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On the Tamagawa number conjecture

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Dr. Xavier Xarles i Ribas

Era 7 de Setembre de 1998.
Aterrava plè d'il.lusions
en una ciutat alemana,
anomenada Münster.
Feia plugim i estava ennuvolat.
Arribava a un bressol matemàtic
on en l'ambient s'hi respiraven
lloables grans teories: L-funcions,
teoria d'Iwasawa, teoria Hodge p-àdica,
geometria rígida,
polilogaritmes, . . .
Heus aquí que sota aquesta
atmosfera començava la següent
fascinant aventura d'amors i
desamors en:
"The Tamagawa number conjecture."

Contents

Introduction

The Tamagawa number and its conjecture appears first in the '60 in the works of T. Ono (see for example [45]) for algebraic groups G defined over a number field F. The conjecture relates the value at 1 of the L functions of the algebraic group with some arithmetical objects associated to the algebraic group: the torsion Picard group of the algebraic group and a Tate-Shafarevich group, which is defined by a local-global principle for the Galois cohomology group associated to the $Gal(\overline{F}/F)$-module $G(\overline{F})$. Bloch in [4] relates the above conjecture with the Birch-Swinnerton-Dyer conjecture when G is an abelian torus of dimension 1 and $F = \mathbb{Q}$.

For a long time there have been conjectures and theorems relating L-functions of algebraic objects defined over a number field F to its arithmetic properties. The oldest example is the Dedekind zeta function of an algebraic number field, which has a simple pole at 1 and its residue is given in terms of the class number and the regulator of the field F. In 1963, Birch and Swinnerton-Dyer obtain the famous conjecture which predicts in terms of arithmetic properties of an elliptic curve defined over $\mathbb{Q}$, the exact integer value of the first non-zero Fourier coefficient of the the L-function at 1.

In order to globalize the conjectures, Bloch and Kato in 1990 defined the Tamagawa number associated to any motive with negative weight over $\mathbb{Q}$ in [6]. They also define a Tate-Shafarevich group associated to the motive and they predict the value of this number in terms of arithmetic objects associated to the motive. We note that the Tamagawa number of the motive is related with the value of the L-function of the motive at 0. Latter appeared the work of Fontaine and Perrin-Riou [21][22] generalizing the above conjectures and the work of Kato [33][34], who wrote the above conjecture for Chow motives over an arbitrary number field F.

The value of the L-function at integer points for Chow motives modulo $\mathbb{Q}^*$, was conjecturally understood in 1985 by the works of Deligne [8] and Beilinson [2]; their conjectures predict the exact value mod $\mathbb{Q}^*$ at integer points for the L-function associated to a Chow motive. Beilinson [2] defines a regulator map in order to explain the order of vanishing for the L-function of the Chow motive at 0 and also predicts the value modulo $\mathbb{Q}^*$ of the first non-zero Fourier coefficient at 0 of the L-function associated to this Chow motive. To verify this conjecture, one needs to construct elements in Quillen's K-group of our motive and compute its image by his regulator map.

To obtain the Tamagawa number conjecture from the Beilinson conjecture, it remains to control, for every prime p of $\mathbb{Q}$, the valuation at p of the number which we control modulo $\mathbb{Q}^*$ using the Beilinson conjecture. This control for a fixed prime p is called the local Tamagawa number conjecture for the motive. The control for any prime number p of $\mathbb{Q}$ of the local Tamagawa number conjecture is equivalent to the Tamagawa number conjecture for the motive. Here the Soulé regulator map gives the key technique to remove the ambiguity from $\mathbb{Q}^*$ to $\mathbb{Z}^*$. Soulé [60] constructed, for every finite prime p of $\mathbb{Q}$, a regulator map for any Chow motive which maps the K theory with $\mathbb{Z}_p$-coefficients to the first Galois cohomology with respect to a convenient Galois group applied to a particular p-adic realization associated to the Chow motive.

In the Kato's reformulation [34] of the conjecture, the value of the L-function for a Chow motive is exactly (modulo $\mathbb{Z}^*$) the one given by the Beilinson conjecture if the cardinal of the coimage of the elements in K-theory by the p-Soulé regulator map for every finite prime p of $\mathbb{Q}$, is equal to the number of elements of the second cohomology group of the Galois group $Gal(F_S/F)$ acting on the p-adic integer realization of the Chow motive, where F is the field of definition of the Chow motive, and F_S is the maximal unramified outside S of F where S contains the prime of bad reduction of the Chow motive and the primes above p. When this is checked only for a prime p, we speak of the p-local Tamagawa number conjecture for the motive.

The L-function for any motive is conjecturally extended holomorphically to $\mathbb{C}$, and defined by an Euler product for $Re(s) > i/2+1$ in the case of a pure Chow motive of the form $h^i(X)$. In this last situation, there is a functional equation relating the values of the L-function at s and at $i+1-s$. As a consequence, the Tamagawa number conjecture predicts also the value of the L-functions where the Euler product of the L-function does not takes sense. There is really few knowledge of these values in terms of the arithmetic of the motive which defines the L-function, and there are classically a lot of works in the study for the critical strip $i/2 \leq m \leq i/2+1$; for example, if $i = 1$ and X an elliptic curve defined over $\mathbb{Q}$, it is the Birch and Swinnerton-Dyer conjecture. Our work concentrates in the values of L-functions where the Euler product take sense. In the case of a pure Chow motives $h^i(X)$, we restrict hence to motives of the form $h^i(X)(r)$ with $r > i/2+1$, this means $i - 2r \leq -3$.

There are few situations where attempts to the conjecture had been proved for values at the convergent part of the Euler product. Let me list these situations. For motives $\mathbb{Q}(r)$, which corresponds to the Riemann zeta function, Kato in §6[6] proves the Tamagawa number conjecture for $r \geq 2$. The conjecture corresponding to Dirichlet characters have been proved recently by Hubber and Kings [29]. These cases corresponds to 0-dimensional motives. For varieties of higher dimension we only know the Beilinson conjecture for: elliptic curves with CM proved by Deninger [9][10], (moreover, Deninger proves the Beilinson conjecture in generality for Hecke characters over an imaginary quadratic field and also in the work of S.Feil [17] is obtained the Beilinson conjecture for these characters when they descends over a real field), and for modular curves over $\mathbb{Q}$ proved by Beilinson [54]. There are no more known cases on this first step on the Tamagawa number conjecture. Thus, we will center our attention in the previous situations. Our work concentrates in the case of elliptic curves with CM and on Hecke characters.

K. Kato has made some progress recently in the case of modular curves, but in the critical case. Then, with this work and with the proved modularity for any elliptic curve over $\mathbb{Q}$ (Wiles-Taylor- Darmon-Diamond-B.Conrad) seems that Kato would obtain some progress on the Birch- Swinnerton-Dyer conjecture.

Let us concentrate first to elliptic curves E with CM. We mention first that for the critical case $h^1(E)(1)$, it is known the result of Coates-Wiles proving the conjecture if the $L(h^1(E)(1), 0) \neq 0$ and if E is defined over $\mathbb{Q}$ or over the imaginary quadratic field which is the field of fractions of the endomorphism ring of E. We study the values where the Euler product take sense for the L-function associated to E, i.e. we study the values at zero for the L-function of $h^1(E)(k+2)$ for $k \geq 0$. Observe that these motives have weights ≤ -3, which correspond to the value at $k+2$ of the L-function $L(E, s)$. Kato in [6] proves, under some hypothesis for the primes p, the Tamagawa number conjecture for the pure motive $h^1(E)(2)$ when E is defined over $\mathbb{Q}$. Kings [36] computes the specialization of the elliptic polylogarithm which implies a comparison between the image of the Soulé regulator map with the image of a map between modules on Iwasawa theory to the group which maps the Soulé regulator map (this map was also constructed by Soulé). With this ideas, Kings partially proves the Tamagawa number conjectures for $h^1(E)(k+2)$ for any positive integer k, with E a CM elliptic curve defined over the field K of the endomorphism ring, which is an imaginary quadratic field. In order to generalize the result of Kato from $h^1(E)(2)$ to $h^1(E)(k+2)$ when E is defined over $\mathbb{Q}$, we need to descend the work of Kings. Using the functional equational which we have for $h^1(E)$, the conjecture is related with the value of the first non-zero Fourier coefficient of $L(E, s)$ at $s = -k$. We obtain then the desired result with the same hypothesis of Kato in §7[6], which generalizes the case of $h^1(E)(2)$ to $h^1(E)(k+2)$ for any integer $k \geq 0$,

Theorem A. *Let $\mathfrak{f}$ be the conductor of the elliptic curve E with CM by $\mathcal{O}_K$, the ring of integers of an imaginary quadratic field, and defined over $\mathbb{Q}$. Let $v \in \mathcal{O}_K$ be the element which multiplied with the complex period gives the real period for the elliptic curve E. Consider a prime number $p > 3$ of $\mathbb{Q}$ such that $(p, N_{K/\mathbb{Q}} v\mathfrak{f}) = 1$. Let S be the set of primes of $\mathbb{Q}$ which divide $pN_{K/\mathbb{Q}}\mathfrak{f}$. Let's suppose that the $\text{rank}_{\mathbb{Q}} K_{2k+2}(E)^{(k+2)} \geq 1$ is one, and that $H^2(Spec(\mathbb{Z}[1/S]), H^1_{et}(E \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p(k+2)))$ is finite. Then, for such prime numbers p, the local Tamagawa number conjecture at p for the motive $h^1(E)(k+2)$ is true.*

We study now in our work the Tamagawa number conjecture for the Hecke character over an imaginary quadratic field K given by $\psi_\theta := \psi^a \overline{\psi}^b$, where ψ is the Hecke character associated to the elliptic curve E with CM defined over the field of fractions of the endomorphism ring of E. Let us denote by $w = a + b$. Our work corresponds to the study for every natural $k \geq 0$ of the conjecture for the pure motive

$$e_\theta \left((\otimes^w h^1(E))(w+k+1) \right) \subset h^w(E^w)(w+k+1),$$

where e_θ means an idempotent in the correspondences of the Chow group. The work of Kings for type $(1,0)$ [36] give us the basic ingredients to generalize the Tamagawa number conjecture to other types of Hecke characters over imaginary quadratic fields. The main result we obtain is the following. We write for simplicity only a result in the case p a prime of $\mathbb{Q}$ which splits in the imaginary quadratic field K.

Theorem B. *Let ψ_θ be a Hecke character over an imaginary quadratic field K with infinite type $(w,0)$ or $(0,w)$, with w a natural number ≥ 1. Let $\mathfrak{f}_\theta$ be the conductor of this Hecke character, and let $\mathfrak{f}$ be the conductor of the elliptic curve E with CM defined over K.*

Let k be an integer such that $-w - 2k \leq -3$. Fix a finite prime number $p > 3$, which splits in K and such that $(p, N_{K/\mathbb{Q}}\mathfrak{f}) = 1$. Suppose that $(p-1, w) = 1$ and $(|\mathcal{O}_K^*|, w) = 1$. Suppose moreover that the cohomology group $H^2(Spec(\mathcal{O}_K[1/S]), e_\theta(\otimes^w T_p E)(k+1))$ is finite, where S is the set of finite places of K which divide $pN_{K/\mathbb{Q}}\mathfrak{f}_\theta$, and that the K-theory group corresponding to the motive $e_\theta(\otimes^w h^1(E))(w+k+1)$ is one dimensional as K-vector space. Then, the local Tamagawa number conjecture at p is true for $e_\theta(\otimes^w h^1(E))(w+k+1)$.

In the above two theorems on the Tamagawa number conjecture is not proved the finiteness for the second Galois cohomology group

$$H^2(Spec(\mathcal{O}_K[1/S]), e_\theta(\otimes^w T_p E)(k+1)) \tag{1}$$

with $w = 1$ in theorem A. A conjecture of Jannsen asserts these finiteness; this conjecture is included in the general Tamagawa number conjecture in the formulation of Kato [34]. This conjecture of Jannsen is proved for the regular primes p of any elliptic curve with CM defined over the endomorphism ring by Wingberg [62]. With this techniques we can also prove, for the regular primes of E, the conjecture corresponding to Hecke characters with $w > 1$. Some other results on the Jannsen conjecture are in the case that we fix w in (1) and we do not fix the Tate twist k. Then, for almost all k, this Galois cohomology groups would be finite if we can check that the corresponding motives satisfies the conditions of lemma 8b) [32]. This have been checked if $w = 1$.

The Tamagawa number conjecture try to explain in terms of the arithmetic of the motive the integer values of the L-function. It is natural idea, if we know how to define p-adic L-functions, to use them to take out the p-adic ambiguity modulo $\mathbb{Q}^*$ for the values of the L-function at the integers. Thus, one can do a conjecture for these p-adic L-function compatible with the Tamagawa number conjecture; in this way, there are the works of Perrin-Riou [46] and Colmez [7]. For Hecke character over an imaginary quadratic field K there is a well-known definition of their p-adic L-functions. Moreover, in this situation, it is proved an interpolation formula in the critical band (see de Shalit chapter IV 4.16(50) [14]). Geisser in [25] computes the coimages with respect to a natural Soulé map of certain Iwasawa module to the first cohomology group at a place $\mathfrak{p}$ over p of K (we restrict to the case that p split in K) of the local Galois group $K_{\mathfrak{p}}$ acting on the p-adic integer étale realization of the motive associated to a Hecke character. His result is that the length of the coimage is equal to the p-valuation of the p-adic L-functions of the Hecke character. The work of Kings [36] relates the image with respect to the Soulé regulator with the the image with respect to a map from certain Iwasawa module as above but using the global Galois group of K instead of a local Galois group. In order to relate this two works, we prove for a power of the grossencharacter ψ, that the subspace on the Iwasawa modules constructed by Geisser can be replace for the subspace on K-theory of these Hecke characters, constructed by Deninger in the proof of the Beilinson conjecture. The result is contained in appendix C.

Let me made briefly a sketch of the contents of this work. In every chapter and in the appendix there is a more extended explanation. In chapter 1 we rewrite the original Tamagawa number conjecture for any motive defined over $\mathbb{Q}$ of Bloch and Kato. For our interest later on, we restrict to pure motives and we write the formulation of the conjecture of Kato for any pure motive over a number field K. Here, some conjectures on the arithmetic geometric regulators (the Beilinson regulator and the Soulé regulator) appears naturally.

In chapter 2 we prove the local Tamagawa number conjecture for the motive $h^1(E^+)(k+2)$ with E^+ an elliptic curve with CM defined over $\mathbb{Q}$, and such that E^+ the endomorphism ring in $\overline{\mathbb{Q}}$ correspond to the ring of integers of an imaginary quadratic field K. Here we need to choose the convenients elements in K-theory for our motive in order to check the conjecture on the image of these elements by the regulator maps. We compare these elements with the ones defined by Kings used to prove the conjecture for $h^1(E^+ \times_{\mathbb{Q}} K)(k+2)$. We need to study the descend problem for the Soulé regulator map, and the descend problem for the Galois cohomology groups which come from the action of a global Galois group on the p-adic integer realization of the motive $h^1(E^+ \times_{\mathbb{Q}} K)$.

In the third chapter we define the natural motive associated to Hecke characters over an imaginary number field of type $\psi_\theta = \psi^a \overline{\psi}^b$ where ψ is the associated Hecke character associated to an elliptic curve with CM $\mathcal{O}_K$ which is also defined over K. We construct the elements in K-theory which we will use to check under some hypotheses the local Tamagawa number conjecture. We define the Soulé map between Iwasawa modules and the first Galois cohomology group for the p-adic integer realization of the motive associated to these Hecke characters. Here it plays an important role the main conjecture for the Iwasawa theory for imaginary quadratic fields. Then we compare the image of this map with the Soulé regulator map applied to the elements we choose in the K-theory group, to check the conjecture in this situation. In this last part plays a key role the specialization of the polylogarithm sheaf for the elliptic curve E.

In appendix A we remind some properties of theta functions which we need for the definition of elliptic units; these properties play an important role in order to compare the Soulé regulator map with the map between modules on Iwasawa theory with Galois cohomology groups in chapter 3. In appendix B we make an approach to the general definition of regularity following the Kummer criterium. Here it is also presented some properties of these definition for imaginary quadratic fields, which is related to the question of when a global Galois group is isomorphic to a local Galois group (Galois group of local type). Here we prove the Jannsen conjecture for (1) with $w > 1$ under the restriction that the prime p is regular for the elliptic curve E.

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Chapter 1

The Tamagawa number conjecture

1.1 Bloch-Kato's formulation of the Tamagawa number conjecture

Introduction.

In this section we formulate the original conjecture for the Tamagawa number of a motive. The Tamagawa number was defined for algebraic groups and Bloch-Kato in [6] made the first approach to a general definition of the Tamagawa number for motives M over $\mathbb{Q}$. The Tamagawa number (which is conjecturally a rational number) tries to explain the value at 0 of the L-function. Or, more precisely, we need to know the first Fourier coefficient of the L-function at the value 0. Observe that, if we take a Chow motive $M = h_m(X)(r)$, the behavior at 0 of the L-function associated to M corresponds to the behavior at the value r of the L-function associated to $h_m(X)$; thus, using the Tate twist, the conjecture on the Tamagawa number tries to understand the behavior at integer values of the L-functions. We present the conjecture for motivic pairs having weights ≤ -3 because the motives we are interested in, satisfy this condition. We refer to the Bloch-Kato original article §5 in [6] for the definition of the Tamagawa number for weight ≤ -1. We note that the restriction to negative weights is not too strict, since it is expected a functional equation relating the L-functions of M and $M^*(1)$, where M^* denotes the dual Chow motive. Hence, if M has positive weight, the control for negative weights of the value of the L-function for M would control the value of the L-function for $M^*(1)$.

Let me briefly present and overview of the diferent subsections of this section.

The first three ones deal with general facts on arbitrary local fields. In the first we remind the rings B_{crys} and B_{dR} which are the key to define some measures in the case (p, p) needed in the definition of the Tamagawa number. We present these rings in full generality, without the assumption that the residue field of the local field is perfect. For our interest later on, however, we only use them in the case that the residue field is perfect. In the second subsection we present the exponential map mapping the de Rham realization of a motive at a place p to a Galois cohomology group with coefficients in a p-adic vector space. This gives a way to translate a measure to the de Rham realization of a motive. In the third subsection, we define, for any l-adic representation of a local Galois group which residue field is a finite field of characteristic p, some particular subgroups of the Galois cohomology of the l-adic realization for (p, l) with $p \neq l$. We define also the local Euler factor of any p-adic vector space with a Galois action by a local Galois group.

The next subsection deals with motives over $\mathbb{Q}$. Here we define the Tamagawa number as a measure of a lattice inside the diferent realizations of the motive. We work via the realizations of the motive, because we can not work directly with the motive yet. We present also the conjecture for the Tamagawa number. In the last subsection we try to understand the l-value of the Tamagawa number. This conjecture needs less conditions and assumptions for its formulation. We will proof in the next chapters results on this local Tamagawa number conjecture.

1.1.1 The rings B_{crys} and B_{dR}.

In this section, we denote by L a complete valuation ring of characteristic 0 and by $\mathcal{O}_L$ its valuation ring. Let $\overline{L}$ be a fixed algebraic closure of L. We suppose that the residue field, k, has characteristic $p > 0$. We suppose $[k : k^p] = p^e < \infty$. We follow the construction of Kato in [35].
Let L' be a subfield of L. Denote by $\mathcal{O}_{L'}$ the discrete valuation ring $L' \cap \mathcal{O}_L$. Let

$$B_{n,\overline{L}/L'} := H^0_{crys}(Spec(\mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}})_{crys}, \mathcal{O}_{crys}),$$

considering $Spec(\mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}})$ as a $Spec(\mathcal{O}_{L'}/p^n\mathcal{O}_{L'})$-scheme. Here $\mathcal{O}_{crys}$ is the structural ring in the crystalline cohomology taking the standard PD-structure of the ideal (p). For the definition and properties of crystalline cohomology, we refer to Berthelot [3].

Let L'' be another subfield of L with $L' \subset L''$. We have a canonical morphism $B_{n,\overline{L}/L'} \to B_{n,\overline{L}/L''}$. Let $J_{n,\overline{L}/L'}$ be the ideal

$$J_{n,\overline{L}/L'} := Ker(B_{n,\overline{L}/L'} \overset{\varphi_{n,L'}}{\to} \mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}}),$$

where $\varphi_{n,L'}$ is the natural map, which is surjective (see 2.1.2 in [35]). For any $q \geq 0$, the ideal $J^{[q]}_{n,\overline{L}/L'}$ of $B_{n,\overline{L}/L'}$ is the q-th divided power of $J_{n,\overline{L}/L'}$. Define

$$B_{\infty,\overline{L}/L'} := \varprojlim_n B_{n,\overline{L}/L'}, \quad J^{[q]}_{\infty,\overline{L}/L'} = \varprojlim_n J^{[q]}_{n,\overline{L}/L'},$$

and

$$B^+_{crys,\overline{L}/L'} := B_{\infty,\overline{L}/L'} \otimes \mathbb{Q}.$$

Remark 1.1.1. *When the residue field k is perfect, the above construction can be made explicitly by using the Witt ring of the residue field of $\mathcal{O}_{\overline{L}}$ (see for example §1 in [6], or originally, Fontaine-Messing [20]). In the non perfect case, it could be obtained from a proper use of Cohen rings (see Bourbaki, Alg. comm. Chp.9 for the precise definition of Cohen rings, and I.2.3[3] for the properties we need) but, in this case, we have no uniqueness for the discrete valuation rings with k as residue field and p as generator of the maximal ideal (these rings are the Cohen rings that interest us). When k is perfect, this Cohen ring corresponds to the Witt ring of the (finite) field k.*

As usual, let $\mathbb{Z}/p^n(1)$ be the group of p^n-roots of 1 in $\overline{L}$ and let $\mathbb{Z}_p(1)$ be the projective limit $\varprojlim_n \mathbb{Z}/p^n(1)$. We have a canonical homomorphism

$$t : \mathbb{Z}_p(1) \to J_{\infty,\overline{L}/\mathbb{Q}_p},$$

obtained by taking inverse limit of

$$\mathbb{Z}/p^n(1) \to J_{n,\overline{L}/\mathbb{Q}_p}.$$

Let us describe t in more detail. Let x be the image of a p^n-th root of 1, α, in $\mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}}$. Let y^{p^n} be any lifting of x, with $y \in B_{n,\overline{L}/\mathbb{Q}_p}$. This element y^{p^n} belongs to the ideal

$$ker((B_{n,\overline{L}/\mathbb{Q}_p}) \to (\mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}})^*) = 1 + J_{n,\overline{L}/\mathbb{Q}_p}.$$

The map t is then defined by $\alpha \mapsto log(y^{p^n})$. This map t is injective (2.1.3 [35]).

The canonical injective morphism $B_{n,\overline{L}/L'} \to B_{n,\overline{L}/L''}$, when $L' \subset L'' \subset L$, defines an injective map

$$J_{n,\overline{L}/L'} \to J_{n,\overline{L}/L''}.$$

We have then an injective map

$$\mathbb{Q}_p(1) \to J_{\infty,\overline{L}/L'} \otimes \mathbb{Q}, \tag{1.1}$$

obtained as tensoring by $\mathbb{Q}$ the $\mathbb{Z}_p$-linear map t, and denote also by t the above map (1.1) composed by the natural inclusion in $\subset B^+_{crys,\overline{L}/L'}$.

We define

$$B_{crys,\overline{L}/L'} := B^+_{crys,\overline{L}/L'}[t^{-1}].$$

Let's now study the relation between $B^+_{crys,\overline{L}/L}$ and $B^+_{crys,\overline{L}/\mathbb{Q}_p}$. Observe that we have natural inclusions

$$B^+_{crys,\overline{L}/\mathbb{Q}_p} \subseteq B^+_{crys,\overline{L}/L'} \subseteq B^+_{crys,\overline{L}/L}.$$

Let $(b_i)_{1\leq i\leq e}$ be a lifting of a p-basis of k to $\mathcal{O}_L$ (the images of b_i in k generate k over k^p), and fix a p^n-th root $b_{i,n} \in \overline{L}$ for each i and $n \geq 0$ satisfying $(b_{i,n+1})^p = b_{i,n}$ for any i and n. We take a lifting $y_{i,n} \in B_{n,\overline{L}/\mathbb{Q}_p}$ of $b_{i,n}$ modulo p^n in $\mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}}$ and define

$$c_{i,n} = (y_{i,n})^{p^n} \in B_{n,\overline{L}/\mathbb{Q}_p},\ c_i = (c_{i,n})_n \in B_{\infty,\overline{L}/\mathbb{Q}_p}.$$

Then $b_i - c_i \in J_{\infty,\overline{L}/L}$ (c.f. 2.1.7 [35]). We have the following results:

Proposition 1.1.2 (Faltings[16], Hyodo[30]). *The elements t and $b_i - c_i$ ($1 \leq i \leq e$) form a basis of the C_p-vector space*

$$\mathbb{Q} \otimes (J_{\infty,\overline{L}/L}/J^{[2]}_{\infty,\overline{L}/L}),$$

where C_p is the completion $\hat{\overline{L}}$ of $\overline{L}$.

Proposition 1.1.3 (Kato, c.f.2.1.13 [35]). *The canonical surjection*

$$Sym^i_{C_p}(\mathbb{Q} \otimes (J_{\infty,\overline{L}/L}/J^{[2]}_{\infty,\overline{L}/L})) \to \mathbb{Q} \otimes (J^{[i]}_{\infty,\overline{L}/L}/J^{[i+1]}_{\infty,\overline{L}/L})$$

is a bijection for all $i \geq 1$.

Corollary 1.1.4. *Let $p \neq 2$ be a prime number, and suppose that L has perfect residue field (i.e. L is a finite extension of $\mathbb{Q}_p$). Then*

$$B^+_{crys,\overline{L}/\mathbb{Q}_p} = B^+_{crys,\overline{L}/L'} = B^+_{crys,\overline{L}/L},$$

for all $L' \subseteq L$.

Proof. If the residue field of L is perfect, $e = 0$. Then, by proposition 1.1.2, t generates the $\mathbb{C}_p$-space $\mathbb{Q} \otimes J_{\infty,\overline{L}/L}/J^{[2]}_{\infty,\overline{L}/L}$. We know that the ideal $J_{\infty,\overline{L}/\mathbb{Q}_p}$ injects in $J_{\infty,\overline{L}/L}$, and that the quotient $\mathbb{Q} \otimes J_{\infty,\overline{L}/\mathbb{Q}_p}/J^{[2]}_{\infty,\overline{L}/\mathbb{Q}_p}$ is also generated by t (c.f. proposition 1.1.2). We claim that the ideals $J_{\infty,\overline{L}/L}$ and $J_{\infty,\overline{L}/\mathbb{Q}_p}$ are equal after tensoring with $\mathbb{Q}$. In fact, consider the following commutative diagram for all i:

$$\begin{array}{ccccccccc} 0 & \to & J^{[i+1]}_{\infty,\overline{L}/\mathbb{Q}_p} \otimes \mathbb{Q} & \to & J^{[i]}_{\infty,\overline{L}/\mathbb{Q}_p} \otimes \mathbb{Q} & \to & (J^{[i]}_{\infty,\overline{L}/\mathbb{Q}_p}/J^{[i+1]}_{\infty,\overline{L}/\mathbb{Q}_p}) \otimes \mathbb{Q} & \to & 0 \\ & & \downarrow \alpha_{i+1} & & \downarrow \alpha_i & & \downarrow\cong & & \\ 0 & \to & J^{[i+1]}_{\infty,\overline{L}/L} \otimes \mathbb{Q} & \to & J^{[i]}_{\infty,\overline{L}/L} \otimes \mathbb{Q} & \to & (J^{[i]}_{\infty,\overline{L}/L}/J^{[i+1]}_{\infty,\overline{L}/L}) \otimes \mathbb{Q} & \to & 0 \end{array}.$$

We obtain by the snake lemma that $coker(\alpha_i) = coker(\alpha_{i+1})$. Then, using corollary 3.2.1 [2] (here we need $p \neq 2$) we have that $0 = \cap_i J^{[i]}_{\infty,\overline{L}/L} \otimes \mathbb{Q}$, and this implies that $J_{\infty,\overline{L}/L} \otimes \mathbb{Q} = J_{\infty,\overline{L}/\mathbb{Q}_p} \otimes \mathbb{Q}$.

To obtain the result, take projective limit with respect to n of the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \to & J_{n,\overline{L}/\mathbb{Q}_p} & \to & B_{n,\overline{L}/\mathbb{Q}_p} & \to & \mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}} & \to & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \to & J_{n,\overline{L}/L} & \to & B_{n,\overline{L}/L} & \to & \mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}} & \to & 0 \end{array}$$

(observe that $\varprojlim_n^1 J_n = 0$ because is a surjective system) and tensor with $\mathbb{Q}$. The diagram we get has the left and right vertical arrows being isomorphisms. Then use the five lemma to conclude. □

In general, there is always a natural isomorphism (if $p \neq 2$):

$$B^+_{crys,\overline{L}/\mathbb{Q}_p}[T_1, \ldots, T_e] \to B^+_{crys,\overline{L}/L}, \qquad (1.2)$$

given by $T_i \mapsto b_i - c_i$. The proof runs parallel to that of Corollary 1.1.4.

In $B^+_{crys,\overline{L}/L'}$ we have naturally associated a Galois action and a Frobenius operator, f. Using the element t we can define on the ring $B^+_{crys,\overline{L}/L'}$ a filtration too. Let us precise this a little.

The Frobenius is the p-power in the structural sheaf $\mathcal{O}_{crys}$ in the definition of B_n. Thus, the action of Frobenius on an element of B_n consists in raising to the p-power. This makes sense because we consider the divided structure given by the ideal (p).

The Galois action comes naturally from the Galois action on the structural sheaf and on $Spec(\mathcal{O}_{\overline{L}}/p^n\mathcal{O}_{\overline{L}})$. In the perfect residue field situation we have (c.f. 1.12, 1.13 in [6]):

$$H^0(Gal(\overline{L}/L), B^+_{crys,\overline{L}/L}) = H^0(Gal(\overline{L}/L), B_{crys,\overline{L}/L}) = W(k) \otimes \mathbb{Q},$$

where $W(k)$ is the Witt group of the residue field k, which is finite, and we have, for all $r \geq 0$, the exact sequence

$$0 \to \mathbb{Q}_p(r) \to \mathbb{Q} \otimes J^{[r]}_{\infty,\overline{L}/L} \stackrel{1-p^{-r}f}{\to} \mathbb{C}_p \to 0.$$

Observe that $B^+_{crys,\overline{L}/L}$ is a valuation ring with parameter given by t. This gives a filtration on these rings.

In general, we define a filtration in $B_{crys,\overline{L}/L'}$ by

$$(B_{crys,\overline{L}/L'})^i := \cup_{j\geq 0} t^{-j} J^{[i+j]}_{\infty,\overline{L}/L'}.$$

We remark here that in the non-perfect residue field situation we have more structure in our rings; it appears a connection in a natural way. We will describe this structure for the rings B_{dR}.

Let's recall the definition of the ring B_{dR}. Define

$$B^+_{dR,\overline{L}/L'} := \lim_{\substack{\longleftarrow \\ i}} (\mathbb{Q} \otimes (B_{\infty,\overline{L}/L'}/J^{[i]}_{\infty,\overline{L}/L'}))$$

and

$$J^{[q]}_{dR,\overline{L}/L'} := \lim_{\substack{\longleftarrow \\ i}} (\mathbb{Q} \otimes (J^{[q]}_{\infty,\overline{L}/L'}/J^{[i]}_{\infty,\overline{L}/L'})).$$

It is known that $B^+_{dR,\overline{L}/L'}$ is an integral domain. We have a natural Galois action on them, coming from the action on the ring B_{crys}. We define also

$$B_{dR,\overline{L}/L'} = B^+_{dR,\overline{L}/L'}[t^{-1}].$$

This allows us to define a filtration by

$$B^i_{dR,\overline{L}/L'} := \cup_{j\geq 0} t^{-j} J^{[i+j]}_{dR,\overline{L}/L'}.$$

We have also a comparison map between the diferent B_{dR}-rings in $\mathbb{Q}_p \subset L' \subset L$: we have (2.1.7 in [35])

$$B^+_{dR,\overline{L}/\mathbb{Q}_p}[[T_1, \ldots, T_e]] \stackrel{\cong}{\to} B^+_{dR,\overline{L}/L}, \tag{1.3}$$

hence in the perfect residue field situation all the B_{dR}-rings are the same.

We write B_{dR} when $L' = L$, and the same for B_{crys}. We have in this case the existence of a unique $B_{dR,\overline{L}/\mathbb{Q}_p}$-linear map $d : B_{dR} \to \hat{\Omega}^1_L \otimes B_{dR}$ satisfying the additive and multiplicative property that satisfies any commutative derivation. Moreover, the restriction of d to $L \subset B_{dR}$ coincides with $L \to \hat{\Omega}^1_L$; $a \mapsto da$, and

$$d(J^{[q]}_{dR,\overline{L}/L}) \subset \hat{\Omega}^1_L \otimes_L J^{[q-1]}_{dR,\overline{L}/L}$$

for all natural numbers $q \geq 1$. Here $\hat{\Omega}^1_L$ is a L-vector space with basis $(db_i)_{1 \leq i \leq e}$.

For our later interest, we concentrate from now on in the perfect residue field situation. We have (c.f. 1.16 [6])

$$H^0(Gal(\overline{L}/L), B^+_{dR}) = H^0(Gal(\overline{L}/L), B_{dR}) \cong L.$$

We have also that B_{dR} is a complete valuation ring with $B^0_{dR} = B^+_{dR}$, and with residue field $\mathbb{C}_p$. Moreover, as a $Gal(\overline{L}/L)$-module,

$$B^i_{dR}/B^{i+1}_{dR} \cong \mathbb{C}_p(i).$$

Remark 1.1.5. *Let X be a variety over L. Then, for any $m \geq 0$, we have a perfect pairing*

$$H^m_B(X(\mathbb{C}), \mathbb{Q}) \times H^m_{dR}(X/L) \to B_{dR}$$

$$(\gamma, w) \mapsto \int_\gamma w.$$

Moreover, if X has good reduction at p we can use the ring B_{crys} instead of B_{dR}. These results come from Artin's theorem, which establishes an isomorphism between étale cohomology and Betti cohomology, and Falting's theorem establishing the comparison between the étale cohomology in p-coefficients and the de Rham cohomology of the variety. Thus, we can interpret B_{dR} (resp. B_{crys}) as the ring of all p-adic periods associated to varieties (resp. with good reduction at p).

The following result prop.1.17 [6] is the key to define in the next section an exponential map which maps to some particulars Galois cohomology groups, and to put a mesure on these Galois cohomology groups that will control the value at 0 of the L-function associated to a motive.

Proposition 1.1.6. *The following sequences are exact*

$$0 \to \mathbb{Q}_p \xrightarrow{\alpha} B^{f=1}_{cris} \xrightarrow{\beta} B_{dR} \to 0,$$

$$0 \to \mathbb{Q}_p \xrightarrow{\alpha} B_{cris} \oplus B^+_{dR} \xrightarrow{\gamma} B_{cris} \oplus B_{dR} \to 0,$$

where $\alpha(x) = (x, x)$, $\beta(x, y) = x - y$ and $\gamma(x, y) = (x - f(y), x - y)$.

1.1.2 The exponential map

Let L be a complete valuation ring of characteristic 0 with residue field k of characteristic $p > 0$. We suppose in this section that L is a finite extension of $\mathbb{Q}_p$. We fix once and for all an embedding of L in $\mathbb{C}_p$ and we think all extensions of $\mathbb{Q}_p$ as subfields of this fixed extension. We recall the classical definition of the exponential function as the power series

$$exp(X) := \sum_{n\geq 0} X^n/(n!) \in \mathbb{Q}_p[[X]].$$

This function has convergence radius $|X| < p^{\frac{-1}{p-1}}$. We have also a logarithm function with convergence radius 1:

$$log(1+X) := \sum_{n\geq 1} (-1)^{n+1} X^n/n \in \mathbb{Q}_p[[X]].$$

Let L_m be any finite extension of $\mathbb{Q}_p$. We define the exponential map:

$$\begin{aligned} exp : L_m &\to L_m^* \otimes \mathbb{Q}_p, \\ x &\mapsto exp(p^i x) \otimes p^{-i} \end{aligned} \tag{1.4}$$

where i is big enough; for $i >> 0$ this definition does not depend on i.

We define now an exponential map for p-adic representations, which includes the previous one as a particular case. We begin with the key exact sequences (proposition 1.1.6):

$$0 \to \mathbb{Q}_p \to B_{crys}^{f=1} \oplus B_{dR}^+ \to B_{dR} \to 0$$

$$0 \to \mathbb{Q}_p \to B_{crys} \oplus B_{dR}^+ \to B_{crys} \oplus B_{dR} \to 0.$$

We tensor by $\mathbb{Q}_p(1)$ the first exact sequence (i.e. we make a Tate twist (1) which is given by t) and we obtain the following commutative diagram as $Gal(\overline{L}/L)$-modules:

$$\begin{array}{ccccccccc} 0 & \to & \mathbb{Q}_p(1) & \to & B_{crys}^{f=p} \oplus B_{dR}^1 & \to & B_{dR} & \to & 0 \\ & & \uparrow id & & \cup & & \cup & & \\ 0 & \to & \mathbb{Q}_p(1) & \to & (B_{crys}^{f=p} \cap B_{dR}^+) \oplus B_{dR}^1 & \to & B_{dR}^+ & \to & 0 \\ & & \downarrow id & & \downarrow pr_1 & & \downarrow & & \\ 0 & \to & \mathbb{Q}_p(1) & \to & B_{crys}^{f=p} \cap B_{dR}^+ & \to & \mathbb{C}_p & \to & 0 \end{array} \tag{1.5}$$

where pr_1 means the projection in the first component. To check the commutativity one uses the properties of the rings B_{dR} and B_{crys} and that $(B_{dR})^i/(B_{dR})^{i+1} = \mathbb{C}_p(i)$.

We have moreover the following commutative diagram (p.358 [6]):

$$
\begin{array}{ccccccccc}
0 & \to & \mathbb{Z}_p(1) & \to & \varprojlim_{p-power} \overline{L}^* & \overset{pr_1}{\to} & \overline{L}^* & \to & 0 \\
 & & \uparrow = & & \cup & & \cup & & \\
0 & \to & \mathbb{Z}_p(1) & \to & \varprojlim (1+p\mathcal{O}_{\overline{L}})^{*p^{-n}} & \overset{pr_1}{\to} & (1+p\mathcal{O}_{\overline{L}})^* & \to & 0 \\
 & & \downarrow i & & \downarrow log[\ ] & & \downarrow log & & \\
0 & \to & \mathbb{Q}_p(1) & \to & B_{crys}^{f=p} \cap B_{dR}^+ & \to & \mathbb{C}_p & \to & 0
\end{array}, \quad (1.6)
$$

where i is the inclusion and $log[\ ]$ means the logarithm of the Teichmüller representative.

We recall briefly how this comparison representative is constructed. We know by 1.1.1 that the rings B_n can be constructed using Witt rings and Witt vectors, let us remind a little this construction. For any ring R we have a comparison map $[\ ] : R \to W(R)$, where $W(R)$ means the Witt ring of R. Consider in our case

$$R = \varprojlim_{p-power} (\mathcal{O}_{\overline{L}});$$

we can think

$$\varprojlim_{inclusion} (1+p\mathcal{O}_{\overline{L}})^{*p^{-n}} \subset R \quad (1.7)$$

and then we can apply on it the Teichmüller map. Furthermore, we can take the logarithm of this elements because $a_0 \in 1+p\mathcal{O}_{\overline{L}}$, ($a_0$ is the first component as a element in the projective limit $(a_0, \ldots) \in \varprojlim (1+p\mathcal{O}_{\overline{L}})^{*p^n}$). The projective limit in 1.7 maps to B_{crys} because the ring $W(R)$ maps to the Witt vector $W_n(A_1 := \mathcal{O}_{\overline{L}}/p)$, because

$$R = \varprojlim_{p-power} (A_1)$$

and the rings $B_{n,\overline{L}/L}$ are obtained from the divided power of the Witt ring $W_n(A_1)$.

Taking invariants by the Galois group $G_L = Gal(\overline{L}/L)$, we obtain from diagram 1.6:

$$
\begin{array}{cccccccc}
0 & \to & \varprojlim_{p-power} L^* & \overset{pr_1}{\to} & L^* & \overset{\delta_1}{\to} & H^1(Gal(\overline{L}/L), \mathbb{Z}_p(1)) & \to \\
 & & \uparrow & & \cup & & \uparrow = & \\
0 & \to & \varprojlim (1+p\mathcal{O}_L)^{*p^{-n}} & \overset{pr_1}{\to} & (1+p\mathcal{O}_L)^* & \to & H^1(Gal(\overline{L}/L), \mathbb{Z}_p(1)) & \to \\
 & & \downarrow log[\ ] & & \downarrow log & & \downarrow & \\
0 & \to & (B_{crys}^{f=p} \cap B_{dR}^+)^{Gal(\overline{L}/L)} & \to & L & \to & H^1(Gal(\overline{L}/L), \mathbb{Q}_p(1)) & \to
\end{array} \quad (1.8)
$$

We obtain in every row the long exact sequence for the Galois cohomology because L is a complete local field and the Galois cohomology is obtained as the derived functor of taking Galois invariants (see appendix in [53]). Observe that the connecting morphism of the lower short exact sequence of diagrams 1.5 and 1.6 corresponds to

$$\delta : H^0(L, \mathbb{C}_p) = L \to H^1(L, \mathbb{Q}_p(1)) \cong (\varprojlim L^*/L^{*p^n}) \otimes \mathbb{Q}), \qquad (1.9)$$

where the last isomorphism on the right in 1.9 comes from Kummer theory.

Proposition 1.1.7 (Bloch-Kato). *The conecting morphism δ is the classical exponential map, i.e. the map defined by 1.4 taking the image in* $\varprojlim_n L^*/L^{*p^n} \otimes \mathbb{Q}$.

Proof. In the diagram 1.8, we fix our attention on the commutative diagram:

$$\begin{array}{ccc} (1+p\mathcal{O}_L)^* & \overset{\delta_1}{\to} & \varprojlim L^*/L^{*p^n} \\ \downarrow log & & \downarrow id \\ L & \overset{\delta}{\to} & \varprojlim L^*/L^{*p^n} \otimes \mathbb{Q} \end{array}.$$

Consider any $x \in L$. Then, there exists i with $v_p(p^i x) > 1/(p-1)$ and $v_p(exp(p^i x) - 1) > 0$; thus, $\delta(x) = \delta(p^i x) \otimes p^{-i} = \delta_1(exp(p^i x)) \otimes p^{-i}$ because δ is a $\mathbb{Q}_p$-linear morphism. By the commutative diagram 1.8, the connecting δ_1 can be read from the upper sequence that appears in 1.8:

$$0 \to \varprojlim_{p-power} L^* \to L^* \overset{\delta_1'}{\to} H^1(L, \mathbb{Z}_p(1)) \to 0,$$

because $(1+p\mathcal{O}_L)^* \subseteq L^*$. But δ_1' corresponds to the natural projection of the elements of L^* in $\varprojlim_n L^*/L^{*,p^n}$. Denoting by $proj$ this projection, we obtain:

$$\delta(x) = proj(exp(p^i x)) \otimes p^{-i}.$$

□

The last result says that, for the $\mathbb{Q}_p$-vector space $\mathbb{Q}_p(1)$ with a continuous G_L-action, we obtain the exponential map as the connecting morphism of taking Galois invariant for the first exact sequence in proposition 1.1.6. Let's give a general definition.

Definition 1.1.8. *Let V be a $\mathbb{Q}_p$-vector space of finite dimension with a continuous G_L-action. Define the subgroup:*

$$H^1_e(L,V) := Ker(H^1(L,V) \to H^1(L, B^{f=1}_{crys} \otimes V)$$

as the exponential Galois group of V,

$$H^1_f(L,V) := Ker(H^1(L,V) \to H^1(L, B_{crys} \otimes V)$$

as the finite part, and the geometrical part as

$$H^1_g := Ker(H^1(L,V) \to H^1(L, B_{dR} \otimes V).$$

Definition 1.1.9. *Let V be a $\mathbb{Q}_p$-vector space of finite dimension with a comparison G_L-action. We denote:*

$$DR(V) := (V \otimes B_{dR})^{G_L}, \quad Crys(V) := (V \otimes B_{crys})^{G_L}.$$

We say that the representation V is a de Rham representation if $dim_L DR(V) = dim_{\mathbb{Q}_p}(V)$; and that it is crystalline if $dim_{W(k)\otimes\mathbb{Q}} Crys(V) = dim_{\mathbb{Q}_p}(V)$.

It is known that a crystalline representation is a de Rham representation (c.f. 5.1 [18]).

Remark 1.1.10. *Let $\alpha \in H^1(L,V)$ be represented by a short exact sequence*

$$0 \to V \to E \to \mathbb{Q}_p \to 0.$$

If V is a crystalline representation then: E is a crystalline representation if and only if $\alpha \in H^1_f(L,V)$. If V is a de Rham representation then: E is a de Rham representation if and only if $\alpha \in H^1_g(L,V)$.

Tensoring the exact sequences of proposition 1.1.6 with V and taking G_L-invariants we obtain (corollary 3.8.4 in [6]) the following result.

Proposition 1.1.11. *Let V be a de Rham representation. Then there is a commutative diagram of exact sequences as follows:*

$$\begin{array}{ccccccccc} 0 & \to & H^0(L,V) & \xrightarrow{\alpha} & Crys(V)^{f=1} \oplus DR(V)^0 & \xrightarrow{\beta} & DR(V) & \xrightarrow{\delta} & H^1_e(L,V) & \to & 0 \\ & & \downarrow id & & \cap & & \downarrow (0,id) & & \cap & & \\ 0 & \to & H^0(L,V) & \xrightarrow{\alpha} & Crys(V) \oplus DR(V)^0 & \xrightarrow{\gamma} & Crys(V) \oplus DR(V) & \to & H^1_f(L,V) & \to & 0 \end{array}, \tag{1.10}$$

where $Crys(V)^{f=1} = \{a \in Crys(V) | f(a) = a\}$, the action of Frobenius f on $Crys(V)$ is given by the action of the Frobenius in B_{crys} and by the trivial action on V. We note also that $DR(V)$ has a natural filtration coming from the filtration of B_{dR} and $DR(V)^0 = (B^0_{dR} \otimes V)^{Gal(\overline{L}/L)}$.

Definition 1.1.12 (Exponential map). *Let V be a de Rham representation of $Gal(\overline{L}/L)$. The exponential map is the connecting morphism δ of the upper arrow on the above commutative diagram 1.10*

$$DR(V)/DR(V)^0 \to H^1_e(L,V).$$

From prop.1.1.11 it can be seen easily that the exponential map is surjective and it has kernel $Crys(V)^{f=1}/H^0(L,V)$.

Remark 1.1.13. *Can we make a similar construction in the case of non-perfect residue field? There is an exponential function for L also in the case of non-perfect residue field defined by the same power series. Moreover we can try to find an exact sequence as in prop.1.1.6, which could be possible by the relation as polynomial rings on B_{crys}, 1.2, between perfect an non-perfect residue field, and similarly on B_{dR}, 1.3. The first arising problem is due to the fact that the Galois invariants do not give necessarily the Galois cohomology. I wonder if it is possible to find a definition of exponential maps for G_L-representations generalizing the idea of the perfect residue field case, and to relate it with the definition that appears in the literature using K-groups. For this last point of view see the work of Kurihara [39].*

1.1.3 Euler factors of the L-function.

The aim of this section is to obtain a relation between the local Euler factors of the L-function associated to an l-representation and measures on Galois cohomology groups. Let L be a finite extension of $\mathbb{Q}_p$. Fix a $\mathbb{Q}_l$-representation V of finite dimension with a comparison $Gal(\overline{L}/L) = G_L$-action. We associate to the representation some subgroups of the Galois group $H^1(L,V)$. In the previous section we have defined the exponential, finite and geometrical subgroups of the Galois group $H^1(L,V)$ for the case $l = p$ (c.f. definition 1.1.8). For $l \neq p$ they are defined by

$$H^1_e(L,V) := 0, \tag{1.11}$$
$$H^1_g(L,V) := H^1(L,V) \tag{1.12}$$
$$H^1_f(L,V) := Ker(H^1(L,V) \to H^1(L_{nr},V)), \tag{1.13}$$

where L_{nr} is the maximal unramified extension of L.

Definition 1.1.14. *Let be $l \neq p$ and denote by V a finite $\mathbb{Q}_l$-dimensional vector space with a comparison $Gal(\overline{L}/L)$-action. V is said to be unramified if and only if the inertia group of $Gal(\overline{L}/L)$ acts trivially on V.*

Remark 1.1.15. *Let be $l \neq p$ and consider $\alpha \in H^1(L,V)$ corresponding to the extension*

$$0 \to V \to E \to \mathbb{Q}_l \to 0.$$

Suppose that V is an unramified representation. Then $\alpha \in H^1_f(L,V)$ if and only if E is an unramified representation.

Now, if we consider a free $\mathbb{Z}_l$-submodule T of V with a comparison action of G_L, we define

$$H^1_*(L,T) := \iota^{-1}(H^1_*(L,T\otimes\mathbb{Q})),$$

where $\iota : H^1(L,T) \to H^1(L,T\otimes\mathbb{Q})$ and $*$ corresponds to e, f or g.

For the local Galois group we have the perfect local Tate pairings (c.f. 3.8 [6]):

$$H^1(L,V) \times H^1(L,V^*(1)) \to H^2(L,\mathbb{Q}_l(1)) \cong \mathbb{Q}_l, \tag{1.14}$$

where $V^* := Hom_{\mathbb{Q}_l}(V,\mathbb{Q}_l)$. We have also the perfect pairing with T as above,

$$H^1(L,T) \times H^1(L,T^*\otimes\mathbb{Q}_l/\mathbb{Z}_l(1)) \to H^2(L,\mathbb{Q}_l/\mathbb{Z}_l(1)) \cong \mathbb{Q}_l/\mathbb{Z}_l, \tag{1.15}$$

where $T^* = Hom_{\mathbb{Z}_l}(T,\mathbb{Z}_l)$. The exponential, finite or geometrical groups appear naturally as the exact annihilator of each other in these pairings:

Proposition 1.1.16 (Bloch-Kato, prop.3.8 [6]). *Let l be a prime number, and let V be a finite dimensional $\mathbb{Q}_l$-vector space with a comparison Galois action by $Gal(\overline{L}/L)$, where L is a finite extension of $\mathbb{Q}_p$. If $l = p$ we assume that V is a de Rham representation.*

Then, in the perfect pairing (1.14), $H^1_e(L,V)$ and $H^1_g(L,V^*(1))$ are the exact annihilators of each other. The same happens with $H^1_e(L,T)$ and $H^1_g(L,T^*(1)\otimes\mathbb{Q}_l/\mathbb{Z}_l)$ for the pairing (1.15). The same statement holds with e replaced by g and g by e and also when e and g are both replaced by f.

Definition 1.1.17. *Let us denote by $L_0 = W(k)\otimes\mathbb{Q}$, which is the maximal unramified subfield of L over $\mathbb{Q}_p$. For a finite $\mathbb{Q}_l$-dimensional $G_L = Gal(\overline{L}/L)$-representation V, we define the Euler local factors by*

$$P_v(V,s) := \begin{cases} det_{\mathbb{Q}_l}(1 - f_L s | H^0(L_{nr},V)) \in \mathbb{Q}_l[s] & if\ l \neq p \\ det_{L_0}(1 - f_L s | Crys(V)) \in L_0[s] & if\ l = p \end{cases},$$

where f_L denotes the L_0-linear map $f^{[L_0:\mathbb{Q}_p]}$ if $l = p$. If $l \neq p$, f_L denotes the geometric Frobenius in G_L, which acts on $\mathbb{Z}_l(-1)$ by $p^{[L_0:\mathbb{Q}_p]}$. Here v is the place of the local field L.

We relate now a measure in some subgroups of the Galois group with the value at 1 of the Euler factors.

Theorem 1.1.18 (thm. 4.1, Bloch-Kato). *Let l and V as above, and assume $P_p(V,1) \neq 0$.*

1. *Assume $l \neq p$. Then $(0) = H^1_e(L,V) = H^1_f(L,V)$. If V is unramified and T is a $Gal(\overline{L}/L)$-stable $\mathbb{Z}_l$-sublattice in V, then*

$$\# H^1_f(L,T) = |P_p(V,1)|_l^{-1},$$

where the absolute value is the normalized one on $\mathbb{Q}_l$.

2. *Assume $l = p$ and V is a de Rham representation. Then*

$$DR(V)/DR(V)^0 \overset{exp}{\underset{\cong}{\rightarrow}} H^1_e(L,V) = H^1_f(L,V).$$

3. *Assume $l = p$, L is unramified over $\mathbb{Q}_p$, V is a crystalline representation, and the following conditions (*) holds:*

(*) There exists $i \leq 0$ and $j \geq 1$ with $j - i < p$ such that $DR(V)^i = DR(V)$ and $DR(V)^j = 0$.

Let $D \subset Crys(V) = DR(V)$ be a strongly divisible lattice (i.e., a finitely generated $\mathcal{O}_L$-submodule of $DR(V)$ such that

$$D = \sum p^{-i} f(D^i); \text{ with } D^i = D \cap DR(V)^i \text{).}$$

Let $T \subset V$ be the Galois stable sublattice constructed from D in [19]. Then

$$\mu(H^1_f(L,T)) = |P_p(V,1)|_p^{-1}$$

where μ is the Haar measure of $H^1_f(L,V)$ induced from the Haar measure of D/D^0 having total measure 1 via the exponential map. Here the absolute value is the one on $L = L_0$ such that $|p|_p = p^{-1}$.

1.1.4 The global conjecture

The aim of this paragraph is to present the original conjecture [6] that predicts the values of the L function associated to any geometric-arithmetic object defined over $\mathbb{Q}$. Following ideas of Grothendieck will be express in the language of motives. Nowadays, it is not yet clear the existence of this universal category, containing a global element associated to any geometric object, from which one obtains via realizations all cohomologies that can be associated to geometric-arithmetic objects. Thus, we have to formulate a general conjecture using the realizations in diferent cohomologies that we know, and impose some conditions on them. We present this definition for motives defined over $\mathbb{Q}$.

Definition 1.1.19. *A motivic pair (V, D) is a pair of finite dimensional $\mathbb{Q}$-vector spaces with the following extra structure (1)-(3) satisfying axioms (P1)-(P4).*

1. *$V \otimes \mathbb{A}_f$ has a comparison $\mathbb{A}_f$-linear Galois action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ such that $V \subset V \otimes \mathbb{A}_f$ is stable under $Gal(\mathbb{C}/\mathbb{R}) \subset Gal(\overline{\mathbb{Q}}/\mathbb{Q})$. Here $\mathbb{A}_f$ means the finite ideles on $\mathbb{Q}$.*

2. *D has a decreasing filtration $(D^i)_{i\in\mathbb{Z}}$ by $\mathbb{Q}$-subspaces such that $D^i = (0)$ for $i >> 0$ and $D^i = D$ for $i << 0$.*

3. *If we denote $D_p := D \otimes_{\mathbb{Q}} \mathbb{Q}_p$ and $V_p := V \otimes_{\mathbb{Q}} \mathbb{Q}_p$, then for $p < \infty$ we are given an isomorphism of $\mathbb{Q}_p$-vector spaces*

$$\theta_p : D_p \cong DR(V_p)$$

preserving filtrations. For $p = \infty$, we are given an isomorphism of $\mathbb{R}$-vector spaces

$$\theta_\infty : D_\infty \cong (V_\infty \otimes_{\mathbb{R}} \mathbb{C})^+.$$

Here $DR(V_p)$ is defined with respect to the action of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$. We denote the $Gal(\mathbb{C}/\mathbb{R})$ fixed part by $(\ )^+$, where the action of $\sigma \in Gal(\mathbb{C}/\mathbb{R})$ on $V_\infty \otimes \mathbb{C}$ is $\sigma \otimes \sigma$.

These data are subject to the following axioms:
(P1) There exists a non-empty open set U of $Spec(\mathbb{Z})$ such that for any $p \in U$, V_l is unramified at p for $l \neq p$ and V_p is a crystalline representation.
(P2) Let M be a $\mathbb{Z}$-lattice in V and let L be a $\mathbb{Z}$-lattice in D. Then there exists a finite set S of primes of $\mathbb{Q}$ ("bad primes") with $\infty \in S$ and such that for all $p \notin S$, V_p is a crystalline representation of $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$, condition (*) in 1.1.18 is satisfied by the filtration on $DR(V)$, $L \otimes \mathbb{Z}_p$ is a strongly divisible lattice in $D_p = Crys(V_p)$, and $M \otimes \mathbb{Z}_p$ coincides with the $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ -stable lattice in V_p corresponding to $L \otimes \mathbb{Z}_p$ via (3).
(P3) Let $p < \infty$, and $P_p(V_l, s)$ be the Euler local factor for the $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-module V_l. Then $P_p(V_l, s) \in \mathbb{Q}[s]$ for all l and these polynomials are independent of l. We denote this polynomial by $P_p(V, s)$.
(P4) If $p < \infty$, there exists a $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$-stable $\mathbb{Z}$-lattice $T \subset V \otimes \mathbb{A}_f$ such that $H^0(\mathbb{Q}_p^{nr}, T \otimes \mathbb{Q}_l/\mathbb{Z}_l)$ is divisible for almost all l.

Definition 1.1.20. *A motivic pair (V, D) has weights $\leq w$ if for each $p < \infty$, the polynomial $P_p(V, s)$ has the form $\prod(1 - \alpha_i s)$ with $|\alpha_i| \leq p^{w/2}$ in $\mathbb{C}[s]$, and if $D^i_\infty \cap V^+_\infty = (0)$ for $i > w/2$.*

If (V, D) has weights $\leq w$, and S is a finite set of places of $\mathbb{Q}$ containing ∞, the L function $L_S(V, s)$ is defined by

$$L_S(V, s) := \prod_{p \notin S} P_p(V, p^{-s})^{-1}, \tag{1.16}$$

which converges absolutely for $Re(s) > 1 + w/2$.

Remark 1.1.21. *For a general motive over a number field F, we expect the same properties in definition 1.1.19 to be fulfilled. We have then associated to any prime p of $\mathbb{Q}$ a finite dimensional $\mathbb{Q}_p$-representation V_p by property (1) with a $Gal(\overline{F}/F)$-action. Thus, for every finite place v of F the local Euler factor for $P_v(V_p, s)$ is well-defined (1.1.17)and by the property (P3) it has to be independent of p and with $\mathbb{Q}$-coefficients. We denote this Euler factor by $P_v(V, s)$. Then it is possible to define the function L_S by*

$$L_S(V, s) := \prod_{v \notin S} P_v(V, Norm_{F_v/\mathbb{Q}_l}(v)^{-s})^{-1},$$

where S is a finite set of places of F containing the archimedean places and l is the place of $\mathbb{Q}$ where v lies. This L-function converges absolutely for $Re(s) > 1 + w/2$, where w comes from $|\alpha_i| \leq Norm(v)^{w/2}$, if we write $P_v(V, s) = \prod(1 - \alpha_i t)$.

It is expected that a motive over $\mathbb{Q}$ gives a motivic pair. The Chow motives of the form $h_m(X)(r)$ with X a smooth, proper scheme over $\mathbb{Q}$ of pure dimension l, and with the diagonal Δ_X as an algebraic cycle on $X \times_{\mathbb{Q}} X$, should define a natural motivic pair as follows. Define

$$V = H^m(X(\mathbb{C}), \mathbb{Q}(2\pi i)^r), \qquad D = H^m_{dR}(X/\mathbb{Q}).$$

By Artin's theorem, $V \otimes \mathbb{A}_f \cong H^m_{et}(X_{\overline{\mathbb{Q}}}, \mathbb{A}_f)(r)$, whence (1). The filtration on D is deduced from the Hodge filtration on H_{dR} by $D^i := Fil^{r+i} H^m_{dR}(X/\mathbb{Q})$. The isomorphism θ_∞ is standard and the existence of θ_p is proved by Faltings. We take $M = H^m(X(\mathbb{C}), \mathbb{Z}(r))/(torsion)$. (P1) and (P2) hold from the work of Fontaine and Messing. (P3) holds for almost all p. (P3) and (P4) hold if $m = 1$. Then in this situation the weight w for $h_m(X)(r)$ takes sense and is equal to $m - 2r$. We remark that (P1)-(P4) involve unproven properties of motives.

We define the Tamagawa measure on $\prod_{p \leq \infty} A(\mathbb{Q}_p)$ assuming that the motivic pair has weights ≤ -3. We follow the notation introduced in the definition 1.1.19 of a motivic pair. Let us first define $A(\mathbb{Q}_p)$.

$$A(\mathbb{Q}_p) := \begin{cases} H^1_f(\mathbb{Q}_p, M \otimes \hat{\mathbb{Z}}) & if\ p < \infty \\ ((D_\infty \otimes_{\mathbb{R}} \mathbb{C})/((D^0_\infty \otimes_{\mathbb{R}} \mathbb{C}) + M))^+ & if\ \ p = \infty. \end{cases} \tag{1.17}$$

(The inclusion $M \to D_\infty \otimes_{\mathbb{R}} \mathbb{C}$ is given by the identification $D_\infty \otimes_{\mathbb{R}} \mathbb{C} = V_\infty \otimes \mathbb{C}$). We regard $A(\mathbb{Q}_p)$ for $p < \infty$ as a compact group with the natural topology, and $A(\mathbb{R})$ as a locally compact group.

For $p \leq \infty$, we have the exponential homomorphism (1.1.12)

$$exp : D_p/D^0_p - -- \to A(\mathbb{Q}_p)$$

which is not defined in all D_p/D^0_p if $p < \infty$. Observe that exp is a local isomorphism on a neighborhood of zero in D_p/D^0_p, because $A(\mathbb{Q}_p)/H^1_f(\mathbb{Q}_p, M \otimes \mathbb{Z}_p)$ is finite, equal to $\prod_{l \neq p} H^0(\mathbb{Q}_p, M \otimes \mathbb{Q}_l/\mathbb{Z}_l)$. In the archimedean case, the exponential map is the canonical map, which is defined on the total space.

We fix an isomorphism

$$w : det_{\mathbb{Q}}(D/D^0) \cong \mathbb{Q}, \tag{1.18}$$

which for each $p \leq \infty$ gives

$$det_{\mathbb{Q}_p}(D_p/D^0_p) \cong \mathbb{Q}_p.$$

This trivialization of the determinant furnishes a Haar measure on the p-adic space D_p/D^0_p and hence a Haar measure $\mu_{p,w}$ on $A(\mathbb{Q}_p)$ via the exponential map, since $exp : D_p/D^0_p \cong H^1_f(\mathbb{Q}_p, M \otimes \mathbb{Q}_p)$. By Theorem 1.1.18 and (P1) there exists a sufficiently large finite set S of places of $\mathbb{Q}$, containing ∞, such that for $p \notin S$:

$$\mu_{p,w}(A(\mathbb{Q}_p)) = P_p(V, 1).$$

Since the weights are ≤ -3, the product

$$L_S(V, 0)^{-1} = \prod_{p \notin S} \mu_{p,w}(A(\mathbb{Q}_p))$$

converges, so that the product measure $\mu = \prod_{p \leq \infty} \mu_{p,w}$ on $\prod_{p \leq \infty} A(\mathbb{Q}_p)$ is well-defined and μ does not depend of w.

Definition 1.1.22. *μ is the Tamagawa measure of the motivic pair (D, V).*

We define now the set of global points of a motivic pair. To define it, we need to have a finite dimensional $\mathbb{Q}$-vector space Φ endowed with an isomorphism of $\mathbb{R}$-vector spaces

$$r_{\mathcal{D}} : \Phi \otimes \mathbb{R} \cong D_\infty/(D^0_\infty + V^+_\infty) = A(\mathbb{R})/A(\mathbb{R})_{cpt}$$

and an isomorphism of $\mathbb{A}_f$-modules

$$r_S : \Phi \otimes \mathbb{A}_f \cong H^1_{f,Spec(\mathbb{Z})}(\mathbb{Q}, V \otimes \mathbb{A}_f),$$

where $H^1_{f,Spec(\mathbb{Z})}$ is the set of elements of $H^1(\mathbb{Q}, V \otimes \mathbb{A}_f)$ that are mapped to $H^1_f(\mathbb{Q}_l, V \otimes \mathbb{A}_f)$ for any finite place of $\mathbb{Q}$ (see next section for the conjectural definition of Φ for motives of the type $h_m(X)(r)$). Fix a $\mathbb{Z}$-lattice M in V such that $M \otimes \hat{\mathbb{Z}}$ is Galois stable in $V \otimes \mathbb{A}_f$. Define the set of global points $A(\mathbb{Q}) \subset H^1_{f,Spec(\mathbb{Z})}(\mathbb{Q}, M \otimes \hat{\mathbb{Z}})$ to be the inverse image of $r_S(\Phi)$. There are natural homomorphisms $A(\mathbb{Q}) \to A(\mathbb{Q}_p)$ for finite p and for the infinite place as well. We define

$$Tam(M) = \mu((\prod A(\mathbb{Q}_p))/A(\mathbb{Q})), \tag{1.19}$$

which is well-defined because the coimage of $A(\mathbb{Q})$ in $A(\mathbb{R})/A(\mathbb{R})_{cpt}$ is co-compact.

We define now a Tate-Shafarevich group. Consider the map

$$\alpha_M : \frac{H^1(\mathbb{Q}, M \otimes \mathbb{Q}/\mathbb{Z})}{A(\mathbb{Q}) \otimes \mathbb{Q}/\mathbb{Z}} \to \oplus_{p \leq \infty} \frac{H^1(\mathbb{Q}_p, M \otimes \mathbb{Q}/\mathbb{Z})}{A(\mathbb{Q}_p) \otimes \mathbb{Q}/\mathbb{Z}}. \tag{1.20}$$

We define

$$Ш(M) = Ker(\alpha_M).$$

Proposition 1.1.23 (Bloch-Kato, prop. 5.14 [6]). *Let l be a prime number, U a non-empty open set of $Spec(\mathbb{Z})$ not containing l and $V = M \otimes \mathbb{Q}_l$. Assume conditions (a)-(d) below hold:*
(a) V is unramified on U.
(b) V is a de Rham representation of $Gal(\overline{\mathbb{Q}_l}/\mathbb{Q}_l)$.
(c) $P_p(V, 1) \neq 0$ for any $p \notin U$, $p \neq \infty$.
(d) $P_p(V(-1), 1) \neq 0$ for any $p \in U$.
Then,

1. *For any prime number l, the l-primary part of $Ш(M)\{l\}$ is finite.*
2. *$Coker(\alpha_M)$ is finite and it is isomorphic to the Pontryagin dual of the finite group $H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))$, with $M^* = Hom(M, \mathbb{Z})$.*
3. *Assume that $Ш(M)$ is finite and define $\delta(M)$ by*

   $$\delta(M) = Tam(M)\#(Ш(M))\#(H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1)))^{-1}.$$

   Then, for any $\mathbb{Z}$-lattice $M' \subset V$ such that $M' \otimes \hat{\mathbb{Z}}$ is Galois stable in $V \otimes \mathbb{A}_f$, we have that $Ш(M')$ is also finite and $\delta(M) = \delta(M')$.

Conjecture 1.1.24 (Tamagawa number conjecture). *Assume that the triple (V, D, Φ) comes from a motive. Let M be a $\mathbb{Z}$-lattice in V such that $M \otimes \hat{\mathbb{Z}}$ is Galois stable in $V \otimes \mathbb{A}_f$. Then $\text{Ш}(M)$ is finite and*

$$Tam(M) = \frac{\#(H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))}{\#(\text{Ш}(M))}.$$

The conjecture can also be written as

$$L_S(V, 0) = \frac{\#\text{Ш}(M)}{\#(H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}/\mathbb{Z}(1))} \mu_{\infty,w}(A(\mathbb{R})/A(\mathbb{Q})) \prod_{p \in S \setminus \infty} \mu_{p,w}(A(\mathbb{Q}_p)),$$

which depens on w.

1.1.5 The local conjecture

We want to work out the equality of the Tamagawa number conjecture modulo $\mathbb{Z}^*_{(l)} = \{\frac{a}{b} | a, b \in \mathbb{Z};\ l \nmid ab\}$. In this situation we can forget all the l'-realization of the motive for $l' \neq l$. Here we assume $P_l(V_l, 1) \neq 0$ and we need to find Φ such that

$$r_{\mathcal{D}} : \Phi \otimes \mathbb{R} \cong D_\infty/(D^0_\infty + V^+_\infty); \tag{1.21}$$

$$r_{S,l} : \Phi \otimes \mathbb{Q}_l \cong ker(H^1(\mathbb{Q}, V_l) \to (H^1(\mathbb{Q}_l, V_l)/H^1_f(\mathbb{Q}_l, V_l)) \oplus \prod_{p \neq l} H^1(\mathbb{Q}_p, V_l)). \tag{1.22}$$

This Φ is expected to be taken in the K-theory for the motive that should be attached to any motivic pair. In the case of motivic pairs of the form $h_m(X)(r)$ with X a projective variety over $\mathbb{Q}$, it is conjectured that Φ comes from the intersection of the image of the localization map in K-theory

$$loc : K_{2r-m-1}(\mathfrak{X}) \to K_{2r-m-1}(X)$$

with $K_{2r-m-1}(X)^{(r)}$ (see precisely c.f. 1.2.4 and 1.2.5). Here $\mathfrak{X}$ denotes a proper flat model of X over $\mathbb{Z}$ and the image of loc does not depend, conjecturally, on the election of $\mathfrak{X}$. If we impose that the scheme is regular, then we positively know the independence on its election (c.f. remark on p.13 [57]), but the problem is its existence (see for example [41]). The above maps $r_{\mathcal{D}}$ and r_S will correspond conjecturally to the Deligne or Soulé regulator maps respectively (we refer to §1.2.2 and 1.2.3 for a general explanation of these maps).

In the local case we know the finiteness of the $ker(\alpha_M)\{l\}$ and $coker(\alpha_M)\{l\}$ (c.f. 1.1.23) and the definition of the objects in the conjecture needs less assumptions. We precise the objects and the above claims in this case.

The Tamagawa number is defined as $\mu_{\infty,w}(A(\mathbb{R})/A(\mathbb{Q}))\prod_p \mu_{p,w}(A(\mathbb{Q}_p))$. We are interested in this number on $\mathbb{R}^*/\mathbb{Z}^*_{(l)}$. Then, in order to define this number we use the groups

$$A^{(l)}(\mathbb{Q}_p) := \begin{cases} H^1_f(\mathbb{Q}_l, M\otimes\mathbb{Z}_l) & if\ p = l \\ H^1(\mathbb{Q}_l, M\otimes\mathbb{Z}_l)_{tor} & if\ p\neq l,\ p<\infty \end{cases}$$

instead of A in the definition of the Tamagawa number. We obtain then the l-Tamagawa number $Tam^{(l)}(M)\in\mathbb{R}/\mathbb{Z}^*_{(l)}$, which corresponds to $Tam(M)$ modulo $\mathbb{Z}^*_{(l)}$.

By prop. 1.1.23, $coker(\alpha_M)$ is a finite group. The conjecture 1.1.24 asserts that $ker(\alpha_M)$ is finite but from prop.1.1.23 we know that $coker(\alpha_M)\{l\}$ and $ker(\alpha_M)\{l\}$ are always finite. Then it has sense to ask

Conjecture 1.1.25 (The local Tamagawa number conjecture). *Given a prime l, we have the following equality in $\mathbb{R}^*/\mathbb{Z}^*_{(l)}$:*

$$Tam^{(l)}(M)\#(ker(\alpha_M)\{l\})\#(coker(\alpha_M)\{l\})^{-1} = 1.$$

The local conjecture is independent of the lattice M (by prop.1.1.23). For a motivic pair (with the conditions imposed in the definition 1.1.19) M only has to satisfy: $M\otimes\hat{\mathbb{Z}}$ is Galois stable in $V\otimes\mathbb{A}_f$, the conditions on prop. 1.1.23 with $l\notin U$ and the existence of Φ verifying 1.21 and 1.22.

The study of ker and $coker$ of the map α_M modulo $\mathbb{R}^*/\mathbb{Z}^*_{(l)}$ can be reduced to the study of the ker and $coker$ of

$$H^1(\mathbb{Q}, M\otimes\mathbb{Q}_l/\mathbb{Z}_l)/image(\Phi)\otimes\mathbb{Q}_l/\mathbb{Z}_l \overset{\alpha_{M,l}}{\to} \frac{H^1(\mathbb{Q}_l, M\otimes\mathbb{Q}_l/\mathbb{Z}_l)}{H^1_f(\mathbb{Q}_l, M\otimes\mathbb{Z}_l)\otimes\mathbb{Q}/\mathbb{Z}}\oplus\oplus_{p\neq l}H^1(\mathbb{Q}_p, M\otimes\mathbb{Q}_l/\mathbb{Z}_l). \tag{1.23}$$

Bloch and Kato reformulate the above conjecture for Chow motives via the value of the first Fourier coefficient for at zero of the L-function of $M^*(1)$. To do this, they use the hypothetical functional equation between the L-function of M and $M^*(1)$.

Proposition 1.1.26 (Bloch-Kato, lemma 7.10 [6]). *Let X be a smooth scheme over $\mathbb{Q}$ having potentially good reduction at all finite places of $\mathbb{Q}$. Consider the motive $h_m(X)(r)$ with $m, r\in\mathbb{Z}$, $r > sup(m,1)$, and let V and D be the associated $\mathbb{Q}$-vector spaces. Fix a prime number $l > r+1$ such that X has good reduction at l.*

Assume that $P_p(H^m_{et}(X\times_{\mathbb{Q}}\overline{\mathbb{Q}}, \mathbb{Q}_l), s)$ has $\mathbb{Q}$-coefficients for all finite p, and that

$$L(h_m(X), s) = \prod_{p<\infty} P_p(H^m_{et}(\overline{X}, \mathbb{Q}_l), p^{-s})^{-1}$$

has a meromorphic analytic continuation to the whole complex plane satisfying the conjectural functional equation. Assume further that we have a $\mathbb{Q}$*-vector subspace* Φ *in*

$$(K_{2r-m-1}(X) \otimes \mathbb{Q})^{(r)}$$

such that

$$\Phi \otimes \mathbb{R} \xrightarrow{\cong} D_{\mathbb{R}}/V_{\mathbb{R}}^{+}, \quad \Phi \otimes \mathbb{Q}_l \xrightarrow{\cong} H^1_{f,Spec(\mathbb{Z})}(\mathbb{Q}, V_l).$$

Let S *be a finite set of places of* $\mathbb{Q}$ *containing* ∞*,* l *and all finite places at which* X *has bad reduction, and let* $U = Spec(\mathbb{Z}) \setminus S$*. Then for any* $\mathbb{Z}$*-lattice* M *in* V *such that* $M \otimes \hat{\mathbb{Z}}$ *is* $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$*-stable,* $H^2(U, M \otimes \mathbb{Z}_l)$ *is finite and we have an equation in* $\mathbb{R}^*/\mathbb{Z}^*_{(l)}$*:*

$$Tam^{(l)}(M)\#ker(\alpha_M)\{l\}(\#H^0(\mathbb{Q}, M^* \otimes \mathbb{Q}_l/\mathbb{Z}_l(1))^{-1} =$$

$$(\lim_{s\to m+1-r}(L_S(H^m_{et}(X,\mathbb{Q}_l))^{-1}(s-m-1+r)^n)\cdot$$

$$vol((D_\infty/V_\infty^+)/Image(A^{(l)}(\mathbb{Q})))\#(A^{(l)}(\mathbb{Q})_{tor})^{-1}\#H^2(U, M\otimes\mathbb{Z}_l),$$

where n *is the order of* $L_S(H^m_{et}(\overline{X},\mathbb{Q}_l), s)$ *at* $s = m+1-r$*.*

Let us include a sketch of the proof of the previous result. By the functional equation we have the equality in $\mathbb{R}^*/\mathbb{Z}^*_{(l)}$:

$$L_S(H^m_{et}(\overline{X},\mathbb{Q}_l), r)\prod_{p\in S} P_p(H^m_{et}(\overline{X},\mathbb{Q}_l), r) =$$

$$lim_{s\to m+1-r} L_S(H^m_{et}(\overline{X},\mathbb{Q}_l), s)\prod_{p\in S} P_p(H^m_{et}(\overline{X},\mathbb{Q}_l, s),$$

where L_∞ is the associated Gamma factor that comes from the Hodge-Tate decomposition (see for their definition Serre [56] or §1.2.2; for more general situations we refer to Deligne [8] and Deninger [11] [13]).

The result is reduced to check in $\mathbb{R}^*/\mathbb{Z}^*_{(l)}$ that:

1. for all finite places p,

   $$L_p(r)^{-1}L_p(m+1-r) =$$

   $$\mu_{p,w}(A^{(l)}(\mathbb{Q}_p))\#(H^1(\mathbb{Q}_p, M\otimes\mathbb{Q}_l/\mathbb{Z}_l)/(A^{(l)}(\mathbb{Q}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l))^{-1},$$

   where we denote for simplicity $L_p(s) = L_p(H^m_{et}(\overline{X},\mathbb{Q}_l), s)$.

   In fact, if $l \neq p$, then $\mu_{p,w}(A^{(l)}) = \#H^1(\mathbb{Q}_p, M\otimes\mathbb{Z}_l)_{tor}$, which corresponds to $H^1_f(\mathbb{Q}_p, M\otimes\mathbb{Z}_l)$ since $l\neq p$. We can use the same argument for the dual motive $M^*(-1)$, which corresponds to the value at

$m+1-r$ of the Euler factor $L_p(s)$. The perfect Tate pairing claims that $H^1_f(\mathbb{Q}_p, M^*(-1)\otimes\mathbb{Z}_l)$ is the exact annihilator of $H^1_f(\mathbb{Q}_p, M\otimes\mathbb{Q}_l/\mathbb{Z}_l) = A^{(l)}(\mathbb{Q}_p)$, which are in Pontryagin duality. Thus, we have

$$\#H^1_f(\mathbb{Q}_p, M^*(-1)\otimes\mathbb{Z}_l) = \#\frac{H^1(\mathbb{Q}_p, M\otimes\mathbb{Q}_l/\mathbb{Z}_l)}{A^{(l)}(\mathbb{Q}_p)\otimes\mathbb{Q}_l/\mathbb{Z}_l}.$$

The case $l = p$ uses also Tate duality in the context of the theory of Fontaine-Laffaille [19], see theorem 1.1.18(3).

2. for the infinite places, observe that (c.f.lemma 7.11 [6])

$$L_\infty(r)^{-1}\lim_{s\to m+1-r} L_\infty(s)(s-m-1+r)^n =$$

$$\mu_{\infty,w}(A(\mathbb{R})/A^{(l)}(\mathbb{Q}))vol((D_\infty/V_\infty^+)/Image(A^{(l)}(\mathbb{Q})))^{-1}\#(A^{(l)}(\mathbb{Q})_{tor}).$$

1.2 Kato's reformulation of the Tamagawa number conjecture

Introduction

The original Tamagawa number conjecture, presented in the above section, applies to any motive over $\mathbb{Q}$. For a fixed prime number p, we recall in this section the local Tamagawa number conjecture for any pure motive over a number field F. To define the Tamagawa number we need to construct some subspace in a global space, with two isomorphisms, one mapping to the Deligne cohomology and the other to the Galois cohomology of the p-realization of the motive. We have seen that this space comes conjecturally from the K-theory of Quillen. Thus, the maps are conjecturally the Deligne regulator map and the Soulé regulator map. In this section we precise both regulators maps and the conjectures on them. They are studied in the first two subsections. In the third subsection we state the local Tamagawa number conjecture for a general pure motive over an arbitrary number field F. Following the original statement of the Tamagawa number conjecture, we could try to define a motivic pair for an arbitrary number field F and define the diferent $A(F_v)$ at any place v of F and on them define a general Tamagawa measure. Nevertheless, in this section, the conjecture is written in the language of determinants of certain modules on the diferent realizations of a pure motive and the control of some elements via the regulator maps. This is the new reformulation of Kato.

1.2.1 Beilinson regulator and Beilinson's conjecture.

To any motive M of weight w over a number field F (we can think it like a motivic pair) we can associate an L-function (see remark 1.1.21) and certain Γ-factor corresponding to the infinity places. We can see that $L(M, s)$ converges for $Re(s) > 1 + w/2$. We know for any motive M the following equality

$$L(M(r), s) = L(M, r + s),$$

where $M(r)$ is the r-twisted Tate of M. Thus, the study at the integer values of the behavior of the L-function of a motive M is equivalent to the study at the integer values of the L-function of the motive $M(r)$. In the following, we study the value at zero (following 1.16) of the L-function of a Chow motive of the form $M = h_i(X)(r)$, with X a smooth proper scheme over a number field F. This value is equal to the value at r of the L-function of the pure motive $h_i(X)$.

Imposed hypotheses 1.2.1. *The L-function for $h_i(X)$ has a meromorphic continuation to the whole complex plane $\mathbb{C}$. The function*

$$L(s)L_\infty(s)(some\ exponential\ factor)$$

has a functional equation with respect to $s \longleftrightarrow i + 1 - s$. Moreover, all poles on $L(s)$ occur only in the line $Re(s) = 1 + i/2$. Here L_∞ means the function of the archimedean Euler factors, which are built, using the Γ-function.

Assuming 1.2.1, we know the multiplicity of the zeroes of the L-function at the integer values. Indeed, the function L_∞ is defined via Γ-factors in which we can control the poles at the integer values; using the expected functional equation we obtain the vanishing order of the L-function at these values. For the definition of the Γ-factors and the L_∞-function we refer to Serre [56], Deligne [8] and Deninger [11] [13]. The definition uses the Hodge decomposition of the motive, which in the case $h_i(X)$ corresponds to the Hodge decomposition

$$H^i_{dR}(X \otimes_{\mathbb{Q}} \mathbb{C}/\mathbb{C}) = H^i_B(X \otimes_{\mathbb{Q}} \mathbb{C}, \mathbb{C}) = \oplus_{p+q=i_{p,q\geq 0}} H^{p,q}.$$

This decomposition on $\mathbb{C}$-subspaces gives the Hodge structure, with the $\mathbb{C}$-linear involution F_∞ on $H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{C})$ induced by complex conjugation on $X \times_{\mathbb{Q}} \mathbb{C}$. This involution satisfies $F_\infty(H^{p,q}) = H^{q,p}$. We write $h^{p,q} := dim_{\mathbb{C}} H^{p,q}$ and $h^{p\pm} := dim_{\mathbb{C}} H^{p\pm}$, where $H^{p,p} = H^{p+} \oplus H^{p-}$ is the decomposition into eigenspaces with respect to F_∞. For example, when X is defined over $\mathbb{Q}$ we define L_∞ with i even, by:

$$\prod_{p>q, p+q=i} \Gamma_{\mathbb{C}}(s-p)^{h^{p,q}} \Gamma_{\mathbb{R}}(s-i/2)^{h^{i/2+}} \Gamma_{\mathbb{R}}(s+1-i/2)^{h^{i/2-}},$$

where $\Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$ and $\Gamma_{\mathbb{C}}(s) := 2(2\pi)^{-s}\Gamma(s)$. We have control on the residues at the integer values of the Γ-functions. In particular, we know that $L_\infty(h_i(X), s)$ has poles only at the integer points $m \leq i/2$, with multiplicity equal to (c.f. p.5[57])

$$dim_{\mathbb{C}} H^i(X(\mathbb{C}), \mathbb{C})^{(-1)^{r-1}} - dim_{\mathbb{C}}(F^r H^i_{dR}(X(\mathbb{C})). \tag{1.24}$$

where $r := i + 1 - m$ and the exponent denotes the eigenspace with respect to F_∞. The subspace $F^r H^i_{dR}(X \times_F \mathbb{C})$ can be read in the Betti cohomology as $\oplus_{p' \geq n} H^{p',q}$. Observe that $F^r H^i_{dR}(X \times_F \mathbb{C}) = 0$ if $r > min(i, dimX)$.
Thus, by taking invariants, we see that, if $r > min(i, dimX)$, the dimension of the Betti cohomology coincides with the vanishing order of the L-function at the value m where m is an integer $\leq i/2$ (c.f.1.24).

Definition 1.2.2. *An integer m is called critical for the L-function of a Chow motive $h_i(X)$ if $L(M, m) \neq 0$.*

Beilinson proposed a map from K-theory to Deligne cohomology which would determine the value of the first Fourier coefficient of the L-function at m modulo $\mathbb{Q}^*$ in the non-critical case with $m < i/2$. Deligne proposed a conjecture for the values at the integer values m of the L-function in the critical case with $m < i/2$. Let us recall briefly these conjectures. We begin by defining the Deligne cohomology. The "real" Deligne cohomology $H^i_{\mathcal{D}}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(p))$ of $X \times_{\mathbb{Q}} \mathbb{C}$ is defined to be the cohomology of the complex

$$\mathbb{R}(p)_{\mathcal{D}} : \mathbb{R}(p) \to \mathcal{O}_{X \times_{\mathbb{Q}} \mathbb{C}} \to \Omega^1 \to \ldots \to \Omega^{p-1} \to 0,$$

where Ω^j are the sheaves of j-holomorphic differential forms of X. Since the cohomology of the complex of sheaves $\Omega^{\cdot} : \mathcal{O}_{X \times_{\mathbb{Q}} \mathbb{C}} \to \Omega^1 \to \Omega^2 \to$ converges to the de Rham cohomology, we have the following long exact sequence

$$\to H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(p)) \to H^i_{dR}(X \times_{\mathbb{Q}} \mathbb{C})/F^p \to H^{i+1}_{\mathcal{D}}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(p)) \to$$
$$H^{i+1}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(p)) \to \ldots. \tag{1.25}$$

Observe that the "real" Deligne cohomology could be interpreted as the difference between the real structure on the singular cohomology and the de Rham filtration.

We define the real Deligne cohomology for $X \times_{\mathbb{Q}} \mathbb{R}$ by

$$H^i_{\mathcal{D}}(X \times_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(p)) := H^i_{\mathcal{D}}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(p))^{DR-conj} \tag{1.26}$$

where the upper script DR-conj indicates the subspace invariant by the DR-conjugation that comes from complex conjugation on the pair $(X \times_{\mathbb{Q}} \mathbb{C}, \Omega^{\cdot})$.

This conjugation induces the complex conjugation on the Betti cohomology acting in the coefficients and in the scheme $X\times_{\mathbb{Q}}\mathbb{C}$. For $i+1<2r$, the natural map $H^{i+1}(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(p))\to H^{i+1}_{dR}(X\times_{\mathbb{Q}}\mathbb{C})/F^p$ is injective (c.f.lemma,p.8[57]); hence, the long exact sequence becomes

$$0\to H^i(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(r))\to H^i_{dR}(X\times_{\mathbb{Q}}\mathbb{C})/F^r\to H^{i+1}_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(r))\to 0, \tag{1.27}$$

and, using the decomposition $\mathbb{C}=\mathbb{R}(r)\oplus\mathbb{R}(r-1)$, we obtain the long exact sequence:

$$0\to F^pH^{i-1}_{dR}(X\times_{\mathbb{Q}}\mathbb{C}))\to H^{i-1}(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(p-1))\to H^i_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(p))\to 0. \tag{1.28}$$

We take DR-conjugation to obtain (remembering that $r:=i+1-m$) from 1.27 that

$$0\to H^i(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(r))^+\to H^i_{dR}(X\times_{\mathbb{Q}}\mathbb{R})/F^r\to H^{i+1}_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{R},\mathbb{R}(r))\to 0, \tag{1.29}$$

and, from 1.28, that

$$0\to F^nH^i_{dR}(X\times_{\mathbb{Q}}\mathbb{R})\to H^i(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(n-1))^+\to H^{i+1}_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{R},\mathbb{R}(n))\to 0. \tag{1.30}$$

where the upper script $+$ means the fixed subspace by the action on $X\times_{\mathbb{Q}}\mathbb{C}$ and on $\mathbb{R}(p)$ in the Betti cohomologies above. Note that, in the above short exact sequence, $F^rH^i_{dR}(X\times_{\mathbb{Q}}\mathbb{R})=0$ if $r>min(i,dimX)$ with $i+1<2r$.

We obtain as a consequence of the hypotheses about the Γ-factors of the L-functions that

Proposition 1.2.3. *Assume the conditions 1.2.1. For $m<i/2$ (or equivalently $i-2r<-2$), we have that*

$$dim_{\mathbb{R}}H^{i+1}_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{R},\mathbb{R}(r))=ord_{s=m}L(h^i(X),s).$$

When m is critical and $m<i/2$, the Deligne cohomology group $H^{i+1}_{\mathcal{D}}(X\times_{\mathbb{Q}}\mathbb{R},\mathbb{R}(i+1-m))$ is 0. This implies an isomorphism

$$F^rH^i_{dR}(X\times_{\mathbb{Q}}\mathbb{R})\cong H^i(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{R}(r-1))^+. \tag{1.31}$$

Observe that both sides carry natural $\mathbb{Q}$-structures: in the left we consider de Rham cohomology of $X\times_{\mathbb{Q}}\mathbb{Q}$ and in the right side the singular cohomology with $\mathbb{Q}(n-1)$-coefficients. Therefore, the determinant of this isomorphism calculated in these $\mathbb{Q}$-rational bases defines a number $c_X(m)\in\mathbb{R}^*/\mathbb{Q}^*$ and Deligne conjectured that it corresponds to $L(M,m)$ modulo $\mathbb{Q}^*$, when $m<i/2$ is critical.

In the non-critical case the Deligne cohomology is non-trivial, and we want some conjecture similar to Deligne's above. One could consider a $\mathbb{Q}$-structure in the Deligne cohomology and use the short exact sequence relating de Rham-singular and Deligne cohomology (1.30). However, there is no way to define a "good" $\mathbb{Q}$-structure in Deligne cohomology. Observe moreover, that in the non-critical case with $i - 2m > 0$, the value $L(M, r)$ is no zero for weights reasons (and the first Fourier coefficients of the function L at r and m are related by the hypothetical functional equation in 1.2.1).

Beilinson defines the regulator map to solve this problem. In order to explain this value and the dimension of the real Deligne cohomology, one has to understand the order of vanishing at m of the L-function. Beilinson constructs a Chern class map that determines a regulator map between

$$r : K_{2r-i-1}(X)^{(r)} \to H^i_{\mathcal{D}}(X \times_{\mathbb{Q}} \mathbb{R}, \mathbb{R}(r)), \tag{1.32}$$

where K_* are the Quillen groups tensored by $\mathbb{Q}$ [47], and the upperscript means the r part of Adams filtration (see [2] for the explicit construction). Observe that the $\mathbb{R}$-dimension of the space on the right is the order of zero at m of $L(h_i(X), s)$ (c.f. proposition 1.2.3). The above map tensor $\mathbb{R}$ is not an isomorphism (see for example [5]). To solve this problem we need to found a $\mathbb{Q}$-subspace in the K-group $K_{2r-i-1}(X)^{(r)}$ such that r is an isomorphism when we restrict it to this subspace.

Take $\mathcal{X}_{\mathcal{O}_F}$ a proper flat regular model of X over $\mathcal{O}_F$ if it exists. If not, we take a flat proper model over $\mathcal{O}_F$ and we define

$$K_*(X/\mathcal{O}_F)^{(l)} := image(K_*(\mathcal{X})^{(l)} \otimes \mathbb{Q} \to K_*(X)^{(l)} \otimes \mathbb{Q}). \tag{1.33}$$

If $\mathcal{X}$ is regular we obtain that this group is independent of the choice of the model. Denote by

$$H^i_{\mathcal{M}}(X, j) := K_{2j-i}(X)^{(j)} \otimes \mathbb{Q} \quad and \quad H^i_{\mathcal{M}}(X, j)_{\mathcal{O}_F} := K_{2j-i}(X/\mathcal{O}_F)^{(j)} \otimes \mathbb{Q}.$$

We have the following conjecture

Conjecture 1.2.4 (c.f. p.13[57]). *Suppose that X is defined over $\mathbb{Q}$, i.e. $F = \mathbb{Q}$. Then,*

1. $H^i_{\mathcal{M}}(X, r)_{\mathbb{Z}} = H^i_{\mathcal{M}}(X, r)$ *except for (i, r) with $r \leq i \leq 2r - 1$ and $r \leq dimX + 1$.*

2. $H^i_{\mathcal{M}}(X, r)/H^i_{\mathcal{M}}(X, r)_{\mathbb{Z}}$*, for $i \leq 2r - 2$, only depends on the bad fibers of $\mathcal{X}_{\mathbb{Z}}$.*

Now, we can formulate the Beilinson conjecture using the $\mathbb{Q}$-subspace $H^i_{\mathcal{M}}(X,r)_{\mathcal{O}_F}$

Conjecture 1.2.5 (Beilinson). *For $i < 2r-2$ the regulator map*

$$r : H^{i+1}_{\mathcal{M}}(X,r)_{\mathcal{O}_F} \otimes_{\mathbb{Q}} \mathbb{R} \to H^i_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(r-1))^+/F^r$$

is an isomorphism, and the dimension, e, of these vector spaces, is the order of zero of the L function $L(H^i(\overline{X},\mathbb{Q}_l),s)$ at $s = m = i+1-r$. Moreover, given η a $\mathbb{Q}$-basis for $det_{\mathbb{Q}}(H^i(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{Q}(r-1))^+/(F^n \cap H^i(X\times_{\mathbb{Q}}\mathbb{C},\mathbb{Q}(r-1))^+)$ and if S is a finite set of places such that the local Euler factors in the places of S are not zero at the value m, there exists an element ξ of $det_{\mathbb{Q}}(H^{i+1}_{\mathcal{M}}(X,r)_{\mathcal{O}_F})$ such that

$$r_{\mathcal{D}}(\xi) = \left(\lim_{s \to i+1-r} s^{-e} L_S(H^i(\overline{X},\mathbb{Q}_l),s) \right) \eta.$$

Remark 1.2.6. *In the conjecture 1.2.5 we have used 1.30 to obtain that r maps in $H^i_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(r-1))^+/F^r$. Using 1.29 instead of 1.30, r maps in $H^i_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{C})^+/(F^r + H^i_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(r))^+)$, for $i+1 \neq 2r$.*

Remark 1.2.7. *Observe that, for $r > min(i, dimX)$, we have $F^r H^i_{dR}(X \times_{\mathbb{Q}} \mathbb{C}) = 0$ and the regulator map is mapping to:*

$$r : H^{i+1}_{\mathcal{M}}(X,r)_{\mathcal{O}_F} \otimes \mathbb{Q} \to H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(r-1))^+.$$

Remark 1.2.8. *Following the conjecture 1.2.4, the motivic group $H^i_{\mathcal{M}}(X,r)_{\mathcal{O}_F}$ would coincide with the motivic group $H^i_{\mathcal{M}}(X,r)$ if $r > i$ or $r > dimX+1$. In these situation we can replace $H^i_{\mathcal{M}}(X,r)_{\mathcal{O}_F}$ by $H^i_{\mathcal{M}}(X,r)$ in the conjecture 1.2.5.*

1.2.2 Soulé regulator

We describe now an l-adic analogue of the Beilinson regulator. We consider also Chow motives $M = h_i(X)(r)$, with X a smooth proper variety defined over a number field F. We want to construct a Chern class map in the l-adic site. To do this, we construct first the Chern class maps for finite coefficients by using algebraic K-theory with finite coefficients.

The K-groups of Quillen are defined as the homotopy groups of a suitable topological space [47]. We define the K-groups with finite coefficients $\mathbb{Z}/n$ as the homotopy group $\pi_a(T,\mathbb{Z}/n) = [M^a, T]$ for $a \geq 2$, where $M^a = S^{a-1} \cup_n D^a$, glueing the $a-1$-sphere S^{a-1} to the a-ball D^a via a map of degree n, and T is the same topological space constructed for the general K-theory by Quillen [47]. We have then a short exact sequence:

$$0 \to \pi_a(T)/n \to \pi_a(T,\mathbb{Z}/n) \to \pi_{a-1}(T)[n] \to 0.$$

Moreover, the K-theory with $\mathbb{Z}_p$-coefficients is defined by

$$K_*(X, \mathbb{Z}_p) = \lim_{\substack{\longleftarrow \\ j}} K_*(X, \mathbb{Z}/p^j). \tag{1.34}$$

If we tensor the above group by $\otimes_{\mathbb{Z}_p}\mathbb{Q}_p$, we obtain the K-theory with $\mathbb{Q}_p$-coefficients. We can also consider the K-groups $K_*(X)$ and complete them by p, i.e. consider

$$\lim_{\substack{\longleftarrow \\ j}} K_*(X)/p^j = K_*(X) \otimes_{\mathbb{Z}} \mathbb{Z}_p.$$

We have then the following short exact sequence:

$$0 \to K_*(X) \otimes \mathbb{Z}_p \to K_*(X, \mathbb{Z}_p) \to T_p K_{*-1}(X) \to 0,$$

where $T_p K_{*-1}(X)$ is the p-divisible torsion subgroup of $K_{*-1}(X)$. The vanishing of $T_p K_{*-1}(X)$ would follows from Bass' conjecture ($K_*(Y)$ is finitely generated for any regular scheme Y of finite type over $\mathbb{Z}$) and the localization sequence (c.f. p.329 [32]).

Soulé defines in [59] some Chern class maps

$$c_{i,r} : \lim_{\substack{\longleftarrow \\ j}} K_{2r-i-1}(X, \mathbb{Z}/p^j) \to \lim_{\substack{\longleftarrow \\ j}} H^{i+1}_{et}(X, \mathbb{Z}/p^j). \tag{1.35}$$

Now, we want to use a Hochschild-Serre spectral sequence, and then tensor by $\mathbb{Q}$ in order to define a map from K-theory to $H^1(Gal(\overline{F}/F), H^i(\overline{X}, \mathbb{Q}_p(r))$. But there is a problem: the étale cohomology with $\mathbb{Z}_p$-coefficients does not have in general a Hochschild-Serre spectral sequence. There are basically two approaches to this problem, one made by Jannsen and the other by Kato. Let us remind Kato's solution [34](2.4,2.5) and briefly Jannsen's [31].
We need first to fix some notation. For a scheme Y of finite type over $\mathcal{O}_F$ where p is invertible and for a smooth $\mathbb{Z}_p$-sheaf $\mathcal{F}$ on Y_{et} (i.e. unramified in Y), the étale cohomology group

$$H^l_{et}(Y, \mathcal{F}) = \lim_{\substack{\longleftarrow \\ n}} H^l_{et}(Y, \mathcal{F}/p^n\mathcal{F})$$

is a finitely generated $\mathbb{Z}_p$-module. For such Y and for a smooth $\mathbb{Q}_p$-sheaf $\mathcal{F}$ on Y_{et}, define $H^l(Y, \mathcal{F}) = H^l(Y, \mathcal{F}') \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, where $\mathcal{F}'$ is a smooth $\mathbb{Z}_p$-sheaf such that $\mathcal{F}' \otimes_{\mathbb{Z}_p} \mathbb{Q}_p = \mathcal{F}$. For a scheme Y of finite type over F and for a smooth $\mathbb{Z}_p$(resp.$\mathbb{Q}_p$)-sheaf $\mathcal{F}$ on Y_{et} which comes from some smooth $\mathbb{Z}_p$(resp.$\mathbb{Q}_p$)-sheaf on a scheme Y' of finite type over $\mathcal{O}_F$ such that $Y = Y' \otimes_{\mathcal{O}_F} F$, consider

$$H^l_{lim}(Y, \mathcal{F}) := \lim_{\substack{\longrightarrow \\ U}} H^l_{et}(Y' \times_{\mathcal{O}_F} U, F')),$$

where U ranges over all non-empty open subsets of $Spec(\mathcal{O}_F)$. This definition does not depend of the choices of Y' and F'. We have then a Leray spectral sequence:

$$E_2^{l,j} = H^l_{lim}(F, H^j_{et}(\overline{X}, \mathbb{Q}_p(n))) \Rightarrow H^{l+j}_{lim}(X, \mathbb{Q}_p(n)).$$

Considering $\{0\}$ as an open subset we have clearly $H^l_{et}(Y, \mathcal{F}) \subseteq H^l_{lim}(Y, \mathcal{F})$. Then, by taking projective limit, the Chern classes defined by Soulé determine a Chern class map:

$$K_{2r-i-1}(X, \mathbb{Q}_p) := K_{2r-i-1}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to H^{i+1}_{lim}(X, \mathbb{Q}_p(r)).$$

Since we have an Hochschild-Serre (or Leray) spectral sequence, by using the map

$$\beta : (ker(H^{i+1}_{lim}(X, \mathbb{Q}_p(r)) \to H^{i+1}_{et}(\overline{X}, \mathbb{Q}_p(r))) \to H^1_{lim}(F, H^{v-1}_{et}(\overline{X}, \mathbb{Q}_p(n)),$$

we obtain the regulator map by composing the Chern class map with the map β

$$r_{S,p} : K_{2r-i-1}(X, \mathbb{Z}_p) \otimes \mathbb{Q}_p \to H^1_{lim}(F, H^i_{et}(\overline{X}, \mathbb{Q}_p(r)),$$

if $i + 1 \neq 2r$.

We know that $K_*(X) \otimes_{\mathbb{Z}} \mathbb{Q}_p$ injects in $K_*(X, \mathbb{Z}_p) \otimes \mathbb{Q}_p$, and the same happens when we consider in the K-groups the subspaces given by Adams filtration. All these maps of these K-groups to $H^1_{lim}(F, H^i_{et}(\overline{X}, \mathbb{Q}_p(r))$ will be denoted by $r_{S,p}$.

In the literature we find the following fact (see 2.2.6(4) [33] and proposition 3.6 [34]):

Lemma 1.2.9. *Let S be any finite set of primes of F, containing p and the primes of bad reduction of X. We consider $M_p = H^i_{et}(\overline{X}, \mathbb{Q}_p(r))$. Then, for $r > inf(i, dim(X))$ and $(i, r) \neq (1, 0), (2dim(X), dim(X) + 1)$, we have that*

$$H^l_{lim}(F, M_p) \cong H^l_{et}(Spec(\mathcal{O}_F[1/S]), M_p). \tag{1.36}$$

for any l.

In the case $l = 1$ we have the following result.

Lemma 1.2.10. *Let X be a scheme over F and consider S a finite set of places of F containing the places above p and the places of bad reduction of the scheme X. Then, for $i - 2r \neq 0$ and $i + 2 - 2r \neq 0$ we have the equality:*

$$H^1(F, H^i_{et}(\overline{X}, \mathbb{Q}_p(r)) = H^1_{et}(Spec(\mathcal{O}_F[1/S]), H^i_{et}(\overline{X}, \mathbb{Q}_p(r))),$$

where $\overline{X} = X \times_F \overline{F}$.

Proof. Denote for simplicity $M = H^i(\overline{X}, \mathbb{Q}_p)$ and consider the localization sequence

$$0 \to H^1(Spec(\mathcal{O}_F[1/S]), M(r)) \to H^1(F, M(r)) \to \oplus_{w \notin S} H^0(k(w), H^1(I_w, M(r))),$$

where $k(w)$ is the residue field at w and I_w is the inertia group of the place w of F. By purity, $H^0(k(w), H^1(I_w, M(r)) = H^0(k(w), M(r-1))$, which is zero because $i - 2(r-1) \neq 0$. □

Using lemma 1.2.10, the regulator map is mapped to $H^1(F, H^i(\overline{X}, \mathbb{Q}_p(r)))$ and it does not depend of S whenever the primes above p are in S and $H^i(\overline{X}, \mathbb{Q}_p(r))$ is smooth outside S.

Conjecture 1.2.11. *The regulator map*

$$r_p : K_{2r-i-1}(X)^{(r)} \otimes \mathbb{Q}_p \to H^1(F, H^i_{et}(X \times_F \overline{F}, \mathbb{Q}_p(r)))$$

is an isomorphism for $i + 1 < r$.

Jannsen substitutes the cohomology groups H^i_{lim} by comparison étale cohomology groups defined as the right derived functors of the projective limit of the functor taking étale global sections in $\mathbb{Z}/p^n$-coefficients. We have in this context a Hochschild-Serre spectral sequence

$$E_2^{p,q} = H^p(Gal(\overline{F}/F), H^q_{et}(\overline{X}, \mathbb{Z}_p(n)) \Rightarrow H^{p+q}_{cont,et}(X, \mathbb{Z}_p(n)).$$

The Chern class maps of Soulé map to these comparison étale cohomology groups, and the definition of the Soulé regulator is defined exactly in the same way as above. We refer to Jannsen [31] for the details.

Remark 1.2.12 (Jannsen, p.328 [32]). *Observe first that*

$$H^1_{et}(\mathcal{O}_S, H^i(\overline{X}, \mathbb{Z}_p(r))) = H^1(Gal(F_S/F), H^i(\overline{X}, \mathbb{Z}_p(r)),$$

where F_S is the maximal field extension of F unramified outside S. Write $G_S = Gal(F_S/F)$. For these Galois group with F a global field we have

$$\varprojlim_j H^1(G_S, H^i(\overline{X}, \mathbb{Z}/p^j(r))) = H^1(G_S, H^i(\overline{X}, \mathbb{Z}_p(r))).$$

In particular this is true for the absolute Galois group $H^1(G_F, H^i(\overline{X}, \mathbb{Z}_p(r)))$. Then, in the H-S spectral sequence, the surjective map β factorizes through

$$\varprojlim_j (H^{i+1}_{et}(X, \mathbb{Z}/p^j(r)))_0$$

where

$H^{i+1}_{et}(X, \mathbb{Z}/p^j(r))_0 = Ker(res : H^{i+1}(X, \mathbb{Z}/p^j(r)) \to H^{i+1}(X \times_F \overline{F}, \mathbb{Z}/p^j(r)))$.

Then the regulator map can be also obtained by taking projective limit and tensoring with $\mathbb{Q}_p$ *from the maps*

$$c_{i,r,j} : K_{2r-i-1}(X, \mathbb{Z}/p^j) \to H^1(G_F, H^i_{et}(\overline{X}, \mathbb{Z}/p^j(r))),$$

because the Chern class maps $c_{i,r}$ *can also be defined for every* j *[61][23]*

$$c_{i,r,j} : K_{2r-i-1}(X, \mathbb{Z}/p^j) \to H^{i+1}(X, \mathbb{Z}/p^j(r)).$$

1.2.3 Kato's approach to the local conjecture.

We reformulate now the conjecture for pure motives over a number field F.

A pure motive of weight $w \in \mathbb{Z}$ over a number field F is a finite family of 4-ples $\{(X_j, i_j, r_j, \epsilon_j)\}$ where X_j is a smooth proper scheme of pure dimension m_j over F and r_j are integers with $w = i_j - 2r_j$, and ϵ_j is an idempotent in the ring of algebraic cycles on $X_j \times_K X_j$ with $\mathbb{Q}$-coefficients modulo rational equivalence.

Remark 1.2.13. *The weight of* (X, i, r, ϵ) *is well-defined if we assume that the* p*-adic realization of this pure motive satisfies condition (P3) in the definition of a motivic pair 1.1.19, which in the case of motives defined over a number field* F *translates exactly into the observation made in 1.1.21.*

We denote by $h_i(X)(r)$ the pure motive determined by the 4-tuple (X, i, r, Δ_X), where Δ_X denotes the diagonal, regarded as the identity correspondence. We interpret $\{(X, i, r, \epsilon)\}$ as the direct sumand of $h_i(X)(r)$ corresponding to ϵ. For simplicity and for our interest later on we restrict from now on our analysis to pure motives M of the form (X, i, r, ϵ).

We fix notations for various realizations of M. It is more easy to define everything for the case $\epsilon = \Delta_X$ and then obtain the realization by the direct sum corresponding to the idempotent ϵ. With this point of view take $M = h_i(X)(r)$. Let $V_p(M)$ be the p-adic étale realization of M, which corresponds to $H^i_{et}(X \times_F \overline{F}, \mathbb{Q}_p(r))$. Let M_h be the $\mathbb{Q}$-structure in the Hodge structure of M, which we regard as a sheaf of $\mathbb{Q}$-vector spaces on $Spec(F \times_{\mathbb{Q}} \mathbb{R})$. Let $D(M)$ be the Hodge filtration $\{D^j(M)\}_i$ defined by $D^j(M) = Fil^{j+r} H^i_{dR}(X/F)$ and $D(M) = H^i_{dR}(X/F)$. For a general pure motive, a realization of M is defined as the direct sumand of the realization for $h_i(X)(r)$.

We suppose from now on in this section that the weight of M is ≤ -3 and that p is a fixed prime number. Define some $\mathbb{Q}$ and $\mathbb{Q}_p$-vector spaces:

$$H_h = H^0(F \times_{\mathbb{Q}} \mathbb{R}, M_h), \quad H_d := (D(M)/D^0(M)), \quad H_k := H^1_f(F, M),$$

$$H^l_p := H^l_{lim}(F, V_p(M)^*(1)).$$

Here $V_p(M)^* = Hom(V_p(M), \mathbb{Q}_p)$. The $\mathbb{Q}$-subspace H_k is defined as follows: first we define the subspace of $H^1(F, V_p(M))$ consisting of the elements whose images in $H^1(F_v, V_p(M))$ belong to $H^1_f(F_v, V_p(M))$ for all finite places v of F. Then H_k is the inverse image of $H^1_f(F, V_p(M))$ with respect to the Soulé regulator map.

Observe that H^l_p are zero for $l \geq 3$. For example, when the groups are equal to the Galois cohomology of $V_p(M)^*(1)$ considered as a $Gal(F_S/F)$-module, the vanishing of H^l_p for $l \geq 3$ is a consequence of the fact that $cd_p Gal(F_S/F) \leq 2$ (see §X in [44]).

If $M = h_i(X)(r)$, we have

$$H_h = H^i_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Q}(r))^+$$

where H_B means the Betti or also said singular cohomology, and the upperscript $+$ means the fixed part by the $Gal(\mathbb{C}/\mathbb{R})$-action acting simultaneously on $\mathbb{C}$ and on $\mathbb{Q}(r)$.

We generalize now the Tamagawa number conjecture for pure motives not necessarily defined over $\mathbb{Q}$. Let me remind some facts on determinant theory; for more explanations we refer to [37]. Let R be a commutative ring. If L is a finitely generated projective R-module, $det_L R$ is defined as the exterior power $\bigwedge^r_R L$, where r is the rank of L, which is a locally constant function on $Spec(R)$. This definition generalizes to perfect complexes. Let $\mathcal{C}$ be the derived category of the category of R-modules. An object of $\mathcal{C}$ is called perfect if it is represented by a bounded complex of R-modules consisting of finitely generated projective R-modules. For a perfect complex C, the determinant module $det_R C$ is the invertible R-module defined by:

$$det_R C := \otimes_{i \in \mathbb{Z}} \{det_R(L_i)\}^{\otimes(-1)^i}$$

where $\ldots \to L_i \to L_{i-1} \to \ldots$ is a bounded complex in the class of C in $\mathcal{C}$. This definition is independent, modulo canonical isomorphisms, of the choice of a representative as above.

The ring R will be in the following $\mathbb{Q}$, $\mathbb{Q}_p$, $\mathbb{Z}_p$ or $\mathbb{Z}_{(p)} = \mathbb{Z}_p \cap \mathbb{Q}$. For an R-module N, we denote $N^* = Hom_R(N, R)$.

Definition 1.2.14. *We define the motivic $\mathbb{Q}$-space by*

$$\Phi^{mot} := det_{\mathbb{Q}}(H_h) \otimes_{\mathbb{Q}} det_{\mathbb{Q}}(H^*_d) \otimes_{\mathbb{Q}} det_{\mathbb{Q}}(H_k).$$

And the p-adic realization $\mathbb{Q}_p$-vector space, for $p \neq 2$ by

$$\Phi^{ar}_p := det_{\mathbb{Q}}(H_h) \otimes_{\mathbb{Q}} det_{\mathbb{Q}_p} R\Gamma_{lim}(K, V_p(M)^*(1))^*.$$

By (1.29) we reformulate the Beilinson conjecture as the following exact sequence ($w \leq -3$):

$$0 \to H_k \otimes_{\mathbb{Q}} \mathbb{R} \to (H_d \otimes_{\mathbb{Q}} \mathbb{R})/(H_h \otimes_{\mathbb{Q}} \mathbb{R}) \to 0, \tag{1.37}$$

because conjecturally H_k has to coincide with the subspace in K-theory defined by Beilinson (c.f. conjecture 2.6 [34]):

Conjecture 1.2.15. *1. $H^1_f(F, M)$ and $H^0(F, M)$ are finite dimensional $\mathbb{Q}$-vector spaces.*

2. Via the Soulé regulator map, $H^1_f(F, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^1_f(F, V_p(M))$.

3. (Tate) $H^0(F, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^0(F, V_p(M))$.

4. (Beilinson, 1.2.5) If $M = h_i(X)(r)$ and $\mathcal{X}$ is a proper flat regular scheme over $\mathcal{O}_F$ such that $X = \mathcal{X} \times_{\mathcal{O}_F} F$, then $H^1_f(F, M) \subset K_{2r-i-1}(X)$ coincides with the intersection of $K_{2r-i-1}(X)^{(r)} \otimes \mathbb{Q}$ with the image of $K_{2r-i-1}(\mathcal{X}) \otimes \mathbb{Q}$ in $K_{2r-i-1}(X) \otimes \mathbb{Q}$.

Observe that, for weight reasons, $H^0(F, M) = 0$. We will suppose in the following that conjecture 1.2.15 is true.

From (1.37) we have an isomorphism of $\mathbb{R}$-modules

$$\Phi^{mot} \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\cong} \mathbb{R}. \tag{1.38}$$

Next we consider the p-adic side. Assume that $V_p(M)$ is a de Rham representation for the places of F above p (condition (P1) for a motivic pair) and S is a finite set of places of F where $V_p(M)$ is smooth on S. Tate-Poitou global-local duality leads to (c.f. cor. 3.2 [34]):

$$0 \to H^0(F, V_p(M)) \to \oplus_{v \in S} H^0(F_v, V_p(M)) \to H^2(\mathcal{O}_F[1/S], V_p(M)^*(1))^*$$
$$\to H^1_f(F, V_p(M)) \to \oplus_{v \in S} H^1_f(F_v, V_p(M)) \to H^1(\mathcal{O}_F[1/S], V_p(M)^*(1))^*$$
$$\to H^1_f(F, V_p(M)^*(1))^* \to 0.$$

We need now to use: the weight of our motive M, the isomorphism given by the exponential map (which gives an isomorphism between $D(M)$ and $H^1_f(L_v, V_p(M))$ for the local places of F over p), the conjecture for the Soulé regulator map which gives an isomorphism between $H^1_f(K, M) \otimes \mathbb{Q}_p$ and $H^1_f(F, V_p(M))$ (c.f. (2) in 1.2.15), to obtain the following exact sequence of finite dimensional $\mathbb{Q}_p$-vector spaces (c.f. prop. 3.6 [34]):

$$0 \to (H^2_p)^* \to H_k \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H_d \otimes_{\mathbb{Q}} \mathbb{Q}_p \to (H^1_p)^* \to 0.$$

For weight reasons, $H^0_p = 0$, and this gives an isomorphism of $\mathbb{Q}_p$-modules

$$\Phi^{mot} \otimes_{\mathbb{Q}} \mathbb{Q}_p \xrightarrow{\cong} \Phi^{ar}_p. \tag{1.39}$$

Define the analytic zeta element for a pure motive M by

$$\eta^{an}_{K,S}(M) := \prod_{v \notin S} P_v(V_p(M), 1)^{-1} \in \mathbb{R}.$$

Here S is a finite set of primes of F which contains the primes above p and the primes of bad reduction of M.

Conjecture 1.2.16. *The image of $\eta^{an}_{F,S}(M)$ under the isomorphism 1.38 is contained in*

$$\Phi^{mot} \subset \Phi^{mot} \otimes_{\mathbb{Q}} \mathbb{R}.$$

Assuming this conjecture, we denote by $\eta^{mot}_{F,S}(M)$ the element of Φ^{mot} corresponding to $\eta^{an}_{F,S}(M)$ via the isomorphism (1.38). We call $\eta^{ar}_{F,S}(M)_p$ its image in Φ^{ar}_p via the isomorphism (1.39).

Fix a $\mathbb{Z}_p$-sheaf T in $V_p(M)$ such that $T \otimes \mathbb{Q}_p = V_p(M)$. Let $H_{h,T} \subset H_h$ be the inverse image of $H^0(F \otimes_{\mathbb{Q}} \mathbb{C}, T) \subset H^0(F \otimes_{\mathbb{Q}} \mathbb{C}, V_p(M))$ under the composite map

$$H_h \to H^0(F \otimes_{\mathbb{Q}} \mathbb{C}, M_h) \otimes_{\mathbb{Q}} \mathbb{Q}_p.$$

For example if $M = h_m(X)(r)$, we can take, $T = H^m_{et}(\overline{X}, \mathbb{Z}_p(r))/torsion \subset V_p(M)$ and $H_{h,T} = (H^m_B(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(r))/torsion)^+ \otimes_{\mathbb{Z}} \mathbb{Z}_{(p)}$. We have the equalities:

$$H_{h,T} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} = H_h, \qquad H_{h,T} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p = H^0(K \otimes_{\mathbb{Q}} \mathbb{R}, T).$$

Suppose $p \neq 2$ and denote $T^* = Hom_{\mathbb{Z}_p}(T, \mathbb{Z}_p)$ and $H^*_{h,T} = Hom_{\mathbb{Z}_{(p)}}(H_{h,T}, \mathbb{Z}_{(p)})$. Let us define

$$\Phi^{ar}_{p,S,T} := det_{\mathbb{Z}_{(p)}}(H_{h,T}) \otimes_{\mathbb{Z}_{(p)}} \{det_{\mathbb{Z}_p} R\Gamma(\mathcal{O}_F[1/S]), T^*(1))^*\}. \tag{1.40}$$

Remark 1.2.17. *(c.f. prop.4.17 [34]) The above determinant 1.40 has sense. In fact, assume $p \neq 2$, and take S a finite set of finite places of F containing all prime divisors of p in F. Also, assume that $\mathcal{F}$ is a smooth $\mathbb{Z}_p$-sheaf on $Spec(\mathcal{O}_F)_{et}$. Then:*

$$R\Gamma(\mathcal{O}_F[1/S], \mathcal{F})$$

is a perfect complex in the derived category of $\mathbb{Z}_p$-modules.

Conjecture 1.2.18 (Kato). *Assume $p \neq 2$ and that S contains all places of F lying over p and all places of F at which M has bad reduction. Then the arithmetic zeta element*

$$\eta^{ar}_{F,S}(M)_p \in \Phi^{ar}_p = \Phi^{ar}_{p,S,T} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

is a $\mathbb{Z}_p$-basis of $\Phi^{ar}_{p,S,T}$.

Remark 1.2.19. *Kato proposes a general conjecture for a general finite abelian extension L of F. He define H_h, H_d, H_k above as a $\mathbb{Z}_p[Gal(L/F)]$-modules, and he constructs a motivic element as $\mathbb{Z}_p[Gal(L/F)]$-modules. Kato calls this conjecture the Iwasawa main conjecture. He proves in §6 [34] the equivalence of his conjecture with the classical Iwasawa theory in the case when $F = \mathbb{Q}$, $M = \mathbb{Q}(r)$ and L is the maximal real subfield of $\mathbb{Q}(\xi)$ where ξ is a root of 1 of order a power of p.*

We claim that the above conjectures 1.2.16, 1.2.18 correspond to the local Tamagawa number conjecture at p , conjecture 1.1.25. Let us go to show this.

Suppose that M is a pure motive over $\mathbb{Q}$ ($F = \mathbb{Q}$) whose realizations satisfy the expected conditions for a motivic pair. For instance, $M = h_m(X)(r)$. We assume that the regulator maps are isomorphism, and for the Soulé regulator map we have that $H^1_f(\mathbb{Q}, M) \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong H^1_f(\mathbb{Q}, V_p(M))$, with $H^1_f(\mathbb{Q}, M)$ coming from the K-theory group predicted by Beilinson (4) 1.2.15.

Let $H_{k,T} \subset H^1_f(\mathbb{Q}, T)$ be the inverse image of $H_k \subset H^1_f(\mathbb{Q}, V_p(M))$. Observe that $H_{k,T}$ satisfies

$$H_{k,T} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q} \cong H_k, \quad H_{k,T} \otimes_{\mathbb{Z}_{(p)}} \mathbb{Z}_p \cong H^1_f(\mathbb{Q}, T).$$

Take a $\mathbb{Z}_{(p)}$-lattice Δ on H_d. Supposing the isomorphism in the Beilinson regulator 1.37, we have an isomorphism

$$H_{k,T} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R} \cong (\Delta \otimes_{\mathbb{Z}_{(p)}} \mathbb{R})/(H_{h,T} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R}).$$

Define the motivic element by

$$\Phi^{mot}_{T,\Delta} = det_{\mathbb{Z}_{(p)}}(H_{h,T}) \otimes_{\mathbb{Z}_{(p)}} det_{\mathbb{Z}_{(p)}}(H_{k,T}) \otimes_{\mathbb{Z}_{(p)}} det_{\mathbb{Z}_{(p)}}(\Delta)^*. \tag{1.41}$$

The Beilinson regulator induces then the isomorphism

$$\Phi^{mot}_{T,\Delta} \otimes_{\mathbb{Z}_{(p)}} \mathbb{R} \cong \mathbb{R}. \tag{1.42}$$

Let $\alpha \in \mathbb{R}^*/\mathbb{Z}^*_{(p)}$ be the image of a $\mathbb{Z}_{(p)}$-basis of $\Phi^{mot}_{T,\Delta}$ under 1.38 ($\alpha = \mu_{\infty,\Delta}(A(\mathbb{R})/A(\mathbb{Q}))$ and Δ corresponds to the choice of $det_{\mathbb{Q}}(H_d) \overset{w}{\cong} \mathbb{Q}$ in

1.18). Furthermore, the exponential map induces an isomorphism (via the hypotheses of the Soulé regulator map 1.2.15)

$$det_{\mathbb{Z}_{(p)}}\Delta \otimes_{\mathbb{Z}_{(p)}} \mathbb{Q}_p \cong det_{\mathbb{Z}_p} H^1_f(\mathbb{Q}_p, T) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p. \tag{1.43}$$

Then the $\mathbb{Z}_{(p)}$-basis of Δ maps to $\mu_{p,\Delta} A(\mathbb{Q}_p)$ modulo $\mathbb{Z}^*_{(p)}$ times a $\mathbb{Z}_p$-basis of $H^1_f(\mathbb{Q}_p, T)$. Denote this number by μ_p. Let S be a finite set of places of $\mathbb{Q}$ containing ∞, p and all the finite places at which M has bad reduction. Define by

$$\mu_{S,\Delta} := \mu_p \prod_{v \in S \setminus p,\infty} \#(H^0(\mathbb{Q}_v, T \otimes \mathbb{Q}_p/\mathbb{Z}_p)).$$

We know by definition that

$$Tam(M,T)^{(p)} = \mu_{S,\Delta} \alpha L_S(M,0)^{-1}.$$

Proposition 1.2.20 (lemma 7.3 [34]). *The image of $L_S(M,0) \in \mathbb{R}$ under the isomorphism 1.41 is equal to (a representative in $\mathbb{Q}^*$ of) $\mu_{\Delta,S} Tam(M,T)^{-1}$ times a $\mathbb{Z}_{(p)}$-basis of $\Phi^{mot}_{T,\Delta}$.*

By conjecture 1.2.16, we have then that $\mu_{S,\Delta} Tam(M,T)^{-1}$ is an element in $\mathbb{Q}^*$. Therefore,

Proposition 1.2.21 (prop. 7.8 [34]). *Under the isomorphism given by the Soulé regulator map (1.39):*

$$\Phi^{mot} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong \Phi^{ar}_p$$

the image of the zeta element $\eta^{mot}_{\mathbb{Q},S}(M)$ is

$$\#(Coker(\alpha_M)\{p\})(\#(ker(\alpha_M)\{p\})^{-1} Tam(M,T)^{-1}$$

times a $\mathbb{Z}_p$-basis of

$$\Phi^{ar}_{p,T} = det_{\mathbb{Z}_{(p)}}(H_{h,T}) \otimes_{\mathbb{Z}_{(p)}} \{det_{\mathbb{Z}_p}(R\Gamma(\mathbb{Z}[1/S], T^*(1)))^*\}.$$

Thus, from the conjecture 1.2.18 proposed by Kato we obtain the initial local Tamagawa number conjecture.

With this point of view we formulate the local Tamagawa number conjecture for an arbitrary pure motive M over a number field F as follows.

Conjecture 1.2.22 (local Tamagawa number conjecture). *Let M be a pure motive over F a number field which is the direct sumand corresponding to an idempotent ϵ of the Chow motive $h_i(X)(r)$ with X a smooth , proper scheme over F. Fix a prime p diferent from 2. Let S be a finite set of primes of F containing the primes above p and the primes where M has bad reduction. Let Δ, $H_{h,T}$, $V_p(M)$ and T be as above and consider $H_{\mathcal{M}}$ as in conjecture (4)1.2.15 which is $H^1_f(F,V)$ in the above notation. Suppose M has weight $w \leq -3$ and satisfies $r > inf(i, dimX)$. Then:*

1. *The regulators maps $r_{\mathcal{D}}$ in 1.2.5 and $r_{p,S}$ in 1.2.11 are isomorphisms.*

2. *There is an element $\xi \in det_{\mathbb{Q}} H_{\mathcal{M}}$ such that*

$$r_{\mathcal{D}}(\xi) = L_S(V_p(M), 0)\eta,$$

where η is a $\mathbb{Z}_{(p)}$-base for $det_{\mathbb{Z}_{(p)}}\Delta \otimes det_{\mathbb{Z}_{(p)}}(H_{h,T})^{-1}$.

3. *$r_p(\xi)$ is a basis for the $\mathbb{Z}_p$-lattice*

$$det_{\mathbb{Z}_{(p)}}(\Delta) \otimes det_{\mathbb{Z}_p}((R\Gamma(\mathcal{O}_F[1/S], T^*(1))^*)^{-1}$$
$$\subset det_{\mathbb{Q}_p}(H_d) \otimes det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_F[1/S], V_p(M)^*(1))^*[1].$$

Using the hypothetical functional equation we can rewrite this conjecture via the value of the first non-vanishing Fourier coeficient at 0 for $V_p(M)^*(1)$, like in proposition 1.1.26. We obtain in this way the following formulation for the local Tamagawa number conjecture.

Conjecture 1.2.23. *(c.f. 2.2.7 [33], c.f. 1.1.1 [36]) Let $p \neq 2$ be a fix prime number. Let M be a pure motive over a number field F coming from some idempotent of a Chow motive of the form $h_i(X)(r)$, with X a proper smooth scheme over F. Denote by $V_p(M)$ the p-adic realization of the motive M. Suppose that M has weight $w \leq -3$ and satisfies $r > inf(i, dimX)$. Consider S a finite set of primes of F containing the primes above p and the primes of bad reduction of M.*
Observe in this situation that the Beilinson regulator maps the K-theory group $H_{\mathcal{M}} = H^1_f(K, V)$ defined above to the part of $H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{R}(r-1))^+$ associated to the idempotent 1.2.7. Take the part corresponding to the idempotent of the motive M for $H^i(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(r-1))^+$, and denote it by $H_{h,\mathbb{Z}}$. Suppose moreover that $L_S(V_p(M)^*(1), s)$ has an analytic continuation to all $\mathbb{C}$ and for all $\mathfrak{p} \in S$ the local Euler factors at 1 satisfy

$$P_{\mathfrak{p}}(V_p(M)^*(1), 1) \neq 0.$$

Then:

1. *The regulators $r_{\mathcal{D}}$ in 1.2.5 and $r_{p,S}$ in 1.2.11 maps are isomorphisms. Moreover $H^2(\mathcal{O}_F[1/S], V_p(M)) = 0$.*

2. *$dim_{\mathbb{Q}}(H_{h,\mathbb{Z}} \otimes \mathbb{Q}) = ord_{s=0} L_S(V_p^*(1), s)$. Denote by e this dimension.*

3. *Let $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ be a $\mathbb{Z}$-basis. Then there is an element $\xi \in det_{\mathbb{Q}} H_{\mathcal{M}}$ such that*

$$r_{\mathcal{D}}(\xi) = (\lim_{s \to 0} s^{-e} L_S(V_p^*(M)(1), s))\eta.$$

4. *Consider* $r_{p,S}(\xi) \in det_{\mathbb{Q}_p}(H^1(\mathcal{O}_F[1/S], V_p(M))$. *Then* $r_{p,S}(\xi)$ *is a basis for the* $\mathbb{Z}_p$*-lattice*

$$det_{\mathbb{Z}_p}((R\Gamma(\mathcal{O}_F[1/S], T)^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_F[1/S], V_p(M))[-1]),$$

where T *corresponds to the part of the lattice* $H^i_{et}(X \times_F \overline{F}, \mathbb{Z}_p(r))$ *associated to the idempotent.*

Remark 1.2.24. *In this new formulation one impose that*

$$H^2_p(\mathcal{O}_F[1/S], V_p(M)) = 0.$$

This vanishing is consequence of a general conjecture of Jannsen [32]. See appendix B for more explanations, the precise formulation and some examples of this conjecture.

Chapter 2

On the local Tamagawa conjecture for CM elliptic curves over $\mathbb{Q}$

Introduction

The Tamagawa number conjecture (or Bloch-Kato conjecture), presented by Bloch and Kato in [6] and reminded in §1.1 , describes the integer values of the L-function of a $\mathbb{Q}$-pure motive, or more generally of a motivic pair (motivic pairs should include the $\mathbb{Q}$-mixed motives). With the point of view of the Beilinson conjecture, the local Tamagawa number conjecture is rewritten (§1.2) in terms of the construction of a module inside K-theory and the computation of the Beilinson and Soulé regulators on this module. We follow this formulation for the conjecture.

We concentrate in this chapter on the motives $h_1(E)(k+2)$ with k an integer ≥ 0, where E is an elliptic curve with CM. Observe that, by the functional equation, the understanding of the value at zero of L-function of $h_1(E)(k+2)$ gives the understanding of the first Fourier non-zero of the L-function of E at $-k$. The case which does not cover this study is the case $k=-1$, which corresponds to the statement of the Birch-Swinnerton-Dyer conjecture, if we suppose moreover that E is defined over $\mathbb{Q}$. A result of Coates and Wiles proves partially the Tamagawa number conjecture for $h_1(E)(1)$ when $L(h_1(E)(1),0)\neq 0$ and if E is defined over $\mathbb{Q}$.

Let's consider now $k\geq 0$. For weight reasons we have that $L(h_1(E)(k+1),0)\neq 0$. We study the Tamagawa number conjecture for these values. Kato proves partially the local conjecture for $h_1(E)(2)$, for E defined over $\mathbb{Q}$ and with CM $\mathcal{O}_K$ the ring of integers of an imaginary quadratic field; in this case he can work in $K_2(E)$. For $k > 0$ we need to work with higher K-groups.

In a recent work, Kings [36] proves the local conjecture for the L-value at $2+k$, for all positive integers k, for CM elliptic curves defined over the field of their endomorphism ring, and for primes where their Soulé regulators are injective on the constructed module in K-theory. Thus, to obtain the result for $h_1(E)(k+2)$ when E is defined over $\mathbb{Q}$, we need to descend the result of Kings. We obtain hence the generalization of the result of Bloch and Kato in §7 [6] for CM elliptic curves defined over $\mathbb{Q}$. With this we mean that we prove the local Tamagawa conjecture at p for the L-values at $2+k$, for all positive integers $k \geq 0$, of CM elliptic curves define over $\mathbb{Q}$, for primes p under some assumptions, which are similar conditions that appears in the work of Kings [36] and also in the work of Kato in §7[6]. For the exact statement of the result see Theorem 2.1.9.

Let's give a sketch of the contents of this chapter. In the first section we present a weak version of the local Tamagawa number conjecture, due essentially to our lack of information about the dimension of the K-groups for the algebraic varieties involved. We study some properties on the modules which relates the conjecture and we write the main result. In §2 we construct the elements in higher K-theory which correspond to the part of the motive $h_1(E)(k+2)$. We compare these elements with the ones used to check the local Tamagawa number conjecture for $h_1(E \times_{\mathbb{Q}} K)(k+2)$. We change also the K-subspace defined over K by Kings to a subspace on the K-theory defined over $\mathbb{Q}$. In §3, we study the descend problem for the Soulé regulator. In §4, we compare the Galois cohomology group of the Tate module $T_p(E \times_{\mathbb{Q}} K)$ and $T_p(E)$ to obtain a general relation in both. Then, by using determinant techniques, we obtain the local Tamagawa number conjecture. In §6 is study the conditions imposed to prove the local Tamagawa number conjecture for $h_1(E)(k+2)$. Essentially, this means to study the bijectivity of the Soulé regulator map for the finite primes p.

2.1 The Tamagawa number conjecture for CM elliptic curves

Let us write another time the local Tamagawa number conjecture for a Chow motive of the form $h_m(X)(r)$ (cf. 1.2.23). Let X/F be a smooth proper variety defined over a number field F with ring of integers $\mathcal{O}_F$. Fix integers $m \geq 0$ and r such that $m - 2r \leq -3$ and $r > \inf(m, dim(X))$. Let p be an odd prime number and S be a set of finite primes of F containing the primes lying over p and the ones where X has bad reduction. Write $\mathcal{O}_S = \mathcal{O}_F[1/S]$. Define the $\mathrm{Gal}(\overline{F}/F)$-modules:

$$V_p := H^m_{et}(X_F \times_F \overline{F}, \mathbb{Q}_p(r)),$$

$$T_p := H^m_{et}(X_F \times_F \overline{F}, \mathbb{Z}_p(r)).$$

Consider $j : Spec(F) \to Spec\ \mathcal{O}_S$ the natural map, and define the p-adic realizations

$$H^i_p := H^i_{et}(\mathcal{O}_S, j_* T_p).$$

We omit the j_* if no confusion is likely. Write

$$H_{h,\mathbb{Z}} := H^m_{sing}(X \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Z})^+.$$

Denote $H_{\mathcal{M}} := (K_{2r-m-1}(X) \otimes \mathbb{Q})^{(r)}$. There are regulator maps due to Beilinson and Soulé:

$$r_{\mathcal{D}} : H_{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{R} \to H_{h,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{R} \quad ,\S 1.2.1$$

$$r_p : H_{\mathcal{M}} \otimes_{\mathbb{Q}} \mathbb{Q}_p \to H^1_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \quad \S 1.2.2.$$

We have local Euler factors for a prime $\mathfrak{p} \nmid p$ in $\mathcal{O}_F$, which are defined here by

$$P_{\mathfrak{p}}(V_p, s) := det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{p}} N\mathfrak{p}^{-s} | V_p^{I_{\mathfrak{p}}}),$$

where $Fr_{\mathfrak{p}}$ is the geometric Frobenius at $\mathfrak{p}$ and $I_{\mathfrak{p}}$ is the inertia group at $\mathfrak{p}$. For $\mathfrak{p} \mid p$, is defined by

$$P_{\mathfrak{p}}(V_p, s) := det_{\mathbb{Q}_p}(1 - \psi_{\mathfrak{p}}^{-1} N\mathfrak{p}^{-s} | D_{cris}(V_p)),$$

where $\psi_{\mathfrak{p}}$ is the arithmetic Frobenius. We note that a conjecture claims that $P_{\mathfrak{p}}(V_p, s)$ is independent of the choice of the finite prime p. This conjecture is proved for $m = 1$.
The L-function of X is defined by

$$L_S(V_p, s) := \prod_{\mathfrak{p} \notin S} P_{\mathfrak{p}}(V_p, s)^{-1}.$$

Conjecture 2.1.1. *(cf. §4[34], conj. 2.2.7. in [33]) Let $V_p^* = Hom(V_p, \mathbb{Q}_p)$ be the dual Galois module. Let $p \neq 2$, r, m and S be as above. Assume that*

$$P_{\mathfrak{p}}(V_p^*(1), 0) \neq 0$$

for all $\mathfrak{p} \in S$ and that $L_S(V_p^*(1), s)$ has an analytic continuation to all $\mathbb{C}$, then:

1. *The maps $r_{\mathcal{D}}$ and r_p are isomorphisms and H_p^2 is finite.*

2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0}L_S(V_p^*(1), s)$*; write this number* l.

3. *Let $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ be a $\mathbb{Z}$-basis. There is an element $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M}})$ such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s\to 0} s^{-l} L_S(V_p^*(1), s))\eta.$$
This is the "Beilinson conjecture".

4. *Consider $r_p(\xi) \in det_{\mathbb{Q}_p}(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Then $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1]).$$
i.e.
$$[det_{\mathbb{Z}_p}(H_p^1) : r_p(\xi)\mathbb{Z}_p] = \#(H_p^2) = det_{\mathbb{Z}_p}(H_p^2).$$

Remark 2.1.2. *The assumption in the conjecture is true for abelian varieties with CM. So, in particular, in dimension 1, for elliptic curves with CM, which is the case that we are interested on.*

As our knowledge of K-theory is limited, we take a weak version of the above conjecture 1.2.23.

Conjecture 2.1.3. *(cf. conj.1.1.2.[36]) Suppose the same hypothesis as in conjecture 2.1.1 are satisfied. Then, there is a subspace $H_{\mathcal{M}}^{constr}$ in $H_{\mathcal{M}}$ such that:*

1. *$r_{\mathcal{D}}$ and r_p restricted to $H_{\mathcal{M}}^{constr}$ are isomorphisms and H_p^2 is finite.*

2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0}L_S(V_p^*(1), s)$*; write this number* l.

3. *There is an element $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M}}^{constr})$ such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s\to 0} s^{-l} L_S(V_p^*(1), s))\eta,$$
where η is a $\mathbb{Z}$-basis of $det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$.

4. *The element $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_p)[-1])$$

Remark 2.1.4. *We call the conjecture 2.1.3, with the missing part for the finiteness of H_p^2 and the exhaustivity condition for r_p, the p-part of the local Bloch-Kato conjecture for the first Fourier coeficient of $L(V_p^*(1), s)$ at $s = 0$.*

We are going to state the main result of the chapter. We will fix the realizations for which we prove some part of the conjecture 2.1.3. We are interested in the rational value of $L(E^+, n)$, with n an integer bigger than or equal to 2; then the representations and the regulator maps would come from those of the pure motive $h_1(E^+)(n)$. With this in mind, we take $X = E^+$ a CM elliptic curve defined over $\mathbb{Q}$ with the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K as endomorphism ring. Let be $E := E^+ \times_{\mathbb{Q}} K$. Denote by

$$\psi : \mathbb{A}_K^* \to K^* \subset \mathbb{C}^*$$

the CM-character (Serre-Tate character) of E and denote by $\mathfrak{f}$ its conductor. We have also an odd prime number p fixed. Let S_E be the set of primes of K dividing $p\mathfrak{f}$ and S_{E^+} the set of primes of $\mathbb{Q}$ dividing $p\mathfrak{f}$. The primes dividing the discriminant d_K of $K/\mathbb{Q}$ divide $\mathfrak{f}$ (prop.1, [48]) and this implies that the set of bad reduction does not change in the extension $K/\mathbb{Q}$, (see for example ex. II.2.31 in [58]). For this reason, we denote S_E or S_{E^+} by S, the context making the meaning clear.
We recall the following result of Deuring:

Theorem 2.1.5. *(see [58] II 10.5) Let E^+ be a CM elliptic curve as above defined over $\mathbb{Q}$ and let S be the set as above.*

1. *Let $L_S(E^+/\mathbb{Q}, s) := L_S(V_p, s)$ be the L-series of the Galois representation $V_p := H^1(E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p)$; then*

$$L_S(E^+/\mathbb{Q}, s) = L_S(\psi, s),$$

where $L_S(\psi, s) = \prod_{\mathfrak{p} \nmid p\mathfrak{f}} \frac{1}{1 - \frac{\psi\mathfrak{p}}{N\mathfrak{p}^s}}$ for $Re(s) > 1$ and has analytic continuation.

2. *Let $L_S(E/K, s) := L_S(V_p, s)$ be the L-series of the Galois representation $V_p := H^1(E \times_K \overline{\mathbb{Q}}, \mathbb{Q}_p)$; then*

$$L_S(E/K, s) = L_S(\psi, s) L_S(\overline{\psi}, s) = L_S(\psi, s)^2,$$

where the last equality is because, in our situation, we have $L_S(\psi, s) = L_S(\overline{\psi}, s)$.

Let $T_p E^+ = \varprojlim E^+[p^n]$ be the Tate-module of E^+, that is, a $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$-module. Then, by Kummer theory, we have that

$$H^1(E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Z}_p) \cong Hom(T_p E^+, \mathbb{Z}_p) \cong T_p E^+(-1).$$

In order to state the result and to fix notation for the rest of the paper, we take $m = 1$, $r = k + 2$, with $k \geq 0$, and

$$H_p^i = H^i(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)),$$

$$H_{h,\mathbb{Z}} = H^1_{sing}(E^+ \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{r-1}\mathbb{Z})^+,$$

$$H_{\mathcal{M}} = H^2_{\mathcal{M}}(E^+, k+2),$$

where $H^i_{\mathcal{M}}(X, j) := (K(X)_{2j-i} \otimes \mathbb{Q})^{(j)}$. Note that $H_{h,\mathbb{Z}}$ is a free $\mathbb{Z}$-module of rank 1. Moreover, for weights reasons, $H_p^0 = 0$. The Jannsen conjecture [32](see Appendix B.3), claims the finiteness of H_p^2. Let us go to describe a little the group H_p^1, which is a finitely generated $\mathbb{Z}_p$-module. To study this group, we will see that is enough to study the $\mathcal{O}_K$-structure of the cohomology group $H^1(Gal(K_S/K), T_pE(k+1))$ (see lemma 2.4.1).

Lemma 2.1.6. *The torsion group of $H^1(Gal(K_S/K), T_pE(k+1))$ with $k \geq 0$ is equal to the group*

$$(V_pE(k+1)/T_pE(k+1))^{Gal(K_S/K)}.$$

Proof. Consider the short exact sequence

$$0 \to T_pE(k+1) \to V_pE(k+1) \to V_pE(k+1)/T_pE(k+1) \to 0.$$

Taking Galois invariant by $Gal(K_S/K)$, we obtain the exact sequence

$$0 \to (V_pE(k+1)/T_pE(k+1))^{Gal(K_S/K)} \to H^1(Gal(K_S/K), T_pE(k+1))$$
$$\to H^1(Gal(K_S/K), V_pE(k+1)),$$

where the last group is a $\mathbb{Q}_p$-module, hence torsion free, proving the result. □

Lemma 2.1.7. *Consider $p > k+3$ and suppose that p splits in two diferent primes in K. Then $H^1(Gal(K_S/K), T_pE(k+1))$ (and in particular H_p^1 cf. 2.4.1) has no torsion and then is a free module.*

Proof. Let's write $(p) = \mathfrak{p}\mathfrak{p}^*$ in $Spec(\mathcal{O}_K)$, and, as $K = K(1)$, we write $\mathfrak{p} = (\pi)$. Let's consider the field $K_\infty := \cup_n K(E[\pi^n])$. Let denote by $G_\infty = Gal(K_S/K_\infty)$. Observe that

$$K_\infty \subset (K(E[p^\infty]) \subset K_S$$

by [14]). Since p is split, we have a decomposition of the Tate module. Hence, the Galois invariants we need to study using the above lemma correspond to $T_{\mathfrak{p}}E(k+1) \otimes \mathbb{Q}_p/\mathbb{Z}_p$ and $T_{\mathfrak{p}^*}E(k+1) \otimes \mathbb{Q}_p/\mathbb{Z}_p$.

In K_∞, $T_\mathfrak{p}E(k+1) = \mathbb{Q}_p/\mathbb{Z}_p(k+1)$, because $K(\mu_{p^\infty}) \cap K_\infty = K$. Using that $Gal(K_S/K_\infty)$ acts by the cyclotomic character on $T_{\mathfrak{p}^*}E$, we obtain then that, over K_∞, the module $T_{\mathfrak{p}^*}E(k+1)\otimes\mathbb{Q}_p/\mathbb{Z}_p$ is isomorphic to $\mathbb{Q}_p/\mathbb{Z}_p(k+2)$.

Now, we will compute the $G_\infty := Gal(K_S/K_\infty)$-invariants of

$$V_pE/T_pE(k+1) = \mathbb{Q}_p/\mathbb{Z}_p(k+1) \oplus \mathbb{Q}_p/\mathbb{Z}_p(k+2),$$

where the equality is as $Gal(K_S/K_\infty)$-modules.

We think $\mathbb{Q}_p/\mathbb{Z}_p(l) = \varinjlim_j \mathbb{Z}/p^j\mathbb{Z}(l)$, and then the elements fixed by G_∞ corresponds to the p-roots of unity μ such that $\mu^{\otimes i} = 1$. That is, if we denote p^n the bigger power of p such that $[K_\infty(\mu_{p^n}) : K_\infty]|l$, then the G_∞-invariants of $\mathbb{Q}_p/\mathbb{Z}_p(l)$ are $\mathbb{Z}/p^n\mathbb{Z}$. Using now that $l = k+1$ or $l = k+2$ and $p > k+3$, we have that the torsion is empty since $K(\mu_{p^\infty}) \cap K_\infty = K$. $\square$

For the rank of free part for H^1_p as $\mathbb{Z}_p$-modules, which coincides with the dimension as $\mathbb{Q}_p$ vector space of $H^1(Spec(\mathbb{Z}[1/S]), T_pE(k+1)\otimes\mathbb{Q})$, we have the following result.

Lemma 2.1.8 (Jannsen, corollary 1 [32]).
$dim_{\mathbb{Q}_p} H^1(Gal(\mathbb{Q}_S/\mathbb{Q}), H^m_{et}(X_F \times_F \overline{F}, \mathbb{Q}_p(r))) \geq dim_{\mathbb{R}} H^m_{sing}(X \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(r-1))^+ \otimes \mathbb{R}$ *with equality if and only if*

$$H^0(Gal(\mathbb{Q}_S/\mathbb{Q}), H^m_{et}(X_F \times_F \overline{F}, \mathbb{Q}_p(r))$$

and

$$H^2(Gal(\mathbb{Q}_S/\mathbb{Q}), H^m_{et}(X_F \times_F \overline{F}, \mathbb{Q}_p(r)))$$

are both 0.

The above lemma says in our case

$$H^m_{et}(\overline{X}, \mathbb{Q}_p(r)) = T_pE^+(k+1) \otimes \mathbb{Q}$$

that $dim_{\mathbb{Q}_p} H^1_p \geq 1$, because the Deligne cohomology has rank 1. Observe that, if H^2_p is finite, then H^1_p has rank exactly 1. Moreover, let us note that we know that the Beilinson conjecture for elliptic curves with CM defined over $\mathbb{Q}$ is true by Deninger §4 [9] if the dimension of the $\mathbb{Q}$-vector space $H_{\mathcal{M}}$ is one. It is expected that the dimension of this K-group is exactly one, since this fact is equivalent to the Beilinson's conjecture. So, if the Beilinson conjecture is true for CM elliptic curves defined over $\mathbb{Q}$, then H^1_p has rank 1 if and only if the dimensions in the both spaces which relates the Soulé regulator map have the same dimension.

Let us write the main result of this chapter, which corresponds to the theorem A in the introduction.

Theorem 2.1.9. *With the previous notations, consider $p \neq 2, 3$, $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$, and $(p, N_{K/\mathbb{Q}}v) = 1$ (for the construction of this $v \in \mathcal{O}_K$ we refer to the next section) and let k be an integer greater than or equal to 0.*

Then, there is a submodule $\mathcal{R}_\psi \subset H_\mathcal{M}$ of rank 1 such that:

1. *In $det_\mathbb{R}(H_{h,\mathbb{Z}} \otimes \mathbb{R})$:*

$$det_\mathbb{Z}(r_\mathcal{D}(\mathcal{R}_\psi)) \cong L^*_S(\psi, -k) det_\mathbb{Z}(H_{h,\mathbb{Z}}) = L^*_S(E^+, -k) det_\mathbb{Z}(H_{h,\ Z})$$

and

2. *The map r_p induces an isomorphism:*

$$det_{\mathbb{Z}_p}(\mathcal{R}_\psi) \cong det_{\mathbb{Z}_p} R\Gamma(Spec(\mathbb{Z}[1/S], T_pE^+(k+1)))^{-1}.$$

Here $L^*(\psi, -k) = \lim\limits_{s \to -k} \frac{L(\psi,s)}{s+k}$. If r_p is injective on $\mathcal{R}_\psi$, then

$$det_{\mathbb{Z}_p} H^1_p / r_p(\mathcal{R}_\psi) \cong det_{\mathbb{Z}_p} H^2_p.$$

Remark 2.1.10. *The above result, with the same hypothesis and with the condition of regularity on the prime p (see §2.6 for a short definition and appendix B for a long explanation) is proved at $k = 0$ by Bloch and Kato in §7 [6]. They use an ad-hoc method which does not extend to higher K-groups. The condition p regular shows the finiteness of the H^2_p and the injectivity on $\mathcal{R}_\psi$ for the Soulé regulator map, proving then in generality the conjecture 2.1.3*

Remark 2.1.11. *Part 1) of the theorem is proved by Deninger in §4 [10]. This is the Beilinson conjecture for elliptic curves with CM which has descend over the real field, in our situation $\mathbb{Q}$.*

The last theorem is the p-part of the local Bloch-Kato conjecture for the value $L(E^+, k+2)$ under the assumption that the Soulé regulator is injective. In §5 we show that H^2_p finite implies the conjecture 2.1.3 in our situation.

Moreover, the reason that we call this the conjecture at the concrete value $k+2$ of the L-function, instead of $-k$, is because it is equivalent to the statement of the conjecture for the pure motive $h^1(E^+)(k+2)$ (see conjecture 1.2.23). This fact comes from the functional equation relating the L-functions of M and $M^*(1)$; the existence of this functional equation was already proved in our situation by Hecke and Tate, and it relates the first Fourier coeficient of $L(E^+, s)$ at $s = -k$ with the non-zero value (for weight reasons) of $L(E^+, k+2)$.

The proof of the theorem is completed in the following sections. The idea is to descend over E^+ the statement of the theorem for E proved over K by Kings [36].

In the next section we define the subspace $\mathcal{R}_\psi$, using the subspace constructed by Kings to prove the Tamagawa conjecture for the elliptic curve E. In sections 3 and 4 we check that the problem descends: §3 is related to the Soulé regulator and §4 to the étale cohomology of the Tate module and the relation with the determinant theory. The last section is a study of the injectivity condition for the Soulé regulator in our subspace $\mathcal{R}_\psi$.

2.2 The construction of $\mathcal{R}_\psi$ and its first properties.

The initial point of our work is the following result of G. Kings.

Theorem 2.2.1 (Kings[36] thm 1.1.5). *With the same notations as in §2, write $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$. Consider $p \neq 2, 3$ and $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$ and $k \geq 0$. Then, there is an $\mathcal{O}_K$ submodule $\tilde{\mathcal{R}_\psi}' \subset H^2_{\mathcal{M}}(E, k+2)$ of rank 1 such that*

1. *In $det_{\mathcal{O}_K \otimes \mathbb{R}}(H^1(E \times_\mathbb{Q} \mathbb{C}, (2\pi i)^{k+1}\mathbb{Z})^+ \otimes \mathbb{R})$:*

$$det_{\mathcal{O}_K}(r_\mathcal{D}(\tilde{\mathcal{R}_\psi}')) \cong L^*_S(\overline{\psi}, -k) det_{\mathcal{O}_K}(H^1(E \times_\mathbb{Q} \mathbb{C}, (2\pi i)^{k+1}\mathbb{Z})^+).$$

2. *The map r_p induces an isomorphism*

$$det_{\mathcal{O}_p}(\tilde{\mathcal{R}_\psi}') \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, T_pE(k+1)))^{-1}.$$

   If r_p is injective on $\tilde{\mathcal{R}_\psi}'$, then

$$det_{\mathcal{O}_p} H^1(\mathcal{O}_S, T_pE(k+1))/r_p(\tilde{\mathcal{R}_\psi}') \cong det_{\mathcal{O}_p} H^2(\mathcal{O}_S, T_pE(k+1)).$$

Remark 2.2.2. *This theorem corresponds to the local Tamagawa number conjecture for $L(E, k+2)$, corollary 1.1.6 [36], since*

$$Norm_{\mathcal{O}_K \otimes \mathbb{R}/\mathbb{R}}(L^*(\overline{\psi}, -k)) = \lim_{s \to -k} \frac{L(E, s)}{(s+k)^2}.$$

We will review first the construction of the element that generates the $\mathcal{O}_K$-subspace $\tilde{\mathcal{R}_\psi}'$. This subspace was constructed for the first time by Deninger to prove the Beilinson conjecture. We will define $\mathcal{R}_\psi$ in theorem 2.1.9 to be the norm of the space $\tilde{\mathcal{R}}'_\psi$ in the theorem above, with some normalization needed to relate it to the elements used to prove the Beilinson conjecture in the real descent situation (see §4 in [10]).

Fix an algebraic differential $\omega \in H^0(E, \Omega^1_{E/K})$ that we suppose lies in $H^0(E^+, \Omega^1_{E^+/\mathbb{Q}})$. Let Γ be its period lattice. We have

$$E^+(\mathbb{C}) = E(\mathbb{C}) \to \mathbb{C}/\Gamma,$$

$$z \mapsto \int_0^z \omega,$$

with an embedding $K \subset \mathbb{C}$ fixed once and for all. We have $\Gamma = \Omega\mathcal{O}_K$ for some $\Omega \in \mathbb{C}^*$.

Denote by $\mathbb{Z}[E[\mathfrak{f}] \setminus 0]$ the group of divisors with support in the $\mathfrak{f}$-torsion points defined over K. Beilinson defines a non-zero Eisenstein symbol map

$$\mathcal{E}^{2k+1}_{\mathcal{M}} : \mathbb{Z}[E[\mathfrak{f}] \setminus 0] \to H^{2k+2}_{\mathcal{M}}(E^{2k+1}, 2k+2),$$

where $E^n = E \times_K \cdots \times_K E$, and Deninger constructs (§11[9]) a projector map

$$\mathcal{K}_{\mathcal{M}} : H^{2k+2}_{\mathcal{M}}(E^{2k+1}, 2k+2) \to H^2_{\mathcal{M}}(E, k+2).$$

Hence we get a map:

$$\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}^{2k+1}_{\mathcal{M}} : \mathbb{Z}[E[\mathfrak{f}] \setminus 0] \to H^2_{\mathcal{M}}(E, k+2).$$

We define now an element on $\mathbb{Z}[E[\mathfrak{f}] \setminus 0]$. Let $K(\mathfrak{f}) = K(E[\mathfrak{f}])$ be the ray class field modulo $\mathfrak{f}$, and let f be a generator of $\mathfrak{f}$. Observe

$$\Omega f^{-1} \in \mathfrak{f}^{-1}\Gamma$$

defines a divisor over $K(\mathfrak{f})$. Define

$$\beta := N_{K(\mathfrak{f})/K}((\Omega f^{-1})).$$

The element on K-theory $\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}^{2k+1}_{\mathcal{M}}(\beta)$ is the sought-after element. The work of Deninger on the Beilinson conjecture controls the image of the Deligne regulator of this element. For the precise statement, take γ to be the $\mathcal{O}_K$ generator of $H^1(E(\mathbb{C}), \mathbb{Z})$ such that Ω is obtained by $\Omega = \int_\gamma \omega$.
Denote by η the $\mathcal{O}_K$ generator of $H^1(E \times_{\mathbb{Q}} \mathbb{C}, (2\pi i)^{k+1}\mathbb{Z})^+$ corresponding to $(2\pi i)^k\gamma$ under the isomorphism:

$$H^1(E(\mathbb{C}), (2\pi i)^{k+1}\mathbb{Z}) \cong H_1(E(\mathbb{C}), (2\pi i)^k\mathbb{Z}).$$

Theorem 2.2.3 (Deninger [9] thm 11.3.2). *Let β and η be as above. Define*

$$\xi := (-1)^{k-1}\frac{(2k+1)!}{2^{k-1}}\frac{L_p(\overline{\psi},-k)^{-1}|\tilde{u}|^{2k}}{\psi(f)N_{K/\mathbb{Q}}(\tilde{u}\mathfrak{f})^k}\mathcal{K}_{\mathcal{M}}\circ\mathcal{E}_{\mathcal{M}}^{2k+1}(\beta)\in H^2_{\mathcal{M}}(E,k+2),$$

for some $\tilde{u}\in\mathcal{O}_K$, where $L_p(\overline{\psi},-k)$ is the Euler factor of $\overline{\psi}$ at p evaluated at $-k$. Then

$$r_{\mathcal{D}}(\xi)=L_S^*(\overline{\psi},-k)\eta\in H^1(E\times_K\mathbb{C},(2\pi i)^{k+1}\mathbb{R}).$$

Kings defines the subspace on K-theory used in the theorem2.2.1 as

$$\tilde{\mathcal{R}}'_\psi := \xi\mathcal{O}_K$$

We need a diferent version of theorem 2.2.3 for an element of $H^2_{\mathcal{M}}(E^+,k+2)$ and for a generator η^+ of $H^1(E^+\times_{\mathbb{Q}}\mathbb{C},\mathbb{Z}(2\pi i)^{k+1})^+$. This is done by Deninger in §4[10] for rational coeficients. Let $\gamma_+\in H_1(E_{\mathbb{R}},\mathbb{Z})$ be the homology class of the real cycle $E^+(\mathbb{R})^0$(where ^{0} means the connected component) and let Ω^+ be its period.

Define v, the element of $\mathcal{O}_K$ in the formulation of theorem 2.1.9, by:

$$\Omega^+ = v\Omega.$$

Since we have the relation $\overline{\Omega}=u\Omega$, for some $u\in\mu(K)$, v satisfies $v=\overline{v}u$. Define

$$\eta' := \overline{v}\sqrt{d_K}^{\epsilon_k}\eta,$$

where ϵ_k takes the value 1 if k is odd and 0 otherwise. We have that η' is a generator of

$$H^1(E^+\times_{\mathbb{Q}}\mathbb{C},\mathbb{Q}(k+1))^+ = H^1(E^+_{\mathbb{R}},\mathbb{Q}(k+1)),$$

(see p.153 in [10]) as it is fixed by the complex conjugation $+$ on $H^1(E(\mathbb{C}),\mathbb{Q}(k+1))$.

Since η has $\mathbb{Z}$-coefficients and $v\sqrt{d_K}^{\epsilon_k}\in\mathcal{O}_K$, we have that

$$\eta'\in H^1(E^+\times_{\mathbb{Q}}\mathbb{C},\mathbb{Z}(k+1))^+.$$

To obtain an integer statement, write m for the biggest natural number such that $\overline{v}\sqrt{d_K}^{\epsilon_k}/m\in\mathcal{O}_K$. Put $\overline{v}' := \overline{v}\sqrt{d_K}^{\epsilon_k}/m\in\mathcal{O}_K$. I do not compute the exact value of m because, under the hypotheses of theorem 2.1.9 (p prime to the discriminant), we do not need to control this value, but let me write the following result that clarifies a little the value m in some particular situations.

Lemma 2.2.4. *The value $v \in \mathcal{O}_K$ divides 2 or 3 or d_K in $\mathcal{O}_K$.*

Proof. We know that $\Omega = \overline{\Omega} u$ with $u \in \mu(K)$ verifying that $N_{K/\mathbb{Q}}(u) = 1$, so u is a unit in $\mathcal{O}_K$. We have moreover the equality $\overline{v} = uv$. Let us consider the different situations.

If $u = 1$ then $v = 1$ by minimality. If $u = -1$ then v is purely imaginary and then $v = \sqrt{d_K}$ by minimality. If $u = \pm i$, then $K = \mathbb{Q}(i)$, and we obtain that $N_{K/\mathbb{Q}} v = 2$ by minimality, and then $v|2$ in $\mathcal{O}_K$. Similarly, if $u^3 = 1$, we consider then $K = \mathbb{Q}(\sqrt{-3})$ and in this situation by minimality we obtain $v|3$. □

Observe that with the above lemma 2.2.4 the hypothesis $(p, N_{K/\mathbb{Q}} v) = 1$ in theorem A or theorem 2.1.9 can be removed.

Define

$$\eta^+ := \overline{v}' \eta.$$

This element is a base for $H^1(E^+ \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(k+1))^+$, a $\mathbb{Z}$-module of rank one.

Theorem 2.2.5 (Deninger,[10]§4). *With the above notation, we have that*

$$r_{\mathcal{D}}(\xi \overline{v}') = L_S^*(\overline{\psi}, -k) \eta^+$$

and $\xi \overline{v}' \in H^2_{\mathcal{M}}(E^+, k+2)$.

Proof. The last formulae in §4[10] claims that there exists a $\xi' \in H^2_{\mathcal{M}}(E^+, k+2)$ satisfying

$$r_{\mathcal{D}}(\xi') = (-1)^{k-1} \frac{2^{k-1}}{(2k+1)!} \frac{\Phi(\mathfrak{m})}{\Phi(\mathfrak{f})} N_{K/\mathbb{Q}}(v\mathfrak{f})^k \psi(f) L_p(\overline{\psi}, -k) L_S^*(\overline{\psi}, -k) \eta',$$

where $\Phi(\mathfrak{a}) := |(\mathcal{O}_K/\mathfrak{a})^*|$, because $e = \tilde{e}$ in our situation (we have only one idempotent) and the construction of our divisor β is made for $a \not\equiv b \; mod |\mathcal{O}_k^*|$ (p.142 in [10]), with $a = 1$ and $b = 0$. Moreover, the element β is constructed for $\mathfrak{m} = \mathfrak{f}$ (see the proof of theorem 11.3.2. in [9]). The explicit value of ξ' is given by

$$\xi' = |v|^{2k} \overline{v} \sqrt{d_K}^{\epsilon_k} \mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}(\beta)$$

and it is seen in §4[10] that this is defined in $H^2_{\mathcal{M}}(E^+, k+2)$. Take $\tilde{u} := v$ in the definition of ξ; we then obtain

$$r_{\mathcal{D}}(\overline{v} \sqrt{d_K}^{\epsilon_k} \xi) = L_S^*(\overline{\psi}, -k) \eta'$$

and hence the result. □

After this result, one sees that the subspace $\tilde{\mathcal{R}_\psi}' \subset H^2_{\mathcal{M}}(E, k+2)$ is not at all good for our purposes. We modify it to the subspace $\tilde{\mathcal{R}_\psi} := \xi\overline{v}'\mathcal{O}_K \subset \tilde{\mathcal{R}_\psi}'$. Observe that, under our hypothesis for the primes p in theorem 2.1.9, we have that

$$det_{\mathcal{O}_p}(\tilde{\mathcal{R}_\psi}') = det_{\mathcal{O}_p}(\tilde{\mathcal{R}_\psi}),$$

and then we can use the subspace $\tilde{\mathcal{R}_\psi}$ instead of $\tilde{\mathcal{R}_\psi}'$ obtaining the same result that in the second part of theorem 2.2.1.

Definition 2.2.6. *By using the norm map in $H^2_{\mathcal{M}}(E, k+2) \to H^2_{\mathcal{M}}(E^+, k+2)$, we define*

$$\mathcal{R}_\psi := Norm(\tilde{\mathcal{R}_\psi}) = \xi\overline{v}'\mathbb{Z}.$$

Now the first claim of theorem 2.1.9 follows.

Corollary 2.2.7. *With the above notation*

$$r_{\mathcal{D}}(det_{\mathbb{Z}}(\mathcal{R}_\psi)) = L^*_S(E^+/\mathbb{Q}, -k)det_{\mathbb{Z}}(H^1(E^+(\mathbb{C}), (2\pi i)^{k+1}\mathbb{Z}))^+).$$

2.3 Galois descent for the Soulé regulator

Consider now the following Soulé regulator maps:

$$r_{p,K} : K_{2n-2}(E)^{(n)} \otimes \mathbb{Q}_p \to H^1(G_K, H^1(E \times_K \overline{K}, \mathbb{Q}_p(n))),$$

$$r_{p,\mathbb{Q}} : K_{2n-2}(E^+)^{(n)} \otimes \mathbb{Q}_p \to H^1(G_{\mathbb{Q}}, H^1(E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_p(n))),$$

with $n = k+2 \geq 2$ and where G_F denotes $Gal(\overline{F}/F)$. Put $G = Gal(K/\mathbb{Q})$. G acts on the two groups that $r_{p,K}$ relates. We are going to check that this action commutes with the Soulé regulator and then relate the two regulator maps via the norm maps.

We describe first the construction of the higher regulator maps or Soulé regulator maps. First, recall the construction done by Friedlander of Chern class maps $ch_{n,2}$ between K-theory and étale cohomology with finite coefficients:

$$ch_{n,2} : K_{2n-2}(E, \mathbb{Z}/p^m) \to H^2(E_{et}, \mathbb{Z}/p^m(n)) \quad (2.1)$$

which coincides taking projective limit in m with the Chern class map (1.35) in Chapter 1.

Proposition 2.3.1 (Friedlander, prop. 3.10 [23]). *The chern class map $ch_{n,2}$ has Galois descent.*

In our situation, this means that, for all $\sigma \in G = Gal(K/\mathbb{Q})$, we have that

$$\sigma^{-1} ch_{n,2} \sigma = ch_{n,2},$$

where the group G acts on $E = E^+ \times_{\mathbb{Q}} K$ and induces a natural action on $H^2_{et}(E, \mathbb{Z}/p^m(n))$ and on $K_{2n-2}(E, \mathbb{Z}/p^m(n))$.

To study the étale cohomology, consider the HS-spectral sequence,

$$E_2^{l,q} = H^l(G_K, H^q(\overline{E}, \mathbb{Z}/p^m(n))) \Rightarrow H^{l+q}(E, \mathbb{Z}/p^m(n)),$$

where $\overline{E}$ is $E \times_K \overline{K}$ or $E^+ \times_{\mathbb{Q}} \overline{\mathbb{Q}}$, because we have fixed the embedding of K in $\overline{\mathbb{Q}}$, (§3).

For weight reasons ($2 \neq 2n$), we have a map

$$\phi_{K,m} : H^2_{et}(E, \mathbb{Z}/p^m(n)) \to H^1(G_K, H^1(\overline{E}, \mathbb{Z}/p^m(n)))$$

coming from the spectral sequence.

Lemma 2.3.2. *The map $\phi_{K,m}$ is $Gal(K/\mathbb{Q})$-equivariant, i.e. $\sigma^{-1}\phi_{K,m}\sigma = \phi_{K,m}$ for all $\sigma \in Gal(K/\mathbb{Q})$.*

Proof. The map $\phi_{K,m}$ can be defined from the HS-spectral sequence, constructed as follows (we rewrite Grothendieck's construction). Take a resolution of sheaves of $\mathbb{Z}/p^m(n)$, say $I^\bullet$, on the étale cohomology of E^+. Consider G_K as the projective limit of $H_i = Gal(K_i/K)$ where K_i is a finite extension of K. Put $H_i^f = Gal(K_i/\mathbb{Q})$. Now, take, for every $I^l(E^+ \times_{\mathbb{Q}} K_i = E \times_K K_i)$, an H_i^f-resolution. We take H_i invariants of the constructed bicomplex to obtain another bicomplex. In this last bicomplex, two filtrations that give the spectral sequence are defined, and since the maps on the bicomplex commute with the action of G , we have the same property for the filtrations, and hence for the spectral sequence

$$H^p(H_i, H^q(E \times_K K_i, \mathbb{Z}/p^m(n))) \Rightarrow H^{p+q}(E, \mathbb{Z}/p^m(n)).$$

Taking the direct limit over these spectral sequences, and since we can construct the bicomplex with G-compatibility of the groups H_i, we have that in the spectral sequence

$$E_2^{l,q} \Rightarrow H^{l+q}(E, \mathbb{Z}/p^m(n))$$

the maps are G-equivariant. Since the map $\phi_{K,m}$ comes from this spectral sequence, we obtain the result. □

The regulator map with finite coefficients

$$r_{p,K,m} : K_{2n-2}(E, \mathbb{Z}/p^m(n)) \to H^1(K, H^1(\overline{E}, \mathbb{Z}/p^m(n)))$$

coincides, by composing with the Chern class map, with $\phi_{K,m}$ (see remark 1.2.12). The same result holds also for $r_{p,\mathbb{Q},m}$. We have hence that $r_{p,K,m}$ is G-equivariant.

Lemma 2.3.3. *There is a commutative diagram*

$$\begin{array}{ccc} K_{2n-2}(E, \mathbb{Z}/p^m(n)) & \overset{r_{p,K,m}}{\to} & H^1(K, H^1(\overline{E}, \mathbb{Z}/p^m(n))) \\ \downarrow & & \downarrow \\ K_{2n-2}(E^+, \mathbb{Z}/p^m(n)) & \underset{r_{p,\mathbb{Q},m}}{\to} & H^1(\mathbb{Q}, H^1(\overline{E}, \mathbb{Z}/p^m(n))) \end{array}$$

where the vertical maps correspond to the norm maps.

Proof. First we note that our K-theory groups have G-descent (see [9](8.10)). Since $H^i(\overline{E}, \mathbb{Z}/p^m(n))$ is finite, we have the HS-spectral sequence, which implies, for weight reasons, that the Galois group $H^1(K, H^1(\overline{E}, \mathbb{Z}/p^m(n)))$ has G-descent, i.e. $H^0(G, H^1(K, H^1(\overline{E}, \mathbb{Z}/p^m(n)))) = H^1(\mathbb{Q}, H^1(\overline{E}, \mathbb{Z}/p^m(n)))$. Since the action on G commutes with $r_{p,K,m}$, we obtain the result from the fact that the norm maps are defined by dividing by $\#|G|$ (p prime to $\#|G| = 2$) the sum of the action of all the elements of G. $\square$

The regulator map with $\mathbb{Z}_p$-coeficients is defined by taking the projective limit over m of $r_{p,K,m}$. We obtain then

$$r_{p,K} : H^2_{\mathcal{M}}(E, n) \otimes \mathbb{Z}_p \to H^1(K, H^1(\overline{E}, \mathbb{Z}_p(n))),$$

where the compatibility of the projective limit with our Galois cohomology groups comes from the fact that the $H^0(K, E[p^m](n-1))$ are finite. We tensor with $\mathbb{Q}_p$ to obtain the Soulé regulator map:

$$r_{p,K} : H^2_{\mathcal{M}}(E, n) \otimes \mathbb{Q}_p \to H^1(K, H^1(\overline{E}, \mathbb{Q}_p(n))).$$

As taking projective limits and tensoring commute with norm maps, we obtain the following result.

Lemma 2.3.4. *The following diagram commutes:*

$$\begin{array}{ccc} H^2_{\mathcal{M}}(E, n) \otimes \mathbb{Q}_p & \overset{r_{p,K}}{\to} & H^1(K, H^1(\overline{E}, \mathbb{Q}_p(n))) \\ \downarrow & & \downarrow \\ H^2_{\mathcal{M}}(E^+, n) \otimes \mathbb{Q}_p & \underset{r_{p,\mathbb{Q}}}{\to} & H^1(\mathbb{Q}, H^1(\overline{E}, \mathbb{Q}_p(n))) \end{array}$$

where the vertical maps correspond to the norm maps and, in particular, the right vertical map corresponds to the corestriction map.

We know that the above Galois group coincides with $H^1_p \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ (cf. lemma 1.2.10), and the same proof of lemma 1.2.10 for $\mathbb{Z}_p$-coefficients instead of $\mathbb{Q}_p$ coefficients proves that

$$H^1(K, H^1(\overline{E}, \mathbb{Z}_p(k+2)) \cong H^1_p.$$

Hence we have the following consequence.

Corollary 2.3.5. $r_{p,\mathbb{Q}}(\mathcal{R}_\psi) = r_{p,K}(\tilde{\mathcal{R}_\psi})^{Gal(K/\mathbb{Q})}$

2.4 Relation between determinants

We are going to prove the second part of the main theorem 2.1.9, where p is under the hypotheses of the theorem. We suppose first that r_p is injective. We are going to prove the equality

$$det_{\mathbb{Z}_p}\left((H^1_p := H^1(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)))/r_{p,\mathbb{Q}}(\mathcal{R}_\psi)\right) =$$

$$det_{\mathbb{Z}_p}\left(H^2_p := H^2(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)\right).$$

Observe that

$$H^i_p = H^i(Gal(\mathbb{Q}_S/\mathbb{Q}), T_pE^+(k+1)),$$

for $i = 1, 2$, where $\mathbb{Q}_S$ is the maximal extension of $\mathbb{Q}$ which is unramified outside S. Recall that the set S contains the primes that divide the discriminant of K, so we have that $\mathbb{Q}_S = K_S$.

From theorem 2.2.1, we have a comparison of $\mathcal{O}_p$-determinants that involves the groups

$$H^i(Spec(\mathcal{O}_K[1/S]), T_pE^+(k+1)) = H^i(Gal(K_S/K), T_pE^+(k+1)),$$

for $i = 1, 2$, and $r_{p,K}(\tilde{\mathcal{R}_\psi})$, under the assumption that the Soulé regulator is injective on $\tilde{\mathcal{R}}_\psi$. We relate the groups involved in the $\mathcal{O}_p$-determinant relation with our groups.

Lemma 2.4.1.

$$H^i(Gal(\mathbb{Q}_S/\mathbb{Q}), T_pE^+(k+1)) = H^i(Gal(K_S/K), T_pE(k+1))^{Gal(K/\mathbb{Q})},$$

for all $i \in \mathbb{N}$.

Proof. For $i = 0$ there is nothing to show. Now, for every closed normal subgroup H of the group $Gal(K_S/\mathbb{Q})$, we have the HS-spectral sequence

$$H^l(Gal(K_S/\mathbb{Q})/H, H^q(H, T_pE(k+1))) \Rightarrow H^{l+q}(Gal(K_S/\mathbb{Q}), T_pE(k+1)).$$

The invariants with respect to $Gal(K_S/K)$ of the twisted Tate module are zero for weight reasons. We obtain then the case $i = 1$ by using the spectral sequence for $H = Gal(K_S/K)$.

By using the same spectral sequence, the obstruction for the equality for $i = 2$ comes from $H^1(G, H^1(Gal(K_S/K), T_pE(k+1)))$. Since G is finite and the group $H^1(Gal(K_S/K), T_pE(k+1))$ is a finitely generated $\mathbb{Z}_p$-module, we have that $H^1(G, H^1(Gal(K_S/K), T_pE(k+1)))$ is annihilated by $\#|G| = 2$. And, since $(p, 2) = 1$, this last group is zero. □

We note that the $H^i(Gal(K_S/K), T_pE^+(k+1))$ are $\mathcal{O}_p$-modules with a G-action. Let us remind that the $\mathcal{O}_p$-action on the Tate module is via the complex conjugation; this fact comes from the isomorphism via the Kummer theory between the first étale cohomology group with the Tate module. Also $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$ has a G-action given by $\sigma \otimes 1$ with $\sigma \in G$.

Lemma 2.4.2. *Take* $\alpha_i \in H^i(Gal(K_S/K), T_pE^+(k+1))$, $\delta \in \mathcal{O}_p$, *and* $\sigma \in G$. *Then*

$$\sigma(\delta\alpha_i) = \sigma(\delta)\sigma(\alpha_i).$$

Proof. Denote also by

$$\alpha_i \in Cont(Gal(K_S/\mathbb{Q})^i, Ind_K^{\mathbb{Q}}(E[p^m](k+1)))$$

a representative for the cohomology class α_i. Consider δ (modulo p^m) that acts on $Ind_K^{\mathbb{Q}}E[p^m](k+1)$ by conjugate multiplication on $\mathcal{O}_k/p^m(k+1)$, thinking the induced module as continuous functions from $G_{\mathbb{Q}}$ to $E[p^m](k+1)(\overline{K}) \cong \mathcal{O}_K/p^m(k+1)$. We calculate the action of σ on $\delta\alpha_i$. Take $x \in Gal(K_S/\mathbb{Q})^i$; then

$$\sigma(\delta\alpha_i(x))(y) = \sigma\delta\alpha_i(x)(\sigma^{-1}y) =$$

$$\sigma\delta\sigma^{-1}\sigma(\alpha_i(x)(\sigma^{-1}y)) = (\sigma\delta\sigma^{-1})(\sigma(\alpha_i(x)))(y)$$

for all $y \in G_{\mathbb{Q}}$. Observe also that $\sigma\delta\sigma^{-1} = \overline{\delta}$ modulo p^m: this corresponds to the action of G on $\mathcal{O}_p$ modulo p^m. Then, as we calculate our groups taking projective limits, since $H^{i-1}(Gal(K_S/K), E^+[p^m](k+1))$ are finite for all i, and $\mathcal{O}_p$ is the p-completion of $\mathcal{O}_K$, we obtain the result from the compatibility of the action on the maps of the projective limit. □

Let M be an $\mathcal{O}_p$-module and, moreover, a $\mathbb{Z}_pG$-module such that the G-action satisfies

$$\sigma(rm) = \sigma(r)\sigma(m),$$

for all $\sigma \in G$, $r \in R$ and $m \in M$. Denote by $M^+ = M^G$ the fixed module for the G-action. Write $\mathcal{O}_p = \mathbb{Z}_p[\sqrt{d_K}]$, as $p \neq 2$. We have the following decomposition of M as a $\mathbb{Z}_p$-module:

$$M = \left(\frac{\sigma+1}{2}\right) M \oplus \left(\frac{1-\sigma}{2}\right) M,$$

with σ the non-trivial element of G. It is clear that $M^+ = (\frac{\sigma+1}{2})M$. We write $M^- = (\frac{1-\sigma}{2})M$. The action of $\sqrt{d_K}$ sends $M^+ \to M^-$ bijectively and also $M^- \to M^+$ since $(d_K, p) = 1$.

Lemma 2.4.3. *The following morphism:*

$$\tau : M^+ \otimes_{\mathbb{Z}_p} \mathcal{O}_p \to M,$$

$$m^+ \otimes (a + b\sqrt{d_K}) \mapsto am^+ + b\sqrt{d_K}m^+,$$

is an isomorphism of $\mathcal{O}_p$-modules

Proof. Define a map from M to $M^+ \otimes \mathcal{O}_p$ by

$$m = m^+ + m^- \mapsto m^+ \otimes 1 + \frac{1}{d_K}\sqrt{d_K}m^- \otimes \sqrt{d_K},$$

where $m = m^+ + m^-$ corresponds to the $\mathbb{Z}_p$-decomposition $M = M^+ + M^-$. This last map is $\mathcal{O}_p$-linear, and it defines the inverse of τ. □

In this situation, we have that

$$det_{\mathcal{O}_p} M = (det_{\mathbb{Z}_p}(M^+)) \otimes_{\mathbb{Z}_p} \mathcal{O}_p.$$

Since we have that $det_{\mathcal{O}_p} M = p^j \mathcal{O}_p$ for some integer j, which implies that $det_{\mathbb{Z}_p}(M^+) = p^j\mathbb{Z}_p$, we obtain the equality

$$det_{\mathcal{O}_p} M \cap \mathbb{Q}_p = det_{\mathbb{Z}_p} M^+.$$

Moreover, suppose that we have an exact sequence of $\mathcal{O}_p$-modules

$$0 \to M_1 \to M_2 \to M_2/M_1 \to 0.$$

Then, we have that

$$det_{\mathcal{O}_p}(M_2/M_1) = det_{\mathcal{O}_p}(M_2) det_{\mathcal{O}_p}(M_1)^{-1}$$

by the properties of the determinant. We suppose that M_i has a G-action compatible with the $\mathcal{O}_p$-structure module, as is the case for the above module M; then, since $\mathcal{O}_p$ is flat over $\mathbb{Z}_p$,

$$det_{\mathcal{O}_p}(M_2/M_1) = det_{\mathbb{Z}_p}(M_2^+/M_1^+) \otimes \mathcal{O}_p.$$

We consider the groups from theorem 2.2.1 with the equality:

$$det_{\mathcal{O}_p}(H^1(Gal(K_S/K), T_pE(k+1))/r_p(\tilde{\mathcal{R}_\psi}))$$

$$= det_{\mathcal{O}_p}(H^1(Gal(K_S/K), T_pE(k+1)))/det_{\mathcal{O}_p}(r_{p,K}(\tilde{\mathcal{R}_\psi}))$$
$$= det_{\mathcal{O}_p}(H^2(Gal(K_S/K), T_pE(k+1))).$$

By lemma 2.4.2 all the $\mathcal{O}_p$-modules involved in the previous equation are also G-modules with compatibility with respect to the action of $\mathcal{O}_p$. Note also that $r_{p,K}(\tilde{\mathcal{R}_\psi})$ comes from an $\mathcal{O}_p$-submodule of H_p^1 that it is Galois stable by G.

Corollary 2.4.4. *Under the hypotheses that $r_{p,K}$ is injective on $\tilde{\mathcal{R}}_\psi$, we have that*

$$det_{\mathbb{Z}_p}(H_p^1/r_{p,\mathbb{Q}}(\mathcal{R}_\psi)) = det_{\mathbb{Z}_p}(H_p^2).$$

Proof. The result follows from

$$H_p^i = H^i(Gal(K_S/K), T_pE(k+1))^+,\ i = 1, 2,$$

and

$$r_{p,\mathbb{Q}}(\mathcal{R}_\psi) = r_{p,K}(\tilde{\mathcal{R}_\psi})^+$$

by corollary 2.3.5. □

Remark 2.4.5. *The hypothesis that $r_{p,K}$ is injective on $\tilde{\mathcal{R}}_\psi$ is equivalent to the hypothesis that $r_{p,\mathbb{Q}}$ is injective on $\mathcal{R}_\psi$. This is because, by linearity, this injectivity only depends on*

$$r_{p,K}(\overline{v}\sqrt{d_K}^{\epsilon_k}\xi) = r_{p,\mathbb{Q}}(\overline{v}\sqrt{d_K}^{\epsilon_k}\xi) \neq 0.$$

Moreover, this is equivalent by linearity to $r_{p,K}(\xi) \neq 0$ because the elements are defined on non-torsion groups, since the K-groups and the Galois cohomology groups are tensor by $\mathbb{Q}_p$.

We now consider the situation without the injectivity hypothesis on r_p. The Soulé regulator defined as a map

$$r_{p,\mathbb{Q}} : \mathcal{R}_\psi \to H_p^1 \otimes \mathbb{Q}_p$$

extends to a map to $R\Gamma(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1))[-1] \otimes \mathbb{Q}_p$ because, for weight reasons, $H^0(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1)) = 0$. The second part of theorem 2.1.9 states that the determinant of the complex

$$\mathcal{R}_\psi \to R\Gamma(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1))[-1]$$

is trivial. Using lemma 2.4.3, lemma 2.5.1, the second part of theorem 2.2.1 and the previous results that claim that the $\mathcal{O}_p$-determinant of $R\Gamma(\mathcal{O}_S, T_pE(k+1))[-1]$ comes from the $\mathbb{Z}_p$-determinant of $R\Gamma(Spec(\mathbb{Z}[1/S]), T_pE^+(k+1))[-1]$ by tensoring with $\mathcal{O}_K$, to finish the proof of the theorem 2.1.9 it is enough to show that

$$det_{\mathcal{O}_p}\tilde{\mathcal{R}_\psi} = det_{\mathbb{Z}_p}\mathcal{R}_\psi \otimes_{\mathbb{Z}_p} \mathcal{O}_p.$$

This is clear by definition since $\tilde{\mathcal{R}}_\psi = \mathcal{R}_\psi \otimes \mathcal{O}_K$. Observe that $\tilde{\mathcal{R}}_\psi$ is an $\mathcal{O}_p$-module with a G action compatible with the structure of $\mathcal{O}_p$-module, (see pp.153-155 in [10]), and hence lemma 2.5.3. can be applied.

2.5 On the injectivity of the Soulé regulator

This section is a summary of the known results about the non-vanishing condition for the element that generates $\mathcal{R}_\psi$ through the Soulé regulator.

Proposition 2.5.1 (Kings, 5.25 [36]). *Suppose $H^2(\mathcal{O}_K, T_pE(k+1))$ is a finite group. Then $r_{p,K}$ is injective on $\tilde{\mathcal{R}_\psi}'$.*

The finiteness of the Galois group in the above proposition is enough to prove the injectivity of $r_{p,\mathbb{Q}}$ on $\mathcal{R}_\psi$ or the injectivity of $r_{p,K}$ on $\tilde{\mathcal{R}}_\psi$ or $\tilde{\mathcal{R}}'_\psi$ (see remark 2.4.5), and, moreover, this finiteness is equivalent to the finiteness of the Galois cohomology group $H^2_p = H^2(\mathbb{Z}[1/S], T_pE^+(k+1))$ (lemmas 2.5.1, 2.5.2 and 2.5.3).

We know that the finiteness for H^2_p comes from a general conjecture of Jannsen (see Appendix B).

There are a few situations in which the finiteness of H^2_p can be proved, that we will describe, implying in particular the injectivity of the Soulé regulator on $\mathcal{R}_\psi$ or $\tilde{\mathcal{R}}_\psi$.

Proposition 2.5.2 ([36], thm 1.1.7). *For a fixed p, the group*

$$H^2(\mathcal{O}_K[1/S], T_pE(k+1))$$

is finite for almost all k.

The proof is by checking that the hypotheses of lemma 8b) of [32] are satisfied, and this comes from the works of Rubin ([50], theorem 4.4) and McConnell ([40], theorem 1).

If we want to control its finiteness for a fixed twist, as happens for $k = 0$ in the original paper of Bloch and Kato §7[6], the only situation in which the finiteness of this cohomology group is known is under the regularity condition on the prime p.

Definition 2.5.3. *Let p be a prime of $\mathbb{Q}$ different from $2, 3$ which splits in $\mathcal{O}_K$ as a product of two distint primes $\mathfrak{q}\mathfrak{q}'$, such that E has good reduction in both. Consider*

$$L_\infty(\overline{\psi}^{k+j}, k) := (2\pi/\sqrt{-d_K})^j \Omega^{-(k+j)} L(\overline{\psi}^{k+j}, k),$$

which has p-integral values if $0 \leq j \leq p-1$ and $1 < k \leq p$. We say that p is a regular prime for E when neither $\mathfrak{q}$ nor $\mathfrak{q}'$ divides the numbers $L_\infty(\overline{\psi}^{k+j}, k)$, for $1 \leq j < p-1$ and $1 < k \leq p$.

We note that, when p is regular, $K(E_p)$ admits exactly one extension of degree p unramified outside $\mathfrak{q}$.

We refer to appendix B for more properties on regular primes.

Proposition 2.5.4 ([61] 3.3.2, [62]cor.2). *Let p be a regular prime for E, then*

$$H^2(\mathcal{O}_K[1/S], E[p^\infty](k+1)) = 0.$$

From lemma 1 [32], we then obtain that $H^2(\mathcal{O}_K[1/S], T_pE(k+1))$ is finite.

Then, for regular primes p and all the twists with $k \geq 0$, or for a fixed prime p and almost all twists with $k \geq 0$, the groups H^2_p are finite, proving the local Tamagawa conjecture 2.1.3 in generality. These finiteness imply, in particular, the injectivity of $r_{p,\mathbb{Q}}$ on $\mathcal{R}_\psi$ so, in these situations, there are no conditions on the second part of theorem 2.1.9. Moreover, up to the conjecture of $dim_{\mathbb{Q}} H_{\mathcal{M}} = 1$, the local Tamagawa conjecture 2.1.1 is already proved for a fixed prime p and almost all twist with $k \geq 0$ or for the regular primes p and for all twist $k \geq 0$ because we know the dimension of the free part of H^1_p (remark 2.1.8).

Chapter 3

The local Tamagawa number conjecture on Hecke characters

Introduction

The local Tamagawa number conjecture for an elliptic curve E with CM defined over the field of the endomorphism, proved by Kings [36], corresponds to the local Tamagawa number conjecture for the L-function of certain Hecke character over the imaginary quadratic field K. This is our initial point for the study of the conjecture in the case of Hecke characters over an imaginary quadratic field. We prove the local conjecture for a power of a Grössencharacter in the non-critical values, under some technical restrictions. The basic ingredients used in the proof of King's result are: the specialization of the polylogarithm sheaf and the main conjecture for the Iwasawa theory for imaginary quadratic fields. Both are the key ingredients also for the results of this chapter on the Tamagawa number conjecture for Hecke characters of imaginary quadratic fields.

The main tool in the proof is the exact control of the image of some concrete elements in K-theory under the Soulé and Deligne regulator maps. The work for the Deligne regulator map for our element in K-theory is made by Deninger [10]. The control of the coimage for the Soulé regulator map for, a priori other concrete elements in K-theory coming from motives of Hecke characters is made by Geisser [24]. He constructs a submodule in K-theory such that the length of the coimage in the local Galois cohomology group coincide with the value of the p-adic L-function, in the non-critical values.

We do not know more works in the non-critical situation related to our questions, but a lot of works for the critical situation for the L-function of Hecke characters. We mention only one written in the language of the

Tamagawa number conjecture. If we suppose that the weight of the motive is ≤ -3 there is a result of Li Guo [27] that uses the realizations of the motive constructed by Deninger to check the conjecture in the classical formulation of Bloch-Kato [6]. The key point to check the conjecture is the interpolation formula between the L-function and the p-adic L-function in the critical situation (see 4.16(50)[14]).

Let us finally give a rough sketch of the contents of this chapter. The first section §1, recalls the formulation of the local Tamagawa number conjecture for pure motives. Moreover is presented the modified conjecture for the Künneth product of pure motives for allow us to restrict in the set of constructible elements in the K-theory group. The proof of it for a particular case is the aim of the paper. In section §2 we define the motives that their L-function correspond the L-function of Hecke characters. There we will concentrate with a set of Hecke characters coming from elliptic curves defined over an imaginary quadratic field K. We will proof a Deuring style result for these motives. In §3 we define the constructible elements of Deninger [10] for the motives previously defined. Moreover, we reformulate in the notation of the local Tamagawa number conjecture the Beilinson conjecture for these motives. Next sections study the image of the constructible elements for our motive via the Soulé regulator map. In §4 we define a map from elements in Iwasawa modules to the étale cohomology groups that the conjecture relates. Here the main conjecture of the Iwasawa theory for imaginary quadratic fields plays an important role. Section §5 we use the specialization of the elliptic polylogarithm to compare the image of the map defined in §4 with the p-Soulé regulator map. With this results we obtain the main results of the paper, theorems 3.5.13 and 3.5.14.

3.1 The Tamagawa number conjecture

The local Tamagawa number conjecture for pure motives of the form $h_n(X)(m)$ with X a smooth proper scheme over a number field F, and n, m two integers, conjecture 1.2.23, was explicitly introduced in chapter 2 §2.1 Let follow the notation in §2.1.

Let X/F be a smooth proper variety defined over a number field F with ring of integers $\mathcal{O}_F$. Fix integers $m \geq 0$ and r such that $m - 2r \leq -3$ and $r > \inf(m, dim(X))$. Let p be an odd prime number and S' be a set of finite primes of F containing the primes lying over p and the ones where X has bad reduction. Write $\mathcal{O}_{S'} = \mathcal{O}_F[1/S']$.

Conjecture 3.1.1. *(cf. §4[34], conj. 2.2.7 in [33], conjecture 1.2.23) Let $V_p^* = Hom(V_p, \mathbb{Q}_p)$ be the dual Galois module. Let $p \neq 2$, r, m and S' be as* *above. Assume that*

$$P_{\mathfrak{p}}(V_p^*(1), 1) \neq 0$$

for all $\mathfrak{p} \in S'$ *(we follow the notation of chapter 1) and that* $L_{S'}(V_p^*(1), s)$ *has an analytic continuation to all* $\mathbb{C}$*, then:*

1. *The maps* $r_{\mathcal{D}}$ *and* r_p *are isomorphisms and* H_p^2 *is finite.*

2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0} L_{S'}(V_p^*(1), s)$*; write this number* l*.*

3. *Let* $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z}})$ *be a* $\mathbb{Z}$*-basis. There is an element* $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M}})$ *such that*

   $$r_{\mathcal{D}}(\xi) = (\lim_{s\to 0} s^{-l} L_{S'}(V_p^*(1), s))\eta.$$

   This is the "Beilinson conjecture".

4. *Consider* $r_p(\xi) \in det_{\mathbb{Q}_p}(H_p^1 \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$*. Then* $r_p(\xi)$ *is a basis of the* $\mathbb{Z}_p$*-lattice*

   $$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_{S'}, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_{S'}, V_p)[-1]).$$

   i.e.

   $$[det_{\mathbb{Z}_p}(H_p^1) : r_p(\xi)\mathbb{Z}_p] = \#(H_p^2) = det_{\mathbb{Z}_p}(H_p^2).$$

As our knowledge of K-theory is limited, we take a weak version of the conjecture.

Conjecture 3.1.2. *(cf. conj. (2.1.3)) There is a subspace* $H_{\mathcal{M}}^{constr}$ *in* $H_{\mathcal{M}}$ *such that:*

1. $r_{\mathcal{D}}$ *and* r_p *restricted to* $H_{\mathcal{M}}^{constr}$ *are isomorphisms and* H_p^2 *is finite.*

2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z}}) = ord_{s=0} L_{S'}(V_p^*(1), s)$*; write this number* l*.*

3. *There is an element* $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M}}^{constr})$ *such that*

   $$r_{\mathcal{D}}(\xi) = (\lim_{s\to 0} s^{-l} L_{S'}(V_p^*(1), s))\eta.$$

4. *The element* $r_p(\xi)$ *is a basis of the* $\mathbb{Z}_p$*-lattice*

   $$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_{S'}, T_p))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_{S'}, V_p)[-1]).$$

We want to write a similar conjecture as the conjecture 3.1.2 in the case of the Künneth product of a pure motive , or more precisely some part of it, because we will be interested in the case of Hecke characters of weight not necessarily 1 (with weight 1 is the case developed in [36]). We refer to the next section or much deeply to [10] for the precise definition of a motive associated to a Hecke character over an imaginary quadratic field. This is made in chapter 1 §1.2.3

Consider X/F as above and the fixed integers m, r and an odd prime number p. We want to formulate a conjecture for the pure motive

$$e((\otimes^w h_n(X))(r))$$

where w, n, r are integers with $wn = m$, and we impose on their, the above conditions in m and r. Here e means an idempotent which gives a decomposition in the Künneth product $(\otimes^w h_n(X))(r)$ such that this decomposition is preserved via the regulators maps.

Observe that

$$e((\otimes^w h_n(X))(r)) \subseteq h_m(X^w)(r),$$

then take X^w as our variety X in the above conjecture and we use the previous notation for the pure motive $h_m(X^w)(r)$. We write

$$V_{p,w,r} \subset V_p$$

for the $Gal(\overline{F}/F)$-submodule that comes from the p-adic realization with $\mathbb{Q}_p$-coefficients, $e((\otimes^i h_n(X)(l))(j))$. We now that on V_p acts non-trivially the inertia group in a place v if and only if is in the set of places where X^w has bad reduction, which are the same places where X has bad reduction; these inertia groups could act trivially in $V_{p,w,r}$, then we define S be the set of finite places that contains the places over p and the places which the inertia group acts non-trivially in $V_{p,w,r}$. We denote by $\mathcal{O}_S := \mathcal{O}_F[1/S]$. Let $T_{p,w,r}$ be the $\mathbb{Z}_p$-lattice in $V_{p,w,r}$ which comes from the $\mathbb{Z}_p$-realization for the motive, i.e. taking the étale cohomology a coefficients in $\mathbb{Z}_p$ for our motives. In a similar way we can define on this $\mathbb{Q}_p$-realization the local Euler factors for $V_{p,w,r}$. Let be $eH_{h,\mathbb{Z},w,r} \subset H_{h,\mathbb{Z}}$ the $\mathbb{Z}$-free subgroup that comes from our pure motive from the Künneth product via his realization. Denote by

$$M_{e,X,w,r} := e((\otimes^w h_n(X))(r)).$$

Let be $H_{\mathcal{M},w,r} := (K_{2r-m-1}(M_{e,X,w,r}) \otimes \mathbb{Q})^{(r)}$ and $H^i_{p,\otimes^w} = H^i(\mathcal{O}_S, j_* T_{p,w,r})$.

Conjecture 3.1.3. *(cf. conj. 1.2.23) Let $p \neq 2$, r, m and S be as above. Assume that*

$$P_{\mathfrak{p}}(V^*_{p,w,r}(1), 1) \neq 0$$

for all $\mathfrak{p} \in S$ and that $L_S(V^*_{p,w,r}(1), s)$ has an analytic continuation to all $\mathbb{C}$, then:

1. *The maps $r_{\mathcal{D}}$ and r_p are isomorphisms and $H^2_{p,\otimes^w}$ is finite.*

2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z},w,r}) = ord_{s=0} L_S(V^*_{p,w,r}(1), s)$*; write this number* n.

3. *Let $\eta \in det_{\mathbb{Z}}(H_{h,\mathbb{Z},w,r})$ be a $\mathbb{Z}$-basis. There is an element $\xi \in det_{\mathbb{Q}}(H_{\mathcal{M},w,r})$ such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s\to 0} s^{-n} L_S(V^*_{p,w,r}(1), s))\eta.$$
This is the "Beilinson conjecture".

4. *Consider $r_p(\xi) \in det_{\mathbb{Q}_p}(H^1_{p,\otimes^w} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$. Then $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_{p,w,r}))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_{p,w,r})[-1]).$$
i.e.
$$[det_{\mathbb{Z}_p}(H^1_{p,\otimes^w}) : r_p(\xi)\mathbb{Z}_p] = \#(H^2_{p,\otimes^w}) = det_{\mathbb{Z}_p}(H^2_{p,\otimes^w}).$$

We make a weak conjecture,

Conjecture 3.1.4. *There is a subspace $H^{constr}_{\mathcal{M},w,r}$ in $H_{\mathcal{M},w,r}$ such that:*

1. *$r_{\mathcal{D}}$ and r_p restricted to $H^{constr}_{\mathcal{M},w,r}$ are isomorphisms and $H^2_{p,\otimes^w}$ is finite.*

2. $dim_{\mathbb{Q}}(H_{h,\mathbb{Z},w,r}) = ord_{s=0} L_S(V^*_{p,w,r}(1), s)$*; write this number* n.

3. *There is an element $\xi \in det_{\mathbb{Q}}(H^{constr}_{\mathcal{M},w,r})$ such that*
$$r_{\mathcal{D}}(\xi) = (\lim_{s\to 0} s^{-n} L_S(V^*_{p,w,r}(1), s))\eta.$$

4. *The element $r_p(\xi)$ is a basis of the $\mathbb{Z}_p$-lattice*
$$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_S, T_{p,w,r}))^{-1} \subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_S, V_{p,w,r})[-1]).$$

3.2 The motive associated to Hecke characters

Let K be an imaginary quadratic field and $\mathcal{O}_K$ be its ring of integers. Let E be an elliptic curve over K with CM by $\mathcal{O}_K$. In this section we describe some pure motives coming from a Künneth product of the motive $h_1(E)$ and their realizations, and we prove that the L-functions associated to these motives correspond to Hecke characters. We obtain finally an analog for these motives of a result of Deuring for CM elliptic curves.

Let p be an odd prime, fixed once and for all, such that E has good reduction for all primes over p. Let S' be the set of places that divide the conductor of the elliptic curve $\mathfrak{f}$ (that are the same places where E has bad reduction) and the places that divide p.

Consider the category of Chow motives $\mathcal{M}_{\mathbb{Q}}(K)$ over K with morphisms induced by graded correspondences in Chow theory tensored with $\mathbb{Q}$. We have then a natural covariant functor h from the category of smooth and projective varieties to $\mathcal{M}_{\mathbb{Q}}(K)$. Then, the motive of an elliptic curve E over K has a canonical decomposition $h(E) = h_0E \oplus h_1E \oplus h_2E$ with respect to the zero section, where $h_0E = h(Spec(K))$ and $h_2E = h(Spec(K))(-1)$.

If E is an CM elliptic curve as before, the motive h_1E motive has multiplication by K, [10] 1.3. Consider then the motive $\otimes^w h_1E$, for w a positive integer, which has multiplication by $T_w := \otimes^w_{\mathbb{Q}} K$. If we denote by $\Upsilon = Hom(K, \mathbb{C})$, observe that T_w is equal to $\prod_\theta T_\theta$, where θ runs through the $Aut(\mathbb{C})$-orbits of $\Upsilon^w = Hom(T_w, \mathbb{C})$. Let e_θ be the idempotent corresponding the field component T_θ of T_w, and consider the motive

$$e_\theta(\otimes^w h_1(E)).$$

This construction can be done in fact without tensoring by $\mathbb{Q}$ (by taking in the construction of $\mathcal{M}_{\mathbb{Q}}(K)$ the Chow group without tensoring by $\mathbb{Q}$): we have then the category $\mathcal{M}(K)$, the motive h_1E and so on. To distinguish between integral and rational Chow motives, we will write a subscript $\mathbb{Q}$ in the second case.

Then, on the integral motive $\otimes^w h_1E$ there is a natural multiplication by $\mathcal{O}_w := \otimes^w_{\mathbb{Z}} \mathcal{O}_K$. Considering the idempotent e_θ as before, we get that the integral motive $e_\theta(\otimes^w h_1(E))$ has multiplication by $\mathcal{O}_\theta := e_\theta \mathcal{O}_w$, a ring in T_θ.

Deninger was the first who defined these motives and proved the Beilinson conjecture for them by constructing some elements in K-theory (see [10]) and computing its image by the Deligne regulator map. Our task in the rest of the chapter will be to compute the image of the Soulé regulator for the elements constructed in K-theory by Deninger, and check that the corank is equal to the number of elements of the second group of cohomology $H^2_{p,\otimes^w}$ that comes from the p-adic realization of these motives.

Remark 3.2.1. *In fact, these motives were introduced by Deninger in [10] in a more general setting. He defines motives coming from the Künneth product of $h_1(E)$ without the assumption that E is defined over K. In general, E is defined over a finite extension F of K, with the condition, that we also have, that the field of definition of the torsion points for the elliptic curve over F is an abelian extension of F. In this setting, he constructs a motive for every Hecke character on $\mathbb{A}_K$, such that its L-function corresponds to the one of* *the Hecke character. Under our restrictions, we will work only with a subset of the Hecke characters on $\mathbb{A}_K$.*

Let $\psi : I_K \to K^*$ be the grossencharacter associated to the elliptic curve E. Define the CM character

$$\psi_\theta : I_K \to T_\theta^*$$

by $\psi_\theta = e_\theta \cdot (\otimes^w \psi)$, and denote by $\mathfrak{f}_\theta$ the conductor of ψ_θ. Observe $\mathfrak{f}_\theta | \mathfrak{f}$ since ψ_θ is a sub-character of $\otimes^w \psi$.

The infinity type of this character is obtained as follows: If we fix an embedding $K \to \overline{\mathbb{Q}}$ once and for all, we have then a natural embedding

$$K \to \otimes^w K \to T_w \to T_\theta$$

where the first map corresponds to the diagonal map. For any $\vartheta \in \theta_K := \theta \cap Hom_K(T_\theta, \mathbb{C})$, $\vartheta = (\lambda_1, \ldots, \lambda_w) \in \Upsilon^w$, we set $a_\vartheta = \#\{i | \lambda_i \in Hom_K(K, \mathbb{C})\}$ and $b_\vartheta = w - a_\vartheta$. These numbers do not depend on the element ϑ in θ_K, and they determine the infinity type for the Hecke character ψ_θ (cf. last paragraph 1.3 [10]). We will denote the type of ψ_θ as $(a_\theta, b_\theta) := (a_\vartheta, b_\vartheta)$ where ϑ is any element in θ_K.

Let's denote by

$$M_\theta := e_\theta(\otimes^w h_1(E)),$$

considered as an integral Chow motive, that is an element in the category of $\mathcal{M}(K)$, and $M_{\theta\mathbb{Q}}$ its image in $\mathcal{M}_\mathbb{Q}(K)$. Our objective in this section is to study the p-adic and Betti realizations of this motive twisted by w, and to determine its L function in terms of the Hecke character ψ_θ.

The p-adic realization of the motive $M_{\theta\mathbb{Q}}(w)$ is, by definition, $H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p(w))$. We need to choose a lattice on it.

Lemma 3.2.2. *The integral p-adic realization of $M_\theta(w)$, $H^w_{et}(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w))$, is isomorphic to*

$$e_\theta(\otimes^w T_p E)$$

as free $e_\theta(\otimes^w \mathcal{O}_K)$-modules of rank 1 with G_K-action, with the G_K-action on $e_\theta(\otimes^w T_p E)$ given by $\overline{\psi}_\theta \otimes \mathbb{Z}_p$.

Proof. Observe first that T_pE is isomorphic to $H^1_{et}(h_1(E) \times_K \overline{K}, \mathbb{Z}_p(1))$ by Kummer theory, but with G_K-action given by the complex conjugation by Artin reciprocity.

The claim that $e_\theta(\otimes^w T_p E)$ is a free module of rank 1 follows because T_pE is a free $\mathcal{O}_K$-module of rank 1 and then $e_\theta \cdot (T_pE \otimes \ldots \otimes T_pE)$ is a free $e_\theta \cdot (\mathcal{O}_K \otimes \ldots \otimes \mathcal{O}_K)$-module of rank one.

Now, consider the natural action of G_K on $H^1_{et}(h_1(E) \times_K \overline{K}, \mathbb{Z}_p(1)) = Hom(T_pE, \mathbb{Z}_p(1))$. Since G_K acts on the Tate module by $\psi : G_K \to (\mathcal{O}_K \otimes \mathbb{Z}_p)^*$, then it acts on our G_K-module via the complex conjugate. Using that

$$H^w((h_1(E) \times_K \overline{K})^w, \mathbb{Z}_p(w)) = H^1(h_1(E) \times_K \overline{K}, \mathbb{Z}_p(1))^{\otimes w}$$

by prop. 2.4.3 a) in [24], and taking our idempotent, we obtain the result. □

We can apply this result to the motive $M_\theta(w+l+1)$ any integer $l+1$. We get therefore a $\mathbb{Z}_p$-lattice in the p-adic realization for the motive $M_{\theta,\mathbb{Q}}(w+l+1)$.

We are going now to define a submodule in $H^w_{sing}(E^w \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(2\pi i)^{w+l})^+$ that their elements are a $\mathbb{Z}$-lattice in the Betti realization a $\mathbb{Q}$-coefficients for $M_\theta(w+l+1)$.

Observe first $H^1_B(E(\mathbb{C}), \mathbb{Z}(1)) \cong H^1_B(E \times_{\mathbb{Q}} \mathbb{C}, \mathbb{Z}(1))^+$ is an $\mathcal{O}_K$-module of rank 1, and hence

$$\otimes^w_{\mathbb{Z}} H^1_B(E(\mathbb{C}), \mathbb{Z}(1))$$

is a $\otimes^w \mathcal{O}_K$-module of rank 1. By considering now the idempotent e_θ and taking

$$e_\theta(\otimes^w H^1_B(E(\mathbb{C}), \mathbb{Z}(1)))(l)$$

the corresponding submodule, we get a $\mathcal{O}_\theta := e_\theta(\otimes^w \mathcal{O}_K)$-module of rank 1. This module corresponds to the $\mathbb{Z}$-module $H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l))$, the searched $\mathbb{Z}$-lattice of $H^w(M_{\theta\mathbb{C}}, \mathbb{Q}(w+l))$ for the Betti realization for our motive.

Observe that the K-theory group corresponding to our motive $M_\theta(w+l+1)$ comes from

$$H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1)) = (K_{2(w+l)-w+1}(M_\theta)^{(w+l+1)} \otimes \mathbb{Q}).$$

The main problem is to construct an element inside this group and to control its image for the Beilinson regulator map and its image for the p-Soulé regulator map. We refer to the next section for the definition of this element, and the rest of the chapter for its study.

We observe here that the Soulé and Deligne regulator maps map respectively this K-theory group to the group $H^1_{p,\otimes^w} \otimes \mathbb{Q}_p$ corresponding to the p-adic realization $H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p(w+l+1))$ or to tensoring with $\mathbb{R}$, the $\mathbb{R}$-module for the Betti cohomology group $H^w(M_{\theta\mathbb{C}}, \mathbb{Q}(w+l)) \otimes \mathbb{R}$, see [10] for the Deligne regulator and [24] in the case of the Soulé regulator.

Remark 3.2.3. *Given a motive over $\mathbb{Q}$, we have the realizations given by the Betti cohomology and the de Rham cohomology; both are $\mathbb{Q}$-vector spaces. We obtain that choosing some particular lattices (to give arithmetic information)* *on these two vector spaces, and using some geometric conjectures, they should satisfy the condition of the motivic pair (see the precise definition in 5.5 [6]). I believe that the lattices defined above are the naturals to take for our motive. This affirmation is checked out in our case,i.e. that our motives with our lattices define a motivic pair, by Li Guo [27] in the particular case that E is defined over $\mathbb{Q}$ and the infinity type is $(\lambda, \ldots, \lambda)$, where λ is the fixed immersion of K in $\mathbb{C}$.*

Now, we are going to study the L-function that corresponds to the p-adic rational representation of $M_{\theta\mathbb{Q}}$, $H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p)$. Let be S the places of K that divide $\mathfrak{f}_\theta$ and the places that divide p. Define as usual

$$L_S(M_\theta, s) := \prod_{\mathfrak{l} \notin S} det_{\mathbb{Q}_p}(1 - Frob_{\mathfrak{l}} N\mathfrak{l}^{-s} | (H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p))^{I_{\mathfrak{l}}})^{-1}$$

where $Frob_{\mathfrak{l}}$ means the geometric Frobenius.

Our goal is to compute this determinant and to relate it with the local factors of the L-function of the Hecke character ψ_θ that is defined by,

$$L_S(\psi_\theta, s) := \prod_{\mathfrak{l} \notin S} (1 - \frac{\psi_\theta(\mathfrak{l})}{N\mathfrak{l}^s})^{-1}.$$

Recall that the operation of the decomposition group $D_{\mathfrak{p}}$ for $\mathfrak{p} \nmid p$ on $H^1_{et}(h_1(E) \times_K \overline{K}, \mathbb{Q}_p)$ is given by the operation of $\psi^{-1}|_{K^*_{\mathfrak{p}}}$, and hence $D_{\mathfrak{p}}$ operates on $H^w_{et}(M_{\theta\overline{K}}, \mathbb{Q}_p)$ via ψ_θ^{-1}. On one hand, the inertia group $I_{\mathfrak{p}}$ acts non-trivially if and only if $\mathfrak{p}$ divides the conductor $\mathfrak{f}_\theta$. On the other hand, for $\mathfrak{p} \nmid \mathfrak{f}_\theta$, the geometric Frobenius $Fr_{\mathfrak{p}}$ at $\mathfrak{p}$ acts via $\psi_\theta(\mathfrak{p})$. We obtain hence the following result.

Lemma 3.2.4 (Deninger,§1.3.1[10]). *Let $\mathfrak{l}$ a finite prime of K not over p, then*

$$det_{T_\theta \otimes \mathbb{Q}_p}(1 - Fr_{\mathfrak{l}} N\mathfrak{l}^{-s} | H^w_{et}(M_\theta, \mathbb{Q}_p)^{I_{\mathfrak{l}}}) = (1 - \psi_\theta(\mathfrak{l}) N\mathfrak{l}^{-s})$$

if $\mathfrak{l} \nmid \mathfrak{f}_\theta$, where $\mathfrak{f}_\theta$ is the conductor of the Hecke character ψ_θ.

We fix some restrictions for our motive $M_\theta(w + l + 1)$ once and for all. We suppose $-w - 2l \leq -3$ and $\#|\theta| = 2$, in particular we have $T_\theta \cong K$ and $\mathcal{O}_\theta \cong \mathcal{O}_K$. Moreover, we impose that $S = S'$ in all the rest of the paper.

This last condition can be described in terms of the infinity type (see lemma 3.4.17 for type $(w, 0)$ or $(0, w)$). We observe also that this θ corresponds to all Hecke characters of $\mathbb{A}_K \to K^*$ when $cl(K) = 1$ an K an imaginary quadratic field (see proposition 1.3 [10]).

The L-function for M_θ can be described by using lemma 3.2.4 and by taking the norm map.

Lemma 3.2.5. *Let $\mathfrak{l}$ a prime of K such that $\mathfrak{l} \nmid \mathfrak{f}_\theta$ and prime to p and suppose $\#|\theta| = 2$. We have then the following equality*

$$det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{l}} N\mathfrak{l}^{-s} | (H^w(M_\theta, \mathbb{Q}_p))^{I_{\mathfrak{l}}}) = (1 - \psi_\theta(\mathfrak{l}) N\mathfrak{l}^{-s})(1 - \overline{\psi_\theta}(\mathfrak{l}) N\mathfrak{l}^{-s}).$$

As a corollary we obtain a Deuring type result in our situation.

Theorem 3.2.6. *Suppose $\#|\theta| = 2$, and S be the set of the primes on K dividing $\mathfrak{f}_\theta$ and primes dividing p. Then:*

$$L_S(H^w(M_\theta, \mathbb{Q}_p), s) = L_S(\psi_\theta, s) L_S(\overline{\psi}_\theta, s).$$

3.3 The Beilinson conjecture for Hecke characters

In this section we will review briefly the study of the Beilinson conjecture for the motive $M_{\theta\mathbb{Q}}(w + l + 1)$ done by Deninger in [10]. First of all, recall the main theorem in [10].

Theorem 3.3.1 (Deninger, 1.4.1 [10]). *Let $w = a_\theta + b_\theta \geq 1$. Consider an integer l such that*

$$-l \leq Min(a_\theta, b_\theta) \quad if \ a_\theta \neq b_\theta$$

$$-l < a_\theta = b_\theta = w/2 \quad otherwise.$$

Then the L-series $L(\overline{\psi}_\theta, s)$ has a zero of order 1 in $s = -l$.

Moreover, there exist an element ξ_θ in $H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ such that, under the Deligne regulator map

$$r_{\mathcal{D}} : H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w + l + 1)) \to H^w_B(M_\theta, \mathbb{R}(w + l)),$$

has image

$$r_{\mathcal{D}}(\xi_\theta) = lim_{s \to -l} \frac{L(\overline{\psi_\theta}, s)}{s + l} \eta_\theta \ mod \ T^*_\theta$$

in the free rank one $T_\theta \otimes \mathbb{R}$-mod $H^{w+1}_{\mathcal{D}}(M_\theta, \mathbb{R}(w+l+1) = H^w_B(M_{\theta\mathbb{C}}, \mathbb{R}(w+l))$ where η_θ is a T_θ-generator of $H^w_B(M_{\theta\mathbb{C}}, \mathbb{Q}(w + l))$.

Observe that the L-series $L(\overline{\psi}_\theta, s)$ is equal to the L-function for the dual Beilinson motive of M_θ, i.e. the same motive but with action by T_θ given by its complex conjugation.

Let's recall the construction of ξ_θ, following the results of Deninger. We suppose once for all that $l \geq 0$. Remember that S is the set of finite primes $\{\mathfrak{p} \in Spec(\mathcal{O}_K) | \mathfrak{p} | \mathfrak{f}_\theta \textit{ or } \mathfrak{p} | p\}$.

Fix an algebraic differential form $\omega \in H^0(E, \Omega_{E/K})$. Since we have complex multiplication, we can write the period lattice as $\Gamma = \Omega\mathcal{O}_K$, where $\Omega \in \mathbb{C}^*$ is the complex period. Fix an element γ in $H_1(E(\mathbb{C}), \mathbb{Z})$ such that it is an $\mathcal{O}_K$-generator, and satisfies

$$\Omega = \int_\gamma \omega.$$

By Poincaré duality, we have that to γ it corresponds η_γ, an $\mathcal{O}_K$-generator for $H^1(E(\mathbb{C}), \mathbb{Z}(1))$. Consider now the $\mathcal{O}_\theta$-generator

$$\eta_\theta := (2\pi i)^l e_\theta(\eta_\gamma \otimes \ldots \otimes \eta_\gamma)$$

of

$$H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) = e_\theta\left((\otimes H_B^1(E(\mathbb{C}), \mathbb{Z}(1)))(l)\right).$$

To construct ξ_θ, we will define a divisor on the torsion points of the elliptic curve; its image by the composition of the Eisenstein map $\mathcal{E}_\mathcal{M}$ (§8 [9]) with the Deninger projector map $\mathcal{K}_\mathcal{M}$ (2.8 [10]) will defines our ξ_θ.

Remember that $\mathfrak{f}_\theta$ is the conductor of the Hecke character ψ_θ associated with M_θ, and denote by f a generator of $\mathfrak{f}_\theta$ (it exists since the ideal class group of K is equal to 1). We have that

$$\Omega f^{-1} \in \mathfrak{f}_\theta^{-1}\Gamma$$

and that (Ωf^{-1}) gives a divisor in $\mathbb{Z}[E[\mathfrak{f}_\theta \setminus 0]]$ defined over $K(E[\mathfrak{f}_\theta])$. Since $\mathfrak{f}$ is the conductor of ψ and $\mathfrak{f}_\theta | \mathfrak{f}$, the divisor (Ωf^{-1}) is defined also over $K(E[\mathfrak{f}])$. For technical reasons, we will take our divisor as

$$\beta_\theta := N_{K(E[\mathfrak{f}])/K}((\Omega f^{-1})).$$

We obtain, if $a_\theta \not\equiv b_\theta \bmod |\mathcal{O}_K^*|$, the following equality

$$r_\mathcal{D}(\mathcal{K}_\mathcal{M}\mathcal{E}_\mathcal{M}(\beta_\theta)) = (-1)^{l-1}\frac{2^{l-1}N_{K/\mathbb{Q}}\mathfrak{f}_\theta^{w+2l}\psi_\theta(f)}{(2l+w)!N_{K/\mathbb{Q}}(\mathfrak{f}_\theta)^{l+w}}\frac{\Phi(\mathfrak{f})}{\Phi(\mathfrak{f}_\theta)}L'(\overline{\psi_\theta}, -l)\eta_\theta,$$

where $\Phi(\mathfrak{m}) = |(\mathcal{O}_K/\mathfrak{m})^*|$ for any ideal $\mathfrak{m}$ of $\mathcal{O}_K$ (see pp.142 (2.11)[10]).

This is an analog of theorem 1.2.2 in Kings paper [36], for the Beilinson conjecture for $M_\theta(w+l+1)$. Therefore, it suggests the element that we have to take to prove an analog for the p-Tamagawa number conjecture for $M_\theta(w+l+1)$, that is, to control the coimage of the p-Soulé regulator by the number of elements of a second cohomology group.

Remark 3.3.2. *The p-adic realization* $V_{p,w,w+l+1} = H^w_{et}(M_{\theta\mathbb{Q}} \times_K \overline{K}, \mathbb{Q}_p(w + l + 1))$ *of our motive* M_θ *satisfies that the local Euler factors*

$$P_{\mathfrak{p}}(V_p^*(1), 1) = P_{\mathfrak{p}}(\overline{\psi}_\theta, -l)$$

are different from 0 for all $\mathfrak{p} \in S$. *Hence, it satisfies the hypothesis in the conjecture 3.1.4.*

To show this fact, suppose first $\mathfrak{p}|\mathfrak{f}_\theta$. *Then, the inertia group acts non-trivial on* $V_{p,w,w+l+1}$, *which is a one dimensional* $\mathcal{O}_\theta \otimes \mathbb{Q}$*-module, and hence*

$$L_{\mathfrak{p}}(\overline{\psi}_\theta, s) = 1$$

for all $\mathfrak{p}|\mathfrak{f}_\theta$, *and in particular for* $s = -l$.

If $\mathfrak{p}$ *divides* p, *then the result follows from the fact that the abelian varieties with CM satisfy this condition. i.e.*

$$det_{\mathbb{Q}_p}(1 - Fr_{\mathfrak{p}} N\mathfrak{p}^l | H^w_{et}(E^w, \mathbb{Q}_p)) \neq 0,$$

and therefore, Since the different idempotents e_θ *give a decomposition of* $H^w(E^w, \mathbb{Q}_p)$,

$$L_{\mathfrak{p}}(\overline{\psi}_\theta, -l) \neq 0$$

Theorem 3.3.3 (Deninger,§2[10]). *Suppose that* $a_\theta \not\equiv b_\theta \mod |\mathcal{O}_K^*|$ *and and that* a_θ, b_θ, l *satisfy the hypothesis of the theorem 3.3.1 with* $l \geq 0$. *Define, using the previous notation,*

$$\xi_{\theta,l} := (-1)^{l-1} \frac{(2l+w)! L_p(\overline{\psi_\theta}, -l)^{-1} \Phi(\mathfrak{f})}{2^{l-1} N_{K/\mathbb{Q}} \mathfrak{f}_\theta^l \psi_\theta(f) \Phi(\mathfrak{f}_\theta)} \mathcal{K}_{\mathcal{M}} \circ \mathcal{E}_{\mathcal{M}}^{2l+w}(\beta_\theta) \in H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$$

Then

$$r_{\mathcal{D}}(\xi_{\theta,l}) = L_S^*(\overline{\psi_\theta}, -l)\eta_\theta,$$

where S *are the primes of* K *that divide* $\mathfrak{f}_\theta$ *and* p.

Definition 3.3.4. *For* $a_\theta \not\equiv b_\theta \mod |\mathcal{O}_K^*|$ *we define our constructible space (*$H^{cons}_{M,\omega,r}$ *in conjecture 3.1.4) by*

$$\mathcal{R}_\theta := \xi_{\theta,l}\mathcal{O}_K$$

Remark 3.3.5. *In the situation* $a_\theta \equiv b_\theta \mod |\mathcal{O}_K^*|$, *one can define also a divisor whose image with respect to the composition of the Eisenstein map with the projector map defines* ξ_θ.

As a consequence of Theorem 3.3.3, we have that our submodule $\mathcal{R}_\theta$ verifies the Beilinson conjecture for the motive $M_\theta(w + l + 1)$.

Theorem 3.3.6. *The $\mathcal{O}_K$-submodule $\mathcal{R}_\theta$ of $H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ satisfies that*

$$det_{\mathcal{O}_K}(r_{\mathcal{D}}(\mathcal{R}_\theta)) = L^*_S(\overline{\psi_\theta}, -l) det_{\mathcal{O}_K}(H^w_B(M_\theta, \mathbb{Z}(w+l))$$

in $det_{\mathcal{O}_K\otimes\mathbb{R}}(H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) \otimes \mathbb{R})$. Here

$$L^*_S(\overline{\psi_\theta}, -l) = \lim_{s+l\to 0} L_S(\overline{\psi_\theta}, s)/(s+l),$$

and S is the set of primes dividing $\mathfrak{f}_\theta$ and primes dividing p.

Proof. We only note that η_θ is a $\mathcal{O}_K = \mathcal{O}_\theta$-base for the free $\mathcal{O}_\theta$-module

$$H^w_B(M_\theta \otimes_K \mathbb{C}, \mathbb{Z}(w+l))$$

of rank one, and the result follows. □

Corollary 3.3.7. *The submodule $\mathcal{R}_\theta$ defined above satisfies the Beilinson conjecture inside the p-local Tamagawa number conjecture, that is $\mathcal{R}_\theta$ satisfies the following conditions:*

1. *The map $r_{\mathcal{D}} \otimes \mathbb{R}$ is a isomorphism when restricted to $\mathcal{R}_\theta \otimes \mathbb{R}$.*
2. *$dim_{\mathbb{Q}}(H^w_B(M_\theta, \mathbb{Z}(w+l)) \otimes \mathbb{Q}) = ord_{s=-l} L_S(H^w(M_\theta, \mathbb{Q}_p), s) = 2$.*
3. *We have the following equality*

$$r_{\mathcal{D}}(det_{\mathbb{Z}}(\mathcal{R}_\theta)) = L^*_S(H^w_{et}(M_\theta, \mathbb{Q}_p), -l) det_{\mathbb{Z}}(H^w_B(M_\theta, \mathbb{Z}(w+l)))$$

where $L^*_S(H^w_{et}(M_\theta, \mathbb{Q}_p), -l)$ means $\lim_{s\to -l} L^*_S(H^w_{et}(M_\theta, \mathbb{Q}_p), s)/(s+l)^2$ (this makes sense by using theorem 3.3.1 and theorem 3.2.6).

Proof. The first and the second conditions are clear for the dimensions of the spaces involved in the Deligne regulator map, and the theorem 3.3.6. The third condition comes from the previous theorem using the fact that, if we multiply an $\mathcal{O}_\theta = \mathcal{O}_K$-module with an element $L^*_S(\overline{\psi_\theta}, -l)$ in $\mathcal{O}_\theta \otimes \mathbb{R}$, the determinant is multiplied by the norm

$$N_{\mathcal{O}_\theta\otimes\mathbb{R}/\mathbb{R}}(L^*_S(\overline{\psi_\theta}, -l)) = L^*_S(\overline{\psi_\theta}, -l)\overline{L^*_S(\overline{\psi_\theta}, -l)} = L^*_S(\overline{\psi_\theta}, -l) L^*_S(\psi_\theta, -l).$$

Using theorem 3.2.6, we obtain that this is equal to $L^*_S(H^w_{et}(M_\theta, \mathbb{Q}_p), -l)$. □

3.4 Iwasawa theory

In this section we will define a map that relates the cohomology groups $H^i_{p,\otimes^w}$ for the p-adic lattice in the p-adic realization of $M_\theta(w+l+1)$ defined in §2, with some $\mathbb{Z}_p[[X,Y]]$-modules in Iwasawa theory.

To simplify the notation, we will denote in this section by,

$$M_{\theta\mathbb{Z}_p}(m) = (e_\theta \otimes^w H^1_{et}(E \times_K \overline{K}, \mathbb{Z}_p))(m)$$

the p-adic lattice for the p-adic realization of $M_\theta(m)$.

Suppose in the following and once and for all that $p \nmid \#|\mathcal{O}_K^*|$ (if $p > 3$ this condition is satisfied). Let $K_n := K(E[p^{n+1}])$ be the field of definition of the p^{n+1}-torsion points of E, and let $K_\infty := \varprojlim K_n$ its inverse limit. Denote by $\mathcal{O}_n$ the ring of integers of these fields (respectively $\mathcal{O}_\infty$). Then $\Delta := Gal(K_0/K)$ has order prime to p and $\Gamma := Gal(K_\infty/K_0)$ is isomorphic to $\mathbb{Z}_p^2$.

Let $\mathcal{G}$ be the Galois group $Gal(K_\infty/K)$; then $\mathcal{G} \cong \Delta \times \Gamma$. Let $\mathcal{A}_n$ be the p-part of the ideal class group of K_n, and let $\mathcal{E}_n$ be the group of global units $\mathcal{O}_n^*$ of K_n. Let $\mathcal{U}_n^{\mathfrak{p}}$ be the group of local units of $K_n \otimes_K K_{\mathfrak{p}}$ which are congruent to 1 module the primes above $\mathfrak{p}$, where $\mathfrak{p}$ is a prime of $\mathcal{O}_K$ lying over p.

For every prime v of K_n above $\mathfrak{p}$, there is then an exact sequence

$$1 \to \mathcal{U}_{n,v} \to K^*_{n,v} \to \mathbb{Z} \times \kappa_n^* \to 1$$

and $\mathcal{U}_n^{\mathfrak{p}} = \oplus_{v|\mathfrak{p}} \mathcal{U}_{n,v}$. Here $\mathcal{U}_{n,v}$ are the local units congruent to 1 modulo v and κ_n is the residue class field of $K_{n,v}$.

First of all, let's recall briefly the definition of the elliptic units $\mathcal{C}_n$ in K_n as defined in [51] paragraph 1. For every ideal $\mathfrak{a}$ of K prime to 6 we can define a theta function

$$\theta_{\mathfrak{a}} : E \setminus ker([\mathfrak{a}]) \longrightarrow \mathbb{C}$$

which has divisor $N(\mathfrak{a})(e) - ker([\mathfrak{a}])$ (for the precise definition see 4.2.2 in [36]). The function $\theta_{\mathfrak{a}}(z)$ is in fact a 12-th root of the function defined in II.2.4 [14]. Let's denote by $t_{\mathfrak{f}}$ a generator for $E[\mathfrak{f}]$-torsion points as $\mathcal{O}_K$-module, and let $\mathfrak{a}$ be an ideal prime to $6\mathfrak{f}$.

Definition 3.4.1. *Let C_n be the subgroup of units generated over $\mathbb{Z}[Gal(K_n/K)]$ by*

$$\prod_{\sigma \in Gal(K(\mathfrak{f})/K)} \theta_{\mathfrak{a}}(t_{\mathfrak{f}}^{\sigma} + h_n),$$

where $\mathfrak{a}$ runs through all ideals prime to $6\mathfrak{p}\mathfrak{f}$, $K(\mathfrak{f})$ is the ray class field defined by $\mathfrak{f}$ and h_n is a primitive p^n-torsion point (i.e. a generator of the p^n-torsion points of E as $\mathcal{O}_K$-module). Define the group of elliptic units of K_n as

$$\mathcal{C}_n := \mu_\infty(K_n)C_n.$$

Denote by $\overline{\mathcal{E}}_n$ and $\overline{\mathcal{C}}_n$ the closures of $\mathcal{E}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ respectively $\mathcal{C}_n \cap \mathcal{U}_n^{\mathfrak{p}}$ in $\mathcal{U}_n^{\mathfrak{p}}$. Define

$$\mathcal{A}_\infty := \lim_{\leftarrow} \mathcal{A}_n,\ \overline{\mathcal{E}}_\infty := \lim_{\leftarrow} \overline{\mathcal{E}}_n,\ \overline{\mathcal{C}}_\infty := \lim_{\leftarrow} \overline{\mathcal{C}}_n,\ \mathcal{U}_\infty^{\mathfrak{p}} := \lim_{\leftarrow} \mathcal{U}_n^{\mathfrak{p}}$$

where the limits are taken with respect to the norm maps.

Denote by $M_\infty^{\mathfrak{p}}$ the maximal abelian p-extension of K_∞ which is unramified outside of the primes above $\mathfrak{p}$, and write $\mathcal{X}_\infty^{\mathfrak{p}} := Gal(M_\infty^{\mathfrak{p}}/K_\infty)$. Define the Iwasawa algebra

$$\mathbb{Z}_p[[\mathcal{G}]] := \lim_{\leftarrow} \mathbb{Z}_p[[Gal(K_n/K)]]$$

which has a natural action of $\mathbb{Z}_p[\Delta]$.

For any irreducible $\mathbb{Z}_p$-representation χ of Δ, consider

$$e_\chi := \frac{1}{\#\Delta} \sum_{\tau \in \Delta} Tr(\chi(\tau))\tau^{-1} \in \mathbb{Z}_p[\Delta]$$

and for every $\mathbb{Z}_p[\Delta]$-module Z denote by $Z^\chi := e_\chi Z$.

We define

$$\Lambda^\chi := \mathbb{Z}_p[[\mathcal{G}]]^\chi = R_\chi[[\Gamma]]$$

where R_χ is the ring of integers in the unramified extension of $\mathbb{Z}_p$ of degree $\dim(\chi)$.

We denote by $\Lambda := \mathcal{O}_p[[\Gamma]]$ where $\mathcal{O}_p = \mathcal{O}_K \otimes \mathbb{Z}_p$. The modules $\mathcal{A}_\infty^\chi$ and $\overline{\mathcal{E}}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi$ are torsion Λ^χ-modules. The classical theorem for the main conjecture in Iwasawa theory, using the determinant notation instead of the characteristic ideal (see prop.6.1 in [34]), states:

Theorem 3.4.2 (Rubin, theorem 4.1 in [51]). *Let $p \nmid \#\mathcal{O}_K^*$.*
i) Suppose that p splits in K then

$$det_{\Lambda^\chi}(\mathcal{A}_\infty^\chi) = det_{\Lambda^\chi}(\overline{\mathcal{E}}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi).$$

ii) Suppose that p remains prime or ramifies in K and that χ is nontrivial on the decomposition group of $\mathfrak{p}$ in Δ. Then

$$det_{\Lambda^\chi}(\mathcal{A}_\infty^\chi) = det_{\Lambda^\chi}(\overline{\mathcal{E}}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi).$$

Remark 3.4.3. The above theorem 4.1 of Rubin in [51] uses another definition of elliptic units as the ones defined above. Although theorem 4.1 remains true with our definition of elliptic units, (personal communication with K. Rubin).

We will rewrite to our convenience this result.

Let $\mathcal{X}_\infty$ be the Galois group for the maximal abelian p-extension M_∞^p of K_∞ over K_∞ which is unramified outside of the primes above p. Define also

$$\mathcal{U}_\infty := \mathcal{U}_\infty^{\mathfrak{p}} \times \mathcal{U}_\infty^{\mathfrak{p}^*}$$

if $p = \mathfrak{p}\mathfrak{p}^*$ is split, and

$$\mathcal{U}_\infty := \mathcal{U}_\infty^{\mathfrak{p}}$$

if p is inert or ramified. Similarly, let $\mathcal{Y}_n$ be the p-adic completion of $(K_n \otimes \mathbb{Q}_p)^*$ and $\mathcal{Y}_\infty := \varprojlim \mathcal{Y}_n$. We have an inclusion $\mathcal{U}_\infty \subset \mathcal{Y}_\infty$. Using Class field theory we have an exact sequence

$$0 \to \overline{\mathcal{E}}_\infty / \overline{\mathcal{C}}_\infty \to \mathcal{U}_\infty / \overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty \to \mathcal{A}_\infty \to 0, \tag{3.1}$$

where $\overline{\mathcal{C}}_\infty$ is diagonally embedded into $\mathcal{U}_\infty^{\mathfrak{p}} \times \mathcal{U}_\infty^{\mathfrak{p}^*}$ if p is split.

We can consider moreover the representations $\chi : \Delta \to \mathcal{O}_K^*$. For any topological group B, we denote by $\hat{B}$ the completion by the ideal p. Denote then by

$$B^\chi := \{b \in \hat{B} \otimes \mathcal{O}_K | \sigma b = \chi(\sigma) b \quad \forall \sigma \in \Delta\}.$$

Using this notation, we have also an idempotent

$$e_\chi := \frac{1}{\#\Delta} \sum_{\tau \in \Delta} Tr(\chi(\tau))\tau^{-1} \in \mathcal{O}_K \otimes \mathbb{Z}_p[\Delta].$$

Using the sequence (3.1) one gets easily cor.2.1.5 in [36]:

Corollary 3.4.4. *Let χ with the same conclusions as in theorem 3.4.2, and if $\mathcal{X}_\infty^\chi$ denotes $e_\chi \mathcal{X}_\infty$, we have that*

$$det_{\Lambda^\chi}(\mathcal{X}_\infty^\chi) = det_{\Lambda^\chi}(\mathcal{U}_\infty^\chi / \overline{\mathcal{C}}_\infty^\chi).$$

To control the coimage of the inclusion of $\mathcal{U}_\infty \subset \mathcal{Y}_\infty$ we will use the following result, lemma 2.1.6 in [36].

Lemma 3.4.5. *Let p be a prime such that $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$. If p splits in K, the inclusion $\mathcal{U}_\infty \to \mathcal{Y}_\infty$ is an isomorphism and if p is inert or ramified in K, there is an exact sequence*

$$0 \to \mathcal{U}_\infty \to \mathcal{Y}_\infty \to \mathbb{Z}_p[\Delta/\Delta_p] \to 0$$

where Δ_p is the decomposition group of p in Δ.

This lemma controls the difference between $\mathcal{U}_\infty$ and $\mathcal{Y}_\infty$ in order to chance one for the other one in the corollary 3.4.4. The above lemma claims always that we can change one for the other one when we restrict to primes p which split in K.

Recall that $\mathcal{O}_S = \mathcal{O}_K[1/S]$. Here and in the following S means the primes of K which divide $\mathfrak{f}_\theta$ with the primes of K above p. S' means the primes of K above p and the ones which divide the conductor $\mathfrak{f}$ of the elliptic curve E. Denote by S_p the set of finite primes over p, and by $\mathcal{O}_{n,S_p}$ the ring of integers in K_n where all the primes above p are inverted. We define then $\mathcal{O}_{\infty,S_p} := \varinjlim \mathcal{O}_{n,S_p}$. Similarly we define $\mathcal{O}_{n,S}$ and $\mathcal{O}_{\infty,S}$, where the primes above S are inverted in K_n or K_∞ respectively.

After introducing all the above notations, we are going to define a map in the spirit of Soulé:

$$e_p : \overline{\mathcal{C}}_\infty \otimes_{\mathbb{Z}_p} (e_\theta \otimes^w T_pE)(l) \to H^1(\mathcal{O}_S, (e_\theta \otimes^w T_pE)(l+1)).$$

All makes sense because $M_{\theta\mathbb{Z}_p}(w+l)$ is unramified outside S and it is consider as $\mathcal{O}_K[1/S]$-sheaf. Moreover, remembering that $(e_\theta(\otimes^w T_pE))(l)$ is a $\mathcal{O}_\theta \otimes \mathbb{Z}_p$-module of rank 1. Denote in the following by $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$ and $\mathcal{O}_{\theta,p} := \mathcal{O}_\theta \otimes \mathbb{Z}_p$.

Using the definition of $M_{\theta\mathbb{Z}_p}(w)(l+1)$, we have that

$$H^1(\mathcal{O}_S, (e_\theta \otimes^w T_pE)(l+1)) = \varprojlim H^1(\mathcal{O}_S, (e_\theta \otimes^w E[p^{r+1}])(l+1)).$$

Define e_p in the following way. Consider $(\theta_r)_r$ a norm compatible system of elliptic units and an element $(t_r)_r$ of $\varprojlim(e_\theta(\otimes^w E[p^{r+1}]))(l)$, then we define

$$e_p((\theta_r \otimes t_r)_r) := (Norm_{K_r/K}(\theta_r \otimes t_r))_r.$$

It is well defined because $\theta_r \otimes t_r$ is an element in

$$\mathcal{O}^*_{r,S}/(\mathcal{O}_{r,S})^{p^{r+1}} \otimes (e_\theta(\otimes^w E[p^{r+1}]))(l) \subset H^1(\mathcal{O}_{r,S}, (e_\theta(\otimes^w E[p^{r+1}]))(l+1))$$

Definition 3.4.6. *The Soulé elliptic elements are the elements in the image of the map*

$$e_p : (\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_{\mathcal{G}} \to H^1(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1))$$

where $\mathcal{G} = Gal(K(E[p^\infty])/K)$.

For some technical reasons we need to consider a finite extension field K_0^1 of K such that $M_{\theta\mathbb{Z}_p}(w+m)$ is unramified in all places outside p. Then, we know that the elliptic curve has good reduction over K_0 at all the places not dividing p. And hence, using Serre-Tate theorem, the Tate module of the elliptic curve is unramified in those places. In particular, our tensor product of Tate modules twisted by m, corresponding to $M_{\theta\mathbb{Z}_p}(w+m)$, is unramified in all the places not dividing p in K_0. Therefore, we can take $K_0^1 = K_0$.

For a G_K-module M, we define

$$H^1(K_\infty \otimes \mathbb{Q}_p, M) := \lim_{\leftarrow} H^1(K_n \otimes \mathbb{Q}_p, M) = \lim_{\leftarrow} \oplus_{v|p} H^1(K_{n,v}, M).$$

Suppose that M is a $\mathcal{O}_p := \mathcal{O}_K \otimes \mathbb{Z}_p$-module. We will use the following notation: If M has and action of absolute Galois group of K, G_K,

$$M' := Hom_{\mathcal{O}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathcal{O}_p)$$

with the natural action of G_K; and if M has an action of a group G

$$M^* := Hom_{\mathcal{O}_p}(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes_{\mathbb{Z}_p} \mathcal{O}_p)$$

with the natural action of G.

Proposition 3.4.7. *There are isomorphisms of $\mathcal{O}_p[[\mathcal{G}]]$-modules*

$$\mathcal{X}_\infty \otimes_{\mathbb{Z}_p} M_{\theta\mathbb{Z}_p}(w+l) \cong H^1(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')^*$$

$$\mathcal{Y}_\infty \otimes_{\mathbb{Z}_p} M_{\theta\mathbb{Z}_p}(w+l) \cong H^1(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*.$$

Proof. By using that $Gal(M_\infty^p/K_\infty)$ acts trivially on $M_{\theta\mathbb{Z}_p}(w+l+1)'$, we have that

$$H^1(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')^* =$$
$$Hom(Gal(M_\infty^p/K_\infty), M_{\theta\mathbb{Z}_p}(w+l+1)')^* = \mathcal{X}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l).$$

To show the second isomorphism, denote by

$$M_{n,\theta\mathbb{Z}_p}(w) := H^w(M_\theta \times_K \overline{K}, \mathbb{Z}/p^n(w)) = e_\theta(\otimes^w H^1(E(\mathbb{C}), \mathbb{Z}/p^n(1))).$$

We have that

$$H^1(K_n \otimes \mathbb{Q}_p, M_{n,\theta\mathbb{Z}_p}(w+l+1)') = \oplus_{v|p} H^1(K_{n,v}, M_{n,\theta\mathbb{Z}_p}(w+l+1)') =$$

$$\oplus_{v|p} Hom(Gal(\overline{K_{n,v}}/K_{n,v})^{ab}, M_{n,\theta\mathbb{Z}_p}(w+l+1)')$$

since $Gal(\overline{K_{n,v}}/K_{n,v})$ acts trivially on $M_{n,\theta\mathbb{Z}_p}(w+l+1)'$. Now, by local class field theory

$$Hom(Gal(\overline{K_{n,v}}/K_{n,v})^{ab}, M_{n,\theta\mathbb{Z}_p}(w+l+1)')^* \cong K_{n,v}^*/p^n \otimes M_{n,\theta\mathbb{Z}_p}(w+l)$$

Taking projective limits on both sides we obtain the result. □

Proposition 3.4.8. *The groups*

$$H^2(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)') \quad \textit{and } H^2(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')$$

are zero.

Proof. By local Tate duality, we have that

$$H^2(K_n \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^* \cong H^0(K_n \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)) = 0,$$

the last group being trivial for weights considerations, since $l \geq 0$ and $-w-2l \leq -3$.

Thus, we know by Soulé (see the proof of prop.2.2.4 in [36]) that

$$H^2(\mathcal{O}_{\infty,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(-l-1)) = 0.$$

Hence, we only have to notice that

$$H^2(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w)'(-l-1)) = H^2(\mathcal{O}_{\infty,S_p}, \mathbb{Q}_p/\mathbb{Z}_p(-l-1)) \otimes M_{\theta\mathbb{Z}_p}(w).$$

□

Let's now recall lemma 2.2.6 in [36].

Lemma 3.4.9. *Let M be a perfect complex of $\Lambda = \mathcal{O}_p[[\Gamma]]$-modules. Then here are canonical isomorphism*

$$M^* \otimes_\Lambda^{\mathbb{L}} \mathcal{O}_p \cong R\Gamma(\Gamma, M)^*$$

where the right band side is the (continuous) group cohomology of Γ and $M^* = Hom(M, \mathbb{Q}_p/\mathbb{Z}_p \otimes \mathcal{O}_p)$.

Using this lemma, we obtain the following result.

Corollary 3.4.10. *There are exact triangles*

$$(\mathcal{Y}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes_\Lambda^{\mathbb{L}} \mathcal{O}_p \to R\Gamma\left(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)'\right)^*[-1]$$

$$\to R\Gamma\left(\Gamma, H^0(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')\right)^*[-1]$$

and

$$(\mathcal{X}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes_\Lambda^{\mathbb{L}} \mathcal{O}_p \to R\Gamma(\mathcal{O}_{0,S_p} \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1]$$

$$\to R\Gamma\left(\Gamma, H^0(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')\right)^*[-1].$$

If in the definition of e_p only take the norm maps to K_0, we get a map of complexes

$$e_p : \overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l) \to H^1(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))$$

Recall that, for weight reasons, we have that $H^0(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)) = 0$. So, we obtain a map of complexes

$$e_p : (\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p \to R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))[1]$$

Lemma 3.4.11. *-The following diagram is commutative*

$$\begin{array}{ccc}
(\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p & \xrightarrow{e_p} & R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))[1] \\
\downarrow & & \downarrow \\
(\mathcal{Y}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p & \longrightarrow & R\Gamma(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1] \\
\alpha \downarrow & & \downarrow \\
(\mathcal{X}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p & \longrightarrow & R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1]
\end{array}$$

where the map α is induced by the natural map $\mathcal{Y}_\infty/\overline{\mathcal{C}}_\infty \to \mathcal{X}_\infty$.

Proof. Let's proves first the commutativity of the upper square. The map

$$(\mathcal{Y}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma \to H^1(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*$$

is the dual of the corestriction

$$H^1(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)') \to H^1(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^\Gamma.$$

By Tate local duality the corestriction is dual to the norm map:

$$H^1(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1))_\Gamma \to H^1(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)).$$

These results together with the definition of e_p proves the commutative of the upper square, where the maps above were taken in the first component in the complex of our objects inside the derivative category. The commutativity of the lower square follows easily. □

We consider in the following the representation χ of the group Δ given by the action of Δ in

$$Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p).$$

We impose that the above representation is irreducible. In particular if p splits in K, and (a_θ, b_θ) is the infinite type for ψ_θ is irreducible for example when $p-1 \nmid a_\theta - b_\theta$, and if p is inert it is irreducible when $p+1 \nmid a_\theta - b_\theta$.

Definition 3.4.12. *A character χ of the group Δ is called a good character if it is irreducible and it satisfies that*

$$\mathcal{U}_\infty^\chi \cong \mathcal{Y}_\infty^\chi$$

and the Iwasawa main conjecture. That is, χ satisfies the conclusions of Theorem 3.4.2.

These conditions are always satisfied when the prime p splits in K. If p is inert or ramified, we have to study the condition for χ in the Rubin theorem 3.4.2 and in the exact sequence of lemma 3.4.5. We impose once for all that χ is not the cyclotomic character and that χ is a good character.

Lemma 3.4.13. *Observe that our character χ is equal to $(\overline{\psi}_\theta \kappa^l)^{-1}$ where κ is the cyclotomic character. Suppose that p splits in K and suppose that $p-1 \nmid a_\theta + l + 1$ or $p-1 \nmid b_\theta + l + 1$ or $p-1 \nmid a_\theta - b_\theta$. Then χ is not the cyclotomic character.*

Proof. Since p is split in K we have that $p = \mathfrak{p}\mathfrak{p}^*$, where $\mathfrak{p}$ is not equal to $\mathfrak{p}^*$. Let $\Delta_\mathfrak{p}$ be the Galois group $Gal(K(E[\mathfrak{p}])/K)$; it is a subgroup of the decomposition group since $\mathfrak{p}$ is totally ramified in $\Delta_\mathfrak{p}$. Observe that $\overline{\psi}_\theta \otimes \mathbb{Z}_p = \psi_{\Omega_1} \oplus \psi_{\Omega_2}$, since p is split. It is known $\overline{\psi}_{\Omega_1}|_{\Delta_\mathfrak{p}} = \kappa^{b_\theta}$ (see for example §2.5 [24]), so we get that our character is different for κ as long as $\#\Delta_\mathfrak{p} = p-1 \nmid b_\theta + l + 1$, since κ is a generator for the character group of $\Delta_\mathfrak{p}$.

Using the same kind of argument for $\mathfrak{p}^*$ applied also to the character ψ_{Ω_1} we obtain the same result but with a_θ instead of b_θ. Thus, we will obtain the cyclotomic character only in the case that $p-1 \mid a_\theta + l + 1$ and $p-1 \mid b_\theta + l + 1$.

We can also use the same argument for ψ_{Ω_2}, obtaining the same simultaneous arithmetic conditions, i.e. $p-1 | l + b_\theta + 1$ and $p-1 | a_\theta + l + 1$. We refer p.220 pp.223-224 in [25] for more details on the above characters. □

The goal of the rest of the section is to show that the map e_p induces an isomorphism of $\mathcal{O}_p$-modules

$$det_{\mathcal{O}_p}((\overline{\mathcal{C}}_\infty^\chi \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes_\Lambda^{\mathbb{L}} \mathcal{O}_p) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}.$$

Proposition 3.4.14. *With the same hypothesis as above, we have that*

$$det_{\mathcal{O}_p}(R\Gamma(\mathcal{G}, H^0(K_\infty \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)'))) \cong \mathcal{O}_p$$

$$det_{\mathcal{O}_p}(R\Gamma(\mathcal{G}, H^0(\mathcal{O}_{\infty,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)'))) \cong \mathcal{O}_p$$

Proof. It follows from prop. 2.4.6 in [24], that the action of $\mathcal{G}$ on

$$M_{\theta\mathbb{Z}_p}(w+l) \cong (e(\mathcal{O}_K \otimes \cdots \mathcal{O}_K) \otimes \mathbb{Q}_p) \cong \mathcal{O}_{\theta,p}$$

is via the character

$$\psi_\theta : \mathcal{G} \to \mathcal{O}^*_{\theta,p}.$$

We have hence a surjection of $\mathcal{O}_p$-modules $\rho : \mathcal{O}_p[[\Gamma]] \to M_{\theta\mathbb{Z}_p}(w+l)$. Thus $ker(\rho)$ is an ideal of height 2 because $\Gamma \cong \mathbb{Z}_p^2$. We know that det_R is determined by the ideals of height 1 for the ring R (cf. 2.1.4 [33]) then

$$det_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p) \cong \mathcal{O}_p. \tag{3.2}$$

In fact, since Δ is finite and $\mathcal{G} \cong \Gamma \times \Delta$ we have the equality, $M_{\theta\mathbb{Z}_p}(w+l) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p \cong (M_{\theta\mathbb{Z}_p}(w+l))_\Delta \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\Gamma]]} \mathcal{O}_p$. Since we know that $ker(\rho)$ has height 2, we have $det_{\mathcal{O}_p[[\Gamma]]}((M_{\theta\mathbb{Z}_p}(w+l))_\Delta) \cong \mathcal{O}_p[[\Gamma]]$ and so $det_{\mathcal{O}_p}((M_{\theta\mathbb{Z}_p}(w+l))_\Delta \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\Gamma]]} \mathcal{O}_p) \cong \mathcal{O}_p$. This checks (3.2). We conclude by using lemma 3.4.9. □

Using the isomorphism

$$\mathcal{Y}^\chi_\infty \cong \mathcal{U}^\chi_\infty$$

Corollary 3.4.15. *Applying the derived functor $R\Gamma(\Delta,)$ to the sequence in corollary 3.4.10 we obtain that*

$$det_{\mathcal{O}_p}((\mathcal{U}^\chi_\infty \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p) \cong$$

$$det_{\mathcal{O}_p}\left(H^0(\Delta, R\Gamma(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1])\right)$$

and that

$$det_{\mathcal{O}_p}\left((\mathcal{X}^\chi_\infty \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p\right) \cong$$

$$det_{\mathcal{O}_p}\left(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p} \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)')^*[-1])\right)$$

So, if we apply the functor $R\Gamma(\Delta, \ )$ to the triangle

$$R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)) \to R\Gamma(K_0 \otimes \mathbb{Q}_p, M_{\theta\mathbb{Z}_p}(w+l+1)^{'})^*[-2]$$

$$\to R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1)^{'})^*[-2]$$

and we obtain the following result.

Corollary 3.4.16. *There is an isomorphism of determinants*

$$det_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))))^{-1} \cong$$

$$det_{\mathcal{O}_p}((\mathcal{U}^\chi_\infty \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p) det_{\mathcal{O}_p}((\mathcal{X}^\chi_\infty \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_\Lambda \mathcal{O}_p)^{-1}$$

Remember that S is formed by the primes that divide p and the ones such that the inertia on them acts non-trivially on $M_{\theta\mathbb{Z}_p}(w+l+1)$ (that are the ones dividing the conductor $\mathfrak{f}_\theta$). Observe that the places $S \setminus S_0$ are the ones that divide the conductor $\mathfrak{f}$ of the elliptic curve, under our assumption $S = S'$. With respect to this assumption we remark the following result.

Lemma 3.4.17. *Suppose that our fixed ψ_θ has with infinity type (a_θ, b_θ). Suppose also that $\#|\theta| = 2$ and moreover that $a_\theta \not\equiv b_\theta \ mod|\mathcal{O}_K^*|$. Restrict us with the infinity type is $(w,0)$ or $(0,w)$. If we suppose also that $(w, p-1) = 1$, then $S = S'$, i.e. the maximal ideals in K that divide the conductor of the elliptic curve, divide also the conductor of the Hecke character ψ_θ.*

Proof. Let v be a prime dividing $\mathfrak{f}$. Let v_0 be a prime in K_0 dividing v. Denote by Δ_{v_0} the stabilizer of v_0 in K_0. We have then that $I_{v_0} \subset \Delta_{v_0} \subset \Delta$ acts non-trivially in the Tate module, since has E bad reduction on v. Now, since $\#|\Delta| = (p-1)^2$, we have that I_{v_0} acts non-trivially on $e_\theta \otimes^w T_pE$ if we impose that $(w, p-1) = 1$. We obtain therefore that the inertia on v acts non-trivially on $M_\theta(m)$ for any twist($(v,p) = 1$), proving hence that $v|\mathfrak{f}_\theta$. □

We need now to find the relation between $\mathcal{O}_{0,S_p}$ and $\mathcal{O}_{0,S}$.

Lemma 3.4.18. *The restriction map induces an equality of determinants*

$$det_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l+1))) \cong$$

$$det_{\mathcal{O}_p}(H^0(\Delta, R\Gamma(\mathcal{O}_{0,S}, M_{\theta\mathbb{Z}_p}(w+l+1)))) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1))).$$

Proof. Consider the exact triangle

$$R\Gamma(\mathcal{O}_{0,S_p}, M_{\theta\mathbb{Z}_p}(w+l)) \to R\Gamma(\mathcal{O}_{0,S}, M_{\theta\mathbb{Z}_p}(w+l))$$
$$\to \oplus_{v\in S\setminus S_p} R\Gamma_{k(v)}(\mathcal{O}_v, M_{\theta\mathbb{Z}_p}(w+l))[1]$$

where $\mathcal{O}_v$ is the local ring at v. Since T_pE is unramified at the places of K_0 in $S \setminus S_p$, the same is true for $e_\theta(\otimes T_pE)(l)$. By purity we have that

$$R\Gamma_{k(v)}(\mathcal{O}_v, M_{\theta\mathbb{Z}_p}(w+l)) \cong R\Gamma(k(v), M_{\theta\mathbb{Z}_p}(w+l)).$$

But we have that

$$H^0(\Delta, \oplus_{v\in S\setminus S_p} R\Gamma(k(v), M_{\theta\mathbb{Z}_p}(w+l)) = 0.$$

To show this result, observe that $H^1(k(v), M_{\theta\mathbb{Z}_p}(w+l)) \cong M_{\theta\mathbb{Z}_p}(w+l)_{Gal(\overline{k(v)}/k(v))}$ and $H^0 = 0$ since we are under the hypothesis that $-w-2l \leq 3$. Now, let v_o be a prime dividing v in K_0 and let Δ_{v_0} be the stabilizer of v_o. Since $I_{v_0} \subset \Delta_{v_0}$ acts non trivially in the coinvariants

$$M_{\theta\mathbb{Z}_p}(w+l)_{Gal(\overline{k(v)}/k(v))}$$

because $v_0|\mathfrak{f}_\theta$, we obtain the result. □

As a consequence we have the main result of this section.

Theorem 3.4.19. *Suppose that p is an odd prime, prime to $N_{K/\mathbb{Q}}\mathfrak{f}_\theta$ and to $\#|\mathcal{O}_K^*|$. Let χ the Δ-representation on $Hom_{\mathcal{O}_p}(M_\theta(w+l),\mathcal{O}_p)$ be a good representation. Then, the map e_p induces an isomorphism of $\mathcal{O}_p$-modules*

$$det_{\mathcal{O}_p}((\overline{\mathcal{C}}_\infty^\chi \otimes_{\mathcal{O}_p} M_{\theta\mathbb{Z}_p}(w+l)) \otimes_\Lambda^{\mathbb{L}} \mathcal{O}_p) \cong det_{\mathcal{O}_p}(R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}.$$

Proof. On one hand, by the corollary 3.4.4 of Rubin's theorem on the Iwasawa main conjecture, we have that

$$det_{\mathcal{O}_p}((\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi)\otimes M_{\theta\mathbb{Z}_p}(w+l)\otimes_\Lambda^{\mathbb{L}}\mathcal{O}_p) \cong det_\Lambda((\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi)\otimes M_{\theta\mathbb{Z}_p}(w+l))\otimes_\Lambda \mathcal{O}_p$$

$$\cong det_\Lambda(\mathcal{X}_\infty^\chi \otimes M_{\theta\mathbb{Z}_p}(w+l))\otimes_\Lambda \mathcal{O}_p \cong det_{\mathcal{O}_p}(\mathcal{X}_\infty^\chi \otimes M_{\theta\mathbb{Z}_p}(w+l)\otimes_\Lambda^{\mathbb{L}} \mathcal{O}_p),$$

where the first and third isomorphism comes from a general theory of determinant in the derivative tensor product, and the second comes from corollary 3.4.4, using that the archimedean of determinant does not chance if we multiply the both modules by a same module, in our situation by $M_{\theta\mathbb{Z}_p}(w+l)$.

On the other hand,

$$det_{\Lambda^\chi}(\mathcal{U}_\infty^\chi/\overline{\mathcal{C}}_\infty^\chi) \cong det_{\Lambda^\chi}(\mathcal{U}_\infty^\chi)det_{\Lambda^\chi}(\overline{\mathcal{C}}_\infty^\chi)^{-1}.$$

By using this result with results 3.4.16 and 3.4.18, we conclude. □

3.5 The comparison between the map r_p and e_p in the constructible K-elements

Let's start recalling the result of Kings on the specialization of the elliptic polylogarithm sheaf, which is a new key in the proof of the Tamagawa number conjecture.

Let E be an elliptic curve over a base scheme T, and denote by $\overline{\pi}$ the structural morphism, i.e. $\overline{\pi}: E \to T$ which is proper and smooth. Consider $U = E \setminus e$, where e is the zero section of E. There exist on U a lisse pro-sheaf (i.e. a projective limits of lisse sheaves), which is called the elliptic polylogarithm sheaf and denoted by $\mathcal{P}ol_{\mathbb{Q}_p}$ (see §3.2 [36]). Let us consider t a N-torsion point in E diferent of e. Then are defined too, some projections pr_t and σ (see §3.5.1, §3.5.3 [36] respectively), which define the p-adic k-Eisenstein class associated to a torsion point t by

$$(\sigma^k pr_t t^* \mathcal{P}ol_{\mathbb{Q}_p}),$$

for any integer k. This element is defined inside the cohomology group $H^1(T, Sym^k\mathcal{H}_{\mathbb{Q}_p})$, where $\mathcal{H}_{\mathbb{Q}_p} := \underline{Hom}_T(R^1\overline{\pi}_*\mathbb{Q}_p, \mathbb{Q}_p)$. The definition of the p-adic Eisenstein classes is extended by linearity to any divisor formed by N-torsion points (see 3.5.9 [36]). The main part of the result of Kings is the explicit computation of these Eisenstein classes. He compares this classes with the elements coming from a connecting map of the cohomology of Torus which appears when we consider torsors in the elliptic curve. Let us be a little more precise. Consider $H_n := ker[p^n]$ as a scheme over T. Let us consider the map multiplication by p^n, $p_n : E_n \to E$, where E_n is the elliptic curve E but which has p^n-torsors. Consider the characteristic group $I[H_n] := ker(p_{n,*}\mathbb{Z} \to \mathbb{Z})$ which corresponds to the torus T_{H_n}. In this situation we have a connecting map δ as follows ((12)§4.1 [36]):

$$\delta : H^0(H_n, T_{H_n}) \to H^1(H_n, T_{H_n}[p^r]). \tag{3.3}$$

Using this connecting morphism, we can express the Eisenstein classes explicitly.

Theorem 3.5.1 (Kings, 4.2.9 in [36]). *Let p be a prime number, and let E be an elliptic curve over a base scheme T where p is invertible*

Let β be any divisor in E of the form

$$\beta := \sum_{t \in E[N](T)\setminus e} n_t(t)$$

and consider $[\mathfrak{a}] : E \to E$ any isogeny relatively prime to Np.

Then, for any $m > 0$, the p-adic Eisenstein class

$$N\mathfrak{a}(\mathfrak{a}^{\otimes m}N\mathfrak{a} - 1)(\beta^*\mathcal{P}ol_{\mathbb{Q}_p})^m \in H^1(T, Sym^m\mathcal{H}_{\mathbb{Q}_p}(1))$$

is given by

$$\pm\frac{1}{k!}(\delta \sum_{t \in E[N](T)\setminus e} n_t \sum_{[p^n]t_n = t} \theta_{\mathfrak{a}}(-t_n)\tilde{t_n}^{\otimes m})_n$$

where $\tilde{t_n}$ is the projection of t_n to $E[p^n]$ and δ is the Sym-extension of the boundary map $H^0(H_n, T_{H_n}) \to H^1(H_n, T_{H_n}[p^r])$ where $H_n := ker[p^n]$ is consider as a scheme over T and T_{H_n} is the torus with characteristic group $I[H_n] := ker(p_{n\,*}\mathbb{Z} \to \mathbb{Z})$ (3.3).

The following result relates the image of the Soulé regulator of $\mathcal{E}^m_{\mathcal{M}}(\beta)$ with the polylogarithmic sheaf.

Theorem 3.5.2. *Let β be as in the previous theorem. Then we have that*

$$r_p(\mathcal{E}^m_{\mathcal{M}}(\beta)) = -N^{2m}(\beta^*\mathcal{P}ol_{\mathbb{Q}_p})^m$$

in $H^1(T, Sym^m\mathcal{H}_{\mathbb{Q}_p}(1))$.

Proof. This formula can be deduced form the combination of two results: Theorem 2.2.4 in [28], which states that

$$r_p(\mathcal{E}is^m_{\mathcal{M}}(\rho\beta)) = -N^{m-1}(\beta^*\mathcal{P}ol_{\mathbb{Q}_p})^m$$

where ρ is the horospherical map and $\mathcal{E}is$ is Beilinson's Eisenstein symbol; and the formula 3.35 in [12], which says that

$$\mathcal{E}^m_{\mathcal{M}}(\beta) = N^{m+1}\mathcal{E}is^m_{\mathcal{M}}(\rho\beta).$$

□

We are going to apply these results above to the divisor $\beta_\theta = N_{K(\mathfrak{f})/K}((t))$, where $t := \Omega f^{-1}$ is a $\mathfrak{f}_\theta$-torsion point. Take $N = N_{K/\mathbb{Q}}\mathfrak{f}_\theta$, $m = w+2l$, $T = \mathcal{O}_S$ and $\mathcal{H}_{\mathbb{Q}_p} = T_pE \otimes \mathbb{Q}_p$, using the notations of the previous theorem 3.5.1. Let $\mathfrak{a} \subset \mathcal{O}_K$ be an ideal prime to $6p\mathfrak{f}$, and consider the isogeny given by $\psi(\mathfrak{a})$. Finally, $\theta_\mathfrak{a}$ means the classical theta function.

To simplify the notation, define for any $\tilde{t_r} \in E[p^r]$

$$\gamma(\tilde{t_r})^m :=< \tilde{t_r}, \sqrt{d_K}\tilde{t_r} >^{\otimes m}$$

where $<,>$ means the Weil pairing. Our objective is the computation of

$$\mathcal{K}_{\mathcal{M}} \circ \mathcal{E}^{w+2l}_{\mathcal{M}}(\beta_\theta)$$

Remember that we are in the situation $\#|\theta| = 2$ and $a_\theta \not\equiv b_\theta \; mod|\mathcal{O}^*_K|$, i.e. $a_\theta = w, b_\theta = 0$ or $a_\theta = 0, b_\theta = w$ for the type $(a_\theta, b_\theta) \in \theta_K$.

Considering the diagram (2.8) in [10]

$$\begin{array}{ccc} H^{2l+w+1}_{\mathcal{M}}(Sym^{2l+w}h_1E, \mathbb{Q}(w+2l+1)) & \overset{(\delta^l\times id)^*}{\longrightarrow} & H^{2l+w+1}_{\mathcal{M}}(E^{l+w}, \mathbb{Q}(2l+w+1)) \\ \mathcal{K}_{\mathcal{M}} \downarrow & & \downarrow \\ H^{w+1}_{\mathcal{M}}(M_{\theta\mathbb{Q}}, \mathbb{Q}(w+l+1)) & \overset{e_\theta}{\longleftarrow} & H^{w+1}_{\mathcal{M}}(h_1(E)^{\otimes w}, \mathbb{Q}(l+w+1)) \end{array}$$

we obtain a map in Galois cohomology given by

$$H^1(\mathcal{O}_S, Sym^{2l+w}\mathcal{H}_{\mathbb{Q}_p}(1)) \rightarrow$$

$$H^1(\mathcal{O}_S, (e_\theta \mathcal{S}ym^w H_{\mathbb{Q}_p})(l+1)) = H^1(\mathcal{O}_S, H^w(M_\theta \times_K \overline{K}, \mathbb{Q}_p(w+l+1)))$$

such that

$$\mathcal{K}_{\mathcal{M}}(\psi(\mathfrak{a})^{\otimes 2l+w}Sym^{2l+w}\mathcal{H}_{\mathbb{Q}_p}(1)) = e_\theta(\otimes^w\psi(\mathfrak{a}))N\mathfrak{a}^l Sym^w\mathcal{H}_{\mathbb{Q}_p}(l+1).$$

Theorem 3.5.3. *Let p be a prime number such that $p \nmid 6N\mathfrak{f}_\theta$. Let θ be the idempotent satisfying $\#|\theta| = 2$ and such that the infinity type (a_θ, b_θ) satisfies $a_\theta \not\equiv b_\theta \mod |\mathcal{O}_K^*|$. For a $p^r N\mathfrak{f}_\theta$-torsion point t_r, denote by $\tilde{t_r}$ its projection to $E[p^r]$. Then, if $t = \Omega f^{-1}$, we have the following equality*

$$N\mathfrak{a}\left(\psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1} - 1\right) r_p(\xi_{\theta,l}) =$$

$$\frac{(-1)^l L_p(\overline{\psi}_\theta, -l)^{-1} N_{T_\theta/\mathbb{Q}}\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})}\left(\delta N_{K(\mathfrak{f})/K}\sum_{p^r t_r = t}\theta_\mathfrak{a}(-t_r)\otimes e_\theta(\otimes^w \tilde{t_r})\otimes\gamma(\tilde{t_r})^l\right)_r$$

Proof. Using theorem 3.5 and the theorems above, we have that

$$r_p(\xi_{\theta,l}) = \frac{(-1)^{l-1}(2l+w)! L_p(\overline{\psi}_\theta, -l)^{-1}\Phi(\mathfrak{f}_\theta)}{2^{l-1}N_{T_\theta/\mathbb{Q}}\mathfrak{f}_\theta^l\psi_\theta(f)\Phi(\mathfrak{f})}\mathcal{K}_\mathcal{M}(\mathcal{E}_\mathcal{M}^{2l+w}(\beta))$$

$$= \frac{(-1)^l(2l+w)! L_p(\overline{\psi}_\theta, -l)^{-1} N_{T_\theta/\mathbb{Q}}\mathfrak{f}_\theta^{3l+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})}\mathcal{K}_\mathcal{M}(\beta^*\mathcal{P}ol_{\mathbb{Q}_p})^{w+2l}.$$

Using the same notation as above, we have

$$\mathcal{K}_\mathcal{M}(\tilde{t_r}^{\otimes 2l+w}) = e_\theta(\otimes^w \tilde{t_r})\otimes\gamma(\tilde{t_r})^l$$

Finally, applying Kings' theorem, we obtain the desired identity. □

We want to rewrite the previous formula in terms of the norm map of the extension $K(\mathfrak{f}_\theta)K(E[p^n])$. For technical reasons, we will work with $\mathfrak{f}$ instead of $\mathfrak{f}_\theta$ since then we can use that $K(E[\mathfrak{p}^n\mathfrak{f}]) = K(\mathfrak{p}^n\mathfrak{f})$, the class field field, because $\mathfrak{f}$ is the conductor of E and divides the ideal $\mathfrak{f}\mathfrak{p}^n$ (see prop. 1.6 in [14]).

Fix a prime $\mathfrak{p}$ of K where E has good reduction, and take $\pi = \psi(\mathfrak{p})$. Denote by

$$H_{r,t}^\mathfrak{p} := \{t_r \in E[\mathfrak{p}^r\mathfrak{f}] | \pi^r t_r = t\}.$$

We write $t_r = (\tilde{t_r}, \pi^{-r}t) \in E[\mathfrak{p}^r\mathfrak{f}] = E[\mathfrak{p}^r]\oplus E[\mathfrak{f}]$. Define a filtration of $H_{r,s}^\mathfrak{p}$ as

$$F_{r,t}^i := \{t_r \in H_{r,s}^\mathfrak{p} | \pi^{r-i}\tilde{t_r} = 0\}.$$

Theorem 3.5.4. *Let $\mathfrak{p}$ be as above and $t_r = (\tilde{t_r}, \pi^{-r}t) \in F_{r,s}^0 \setminus F_{s,t}^1$. Suppose $\mathcal{O}^* \to (\mathcal{O}/\mathfrak{f}_\theta)^*$ is injective. Denote the Euler factor of ψ_θ at $\mathfrak{p}$ evaluated at $-l$ by $L_\mathfrak{p}(\overline{\psi}_\theta, -l)$. Then*

$$L_\mathfrak{p}(\overline{\psi}_\theta, -l)^{-1}\left(N_{K(\mathfrak{f})/K}\sum_{s_r\in H_{r,t}^\mathfrak{p}}\theta_\mathfrak{a}(-s_r)\otimes e_\theta(\otimes^w\tilde{s_r})\otimes\gamma(\tilde{s_r})^l\right)_r =$$

$$(N_{K(\mathfrak{p}^r\mathfrak{f})/K}\left(\theta_\mathfrak{a}(-t_r)\otimes e_\theta(\otimes^w\tilde{t_r})\otimes\gamma(\tilde{t_r})^l\right))_r$$

in $H^1(\mathcal{O}_S, e_\theta(T_\mathfrak{p}E(1))(l)\otimes\mathbb{Q}_p)$ for all $\mathfrak{a}$ relatively prime to $\mathfrak{p}\mathfrak{f}$.

Proof. The identification $Hom_{\mathcal{O}_p}(T_pE, \mathcal{O}_p) \cong T_pE(-1)$ is via the conjugate linear $\mathcal{O}_p$-action on the right factor. Hence $\overline{\psi(\mathfrak{p})}t_r = t_{r-1}$. We have the equality

$$(\overline{\psi_\theta(\mathfrak{p})}/N\mathfrak{p}^{-l})^i N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}(\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}(\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \overline{\psi(\mathfrak{p})}^i \tilde{t_r}) \otimes \gamma(\overline{\psi(\mathfrak{p})}^i \tilde{t_r})^l) =$$

$$(N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-i}\mathfrak{f})}(\theta_\mathfrak{a}(-t_r))) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-1}})^l) =$$

$$\theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-i}})^l,$$

where the last equality uses the distribution relation for $\theta_\mathfrak{a}$ (see [14] II 2.5, or more precisely A.2.1).

The Galois group of $K(\mathfrak{p}^{r-i}\mathfrak{f})/K(\mathfrak{f})$ acts simply transitively on $F^i_{r,t} \setminus F^{i+1}_{r,t}$. We get hence that

$$(\overline{\psi_\theta(\mathfrak{p})}/N\mathfrak{p}^{-l})^i N_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{f})}(\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$\sum_{t_{r-i} \in F^i_{r,t} \setminus F^{i+1}_{r,t}} \theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-i}})^l.$$

We claim that we have the equality $\theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) = \theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{-r}t))^{\sigma^i_\mathfrak{p}}$ with $\sigma_\mathfrak{p}$ is the Frobenius at $\mathfrak{p}$ in the Galois group of $K(\mathfrak{f})/K$. This claim and the fact that $N_{K(\mathfrak{f})/K}$ is the sum over all Galois translates, which act trivially on $\tilde{t_{r-i}}$, gives that

$$(\overline{\psi_\theta(\mathfrak{p})}/N\mathfrak{p}^{-l})^i N_{K(\mathfrak{p}^r\mathfrak{f})/K}(\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$N_{K(\mathfrak{f})/K}\left(\sum_{t_{r-i} \in F^i_{r,t} \setminus F^{i+1}_{r,t}} \theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{-r}t)) \otimes e_\theta(\otimes^w \tilde{t_{r-i}}) \otimes \gamma(\tilde{t_{r-i}})^l\right),$$

Adding this equalities with respect to i and increasing r if necessary we get the result.

It remains to prove the claim

$$\theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{i-r}t)) = \theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^{-r}t))^{\sigma^i_\mathfrak{p}}.$$

We know by pag.21 in [36] that we have the following equality

$$\theta_\mathfrak{a}(-(\tilde{m_{r-i}}, \pi^{i-r}t_\mathfrak{f})) = \theta_\mathfrak{a}(-(\tilde{m_{r-i}}, \pi^{-r}t_\mathfrak{f}))^{\sigma^i_\mathfrak{p}}$$

where $t_\mathfrak{f}$ is the elected primitive $\mathfrak{f}$-torsion element, and $\tilde{m_{r-i}}$ denotes a $E[\mathfrak{p}^{r-i}]$ torsion point. Consider the norm map for $K(\mathfrak{f})/K(\mathfrak{f}_\theta)$; since the Galois group of $K(\mathfrak{f})/K(\mathfrak{f}_\theta)$ is commutative, this norm map commutes with $\sigma_\mathfrak{p}$. Using now the norm relation for the theta functions, and that the extensions between the field of $K(\mathfrak{f})$ with the one of the $\mathfrak{p}^n$-torsion points are disjoint over K, we obtain that

$$N_{K(\mathfrak{f})/K(\mathfrak{f}_\theta)}\theta_\mathfrak{a}(-(\tilde{m_{r-i}}, \pi^m t_\mathfrak{f})) = \theta_\mathfrak{a}(-(\tilde{t_{r-i}}, \pi^m t))$$

(see for example cor. 7.7 in [52] or much precisely A.2.2). We obtain now the result because $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective, where the morphism $\mathfrak{l} = \mathfrak{f}/\mathfrak{f}_\theta$ (we suppose $S = S'$) maps $\tilde{m_{r,i}}$ to $\tilde{t_{r,i}}$ on $E[\mathfrak{p}^{r-i}]$. □

Lemma 3.5.5. *Suppose that the infinite type of θ is $(w, 0)$ or $(0, w)$. Suppose moreover that $(\#|\mathcal{O}_K^*|, w) = 1$. Then*

$$\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$$

is injective.

Proof. Let u be and element in $\mathcal{O}_K^*$, $u \neq 1$ and consider the idele defined by $x_\infty = 1$ and $x_\mathfrak{p} = u$ for all finite places $\mathfrak{p}$ of K. Then $\psi^w(x) = \psi^w(u^{-1}x) = u^w \neq 1$ if $(w, |\mathcal{O}_K^*|) = 1$. So, by definition of the conductor of $\psi_\theta = \psi^w$, we obtain that $u \not\equiv 1 (mod\ \mathfrak{f}_\theta)$, hence the result desired for the type $(w, 0)$. For the type $(0, w)$ the proof is the same but with $\overline{\psi}$ instead of ψ. □

Corollary 3.5.6. *With the same hypothesis of theorem 3.5.3 and if $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective, we have the equality*

$$N\mathfrak{a}(\psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1} - 1)r_p(\xi_{\theta,l}) =$$

$$\pm\frac{N\mathfrak{f}_\theta^{3k+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})}\delta\left(N_{K(E[p^r])K(\mathfrak{f})/K}\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(t_r)^l\right)_r =$$

$$\pm\frac{N\mathfrak{f}_\theta^{3k+2w}\Phi(\mathfrak{f}_\theta)}{2^{l-1}\psi_\theta(f)\Phi(\mathfrak{f})}|Gal(K(\mathfrak{f})/K(\mathfrak{f}_\theta))|\delta\left(N_{K(E[p^r])K(\mathfrak{f}_\theta)/K}\theta_\mathfrak{a}(-t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(t_r)^l\right)_r$$

where $p^r t_r = t$ and t_r is a primitive $p^r\mathfrak{f}_\theta$-torsion point.

Proof. If p is inert or prime the first equality is deduced from the previous theorem. If p split, it decomposes in a $\mathfrak{p}$ and a $\mathfrak{p}^*$ part. Putting together the previous result with $\mathfrak{p}$ and with $\mathfrak{p}^*$, we have the first equality.

For the second equality, we have that

$$N_{K(E[p^r])K(\mathfrak{f})/K(E[p^r])K(\mathfrak{f}_\theta)}\theta_\mathfrak{a}(-t_r) =$$

$$\prod_{\sigma \in Gal(K(\mathfrak{f})K(E[p^r])K(p^r)/K(\mathfrak{f}_\theta)K(E[p^r])K(p^r))} \theta_\mathfrak{a}(-t_r)^\sigma$$

since $K = K(1)$ and hence $K(\mathfrak{f})$ is disjoint with $K(p^r)$ over K, and we also know that $K(\mathfrak{f}) = K(E[\mathfrak{f}])$ is disjoint with $K(E[p^r])$ over K. Moreover, since $\theta_{\mathfrak{a}}(-t_r) \in K(\mathfrak{f})K(p^r) = K(\mathfrak{f}p^r)$ and $(\mathfrak{f}, p) = 1$, we have that the norm is equal to

$$\prod_{\tau \in Gal(K(\mathfrak{f}p^r)/K(\mathfrak{f}_\theta p^r))} \theta_{\mathfrak{a}}(-t_r)^\tau.$$

But $\theta_{\mathfrak{a}}(-t_r) \in K(\mathfrak{f}_\theta p^r)$ because $-t_r$ is a point of $\mathfrak{f}_\theta p^r$-torsion. Hence we get the second equality. □

Now we want to show that the elements

$$(N_{K(E[p^r])K(\mathfrak{f}_\theta)/K}\theta_{\mathfrak{a}}(t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l)_r$$

generate $(\overline{\mathcal{C}}^\chi_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma$, where $\mathfrak{a}$ is prime to $6p\mathfrak{f}_\theta$ and where χ is the representation of Δ on $Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p)$, that we suppose it is a good representation. Here we use $M_{\theta\mathbb{Z}_p}$ with the same notation as in section 4.

For technical reasons, due to how are defined the elliptic units in the work of Rubin [52], we need to compare our elliptic units with the ones of Rubin in [52], that we had already defined in 4.1. This is done in the following lemma.

Lemma 3.5.7. *Let t_r be a primitive $p^r\mathfrak{f}_\theta$ torsion point such that $p^r t_r = t$ and let denote by $\tilde{t_r}$ the projection of t_r on $E[p^r]$. Let t'_r be a primitive $p^r\mathfrak{f}$ torsion point, such that $p^r t'_r = t'$ with t' a generator for $\mathfrak{f}$. Define the ideal $\mathfrak{l}$ of $\mathcal{O}_K$ as the product of the prime ideals $\mathfrak{q}$ which compose S every one with the corresponding power $v_{\mathfrak{q}}(\mathfrak{f}) - v_{\mathfrak{q}}(\mathfrak{f}_\theta)$. We impose that $[l]t' = t$. Let suppose that $S = S'$ and $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ injective. Then,*

$$(N_{K(E[p^r])K(\mathfrak{f}_\theta)/K}\theta_{\mathfrak{a}}(t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l) =$$

$$(N_{K(p^r\mathfrak{f})/K}\theta_{\mathfrak{a}}(t'_r) \otimes e_\theta(\otimes^w \tilde{t'_r}) \otimes \gamma(\tilde{t'_r})^l)$$

for all integer r. Since $(p, \mathfrak{f}) = 1$, we can moreover choose $\tilde{t_r} = \tilde{t'_r}$.

Proof. First, we are going to calculate

$$(N_{K(p^r\mathfrak{f})/K(E[p^r])K(\mathfrak{f}_\theta)}\theta_{\mathfrak{a}}(t'_r) \otimes e_\theta(\otimes^w \tilde{t'_r}) \otimes \gamma(\tilde{t'_r})^l).$$

We know that $K(E[p^r])$ and $K(\mathfrak{f})$ are disjoint extensions whose composition gives $K(p^r\mathfrak{f})$ (see cor 1.7 in [14]). Consider then $\sigma \in Gal(K(p^r\mathfrak{f})/K(E[p^r])K(\mathfrak{f}_\theta))$, which fixes $\tilde{t'_r}$. Then the element to be calculated can be written as

$$(N_{K(p^r\mathfrak{f})/K(E[p^r])K(\mathfrak{f}_\theta)}\theta_{\mathfrak{a}}(t'_r)) \otimes e_\theta(\otimes^w \tilde{t'_r}) \otimes \gamma(\tilde{t'_r})^l.$$

Using the hypothesis and lemma A.2.2, we obtain,

$$N_{K(p^r\mathfrak{f})/K(E[p^r])K(\mathfrak{f}_\theta)}\theta_{\mathfrak{a}}(t'_r) = \theta_{\mathfrak{a}}(t_r)$$

and hence the claim. □

We suppose from now on that the natural map $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ is injective.

Proposition 3.5.8. *Consider $p \nmid 6N\mathfrak{f}$ and $\mathfrak{a}$ an ideal in $\mathcal{O}_p$, which is prime to $6p\mathfrak{f}$ and such that $\psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1} \not\equiv 1(mod\ p)$. Then the $\mathcal{O}_p[[\Gamma]]$-module*

$$\overline{\mathcal{C}}_\infty^\chi \otimes_{\mathcal{O}_p} (e_\theta(\otimes^w T_p E)(l))$$

is generated by $(\theta_\mathfrak{a}(t_r) \otimes e_\theta(\otimes^w \tilde{t_r}) \otimes \gamma(\tilde{t_r})^l)_r$, where t_r is a primitive $p^r\mathfrak{f}$-division point.

Remark 3.5.9. *Before we begin with the proof, let's remark that the existence of an ideal $\mathfrak{a}$ satisfying the conditions of the proposition 3.5.8 is equivalent to the condition that the Δ-representation χ is not the cyclotomic character (which is one of the hypothesis that we imposed before for χ). To show this fact, just notice that the $\mathcal{O}_p$-action on $\mathcal{O}_p[[\Gamma]]$ is given by complex conjugation because the $\mathcal{O}_p$-action on $e_\theta(\otimes^w T_p E)$ is given by complex conjugation on $\mathcal{O}_p$.*

Proof. Let $\mathfrak{b}$ be another ideal prime to $6p\mathfrak{f}$. Take $\sigma_\mathfrak{a} = [\mathfrak{a}, K_n/K]$ and $\sigma_\mathfrak{b} = [\mathfrak{b}, K_n/K]$. Then, by the properties of the theta function, we have that

$$(\sigma_\mathfrak{a} - \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1})(\theta_\mathfrak{b}(t_n) \otimes e_\theta(\otimes^w \tilde{t_n}) \otimes \gamma(\tilde{t_n})^l) =$$

$$\psi_\theta(\mathfrak{a})N\mathfrak{a}^l(\theta_\mathfrak{b}(t_n)^{\sigma_\mathfrak{a} - N\mathfrak{a}} \otimes e_\theta(\otimes^w \tilde{t_n}) \otimes \gamma(\tilde{t_n})^l) =$$

$$\psi_\theta(\mathfrak{a})N\mathfrak{a}^l(\theta_\mathfrak{a}(t_n)^{\sigma_\mathfrak{b} - N\mathfrak{b}} \otimes e_\theta(\otimes^w \tilde{t_n}) \otimes \gamma(\tilde{t_n})^l).$$

Now, it is enough show that $(\sigma_\mathfrak{a} - \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1})$ is invertible in $\mathcal{O}_p[[\Gamma]] = \Lambda$, because $\overline{\mathcal{C}}_\infty^\chi$ is a torsion free Λ-module since $\mathcal{U}_\infty$ is torsion free (see prop.11.4 [52]), and we have an inclusion of $\mathcal{C}_\infty$ in $\mathcal{U}_\infty$.

But the element $\sigma_\mathfrak{a}$ corresponds to 1 on $\mathcal{O}_p/p$ and thus $\sigma_\mathfrak{a} - \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1}$ is invertible in Λ because $1 \not\equiv \psi_\theta(\mathfrak{a})N\mathfrak{a}^{l+1}$ *mod p*.

It remains prove that $e_\theta(\tilde{t_r}) \otimes \gamma(\tilde{t_r})$ generates $M_{\theta\mathbb{Z}_p}(w + l)$. This follows since $M_{\theta\mathbb{Z}_p}(w)$ is one dimensional and hence to prove that it generates $\mathbb{Z}_p(l)$ one can use the same proof as in [36] p.56. □

Corollary 3.5.10. *The image of $\mathcal{R}_\theta$ by r_p in $H^1(\mathcal{O}_S, e_\theta(\otimes^w T_p E)(l+1)) \otimes \mathbb{Q}_p$ coincides with*

$$e_p((\overline{\mathcal{C}}_\infty^\chi \otimes e_\theta(\otimes^w T_p E)(l))_\Gamma).$$

Proof. As

$$|Gal(K(\mathfrak{f})/K(\mathfrak{f}_\theta))| N\mathfrak{f}_\theta^{3k+2w}\Phi(\mathfrak{f})/2^{k-1}\psi_\theta(f)\Phi(\mathfrak{f}_\theta)$$

is prime to p, it follows from the definition of e_p and Corollary 3.5.6. □

Let's note the following lemma.

Lemma 3.5.11. *The canonical map*

$$(\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l)) \otimes^{\mathbb{L}}_{\mathcal{O}_p[[\mathcal{G}]]} \mathcal{O}_p \to (\overline{\mathcal{C}}_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_{\mathcal{G}} \cong (\overline{\mathcal{C}}^\chi_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma$$

is an isomorphism and moreover $(\overline{\mathcal{C}}^\chi_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma \cong \mathcal{O}_p$.

Proof. We observe that the proof of proposition 3.5.8 shows that $\overline{\mathcal{C}}^\chi_\infty \cong \Lambda^\chi = \mathcal{O}_p[[\Gamma]]$ is a free Λ^χ-module of rank 1. This implies, as in lemma 5.2.3 in [36], that $(\overline{\mathcal{C}}^\chi_\infty \otimes M_{\theta\mathbb{Z}_p}(w+l))_\Gamma \cong \mathcal{O}_p$. The claim follows since the previous module is induced and hence the higher Tor-terms vanish. □

As a consequence, we get the part 4 of conjecture 3.1.2 for the constructible subspace $\mathcal{R}_\theta$.

Corollary 3.5.12. *The map*

$$\mathcal{R}_\theta \otimes \mathbb{Z}_p \to R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1) \otimes \mathbb{Q}_p)[1]$$

induced by r_p*, gives an isomorphism*

$$det_{\mathcal{O}_p}\mathcal{R}_\theta \cong det_{\mathcal{O}_p} R\Gamma(\mathcal{O}_S, M_{\theta\mathbb{Z}_p}(w+l+1))^{-1}$$

This proves, taking $Norm_{K/\mathbb{Q}}$, the conjecture 3.1.2 under the hypothesis that the Soulé regulator is not zero for the elements in K-theory constructed for the motive $M_\theta(w+l+1)$. Let us first write the same result but over $\mathcal{O}_K$.

Theorem 3.5.13. *Let* p *be a prime different from* 2 *and* 3 *(hence, in particular,* $p \nmid \#|\mathcal{O}_K^*|$*), and* $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$*. Consider* l *a strictly positive integer. Suppose that* ψ_θ *has infinity type* $(w,0)$ *or* $(0,w)$*, where* $w \geq 1$ *verifies* $-w-2l \leq -3$ *and* $w \not\equiv 0 (mod|\mathcal{O}_K^*|)$*. Suppose that* $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ *is injective and that* $Spec_{max}(\mathfrak{f}) = Spec_{max}(\mathfrak{f}_\theta)$*, where the notation* $Spec_{max}$ *of an ideal means the set of finite places of* $\mathcal{O}_K$ *that divide the ideal.*

Suppose moreover that the representation χ *of* Δ *in* $Hom_{\mathcal{O}_p}(H^w(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+l)), \mathcal{O}_p)$ *is a good representation (see the definition in 3.4.12) which is not the cyclotomic character.*

If we denote by $M_{\theta\mathbb{Z}_p}(w+m) = H^w(M_\theta \times_K \overline{K}, \mathbb{Z}_p(w+m))$*, then, there is an* $\mathcal{O}_K$*-submodule* $\mathcal{R}_\theta \subset H^{w+1}_{\mathcal{M}}(M_\theta, \mathbb{Q}(w+l+1))$ *of rank 1 such that:*

1. $det_{\mathcal{O}_K}(r_{\mathcal{D}}(\mathcal{R}_\theta)) \cong L^*_S(\overline{\psi}_\theta, -l) det_{\mathcal{O}_K}(H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)))$
   in $det_{\mathcal{O}_K \otimes \mathbb{R}}(H^w_B(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) \otimes \mathbb{R})$.

2. *The map* r_p *induces an isomorphism*

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(\mathcal{R}_\theta) \cong det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))^{-1}.$$

Here

$$L_S^*(\overline{\psi}_\theta, -l) = \lim_{s\to -l} \frac{L_S(\overline{\psi}_\theta, s)}{s+l},$$

and S is the set of primes of K dividing p and the ones dividing $\mathfrak{f}_\theta$.

Moreover, if r_p is injective on $\mathcal{R}_\theta$, the second part can be written as

$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}(H^1(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))/r_p(\mathcal{R}_\theta)) \cong$$
$$det_{\mathcal{O}_K\otimes\mathbb{Z}_p}H^2(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)).$$

Proof. It is direct consequence of the theorem 3.3.6 and the above corollary 3.5.12. □

As a consequence we obtain part of the local Tamagawa conjecture 3.1.4 for the motive $M_\theta(w+l+1)$, which corresponds to a more general version of Theorem B in the introduction.

Theorem 3.5.14. *Let p be a prime different from 2 and 3 (hence, in particular, $p \nmid \#|\mathcal{O}_K^*|$), and $p \nmid N_{K/\mathbb{Q}}\mathfrak{f}$. Consider l a strictly positive integer. Suppose that ψ_θ has infinity type $(w,0)$ or $(0,w)$, where $w \geq 1$ verifies $-w-2l \leq -3$ and $w \not\equiv 0 (mod|\mathcal{O}_K^*|)$. Suppose $\mathcal{O}_K^* \to (\mathcal{O}_K/\mathfrak{f}_\theta)^*$ in injective and that $Spec_{max}(\mathfrak{f}) = Spec_{max}(\mathfrak{f}_\theta)$.*

Moreover, suppose that χ, the representation of Δ in $Hom_{\mathcal{O}_p}(M_{\theta\mathbb{Z}_p}(w+l), \mathcal{O}_p)$, is a good representation which is not the cyclotomic character.

Then, there is a submodule $\mathcal{R}_\theta$ in $H_{\mathcal{M}}^{w+1}(M_\theta, \mathbb{Q}(w+l+1))$ such that:

1. *The map $r_{\mathcal{D}} \otimes \mathbb{R}$ is an isomorphism restricted to $\mathcal{R}_\theta \otimes \mathbb{R}$.*
2. *$dim_{\mathbb{Q}}(H_B^w(M_{\theta\mathbb{C}}, \mathbb{Z}(w+l)) \otimes \mathbb{Q}) = ord_{s=-l}L_S(H^w(M_\theta, \mathbb{Q}_p), s) = 2$.*
3. *We have the equality*

   $$r_{\mathcal{D}}(det_{\mathbb{Z}}(\mathcal{R}_\theta)) = L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l) det_{\mathbb{Z}}(H_B^w(M_\theta, \mathbb{Z}(w+l)))$$

   where

   $$L_S^*(H_{et}^w(M_\theta, \mathbb{Q}_p), -l) = \lim_{s\to -l} \frac{L_S(H_{et}^w(M_\theta, \mathbb{Q}_p), s)}{(s+l)^2}$$

   where S is the set of places of K that divides p and the places dividing the conductor $\mathfrak{f}_\theta$.
4. *We have that*

   $$det_{\mathbb{Z}_p}(\mathcal{R}_\theta) = det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}.$$

   If r_p is injective on $\mathcal{R}_\theta$, then $r_p(det_{\mathbb{Z}}(\mathcal{R}_\theta))$ is a basis of the $\mathbb{Z}_p$-lattice

   $$det_{\mathbb{Z}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1)))^{-1}$$
   $$\subset det_{\mathbb{Q}_p}(R\Gamma(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1) \otimes \mathbb{Q})[-1]).$$

Remark 3.5.15. *Theorem 3.5.14 corresponds to the p-local Tamagawa number conjecture 3.1.4 for Hecke characters. To prove the general assertion for the conjecture 3.1.4 remains to prove the finiteness of the second group of cohomology and the bijectivity of the p-Soulé regulator map.*

1. *The finiteness of $H^2(\mathcal{O}[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))$ (denote it by H^2_p) follows from a general conjecture of Jannsen [32], which affirms that*

$$H^2(\mathcal{O}_K[1/S'], H^w_{et}(E^w, \mathbb{Z}_p(w+l+1))) \text{ is finite,} \tag{3.4}$$

which implies the finiteness for our second Galois group. It is easy to prove following the proof of Wingberg in [62] that, if p is a regular prime for the field $K(E[p])$, then it satisfies (3.4) (for an explicit proof see B.3.7).

2. *Moreover, on the bijectivity of the Soulé regulator we make the following remarks.*

   (a) *Deninger proves the Beilinson conjecture for the above Hecke characters. He computes the dimension of the space $H^w_B(M_\theta, \mathbb{Q}(w+l))$ which it is mapped the Beilinson regulator map. He constructs elements on the K-theory group $H_\mathcal{M}$, generating a subspace which has the same dimension of $H^w_B(M_\theta, \mathbb{Q}(w+l))$. These subspace of constructed elements are denoted in the paper by $H^{constr}_\mathcal{M}$. It remains to prove on the Beilinson conjecture on Hecke characters of imaginary quadratic fields that the K-theory group $H_\mathcal{M}$ coincides with $H^{constr}_\mathcal{M}$. On the Soulé regulator map, the image of $H^{constr}_\mathcal{M}$ tensor by $\mathbb{Q}$ should map (if the Soulé regulator map is an isomorphism) to a free group $H^1(\mathcal{O}_K[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))\otimes\mathbb{Q}$ of the same rank as $H^{constr}_\mathcal{M}$. We note that the group $H^1(\mathcal{O}[1/S], M_{\theta\mathbb{Z}_p}(w+l+1))\otimes\mathbb{Q}$ has exactly this rank when H^2_p is finite, and bigger if not, see corollary 1 [32].*

   (b) *We observe that, if e_p is injective, then r_p is injective on $\mathcal{R}_\theta$, because the image of e_p is an $\mathcal{O}_\theta\otimes\mathbb{Z}_p$-module of rank 1 and $\mathcal{R}_\theta$ is a $\mathcal{O}_\theta\otimes\mathbb{Z}_p$-module of rank 1. Let us suppose that H^2_p is finite. Then, similar arguments as in §5.2.2 [36] would give the injectivity for e_p like in the situation of a Hecke character of type $(1,0)$ which is made in [36].*

As a conclusion, for regular primes p, we obtain in full generality the conjecture 3.1.4 for Hecke characters of imaginary quadratic fields.

Remark 3.5.16. *I can proof with a better definition of the elliptic units gives the above result without the hypothesis $S = S'$, i.e. without the hypothesis $Spec_{max}(\mathfrak{f}) = Spec_{max}(\mathfrak{f}_\theta)$, then I have to change the hypothesis $p \nmid N\mathfrak{f}$ by $p > N\mathfrak{f}$ to control the p-part of the factor appearing in corollary 3.5.10.*

Remark 3.5.17. *Moreover the above theorems 3.5.13 and 3.5.14, can be formulate in general for any Hecke character with θ with $\#|\theta| = 2$, which corresponds to any Hecke character $\psi_\theta : I_K \to K^*$ with $cl(K) = 1$ with K an imaginary quadratic field. The proof is exactly the same as above, under the condition if (a_θ, b_θ) is the infinite type then $a_\theta \not\equiv b_\theta (mod|\mathcal{O}_K^*|)$ instead of $w := a_\theta + b_\theta \not\equiv 0$. We obtain the same result, the local Tamagawa number conjecture, for all these characters.*

Appendix A

Elliptic units

Introduction

The elliptic units are generated by theta functions evaluated at torsion points with the root of unity. This appendix introduce theta functions making the text more self contain and write some properties on them which comes from the general classical properties on theta functions but not explicit write down.

A.1 Definition

Let's consider E an elliptic curve over a base scheme S, i.e. a proper smooth morphism $\overline{\pi} : E \to S$ together with a section $e : S \to E$, such that the geometric fibers $E_{\overline{s}}$ of $\overline{\pi}$ are connected curves of genus 1.

Theorem A.1.1 (see [55] theorem 1.2.1). *Let $a \in End_{\mathcal{O}_S}(E)$ be an endomorphism with $(6, a) = 1$(i.e. $ker(a) \cap ker(6) = e$). There is an unique section*

$$\theta_a \in \mathcal{O}^*(E \setminus ker\ a)$$

compatible with base change in S, with the following properties:

1. *$Div(\theta_a) = deg(a)(e) - ker\ a$,*
2. *for any $b \in End_{\mathcal{O}_S}(E)$ with $(a, b) = 1$*
$$b_*\theta_a = \theta_a.$$
3. *Moreover, for any $b \in End_{\mathcal{O}_S}(E)$ with $(6, b) = 1$ and $ab = ba$*
$$\frac{\theta_a \circ b}{\theta_a^{deg(b)}} = \frac{\theta_b \circ a}{\theta_b^{deg(a)}}.$$

Definition A.1.2. *The values of* θ_a *at torsion sections* $t : \mathcal{O}_S \to E \setminus ker\ a$ *are called elliptic units.*

Let's consider now an elliptic curve E over a number field F with CM by a ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K, with $K \subseteq F$ and $F(E_{tor})$ is an abelian extension of K. If follows that $K(1) \subset F$. Denote by $H = K(1)$. In this case, we can write explicitly the function $\theta_{\mathfrak{a}}$ with $\mathfrak{a} \in \mathcal{O}_K = End_F(E)$ and $(\mathfrak{a}, 6) = 1$ as follows:

$$\theta_{\mathfrak{a}}(z) = (\frac{\Delta(L)^{N(\mathfrak{a})}}{\Delta(\mathfrak{a}^{-1}L)})^{1/12} \prod_{P \in E[\mathfrak{a}] \setminus 0} (\mathcal{P}(z; L) - \mathcal{P}(u; L))^{-1},$$

where L is the period lattice for the elliptic curve E, Δ is the usual Ramanujan Δ-function and $\mathcal{P}$ is the Weierstrass $\mathcal{P}$-function. This is independent of the choice of the Weierstrass model (lemma 7.2 [52]), and, if E is defined over H, $\theta_{\mathfrak{a}}$ is also defined over H (see page 28 [51] or lemma in §II2.3 [14]). The function $\theta_{\mathfrak{a}}$ above is a 12-th root of the function $\theta(z; L, \mathfrak{a})$ in §II.2 [14]. Since $\theta_{\mathfrak{a}}$ is defined over H (we assume in the following that E is defined over H), we have the following results

Lemma A.1.3. *Let* T *be a point of* $\mathfrak{b}$*-torsion with* $(\mathfrak{a}, \mathfrak{b}) = 1$*. Then*

$$\theta_{\mathfrak{a}}(T) \in H(E[\mathfrak{b}]).$$

Lemma A.1.4 (Proposition 2.4 [14]). *Let* $\mathfrak{m}$ *be a non-trivial integral ideal of* K*, and* v *a primitive* $\mathfrak{m}$*- division point of* L*(i.e.* $v \in \mathfrak{m}^{-1}L$*, but* $v \notin \mathfrak{n}^{-1}L$ *for any proper divisor* $\mathfrak{n}$ *of* $\mathfrak{m}$*). Then, if* $(\mathfrak{a}, \mathfrak{m}) = 1$*,*

$$\theta_{\mathfrak{a}}(v) \in K(\mathfrak{m}).$$

We have also, using generators, an Euler system as follows, using an appropriate election of the 12-th root of unity taken in the definition of $\theta_{\mathfrak{a}}$.

Proposition A.1.5 (Proposition II-2.5[14]). *Let* $\mathfrak{m}$ *be a non-trivial integral ideal of* K*, and* $\mathfrak{n} = \mathfrak{m}\mathfrak{l}$*, with a prime* $\mathfrak{l}$*. Let* $e = w_{\mathfrak{m}}/w_{\mathfrak{n}}$ *where* $w_{\mathfrak{b}}$ *for an ideal* $\mathfrak{b}$ *are the roots of unity in* K *which are congruent to 1* $\mod \mathfrak{b}$*. Then, if* v *is a primitive* $\mathfrak{n}$*-division point of* L *and* $(\mathfrak{a}, \mathfrak{n}) = 1$*,*

$$N_{K(\mathfrak{g})/K(\mathfrak{f})}\theta_{\mathfrak{a}}(v)^e = \begin{cases} \theta_{\mathfrak{a}}(v)^{1-Frob_{\mathfrak{l}}^{-1}} & if\ \mathfrak{l} \not| \mathfrak{m} \\ \theta_{\mathfrak{a}}(v) & if\ \mathfrak{l} | \mathfrak{m}, \end{cases}$$

where $Frob_{\mathfrak{l}}$ *is the Frobenius of* $\mathfrak{l}$ *in* $Gal(K(\mathfrak{n})/K)$*.*

A.2 Some results on elliptic units

We write down some properties of elliptic units. Here we restrict to elliptic curves E with CM the ring of integers $\mathcal{O}_K$ of a quadratic imaginary field K. Moreover we suppose that E is defined over K. Then, with this restrictions, we have that $K = K(1)$.

Lemma A.2.1. *Let u_r denote a torsion point of E of exact order $\mathfrak{f}_\theta p^r$ for some fix integer $r > 1$ with $\mathfrak{f}_\theta|\mathfrak{f}$, and let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime with $\mathfrak{f}p$. Then,*

$$Norm_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-1}\mathfrak{f})}\theta_{\mathfrak{a}}(u_r) = \theta_{\mathfrak{a}}(\pi u_r),$$

where $\pi = \psi(\mathfrak{p})$ corresponds to endomorphism given by the ideal $\mathfrak{p}$.

Proof. We know that $\sigma \in Gal(K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-1}\mathfrak{f})$ act trivially in all point of $\mathfrak{f}$-torsion because $K(E[\mathfrak{f}]) = K(\mathfrak{f})$. Then, if we write $u_r = \tilde{u_r} + \overline{u_r} \in E[\mathfrak{p}^r] \oplus E[\mathfrak{f}]$, we have,

$$Norm_{K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-1}\mathfrak{f})}\theta_{\mathfrak{a}}(u_r) = \prod_{\sigma\in Gal(K(\mathfrak{p}^r\mathfrak{f})/K(\mathfrak{p}^{r-1}\mathfrak{f})} \theta_{\mathfrak{a}}(\tilde{u_r}^\sigma + \overline{u_r})$$

We can change the previous Galois group in the equality by the Galois group $Gal(K(E[\mathfrak{p}^r])K(\mathfrak{f})/K(E[\mathfrak{p}^{r-1}])K(\mathfrak{f})$, since $K(\mathfrak{f})$ and $K(E[\mathfrak{p}^r])$ are disjoint. Moreover, we can change this last Galois group, using cor.5.20 iii)[52], for the Galois group $Gal(K(E[\mathfrak{p}^r])/K(E[\mathfrak{p}^{r-1}])$. Since $\mathfrak{p}$ is prime to $\mathfrak{f}$ (cor.5.200 ii)[52]), we obtain that the norm is equal to (observe $\mathfrak{p}|(\mathfrak{p}^{r-1}\mathfrak{f})$),

$$\prod_{R\in E[\mathfrak{p}]} \theta_{\mathfrak{a}}(\tilde{u_r} + R + \overline{u_r}).$$

The distribution relation (see for example theorem 7.6 in [52]) implies the result. □

Lemma A.2.2. *Let t_r be a point of exact order $\mathfrak{f}p^r$. Write it as $t_r = t_{\mathfrak{f}} + \tilde{t_r} \in E[\mathfrak{f}] \oplus E[p^r]$. Let v be a generator of the ideal $\mathfrak{f}/\mathfrak{f}_\theta$ with $S = S'$. Let $\mathfrak{a}$ be an ideal of $\mathcal{O}_K$ prime to $6p\mathfrak{f}$. Then*

$$N_{K(\mathfrak{f})K(E[p^r])/K(\mathfrak{f}_\theta)K(E[p^r])}\theta_{\mathfrak{a}}(t_r) = \theta_{\mathfrak{a}}(vt_r).$$

Proof. Is enought to prove the result when $\mathfrak{f}/\mathfrak{f}_\theta$ is a prime $\mathfrak{b}$ of $\mathcal{O}_K$, the general case is obtained iterating this case. Since $K(E[p^n])$ and $K(\mathfrak{f}) = K(E[\mathfrak{f}])$ are disjoint and moreover $K(p^n)$ is disjoint with $K(\mathfrak{f})$ since $(\mathfrak{f}, p) = 1$, we can write the norm map as

$$N_{K(\mathfrak{f})K(E[p^r])/K(\mathfrak{f}_\theta)K(E[p^r])}\theta_{\mathfrak{a}}(t_r) =$$

$$\prod_{\sigma\in Gal(K(\mathfrak{f})K(E[p^r])K(p^r)/K(\mathfrak{f}_\theta)K(E[p^r])K(p^r))} \theta_{\mathfrak{a}}(t_{\mathfrak{f}}+\tilde{t_r})^{\sigma}.$$

We can write the last element as

$$\prod_{\sigma\in Gal(K(\mathfrak{f})K(p^r)/K(\mathfrak{f}_\theta)K(p^r))} \theta_{\mathfrak{a}}(t_r)^{\sigma},$$

since $\theta_{\mathfrak{a}}(t_r)\in K(\mathfrak{f})K(p^r)$. We use then cor.7.7 in [52], (since $K=K(1)$ and then $K(\mathfrak{f})K(p^r)=K(\mathfrak{f}p^r)$ because $(\mathfrak{f},p)=1$), to obtain then the result. □

Appendix B

On the local type of the Galois group and regularity for imaginary quadratic fields.

Introduction

This appendix presents the definition of regularity for imaginary quadratic fields under the important characterization that a global Galois group is isomorphic to a local Galois group, following the ideas of Yager and Wingberg. This definition follows also ideas to generalize the Kummer criterium for regular primes over $\mathbb{Q}$ to another number fields. Our definition of regularity implies that a global Galois group is local. Then we make an overview on the theory about how global Galois group of the form $Gal(F_S/F)$, where F_S is the maximal unramified extension of F, express in terms of local Galois groups; here appears naturally the definition on Galois group of local type. When we restrict over a quadratic imaginary field K and with p prime of $\mathbb{Q}$ which splits in K and there are regular, we can prove the Jannsen conjecture, which is presented in B.3, because some global Galois group is local, then we prove the conjecture on regular primes p for the motive $h^w(E^w)(w+k+1)$ with E a elliptic curve with CM $\mathcal{O}_K$ the ring of integers of an imaginary quadratic field, and we suppose also that E is defined over K. Here w is any integer $w \geq 1$ and $w \neq 2k$.

B.1 The definition of regularity, following the Kummer criterium, for imaginary quadratic fields.

Let first remember the definition of regularity for finite primes over $\mathbb{Q}$.

Definition B.1.1. *A finite prime p of $Spec(\mathbb{Q})$ is regular if the class number of the field $\mathbb{Q}(\mu_p)$ is prime to p, where μ_p is a primitive p-root of unity.*

Under this condition on the prime p for the diophantine equation $x^p+y^p = z^p$, Kummer proves the Fermat last Theorem. The general proof without no assumption of regularity is proved recently by Wiles, using completely different techniques.
The notion of regularity is related to the existence of unramified extensions, values of L-functions,.. this equivalent point of view of regularity is called "the Kummer criterium". Remember that for a one dimensional pure motives over $\mathbb{Q}$, the Riemann zeta function $\zeta(s)$ appears in the L-function associated to these motives. Let define

$$\zeta_\infty(k) := (k-1)!(2\pi i)^{-k}\zeta(k).$$

We known $\zeta_\infty(k) = -B_k/2k$ are p-integral values $(1 < k < p-1)$ with p an odd prime number, and B_k denote the Bernoulli number.

Theorem B.1.2 (Kummer criterium). *The following are equivalent:*

- *p is a regular prime for $Spec(\mathbb{Q})$.*
- *There is no unramified cyclic extension of $\mathbb{Q}(\mu_p)$ of degree p.*
- *Exist an unique cyclic extension of $\mathbb{Q}(\mu_p)^+$ of degree p which is unramified outside the prime dividing p, where $\mathbb{Q}(\mu_p)^+$ is the maximal real extension of $\mathbb{Q}$ inside $\mathbb{Q}(\mu_p)$.*
- *The numbers $\zeta_\infty(k)$ with k even and $1 < k < p-1$ are units in $\mathbb{Z}_p$.*

Let's consider now, K an imaginary quadratic field with $cl(K) = 1$. We also fix an elliptic curve E defined over K with complex multiplication $\mathcal{O}_K$, the ring of integers of K. Let's fix a prime $p > 3$ once and for all, such that it splits in K, $p = \mathfrak{p}\mathfrak{p}^*$, and E has good reduction on p. We denote by $\mathcal{F} := K(E[p])$, and let F be a Galois extension of K inside $\mathcal{F}$.

Definition B.1.3. *A finite prime $\mathfrak{p}$ of $SpecF$ is irregular if exist a cyclic extension of F of degree p which is unramified outside the primes of F dividing $\mathfrak{p}$ and which is disjoint with the $\mathbb{Z}_p$-extension K_∞ of K unramified outside $\mathfrak{p}$. Let define $\mathfrak{p}$ regular if it no satisfies the irregularity condition.*

Remark B.1.4. *Remind that the Leopoldt's conjecture is true for abelian extensions of imaginary quadratic fields. In particular for imaginary quadratic fields the conjecture claims the existence of exactly two independent $\mathbb{Z}_p$-extensions both unramified outside p, and for the case p split, one unramified outside $\mathfrak{p}$ and the other outside $\mathfrak{p}^*$ (see for example Ch.X § 7 theorem 10.3.6 and proposition 10.3.20(ii) [44]).*

We want to relate for this definition of regularity some kind of "Kummer criterium". Associated to the elliptic curve, we have the Grössencharacter ψ. In chapter 3 we proved that the associated motive $h_1(E)$ and the Künneth product of itself composing with an idempotent (here we restrict to infinite type $(m, 0)$ or $(0, m)$) defines naturally 1-dimensional motives with K-multiplication, and the L-functions associated to them corresponds to

$$L(\overline{\psi}^m, s)$$

with m an integer. We can ask, like in the case of the Riemann zeta function, for the integrality of the special values in the critical band (we remember the above L-functions satisfies a functional equation and they extend to all $\mathbb{C}$ as meromorphic functions).

Theorem B.1.5 (Damerell's theorem). *The numbers*

$$L_\infty(\overline{\psi}^{k+j}, k) := (2\pi/\sqrt{d_K})^j \Omega_\infty^{-(k+j)} L(\overline{\psi}^{k+j}, k),$$

$k \geq 1$, $j \geq 1$ belong to $\overline{K}$. Moreover if $0 \leq j < k$ they belong to K. Here d_K means the absolute value of the discriminant of $K/\mathbb{Q}$, and Ω_∞ means the complex period of the elliptic curve.

In order to obtain an analog of the Kummer criterium relating the no existence of unramified extension of degree p disjoint with a $\mathbb{Z}_p$-extension (definition of regularity B.1.3) with the value at some critical integers for the L-function (last point in Kummer criterium B.1.2), let's write χ_1, χ_2 the canonical characters with values in $\mathbb{Z}_p^*$ giving the action of $Gal(\overline{K}/K)$ on the $\mathfrak{p}$ and $\mathfrak{p}^*$-torsion points of E. It can be seen that χ_1, χ_2 generate $Hom(Gal(\overline{K}/K), \mathbb{Z}_p^*)$.
We have the following result in the direction to a Kummer criterium for imaginary quadratic fields.

Theorem B.1.6 (Yager, theorem 3 [64]). *Let F be any extension of K contained in $\mathcal{F}$. Then, the prime $\mathfrak{p}$ is irregular for F if and only if there exists integers k and j with $0 \leq j < p-1$, $1 < k \leq p$ with $j < k$ such that $\chi_1^k \chi_2^{-j}$ is a non-trivial character belonging to F (i.e. is trivial when is restricted to $Gal(\mathcal{F}/F)$) and $L_\infty(\overline{\psi}^{k+j}, k)$ is not a unit in $K_\mathfrak{p}$.*

The above result allow us define the regularity condition by (Wingberg [62]),

Definition B.1.7. *$\mathfrak{p}$ is regular for E and F if $\mathfrak{p}$ does not divide the numbers $L_\infty(\overline{\psi}^{k+j}, k)$ for all integers j, k with $1 \leq j < p-1$ and $1 < k \leq p$ such that $\chi_1^k \chi_2^{-j}$ is a non-trivial character belonging to F.*

Remark B.1.8. *The structure theorem for the field extensions $K(E[p^n])/K$, (see proposition 1.9 in [14]) affirms that $\mathfrak{p}$ is a prime ideal of F, and also $\mathfrak{p}^*$, because are primes of good reduction for the elliptic curve E.*

Let $F_{S_\mathfrak{p}}(p)$ be the maximal p-extension of F unramified outside $S_\mathfrak{p} := \{v| \ \ v \text{ divides } \mathfrak{p}\}$. From the first definition for regularity on primes of F we have,

Lemma B.1.9. *$\mathfrak{p}$ is regular for E and F if and only if $F_{S_\mathfrak{p}}(p)$ is a $\mathbb{Z}_p$-extension of F.*

Remark B.1.10. *Observe if $\mathfrak{p}$ is regular for E and F and $\mathfrak{p}^*$ is also regular for E and F, we obtain that the other r_2-1 independent $\mathbb{Z}_p$-extensions (which are unramified outside p) ramifies in both primes $\mathfrak{p}$ and $\mathfrak{p}^*$, where r_2 are the complex places of F that corresponds, since K is an imaginary quadratic field, to the value $[F : K]$, (see proposition 10.3.20 in [44]).*

Definition B.1.11. *Let p be a prime of $\mathbb{Q}$. Then p is regular for E and F if $\mathfrak{p}$ and $\mathfrak{p}^*$ are regular for E and F.*

Question B.1.12. *After this last definition, could we rewrite the condition of regularity for p for an elliptic curve E and a field F, in terms of p be prime to the class field number of a certain field? See corollary B.2.16 and question B.2.17.*

We also observe that the lemma B.1.9 gives us the possibility to extend the definition of regularity for any field. We extend the definition thinking that the Leopoldt's conjecture is true.

Definition B.1.13. *Let F be any number field such that is an abelian extension of $\mathbb{Q}$ or K, an arbitrary imaginary quadratic field (not necessarily of class 1). Let $\mathfrak{p}$ be a finite prime of F. We say that $\mathfrak{p}$ is a regular prime for F if and only if $F_{S_\mathfrak{p}}(p)$ is a $\mathbb{Z}_p$-extension. A prime of $\mathbb{Q}$, p, is regular for F if all the primes of F above p are regular for F.*

B.2 The regularity condition in the theory of Galois groups of local type

We are interested on the study of the cohomology of some $\mathbb{Z}_p$-modules for the Galois group $G_S := Gal(F_S(p)/F)$, with S a finite set of places that contains the primes of F dividing p, where $F_S(p)$ means the maximal p-extension unramified outside the primes of S. In this section F means an arbitrary number field if we do not say anything. Let's concentrate in this section on the study of G_S as a pro-p-group. Precisely, we ask when this group could be understand as a Galois group depending only on a finite place of F i.e. is of "local type".

Remark B.2.1. *We can change S taking out or adding the complex primes, the real primes if $p \neq 2$, and the primes $\mathfrak{a} \nmid p$ with $Norm(\mathfrak{a}) \not\equiv 1$ mod p, because these places cannot ramify in a p-extension.*

Observe for a non-archimedian place v of F we have the inclusion of the decomposition group of G_S with respect to the prime v in G_S. Write it by

$$\varphi_v : G_v := Gal(F_S(p)_v/F_v) \xrightarrow{\subseteq} G_S.$$

Definition B.2.2. *G_S is of local type if there exists a prime $v \in S$ such that $G_v = G_S$.*

Denote by $\mathcal{G}_v := Gal(F_v(p)/F_v)$ the Galois group of the maximal p-extension $F_v(p)$ of the local field F_v. We have the surjective natural map

$$i_{v,S} : \mathcal{G}_v \to G_v.$$

Definition B.2.3. *G_S is of maximal local type if G_S is of local type with respect some prime $v \in S$ and moreover $i_{v,S}$ is bijective.*

The bijectivity of the map $i_{v,S}$, is equivalent to $F(\mathfrak{c})_v = F_v(\mathfrak{c})$ for the particular $\mathfrak{c}$ equals to p; where $F(\mathfrak{c})$ means the maximal $\mathfrak{c}$-extension of F, i.e. the composite of all finite Galois extensions $L \mid F$ such that $Gal(L/F) \in \mathfrak{c}$ where $\mathfrak{c}$ means any full class of finite groups. About this question we have the Grunwald-Wang theorem (see a proof in 9.3.1 in [44]),

Theorem B.2.4 (Grunwald-Wang theorem by Neukirch). *Let F be a number field and let $\mathfrak{c}$ be a full class of finite groups. Let $\mathfrak{M}$ be a set of primes of F containing all the primes minus a finite set of primes, with $S_\infty \cup S_p = \{v \in F \mid v \text{ divides } p\} \subset \mathfrak{M}$ for all prime numbers p with $\mathbb{Z}/p\mathbb{Z} \in \mathfrak{c}$(more generally, is enough to suppose, that $S_p \cup S_\infty \subseteq \mathfrak{M}$, and that the*

Dirichlet density of $\mathfrak{M}$ is one. For the definition of the Dirichlet density, see for example §13 [43]). Then, for the maximal $\mathfrak{c}$-extension $F_{\mathfrak{M}}(\mathfrak{c})$ of F unramified outside $\mathfrak{M}$ and a prime $v \in \mathfrak{M}$, we have

$$(F_{\mathfrak{M}}(\mathfrak{c}))_v = F_v(\mathfrak{c})$$

or equivalently, the canonical surjective map $i_{v,\mathfrak{M}}$ is injective.

For the maximal local type we ask the truth of the previous theorem in the case that $\mathfrak{M}$ is finite for a particular v.
We can consider the composition

$$\varphi_v \circ i_{v,S} : \mathcal{G}_v \to G_S.$$

Definition B.2.5. *G_S is of purely local type if the map $\varphi_v \circ i_{v,S}$ is an isomorphism.*

Remark B.2.6. *This definition is agree with the original one of Wingberg in [62]. His definition is that the composition*

$$\mathcal{G}_v \stackrel{i_{v,}}{\hookrightarrow} Gal(F(p)_v/F_v) \stackrel{\alpha_v}{\hookrightarrow} Gal(F(p)/F) \stackrel{res}{\to} G_S,$$

where $F(p)$ the maximal p-extension of F and pr means the projection, is a bijection. Observe that we have the following commutative diagram:

$$\begin{array}{ccccc} \mathcal{G}_v & \stackrel{i_{v,S}}{\to} & Gal(F_S(p)_v/F_v) & \stackrel{\varphi_v}{\to} & Gal(F_S(p)/F) \\ \| & & \uparrow & & \uparrow \\ \mathcal{G}_v & \stackrel{i_{v,\emptyset}}{\to} & Gal(F(p)_v/F_v) & \stackrel{\alpha_v}{\to} & Gal(F(p)/F) \end{array}$$

where the vertical maps are the projection ones in the Galois groups.

From these definitions, and since $i_{v,S}$ is always surjective, we have the following result.

Corollary B.2.7. *1. If G_S is of maximal local type then it is of purely of local type.*

2. If G_S is of purely local type then it is of maximal local type.

We want to characterize algebraically these conditions of locality for the group G_S, following the work of Wingberg in [63].

Let's introduce some notation. We had already fixed a finite prime ideal p of $\mathbb{Q}$. Denote by S^f the set of finite primes in S.
We set

$$\delta = \begin{cases} 1 & , \quad \mu_p \subseteq F, \\ 0 & , \quad \mu_p \not\subseteq F, \end{cases} \quad , \quad \delta_v = \begin{cases} 1 & , \quad \mu_p \subseteq F_v, \\ 0 & , \quad \mu_p \not\subseteq F_v, \end{cases}$$

where v is a prime of F. Let $\mathcal{G}_v$ be the p-full local group $Gal(F_v(p)/F_v)$, and let $\mathcal{T}_v$ denote the inertia subgroup of $\mathcal{G}_v$. Suppose that we are given a subset $S_0 \subseteq S$. We define $V^S_{S_0}$ by

$$\{a \in K^* | a \in K_v^{*p} \ for \ v \in S_0 \ and \ a \in U_v K_v^{*p} \ for \ v \notin S\}/K^{*p},$$

where U_v is the unit group of the local field K_v (by convention $U_v = K_v^*$ if v is archimedian). Define

$$\mathfrak{B}^S_{S_0} = Hom(V^S_{S_0}, \mathbb{Z}/p\mathbb{Z}).$$

With this notation we obtain the following result of Kuz'min in Neukirch-Schmidt-Wingberg book [44],

Theorem B.2.8. *([44] theorem 10.7.2.) Let S_0 be a subset of S^f. The following assertions are equivalent.*

1. *There exists a finite set of primes $T \supseteq S$ such that the canonical homomorphism*
$$*_{v \in S\setminus S_0} \mathcal{G}_v * *_{v \in T \setminus S} \mathcal{G}_v/\mathcal{T}_v \rightarrow G_S$$
is an isomorphism.

2. $\mathfrak{B}^S_{S_0} = 0$ *and* $\sum_{v \in S_0} \delta_v = \delta$.

Furthermore, we have

$$\#(T \setminus S) = 1 + \sum_{v \in S_0 \cap S_p} [K_v : \mathbb{Q}_p] - \#(S \setminus S_0),$$

where S_p are the primes of K over p.

Corollary B.2.9 (Wingberg). *The group G_S is of purely local type if and only if F is totally imaginary if $p = 2$ and there exists a prime $\mathfrak{p}_0 \in S^f$ such that*

$$\sum_{S^f \setminus \{\mathfrak{p}_0\}} \delta_v = \delta, \quad \mathfrak{B}^S_{S\setminus\{\mathfrak{p}_0\}} = 0, \ and \ r_2 = \begin{cases} n_{\mathfrak{p}_0}, & , \quad \mathfrak{p}_0 | p, \\ 0, & , \quad \mathfrak{p}_0 \nmid p, \end{cases}$$

where r_2 is the number of complex primes of F, and n_v for a place v means the local degree $[F_v : \mathbb{Q}_l]$ of a non-archimedian prime $v|l$ of F.

Proof. First notice that for $p \neq 2$ or F totally imaginary, we have $\mathfrak{B}^S_{S^f\setminus\{\mathfrak{p}_0\}} = \mathfrak{B}^S_{S\setminus\{\mathfrak{p}_0\}}$ because the archimedian places cannot ramify in a p-extension. Applying the previous theorem, G_S is of purely local type if and only if in (i) on the above theorem exist $\mathfrak{p}_0$ which $\mathcal{G}_{\mathfrak{p}_0} \xrightarrow{\sim} G_S$. Then it satisfy $S \setminus S_0 = \{\mathfrak{p}_0\}$ and $\#(T \setminus S) = 0$. The assertion $\#(T \setminus S) = 0$ is now equivalent to

$$\#(S \setminus S_0) = r_1 + r_2 + 1 = 1 + \sum_{v \in S_p \setminus \{\mathfrak{p}_0\}} n_v = 1 + r_1 + 2r_2 - \begin{cases} n_{\mathfrak{p}_0}, & , \quad \mathfrak{p}_0 | p, \\ 0, & , \quad \mathfrak{p}_0 \nmid p, \end{cases}$$

where r_1 are the real primes of F. Using the equivalence on the above theorem between (i) and (ii) we obtain the result. □

We have a characterization of local type.

Theorem B.2.10 (Wingberg, theorem 1.6 in [63]). *The group G_S is of local type if and only if there exists a prime $\mathfrak{p}_0 \in S^f$ such that*

$$\sum_{S^f \setminus \{\mathfrak{p}_0\}} \delta_v - \delta + dim_{\mathbb{F}_p} \mathfrak{B}^S_{S \setminus \{\mathfrak{p}_0\}} + r_2 = \begin{cases} n_{\mathfrak{p}_0}, & , \quad \mathfrak{p}_0 | p, \\ 0, & , \quad \mathfrak{p}_0 \nmid p. \end{cases}$$

The most interesting example of local type is given by Wingberg in [62].

Theorem B.2.11 (Wingberg). *Let F be a number field between K and $\mathcal{F}$ following the notation on §1 in this appendix. The prime $\mathfrak{p}$ (split prime in K of p which E has good reduction in p) is regular for E and F if and only if $Gal(F_{S_p}(p)/F)$ is purely local with respect to $\mathfrak{p}^*$. Here S_p are the primes of F over p.*

Proof. (sketch) Consider the following commutative exact diagram:

$$\begin{array}{ccccccccc} 1 & \rightarrow & G(F_{S_p}(p)/F_{S_\mathfrak{p}}(p)) & \rightarrow & G(F_{S_p}(p)/F) & \rightarrow & Gal(F_{S_\mathfrak{p}}/F) & \rightarrow & 1 \\ & & \uparrow & & \varphi_{\mathfrak{p}^*}\uparrow & & \phi_{\mathfrak{p}^*}\uparrow & & \\ 1 & \rightarrow & I(F_{\mathfrak{p}^*}(p)/F_{\mathfrak{p}^*}) & \rightarrow & G(F_{\mathfrak{p}^*}(p)/F_{\mathfrak{p}^*}) & \rightarrow & Gal(F^{nr}_{\mathfrak{p}^*}(p)/F_{\mathfrak{p}^*}) & \rightarrow & 1 \end{array}$$

Observe that $\varphi_{\mathfrak{p}^*}$ is an isomorphism. Hence $\phi_{\mathfrak{p}^*}$ is surjective, implying that $F_{S_\mathfrak{p}}(p)/F$ is a $\mathbb{Z}_p$-extension. Then, for lemma B.1.9, we have that $\mathfrak{p}$ is regular for E and F.
Supose $\mathfrak{p}$ is regular, then $\phi_{\mathfrak{p}^*}$ is isomorphism. This implies that $\varphi_{\mathfrak{p}^*}$ is surjective. Let R be the kernel of $\varphi_{\mathfrak{p}^*}$. The H-S spectral sequence gives

$$0 \rightarrow H^1(G(F_{S_p}(p)/F), \mathbb{Q}_p/\mathbb{Z}_p) \rightarrow H^1(G(F_{\mathfrak{p}^*}(p)/F_{\mathfrak{p}^*}), \mathbb{Q}_p/\mathbb{Z}_p)$$

$$\rightarrow H^1(R, \mathbb{Q}_p/\mathbb{Z}_p)^{Gal(F_{S_p}(p)/F)} \rightarrow 0$$

because $H^2(G(F_{S_p}(p)/F), \mathbb{Q}_p/\mathbb{Z}_p) = 0$ since the Leopoldt conjecture is true for abelian extension of an imaginary quadratic field K. We have the (in)-equalities

$$corank_{\mathbb{Z}_p} H^1(G(F_{S_p}(p)/F), \mathbb{Q}_p/\mathbb{Z}_p) =$$

$$[F:K]+1 = corank_{\mathbb{Z}_p} H^1(G(F_{\mathfrak{p}^*}(p)/F_{\mathfrak{p}^*}), \mathbb{Q}_p/\mathbb{Z}_p)$$

and

$$dim_{\mathfrak{f}_p} H^1(G(F_{S_p}(p)/F), \mathbb{Z}/p\mathbb{Z}) \geq$$
$$[F:K]+1+\delta_{\mathfrak{p}^*} = dim_{\mathbb{F}_p} H^1(G(F_{\mathfrak{p}^*}(p)/F_{\mathfrak{p}^*}), \mathbb{Z}/p\mathbb{Z}).$$

We obtain then $H^1(R, \mathbb{Q}_p/\mathbb{Z}_p)^{Gal(F_{S_p}(p)/F)} = 0$ using results on pro-p-groups ([38]), and therefore $R = 0$. □

Remark B.2.12. *The last result only uses the characterization of regularity via the general definition in §1, plus that p splits in two primes in F, $\mathfrak{p}$ and $\mathfrak{p}^*$. Then we can extend the previous result to any $\mathfrak{p}$ prime of F regular such that if p is the prime in $\mathbb{Q}$ where $\mathfrak{p}$ lies, there are only two primes that lies above p in F, say them $\mathfrak{p}$ and $\mathfrak{p}^*$.*

From now on F denotes a field under the hypothesis of §1 on this appendix.

Corollary B.2.13 (Grunwald-Wang theorem for regular primes). *If $\mathfrak{p}$ is regular for E and F, then the maximal p-extension $F_{\mathfrak{p}^*}(p)$ of $F_{\mathfrak{p}^*}$ coincides with the completion of the maximal p-extension $F_{S_p}(p)$ of F_{S_p} with respect to $\mathfrak{p}^*$.*

Corollary B.2.14. *Suppose F does not have real primes. Then, $\mathfrak{p}$ is regular for E and F, if and only if*

$$\sum_{S_p \setminus \{\mathfrak{p}^*\}} \delta_v = \delta, \quad \mathfrak{B}_{\mathfrak{p}}^{S_p} = 0, \ \text{and} \ \ [F:K] = [F_{\mathfrak{p}^*} : \mathbb{Q}_p].$$

Proof. It is a direct consequence of the characterization of purely local type Galois groups B.2.9, applied to the purely local type Galois group $Gal(F_{S_p}(p)/F)$ when $\mathfrak{p}$ is regular (see theorem B.2.11). □

Corollary B.2.15. *Suppose $\mu_p \subset F$ and F does not have real primes (for example for $F = \mathcal{F}$). Then p is regular for E and F if and only if $Gal(F_{S_p}(p)/F)$ is purely of local type for $\mathfrak{p}$ and for $\mathfrak{p}^*$, and if and only if $\delta_{\mathfrak{p}} = \delta_{\mathfrak{p}^*} = 1$ and $\mathfrak{B}_{\mathfrak{p}}^{S_p} = \mathfrak{B}_{\mathfrak{p}^*}^{S_p} = 0$.*

Proof. The first afirmation is clear from theorem B.2.11 taking the primes $\mathfrak{p}$ and $\mathfrak{p}^*$. The second is clear using the characterization of the purely local type groups B.2.9, and using that the condition of local degree is always satisfied in this situation,(see II1.9 in [14]) and that for the Weil pairing we know that the p-roots of unity are in $\mathcal{F}$. □

We note that the objects $\mathfrak{B}$ are the objects that appears for the Kummer criterium in the case of an imaginary quadratic field that substitutes be prime to the class field number. Let me write this down in a particular case.

Corollary B.2.16. *Let p be a regular prime for E and F, such that $\mu_p \subset F$. Then $(p, Cl_{S_p}(F)) = 1$, where $Cl_{S_p}(F)$ means the S_p-class group for the field F.*

Proof. Since p is regular, we have that $\mathfrak{B}_{\mathfrak{p}}^{S_p}$ and $\mathfrak{B}_{\mathfrak{p}^*}^{S_p}$ are zero and $\mu_p \subset F_{\mathfrak{p}}$ and $F_{\mathfrak{p}^*}$. This implies in particular that $V_{\mathfrak{p}}^{S_p}$ and $V_{\mathfrak{p}^*}^{S_p}$ are prime to p, in particular $V_{S_p}^{S_p} = V_{\mathfrak{p}}^{S_p} \cap V_{\mathfrak{p}^*}^{S_p}$ is prime to p. Then, by 10.7.1 [44], we have

$$V_{S_p}^{S_p} = ker(H^1(Gal(F_{S_p}/F), \mu_p) \to \oplus_{v \in S_p} H^1(F_v, \mu_p)).$$

Since μ_p is contained in the local fields and in the field F, $V_{S_p}^{S_p}$ can be computed as the kernel of the previous groups for the module $\mathbb{Z}/p\mathbb{Z}$ instead of μ_p. Then using lemma 8.6.3 in [44], we have the equality

$$V_{S_p}^{S_p} = Hom(Cl_{S_p}(F), \mathbb{Z}/p\mathbb{Z})$$

implying that $(Cl_{S_p}(F), p) = 1$. □

Question B.2.17. *Let p be a prime such that $\mu_p \subseteq F$, $F_{\mathfrak{p}}$ and $F_{\mathfrak{p}^*}$, and suppose we are under the hypothesis that we need for the definition of regularity.*

Is it true that $(CL_{S_p}(F), p) = 1$ implies that p is regular for E and F?

A positive answer to this question would imply in particular the first part of the Kummer criterium for $\mathbb{Q}$ in the case of the extension of regularity for imaginary quadratic fields.

B.3 The Jannsen conjecture on local type Galois group

Jannsen in [32] presents a conjecture on the cohomology of the Galois groups G_S generalizing the weak Leopoldt's conjecture, which is related with the finiteness of the Tate-Shafarevich group for motives in the statement of the Tamagawa number conjecture [6]. Let's introduce the notation for the formulation of the conjecture. Let F be a number field with algebraic closure $\overline{F}$. Let X be a smooth, projective variety of pure dimension d over F. Let p be a prime number, and S a finite set of places of F, containing all places above ∞ and p, and all primes where X has bad reduction. Let G_S be the Galois group over F of the maximal S-ramified (unramified outside of S) extension F_S of F.

Conjecture B.3.1 (Jannsen). *If $\overline{X} = X \times_F \overline{F}$, then*

$$H^2(G_S, H^i_{et}(\overline{X}, \mathbb{Q}_p(n))) = 0 \; if \; \begin{cases} a) \; i+1 < n, & or \\ b) \; i+1 > 2n. \end{cases}$$

Observe that the nullity of the previous Galois cohomology groups for the étale cohomology with $\mathbb{Q}_p$-coefficients, can be rewritten in terms of the étale cohomology with $\mathbb{Z}_p$ or $\mathbb{Q}_p/\mathbb{Z}_p$-coefficients, as follows:

Lemma B.3.2 (lemma 1 [32]). *The following statements are equivalent:*

1. $H^2(G_S, H^i(\overline{X}, \mathbb{Q}_p(n))) = 0$.
2. $H^2(G_S, H^i(\overline{X}, \mathbb{Z}_p(n)))$ *is finite.*
3. $H^2(G_S, H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$ *is finite.*
4. *(if $p \neq 2$ or F is totally imaginary)* $H^2(G_S, \tilde{H}^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n))) = 0$ *where* $\tilde{H}^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)) = p - Div(H^i(\overline{X}, \mathbb{Q}_p/\mathbb{Z}_p(n)))$.

Observe that, as the étale cohomology groups gives us $\mathbb{Z}_p$-modules, the cohomology group do not change if we change G_S by $Gal(F_S(p)/F)$ where $F_S(p)$ is the maximal p-extension of F inside F_S. We denote this last group also with the same notation G_S, and from now on G_S is the Galois group of the maximal p-extension inside F_S(it agrees then with the notation on §2 in this appendix).

Lemma B.3.3. *Let $S' \subseteq S$ that contain some finite places. We suppose that the Galois group $Gal(F_{S'}(p)/F)$ is purely of local type for a place $w \in S'$ of F. Furthermore let M be a p-primary divisible $Gal(F_{S'}(p)/F)$-module of cofinite type such that, for all $v \in S \setminus (S' \setminus \{w\})$ with $\mu_p \in F_v$, the $Gal(F_v(p)/F_v)$-coinvariants of $M(j-1)$, $M(j-1)_{Gal(F_v(p)/F_v)}$, are zero. Then*

$$H^2(G_S, M(j)) = 0.$$

Proof. The H-S spectral sequence gives the exact sequence:

$$H^2(Gal(F_{S'}(p)/F), M(j)) \to H^2(G_S, M(j))$$

$$\to H^1(Gal(F_{S'}(p)/F), H^1(Gal(F_S(p)/F_{S'}(p)), M(j))).$$

Denote by A this last Galois cohomology group. Since, for hypothesis, $M(j)$ is a trivial $Gal(F_S(p)/F_{S'}(p))$-module, we obtain (using Shapiro lemma and Satz 4.1 in [42]) that

$$A = \oplus_{v \in S \setminus S'} H^1(Gal(F_v^{nr}(p)/F_v), H^1(I(F_v(p)/F_v), M(j)))$$

where I denotes the inertia subgroup. Then, from the HS-spectral sequence, this cohomology group is equal to

$$\oplus_{v \in S \setminus S'} H^2(Gal(F_v(p)/F_v), M(j)).$$

The problem reduces now to a problem of cohomology of local fields, very well understood, see chapter. VII in [44]. If $\mu_p \not\subseteq F_v$, then $Gal(F_v(p)/F)$ is free, hence its second cohomology group is zero; otherwise it is a Poincaré group of dimension two with dualising module $\mathbb{Q}_p/\mathbb{Z}_p(1)$. Hence, using local Tate duality, we have that

$$H^2(Gal(F_v(p)/F_v), M(j)) = \varprojlim_n H^0(Gal(F_v(p)/F_v), Hom(M_{p^n}(j), \mathbb{Q}_p/\mathbb{Z}_p(1))^*$$
$$= M(j-1)_{Gal(F_v(p)/F_v)},$$

where $M_{p^n} = \{x \in M| \ p^n x = 0\}$, and $*$ denotes $Hom(-, \mathbb{Q}_p/\mathbb{Z}_p)$. Observe that, for hypothesis, for all $v \in S \setminus S'$ with $\mu_p \subset F_v$ we have $M(j-1)_{Gal(F_v(p)/F_v)} = 0$ proving that $A = 0$.
Let's study the first factor on the exact sequence coming from the H-S spectral sequence. Since $Gal(F_{S'}(p)/F)$ is purely of local type with respect to w we obtain

$$H^2(Gal(F_{S'}(p)/F), M(j)) = H^2(Gal(F_w(p)/F_w), M(j)).$$

We have reduced to proof that this local group is zero, the proof is exactly the same made before with v instead of w. □

Consider F and p under the condition in §1 of this appendix. We suppose also $p > 3$. Consider $X = E$ the elliptic curve fixed on §1 with CM by the ring $\mathcal{O}_K$ of integers of an imaginary quadratic field $K \subseteq F$, which is defined over K, and let S_K be the places of K where E has bad reduction. We denote by S_F the places of F where $E \times_K F$ has bad reduction. Observe that if $\mathcal{F} = K(E[p])$ we know that $E \times_K \mathcal{F}$ has good reduction for all finite prime outside p, which is also of good reduction for hypothesis; in particular, it has good reduction everywhere.

Corollary B.3.4 (Wingberg, [62]). *Let p be a regular prime for E and $\mathcal{F} = K(E[p])$, i.e. $\mathfrak{p}$ and $\mathfrak{p}^*$ are regular for E and $\mathcal{F}$. Let F be an extension on K inside $\mathcal{F}$. Denote by S^* the primes of F containing the primes over p and the ones of bad reduction for the elliptic curve $E \times_K F$. Then,*

$$H^2(Gal(F_{S^*}/F), H^1(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(j+1))) = 0$$

for all integer j.

Proof. By Kummer theory, we have that

$$H^1_{et}(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(1)) = E[p^\infty] = E[\mathfrak{p}^\infty] \oplus E[(\mathfrak{p}^*)^\infty]$$

where $E[a^\infty] = \varinjlim_n n E[a^n]$ is the inductive limit of a^n-torsion points for the elliptic curve E. Since $K \subseteq F \subseteq \mathcal{F}$, is enough to prove

$$H^2(Gal(\mathcal{F}_{S^*}(p)/F), E[p^\infty](j)) = 0,$$

because $F \subseteq \mathcal{F}$ is unramified outside S^* (lemma II.1.9 [14]) and hence $\mathcal{F}_{S^*}(p) = F_{S^*}(p)$. Observe that $E[\mathfrak{p}^\infty]$ and $E[(\mathfrak{p}^*)^\infty]$ are trivial $Gal(\mathcal{F}_{S^*}(p)/\mathcal{F}_{S_p}(p))$-modules, where S_p are the places of $\mathcal{F}$ above p (see for example II 1.9 in [14], that controls the unramified extension $\mathcal{F}(E[p^\infty])/\mathcal{F}$ and the fact that, over $\mathcal{F}$, the elliptic curve has good reduction for all prime outside the ones that divide p (we use here that $p > 3$, see for example 1.3 in [49])). From the H-S spectral sequence we have that, (where we write $M = E[p^\infty]$ to simplify notation):

$$H^2(Gal(\mathcal{F}_{S_p}(p)/F), M(j)) \to H^2(Gal(\mathcal{F}_{S^*}(p)/F), M(j))$$

$$\to H^1(Gal(\mathcal{F}_{S_p}(p)/F), H^1(Gal(\mathcal{F}_{S^*}(p)/\mathcal{F}_{S_p}(p)), M(j))).$$

We want to see that the factor in the middle is zero.
Let's begin with the factor on the right. Applying the H-S spectral sequence we have, denoting $M' := H^1(Gal(F_{S^*}(p)/F_{S_p}(p)), M(j))$ and $H := Gal(\mathcal{F}_{S_p}(p)/\mathcal{F})$, that:

$$H^1(Gal(\mathcal{F}/F), (M')^H) \to H^1(Gal(\mathcal{F}_{S_p}/F), M') \to$$

$$H^1(H, M')^{Gal(\mathcal{F}/F)} \to H^2(Gal(\mathcal{F}/F), (M')^H).$$

Since $Gal(\mathcal{F}/F)$ is a finite group of order prime to p, the first and the last factor vanish. We prove that $H^1(H, M') = 0$. Using the proof of the last lemma we have that

$$H^1(H, M') = \oplus_{v \in S^* \setminus S_p} \delta(M(j-1)_{Gal(\mathcal{F}_v(p)/\mathcal{F}_v)}),$$

where δ is the Dirac operator such that $\delta(M(j-1)_{Gal(\mathcal{F}_v(p)/\mathcal{F}_v)}) = 0$ if $\mu_p \not\subseteq \mathcal{F}_v$ and $\delta(M(j-1)_{Gal(\mathcal{F}_v(p)/\mathcal{F}_v)}) = M(j-1)_{Gal(\mathcal{F}_v(p)/\mathcal{F}_v)}$ if $\mu_p \subseteq \mathcal{F}_v$. Since $E \times_F \mathcal{F}$ has good reduction everywhere, in particular in the places of $S^* \setminus S_p$, we have from the Weil conjectures

$$M(j-1)_{Gal(\mathcal{F}_v(p)/\mathcal{F}_v)} = 0,$$

that here we have used the local Tate duality because the above coinvariants correspond to a second group of cohomology, then by Tate duality, dual to a zero group of cohomology on a local Galois group. Then we use that in this dual Tate group: the weight of the Galois module which we take invariants is different of zero, $1 \neq 2j$, obtaining the result.

Thus,

$$H^1(Gal(\mathcal{F}_{S_p}/F), H^1(Gal(\mathcal{F}_{S^*}(p)/F_{S_p}(p)), M(j))) = 0.$$

To prove that

$$H^2(Gal(F_{S^*}/F), M(j)) = H^2(Gal(F_{S^*}(p)/F), M(j)) =$$

$$H^2(Gal(\mathcal{F}_{S^*}(p)/F, M(j))) = 0,$$

it remains, using the HS-spectral sequence above, to show that

$$H^2(Gal(\mathcal{F}_{S_p}(p)/F), M(j)) = H^2(Gal(\mathcal{F}_{S_p}(p)/\mathcal{F}), M(j)) = 0.$$

The equality on the above two Galois cohomology groups is because $\#|Gal(\mathcal{F}/F)|$ is prime to p.

We know that $\mathfrak{p}^*$ is regular in $\mathcal{F}$. Then we have by theorem B.2.11 that

$$Gal(\mathcal{F}_{S_p}(p)/\mathcal{F}) \cong Gal(\mathcal{F}_{\mathfrak{p}}(p)/\mathcal{F}_{\mathfrak{p}}).$$

On the other hand, we know from corollary B.2.9 that, since $\mu_p \subseteq \mathcal{F}_{\mathfrak{p}}$,

$$H^2(Gal(\mathcal{F}_{S_p}(p)/\mathcal{F}), M(j)) = H^2(Gal(\mathcal{F}_{\mathfrak{p}}(p)/\mathcal{F}_{\mathfrak{p}}), M(j)) =$$

$$= M(j-1)_{Gal(\mathcal{F}_{\mathfrak{p}}(p)/\mathcal{F}_{\mathfrak{p}})}.$$

Then we take the coinvariants of

$$M(j-1)_{Gal(\mathcal{F}_{\mathfrak{p}}(p)/\mathcal{F}_{\mathfrak{p}})}$$

that are zero since $E \times_F \mathcal{F}$ has good reduction on the prime $\mathfrak{p}$, and then from the Weil conjectures we conclude since the weight is not zero (here we use also the local Tate duality to relate the coinvariants with the invariants of a $\mathbb{Z}_p$-module with Galois action with weight $1-2j$, not zero). □

Remark B.3.5. *The proof of the above proposition only uses that $\mathfrak{p}^*$ is a regular prime for E and $\mathcal{F}$. We note that applying the complex multiplication action, $\mathfrak{p}^*$ regular for E and $\mathcal{F}$ implies the regularity for $\mathfrak{p}$ for E and $\mathcal{F}$ (using the structure of $K \subset \mathcal{F}$).*

Remark B.3.6. *The general conjecture of Jannsen does not consider the values $j = 0$ and 1 in the case where X corresponds to an elliptic curve defined over F. Let's make some remarks for these obstructions for elliptic curves with CM. For $j = 1$ we observe that (cf. [32])*

$$H^2(Gal(\overline{F}/F), H^1(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(2))) \cong \oplus_{\mathfrak{a}\in S_0}(E[p^\infty])_{Gal(\overline{F_{\mathfrak{a}}}/F_{\mathfrak{a}})},$$

where S^0 runs the finite set of places where E does not have potentially good reduction. In the CM case we know that all elliptic curves have potentially good reduction, then this obstruction for $j = 1$ disappears when we restrict in the CM situation. The case $j = 0$, we can compute the Euler characteristic for the G_S-module $H^1(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(1))$. Then we obtain the corank of the group $H^2(G_S, H^1(\overline{E}, \mathbb{Q}_p/\mathbb{Z}_p(1)))$ is bigger or equal to $rank_{\mathbb{Q}} E(F) - [F : \mathbb{Q}]$. The above result of Wingberg implies in particular that for CM elliptic curves defined over F we have $rank_{\mathbb{Q}} E(F) \leq [F : \mathbb{Q}]$.

With the same proof of the previous corollary B.3.4 but with $M = H^w((E \times_K F)^w \times_F \overline{F}, \mathbb{Q}_p/Z_p(w))$ instead of $H^1(E \times_K F, \mathbb{Q}_p/\mathbb{Z}_p(1))$, where $(E \times_K F)^w = (E \times_K F) \times_F \overset{w}{\ldots} \times_F (E \times_K F)$, we obtain the following result.

Corollary B.3.7. *Let p be a regular prime for E and $\mathcal{F} = K(E[p])$. Let F be an extension of K inside $\mathcal{F}$. Let S^* be the set of primes of F plus the primes over p and the ones of bad reduction of the elliptic curve $E \times_K F$. Fix an integer $w \geq 1$. Then*

$$H^2(Gal(F_{S^*}/F), H^w(\overline{E}^w, \mathbb{Q}_p/\mathbb{Z}_p(w + j))) = 0$$

for all integer j such that $w \neq 2j$.

Appendix C

Relation with p-adic L-functions

Introduction

This appendix is the natural prolongation of chapter 3. We follow the notations of chapter 3. In this appendix we compare the global result which gives the local Tamagawa number conjecture for power series of the Hecke character ψ associated to a CM elliptic curve E defined over an imaginary quadratic field K, with the local result given by the work of Geisser [25]. In Geisser's work it is studied the coimage of the map e_p in the local Galois cohomology instead of global Galois cohomology that we did in chapter 3. Using as the key point the specialization of the polylogarithm map, we compare the elements on K-theory with the elements on the Iwasawa modules that appear when we study the image by the Soulé regulator map or e_p respectively. This appendix compares our computation in the global Galois cohomology with the study made for e_p in Geisser's work [25].

C.1 Relation with Geisser's p-adic analogue of Beilinson's conjectures

Suppose in this section that the prime p split in K as $p = \mathfrak{p}\mathfrak{p}^*$, with $\mathfrak{p} \neq \mathfrak{p}^*$. We want relate our theorem 3.5.14 in chapter 3 with the result of Geisser (see [24][25]), in the case that $\#|\theta| = 2$. Observe that, in our situation, the motive $M_\theta \otimes \mathbb{Z}_p$ has multiplication by $(\mathcal{O}_\theta \otimes \mathbb{Q}) \otimes \mathbb{Z}_p$ and it decomposes in two local fields. We obtain two idempotents Ω_1 and Ω_2 such that they give the direct sum $M_\theta \otimes \mathbb{Z}_p = M_{\Omega_1} \oplus M_{\Omega_2}$. We obtain also a decomposition in direct sum of the map $e_p = e_{\Omega_1} \oplus e_{\Omega_2}$ as follows (we use the same notation that in Chapter 3, §4)

$$(\overline{\mathcal{C}}_\infty \otimes M_{\Omega_1 \mathbb{Z}_p}(w+l))_{\mathcal{G}} \oplus (\overline{\mathcal{C}}_\infty \otimes M_{\Omega_2 \mathbb{Z}_p}(w+l))_{\mathcal{G}} \to$$

$$H^1(K, M_{\Omega_1 \mathbb{Z}_p}(w+l+1)) \oplus H^1(K, M_{\Omega_2 \mathbb{Z}_p}(w+l+1))$$

where $M_{\Omega_i \mathbb{Z}_p}$ means the p-adic lattice corresponding to $H^w_{et}(\overline{M}_{\Omega_i}, \mathbb{Z}_p)$.

Remark C.1. *To unify notation on both regulators, we note the following equality*

$$H^1(K, M_{\theta \mathbb{Z}_p}(w+l+1)) = H^1(\mathcal{O}_K[1/S], M_{\theta \mathbb{Z}_p}(w+l+1)).$$

To obtain this equality, one uses the localization exact sequence, that the inertia acts trivially at the places not in S, and an argument using weights.

Let's consider the local map in Galois cohomology $\iota : H^1(K, \quad) \to H^1(K_{\mathfrak{p}}, \quad)$. Observe that $\psi_\theta \otimes \mathbb{Z}_p = \psi_{\Omega_1} \oplus \psi_{\Omega_2}$ and satisfy $\overline{\psi_{\Omega_1}} = \psi_{\Omega_2}$. If (a_θ, b_θ) denotes the infinity type of ψ_θ, and we suppose that Ω_1 comes from $\mathfrak{p}$ and Ω_2 from $\mathfrak{p}^*$ by using the identification $\Omega_\theta = \mathcal{O}_K$, then the infinity types for ψ_{Ω_1} and ψ_{Ω_2} are (a_θ, b_θ) and (b_θ, a_θ) respectively. Here the infinity types comes from morphisms to $\mathbb{C}_p$, we refer to [25] for more details.

Theorem C.2 (Geisser, theorem 9.1 [25]). *Suppose that $p > 3w + 2l + w + 1$, $w + l > 0$, $l > 0$ and $\#|\theta| = 2$. Then, the length as an $\mathcal{O}_{\Omega_i} \cong \mathbb{Z}_p$-module of the coimage of the Geisser elliptic units $\overline{\mathcal{C}}^{Rob}$ (defined below) via the map $\iota \circ e_{\Omega_i}$ is equal to the p-adic valuation of the p-adic L-function*

$$G(\psi_{\Omega_i} \kappa^l, u_1^{-a_i - 1} - 1, u_2^{-b_i - 1} - 1)$$

where (a_i, b_i) is the infinity type for ψ_{Ω_i}, and G is the $\psi_{\Omega_i}^{-1}$-component of the two variable p-adic L-function (see p.227 [25] for an explicit definition).

Let's now define the Geisser elliptic units; we will prove that their coincide with our elliptic units. The Geisser elliptic units are a modification of the elliptic units in the book of de Shalit [14].

We follow III§1 in [14]. Let's consider the ideal of K given by $\mathfrak{g} := \mathfrak{f}\mathfrak{p}^{*\mathfrak{n}}$. We define by $C_{n,m}$ the group generated by the primitive Robert units of conductor $\mathfrak{g}\mathfrak{p}^{\mathfrak{m}}$. They are given by $\theta_{\mathfrak{a}}(t_{\mathfrak{f}} + \mathfrak{h}_{n,m})$ where $\mathfrak{h}_{n,m}$ is a point of $\mathfrak{p}^{*n}\mathfrak{p}^m$-torsion, with $(\mathfrak{a}, 6\mathfrak{g}\mathfrak{p}) = 1$, times the roots of unity in $K(\mathfrak{g}\mathfrak{p}^m)$. Define $C_{\mathfrak{g}} = C_{\mathfrak{f}\mathfrak{p}^{*\mathfrak{n}}}$ by the projective limit via Norm maps of $C_{n,m}$ varying m. We take the clausure of $C_{\mathfrak{g}}$ inside the local units and we denote it with a bar.

We define

$$C(\mathfrak{f}) := \varprojlim_n \overline{C}_{\mathfrak{f}\mathfrak{p}^{*n}}.$$

Then the Geisser elliptic units over K are defined by taking the norm map from $K(\mathfrak{f})$ to K of $C(\mathfrak{f})$. We will denote this elliptic units by $\overline{C}^{Rob}$.

Lemma C.3. *The Geisser elliptic units coincide with our elliptic units.*

Proof. First of all we have to notice that $Norm_{K(\mathfrak{f})/K}$ commutes with the projective limit. This is because we have

$$N_{K(\mathfrak{f})/K}\varprojlim = \varprojlim N_{K(\mathfrak{f})/K},$$

where the projective limit on the left is over $K(\mathfrak{f}\mathfrak{p}^{\mathfrak{m}}(\mathfrak{p}^*)^{\mathfrak{n}})$ running n, m, and on the right over $K(E[\mathfrak{p}^m\mathfrak{p}^{*n}])$ running also n, m; and this equality comes from the fact that $K(E[\mathfrak{g}]) = K(\mathfrak{g})$ if $\mathfrak{f}$ divides $\mathfrak{g}$.

We need to proof then that $\varprojlim_n \varprojlim_m N_{K(\mathfrak{f})/K} C_{n,m}$ is equal to $\varprojlim_k N_{K(\mathfrak{f})/K} C_{k,k}$, because our elliptic units come taking limit from $N_{K(\mathfrak{f})/K} C_{k,k}$. But we claim that the equality of the both projective limits coincide, because the diagonal gives all the information. □

We remember now some facts on p-adic L-functions in our situation. Let's denote $\mathbb{D}$ the ring of integers of the maximal unramified extension of $K_{\mathfrak{p}}$; we have that all characters on finite groups of order prime to p have values in $\mathbb{D}$.

Observe that $\Gamma \cong \Gamma_1 \times \Gamma_2$ with $\Gamma_i \cong \mathbb{Z}_p$ and $\Gamma_1 = Gal(K(E[\mathfrak{p}^\infty]))/K(E[\mathfrak{p}])$ and Γ_2 with $\mathfrak{p}^*$ instead of $\mathfrak{p}$. Let γ_i be a generator of Γ_i, and let κ_i be the character of Γ_i giving the action on the torsion points of the elliptic curve. Denote by u_i the image of γ_i in $\mathbb{Z}_p$.

We consider the Iwasawa algebra

$$\Lambda(\Gamma_1 \times \Gamma_2, \mathbb{D}) \cong \mathbb{D}[[T_1, T_2]],$$

mapping a mesure μ to the power series

$$G(T_1, T_2) := \int_\Gamma (1 + T_1)^\alpha (1 + T_2)^\beta d\mu(\alpha, \beta).$$

In particular $G(u_1^a - 1, u_2^b - 1)$ is equal to

$$\int (u_1^a)^\alpha (u_2^b)^\beta d\mu(\alpha, \beta) = \int \kappa_1^a \kappa_2^b d\mu.$$

Let μ be a measure in the Iwasawa algebra for $\mathcal{G} = Gal(K(E[p^\infty])/K)$, and let χ be a character of $Gal(K(E[p])/K)$. We denote the power series associated to the χ-component of μ by $G(\chi^{-1}, T_1, T_2)$. Then we obtain that

$$\int_{\mathcal{G}} \kappa_1^b \kappa_2^b \chi\mu = \int_{\Gamma_1\times\Gamma_2} \kappa_1^a \kappa_2^b d\chi(\mu).$$

By the interpolation theorem 4.14 [14], $G(\chi, u_1^a - 1, u_2^b - 1)$ is a p-adic interpolation of $L(\chi\psi^{b-a}, -b)$, at least for $0 \leq -b \leq a$.

Corollary C.4. *With the hypothesies of theorems 3.5.14 and C.2, we have that the length of the coimage of $\iota \circ r_p(\mathcal{R}_\theta)$ in $H^1(K_\mathfrak{p}, M_{\theta\mathbb{Z}_p}(w + l + 1))$ is equal to the p-adic valuation of*

$$G(\psi_{\Omega_1}\kappa^l, u_1^{-a_\theta-1} - 1, u_2^{-b_\theta-1} - 1)G(\psi_{\Omega_2}\kappa^l, u_1^{-b_\theta-1} - 1, u_2^{-a_\theta-1} - 1).$$

Proof. The proof of proposition 3.5.8 shows that the elliptic units inside the module $(\overline{\mathcal{C}}_\infty \otimes M_{\Omega_i}(w + l))_{\mathcal{G}}$ is all the module. Then the comparison map between e_p and r_p implies that

$$\iota \circ e_p(\overline{\mathcal{C}}_\infty) = \iota \circ r_p(\mathcal{R}_\theta).$$

Using the direct decomposition of e_p and since we know it factor by factor as $\mathbb{Z}_p$-module, we obtain the result by using the previous theorem of Geisser. □

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