrt{\left|\det g|_{\tilde{\xi}}\right|} \right) \ , \\ S_{\mathrm{WZ}} &\to \sum_{Nf} \int \Xi_2 \wedge C_8 \ . \end{aligned} \tag{5.64}$$

One consequence of this smearing is that the Bianchi identities are modified

$$\mathrm{d}F_1 = \Xi_2 \ , \qquad \mathrm{d}F_5 = 0 \ . \tag{5.65}$$

Note that the coefficient $2\kappa_{10}^2 T_{Dp}$ does not appear in the Bianchi identities as in (4.69). This is because we are not considering the coefficient T_{Dp} of the DBI-WZ actions as well as the coefficient $1/2\kappa_{10}^2$ of the supergravity action. They can be easily recovered by rescaling in the Bianchi identities

$$\Xi_p \to 2\kappa_{10}^2 T_{D9-p} \Xi_p \ . \tag{5.66}$$

The D7-brane backreaction is accommodated by the following ansatz (as above we

[7] This smearing procedure overcomes the bound on the number of D7 branes that comes from looking the deficit angle of the D7 solution so N_f may indeed be taken large.

work in string frame)

$$
\begin{aligned}
\mathrm{d}s^2 = {} & \frac{e^{\frac{\Phi}{2}}}{\sqrt{h}}\mathrm{d}y_{1,3}^2 \\
& + e^{\frac{\Phi}{2}}\sqrt{h}\Big(\mathrm{d}r^2 + \lambda_1^2 e^{2g}(\sin^2\theta \mathrm{d}\varphi^2 + \mathrm{d}\theta^2) + \lambda_2^2 e^{2g}(\sigma_1^2+\sigma_2^2) + \lambda^2 e^{2f}(\sigma_3 + \cos\theta \mathrm{d}\varphi)^2\Big) \ , \\
F_1 = {} & \frac{N_f}{4\pi}(\sigma_3 + \cos\theta\mathrm{d}\varphi) \ , \qquad F_5 = (1+\star)\mathrm{d}t\wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge K\mathrm{d}r \ ,
\end{aligned}
\tag{5.67}
$$

where the warp factors f, g, h and the dilaton Φ are functions of the radial variable r and $\lambda_1^2 = \lambda_2^2 = 1/6$, $\lambda^2 = 1/9$ and as a consequence of the Bianchi identities $Kh^2e^{4g+f} = 27\pi N_c$. [8] A convenient basis of vielbeins is given by:

$$
\begin{aligned}
e^{y^\mu} &= e^{\Phi/4}h^{-1/4}\mathrm{d}y^\mu \ , & e^r &= e^{\Phi/4}h^{1/4}\mathrm{d}r \ , \\
e^{\phi} &= \lambda_1 e^{g+\Phi/4}h^{1/4}\sin\theta \, d\varphi \ , & e^\theta &= \lambda_1 e^{g+\Phi/4}h^{1/4}\mathrm{d}\theta \ , \\
e^1 &= \lambda_1 e^{g+\Phi/4}h^{1/4}\sigma_1 \ , & e^2 &= \lambda_1 e^{g+\Phi/4}h^{1/4}\sigma_2 \ , \\
e^3 &= \lambda h^{1/4}e^{f+\Phi/4}(\sigma_3 + \cos\theta\mathrm{d}\varphi) \ .
\end{aligned}
\tag{5.68}
$$

Like the unflavored version, this solution supports an $SU(3)$-structure:

$$
\begin{aligned}
J &= -\left(e^{r3} + e^{\phi\theta} + e^{12}\right) = -\frac{4\pi\sqrt{h}}{3N_f}e^{\frac{\Phi}{2}}\left(\frac{1}{2}e^{2g}\Xi_2 + e^f\mathrm{d}r\wedge F_1\right) \ , \\
\Omega &= (e^2 + ie^1)\wedge(e^\theta + ie^\phi)\wedge(e^3 + ie^r) \ .
\end{aligned}
\tag{5.69}
$$

With these and the structure conditions for $SU(3)$, it is possible to derive a set of first order BPS equations for the various functions introduced thus far:

$$
\begin{aligned}
f' &= e^{-f}(3 - 2e^{2f-2g}) - \frac{3N_f}{8\pi}e^{\Phi - f} \ , & g' &= e^{f-2g} \ , \\
h' &= -27\pi N_c e^{-f-4g} \ , & \Phi' &= \frac{3N_f}{4\pi}e^{\Phi-f} \ .
\end{aligned}
\tag{5.70}
$$

The RR potentials can be expressed in terms of the $SU(3)$-structure forms as:

$$
C_8 = -\frac{1}{2}e^{-\Phi}\left(\frac{e^\Phi}{h}\mathrm{vol}_4\right)\wedge J\wedge J \ , \qquad C_4 = e^{-\Phi}\left(\frac{e^\Phi}{h}\mathrm{vol}_4\right) \ , \tag{5.71}
$$

where $F_9 = \star F_1$. The reason why we did not cancel both factors of the dilaton is just for comparison with formulas below.

Finally for the brane embedding to be supersymmetric it must obey the calibration condition (see appendix 5.B):

$$
\sqrt{-\det g|_{\xi}}\,\mathrm{d}^8\xi = -\frac{1}{2}\left(\frac{e^\Phi}{h}\mathrm{vol}_4\right)\wedge J\wedge J\Big|_{\xi} \ , \qquad \sqrt{-\det g|_{\tilde\xi}}\,\mathrm{d}^8\tilde\xi = -\frac{1}{2}\left(\frac{e^\Phi}{h}\mathrm{vol}_4\right)\wedge J\wedge J\Big|_{\tilde\xi} \ . \tag{5.72}
$$

[8] The unflavored Klebanov-Witten can be recovered with the following substitution:

$$
y^\mu \to \frac{1}{\sqrt{g_s}}y^\mu \ , \quad N_f = 0 \ , \quad h = \frac{L^4}{g_s r^4} \ , \quad e^{2f} = e^{2g} = r^2 \ , \quad K = \frac{4r^3 g_s}{L^4} \ , \quad e^\Phi = g_s \ .
$$

This allows the DBI and WZ actions of the smeared brane embedding to be expressed as:

$$S_{\rm DBI} = \frac{1}{2}\int_{M_{10}} e^{-\Phi}\left(\frac{e^{\Phi}}{h}\mathrm{vol}_4\right)\wedge J\wedge J\wedge \Xi_2\ , \qquad S_{\rm WZ} = \int_{M_{10}} C_8\wedge \Xi_2\ , \tag{5.73}$$

from which it is immediate that $S_{\rm DBI} + S_{\rm WZ} = 0$, as required by supersymmetry. As the sources are calibrated the dilaton equation of motion, Einstein's equations and the flux equation for H are all satisfied once the Bianchi identities are imposed.

We will now find the non-Abelian T-dual of this system involving metric, fluxes and sources. The interest of this problem is two-fold. On the one hand, it teaches us the effect of the non-Abelian duality on the Born-Infeld-Wess-Zumino action. On the other hand, it will tell us how to find the new smearing forms. Both these points give clues to a generic procedure.

5.3.2 The T-dual of the flavored Klebanov-Witten model

We perform the non-Abelian T-duality along the $SU(2)$ directions as before. To compactly display the results it is convenient to perform a supplementary rotation as detailed in equation (3.21) of [133]. We find the frame fields for the T-dual metric to be

$$\begin{aligned}
\hat{e}^1 &= -\frac{\lambda_1}{\Delta}e^{g+\frac{\Phi}{4}}h^{1/4}\left((\lambda_1^2\lambda^2 h e^{2f+2g+\Phi} + x_1^2)\mathrm{d}x_1 + x_1x_2(\mathrm{d}x_2 + \lambda^2\sqrt{h}e^{2f+\frac{\Phi}{2}}\hat{\sigma}_3)\right)\ ,\\
\hat{e}^2 &= \frac{\lambda_1}{\Delta}e^{g+\frac{3}{4}\Phi}h^{3/4}\left(\lambda^2 x_2 e^{2f}\mathrm{d}x_1 - \lambda_1^2 x_1 e^{2g}(\mathrm{d}x_2 + \lambda^2\sqrt{h}e^{2f+\frac{\Phi}{2}}\hat{\sigma}_3)\right)\ ,\\
\hat{e}^3 &= -\frac{\lambda}{\Delta}e^{f+\frac{\Phi}{4}}h^{1/4}\left(x_1x_2\mathrm{d}x_1 + (\lambda_1^4 h e^{4g+\Phi} + x_2^2)\mathrm{d}x_2 - \lambda_1^2\sqrt{h}x_1^2 e^{2g+\frac{\Phi}{2}}\hat{\sigma}_3\right)\ .
\end{aligned} \tag{5.74}$$

Where we recall $\hat{\sigma}_3 = \cos\theta\mathrm{d}\phi + \mathrm{d}\psi$ and

$$\Delta = \sqrt{h}e^{\frac{\Phi}{2}}\left(\lambda_1^4\lambda^2 h e^{2f+4g+\Phi} + \lambda_1^2 x_1^2 e^{2g} + \lambda^2 x_2^2 e^{2f}\right)\ . \tag{5.75}$$

The T-dual NS sector is then given by

$$\begin{aligned}
\mathrm{d}\hat{s}^2 &= (e^{y_\mu})^2 + (e^r)^2 + (e^\phi)^2 + (e^\theta)^2 + (\hat{e}^1)^2 + (\hat{e}^2)^2 + (\hat{e}^3)^2\ ,\\
\hat{B} &= \frac{\lambda e^{f-g}x_2}{\lambda_1 x_1}\hat{e}^{13} + \frac{\lambda\lambda_1 e^{f+g+\frac{\Phi}{2}}\sqrt{h}}{x_1}\hat{e}^{23}\ ,\\
H &= \mathrm{d}\hat{B}\ ,\\
e^{-2\hat{\Phi}} &= 8\Delta e^{-2\Phi}\ .
\end{aligned} \tag{5.76}$$

This geometry is supported by RR fluxes, obtained using the general formula equation (5.19),

$$\begin{aligned}
F_0 &= \frac{N_f}{\sqrt{2}\pi}x_2\ ,\\
F_2 &= \frac{\lambda_1 e^{g-f-\frac{\Phi}{2}}}{\sqrt{2}\lambda\pi}\left(4\pi\lambda_1\lambda^2 K e^{2f+g}h^{3/2}e^{\phi\theta} + \lambda\lambda_1 N_f e^{f+g+\Phi}\sqrt{h}\hat{e}^{12} - x_1 N_f e^{\frac{\Phi}{2}}\hat{e}^{13}\right)\ ,\\
F_4 &= -2\sqrt{2}e^{-\Phi}hKe^{\phi\theta}\wedge\left(\lambda x_2 e^f\hat{e}^{12} + \lambda_1 x_1 e^g\hat{e}^{23}\right)\ .
\end{aligned} \tag{5.77}$$

Although there is an F_0, it is possible that one should not regard this as a solution of Massive IIA – the would-be mass parameter is neither constant nor quantized— but rather, as we shall discuss, this should be thought of as a solution to type IIA in the presence of D8 sources. Now since the original Bianchi identities were not satisfied (due to D7 source) one would not expect these new fluxes in equation (5.77) to obey standard Bianchi identities after the non-Abelian T-duality. Indeed, one finds T-dual smearing forms enter the game

$$\begin{aligned} \mathrm{d}F_0 &= \Xi_1 \ , \\ \mathrm{d}F_2 - F_0 H &= \Xi_1 \wedge B + \Xi_3 \ , \\ \mathrm{d}F_4 - H \wedge F_2 &= \frac{1}{2}\Xi_1 \wedge B \wedge B + B \wedge \Xi_3 \ . \end{aligned} \tag{5.78}$$

We find a rather nice result: the T-dual smearing forms can be calculated directly as

$$\begin{aligned} \Xi_1 &= -\frac{N_f e^{-g-\frac{\Phi}{4}}}{\sqrt{2\pi}\lambda_1 h^{1/4}}\left(x_1\hat{e}^2 + \lambda\lambda_1\sqrt{h}e^{f+g+\frac{\Phi}{2}}\hat{e}^3\right) = \frac{N_f}{\sqrt{2\pi}}\mathrm{d}x_2 \ , \\ \Xi_3 &= \frac{N_f e^{-2g-\frac{\Phi}{4}}}{\pi h^{1/4}} e^{\phi\theta} \wedge \left(\sqrt{3}x_1 e^g \hat{e}^1 + \sqrt{2}x_2 e^f \hat{e}^3\right) \\ &= \frac{N_f}{\sqrt{2\pi}} \sin\theta \left(x_1 \mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}x_1 + x_2 \mathrm{d}\theta \wedge \mathrm{d}\phi \wedge \mathrm{d}x_2\right) \ . \end{aligned} \tag{5.79}$$

These may be obtained equally using a transformation rule much like that of the RR fields,

$$e^{\Phi}\not{\Xi}_2 \Omega^{-1} = e^{\hat{\Phi}}\hat{\not{\Xi}}_B \ , \tag{5.80}$$

where $\hat{\Xi}_B = e^B \wedge (\Xi_1 + \Xi_3)$. The active smearing forms indicate sources for both D6 and D8 branes.

5.3.3 A nice subtlety

There is a subtlety here. A naive reasoning would lead us to believe that when the non-Abelian T-dual is applied to D7 sources, it will generate charge for D8, D6, D4 branes, whilst in equation (5.79) we only have D8, D6 charges, since Ξ_5, the smearing form for D4 charges is absent in equation (5.78). Below, we will solve this apparent contradiction.

If we consider the Bianchi identity of the RR polyform

$$\mathrm{d}_H F = \hat{\Xi} \wedge e^B \ , \tag{5.81}$$

it is clear that, since the LHS of this equation is gauge invariant, the RHS must also be. Throughout this chapter we have set to zero gauge fields on the world-volume, however, one should remember that they occur in conjunction with the NS two-form in the gauge-invariant combination $\mathcal{F} = B + 2\pi\alpha'\mathrm{d}A$. Then the most conservative view is that performing a gauge transformation on the NS B-field simply activates appropriate compensating world-volume gauge field. There is however another point of view which is to keep the world-volume gauge fields turned off and instead compensate for a B-field transformation with an appropriate redefinition of the smearing form $\hat{\Xi}$. This is best thought of not as a gauge transformation but rather as a mapping. In this picture the transformation of the NS potential, $B \to B + \Delta B$, mediates a redistribution of source charge between the D4 and D6 branes. The reason to prefer this second viewpoint is that turning on a one-form gauge field on the brane would break either the $SU(N_f)$ or the $U(1)^{N_f}$ symmetry.

To explain this second viewpoint, we consider the transformation $B \to B' = B + \Delta B$. Such a transformation must be supplemented by a transformation of the smearing polyform $\hat{\Xi} \to \hat{\Xi}'$ so that the Bianchi identity of the RR polyform is unchanged. This requires that

$$\hat{\Xi}' \wedge e^{B'} = \hat{\Xi} \wedge e^{B} \ . \tag{5.82}$$

As an example, consider a transformation for which $\Xi_1 \wedge \Delta B = 0$. Then we still have

$$\mathrm{d}F_0 = \Xi_1 \ , \qquad \mathrm{d}F_2 - HF_0 = \Xi_3 + B \wedge \Xi_1 \ . \tag{5.83}$$

The final Bianchi identity of the RR sector then becomes

$$\mathrm{d}F_4 - H \wedge F_2 = \Xi_5 + B \wedge \Xi_3 + \frac{1}{2} B \wedge B \wedge \Xi_1 \ , \tag{5.84}$$

where $\Xi_5 = \Delta B \wedge \Xi_3$. So we generate an explicit source for D4-branes under such a transformation. Clearly there are always source D8-branes but whether we have explicit source D6's or source D6 and D4's is a gauge-dependent statement. We do not believe it is possible to find a gauge in which we only have explicit D8 sources. This appears to be related to the fact that the original type IIB D7-brane embedding has two branches. This may seem rather mysterious, however one should understand that the total DBI and WZ actions of the source branes depend only on the sources through the gauge-invariant quantity $\Xi \wedge e^B$. The higher potentials in the WZ action, C_5, C_7 and C_9, are gauge invariant as consequence of the $SU(2)$ SUSY conditions (see appendix 5.A for details on this). So, it is only the "portion" of the sources that are viewed as being explicit rather than induced that changes, the equations of motion, the Bianchi identities and the total Maxwell charge are all invariants.

In summary, we advocated a picture in which gauge transformations mediate a redistribution of the source charge between the D4 and D6 branes.

To emphasize these points above, we can consider their Page charges [160] defined as

$$\begin{aligned} Q^{D6}_{\text{Page}} &= \int_{\mathcal{M}_2} (F_2 - F_0 B) \ , \\ Q^{D4}_{\text{Page}} &= \int_{\mathcal{M}_4} (F_4 - B \wedge F_2 + \frac{1}{2} F_0 B \wedge B) \ . \end{aligned} \tag{5.85}$$

The Maxwell charges are invariant under a shift in the B-field described above. While the shift of the Page charges is given by

$$\begin{aligned} \Delta Q^{D6}_{\text{Page}} &= \int_{\mathcal{M}_2} F_0 \Delta B \ , \\ \Delta Q^{D4}_{\text{Page}} &= \int_{\mathcal{M}_4} (-\Delta B \wedge (F_2 - F_0 B) + \frac{1}{2} F_0 \Delta B \wedge \Delta B) \ . \end{aligned} \tag{5.86}$$

As these these integrals are defined over compact manifolds these quantities are invariant for small gauge transformations. The integrands are exact so the integrals are zero. It is of course a generic feature of Page charges that they are only defined up to quantized shifts under large gauge transformations[9]. This is generally interpreted in the literature as a Seiberg duality in the dual gauge theory as in [158, 161].

5.3.4 Potentials, $SU(2)$-structure and Calibration

We may use the formula for the T-dual RR potential in equation (5.20) to find the RR potentials. These are given in coordinate frame by (for alternative expressions see below)

$$\begin{aligned}
C_5 &= e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h}\mathrm{vol}_4\right)\wedge\left(\frac{\lambda\lambda_1^2 e^{f+2g+\Phi}h\mathrm{d}r-(x_1\mathrm{d}x_1+x_2\mathrm{d}x_2)}{\sqrt{\Delta}}\right),\\
C_7 &= e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h}\mathrm{vol}_4\right)\wedge\left(\frac{\lambda_1^2 e^{2g+\Phi}h\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\phi\wedge(\lambda e^f x_2\mathrm{d}r+\lambda_1^2 e^{2g}\mathrm{d}x_2)}{\sqrt{\Delta}}+\right.\\
&\left.\frac{\lambda\lambda_1^2 x_1 e^{f+2g+\frac{3\Phi}{2}}h^{\frac{3}{2}}(\lambda_1^2 e^{2g}(x_1\mathrm{d}r\wedge\mathrm{d}x_2+\lambda e^f\mathrm{d}x_1\wedge\mathrm{d}x_2)-\lambda^2 e^{2f}x_2\mathrm{d}r\wedge\mathrm{d}x_1)\wedge\hat{\sigma}_3}{\Delta^{3/2}}\right),\\
C_9 &= e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h}\mathrm{vol}_4\right)\wedge\left(\lambda\lambda_1^4 e^{f+4g+\frac{3\Phi}{2}}x_1 h^{\frac{3}{2}}\sin\theta\mathrm{d}\theta\wedge\mathrm{d}\varphi\wedge\hat{\sigma}_3\right)\wedge\\
&\left(\frac{(h\lambda^2\lambda_1^2 e^{2f+2g+\Phi}+x_1^2)\mathrm{d}r\wedge\mathrm{d}x_1+x_1x_2\mathrm{d}r\wedge\mathrm{d}x_2+\lambda e^f x_2\mathrm{d}x_1\wedge\mathrm{d}x_2}{\Delta^{3/2}}\right).
\end{aligned}\tag{5.87}$$

This background is again of $SU(2)$-structure where the defining forms $v+iw$, j, ω are the same as in the unflavored case – see equations (5.59) – the only difference being that the parameters entering the rotation matrix used in equation (5.51) become

$$\zeta^1=\frac{e^{-f-g-\frac{\Phi}{2}}x_1\cos\psi}{\lambda\lambda_1\sqrt{h}},\qquad \zeta^2=\frac{e^{-f-g-\frac{\Phi}{2}}x_1\sin\psi}{\lambda\lambda_1\sqrt{h}},\qquad \zeta^3=\frac{e^{-2g-\frac{\Phi}{2}}x_2}{\lambda_1^2\sqrt{h}}.\tag{5.88}$$

This rotation leads to the following simple vielbeins for the dual geometry

$$\begin{aligned}
\tilde{e}^r &= \frac{h\lambda\lambda_1^2 e^{f+2g+\Phi}\mathrm{d}r-(x_1\mathrm{d}x_1+x_2\mathrm{d}x_2)}{\sqrt{\Delta}},\\
\tilde{e}^\phi &= h^{\frac{1}{4}}\lambda_1 e^{g+\frac{\Phi}{4}}\sin\theta\mathrm{d}\phi,\qquad \tilde{e}^\theta = h^{\frac{1}{4}}\lambda_1 e^{g+\frac{\Phi}{4}}\mathrm{d}\theta,\\
\tilde{e}^1 &= \sqrt{h}\lambda_1 e^{g+\Phi/2}\frac{-x_1\cos\psi\,\mathrm{d}r-e^f\lambda(\cos\psi\,\mathrm{d}x_1-x_1\sin\psi\,\hat{\sigma}_3)}{\sqrt{\Delta}},\\
\tilde{e}^2 &= -\sqrt{h}\lambda_1 e^{g+\Phi/2}\frac{x_1\sin\psi\,\mathrm{d}r+e^f\lambda(\sin\psi\,\mathrm{d}x_1+x_1\cos\psi\,\hat{\sigma}_3)}{\sqrt{\Delta}},\\
\tilde{e}^3 &= -\sqrt{h}e^{\frac{\Phi}{2}}\frac{\lambda e^f x_2\mathrm{d}r+\lambda_1^2 e^{2g}\mathrm{d}x_2}{\sqrt{\Delta}}.
\end{aligned}\tag{5.89}$$

This whole background is indeed a solution to the combined (massive)-IIA supergravity plus DBI plus WZ action (the details are explicit in appendix 5.B):

$$S=S_{\text{Massive IIA}}+S_{\text{DBI}}+S_{\text{WZ}}.\tag{5.90}$$

[9] Large gauge transformations are topological in nature and always induce quantized shifts.

In the gauge in which the B-field is given by equation (5.76) and there are no explicit D4 sources, the appropriate WZ terms are given by

$$\begin{aligned} S_{\mathrm{WZ}} &= S_{\mathrm{WZ}}^{D8} + S_{\mathrm{WZ}}^{D6} \ , \\ S_{\mathrm{WZ}}^{D6} &= \int_{M_{10}} \Big(C_7 - B \wedge C_5 \Big) \wedge \Xi_3 \ , \\ S_{\mathrm{WZ}}^{D8} &= - \int_{M_{10}} \Big(C_9 - B \wedge C_7 + \frac{1}{2} B \wedge B \wedge C_5 \Big) \wedge \Xi_1 \ , \end{aligned} \tag{5.91}$$

whilst the DBI action, expressed in terms of the D8 and D6 calibrations – c.f. (5.73) – is given by

$$\begin{aligned} S_{\mathrm{DBI}} &= S_{\mathrm{DBI}}^{D8} + S_{\mathrm{DBI}}^{D6} \ , \\ S_{\mathrm{DBI}}^{D6} &= - \int_{M_{10}} e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} \mathrm{vol}_4 \right) \wedge \Big(v_1 \wedge j_2 - w_1 \wedge B \Big) \wedge \Xi_3 \ , \\ S_{\mathrm{DBI}}^{D8} &= - \int_{M_{10}} e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} \mathrm{vol}_4 \right) \wedge \Big(\frac{1}{2} w_1 \wedge j_2 \wedge j_2 + v_1 \wedge j_2 \wedge B - \frac{1}{2} w_1 \wedge B \wedge B \Big) \wedge \Xi_1 \ . \end{aligned} \tag{5.92}$$

Operating with the $SU(2)$ structure we can recast the RR potentials as

$$\begin{aligned} C_5 &= e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} \mathrm{vol}_4 \right) \wedge w_1 \ , \\ C_7 &= e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} \mathrm{vol}_4 \right) \wedge j_2 \wedge v_1 \ , \\ C_9 &= -\frac{1}{2} e^{-\hat{\Phi}} \left(\frac{e^{\Phi}}{h} \mathrm{vol}_4 \right) \wedge j_2 \wedge j_2 \wedge w_1 \ . \end{aligned} \tag{5.93}$$

This makes it clear that on shell, as is required by sypersymmetry, $S_{\mathrm{DBI}} + S_{\mathrm{WZ}} = 0$. This reflects the fact that the branes are calibrated, a fact that we now discuss in some detail.

5.3.5 Analysis of the dualized geometry

As we saw in the previous chapter, one is often interested, particularly in the context of the AdS/CFT correspondence, in the possibility that D-branes may wrap certain submanifolds of the geometry in a way that preserves supersymmetry. One approach to check whether a brane embedding is supersymmetric is the use of calibrations. $SU(3) \times SU(3)$ backgrounds admit a rich structure of supersymmetric cycles and the polyforms $\Psi_\pm$ (or rather the appropriate imaginary parts) serve as generalized calibrations as detailed at the end of chapter 3.

For the case of $SU(2)$-structure backgrounds with non-trivial NS three-form the calibrations for space-time filling D-branes wrapping odd cycles are given by[10]

$$\mathcal{C}_{\text{Cal. odd}} = -8 h^{\frac{1}{4}} e^{-\frac{\Phi}{4}} \, \mathrm{Im}(\Psi_-) \wedge e^B \ , \tag{5.94}$$

[10] Here we assume no gauge field on the brane world-volumes and then, we have absorbed the factor $\exp \mathcal{F} = \exp B|_\Sigma$ appearing in (3.110) into the definition of the calibration and we have also canceled out

while those for the even cycles by

$$\mathcal{C}_{\text{Cal. even}} = -8h^{\frac{1}{4}}e^{-\frac{\Phi}{4}}\,\mathrm{Im}(\Psi_+)\wedge e^B \ , \tag{5.95}$$

where the pure spinors are given by equation (3.94) for $|ab| = e^A = \frac{e^{\frac{\Phi}{4}}}{h^{\frac{1}{4}}}$. Specifically this gives the calibrations:

$$\begin{aligned}
\mathcal{C}_1 &= -w_1 \ , \\
\mathcal{C}_2 &= -\mathrm{Re}(\omega_2) \ , \\
\mathcal{C}_3 &= v_1\wedge j_2 - w_1\wedge B \ , \\
\mathcal{C}_4 &= -v_1\wedge w_1\wedge \mathrm{Im}(\omega_2) - B\wedge \mathrm{Re}(\omega_2) \ , \\
\mathcal{C}_5 &= \frac{1}{2}w_1\wedge j_2\wedge j_2 + v_1\wedge j_2\wedge B - \frac{1}{2}w_1\wedge B\wedge B \ , \\
\mathcal{C}_6 &= -v_1\wedge w_1\wedge \mathrm{Im}(\omega_2)\wedge B - \frac{1}{2}\mathrm{Re}(\omega_2)\wedge B\wedge B \ .
\end{aligned} \tag{5.96}$$

A cycle in the internal space is supersymmetric if it saturates the bound (3.110)

$$\sqrt{|\det(g+B)|_\xi|}d^i\xi = \mathcal{C}_i\big|_\xi \ . \tag{5.97}$$

This makes it clear that an $SU(2)$-structure in six dimensions in type IIA supergravity can potentially support Minkowski space-time filling D4, D6 and D8-branes wrapping one, three and five-cycles, respectively. Similarly, $SU(2)$-structures in six dimensions in type IIB supergravity might support space-time filling D5, D7 and D9-branes wrapping two, five and seven-cycles.

Indeed, one can explicitly check that in our case, space-time filling D4, D6 and D8 branes wrapping the following cycles are supersymmetric:

$$\begin{aligned}
\Sigma_{D4} &= (y^\mu, r) && \text{with} \quad x_1 = x_2 = 0 \ , \\
\Sigma_{D6} &= (y^\mu, r, \psi, x_1) && \text{with} \quad x_1^2 + x_2^2 = \text{constant} \ , \\
\Sigma_{D8} &= (y^\mu, r, \psi, \theta, \phi, x_1) \ .
\end{aligned} \tag{5.98}$$

The task of finding other supersymmetric cycles is left as an open problem.

5.4 Comments

In this chapter we have clarified the action of non-Abelian T-duality in the context of backgrounds possessing $SU(3)\times SU(3)$ structure and $\mathcal{N}=1$ supersymmetry.

We saw that rather generically the effect of performing a dualization along an $SU(2)$ isometry group is to map an $SU(3)$-structure background to an $SU(2)$-structure background. A heuristic reason for this can be found by looking at the abelian case following

the $\exp(4A-\Phi)$ factors appearing in (3.110) and (3.121), so that the relation between $\mathcal{C}$ and $\tilde{\varpi}$ is given by

$$\mathcal{C} = e^{-4A+\Phi}\left(\tilde{\varpi}\wedge e^B\right)\Big|_\Sigma \ .$$

[154]. After T-duality, left and right movers couple to different set of frame fields for the same geometry, $\hat{e}^i_+$ and $\hat{e}^i_-$. In the simplest case we can understand this T-duality as a reflection on right movers so that in directions dualized $\hat{e}^i_+ = -\hat{e}^i_-$. The J and Ω of the starting $SU(3)$-structure give rise, after dualization, to a $\hat{J}$ and $\hat{\Omega}$ which may be expressed in terms of either the left or right moving frame fields giving a corresponding $\hat{J}_\pm$ and $\hat{\Omega}_\pm$. Suppose that the expression for $\hat{J}$ is

$$\hat{J}_\pm = \hat{e}^1_\pm \wedge \hat{e}^2_\pm + \hat{e}^3_\pm \wedge \hat{e}^4_\pm + \hat{e}^5_\pm \wedge \hat{e}^6_\pm \ . \tag{5.99}$$

Consider the case where the dualized directions are 1 and 2, then $\hat{J}_+ = \hat{J}_-$ and in this case the T-dual also has $SU(3)$-structure. Now consider the dualization of two directions that are not paired by the complex structure, say 1 and 3, in this case $\hat{J}_+ \neq \hat{J}_-$ and type changing has occurred; the $SU(3)$-structure gives rise to a T-dual $SU(2)$-structure after T-dualization. Since the non-Abelian T-dualizations performed here involve three directions they cannot respect the paring of the complex structure and so they have to be type changing.

These $SU(2)$-structure geometries remain an interesting sector of compactifications which are much less well explored than their IIB $SU(3)$-structure cousins. The results presented here then open the door to constructing a rich class of such geometries. Indeed although we have illustrated this work with the Klebanov-Witten geometry, everything we have said holds true for the wide variety of $\mathcal{N} = 1$ backgrounds presented in [133]. A particularly noteworthy direction is to consider the dualization of more general toric Calabi-Yau geometries.

One feature of the geometries presented above was that they possess static $SU(2)$-structure. An interesting question from the point of view of generalized complex geometry is whether backgrounds with a dynamical $SU(2)$-structure can be found using these techniques. For this to be the case one would have to substantially change the relationship between the isometry group dualized and the initial complex structure.

Establishing a clear dictionary between the geometries [133] discussed here and a dual field theoretic description remains the most pressing physical question. In this chapter we showed how to readily add flavor branes to the picture and this will provide further insight into any putative dual field theoretic description. Indeed, this geometrical approach could be extended with interesting subtleties to the Klebanov-Strassler baryonic branch solution (including the wrapped D5 system) [162]. This viewpoint will make clear the way to calculate some physical observables, like domain walls and other topological defects corresponding to branes wrapping calibrated sub-manifolds. On the other hand, it is likely that this geometric view might help address important questions, like the periodicity of the new coordinates x_1, x_2, the existence of different cycles on which to integrate fluxes, a clear interpretation of the background in terms of color/flavor branes, etc. All these points remain for future study. A long but somewhat clear path needs be travelled, to use the Maldacena conjecture and define strongly coupled field theories based on these backgrounds.

5.A On the $SU(2)$-structure of the T-dual background

In this section, we give further details regarding the $SU(2)$-structure that are used through out the main body of this chapter. We sketch the derivation of the conditions that the $SU(2)$-structure must satisfy for $\mathcal{N} = 1$ SUSY in type IIA. We will also use these to define potentials for the space-time filling RR-fluxes. We assume a string frame metric of the form:

$$\mathrm{d}s^2 = e^{2A}\mathrm{d}y_{1,3} + \mathrm{d}s_6^2 \tag{5.100}$$

with a dilaton Φ and a NS three form $H = \mathrm{d}B$. We further assume that $\Phi(z), A(z)$ with z any coordinate in $\mathrm{d}s_6^2$. Expanding out the $SU(2)$ pure spinors in (3.94) gives:

$$\begin{aligned}
\Psi_+ &= i\frac{|ab|}{8}\Big[\omega_2 - i\omega_2 \wedge v_1 \wedge w_1 - \frac{1}{2}\omega_2 \wedge v_1 \wedge w_1 \wedge v_1 \wedge w_1\Big] \ , \\
\Psi_- &= \frac{|ab|}{8}(1 - ij_2 - \frac{1}{2}j_2 \wedge j_2) \wedge (v_1 + iw_1) \ , \\
\bar{\Psi}_- &= \frac{|ab|}{8}\Big[v_1 - iw_1 + j_2 \wedge (w_1 + iv_1) - \frac{1}{2}j_2 \wedge j_2 \wedge (v_1 - iw_1)\Big] \ .
\end{aligned} \tag{5.101}$$

Supersymmetry requires that $|a| = |b|$, we define:

$$|ab| = |a|^2 = e^A \ . \tag{5.102}$$

Plugging (5.101) into (3.99), equating forms with equal number of legs and separating real and imaginary parts gives

$$\begin{aligned}
&\mathrm{d}\Big[e^{3A-\Phi}\omega_2\Big] = 0 \ , \\
&\mathrm{d}\Big[e^{3A-\Phi}\omega_2 \wedge v_1 \wedge w_1\Big] + ie^{3A-\Phi}H \wedge \omega_2 = 0 \ .
\end{aligned} \tag{5.103}$$

For two-forms,

$$\begin{aligned}
&\mathrm{d}\Big[e^{3A-\Phi}v_1\Big] - e^{3A-\Phi}\mathrm{d}A \wedge v_1 = 0 \ , \\
&\mathrm{d}\Big[e^{3A-\Phi}w_1\Big] + e^{3A-\Phi}\mathrm{d}A \wedge w_1 = -e^{3A} \star_6 F_4 \ .
\end{aligned} \tag{5.104}$$

For four-forms,

$$\begin{aligned}
&-\mathrm{d}\Big[e^{3A-\Phi}j_2 \wedge w_1\Big] - e^{3A-\Phi}H \wedge v_1 + e^{3A-\Phi}\mathrm{d}A \wedge j_2 \wedge w_1 = 0 \ , \\
&\mathrm{d}\Big[e^{3A-\Phi}j_2 \wedge v_1\Big] - e^{3A-\Phi}H \wedge w_1 + e^{3A-\Phi}\mathrm{d}A \wedge j_2 \wedge v_1 = e^{3A} \star_6 F_2 \ ,
\end{aligned} \tag{5.105}$$

while for the six-form,

$$\begin{aligned}
&-\frac{1}{2}\mathrm{d}\Big[e^{3A-\Phi}j_2 \wedge j_2 \wedge v_1\Big] + e^{3A-\Phi}H \wedge j_2 \wedge w_1 + \frac{1}{2}e^{3A-\Phi}\mathrm{d}A \wedge j_2 \wedge j_2 \wedge v_1 = 0 \ , \\
&\frac{1}{2}\mathrm{d}\Big[e^{3A-\Phi}j_2 \wedge j_2 \wedge w_1\Big] + e^{3A-\Phi}H \wedge j_2 \wedge v_1 + \frac{1}{2}e^{3A-\Phi}\mathrm{d}A \wedge j_2 \wedge j_2 \wedge w_1 = e^{3A} \star_6 F_0 \ .
\end{aligned} \tag{5.106}$$

Finally, we have for the zero-form

$$\star_6 F_6 = 0 \tag{5.107}$$

where the fluxes F_0, F_2 and F_4 are understood to have legs in the six-dimensional internal space only. These equations can be further simplified as follows:

$$\begin{aligned}
&\mathrm{d}\Big[e^{3A-\Phi}\omega_2\Big] = 0\\
&\omega_2\wedge\Big[\mathrm{d}(v_1\wedge w_1)+iH\Big] = 0\\
&\mathrm{d}\Big[e^{2A-\Phi}v_1\Big] = 0\\
&\mathrm{d}\Big[e^{4A-\Phi}w_1\Big] = -e^{4A}\star_6 F_4\\
&\mathrm{d}\Big[e^{2A-\Phi}j_2\wedge w_1\Big] + e^{2A-\Phi}H\wedge v_1 = 0\\
&\mathrm{d}\Big[e^{4A-\Phi}j_2\wedge v_1\Big] - e^{4A-\Phi}H\wedge w_1 = e^{4A}\star_6 F_2\\
&\frac{1}{2}\mathrm{d}\Big[e^{2A-\Phi}j_2\wedge j_2\wedge v_1\Big] - e^{2A-\Phi}H\wedge j_2\wedge w_1 = 0\\
&\frac{1}{2}\mathrm{d}\Big[e^{4A-\Phi}j_2\wedge j_2\wedge w_1\Big] + e^{4A-\Phi}H\wedge j_2\wedge v_1 = e^{4A}\star_6 F_0\\
&\star_6 F_6 = 0.
\end{aligned}\tag{5.108}$$

We clearly now have a definition of the Minkowski space-time filling RR-sector in terms of the $SU(2)$-structure:

$$\begin{aligned}
F_6 &= \mathrm{d}\Big[e^{4A-\Phi}\mathrm{vol}_4\wedge w_1\Big]\\
F_8 &= \mathrm{d}\Big[e^{4A-\Phi}\mathrm{vol}_4\wedge j_2\wedge v_1\Big] - e^{4A-\Phi}H\wedge\mathrm{vol}_4\wedge w_1\\
F_{10} &= -\frac{1}{2}\mathrm{d}\Big[e^{4A-\Phi}\mathrm{vol}_4\wedge j_2\wedge j_2\wedge w_1\Big] + e^{4A-\Phi}H\wedge\mathrm{vol}_4\wedge j_2\wedge v_1,
\end{aligned}\tag{5.109}$$

where the remaining fluxes can be obtained from the duality condition $F_{2n} = (-)^n\star F_{10-2n}$. With these equations it is possible to derive expressions for the potentials associated with these fluxes. They take the most compact form when the space-time filling part of the RR flux ployform is expressed as[11]

$$F_{\mathrm{Mink}} = \mathrm{d}C_{\mathrm{Mink}} - H\wedge C_{\mathrm{Mink}}\ .\tag{5.110}$$

We must have $-H\wedge C_3+\frac{1}{3!}F_0B^3 = 0$ for $\mathcal{N}=1$ SUSY, otherwise the final line in equation (5.108) cannot hold. This allows the derivation of canonical potentials in terms of the $SU(2)$-structure,

$$\begin{aligned}
C_5 &= e^{4A-\Phi}\mathrm{vol}_4\wedge w_1\ ,\\
C_7 &= e^{4A-\Phi}\mathrm{vol}_4\wedge j_2\wedge v_1\ ,\\
C_9 &= -\frac{1}{2}e^{4A-\Phi}\mathrm{vol}_4\wedge j_2\wedge j_2\wedge w_1\ .
\end{aligned}\tag{5.111}$$

[11] We are assuming B is defined only on the internal space so that $B^4=0$.

5.B Some details of the flavored $SU(3)$ and $SU(2)$-structure solutions

We will start analyzing the case of the addition of flavors to the Klebanov-Witten field theory [158]. This will be explicitly dealt with using the language of $SU(3)$-structures. Then, we will extend the analysis to the background generated in section 5.3. This will require the full $SU(2)$-structure formalism, developed above.

We consider the addition of Minkowski space-time filling sources to an $SU(3)$-structure background in type-IIB. The action of type-IIB in string frame is modified as:

$$S = S_{\text{IIB}} + S_{\text{DBI}} + S_{\text{WZ}} \ . \tag{5.112}$$

With pure spinors defined as in equation (3.91) the calibration condition is given by:

$$\Psi_{\text{cal. IIB}} = -\frac{8e^{4A-\Phi}}{|a|^2}\,\text{Im}\,\Psi_+ = e^{-\Phi}\left(\frac{e^{\Phi}}{h}\right)\left(1-\frac{1}{2}J\wedge J\right) \ , \tag{5.113}$$

which is compatible with source D3 and D7-branes. We are assuming, as it is true for the Klebanov-Witten model with massless flavors, that $H=0$. The combined DBI action of such a system will be given by:

$$\begin{aligned}
S_{\text{DBI}} &= S_{\text{DBI}}^{D3} + S_{\text{DBI}}^{D7} \ , \\
S_{\text{DBI}}^{D3} &= -\int_{M_{10}} e^{-\Phi}\left(\frac{e^{\Phi}}{h}\right)\text{vol}_4\wedge\Xi_6 \ , \\
S_{\text{DBI}}^{D7} &= \frac{1}{2}\int_{M_{10}} e^{-\Phi}\left(\frac{e^{\Phi}}{h}\right)\text{vol}_4\wedge J\wedge J\wedge\Xi_2 \ .
\end{aligned} \tag{5.114}$$

While the WZ terms will be given by:

$$\begin{aligned}
S_{\text{WZ}} &= S_{\text{WZ}}^{D3} + S_{\text{WZ}}^{D7} \ , \\
S_{\text{WZ}}^{D3} &= -\int_{M_{10}} C_4\wedge\Xi_6 \ , \\
S_{\text{WZ}}^{D7} &= \int_{M_{10}} C_8\wedge\Xi_2 \ .
\end{aligned} \tag{5.115}$$

The fluxes, in the presence of sources – for the case of $B=0$, should be defined as,

$$H=\text{d}B \ , \qquad F_1=\text{d}C_0 \ , \qquad F_3=\text{d}C_2 \ , \qquad F_5=\text{d}C_4 \tag{5.116}$$

and the Bianchi identities are modified as follows:

$$\begin{aligned}
&\text{d}H=0, \quad \text{d}F_1=\Xi_2, \quad \text{d}F_3-H\wedge F_1=0 \ , \\
&\text{d}F_5-H\wedge F_3=\Xi_6 \ ,
\end{aligned} \tag{5.117}$$

where the Ξ_i's that are non zero are determined by the specific source brane content. The dual fluxes, related by the expression $F_{2n+1}=(-)^n\star F_{9-2n}$, are defined as:

$$\star F_5 = F_5, \quad F_7=\text{d}C_6, \quad F_9=\text{d}C_8 \tag{5.118}$$

and the fluxes have the following equations of motion:

$$\mathrm{d}\star F_1 = 0, \qquad \mathrm{d}\star F_3 = 0\ . \tag{5.119}$$

For Klebanov-Witten with massless flavors we should set $\Xi_6 = 0$ and then the equation of motion of the dilaton and Einstein's equations can be shown to be satisfied also as in [163].

5.B.1 Analysis of the generated background

In this work we generated a flavored type-IIA solution which supports an $SU(2)$-structure and non closed B. The action of (massive) type IIA in string frame, is now modified,

$$S = S_{\text{Massive IIA}} + S_{\text{DBI}} + S_{\text{WZ}}\ . \tag{5.120}$$

As shown around equation (5.96), an $SU(2)$-structure can in general support smeared source D4, D6 and D8-branes that extend in the Minkowski directions. The combined DBI and WZ actions of this system are given by:

$$\begin{aligned}
S_{\text{DBI}} &= S^{D8}_{\text{DBI}} + S^{D6}_{\text{DBI}} + S^{D4}_{\text{DBI}}\ ,\\
S^{D4}_{\text{DBI}} &= \int_{M_{10}} e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h}\right)\text{vol}_4\wedge w_1\wedge\Xi_5\ ,\\
S^{D6}_{\text{DBI}} &= -\int_{M_{10}} e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h}\right)\text{vol}_4\wedge\Big(v_1\wedge j_2 - w_1\wedge B\Big)\wedge\Xi_3\ ,\\
S^{D8}_{\text{DBI}} &= -\int_{M_{10}} e^{-\hat{\Phi}}\left(\frac{e^{\Phi}}{h}\right)\text{vol}_4\wedge\left(\frac{1}{2}w_1\wedge j_2\wedge j_2 + v_1\wedge j_2\wedge B - \frac{1}{2}w_1\wedge B\wedge B\right)\wedge\Xi_1\ ,
\end{aligned} \tag{5.121}$$

and

$$\begin{aligned}
S_{\text{WZ}} &= S^{D8}_{\text{WZ}} + S^{D6}_{\text{WZ}} + S^{D4}_{\text{WZ}}\ ,\\
S^{D4}_{\text{WZ}} &= -\int_{M_{10}} C_5\wedge\Xi_5\ ,\\
S^{D6}_{\text{WZ}} &= \int_{M_{10}}\Big(C_7 - B\wedge C_5\Big)\wedge\Xi_3\ ,\\
S^{D8}_{\text{WZ}} &= -\int_{M_{10}}\left(C_9 - B\wedge C_7 + \frac{1}{2}B\wedge B\wedge C_5\right)\wedge\Xi_1\ .
\end{aligned} \tag{5.122}$$

In the presence of such sources we should define the RR-potentials as:

$$F_0\ , \qquad F_2 = \mathrm{d}C_1 + F_0 B\ , \qquad F_4 = \mathrm{d}C_3 + B\wedge\mathrm{d}C_1 + \frac{F_0}{2}B\wedge B\ , \tag{5.123}$$

this ensures that we have no ill-defined potential terms appearing explicitly. We note that source D8-branes imply that F_0 will no longer be quantized. In general the Bianchi identities are given by

$$\begin{aligned}
&\mathrm{d}F_0 = \Xi_1\ , \qquad \mathrm{d}F_2 - F_0 H = \Xi_3 + B\wedge\Xi_1\ ,\\
&\mathrm{d}F_4 - H\wedge F_2 = \Xi_5 + B\wedge\Xi_3 + \frac{1}{2}B\wedge B\wedge\Xi_1\ .
\end{aligned} \tag{5.124}$$

The dual fluxes, related by the expression $F_{2n} = (-)^n \star F_{10-2n}$, are defined as:

$$\begin{aligned} &F_6 = \mathrm{d}C_5 \ , \qquad F_8 = \mathrm{d}C_7 - H \wedge C_5 \ , \\ &F_{10} = \mathrm{d}C_9 - H \wedge C_7 \ . \end{aligned} \tag{5.125}$$

Here, we did not write the terms that are zero due to the $SU(2)$ SUSY conditions in six dimensions. The flux equations of motion for the RR sector are given by:

$$\mathrm{d} \star F_2 + H \wedge \star F_4 = 0 \ , \qquad \mathrm{d} \star F_4 + H \wedge F_4 = 0 \ , \tag{5.126}$$

while for the NS sector we find:

$$\begin{aligned} \mathrm{d}\left(e^{-2\hat{\Phi}} \star H\right) =& F_0 \star F_2 + F_2 \wedge \star F_4 + \frac{1}{2} F_4 \wedge F_4 - \\ & \frac{e^{\Phi - \hat{\Phi}}}{h}\Big[\mathrm{vol}_4 \wedge (w_1 \wedge B - v_1 \wedge j_2) \wedge \Xi_1 + \mathrm{vol}_4 \wedge w_1 \wedge \Xi_3\Big] \ . \end{aligned} \tag{5.127}$$

A careful calculation shows that the potentials do not enter into this equation explicitly [57]. We can express the variation of the dilaton as an integral for compactness,

$$S_{\mathrm{DBI}} = -\int 8 e^{-2\hat{\Phi}} (\mathrm{d} \star \mathrm{d}\hat{\Phi} + \star \frac{R}{4} - \mathrm{d}\hat{\Phi} \wedge \star \mathrm{d}\hat{\Phi} - \frac{1}{8} H \wedge \star H) \ . \tag{5.128}$$

It is useful at this stage the following identity,

$$\int \omega_{(p)} \wedge \lambda_{(10-p)} = -\int \sqrt{-\det g}\, \lambda \lrcorner (\star \omega) \ . \tag{5.129}$$

Then Einstein's equations can be expressed in a gauge-invariant fashion as:

$$\begin{aligned} R_{\mu\nu} =& -2 D_\mu D_\nu \hat{\Phi} + \frac{1}{4} H^2_{\mu\nu} + e^{2\hat{\Phi}}\left[\frac{1}{2}(F_2^2)_{\mu\nu} + \frac{1}{12}(F_4^2)_{\mu\nu} - \frac{1}{4} g_{\mu\nu}(F_0^2 + \frac{1}{2}F_2^2 + \frac{1}{4!}F_4^2)\right] + \\ & \frac{e^{\Phi + \hat{\Phi}}}{h}\Bigg[\frac{1}{48}(\Xi_5 + \Xi_3 \wedge B + \frac{1}{2} B \wedge B \wedge \Xi_1)_{\mu\alpha_1 \ldots \alpha_4} \star (\mathrm{vol}_4 \wedge w_1)_\nu^{\alpha_1 \ldots \alpha_4} - \\ & \quad \frac{1}{4}(\Xi_3 + B \wedge \Xi_1)_{\mu\alpha_1\alpha_2} \star (\mathrm{vol}_4 \wedge v_1 \wedge j_2)_\nu^{\alpha_1\alpha_2} - \frac{1}{4}\Xi_{1\,\mu} \star (\mathrm{vol}_4 \wedge w_1 \wedge j_2 \wedge j_2)_\nu \\ & \quad - \frac{1}{4} g_{\mu\nu}\Big((\Xi_5 + \Xi_3 \wedge B + \frac{1}{2} B \wedge B \wedge \Xi_1) \lrcorner \star (\mathrm{vol}_4 \wedge w_1) - \\ & \quad (\Xi_3 + B \wedge \Xi_1) \lrcorner \star (\mathrm{vol}_4 \wedge v_1 \wedge j_2) - \frac{1}{2} \Xi_1 \lrcorner \star (\mathrm{vol}_4 \wedge w_1 \wedge j_2 \wedge j_2)\Big)\Bigg] \ . \end{aligned} \tag{5.130}$$

The equations (5.124)-(5.130) are solved by the system in section 5.3 after the BPS equations (5.70) are imposed.

Chapter 6

Supersymmetric superconductors

In the previous chapters, we have already seen various applications of supersymmetry. In this chapter, we are going to use supersymmetry in a rather different environment, we are going to use it within the context of condensed matter physics, in particular, superconductivity.

Superconductivity is a common phenomenon realizing spontaneous symmetry breaking of a local $U(1)$ symmetry. In particular, Bardeen-Cooper-Schrieffer theory of superconductivity (BCS), [164, 165], describes this spontaneous symmetry breaking, where one starts with a theory with a local $U(1)$ symmetry at finite temperature and a chemical potential that generates a Fermi surface. When the temperature is lowered enough, quantum effects generate a spontaneous symmetry breaking vacuum by fermion condensation. The IR choice of vacuum can be described in terms of an effective Ginzburg-Landau theory (GL), which can be derived from BCS theory.

Our main purpose in this chapter is to implement relativistic BCS theory with the simplest supersymmetric field theory model and to explore the new features that supersymmetry introduces in comparison to the general features that standard non-supersymmetric superconductors have. To be precise, we look for the simplest field theory model that at zero temperature and zero chemical potential enjoys $\mathcal{N} = 1$ supersymmetry (since any theory with higher supersymmetry can be viewed as a particular $\mathcal{N} = 1$ supersymmetric field theory, it is convenient to use the $\mathcal{N} = 1$ framework to provide a general picture of the conditions under which $U(1)$ spontaneous symmetry breaking can arise by BCS fermion condensation). We also require that when temperature and chemical potential are turned on, a Fermi surface is generated and for temperatures below a critical one quantum effects produce a Fermion condensate that spontaneously breaks a $U(1)$ symmetry, just as it occurs in the BCS theory.

Having a supersymmetric version of superconductivity might be interesting, since supersymmetric field theories are more stable and less sensitive to radiative corrections, as a result, the theory is less sensitive to the UV cutoff, which in BCS theory must be put by hand by introducing a Debye energy as a phenomenological input of the model.

Although the BCS theory of superconductivity was a fundamental step towards the microscopic understanding of conventional superconductivity, it fails to describe other exotic behaviors like high T_c superconductivity. If the type of non-perturbative results we have seen in the previous chapters in other contexts could be extended to the field of superconductivity, this would allow us to gain insight into how the superconducting mechanism behaves in the strong coupling regime, a regime which is believed to be involved in this high T_c superconductivity [166]. To this purpose, some models of superconductivity have been implemented in the context of the gauge/gravity duality, the so-called holographic superconductors, (some of the earliest approaches are [11, 167] and some reviews can be found at [168, 169]). These holographic superconductors are models of superconductivity built in the gravity side of the duality, so that, applying the AdS/CFT dictionary, they should correspond to a superconducting field theory on the field theory side of the duality. In top-down holographic superconductor models supersymmetry plays a central role, thus, having implemented superconductivity in a supersymmetric field theory might help to provide field-theoretical understanding of the possible mechanisms underlying holographic superconductivity.

On the other hand, the introduction of a chemical potential generating a Fermi surface in supersymmetric field theories is not straightforward, as it can lead to undesired Bose-Einstein condensation of the scalar superpartners, which spoils BCS mechanism. Hence, this model illustrates the difficulties that the introduction of a chemical potential in a supersymmetric theory brings up. However, despite this difficulty, the fact of having scalar superpartners for fermions in the supersymmetric version of BCS theory might be of relevance for certain real condensed matter systems where scalar and fermionic excitations arise in a quasi-supersymmetric way. This quantum mechanical quasi-supersymmetry arises in BCS theory itself, as noticed by Nambu [170], in terms of which one can describe the Interacting Boson Model [12], which describes the low-lying states of intermediate and heavy atomic nuclei.

As we see, there is a lot of interest in having a supersymmetric model of superconductivity. For this reason, we can find some attempts in the literature to build such a model [171–173]. However, these models use explicit supersymmetry breaking terms and, therefore, the Lagrangian does not describe a supersymmetric theory. Some other studies of phase transitions in supersymmetric field theories, for example [174], have neither found superconducting phases, despite the fact that its gravity dual shows a rich phenomena to be taking place at strong coupling, [175].

The outline of this chapter is the following. We will start by reviewing the main properties of superconducting systems and the main models describing them following an historical approach. Next, we will propose a supersymmetric model describing a superconducting phase transition a la BCS after some abortive, although instructive, attempts. The third section is devoted to the comparison of some physical quantities obtained for standard relativistic BCS theory and those computed for the presented supersymmetric model. These physical quantities are the gap, the specific heat, the magnetic and coherence lengths and critical magnetic fields, all of them reviewed in the first section. Finally, we conclude with some comments.

In this chapter we will follow the notation of [176]. We also will work in units in which the vacuum permeability is one.

6.1 Brief review of superconductivity

Superconductivity was discovered in 1911 when Kamerlingh Onnes observed that the electrical resistance of various metals drops to zero when their temperature is lowered below a certain critical one, T_c, characteristic of the material.

The second distinctive sign of superconductivity is perfect diamagnetism, found in 1933 by Meissner and Ochsenfeld, a phenomenon nowadays known as the Meissner effect. This is a completely different phenomenon from perfect conductivity as, not only does it imply that the magnetic field cannot penetrate the superconductor, which could be explained by perfect superconductivity, but also that the magnetic field in an originally normal conductor is expelled from it when the material becomes superconducting as its temperature is cooled below T_c.

The existence of this reversible Meissner effect implies that a high enough magnetic field can destroy superconductivity. The critical field, B_c, above which superconductivity would be lost can be determined by equating the condensation energy with the magnetic energy needed to maintain the field out of the sample, i.e.

$$\frac{1}{2}B_c^2(T) = F_n(T) - F_s(T) \ , \tag{6.1}$$

where F_n and F_s are the free energies per unit volume in the normal and superconducting phases at zero magnetic field.

At zero magnetic field, the phase transition at T_c is second order, but when an external field is turned on the phase transition taking place at B_c becomes first order, since the free energy is discontinuous.

6.1.1 The London equations

These two features of superconductivity, perfect conductivity and perfect diamagnetism, were first described by the London brothers in 1935, who proposed the following relations between the superconducting current and the electric and magnetic field,

$$\partial_t \vec{j}_s = c\vec{E} \ , \qquad \vec{\nabla} \times \vec{j}_s = -c\vec{B} \ . \tag{6.2}$$

These two equations can be summarized in a single equation in terms of the vector potential, $\vec{A}$,

$$\vec{j}_s = -c\vec{A} \ , \tag{6.3}$$

where c is a phenomenological parameter. The first equation in (6.2) describes perfect superconductivity because the electric field accelerates the superconducting electrons rather than maintaining their velocity constant, as it would happen in a normal conductor as described by Ohm's law. On the other hand, the second equation in (6.2), together with Maxwell equations, predicts an exponential decay in the magnetic field as it penetrates the superconducting sample, characterized by a magnetic penetration length λ. It turns out that the parameter c is related with the magnetic penetration length, $\lambda = 1/\sqrt{c}$.

6.1.2 The Ginzburg-Landau theory of superconductivity

The next successful theory of superconductivity was elaborated by Ginzburg and Landau in 1950. This theory still provides a phenomenological, macroscopic description of superconductivity, nevertheless it is a powerful theory which embodies in a simple way the macroscopic quantum mechanical nature of superconductors. Afterwards, the BCS theory will provide a microscopical completion, from which the Ginzburg-Landau theory of superconductivity can be derived and this will show that the Ginzburg-Landau theory is valid in the regime of temperatures close to T_c and slowly varying fields.

Within the framework of Landau theory of phase transitions, Ginzburg and Landau introduced a complex order parameter, $\Delta(x)$, related to the local density of superconducting charge carriers by

$$\varepsilon = |\Delta(x)|^2 \ . \tag{6.4}$$

The Ginzburg-Landau theory assumes that the free energy can be expanded around its non-superconducting value, F_n, as a power series of the carrier density, which is supposed to be small. If we only consider the first two terms of this series expansion, characterized by phenomenological coefficients $a(T-T_c)$ and b, the free energy in the GL theory is, thus, given by

$$F_s - F_n = a\,(T-T_c)\,|\Delta|^2 + b|\Delta|^4 + \frac{1}{2m}\left|\left(\vec{\nabla} - iq\vec{A}\right)\Delta\right|^2 + \frac{1}{2}(\vec{B}-\vec{B}_{ext})^2 \ , \tag{6.5}$$

where m and q are the mass and the electric charge of the superconducting carriers. Although at that time the nature of the superconducting carriers was to be determined, BCS theory will show that these carriers are paired electrons so that $m = 2m_e$ and $q = -2|e|$, being m_e and e the electron mass and electric charge, respectively.

Minimizing the free energy with respect to the order parameter and the vector potential one gets the Ginzburg-Landau equations,

$$\frac{1}{2m}\left(\vec{\nabla} - iq\vec{A}\right)^2\Delta - 2b|\Delta|^2\Delta = a(T-T_c)\Delta \ , \tag{6.6}$$

$$\vec{j}_s = \vec{\nabla}\times\vec{B} = -i\frac{q}{2m}(\Delta^*\vec{\nabla}\Delta - \Delta\vec{\nabla}\Delta^*) - \frac{q^2}{m}|\Delta|^2\vec{A} \ , \tag{6.7}$$

the former of which can be viewed as an analogue to the Schrödinger equation for a free particle, but with a non-linear term, and the later generalizes the London superconducting current. This improves the London theory as it allows to consider non-linear effects of fields able to change the density of superconducting carriers and it also allows to consider the case in which this density varies in space.

Apart from the magnetic penetration length, the Ginzburg-Landau theory allows to define another length, the coherence length ξ, which characterizes the distance over which the order parameter can vary without much energy increase. Together with the magnetic penetration length, their expressions are

$$\lambda^{-2} = \frac{aq^2}{2bm}(T-T_c) \ , \qquad \xi^{-2} = 2am(T-T_c) \ . \tag{6.8}$$

The quotient between these two lengths defines the Ginzburg-Landau parameter, $\kappa = \lambda/\xi$. At that time, known superconductors, today called type I, had a GL parameter in the regime $\kappa \ll 1$. However, in 1957 Abrikosov studied what would happen to a superconductor described by the GL theory in the regime $\kappa \gg 1$. He found what he called type II superconductors to distinguish them from the already known $\kappa \ll 1$ superconductors. In particular, the exact breakpoint between type I and type II superconductors occurs at $\kappa = 1/\sqrt{2}$. Instead of a discontinuous breakdown of superconductivity in a first order phase transition at B_c like in type I superconductors, this new type of superconductor exhibits an intermediate vortex state where the magnetic field penetrates in regular arrays of flux tubes of non-superconducting material surrounded by a superconducting current. This Abrikosov vortex state takes place in between two critical magnetic fields,

$$B_{c1} \approx \frac{\phi_0}{2\pi\lambda^2} \ , \qquad B_{c2} \approx \frac{\phi_0}{2\pi\xi^2} \ , \tag{6.9}$$

where B_{c1} is the value of the magnetic field for which a single vortex with flux quantum $\phi_0 = \pi/e$ would appear, whereas near B_{c2} vortices are as closely packaged as the coherence length allows. Superconductivity disappears completely for magnetic fields above B_{c2}.

6.1.3 The (relativistic) Bardeen-Cooper-Schrieffer theory of superconductivity

In 1957 Bardeen, Cooper and Schrieffer proposed a microscopic theory of superconductivity. Even though it fails to explain high T_c superconductivity and other exotic superconducting materials, it provides theoretical understanding of the microscopic physics behind standard superconducting experimental results and it is the starting point for more complex theories that aim to describe these other exotic behaviors.

The cornerstone of BCS mechanism is the fact that even a weak attractive interaction between electrons due to phonons is able to produce an instability in the Fermi sea of conducting electrons. This instability leads to the production of bound states of paired electrons able to condense. These are the so called Cooper pairs, which have, roughly speaking, a spatial extension given by the coherence length introduced in the GL theory. Cooper pairs clarify then the nature of the superconducting charge carriers, which remained an obscure point in the previous theories.

Another important feature of superconductivity that the BCS theory can explain is the existence of an energy gap between the Fermi level and the first excited states, this corresponds, from the BCS point of view, to the energy necessary to break a Cooper pair. This energy gap increases from zero at T_c to a limiting value at zero temperature. The quantitative agreement with existing experiments measuring the energy gap was one of the decisive validating arguments of the theory, see fig. 6.1 obtained from [177].

Instead of presenting here the original BCS formulation, we are going to show its relativistic generalization (rBCS), as it is going to be more appropriate for the later supersymmetric extension and it is conceptually similar from the point of view of quantum field theory. The simplest realization of BCS mechanism, which we will be considering here, is s-wave pairing, i.e. the paired electrons form a singlet anti-parallel spin state where the gap is constant in momentum space.

To build the relativistic BCS Lagrangian just start with the standard Dirac one

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi) - m\bar{\psi}\psi \ , \tag{6.10}$$

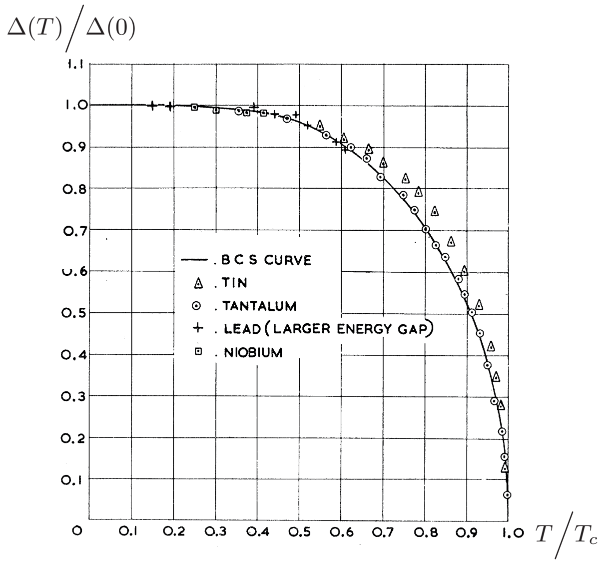

Figure 6.1: Comparison between experimental results and the BCS prediction for the gap as a function of the temperature for tin, tantalum, lead and niobium. Extracted from [177].

where, as usual, $\bar{\psi} = \psi^\dagger\gamma^0$. This Lagrangian enjoys a global $U(1)$ symmetry, this allows us to introduce a chemical potential, μ, associated with the corresponding conserved charge. This is done by adding the term $\mu\psi^\dagger\psi$, as it is described in appendix 6.A. Strictly speaking, if the symmetry is global we will be discussing superfluidity instead of superconductivity, although transport properties are similar in both cases and one can consider the global $U(1)$ symmetry model as that of a superconductor with weakly gauged symmetry. Below we will consider the addition of the corresponding gauge field, however, we maintain for the moment the discussion with the global symmetry for the sake of simplicity.

The next term of higher dimension we can add to the Dirac Lagrangian is a four-fermion interaction term. As shown in [178], the attractive interaction between electrons can be described in the relativistic BCS theory by the term $(\bar{\psi}_c\gamma_5\psi)^\dagger(\bar{\psi}_c\gamma_5\psi)$, hence we are left with the Lagrangian

$$\mathcal{L} = \frac{i}{2}(\bar{\psi}\gamma^\mu\partial_\mu\psi - \partial_\mu\bar{\psi}\gamma^\mu\psi) - m\bar{\psi}\psi + \mu\psi^\dagger\psi + g^2(\bar{\psi}_c\gamma_5\psi)^\dagger(\bar{\psi}_c\gamma_5\psi) \ . \tag{6.11}$$

ψ_c is the charge conjugate of ψ. Other quartic terms might be worth considering, but only the $(\bar{\psi}_c\gamma_5\psi)^\dagger(\bar{\psi}_c\gamma_5\psi)$ term respects the s-wave pairing.

The Lagrangian (6.11) is not renormalizable, the four-fermion interaction typically represents an irrelevant operator, but the dynamics of BCS superconductivity is such that for fermions which are close to the Fermi surface this attractive four-fermion interaction becomes strong. At weak coupling the scaling dimension of the fermionic fields must be very close to that of the 3/2 for a free field. Hence, on dimensional grounds, the interaction term is irrelevant in the IR. Naively it would seem that this theory cannot lead to any interesting IR physics. The phenomenon that actually takes place is explained in [179].

The key observation is that, in the presence of a chemical potential there is a Fermi surface which can change the naive scaling dimensions for the operators in such a way that the otherwise irrelevant interaction becomes indeed marginal. This is the seed for the possibility of a non-trivial IR physics such as superconductivity.

As we said, BCS is a theory describing a spontaneous symmetry breaking of the aforementioned $U(1)$ symmetry driven by the temperature. For this reason, we want to study the theory at finite temperature and to do that we have to consider the Lagrangian (6.11) in Euclidean space (as described in appendix 6.A). At this point, we have a Lagrangian which is not quadratic in the fields, to alleviate this, one has to perform a Hubbard-Stratonovich transformation by introducing an auxiliary field, Δ. At the end of the day we arrive at the Lagrangian

$$\begin{aligned}\mathcal{L}_E = \frac{1}{2}(\psi^\dagger\partial_\tau\psi - \partial_\tau\psi^\dagger\psi) - \frac{i}{2}(\bar{\psi}\gamma^i\partial_i\psi - \partial_i\bar{\psi}\gamma^i\psi) + m\bar{\psi}\psi - \mu\psi^\dagger\psi \\ + g^2|\Delta|^2 - g^2\left[\Delta^\dagger(\bar{\psi}_c\gamma_5\psi) + \Delta(\bar{\psi}_c\gamma_5\psi)^\dagger\right] \ .\end{aligned} \tag{6.12}$$

In this way, if we eliminate the auxiliary field through its equations of motion, we recover the original Lagrangian (6.11) (in Euclidean space). The equations of motion set $\Delta = (\bar{\psi}_c\gamma_5\psi)$, where we see that Δ is a measure of the density number of Cooper pairs and it is analogous to the order parameter that appeared in the GL theory.

Once we have a Lagrangian which is quadratic in fermions, our purpose is to integrate them out to obtain a one-loop effective potential for the auxiliary field Δ. After performing the Matsubara thermal sums, this is described in appendix 6.A, the effective potential is given by

$$\begin{aligned}V_{\text{eff}} &= g^2|\Delta|^2 \\ &\quad - \int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\omega_-(p) + \omega_+(p)\right) \\ &\quad - \frac{2}{\beta}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\left(\log(1+e^{-\beta\omega_-(p)}) + \log(1+e^{-\beta\omega_+(p)})\right) \ ,\end{aligned} \tag{6.13}$$

where we have written in separate lines, the classical potential, the Coleman-Weinberg potential and the thermal potential, from top to bottom respectively. The $\omega_\pm$ appearing in the effective potential are the energy eigenvalues coming from the Lagrangian (6.12). In appendices 6.A and 6.B we explain how they are obtained. The expression for the eigenvalues is given by

$$\omega_\pm = \sqrt{(\omega_0(\vec{p}) \pm \mu)^2 + 4g^4|\Delta|^2} \ , \tag{6.14}$$

where $\omega_0 \equiv \sqrt{p^2 + m^2}$. From the expression of the energy eigenvalues it is clear that BCS theory predicts an energy gap characterized by Δ.

The whole one-loop effective potential is identified with the free energy density in the grand canonical ensemble. It is UV divergent due to the Coleman-Weinberg contribution, for which we must introduce a cut-off, Λ, appearing here like a “Debye energy”. The cut-off as usual represents the energy where new physics emerges.

Considering the thermal potential at low temperatures, one can see that the dominant contribution comes from the minimum of the energy eigenvalues. Supposing $\mu > 0$, this minimum is reached for ω_- around $\omega_0 = \mu$, these contributions are identified with those coming from the particle, and contributions from ω_+ are those coming from the anti-particle. The momentum space location of this minimum defines a Fermi surface, $p_F^2 = \mu^2 - m^2$, from which it is clear that the condition $\mu > m$ is required for the Fermi surface to exist.

The minimum of the effective potential (6.13) as a function of the gap for each temperature defines a curve $\Delta(T)$. This curve can be found by solving the gap equation, $\partial_\varepsilon V_{\text{eff}} = 0$, where $\varepsilon = |\Delta|^2$. Explicitly, the gap equation is

$$1 = \frac{g^2}{\pi^2} \int_0^\Lambda \mathrm{d}p \; p^2 \left(\frac{\tanh\left(\frac{1}{2}\beta\omega_-(p,\Delta)\right)}{\omega_-(p,\Delta)} + \frac{\tanh\left(\frac{1}{2}\beta\omega_+(p,\Delta)\right)}{\omega_+(p,\Delta)} \right) . \tag{6.15}$$

This describes the usual critical curve for a second order phase transition for the order parameter Δ as a function of the temperature. Solving this equation at $\Delta = 0$ one can obtain the value of the critical temperature, T_c, at which the phase transition takes place, below which the fermion condensate appears. There are standard approximations that one can do. The second term inside the integrand comes from the antiparticle and can be neglected as $\omega_- \ll \omega_+$ near the Fermi surface. In doing so one connects with the expressions of the standard non-relativistic BCS theory. Near the Fermi surface one may also approximate the factor p^2 in the integrand by $\mu^2 - m^2$.

6.1.4 Considering a gauge field and fluctuations of the gap

If one is interested in studying electromagnetic properties, such as the Meissner effect, one must include in (6.12) a $U(1)$ gauge field, A_μ. We will do so treating this gauge field as an external field.

One can also consider fluctuations of the gap, $\Delta(\vec{x}) = \Delta_0 + \bar{\Delta}(\vec{x})$, around its equilibrium position, Δ_0, determined by eq. (6.15). This will allow us to compute the coherence length. For simplicity, we will consider static fluctuations and suppose Δ to be real.

We are going to treat the inclusion of the gauge field and the fluctuations of the gap as static perturbations. To this purpose, we write the Lagrangian (6.12) in the form $\mathcal{L} = \bar{\Psi} O_F \Psi$, where $\bar{\Psi} = (\bar{\psi}, \bar{\psi}_c)$, and we split the matrix O_F as $O_F(\Delta, A) = O_{F0}(\Delta_0) + \delta O_F(\bar{\Delta}, A)$. In the path integral formalism this amounts to consider the saddle point approximation, which is the approach that Gor'kov [180] followed to derive the Ginzburg-Landau effective action from the BCS theory. In this way, the free energy can be expanded as[1]

$$\Omega = \int \mathrm{d}^3x \, V_{\text{cl}}(\Delta_0 + \bar{\Delta}(\vec{x})) \; - \frac{1}{2\beta} \log\det O_F = \Omega_0 + \Omega_1 + \Omega_2 + \ldots \tag{6.16}$$

$$\begin{aligned}
\Omega_0 &= g^2 \int \mathrm{d}^3x \, \Delta_0^2 \; - \frac{1}{2\beta} \log\det O_{F0} \; , \\
\Omega_1 &= g^2 \int \mathrm{d}^3x \, 2\Delta_0 \bar{\Delta}(\vec{x}) \; - \frac{1}{2\beta} \mathrm{Tr}[O_{F0}^{-1} \delta O_F] \; , \\
\Omega_2 &= g^2 \int \mathrm{d}^3x \, \bar{\Delta}^2(\vec{x}) \; + \frac{1}{4\beta} \mathrm{Tr}[(O_{F0}^{-1} \delta O_F)^2] \; ,
\end{aligned}$$

[1] A note on notation, the effective potential is identified with the free energy density so we will equally use V_{eff} or F. We will use Ω for the free energy of the whole space.

where $V_{\rm cl}$ stands for the classical potential and

$$\mathrm{Tr}[O_0^{-1}\delta O] \equiv \int \mathrm{d}^4x \int \mathrm{d}^4x_1\, \mathrm{tr}[O_0^{-1}(x,x_1)\delta O(x_1,x)] \;, \tag{6.17}$$

$$\mathrm{Tr}[(O_0^{-1}\delta O)^2] \equiv \int \mathrm{d}^4x\, \mathrm{d}^4x_1\, \mathrm{d}^4x_2\, \mathrm{d}^4x_3\, \mathrm{tr}[O_0^{-1}(x,x_1)\delta O(x_1,x_2)O_0^{-1}(x_2,x_3)\delta O(x_3,x)] \;, \tag{6.18}$$

$\delta O(x,y) = \delta^{(4)}(x-y)\delta O(x)$ and tr[.] is the usual trace over matrix elements. The first term of the expansion (6.16), Ω_0, is just the free energy corresponding to eq. (6.13), which fixes the value of the gap. The second term, Ω_1, has two contributions, one corresponding to $\bar{\Delta}$ (which vanishes, as it is proportional to the gap equation) and another one due to the gauge field A. In momentum space we have

$$\mathrm{Tr}[O_{F0}^{-1}\delta O_F] = \int d^4x\, \frac{1}{\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} \mathrm{tr}[O_{F0}^{-1}(\omega_n,\vec{p})\delta O_F(x)] \;, \tag{6.19}$$

which gives rise to terms of the form

$$\int d^4x\, \frac{1}{\beta}\sum_n \int \frac{d^3p}{(2\pi)^3} h(\omega_n,p^2)\vec{p}\cdot\vec{A}(x) \;, \tag{6.20}$$

for some function h depending on ω_n and p^2. This term does not survive the momentum integration, so this just leaves contributions involving the temporal component of the gauge field, which are interpreted as fluctuations or space inhomogeneities of the chemical potential.

Once the effective potential is computed we have to add the kinetic term for the gauge field, the complete free energy is then

$$\begin{aligned}\Omega_{\rm tot} &= \int \mathrm{d}^3x \left[\frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2\right] + \Omega(\Delta,A) \\ &= \int \mathrm{d}^3x \left[f_1\left|\left(\vec{\nabla} - ie\vec{A}\right)\Delta\right|^2 + m^{-2}|\bar{\Delta}|^2 + f_2\vec{E}^2 + f_3\vec{B}^2\right] + \Omega_0 + \dots \;,\end{aligned} \tag{6.21}$$

where the coefficients f_1, f_2, f_3 and m^{-2} have been introduced to account for the contributions coming from the Ω_1 and Ω_2 terms of the expansion of the free energy. Near T_c, we can expand Ω_0 in eq. (6.21) as

$$\Omega_0(\Delta_0,T) = \int \mathrm{d}^3x \left[a(T-T_c)\Delta_0^2 + b\Delta_0^4 + \dots\right] \;, \tag{6.22}$$

where

$$a = \frac{g^4}{2\pi^2}\beta_c^2 \int_0^\Lambda p^2\mathrm{d}p \left(\mathrm{sech}^2\left(\frac{1}{2}\beta_c(\omega_0(p)+\mu)\right) + (\mu\to-\mu)\right) \;,$$

$$b = \frac{g^8}{2\pi^2}\int_0^\Lambda p^2\mathrm{d}p \left(-\beta_c\frac{\mathrm{sech}^2\left(\frac{1}{2}\beta_c(\omega_0(p)+\mu)\right)}{(\omega_0(p)+\mu)^2} + 2\frac{\tanh\left(\frac{1}{2}\beta_c(\omega_0(p)+\mu)\right)}{(\omega_0(p)+\mu)^3} + (\mu\to-\mu)\right) \,.$$

Eq. (6.22) is just the Ginzburg-Landau (GL) free energy, which is valid near the critical temperature, so (6.21) must be considered as a generalization of the GL free energy, valid for all temperatures well below the cut-off scale. Thus, as in the GL case, the equations of motion for the gauge field and the gap define a magnetic penetration length and a coherence length.

From the equations of motion for $\vec{A}$ (in the London limit, i.e. $\bar{\Delta} = 0$, and in the Coulomb gauge, $\vec{\nabla} \cdot \vec{A} = 0$), we get the following magnetic penetration length, λ:

$$\nabla^2 \vec{A} = \frac{e^2 f_1}{f_3} |\Delta_0|^2 \vec{A} \qquad \Rightarrow \qquad \frac{1}{\lambda^2} = \frac{e^2 f_1}{f_3} |\Delta_0|^2 \ . \tag{6.23}$$

And similarly, the explicit expression for the coherence length, ξ, is obtained from the equations of motion for $\bar{\Delta}(x)$ obtained from (6.21). In absence of gauge field, we get

$$\nabla^2 \bar{\Delta} = \frac{1}{m^2 f_1} \bar{\Delta} \qquad \Rightarrow \qquad \frac{1}{\xi^2} = \frac{1}{m^2 f_1} \ . \tag{6.24}$$

Hence, to determine these lengths we must be able to identify the coefficients f_1, f_3 and m^{-2} in (6.21). The identification of these coefficients is explained in appendix 6.C.

After presenting the supersymmetric model of BCS superconductivity we will give expressions with the explicit temperature dependence of these quantities, both in the non-supersymmetric case as well as in the supersymmetric model. We will do likewise with the specific heat and the critical magnetic fields.

We will compute the specific heat at constant μ, instead of constant charge density. This way is more convenient because, when neutral scalars are considered in the SUSY case, they do not contribute to the charge density constraint $\rho = \mathrm{d}F/\mathrm{d}\mu$. The specific heat is computed through the formula

$$S = -\left(\frac{\partial V_{\text{eff}}}{\partial T}\right)_\mu \ , \qquad c = T\left(\frac{dS}{dT}\right)_\mu = -T\left(\frac{\partial^2 V_{\text{eff}}}{\partial T^2} + \frac{\partial^2 V_{\text{eff}}}{\partial T \partial \varepsilon}\frac{\partial \varepsilon}{\partial T}\right) \ , \tag{6.25}$$

where

$$\frac{\partial \varepsilon}{\partial T} = -\frac{\partial_T \partial_\varepsilon V_{\text{eff}}}{\partial_\varepsilon^2 V_{\text{eff}}} \ . \tag{6.26}$$

Finally, the critical magnetic field, B_c, which corresponds to a type I superconductor, and those corresponding to a type II superconductor, B_{c1} and B_{c2}, will be computed with (6.1) and (6.9), once we have explicit expressions for the free energy, magnetic penetration length and coherence length.

6.2 Supersymmetric BCS

Let us now try to design a supersymmetric Lagrangian which incorporates these basic features. But, as we did in the presentation of the relativistic BCS theory, we will first consider the simpler case in which we do not care about the gauge field and possible spatial fluctuations of the gap.

We are interested in a supersymmetric theory with a global $U(1)$ symmetry which undergoes spontaneous symmetry breaking. This symmetry can be a baryonic $U(1)_B$

symmetry or, thanks to supersymmetry, a $U(1)_R$ symmetry. In general, $U(1)$ symmetry breaking is easy to achieve by a suitable choice of the superpotential W. However, here we are looking for a BCS type mechanism, where the breaking is caused by fermion condensation triggered by quantum effects. $\mathcal{N}=1$ supersymmetric models with a canonical Kähler potential do not contain any quartic fermion interaction for any choice of superpotential W. Quartic fermion interactions arise by means of the following choice of Kähler potential:

$$K(\Phi, \Phi^\dagger) = \Phi^\dagger \Phi + g^2 (\Phi^\dagger \Phi)^2 \ . \tag{6.27}$$

We would like to construct a supersymmetric BCS theory with Dirac fermions and in $\mathcal{N}=1$ supersymmetric theories this requires at least two chiral superfields (a single chiral superfield describes a Weyl fermion, although a theory of BCS superconductivity for Weyl fermions can also be implemented). The simplest theory consists of two chiral superfields X and Y with the Kähler potential

$$K(X, Y, X^\dagger, Y^\dagger) = X^\dagger X + Y^\dagger Y + g^2 (X^\dagger X)^2 + g^2 (Y^\dagger Y)^2 \ . \tag{6.28}$$

The coupling g could in principle be different for the interaction terms involving X and Y superfields. One could also add, for example, a term $X^\dagger X Y^\dagger Y$ (used in [181]). However we shall consider the above simple choice which already illustrates the essential points.

6.2.1 Chemical potential for $U(1)_B$

We first consider the $\mathcal{N}=1$ supersymmetric model defined in terms of two chiral superfields with Kähler potential (6.28) and superpotential:

$$W = mXY \ . \tag{6.29}$$

This gives masses to scalars and fermions. It will be shown that this model is not suitable to implement BCS mechanism in supersymmetric theories. The model will illustrate the typical problems that one has to deal with.

We first consider the Lorentzian theory on $\mathbb{R}^4$. For the finite temperature theory, we shall later consider the Euclidean theory on $S^1 \times \mathbb{R}^3$, and eventually on $S^1 \times S^3$. In components, the Lagrangian reads

$$\mathcal{L}_S = (1 + 4g^2 |\phi_x|^2) \partial_\mu \phi_x^* \partial^\mu \phi_x - \frac{m^2 |\phi_y|^2}{1 + 4g^2 |\phi_x|^2} + (x \leftrightarrow y) \tag{6.30}$$

$$\begin{aligned}\mathcal{L}_F &= i(1 + 4g^2 |\phi_x|^2)(\psi_x^\dagger \bar{\sigma}^\mu \partial_\mu \psi_x) + 4ig^2 (\psi_x^\dagger \bar{\sigma}^\mu \psi_x) \phi_x^* \partial_\mu \phi_x + \frac{g^2 (\psi_x \psi_x)(\psi_x^\dagger \psi_x^\dagger)}{1 + 4g^2 |\phi_x|^2} \\ &\quad + \left(\frac{2mg^2 \phi_y \phi_x^*}{1 + 4g^2 |\phi_x|^2} (\psi_x \psi_x) - \frac{1}{2} m \psi_x \psi_y + h.c. \right) + (x \leftrightarrow y) \ . \end{aligned} \tag{6.31}$$

Note the presence of the (non-renormalizable) quartic fermion interaction. The choice of sign of g^2 was made in order to have the same type of interaction as in BCS. One can check that the opposite sign does not lead to fermion condensation by quantum effects. For $g^2 < 0$ there is no consistent solution to the gap equation for the vacuum condensate.

The effective potential is unstable and cannot be consistently minimized in the one-loop approximation. Therefore in what follows we assume $g^2 > 0$.

We need to introduce a chemical potential and consistency demands that this is coupled to a conserved non-anomalous $U(1)$ current. The superfields X and Y carry opposite $U(1)$ charge so the baryonic $U(1)_B$ current is non-anomalous. Turning on a chemical potential corresponds to turning on a background $U(1)_B$ gauge field component $A_0 = \mu$ as explained in appendix 6.A. In order to have a Lagrangian quadratic in fermion fields, one can make a Hubbard-Stratonovich transformation in the component Lagrangian by introducing two auxiliary fields, Δ_x and Δ_y,

$$\mathcal{L}_S = (1+4g^2|\phi_x|^2)D_\mu\phi_x^* D^\mu\phi_x - \frac{m^2|\phi_y|^2}{1+4g^2|\phi_x|^2} - g^2(1+4g^2|\phi_x|^2)|\Delta_x|^2 + (x\leftrightarrow y) \quad (6.32)$$

$$\begin{aligned}\mathcal{L}_F &= i(1+4g^2|\phi_x|^2)(\psi_x^\dagger\bar\sigma^\mu D_\mu\psi_x) + 4ig^2\phi_x^* D_\mu\phi_x(\psi_x^\dagger\bar\sigma^\mu\psi_x)\\ &+\left(\left(\frac{2mg^2\phi_x^*\phi_y}{1+4g^2|\phi_x|^2}+g^2\Delta_x\right)(\psi_x\psi_x) - \frac{1}{2}m\psi_x\psi_y + h.c.\right) + (x\leftrightarrow y)\ ,\end{aligned} \quad (6.33)$$

where we have already introduced the chemical potential through the covariant derivatives

$$D_\nu = \partial_\nu - iq\mu\delta_{\nu 0}\ . \quad (6.34)$$

With no loss of generality we can set the $U(1)$ charge of the X chiral superfield equal to one, $q_X = 1$, as it can be absorbed into a redefinition of μ; in this way the superfield Y has charge $q_Y = -1$. After the Hubbard-Stratonovich transformation the Lagrangian has become quadratic in the fermion fields, no quartic fermion interaction is left. As a result, the functional integral over fermions can be directly performed.

Next, we expand the scalar fields, $\phi = v + \varphi$, around their vacuum expectation values, v, and retain only up to quadratic terms in the scalar fields (we assume real v). We find

$$\begin{aligned}\mathcal{L}_S &= (1+4g^2v_x^2)\partial_\mu\varphi_x^*\partial^\mu\varphi_x + 4g^2v_x^2\left(\mu^2 - \frac{4g^2m^2v_y^2}{(1+4g^2v_x^2)^3}\right)(\varphi_x^2+\varphi_x^{*2})\\ &+\frac{4g^2m^2v_xv_y}{(1+4g^2v_x^2)^2}(\varphi_x\varphi_y+\varphi_x^*\varphi_y+\varphi_x\varphi_y^*+\varphi_x^*\varphi_y^*)\\ &+\left(\left(1+16g^2v_x^2\right)\mu^2 - 4g^4|\Delta_x|^2 - \frac{4g^2m^2\left(-1+4g^2v_x^2\right)v_y^2}{(1+4g^2v_x^2)^3}\right)|\varphi_x|^2 - \frac{m^2}{1+4g^2v_x^2}|\varphi_y|^2\\ &+ i\mu(1+8g^2v_x^2)(\varphi_x^*\partial_t\varphi_x - \varphi_x\partial_t\varphi_x^*) - 4i\mu g^2v_x^2(\varphi_x^*\partial_t\varphi_x^* - \varphi_x\partial_t\varphi_x)\\ &+(x\leftrightarrow y,\ \mu\to-\mu)\end{aligned} \quad (6.35)$$

$$\begin{aligned}\mathcal{L}_F &= i(1+4g^2v_x^2)(\psi_x^\dagger\bar\sigma^\mu\partial_\mu\psi_x) + \mu(1+8g^2v_x^2)(\psi_x^\dagger\bar\sigma^0\psi_x)\\ &+\left(\left(\frac{2mg^2v_xv_y}{1+4g^2v_x^2}+g^2\Delta_x\right)(\psi_x\psi_x) - \frac{1}{2}m\psi_x\psi_y + h.c.\right) + (x\leftrightarrow y,\ \mu\to-\mu)\end{aligned} \quad (6.36)$$

with classical potential

$$V_{\rm cl} = \frac{m^2v_y^2}{1+4g^2v_x^2} + (1+4g^2v_x^2)(g^2|\Delta_x|^2 - \mu^2v_x^2) + (x\leftrightarrow y)\ . \quad (6.37)$$

To have canonically normalized kinetic terms, one can redefine fields as follows:

$$\varphi \to \frac{\varphi}{\sqrt{1+4g^2v^2}} \ , \qquad \psi \to \frac{\psi}{\sqrt{1+4g^2v^2}} \ . \tag{6.38}$$

Integrating over ψ, $\psi^\dagger$, φ and φ^* leads to a one-loop potential depending on g, v, Δ, μ, m. Since the model is not renormalizable just like BCS, the integrals will be regularized, as usual, by a momentum cut-off.

We proceed as follows. Calling O_S and O_F to the resulting 4×4 scalar and fermion matrices for the quadratic terms in momentum space, we shall write the determinants as:

$$\det O_S = \prod_{i=1}^{4}\left(\omega^2+\omega_{Si}^2\right) \ , \qquad \det O_F = \prod_{i=1}^{4}\left(\omega^2+\omega_{Fi}^2\right) \ , \tag{6.39}$$

where

$$\omega_{Si} = \omega_{Si}(\mu, |\vec{p}|, g, m, v_x, v_y, \Delta_x, \Delta_y) \ , \qquad \omega_{Fi} = \omega_{Fi}(\mu, |\vec{p}|, g, m, v_x, v_y, \Delta_x, \Delta_y) \ . \tag{6.40}$$

The expressions for O_S and O_F are shown in appendix 6.B. The eigenvalues for the frequencies have complicated expressions when v_x and v_y are non-vanishing. The strategy is to look for non-trivial minima at $v_x = v_y = 0$ with $\Delta_x,\ \Delta_y \neq 0$, assuming them to be real. Next, we shall check that the one-loop effective potential is locally stable in v_x and v_y directions, a property that will be ensured by the presence of a mass term.

When $v_x = v_y = 0$ the scalar and fermion quadratic terms greatly simplify. At this point, we find the following eigenvalues for the frequency.

$$\begin{aligned} \omega_{S\ 1,2} &= \sqrt{4g^4\Delta_x^2+m^2+p^2} \pm \mu \ , \\ \omega_{S\ 3,4} &= \sqrt{4g^4\Delta_y^2+m^2+p^2} \pm \mu \ , \end{aligned} \tag{6.41}$$

$$\begin{aligned} \omega_{F\ 1,2}^2 &= 2g^4\Delta_x^2+2g^4\Delta_y^2+\mu^2+m^2+p^2 \pm \mathcal{E}_+ \ , \\ \omega_{F\ 3,4}^2 &= 2g^4\Delta_x^2+2g^4\Delta_y^2+\mu^2+m^2+p^2 \pm \mathcal{E}_- \ , \end{aligned} \tag{6.42}$$

$$\mathcal{E}_\pm = 2\sqrt{\mu^2\left(m^2+p^2\right)+g^8\left(\Delta_x^2-\Delta_y^2\right)^2+g^4\left(m^2\left(\Delta_x+\Delta_y\right)^2 \pm 2\mu p\left(\Delta_x^2-\Delta_y^2\right)\right)} \ . \tag{6.43}$$

For configurations with $\Delta_x = \Delta_y \equiv \Delta$, the fermion frequencies become

$$\omega_F = \sqrt{\left(\sqrt{p^2+m^2+4g^4\Delta^2\frac{m^2}{\mu^2}} \pm \mu\right)^2 + 4g^4\Delta^2\left(1-\frac{m^2}{\mu^2}\right)} \ . \tag{6.44}$$

On the other hand, for $\Delta_x = -\Delta_y \equiv \Delta$, we find

$$\omega_F = \sqrt{\left(\sqrt{p^2+m^2} \pm \mu\right)^2 + 4g^4\Delta^2} \ . \tag{6.45}$$

This is the same dispersion relation as in the relativistic BCS system of section 6.1.3. This might suggest that the BCS mechanism can be implemented in a similar way. But the presence of charged scalars demands some care. We first need to identify the Fermi surfaces. For $\Delta_x = \Delta_y = 0$, they lie on the region where $\omega_{F\ 2,4}$ vanish, i.e. at

$$\sqrt{p_F^2 + m^2} = \mu \ . \tag{6.46}$$

As in the standard relativistic BCS case, the existence of a Fermi surface would require $\mu > m$. However, in the present supersymmetric system we cannot set $\mu > m$, since the scalar contribution to the thermal partition function,

$$\frac{1}{\beta}\sum_i \int \frac{\mathrm{d}^3 p}{(2\pi)^3} \log\left(1 - e^{-\beta\omega_{Si}}\right) \ , \tag{6.47}$$

is ill-defined, because $\omega_{S\ 2,4}$ become negative below some momentum. The system presents Bose-Einstein condensation, the occupation number of scalars with zero momentum goes to infinity as μ approaches m from below. This spoils the BCS mechanism.

One possible approach to elude this problem while maintaining supersymmetry is to put the theory on a curved manifold, such that scalar fields couple to the curvature. This coupling between scalars and the curvature provides an extra mass term for the scalar fields, which might allow for regions in parameter space with Fermi surfaces and without problems of Bose-Einstein condensation of scalars. In particular, we will consider the theory on $S^1 \times S^3$, (see e.g. [36]). The mass term, when the R-charge of the scalars is one, is now of the form

$$\left(m^2 + R^{-2}\right)\left(\phi_x^*\phi_x + \phi_y^*\phi_y\right) \ ,$$

where R is the radius of the three-sphere. The scalar contribution would be negligible if one could assume that $1/R > \Lambda$. However, having put the theory on S^3, the integral over momentum is replaced by a discrete sum coming from the Kaluza-Klein modes of S^3. This replacement is achieved by (see for example [174])

$$\text{Scalars}: \qquad p^2 \longrightarrow l(l+2)R^{-2} \ , \tag{6.48}$$

$$\text{Fermions}: \qquad p^2 \longrightarrow (l+1/2)^2 R^{-2} \ , \tag{6.49}$$

with $l = 0, 1, 2\ldots$ One must also take into account the degeneracy of each mode:

$$\text{Scalars}: \qquad d_l^S = (l+1)^2, \tag{6.50}$$

$$\text{Fermions}: \qquad d_l^F = l(l+1) \ . \tag{6.51}$$

In particular, for the fermions, $l = 0$ does not contribute. For the scalars, in addition we must add the mass term R^{-2}. This is effectively incorporated by the replacement

$$\text{Scalars}: \qquad p^2 \longrightarrow l(l+2)R^{-2} + R^{-2} = (l+1)^2 R^{-2} \tag{6.52}$$

instead of (6.48). These formulas show that one cannot assume $1/R > \Lambda$, since such cut-off would leave no excitation in the system. Therefore it is not possible to separate the scalar mass scale from the Fermi surface.

In order to see if the system can have Fermi surfaces, we need the detailed form of the Lagrangian on S^3. This depends on the R charges of the fields. We denote by q the R charge of ϕ_x so that the charge of ϕ_y is $2-q$. From the expressions given in [36], we find

$$\begin{aligned}
\mathcal{L}_S &= (1+4g^2|\phi_x|^2)\partial_\mu\phi_x^*\partial^\mu\phi_x \\
&+ \left(\frac{q(q-2)}{R^2} + 2\mu\frac{q-1}{R} + \mu^2\right)|\phi_x|^2 + 4g^2\left(\frac{q(q-1)}{R^2} + \mu\frac{2q-1}{R} + \mu^2\right)|\phi_x|^4 \\
&+ i\left(\frac{q-1}{R} + \mu\right)(\phi_x^*\partial_t\phi_x - \phi_x\partial_t\phi_x^*) + 2ig^2\left(\frac{2q-1}{R} + 2\mu\right)|\phi_x|^2(\phi_x^*\partial_t\phi_x - \phi_x\partial_t\phi_x^*) \\
&- \frac{m^2|\phi_y|^2}{1+4g^2|\phi_x|^2} - g^2(1+4g^2|\phi_x|^2)|\Delta_x|^2 + (x\leftrightarrow y, \mu\to-\mu, q\to 2-q)
\end{aligned} \tag{6.53}$$

$$\begin{aligned}
\mathcal{L}_F &= i(1+4g^2|\phi_x|^2)(\psi_x^\dagger\bar{\sigma}^\mu\partial_\mu\psi_x) + 4ig^2(\phi_x^*\partial_\mu\phi_x)(\psi_x^\dagger\bar{\sigma}^\mu\psi_x) \\
&+ \left(\frac{2q-1}{2R} + \mu\right)(\psi_x^\dagger\bar{\sigma}^0\psi_x) + 4g^2\left(\frac{4q-1}{2R} + 2\mu\right)|\phi_x|^2(\psi_x^\dagger\bar{\sigma}^0\psi_x) \\
&+ \left(\frac{2mg^2\phi_x^*\phi_y}{1+4g^2|\phi_x|^2} + g^2\Delta_x\right)(\psi_x\psi_x) + \left(\frac{2mg^2\phi_x\phi_y^*}{1+4g^2|\phi_x|^2} + g^2\Delta_x^*\right)(\psi_x^\dagger\psi_x^\dagger) \\
&- \frac{1}{2}m(\psi_x\psi_y + \psi_x^\dagger\psi_y^\dagger) + (x\leftrightarrow y, \mu\to-\mu, q\to 2-q)
\end{aligned} \tag{6.54}$$

We shall demand that in the unbroken phase the theory has well-defined thermodynamical potentials. So we begin by considering the case $\Delta_x = \Delta_y = 0$, $v_x = v_y = 0$. We will now see that Bose-Einstein condensation is inevitable in this case, which is sufficient to rule out the model. Consider first the case $q = 1$, i.e. the $U(1)$ charges of X and Y are equal to 1. The scalar contribution is now given in terms of the frequencies

$$\omega_S = \sqrt{(l+1)^2R^{-2} + m^2} \pm \mu\ , \qquad l = 0, 1, 2, \ldots \tag{6.55}$$

If both X and Y had the same baryon charge, the Fermi surface would just be determined by the replacement (6.49) in the flat expression (6.46), and shifting the chemical potential by $\mu \to \mu + 1/(2R)$. As X and Y have opposite baryon charges, this is more involved. By explicitly computing ω_F from the above Lagrangian, we obtain that the Fermi surface $\omega_F = 0$ lays at

$$\sqrt{l_F^2R^{-2} + m^2} = \mu, \qquad l_F = 1, 2, \ldots \tag{6.56}$$

For a given choice of l_F, one can determine μ. Substituting μ in the lowest ($l = 0$) scalar frequency, we see that the scalar frequency cannot be positive as long as $l_F = 1, 2, \ldots$,

$$\sqrt{R^{-2} + m^2} - \sqrt{l_F^2R^{-2} + m^2} \le 0\,. \tag{6.57}$$

Therefore, even on S^3, it is not possible to separate the Fermi surface from the region of Bose-Einstein condensation. The underlying reason being that the extra mass term for the scalar provided by the coupling to the curvature of the space is of the same order as the quantized fermion momentum values. The same problem arises for any choice of q.

6.2.2 A simple supersymmetric BCS model: Chemical potential for $U(1)_R$

Let us now consider an $\mathcal{N}=1$ supersymmetric model with two chiral superfields X and Y with Kähler potential given by (6.28) and superpotential $W=0$. The Lagrangian has a $U(1)_R$ symmetry for arbitrary $U(1)_R$ charges of the X and Y superfields. It is convenient to consider the $U(1)_R$ symmetry under which scalars ϕ_x and ϕ_y are neutral, so that fermions ψ_x and ψ_y have charge -1. The advantage of this choice is that we can avoid problems of Bose-Einstein condensation even in $\mathbb{R}^4$. Note that with this charge assignation the $U(1)_R$ symmetry is anomalous. However, this can be easily cured by adding to the theory free superfields with canonical Kähler potential with the required $U(1)_R$ charges to cancel the anomaly. For example, one may add Z_i, $i=1,2$ with R-charges $R(Z_i)=2$ so that ψ_{Z_1}, ψ_{Z_2} have charges $+1$. The scalars in Z_i would then couple to the chemical potential and may undergo Bose-Einstein condensation. However, this sector is completely decoupled and therefore does not participate in the thermodynamics governing the X, Y sector.

The component Lagrangian with chemical potential included can be obtained from the previous case, (6.32), (6.33), by setting $m=0$, vanishing $U(1)$ charges for the scalar fields (which amounts to replace covariant derivatives of the scalar fields by ordinary derivatives) and taking into account that fermions ψ_x and ψ_y now have the same charge -1. The quadratic Lagrangian for the fluctuations (after expanding around expectation values) is given by

$$\mathcal{L}_S = \partial_\mu \varphi_x^* \partial^\mu \varphi_x + \partial_\mu \varphi_y^* \partial^\mu \varphi_y - \frac{4g^4|\Delta_x|^2}{1+4g^2v_x^2}|\varphi_x|^2 - \frac{4g^4|\Delta_y|^2}{1+4g^2v_y^2}|\varphi_y|^2 \ , \tag{6.58}$$

$$\begin{aligned}\mathcal{L}_F &= i(\psi_x^\dagger \bar\sigma^\mu \partial_\mu \psi_x) + i(\psi_y^\dagger \bar\sigma^\mu \partial_\mu \psi_y) - \mu(\psi_x^\dagger \bar\sigma^0 \psi_x) - \mu(\psi_y^\dagger \bar\sigma^0 \psi_y) \\ &+ \left(\frac{g^2\Delta_x}{1+4g^2v_x^2}(\psi_x\psi_x) + \frac{g^2\Delta_y}{1+4g^2v_y^2}(\psi_y\psi_y) + h.c. \right) \ ,\end{aligned} \tag{6.59}$$

where we have rescaled the fields to have canonical kinetic terms. The classical potential is given by

$$V_{\rm cl} = g^2\left(4g^2v_x^2+1\right)|\Delta_x|^2 + (x \leftrightarrow y) \ . \tag{6.60}$$

The equations of motion for Δ_x, Δ_y give (setting the scalar fluctuations $\varphi_x, \varphi_y \to 0$)

$$\Delta_x = \frac{\psi_x^\dagger \psi_x^\dagger}{(1+4g^2v_x^2)^2} \ , \qquad \Delta_y = \frac{\psi_y^\dagger \psi_y^\dagger}{(1+4g^2v_y^2)^2} \ . \tag{6.61}$$

Δ_x, Δ_y have both $U(1)_R$ charges equal to 2. Vacuum expectation values for them thus spontaneously break $U(1)_R$ and represent a measure of the fermion condensate.

By proceeding in a similar way as in the previous case with the baryonic symmetry, we now find the following frequencies for scalars and fermions

$$\omega_{S\ 1,2}^2 = p^2 + \frac{4g^4\Delta_x^2}{1+4g^2v_x^2} \ , \qquad \omega_{S\ 3,4}^2 = p^2 + \frac{4g^4\Delta_y^2}{1+4g^2v_y^2} \ , \tag{6.62}$$

$$\omega_{F\ 1,2}^2 = (p\pm\mu)^2 + \frac{4g^4\Delta_x^2}{(1+4g^2v_x^2)^2} \ , \qquad \omega_{F\ 3,4}^2 = (p\pm\mu)^2 + \frac{4g^4\Delta_y^2}{\left(1+4g^2v_y^2\right)^2} \ . \tag{6.63}$$

Here we have chosen real $\Delta_x,\ \Delta_y$, as one-loop potential depends only on their moduli. We stress that these simple dispersion relations are a consequence of the extreme simplicity of this supersymmetric model; generic models (even with simple superpotentials) typically lead to very complicated eigenvalues for the frequencies.

In the present case, the dynamics of the X and Y fields are decoupled. It is clear that the same configuration that minimizes the one-loop potential in the x direction also minimizes the one-loop potential in the y direction. Therefore, with no loss of generality and for the sake of simplicity we can consider just one chiral superfield, this would correspond to consider a model of superconductivity with one Weyl fermion an one complex scalar field.

For one chiral superfield, the complete one-loop thermodynamic potential is given by

$$\begin{aligned} V_{\text{eff}} = g^2\left(1+4g^2v^2\right)\Delta^2 \\ + \frac{1}{2\pi^2\beta}\int_0^\Lambda \mathrm{d}p\; p^2 \Bigg(2\log\left[\sinh\frac{\beta}{2}\sqrt{p^2+\frac{4g^4\Delta^2}{1+4g^2v^2}}\right] \\ - \log\left[\cosh\frac{\beta}{2}\sqrt{(p+\mu)^2+\frac{4g^4\Delta^2}{\left(1+4g^2v^2\right)^2}}\right] \\ - \log\left[\cosh\frac{\beta}{2}\sqrt{(p-\mu)^2+\frac{4g^4\Delta^2}{\left(1+4g^2v^2\right)^2}}\right]\Bigg)\ . \end{aligned} \tag{6.64}$$

When the vacuum lies at $\Delta \neq 0$, then $v = 0$ is a local minimum. When the vacuum lies at $\Delta = 0$, then there is a flat direction in v, because in this case the frequencies do not depend on v. This is confirmed by the evaluation of the one-loop potential. The effective potential at $v = 0$ is then

$$\begin{aligned} V_{\text{eff}} = g^2\Delta^2 \\ - \frac{1}{2}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\sqrt{(p+\mu)^2+4g^4\Delta^2}+\sqrt{(p-\mu)^2+4g^4\Delta^2}\right) \\ - \frac{1}{\beta}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\left(\log\left[1+e^{-\beta\sqrt{(p+\mu)^2+4g^4\Delta^2}}\right]-\log\left[1+e^{-\beta\sqrt{(p-\mu)^2+4g^4\Delta^2}}\right]\right) \\ + \int\frac{\mathrm{d}^3p}{(2\pi)^3}\sqrt{p^2+4g^4\Delta^2}+\frac{1}{\beta}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\log\left[1-e^{-\beta\sqrt{p^2+4g^4\Delta^2}}\right]\ , \end{aligned} \tag{6.65}$$

to be compared with (6.13). Up to redefinitions of the field Δ and the coupling constant g, they are the same expression except for a couple of differences. One difference is that now fermions (and also scalar fields) are massless, since a mass term would not be consistent with scalars neutral under $U(1)_R$. The second and more fundamental difference is the last line of equation (6.65), corresponding to the scalar contribution, not present in (6.13).

Now if we want to study the magnetic response of the superconductor, we have to turn on an external $U(1)$ gauge field. We have two possibilities: the gauge field can be turned on for the baryonic $U(1)_B$ symmetry or the $U(1)_R$ symmetry[2]. In either case

[2] R-symmetry can only be gauged within the context of supergravity. The present model can be easily embedded in $\mathcal{N}=1$ supergravity, in such a way that in the usual laboratory set up, where energy configurations are much lower than the Planck scale, the supergravity multiplet can be ignored.

supersymmetry is broken by the background. In principle, the magnetic response could depend on this choice, but as explained in appendix 6.C, this is not the case, at least for a small enough gauge coupling[3]. For definiteness we introduce the gauge field through the baryonic $U(1)_B$ symmetry, therefore the Lagrangian (6.58), (6.59) is modified to (once we set $v = 0$)

$$\mathcal{L}_E = \left(\partial^a \phi \partial^a \phi^* + \bar{\psi}\bar{\sigma}^0 \left(\partial^\tau - ieA^\tau + \mu\right)\psi + i\bar{\psi}\bar{\sigma}^i \left(\partial^i - ieA^i\right)\psi\right) \\ - ieA^a \phi \partial^a \phi^* + ieA^a \phi^* \partial^a \phi + e^2 A^a A^a |\phi|^2 + 4g^4 |\Delta|^2 |\phi|^2 - g^2 \Delta(\psi\psi) - g^2 \Delta^* (\bar{\psi}\bar{\psi}) \,. \tag{6.66}$$

Given this Lagrangian, we can construct the bosonic and fermionic matrices (whose explicit form is written in appendix 6.B) and split them as explained in section 6.1.4, so that we have to add to eq. (6.16), the scalar contribution

$$+\frac{1}{2\beta}\log\det O_{S0} + \frac{1}{2\beta}\mathrm{Tr}[O_{S0}^{-1}\delta O_S] - \frac{1}{4\beta}\mathrm{Tr}[(O_{S0}^{-1}\delta O_S)^2] + \ldots \tag{6.67}$$

The determination of the quantities explained in the case of relativistic BCS theory is now completely analogous for the supersymmetric case. We are going to examine them in detail in the next section.

6.3 Comparison between the SUSY model and relativistic BCS

In this section we want to compare some physical quantities when obtained for the rBCS theory and the supersymmetric model presented in the previous section. In particular, we will study the gap, the specific heat, the penetration length and the coherence length and the critical magnetic fields, B_c, B_{c1} and B_{c2}. To be precise, we will compare these quantities obtained for (6.65) with and without the scalar contribution. This amounts to compare the relativistic BCS theory in the massless case with the supersymmetric version (sBCS).

6.3.1 Gap

The gap as a function of the temperature, $\Delta_0 = \Delta_0(T)$, is determined by solutions to the gap equation $\partial_\varepsilon V_{\text{eff}} = 0$. Taking this derivative in (6.64) we obtain that the explicit form

[3]However, things are different if we turn on the gauge fields associated with the $U(1)_B$ and $U(1)_R$ symmetries at the same time. In this case, there is a linear combination of the previously introduced gauge fields which defines a massless rotated gauge field. Therefore, the magnetic field associated to this rotated gauge field can penetrate the superconductor without being subject to the Meissner effect. This case is similar to that of [182–185], so it would be interesting to study how the phenomenology presented there translates into the present supersymmetric model.

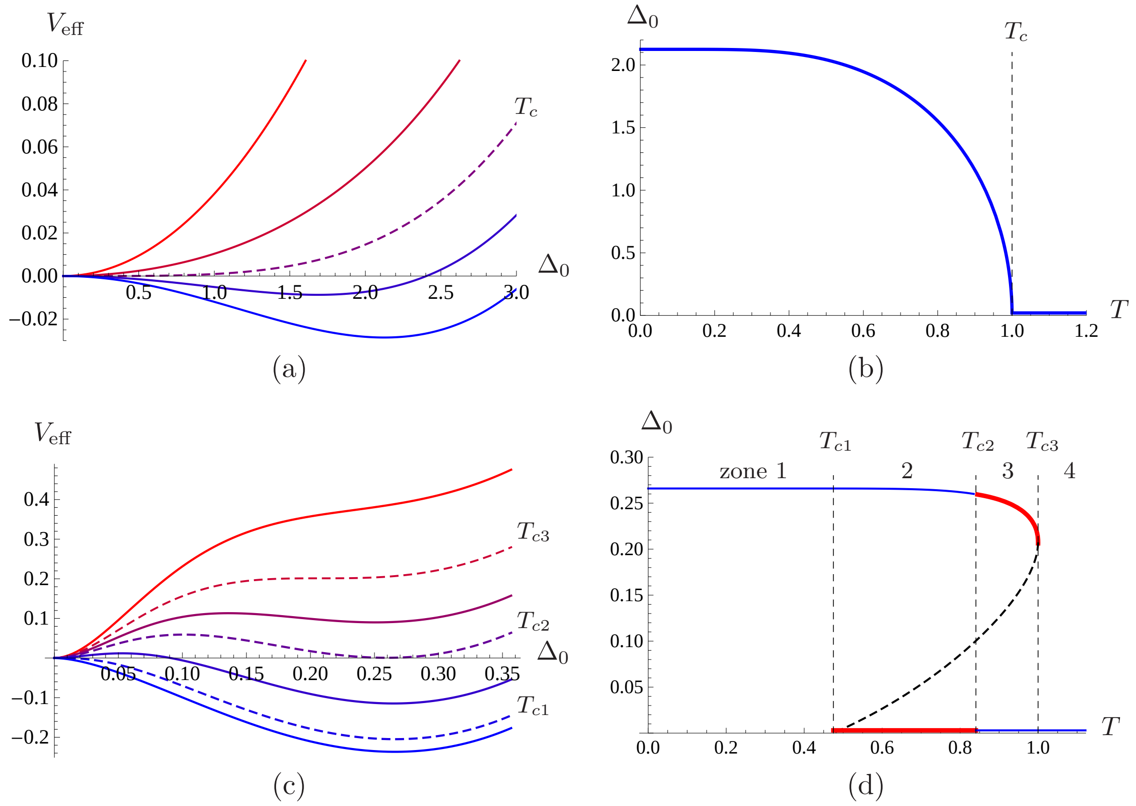

Figure 6.2: (a) Effective potential as a function of the gap for different temperatures ($T = 2,\ 1.3,\ 1,\ 0.75,\ 0$ from top to bottom) in the relativistic BCS case. (b) Corresponding gap as a function of the temperature ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (c) ($T = 1.1,\ 1,\ 0.92,\ 0.84,\ 0.7,\ 0.47,\ 0$) and (d) analogous figures for the SUSY case ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$). The thicker red lines in (d) correspond to metastable solutions and the curved dashed line represents the maximum that separates both minimums in the effective potential. Data given in units of T_c.

of the gap equation at $v = 0$ is

$$
\begin{aligned}
1 = \frac{g^2}{2\pi^2} \int_0^\Lambda dp\, p^2 \Bigg(& \frac{\tanh\left(\frac{1}{2}\beta\sqrt{4g^4\Delta_0^2 + (p-\mu)^2}\right)}{\sqrt{4g^4\Delta_0^2 + (p-\mu)^2}} + \frac{\tanh\left(\frac{1}{2}\beta\sqrt{4g^4\Delta_0^2 + (p+\mu)^2}\right)}{\sqrt{4g^4\Delta_0^2 + (p+\mu)^2}} \\
& - \frac{2\coth\left(\frac{1}{2}\beta\sqrt{4g^4\Delta_0^2 + p^2}\right)}{\sqrt{4g^4\Delta_0^2 + p^2}} \Bigg) , \qquad (6.68)
\end{aligned}
$$

where the second line is the new contribution due to the scalar.

The IR physics produced by the scalar sector has a striking effect: the superconducting transition becomes first-order, instead of second-order, as it would be in standard BCS. The IR physics of the scalar sector is important at the onset of the transition, where Δ_0

is small. To see the nature of the transition, we need to compute $\mathrm{d}\Delta_0/\mathrm{d}T$. Writing the gap equation in the form $1 = f(\Delta_0^2, T)$, one has

$$\frac{\mathrm{d}\Delta_0}{\mathrm{d}T} = -\frac{1}{2\Delta_0}\frac{\partial f}{\partial T}\Big/\frac{\partial f}{\partial(\Delta_0^2)} \,. \tag{6.69}$$

In a second-order phase transition, $\mathrm{d}\Delta_0/\mathrm{d}T$ is singular at the critical temperature, where $\Delta_0 = 0$. This is because $\partial f/\partial T$ and $\partial f/\partial(\Delta_0^2)$ are regular at $\Delta_0 = 0$. While the scalar contribution to the one-loop potential is regular at $\Delta_0 = 0$, its second derivative with respect to Δ_0^2 has a singularity near $\Delta_0 = 0$ originating from the region near $p = 0$. We have

$$\frac{\partial f}{\partial(\Delta_0^2)} \approx \frac{8g^6T}{\pi^2}\int_0 \mathrm{d}p\, p^2 \frac{1}{(p^2+4g^4\Delta_0)^2} \approx \frac{g^4T}{\pi}\,\frac{1}{\Delta_0} \,. \tag{6.70}$$

As a result, $\mathrm{d}\Delta_0/\mathrm{d}T$ is now finite at $\Delta_0 = 0$. The superconducting phase transition is therefore first-order. This significant change coming from the $p = 0$ region would obviously not take place if the scalar field was massive. In such a case, the phase transition would still be second-order. But, as explained, in the present model it is not possible to add a mass term.

First order phase transitions in usual superconductivity have already been explained in [186], where a gauge field takes a non zero vacuum expectation value appearing as a $\langle A^2\rangle|\Delta_0|^3$ term in the free energy, which inevitably leads to a first order phase transition. This gauge field plays the same role as the scalar in the supersymmetric case. Near $\Delta_0 = 0$ and at low momentum, we can approximate the contribution of the scalar as

$$\partial_\varepsilon V_{\mathrm{eff}} \approx \frac{2g^4T}{\pi^2}\int_0 \mathrm{d}p\, p^2 \frac{1}{(p^2+4g^4\Delta_0^2)} \approx -\frac{2g^6T}{\pi}\Delta_0 \,, \tag{6.71}$$

which leads to the analogous $\mathcal{O}(|\Delta_0|^3)$ term in the free energy.

In fig. 6.2 we compare the effective potential and the gap as a function of the temperature obtained for relativistic BCS and supersymmetric BCS. For the former, the effective potential develops a non-trivial minimum at $\Delta_0 \neq 0$, which will approach the origin as we increase the temperature, reaching the zero gap value at the critical temperature T_c, (fig. 6.2 (a) and (b)). The phase transition is second order. For the sBCS theory, at low temperatures, $T < T_{c1}$, the effective potential has a unique non-trivial minimum (zone 1 of fig. 6.2 (c) and (d)), above T_{c1} (zone 2) a zero gap metastable minimum appears becoming the dominant one for $T > T_{c2}$ (zone 3), so that the minimum with non-vanishing gap becomes metastable until it disappears at T_{c3}, jumping to the non-superconducting zero gap solution for $T > T_{c3}$ (zone 4). Therefore, the phase transition is first order in the supersymmetric case. For short, we will be calling $T_c \equiv T_{c3}$ and $\Delta_c \equiv \Delta(T_{c3})$ from now on.

We can also study the dependence with the chemical potential. $\Delta_0(T)$ is shown in fig. 6.3 for different values of the chemical potential. We see that, as the chemical potential gets smaller, the transition approaches a second-order phase transition. In general, the scalar field has the effect of decreasing the critical temperature with respect to the relativistic BCS case to the extent that at small chemical potential and small temperatures the sBCS system does not admit a superconducting phase, something that it is allowed in the rBCS theory. This can be seen from the phase diagram $T - \mu$, shown in figs. 6.4 (a) and (b), or from the gap equation (6.68). At low temperatures and small chemical potential (6.68) can be expanded as

$$1 = \frac{g^2}{2\pi^2}\int_0^{\Lambda} dp\; p^2 \Bigg[\frac{2}{\sqrt{p^2+4g^4\Delta_0^2}} + \mathcal{O}\left(\exp\left[-\beta\sqrt{p^2+4g^4\Delta_0^2}\right]\right) + \mathcal{O}(\mu^2) \\ - \frac{2}{\sqrt{p^2+4g^4\Delta_0^2}} + \mathcal{O}\left(\exp\left[-\beta\sqrt{p^2+4g^4\Delta_0^2}\right]\right) \Bigg] , \quad (6.72)$$

where, again, the second line corresponds to the scalar contribution. It is clear from the previous equation that the equality cannot hold if the scalar contribution is included.

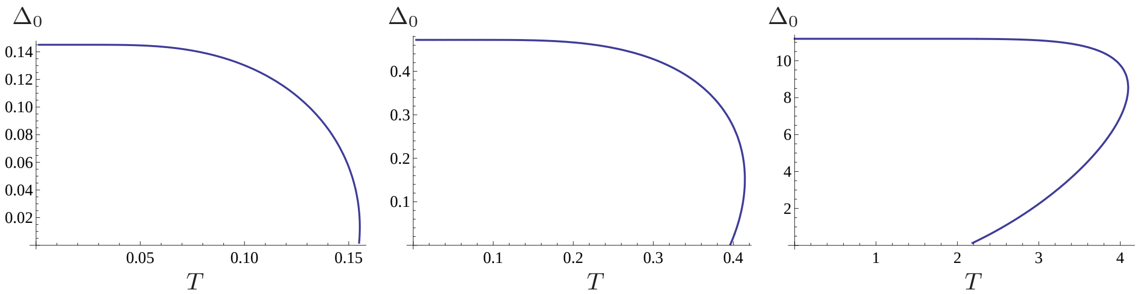

Figure 6.3: Gap as a function of the temperature for different values of the chemical potential ($\mu = 1.4,\ 1.6,\ 3.6$) and $\Lambda = 100$. At low chemical potential the phase transition approaches a second order. Data given in units of g.

Let us study now the analytic behavior of the gap for temperatures near the phase transition, $T \approx T_c$, and at zero temperature.

At low temperatures, we can neglect in (6.68) the antiparticle and the scalar contributions for momenta near the Fermi surface, where the main contribution to the integral comes from. By doing so, we are taking the non-relativistic limit and connecting with the standard BCS result. Specifically, one performs the *ad hoc* approximation by which one substitutes $\mathrm{d}p^3$ by $4\pi p_F^2 \mathrm{d}p$ and the integral is done in the interval $|p - p_F| < \Lambda$ around the Fermi surface. After these approximations and at large cut-off ($\Lambda/T_c \gg 1$), one can show that the behavior of the gap near the critical temperature T_c follows a universal behavior

$$\frac{\Delta_0(T)}{\Delta_0(0)} \approx \eta\sqrt{1-\frac{T}{T_c}} , \quad (6.73)$$

where $\eta = 1.74$ can be computed numerically from eq. (6.22). In the supersymmetric case we obtain numerically the following behavior near the critical temperature:

$$\frac{\Delta_0(T) - \Delta_c}{\Delta_0(0)} \approx \eta\left(1-\frac{T}{T_c}\right)^{\alpha} , \quad (6.74)$$

where $\alpha \approx 0.5$ and now η depends on the parameters g, μ and Λ.

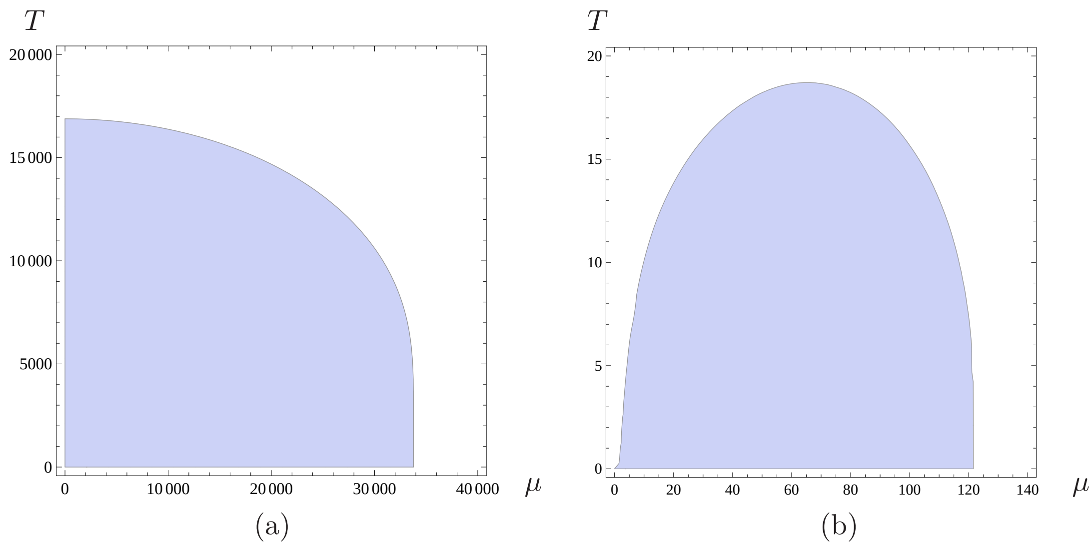

Figure 6.4: Phase diagram $T - \mu$ ($\Lambda = 100$) for rBCS in (a) and sBCS in (b). The shaded region represents the $U(1)$ symmetry breaking phase by a fermion condensate. The curve represents the critical temperature as a function of the chemical potential μ. Data given in units of g.

We can also determine the expressions for the gap at zero temperature. The non-relativistic and relativistic BCS expressions are:

$$|\Delta_0(0)_{\rm BCS}| \approx \frac{\Lambda}{g^2} e^{-\frac{2\pi^2}{g^2\mu^2}} \approx \frac{1}{2}\pi e^{-\gamma} \frac{T_c}{g^2} \ , \qquad |\Delta_0(0)_{\rm rBCS}| \approx \frac{\Lambda^3}{6\pi^2} \approx \frac{T_c}{g^2} \ , \tag{6.75}$$

where in the non-relativistic limit we have considered the aforementioned approximation and γ is the Euler-Mascheroni constant. Both the gap at zero temperature and the critical temperature depend on the cut-off (the Debye energy in the standard BCS case), in such a way that the cut-off dependence disappears from the formula for the gap once it is expressed in terms of the critical temperature. We have also shown here the dependence on the cut-off because we want to compare these expressions with those obtained for sBCS, since one of the virtues of supersymmetry is the softening of divergences due to cancellations between fermionic and bosonic contributions. As the thermal integrals in (6.13) and (6.65) are convergent, it is sufficient to study the dependence of the gap with the cut-off at zero temperature. Comparing the relativistic expression with the supersymmetric one,

$$|\Delta_0(0)_{\rm sBCS}| \approx \frac{\Lambda}{g^2} e^{-\frac{\pi^2}{g^2\mu^2} - \frac{3}{2}} \ , \tag{6.76}$$

we observe that this softening in the cut-off dependence appears in the present system. However, this reduction in the power of the cut-off does not appear if we compare the supersymmetric expression with the non-relativistic one, where the dependence is linear rather than cubic as in the relativistic theory. This is because the power of the cut-off has already been reduced after performing the substitution $\mathrm{d}p^3 \to 4\pi p_F^2 \mathrm{d}p$ in the non-relativistic case. Finally, just as in the non-supersymmetric cases (6.75), one can check that the supersymmetric case (6.76) is also linear with the critical temperature with no dependence on the cut-off.

6.3.2 Specific heat

Let us now study the specific heat. It is instructive to examine the different contributions to the thermodynamic potential closely. Consider first the symmetric phase $\Delta_0 = 0$. As we shall be interested in derivatives with respect to the temperature, we can subtract the Coleman-Weinberg contribution, so that integrals are convergent. We write

$$F = F_{\text{scalar}} + F_{\text{fermion}} + F_{\text{anti-fermion}} \ , \tag{6.77}$$

where (after integration by parts)

$$F_{\text{scalar}}\big|_{\Delta_0=0} = -\frac{1}{3\pi^2}\int_0^\infty \mathrm{d}p\ p^3\ \frac{e^{-\frac{p}{T}}}{1-e^{-\frac{p}{T}}} \ , \tag{6.78}$$

$$F_{\text{fermion}}\big|_{\Delta_0=0} = -\frac{1}{6\pi^2}\int_0^\infty \mathrm{d}p\ p^3\ \frac{e^{-\frac{p-\mu}{T}}}{1+e^{-\frac{p-\mu}{T}}} \ , \tag{6.79}$$

$$F_{\text{anti-fermion}}\big|_{\Delta_0=0} = -\frac{1}{6\pi^2}\int_0^\infty \mathrm{d}p\ p^3\ \frac{e^{-\frac{p+\mu}{T}}}{1+e^{-\frac{p+\mu}{T}}} \ . \tag{6.80}$$

We get

$$F_{\text{scalar}}\big|_{\Delta_0=0} = -\frac{1}{45}\pi^2 T^4 \ , \tag{6.81}$$

as expected, since when $\Delta_0 = 0$ the scalar contribution describes a relativistic boson particle (there is an extra factor of 2 as compared with the usual single scalar contribution, because we have a complex scalar field). The integrals for the fermionic contribution can be computed analytically in terms of polylogarithmic functions.

Let us now compute the different contributions to the specific heat. These can be computed with the formulas in (6.25). We obtain

$$c_{\text{scalar}}\big|_{\Delta_0=0} = \frac{4\pi^2 T^3}{15} \ , \tag{6.82}$$

as usual for relativistic bosons. Consider now the fermion contributions. At large T, the dependence on μ disappears and one gets the usual behavior of a relativistic fermion

$$c_{\text{fermion}}\big|_{\Delta_0=0} = c_{\text{anti-fermion}}\big|_{\Delta_0=0} = \frac{7\pi^2 T^3}{60} \ , \qquad \text{for} \quad T \gg \mu \ . \tag{6.83}$$

At low temperatures the antiparticle and the scalar can be neglected and the main contribution comes from the region near the Fermi surface $p \sim \mu$, so that the behavior of the specific heat in the normal phase and in the superconducting phase is given by

$$c_{\text{fermion}}\big|_{\Delta_0=0} \sim \frac{\mu^2 T}{6} \ , \qquad\qquad c_{\text{fermion}}\big|_{\Delta_0\neq 0} \sim e^{-\frac{2g^2\Delta_0(T=0)}{T}} \ . \tag{6.84}$$

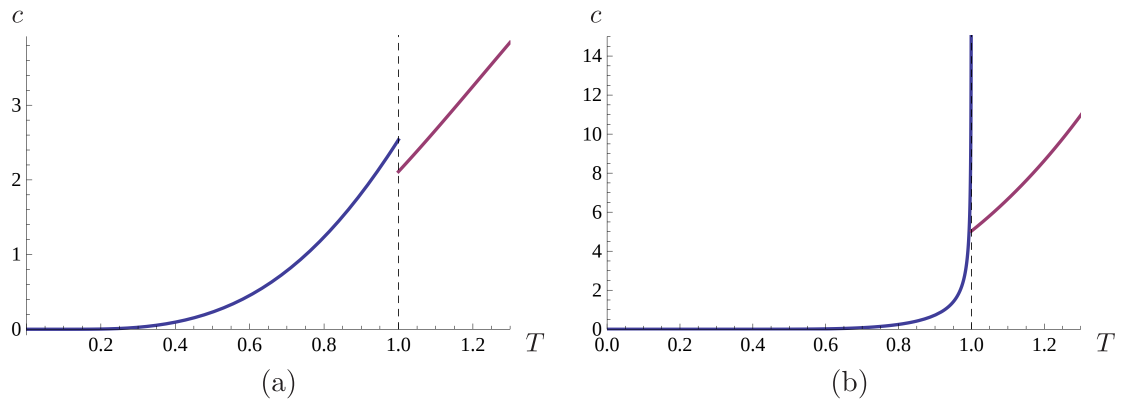

Figure 6.5: Specific heat a function of the temperature. (a) Relativistic BCS ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (b) SUSY BCS ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$). Data given in units of T_c.

Let us now compute the full $c(T)$ including the region $T < T_{c2}$ where $\Delta_0 \neq 0$. The comparison of the full specific heat between the sBCS and rBCS theories is shown in fig. 6.5. When only fermions are considered the jump in the specific heat at the critical temperature is finite, as it is characteristic for second order phase transitions, whereas it is infinite when the scalar is included.

The last expression of equation (6.84) shows a way to compute the value of the gap at zero temperature. Indeed, by performing a fit of the plots in fig. 6.5 one can obtain the value of the gap at zero temperature shown in fig. 6.2.

6.3.3 Magnetic penetration length and coherence length

The magnetic penetration length, obtained from eq. (6.23), is plotted in fig. 6.6.

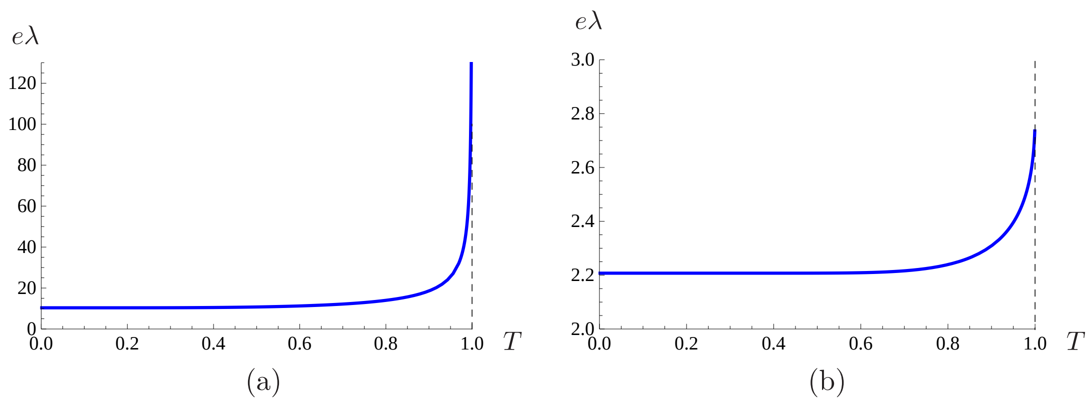

Figure 6.6: Magnetic penetration length as a function of the temperature. (a) Relativistic BCS ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (b) SUSY BCS ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$). Data given in units of T_c.

In both the relativistic and the SUSY BCS theory, the magnetic penetration length is a monotonically increasing function[4].

For the fermionic contribution alone, the magnetic penetration length diverges near the critical temperature. Whereas in the supersymmetric case, it reaches a finite value at T_c and it jumps to infinity for $T > T_c$, which is the expected behavior for a first order phase transition. This behavior is explained basically from the gap dependence. Expanding the f_1 coefficient as a power series of the temperature and the gap, one obtains

$$f_{1\ \mathrm{rBCS}} = a_0 + a_1\Delta_0^2 + a_2(T - T_c) + \dots \ , \qquad f_{1\ \mathrm{sBCS}} = \alpha_0 + \alpha_1(\Delta_0^2 - \Delta_c^2) + \dots \ , \tag{6.85}$$

where one can check that the coefficients shown do not vanish. The ellipsis stands for higher powers of the temperature, taking into account the temperature dependence of the gap, (6.73) or (6.74). Substituting this expansion in (6.23), and using (6.73) or (6.74), one finds the behavior

$$\lambda_{\mathrm{rBCS}} \sim \left(1 - \frac{T}{T_c}\right)^{-1/2} \ , \qquad (\lambda - \lambda_c)_{\mathrm{sBCS}} \sim \left(1 - \frac{T}{T_c}\right)^{\alpha} \ . \tag{6.86}$$

The behavior of the magnetic penetration length at zero temperature can be computed analytically. According to the dependence of the coefficients f_1 and m^{-2} with the cut-off, given in appendix 6.C, and that of the gap, we find the following expressions for the magnetic penetration length,

$$\lambda_{\mathrm{rBCS}} \approx \frac{4g^3}{3\sqrt{3}\pi^2 e}\Lambda^2 \ , \qquad \lambda_{\mathrm{sBCS}} \approx \frac{2\pi}{ec}(1 + 4c^2)^{3/4}\Lambda^{-1} \ , \qquad \text{where } c = \exp\left[-\frac{\pi^2}{g^2\mu^2} - \frac{3}{2}\right] \ . \tag{6.87}$$

The coherence length, ξ, obtained from eq. (6.24), is plotted in fig. 6.7. It is a monotonically increasing function of the temperature for both the relativistic and the SUSY BCS theory. The behavior near the critical temperature is the same as for the magnetic penetration length. To see this, expand the m^{-2} coefficient as we did with the f_1 coefficient. Using the gap equation, one can see that the m^{-2} coefficient has a global Δ_0^2 factor so that the expansions are

$$m^{-2}_{\mathrm{rBCS}} = \Delta_0^2(b_0 + b_1\Delta_0^2 + b_2(T - T_c) + \dots) \ , \qquad m^{-2}_{\mathrm{sBCS}} = \Delta_0^2(\beta_0 + \beta_1(\Delta_0^2 - \Delta_c^2) + \dots) \ . \tag{6.88}$$

Inserting the expansions (6.85) and (6.88) in the expression for the coherence length, (6.24), we find

$$\xi_{\mathrm{rBCS}} \sim \left(1 - \frac{T}{T_c}\right)^{-1/2} \ , \qquad (\xi - \xi_c)_{\mathrm{sBCS}} \sim \left(1 - \frac{T}{T_c}\right)^{\alpha} \ . \tag{6.89}$$

Thus, the coherence length exhibits the same behavior as the magnetic penetration length. However, the behavior of the coherence length at zero temperature is different from that

[4] For high enough values of the chemical potential and the cut-off a counterintuitive non-trivial minimum can appear before reaching the critical temperature, which would mean that there is a range of temperatures where the Meissner effect is enhanced with increasing temperature. This odd behavior can be avoided by restricting the parameter range of validity to not very high values of the chemical potential and the cut-off. However, this restriction is relaxed in the SUSY case where the cut-off dependence is softened.

of the magnetic penetration length. Now, the dependence on the cut-off is

$$\xi_{\rm rBCS} \approx \frac{9\sqrt{6}\pi^4}{4g^4}\Lambda^{-5} \ , \tag{6.90}$$

$$\xi_{\rm sBCS} \approx \frac{g}{2(1+4c^2)^{3/4}}\left(2\pi^2 + g^2\mu^2\left(\frac{32c^4+16c^2+5}{(1+4c^2)^{5/2}} - 2\log\frac{1+\sqrt{1+4c^2}}{2c}\right)\right)^{-1/2} + \mathcal{O}(\Lambda^{-2}) \ . \tag{6.91}$$

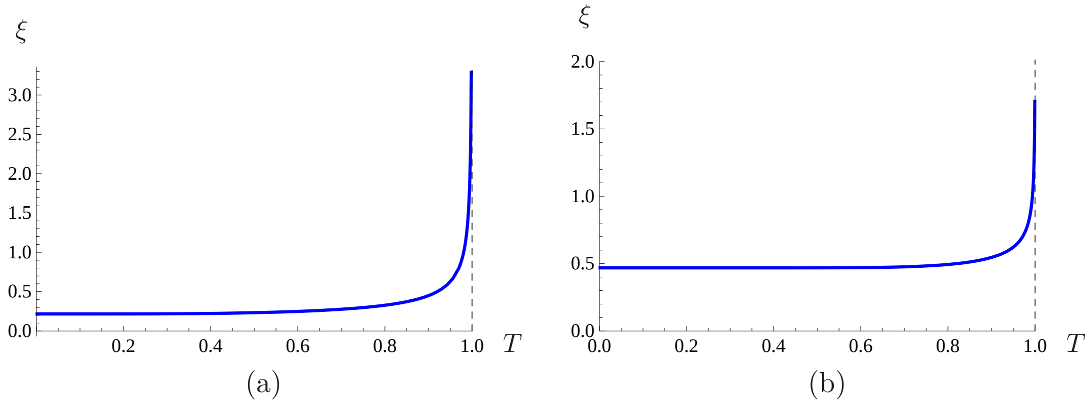

Figure 6.7: Coherence length as a function of the temperature. (a) Relativistic BCS ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (b) SUSY BCS ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$). Data given in units of T_c.

If we take the quotient between these two characteristic lengths,

$$\kappa = \frac{\lambda}{\xi} = \frac{1}{\sqrt{2}e\, m\, f_1 \Delta_0} \ , \tag{6.92}$$

we get the Ginzburg-Landau parameter, shown in fig. 6.8. As the coherence length behaves in the same way as the magnetic penetration length near the phase transition, at leading order, the GL parameter will take a finite constant value. Depending on the value of the GL parameter one has a type I ($\kappa \ll 1$) or a type II superconductor ($\kappa \gg 1$). In the GL theory κ is defined near the critical temperature and the critical value differentiating between the two types of superconductor is $\kappa = 1/\sqrt{2}$. As shown in fig. 6.8, $\kappa \gg 1$, since we are considering small values of gauge coupling e, then the superconductors are type II in both the rBCS and the sBCS cases.

6.3.4 Critical magnetic fields

For a type I superconductor, the critical magnetic field is obtained by equating the energy per unit volume, associated with holding the field out against the magnetic pressure, with the condensation energy. That is eq. (6.1),

$$\frac{1}{2}B_c^2(T) = V_n(T) - V_s(T) \ .$$

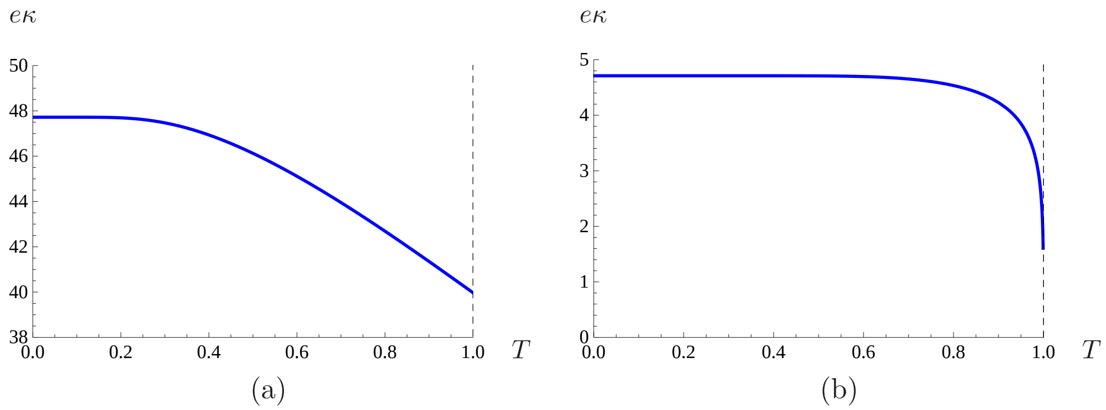

Figure 6.8: Ginzburg-Landau parameter as a function of temperature. (a) Relativistic BCS ($\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$). (b) SUSY BCS ($\mu = 0.65$, $g = 3.9$, $\Lambda = 52$). Data given in units of T_c.

The behavior near the critical temperature is found by preforming expansions similar to those for the f_1 and m^{-2} coefficients. In the rBCS theory the gap is expanded around $\Delta_0 = 0$. On the other hand, when the scalar is considered, B_c does not make sense above T_{c2}, since the superconducting minimum in the effective potential becomes metastable, but we can perform the expansion around $\Delta_{c2} \equiv \Delta_0(T_{c2})$. According to the dependence of the gap with the temperature (6.73) in the rBCS theory and due to the fact that Δ_0 is linear with the temperature near T_{c2} in the sBCS theory, we have

$$\begin{aligned} \frac{1}{2}B^2_{c\ \text{rBCS}} &= \partial_T\partial_\varepsilon(V_n(T) - V_s(T))\Big|_{\substack{T=T_c\\ \Delta_0=0}}(T-T_c)\Delta_0^2 \\ &\quad + \frac{1}{2}\partial_\varepsilon^2(V_n(T) - V_s(T))\Big|_{\substack{T=T_c\\ \Delta_0=0}}\Delta_0^4 + \dots \end{aligned} \tag{6.93}$$

$$\frac{1}{2}B^2_{c\ \text{sBCS}} = \partial_T(V_n(T) - V_s(T))\Big|_{\substack{T=T_{c2}\\ \Delta_0=\Delta_{c2}}}(T-T_{c2}) + \dots \tag{6.94}$$

from which we find a linear and square root behavior near the critical temperatures,

$$B_{c\ \text{rBCS}} \sim \left(1 - \frac{T}{T_c}\right) , \qquad B_{c\ \text{sBCS}} \sim \sqrt{1 - \frac{T}{T_{c2}}} , \tag{6.95}$$

as shown in fig. 6.9 (a) and (b), respectively. The expressions for the critical magnetic field at zero temperature are

$$B_{c\ \text{rBCS}} \approx \frac{\sqrt{2}g}{6\pi^2}\Lambda^3 , \qquad B_{c\ \text{sBCS}} \approx \sqrt{2}\sqrt{-1 - \frac{g^2\mu^2}{\pi^2}\left(\frac{1}{\sqrt{1+4c^2}} - \log\frac{1+\sqrt{1+4c^2}}{2c}\right)}\frac{c}{g}\Lambda . \tag{6.96}$$

As expected the dependence on the cut-off is milder in the supersymmetric case.

As explained at the end of sec. 6.1.2, for type II superconductors, which are characterized by the appearance of Abrikosov vortices in a mixed superconductor-normal state, there are two critical magnetic fields, B_{c1} and B_{c2},

$$B_{c1} \approx \frac{\phi_0}{2\pi\lambda^2} , \qquad\qquad B_{c2} \approx \frac{\phi_0}{2\pi\xi^2} .$$

These are plotted in fig. 6.9 together with B_c. It is easy to obtain the behavior of these two critical magnetic fields in the different regimes using the expressions for the magnetic penetration length and the coherence length.

If $B_c \ll B_{c2}$, we will be able to see the intermediate vortex state as we decrease the applied magnetic field, i.e. the superconductor is type II. On the contrary if $B_c \gg B_{c2}$, we reach the pure superconducting state without the formation of any vortex, and we have type I superconductivity. According to fig. 6.9, we have type II superconductivity in both the relativistic and the supersymmetric BCS theory, in agreement with the prediction obtained in the previous section by computing the GL parameter.

Given that B_c ends at T_{c2}, instead of T_{c3} as B_{c1} and B_{c2}, and due to the fact that there is a range of parameters where $B_{c2} < B_c$ at zero temperature, one can find a crossing between the two magnetic fields and a crossover between type I and type II behavior as we increase the temperature. However, this crossing effect disappears if the gauge coupling is sufficiently small.

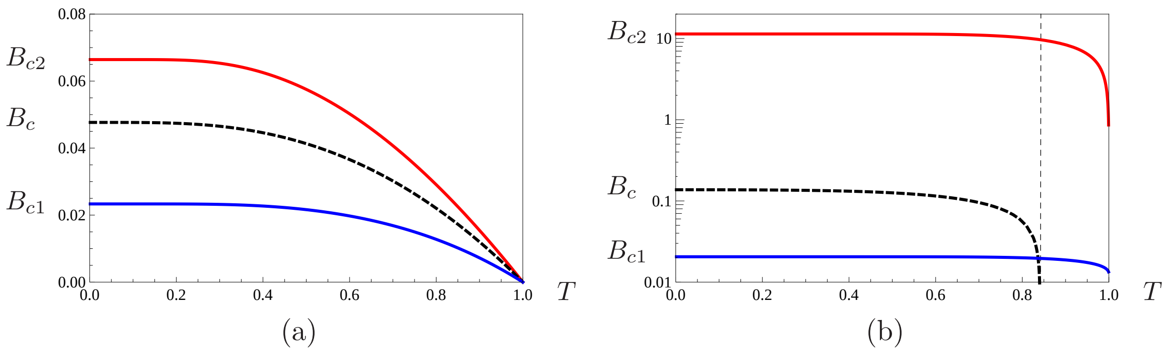

Figure 6.9: Critical magnetic fields B_c (black), B_{c1} (blue) and B_{c2} (red) as a function of the temperature. (a) Relativistic BCS ($e = 0.2$, $\mu = 0.0065$, $g = 0.54$, $\Lambda = 8.4$), for clarity B_{c1} and B_{c2} have been rescaled by a factor 25 and 0.00125 respectively). (b) SUSY BCS ($e = 0.2$, $\mu = 0.65$, $g = 3.9$, $\Lambda = 52$). Data given in units of T_c.

6.4 Comments

After reviewing the main features of superconductivity and the BCS paradigm, we have shown how to engineer a supersymmetric model realizing these features by means of a non-canonical Kähler potential of the form

$$K = X^\dagger X + g^2 (X^\dagger X)^2 , \tag{6.97}$$

with no superpotential and with a chemical potential coupled to the $U(1)_R$ current, necessary to produce a Fermi surface without the presence of Bose-Einstein condensation of the scalar superpartners.

The salient aspects of this model are that the equations determining the temperature dependence of the gap are very similar to those of BCS theory with the main difference represented by the contribution coming from scalar fluctuations. One important effect of this contribution is the change in the character of the phase transition from second to first-order. The main differences between standard relativistic BCS results and those from the supersymmetric model lie in the different orders of the phase transition. For example, the infinite jumps found in the supersymmetric model for the specific heat, magnetic penetration length and coherence length are those characteristic of a first order phase transition. Another effect due to the scalar superpartner is a drastic reduction of the dependence on the UV cutoff in all quantities we have computed.

As it was already pointed out, our aim here was to design a simple supersymmetric model of superconductivity. Therefore, there are various ways in which one would desire to extend the model presented here:

- One thing one would like to implement is the addition of mass. In section 6.2.1 we discussed a model with a superpotential giving mass to fermions and hence, to scalars, but it led to Bose-Einstein condensation of the later, spoiling in this way the BCS mechanism.

- This difficulty in dealing with superpotential terms is also reproduced when trying to realize the BCS mechanism in models with canonical Kähler potential when all the fields are dynamical. In this type of models fermion condensation needs to be triggered by interactions contained in a superpotential W.

  It is easy to implement the BCS mechanism if not all the fields are dynamical. In [16] different models were considered by starting from the previous one with non-canonical Kähler potential (6.97) and integrating in some fields. For example, the model
  $$K = X^\dagger X + Y^\dagger Y \ , \qquad W = mZ(X - gY^2) \ , \tag{6.98}$$
  sets $X = gY^2$ after integrating out the non-dynamical chiral superfield Z. Substituting this relation into the Kähler potential, one recovers the $K = Y^\dagger Y + g^2(Y^\dagger Y)^2$ model.

  If we consider that all the superfields X, Y and Z are dynamical and they have canonical Kähler potential, things are not so easy as we will see in a moment. For the models considered in [16], the functional integral over fermions can be explicitly carried out, since the Lagrangians already are quadratic in fermions, while, as usual, we have to expand scalar fields around their vacuum expectation values, e.g. $\phi_x = v_x + \varphi_x$, $\phi_y = v_y + \varphi_y$ and $\phi_z = v_z + \varphi_z$. In these models, the chiral superfield X represents the supersymmetric analog of the Hubbard-Stratonovich field, Δ. We summarize here the results of the models studied in [16]:

  - $W = m(X - gY^2)^2$
    This model has only $U(1)_R$ symmetry with $(q_X, q_Y) = (1, 1/2)$. Then the chemical potential is introduced for this $U(1)_R$. Like in the model of section 6.2.1 with baryonic $U(1)_B$ chemical potential, scalar particles are charged. As a result on $\mathbb{R}^4$ we cannot have Fermi surfaces because they would overlap with

regions of Bose-Einstein condensation. The theory on three sphere produces analogue negative results to those in section 6.2.1.

– $W = mZ(X - gY^2) + MZ^2$
This is a renormalizable model and can be viewed as a UV completion of the previous case. It has a $U(1)_R$ symmetry with charge assignation $(q_X, q_Y, q_Z) = (1, 1/2, 1)$ and a similar IR physics to the previous model, with M playing the role of the UV cutoff Λ.

– $W = mZ(X - gY^2)$
This model has a baryonic $U(1)_B$ as well as $U(1)_R$ symmetry. If chemical potential is introduced for $U(1)_B$, again we find that Fermi surfaces cannot be separated from the region of Bose-Einstein condensation (irrespective of the R-charge assignation).
Consider now a chemical potential coupled to the $U(1)_R$ current. One can assign charges $(q_X, q_Y, q_Z) = (2 - q, 1 - q/2, q)$. Unlike the model of section 6.2.2, now it is not possible to have only neutral scalars. Nonetheless, for $q = 2$, i.e. when $(q_X, q_Y, q_Z) = (0, 0, 2)$, in the unbroken phase $v_x = v_y = v_z = 0$, it is possible to have a Fermi surface without Bose-Einstein condensation even in flat space. There is a Fermi surface at $p_F = \mu$. The scalar frequencies are

$$\omega_S = \{\sqrt{p^2 + m^2} \pm 2\mu,\ \sqrt{p^2 + m^2},\ \sqrt{p^2 + m^2},\ p,\ p\}\ , \tag{6.99}$$

which are always positive definite for $\mu < m/\sqrt{2}$. The problem is that Bose-Einstein condensation reappears in an infinitesimal neighborhood of $v_x = v_y = v_z = 0$. After turning on v_z and v_x, the last frequency in (6.99) becomes

$$\omega_S = \sqrt{p^2 + 4g^2m^2v_z^2 - 2gm^2v_x}\ ,$$

which becomes complex at low momenta in the region $v_x > 2gv_z^2$. Thus the model is not protected from Bose-Einstein condensation.
It would be interesting to further explore the possibility of introducing masses and the existence of models with canonical Kähler potential despite the difficulties presented here.

- Since one of the motivations for building a supersymmetric model of superconductivity is high T_c superconductivity, it would be very interesting to generalize the present model to d-wave superconductivity, which is substantial for high T_c superconductors [187]. To this purpose we would need to consider terms like $\bar{\psi}_c\psi$ and $\bar{\psi}_c\gamma^\mu\gamma^5\psi$ in the Lagrangian.

- In addition, it would be worth considering extensions of the present supersymmetric model to describe supersymmetric color superconductivity and compare the resulting dynamics with the standard phenomenology of QCD color superconductivity [188–190].

- Finally, we would like to see if there are realistic condensed matter systems with an effective quasi supersymmetric dynamics with thermodynamic and response properties similar to those presented in section 6.3, summarized in the different figures.

6.A The effective potential at finite temperature and density

In this appendix, we review some basic concepts of thermal field theory and what are the steps to obtain the effective potential, containing the thermal and the Coleman-Weinberg contributions.

To study the statistical behavior of a quantum system in thermal equilibrium one defines the density matrix

$$\rho(\beta) = e^{-\beta\mathcal{H}} , \tag{6.100}$$

where $\mathcal{H}$ is the appropriate Hamiltonian for the corresponding ensemble.

If our system possesses a certain global symmetry, let us consider for definiteness a $U(1)$ symmetry like in the main text, by the standard Noether procedure we can compute the associated conserved current, J_μ, whose temporal component defines a conserved charge,

$$q = \int \mathrm{d}^3x J_0(t,x) , \tag{6.101}$$

to which we can associate a chemical potential, μ.

We will work at fixed chemical potential μ, therefore it is convenient to work in the grand canonical ensemble, where $\mathcal{H} = H - \mu q$, and H is the dynamical Hamiltonian. The partition function of the system is defined from this density matrix

$$Z = \mathrm{Tr}\ \rho(\beta) = \mathrm{Tr}\ e^{-\beta(H-\mu q)} , \tag{6.102}$$

where Tr represents the sum over expectation values in a complete basis and β is the usual notation for the inverse of the temperature.

Working out the path integral representation of this partition function, one sees that (let us set $\mu = 0$ for the moment) it is analogue to a transition amplitude between states separated by an imaginary time $\Delta\tau = \beta$ (with $t = -i\tau$) and with periodic boundary conditions

$$\phi(\tau,x) = \phi(\tau+\beta,x) \tag{6.103}$$

for bosonic degrees of freedom and antiperiodic boundary conditions

$$\psi(\tau,x) = -\psi(\tau+\beta,x) \tag{6.104}$$

for fermions.

Turning on the operator $-\mu q$ and adding it to the Hamiltonian H is equivalent, at the level of the action, to add the same terms we would obtain by gauging the $U(1)$ symmetry with a non-dynamical gauge field with only non-vanishing temporal component,

$$A_\nu = (\mu,0,0,0) , \tag{6.105}$$

i.e. to account for the addition of the chemical potential we just have to replace normal derivatives with covariant derivatives $D_\nu = \partial_\nu - iq\mu\delta_{\nu 0}$.

Therefore, the whole prescription to build the partition function at finite temperature and finite chemical potential in the path integral formalism is:

- First add the chemical potential μ by considering covariant derivatives $D_\nu = \partial_\nu - iq\mu\delta_{\nu 0}$ in your Lagrangian.

- Then, to consider the system at a finite temperature, $T = 1/\beta$, go to Euclidean space by performing the Wick rotation to imaginary time, $\mathcal{L}_E(\tau) = -\mathcal{L}(t = -i\tau)$.
- Restrict the time to the interval $\tau \in (0, \beta)$.
- Require (anti)periodicity over τ for bosons (fermions).

This can be summarized in the formula

$$Z = \int_{\phi(0)\,=\,\phi(\beta)} \mathcal{D}\phi\mathcal{D}\phi^* \quad \int_{\psi(0)\,=\,-\psi(\beta)} \mathcal{D}\psi\mathcal{D}\psi^\dagger \quad \exp\left[-\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_E\left(\phi, \phi^*, \psi, \psi^\dagger; \partial_\mu \to D_\mu\right)\right] . \tag{6.106}$$

Once we have the partition function describing the system at finite temperature and finite density, it is desirable to integrate the fields to get an effective potential. We only now how to perform Gaussian path integrals, hence we should manipulate $\mathcal{L}_E$ in (6.106) in case it was not quadratic in the fields, in order to get a Gaussian integral. For a Lagrangian made of scalar fields, one usually expands the scalars up to quadratic order in fluctuations around their vacuum expectation values; whereas for fermions one has to carry out tricks like the Hubbard-Stratonovich transformation explained in the main text, which provides a quadratic Lagrangian by means of the introduction of extra non-dynamical auxiliary fields.

As an example, consider the Lagrangian for BCS theory (6.12) after the previous manipulations are carried out. We rewrite it here for the reader comfortability

$$\begin{aligned}\mathcal{L}_E = \frac{1}{2}\left(\psi^\dagger\partial_\tau\psi - \partial_\tau\psi^\dagger\psi\right) - \frac{i}{2}\left(\bar\psi\gamma^i\partial_i\psi - \partial_i\bar\psi\gamma^i\psi\right) + m\bar\psi\psi - \mu\psi^\dagger\psi \\ + g^2|\Delta|^2 - g^2\left[\Delta^\dagger(\bar\psi_c\gamma_5\psi) + \Delta(\bar\psi_c\gamma_5\psi)^\dagger\right] .\end{aligned} \tag{6.107}$$

It is clear that the classical part of the effective potential (6.13) is given by the non-dynamical term

$$V_{\mathrm{cl}} = g^2|\Delta|^2 . \tag{6.108}$$

The remaining terms in (6.13), i.e the thermal potential and the Coleman-Weinberg potential, are obtained after integration. At this point, it is convenient to go to momentum space, we take the Fourier transform

$$\psi(\tau, x) = \frac{\mathrm{vol}}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\psi(\omega_n, p)e^{i\omega_n\tau + ip\cdot x} , \tag{6.109}$$

$$\psi_c(\tau, x) = \frac{\mathrm{vol}}{\beta}\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\psi_c(\omega_n, p)e^{-i\omega_n\tau - ip\cdot x} , \tag{6.110}$$

where vol is a factor regularizing the volume of the four-dimensional coordinate space. Due to anti-periodic boundary conditions (6.104), the Matsubara frequencies, ω_n, run over the values

$$\text{Fermions:} \qquad \omega_n = \frac{\pi(2n+1)}{\beta} . \tag{6.111}$$

Now, making use of the identities

$$\bar{\psi}\gamma^\mu\partial_\mu\psi = \bar{\psi}_c\gamma^\mu\partial_\mu\psi_c \ , \qquad \bar{\psi}\psi = \bar{\psi}_c\psi_c \ , \qquad \bar{\psi}\gamma^0\psi = -\bar{\psi}_c\gamma^0\psi_c \ , \qquad (\bar{\psi}_c\gamma^5\psi)^\dagger = -\bar{\psi}\gamma_5\psi_c \ , \tag{6.112}$$

we can write the quadratic Lagrangian (6.107), in momentum space, in the form

$$\int \mathrm{d}\tau \int \mathrm{d}^3x\, \mathcal{L}_E = \frac{\mathrm{vol}}{\beta}\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\bar{\Psi}O\Psi \ , \tag{6.113}$$

where

$$\bar{\Psi} = \left(\bar{\psi}, \bar{\psi}_c\right) \ , \qquad \Psi = \begin{pmatrix} \psi \\ \psi_c \end{pmatrix} \ ,$$
$$O = \frac{1}{2}\begin{pmatrix} (i\omega_n - \mu)\gamma^0 + p_i\gamma^i + m & 2g^2\Delta\gamma^5 \\ -2g^2\Delta^\dagger\gamma^5 & (i\omega_n + \mu)\gamma^0 + p_i\gamma^i + m \end{pmatrix} \ . \tag{6.114}$$

In general, the determinant of the O-matrix can be written as

$$\det O = \prod_i (\omega_n^2 + \omega_i^2) \ , \tag{6.115}$$

up to an innocuous constant. In the present case we have[5]

$$\det O = (\omega_n^2 + \omega_+^2)(\omega_n^2 + \omega_-^2) \ , \tag{6.116}$$

where the eigenvalues obtained in this way are exactly those appearing in (6.14),

$$\omega_\pm^2 = \left(\sqrt{p^2 + m^2} \pm \mu\right)^2 + 4g^4|\Delta|^2 \ . \tag{6.117}$$

We know that the Gaussian integral over Grassmann variables gives

$$\begin{aligned} \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\exp\left[-\frac{\mathrm{vol}}{\beta}\sum_n\int\frac{\mathrm{d}^3p}{(2\pi)^3}\bar{\Psi}O\Psi\right] &= \prod_{p,\,\omega_n}\sqrt{\det O} \\ &= \exp\left[\frac{\mathrm{vol}}{2\beta}\sum_i\sum_{n=-\infty}^{\infty}\int\frac{\mathrm{d}^3p}{(2\pi)^3}\log(\omega_n^2 + \omega_i^2)\right] \ . \end{aligned} \tag{6.118}$$

It only remains to perform the sum over the Matsubara modes,

$$f(\omega_i) = \frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\log\left(\frac{\pi^2}{\beta^2}(2n+1)^2 + \omega_i^2\right) = \frac{1}{\beta}\log\cosh\left(\frac{\beta}{2}\omega_i\right) \ , \tag{6.119}$$

which can be obtained, up to a constant, by taking the derivative $\partial_{\omega_i} f$, doing the sum there and integrating back to obtain the result shown. Finally, rearranging the expression (6.119) and substituting

$$\mathrm{vol} = \int \mathrm{d}\tau \int \mathrm{d}^3x \ , \tag{6.120}$$

[5] O is an 8×8 matrix and, hence, (6.116) actually is the square root of the determinant, otherwise we would be overcounting the number of fermionic degrees of freedom. We must have as many powers of ω_n as degrees of freedom.

the partition function becomes

$$Z = \exp\left[-\int \mathrm{d}\tau \int \mathrm{d}^3x \left(V_{\mathrm{cl}} + V_{\mathrm{CW}} + V_{\mathrm{thermal}}\right)\right] \tag{6.121}$$

where we identify the effective potential, given by the sum of the following expressions

$$V_{\mathrm{cl}} = g^2|\Delta|^2 \ , \tag{6.122}$$

$$V_{\mathrm{CW}} = -\frac{1}{2}\sum_{i=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\omega_i \ , \tag{6.123}$$

$$V_{\mathrm{thermal}} = -\frac{1}{\beta}\sum_{i=\pm}\int \frac{\mathrm{d}^3p}{(2\pi)^3}\log\left(1 + e^{-\beta\omega_i}\right) \ . \tag{6.124}$$

The Lagrangian (6.107) does not include any dynamical scalar field, however, in case it had, expressions would be similar, with the differences:

- The Matsubara frequencies for bosons are

$$\text{Bosons:} \qquad \omega_n = \frac{2\pi n}{\beta} \ , \tag{6.125}$$

  due to periodic boundary conditions (6.103).

- The Gaussian path integral for complex scalar fields is

$$\int \mathcal{D}\phi\mathcal{D}\phi^* \exp\left[-\frac{\mathrm{vol}}{\beta}\sum_n \int \frac{\mathrm{d}^3p}{(2\pi)^3}\Phi^\dagger O \Phi\right] = \prod_{p,\,\omega_n}\frac{1}{\sqrt{\det O}} \ , \quad \text{with} \quad \Phi = \begin{pmatrix}\phi \\ \phi^*\end{pmatrix} \ . \tag{6.126}$$

- And the Matsubara sums are then

$$\frac{1}{2\beta}\sum_{n=-\infty}^{\infty}\log\left(\frac{4\pi^2n^2}{\beta^2} + \omega_i^2\right) = \frac{1}{\beta}\log\sinh\left(\frac{\beta}{2}\omega_i\right) \ . \tag{6.127}$$

Therefore, if we can write the Lagrangian in the quadratic form (6.113) and its determinant can be expressed in the form (6.115), then the general prescription to obtain the effective potential is

$$V_{\mathrm{eff}} = V_{\mathrm{cl}} + \frac{1}{2}\sum_i \int \frac{\mathrm{d}^3p}{(2\pi)^3}\eta_i\omega_i + \frac{1}{\beta}\sum_i \int \frac{\mathrm{d}^3p}{(2\pi)^3}\eta_i \log\left(1 - \eta_i e^{-\beta\omega_i}\right) \ , \tag{6.128}$$

where the sum over i accounts for fermionic ($\eta_i = -1$) and scalar ($\eta_i = 1$) degrees of freedom.

6.B Matrix elements

In this section we show the remaining matrix elements and the corresponding eigenvalues used in the main text.

Matrices for chemical potential coupled to a $U(1)_B$ current

Consider now the expressions (6.35) and (6.36) describing the quadratic terms in the fluctuation Lagrangian when the chemical potential is coupled to a $U(1)_B$ current. In a completely analogous way to the previous case, we can write these Lagrangians in momentum space in the matrix form

$$\mathcal{L}_S = \Phi^\dagger O_S \Phi \ , \qquad \mathcal{L}_F = \Psi^\dagger O_F \Psi \ ,$$
$$\Phi^\dagger = \left(\phi_x^*, \phi_y^*, \phi_x, \phi_y\right) \ , \qquad \Psi^\dagger = \left(\psi_{x1}^\dagger, \psi_{x2}^\dagger, \psi_{y1}^\dagger, \psi_{y2}^\dagger, \psi_{x1}, \psi_{x2}, \psi_{y1}, \psi_{y2}\right) \ ,$$

where now, either ψ_x or ψ_y represents a two component Weyl spinor. When the vacuum expectation value of the scalar fields is zero, $v_x = v_y = 0$, the scalar matrix, O_S, takes the diagonal form

$$O_S = \begin{pmatrix} (\omega+i\mu)^2+p^2+m^2+4g^4\Delta_x^2 & & & 0 \\ & (\omega-i\mu)^2+p^2+m^2+4g^4\Delta_x^2 & & \\ & & (\omega-i\mu)^2+p^2+m^2+4g^4\Delta_x^2 & \\ 0 & & & (\omega+i\mu)^2+p^2+m^2+4g^4\Delta_x^2 \end{pmatrix} \tag{6.129}$$

and the fermionic matrix, O_F, is given by

$$O_F = \begin{pmatrix} A_+ & B \\ -B & A_- \end{pmatrix} \ , \tag{6.130}$$

$$A_\pm = \begin{pmatrix} \frac{1}{2}(i\omega-p\mp\mu) & 0 & 0 & 0 \\ 0 & \frac{1}{2}(i\omega+p\mp\mu) & 0 & 0 \\ 0 & 0 & \frac{1}{2}(i\omega-p\pm\mu) & 0 \\ 0 & 0 & 0 & \frac{1}{2}(i\omega+p\pm\mu) \end{pmatrix} \ , \tag{6.131}$$

$$B = \begin{pmatrix} 0 & -g^2\Delta_x & 0 & \frac{m}{2} \\ g^2\Delta_x & 0 & -\frac{m}{2} & 0 \\ 0 & \frac{m}{2} & 0 & -g^2\Delta_y \\ -\frac{m}{2} & 0 & g^2\Delta_y & 0 \end{pmatrix} \ . \tag{6.132}$$

We omit the (long) general expressions with $v_x, v_y \neq 0$ as these are not used in the discussion. Writing the determinants of these matrices in an analogous way to (6.115),

$$\det O_S = \prod_{i=1}^{4}(\omega^2+\omega_{Si}^2) \ , \qquad \det O_F = \prod_{i=1}^{4}(\omega^2+\omega_{Fi}^2) \ , \tag{6.133}$$

we find the following eigenvalues

$$\begin{aligned} \omega_{S\ 1,2} &= \sqrt{4g^4\Delta_x^2+m^2+p^2} \pm \mu \ , \\ \omega_{S\ 3,4} &= \sqrt{4g^4\Delta_y^2+m^2+p^2} \pm \mu \ , \end{aligned} \tag{6.134}$$

$$\begin{aligned} \omega_{F\ 1,2}^2 &= 2g^4\Delta_x^2+2g^4\Delta_y^2+\mu^2+m^2+p^2 \pm \mathcal{E}_+, \\ \omega_{F\ 3,4}^2 &= 2g^4\Delta_x^2+2g^4\Delta_y^2+\mu^2+m^2+p^2 \pm \mathcal{E}_-, \end{aligned} \tag{6.135}$$

$$\mathcal{E}_\pm = 2\sqrt{\mu^2\left(m^2+p^2\right)+g^8\left(\Delta_x^2-\Delta_y^2\right)^2+g^4\left(m^2\left(\Delta_x+\Delta_y\right)^2 \pm 2\mu p\left(\Delta_x^2-\Delta_y^2\right)\right)} \ . \tag{6.136}$$

Matrices for chemical potential coupled to a $U(1)_R$ current

Now we turn to the case in which the chemical potential couples to a $U(1)_R$ symmetry, corresponding to Lagrangians (6.58) and (6.59). If we also introduce a baryonic $U(1)_B$ gauge field (see Lagrangian (6.66)) and consider spatial fluctuations of the gap, $\Delta = \Delta_0 + \bar{\Delta}(\vec{x})$, after the splitting $O(\Delta, A) = O_0(\Delta_0) + \delta O(\bar{\Delta}, A)$ the scalar and fermionic matrices are given in momentum space by

$$O_{S0} = \begin{pmatrix} \frac{1}{2}\left(\omega^2+p^2\right)+2g^4\Delta_0^2 & 0 \\ 0 & \frac{1}{2}\left(\omega^2+p^2\right)+2g^4\Delta_0^2 \end{pmatrix} ,$$

$$\delta O_S = \begin{pmatrix} \begin{matrix} e(A_\tau\omega+\vec{A}\cdot\vec{p})+\frac{1}{2}e^2\left(A_\tau^2+\vec{A}^2\right) \\ +2g^4\left(2\bar{\Delta}\Delta_0+\bar{\Delta}^2\right)\end{matrix} & 0 \\ 0 & \begin{matrix} -e(A_\tau\omega+\vec{A}\cdot\vec{p})+\frac{1}{2}e^2\left(A_\tau^2+\vec{A}^2\right) \\ +2g^4\left(2\bar{\Delta}\Delta_0+\bar{\Delta}^2\right)\end{matrix} \end{pmatrix} ,$$

$$O_{F0} = \begin{pmatrix} \frac{i}{2}\omega+\frac{1}{2}p+\frac{1}{2}\mu & 0 & 0 & -g^2\Delta_0 \\ 0 & \frac{i}{2}\omega-\frac{1}{2}p+\frac{1}{2}\mu & g^2\Delta_0 & 0 \\ 0 & g^2\Delta_0 & \frac{i}{2}\omega+\frac{1}{2}p-\frac{1}{2}\mu & 0 \\ -g^2\Delta_0 & 0 & 0 & \frac{i}{2}\omega-\frac{1}{2}p-\frac{1}{2}\mu \end{pmatrix} ,$$

$$\delta O_F = \begin{pmatrix} -\frac{i}{2}eA_\tau-\frac{1}{2}eA & 0 & 0 & -g^2\bar{\Delta} \\ 0 & -\frac{i}{2}eA_\tau+\frac{1}{2}eA & g^2\bar{\Delta} & 0 \\ 0 & g^2\bar{\Delta} & \frac{i}{2}eA_\tau+\frac{1}{2}eA & 0 \\ -g^2\bar{\Delta} & 0 & 0 & \frac{i}{2}eA_\tau-\frac{1}{2}eA \end{pmatrix} . \tag{6.137}$$

If instead of introducing a baryonic $U(1)_B$ gauge field, we had introduced $U(1)_R$ gauge field, the scalar and fermionic matrices would have been the following ones:

$$\delta O_S = \begin{pmatrix} 2g^4\left(2\bar{\Delta}\Delta_0+\bar{\Delta}^2\right) & 0 \\ 0 & 2g^4\left(2\bar{\Delta}\Delta_0+\bar{\Delta}^2\right) \end{pmatrix} ,$$

$$\delta O_F = \begin{pmatrix} \frac{i}{2}eA_\tau+\frac{1}{2}eA & 0 & 0 & -g^2\bar{\Delta} \\ 0 & \frac{i}{2}eA_\tau-\frac{1}{2}eA & g^2\bar{\Delta} & 0 \\ 0 & g^2\bar{\Delta} & -\frac{i}{2}eA_\tau-\frac{1}{2}eA & 0 \\ -g^2\bar{\Delta} & 0 & 0 & -\frac{i}{2}eA_\tau+\frac{1}{2}eA \end{pmatrix} . \tag{6.138}$$

The energy eigenvalues computed for the O_{S0} and O_{F0} matrices are

$$\omega_{S\ 1,2} = \sqrt{p^2+4g^4\Delta_0^2} , \qquad \omega_{F\pm} = \sqrt{(p\pm\mu)^2+4g^4\Delta_0^2} . \tag{6.139}$$

6.C m^{-2}, f_1 and f_3 coefficients

In momentum space, the m^{-2} and f_1 terms in (6.21) are given by

$$\int \mathrm{d}^3x\, m^{-2}\bar{\Delta}\bar{\Delta}^* = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, m^{-2}\bar{\Delta}^*(\vec{k})\bar{\Delta}(\vec{k}) , \tag{6.140}$$

$$\int \mathrm{d}^3x\, f_1\partial^i\Delta\,\partial^i\Delta^* = \int \frac{\mathrm{d}^3k}{(2\pi)^3}\, f_1\vec{k}^2\bar{\Delta}^*(\vec{k})\bar{\Delta}(\vec{k}) , \tag{6.141}$$

where we have considered time independent perturbations. Thus, we have to find in (6.16) a term quadratic in $\bar{\Delta}$ and expand its coefficient up to quadratic order in momentum. The zero order term will correspond to m^{-2} and the coefficient of the quadratic term in momentum will be identified with f_1. Terms quadratic in $\bar{\Delta}$ are found in Ω_2 and, if scalars are considered, in Ω_1. The Ω_1 term only contributes to the m^{-2} coefficient, which in momentum space becomes

$$\frac{1}{2\beta}\mathrm{Tr}\left[O_{S0}^{-1}\delta O_S\right] = \frac{1}{2\beta^2}\int \mathrm{d}^4x \sum_n \int \frac{\mathrm{d}^3K}{(2\pi)^3}\mathrm{tr}\left[O_{S0}^{-1}(\omega_n,\vec{K})\delta O_S(\vec{x})\right] \ , \tag{6.142}$$

where we have to sum over Matsubara frequencies, $\omega_n = 2n\pi/\beta$ for bosonic frequencies and $\omega_n = (2n+1)\pi/\beta$ for fermionic ones. Once the Matsubara sums are done, we have to consider the piece quadratic in $\bar{\Delta}$ (supposing $\bar{\Delta}$ to be real)

$$\begin{aligned}\frac{1}{2\beta}\mathrm{Tr}\left[O_{S0}^{-1}\delta O_S\right]_{\bar{\Delta}\bar{\Delta}} &= \frac{1}{\beta^2}\int \mathrm{d}^4x \int \frac{\mathrm{d}^3K}{(2\pi)^3}\mathfrak{B}_{\bar{\Delta}\bar{\Delta}}(\vec{K})\bar{\Delta}(\vec{x})\bar{\Delta}(\vec{x}) \\ &= \frac{1}{\beta}\int \frac{\mathrm{d}^3K}{(2\pi)^3}\mathfrak{B}_{\bar{\Delta}\bar{\Delta}}(\vec{K})\int \frac{\mathrm{d}^3k}{(2\pi)^3}\bar{\Delta}^*(\vec{k})\bar{\Delta}(\vec{k}) \ .\end{aligned} \tag{6.143}$$

Hence, we identify the first contribution to m^{-2} as

$$m^{-2} = \frac{1}{\beta}\int \frac{\mathrm{d}^3K}{(2\pi)^3}\mathfrak{B}_{\bar{\Delta}\bar{\Delta}}(\vec{K}) + \dots \tag{6.144}$$

Let us elaborate now on the Ω_2 contribution,

$$\frac{1}{4\beta}\mathrm{Tr}[(O_{F0}^{-1}\delta O_F)^2] = \frac{1}{4\beta}\int \mathrm{d}^4x_1 \int \mathrm{d}^4x_2\, \mathrm{tr}[\delta O_F(\vec{x}_1)O_{F0}^{-1}(x_1,x_2)\delta O_F(\vec{x}_2)O_{F0}^{-1}(x_2,x_1)] \ , \tag{6.145}$$

plus the analogous scalar term if one considers the supersymmetric case, and the extra term $\int \mathrm{d}^3x\, g^2\bar{\Delta}^2$ for m^{-2}. The previous expression in momentum space is

$$\begin{aligned}\frac{1}{4\beta}\mathrm{Tr}[(O_{F0}^{-1}\delta O_F)^2] &= \frac{1}{4\beta^3}\int \mathrm{d}^4x_1\mathrm{d}^4x_2 \sum_{m,n}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\frac{\mathrm{d}^3q_1}{(2\pi)^3}\frac{\mathrm{d}^3q_2}{(2\pi)^3} \\ &\qquad \times e^{-i(\omega_m-\omega_n)(\tau_1-\tau_2)}e^{-i(\vec{k}_1-\vec{k}_2+\vec{q}_1)\vec{x}_1}e^{-i(-\vec{k}_1+\vec{k}_2+\vec{q}_2)\vec{x}_2} \\ &\qquad \times \mathrm{tr}[\delta O_F(\vec{q}_1)O_{F0}^{-1}(\omega_m,\vec{k}_1)\delta O_F(\vec{q}_2)O_{F0}^{-1}(\omega_n,\vec{k}_2)] \\ &= \frac{1}{4\beta}\sum_n \int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\mathrm{tr}[\delta O_F(\vec{k}_2-\vec{k}_1)O_{F0}^{-1}(\omega_n,\vec{k}_1)\delta O_F(\vec{k}_1-\vec{k}_2)O_{F0}^{-1}(\omega_n,\vec{k}_2)] \\ &\equiv \frac{1}{\beta}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\mathcal{F}(\vec{k}_1,\vec{k}_2) \ .\end{aligned} \tag{6.146}$$

Taking the piece quadratic in $\bar{\Delta}$ in (6.146), we have

$$\Omega_2|_{\bar{\Delta}\bar{\Delta}} = \frac{1}{\beta}\int \frac{\mathrm{d}^3k_1}{(2\pi)^3}\frac{\mathrm{d}^3k_2}{(2\pi)^3}\,\bar{\Delta}^*(\vec{k}_2-\vec{k}_1)\bar{\Delta}(\vec{k}_2-\vec{k}_1)\mathcal{F}_{\bar{\Delta}\bar{\Delta}}(\vec{k}_1,\vec{k}_2) \ . \tag{6.147}$$

Assuming that second order corrections are located close to each other in momentum space, we can expand the momenta around their average value, $\vec{k}_1 = \vec{K} - \vec{k}/2$, $\vec{k}_2 = \vec{K} + \vec{k}/2$, so that the corresponding free energy term is

$$\Omega_2|_{\bar{\Delta}\bar{\Delta}} = \frac{1}{\beta}\int \frac{\mathrm{d}^3K}{(2\pi)^3}\frac{\mathrm{d}^3k}{(2\pi)^3}\,\bar{\Delta}^*(\vec{k})\bar{\Delta}(\vec{k})\mathcal{F}_{\bar{\Delta}\bar{\Delta}}(\vec{K},\vec{k})\ , \tag{6.148}$$

Expanding $\mathcal{F}_{\bar{\Delta}\bar{\Delta}}$ up to quadratic order in k, we identify the m^{-2} and f_1 coefficients with

$$m^{-2} = g^2 + \frac{1}{\beta}\int \frac{\mathrm{d}^3K}{(2\pi)^3}\left(\mathcal{F}_{\bar{\Delta}\bar{\Delta}}(\vec{K},0) - \mathcal{B}_{\bar{\Delta}\bar{\Delta}}(\vec{K},0) + \mathfrak{B}_{\bar{\Delta}\bar{\Delta}}(\vec{K})\right)\ , \tag{6.149}$$

$$f_1 = \frac{1}{2\beta}\int \frac{\mathrm{d}^3K}{(2\pi)^3}\left(\partial_k^2\mathcal{F}_{\bar{\Delta}\bar{\Delta}}(\vec{K},0) - \partial_k^2\mathcal{B}_{\bar{\Delta}\bar{\Delta}}(\vec{K},0)\right)\ , \tag{6.150}$$

once the analogous bosonic contribution is included.

The f_3-term in the Ginzburg-Landau free energy (6.21) has a contribution coming from the gauge field kinetic term plus contributions coming from the part of Ω_2 quadratic in the gauge field, which will be proportional to the square of the gauge coupling, e^2,

$$f_3 = \frac{1}{2} + \mathcal{O}(e^2)\ . \tag{6.151}$$

As the gauge coupling is assumed to be small we can simply take $f_3 = 1/2$. From this identification for the f_3 coefficient we see that there is no significant difference between turning on a baryonic gauge field (6.137) or an R-symmetry (6.138) gauge field, since differences would appear to order $\mathcal{O}(e^2)$.

Once these coefficients are computed, one can study their cut-off dependence in both the relativistic and supersymmetric case. We will restrict ourselves to the zero temperature regime, where integrals can be performed analytically. In the zero temperature limit, the explicit forms of the coefficients f_1 and m^{-2} are

$$\begin{aligned} f_1|_{T=0} = {}& \frac{128g^{12}\Delta_0^6\mu + 80g^8\Delta_0^4\mu(\Lambda+\mu)^2 + \mu^2(\Lambda+\mu)^5 + 2g^4\Delta_0^2(\Lambda+\mu)^3\left(3\Lambda^2+6\Lambda\mu+8\mu^2\right)}{96\pi^2\Delta_0\left(4g^4\Delta_0^2+(\Lambda+\mu)^2\right)^{5/2}} \\ &+ (\mu\to-\mu) \\ &+ \frac{g^8\Delta_0^2\Lambda^3}{2\pi^2\left(4g^4\Delta_0^2+\Lambda^2\right)^{5/2}}\ , \end{aligned} \tag{6.152}$$

$$\begin{aligned} m^{-2}|_{T=0} = g^2 + \frac{g^4}{4\pi^2}\Bigg(&\frac{(\Lambda+\mu)(5\mu^2+2\Lambda\mu-\Lambda^2)+4g^4(5\mu-3\Lambda)\Delta_0^2}{\sqrt{4g^4\Delta_0^2+(\Lambda+\mu)^2}} \\ &+ 2\left(\mu^2-6g^4\Delta_0^2\right)\log\left(\frac{\mu+\sqrt{4g^4\Delta_0^2+\mu^2}}{\Lambda+\mu+\sqrt{4g^4\Delta_0^2+(\Lambda+\mu)^2}}\right) + (\mu\to-\mu)\Bigg) \\ &+ \frac{g^4}{2\pi^2}\left(\frac{12g^4\Delta_0^2\Lambda+\Lambda^3}{\sqrt{4g^4\Delta_0^2+\Lambda^2}} - 12g^4\Delta_0^2\mathrm{csch}^{-1}\left(\frac{2g^2\Delta_0}{\Lambda}\right)\right)\ , \end{aligned} \tag{6.153}$$

where the last line in (6.152) or (6.153) corresponds to the scalar contribution. Taking into account the cut-off dependence of the gap at zero temperature, (6.75), we find the following expressions for the f_1 and m^{-2} coefficients at leading order in Λ:

$$f_{1\ \mathrm{rBCS}} = \frac{3^5\pi^8}{8g^6}\Lambda^{-10} \ , \qquad f_{1\ \mathrm{sBCS}} = \frac{g^4}{8\pi^2}(1+4c^2)^{-3/2} + \mathcal{O}(\Lambda^{-2}) \ , \tag{6.154}$$

$$\begin{aligned} m^{-2}_{\mathrm{rBCS}} = g^2 + \mathcal{O}(\Lambda^{-4}) \ , \quad m^{-2}_{\mathrm{sBCS}} &= g^2 + \frac{g^4\mu^2}{2\pi^2}\left(\frac{32c^4+16c^2+5}{(1+4c^2)^{5/2}} - 2\log\frac{1+\sqrt{1+4c^2}}{2c}\right) \\ &+ \mathcal{O}(\Lambda^{-2}) \ , \end{aligned} \tag{6.155}$$

where $c = \exp\left[-\frac{\pi^2}{g^2\mu^2} - \frac{3}{2}\right]$. We must stress that the coefficient f_1 in the relativistic BCS theory does not vanish at zero temperature, because Λ is a physical cut-off acting like a "Debye energy", which takes a finite value.

Summary

The study of supersymmetry has led us to a better understanding of field theories, specially in the strong coupling regime. In this thesis we have tried to show this through several examples. These are:

- The first of these examples has been the application of localization techniques in supersymmetric theories. Specifically, we have shown how to compute the partition function of $\mathcal{N} = 2$ supersymmetric Chern-Simons theory with gauge group $U(N)$ and $2N_f$ flavors, i.e N_f chiral multiplets transforming in the fundamental representation of the gauge group and N_f more transforming in the anti-fundamental. To regularize the theory, it is necessary to make the computation in a three sphere whose radius, R, serves as an IR regulator which can be taken to infinity at the end of the computation.

  Once we have the exact partition function in terms of a matrix integral, although difficult to compute, it is much easier than the original path integral one starts with before applying localization. Then we can solve the integral by means of a saddle-point approximation. This approximation becomes exact in the large N limit.

  Hence, solving the integral by saddle-point in the large N limit, we can consider a continuous distribution of eigenvalues of the $N \times N$ matrix that represents the scalar field of the $\mathcal{N} = 2$ vector hypermultiplet in three dimensions. This leads to an equation that can be solved exactly and that in the decompactification limit, $R \to \infty$, it shows different phases depending on the value of the 't Hooft coupling. This coupling also has to be rescaled with the radius because otherwise, matter fields would simply decouple and we would end with pure Chern-Simons theory. In this way, depending on the value that the rescaled 't Hooft coupling, λ, takes in comparison to the quotient $\zeta = N_f/N$, we find one of the three phases described by the eigenvalue distributions shown in figures 2.3, 2.4 and 2.5 for the intervals:

  $$\text{Phase I: } \lambda < 1 \ , \qquad \text{Phase II: } 1 < \lambda < (1-\zeta)^{-1} \ , \qquad \text{Phase III: } (1-\zeta)^{-1} < \lambda \ . \tag{6.156}$$

  If $\zeta \geq 1$, phase III simply disappears and phase II extends up to $\lambda \to \infty$.

  We have also computed the free energy and the vacuum expectation value of a Wilson loop for a big circle of the three sphere. Both of them show discontinuities in their derivatives, in particular, the discontinuity in the free energy appears in the third derivative and thus, both phase transitions are third order.

- Other application that we have seen consists of the use of the gauge/gravity duality to build supergravity solutions dual to supersymmetric gauge theories that allow us to obtain information about these gauge theories in the strong coupling regime, where other tools are no longer useful.

In particular, starting from the gravity dual to $\mathcal{N} = 1$ super Yang-Mills, proposed by Maldacena and Núñez, we have reviewed how to add flavors (quarks) to this theory, without mass first and with mass later. We have also seen how to extract information about the field theory from these gravity duals. In particular, we have paid special attention to how the β-function of the field theory dual is obtained from the gravity background proposed by Conde, Gaillard and Ramallo, dual to $\mathcal{N} = 1$ super Yang-Mills field theory with N_f massive flavors and a quartic superpotential.

The gravity dual proposed by Conde, Gaillard and Ramallo is based on the addition of N_f backreacting flavor D5-branes to the background generated by N color D5-branes. The distribution of the flavor branes is governed by a function $S(r)$. Here we have used a simple function $S(r)$, determined by some physical and computational requirements.

Once we have a particular $S(r)$ function, we can solve the BPS system of equations, which admits various solutions. Under some physical criteria we are able to choose a single relevant solution for each value of the quotient N_f/N.

The main result from the point of view of the field theory corresponding to this solution is that, in the case $N_f = 2N$, the β-function (figure 4.6) shows a non-trivial UV fixed point. Taking into account the perturbative behavior of the β-function, the simplest interpolation between the weak and strong coupling regimes of the β-function implies a new non-trivial IR fixed point. In the cases $N_f < 2N$ and $N_f > 2N$ we do not find any evidence of any non-trivial fixed point.

This hint of a new IR fixed point is in agreement with Seiberg proposal of a conformal window for $3N/2 < N_f < 3N$ although in our case it is reduced to a single point, $N_f = 2N$, maybe because of the presence of a quartic superpotential.

In addition, we have seen how Seiberg duality is realized in these gravity backgrounds. Indeed, the β-function of the case $N_f > 2N$ seems to correspond to that of the Seiberg-dual theory.

- Again, in the context of the gauge/gravity duality, we have studied how to generate new supergravity solutions applying T-duality and how this affects the G-structures that describe the supersymmetry of these solutions.

  In particular, we have applied T-duality to the IIB supergravity solution of Klebanov and Witten, obtained by placing N D3-branes at the tip of the conifold. Firstly, we have done this without flavors, as an example; and later we have included flavors by adding N_f D7-branes. Both cases, with and without flavors, posses an $SU(3)$-structure before the application of T-duality. After we T-dualize in some $SU(2)$ isometry directions of the background, we find new IIA supergravity solutions with an $SU(2)$-structure.

  When we consider the flavored background, flavor branes act as sources which produce the violation of Bianchi identities. Studying how these identities transform under T-duality we obtain that the new supergravity solution is able to accommodate sources for D4, D6 and D8-branes.

- Finally, we have presented an $\mathcal{N}=1$ supersymmetric model that exhibits a superconducting phase transition. This model is based on the following Kähler potential

$$K = \Phi^\dagger\Phi + g^2(\Phi^\dagger\Phi)^2 \ , \tag{6.157}$$

for chiral multiplets Φ and no superpotential. We also have to include temperature and chemical potential. Since this system has two global $U(1)$ symmetries, a baryonic one and the $U(1)_R$-symmetry, in principle, we can couple the chemical potential to the associated current of one of these symmetries. However, as the appearance of a Fermi surface is a necessary condition, the chemical potential can be only introduced for the $U(1)_R$ charge. In this way the scalars of the chiral multiplet do not suffer from Bose-Einstein condensation and do not spoil the BCS mechanism at the same time that a Fermi surface is generated.

Moreover, the scalars, which do not appear in the usual BCS theory, have the effect of making the phase transition first order rather than second order (see figure 6.2).

We have also considered the effect of an external gauge field as well as possible gap fluctuations. This allows us to study the Meissner effect, penetration and coherence lengths and critical magnetic fields. Then we have been able to determine that the superconducting model studied here is type II.

We have compared different quantities (figures 6.2-6.9) for our model and the relativistic BCS model and the main differences are due to the difference in the order of the phase transition and the fact that the dependence on the cut-off is milder in the supersymmetric model.

Resumen

El estudio de supersimetría nos ha permitido un mejor entendimiento de las teorías de campos, especialmente en el régimen de acoplamiento fuerte. En esta tesis hemos tratado de mostrar esto a través de varios ejemplos. A saber:

- El primero de estos ejemplos ha sido la aplicación de técnicas de localización en teorías supersimétricas. En particular, hemos mostrado cómo calcular de manera exacta la función de partición de la teoría de Chern-Simons supersimétrica $\mathcal{N}=2$ con grupo gauge $U(N)$ y $2N_f$ sabores, es decir N_f multipletes quirales transformando en la representación fundamental del grupo gauge y otros N_f, en la antifundamental. Para regularizar esta teoría es necesario hacer el cálculo en una tres esfera cuyo radio, R, sirve como regulador IR que puede tomarse infinito al terminar los cálculos.

  Una vez que tenemos la función de partición exacta en términos de una integral de matrices, que aunque complicada de calcular, es mucho más fácil que la integral de camino original de la que uno parte antes de aplicar localización, podemos resolverla por medio de la aproximación de punto silla. Esta aproximación deja de serlo y se vuelve exacta cuando tomamos el límite de N grande.

  Resolviendo, por tanto, la integral por medio del punto silla, en el límite de N grande podemos considerar una distribución continua de los autovalores de la matriz $N \times N$ que representa el campo escalar del hipermultiplete vector de $\mathcal{N}=2$ en tres dimensiones. Esto da lugar a una ecuación que puede resolverse de manera exacta y que en el límite de descompactificación $R \to \infty$ tiene como solución diferentes fases dependiendo del valor del acoplamiento de 't Hooft. Este acoplamiento también debe rescalearse con el radio ya que de otra manera, los campos de materia simplemente se desacoplan y acabaríamos con una teoría de Chern-Simons pura. De forma que, dependiendo del valor que tome el acoplamiento de 't Hooft rescaleado, λ, en relación al cociente $\zeta = N_f/N$, encontramos tres fases descritas por las distribuciones de autovalores mostradas en las figuras 2.3, 2.4 y 2.5 para los intervalos:

  $$\text{Fase I: } \lambda < 1 \,, \qquad \text{Fase II: } 1 < \lambda < (1-\zeta)^{-1} \,, \qquad \text{Fase III: } (1-\zeta)^{-1} < \lambda \,, \tag{6.158}$$

  Si $\zeta \geq 1$ la fase III simplemente desaparece y la fase II se extiende hasta $\lambda \to \infty$.

  También hemos calculado la energía libre y el valor de expectación de un lazo de Wilson correspondiente a un círculo máximo de la tres-esfera. Ambos exhiben discontinuidades en sus derivadas, en concreto, la discontinuidad para la energía libre es en la tercera derivada y, por tanto, las dos transiciones de fase son de tercer orden.

- Otra aplicación que hemos visto consiste en el uso de la dualidad gravedad/gauge para construir soluciones de supergravedad duales a teorías gauge supersimétricas que nos permitan obtener información acerca de estas últimas en el régimen de acoplamiento fuerte, donde otras técnicas dejan de ser útiles.

  En particular, partiendo del dual gravitatorio de la teoría $\mathcal{N} = 1$ super Yang-Mills, propuesto por Maldacena y Núñez, hemos revisado cómo añadir sabores (quarks) a esta teoría, con y sin masa. Hemos visto también cómo extraer de los duales gravitatorios información sobre la teoría de campos. En concreto, hemos puesto especial atención en cómo obtener la función β de la teoría de campos dual al fondo gravitatorio propuesto por Conde, Gaillard y Ramallo, dual a la teoría de campos $\mathcal{N} = 1$ super Yang-Mills con N_f sabores masivos y un superpotencial cuártico.

  El dual gravitatorio propuesto por Conde, Gaillard y Ramallo se basa en añadir N_f D5-branas de sabor al fondo generado por N D5-branas de color, alterándolo a su vez. La distribución de las branas de sabor viene caracterizada por una función $S(r)$. Nosotros hemos elegido una función $S(r)$ simple, determinada por varios requerimientos físicos y computacionales.

  Una vez que tenemos una función $S(r)$ particular, podemos resolver el sistema de ecuaciones BPS, que admite varias soluciones. Bajo determinados criterios físicos podemos elegir una única solución relevante para cada valor del cociente N_f/N.

  El principal resultado desde el punto de vista de la teoría de campos que corresponde a dicha solución, consiste en que, en el caso $N_f = 2N$, la función β (figura 4.6) exhibe un punto fijo UV no trivial. Teniendo en cuenta el comportamiento perturbativo de la función β, la forma más simple de interpolar entre el comportamiento perturbativo y el de acoplamiento fuerte es a través de la aparición de un nuevo punto fijo no trivial IR. En los casos $N_f < 2N_c$ y $N_f > 2N_c$ no encontramos evidencia de ningún punto fijo no trivial.

  Este indicio de un nuevo punto fijo IR está de acuerdo con la proposición de Seiberg acerca de la existencia de una venta conforme entre $3N/2 < N_f < 3N$, aunque en nuestro caso se reduce a un único punto $N_f = 2N$, quizá debido a la presencia del superpotencial cuártico.

  También hemos visto como la dualidad de Seiberg aparecía en estas soluciones gravitacionales. De hecho, la función β del caso $N_f > 2N$ parece corresponder con la de la teoría dual de Seiberg.

- También en el contexto de la dualidad gravedad/gauge hemos estudiado cómo generar nuevas soluciones de supergravedad por medio de la aplicación de T-dualidad y como esta afecta a las G-estructuras en términos de las que podemos describir la supersimetría de estas soluciones.

  En particular hemos aplicado T-dualidad a la solución de supergravedad IIB de Klebanov y Witten, obtenida al poner N D3-branas en el conifold, sin sabores primero, a modo de ilustración, y con sabores después, tras añadir N_f D7-branas. Ambos casos, con y sin sabores, posen una estructura $SU(3)$ antes de aplicar la T-dualidad. Después T-dualizamos en las direcciones de la isometría $SU(2)$ de la solución y encontramos nuevas soluciones de supergravedad IIA con una estructura $SU(2)$.

Cuando consideramos el caso con sabores, las branas de sabor actúan como fuentes dando lugar a la violación de las identidades de Bianchi. Estudiando cómo transforman estas identidades bajo T-dualidad obtenemos que la nueva solución es capaz de acomodar fuentes de D4, D6 y D8-branas.

- Finalmente, hemos presentado un modelo supersimétrico que exhibe una transición de fase superconductora. Este modelo está basado en el siguiente potencial de Kähler

$$K = \Phi\Phi^\dagger + g(\Phi\Phi^\dagger)^2 \tag{6.159}$$

para multipletes quirales Φ y sin superpotencial. También debemos incluir en este sistema temperatura y un potencial químico. Como este sistema dispone de dos simetrías globales $U(1)$, una bariónica y otra la simetría $U(1)_R$, podemos, en principio, acoplar el potencial químico a la corriente correspondiente a una u otra simetría. Sin embargo, como es necesaria la aparición de una superficie de Fermi, el potencial químico solo puede ser introducido para la carga a $U(1)_R$, de forma que los escalares del multiplete quiral no condensen y estropeen el mecanismo BCS, a la vez que se genera una superficie de Fermi.

Además, los escalares, que no aparecen en la teoría BCS usual, tienen el efecto de hacer que la transición de fase sea de primer orden en lugar de segundo orden (ver figura 6.2).

También hemos considerado el efecto de un campo gauge externo así como posibles fluctuaciones del gap. Esto permite estudiar el efecto Meissner además de longitudes de penetración y de coherencia y campos magnéticos críticos. Gracias a esto hemos sido capaces de determinar que los superconductores estudiados con este modelos son de tipo II.

Hemos comparado diferentes magnitudes (figuras 6.2-6.9) para nuestro modelo y el modelo BCS relativista y las principales diferencias son debido a la diferencia en el orden de la transición de fase y el hecho de que en el modelo supersimétrico la dependencia en la energía de corte es más suave.

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