Modular Invariants for Manifolds with Boundary

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Certifico que aquesta memòria ha estat realitzada per Maria Immaculada Gálvez Carrillo i supervisada per mi, al Departament de Matemàtiques de la Universitat Autònoma de Barcelona

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Summary

This thesis has had two aims: the practical aim of writing out explicit generalisations of the constructions of Atiyah, Donnelly, Patodi, and Singer [APS75I, APS75II, APS76, ADS84] for formal sums of operators on manifolds, and the more philosophical aim of re-opening the investigations of Hirzebruch and Zagier [Hir66, HZ74] on some important interactions of algebraic topology with number theory and algebraic geometry. We have needed also ingredients from differential geometry and from analysis, and our investigations led to consideration of ideas of Segal on conformal field theory [Seg88].

In particular our definitions lead to a new invariant $\eta_{\mathcal{E}}(q)$ of framed manifolds N^{4k-1}, which arises on considering the formal operator given by twisting the classical signature operator by a certain graded bundle considered by Witten [Wit87] as representing the tangent bundle of the free loop space on a manifold. In this way we obtain an invariant, taking power series in the formal variable q as values, whose constant term is the spectral eta invariant of [ADS84]. The result that this eta invariant coincides with the *signature defect* (i.e., the difference $\varphi_L(M,N) - \operatorname{sign}(M,N)$ between the relative L-genus and the signature, for a closed manifold M^{4k} with $\partial M = N$), generalises to our new invariant to give

$$\eta_{\mathcal{E}}(q) = \overline{\varphi}_{\mathcal{E}}(M,N) - \operatorname{sign}^{S^1}(M,N).$$

Here $\operatorname{sign}^{S^1}$ is the S^1-equivariant signature on the loop space and $\overline{\varphi}_{\mathcal{E}}$ is the (normalised) elliptic genus of [LS88, Och87]; hence the power series $\eta_{\mathcal{E}}$ can also be regarded as a *modular function*, at least modulo the integers. As an illustrative example we note that there are framings of the spheres $S^{4k-1} = \partial D^{4k}$ for which $\eta_{\mathcal{E}}$ is easily expressed in terms of an Eisenstein series G^*_{2k}.

We consider eta invariants arising not only from twisted signature operators, but also from the corresponding Dirac operators. Moreover we define equivariant versions of these invariants, associated to representations of the fundamental group $G = \pi_1 N$ of the manifold. Atiyah–Patodi–Singer give an alternative, more algebraic definition of their equivariant eta invariant in [APS75II], in terms of the K-theory of N and the classifying space BG. We also generalise this to give a definition in terms of elliptic cohomology of our modular eta invariant, inspired by the philosophy that K-theory for loop spaces *is* elliptic cohomology. We give examples of this construction for lens spaces and, at least in the case that G is finite of odd order, show that it takes values in the equivariant elliptic cohomology ring introduced by Devoto [Dev96b].

Contents

Introduction

ATIYAH: Why is the $\hat{A}$-genus an integer for spin manifolds?
SINGER: You know the answer better than I —why do you ask?
ATIYAH: There must be a deeper reason.

In March 1962, Singer suggested a deeper reason: the $\hat{A}$-genus is an integer because it is the index of a Dirac operator...

The Index Theorem for Manifolds with Boundary

In this thesis we develop some of the Atiyah–Patodi–Singer constructions for manifolds with boundary in the context of elliptic genera. At least formally, they will provide a version of the index theorem for the space of free smooth loops on manifolds with boundary. We consider mainly the case of twisted signature operators corresponding to level 2 elliptic genera, and we compute and interpret these in the especially relevant cases of some framed disks and lens spaces. More general results are possible, which will be developed later.

What can elliptic cohomology tell us for manifolds with boundary? We can expect it to generalise classical results for manifolds with boundary to their free smooth loop spaces, following the philosophy that elliptic cohomology may be considered as a sort of K-theory of loop spaces. In this classical case one uses the Chern character to link K-theory to ordinary cohomology; between elliptic cohomology and K-theory we have the Miller character also. Using these tools, we can extend the Atiyah–Patodi–Singer constructions to formal operators on loop spaces on manifolds with boundary.

Recall that the Atiyah–Singer theorem for manifolds without boundary expresses the index of an elliptic operator in terms of characteristic numbers. More precisely, if we consider an elliptic operator D_E^+ on a manifold without boundary X, which may be assumed to be given by the classical signature operator D^+ twisted by some bundle E, then the index is given by

$$\operatorname{ind}\left(D_E^+\right) = \left\{\operatorname{ch}_2(E) \cdot \mathrm{L}(X)\right\}[X], \tag{1}$$

where $\mathrm{L}(X)$ is the Hirzebruch characteristic class on the tangent bundle TX and $\operatorname{ch}_2(E)$ is the Chern character of E up to a power of 2.

For manifolds with boundary, the seminal work of Atiyah–Patodi–Singer [APS75] showed that an extra summand, the *eta invariant*, must be added for formula (1) to hold. The formula obtained becomes

$$\operatorname{ind}\left(D_E^+\right) + \dim\ker D_E^+\,|_Y = \left\{\operatorname{ch}_2(E)\cdot \mathrm{L}(X)\right\}[X] - \eta_{D_E^+|_Y}(0)$$

at least in those cases when we consider a compact oriented Riemannian manifold X of dimension $4k$ with boundary Y and assume that near Y it is isometric to a product. Here $\eta_{D_E^+|_Y}$ is the holomorphic continuation of the eta function

$$\eta(s) = \sum_\lambda \frac{\operatorname{sign}(\lambda)}{|\lambda|^s}, \qquad \operatorname{Re}(s) \gg 0,$$

where the sum is over the non-zero eigenvalues of the operator.

Modular Eta Invariants

In this thesis we define and calculate these invariants for formal operators on loop spaces on manifolds with boundary, and interpret them as modular invariants. In particular, we consider the formal operator corresponding to the signature for a bundle E,

$$R_qE = \bigotimes_{j=1}^{\infty} S_{q^j}E \otimes \bigotimes_{j=1}^{\infty} \Lambda_{q^j}E,$$

where $S_{q^j}E$ and $\Lambda_{q^j}E$ are the formal power series versions of the symmetric and exterior products. Then the index theorem applied to this formal operator will give a formal power series instead of a numerical index. If X is a manifold without boundary and E its complexified tangent bundle, we write $\xi_q = R_qE$ and the index formula becomes

$$\operatorname{ind}\left(D_{\xi_q}^+\right) = \Phi_\varepsilon(X)\ [X].$$

The characteristic class $\Phi_\varepsilon(X)$ is precisely determined by the *elliptic genus*

$$\varphi_\varepsilon\colon \Omega_*^{\mathrm{SO}} \longrightarrow \mathcal{E}\ell\ell^* \cong \mathbb{Z}[\tfrac{1}{2}][\delta,\varepsilon][\Delta^{-1}],$$

whose exponential series is given by the elliptic function $\mathrm{s}^\varepsilon(\tau,x) = (\wp(\tau,x) - \wp(\tau,\pi i))^{-\frac{1}{2}}$. For any class $[X] \in \Omega_*^{\mathrm{SO}}$ in the oriented cobordism ring corresponding to a manifold of dimension $4k$, $\varphi_\varepsilon([X])$ is a modular form of weight $2k$ for the congruence subgroup $\Gamma_0(2) < \mathrm{SL}_2\mathbb{Z}$. Once we normalise it by dividing out by the factor $\varepsilon(\tau)^{\frac{k}{2}}$ it becomes a modular function for the congruence subgroup $\Gamma_0(2)$ which coincides with the formal index of the graded operator $D_{\xi_q}^+$,

$$\operatorname{ind}\left(D_{\xi_q}^+\right) = \sum_{r=0}^{\infty} \operatorname{ind}\left(D_r^+\right) q^r = \Phi_\varepsilon(X)\ [X] = \frac{\varphi_\varepsilon(X)}{\varepsilon(\tau)^{\frac{k}{2}}},$$

with each of the coefficients in its q-expansion corresponding to the index of a twisted signature operator of finite rank over X.

If now X has boundary Y, what can we say about the formal sum

$$\eta_{D^+_{\xi_q}|_Y}(0) = \sum_{r=0}^{\infty} \eta_{D^+_r|_Y}(0) q^r \tag{2}$$

of eta invariants corresponding to the summands of these formal operators? We will show they also give a class of modular invariants; these are the modular invariants of the title of this thesis. Term by term application of the Atiyah–Patodi–Singer theorem, combined with suitable relative versions of elliptic genera, will ensure that the series (2) is well defined, and converges. In particular, we obtain

$$\operatorname{ind}\left(D^+_{\xi_q}\right) + \dim\ker D^+_{\xi_q}|_Y = \Phi_\varepsilon\left(X, Y, \nabla\right)[X, Y] - \eta_{D^+_{\xi_q}|_Y}(0),$$

using the definitions for graded dimensions as defined for instance in [FLM88]. The relative characteristic classes introduce rational coefficients and we see, modulo the integers, that $\eta_{D^+_{\xi_q}|_Y}(0)$ is always congruent to a modular function of level 2 with rational coefficients. Furthermore, if the left-hand side vanishes, then $\eta_{D^+_{\xi_q}|_Y}(0)$ will be a modular function of level 2 with rational coefficients. In terms of the elliptic genus, we have shown that

$$\operatorname{ind}\left(D^+_{\xi_q}\right) + \dim\ker D^+_{\xi_q}|_Y = \frac{\varphi_\varepsilon\left(X, Y, \nabla\right)}{\varepsilon\left(\tau\right)^{\frac{k}{2}}} - \eta_{D^+_{\xi_q}|_Y}(0).$$

Observe that the connection on which the characteristic forms are based appears now explicitly in the expressions. Unlike the case of manifolds without boundary, on a manifold with boundary it is possible to have defined two elliptic operators with the same principal symbol (and hence the same K-theoretic class in the description by Atiyah and Singer), but different indices. In particular, we will generically obtain different invariants whenever we consider operators defined using different connections. Even restricting ourselves to the case of metric-compatible connections for a fixed Riemannian structure on the manifold, the result obtained will depend on the torsion tensor considered.

This allows more analytical invariants to be obtained than in the case of manifolds without boundary, corresponding to operators sharing the same principal symbol but not their total symbol. We will concentrate on the formal operators on loop spaces which extend the signature-based operator considered by Atiyah, Donnelly and Singer in [ADS83]. The usual signature operator on differential forms on a Riemannian manifold is determined by the composite $d^{\nabla^0} = \wedge \circ \nabla^0$ of the Levi-Cività connection ∇^0 and the exterior product $\wedge$; this operator d^{∇^0} agrees with the usual exterior derivative. If instead of the Riemannian connection we use any other metric connection ∇^T with torsion tensor T, we obtain an operator $d^{\nabla^T} = \wedge \circ \nabla^T$ which shares with d^{∇^0} the principal symbol, but which will not have the same index in general for manifolds with boundary. In fact, the operator considered by [ADS83] is a particular case of d^{∇^T}: if one considers a manifold X with framed boundary

(Y, f), the framing determines a flat connection on Y, extending to a connection ∇^f on X compatible with the metric but not necessarily flat. Then, since we fixed the curvature, we know from Riemannian geometry that it will have some non-vanishing torsion tensor T, such that $\nabla^f = \nabla^T$, and $d^{\nabla^f} = d^{\nabla^T}$.

Using a formal loop version of [ADS83] for this operator, we show that the resulting eta invariants are the reduction modulo $\mathbb{Z}[\frac{1}{2}]$ of modular functions of level 2. We compute these invariants explicitly for the case of the framed disks $\left(D^{4k}, S^{4k-1}, \pi\right)$ whose stable tangent bundles generate relative KO-theory coefficients up to the prime 2. These manifolds are especially interesting for algebraic topology, since they generate the relative oriented cobordism groups $\Omega_*^{\mathrm{SO,\,fr}}[\frac{1}{2}]$ and $\Omega_*^{\mathrm{U,\,fr}}$ of oriented and unitary manifolds with respect to framings of their boundaries. Moreover, using the exact sequences

$$0 \to \Omega_{2n}^{\mathrm{U}} \to \Omega_{2n}^{\mathrm{U,\,fr}} \to \Omega_{2n-1}^{\mathrm{fr}} \to 0,$$

the framed disks were used by Conner and Floyd in [CF66] to determine the image of the Adams e-invariant by relating the e-invariant to a relative Todd genus

$$\begin{array}{ccccccccc} 0 & \to & \Omega_{2n}^{\mathrm{U}} & \to & \Omega_{2n}^{\mathrm{U,\,fr}} & \to & \Omega_{2n-1}^{\mathrm{fr}} & \to & 0 \\ & & \downarrow \mathrm{Td}_{\mathrm{U}} & & \downarrow \mathrm{Td}_{\mathrm{U,\,fr}} & & \downarrow e & & \\ 0 & \to & \mathbb{Z} & \to & \mathbb{Q} & \to & \mathbb{Q}/\mathbb{Z} & \to & 0. \end{array}$$

The *odd part* of the image of e is given by the modified Bernoulli numbers, well known to topologists, which arise as the values of the invariant on the framed disks considered above. On the other hand, it was proved in the seventies by Atiyah–Patodi–Singer in their original work [APS75II] that these e-invariants are in fact reduced eta invariants for Dirac operators on framed manifolds.

Now the elliptic genus that we use is generated by Eisenstein series $G_{2k}^*(\tau)$ whose q-expansions have constant term $\left(1 - 2^{2k-1}\right) B_{2k}/4k$. Noting that the twisted signature vanishes, we show that the formal loop space signature of these disks has corresponding eta invariant $4G_{2k}^*(\tau)/\varepsilon(\tau)^k$, whose q-expansion is indeed the Adams e-invariant in its oriented version [see Sto68, p. 215]. This result can be seen as a particular case of a more general statement for Hirzebruch genera determined by formal operators.

Although it is not in the scope of the present thesis, in later work we would like to make more explicit how L-series, Jacobi forms, Rademacher sums, modular forms of half-integral weight, and more general Eisenstein series come into the picture. This is an old wish and, indeed, from Atiyah's work and commentaries in Hirzebruch's Collected Papers, one sees that it arose at the very origin of the subject. It was Hirzebruch who first spotted what he called the *Signaturdefekt* on manifolds with cusps and who gave the hint of the rôle of number theoretic L-series in this area.

Equivariant Elliptic Cohomology

As in the original work [APS75], an important part of our research takes place in the equivariant world. We use the equivariant elliptic cohomology developed by Devoto to construct our invariants, in the same spirit as they arose from equivariant K-theory in the classical case.

We will summarise our result by the test case of lens spaces L_p^{4k-1} obtained as the quotient of $(4k-1)$-dimensional spheres by a cyclic group $G = C_p$ of odd order. We then have two families of invariants from the classical signature operator: the first gives us invariants $\eta_{A_{\xi_q(S^{4k-1})},g}$ for the formal twisted signature operator on $\mathcal{L}S^{4k-1}$ associated to each $g \in G$, and the second gives invariants $\eta_{A_{\xi_q\left(L_p^{4k-1}\right)},\alpha}$ for a formal twisted signature operator on $\mathcal{L}L_p^{4k-1}$ associated to a representation α of G. These two families are related via a *finite Fourier transform* formula, which classically gives the well-known expressions for the lens spaces $Y = L_p^{4k-1}(q_1, \ldots, q_{2k}) = S^{4k-1}/C_p$,

$$\eta_{\varepsilon,\alpha}\left(0,Y\right) = \frac{1}{p}\sum_{r=1}^{p-1}\prod_{j=1}^{2k}\cot\left(\frac{\pi q_j r}{p}\right)\chi_\alpha\left(\zeta_p^r\right)$$

(compare [APS75II, p. 412], [Don78, Thm. 3.3]). The number-theoretic significance of such expressions was known already to Rademacher. Our generalisation gives an expression for the modular eta invariant in terms of the *Teilwerte* $\varphi\left(\tau, 2\pi i q_j r/p\right)$ of the chosen elliptic functions,

$$\eta_{\varepsilon,\alpha}\left(0,Y\right) = \frac{1}{p}\sum_{r=1}^{p-1}\prod_{j=1}^{2k}\frac{\varphi\left(\tau, 2\pi i q_j r/p\right)}{\varepsilon\left(\tau\right)^{\frac{1}{2}}}\chi_\alpha\left(\zeta_p^r\right).$$

The $\varphi\left(\tau, 2\pi i q_j r/p\right)$ are modular for the subgroup $\Gamma = \Gamma_1\left(p\right) \cap \Gamma_0\left(2\right)$ of $\mathrm{SL}_2\left(\mathbb{Z}\right)$, and we see that $\eta_{\varepsilon,\alpha}\left(0,Y\right)$ and $\eta_{\varepsilon,g}\left(0,\widetilde{Y}\right)$ are modular functions for this Γ. We hope to investigate general "elliptic Rademacher expansions" in later work.

We also introduce algebraic elliptic eta invariants based on elements of Devoto's equivariant cohomology ring $\mathcal{E}\ell\ell_G^*$ [Dev96, Dev98]. The definition of these invariants again parallels that of Atiyah–Patodi–Singer for K-theory, and requires some technical results concerning elliptic cohomology with coefficients in $\mathbb{Q}/\mathbb{Z}[\frac{1}{2}]$, which is developed in a section of its own. To describe our algebraic construction of eta invariants, consider a spin manifold Y of dimension $4k-1$, which we may suppose bounds. The invariants in [APS75] associate to a representation of $\pi_1(Y)$ an invariant in $K^*\left(\mathrm{pt};\mathbb{Q}/\mathbb{Z}\right)$. We extend this construction, in the natural way, by giving invariants in the appropriate version of elliptic cohomology.

Next, we use the result of [HKR00, Dev96b] that for a finite group G of odd order there is a completion map c_G which is an isomorphism,

$$\mathcal{E}\ell\ell^*\left(BG\right) \otimes \mathbb{Z}\left[\tfrac{1}{|G|}\right] \overset{c_G}{\cong} \left(\mathcal{E}\ell\ell_G^*\right)^\wedge_{I_G},$$

for I_G the kernel of the augmentation map

$$\epsilon\colon \mathcal{E}\ell\ell_G^* \to \mathcal{E}\ell\ell_{\{e\}}^* \otimes \mathbb{Z}\left[\tfrac{1}{|G|}\right].$$

We apply this to replace $\mathcal{E}\ell\ell_G^*$ by $\mathcal{E}\ell\ell\left[\frac{1}{|G|}\right]^* BG$ and use the connecting map of the short exact sequence of coefficient groups

$$0 \to \mathbb{Z}[\tfrac{1}{2}] \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z}[\tfrac{1}{2}] \to 0$$

to establish an isomorphism

$$\mathcal{E}\ell\ell\left[\tfrac{1}{|G|}\right]^{\text{even}}(BG) \cong \mathcal{E}\ell\ell^{\text{odd}}_{\mathbb{Q}/\mathbb{Z}[\frac{1}{2}]}(BG)\,.$$

We will use this isomorphism to get invariants on manifolds as follows. Consider a $(4k-1)$-dimensional manifold Y with spin structure, which we may assume bounds a manifold X, whose fundamental group $\pi_1(Y) = G$ is finite of odd order. For the corresponding classifying map $f\colon Y \to BG$ its pullback $f^*\colon \mathcal{E}\ell\ell^{\text{odd}}_{\mathbb{Q}/\mathbb{Z}[\frac{1}{2}]}(BG) \to \mathcal{E}\ell\ell^{\text{odd}}_{\mathbb{Q}/\mathbb{Z}[\frac{1}{2}]}(Y)$ gives classes in the elliptic cohomology of the manifold itself. Using the appropriate suitable Gysin map, any such class will give us an element in $\mathcal{E}\ell\ell^{\text{odd}}_{\mathbb{Q}/\mathbb{Z}[\frac{1}{2}]}(\text{pt})$. This is where our invariants live, and so we have extended the [APS75] definition to the framework of elliptic cohomology.

We can develop this construction for the test case of lens spaces, explicitly compute the invariants, and verify that they correspond to the ones given above for equivariant formal signature operators for loop spaces.

A key point in the construction is the mod $\mathbb{Z}[\frac{1}{2}]$ reduction of the invariants. Since $\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}[\frac{1}{2}]$ or $\mathbb{Q}_p/\mathbb{Z}_p$ are not even rings, it does not immediately make sense to consider "modular forms with coefficients mod $\mathbb{Z}$", or "p-adic modular forms with coefficients mod $\mathbb{Z}_p$". Nevertheless, at this point, the work of Katz [Kat75] and its applications by Baker, Clarke, Laures and others is relevant, as well as the description by Hopkins. After defining elliptic cohomology with rational coefficients modulo some ring of localised integers, we can split the coefficient group $\mathbb{Q}/\mathbb{Z}$, at least for the case of finite-dimensional manifolds (and in particular for the point) as the direct sum of coefficient rings of the form $\mathcal{E}\ell\ell^* \otimes \mathbb{Z}/(p^\infty\mathbb{Z})$, p an odd prime, so that any ring of the form $\mathcal{E}\ell\ell^* \otimes \mathbb{Z}/(p^k\mathbb{Z})$ will be in the coefficients. Moreover, our aim is to see how this amounts to considering together all the congruences between modular forms —in our case for $\Gamma_0(2)$— modulo p^k, which hold for every k, for p a fixed odd prime. This puts us in the setting of the Katz divided congruence rings and p-adic modular forms. We will only outline these results, and describe briefly how it is a particular case of a more general construction related to the generalised character constructions of [HKR00].

Representations of Segal Annuli

Finally we consider our constructions and their invariants in the light of conformal field theory proposed in the widely circulated preprint of Graeme Segal [Seg88]. To fill in all the details is beyond the scope of this thesis, but we will recall some particular constructions which play a natural rôle in our theory. We introduce the Segal category $\mathcal{A}^{\text{Spin}}(G)$ of G-Segal annuli with spin structure and consider their representation theory; in particular we study the functoriality, Mackey and Green properties of the functor which associates to each G the set of representations of the corresponding $\mathcal{A}^{\mathrm{Spin}}(G)$. Identification of these representations and their graded characters allows us to define an equivariant version of the result relating the Segal annuli $\mathcal{A}$ and the space $(0,1)\times \mathrm{Diff}^{+}(S^1)\times \mathrm{Diff}^{+}(S^1)/S^1$. The Lie algebra $\mathrm{Vect}\,(S^1)$ of fields over S^1 and its extension, the Virasoro algebra, now enter the picture. Our intention is then to define certain natural representations ρ_{Θ_g} of an equivariant generalisation $\mathrm{Vir}_{G,\,\mathrm{Spin}}$ of the Virasoro algebra, from whose graded characters we recover our elliptic invariants of the previous chapters.

There is a clear relation of these representations with the elliptic objects of Baker and Thomas [BT89] (also motivated by Devoto's work) which parallels the classical relation between representation theory, equivariant bundle and K-theory, and the cohomology of the classifying spaces of finite groups.

Chapter 1

Essential tools from differential geometry and Clifford theory

The goal of this chapter is to set up the frame where we are going to work, to fix notation, and to recall the classical results —mainly from differential geometry, geometrical analysis, and algebraic topology— which are going to be used. In particular, we will need to use the properties of connections with torsion, bundles graded by formal variables, and spinor bundles on spheres.

1.1 Essential tools from differential geometry

In this section we set up the notation for the tools from differential geometry that we will need. We will work mainly with smooth manifolds with boundary, equipped with a fixed Riemannian metric. We consider affine linear connections on bundles naturally obtained from the tangent bundle of these manifolds, usually compatible with the metric, but not necessarily the Levi-Cività connection, since we allow torsion. We give the definitions for the operators from Riemannian geometry adapted to the case of non-vanishing torsion, and we make explicit some expansions in terms of coordinates and moving frames very usual in the Levi-Cività case, but less known in the presence of non-zero torsion. Then we will introduce Tamanoi's generalised differential forms and we will identify the metric and the torsion. We will identify the essential generalised differential parallel forms.

1.1.1 Differential geometry

We briefly review a number of important concepts and definitions of differential and Riemannian geometry, several of which will be needed in greater generality in later sections of this thesis.

Manifolds, connections, tensor fields and forms

Let M^n be an n-dimensional smooth manifold. If $\pi\colon E \to M$ is a vector bundle over M, we write $\Gamma(M, E)$ or just ΓE for the space of global smooth sections s of E. If we take the tangent bundle $E = TM$ or cotangent bundle $E = T^*M = TM^*$, given in each coordinate neighbourhood (U, x) by $\left\langle \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n} \right\rangle$ or $\langle dx_1, \dots, dx_n \rangle$, then ΓE is the space of *vector fields* or of 1-*forms* on M respectively.

The space of vector fields acts on the space $C^\infty(M) = C^\infty(M, \mathbb{R})$ of smooth scalar functions on M,

$$\begin{array}{rcl} C^\infty(M) \otimes \Gamma TM & \to & C^\infty(M) \\ f \otimes X & \mapsto & X(f). \end{array}$$

If X is given in local coordinates by $\sum a_i \frac{\partial}{\partial x_i}$ then $X(f)$ is defined by $\sum a_i \frac{\partial f}{\partial x_i}$. Vector fields may be identified with endomorphisms X of $C^\infty(M)$ which satisfy the Leibniz rule,

$$X(f \cdot g) = X(f) \cdot g + f \cdot X(g).$$

The space of vector fields forms a Lie algebra with bracket

$$[X, Y](f) = X(Y(f)) - Y(X(f)).$$

A *connection* on a vector bundle $E \to M$ is a linear map

$$\nabla\colon \Gamma E \to \Gamma(TM^* \otimes E)$$

satisfying a Leibniz rule

$$\nabla(fs) = f\nabla s + df \otimes s$$

where the *total derivative* df is the 1-form defined by $(df)(X) = X(f)$ or, in local coordinates, by $\sum \frac{\partial f}{\partial x_i} dx_i$. Since elements of ΓTM^* may be evaluated at $X \in \Gamma TM$, one has *covariant derivatives* of sections of E along vector fields:

$$\begin{array}{rcl} \Gamma E \otimes \Gamma TM & \to & \Gamma E \\ s \otimes X & \mapsto & \nabla_X s, \end{array}$$

satisfying $\nabla_X(fs) = f\nabla_X s + X(f)s$ and $\nabla_{fX} s = f\nabla_X s$.

If the vector bundle E is equipped with a metric $g(-,-) = \langle -,- \rangle$, then a connection ∇ on E is *compatible with the metric* if for all sections $s, t \in \Gamma E$ and vector fields X one has

$$X\langle s, t \rangle = \langle \nabla_X s, t \rangle \;+\; \langle s, \nabla_X t \rangle.$$

A *connection on* M is a connection on the tangent bundle TM.

We denote by S^kE, Λ^kE and $E^{\otimes k}$ the symmetric, exterior and tensor product bundles of a vector bundle E. A (k,ℓ)-*tensor field* is a section of $(TM^*)^{\otimes k} \otimes TM^{\otimes \ell}$, and a differential k-*form* is a section of Λ^kTM^*. Given a connection ∇ on M, the covariant derivatives $\nabla_X : \Gamma TM \to \Gamma TM$ on vector fields Y may be *extended* to act on 1-forms ω and (inductively) on arbitrary (k,ℓ)-tensor fields via the following dual and tensor product formulas

$$\begin{aligned} X(\omega(Y)) &= (\nabla_X\omega)(Y) + \omega(\nabla_X Y), \\ \nabla_X(P \otimes Q) &= \nabla_X P \otimes Q + P \otimes \nabla_X Q. \end{aligned}$$

Allowing X to vary, one has for any (k,ℓ)-tensor field P a $(k+1,\ell)$-tensor ∇P defined by

$$\begin{aligned} (\nabla P)(Y \otimes X \otimes \varphi) &= (\nabla_X P)(Y \otimes \varphi) \\ &= X(P(Y \otimes \varphi)) - \sum_{i=1}^{k} P(Y_1 \otimes \cdots \otimes \nabla_X Y_i \otimes \cdots \otimes Y_k \otimes \varphi) \\ &\quad + \sum_{j=1}^{\ell} P(Y \otimes \varphi_1 \otimes \cdots \otimes \nabla_X \varphi_j \otimes \cdots \otimes \varphi_\ell), \end{aligned}$$

for $Y = Y_1 \otimes \cdots \otimes Y_k \in \Gamma TM^{\otimes k}$ and $\varphi = \varphi_1 \otimes \cdots \otimes \varphi_\ell \in \Gamma(TM^*)^{\otimes \ell}$.

Examples 1.1

1. The condition for the connection to be compatible with a metric may be expressed as $\nabla g = 0$.
2. For a "$(0,0)$-tensor" $f \in C^\infty(M)$ we have $(\nabla f)(X) = \nabla_X f = X(f)$, and ∇f coincides with the total derivative df.
3. For a p-form φ we have a $(p+1,0)$-tensor $\nabla\varphi$. Antisymmetrising this tensor gives a $(p+1)$-form $(\wedge \circ \nabla)\varphi$ which satisfies

$$(\wedge \circ \nabla)\varphi(X_0,\ldots,X_p) = \frac{1}{p+1}\left(\sum_i (-1)^i X_i(\varphi(X_0,\ldots,\hat{X}_i,\ldots,X_p)) + \right.$$

$$\left. + \sum_{i<j} (-1)^{i+j} \varphi(\nabla_{X_i}X_j - \nabla_{X_j}X_i, X_0,\ldots,\hat{X}_i,\ldots,\hat{X}_j,\ldots,X_p)\right).$$

Definition 1.2 A *natural vector bundle* over a manifold M is a vector bundle which may be constructed from the tangent bundle TM by taking direct sums, tensor products, duals, and symmetric and exterior products.

Any connection on M extends as above to a connection on any natural bundle over M. Since all the bundles considered in this thesis will be natural bundles, we will often forget the word "natural".

We write $\Omega^k M = \Gamma\Lambda^k TM^*$ for the space of k-forms and, more generally, we write $\Omega^k(M; E) = \Gamma(\Lambda^k TM^* \otimes E)$. The space

$$\Omega^* M = \bigoplus \Omega^k M$$

of all forms on M is a graded-commutative algebra with respect to $\wedge$, with $\Omega^k M = 0$ for $k > n$. The pairing between vector fields and 1-forms extends to *interior multiplication* maps $i(X)\colon \Omega^{k+1}M \to \Omega^k M$ for $X \in \Gamma TM$, defined inductively by

$$i(X)(\omega_1 \wedge \cdots \wedge \omega_k) = \omega_1(X)\omega_2 \wedge \cdots \wedge \omega_k - \omega_1 \wedge i(X)(\omega_2 \wedge \cdots \wedge \omega_k).$$

The *exterior derivative* $d\colon \Omega^* M \to \Omega^{*+1} M$ is the unique linear map extending the total derivative $C^\infty(M) \to \Gamma TM^*$, $f \mapsto df$, and satisfying $d(f\varphi) = (df) \wedge \varphi$. If $d\varphi = 0$, then φ is a *closed* form. The *covariant exterior derivative* $d^\nabla_*\colon \Omega^*(M; E) \to \Omega^{*+1}(M; E)$ associated to a connection $\nabla\colon \Gamma E \to \Gamma(TM^* \otimes E)$ is the unique linear map extending ∇ and satisfying $d^\nabla_p(\varphi \otimes s) = d\varphi \otimes s + (-1)^p \varphi \wedge d^\nabla s$.

$$C^\infty(M) \xrightarrow{d} \Omega^1 M \xrightarrow{d} \Omega^2 M \longrightarrow \cdots \longrightarrow \Omega^k M \xrightarrow{d} \Omega^{k+1} M \longrightarrow \cdots$$

$$\Gamma E \xrightarrow{\nabla} \Omega^1(M; E) \xrightarrow{d^\nabla} \Omega^2(M; E) \longrightarrow \cdots \longrightarrow \Omega^k(M; E) \xrightarrow{d^\nabla} \Omega^{k+1}(M; E) \longrightarrow \cdots$$

The exterior derivative satisfies $d^2 = 0$ and the cohomology of the complex $(\Omega^* M, d)$ is the *de Rham cohomology* of M, which is isomorphic to the real singular cohomology of M. The map $d^\nabla \circ \nabla\colon \Gamma E \to \Omega^2(M; E)$ is *not* always zero; when it is, the connection ∇ is termed *flat*. The *curvature tensor* R of the connection may be defined by $2R = d^\nabla \circ \nabla$ or, evaluating on a pair of vector fields $X, Y \in \Gamma TM$, by

$$R_{X,Y}(s) = \nabla_X(\nabla_Y s) - \nabla_Y(\nabla_X s) - \nabla_{[X,Y]} s.$$

Riemannian manifolds, metrics and torsion

Consider now the special case of the tangent bundle $E = TM$, although our remarks will all carry over to any natural bundle E. The curvature tensor for a connection

$$\nabla\colon \Gamma TM \to \Gamma(TM^* \otimes TM)$$

on M may be regarded as a $(3,1)$-tensor on M. It satisfies the *Bianchi identities*

$$\begin{aligned} \{R_{X,Y}Z - T(T(X,Y),Z) - (\nabla_X T)(Y,Z)\} &= 0 \\ \{(\nabla_X R)_{Y,Z} + R_{T(X,Y),Z}\} &= 0 \end{aligned}$$

where the notation $\{P(X,Y,Z)\} = P(X,Y,Z) + P(Z,X,Y) + P(Y,Z,X)$ refers to the cyclic sum of a $(3,-)$-tensor, and T is the *torsion tensor* associated to the connection ∇ on M, i.e., the $(2,1)$-tensor defined by

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y].$$

If $T = 0$ then the connection is termed *torsion-free* (or *symmetric*), and the Bianchi identities become $\{R_{X,Y}Z\} = \{(\nabla_X R)_{Y,Z}\} = 0$. From Example 1.1.3 one sees also that $\wedge \circ \nabla$ coincides with $d\colon \Omega^p M \to \Omega^{p+1}$ if ∇ is torsion-free.

If the tangent bundle of M is equipped with a metric $\langle\ ,\ \rangle$ then $(M, \langle\ ,\ \rangle)$ is termed a *Riemannian manifold*.

Proposition 1.3 *On a Riemannian manifold there is a unique connection which is both compatible with the metric and torsion-free, termed the* Levi-Cività *or* canonical Riemannian *connection.*

In this case the corresponding (Riemannian) curvature tensor is completely determined by the *sectional curvature*,

$$K(X \wedge Y) = \langle R_{X,Y}Y, X\rangle \,/\, (|X|^2|Y|^2 - \langle X,Y\rangle^2).$$

In fact any connection ∇ on M which is compatible with the Riemannian metric is determined by its torsion tensor T. Explicitly, one finds that

$$\begin{aligned} 2\,\langle \nabla_X Y, Z\rangle \;&=\; X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle X,[Y,Z]\rangle + \langle Y,[Z,X]\rangle \\ &\quad + \langle Z,[X,Y]\rangle - \langle X, T(Y,Z)\rangle + \langle Y, T(Z,X)\rangle + \langle Z, T(X,Y)\rangle. \end{aligned} \tag{1.1}$$

Definition 1.4 We will denote by $\nabla^{g,T}$ the connection on M that is uniquely determined by (1.1), which has torsion given by the antisymmetric tensor T and is compatible with a metric $g = \langle\ ,\ \rangle$.

Taking $T = 0$ in equation (1.1) gives an expression for the Levi-Cività connection $\nabla = \nabla^{g,0}$ on M, and also for the exterior derivative, since $d = \wedge \circ \nabla$ in the torsion-free case.

Oriented manifolds, the Hodge star and coderivatives

A (connected) n-dimensional Riemannian manifold M is *orientable* if the double cover Or of M, with fibres given by the orthonormal frames in TM modulo the action of SO_n, is just the trivial double cover $M \cup M$. In the language of algebraic topology, M is orientable if the *first Stiefel–Whitney class* $w_1 \in H^1(X;\mathbb{Z}_2)$ vanishes, where w_1 may be thought of as classifying the double cover Or via the isomorphism $H^1(X;\mathbb{Z}_2) \cong \mathrm{Hom}(\pi_1 X, \mathbb{Z}_2)$. An orientation on M is then a choice of one sheet of the double cover.

Alternatively, an orientation on M is a nowhere-vanishing global n-form $\Phi \in \Gamma\Lambda TM^*$. Two such orientations Φ_1, Φ_2 are *equivalent* if $\Phi_1 = f\Phi_2$ for some everywhere-positive function $f \in C^\infty(M)$, or *opposite* if f is everywhere negative. The *volume form* of a manifold with orientation Φ is given by the normalisation

$$dv \;=\; \frac{\Phi}{|\Phi|\sqrt{n!}}.$$

The *Hodge star operator* for an oriented Riemannian manifold is the linear isomorphism

$$\begin{aligned} *\colon \Omega^q M &\longrightarrow \Omega^{n-q} M \\ \omega_1 \wedge \cdots \wedge \omega_q &\mapsto \omega_{q+1} \wedge \cdots \wedge \omega_n, \end{aligned}$$

where $(\omega_1, \dots, \omega_n)$ is any (positive) orthonormal coframe of 1-forms, so that in particular

$$dv = \omega_1 \wedge \cdots \wedge \omega_n = *1.$$

For q-forms φ, ψ one has

$$\varphi \wedge *\psi = \langle \varphi, \psi \rangle dv, \qquad\qquad *(*\varphi) = (-1)^{q(n-q)}\varphi.$$

If M is compact, one defines the *inner product* and L^2-*norm* on $\Omega^q M$ by

$$(\varphi, \psi) = \int_M \langle \varphi, \psi \rangle dv, \qquad\qquad \|\varphi\|^2 = (\varphi, \varphi).$$

The *exterior coderivative* $\delta\colon \Omega^q M \to \Omega^{q-1} M$ is then the adjoint of d under the L^2-norm,

$$\int_M \langle \phi, \delta\varphi \rangle dv = \int_M \langle d\phi, \varphi \rangle dv.$$

A differential form φ is called *harmonic* if $\Delta\varphi = 0$, where Δ is the *Laplace operator* $\Delta\colon \Omega^q M \to \Omega^q M$,

$$\Delta = (d + \delta)^2 = d\delta + \delta d.$$

Since $(\Delta\varphi, \varphi) = \|d\varphi\|^2 + \|\delta\varphi\|^2$, we see that φ is harmonic if and only if both $d\varphi$ and $\delta\varphi$ vanish. Let φ be any closed q-form. Then Hodge's theorem on critical points of the L^2-norm implies that there exists a unique $(q-1)$-form ϕ such that $\delta(\varphi + d\phi) = 0$ (in fact, minimising $\|\varphi + d\phi\|^2$), and so $\varphi + d\phi$ is the unique harmonic form in the cohomology class of φ. The de Rham cohomology groups of a compact oriented Riemann manifold M are therefore isomorphic to the spaces of harmonic forms on M.

Proposition 1.5 *The exterior coderivative δ on forms on an oriented, compact Riemannian manifold without boundary can be expressed in terms of d and the Hodge star,*

$$\delta = (-1)^{(q-1)(n-q+1)+q} *d* : \Omega^q(M) \longrightarrow \Omega^{q-1}(M).$$

Proof: Integration over M of the form

$$d(\phi \wedge *\varphi) = d\phi \wedge *\varphi + (-1)^{q-1}\phi \wedge d{*\varphi}$$

and application of Stokes' formula shows that we can take $*\delta\varphi = (-1)^q d{*\varphi}$. □

In particular the Hodge star of a harmonic form is again harmonic and induces isomorphisms

$$*\colon H^q(M) \cong H^{n-q}(M).$$

1.1.2 Parallelisations

This is essential for framed manifolds and connections on them. A *parallelisation* of a smooth n-dimensional manifold M is a section of the bundle LM of n-frames on the tangent bundle TM.

As we next make precise, every parallelisation determines a metric and a connection that is compatible with the metric (since the size of vectors is unaltered by parallel transport). This connection need not be symmetric. For our exposition we follow an approach by Dodson.

Theorem 1.6 *Suppose that an n-dimensional manifold M is parallelisable by a section*

$$\begin{array}{rcl} \mathrm{p}\colon M & \longrightarrow & LM \\ x & \longmapsto & (\mathrm{p}_i)_x . \end{array}$$

Then p *determines a connection ∇^{p} in LM such that*

$$\nabla^{\mathrm{p}}_{\mathrm{p}_i}(\mathrm{p}_j) = 0 \quad \textit{for all } i, j = 1, \ldots, n.$$

If $h\colon M \to \mathrm{GL}(n;\mathbb{R})$, $x \mapsto \left[h^i_j\right]_x$, is a smooth map, then $\mathrm{q}_i = h^k_i \mathrm{p}_k$ defines another parallelisation, and $\nabla^{\mathrm{p}} = \nabla^{\mathrm{q}}$ if and only if h is constant on each connected component of M.

Proof: By hypothesis, the Christoffel symbols for ∇^{p} with respect to the frame $(\mathrm{p}_i)_x$ all vanish, since

$$\nabla^{\mathrm{p}}_{\mathrm{p}_i}(\mathrm{p}_j) = \Gamma^k_{ij}\,\mathrm{p}_k.$$

Locally, the splitting of $T_{\mathrm{u}}LM$ for $\mathrm{u} = \left(x, \left(b^i_j\mathrm{p}_i\right)_x\right) \in LM$ is given as $T_{\mathrm{u}}LM = H_{\mathrm{u}} \oplus G_{\mathrm{u}}$ by $\left(x, \left(b^i_j\mathrm{p}_i\right)_x, X, B\right) = \left(x, \left(b^i_j\mathrm{p}_i\right)_x, X, 0\right) \oplus \left(x, \left(b^i_j\mathrm{p}_i\right)_x, 0, B\right)$. We choose

$$H_{\mathrm{p}(x)} = D_x\mathrm{p}\,(T_xM)$$

and for any $\mathrm{u} \in LM$, with $\mathrm{u} = R_h(\mathrm{p}(x))$,

$$H_{\mathrm{u}} = D_{\mathrm{p}(x)}R_h\left(H_{\mathrm{p}(x)}\right).$$

By the existence of p, we know that

$$LM = M \times \mathrm{GL}(n;\mathbb{R})$$

and the connection ∇^{p} makes the horizontal subspaces look horizontal in this product bundle by

$$\begin{array}{rcl} LM & \rightarrow & M \times G \\ \left(x, \left(b^i_j\mathrm{p}_i\right)_x\right) & \longmapsto & \left(x, \left(\delta^i_j\right)_x\right). \end{array}$$

That is, we locate the identity in G at the frame determined by the parallelisation at each point.

The given q is another parallelisation and

$$\nabla^{\mathrm{p}}_{\mathrm{p}_i}(\mathrm{q}_j) = \nabla^{\mathrm{p}}_{\mathrm{p}_i}\left(h^k_j \mathrm{p}_k\right) = \mathrm{p}_i\left(h^k_j\right)\mathrm{p}_k + h^j_k \nabla^{\mathrm{p}}_{\mathrm{p}_i}\mathrm{p}_k,$$

and $\nabla^{\mathrm{p}}_{\mathrm{p}_i}(\mathrm{q}_j) = 0$ if and only if h^k_j is constant on each connected component of M. □

Corollary 1.7 *The connection ∇^{p} need not be symmetric.*

Corollary 1.8 *The geodesics of ∇^{p} are the integral curves of constant linear combinations like $X\colon M \to TM,\ x \mapsto a^i \mathrm{p}_i$.*

Example: a nontrivial parallelisation of the plane

A nontrivial parallelisation of the two-plane $\mathbb{R}^2$ is given by

$$\begin{aligned} \mathrm{p}\colon \mathbb{R}^2 &\longrightarrow L\mathbb{R}^2 \\ (x,y) &\longmapsto (e^x\partial_1, e^x\partial_2), \end{aligned}$$

with $\partial_1 = \dfrac{\partial}{\partial x}$ and $\partial_2 = \dfrac{\partial}{\partial y}$ giving the standard frame of $T_{(x,y)}\mathbb{R}^2$ via the identity chart on $\mathbb{R}^2$. This p is a parallelisation and it determines a connection ∇^{p} by the conditions, since e^x is never zero, and hence, by linearity of $\nabla^p_v w$ in v,

$$\nabla^{\mathrm{p}}_{\partial_i}(e^x\partial_j) = 0 \quad \text{for } i,j = 1,2,$$

which expands to

$$\begin{aligned} e^x\partial_j + e^x\Gamma^{\mathrm{p}.\partial,k}_{1,j}\partial_k &= 0 \quad \text{for } i = 1, \\ e^x\Gamma^{\partial,k}_{2,j}\partial_k &= 0 \quad \text{for } i = 2. \end{aligned}$$

It follows that

$$\left[\Gamma^{\mathrm{p},\partial,1}_{i,j}\right] = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \left[\Gamma^{\mathrm{p},\partial,2}_{i,j}\right] = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix},$$

and obviously ∇^{p} *fails to be symmetric* because

$$\Gamma^2_{12} \neq \Gamma^2_{21}.$$

Moreover, p determines a Riemannian structure g^{p} which, in standard coordinates (i.e., the ∂-basis), gives

$$\left[g^{\mathrm{p},\partial}_{i,j}\right] = \begin{bmatrix} e^{-2x} & 0 \\ 0 & e^{-2x} \end{bmatrix}.$$

The Ricci Lemma may be used to see that ∇^{p} is compatible with g^{p}. It amounts to check the compatibility equation

$$u\left(g^{\mathrm{p}}\left(v,w\right)\right)=g^{\mathrm{p}}\left(\nabla_u^{\mathrm{p}}v,w\right)+g^{\mathrm{p}}\left(v,\nabla_u^{\mathrm{p}}w\right)$$

for all tangent vector fields u,v,w. In the component form $u=\partial_i$, $v=\partial_j$, $w=\partial_k$, the left-hand side is

$$\begin{aligned}\left[\partial_1 g_{i,j}^{\mathrm{p},\partial}\right] &= \begin{bmatrix} -2e^{-2x} & 0 \\ 0 & -2e^{-2x}\end{bmatrix} \\ \left[\partial_2 g_{i,j}^{\mathrm{p},\partial}\right] &= \begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix},\end{aligned}$$

while the right-hand side is

$$\begin{aligned} & g^{\mathrm{p}}\left(\Gamma_{ki}^{\mathrm{p},\partial,m}\partial_m,\partial_j\right)+g^{\mathrm{p}}\left(\partial_i,\Gamma_{kj}^{\mathrm{p},\partial,m}\partial_m\right) \\ = \; & g_{m,j}^{\mathrm{p},\partial}\Gamma_{ki}^{\mathrm{p},\partial,m}+g_{m,i}^{\mathrm{p},\partial}\Gamma_{kj}^{\mathrm{p},\partial,m} \\ = \; & e^{-2x}\Gamma_{ki}^{\mathrm{p},\partial,j}+e^{-2x}\Gamma_{kj}^{\mathrm{p},\partial,i}\end{aligned}$$

and since

$$\begin{aligned}\left[\Gamma_{1,i}^{\mathrm{p},\partial,j}\right] &= \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}, \quad \left[\Gamma_{2,i}^{\mathrm{p},\partial,j}\right]=\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix}, \\ \left[\Gamma_{1,j}^{\mathrm{p},\partial,i}\right] &= \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}, \quad \left[\Gamma_{2,j}^{\mathrm{p},\partial,i}\right]=\begin{bmatrix} 0 & 0 \\ 0 & 0\end{bmatrix},\end{aligned}$$

we see then that ∇^{p} is indeed compatible with g^{p}. However, since it is not symmetric, it is not the Levi-Cività connection of *any* metric tensor field. In fact, the Levi-Cività connection $\nabla^{g^{\mathrm{p}}}$ determined by the parallelisation metric g^{p} can be found by solving the equation of the Ricci Lemma. In coordinates, one has that its components $\Gamma_{i,j}^{g^{\mathrm{p}},\partial,k}$ satisfy

$$\begin{aligned}\nabla_{\partial_i}^{g^{\mathrm{p}}}\partial_j &= \Gamma_{i,j}^{g^{\mathrm{p}},\partial,k}\partial_k, \text{ by definition, and} \\ g_{k,m}^{\mathrm{p},\partial}\Gamma_{ij}^{g^{\mathrm{p}},\partial,k} &= \frac{1}{2}\left(\partial_i g_{j,m}^{\mathrm{p},\partial}+\partial_j g_{i,m}^{\mathrm{p},\partial}-\partial_m g_{i,j}^{\mathrm{p},\partial}\right), \text{ by the Ricci Lemma,}\end{aligned}$$

so that

$$\left[\Gamma_{i,j}^{g^{\mathrm{p}},\partial,1}\right]=\begin{bmatrix} -1 & 0 \\ 0 & 1\end{bmatrix}, \quad \left[\Gamma_{i,j}^{g^{\mathrm{p}},\partial,2}\right]=\begin{bmatrix} 0 & -1 \\ -1 & 0\end{bmatrix},$$

with compatibility

$$\begin{aligned}\left[\Gamma_{1,i}^{g^{\mathrm{p}},\partial,j}\right] &= \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}, \quad \left[\Gamma_{2,j}^{g^{\mathrm{p}},\partial,i}\right]=\begin{bmatrix} 0 & -1 \\ 1 & 0\end{bmatrix}, \\ \left[\Gamma_{1,j}^{g^{\mathrm{p}},\partial,i}\right] &= \begin{bmatrix} -1 & 0 \\ 0 & -1\end{bmatrix}, \quad \left[\Gamma_{2,j}^{g^{\mathrm{p}},\partial,i}\right]=\begin{bmatrix} 0 & 1 \\ -1 & 0\end{bmatrix},\end{aligned}$$

so that, for $k=1,2$,

$$\partial_k g_{i,j}^{\mathrm{p},\partial}=e^{-2x}\left(\Gamma_{k,i}^{g^{\mathrm{p}},\partial,j}+\Gamma_{k,j}^{g^{\mathrm{p}},\partial,i}\right).$$

Jet bundles and stable jet bundles

Let (M, g) be a Riemannian manifold of dimension m, and identify TM and T^*M through g. For a vector bundle E over M, define the *jet bundle*

$$J(E) = E \oplus (E \otimes TM)$$

and the *iterated jet bundles*

$$J^i(E) = J\left(J^{i-1}(E)\right).$$

Likewise, define the *stable jet bundle*

$$(J(E))^{\mathrm{s}} = \mathbf{1} \oplus J(E) = \mathbf{1} \oplus E \oplus (E \otimes TM)$$

and the *iterated stable jet bundles*

$$\left(J^i(E)\right)^{\mathrm{s}} = \mathbf{1} \oplus J^i(E) = \mathbf{1} \oplus J\left(J^{i-1}(E)\right).$$

Remark that

$$\begin{aligned} (J(E))^{\mathrm{s}} &= \mathbf{1} \oplus E \oplus (E \otimes TM) \\ &\cong \mathbf{1} \oplus (E \otimes \mathbf{1}) \oplus (E \otimes TM) \\ &\cong \mathbf{1} \oplus E \otimes (\mathbf{1} \oplus TM) \\ &\cong \mathbf{1} \oplus E \otimes T^{\mathrm{s}}M, \end{aligned}$$

where $T^{\mathrm{s}}M$ is the stable tangent bundle of M. In that formalism,

$$\begin{aligned} (J(E))^{\mathrm{s}} &= \mathbf{1} \oplus (E \otimes T^{\mathrm{s}}M) \\ \left(J^2(E)\right)^{\mathrm{s}} &= \mathbf{1} \oplus (J(E) \otimes T^{\mathrm{s}}M) = \mathbf{1} \oplus \left(E \otimes T^{\mathrm{s}}M^{\otimes 2}\right) \\ \left(J^3(E)\right)^{\mathrm{s}} &= \mathbf{1} \oplus \left(J^2(E) \otimes T^{\mathrm{s}}M\right) = \mathbf{1} \oplus \left(E \otimes T^{\mathrm{s}}M^{\otimes 3}\right) \\ &\cdots \\ \left(J^i(E)\right)^{\mathrm{s}} &= \mathbf{1} \oplus \left(J^{i-1}(E) \otimes T^{\mathrm{s}}M\right) = \mathbf{1} \oplus \left(E \otimes T^{\mathrm{s}}M^{\otimes i}\right), \end{aligned}$$

and in particular

$$J^i(M) = J^i(TM) = \mathbf{1} \oplus \left(TM \otimes (\mathbf{1} \oplus TM)^{\otimes i}\right).$$

A vector bundle E is said to be *flat* if it admits a connection whose curvature vanishes identically. Vector bundles can admit inequivalent flat structures. A manifold M is said to be flat if TM is flat, and M is called i-th order flat if $J^i(M)$ admits a flat structure. We denote by $\alpha(M)$ the smallest integer i for which $J^i(M)$ admits a flat structure; otherwise, $\alpha(M) = \infty$. If $J^i(M)$ admits a flat structure for some i, then the rational Pontrjagin classes of M are zero in positive degrees and TM is rationally trivial. For $n > 1$, $J^i(\mathbb{C}\mathrm{P}^n)$ is never flat, so that, if $n > 1$, $\alpha(\mathbb{C}\mathrm{P}^n) = \infty$.

For the sphere S^n, one has $J(S^n) = J^1(S^n) = T(S^n) \otimes (\mathbf{1} \oplus T(S^n))$.

1.1.3 On connections

We will consider a linear connection on a bundle ξ over a compact smooth manifold M with boundary, as a map

$$\nabla \colon \Gamma(M,\xi) \to \Gamma(M, T^*M \otimes \xi)\,.$$

A linear connection ∇ on a manifold provides a covariant way to differentiate tensor fields. It provides a type-preserving derivation on the algebra of tensor fields that commutes with contractions. Given an arbitrary local basis of vector fields $\{X_a\}$, the most general linear connection is specified locally by a set of n^2 1-forms $\omega^a{}_b$ where n is the dimension of the manifold,

$$\nabla_{X_a} X_b = \omega^c{}_b(X_a)\, X_c.$$

Generally, we will be given a metric on the manifold and we will restrict ourselves to *metric-compatible connections*. However, we do not ask our connection to be torsion-free. In particular, we will deal mainly with connections on bundles constructed from the tangent bundle of the manifold.

So, for our connections, the following will hold in general, for fields X, Y, Z:

$$X(g(Y,Z)) = S(X,Y,Z) + g(\nabla_X Y, Z) + g(Y, \nabla_X Z),$$

but

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \neq 0$$

in general. That amounts to the connection be determined uniquely by

$$\begin{aligned} 2g(Z, \nabla_X Y) \;\; &= \;\; X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) \\ &\quad - g(X,[Y,Z]) - g(Y,[X,Z]) - g(Z,[Y,X]) \\ &\quad - g(X,T(Y,Z)) - g(Y,T(X,Z)) - g(Z,T(Y,X)). \end{aligned}$$

The general curvature operator $R_{X,Y}$ defined in terms of ∇ by

$$R_{X,Y} Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

is also a type-preserving tensor derivation on the algebra of tensor fields. The general $(3,1)$ curvature tensor R of ∇ is defined by

$$R(X,Y,Z,\beta) = \beta(R_{X,Y} Z),$$

where β is an arbitrary 1-form. This tensor gives rise to a set of local curvature 2-forms $R^a{}_b$:

$$R^a{}_b(X,Y) = \frac{1}{2}\, R(X,Y,X_b,e^a),$$

where $\{e^c\}$ is any local basis of 1-forms dual to $\{X_c\}$. In terms of the contraction operator i_X with respect to X, one has $i_{X_b}\, e^a \equiv i_b\, e^a = e^a(X_b) = \delta^a{}_b$. In terms of the connection forms, $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b$. It is customary as well to use Ω for the matrix of 2-forms $\Omega^a{}_b = \frac{1}{2} R^a{}_b$.

Determination of the connection from the relevant tensor fields

Such a connection can be fixed by specifying a $(2,0)$ symmetric metric tensor $\mathbf{g}$, a $(2,1)$ antisymmetric tensor $\mathbf{T}$ and a $(3,0)$ tensor $\mathbf{S}$, symmetric in its last two arguments. If we require that $\mathbf{T}$ be the torsion of ∇ and $\mathbf{S}$ be the gradient of $\mathbf{g}$, then it is straightforward to determine the connection in terms of these tensors. Indeed, since ∇ is defined to commute with contractions and reduce to differentiation on scalars, it follows from the relation

$$X(\mathbf{g}(Y,Z)) = \mathbf{S}(X,Y,Z) + \mathbf{g}(\nabla_X Y, Z) + \mathbf{g}(Y, \nabla_X Z)$$

that

$$\begin{aligned} 2\mathbf{g}(Z,\nabla_X Y) &= X(\mathbf{g}(Y,Z)) + Y(\mathbf{g}(Z,X)) - Z(\mathbf{g}(X,Y)) \\ &\quad - \mathbf{g}(X,[Y,Z]) - \mathbf{g}(Y,[X,Z]) - \mathbf{g}(Z,[Y,X]) \\ &\quad - \mathbf{g}(X,\mathbf{T}(Y,Z)) - \mathbf{g}(Y,\mathbf{T}(X,Z)) - \mathbf{g}(Z,\mathbf{T}(Y,X)) \\ &\quad - \mathbf{S}(X,Y,Z) - \mathbf{S}(Y,Z,X) + \mathbf{S}(Z,X,Y), \end{aligned}$$

where X, Y, Z are any vector fields.

1.1.4 Invariance theory for metric connections with torsion

Now we want to see that the invariants calculated by generalising [APS75II] are of the same kind as the ones defined in [ADS83]. What has to be done is to prove that the classes of the pullback bundles in the former agree with the ones given by the connections in the latter. For doing so, one needs to know about invariant polynomials in characteristic classes for connections involving torsion. According to [ADS83], this is done in very much the same way as in [ABS64], with some modifications based on calculations in [Don78, Section 1]. We will begin by recalling these.

Consider a Riemannian manifold (M,g) with a connection ∇ on its tangent bundle which preserves the metric g. Then, relatively to a geodesic coordinate system, the components of the metric tensor have a formal Taylor series whose coefficients may be expressed in terms of the components of the curvature and the torsion of the connection and its covariant derivatives. Donnelly holds this to be well known, but presents in [Don78] a proof along the lines of [ABS64], but taking into account that the connection need not have torsion zero.

Let (x) be the geodesic coordinate system and let e_i be the orthonormal frame obtained from $\dfrac{\partial}{\partial x^i}$ at p by parallel transport along radial geodesics through p. The dual frame to e_i is therefore a frame of 1-forms θ^i well defined near p. The connection forms relative to e_i will be denoted by ω^i_j and the radial field $x^i\dfrac{\partial}{\partial x^i}$ by $\vec{r}$. The structure equations are then

$$\begin{aligned} d\theta^i &= \omega^i_j \wedge \theta^j + T^i_{j,k} dx^j \wedge dx^k \\ d\omega^i_j &= \omega^i_k \wedge \omega^k_j + R^i_{j,k,l} dx^k \wedge dx^l \end{aligned}$$

with $T^i_{j,k}$, $R^i_{j,k,l}$ the components of the torsion and curvature tensors. With $i_{\vec{r}}$ the contraction with respect to the field $\vec{r}$, one obtains formulae:

$$\begin{aligned} i_{\vec{r}}\left(\theta^i\right) &= x^i \\ i_{\vec{r}}\ \left(\omega^i_j\right) &= 0 \\ g_{ij}dx^i \otimes dx_j &= \theta^\alpha \otimes \theta^\alpha \end{aligned}$$

and introducing the change of basis functions

$$\theta^i = a^i_j dx^j$$

since

$$g_{ij} = a^\alpha_i a^\alpha_j$$

it is enough to determine the a's in terms of the curvature and the torsion. To avoid confusion, we will denote by $L_{\vec{r}}$ the Lie derivative associated to field $\vec{r}$. Applying $L_{\vec{r}}$ to θ^i,

$$\begin{aligned} L_{\vec{r}}\,\theta^i &= i_{\vec{r}}\ d\theta^i + dx^i \\ & \quad i_{\vec{r}}\ \left(\omega^i_j \wedge \theta^j\right) + i_{\vec{r}}\ \left(T^i_{j,k}dx^j \wedge dx^k\right) + dx^i \end{aligned}$$

so that

$$L_{\vec{r}}\,\theta^i = -\omega^i_j x^j + 2T^i_{j,k}x^j dx^j \wedge dx^k + dx^i.$$

Write $\hat{a}$, $\hat{R}$, etc, for the formal Taylor series relative to x about p of the function indicated, and $\hat{a}\,[n]$, $\hat{R}\,[n]$ for the corresponding terms of homogeneity n in this expansion. Then, by Euler's formula, $L_{\vec{r}}$ preserves homogeneity and multiplies $\hat{a}\,[n]$ by n. Hence,

$$\left(n^2+n\right)\hat{a}^i_j\,[n] = -2x^j x^k \hat{R}^i_{j,k,l}\,[n-2] + (2n+2)\,\hat{T}^i_{j,k}\,[n-1]\,x^j;$$

compare with the conclusion in [ABS64].

1.2 Essential tools from Clifford theory

In this section we review the definitions and properties from Clifford algebras that we will need later. We recall the definitions of Clifford algebras and Clifford modules and their construction and classification.

1.2.1 Clifford algebra and spinor bundles

Clifford algebras

Definition 1.9 Let $k = \mathbb{R}$ or $\mathbb{C}$, and let V be a finite-dimensional k-vector space with non-degenerate inner product $\langle\ ,\ \rangle$ and corresponding quadratic form $q(v) = \langle v, v\rangle$. Then the *Clifford algebra* $C\ell(V,q)$ is the k-algebra with unit defined as the quotient of the tensor algebra

$$C\ell(V,q) = \bigoplus_{j=0}^{\infty} V^{\otimes j} \Big/ (v \otimes v + q(v))$$

for $v \in V^{\otimes 1}$, $q(v) \in k = V^{\otimes 0}$. The multiplicative structure in $C\ell(V,q)$ induced by the tensor product is termed *Clifford multiplication*.

The Clifford algebra is universal amongst k-algebras $\mathcal{A}$ equipped with a linear map $f\colon V \to \mathcal{A}$ satisfying $(fv)^2 = -q(v)\cdot 1$,

$$\begin{tikzcd} V \arrow[r] \arrow[dr, "\forall f"'] & C\ell(V,q) \arrow[d, "\exists!\, \widetilde{f}"] \\ & \mathcal{A} \end{tikzcd} .$$

The linear map $\alpha(v) = -v$, for example, extends to a unique algebra homomorphism

$$\alpha\colon C\ell(V,q) \to C\ell(V,q)$$

which satisfies $\alpha^2 = \mathrm{Id}$ and induces a decomposition into $(-1)^j$ eigenspaces, $j = 0, 1$,

$$C\ell(V,q) = C\ell^0(V,q) \oplus C\ell^1(V,q),$$

in which $C\ell^0(V,q)$ is in fact a subalgebra. A Clifford algebra also has a filtration

$$k = F^0 \subset V = F^1 \subset F^2 \subset F^3 \subset \cdots \subset C\ell(V,q)$$

induced by the filtration on the tensor algebra. The *associated graded algebra* $G^* = \bigoplus_{j\geq 0} G^j$ is defined by $G^j = F^j/F^{j-1}$. There is a canonical linear isomorphism

$$\Lambda^* V \xrightarrow{\cong} C\ell(V,q)$$

which induces an algebra isomorphism on the associated graded algebra.

Definition 1.10 The groups $\mathrm{Pin}(V,q)$ and $\mathrm{Spin}(V,q)$ are both multiplicative subgroups of $C\ell(V,q)$ generated by unit vectors $v \in V$,

$$\begin{aligned} \mathrm{Pin}(V,q) &= \{v_1 \cdots v_r;\ r \geq 0,\ q(v_j) = \pm 1\ \forall j\} \\ \mathrm{Spin}(V,q) &= \{v_1 \cdots v_r;\ r \text{ even},\ q(v_j) = \pm 1\ \forall j\} \ = \ \mathrm{Pin}(V,q) \cap C\ell^0(V,q). \end{aligned}$$

These groups act on $C\ell(V,q)$ via the *adjoint* and *twisted adjoint* representations

$$\mathrm{Ad}_\varphi(x) = \varphi \cdot x \cdot \varphi^{-1}, \qquad \widetilde{\mathrm{Ad}}_\varphi(x) = \alpha(\varphi)\cdot x \cdot \varphi^{-1}.$$

Restricting the action to $x \in V$, the twisted adjoint representation induces surjective homomorphisms

$$\mathrm{Pin}(V,q) \twoheadrightarrow \mathrm{O}(V,q), \qquad \mathrm{Spin}(V,q) \twoheadrightarrow \mathrm{SO}(V,q),$$

to the orthogonal and special orthogonal groups with respect to the form q.

Consider now the special case of real or complex n-space $V = k^n$ with the canonical inner product, and write $C\ell_n = C\ell(\mathbb{R}^n)$, $\mathbb{C}\ell_n = C\ell(\mathbb{C}^n)$. The *volume elements* ω, $\omega_{\mathbb{C}}$ of these algebras are defined in terms of the canonical basis by

$$\omega = e_1 \cdots e_n, \qquad \omega_{\mathbb{C}} = i^{\lfloor (n+1)/2 \rfloor}\, e_1 \cdots e_n.$$

Theorem 1.11 *For all $n \geq 0$ there is an isomorphism $C\ell^0_{n+1} \cong C\ell_n$ and 'periodicity' isomorphisms*

$$C\ell_{n+8} \cong C\ell_n \otimes M_{16}(\mathbb{R}), \qquad \mathbb{C}\ell_{n+2} \cong \mathbb{C}\ell_n \otimes_{\mathbb{C}} M_2(\mathbb{C}).$$

The linear isomorphism $C\ell_n \cong \Lambda^*\mathbb{R}$ identifies the Clifford multiplication of $v \in \mathbb{R}^n$ and $\varphi \in C\ell_n$ in terms of the wedge $v \wedge -\colon \Lambda^p\mathbb{R} \to \Lambda^{p+1}\mathbb{R}$ and contraction $v^*\colon \Lambda^p\mathbb{R} \to \Lambda^{p-1}\mathbb{R}$, $v^* \in (\mathbb{R}^n)^*$,

$$v \cdot \varphi = v \wedge \varphi - v^*(\varphi).$$

The twisted adjoint action of $\mathrm{Spin}_n = \mathrm{Spin}(\mathbb{R}^n)$ on $x \in \mathbb{R}^n$ gives a non-trivial double cover

$$\mathrm{Spin}_n \longrightarrow \mathrm{SO}_n.$$

which is the universal cover if $n \geq 3$.

Clifford modules

A *representation* of a Clifford algebra $C\ell(V,q)$ is an algebra homomorphism

$$\rho\colon C\ell(V,q) \to \mathrm{End}_k M.$$

The vector space M is termed a *Clifford module* and the action of $\varphi \in C\ell(V,q)$ is called *Clifford multiplication*. Two representations are *equivalent* if there is a linear isomorphism between the modules which commutes with the Clifford multiplication.

Let $\mathfrak{M}_n$, $\mathfrak{M}^{\mathbb{C}}_n$ be the Grothendieck groups of irreducible $C\ell_n$-modules and $\mathbb{C}\ell_n$-modules, respectively; this is just the free abelian group generated by the irreducible representations. An arbitrary representation is decomposable as a direct sum of irreducibles and so corresponds to an element of the Grothendieck group with positive coefficients. From the periodicity isomorphisms of Theorem 1.11 it follows that

$$\mathfrak{M}_{n+8} \cong \mathfrak{M}_n, \qquad \mathfrak{M}^{\mathbb{C}}_{n+2} \cong \mathfrak{M}^{\mathbb{C}}_n.$$

Proposition 1.12 *Up to equivalence there is just one irreducible representation W_n of Cl_n for $n \not\equiv 3 \pmod 4$. For $n \equiv 3 \pmod 4$ there are two irreducible Cl_n-modules $W_n^{\pm}$ given by the splitting*

$$W_{n+1} = W_n^+ \oplus W_n^-, \qquad W_n^{\pm} = (1 \pm \omega)(W_{n+1})$$

of the irreducible representation of $C\ell_{n+1}$ into non-equivalent irreducible representations of $C\ell_{n+1}^0 \cong C\ell_n$. In the complex case there is a unique irreducible representation of $\mathbb{C}l_n$ for n even which splits into two non-equivalent representations for n odd.

Restricting an irreducible representation of $C\ell_n$ to $\mathrm{Spin}_n \subset C\ell^0$ defines the *real spinor representation*

$$\Delta_n \colon \mathrm{Spin}_n \longrightarrow \mathrm{End}_{\mathbb{R}}(S_n),$$

which is irreducible except for the case $n = 4k$ when it splits into two non-equivalent irreducibles,

$$\Delta_{4k} = \Delta_{4k}^+ \oplus \Delta_{4k}^-, \qquad S_{4k}^{\pm} = (1 \pm \omega)(S_{4k}).$$

Analogously one defines the *complex spinor representation*

$$\Delta_n^{\mathbb{C}} \colon \mathrm{Spin}_n^{\mathbb{C}} \longrightarrow \mathrm{End}_{\mathbb{C}}(S_n^{\mathbb{C}}),$$

which is irreducible for n odd and for $n = 2m$ splits as

$$\Delta_{2m}^{\mathbb{C}} = \Delta_{2m}^{\mathbb{C}+} \oplus \Delta_{2m}^{\mathbb{C}-}, \qquad S_{2m}^{\mathbb{C}\pm} = (1 \pm \omega)(S_{4k}^{\mathbb{C}}).$$

Definition 1.13 A $C\ell_n$-module W is $\mathbb{Z}_2$-*graded* if it splits as $W = W^0 \oplus W^1$ with

$$C\ell_n^i \cdot W^j \subseteq W^{i+j \bmod 2} \quad \text{for} \quad i, j \in \{0, 1\}.$$

A $\mathbb{Z}_2$-graded module W is completely determined by the module W^0 over $C\ell_n^0 \cong C\ell_{n-1}$ and we may identify $\mathbb{Z}_2$-graded representations of $C\ell_n$ with ungraded representations of $C\ell_{n-1}$. The advantage of graded representations is that one can tensor $\mathbb{Z}_2$-graded modules V over $C\ell_m$ and W over $C\ell_n$ to obtain a $\mathbb{Z}_2$-graded module $V \widehat{\otimes} W$ over $C\ell_{m+n} \cong C\ell_m \widehat{\otimes} C\ell_n$, where

$$(V \widehat{\otimes} W)^j = V^0 \otimes W^j \oplus V^1 \otimes W^{1-j} \quad \text{for} \quad j \in \{0, 1\}$$

with the Clifford multiplication $(\varphi \otimes \psi)(v \otimes w) = (-1)^{\deg(\psi)\deg(v)} \varphi v \otimes \psi w$.

The Grothendieck groups $\widehat{\mathfrak{M}}_*$ of $\mathbb{Z}_2$-graded representations are given the structure of a graded ring with this tensor product, and similarly for the Grothendieck groups $\widehat{\mathfrak{M}}_*^{\mathbb{C}}$ of complex $\mathbb{Z}_2$-graded representations.

Clifford and spinor bundles

The Clifford algebra and module constructions above can be extended from vector spaces with a quadratic form q to vector bundles E over a Riemannian manifold (M, g), where the fibres of E have an inner product induced from g.

Recall that E is orientable if the first Stiefel–Whitney class $w_1(E)$ vanishes. If

$$w_E\colon H^0(\mathrm{O}_n;\mathbb{Z}_2) \to H^1(M;\mathbb{Z}_2)$$

is the connecting map defined by the fibration

$$\mathrm{O}_n \to P_{\mathrm{O}}(E) \to M$$

of orthonormal frames in E, we can write $w_1(E) = w_E(g_1)$ where g_1 generates $H^0(\mathrm{O}_n;\mathbb{Z}_2)$. If $w_2(E) = 0$ then the orientations correspond to elements of

$$H^0(M, \mathbb{Z}_2) \cong \ker(H^0(P_{\mathrm{O}}(E);\mathbb{Z}_2) \to H^0(\mathrm{O}_n;\mathbb{Z}_2)).$$

Given an orientation we can consider the bundle $P_{\mathrm{SO}}(E)$ of *positively oriented* orthonormal frames. Orientability is used to define Clifford algebras in this context.

Definition 1.14 The *Clifford bundle* of an oriented vector bundle E is given by

$$C\ell(E) = P_{\mathrm{SO}}(E) \times_{\mathrm{SO}_n} C\ell_n,$$

the bundle associated to the canonical action of SO_n on $C\ell(\mathbb{R}^n)$. Alternatively, it is a quotient of the tensor product bundle

$$C\ell(E) = \bigoplus_{j=0}^{\infty} E^{\otimes j} \Big/ (v \otimes v + q(v)) .$$

There is a unique involution α of $C\ell(E)$ extending $v \mapsto -v$ on E and an eigenbundle decomposition

$$C\ell(E) = C\ell^0(E) \oplus C\ell^1(E).$$

Furthermore, one has a vector bundle isometry

$$\Lambda^* E \xrightarrow{\cong} C\ell(E)$$

which identifies $\Lambda^{\mathrm{even}} E$ and $\Lambda^{\mathrm{odd}} E$ with $C\ell^0(E)$ and $C\ell^1(E)$ respectively.

Definition 1.15 A *spin structure* on E is a double cover

$$P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$$

whose restriction to the fibre SO_n of $\pi\colon P_{\mathrm{SO}}(E) \to X$ is non-trivial.

The obstruction to the existence of a spin structure is the second Stiefel–Whitney class $w_2(E)$. If

$$w_E\colon H^1(\mathrm{SO}_n;\mathbb{Z}_2) \to H^2(M;\mathbb{Z}_2)$$

is now the connecting map defined by the fibration

$$\mathrm{SO}_n \to P_{\mathrm{SO}}(E) \to M$$

then $w_2(E) = w_E(g_2)$ where g_2 generates $H^1(\mathrm{SO}_n;\mathbb{Z}_2)$. If $w_2(E)$ vanishes then spin structures correspond to elements of

$$H^1(M,\mathbb{Z}_2) \cong \ker(H^1(P_{\mathrm{SO}}(E);\mathbb{Z}_2) \to H^1(\mathrm{SO}_n;\mathbb{Z}_2)).$$

Definition 1.16 A *spin manifold* is an oriented Riemannian manifold whose tangent bundle admits a spin structure. The *spin cobordism group* Ω_n^{Spin} is the abelian group generated by compact connected n-dimensional spin manifolds modulo the relations $[N_1] + [N_2] = 0$ if there is a spin and orientation preserving diffeomorphism $N_1 \sqcup N_2 \to \delta M$ for some compact connected spin $(n+1)$-manifold M.

Spin structures are necessary to extend Clifford modules to vector bundles.

Definition 1.17 A *real* or *complex spinor bundle* of an oriented vector bundle E with spin structure $P_{\mathrm{Spin}}(E) \to P_{\mathrm{SO}}(E)$ is an induced bundle

$$S_W(E) = P_{\mathrm{Spin}}(E) \times_{\mathrm{Spin}_n} W,$$

where W is a $C\ell_n$- or $\mathbb{C}\ell_n$-module. If W is $\mathbb{Z}_2$-graded, then so is the spinor bundle.

The action of the Clifford algebra on W induces an action of bundles

$$C\ell(E) \oplus S_W(E) \longrightarrow S_W(E).$$

Two spinor bundles are *equivalent* if they are equivalent as bundles of $C\ell(E)$-modules and *irreducible* if the $C\ell(E_x)$-module at each fibre is irreducible. If E is a bundle over a connected n-manifold, it follows from Proposition 1.12 that there is a unique irreducible spinor bundle $S_n(E)$, or $S_n^{\mathbb{C}}(E)$, unless $n+1$ is divisible by 4, or by 2 in the complex case.

The irreducible spinor bundles $S_{4k}(E)$ and $S_{2m}^{\mathbb{C}}$ decompose as a direct sum of irreducible $C\ell^0(E)$-modules $S^{\pm}(E)$ or $S_{\mathbb{C}}^{\pm}(E)$, where

$$\begin{aligned} S^{\pm}(E) &= (1 \pm \omega)(S_{4k}(E)) = P_{\mathrm{Spin}}(E) \times_{\Delta_{4k}^{\pm}} S_{4k}^{\pm}, \\ S_{\mathbb{C}}^{\pm}(E) &= (1 \pm \omega_{\mathbb{C}})(S_{2m}^{\mathbb{C}}(E)) = P_{\mathrm{Spin}}(E) \times_{\Delta_{2m}^{\mathbb{C}\pm}} S_{2m}^{\mathbb{C}\pm}\ . \end{aligned}$$

These correspond to the two irreducible $\mathbb{Z}_2$-graded spinor bundles in these dimensions.

Chapter 2

Some tools from algebraic topology

In this chapter we review some essentials from the theory of characteristic classes, cobordism theory, and elliptic cohomology. In the first section we recall the definition of Hirzebruch genera of elliptic type for manifolds with geometric structure. This is a broad (and fascinating) field and we are by no means exhaustive in our presentation. We also recall the definition of Devoto's equivariant elliptic cohomology and its modularity properties.

We also discuss some approaches to 'relative' versions of the classical theory of characteristic classes and multiplicative sequences, relating them to constructions in relative cobordism and K-theory. Some aspects of the Pontrjagin–Thom construction for framed bordism, collapse maps on disks and relative characteristic classes are reviewed for the benefit of the forthcoming exposition.

2.1 Elliptic genera and cohomology theories

We begin by recalling the definitions of Hirzebruch genera and elliptic genera for various cobordism rings of smooth manifolds with boundary and equipped with a fixed geometric structure. We will concentrate on SO-structures, but the presentation can be easily adapted to other geometric structures such as Spin, Spin^c, U, SU or String. We will also concentrate on the case of the classical level two elliptic genus, which is related to the signature operator.

We next recall the G-equivariant elliptic genus introduced by Devoto [Dev98], for G a finite group of odd order, and the algebraic descriptions of the coefficient and cohomology rings for the associated generalised cohomology theories, which will be relevant later for one of our generalisations of the Atiyah–Patodi–Singer construction.

To avoid too much digression in this section we have omitted many interesting aspects of the theory presented. These include a discussion of important genera of more general elliptic type, such as the Witten genus, a Spin^c elliptic Todd genus, and generalised Eisenstein genera. Tamanoi's description [Tam99] of elliptic genera in terms of vertex operator algebras is also interesting for its relation to the Segal category and Virasoro bundles. In later work we will also consider elliptic genera modulo n, for n an odd integer, since as was already seen by [Dev96a] they give rise to invariants for $\mathbb{Z}/n$-manifolds in the sense of Sullivan, closely related to the invariants we consider. It will then be necessary to consider also the divided congruence rings of [Kat75] and modular forms modulo prime ideals of [Lau99].

2.1.1 Classical elliptic genera and level 2 elliptic cohomology

Definition 2.1 A *genus*, in the sense of [Hir66], is a ring homomorphism from the oriented bordism ring to a commutative $\mathbb{Q}$-algebra with unit,

$$\phi\colon \Omega_*^{\mathrm{SO}} \to R.$$

In [Tho54], Thom showed that modulo torsion the bordism ring is generated by the cobordism classes of the even-dimensional complex projective spaces. More explicitly, the map $[\mathbb{C}\mathrm{P}^{2k}] \mapsto x_{4k}$ defines an isomorphism of graded rings

$$\Omega_*^{\mathrm{SO}} \otimes \mathbb{Z}[\tfrac{1}{2}] \cong \mathbb{Z}[\tfrac{1}{2}][x_4, x_8, x_{12}, \ldots].$$

A genus ϕ is uniquely determined by its values on these generators, and hence by the following formal power series, termed the *logarithm* of the genus:

$$g(x) = x + \sum_{k\geq 1} \phi(\mathbb{C}\mathrm{P}^{2k})\,\frac{x^{2k+1}}{2k+1}.$$

Alternatively, a genus may be specified by

- a total Hirzebruch class $\mathcal{P} \in \prod_{i\geq 0} H^i(B\mathrm{SO}; R)$, or by
- a *characteristic series*

  $$P(u) = 1 + \sum_{k\geq 1} r_k u^{2k}$$

  in $\prod_{i\geq 0} H^i(\mathbb{C}\mathrm{P}^\infty; R) = R[[u]]$, $u \in H^2(\mathbb{C}\mathrm{P}^\infty)$.

These descriptions determine the genus by the formulas

$$\begin{aligned} \phi(X) &= \mathcal{P}(TX)[X] \\ g^{-1}(u) &= u/P(u); \end{aligned}$$

see [Gal96, HBJ92] for further details.

Definition 2.2 [Och87] An *elliptic genus* is a genus $\phi\colon \Omega_*^{\mathrm{SO}} \to R$ whose logarithm satisfies

$$g(x) = \int_0^x (1 - 2\delta t^2 + \varepsilon t^4)^{-\frac{1}{2}}\,dt$$

for some $\delta, \varepsilon \in R$.

Two classical examples of elliptic genera are:

1. The genus $\phi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ defined by taking $\delta = -\frac{1}{8}$, $\varepsilon = 0$ and

$$g(x) = \int_0^x (1 - \tfrac{1}{4}t^2)^{-\frac{1}{2}} dt = 2\sinh^{-1}(x/2)$$

   is the $\widehat{A}$*-genus*. It has characteristic series $P(u) = \dfrac{u/2}{\sinh(u/2)}$.

2. The genus $\phi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}$ defined by taking $\delta = \varepsilon = 1$ and

$$g(x) = \int_0^x (1 - 2t^2 + t^4)^{-\frac{1}{2}} dt = \tanh^{-1}(x)$$

   is the *signature* or L*-genus*. It has characteristic series $P(u) = \dfrac{u}{\tanh(u)}$.

A *universal elliptic genus* is a genus $\Phi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}[\delta, \varepsilon]$ where δ, ε are two algebraically independent indeterminates; it is the unique ring homomorphism satisfying the formal power series identity

$$(1 - 2\delta\, x^2 + \varepsilon\, x^4)^{-\frac{1}{2}} = \sum_{n\geq 0} \Phi(\mathbb{C}\mathrm{P}^{2n})\, x^{2n}. \tag{2.1}$$

The corresponding logarithm may be expressed as

$$g(x) = \sum_{k\geq 0} P_n(\delta/\sqrt{\varepsilon})\, \frac{x^{2k+1}}{2k+1}$$

in terms of the Legendre polynomials $P_n(z)$.

By Quillen's theorem [Qui69] the image $\Phi\Omega_*^{\mathrm{SO}}$ is generated by the coefficients of the corresponding *formal group law*

$$F(x,y) = g^{-1}(g(x) + g(y)) = \frac{x\sqrt{1 - 2\delta y^2 + \varepsilon y^4} + y\sqrt{1 - 2\delta x^2 + \varepsilon x^4}}{1 - \varepsilon\, x^2 y^2}$$

where the second equality is Euler's addition formula for the elliptic integral. Examining the coefficients of the corresponding power series in x and y one concludes

Proposition 2.3 [LRS95] *The universal elliptic genus defines a map*

$$\Phi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon].$$

2.1.2 Modularity

Let $\Gamma_0(2)$ be the subgroup of $\mathrm{SL}_2(\mathbb{Z})$ consisting of the matrices $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with c even, and let $\mathfrak{h}_+ = \{\tau \in \mathbb{C};\ \mathrm{im}(\tau) > 0\,\}$ be the upper half plane on which $\Gamma_0(2)$ and $\mathrm{SL}_2(\mathbb{Z})$ act by Möbius transformations:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}(\tau) = \frac{a\tau + b}{c\tau + d}.$$

The group $\Gamma_0(2)$ has fundamental domain $\{\tau\ :\ |\mathrm{Re}(\tau) - \frac{1}{2}| \le \frac{1}{2}, |\tau - \frac{1}{2}| \ge \frac{1}{2}\}$ and cusps $\tau = i\infty$ and $\tau = 0$.

Definition 2.4 A *modular function* of weight $k \ge 0$ for $\Gamma_0(2)$ is a meromorphic function $\vartheta\colon \mathfrak{h}_+ \to \mathbb{C}$ such that

1. $\vartheta\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau + d)^k \vartheta(\tau)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, $\tau \in \mathfrak{h}_+$,
2. ϑ is meromorphic at both cusps; that is, the functions $\vartheta(\tau)$ and $\vartheta'(\tau) = \tau^{-k}\vartheta(-1/\tau)$ may be written

$$\vartheta(\tau) = \sum_{r \ge K} a_r\, q^r, \qquad \vartheta'(\tau) = \sum_{r \ge K} b_r\, q^{r/2}, \qquad q = e^{2\pi i \tau},$$

for some $K \in \mathbb{Z}$. These are termed the *q-expansions* of ϑ at the cusps $\tau = i\infty, 0$.

A *modular form* is a modular function which is holomorphic on $\mathfrak{h}_+$ and at $\tau = i\infty, 0$.

Since $-\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \in \Gamma_0(2)$, the first property says the weight is always even. Since $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right) \in \Gamma_0(2)$ we have $\vartheta(\tau + 1) = \vartheta(\tau)$ and $\vartheta'(\tau + 2) = \vartheta'(\tau)$ and so the q-expansions make sense.

Landweber and Stong [LS88] and Zagier [Zag88] have shown that the universal elliptic genus may be regarded as taking modular forms as values:

Proposition 2.5 *There is a universal elliptic genus $\Phi\colon \Omega_*^{\mathrm{SO}} \to \mathbb{Q}[[q]]$ whose values on bordism classes $[X^{4k}]$ are the q-expansions at $\tau = i\infty$ of modular forms of weight $2k$ on $\Gamma_0(2)$, with the values of δ and ε given by*

$$\delta = -\frac{1}{8} - 3\sum_{n \ge 1}\Bigg(\sum_{d\ \mathrm{odd},\, d|n} d\Bigg) q^n, \qquad \varepsilon = \sum_{n \ge 1}\Bigg(\sum_{\frac{n}{d}\ \mathrm{odd},\, d|n} d^3\Bigg) q^n, \qquad \textit{with } q = e^{2\pi i\tau}.$$

The corresponding characteristic series may be expressed as

$$P(u) = \exp\left(\sum_{k \ge 1} \frac{2\,\widetilde{G}_{2k}(\tau)\, u^{2k}}{(2k)!}\right) = \frac{u/2}{\sinh(u/2)} \prod_{n \ge 1}\left(\frac{(1 - q^n)^2}{(1 - q^n e^u)(1 - q^n e^{-u})}\right)^{(-1)^n}$$

where $\widetilde{G}_{2k}$ are related to the classical Eisenstein modular forms G_{2k} by

$$\widetilde{G}_{2k}(\tau) = -G_{2k}(\tau) + 2\, G_{2k}(2\tau).$$

Later in this thesis it will sometimes be convenient to identify modular forms for $\Gamma_0(2)$ with their q-expansions at the *other* cusp. The corresponding characteristic series P' is then given by $P'(\tau, u) = P(\frac{-1}{2\tau}, \frac{u}{\tau})$, or explicity:

$$P'(u) = \exp\left(\sum_{k\geq 1} \frac{4\,G^*_{2k}(\tau)\,u^{2k}}{(2k)!}\right) = \frac{u/2}{\tanh(u/2)} \prod_{n\geq 1} \frac{(1+q^n e^u)(1+q^n e^{-u})}{(1-q^n e^u)(1-q^n e^{-u})} \cdot a(q)\,.$$

Here $G^*_{2k}(\tau) = G_{2k}(\tau) - 2^{2k-1} G_{2k}(2\tau)$ and the normalising factor $a(q)$, necessary so that $P'(0) = 1$, may be written in terms of the Dedekind η-function as

$$a(q) = \eta(q)^4 \eta(q^2)^{-2}, \quad \text{where} \quad \eta(q) = q^{1/24} \prod_{n\geq 1} (1-q^n).$$

The q-expansions of the parameters δ, ε at this cusp are

$$\delta = \frac{1}{4} + 6 \sum_{n\geq 1} \left(\sum_{d \text{ odd},\, d|n} d \right) q^n, \qquad \varepsilon = \frac{a(q)^4}{16} = \frac{1}{16} + \sum_{n\geq 1} \left(\sum_{d|n} (-1)^d d^3 \right) q^n.$$

With motivation from physics, and assuming that the manifold X has a spin structure, Witten [Wit88] has shown that P' may be considered as giving an expression for the S^1-equivariant index of the Dirac–Ramond operator on an infinite-dimensional manifold of free smooth loops on X. This 'explains' why modular forms arising as genera of spin manifolds always have q-expansions with *integer* coefficients.

The elements δ, ε are algebraically independent modular forms of weight 2 and 4 respectively and in fact generate the ring of modular forms for $\Gamma_0(2)$. In [LRS95] it is shown that the image $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ of Φ is precisely the subring of modular forms whose q-expansion coefficients at $\tau = i\infty$ lie in $\mathbb{Z}[\frac{1}{2}]$. It is also shown that on inverting the discriminant

$$\Delta = \varepsilon(\delta^2 - \varepsilon)^2$$

of the Jacobi quartic $y^2 = 1 - 2\delta x^2 + \varepsilon x^4$, the ring $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon][\Delta^{-1}]$ coincides with the modular functions with q-expansion coefficients in $\mathbb{Z}[\frac{1}{2}]$ which are holomorphic on $\mathfrak{h}_+$ but not necessarily at the cusps.

2.1.3 Elliptic cohomology theories and the Miller character

For every element ω of positive degree in $\mathbb{Z}[\frac{1}{2}][\delta, \varepsilon]$ there is a functor defined on CW-complexes by

$$(\mathcal{E}\ell\ell^\omega)_*(X) = M\mathrm{SO}_*(X) \otimes_{\Omega^{\mathrm{SO}}_*} \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon, \omega^{-1}].$$

As proved in [LRS95] and [Fra92], this functor satisfies the axioms of a generalised homology theory, since after inverting ω the Landweber Exact Functor Theorem applies. The dual cohomology theory, defined using the Spanier–Whitehead duality operator, may be expressed when X is a finite CW-complex as

$$(\mathcal{E}\ell\ell^{\omega})^*(X) = M\mathrm{SO}^*(X) \otimes_{\Omega^*_{\mathrm{SO}}} \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon, \omega^{-1}].$$

For the usual choice of $\omega = \Delta = \varepsilon(\delta^2 - \varepsilon)^2$ we write simply $\mathcal{E}\ell\ell^*(X)$.

The *Miller character*, defined in [Mil89], is a natural transformation of multiplicative cohomology theories of the form

$$\lambda\colon \mathcal{E}\ell\ell^* \to KO^*[\tfrac{1}{2}][[q]].$$

The importance of the Miller character is that the composite

$$\mathcal{E}\ell\ell^* \xrightarrow{\lambda} KO^*[\tfrac{1}{2}][[q]] \xrightarrow{c} K^*[\tfrac{1}{2}][[q]] \xrightarrow{\mathrm{ch}} H^*_{\mathbb{Q}}[[q]] \tag{2.2}$$

defined via the complexification c from KO- to K-theory and the Chern character from K-theory to ordinary cohomology, is closely related to the elliptic genus.

In the case $X = \mathrm{pt}$, the Miller character on the coefficient rings is just the graded ring homomorphism

$$\begin{array}{ccc} \mathbb{Z}[\tfrac{1}{2}][\delta, \varepsilon][\Delta^{-1}] & \xrightarrow{\lambda_*} & \mathbb{Z}[\tfrac{1}{2}][v^2][[q]] \\ \vartheta(\tau) & \mapsto & v^{2k}\,\tilde{\vartheta}(q) \end{array}$$

which sends a modular function $\vartheta(\tau)$ of weight $4k$ to its q-expansion at the cusp $\tau = i\infty$; in particular the images of δ and ε are the formal power series given in described in Proposition 2.5. Here $v^2 = y$ is the usual generator for $KO[\frac{1}{2}]$-theory, in degree 4.

For $X = \mathbb{C}\mathrm{P}^\infty$ we have the complex orientation class $x^{\mathcal{E}} \in \mathcal{E}\ell\ell^2(\mathbb{C}\mathrm{P}^1)$, with

$$\mathcal{E}\ell\ell^*(\mathbb{C}\mathrm{P}^\infty) \cong \mathcal{E}\ell\ell^*[[x^{\mathcal{E}}]],$$

and the Miller character is determined by its value $t^{\mathcal{K}}_{\mathcal{E}}(x^{\mathcal{E}})$ on $x^{\mathcal{E}}$,

$$t^{\mathcal{K}}_{\mathcal{E}}(x^{\varepsilon}) \;=\; x^{\mathcal{K}} \prod_{n\geq 1} \left(1 - \frac{q^n v^2 (x^{\mathcal{K}})^2}{(1-q^n)^2}\right)^{(-1)^n}.$$

On applying the Chern character to the complexification we obtain

$$\mathrm{ch}\left(\mathrm{t}^{\mathcal{K}}_{\mathcal{E}}(x^{\varepsilon})\right) = \left(e^{\frac{x}{2}} - e^{\frac{-x}{2}}\right) \prod_{n\geq 1} \left(1 - \frac{q^n v^2\left(e^x + e^{-x} - 2\right)}{\left(1-q^n\right)^2}\right)^{(-1)^n}$$

which is just the class $x/P(x)$ where P is the characteristic series for the elliptic genus given in Proposition 2.5.

In fact this series corresponds to the q-expansion of the Jacobi elliptic sine $\mathrm{s}^{\mathcal{E}}$

$$\mathrm{ch}\left(\mathrm{t}^{\mathcal{K}}_{\mathcal{E}}(x^{\varepsilon})\right) = \mathrm{s}^{\mathcal{E}}(\tau, x) = \left(\wp(\tau, x) - e_{\varepsilon}(\tau)\right)^{-\frac{1}{2}}$$

which is known to be a Jacobi meromorphic form of weight -1 and index 0. From the general theory of Hirzebruch genera [Hir66], computation of the universal elliptic genus of any projective space $\mathbb{C}\mathrm{P}^{2k}$ involves only taking derivatives and evaluation, and in general we may identify the modular forms obtained with Jacobi forms of index 0:

$$\mathcal{E}\ell\ell^*\left(\mathbb{C}\mathrm{P}^\infty\right) \cong \mathcal{E}\ell\ell^*\left[[x^\varepsilon]\right] \hookrightarrow \mathcal{J}^{\mathrm{mer}}_{*,0}\left(\Gamma_0(2); \mathbb{Z}[\tfrac{1}{2}]\right).$$

In terms of theta functions we may also write

$$\mathrm{s}^{\mathcal{E}}(\tau, x)^{-1} = \varepsilon^{\frac{1}{4}} \frac{\vartheta_{\left(\frac{1}{2},0\right)}\left(\tau, \frac{x}{2\pi i}\right)}{\vartheta_{\left(\frac{1}{2},\frac{1}{2}\right)}\left(\tau, \frac{x}{2\pi i}\right)};$$

see [Dev96b, EZ85] for more details.

2.1.4 Equivariant elliptic cohomology

In [Dev98], Devoto showed that for any finite group G of odd order one may define a stable G-equivariant cohomology theory on finite G-CW-complexes, termed *equivariant elliptic cohomology*, by

$$\mathcal{E}\ell\ell^*_G(X) = M\mathrm{SO}^*_G(X) \otimes_{M\mathrm{SO}^*_G} \mathcal{E}\ell\ell^*_G \tag{2.3}$$

where $M\mathrm{SO}^*_G$ is $\mathbb{Z}$-graded oriented equivariant cobordism theory [CW89]. The graded ring $\mathcal{E}\ell\ell^*_G = \mathcal{E}\ell\ell^*_G(\mathrm{pt})$ is related to the moduli space of G-coverings of Jacobi quartics, and comes equipped with a universal *twisted elliptic genus*

$$\Phi_G : M\mathrm{SO}^*_G \longrightarrow \mathcal{E}\ell\ell^*_G. \tag{2.4}$$

This is the G-equivariant version of the definition of ordinary elliptic cohomology by

$$\mathcal{E}\ell\ell^*_G(X) = M\mathrm{SO}^*(X) \otimes_{M\mathrm{SO}^*} \mathcal{E}\ell\ell^*$$

where the coefficient ring $\mathcal{E}\ell\ell^* \cong \mathbb{Z}[\frac{1}{2}][\delta, \epsilon][\Delta^{-1}]$ is the graded ring of modular functions which are holomorphic away from the cusps and have q-expansion coefficients in $\mathbb{Z}[\frac{1}{2}]$, as above. If X has a *free* G-action, then $\mathcal{E}\ell\ell^*_G X$ will be isomorphic to $\mathcal{E}\ell\ell^*(X/G) \otimes \mathbb{Z}\left[\frac{1}{|G|}\right]$.

For the equivariant case, Devoto makes an appropriate generalisation of the notion of "modular form". Let $TG = \{\,(x,y) \in G^2\,;\,[x,y] = 1\,\}$ be the set of pairs of commuting elements of G. Then the usual action of $\Gamma_0(2)$ on $\mathfrak{h}_+$ as usual and the conjugation action of G on TG are are combined to give actions ρ_k of $\Gamma_0(2) \times G$ on the ring of functions $\vartheta\colon TG \times \mathfrak{h}_+ \to \mathbb{C}$,

$$\rho_k(A, g)\,\vartheta : ((x,y), \tau) \longmapsto (c\tau + d)^{-k}\,\vartheta\left(\left(\,gx^d y^{-c} g^{-1}, gx^{-b} y^a g^{-1}\right), \frac{a\tau + b}{c\tau + d}\right) \tag{2.5}$$

for $k \in \mathbb{Z}$, $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(2)$, $g \in G$ and $((x,y),\tau) \in TG \times \mathfrak{h}_+$.

We write ζ_j for the primitive jth root of unity $e^{2\pi i/j} \in \mathbb{C}$.

Definition 2.6 The graded ring $\mathcal{E}\ell\ell^*_G = \bigoplus \mathcal{E}\ell\ell^{-2k}_G$ is the ring of functions $\vartheta \colon TG \times \mathfrak{h}_+ \to \mathbb{C}$ satisfying the following conditions, for some $k \in \mathbb{Z}$:

1. $\rho_k(A,g)\vartheta = \vartheta$ for all $(A,g) \in \Gamma_0(2) \times G$,

2. for each $(x,y) \in TG$ the function $\vartheta((x,y),\tau)$ is holomorphic, and is meromorphic at the cusps; that is, the functions $\vartheta((x,y),\tau)$ and $\vartheta'((x,y),\tau) = \tau^{-k}\vartheta((x,y),-\frac{1}{\tau})$ have q-expansions
$$\vartheta((x,y),\tau) = \sum_{r\geq K} a_r\, q^{r/|x|}, \qquad \vartheta'((x,y),\tau) = \sum_{r\geq K} b_r\, q^{r/|x|}, \qquad q = e^{2\pi i\tau}, \tag{2.6}$$
for some $K \in \mathbb{Z}$,

3. the coefficients $a_r(x,y), b_r(x,y)$ lie in $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{|G|}, \zeta_{|xy|}\right]$ and satisfy
$$\sigma_n(a_r(x,y)) = a_r(x,y^n), \qquad \sigma_n(b_r(x,y)) = b_r(x,y^n) \tag{2.7}$$
for σ_n a ring automorphism of $\mathbb{Z}\left[\frac{1}{2}, \frac{1}{|G|}, \zeta_m\right]$ given by $\sigma_n(\zeta_m) = {\zeta_m}^n$ for any n coprime to m, where m is the order of the centraliser $C_G(x)$.

The third condition says that for each $x \in G$ the coefficient functions $a_r(x,-), b_r(x,-) : C_G(x) \to \mathbb{C}$ are elements of $R(C_G(x)) \otimes \mathbb{Z}[\frac{1}{2}]$, where $R(C_G(x))$ is the character ring of the centraliser.

In Devoto's papers [Dev96a, Dev96b, Dev98], the identification of coefficient rings $\mathcal{E}\ell\ell^*_G$ as modular forms ones is extensively developed. We will give a partial account only without introducing the formalism of schemes, but instead by applying the following result by Eichler and Zagier [EZ85].

Theorem 2.7 *Let ϕ be a Jacobi form on Γ of weight k and index m and λ, μ rational numbers. Then, the function*
$$f(\tau) = e^{2\pi i\lambda^2\tau}\phi(\tau, \lambda\tau + \mu)$$
is a modular form of weight k and on some subgroup Γ' of finite index depending only on Γ and on λ, μ. In particular, for $\lambda = \mu = 0$, $f(\tau)$ is a modular form for Γ.

According to the description in the proof of this theorem, the group Γ' can be written explicitly as
$$\Gamma' = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \ : \ (a-1)\lambda + c\mu, b\lambda + (d-1)\mu, m(c\mu^2 + (d-a)\lambda\mu - b\lambda^2) \in \mathbb{Z} \right\}$$
and hence this group contains $\Gamma \cap \Gamma\left(\frac{N^2}{(N,m)}\right)$ if $N(\lambda,\mu) \in \mathbb{Z}^2$. We are interested in particular in generalised Jacobi forms and functions coming from the Weierstrass $\wp$ function, for which one obtains the *Teilwerte* [Ogg69]
$$f(\omega_1, \omega_2, N, a_1, a_2) \ = \ \wp\left(a_1\frac{\omega_1}{N} + a_2\frac{\omega_2}{N}, \omega_1, \omega_2\right),$$
which is a modular form of weight 2 and level N, for a given lattice (ω_1,ω_2) and integers N, $(a_1,a_2) \neq (0,0)$.

2.2 Relative cobordism theories and classes

In this section we review some fundamental constructions from cobordism theory and the related differential topology. We recall the notions of cobordism categories and (B, f)-constructions, and we describe their 'relative' versions, especially for the case of relative cobordism of oriented (or almost complex manifolds) with framed boundary.

We discuss the Pontrjagin–Thom construction and Kervaire's account of relative Chern classes. The corresponding relative multiplicative sequences turn out to be just reduced multiplicative sequences, leading to relative characteristic numbers following Stong's definition.

2.2.1 Relative characteristic classes

We recall Kervaire's account of relative characteristic classes in ordinary cohomology [Ker57] and summarise their properties.

Definition 2.8 Let

$$\mathcal{B} = \left(E_{\mathrm{U}(n)}, p, B_{\mathrm{U}(n)}, \mathrm{U}(n)\right)$$

be the classifying bundle for $\mathrm{U}(n)$. Suppose that a cross section θ^r over a closed subset A of $B_{\mathrm{U}(n)}$ is given in the associated bundle

$$\mathcal{B}^r = \left(E_{\mathrm{U}(n)}, p, B_{\mathrm{U}(n)}, \mathrm{U}(n), \mathrm{W}_{n,\,n-r}\right)$$

with fibre $\mathrm{W}_{n,\,n-r}$, the complex Stiefel manifold of $n-r$ complex vectors in $\mathbb{C}^n$. Then, for $j \geq r$, the *relative Chern classes*

$$\mathrm{c}_j^R\left(\mathcal{B}^r\right) \in H^{2(j+1)}\left(B_{\mathrm{U}(n)}, A; \mathbb{Z}\right)$$

corresponding to the cross section θ^r will be defined by the properties:

1. For the natural homomorphism $a^*\colon H^*\left(B_{\mathrm{U}(n)}, A; \mathbb{Z}\right) \to H^*\left(B_{\mathrm{U}(n)}; \mathbb{Z}\right)$ induced by the inclusion $a\colon \left(B_{\mathrm{U}(n)}, 0\right) \to \left(B_{\mathrm{U}(n)}, A\right)$ one has

$$a^*\left(\mathrm{c}_{j+1}^R\left(\mathcal{B}^r\right)\right) = \mathrm{c}_{j+1}\left(\mathcal{B}^r\right)$$

   the usual (i.e., absolute) Chern classes.

2. For the homomorphism $\rho_{j,n}^*\colon H^*\left(B_{\mathrm{U}(n)}, A; \mathbb{Z}\right) \to H^*\left(B_{\mathrm{U}(j)}, \theta^j A; \mathbb{Z}\right)$ induced by the *Borel map* $\rho\left(\mathrm{U}(j), \mathrm{U}(n)\right)$ one has

$$\rho_{j,n}^*\left(\mathrm{c}_{j+1}^R\left(\mathcal{B}^r\right)\right) = 0.$$

Consider the diagram

$$\begin{array}{ccccccccc}
\cdots & H^j\left(B_{\mathrm{U}(j)};\mathbb{Z}\right) & \to & H^j\left(\theta^j A;\mathbb{Z}\right) & \xrightarrow{\delta} & H^{j+1}\left(B_{\mathrm{U}(j)},\theta^j A;\mathbb{Z}\right) & \xrightarrow{\overline{a}^*} & H^{j+1}\left(B_{\mathrm{U}(j)};\mathbb{Z}\right) & \cdots \\
& \uparrow_{\alpha^*} & & \uparrow_{\theta^*} & & \uparrow_{\rho^*_{j,n}} & & \uparrow_{\alpha^*} & \\
\cdots & H^j\left(B_{\mathrm{U}(n)};\mathbb{Z}\right) & \to & H^j\left(A;\mathbb{Z}\right) & \xrightarrow{\delta} & H^{j+1}\left(B_{\mathrm{U}(n)},A;\mathbb{Z}\right) & \xrightarrow{a^*} & H^{j+1}\left(B_{\mathrm{U}(n)};\mathbb{Z}\right) & \cdots
\end{array}$$

Then α^* is an epimorphism in every dimension and a monomorphism in dimensions not exceeding j. As a consequence it follows that if $\rho^*_{j,n}z = 0$ and $a^*z = 0$ for some $z \in H^*\left(B_{\mathrm{U}(n)},A;\mathbb{Z}\right)$, then $z = 0$.

Kervaire goes on to prove the existence of cohomology classes with the required properties as follows: given a cross section in A over $\mathcal{B}^r$, the restriction of $\mathrm{c}_{j+1}\left(\mathcal{B}^r\right)$ to A will be zero. Let $\mathrm{c}_{j+1}\left(\mathcal{B}^r\right) = a^*x$, then since $0 = \alpha^*a^*x = \overline{\alpha}^*\rho^*_{j,n}x$ we have $\rho^*_{j,n}\left(x\right) = \delta\theta^{*-1}y$ for some $y \in H^j\left(A;\mathbb{Z}\right)$.

Thus, the two properties together define the relative Chern classes uniquely for the classifying $\mathrm{U}(n)$ bundle.

Now consider the more general case of a $\mathrm{U}(n)$ bundle over some compact finite dimensional space X induced by some map $g\colon X \to B_{\mathrm{U}(n)}$ and let

$$\mathcal{B}^r = \left(E^r, p, X, \mathrm{U}(n), \mathrm{W}_{n,\,n-r}\right)$$

be the associated bundle with fibre $\mathrm{W}_{n,\,n-r}$, and consider a cross section

$$\theta^r\colon A \to E^r$$

given over the closed subset A in X. We may assume that there exists an *injective* map $f\colon X \to B_{\mathrm{U}(n)}$ homotopic to g, and denote the bundle induced by f (equivalent to that induced by g) also by $\mathcal{B}^r$.

Let S be a closed subset in $B_{\mathrm{U}(n)}$ containing $f\left(A\right)$ and such there that exists a cross section $\psi\colon S \to B_{\mathrm{U}(r)}$ in $\mathcal{B}^r$ with the property $\psi\left(a\right) = \overline{f}\left(\theta\left(a\right)\right)$ where $\overline{f}\colon E^r \to B_{\mathrm{U}(r)}$ is the bundle map covering f. Let $\mathrm{c}^R_{j+1}\left(\mathcal{B}^r\right)$ be the $(j+1)$-dimensional relative Chern class of the classifying bundle $\mathrm{mod}\left(S\right)$ obtained using the cross section $\psi, j \geq r$. and define the *relative Chern class* of dimension $2\left(j+1\right)$, defined for $j \geq r$, $\mathrm{mod}\left(A\right)$ of the bundle $\left(E, \pi, X\right)$ corresponding to the cross section θ^r by

$$f^*\left(\mathrm{c}^R_{j+1}\left(\mathcal{B}^r\right)\right) = \mathrm{c}^R_{j+1}\left(E^r\right).$$

Kervaire proves that $f^*\left(\mathrm{c}^R_{j+1}\left(\mathcal{B}^r\right)\right)$ depends only on the homotopy class of g and on θ^r. More precisely, one has [Ker57, 11.4]

Lemma 2.9 *Let $\left(E,\pi,X\right)$ be a $\mathrm{U}(n)$-bundle and $\left(E',\pi',X'\right)$ the $\mathrm{U}(n)$-bundle induced by some map $g\colon X' \to X$. We denote by $\mathrm{c}^R_{j+1}\left(E\right)$, $\mathrm{c}^R_{j+1}\left(E'\right)$ the corresponding $2\left(j+1\right)$-dimensional relative Chern classes of those bundles, respectively, modulo closed sets $A \subset X$, $A' \subset X'$ such that $g\left(A'\right) \subset A$ and corresponding to cross sections θ, θ', such that $g\left(\theta'\left(a\right)\right) = \theta\left(g\left(a'\right)\right)$ in the associated bundles with fibre $W_{n,\,n-r}$, $j \geq r$. Then, $\mathrm{c}^R_{j+1}\left(E\right) = \mathrm{c}^R_{j+1}\left(E'\right)$.*

Let $\mathcal{B}_1$ and $\mathcal{B}_2$ be two principal bundles with bundle groups $\mathrm{U}(n_1)$ and $\mathrm{U}(n_2)$ respectively, over the same base space X and let $\theta^{r_1}, \theta^{r_2}$ be cross sections over closed subsets A_1, A_2 of X in the associated bundles $\mathcal{B}_1^{r_1}$ and $\mathcal{B}_2^{r_2}$ with fibres $\mathrm{W}_{n_1, n_1-r_1}, \mathrm{W}_{n_2, n_2-r_2}$, respectively. Then $\theta^{r_1}, \theta^{r_2}$ determine a cross section θ^r, $r = r_1+r_2$, over $A = A_1 \cap A_2$ in the bundle $\mathcal{B}^{r_1}$ with fibre $\mathrm{W}_{n,n-r}$, $n = n_1+n_2$ associated to the Whitney sum $\mathcal{B} = \mathcal{B}_1 \oplus \mathcal{B}_2$. Let $\mathrm{c}_{j+1}^R(\mathcal{B}_1), \mathrm{c}_{j+1}^R(\mathcal{B}_2)$ be the relative Chern class of $\mathcal{B}_1, \mathcal{B}_2$, defined for $j \geq r_1, r_2$, respectively, and let $\mathrm{c}_{j+1}^R(\mathcal{B})$ be the relative Chern class of $\mathcal{B}$, defined for $j \geq r$. For the relative Chern classes, Whitney duality takes the form

$$\mathrm{c}_{j+1}^R(\mathcal{B}^r) = \mathrm{c}_{j+1}^R(\mathcal{B}_1) + \cdots + \mathrm{c}_j^R(\mathcal{B}_1)\,\mathrm{c}_1(\mathcal{B}_2) + \cdots + \mathrm{c}_{j+1}^R(\mathcal{B}_2)$$

where some absolute Chern classes occur. However, since each product contains at least one relative class, it is itself a relative class.

2.2.2 Relative multiplicative sequences

Let $\mathcal{K}$ be a multiplicative sequence in the sense of [Hir66] in Chern classes of bundles. Denote by $\widetilde{\mathcal{K}}$ the corresponding reduced classes, defined as follows. For a bundle ξ,

$$\mathcal{K}_\bullet(\xi) = 1 + \widetilde{\mathcal{K}}_\bullet(\xi).$$

Because of the multiplicativity of $\mathcal{K}$, one has

$$\mathcal{K}_\bullet(\xi_1 \oplus \xi_2) = \mathcal{K}_\bullet(\xi_1) \cdot \mathcal{K}_\bullet(\xi_2)$$

$$1 + \widetilde{\mathcal{K}}_\bullet(\xi_1 \oplus \xi_2) = \left(1 + \widetilde{\mathcal{K}}_\bullet(\xi_1)\right) \cdot \left(1 + \widetilde{\mathcal{K}}_\bullet(\xi_2)\right)$$

and hence

$$\widetilde{\mathcal{K}}_\bullet(\xi_1 \oplus \xi_2) = \widetilde{\mathcal{K}}_\bullet(\xi_1) + \widetilde{\mathcal{K}}_\bullet(\xi_2) + \widetilde{\mathcal{K}}_\bullet(\xi_1)\,\widetilde{\mathcal{K}}_\bullet(\xi_2).$$

Now, if one considers relative characteristic classes in the sense of Kervaire as described before, in the particular case of $r = 0$, then θ^0 will determine further relative characteristic classes given by

$$\widetilde{\mathcal{K}}_\bullet(\xi_1) = \mathcal{K}_\bullet^R(\xi_1).$$

Hence

$$\mathcal{K}_\bullet^R(\xi_1 \oplus \xi_2) = \mathcal{K}_\bullet^R(\xi_1) + \mathcal{K}_\bullet^R(\xi_2) + \mathcal{K}_\bullet^R(\xi_1)\,\mathcal{K}_\bullet^R(\xi_2).$$

2.2.3 The relative cobordism description

We recall now some facts from cobordism theories. We refer the reader to [Sto68] or [CF66] for further details.

A *parallelism* is a trivialisation of the tangent bundle. A *framing* means a trivialisation of the stable normal bundle and up to homotopy this is equivalent to a trivialisation of the stable tangent bundle.

If N^n is a differentiable manifold and τ its tangent bundle, the stable tangent bundle is

$$\tau_s = \tau \oplus (2k-n)\,, \qquad 2k-n \geq 2.$$

By Whitney's theorem every manifold admits an embedding into some $\mathbb{R}^N$ as a submanifold, so that

$$\tau \oplus \nu = \tau\mathbb{R}^N$$

and it follows that the existence of framings for tangent and normal stable bundles are equivalent.

Definition 2.10 A *stable framing* θ of N^n is a homotopy class of maps $\varphi\colon E_{\tau_s} \to \mathbb{R}^{2k}_{M^m}$ each of which maps every fibre of τ_s onto $\mathbb{R}^{2k}$ linearly. Here $\mathbb{R}^{2k}_{M^m}$ denotes the total space of the trivial real $2k$-dimensional vector bundle over M^m. It is independent of k as long as $2k-n \geq 2$.

A (*normally*) *framed* n-submanifold of an m-manifold M is a submanifold N with a given framing $f\colon \nu_N M \cong N \times \mathbb{R}^{m-n}$ of the normal bundle.

Definition 2.11 Two framed n-submanifolds (N_j, f_j) are *bordant* is there is a framed $(n+1)$-submanifold (B, h) of $[0,1]\times M$ with $\partial B = \{0\}\times N_0 \cup \{1\}\times N_1$, $\partial h = f_0 \cup f_1$. For $m-n \geq 2$ the bordism classes $[N, f]$ of framed n-submanifolds of M form an abelian group $\Omega^{\mathrm{fr}}_n M$.

In particular, one considers the special case $M = S^m$ and we write

$$\Omega^{\mathrm{fr}}_{m,n} = \Omega^{\mathrm{fr}}_n S^m.$$

Freudenthal's Theorem and the Pontrjagin–Thom construction tell us the following:

Theorem 2.12 *There is a commutative diagram of group homomorphisms, in which the vertical maps are all isomorphisms and the horizontal maps are isomorphisms for $k \geq n+2$,*

$$\begin{tikzcd}
\Omega^{\mathrm{fr}}_{n+2,n} \arrow[r, "i_2"] \arrow[d, "\cong"'] & \Omega^{\mathrm{fr}}_{n+3,n} \arrow[r, "i_3"] \arrow[d, "\cong"'] & \Omega^{\mathrm{fr}}_{n+4,n} \arrow[r] \arrow[d, "\cong"'] & \cdots \arrow[r] & \Omega^{\mathrm{fr}}_{n+k,n} \arrow[r, "i_k", "\cong"'] \arrow[d, "\cong"'] & \Omega^{\mathrm{fr}}_{n+k+1,n} \arrow[r] \arrow[d, "\cong"'] & \cdots \\
\pi_{n+2}S^2 \arrow[r, "s_2"] & \pi_{n+3}S^3 \arrow[r, "s_3"] & \pi_{n+4}S^4 \arrow[r] & \cdots \arrow[r] & \pi_{n+k}S^k \arrow[r, "s_k", "\cong"'] & \pi_{n+k+1}S^{k+1} \arrow[r] & \cdots
\end{tikzcd}$$

There are therefore isomorphisms between the framed bordism groups $\Omega_n^{\mathrm{fr}} = \varinjlim_k \Omega_{n+k,n}^{\mathrm{fr}}$ *and the stable homotopy groups of spheres,*

$$\Omega_n^{\mathrm{fr}} \;\; \cong \;\; \pi_n^{\mathrm{s}} = \varinjlim_k \pi_{n+k} S^k.$$

In this theorem the maps s_k are given by suspension, and $i_k[N,f] = [N, f \times \mathbb{R}]$ since for $N \subset S^m$ one can identify the normal bundle in S^{m+1} as the normal bundle in S^m plus a copy of the trivial line bundle. The vertical maps are the Pontrjagin–Thom construction, which uses transversality arguments in differential topology; in general $\Omega_n^{\mathrm{fr}} M^{n+k} \cong [M^{n+k}, S^k]$, identifying framed n-submanifolds of M^{n+k} with the zero-sets of maps $f\colon M^{n+k} \to S^k$ which are smooth on a neighbourhood of $f^{-1}(0)$ and have 0 as a regular value.

Definition 2.13 The *J-homomorphism* is the map

$$J\colon \pi_n(\mathrm{O}(k)) \to \Omega_{n+k,n}^{\mathrm{fr}} \cong \pi_{n+k} S^k$$

defined by twisting the normal framing of S^n in S^{n+k} by a representation of S^n in the orthogonal group of the fibre. The *stable J-homomorphism* $J\colon \pi_n(\mathrm{O}) \to \pi_n^{\mathrm{s}}$ is given by taking $\varinjlim_k$.

Other bordism theories (G,f) are constructed by requiring further structure on the normal bundle of a smooth manifold. Let G be a sequence of topological groups G_n with compatible maps $G_n \to \mathrm{O}(n)$. For example, $G_n = \mathrm{O}(n)$, $\mathrm{SO}(n)$, $\mathrm{Spin}(n)$, $\mathrm{U}(n/2)$ will give ordinary, oriented, spin and complex bordism theories respectively. The *stable bordism group* $\Omega_n^G(X)$ is the group of bordism classes of n-submanifolds of S^{n+k} with maps to X and stable G-structures on the normal bundle. The Pontrjagin–Thom construction generalises to give isomorphisms

$$\Omega_n^G(X) \cong MG_n(X)$$

where MG is the Thom spectrum of G, with $MG_n = D(EG_n)/S(EG_n)$ the Thom space of the principal G-bundle $EG_n \to BG_n$, and

$$MG_n(X) = \varinjlim_k \pi_{n+k}(X_+ \wedge MG_k).$$

In the case of framed cobordism one takes $G_n = 1$ for all n, and MG is then the sphere spectrum.

An important point for us to recall is how every stable framing θ of N^n defines a U-structure. For given $\varphi\colon E_{\tau_s} \to \mathbb{R}^{2k}_{M^m}$, the natural operator $J\colon \mathbb{R}^{2k} \to \mathbb{R}^{2k}$ given by

$$J\left(x_1, x_{2,} \ldots, x_{2k-1}, x_{2k}\right) = (-x_2, x_{1,} \ldots, -x_{2k}, x_{2k-1})$$

pulls back to an operator

$$J\colon E_{\tau_s} \to E_{\tau_s}$$

representing a U-structure θ^{U} on N^n. This leads to a homomorphism

$$r\colon \Omega_n^{\mathrm{fr}} \to \Omega_n^{\mathrm{U}}$$

by

$$[N^n, \theta]_{\mathrm{fr}} \longmapsto \left[N^n, \theta^{\mathrm{U}}\right]_{\mathrm{U}}.$$

It is clear that, for $n > 0$, Ω_n^{fr} finite and Ω_n^{U} free abelian implies that $r = 0$. So, given a closed stably framed manifold N^n with $n > 0$, then N^n is as well an U-manifold, and one such that $[N^n]_{\mathrm{U}} = 0$. Hence, there exists a compact U-manifold M^{n+1} with $\partial M^{n+1} = N^n$. So, from now on we are going to consider such pairs (M^{n+1}, N^n), or, more precisely, (M^{n+1}, N^n, π), where $\pi = \partial\theta$, from θ^{U}. Such a triple is a (U-fr)-*manifold*.

Consider τ_s the stable tangent bundle of $(M^m, \partial M^m)$, which is a bundle of k-dimensional complex vector spaces and one with a trivialisation on its restriction to the boundary: we are given an isomorphism

$$\varphi\mid_{\partial M^m}\colon E_{(\tau_s|_{\partial M^m})} \to \mathbb{R}^{2k}_{|_{\partial M^m}}.$$

Recall the definitions for difference elements in K-theory by Atiyah (see [APS75II] or [LM89]). The isomorphism $\varphi\mid_{\partial M^m}$ determines a *difference class*

$$\mathrm{d}\,(\tau_s, k, \varphi) \in K\,(M^n, \partial M^n)$$

which is denoted as *the stable tangent bundle of the* (U-fr)-*manifold* M^n. Sometimes, however, we will denote this element as

$$\tau = \tau\,(M, \partial M) \in K\,(M^n, \partial M^n)\,.$$

For any (G, f)-theory, there is an exact couple

$$\begin{array}{ccc} \Omega_*^{\mathrm{fr}} & \rightarrow & \Omega_*^{\mathrm{G}} \\ \nwarrow & & \swarrow \\ & \Omega_*^{G,\,\mathrm{fr}} & \end{array}$$

where $\Omega_*^{G,\,\mathrm{fr}}$ are the *relative cobordism groups* of manifolds with a G structure whose boundary is given a framing. So, the construction can be performed for $G = \mathrm{SO}$, Spin, $\mathrm{Spin^c}$ etc, but it turns out that the simplest statement comes out by using

$$\begin{array}{ccc} \Omega_*^{\mathrm{fr}} & \rightarrow & \Omega_*^{\mathrm{U}} \\ \nwarrow & & \swarrow \\ & \Omega_*^{\mathrm{U},\,\mathrm{fr}} & \end{array}$$

because in that case for every $n > 0$, one obtains a short exact sequence

$$0 \to \Omega_*^{\mathrm{U}} \to \Omega_*^{\mathrm{U},\,\mathrm{fr}} \to \Omega_*^{\mathrm{fr}} \to 0$$

and elements in $\Omega_*^{\mathrm{U,\,fr}}$ are called cobordism classes of (U, fr).

If M is a (U-fr)-manifold as described above, then it has Chern classes, which we may write as

$$c_k\left(M\right) = c_k\left(\tau\right) \in H^{2k}\left(M^n, \partial M^n\right)$$

as defined in [CF66, p. 93]. Then, we may define the Chern numbers of a compact (U-fr)-manifold by

$$\begin{aligned} c_I\left(M^{2n}\right) &= c_{i_1}\cdots c_{i_\chi}\left[M^{2n}\right] = \\ &\quad \left\langle c_{i_1}\left(M\right)\cdots c_{i_\chi}\left(M\right), \sigma\left(M\right)\right\rangle, \end{aligned}$$

where

$$\sigma\left(M\right) \in H_n\left(M^n, \partial M^n\right)$$

denotes the orientation class of M^n, M^n being compact. For convenience we remember the definition for the Chern classes of a vector bundle.

The *relative characteristic numbers* so obtained may be seen to be consistent with what is obtained using Kervaire's description using the more general Stong's definitions for quite general theories as explained in [Sto68, p. 32].

Given fibration sequences

$$\overline{B} \xrightarrow{h} B \xrightarrow{f} B\mathrm{O}$$

one may think of

$$y \in H^*(B, \overline{B}; \underset{\sim}{A})$$

as a *relative characteristic class*, by means of the following procedure. Let M be a (B, f)-manifold with $\left(\overline{B}, f \circ h\right)$ a structure on its boundary ∂M. In that case, one has defined a *relative characteristic number*

$$y\left[M, \partial M\right] \in H^*(\mathrm{pt}; \underset{\sim}{A})$$

since the normal map gives $(M, \partial M) \to \left(B, \overline{B}\right)$.

From the algebraic topology point of view, a very interesting feature of such numbers is that they are *relative cobordism invariants*. To prove it, Stong uses the fact one may suppose by additivity that there is a (B, f)-manifold W with $\partial W = M \cup (-U)$ joined along $\partial M \cong \partial U$, with U a $\left(\overline{B}, f \circ h\right)$-manifold and so one obtains the sequence

$$(W, \partial W) \xrightarrow{d} \Sigma\left(\partial W/\varnothing\right) \xrightarrow{\Sigma j} \Sigma\left(\partial W/U\right) \xleftarrow{\Sigma p} \Sigma\left(M/\partial M\right)$$

which gives out

$$p_*\left[M, \partial M\right] = j_* \partial\left[W, \partial W\right]$$

by the orientation assumption in the decomposition of ∂W and $y\left(M,\partial M\right)=p^{*}q^{*}y\left(W,U\right)$ where

$$\left(M,\partial M\right)\xrightarrow{p}\left(\partial W,U\right)\xrightarrow{q}\left(W,U\right)$$

and

$$y\left[M,\partial M\right]=\left\langle q^{*}y,j_{*}\partial\left[W,\partial W\right]\right\rangle=\left\langle\delta j^{*}q^{*}y,\left[W,\partial W\right]\right\rangle.$$

However, from the exact sequence of the triple $\left(W,\partial W,U\right)$ the composition

$$H^{*}\left(W,U\right)\xrightarrow{q^{*}}H^{*}\left(\partial W,U\right)\xrightarrow{p^{*}}H^{*}\left(M,\partial M\right)$$

is zero, and hence

$$y\left[M,\partial M\right]=0.$$

Observe as well that taking $\overline{B}=\varnothing$, this reduces to the closed case.

2.2.4 The case of the disk bundles

Suppose we are given M^{n} a compact smooth orientable manifold with boundary ∂M^{n}, with a class $x\in K\left(M^{n},\partial M^{n}\right)$ such that the composition $K\left(M^{n},\partial M^{n}\right)\to\widetilde{K}\left(M^{n}\right)\to KO\left(M^{n}\right)$ maps x into the class of the stable tangent bundle in $KO\left(M^{n}\right)$. Then x can be used to give to τ_{s}, the stable tangent bundle of $\left(M^{n},\partial M^{n}\right)$, the structure of a complex vector bundle and one with a trivialisation on its restriction to the boundary. That amounts to give a (U-fr)-manifold structure (not in a unique way!) such that $\tau\left(M,\partial M\right)=x\in K\left(M^{n},\partial M^{n}\right)$. This will give $c_{k}^{H}\left(x\right)=c_{k}^{H}\left(M\right)$ with the definitions in [CF66].

All this may be applied to $\left(D^{2n},S^{2n-1}\right)$. In this case, there is a class $x\in K\left(D^{2n},\partial S^{2n-1}\right)$ such that

$$\left\langle c_{k}^{H}\left(x\right),\sigma\left(D^{2n}\right)\right\rangle=\left(n-1\right)!$$

This can be seen by considering ξ a complex vector bundle arising from the principal spin bundle on S^{2n} determined by any of the half spin irreducible representations of $\mathrm{Spin}\left(2n\right)$, so that $\left\langle c_{k}^{H}\left(\xi\right),\sigma\left(S^{2n}\right)\right\rangle=\left(n-1\right)!$ and then considering its pullback under the collapse map $c\colon D^{2n}\twoheadrightarrow D^{2n}/S^{2n-1}\sim S^{2n}$ topologically and using the naturality of the Chern classes. We are allowed to consider $x=\left[c^{*}\left(\xi\right)\right]$ as an element in $K\left(D^{2n},\partial S^{2n-1}\right)$ giving the class of τ_{s}, since $\widetilde{KO}\left(M^{n}\right)=0$ in this case and hence the short exact sequence

$$K\left(M^{n},\partial M^{n}\right)\to\widetilde{K}\left(M^{n}\right)\to\widetilde{KO}\left(M^{n}\right)$$

is in fact

$$K\left(M^{n},\partial M^{n}\right)\to\widetilde{K}\left(M^{n}\right)\to0$$

so any nontrivial element in it will be suitable to become a stable tangent bundle for the disk. Anyway, we are allowed to take D^{2n} as a (U-fr)-manifold with $\tau = x$. Hence, cobordism theory ensures the existence of a compact (U-fr)-manifold D^{2n} with $c_k^H\left[D^{2n}\right] = (n-1)!$ and all the other Chern numbers are zero because of vanishing of all non-extreme-dimensional cohomologies in D^{2n}.

According to [Smi71, p. 241], the results from applying ∂_* to such cells give us an element in $\Omega^{\mathrm{fr}}_{2n-1} = \pi^{\mathrm{s}}_{2n-1}$ which is a generator for the image of

$$J_{\mathbb{C}} \colon \pi_{2n-1}(U) \to \pi^{\mathrm{s}}_{2n-1}.$$

The usual generator $\sigma \in K\left(S^{2n}\right)$ (see for instance [LM89] or [ABS64]) corresponds to the spin bundle before described and for it one has

$$c_n(\sigma) = (n-1)!\iota_{2n} \in H^{2n}\left(S^{2n};\mathbb{Z}\right)$$

where ι_{2n} is the corresponding usual generator for cohomology. Use of the Chern character tells us then that

$$\mathrm{ch}(\tau) = e^{2n} \in H^{2n}\left(D^{2n}, S^{2n-1};\mathbb{Q}\right).$$

Now, for $\mu \in K\left(D^{2n}, S^{2n-1}\right)$ the element corresponding to σ, one has

$$c_n(\sigma) = (n-1)!e^{2n} \in H^{2n}\left(D^{2n}, S^{2n-1};\mathbb{Q}\right)$$

and, since $\widetilde{KO}\left(D^{2n}\right) = 0$, we may choose μ as a stable tangent bundle for $\left(D^{2n}, S^{2n-1}\right)$, providing it with the required structure of a (U-fr)-manifold. The K-theoretical characteristic numbers as described in Stong are computed by [Smi71, p. 242]. For $H \in K\left(S^2\right)$ the canonical line bundle, we have $\sigma_2 = H - 1$.

As a consequence, an (SO-fr)-manifold structure on D^{4k} exists such that the Newton characteristic numbers are

$$s_k\left[D^{4k}\right] = (-1)^k kp_k\left[D^{4k}\right] = (2k)!$$

Consider now $\left(S^{4k-1}, f\right)$ any framed sphere of dimension $4k-1$. The Pontrjagin and Stiefel–Whitney classes of S^{4k-1} vanish, so there is a compact oriented disk manifold D^{4k} with $\partial D^{4k} = S^{4k-1}$, and a Riemannian one such that on a neighbourhood of the boundary the metric on the disk is the product metric from the one induced on the sphere by considering the given framing as orthonormal. Since S^{4k-1} is framed, the tangent bundle of D^{4k} is pulled back from a bundle on the quotient space D^{4k}/S^{4k-1}, which in this very special case happens to be a smooth manifold itself, an S^{4k}. This allows one to define the Pontrjagin classes of D^{4k} as relative classes $p_i \in H^{4i}\left(D^{4k}, S^{4k-1}\right)$ and to see them as elements in the reduced cohomology group $\widetilde{H}^{4i}\left(S^{4k}\right)$.

In fact, there is as choice for $\left(D^{4k}, S^{4k-1}, f_0\right)$ such that D^{4k} is the unit ball in $\mathbb{C}^{2k}$, so an almost complex manifold with boundary $M = S^{4k-1}$, and in fact the almost complex structure is a product near the boundary, as may be seen recalling that the almost complex J_0 considered on $\mathbb{C}^{2k}$ commutes with dilations and with the map $M \times I \to B$ via $(m,t) \longmapsto e^{-tm}$. One knows as well that this complex manifold $\left(\mathbb{C}^{2k}, J_0\right)$ is Kähler. Other related to non-Kähler structure and Hopf manifolds. In fact, so considered the disks are Kähler manifolds and hence have no torsion. This case is neither the one we want, but it is different from the usual Riemannian structure.

One considers in S^3 the standard parallelism π_0, i.e., induced by considering S^3 as the unit quaternions in $\mathbb{H}$. Then, one has a relative Pontrjagin number $p_1\left(D^4, \pi_0\right)$ given on D^4 by the parallelism π_0 on its boundary S^3. Atiyah–Patodi–Singer observe in [APS75II, p. 425] that this is the same as the Pontrjagin number of the standard 4-dimensional bundle over S^4 (underlying the quaternionic Hopf bundle) and is hence equal to -2 (sign may vary because of chosen orientation); cf. [Sto68].

Namely, if $\lambda_{\mathbb{H}}$ is the canonical quaternionic (also called symplectic in the old sense, [CF66, p. 95]) line bundle over $S^4 \cong \mathbb{H}\mathrm{P}^1$, with total space $E_{\lambda_{\mathbb{H}}}$ and associated sphere bundle $S\left(\lambda_{\mathbb{H}}\right) \cong S^7$, with fibre S^3, giving in this case the Hopf fibration. So, in that case $S^3 \hookrightarrow D^4 \hookrightarrow \mathbb{R}^4 \cong \mathbb{H}$ and the pullback bundle of $\lambda_{\mathbb{H}}$ by the collapse map $c\colon \left(D^4, S^3\right) \twoheadrightarrow S^4$ $c^*\left(\lambda_{\mathbb{H}}\right)$ over the disk has a non-vanishing section over S^3 given by the parallelism. This is how they correspond one to each other. Remark that this construction gives us $\tau_{\mathrm{U}} D^4 = c^*\left(\lambda_{\mathbb{H}}\right)$, a bundle over D^4. The connection should be one agreeing with the one already given on the boundary and which goes to the one for $\lambda_{\mathbb{H}}$ (the instanton bundle) giving the right classes. We know that classes for bundles on S^4 are invariant in many ways. Any bundle has a connection and the classes known can be seen and given by the curvature. The reason why we care about all this is the use we will do in the next chapters of constructions in the style of the one which follows: Let (W, X, f) be a Riemannian manifold with boundary, oriented, so that $[(W, X, f)] \in \Omega_*^{\mathrm{SO,\,fr}}\left[\frac{1}{2}\right]$. Then, as such a manifold, its tangent bundle τW can be considered as the pullback of (the K-class of) some bundle $\xi_{W/X}$ on the quotient space W/X. Usually, this space will not be a manifold, but it is in very important cases: namely, for $[(W, X, f)] = \left[\left(D^n, S^{n-1}, \pi\right)\right]$, where $D^n/S^{n-1} \cong S^n$. Work of [Smi71] and others shows that those disks essentially generate the groups $\Omega_*^{\mathrm{SO,\,fr}}\left[\frac{1}{2}\right]$.

However, since unless the boundary of W be connected, W/X would not be a manifold for sure and the collapsing map will fail to be smooth at least at the boundary points as soon as it has more than a connected component. This may be not relevant from the point of view of homotopy classes of bundles, but it will certainly be when considering the construction from a differential-geometric point of view. The most illustrative example will certainly be the collapse map from the disk to the sphere by collapse of the boundary as described for instance in [Ker57, p. 33]. One defines the collapse map by

$$\begin{array}{rcl} D^n & \twoheadrightarrow & S^{n-1} \\ (y_1, \ldots, y_n) & \to & \left(1 - 2y^2, 2y_1\sqrt{1-y^2}, \ldots, 2y_n\sqrt{1-y^2}\right) \end{array}$$

which is certainly continuous everywhere and sends all of $\partial D^n = S^{n-1}$ to the south pole $\mathrm{s} = (-1, 0, \ldots, 0)$. However, it fails clearly to be differentiable at those same points.

The obvious question arising at this moment is if it is possible to see Pontrjagin classes as the ones considered for $\left(D^{4k}, S^{4k-1}, f\right)$ as relative classes $p_i \in H^{4i}\left(D^{4k}, S^{4k-1}\right)$as coming from the use of the Chern–Weil construction for some connection ∇. The answer is essentially "yes" and the procedure is described for instance in [ADS83] and its offspring by Ogasa. The idea is that the framing on the boundary determines a metric and a flat metric connection with torsion on it, and a procedure to extend it to the whole manifold is given there under some mild assumptions.

The case of quaternionic plane bundles

However, other bundles should be used to obtain a full description of our invariants. Consider the disk bundle $p\colon D\left(\lambda_{\mathbb{H}}\right) \downarrow S^4$, for $\lambda_{\mathbb{H}}$ the canonical quaternionic line bundle over $S^4 \cong \mathbb{HP}^1$, so that we know that $D\left(\lambda_{\mathbb{H}}\right)/S\left(\lambda_{\mathbb{H}}\right) \cong \mathbb{HP}^2$. We consider $D\left(\lambda_{\mathbb{H}}\right)$ with stable tangent bundle $p^*\left(\lambda_{\mathbb{H}} - 2\right) \in K\left(D\left(\lambda_{\mathbb{H}}\right)\right)$. Over $\mathbb{HP}^2$ there is a *symplectic* (in the sense of [Sto68]) Hopf line bundle $\lambda'_{\mathbb{H}}$. One can see that

$$\widetilde{K}\left(\mathbb{HP}^2\right) \cong K\left(D\left(\lambda_{\mathbb{H}}\right), S\left(\lambda_{\mathbb{H}}\right)\right) \to \widetilde{K}\left(D\left(\lambda_{\mathbb{H}}\right)\right)$$

which maps $\lambda'_{\mathbb{H}} - 2 \longmapsto p^*\left(\lambda_{\mathbb{H}} - 2\right)$. This is how one may consider $D\left(\lambda_{\mathbb{H}}\right)$ as a compact (U-fr)-manifold with stable tangent bundle $\lambda'_{\mathbb{H}} - 2$. Then,

$$\left\langle c_2^2\left(D\left(\lambda_{\mathbb{H}}\right)\right), S\left(D\left(\lambda_{\mathbb{H}}\right)\right)\right\rangle = 1$$

for an appropriate orientation, and all the other Chern numbers are 0. This bundle is going to prove itself very significant in our context.

2.2.5 Relative genera on framed manifolds

We will now briefly describe relative genera on framed manifolds. This is a particular case of a more general construction for relative cobordism groups. The foundational example may be called the Todd relative genus considered by Conner, Floyd and Smith [CF66], [Smi71]. In that case, they prove that the usual Todd genus

$$\varphi_{\mathrm{Td}}\colon \Omega_*^{\mathrm{U}} \to \mathbb{Z}$$

may be extended to a map

$$\varphi_{\mathrm{Td,\,fr}}\colon \Omega_*^{\mathrm{U,\,fr}} \to \mathbb{Q}$$

is such a way that the following diagram of short exact sequences is commutative:

$$\begin{array}{ccccccccc}
0 & \to & \Omega_*^{\mathrm{U}} & \to & \Omega_*^{\mathrm{U,\,fr}} & \to & \Omega_*^{\mathrm{fr}} & \to & 0 \\
 & & \downarrow \varphi_{\mathrm{Td}} & & \downarrow \varphi_{\mathrm{Td,\,fr}} & & \downarrow e_{\mathrm{Td}} & & \\
0 & \to & \mathbb{Z} & \to & \mathbb{Q} & \to & \mathbb{Q}/\mathbb{Z} & \to & 0.
\end{array}$$

In particular, for the disks $\left(D^{4k}, S^{4k-1}, \pi\right)$ before described one has

$$\begin{aligned}
\mathrm{Td}_{4k} &= \frac{(-1)^{k-1} B_{4k}}{(2k-1)!2k} c_{2k} + \text{decomposables} \\
\varphi_{\mathrm{Td,\,fr}}\left(D^{4k}, S^{4k-1}, \pi\right) &= \mathrm{Td}_{4k}\left[\left(D^{4k}, S^{4k-1}, \pi\right)\right] = (-1)^{k-1}\frac{B_{2k}}{2k}
\end{aligned}$$

(see [CF66], [Smi71, p. 252]) where they are used to describe the image of Adams e-invariant. However, one cannot expect the map $\varphi_{\mathrm{Td,\,fr}}\colon \Omega_*^{\mathrm{U,\,fr}} \to \mathbb{Q}$ to be a ring homomorphism, because $\oplus_{n=0}\,\Omega_n^{\mathrm{U,\,fr}}$ is not a ring, since because of

$$0 = \Omega_1^{\mathrm{U}} \overset{\cong}{\to} \Omega_1^{\mathrm{U,\,fr}} \to \Omega_0^{\mathrm{fr}} \overset{\cong}{\to} \Omega_0^{\mathrm{U}} \to \Omega_0^{\mathrm{U,\,fr}} \overset{\cong}{\to} \Omega_{-1}^{\mathrm{fr}}$$

one does not have a unit in $\Omega_0^{\mathrm{U,\,fr}} = \{e\}$. However, the restriction of the abelian group homomorphism to the strictly positive dimensions $\varphi_{\mathrm{Td,\,fr}}\colon \Omega_{2n}^{\mathrm{U,\,fr}} \to \mathbb{Q}$ ensures that the operation induced by connected sum on classes will be respected. Products become more complicated, particularly because the Cartesian product of two manifolds with boundary is not a manifold with boundary itself, but only what is called a *manifold with corners.*

The version of the same construction for the case of Hirzebruch's $\widehat{\mathrm{L}}$-genus will be

$$\begin{array}{ccccccccc}
0 & \to & \Omega_*^{\mathrm{SO}} & \to & \Omega_*^{\mathrm{SO,\,fr}} & \to & \Omega_*^{\mathrm{fr}} & \to & 0 \\
 & & \downarrow \varphi_{\widehat{\mathrm{L}}} & & \downarrow \varphi_{\widehat{\mathrm{L}},\,\mathrm{fr}} & & \downarrow e_{\widehat{\mathrm{L}}} & & \\
0 & \to & \mathbb{Z} & \to & \mathbb{Q} & \to & \mathbb{Q}/\mathbb{Z}\left[\frac{1}{2}\right] & \to & 0.
\end{array}$$

Here $\varphi_{\widehat{\mathrm{L}}}$ sends the 2-torsion in Ω_*^{SO} to 0 and it is not a diagram of exact sequences anymore. However, the forgetful homomorphism

$$F_*\colon \Omega_*^{\mathrm{fr}} \to \Omega_*^{\mathrm{SO}}$$

is zero for $n > 0$ and iso for $n = 0$. One has in this case

$$\begin{array}{ccccccccccc}
0 & \to & \begin{array}{c}\Omega_1^{\mathrm{SO}} \\ = 0\end{array} & \to & \begin{array}{c}\Omega_1^{\mathrm{SO,\,fr}} \\ = 0\end{array} & \to & \begin{array}{c}\Omega_0^{\mathrm{fr}} \\ = \mathbb{Z}\end{array} & \to & \begin{array}{c}\Omega_0^{\mathrm{SO}} \\ = \mathbb{Z}\end{array} & \to & \begin{array}{c}\Omega_{-1}^{\mathrm{fr}} \\ = 0\end{array}
\end{array}$$

and in fact the forgetful homomorphism factors through the complex theory. When one sees it restricted to dimension 4 we may write

$$\begin{array}{ccccccccc}
0 & \to & \Omega_4^{\mathrm{SO}} & \to & \Omega_4^{\mathrm{SO,\,fr}} & \to & \Omega_3^{\mathrm{fr}} & \to & 0 \\
 & & \downarrow \varphi_{\widehat{\mathrm{L}}} & & \downarrow \varphi_{\widehat{\mathrm{L}},\,\mathrm{fr}} & & \downarrow e_{\widehat{\mathrm{L}}} & & \\
0 & \to & \mathbb{Z}\left[\frac{1}{2}\right] & \to & \mathbb{Z}\left[\frac{1}{24}\right] & \to & \mathbb{Z}\left[\frac{1}{24}\right]/\mathbb{Z}\left[\frac{1}{2}\right] & \to & 0.
\end{array}$$

When the $\varphi_{\widehat{\mathrm{L}}}$-related elliptic genus φ_ε is considered, the equivalent diagram is the following, where $\delta = \varphi_\varepsilon\left(\mathbb{C}\mathrm{P}^2\right)$ and one uses the fact that $\varphi_\varepsilon\left(V^4\right) = 16\delta$ for V^4 the Kummer

$K3$-surface, which is spin:

$$\begin{array}{ccccccccc} 0 & \to & \Omega_4^{\mathrm{SO}} & \to & \Omega_4^{\mathrm{SO,\,fr}} & \to & \Omega_3^{\mathrm{fr}} & \to & 0 \\ & & \downarrow\ \varphi_\varepsilon & & \downarrow \varphi_{\varepsilon,\,\mathrm{fr}} & & \downarrow e_\varepsilon & & \\ 0 & \to & \mathbb{Z}\left[\frac{1}{2}\right]\langle 16\delta\rangle & \to & \mathbb{Q}\left\langle \frac{-2}{3}\delta\right\rangle & \to & \mathbb{Q}\left\langle \frac{-2}{3}\delta\right\rangle/\mathbb{Z}\left[\frac{1}{2}\right]\langle 16\delta\rangle & \to & 0 \end{array}$$

where $\varphi_{\varepsilon,\,\mathrm{fr}}\left(D^4, S^3, \pi\right) = -\frac{2}{3}\delta$.

For dimension 8, one has respective diagrams

$$\begin{array}{ccccccccc} 0 & \to & \Omega_8^{\mathrm{SO}} & \to & \Omega_8^{\mathrm{SO,\,fr}} & \to & \Omega_7^{\mathrm{fr}} & \to & 0 \\ & & \downarrow\ \varphi_{\widehat{\mathrm{L}}} & & \downarrow \varphi_{\widehat{\mathrm{L}},\,\mathrm{fr}} & & \downarrow e_{\widehat{\mathrm{L}}} & & \\ 0 & \to & \mathbb{Z}\left[\frac{1}{2}\right] & \to & \mathbb{Z}\left[\frac{1}{240}\right] & \to & \mathbb{Z}\left[\frac{1}{240}\right]/\mathbb{Z}\left[\frac{1}{2}\right] & \to & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \to & \Omega_8^{\mathrm{SO}} & \to & \Omega_8^{\mathrm{SO,fr}} & \to & \Omega_7^{\mathrm{fr}} & \to & 0 \\ & & \downarrow\ \varphi_\varepsilon & & \downarrow \varphi_{\varepsilon,\mathrm{fr}} & & \downarrow e_\varepsilon & & \\ 0 & \to & \mathbb{Z}\left[\frac{1}{2}\right]\left\langle \delta^2, \varepsilon\right\rangle & \to & \mathbb{Q}\left\langle \frac{-8}{15}\delta^2 + \frac{12}{5}\varepsilon, \frac{4}{15}\delta^2 - \frac{1}{5}\varepsilon\right\rangle & \to & \begin{array}{l}\mathbb{Q}\left\langle \frac{-8}{15}\delta^2 + \frac{12}{5}\varepsilon, \frac{4}{15}\delta^2 - \frac{1}{5}\varepsilon\right\rangle/ \\ \mathbb{Z}\left[\frac{1}{2}\right]\left\langle \delta^2, \varepsilon\right\rangle\end{array} & \to & 0 \end{array}$$

where $\varepsilon = \varphi_\varepsilon\left(\mathbb{H}\mathrm{P}^2\right)$, $\frac{-8}{15}\delta^2 + \frac{12}{5}\varepsilon = \varphi_{\varepsilon,\mathrm{fr}}\left(D^8, S^7, \pi\right)$, and

$$\frac{4}{15}\delta^2 - \frac{1}{5}\varepsilon = \varphi_{\varepsilon,\mathrm{fr}}\left(\left(D\left(\lambda_{\mathbb{H}}\right)\right), S\left(D\left(\lambda_{\mathbb{H}}\right)\right), \pi_{\mathbb{H}}\right).$$

A very suggestive interpretation in terms of modular forms and functions of half weight with *Nebentypus* **1** for the modular subgroup $\Gamma_0(4)$ is to be developed from this. For $\delta = \frac{1}{4}H\rho$ and $\varepsilon = \frac{1}{16}H^2$, given in terms of the Dedekind eta function, one has $\rho = 1 - \dfrac{32\eta(4\tau)^8}{\eta(\tau)^8}$ and $H = \dfrac{\eta(\tau)^8}{\eta(2\tau)^4}$. Moreover, one sees that

$$\begin{aligned} -\frac{8}{15}\delta^2 + \frac{12}{5}\varepsilon &= -4G_4^* = \varphi_{\varepsilon,\,\mathrm{fr}}\left(D^8, S^7, \pi\right) \\ \frac{4}{15}\delta^2 - \frac{1}{5}\varepsilon &= G_4 = \varphi_{\varepsilon,\,\mathrm{fr}}\left(\left(D\left(\lambda_{\mathbb{H}}\right)\right), S\left(D\left(\lambda_{\mathbb{H}}\right)\right), \pi_{\mathbb{H}}\right) \end{aligned}$$

for G_4, G_4^* defined as in [Zag88]. Despite the problems which arise with products, the fact that the product of a manifold with boundary by a manifold without boundary is a manifold with boundary itself can be used to construct generators for the image of the relative elliptic genus out of manifolds with boundary.

Chapter 3

The Atiyah–Patodi–Singer theorem, elliptic genera and eta invariants

3.1 Essential tools from geometric analysis: Dirac operators and the index theorem

In this section we discuss the essential properties of elliptic operators on manifolds with boundary. We concentrate on first-order elliptic operators, and especially on those of Dirac type. We recall the definitions of generalised and compatible Dirac operators, and their ellipticity properties. We will generalise the definition of the latter, defining torsion-compatible Dirac operators, and establish self-adjointness properties and Green–Palais theorems for them. We review the spectral properties of elliptic operators and examine some of the analytical aspects of the construction of related operators on a manifold and its double, as in Booß-Bavnbek–Wojciechowski [BBW93].

We briefly recall the classical formulation of the Atiyah–Singer index theorem for manifolds without boundary and the expression of their integrands as characteristic classes for the operators which interest us. We give explicit expressions for these classes in terms of the curvature and the torsion of the connection involved.

3.1.1 First-order elliptic operators on manifolds with boundary

Let M be a compact n-manifold with boundary N. A *differential operator of order* r between real or complex vector bundles E, F over M is a linear map

$$D\colon \Gamma E \to \Gamma F$$

which can be expressed, with respect to local coordinates $(U, (x^1, \ldots, x^n))$ and local trivialisations of E, F, by

$$D = \sum_{\substack{\alpha=(\alpha_1,\ldots,\alpha_n)\\ \alpha_1+\cdots+\alpha_n\leq r}} A^\alpha(x^1, \ldots, x^n)\, \frac{\partial^{\alpha_1}}{\partial x^1} \cdots \frac{\partial^{\alpha_n}}{\partial x^n}.$$

Here the coefficients A^α are $\operatorname{rank}(F) \times \operatorname{rank}(E)$ matrices of smooth real-valued or complex-valued functions; invariance under change of local trivialisations and coordinates implies that the collection of coefficients $\{i^r A^\alpha\}$ with maximum degree $\alpha_1 + \cdots + \alpha_n = r$ define a (nonzero) section

$$\sigma(D) \in \Gamma(S^r TM \otimes \operatorname{Hom}(E, F)),$$

termed the *principal symbol* of the differential operator D.

Definition 3.1 A differential operator $D \colon \Gamma E \to \Gamma F$ of order r is *elliptic* if evaluation of the principal symbol at any nonzero cotangent vector $\xi \in T_x^* M$ gives a linear isomorphism of the fibres,

$$\sigma_\xi(D) \colon E_x \xrightarrow{\cong} F_x.$$

An operator $D : \Gamma E \to \Gamma E$ is of *Dirac type* if its square has order 0 with symbol given by the metric,

$$\sigma_\xi(D^2) \colon v \mapsto |\xi|^2 v.$$

In particular, Dirac-type operators are elliptic.

If S is a $C\ell(M)$-module with a connection ∇^S, then there is a Dirac-type operator

$$D^\nabla \colon \Gamma(S) \xrightarrow{\nabla^S} \Gamma(TM^* \otimes S) \xrightarrow{\flat} \Gamma(TM \otimes S) \xrightarrow{c} \Gamma(S),$$

where c is left Clifford multiplication and the isomorphism $\flat$ is given by the metric on TM. In terms of a local orthonormal basis $\{e_j\}$ for $T_x M$, we have

$$D^\nabla \colon s \mapsto \sum_{j=1}^n e_j \cdot \nabla_{e_j} s$$

and the principal symbol is given by $\sigma_\xi(D^\nabla) \colon v \mapsto c(\xi)v$.

The module S and its connection ∇^S are *compatible* with the connection ∇^{TM} on M if ∇^S is a module derivation extending $\nabla^T M$,

$$\nabla^S(c(v)s) = c(\nabla^{TM} v)s + c(v)\nabla^S s.$$

We do *not* assume, as they do in [BBW93], that the connection on M is the Levi-Cività connection; it will be compatible with the Riemannian metric but it may have torsion.

An operator F between two Hilbert spaces is *Fredholm* if its kernel and cokernel are finite dimensional and its image is a closed subspace. It follows that all eigenspaces of a Fredholm operator are finite dimensional.

Lemma 3.2 *An elliptic operator P on a compact Riemannian manifold may be completed to Fredholm operators P_s with respect to the Sobolev norms $\| \; \|_s$, for which*

$$\ker(P) = \ker(P_s), \qquad \ker(P^*) = \ker(P_s^*) = \mathrm{coker}(P_s)$$

for all s.

Proof: See Theorem 5.2 in chapter III of [LM89]. $\square$

For any elliptic operator $P\colon \Gamma E \to \Gamma F$ over a compact Riemannian manifold, the composition of P with its formal adjoint P^* gives self-adjoint elliptic operators P^*P, PP^* on ΓE, ΓF respectively, termed the *associated Laplacians* of P. Write E_λ, F_λ for the eigenspaces of these Laplacians for each $\lambda \in \mathbb{R}$.

Lemma 3.3 (Hodge) *The eigenspaces E_λ, F_λ are finite dimensional and are zero except for a discrete set of $\lambda \geq 0$. For $\lambda = 0$ one has the Hodge formulae*

$$E_0 = \ker P, \qquad F_0 = \mathrm{coker} P,$$

and one has isomorphisms $P\colon E_\lambda \to F_\lambda$ for all $\lambda > 0$.

3.1.2 The Atiyah–Singer index theorem

The index of an elliptic operator $P\colon \Gamma E \to \Gamma F$ may be defined as

$$\mathrm{ind}(P) = \dim\ker(P) - \dim\mathrm{coker}(P).$$

In fact, it depends only on the homotopy class of the principal symbol $\sigma(P)$.

The pullback of the principal symbol along $\pi\colon TM^* \to M$ is a map

$$\sigma(P)\colon \pi^*E \to \pi^*F$$

which, by the *difference bundle construction* of Atiyah–Bott–Shapiro [ABS64], defines a class $[\sigma(P)] \in K(DM, SM)$ in the K-theory of the Thom space of TM. Choosing a smooth embedding j of M in some $\mathbb{R}^N$, we now define the *topological index* of P as the image of the symbol class $[\sigma(P)]$ under the composite map

$$K(DM, SM) \xrightarrow{\cong} K_{\mathrm{cpt}}(TM) \xrightarrow{j_!} K_{\mathrm{cpt}}(T\mathbb{R}^N) \xrightarrow{\cong} K(S^{2N}) \xrightarrow{\cong} \mathbb{Z}.$$

Then the Atiyah–Singer index theorem says that the topological index coincides with the analytic index $\mathrm{ind}(P)$, and may also be expressed in terms of cohomology classes, as follows.

Theorem 3.4 *Let M be a compact Riemannian manifold of dimension $4n$ and $P\colon \Gamma E \to \Gamma E$ a Dirac-type operator on M with coefficients in a vector bundle E. Then*

$$\mathrm{ind}(P) = \int_M \mathrm{ch}(E)\, \hat{A}(M),$$

where $\hat{A}$ is a polynomial on the Pontrjagin classes of M, and $\mathrm{ch}(E)$ is the Chern character of the bundle E.

3.2 The Atiyah–Patodi–Singer theorem

In this chapter we state a version of the Atiyah–Patodi–Singer theorem for manifolds with boundary and formal operators on infinite-dimensional natural bundles derived from the tangent bundle.

The interest of this formal construction arises when one shows that the formal sums involved not only converge but take values in suitable rings of modular forms from elliptic cohomology, generalising the classical eta invariant.

We state first the classical Atiyah–Patodi–Singer theorem for manifolds with boundary, under the assumption that the metric near the boundary is of product type. This extends the theory for closed manifolds by the addition of a term —the eta invariant— whose definition and properties we review.

Then we recall Gilkey's modification of the Atiyah–Patodi–Singer formula to include the case that the metric near the boundary is not of product type. A further integrand now appears in the formula, which can be expressed in terms of transgression of the characteristic class involved.

We apply these results for operators arising from elliptic genera on manifolds with boundary, considering both the classical setting and Gilkey's extension to more general boundary metrics. We derive expressions for the integrands that arise when generalising from classical operators to those of elliptic genera.

3.2.1 The classical APS theorem and the eta invariant

Hirzebruch's signature theorem relates a cohomological invariant, the signature, which may also be defined as the index of a Dirac-type operator, with an analytical invariant given by an integral of certain differential forms.

Theorem 3.5 *Let M be a $4k$-dimensional compact oriented Riemannian manifold. Then*

$$\operatorname{sign}(M) = \operatorname{ind}(A^+) = \int_M L_k(p_1, \ldots, p_k),$$

where

- $\operatorname{sign}(M)$ *is the signature of the non-degenerate intersection form $q(a) = (a \cup a)[M]$ on $H^{2k}(X;\mathbb{R})$, given by the difference between the number of positive and negative entries in the diagonalisation of q,*
- *the signature operator $A^+ \colon \Omega^* M^+ \to \Omega^* M^-$ is the restriction of $A = d + \delta = d + {*d*}$ to the ± 1 eigenspaces of the involution given by Clifford multiplication by $(-1)^k \omega$ on $\Omega^* M$,*
- *the integrand is the Hirzebruch L-polynomial in differential j-forms p_j, $j = 1, \ldots, k$, representing the Pontrjagin classes of M.*

Proof: The first equality is a straightforward example of Hodge theory. The index of $d + \delta$ counts the dimensions of harmonic forms in all degrees, but everything cancels except in the middle dimension, leaving the difference in dimensions of the positive and negative definite parts. The second equality is the cohomological formula from the Atiyah–Singer index theorem. $\square$

If M is a manifold with boundary $\partial M = N$, then Atiyah–Patodi–Singer show that an extra term enters the equation, a new *spectral invariant* of N, given by a certain number-theoretic function of the eigenvalues of the signature operator on the boundary.

Definition 3.6 Let A be any endomorphism with a discrete spectrum of eigenvalues $\lambda \in \mathrm{Spec}(A)$ and finite dimensional eigenspaces. Then the *eta function* associated to A is given by

$$\eta_A(s) = \sum_{\lambda \neq 0} \frac{\mathrm{sign}(\lambda)}{|\lambda|^s}$$

where the sum is over the positive and negative eigenvalues, repeated according to their multiplicities.

Alternatively, one can write

$$\eta_A(s) = \sum_{\lambda > 0} \frac{\dim \ker(A - \lambda)}{\lambda^s} - \sum_{\lambda < 0} \frac{\dim \ker(A - \lambda)}{(-\lambda)^s} .$$

This function will be absolutely convergent for $\mathrm{Re}(s)$ sufficiently large. If it has an analytic continuation such that the value at $s = 0$ is finite, we write $\eta(A) = \eta_A(0)$, the *eta invariant* of A.

Theorem 3.7 *Let M be a $4k$-dimensional compact oriented Riemannian manifold with boundary N, such that the inclusion $N \subset M$ extends to an isometric inclusion in a neighbourhood of the boundary, $N \times [0, \varepsilon] \hookrightarrow M$. Then*

$$\mathrm{sign}(M) = \mathrm{ind}(A) + h = \int_M L_k(p_1, \ldots, p_k) \ - \ \eta(B),$$

where A the is signature operator as before, and

- $\mathrm{sign}(M)$ *is now the signature of the non-degenerate quadratic form q', given by restricting the intersection form q to the image of $H^{2k}(M, N)$ in $H^{2k}(M)$,*
- *the differential forms p_j represent relative Pontrjagin classes,*
- *$\eta(B) = \eta_B(0)$ is the eta invariant of the self-adjoint operator B^{ev} on the even forms on N given by*

$$B^{\mathrm{ev}} \varphi_{2p} = (-1)^{k+p-1} (*d\varphi - d{*}\varphi), \qquad \varphi \in \Omega^{2p}(N).$$

The eta function $\eta_B(s)$ is holomorphic on the half-plane $\mathrm{Re}(s) > -\frac{1}{2}$.

- $h = \dim \ker B$, *the multiplicity of the zero eigenvalue of* B.

This result was proved using, and was the motivation for, the following important theorem of Atiyah–Patodi–Singer [APS75I, Theorem $3 \cdot 10$].

Theorem 3.8 *Let* M *be a compact manifold with boundary* N *and* $D \colon \Gamma E \to \Gamma F$ *a first-order differential operator on* M. *Assume that, in a neighbourhood* $N \times I$ *of the boundary,* D *has the special form*

$$D = \sigma \left(\frac{\partial}{\partial u} + A \right),$$

where u *is the inward normal coordinate,* σ *is a bundle isometry given by the symbol* $\sigma_{du}(D)$, *and* A *is a self-adjoint elliptic operator on the boundary* N *(independent of* u*). Let* $\Gamma(E; P)$ *be the space of sections* f *of the bundle* $E \to M$ *satisfying the global boundary condition*

$$P(f|_N) = 0 \tag{3.1}$$

where P *is the spectral projection of* A *corresponding to eigenvalues* ≥ 0. *Then the restriction*

$$D \colon \Gamma(E; P) \to \Gamma F$$

has a finite index given by

$$\operatorname{ind}(D) = \int_M \alpha_0(x)\, dx - \frac{1}{2}(h + \eta(0)),$$

in which α_0 *is the constant term in the expansion as* $t \to 0$ *of*

$$\sum e^{-t\mu'} \left|\phi'_\mu(x)\right|^2 - \sum e^{-t\mu''} \left|\phi''_\mu(x)\right|^2 \tag{3.2}$$

where μ', ϕ'_μ *and* μ'', ϕ''_μ *denote the eigenvalues and eigenfunctions of* D^*D *and* DD^* *on the double of* M, *and* $\eta(A) = \eta_A(0)$ *is the eta invariant of the eta-series of the operator* A *on* N *and* h *is the multiplicity of the eigenvalue* $\lambda = 0$ *of* A. *The series* $\eta_A(s)$ *converges absolutely for* $\mathrm{Re}(s)$ *large and extends to a meromorphic function with finite value at* $s = 0$*; it extends to a holomorphic function for* $\mathrm{Re}(s) > -\frac{1}{2}$ *if the expansion* (3.2) has no negative powers of t.

3.2.2 Generalisations of Gilkey, Donnelly and Nicolaescu

There is a version of the Atiyah–Patodi–Singer theorem without the assumption on the product metric, due to Gilkey.

Theorem 3.9 [Gil75, Theorem 3.1] *For the signature of an* m*-dimensional manifold* M, *where* $m = 4k$, *the following holds:*

$$\operatorname{sign}(M) = \int_M B^s_m + \int_{\partial M} C^s_m + \eta(\partial M).$$

More precisely, there is an element $TG(L_k)$ *in* $\mathcal{P}^b_{m-1,m,m-1}$ *such that for any metric on* M*,*

$$\operatorname{sign}(M) = \int_M L_k + \int_{\partial M} TG(L_k) + \eta(\partial M),$$

with

$$\begin{aligned} TG(P) &= \frac{1}{2k}\int_0^1 P(\theta, \Omega_t, \ldots, \Omega_t)\, dt, \\ \theta &= \nabla_1 - \nabla_0, \qquad \nabla_t = t\nabla_1 + (1-t)\nabla_0, \\ \Omega_t &= \Omega^{\nabla_t}, \end{aligned}$$

so that $TG(P)$ *is an invariantly defined* $(m-1)$*-form with*

$$d(TG(P)) = P(\nabla_1) - P(\nabla_0).$$

Fix a Riemannian manifold (M, g), and let T be a tensor $(1,2)$ on (M, g); that is, a skew symmetric linear map $T\colon TM \times TM \to TM$, $T(Y,X) = -T(X,Y)$, and let $\nabla^{\mathrm{g,T}}$ be the only metric-compatible connection on M with torsion tensor T. Let $d^{\mathrm{g,T}}$ be its skewed covariant differential, i.e., given by the composition $d^{\mathrm{g,T}} = \wedge \circ \nabla^{\mathrm{g,T}} = \mathrm{alt} \circ \nabla^{\mathrm{g,T}}$:

$$C^\infty(M, \Lambda^p T^* M) \to C^\infty(M, T^*M \otimes \Lambda^p T^* M) \to C^\infty\left(M, \Lambda^{p+1} T^* M\right).$$

For $T = 0$, $d^{\mathrm{g,T}} = d^{\mathrm{g,0}}$, the usual exterior derivative associated to the underlying smooth manifold structure on M.

In general, d and $d^{\mathrm{g,T}}$ have the same leading order symbol, but their complete symbols differ. More precisely, on p-forms,

$$d_p^{\mathrm{g,T}} = d_p^{\mathrm{g,0}} + E_p,$$

where E_p is an endomorphism depending linearly upon the torsion tensor T of $\nabla^{\mathrm{g,T}}$.

Consider on the relevant bundles or section spaces the induced metrics and define

$$\delta^{\mathrm{g,T}} = \left(d^{\mathrm{g,T}}\right)^*,$$

the *adjoint* of $d^{\mathrm{g,T}}$. Then, on $p+1$ forms, now,

$$\delta_p^{\mathrm{g,T}} = \delta_p^{\mathrm{g,0}} + E_p^*,$$

where $\delta_p^{\mathrm{g,0}}$and E_p^* are the adjoints for $d_p^{\mathrm{g,0}}$and E_p respectively.

Remark 3.10 Miquel points out [Don86] that for a metric-compatible connection with non-vanishing torsion T, it is possible that

$$\delta_p^{\mathrm{g,T}} \neq \pm{*}d^{\mathrm{g,T}}{*}$$

unlike the Levi-Cività case, in which,

$$\delta_p^{\mathrm{g,0}} = \pm{*}d^{\mathrm{g,0}}{*}.$$

For the validity of the classical characteristic class integrands for connections which are compatible with the metric but have torsion, see Nicolaescu's approach [Nic99], who refers to Roe [Roe98, Chapter 11]. However, [Don78] says that, in the case he considers, the integrand is certainly not the Todd form in general (p. 887) for a Dolbeault operator on a manifold with boundary.

In manifolds without boundary, they integrate to the same value. There, in fact, thanks to symbol theory and invariance of characteristic classes with respect to connections and Stokes, we know that they give the same index.

Hence, since the trace appearing in the Atiyah–Patodi–Singer theorem is a trace on the double of the manifold, if we know about traces of heat kernels on manifolds without boundary expressed as characteristic *forms*, we may thereafter restrict them to forms (perhaps not characteristic but only relatively characteristic in the sense of Kervaire) to obtain results for the manifolds with boundary.

For the case of manifolds without boundary, all operators can be seen as twisted signature or twisted Dirac operators.

As an interesting example, consider any compact Lie group, such as for instance the Lie group as $S^3 \cong \mathrm{SU}(2)$. Lie groups are frameable by their Lie algebras and get from there a natural flat connection with torsion. (E.g., think of any linear group as embedded as an open set in a general matrix group seen as a flat Euclidean space). However, since this is not necessarily the Levi-Civitá connection (in fact, it would never be so if the Lie algebra of the group is not trivial) then we can consider equally the other connection. Now, we get on the Lie group —or, if we prefer, at any compact subspace of it— two metric-compatible connections, one flat and the other torsion-free. Each connection will have associated a covariant exterior derivative on differential forms on the Lie group given as the composition of the connection with the exterior product, which is the same for both. Those operators are known to share the same leading symbol and hence on manifolds without boundary they will give the same K-theoretical class. But on manifolds with boundary, they will originate two different classes both in K and H relative theories. The point is to identify the corresponding differential forms that one has to integrate.

According to Nicolaescu [Nic99], the formula

$$\operatorname{ind}\left(D^g \otimes V\right) = \int_M \hat{A}\left(TM, \nabla^{g,0}\right) \operatorname{ch}\left(V\right) - \frac{1}{2}\eta_{(D^g \otimes V)|_{\partial M}}$$

holds in conditions more general than those considered for instance in [BBG89], namely,

$$\operatorname{ind}\left(D^{g,T} \otimes V\right) = \int_M \hat{A}\left(TM, \nabla^{g,T}\right) \operatorname{ch}\left(V\right) - \frac{1}{2}\eta_{(D^{g,T} \otimes V)|_{\partial M}}$$

holds, whenever $D^{g,T}$ is the Dirac operator associated to the Clifford bundle $C\ell_n M$, but considered not with the connection induced from the Levi-Civitá $\nabla^{g,0}$ on TM, but with any other one compatible with the metric, probably not torsion free, endowed with a torsion tensor T. The quoted author says textually that "the proof for (1) in chapter eleven of

Roe's book, edition 88, extends verbatim to (2)." We will prove the truth of this statement, beginning from [Roe98, p. 136].

We will begin by considering M a spin manifold of even dimension n. We know that M has a canonical Clifford bundle S over it, whose fibre is the spin representation ρ_S, and associated to S there is a Dirac operator $D = \mathrm{c} \circ \nabla^{g,0}$, which is a graded one. However, we are not interested on this D only, which is what we called $D^{g,0}$. We want to consider further operators

$$D^{g,T} = \mathrm{c} \circ \nabla^{g,T}.$$

Lemma 11.30 in [Roe98] (referring to the case of spin even-dimensional manifolds without boundary) says that the signature operator is canonically isomorphic to the classical Dirac operator with coefficients in the spin bundle S. Since, as left Clifford modules,

$$\Lambda^* (T^*M) \otimes \mathbb{C} \cong C\ell_n (T^*M \otimes \mathbb{C}) \cong S \otimes S,$$

and since the first order parts of both considered operators agree under this isomorphism, their difference must be a zeroth order (tensorial) operator, which depends on the connection coefficients in a linear way. For the classical versions considered by Roe, being the connection on TM Riemannian, it is possible, once fixed any point x, to take normal coordinates centred there, so that the components of the connection vanish there, and hence in the classical case the difference must be zero.

3.2.3 Eta invariants of twisted Dirac operators

Let M be an oriented compact Riemannian manifold of dimension $2n$, such that the Riemannian metric coincides in a neighbourhood of the boundary $N = \partial M$ with a product metric on $N \times I$. Suppose that M is a spin manifold. Then the Dirac operator of M is a first order elliptic differential operator on the graded spinor bundles,

$$D^+ \colon \Gamma S^+ \to \Gamma S^- .$$

On the boundary, $S^\pm$ restrict to give the spinor bundle associated to N, and in a neighbourhood of the boundary one has

$$D^+ = \sigma \left(\frac{\partial}{\partial u} + A \right),$$

where σ is Clifford multiplication by the unit inward normal as usual, and D_N is just the Dirac operator on the $(2n-1)$-manifold N. In [ABP73], Atiyah–Bott–Patodi show that the integrand $\alpha_0(x)\,dx$ in Theorem 3.8 can be expressed explicitly as an appropriate Pontrjagin form in this case and that the corresponding expansion (3.2) has no negative powers of t, and one obtains a formula for the index of Dirac operator with the global boundary condition (3.1).

Atiyah–Patodi–Singer themselves point out that the results of [ABP73, Section 6] also apply to twisted Dirac operators:

Theorem 3.11 *Suppose E is a hermitian vector bundle on M with a unitary connection such that near the boundary the metric and the connection are constant in the normal direction. Then the twisted Dirac operator on M,*

$$D_E^+ \colon \Gamma(S^+ \otimes E) \to \Gamma(S^- \otimes E)$$

with the global boundary condition (3.1) *has index*

$$\operatorname{ind}(D_E^+) = \int_M \operatorname{ch}(E)\, \widehat{A}(p) \ - \frac{1}{2}(h_{E|N} + \eta_{E|N}),$$

where $\widehat{A}(p)$ is the Hirzebruch $\widehat{A}$-polynomial $\displaystyle\prod_{i=1}^{n} \frac{x_i/2}{\sinh(x_i/2)}$ with the elementary symmetric functions of the x_i^2 replaced by Pontrjagin forms on M, the form $\operatorname{ch}(E)$ denotes the Chern character of the bundle, $\eta_{E|N}(s)$ is the eta-function of the twisted Dirac operator on N and $h_{E|N}$ is the dimension of its kernel. Moreover, $\eta(s)$ is holomorphic for $\operatorname{Re}(s) > -\frac{1}{2}$.

Theorem 3.7 is a version of this, although M is not required to be spin and the factor of $\frac{1}{2}$ is present only implicitly, since the operator B on the even forms is only 'half' of the restriction $A_{|\partial M}$ of the signature operator.

We can generalise the previous theorem in a purely formal way to *graded bundles* or *q-bundles*. Suppose that E_r, $r \geq 0$, is a sequence of vector bundles as above. We usually write

$$E_q = \bigoplus_{r \geq 0} E_r\, q^r.$$

Here q is simply a formal variable. The formal bundle E_q may be of infinite rank, but each E_r will always be of finite rank. Then the Dirac operator on M may be twisted with E_q. If we write D_r^+ for the twisted Dirac operator $D_{E_r}^+$ then we have

$$D_q^+ = \sum_{r \geq 0} D_r^+\, q^r : \Gamma(S^+ \otimes E_q) \to \Gamma(S^- \otimes E_q).$$

The *index* of D_q^+ is the formal power series

$$\operatorname{ind}(D_q^+) = \sum_{r \geq 0} \operatorname{ind}(D_r^+)\, q^r = \sum_{r \geq 0} \left(\dim\ker(D_r^+) - \dim\operatorname{coker}(D_r^+)\right)\, q^r.$$

The operators D_r^+, D_q^+ restricted to the boundary of M give the twisted Dirac operators $D_{r,N}$ and

$$D_{q,N} = \sum_r D_{r,N}\, q^r$$

on N. We consider the corresponding eta invariants, as follows.

Definition 3.12 The *eta function* of a twisted Dirac operator D_q is the formal series

$$\eta_{D_q}(s,q) = \sum_{r\geq 0} \eta_{D_r}(s)\, q^r.$$

If each eta function $\eta_{D_r}(s)$ has finite value at $s=0$ then the *eta invariant* of D_q is the formal power series

$$\eta_{D_q}(q) = \sum_{r\geq 0} \eta(D_r)\, q^r.$$

One considers similarly the formal power series

$$h(q) = \dim\ker(D_q) = \sum_{r\geq 0} \dim\ker(D_r)\, q^r$$

and the formal Chern character

$$\mathrm{ch}(E_q) = \sum_{r\geq 0} \mathrm{ch}(E_r)\, q^r\,.$$

Corollary 3.13 *The formal twisted Dirac operator D_q^+ with the global boundary condition* (3.1) *has index*

$$\mathrm{ind}(D_q^+) = \int_M \mathrm{ch}(E_q)(q)\, \widehat{A}(p) \;-\; \frac{h_{D_{q,N}}(q) + \eta_{D_{q,N}}(q)}{2}\,,$$

Proof: Apply the previous theorem to each twisted Dirac operator D_r^+ and add up. □

3.2.4 Eta invariants of twisted signature operators

The signature operator $A^+\colon \Omega^* M^+ \to \Omega^* M^-$ may also be generalised to a twisted operator, as follows. Let E be a hermitian vector bundle, with a compatible connection ∇^E, over a compact Riemannian manifold of dimension $2n$. Consider the covariant exterior derivative $d_E = d^\nabla$ on $\Omega^*(M;E)$, the space of differential forms on M with coefficients in the bundle E. As for the usual exterior coderivative, the adjoint $\delta_E = d_E^*$ is given by

$$\delta_E = \pm {*} d_E {*},$$

where the Hodge star acts as the identity on the coefficients. It follows that the self-adjoint operator $A_E = d_E + \delta_E$ splits as

$$A_E^\pm\colon \Omega^\pm(M;E) \to \Omega^\mp(M;E), \qquad \Omega^\pm(M;E) = (1 \pm i^n\omega)\Omega(M;E).$$

The operator A_E^+ is the twisted signature operator, and one has a generalised Hirzebruch signature theorem:

Theorem 3.14 *Let* $\operatorname{sign}(M;E)$ *be the signature of the quadratic form given by wedge product on* $H^n(M;E)$*. Then*

$$\operatorname{sign}(M;E) = \operatorname{ind}(A_E^+) = 2^n \operatorname{ch}(E)\,\mathrm{L}(M)[M],$$

where L *denotes the Hirzebruch characteristic class* $\displaystyle\prod_{i=1}^{n} \frac{x_i/2}{\tanh(x_i/2)}$ *with the elementary symmetric functions of the* x_i^2 *replaced by Pontrjagin classes of* M*.*

Proof: This is completely parallel to the usual Signature Theorem 3.5; see [ABP73, p. 313] for more details. $\square$

We can also take E to be a graded vector bundle $E_q = \bigoplus E_r q^r$ as discussed above, and we will then interpret the twisted signature theorem as giving equality of power series in the formal variable q. The main example is given by the graded bundle $\mathcal{L}E_q$, defined as follows. Let E be any vector bundle over M, and define

$$\mathcal{L}E_q = \bigotimes_{j=1}^{\infty} S_{q^j} E \otimes \bigotimes_{j=1}^{\infty} \Lambda_{q^j} E,$$

where we use the notation S_t, Λ_t for the formal power series versions of the symmetric and exterior products S^*, Λ^*. Explicitly, we set

$$S_t E = \sum_{\ell=0}^{\infty} S^\ell(E)\cdot t^\ell, \qquad \Lambda_t E = \sum_{\ell=0}^{\infty} \Lambda^\ell(E)\cdot t^\ell.$$

In the case of the complex tangent bundle $E = T_{\mathbb{C}}M = TM \otimes \mathbb{C}$, we sometimes write $\mathcal{L}_M = \mathcal{L}T_{\mathbb{C}}M_q$. This bundle is termed the *free loop bundle* on M and the corresponding twisted signature is usually interpreted as the S^1*-equivariant signature of the free loop space* $\mathcal{L}M$,

$$\operatorname{sign}^{S^1}(M) = \operatorname{sign}(M;\mathcal{L}_M) = \sum_{r\geq 0} \operatorname{sign}(M;\mathcal{L}TM_r)\,q^r.$$

Corollary 3.15 *The* S^1*-equivariant signature is related to the universal elliptic genus by*

$$\operatorname{sign}^{S^1}(M^{4k})\,\varepsilon^{k/2} = \varphi_{\mathcal{E}\ell\ell}(M^{4k}).$$

Proof: The Chern character of S_tE is just $\prod_{i=1}^n \sum_{k\geq 0} t^k e^{kx_i} = \prod_{i=1}^n (1-te^{x_i})^{-1}$ and similarly

$$\operatorname{ch}(\mathcal{L}_M) = \prod_{i=1}^{n}\prod_{j=1}^{\infty} \frac{(1+q^n e^{x_i})(1+q^n e^{-x_i})}{(1-q^n e^{x_i})(1-q^n e^{-x_i})}.$$

The result follows from Theorem 3.14 on comparing with the characteristic class for the universal elliptic genus; compare [HBJ92, Theorem I.5.6 and Section 6.1]. $\square$

The results of Atiyah–Patodi–Singer also imply a generalisation of Theorem 3.14. Suppose that M has boundary $N = \partial M$, dimension $2n-1$, and that the metric on M is a product metric in a neighbourhood $N \times I$ of the boundary. The restrictions to the neighbourhood $N \times I$ of the bundles $\Lambda^{\pm}TM^* \otimes E$ are isomorphic to $p^*\Lambda^*TN^* \otimes E$, and the twisted signature operator here takes the form

$$A_E^+ \cong \left(\frac{\partial}{\partial u} + B_E\right).$$

Here $B = B^{\mathrm{ev}} \oplus B^{\mathrm{od}}$ where B^{ev} and B^{od} act on even and odd forms respectively. Analogously to Theorem 3.7 the even part is given by $B_E^{\mathrm{ev}} = \pm i^n(*d_E - d_E*)$, a self-adjoint elliptic operator, and we have

Theorem 3.16 *The signature* $\mathrm{sign}(M, N; E)$ *of the quadratic form given by cup product on* $H^n(M, N; E)$ *satisfies*

$$\mathrm{sign}(M, N; E) = \mathrm{ind}(A_E^+) + \dim\ker B_E^{\mathrm{ev}} = \int_M 2^n \mathrm{ch}(E)\mathrm{L}(M) \; - \; \eta(B_E^{\mathrm{ev}}).$$

3.3 Elliptic invariants for framed manifolds

In this section we consider the version of the Atiyah–Patodi–Singer formula given by Atiyah, Donnelly and Singer in [ADS83] for operators on framed manifolds. We summarise the results of this paper that will be needed for our constructions later, and in particular recall the general signature operator construction associated to a given metric connection with torsion. We study the relation between the operators on a manifold and on its boundary and give the integrands for the corresponding Atiyah–Patodi–Singer formula.

We apply the Atiyah–Donnelly–Singer construction to the case of the disks with the framings of Conner and Floyd [CF66], both in a classical context and for formal loop space operators. For classical operators, the Bernoulli numbers appear, and it is interesting that in our generalisation one obtains Eisenstein forms. We discuss the results in the context of relative cobordism and higher e-invariants, and relate the modularity of the invariants to the string manifold genera considered, for example, by Mahowald. We also interpret our results in terms of spectral flow on Hopf manifolds. Finally, we suggest some possible directions for further work, related to the Eichler–Kohnen–Zagier correspondence [EZ85] between Jacobi forms and half-integral weight modular forms.

3.3.1 Classical eta invariants of framed manifolds

Suppose (N, f) is a framed compact closed manifold of dimension $4k-1$. Then there exists a compact oriented manifold M with $N = \partial M$ whose tangent bundle $TM \to M$ is the pullback of some bundle on the quotient space M/N. Thus we have relative Pontrjagin classes $p_j \in H^{4j}(M, N)$ and we can consider

$$L(p_1, \ldots, p_k)[M, N] \; - \mathrm{sign}(M), \tag{3.3}$$

the difference between the Hirzebruch L-polynomial in the Pontrjagin classes, evaluated at the fundamental class $[M, N] \in H_{4k}(M, N)$, and the signature of M. In fact this expression is independent of the choice of M and depends only on (N, f). We remark that it is the framing of the boundary which enables us to replace the differential-geometric expression $\int_M L(p)$ of Theorem 3.7 with the cohomological expression $L(p)[M, N]$ here.

Alternatively, the framing f on N determines a Riemannian metric, a Hodge star and a flat metric-compatible connection ∇^f. Consider the *skewed covariant differential* $d^f = \wedge \circ \nabla^f$ on the differential forms on N,

$$d^f : \Gamma\Lambda^p T^*N \xrightarrow{\nabla} \Gamma(\Lambda^p T^*N \otimes T^*N) \xrightarrow{\wedge} \Gamma\Lambda^{p+1}T^*N, \tag{3.4}$$

where the final map is antisymmetrisation or exterior multiplication. We remark that ∇^f will not in general be torsion-free; if it is, then $d^f = d$, the usual covariant differential. Associated to d^f there is a self-adjoint elliptic operator

$$B^f \varphi_{2p} = (-1)^{k+p-1}(*d^f\varphi - d^f*\varphi), \qquad \varphi \in \Omega^{2p}(N) \tag{3.5}$$

on even forms.

Theorem 3.17 [ADS83, Theorem 4.3] *The 'signature defect'* (3.3) *coincides with the eta invariant of B^f,*

$$\eta(B^f) = L(p_1, \ldots, p_k)[M, N] - \mathrm{sign}(M).$$

Proof: For the special case that the connection ∇^f is torsion-free, this follows from Theorem 3.7, with the integral over Pontrjagin forms replaced with evaluation of cohomology classes. The general case is more complicated; details may be found in Sections 15–17 of [ADS83]. $\square$

Similarly, for the twisted signature,

$$\eta(B^f_{E|N}) = 2^{2k}\mathrm{ch}(E)L(p_1, \ldots, p_k)[M, N] - \mathrm{sign}(M; E), \tag{3.6}$$

and for the twisted Dirac operator,

$$(\eta(D^f_{E|N} + \dim\ker(D^f_{E|N}))/2 = \mathrm{ch}(E)\widehat{A}(p_1, \ldots, p_k)[M, N] - \mathrm{ind}(D_E). \tag{3.7}$$

3.3.2 Modular eta invariants for framed manifolds

The above theorem relating the signature defect to the eta invariant can be stated also for twisted signature operators. In particular we consider the free loop bundles

$$\mathcal{L}_M = \sum_{r\geq 0}(\mathcal{L}T_{\mathbb{C}}M)_r\, q^r, \qquad \mathcal{L}_N = \sum_{r\geq 0}(\mathcal{L}T_{\mathbb{C}}N)_r\, q^r$$

which inherit flat connections from the tangent bundles and we can consider the formal covariant exterior derivative $d^f_{\mathcal{L}_N}$ on $\Omega^*(N;\mathcal{L}_N)$ with

$$d^f_{\mathcal{L}_N} \;=\; \sum_{r\geq 0} d^f_{\mathcal{L}T_{\mathbb{C}}N_r} q^r : \Omega^k(N;\mathcal{L}T_{\mathbb{C}}N_r) \to \Omega^{k+1}(N;\mathcal{L}T_{\mathbb{C}}N_r).$$

Then we have operators $B^f_{\mathcal{L}TN_r} = \pm(*d^f_{\mathcal{L}T_{\mathbb{C}}N_r} - d^f_{\mathcal{L}T_{\mathbb{C}}N_r}*)$ as before and a formal operator $B^f_{\mathcal{L}_N} = \sum B^f_{\mathcal{L}TN_r} q^r$ on the even forms on N with coefficients in the free loop bundle on the loop space.

Definition 3.18 Let M be an oriented manifold of dimension $4k$ with framed boundary (N,f) of dimension $4k-1$. Then the *relative S^1-equivariant signature of the free loop space* is given by the signature of the cup product in relative cohomology with coefficients in the free loop bundle $\mathcal{L}_M$,

$$\operatorname{sign}^{S^1}(M,N) = \operatorname{sign}(M,N;\mathcal{L}_M) = \sum_{r\geq 0} \operatorname{sign}(M,N;\mathcal{L}TM_r)\, q^r \quad \in \mathbb{Z}[[q]].$$

The *elliptic eta invariant* of the framed manifold (N,f) is the formal power series

$$\eta_{\mathcal{E}\ell\ell}(N,f) = \sum_{r\geq 0} \eta(B^f_{\mathcal{L}TM_r})\, q^r \quad \in \mathbb{R}[[q]]$$

given by the eta invariants for the twisted operators $B^f_{\mathcal{L}TN_r}$ on $\Omega^{\mathrm{ev}}(N;\mathcal{L}TN_r)$.

We then have an elliptic cohomology generalisation of the Signature Defect Theorem 3.17:

Theorem 3.19 *The relative universal elliptic genus of a framed $(4k-1)$-manifold satisfies*

$$\eta_{\mathcal{E}\ell\ell}(N,f) = \varphi_{\mathcal{E}\ell\ell}(M,N)\,\varepsilon^{-k/2} - \operatorname{sign}^{S^1}(M,N).$$

Proof: Define the relative universal elliptic genus of (N,f), and apply (3.6) to each term in the q-expansions (see [HBJ92, 6.1]). $\square$

Corollary 3.20 *The following relation holds in $(\mathbb{Q}/\mathbb{Z})[[q]]$:*

$$\eta_{\mathcal{E}\ell\ell}(N,f) \equiv \varphi_{\mathcal{E}\ell\ell}(M,N)\,\varepsilon^{-k/2} \qquad (\operatorname{mod} \mathbb{Z})\,.$$

Corollary 3.21 *If the S^1-signature vanishes, then*

$$\eta_{\mathcal{E}\ell\ell}(N,f) = \varphi_{\mathcal{E}\ell\ell}(M,N)\,\varepsilon^{-k/2}\,.$$

If, in addition, k is even, then the elliptic eta invariant is a modular function of weight zero for the congruence subgroup $\Gamma_0(2)$.

Similar results hold for the Dirac operator on spin manifolds twisted by $S_q = \sum S^r TM$, corresponding to the Witten genus rather than the elliptic genus.

We give an example of Corollary 3.21 for the case that N is the sphere. The beauty of this example is the relation between the elliptic eta invariant and the Eisenstein series used in the definition of the universal elliptic genus.

Proposition 3.22 *There is a framing f of the boundary S^{4k-1} of the disk D^{4k} for which the elliptic eta invariant is given by*

$$\eta_{\mathcal{E}\ell\ell}(N, f) = \frac{4\, G^*_{2k}}{\varepsilon^{k/2}}\,.$$

Proof: First note that the signature and relative signature of the disk D^{4k} are zero, since it has zero cohomology in dimension $2k$. This is true for any twisted signature with coefficients in a flat bundle and for the S^1-signature of the free loop space. Thus we are in the situation of Corollary 3.21 and we have

$$\eta_{\mathcal{E}\ell\ell}(N, f) = \frac{\varphi_{\mathcal{E}\ell\ell}(D^{4k}, S^{4k-1})}{\varepsilon^{k/2}}\,.$$

Now recall [Smi71], [CF66] that a framing f may be chosen for the boundary such that

$$c_i = \begin{cases} 0 & \text{for } 0 \le i < 2k\,, \\ (2k-1)! & \text{for } i = 2k\,. \end{cases}$$

Since all but the highest Chern class vanish, the genus will be $c_{2k} \cdot a_{2k}$, where the characteristic series has the form $P(u) = \exp\left(\sum -a_{2k} u^{2k}/2k\right)$ (see [HBJ92, p. 20] for details). For the universal elliptic genus, then, we have

$$\varphi_{\mathcal{E}\ell\ell}(D^{4k}, S^{4k-1}) = (2k-1)!\,(-2k)\,\frac{4G^*_{2k}(\tau)}{(2k)!} = -4\, G^*_{2k},$$

as claimed. □

We will need to consider the reduction modulo Z of the above invariants, where Z is one of the rings $\mathbb{Z}$, $\mathbb{Z}[\frac{1}{2}]$, or more generally $\mathbb{Z}[\frac{1}{n}]$ or $\mathbb{Z}[\frac{1}{2n}]$ for n odd. Recall the elliptic genus φ_ε takes as values homogeneous polynomials in $\mathbb{Q}[\delta, \varepsilon]$, where δ, ε have degree 2, 4 respectively, and so the normalised elliptic genus $\overline{\varphi}_\varepsilon(M^{4k}) = \varphi_\varepsilon(M)/\varepsilon^{k/2}$ is a polynomial in

$$\rho = \delta/\sqrt{\varepsilon} = 1 + 32q + 256q^2 + 1408q^3 + 6144q^4 + 22976q^5 + \cdots$$

taking the q-expansion at the signature cusp $i\infty$. In fact $\rho \in 1 + 32\mathbb{Z}[q]$; in terms of the Dedekind eta function,

$$\begin{aligned} \rho - 1 &= 32\frac{\eta(4\tau)^8}{\eta(\tau)^8} = 32q\left(\prod_{n=1}^{\infty}\left(\frac{1-q^{4n}}{1-q^n}\right)\right) \\ &= 32q\left(\prod_{n=1}^{\infty}\left(1 + q^n + q^{2n} + q^{3n}\right)\right) \end{aligned}$$

Thus the normalised elliptic genus will have a q-expansion with integral coefficients only if as a polynomial in ρ it has coefficients in $\mathbb{Z}[\frac{1}{2}]$.

One can also express the normalised elliptic genus as a modular function of half-integral weight for the congruence subgroup $\Gamma(4, \mathbf{1})$, generated by θ and F_2. Considering

$$H = 4\sqrt{\varepsilon} = \theta^4 + 16F_2$$

one has $\rho = 1 + 32F_2/H$. Both H and F_2 have q-series with integral coefficients, with lowest-order terms 1 and q respectively.

We can also consider the differential operator $D = \dfrac{1}{2\pi i}\dfrac{d}{d\tau}$.

Proposition 3.23 *The modular function of half-integral weight $\rho(\tau)$ satisfies*

$$D\rho = \theta^4\,(\rho - 1).$$

Thus $D\varphi_\varepsilon(M) = \theta^4(\rho - 1)d\varphi_\varepsilon(M)/d\rho$.

Proof: Since $\rho\,(\tau) = 1 - 32\,\eta\,(4\tau)^8\,/\eta\,(\tau)^8$, and $D\,(\eta\,(\tau)) = -G_2\,(\tau)\,\eta\,(\tau)$, $D\,(\eta\,(4\tau)) = -4G_2\,(4\tau)\,\eta\,(4\tau)$. On the other hand, we have $\theta^4\,(\tau) = 8G_2\,(\tau) - 32G_2\,(4\tau)$, and the result follows. $\square$

3.3.3 Divided congruences and elliptic genera

Before introducing divided congruences we will recall a fact we proved in [Gal96]. There we defined a Spin$^{\text{c}}$-genus for which, as a consequence of the module structure of $\Omega_*^{\mathrm{Spin}^c} \otimes \mathbb{Q}$ over $\Omega_*^{\mathrm{Spin}} \otimes \mathbb{Q}$ as seen in [Sto68], we have

$$\phi_c(M)(q) = \sum_{j=0}^{[\frac{m}{2}]} \phi_{c,j}(M)(q)$$

for $[M] \in \Omega_{2m}^{\mathrm{Spin}^c}$ in which $\phi_{c,[\frac{m}{2}]} = \phi_W$ is just the Witten genus and each $\phi_{c,j}(M)$ is an almost-modular form of weight $2[\frac{m}{2}]$. Thus the genus ϕ_c is an inhomogeneous sum of rational modular forms $\sum f_i$ where f_i has weight i. Evaluating at the cusp we see the $\widehat{A}$-genus as a summand of the Todd genus:

$$\mathrm{Td}(M) = \phi_c(M)(0) = \phi_{c,0}(M)(0) + \cdots + \phi_{c,[\frac{m}{2}]-1}(M)(0) + \widehat{A}(M).$$

Suppose now that M is a manifold (such as $\mathbf{C}\mathrm{P}^{2k}$ with its canonical complex structure) with a Spin$^{\text{c}}$-structure but no Spin structure, so that $\mathrm{Td}(M) \in \mathbb{Z}$ but $\widehat{A}(M)$ is only rational, and similarly for their q-series analogues $\phi_c(M)(q)$ and $\phi_W(M)(q)$ respectively. Then we will have an inhomogeneous sum of rational modular forms $\phi_c(M)(q)$ whose q-series at the cusp $i\infty$ has integer coefficients. One says that $\phi_c(M)(q)$ is an element of the *ring of divided congruences* for the quasimodular forms.

In general, following Katz [Kat75] and using the notation of Laures [Lau99], we define the ring of divided congruences associated to a congruence subgroup Γ as the ring of inhomogeneous rational modular forms $\sum f_i$ where f_i has is a form of weight i for γ, such that for each integer $k \neq 0$ the q-expansion of the sum $\sum k^{-i} f_i$ at any cusp has coefficients in $\mathbb{Z}^{\Gamma}[\frac{1}{k}]$, where $\mathbb{Z}^{\Gamma} = \pi(K^{\Gamma})$.

Theorem 3.24 (Clarke–Johnson) *Let D denote the ring of non-homogeneous sums of rational modular forms for $\Gamma_0(2)$ such that the q-series of the sum is in $\mathbb{Z}[\frac{1}{2}][[q]]$. Then*

$$KO_0(\mathcal{E}\ell\ell) \cong D[\Delta^{-1}]$$

where $\mathcal{E}\ell\ell = \mathbb{Z}[\delta, \varepsilon][\frac{1}{2}][\Delta^{-1}]$.

Theorem 3.25 *Let φ be any Hirzebruch genus of elliptic type for a ring of modular forms fulfilling the hypotesis of the Katz theorem. Then the following facts hold:*

1. *The generators for the ring of divided congruences c_n are determined by*

$$\sum_{n=1}^{\infty} \log\left(\frac{1}{1-c_n y^n}\right) = \sum_{m\geq 1}^{\infty} \frac{\varphi(\mathbb{C}\mathrm{P}^m)}{m+1} y^{m+1} = g_{\varphi}(\tau, y)$$

   so that

$$\varphi(\mathbb{C}\mathrm{P}^n) = \sum_{d|n} c_n^{n/d} \cdot d.$$

2. *Let $\mathcal{L}_{\varphi}(x) = \log(Q_{\varphi}(\tau, x))$, whose Taylor expansion in x is*

$$\log(Q_{\varphi}(\tau, x)) = \sum_{m\geq 1}^{\infty} \frac{\varphi\left(D^{4k}, S^{4k-1}, \pi\right)}{(2k)!} x^{2k}$$

   for $\left(D^{4k}, S^{4k-1}, \pi\right)$ the framed disks defined above. Then $\varphi\left(D^{4k}, S^{4k-1}, \pi\right)$ generate the image of $\Omega_*^{G(E_\varphi)}$ up to 2-torsion and writing $G^{\varphi}_{2k_1} = \varphi\left(D^{4k}, S^{4k-1}, \pi\right)$ for the Eisenstein series determined by the genus φ we have

$$\varphi_E(\mathbb{C}\mathrm{P}^{2k}) = \sum_{\mathcal{P}_k} \frac{1}{r_1! \cdots r_t!} \frac{(G^{\varphi}_{2k_1})^{r_1} \cdots (G^{\varphi}_{2k_t})^{r_t}}{k_1^{r_1} \cdots k_t^{r_t} (2k_1-1)!^{r_1} \cdots (2k_t-1)!^{r_t}} (2k+1)^t$$

   where the summation is over partitions $k = \sum r_j k_j$, $r_j \geq 1$ and $0 < k_j < k_{j+1}$.

Whenever we have an elliptic genus, determined by its logarithm $g_{\varphi}(\tau, y)$, we see that the Katz divided congruences definitions may be written as

$$\prod_{n=1}^{\infty} (1-c_n y^n)^{-1} = \sum_{n=0}^{\infty} b_n y^n = \exp(g_{\varphi}(\tau, y)).$$

Observe moreover that for any genus φ_E corresponding to a suitably oriented generalized cohomology theory E_φ by a class x^{E_φ} we have

$$g_\varphi(\tau, y) = g_\varphi\left(\tau, x^{E_\varphi}\right) = x^H$$

where x^H denotes the usual oriented cohomology oriented with genus given by the additive formal group law. Then the expressions above become

$$\prod_{n=1}^{\infty}\left(1 - c_n\left(x^{E_\varphi}\right)^n\right)^{-1} = \sum_{n=0}^{\infty} b_n (x^{E_\varphi})^n = \exp(g_\varphi(\tau, x^{E_\varphi}))$$

and, since the orientation for the multiplicative group law in K-theory is given under the orientation provided by the Todd genus φ_{Td}, by

$$g_{\varphi_{\mathrm{Td}}}(\tau, y) = g_{\varphi_{\mathrm{Td}}}(\tau, x^K) = x^H = \log\left(\frac{1}{1 - x^K}\right).$$

Now suppose we are using the elliptic cohomology theory considered by Devoto, with Miller's character, composed with complexification, so that

$$\Theta\left(\tau, x^K\right) = x^{E_\varphi}.$$

Then the image of the K-theoretical orientation class and the Katz series b are related via

$$\left(b^{-1}\right)\left(\frac{1}{1 - x^K}\right) = x^{E_\varphi} = \Theta\left(\tau, x^K\right).$$

3.4 Equivariant elliptic invariants

We now examine the constructions of the previous section for operators on manifolds with a (finite) group action. First we summarise the results of Donnelly [Don78] for the equivariant Atiyah–Patodi–Singer formula for manifolds with group actions, and recall the definition of the classical Atiyah–Singer invariants from [AB66, AS68I]. We generalise these results to cover the case of Dirac operators associated to elliptic genera, acting on framed manifolds.

We compute the resulting invariants for the classical lens spaces and show that they are precisely the level-two modular functions which generate the coefficient ring of Devoto's equivariant elliptic cohomology [Dev96b].

Let Y be an odd-dimensional spin manifold of dimension $4k - 1$ with finite fundamental group $\pi_1 Y = G$ of odd order and let $E \to Y$ be a unitary flat bundle of dimension m associated to a representation

$$\alpha \colon G \to \mathrm{U}(m)$$

of G. We may denote E by E_α when we want to emphasise this relationship between the bundle and the representation.

We assume a Riemannian metric and connection are given on Y and that G acts by orientation-preserving isometries, and we may consider

$$G \hookrightarrow \mathrm{Isom}^{+}(Y).$$

We give the bundle the usual flat connection. Suppose further that Y bounds some $4k$-dimensional manifold X and that E extends to a bundle W over X, which may not be flat.

We will construct the *R-class* of the manifold X associated to the representation α,

$$R_{q,\alpha}(TX) \in K^{**}(X)[[q]],$$

as follows. Recall the *loop signature class* $R_q = \bigoplus R_n q^n$ for the bundle W by

$$R_q(W) = \bigoplus_{n\geq 0} R_n(W)q^n = \bigotimes_{n=1}^{\infty} S_{q^n}W \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}W.$$

We remark that all the bundles $R_n(W)$ inherit a flat connection. We shall write α_n for the unitary representation of $\pi_1 Y$ which gives $R_n(E)$. Let m_n denote the dimension of $R_n(E)$.

We also have a loop signature class for the tangent bundle,

$$R_q(TX) = \bigoplus_{n\geq 0} R_n(TX)q^n = \bigotimes_{n=1}^{\infty} S_{q^n}(TX\otimes\mathbb{C}) \otimes \bigotimes_{n=1}^{\infty} \Lambda_{q^n}(TX\otimes\mathbb{C}).$$

We shall define a formal power series of bundles by

$$R_{q,\alpha}(TX) = \sum_{n\geq 0} R_{n,\alpha}(TX)q^n = R_q(TX)\otimes R_q(W).$$

Each summand $R_{n,\alpha}(TX)$ can be written in the form

$$\begin{aligned} R_{n,\alpha}(TX) &= \sum_m R_{n,\alpha_{n-m}}(TX) \\ &= \sum_m R_m(TX)\otimes R_{n-m}(W). \end{aligned}$$

Adding all of them up we get a (formal, virtual) bundle over X that we may denote by $R_{q,\alpha}(TX)$. Let $D_+^{R_{q,\alpha}(TX)}$ denote the signature operator on X twisted by the bundle $R_q(TX)$ with respect to the representation α. We shall define $D_{+,n}^{R_q(TX)}$ as the Dirac operator twisted by $R_n(TX)$.

For every twisted signature operator D_+^{ξ} we may consider

Definition 3.26 The eta invariant for the operator D_+^{ξ} is

$$\eta_{D_+^{\xi}}(s) = \sum_{\lambda\neq 0,\ \lambda\in\mathrm{Spec}\left(D_+^{\xi}\right)} \frac{\mathrm{sign}(\lambda)}{|\lambda|^s}.$$

Because of the classical theorem of Atiyah–Patodi–Singer we have that $\eta_{D_+^\xi}(0)$ will be finite. The main result in [APS75] concerning those operators is that, for L the Hirzebruch polynomials in the Pontrjagin classes of X (i.e., the ones of the tangent bundle with the Levi-Cività connection associated to the metrics) in the (relative) characteristic classes of the manifold X and the twisting bundle ξ, the following holds.

Theorem 3.27 [APS75]

$$\operatorname{ind}\left(D_+^\xi\right) = L\left(p_1,\ldots,p_k\right)\operatorname{ch}\left(\Psi_2\left(\xi\right)\right)[X,Y] - \eta_{B^{\xi|_Y}}(0).$$

The APS theorem for the operator D_+^ξ twisted by the representation α then reads

$$\operatorname{ind}\left(D_+^{\xi_\alpha}\right) = L\left(p_1,\ldots,p_k\right)\operatorname{ch}\left(\Psi_2\left(\xi\otimes W_\alpha\right)\right)[X,Y] - \eta_{B^{\xi|_Y\otimes E_\alpha}}(0).$$

Our version of those will be

Definition 3.28 The *elliptic (signature) eta invariant*, $\eta_\alpha^\varepsilon(0)$, associated to the representation α of $\pi_1 Y = G$ with associated flat bundle E_α is the formal power series in the variable q (where $q = e^{2\pi i\tau}$, $\operatorname{Im}(\tau) > 0$) is given by the mod $\mathbb{Z}\left[\frac{1}{|G|}\right]$-reduction

$$\eta_\alpha^\varepsilon(0) = \eta_{B^{R_{q,\alpha}(TY)}}(0) = \sum_{n\ \in\mathbb{Z}} \eta_{B^{\left(R_{q,\alpha}(TY)\right)_n}}(0)\, q^n \in \mathbb{R}/\mathbb{Z}\left[\frac{1}{|G|}\right].$$

Remark 3.29 Formally, at least, we may consider $D_{R_{q,\alpha}(TX)}$. Then, adding formally the equation given by the classical APS theorem for each n, one will have

$$D_+^{R_{q,\alpha}(TX)} = \sum_{n\ \in\mathbb{Z}} D_+^{\left(R_{q,\alpha}(TX)\right)_n} q^n.$$

We will denote by $D_+^{R_{q,\alpha}(TX)}$ this formal sum of operators, and abuse the usual notation for operators by applying it to the objects obtained from it, e.g., index, etc. Let us denote by $\Phi_\varepsilon\left(p_1,\ldots,p_k\right)$ the characteristic polynomial series in the Pontrjagin classes corresponding with the version of the 2-level elliptic genus which agrees with the usual signature in its term in q^0. So we have

Proposition 3.30 *As formal power series in q,*

$$D_+^{R_q(TX)} = \Phi_{\varepsilon,k}\left(p_1,\ldots,p_k\right)[X,Y] - \eta_{B^{R_q(TY)}}(0) \mod \mathbb{Z}\left[\frac{1}{|G|}\right]$$

and

$$\operatorname{ind}\left(D_+^{R_{q,\alpha}(TX)}\right) = \Phi_{\varepsilon,k}\left(p_1,\ldots,p_k\right)\operatorname{ch}\left(\Psi_2\left(W_\alpha\right)\right)[X,Y] - \eta_{B^{R_{q,\alpha}(TY)}}(0) \mod \mathbb{Z}\left[\frac{1}{|G|}\right].$$

Taking into account the signature with coefficients in bundles, we may write

$$\operatorname{sign}\left(X, R_q\left(TX\right)\right) = \varphi_\varepsilon[X,Y](\tau) - \eta^\varepsilon(0;\tau) \mod \mathbb{Z}\left[\frac{1}{|G|}\right],$$

where $\varphi_\varepsilon[X,Y](\tau)$ denotes the relative elliptic genus. Similarly,

$$\operatorname{sign}\left(X, R_{q,\alpha}\left(TX\right)\right) = \varphi_{\varepsilon,\alpha}[X,Y](\tau) - \eta_\alpha^\varepsilon(0;\tau) \mod \mathbb{Z}\left[\frac{1}{|G|}\right].$$

Theorem 3.31 *The elliptic eta invariant $\eta_{\alpha}^{\varepsilon}(0)$ is the power expansion of a $\mathbb{R}/\mathbb{Z}\left[\frac{1}{|G|}\right]$-modular form.*

Suppose Y is a frameable manifold (i.e., it admits a trivialisation of its stable tangent bundle $\tau_s Y$). If one considers on the tangent bundle instead of the Levi-Cività connection any connection extending the flat one given on $\tau_s Y$ by the framing, one may consider the same construction as for the operator consider above, but for the new operator defined by the new connection. Observe that the new connection is going to have torsion. Those where considered in [ADS83]. The expression in the L classes is then twisted by other ones taking into account the torsion form. One get interesting eta invariants for those operators. They can be constructed for all the disks and for the many lens spaces known to be frameable.

Consider the Dirac operator twisted by $S_q TM$. Its index under the global APS boundary condition is related to the Witten genus by

$$\varphi_W(\partial M)\eta(\tau)^{-4k} - \frac{1}{2}(h_W + \eta_W).$$

Since the index and the kernel dimensions h are integers,

$$\frac{1}{2}\eta_W \equiv \frac{1}{2}\varphi_W \eta^{-4k}$$

modulo $\mathbb{Z}$, where η is the Dedekind eta-function.

For example, on (D^{4k}, S^{4k-1}) we have the eta invariant given by

$$\frac{1}{2}\eta_W(S^{4k-2}) \equiv G_{2k}(\tau)\eta^{-4k}(\tau)$$

modulo $\mathbb{Z}$.

Other relevant pairs of the form $\left(D^{4k}(\xi), S^{4k-1}(\xi), \pi_t\right)$ admit a similar treatment. E.g., the generalisation of APS in the other direction yields a corresponding result for $\mathbf{H}\mathrm{P}^k$ bundles, using the constructions of Stong and Conner–Floyd. For the bundle they term η,

$$\phi_c\left(D^{4k}(\eta)\right) = \phi_{\mathcal{W}}\left(D^{4k}(\eta)\right) = G_4(\tau) - DG_2(\tau) = \frac{1}{6}G_4 + 2G_2^2.$$

Remember that $\phi_c(\mathbb{H}\mathrm{P}^2) = \phi_{\mathcal{W}}(\mathbb{H}\mathrm{P}^2) = -DG_2(\tau) = -\frac{5}{6}G_4 + 2G_2^2$, so that

$$\phi_c\left(D^{4k}(\eta)\right) - \phi_c\left(\mathbb{H}\mathrm{P}^2\right) = G_4.$$

From what has been seen for relative multiplicative sequences on disks,

$$\phi_c\left(\left(D^4_\pi\right)^2\right) = 4G_2^2, \qquad \phi_c\left(D^8_\pi\right) = 2G_4, \qquad \phi_c\left(\mathbb{C}\mathrm{P}^2\left(D^4_\pi\right)\right) = 6G_2^2.$$