

PHD DISSERTATION

**Singular Integral Operators on Sobolev Spaces on Domains
and
Quasiconformal Mappings**

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Plus Ultra - Versió diàdica

Allà dellà de l'espai
he vist somriure una fulla
ben amunt del lledoner
com grumet en veient terra,
com al pregon de l'afrau
una efímera lluerna.

– Cub de Whitney – jo li he dit–,
de la mar margada gemma,
de les fites del sender
series tu la darrera?

– No só la darrera, no,
no só més que una llanterna
de la porta del jardí
que creies tu la frontera.
És sols lo començament
lo que prenies per terme.

Lo domini és infinit,
pertot acaba i comença,
i ençà, enllà, amunt i avall,
la immensitat és oberta,
i a on tu veus l'hiperplà
eixams de cubs formiguegen.

Dels camins de l'infinit
són los cubs la polsinera
que puja i baixa a sos peus
quan Peter Jones s'hi passeja.

Jacint Verdaguer (quasi)

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Introduction

The present dissertation studies some problems of geometric function theory, which is an area with great impact in mathematical analysis, relating complex analysis, harmonic analysis, geometric measure theory and partial differential equations. In particular it focuses on the relation between Calderón-Zygmund convolution operators and Sobolev spaces on domains.

The Sobolev space $W^{s,p}(\mathbb{R}^d)$ (or simply $W^{s,p}$) of smoothness $s \in \mathbb{N}$ and order of integrability $1 \leq p \leq \infty$ is the Banach space of L^p functions with distributional derivatives up to order s in L^p as well. This notion can be extended to $0 < s < \infty$ via the so-called Bessel-potential spaces (see Section 1.3). An operator T defined for $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d \setminus \text{supp}(f)$ as

$$Tf(x) = \int_{\mathbb{R}^d \setminus \{x\}} K(x-y)f(y)dy,$$

is called an *admissible convolution Calderón-Zygmund operator of order $s \in \mathbb{N}$* if it is bounded on the Sobolev space $W^{s,p}(\mathbb{R}^d)$ for every $1 < p < \infty$ and its kernel $K \in W^{s,1}_{loc}(\mathbb{R}^d \setminus \{0\})$ satisfies the size and smoothness conditions

$$|\nabla^j K(x)| \leq \frac{C_K}{|x|^{d+j}} \quad \text{for every } 0 \leq j \leq s \text{ and } x \neq 0$$

(see Section 1.6 for more details).

In the complex plane, for instance, the *Beurling transform*, defined as the principal value

$$\mathcal{B}f(z) := -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|w-z|>\varepsilon} \frac{f(w)}{(z-w)^2} dm(w),$$

is an admissible convolution Calderón-Zygmund operator of any order with kernel $K(z) = -\frac{1}{\pi z^2}$.

Along this dissertation some properties of this kind of operators restricted to domains will be unravelled. Let $\Omega \subset \mathbb{R}^d$ be a domain (open and connected) and T an admissible convolution Calderón-Zygmund operator. We are interested in conditions that allow us to infer that the restricted operator defined as $T_\Omega(f) = \chi_\Omega T(\chi_\Omega f)$ is bounded on a certain Sobolev space $W^{s,p}(\Omega)$. In that spaces, the case $sp = d$ is called critical (see Figure 0.1), since the supercritical case $sp > d$ usually implies continuity of the functions involved, and the subcritical case $sp < d$ implies only some degree of integrability, while the functions are in VMO when $sp = d$ and the domain is regular enough (see Proposition 1.11).

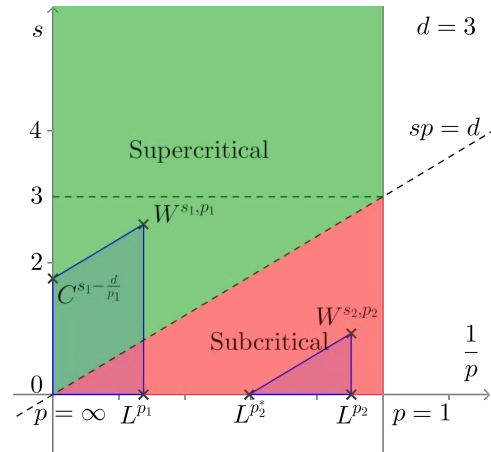


Figure 0.1: Critical, supercritical and subcritical indices and corresponding embeddings for \mathbb{R}^3 . Here, $\frac{d}{p_2} - \frac{d}{p_2^*} = s_2$.

Topics covered in this dissertation

Chapter 1: Background

The first chapter provides the reader with the tools which are common to the whole dissertation. It begins with some examples that will help to illustrate the nature of the problems faced in the thesis. The subsequent sections summarize the notation to be used and some well-known facts, including sections devoted to uniform domains, approximating polynomials and Calderón-Zygmund operators.

Chapter 2: T(P) theorems

In this chapter there are some results for the supercritical case, reducing the boundedness of an operator T_Ω on $W^{s,p}(\Omega)$ to its behavior on test functions, namely polynomials of degree strictly smaller than the considered smoothness. This is in accordance with the pioneering results found in [CMO13]:

Theorem ([CMO13]). *Let $\Omega \subset \mathbb{R}^d$ be a bounded $C^{1,\varepsilon}$ domain (i.e. a Lipschitz domain with parameterizations of the boundary in $C^{1,\varepsilon}$) for a given $\varepsilon > 0$, let $1 < p < \infty$ and $0 < s \leq 1$ such that $sp > d$ and let T be an admissible Calderón-Zygmund operator of order 1 with kernel $K(x) = \frac{\omega(x)}{|x|^d}$ where $\omega \in C^1(S^{d-1})$ is homogeneous of degree 0 with zero integral in the unit sphere S^{d-1} and even. Then the truncated operator T_Ω is bounded on the Sobolev space $W^{s,p}(\Omega)$ if and only if $T(\chi_\Omega) \in W^{s,p}(\Omega)$.*

This theorem is often called $T(1)$ -theorem because the only condition to be checked in order to show that an operator is bounded on $W^{s,p}(\Omega)$ is that $T_\Omega(1) \in W^{s,p}(\Omega)$. The reader will find two main results in that spirit in this chapter. First he or she will find a $T(P)$ -theorem, which deals with $W^{n,p}(\Omega)$ with $n \in \mathbb{N}$ and $p > d$:

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, T an admissible convolution Calderón-Zygmund operator of order $n \in \mathbb{N}$, $d < p < \infty$ and let \mathcal{P}^{n-1} stand for the polynomials of degree smaller than n . Then*

$$\|T_\Omega P\|_{W^{n,p}(\Omega)} < \infty \text{ for every } P \in \mathcal{P}^{n-1} \quad \iff \quad T_\Omega \text{ is bounded on } W^{n,p}(\Omega).$$

This theorem improves the previously known results in the sense that the class of operators considered is wider, the smoothness can be greater than 1 and, moreover, the restrictions on the regularity of the domain are reduced to just asking the domain to be uniform. However, the case $0 < s < 1$ is not covered. The second result of the chapter, Theorem 2.8, solves this gap:

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, T an admissible convolution Calderón-Zygmund operator of order 1, $1 < p < \infty$ and $0 < s < 1$ with $s > \frac{d}{p}$. Then*

$$\|T_\Omega 1\|_{W^{s,p}(\Omega)} < \infty \quad \iff \quad T_\Omega \text{ is bounded on } W^{s,p}(\Omega).$$

Moreover, in Theorem 2.8 the assumptions on the regularity of the kernel are relaxed and it will be extended to the class of Triebel-Lizorkin spaces $F_{p,q}^s$ (see Chapter 2).

The novelty in the approach exposed in this chapter is not only on the aforementioned improvements, but also on the techniques used to reach that results, which rely strongly on the properties of uniform domains and their hyperbolic metric. Furthermore, in both cases a Key Lemma is obtained that provides information even if the condition $sp > d$ is not satisfied.

To prove the second result, some lemmas are used which are proven in Sections 2.3 and 2.4, results which are intuitive according to the literature, but which cannot be found in the precise shape needed. This includes the definition of an equivalent norm in terms of differences for fractional Triebel-Lizorkin spaces in the spirit of [Ste61] (see Corollary 2.12), an extension theorem for Triebel-Lizorkin spaces on uniform domains following the steps of the celebrated paper

[Jon81] by Peter Jones (see Lemma 2.16), and some equivalent norms for $F_{p,q}^s(\Omega)$ introduced by Eero Saksman and the author of the present dissertation (see Corollary 2.22).

Note that we are far from covering all the supercritical cases (see Figure 0.2). This fact is discussed at the end of the dissertation.

The $T(P)$ -theorem reminds the results by Rodolfo H. Torres in [Tor91], where the characterization of some generalized Calderón-Zygmund operators which are bounded on the homogeneous Triebel-Lizorkin spaces in \mathbb{R}^d is given in terms of their behavior over polynomials. It is also in place to remark that in [Väh09] Antti V. Vähäkangas obtained some $T(1)$ -theorem for weakly singular integral operators on domains. Roughly speaking, he showed the image of the characteristic function being in a certain BMO-type space to be equivalent to the boundedness of $T_\Omega : L^p(\Omega) \rightarrow \dot{W}^{m,p}(\Omega)$ where m is the degree of the singularity of T 's kernel.

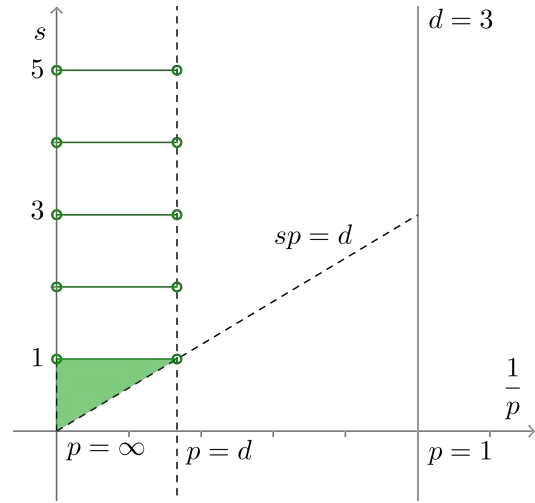


Figure 0.2: Indices for which the $T(P)$ -theorem is valid in $W^{\sigma,p}(\Omega)$ for uniform domains in \mathbb{R}^3 .

Chapter 3: Characteristic functions of planar domains

According to the results above, estimating $\|T_\Omega 1\|_{W^{n,p}(\Omega)}$ is crucial to determine if T_Ω is bounded on $W^{n,p}(\Omega)$ or not. This chapter deals with the question of what conditions on the boundary of the domain imply that this norm is finite. The techniques used rely heavily on complex analysis, so all the results of this chapter are for planar domains.

The first results pointing in this direction were obtained in [CMO13]. Using a result in [MOV09] it is proven that if $\varepsilon > s$ and Ω is a $C^{1,\varepsilon}$ domain then $\mathcal{B}_{\chi_\Omega} \in W^{s,p}(\Omega)$. Thus, assuming the conditions in the $T(1)$ -theorem for Ω , s and p to hold, one always has the Beurling transform bounded on $W^{s,p}(\Omega)$.

Following the thread, Victor Cruz and Xavier Tolsa in [CT12] showed that for $0 < s \leq 1$ and $1 < p < \infty$ with $sp > 1$, if the parameterizations of the domain are Lipschitz and the outward unit normal vector N is in the Besov space $B_{p,p}^{s-1/p}(\partial\Omega)$, then $\mathcal{B}_{\chi_\Omega} \in W^{s,p}(\Omega)$. The Besov spaces of functions are properly defined in 1.2 and 1.3; momentarily the reader unfamiliar with this concept may stick to the right track by noting the fact that $B_{p,p}^\sigma$ with $\sigma \notin \mathbb{N}$ can be defined by means of real interpolation between the Sobolev spaces of integer order $W^{[\sigma],p}$ and $W^{[\sigma]+1,p}$. Their appearance in this context is rather natural since the traces of functions in $W^{s,p}(\Omega)$ are precisely in $B_{p,p}^{s-1/p}(\partial\Omega)$ if certain regularity conditions are satisfied. The condition $N \in B_{p,p}^{s-1/p}(\partial\Omega)$ implies the parameterizations of the boundary of Ω to be in $B_{p,p}^{s+1-1/p}$ and, for $sp > 2$, the parameterizations are in $C^{1,s-2/p}$ as well by the Sobolev Embedding Theorem. In that situation, the $T(1)$ result in [CMO13] implies the boundedness of the Beurling transform in $W^{s,p}(\Omega)$ (see Figure 0.3). Moreover, for $s = 1$, this condition is necessary for Lipschitz domains with small Lipschitz constant such that $\mathcal{B}_{\chi_\Omega} \in W^{1,p}(\Omega)$ as shown in [Tol13].

This chapter deals with the case of Sobolev spaces with smoothness $n \in \mathbb{N}$. It is shown that for $p > 1$, if $N \in B_{p,p}^{n-1/p}(\partial\Omega)$ and the domain has parameterizations in $C^{n-1,1}(\mathbb{R})$, then $\mathcal{B}_{\chi_\Omega} \in W^{n,p}(\Omega)$, in the same spirit of [CT12]. The $T(P)$ -theorem above will be used to deduce that the Beurling transform is bounded on $W^{n,p}(\Omega)$ when, in addition, $p > 2$. Note that in this case again $B_{p,p}^{n+1-1/p}(\mathbb{R}) \subset C^{n-1,1}(\mathbb{R})$.

Theorem. Let $p > 2$, let $n \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain with $N \in B_{p,p}^{n-1/p}(\partial\Omega)$. If $f \in W^{n,p}(\Omega)$, then

$$\|\mathcal{B}(\chi_\Omega f)\|_{W^{n,p}(\Omega)} \leq C \|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)} \|f\|_{W^{n,p}(\Omega)},$$

where C depends on $p, n, \text{diam}(\Omega)$ and the Lipschitz character of the domain.

The proof will be slightly more tricky than the case of smoothness smaller or equal to 1, since the boundary of the domain must be approximated by polynomials instead of straight lines: the derivative of the Beurling transform of the characteristic function of a half-plane is zero (see [CT12]), but the derivative of the Beurling transform of the characteristic function of a domain bounded by a polynomial of degree greater than one is not zero anymore. Moreover, some extra effort will be done in order to get a quantitative result to be used in the subsequent chapter, involving not only the truncated Beurling transform, but its iterates and other related operators.

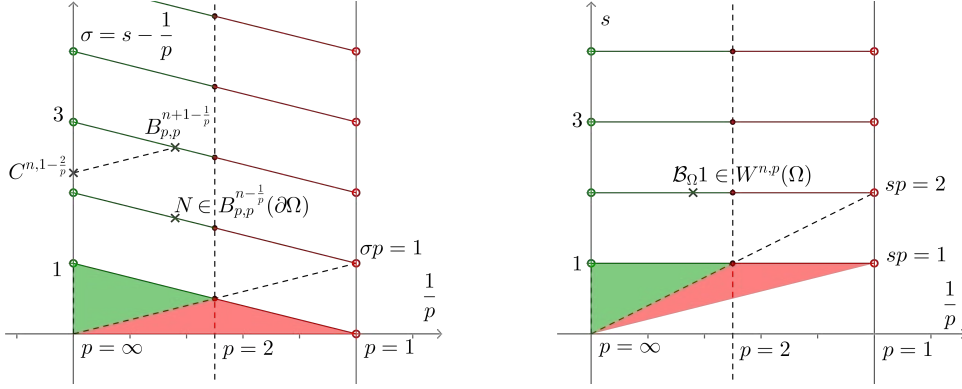


Figure 0.3: The normal vector being in $B_{p,p}^{s-1/p}(\partial\Omega)$ implies that $\mathcal{B}_\Omega 1 \in W^{n,p}(\Omega)$ (green and red regions) and, when $p > 2$ and $sp > 2$ (green region), the $T(P)$ -theorem applies. This is precisely the case where the parameterizations of the boundary belong to $B_{p,p}^{s+1-1/p} \subset C^{s,1-2/p}$. In the figure it is shown the case $s = n \in \mathbb{N}$, but $\frac{1}{p} < s < 1$ follows the same pattern.

Chapter 4: An application to quasiconformal mappings

Let $\mu \in L^\infty$ be compactly supported in \mathbb{C} with $k := \|\mu\|_{L^\infty} < 1$ and consider $K := \frac{1+k}{1-k}$. A function f is a K -quasiregular solution to the Beltrami equation

$$\bar{\partial}f = \mu \partial f$$

with Beltrami coefficient μ if $f \in W_{loc}^{1,2}$, that is, if f and ∇f are square integrable functions in any compact subset of \mathbb{C} , and $\bar{\partial}f(z) = \mu(z)\partial f(z)$ for almost every $z \in \mathbb{C}$. Such a function f is said to be a K -quasiconformal mapping if it is a homeomorphism of the complex plane. If, moreover, $f(z) = z + \mathcal{O}(\frac{1}{z})$ as $z \rightarrow \infty$, then f is the principal solution to the Beltrami equation.

Given a compactly supported Beltrami coefficient μ , the existence and uniqueness of the principal solution is granted by the measurable Riemann mapping Theorem (see [AIM09, Theorem 5.1.2], for instance). The principal solution can be given by means of the Beurling transform. If

$$h := (I - \mu\mathcal{B})^{-1}\mu,$$

then the principal solution of the Beltrami equation satisfies that $\bar{\partial}f = h$ and $\partial f = \mathcal{B}h + 1$.

A natural question is to what spaces h belongs. The key point in finding out answers to that question is inverting the operator $(I - \mu\mathcal{B})$ in some space. Astala showed in [Ast94] that $h \in L^p$

for $1 + k < p < 1 + 1/k$ (in fact, since h is also compactly supported, one can say the same for every $p \leq 1 + k$ even though $(I - \mu\mathcal{B})$ may not be invertible in L^p for that values of p , as shown by Astala, Iwaniec and Saksman in [AIS01]). Clop et al. in [CFM⁺09] and Cruz, Mateu and Orobitg in [CMO13] proved that if μ belongs to the space $W^{s,p}(\mathbb{C})$ with $sp > 2$ then also $h \in W^{s,p}(\mathbb{C})$. One also finds some results in the same spirit for the critical case $sp = 2$ and the undercritical case $sp < 2$ in [CFM⁺09] and [CFR10], but here the space to which h belongs is slightly worse than the space to which μ belongs, that is, either some integrability or some smoothness is lost.

When it comes to dealing with a Lipschitz domain Ω with $\text{supp}(\mu) \subset \bar{\Omega}$, Mateu, Orobitg and Verdera showed in [MOV09] that, if the parameterizations of the boundary of Ω are in $C^{1,\varepsilon}$ with $0 < \varepsilon < 1$, then for every $0 < s < \varepsilon$ one has that

$$\mu \in C^{0,\varepsilon}(\Omega) \implies h \in C^{0,s}(\Omega). \tag{0.1}$$

Furthermore, the principal solution to the Beltrami equation is bilipschitz in that case. The authors allow the domain to have a finite number of holes with tangent boundaries. In [CF12], Giovanna Citti and Fausto Ferrari proved that, if one does not allow this degenerate situation, then (0.1) holds for $s = \varepsilon$. In [CMO13] the authors study also the Sobolev spaces to conclude that for the same kind of domains (i.e., with boundary in $C^{1,\varepsilon}$), when $0 < s < \varepsilon < 1$ and $1 < p < \infty$ with $sp > 2$ one has that

$$\mu \in W^{s,p}(\Omega) \implies h \in W^{s,p}(\Omega). \tag{0.2}$$

A key point is proving the boundedness of the Beurling transform in $W^{s,p}(\Omega)$, something that they do in the same paper as mentioned in the presentation of the previous chapter. The other key point is the invertibility of $I - \mu\mathcal{B}$ in $W^{s,p}(\Omega)$, which is shown using Fredholm theory.

Back to the ideas of the presentation of Chapter 3 above, a domain such that its outward unit normal vector $N \in B_{p_0,p_0}^{s_0-1/p_0}(\partial\Omega)$ with $0 < s_0 < 1$ and $s_0p_0 > 1$ satisfies that the parameterizations of its boundary are in $B_{p_0,p_0}^{s_0+1-1/p_0}$. The aforementioned result by Cruz and Tolsa in [CT12] implies that $\mathcal{B}\chi_\Omega \in W^{s_0,p_0}(\Omega)$. In addition, when $s_0p_0 > 2$, the parameterizations are also in $C^{1,s_0-2/p_0}$ by the Sobolev Embedding Theorem. This fact, combined with the $T(1)$ -theorem in [CMO13], implies the boundedness of the Beurling transform in $W^{s_0,p_0}(\Omega)$. However, (0.2) only allows to infer that for every $2/p < s < s_0 - 2/p_0$ we have that (0.2) holds (see Figure 0.4). Apparently, there is room to improve.

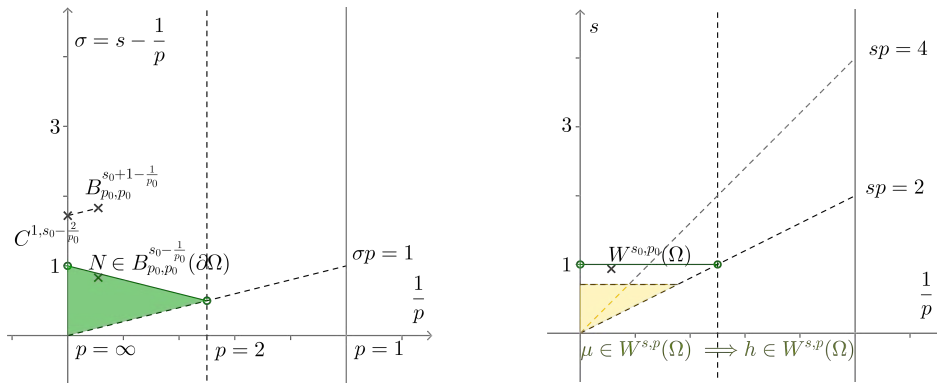


Figure 0.4: Combining the results of [CT12] and [CMO13], if the normal vector is in $B_{p_0,p_0}^{s_0-1/p_0}(\partial\Omega)$ then (0.2) holds for the indices s, p in the yellow region of the second graphic (where $2/p < s < s_0 - 2/p_0$), although the Beurling transform is bounded on $W^{s_0,p_0}(\Omega)$. Note that if $s_0p_0 < 4$ then $p > p_0$, that is, not only smoothness, but also some integrability range is lost.

The natural guess is that if $N \in B_{p_0, p_0}^{s_0 - 1/p_0}(\partial\Omega)$ then (0.2) holds for $s = s_0$. In this chapter we prove that indeed this is the case for natural values of s_0 .

Theorem. *Let $n \in \mathbb{N}$, let Ω be a bounded Lipschitz domain with outward unit normal vector N in $B_{p,p}^{n-1/p}(\partial\Omega)$ for some $2 < p < \infty$ and let $\mu \in W^{n,p}(\Omega)$ with $\|\mu\|_{L^\infty} < 1$ and $\text{supp}(\mu) \subset \bar{\Omega}$. Then, the principal solution f to the Beltrami equation is in the Sobolev space $W^{n+1,p}(\Omega)$.*

Note that this theorem only deals with the natural values of s_0 , but the restriction $s < s_0 - 2/p_0$ is eliminated. For $n = 1$ the author expects this to be a sharp result in view of [Tol13].

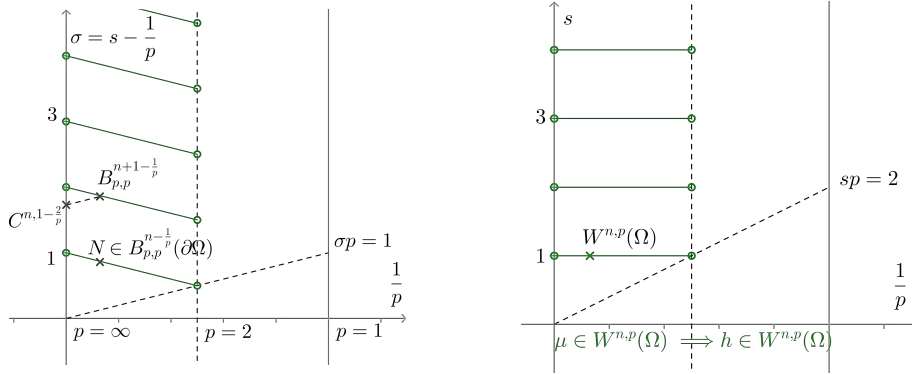


Figure 0.5: The gap presented in the previous diagram is not there anymore, so neither smoothness nor integrability is lost. The argument presented is only valid for natural values for the smoothness index by now.

Chapter 5: Carleson measures on Lipschitz domains

The $T(P)$ -theorems provide useful tools to check if an operator is bounded on $W^{n,p}(\Omega)$ as long as $p > d$. This chapter presents a completely new approach to find a sufficient Carleson condition valid even if $p \leq d$ in the spirit of the celebrated article [ARS02] by N. Arcozzi, R. Rochberg and E. Sawyer. This condition uses polynomials of degree smaller than n as test functions again.

In Section 5.2 the *vertical shadows* $\mathbf{Sh}_v(x)$ and $\widetilde{\mathbf{Sh}}_v(x)$ are defined for every point x in a Lipschitz domain Ω close enough to $\partial\Omega$. These shadows can be understood as Carleson boxes of the domain (see Figure 1.2). A positive and finite Borel measure μ is an s, p -Carleson measure if for every $a \in \Omega$ and close enough to the boundary,

$$\int_{\widetilde{\mathbf{Sh}}_v(a)} \text{dist}(x, \partial\Omega)^{(d-sp)(1-p')} (\mu(\mathbf{Sh}_v(x) \cap \mathbf{Sh}_v(a)))^{p'} \frac{dx}{\text{dist}(x, \partial\Omega)^d} \leq C \mu(\mathbf{Sh}_v(a)).$$

N. Arcozzi, R. Rochberg and E. Sawyer proved in [ARS02] that in the case when Ω coincides with the unit disk $\mathbb{D} \subset \mathbb{C}$, the measure μ is $1, p$ -Carleson if and only if the trace inequality

$$\int_{\mathbb{D}} |f|^p d\mu \leq C |f(0)|^p + C \int_{\mathbb{D}} |f'|^p dm$$

holds for any holomorphic function f on \mathbb{D} . It turns out that the notion of $1, p$ -Carleson measure is also essential for the characterization of the boundedness of Calderón-Zygmund operators of order n in $W^{n,p}(\Omega)$ when $1 < p \leq d$ as the next theorem shows.

Theorem. *Let T be an admissible convolution Calderón-Zygmund operator of order n , and consider a bounded Lipschitz domain Ω and $1 < p \leq d$. If the measure $|\nabla^n T_\Omega P(x)|^p dx$ is a $1, p$ -Carleson measure for every polynomial P of degree at most $n - 1$, then T_Ω is a bounded operator on $W^{n,p}(\Omega)$.*

In connection with Chapter 2, an equivalent formulation for the fractional case $\frac{d}{p} - \frac{d}{2} < s < 1$ is presented. In that case, the gradient notation is explained in Definition 2.9.

Theorem. *Let $1 < p < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{2}$, let T be an admissible convolution Calderón-Zygmund operator of order 1, and consider a bounded Lipschitz domain Ω . If the measure $\mu(x) = |\nabla_2^s T_\Omega 1(x)|^p dx$ is an s, p -Carleson measure, then T_Ω is a bounded operator on $W^{s,p}(\Omega)$.*

The Carleson condition of this theorem is in fact necessary for $n = 1$ (see Figure 0.6):

Theorem. *Let T be an admissible convolution Calderón-Zygmund operator of order 1, and consider a Lipschitz domain Ω and $1 < p < \infty$. Then*

$$T_\Omega \text{ is a bounded operator on } W^{1,p}(\Omega) \iff |\nabla T\chi_\Omega(x)|^p dx \text{ is a } p\text{-Carleson measure for } \Omega.$$

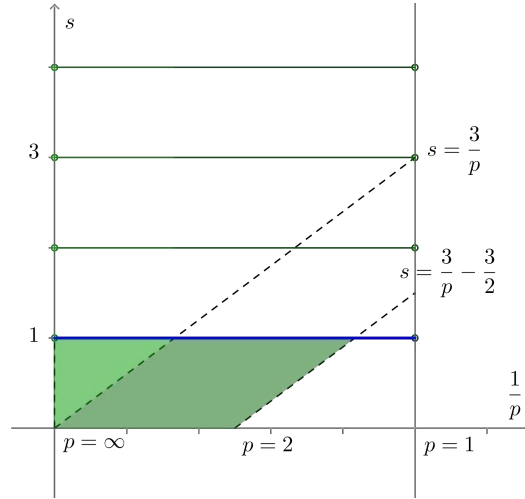


Figure 0.6: Indices for which the Carleson condition is sufficient for T_Ω to be bounded on $W^{s,p}(\Omega)$ for Lipschitz domains in \mathbb{R}^3 . In blue the case $s = 1$, where the Carleson condition is necessary and sufficient.

Final remarks

The results presented in this dissertation are fruit of the PhD studies of the author started in February 2012 under the direction of Xavier Tolsa.

The contents of Chapter 2 are a combination of a joint article with Xavier Tolsa, adapted in the present text to the framework of uniform domains as suggested by the referee during the publication process (see [PT15]), and a joint work with Eero Saksman (see [PS15]). Chapter 3 contains the material of [Pra15b] together with some original proofs (Section 3.6) which are not included in this preprint for the sake of brevity. The results of Chapter 4 can be found in the preprint [Pra15a]. Finally, Chapter 5 contains also material from [PT15], although Section 5.4 cannot be found in any other paper by now.

The chronological order should be Chapter 3, Section 2.1, Chapter 5, Chapter 4 and the remaining sections of Chapter 2.

Chapter 1

Background

We introduce the principles that the other chapters have in common. First of all, in Section 1.1, we provide some basic examples to help the reader to understand the nature of the problems we are facing. Next we set up the notation of the dissertation in Section 1.2 and then we list some well-known facts in Section 1.3. In Section 1.4 we introduce uniform domains. They will appear only in Chapter 2 and, after that chapter, we will restrict our study to Lipschitz domains, which are also uniform. However, we introduce here concepts as “admissible chain” or “shadow”, which will be extremely useful in the subsequent chapters. The reader should be aware that, in the last chapter, we will make some modifications on these concepts to introduce the Carleson measures. Section 1.5 is about Meyers’ approximating polynomials on cubes, which are used in all the chapters to discretize Sobolev functions on domains. Finally, Section 1.6 is devoted to defining admissible Calderón Zygmund operators and proving that, up to a certain order, the weak derivatives of these operators restricted to a domain make sense in our setting.

1.1 Some examples: the Beurling transform as a model

Given a compactly supported smooth function $\phi \in C_c^\infty(\mathbb{C})$, we define its Cauchy transform as

$$\mathcal{C}\phi(z) := \frac{1}{\pi} \int_{\mathbb{C}} \frac{\phi(w)}{z-w} dm(w) \quad \text{for all } z \in \mathbb{C},$$

where m stands for the two-dimensional Lebesgue measure. In other words, the Cauchy transform of ϕ is the convolution of ϕ with the kernel $\frac{1}{z}$. The Beurling transform of ϕ will be defined as the convolution with $\frac{1}{z^2}$, but this kernel is no longer locally integrable. Thus, we must define it via the so-called Cauchy Principal Value, that is,

$$\mathcal{B}\phi(z) := -\frac{1}{\pi} \text{p.v.} \int_{\mathbb{C}} \frac{\phi(w)}{(z-w)^2} dm(w) = -\frac{1}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|w-z|>\varepsilon} \frac{\phi(w)}{(z-w)^2} dm(w) \quad \text{for all } z \in \mathbb{C}. \quad (1.1)$$

Given any $\phi \in C^\infty(\mathbb{C})$, then we write $\partial\phi(z) = \frac{1}{2}(\partial_x\phi - i\partial_y\phi)(z)$ and $\bar{\partial}\phi(z) = \frac{1}{2}(\partial_x\phi + i\partial_y\phi)(z)$. It is well known that for every $\phi \in C_c^\infty(\mathbb{C})$ we have that

$$\bar{\partial}\mathcal{C}\phi(z) = \phi(z) \quad \text{and} \quad \partial\mathcal{C}\phi(z) = \mathcal{B}\phi(z) \quad (1.2)$$

(see, for example, [AIM09, Chapter 4]).

For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ with $\alpha_1, \alpha_2 \geq 0$ and $\phi \in C_c^\infty(\mathbb{C})$, we write $D^\alpha\phi = \partial^{\alpha_1}\bar{\partial}^{\alpha_2}\phi$. For any open set $U \subset \mathbb{C}$ and every distribution $f \in \mathcal{D}'(U)$ the *distributional derivative* $D^\alpha f$ is the distribution

defined by

$$\langle D^\alpha f, \phi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle \quad \text{for every } \phi \in C_c^\infty(U).$$

Given numbers $n \in \mathbb{N}$, $1 \leq p \leq \infty$ an open set $U \subset \mathbb{C}$ and a locally integrable function f defined in U , we say that f is in the *Sobolev space* $W^{n,p}(U)$ of smoothness n and order of integrability p if f has distributional derivatives $D^\alpha f \in L^p$ for every $|\alpha| \leq n$. We write $|\nabla^n f| = \sum_{|\alpha|=n} |D^\alpha f|$. Along this section we will use the norm

$$\|f\|_{W^{n,p}(U)} = \|f\|_{L^p(U)} + \|\nabla^n f\|_{L^p(U)}.$$

The Beurling transform is an isometry in $L^2(\mathbb{C})$ and it is bounded on $L^p(\mathbb{C})$ for $1 < p < \infty$ (see, [AIM09, Chapter 4]), that is, for every $f \in L^p(\mathbb{C})$ we have that

$$\|\mathcal{B}f\|_{L^p(\mathbb{C})} \leq C_p \|f\|_{L^p(\mathbb{C})}.$$

Furthermore, the Beurling transform commutes with derivatives, that is $\mathcal{B} \circ \partial = \partial \circ \mathcal{B}$, and $\mathcal{B} \circ \bar{\partial} = \bar{\partial} \circ \mathcal{B}$, so \mathcal{B} is bounded on the Sobolev space $W^{n,p}(\mathbb{C})$ as long as $1 < p < \infty$ and $n \in \mathbb{N}$. However, this does not apply to $W^{n,p}(U)$ for general open sets U .

Given a domain (open and connected) Ω , we write $\mathcal{B}_\Omega f = \chi_\Omega \mathcal{B}(\chi_\Omega f)$. Among other results, in the present dissertation we will see that if a domain Ω is regular enough then \mathcal{B}_Ω is bounded on $W^{n,p}(\Omega)$.

Let us see some examples. From (1.2) we can deduce the following property for the Beurling transform: for $1 < p < \infty$ and $f \in W^{1,p}(\mathbb{C})$ we have that

$$\mathcal{B}(\bar{\partial}f) = \partial \mathcal{C}(\bar{\partial}f) = \partial \bar{\partial} \mathcal{C}(f) = \partial f. \quad (1.3)$$

Using this result, we can undertake the study of the behavior of the Beurling transform of a polynomial restricted to the unit disk.

Example 1.1. *Consider the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. Then, for every polynomial P of degree $n - 1$, its transform $\mathcal{B}_\mathbb{D} P = \chi_\mathbb{D} \mathcal{B}(\chi_\mathbb{D} P)$ agrees with a polynomial of degree smaller or equal than $n - 1$ in \mathbb{D} so $\nabla^n \mathcal{B}_\mathbb{D} P(z) = 0$ for $z \in \mathbb{D}$.*

Thus, the sufficient conditions of Theorems 2.1 and 5.1 are satisfied. Those theorems can be used to prove the boundedness of $\mathcal{B}_\mathbb{D}$ in $W^{n,p}(\mathbb{D})$ for any $n \in \mathbb{N}$ and $p > 1$ in one stroke.

Proof. We will follow the ideas of [AIM09, page 96]. Consider a given any multiindex $\lambda = (\lambda_1, \lambda_2)$, and let $P_\lambda(z) = z^\lambda = z^{\lambda_1} \bar{z}^{\lambda_2}$.

If $\lambda_1 = 0$, then we define $f_\lambda(z) = \bar{z}^{\lambda_2+1} \chi_\mathbb{D}(z) + z^{-\lambda_2-1} \chi_{\mathbb{D}^c}(z)$. This function is continuous and is in some Sobolev space $W^{1,p}(\mathbb{C})$. Thus, by (1.3) we have that

$$(\lambda_2 + 1) \mathcal{B}(\chi_\mathbb{D} P_\lambda)(z) = \mathcal{B}(\bar{\partial} f_\lambda)(z) = \partial f_\lambda(z) = -(\lambda_2 + 1) \frac{1}{z^{\lambda_2+2}} \chi_{\mathbb{D}^c}(z),$$

so

$$\mathcal{B}_\mathbb{D} P_\lambda = 0.$$

Analogously, if $0 < \lambda_1 < \lambda_2 + 1$ we define $f_\lambda(z) = z^{\lambda_1+1} \chi_\mathbb{D}(z) + z^{\lambda_1-\lambda_2-1} \chi_{\mathbb{D}^c}(z)$, which is also continuous and is in some Sobolev space $W^{1,p}(\mathbb{C})$ because $\lambda_1 - \lambda_2 - 1 < 0$. Again by (1.3) we have that

$$(\lambda_2 + 1) \mathcal{B}(\chi_\mathbb{D} P_\lambda)(z) = \mathcal{B}(\bar{\partial} f_\lambda)(z) = \partial f_\lambda(z) = \lambda_1 z^{\lambda_1+(-1,1)} \chi_\mathbb{D}(z) - (\lambda_2 - \lambda_1 + 1) \frac{1}{z^{\lambda_2-\lambda_1+2}} \chi_{\mathbb{D}^c}(z),$$

so

$$\mathcal{B}_\mathbb{D} P_\lambda = \frac{\lambda_1}{\lambda_2 + 1} P_{\lambda+(-1,1)} \chi_\mathbb{D}.$$

In case, $\lambda_1 = \lambda_2 + 1$ we take $f_\lambda(z) = (z^{\lambda+(0,1)} - 1)\chi_{\mathbb{D}}(z)$. We have that

$$(\lambda_2 + 1)\mathcal{B}(\chi_{\mathbb{D}}P_\lambda)(z) = \mathcal{B}(\bar{\partial}f_\lambda)(z) = \partial f_\lambda(z) = \lambda_1 z^{\lambda+(-1,1)}\chi_{\mathbb{D}}(z),$$

so again

$$\mathcal{B}_{\mathbb{D}}P_\lambda = \frac{\lambda_1}{\lambda_2 + 1}P_{\lambda+(-1,1)}\chi_{\mathbb{D}} = P_{\lambda+(-1,1)}\chi_{\mathbb{D}}.$$

Finally, when $\lambda_1 > \lambda_2 + 1$ we choose $f_\lambda(z) = (z^{\lambda+(0,1)} - z^{\lambda_1-\lambda_2-1})\chi_{\mathbb{D}}(z)$ and we get that

$$(\lambda_2 + 1)\mathcal{B}(\chi_{\mathbb{D}}P_\lambda)(z) = \mathcal{B}(\bar{\partial}f_\lambda)(z) = \partial f_\lambda(z) = \left(\lambda_1 z^{\lambda+(-1,1)} - (\lambda_1 - \lambda_2 - 1)z^{\lambda_1-\lambda_2-2}\right)\chi_{\mathbb{D}}(z),$$

so we have that

$$\mathcal{B}_{\mathbb{D}}P_\lambda = \left(\frac{\lambda_1}{\lambda_2 + 1}P_{\lambda+(-1,1)} - \frac{\lambda_1 - \lambda_2 - 1}{\lambda_2 + 1}P_{(\lambda_1-\lambda_2-2,0)}\right)\chi_{\mathbb{D}}.$$

□

Example 1.2. *The Beurling transform of the characteristic function of a square Q (see Figure 1.1) is not in $W^{n,p}(Q)$ neither for any $n = 1$ and $p \geq 2$ nor for $n \geq 2$ and $p > 1$. Nevertheless, if $1 \leq p < 2$, then $\mathcal{B}\chi_Q \in W^{1,p}(Q)$.*

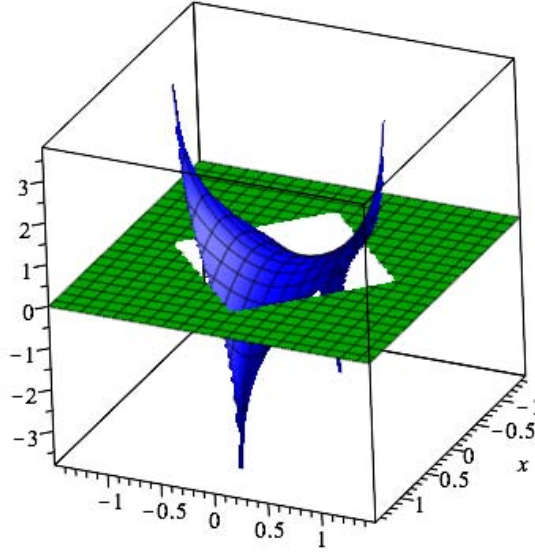


Figure 1.1: Plot of $\text{Re}(\chi_Q \mathcal{B}\chi_Q(x + iy))$. There is a logarithmic singularity at every vertex of Q .

Proof. Consider the square $Q_0 = \{z \in \mathbb{C} : |\text{Re}(z)| + |\text{Im}(z)| < 1\}$ which has vertices $\{1, i, -1, -i\}$. In that case, by [AIM09, (4.122)] one can see that

$$\mathcal{B}(\chi_{Q_0})(z) = \frac{1}{\pi} \log \frac{(z-1)(z+1)}{(z+i)(z-i)},$$

where \log stands for the well-defined branch of the logarithm with argument in $(0, 2\pi)$ (if $z \in Q_0$ then $\text{Re}\left(\frac{z^2-1}{z^2+1}\right) < 0$). Since

$$\partial \mathcal{B}(\chi_{Q_0})(z) = \frac{1}{\pi} \frac{4z}{z^4 - 1}, \quad (1.4)$$

we have that $|\partial\mathcal{B}(\chi_{Q_0})(z)| \approx \frac{1}{|z-1|}$ when $z \in Q_0$ is close enough to 1 and it follows that $\mathcal{B}(\chi_{Q_0}) \notin W^{1,p}(Q_0)$ for $p \geq 2$ and, since $B(\chi_{Q_0})$ is analytic, $\mathcal{B}(\chi_{Q_0}) \in W^{1,p}(Q_0)$ for $1 \leq p < 2$. By the same token, for $n \geq 2$ one has $|\partial^n \mathcal{B}(\chi_{Q_0})(z)| \approx |z-1|^{-n}$ and therefore $\mathcal{B}(\chi_{Q_0}) \notin W^{n,p}(Q_0)$ for any $p > 1$. \square

Example 1.3. *The Beurling transform restricted to the square \mathcal{B}_{Q_0} is not bounded on $W^{n,p}(Q_0)$ neither for $n = 1$ and $p \geq 2$ nor for $n \geq 2$ and $p > 1$.*

However, it is bounded on $W^{1,p}(Q)$ for every $1 < p < 2$.

Proof. Of course (1.2) implies that \mathcal{B}_{Q_0} is not bounded on $W^{n,p}(Q_0)$ when $n \geq 2$ and $p > 1$ or when $n = 1$ and $p \geq 2$.

Nevertheless, when $n = 1$ and $1 < p < 2$, we have seen that $B(\chi_{Q_0}) \in W^{1,p}(Q_0)$. This condition does not suffice to grant the boundedness of \mathcal{B} in $W^{1,p}(Q_0)$ for $p < 2$. By Theorem 5.1 we need to check that the measure

$$\mu(z) = |\nabla B\chi_{Q_0}(z)|^p$$

is a p -Carleson measure, as we sketch below (see Definition 5.16 for the details). Of course since the measure is bounded away from the four vertices, by symmetry, it is enough to check this condition for $|z-1| < \frac{1}{2}$ or, equivalently, for $\mu(z) = \left(\frac{1}{|z|}\right)^p$ in $\Omega = \{z \in \mathbb{C} : \text{Im}(z) > |\text{Re}(z)|\}$.

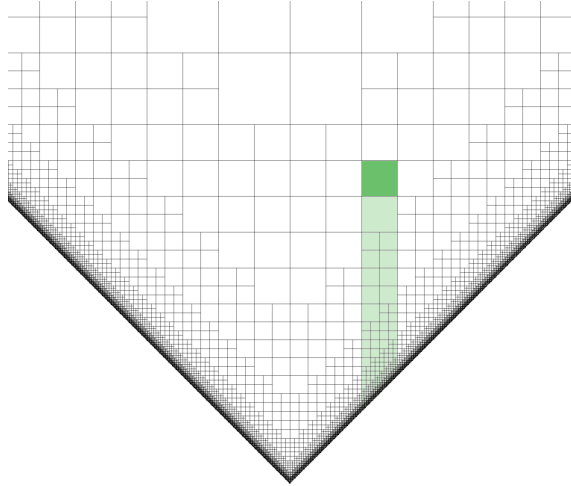


Figure 1.2: The vertical shadow $\mathbf{Sh}_v(P_{-1,2,5})$.

Consider the cubes

$$P_{i,j,k} = \{z \in \mathbb{C} : (|j|-1)2^i < \text{Re}(z) < |j|2^i \text{ and } (|j|+k)2^i < \text{Im}(z) < (|j|+k+1)2^i\} \quad (1.5)$$

for $i \in \mathbb{Z}$, $j \in \mathbb{Z} \setminus \{0\}$, and $k \in \{4, 5, 6, 7, 8\}$ if j is even, $k \in \{4, 5, 6, 7\}$ if j is odd (see Figure 1.2). This collection of cubes is a Whitney covering of Ω , that is, a collection of disjoint cubes with side-length proportional to their distance to $\partial\Omega$ and such that the union of their closures is Ω .

For a cube $P_{i,j,k}$ in this collection, we define its *vertical shadow* as the region of Ω situated beneath it, that is,

$$\mathbf{Sh}_v(P_{i,j,k}) = \{z \in \mathbb{C} : (|j|-1)2^i < \text{Re}(z) < |j|2^i \text{ and } \text{Im}(z) < (|j|+k+1)2^i\}$$

(see Figure 1.2).

Then we say that the measure is p -Carleson if

$$\sum_{Q \subset \mathbf{Sh}_v(P)} \mu(\mathbf{Sh}_v(Q))^{p'} \ell(Q)^{\frac{p-2}{p-1}} \leq C \mu(\mathbf{Sh}_v(P)),$$

where $\frac{1}{p} + \frac{1}{p'} = 1$. For every Whitney cube $P = P_{i_0, j_0, k_0}$, since μ is almost constant in cubes we have that

$$\mu(P) = \int_P d\mu = \int_P \frac{1}{|z|} dm(z) \approx \frac{\ell(P)^2}{\text{dist}(P, 0)^p}.$$

Note that the Lebesgue measure of the shadow $m(\mathbf{Sh}_v(P))$ is comparable to the Lebesgue measure of the cube itself. Assume that $|j_0| > 1$, that is, assume that P is far from the imaginary axis. Then, all the cubes contained in its shadow are essentially at the same distance of the origin, so

$$\mu(\mathbf{Sh}_v(P)) \approx \sum_{Q \subset \mathbf{Sh}_v(P)} \frac{\ell(Q)^2}{\text{dist}(Q, 0)^p} \approx \frac{m(\mathbf{Sh}_v(P))}{\text{dist}(P, 0)^p} \approx \frac{\ell(P)^2}{\text{dist}(P, 0)^p}.$$

If, instead, \bar{P} intersects the imaginary axis (equivalently, if $|j_0| = 1$), then we classify the cubes in its shadow by their size:

$$\mu(\mathbf{Sh}_v(P)) \approx \sum_{i=-\infty}^{i_0} \sum_{\substack{Q \subset \mathbf{Sh}_v(P) \\ \ell(Q)=2^i}} \frac{\ell(Q)^2}{\text{dist}(Q, 0)^p}.$$

For $Q = P_{i,j,k}$ let us define the index $j(Q) := |j| + k$. Then $j(Q) \approx \frac{\text{dist}(Q, 0)}{\ell(Q)}$ by (1.5), and

$$\mu(\mathbf{Sh}_v(P)) \approx \sum_{i=-\infty}^{i_0} \sum_{\substack{Q \subset \mathbf{Sh}_v(P) \\ \ell(Q)=2^i}} \frac{2^{i(2-p)}}{j(Q)^p} = \sum_{i=-\infty}^{i_0} 2^{(2-p)i} \sum_{\substack{Q \subset \mathbf{Sh}_v(P) \\ \ell(Q)=2^i}} \frac{1}{j(Q)^p}.$$

Observe that for a fixed i and $n \in \mathbb{N}$, the number of cubes Q with $\ell(Q) = 2^i$ and $j(Q) = n$ is bounded by 3. Thus, since $p < 2$, we get

$$\mu(\mathbf{Sh}_v(P)) \approx \sum_{i=-\infty}^{i_0} 2^{(2-p)i} \sum_{n=1}^{\infty} \frac{1}{n^p} \approx 2^{(2-p)i_0} = \ell(P)^{2-p}.$$

Summing up, every Whitney cube P satisfies that

$$\mu(P) \approx \mu(\mathbf{Sh}_v(P)) \approx \frac{\ell(P)^2}{\text{dist}(P, 0)^p} \approx \frac{\ell(P)^{2-p}}{j(P)^p}.$$

Thus, since $\frac{p-2}{p-1} = 2 - p'$ and $j(P) > 1$ for all cubes P , we have that

$$\sum_{Q \subset \mathbf{Sh}_v(P)} \mu(\mathbf{Sh}_v(Q))^{p'} \ell(Q)^{\frac{p-2}{p-1}} \approx \sum_{Q \subset \mathbf{Sh}_v(P)} \frac{\ell(Q)^{(2-p)p'}}{j(Q)^{pp'}} \ell(Q)^{2-p'} < \sum_{Q \subset \mathbf{Sh}_v(P)} \frac{\ell(Q)^{(2-p)p'+2-p'}}{j(Q)^p}$$

so, using that $(2-p)p' + 2 - p' = 2 - p$, we have that

$$\sum_{Q \subset \mathbf{Sh}_v(P)} \mu(\mathbf{Sh}_v(Q))^{p'} \ell(Q)^{\frac{p-2}{p-1}} \lesssim \sum_{Q \subset \mathbf{Sh}_v(P)} \frac{\ell(Q)^{2-p}}{j(Q)^p} \approx \mu(\mathbf{Sh}_v(P)).$$

□

1.2 Notation

On inequalities: When comparing two quantities x_1 and x_2 that depend on some parameters p_1, \dots, p_j we will write

$$x_1 \leq C_{p_1, \dots, p_j} x_2$$

if the constant C_{p_1, \dots, p_j} depends on p_1, \dots, p_j . We will also write $x_1 \lesssim_{p_1, \dots, p_j} x_2$ for short, or simply $x_1 \lesssim x_2$ if the dependence is clear from the context or if the constants are universal. We may omit some of these variables for the sake of simplicity. The notation $x_1 \approx_{p_1, \dots, p_j} x_2$ will mean that $x_1 \lesssim_{p_1, \dots, p_j} x_2$ and $x_2 \lesssim_{p_1, \dots, p_j} x_1$.

On polynomials: We write $\mathcal{P}^n(\mathbb{R}^d)$ for the vector space of real polynomials of degree smaller or equal than n with d real variables. If it is clear from the context we will just write \mathcal{P}^n .

On sets: Given two sets A and B , their symmetric difference is $A\Delta B := (A \cup B) \setminus (A \cap B)$ and their long distance is

$$D(A, B) := \text{diam}(A) + \text{diam}(B) + \text{dist}(A, B). \quad (1.6)$$

Given $x \in \mathbb{R}^d$ and $r > 0$, we write $B(x, r)$ or $B_r(x)$ for the open ball centered at x with radius r and $Q(x, r)$ for the open cube centered at x with sides parallel to the axis and side-length $2r$. Given any cube Q , we write $\ell(Q)$ for its side-length, and rQ will stand for the cube with the same center but enlarged by a factor r . We will use the same notation for balls and one dimensional cubes, that is, intervals.

At some point we need to use segments in \mathbb{R}^d : given $x, y \in \mathbb{R}^d$, we call the segment with endpoints x and y

$$[x, y] := \{(1-t)x + ty : t \in [0, 1]\}.$$

We may use the ‘‘open’’ segment $]x, y[:= [x, y] \setminus \{x, y\}$.

On finite differences: Given a function $f : \Omega \subset \mathbb{R}^d \rightarrow \mathbb{C}$ and two values $x, h \in \mathbb{R}^d$ such that $[x, x+h] \subset \Omega$, we call

$$\Delta_h^1 f(x) = \Delta_h f(x) = f(x+h) - f(x).$$

Moreover, for any natural number $i \geq 2$ we define the iterated difference

$$\Delta_h^i f(x) = \Delta_h^{i-1} f(x+h) - \Delta_h^{i-1} f(x) = \sum_{j=0}^{i-1} (-1)^{i-j} \binom{i}{j} f(x+jh)$$

whenever the segment $[x, x+ih] \subset \Omega$.

On domains: We call domain an open and connected subset of \mathbb{R}^d .

Definition 1.4. *Given $n \geq 1$, we say that $\Omega \subset \mathbb{C}$ is a $(\delta, R) - C^{n-1,1}$ domain if given any $z \in \partial\Omega$, there exists a function $A_z \in C^{n-1,1}(\mathbb{R})$ supported in $[-4R, 4R]$ such that*

$$\|A_z^{(j)}\|_{L^\infty} \leq \frac{\delta}{R^{j-1}} \quad \text{for every } 0 \leq j \leq n,$$

and, possibly after a translation that sends z to the origin and a rotation that brings the tangent at z to the real line, we have that

$$\Omega \cap Q(0, R) = \{x + iy : y > A_z(x)\}.$$

In case $n = 1$ the assumption of the tangent is removed (we say that Ω is a (δ, R) -Lipschitz domain). This concept can be extended naturally to every \mathbb{R}^d with dimension $d > 2$. In the present dissertation, we will find only Lipschitz domains in \mathbb{R}^d with $d > 2$ in Chapter 5.

We call window such a cube.

If $n = 1$ and $\Omega = \{x + iy : y > A_0(x)\}$, we say that Ω is a δ -special Lipschitz domain.

On Whitney coverings: Next we present a construction that will be used in all the chapters.

Definition 1.5. *Given a domain Ω , we say that a collection of open dyadic cubes \mathcal{W} is a Whitney covering of Ω if they are disjoint, the union of the cubes and their boundaries is Ω , there exists a constant $C_{\mathcal{W}}$ such that*

$$C_{\mathcal{W}}\ell(Q) \leq \text{dist}(Q, \partial\Omega) \leq 4C_{\mathcal{W}}\ell(Q),$$

two neighbor cubes Q and R (i.e., $\bar{Q} \cap \bar{R} \neq \emptyset$) satisfy $\ell(Q) \leq 2\ell(R)$, and the family $\{50Q\}_{Q \in \mathcal{W}}$ has finite superposition. Moreover, we will assume that

$$S \subset 5Q \implies \ell(S) \geq \frac{1}{2}\ell(Q). \quad (1.7)$$

The existence of such a covering is granted for any open set different from \mathbb{R}^d and in particular for any domain as long as $C_{\mathcal{W}}$ is big enough (see [Ste70, Chapter 1] for instance).

On measure theory: We denote the d -dimensional Lebesgue measure in \mathbb{R}^d by m_d , or simply m when the dimension is clear from the context. At some point we use m also to denote a natural number. When dealing with line integrals in the complex plane, we will write dz for the form $dx + i dy$ and analogously $d\bar{z} = dx - i dy$, where $z = x + i y$. Thus, when integrating a function with respect to the Lebesgue measure of a complex variable z we will always use $dm(z)$ to avoid confusion, or simply dm .

For any measurable set A and any measurable function f , we call $f_A = \int_A f dm$ to the mean of f in A .

On indices: In this text \mathbb{N}_0 stands for the natural numbers including 0. Otherwise we will write \mathbb{N} . We will make wide use of the multiindex notation for exponents and derivatives. For $\alpha \in \mathbb{Z}^d$ its *modulus* is $|\alpha| = \sum_{i=1}^d |\alpha_i|$ and its *factorial* is $\alpha! = \prod_{i=1}^d \alpha_i!$. Given two multiindices $\alpha, \gamma \in \mathbb{Z}^d$ we write $\alpha \leq \gamma$ if $\alpha_i \leq \gamma_i$ for every i . We say $\alpha < \gamma$ if, in addition, $\alpha \neq \gamma$. Furthermore, we write

$$\binom{\alpha}{\gamma} := \prod_{i=1}^d \binom{\alpha_i}{\gamma_i} = \begin{cases} \prod_{i=1}^d \frac{\alpha_i!}{\gamma_i!(\alpha_i - \gamma_i)!} & \text{if } \alpha \in \mathbb{N}_0^d \text{ and } \vec{0} \leq \gamma \leq \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

For $x \in \mathbb{R}^d$ and $\alpha \in \mathbb{Z}^d$ we write $x^\alpha := \prod x_i^{\alpha_i}$. Given any $\phi \in C_c^\infty$ (infinitely many times differentiable with compact support in \mathbb{R}^d) and $\alpha \in \mathbb{N}_0^d$ we write $D^\alpha \phi = \frac{\partial^{|\alpha|}}{\prod \partial x_i^{\alpha_i}} \phi$.

At some point we will use also use roman letter for multiindices, and then, to avoid confusion, we will use the vector notation \vec{i}, \vec{j}, \dots .

On complex notation For $z = x + i y \in \mathbb{C}$ we write $\text{Re}(z) := x$ and $\text{Im}(z) := y$. Note that the symbol i will be used also widely as a index for summations without risk of confusion. The multiindex notation will change slightly: for $z \in \mathbb{C}$ and $\alpha \in \mathbb{Z}^2$ we write $z^\alpha := z^{\alpha_1} \bar{z}^{\alpha_2}$.

We also adopt the traditional Wirtinger notation for derivatives, that is, given any $\phi \in C_c^\infty(\mathbb{C})$, then

$$\partial\phi(z) := \frac{\partial\phi}{\partial z}(z) = \frac{1}{2}(\partial_x\phi - i\partial_y\phi)(z),$$

and

$$\bar{\partial}\phi(z) := \frac{\partial\phi}{\partial \bar{z}}(z) = \frac{1}{2}(\partial_x\phi + i\partial_y\phi)(z).$$

Thus, given any $\phi \in C_c^\infty(\mathbb{C})$ and $\alpha \in \mathbb{N}_0^2$, we write $D^\alpha \phi = \partial^{\alpha_1} \bar{\partial}^{\alpha_2} \phi$.

On Sobolev spaces: For any open set U , every distribution $f \in \mathcal{D}'(U)$ and $\alpha \in \mathbb{N}_0^d$, the *distributional derivative* $D_U^\alpha f$ is the distribution defined by

$$\langle D_U^\alpha f, \phi \rangle := (-1)^{|\alpha|} \langle f, D^\alpha \phi \rangle \quad \text{for every } \phi \in C_c^\infty(U).$$

Abusing notation we will write D^α instead of D_U^α if it is clear from the context. If the distribution is regular, that is, if it coincides with an L^1_{loc} function acting on $\mathcal{D}(U)$, then we say that $D_U^\alpha f$ is a *weak derivative* of f in U . We write $|\nabla^n f| = \sum_{|\alpha|=n} |D^\alpha f|$.

Given numbers $n \in \mathbb{N}$, $1 \leq p \leq \infty$ an open set $U \subset \mathbb{R}^d$ and an $L^1_{loc}(U)$ function f , we say that f is in the Sobolev space $W^{n,p}(U)$ of smoothness n and order of integrability p if f has weak derivatives $D_U^\alpha f \in L^p$ for every $\alpha \in \mathbb{N}_0^d$ with $|\alpha| \leq n$. We will use the norm

$$\|f\|_{W^{n,p}(\Omega)} := \|f\|_{L^p(\Omega)} + \sum_{|\alpha| \leq n} \|D^\alpha f\|_{L^p(\Omega)}.$$

When Ω is a Lipschitz domain, this norm is equivalent to considering only the higher order derivatives. Namely,

$$\|f\|_{W^{n,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \|\nabla^n f\|_{L^p(\Omega)} \approx \|f\|_{L^p(\Omega)} + \sum_{j=1}^d \|\partial_j^n f\|_{L^p(\Omega)} \quad (1.8)$$

(see [Tri78, Theorem 4.2.4]). When Ω is an *extension domain* for $W^{n,p}$, that is, when there exists a bounded operator $\Lambda : W^{n,p}(\Omega) \rightarrow W^{n,p}(\mathbb{R}^d)$ such that $(\Lambda f)|_\Omega \equiv f|_\Omega$ for $f \in W^{n,p}(\Omega)$, then

$$\|f\|_{W^{n,p}(\Omega)} \approx \inf_{F: F|_\Omega \equiv f} \|F\|_{W^{n,p}(\mathbb{R}^d)}.$$

From [Jon81], we know that uniform domains (and in particular, Lipschitz domains) are Sobolev extension domains for any indices $n \in \mathbb{N}$ and $1 \leq p \leq \infty$. One can find deeper results in that sense in [Shv10] and [KRZ15].

On Besov and Triebel-Lizorkin spaces: Next we present a generalization of Sobolev spaces.

Definition 1.6. Let $\Phi(\mathbb{R}^d)$ be the collection of all the families $\Psi = \{\psi_j\}_{j=0}^\infty \subset C_c^\infty(\mathbb{R}^d)$ such that

$$\begin{cases} \text{supp } \psi_0 \subset \mathbb{D}(0, 2), \\ \text{supp } \psi_j \subset \mathbb{D}(0, 2^{j+1}) \setminus \mathbb{D}(0, 2^{j-1}) \quad \text{if } j \geq 1, \end{cases}$$

for all multiindex $\alpha \in \mathbb{N}^d$ there exists a constant c_α such that

$$\|D^\alpha \psi_j\|_\infty \leq \frac{c_\alpha}{2^{j|\alpha|}} \quad \text{for every } j \geq 0$$

and

$$\sum_{j=0}^\infty \psi_j(x) = 1 \quad \text{for every } x \in \mathbb{R}^d.$$

Definition 1.7. Given any Schwartz function $\psi \in \mathcal{S}(\mathbb{R}^d)$, its Fourier transform is

$$F\psi(\zeta) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \zeta} \psi(x) dm(x).$$

This notion extends to the tempered distributions $\mathcal{S}'(\mathbb{R}^d)$ by duality (see [Gra08, Definition 2.3.7]).

Definition 1.8. Let $s \in \mathbb{R}$, $1 \leq p \leq \infty$, $1 \leq q \leq \infty$ and $\Psi \in \Phi(\mathbb{R}^d)$. For any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ we define its non-homogeneous Besov norm

$$\|f\|_{B_{p,q}^s}^\Psi = \left\| \left\{ 2^{sj} \left\| (\psi_j \hat{f})^\vee \right\|_{L^p} \right\}_{j \geq 0} \right\|_{l^q},$$

and we call $B_{p,q}^s \subset \mathcal{S}'$ to the set of tempered distributions such that this norm is finite.

Let $s \in \mathbb{R}$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $\Psi \in \Phi(\mathbb{R}^d)$. For any tempered distribution $f \in \mathcal{S}'(\mathbb{R}^d)$ we define its non-homogeneous Triebel-Lizorkin norm

$$\|f\|_{F_{p,q}^s}^\Psi = \left\| \left\{ 2^{sj} \left(\psi_j \widehat{f} \right)^\vee \right\} \right\|_{l^q} \Big\|_{L^p},$$

and we call $F_{p,q}^s \subset \mathcal{S}'$ to the set of tempered distributions such that this norm is finite.

These norms are equivalent for different choices of Ψ . Usually one works with radial ψ_j and such that $\psi_{j+1}(x) = \psi_j(x/2)$ for $j \geq 1$. Of course we will omit Ψ in our notation since it plays no role (see [Tri83, Section 2.3]).

Remark 1.9. For $q = 2$ and $1 < p < \infty$ the spaces $F_{p,2}^s$ coincide with the so-called Bessel-potential spaces $W^{s,p}$. In addition, if $s \in \mathbb{N}$ they coincide with the usual Sobolev spaces, and they coincide with L^p for $s = 0$ (see [Tri83, Section 2.5.6]). In the present text, we call Sobolev space to any $W^{s,p}$ with $s > 0$ and $1 < p < \infty$, even if s is not a natural number. Note that complex interpolation between Sobolev spaces is a Sobolev space (see [Tri78, Section 2.4.2, Theorem 1]).

On conjugate indices: Given $1 \leq p \leq \infty$ we write p' for its Hölder conjugate, that is $\frac{1}{p} + \frac{1}{p'} = 1$.

1.3 Known facts

On inequalities: For the sake of completeness, we recall the reader Minkowski's integral inequality (see [Ste70, Appendix A1]) which we will use every now and then. It states that for $1 \leq p < \infty$, given a function $F : X \times Y \rightarrow \mathbb{C}$, where X and Y are σ -finite measure spaces, we have that

$$\left(\int_Y \left(\int_X |F(x,y)| dx \right)^p dy \right)^{\frac{1}{p}} \leq \int_X \left(\int_Y |F(x,y)|^p dy \right)^{\frac{1}{p}} dx. \quad (1.9)$$

We will use also the Young inequality. It states that for measurable functions f and g , we have that

$$\|f * g\|_{L^q} \leq \|f\|_{L^r} \|g\|_{L^p} \quad (1.10)$$

for $1 \leq p, q, r \leq \infty$ with $\frac{1}{q} = \frac{1}{p} + \frac{1}{r} - 1$ (see [Ste70, Appendix A2]).

On the Leibniz rule: The Leibniz formula (see [Eva98, Section 5.2.3]) says that given a domain $\Omega \subset \mathbb{R}^d$, a function $f \in W^{n,p}(\Omega)$ and $\phi \in C_c^\infty(\Omega)$, we have that $\phi \cdot f \in W^{n,p}(\Omega)$ with

$$D^\alpha(\phi \cdot f) = \sum_{\gamma \leq \alpha} \binom{\alpha}{\gamma} D^\gamma \phi D^{\alpha-\gamma} f \quad (1.11)$$

for every multiindex $\alpha \in \mathbb{N}_0^d$ such that $|\alpha| = n$.

On Green's formula: The Green Theorem can be written in terms of complex derivatives (see [AIM09, Theorem 2.9.1]). Let Ω be a bounded Lipschitz domain. If $f, g \in W^{1,1}(\Omega) \cap C(\bar{\Omega})$, then

$$\int_\Omega (\partial f + \bar{\partial} g) dm = \frac{i}{2} \left(\int_{\partial\Omega} f(z) d\bar{z} - \int_{\partial\Omega} g(z) dz \right). \quad (1.12)$$

On the Residue Theorem: We say that a function $f : \mathbb{C} \rightarrow \mathbb{C}$ is meromorphic on an open set $U \subset \mathbb{C}$ if f is holomorphic on U but a discrete set of points $\{a_j\}_j$. The singularity of f in

a_j is called a *pole of order* $m_j \geq 1$ when $\lim_{z \rightarrow a_j} f(z)(z - a_j)^m = 0$ if and only if $m \geq m_j + 1$. The well-known Residue Theorem (see [Con78, Chapter V, Theorem 2.2 and Proposition 2.4], for instance), states that given a meromorphic function f in a connected open set U with poles in $\{a_j\}_{j=1}^M$ and no more singularities, and given a closed rectifiable curve γ homologous to 0 in U which does not pass through any a_j , then the line integral

$$\frac{1}{2\pi i} \int_{\gamma} f(z) dz = \sum_{j=1}^M n(\gamma, a_j) \frac{1}{(m_j - 1)!} g^{(m_j-1)}(a_j), \quad (1.13)$$

where $n(\gamma, a_j)$ stands for the winding number of γ around a_j (see [Con78, Chapter IV, Definition 4.2]) and m_j is the order of the pole of f in a_j .

On the Rolle Theorem: We state here also a Complex Rolle Theorem for holomorphic functions [EJ92, Theorem 2.1] that will be a cornerstone of Section 3.5.

Theorem 1.10. [see [EJ92]] *Let f be a holomorphic function defined on an open convex set $U \subset \mathbb{C}$. Let $a, b \in U$ such that $f(a) = f(b) = 0$ and $a \neq b$. Then there exists z in the segment $]a, b[$ such that $\operatorname{Re}(\partial f(z)) = 0$.*

On the Sobolev Embedding Theorem: We state a reduced version of the Sobolev Embedding Theorem for Lipschitz domains (see [AF03, Theorem 4.12, Part II]). For each Lipschitz domain $\Omega \subset \mathbb{R}^d$ and every $p > d$, there is a continuous embedding of the Sobolev space $W^{1,p}(\Omega)$ into the Hölder space $C^{0,1-\frac{d}{p}}(\bar{\Omega})$. That is, writing

$$\|f\|_{C^{0,s}(\bar{\Omega})} = \|f\|_{L^\infty(\Omega)} + \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^s} \quad \text{for } 0 < s \leq 1,$$

we have that for every $f \in W^{1,p}(\Omega)$,

$$\|f\|_{L^\infty(\Omega)} \leq \|f\|_{C^{0,1-\frac{d}{p}}(\bar{\Omega})} \leq C_\Omega \|f\|_{W^{1,p}(\Omega)}. \quad (1.14)$$

If Ω is not Lipschitz but it is an extension domain, then the previous estimate remains true.

On embeddings and equivalent norms for $B_{p,q}^s$ and $F_{p,q}^s$: Next we recall some results on the function spaces that we will use. For a complete treatment we refer the reader to [Tri83].

Proposition 1.11 (See [RS96, Section 2.2]). *The following properties hold:*

1. *Let $1 \leq q_0, q_1 \leq \infty$ and $1 \leq p \leq \infty$, $s \in \mathbb{R}$ and $\varepsilon > 0$. Then*

$$B_{p,q_0}^{s+\varepsilon} \subset B_{p,q_1}^s.$$

2. *Let $1 \leq p_0 < p < p_1 \leq \infty$, $-\infty < s_1 < s < s_0 < \infty$, $1 \leq u \leq p \leq v \leq \infty$ and $1 \leq q < \infty$. Then,*

$$F_{p_0,u}^{s_0} \subset F_{p,q}^s \subset B_{p_1,v}^{s_1} \quad \text{if } s_0 - \frac{d}{p_0} = s - \frac{d}{p} = s_1 - \frac{d}{p_1}.$$

3. *In particular, given $1 \leq p < \infty$, $\frac{d}{p} < s < \infty$ and $1 \leq q \leq \infty$. Then,*

$$F_{p,q}^s \subset B_{\infty,\infty}^{s-\frac{d}{p}} \quad \text{and} \quad B_{p,q}^s \subset B_{\infty,\infty}^{s-\frac{d}{p}}. \quad (1.15)$$

Remark 1.12. When $s - \frac{d}{p} \notin \mathbb{Z}$, the space $B_{\mathcal{X}, \mathcal{X}}^{s - \frac{d}{p}}$ coincides with the Hölder space $C^{s - \frac{d}{p}}$, so this is a generalization of the Sobolev embedding Theorem to the whole double-scale of Besov and Triebel-Lizorkin spaces. Thus, if Ω is an extension domain for $F_{p,q}^s$ and $sp > d$, then $F_{p,q}^s(\Omega) \subset C^{s - \frac{d}{p}}$ if $s - \frac{d}{p} \notin \mathbb{Z}$ and $F_{p,q}^s(\Omega) \subset C^{s - \frac{d}{p} - \varepsilon}$ if $s - \frac{d}{p} \in \mathbb{Z}$.

If we set $j \in \mathbb{Z}$ instead of $j \in \mathbb{N}$ in Definition 1.6, then we get the homogeneous spaces of tempered distributions (modulo polynomials) $\dot{B}_{p,q}^s$. In particular, by [Tri92, Theorem 2.3.3] we have that for $s > 0$

$$\|f\|_{B_{p,q}^s} \approx \|f\|_{\dot{B}_{p,q}^s} + \|f\|_{L^p} \quad \text{for any } f \in \mathcal{S}', \quad (1.16)$$

and the same can be said for Triebel-Lizorkin spaces.

In the particular case of homogeneous Besov spaces with $1 \leq p, q \leq \infty$ and $s > 0$, one can give an equivalent definition in terms of differences of order $M \geq [s] + 1$:

$$\|f\|_{\dot{B}_{p,q}^s} \approx \left(\int_0^\infty \frac{\sup_{|h| \leq t} \|\Delta_h^M f\|_{L^p}^q}{t^{sq}} \frac{dt}{t} \right)^{\frac{1}{q}} \approx \left(\int_{\mathbb{R}^d} \frac{\|\Delta_h^M f\|_{L^p}^q}{|h|^{sq}} \frac{dm(h)}{|h|^d} \right)^{\frac{1}{q}}. \quad (1.17)$$

Consider the boundary of a Lipschitz domain $\Omega \subset \mathbb{C}$. When it comes to the Besov space $B_{p,q}^s(\partial\Omega)$ we can just define it using the arc parameter of the curve, $z : I \rightarrow \partial\Omega$ with $|z'(t)| = 1$ for all t . Note that if the domain is bounded, then I is a finite interval with length equal to the length of the boundary of Ω and we need to extend z periodically to \mathbb{R} in order to have a sensible definition. We also use an auxiliary bump function $\varphi_\Omega : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi_\Omega|_{2I} \equiv 1$ and $\varphi_\Omega|_{(4I)^c} \equiv 0$. Then, if $1 \leq p, q < \infty$, we define naturally the homogeneous Besov norm on the boundary of Ω as

$$\|f\|_{B_{p,q}^s(\partial\Omega)} := \|(f \circ z)\varphi_\Omega\|_{B_{p,q}^s(\mathbb{R})}. \quad (1.18)$$

Consider the boundary of a Lipschitz domain $\Omega \subset \mathbb{C}$. When it comes to the Besov space $B_{p,q}^s(\partial\Omega)$ we can just define it using the arc parameter of the curve, $z : I \rightarrow \partial\Omega$ with $|z'(t)| = 1$ for all t . Note that if the domain is bounded, then I is a finite interval with length equal to the length of the boundary of Ω and we need to extend z periodically to \mathbb{R} in order to have a sensible definition. We also use an auxiliary bump function $\varphi_\Omega : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi_\Omega|_{2I} \equiv 1$ and $\varphi_\Omega|_{(4I)^c} \equiv 0$. Then, if $1 \leq p, q < \infty$, we define naturally the homogeneous Besov norm on the boundary of Ω as

$$\|f\|_{B_{p,q}^s(\partial\Omega)} := \|(f \circ z)\varphi_\Omega\|_{B_{p,q}^s(\mathbb{R})}. \quad (1.19)$$

1.4 On uniform domains

There is a considerable amount of literature on uniform domains and their properties, we refer the reader e.g. to [GO79] and [Väi88].

Definition 1.13. Let Ω be a domain and \mathcal{W} a Whitney decomposition of Ω and $Q, S \in \mathcal{W}$. Given M cubes $Q_1, \dots, Q_M \in \mathcal{W}$ with $Q_1 = Q$ and $Q_M = S$, the M -tuple $(Q_1, \dots, Q_M)_{j=1}^M \in \mathcal{W}^M$ is a chain connecting Q and S if the cubes Q_j and Q_{j+1} are neighbors for $j < M$. We write $[Q, S] = (Q_1, \dots, Q_M)_{j=1}^M$ for short.

Let $\varepsilon \in \mathbb{R}$. We say that the chain $[Q, S]$ is ε -admissible if

- the length of the chain is bounded by

$$\ell([Q, S]) := \sum_{j=1}^M \ell(Q_j) \leq \frac{1}{\varepsilon} \mathsf{D}(Q, S) \quad (1.20)$$

(see (1.6)),

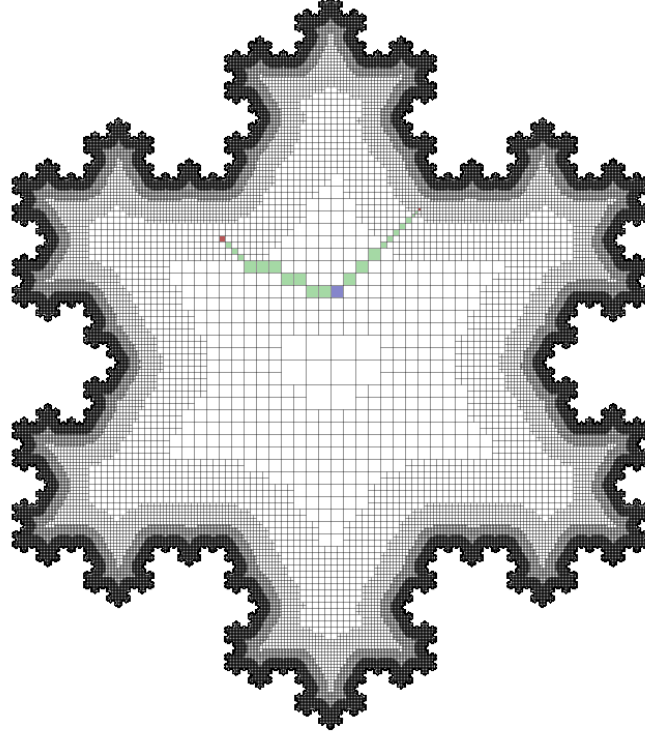


Figure 1.3: A Whitney decomposition of a uniform domain with and an ε -admissible chain. The end-point cubes are colored in red and the central one in blue.

- and there exists $j_0 < M$ such that the cubes in the chain satisfy

$$\ell(Q_j) \geq \varepsilon D(Q_1, Q_j) \text{ for all } j \leq j_0 \quad \text{and} \quad \ell(Q_j) \geq \varepsilon D(Q_j, Q_M) \text{ for all } j \geq j_0. \quad (1.21)$$

The j_0 -th cube, which we call central, satisfies that $\ell(Q_{j_0}) \gtrsim_d \varepsilon D(Q, S)$ by (1.20) and the triangle inequality. We will write $Q_S = Q_{j_0}$. Note that this is an abuse of notation because the central cube of $[Q, S]$ may vary for different ε -admissible chains joining Q and S .

We write (abusing notation again) $[Q, S]$ also for the set $\{Q_j\}_{j=1}^M$. Thus, we will write $P \in [Q, S]$ if P appears in a coordinate of the M -tuple $[Q, S]$. For any $P \in [Q, S]$ we call $\mathcal{N}_{[Q, S]}(P)$ to the following cube in the chain, that is, for $j < M$ we have that $\mathcal{N}_{[Q, S]}(Q_j) = Q_{j+1}$. We will write $\mathcal{N}(P)$ for short if the chain to which we are referring is clear from the context.

Every now and then we will mention subchains. That is, for $1 \leq j_1 \leq j_2 \leq M$, the subchain $[Q_{j_1}, Q_{j_2}]_{[Q, S]} \subset [Q, S]$ is defined as $(Q_{j_1}, Q_{j_1+1}, \dots, Q_{j_2})$. We will write $[Q_{j_1}, Q_{j_2}]$ if there is no risk of confusion.

Next we make some observations on the two subchains $[Q, Q_S]$ and $[Q_S, S]$.

Remark 1.14. Consider a domain Ω with covering \mathcal{W} and two cubes $Q, S \in \mathcal{W}$ with an ε -admissible chain $[Q, S]$. From Definition 1.13 it follows that

$$D(Q, S) \approx_{\varepsilon, d} \ell([Q, S]) \approx_{\varepsilon, d} \ell(Q_S) \approx_{\varepsilon, d} D(Q, Q_S) \approx_{\varepsilon, d} D(Q_S, S). \quad (1.22)$$

If $P \in [Q, Q_S]$, by (1.20) we have that

$$D(Q, P) \approx_{d, \varepsilon} \ell(P). \quad (1.23)$$

On the other hand, by the triangular inequality, (1.19) and (1.20) we have that

$$D(P, S) \lesssim_d \ell([P, S]) \leq \ell([Q, S]) \leq \frac{D(Q, S)}{\varepsilon} \lesssim_d \frac{D(Q, P) + D(P, S)}{\varepsilon} \lesssim_d \frac{\frac{1}{\varepsilon} \ell(P) + D(P, S)}{\varepsilon},$$

that is,

$$D(P, S) \approx_{\varepsilon, d} D(Q, S). \quad (1.24)$$

Definition 1.15. We say that a domain $\Omega \subset \mathbb{R}^d$ is a uniform domain if there exists a Whitney covering \mathcal{W} of Ω and $\varepsilon \in \mathbb{R}$ such that for any pair of cubes $Q, S \in \mathcal{W}$, there exists an ε -admissible chain $[Q, S]$ (see Figure 1.3). Sometimes will write ε -uniform domain to fix the constant ε .

Using (1.23) it is quite easy to see that a domain satisfying this definition satisfies to the one given by Peter Jones in [Jon81] (with $\delta = \infty$ and changing the parameter ε if necessary). It is somewhat more involved to prove the converse implication, but it can be done using the ideas of Remark 1.14. Of course, the definition above is also equivalent to the one given by Gehring and Osgood in [GO79]. In any case it is not transcendent for the present dissertation to prove this fact, which is left for the reader as an exercise.

Now we can define the shadows:

Definition 1.16. Let Ω be an ε -uniform domain with Whitney covering \mathcal{W} . Given a cube $P \in \mathcal{W}$ centered at x_P and a real number ρ , the ρ -shadow of P is the collection of cubes

$$\mathbf{SH}_\rho(P) = \{Q \in \mathcal{W} : Q \subset B(x_P, \rho \ell(P))\},$$

and its “realization” is the set

$$\mathbf{sh}_\rho(P) = \bigcup_{Q \in \mathbf{SH}_\rho(P)} Q$$

(see Figure 1.4).

By the previous remark and the properties of the Whitney covering, we can define $\rho_\varepsilon > 1$ such that the following properties hold:

- For every $P \in \mathcal{W}$, we have the estimate $|\text{diam}(\partial\Omega \cap \overline{\mathbf{sh}_{\rho_\varepsilon}(P)})| \approx \ell(P)$.
- For every ε -admissible chain $[Q, S]$, and every $P \in [Q, Q_S]$ we have that $Q \in \mathbf{SH}_{\rho_\varepsilon}(P)$.
- Moreover, every cube P belonging to an ε -admissible chain $[Q, S]$ belongs to the shadow $\mathbf{SH}_{\rho_\varepsilon}(Q_S)$.

Note that the first property comes straight from the properties of the Whitney covering, while the second is a consequence of (1.22) and the third holds because of the fact that if $P \in [Q, S]$ then $D(P, Q_S) \lesssim_d \ell([Q, S]) \approx D(Q, S) \approx \ell(Q_S)$ by (1.21).

Remark 1.17. Given an ε -uniform domain Ω we will write \mathbf{Sh} for $\mathbf{sh}_{\rho_\varepsilon}$. We will write also \mathbf{SH} for $\mathbf{SH}_{\rho_\varepsilon}$.

For $Q \in \mathcal{W}$ and $s > 0$, we have that

$$\sum_{L: Q \in \mathbf{SH}(L)} \ell(L)^{-s} \lesssim \ell(Q)^{-s} \quad (1.25)$$

and, moreover, if $Q \in \mathbf{SH}(P)$, then

$$\sum_{L \in [Q, P]} \ell(L)^s \lesssim \ell(P)^s \quad \text{and} \quad \sum_{L \in [Q, P]} \ell(L)^{-s} \lesssim \ell(Q)^{-s}. \quad (1.26)$$

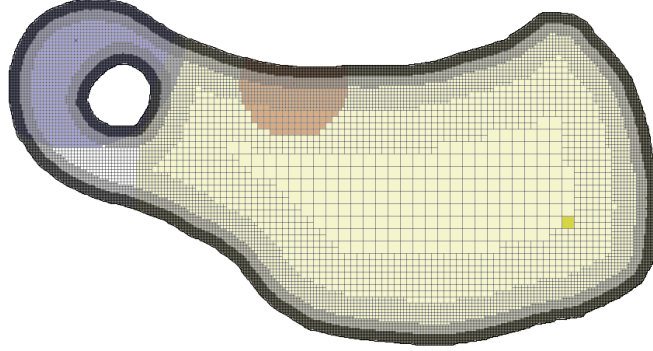


Figure 1.4: A Whitney decomposition of a Lipschitz domain with the shadows of three different cubes (see Definition 1.16).

Proof. Considering the definition of shadow we can deduce that there is a bounded number of cubes with given side-length in the left-hand side of (1.24) and, therefore, the sum is a geometric sum. Again by the definition of shadow we know that the smaller cube in that sum has side-length comparable to $\ell(Q)$.

To prove (1.25), first note that $\ell(Q_P) \approx D(Q, P) \approx \ell(P)$ by (1.21) and Definition 1.16. For every $L \in [Q, P]$, although it may occur that $L \notin \mathbf{SH}(P)$, we still have that by the triangle inequality $D(L, P) \lesssim \ell([Q, P]) \approx D(Q, P)$ and, thus, by the definition of shadow we have that $D(L, P) \lesssim \ell(P)$, i.e.

$$D(L, P) \approx \ell(P). \quad (1.27)$$

When $L \in [Q, Q_P]$, (1.22) reads as

$$\ell(L) \approx D(Q, L),$$

and when $L \in [Q_P, P]$ by (1.22) and (1.26), we have that

$$\ell(L) \approx D(L, P) \approx \ell(P).$$

In particular, the number of cubes in $[Q_P, P]$ is uniformly bounded. Summing up, for $L \in [Q, P]$ we have that $\ell(Q) \lesssim \ell(L) \lesssim \ell(P)$ and all the cubes of a given side-length r contained in $[Q, P]$ are situated at a distance from Q bounded by Cr . so the number of those cubes is uniformly bounded. Therefore, the left-hand side of both inequalities of (1.25) are geometric sums, bounded by a constant times the bigger term. The constant depends on s , but also on the uniformity constant of the domain. \square

We recall the definition of the non-centered Hardy-Littlewood maximal operator. Given $f \in L^1_{loc}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, we define $Mf(x)$ as the supremum of the mean of f in cubes containing x , that is,

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q f(y) dy.$$

It is a well known fact that this operator is bounded on L^p for $1 < p < \infty$. The following lemma will be used repeatedly along the proofs contained in the present text.

Lemma 1.18. *Let Ω be a bounded uniform domain with an admissible Whitney covering \mathcal{W} . Assume that $g \in L^1(\Omega)$ and $r > 0$. For every $\eta > 0$, $Q \in \mathcal{W}$ and $x \in \mathbb{R}^d$, the following inequalities hold:*

1) *The non-local inequality for the maximal operator*

$$\int_{|y-x|>r} \frac{g(y) dy}{|y-x|^{d+\eta}} \lesssim_d \frac{Mg(x)}{r^\eta} \quad \text{and} \quad \sum_{S:D(Q,S)>r} \frac{\int_S g(y) dy}{D(Q,S)^{d+\eta}} \lesssim_d \frac{\inf_{y \in Q} Mg(y)}{r^\eta}. \quad (1.28)$$

2) *The local inequality for the maximal operator*

$$\int_{|y-x|<r} \frac{g(y) dy}{|y-x|^{d-\eta}} \lesssim_d r^\eta Mg(x) \quad \text{and} \quad \sum_{S:D(Q,S)<r} \frac{\int_S g(y) dy}{D(Q,S)^{d-\eta}} \lesssim_d \inf_{y \in Q} Mg(y) r^\eta. \quad (1.29)$$

3) *In particular we have*

$$\int_{|y-x|>r} \frac{dy}{|y-x|^{d+\eta}} \lesssim_d \frac{1}{r^\eta}, \quad \sum_{S \in \mathcal{W}} \frac{\ell(S)^d}{D(Q,S)^{d+\eta}} \lesssim_d \frac{1}{\ell(Q)^\eta} \quad (1.30)$$

and, by Definition 1.16,

$$\sum_{S \in \mathbf{SH}_\rho(Q)} \int_S g(x) dx \lesssim_{d,\rho} \inf_{y \in Q} Mg(y) \ell(Q)^d.$$

Proof. The left-hand side of both inequalities in (1.27) can just be bounded by

$$C \int \frac{g(y) dx}{(|x-y|+r)^{d+\eta}}$$

(in the second one, this bound holds for every $x \in Q$), and this can be bounded separating the integral region in dyadic annuli. The sum in (1.28) can be bounded by an analogous reasoning and (1.29) follows when considering $g \equiv 1$. \square

1.5 Approximating polynomials

Before introducing the approximating polynomials, we need a rather trivial observation, which is proven here for the sake of completeness.

Remark 1.19. *For every polynomial $P \in \mathcal{P}^n$, every cube Q and every $r > 1$, we have that*

$$\|P\|_{L^1(Q)} \approx \ell(Q)^d \|P\|_{L^\infty(Q)},$$

and for $r > 1$, also

$$\|P\|_{L^\infty(rQ)} \lesssim r^n \|P\|_{L^\infty(Q)},$$

with constants depending only on d and n .

Proof. Without loss of generality we can assume that the cube Q is centered at 0. Given a polynomial $P(x) = \sum_{|\gamma| \leq n} a_\gamma x^\gamma$ of degree n , using the linear map ϕ that sends the unit cube $Q_0 = (-\frac{1}{2}, \frac{1}{2})^d$ to Q as a change of coordinates, we have that

$$\|P\|_{L^1(Q)} \approx |Q| \|P \circ \phi\|_{L^1(Q_0)}.$$

It is well known that all norms in a finite dimensional vector space are equivalent (see [Sch02, Theorem 4.2], for instance). In particular the $L^1(Q_0)$ norm, the sum of coefficients and the $L^\infty(Q_0)$ norm are equivalent in \mathcal{P}^n , so we have that

$$\frac{1}{\ell(Q)^d} \|P\|_{L^1(Q)} \approx \sum_{|\gamma| \leq n} \ell(Q)^{|\gamma|} |a_\gamma| \approx \|P\|_{L^\infty(Q)}.$$

By the same token,

$$\|P\|_{L^\infty(rQ)} \approx \sum_{|\gamma| \leq n} (r\ell(Q))^{|\gamma|} |a_\gamma| \lesssim r^n \|P\|_{L^\infty(Q)}.$$

□

Recall that the Poincaré inequality tells us that, given a cube Q and a function $f \in W^{1,p}(Q)$ with 0 mean in the cube, we have the estimate

$$\|f\|_{L^p(Q)} \lesssim \ell(Q) \|\nabla f\|_{L^p(Q)}$$

with universal constants once we fix d and $1 \leq p \leq \infty$ (this well-known result can be shown combining the proof of [Eva98, Theorem 5.8.1/1], the version of Rellich-Kondrachov Theorem in [AIM09, Theorem A.7.1] and a change of variables for the dilatation constant, for instance).

If we want to iterate that inequality, we also need the gradient of f to have 0 mean on Q . That leads us to define the next approximating polynomials.

Definition 1.20. Consider a domain Ω and a cube $Q \subset \Omega$. Given $f \in L^1(Q)$ with weak derivatives up to order n , we define $\mathbf{P}_Q^n(f) \in \mathcal{P}^n$ as the unique polynomial (restricted to Ω) of degree smaller or equal than n such that

$$\int_Q D^\beta \mathbf{P}_Q^n f \, dm = \int_Q D^\beta f \, dm \quad (1.31)$$

for every multiindex $\beta \in \mathbb{N}^d$ with $|\beta| \leq n$.

These polynomials can be understood as a particular case of the projection $L : W^{n+1,p}(Q) \rightarrow \mathcal{P}^n$ introduced by Norman G. Meyers in [Mey78].

Lemma 1.21. Given a cube Q and $f \in W^{n-1,1}(3Q)$, the polynomial $\mathbf{P}_{3Q}^{n-1} f \in \mathcal{P}^{n-1}$ exists and is unique. Furthermore, this polynomial has the next properties:

1. Let x_Q be the center of Q . If we consider the Taylor expansion of $\mathbf{P}_{3Q}^{n-1} f$ at x_Q ,

$$\mathbf{P}_{3Q}^{n-1} f(y) = \sum_{\substack{\gamma \in \mathbb{N}^d \\ |\gamma| < n}} m_{Q,\gamma} (y - x_Q)^\gamma, \quad (1.32)$$

then the coefficients $m_{Q,\gamma}$ are bounded by

$$|m_{Q,\gamma}| \lesssim_n \sum_{j=|\gamma|}^{n-1} \|\nabla^j f\|_{L^1(3Q)} \ell(Q)^{j-|\gamma|-d} \lesssim_n \|f\|_{W^{n-1,\infty}(3Q)} (1 + \ell(Q)^{n-1}). \quad (1.33)$$

2. Let us assume that, in addition, the function f is in the Sobolev space $W^{n,p}(3Q)$ for a certain $1 \leq p \leq \infty$. Given $0 \leq j \leq n$, if we have a smooth function $\varphi \in C^\infty(3Q)$ satisfying $\|\nabla^i \varphi\|_{L^\infty(3Q)} \lesssim \frac{1}{\ell(Q)^i}$ for $0 \leq i \leq j$, then we have the Poincaré inequality

$$\left\| \nabla^j \left((f - \mathbf{P}_{3Q}^{n-1} f) \varphi \right) \right\|_{L^p(3Q)} \leq C \ell(Q)^{n-j} \|\nabla^n f\|_{L^p(3Q)}. \quad (1.34)$$

3. Given a domain with a Whitney covering \mathcal{W} , two Whitney cubes $Q, S \in \mathcal{W}$, a chain $[S, Q]$ as in Definition 1.13, and $f \in W^{n,p}(\Omega)$, we have that

$$\|f - \mathbf{P}_{3Q}^{n-1} f\|_{L^1(S)} \lesssim \sum_{P \in [S, Q]} \frac{\ell(S)^d D(P, S)^{n-1}}{\ell(P)^{d-1}} \|\nabla^n f\|_{L^1(3P)}. \quad (1.35)$$

Proof. Note that (1.30) is a triangular system of equations on the coefficients of the polynomial. Indeed, for γ fixed, if the polynomial exists and has Taylor expansion (1.31), then

$$D^\gamma \mathbf{P}_{3Q}^{n-1} f(y) = \sum_{\beta \geq \gamma} m_{Q,\beta} \frac{\beta!}{(\beta - \gamma)!} (y - x_Q)^{\beta - \gamma}.$$

When we take means on the cube $3Q$,

$$\begin{aligned} \int_{3Q} D^\gamma f \, dm &= \int_{3Q} D^\gamma \mathbf{P}_{3Q}^{n-1} f \, dm \\ &= \sum_{\beta \geq \gamma} m_{Q,\beta} \frac{\beta!}{(\beta - \gamma)!} \left(\frac{3}{2}\ell(Q)\right)^{|\beta - \gamma|} \int_{Q(0,1)} y^{\beta - \gamma} \, dy \\ &= \sum_{\beta \geq \gamma} C_{\beta,\gamma} m_{Q,\beta} \ell(Q)^{|\beta - \gamma|}, \end{aligned}$$

which is a triangular system of equations on the coefficients $m_{Q,\beta}$.

Solving for $m_{Q,\gamma}$, since $C_{\gamma,\gamma} \neq 0$ we obtain the explicit expression

$$m_{Q,\gamma} = \frac{1}{C_{\gamma,\gamma}} \int_{3Q} D^\gamma f \, dm - \sum_{\beta > \gamma} C_{\beta,\gamma} m_{Q,\beta} \ell(Q)^{|\beta - \gamma|}. \quad (1.36)$$

For $|\gamma| = n - 1$ this gives the value of $m_{Q,\gamma}$ in terms of $D^\gamma f$,

$$m_{Q,\gamma} = \frac{1}{C_{\gamma,\gamma}} \int_{3Q} D^\gamma f \, dm.$$

Using induction on $n - |\gamma|$ we get the existence and uniqueness of $\mathbf{P}_{3Q}^{n-1} f$. Taking absolute values we obtain (1.32).

The equality (1.30) allows us to iterate the Poincaré inequality

$$\|f - \mathbf{P}_{3Q}^{n-1} f\|_{L^p(3Q)} \leq C \ell(Q) \|\nabla (f - \mathbf{P}_{3Q}^{n-1} f)\|_{L^p(3Q)} \leq \dots \leq C^n \ell(Q)^n \|\nabla^n f\|_{L^p(3Q)}.$$

Therefore, by the Leibniz rule (1.11) we have that

$$\begin{aligned} \|\nabla^j ((f - \mathbf{P}_{3Q}^{n-1} f) \varphi)\|_{L^p(3Q)} &\lesssim \sum_{i=0}^j \|\nabla^i \varphi\|_{L^\infty(3Q)} \|\nabla^{j-i} (f - \mathbf{P}_{3Q}^{n-1} f)\|_{L^p(3Q)} \\ &\lesssim \sum_{i=0}^j \frac{\ell(Q)^i}{\ell(Q)^i} \|\nabla^j (f - \mathbf{P}_{3Q}^{n-1} f)\|_{L^p(3Q)} \lesssim \ell(Q)^{n-j} \|\nabla^n f\|_{L^p(3Q)}, \end{aligned}$$

proving (1.33).

To prove (1.34), we consider the chain $[Q, S]$ to write

$$\|f - \mathbf{P}_{3Q}^{n-1} f\|_{L^1(S)} \leq \|f - \mathbf{P}_{3S}^{n-1} f\|_{L^1(S)} + \sum_{P \in [S, Q]} \|\mathbf{P}_{3P}^{n-1} f - \mathbf{P}_{3N(P)}^{n-1} f\|_{L^1(S)}, \quad (1.37)$$

where we write $\mathcal{N}(P)$ instead of $\mathcal{N}_{[S,Q]}(P)$ from Definition 1.13. Applying Remark 1.19 to the polynomial $\mathbf{P}_{3P}^{n-1}f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1}f$, the cubes S and P and with $r \approx \frac{D(P,S)}{\ell(P)}$, it follows that

$$\begin{aligned} \left\| \mathbf{P}_{3P}^{n-1}f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1}f \right\|_{L^1(S)} &\approx \left\| \mathbf{P}_{3P}^{n-1}f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1}f \right\|_{L^\infty(S)} \ell(S)^d \\ &\lesssim \left\| \mathbf{P}_{3P}^{n-1}f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1}f \right\|_{L^\infty(3P \cap 3\mathcal{N}(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1}} \\ &\approx \left\| \mathbf{P}_{3P}^{n-1}f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1}f \right\|_{L^1(3P \cap 3\mathcal{N}(P))} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{n-1} \ell(P)^d}. \end{aligned}$$

Using this estimate in (1.36), and then (1.33) with $\varphi \equiv 1$, we get

$$\begin{aligned} \left\| f - \mathbf{P}_{3Q}^{n-1}f \right\|_{L^1(S)} &\lesssim \sum_{P \in [S,Q]} \left(\left\| \mathbf{P}_{3P}^{n-1}f - f \right\|_{L^1(3P)} + \left\| f - \mathbf{P}_{3\mathcal{N}(P)}^{n-1}f \right\|_{L^1(3\mathcal{N}(P))} \right) \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d+n-1}} \\ &\lesssim \sum_{P \in [S,Q]} \left\| f - \mathbf{P}_{3P}^{n-1}f \right\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d+n-1}} \\ &\lesssim \sum_{P \in [S,Q]} \left\| \nabla^n f \right\|_{L^1(3P)} \frac{\ell(S)^d D(P,S)^{n-1}}{\ell(P)^{d-1}}. \end{aligned}$$

□

1.6 Calderón-Zygmund operators

Definition 1.22. We say that a measurable function $K \in L_{loc}^1(\mathbb{R}^d \setminus \{0\})$ is a convolution Calderón-Zygmund kernel if it satisfies the size condition

$$|K(x)| \leq \frac{C_K}{|x|^d} \quad \text{for } x \neq 0, \quad (1.38)$$

and the Hörmander condition

$$\sup_{y \neq 0} \int_{|x| \geq 2|y|} |K(x+y) - K(x)| dx = C_K, \quad (1.39)$$

and that kernel can be extended to a convolution with a tempered distribution W_K in \mathbb{R}^d in the sense that for all Schwartz functions $f, g \in \mathcal{S}$ with $\text{supp}(f) \cap \text{supp}(g) = \emptyset$, one has

$$\langle W_K * f, g \rangle = \int_{\mathbb{R}^d \setminus \{0\}} K(x) (f_- * g)(x) dx, \quad (1.40)$$

where $f_-(x) = f(-x)$.

Remark 1.23. We are using the notion of distributional convolution. Given Schwartz functions f and g , the convolution coincides with multiplication at the Fourier side, that is, $f * g(x) = (\hat{f} \cdot \hat{g})^\vee$. Given a tempered distribution W , a function $f \in \mathcal{S}$ and $x \in \mathbb{R}^d$, the tempered distribution $W * f$ is defined as

$$\langle W * f, g \rangle := \langle (\widehat{W} \cdot \hat{f})^\vee, g \rangle = \langle \widehat{W}, \hat{f} \cdot \hat{g} \rangle = \langle W, f_- * g \rangle \quad \text{for every } g \in \mathcal{S}.$$

Note that $f_- * g(x) = \int f(-y)g(x-y) dy$, so in case $\text{supp}(f) \cap \text{supp}(g) = \emptyset$ then $f_- * g \equiv 0$ in a neighborhood of 0 and, therefore, the integral in (1.39) is absolutely convergent by (1.37).

In any case, the distribution $W * f$ is regular (i.e., it can be expressed as an L^1_{loc} function) and it coincides with the C^∞ function $W * f(x) = \langle W, \tau_x f_- \rangle$, where $\tau_x f_-(y) = f_-(y-x)$ (see [SW71, Chapter I, Theorem 3.13]).

There are some cancellation conditions that one can impose to a kernel satisfying the size condition (1.37) to grant that it can be extended to a convolution with a tempered distribution. For instance, if K satisfies (1.37) and W_K is a principal value operator in the sense that

$$\langle W_K, \varphi \rangle = \lim_{j \rightarrow \infty} \int_{|x| \geq \delta_j} K(x) \varphi(x) dx \quad \text{for all } \varphi \in \mathcal{S},$$

for a certain sequence $\delta_j \searrow 0$, then W_K satisfies (1.39) (see [Gra08, Section 4.3.2]).

Definition 1.24. We say that an operator $T : \mathcal{S} \rightarrow \mathcal{S}'$ is a convolution Calderón-Zygmund operator with kernel K if

1. K is an admissible convolution Calderón-Zygmund kernel which can be extended to a convolution with a tempered distribution W_K ,
2. T satisfies that $Tf = W_K * f$ for all $f \in \mathcal{S}$ and
3. T extends to an operator bounded on L^2 .

Remark 1.25. Using the Calderón-Zygmund decomposition one can see that T is also bounded on L^p for $1 < p < \infty$ (see [Gra08, proof of Theorem 4.3.3]).

The Fourier transform of an admissible convolution Calderón-Zygmund operator T is a Fourier multiplier for L^2 , and this implies that $\widehat{W_K} \in L^\infty$ (see [SW71, Chapter I, Theorem 3.18]).

It is a well-known fact that the Schwartz class is dense in L^p for $1 \leq p < \infty$. Thus, if $f \in L^p$ and $x \notin \text{supp}(f)$, then

$$Tf(x) = \int K(x-y)f(y)dy. \quad (1.41)$$

Definition 1.26. We say that an operator $T : \mathcal{S} \rightarrow \mathcal{S}'$ is an admissible convolution Calderón-Zygmund operator of order $n \in \mathbb{N}$ with kernel $K \in W^{n,1}_{loc}(\mathbb{R}^d \setminus \{0\})$ if T satisfies Definition 1.24 and, moreover, the kernel K satisfies the higher order smoothness condition

$$|\nabla^n K(x)| \leq \frac{C_K}{|x|^{d+n}} \quad \text{for all } x \neq 0. \quad (1.42)$$

Example 1.27. The Beurling transform (1.1) is an admissible Calderón-Zygmund operator of order ∞ .

Let T be an admissible convolution Calderón-Zygmund operator of order n , let $\Omega \subset \mathbb{R}^d$ be a domain and let $1 < p < \infty$. Given a function $f \in W^{n,p}(\Omega)$, we want to study in what situations its transform $T_\Omega f = \chi_\Omega T(\chi_\Omega f)$ is in some Sobolev space, so we need to check that its weak derivatives exist up to order n . Indeed that is the case.

Lemma 1.28. Given $f \in W^{n,p}(\Omega)$, the weak derivatives of $T_\Omega f$ in Ω exist up to order n .

Before proving this, we consider the functions defined in all \mathbb{R}^d .

Remark 1.29. Since T is a bounded linear operator in $L^2(\mathbb{R}^d)$ that commutes with translations, for Schwartz functions the derivative commutes with T (see [Gra08, Lemma 2.5.3]). Using that \mathcal{S} is dense in $W^{n,p}(\mathbb{R}^d)$ (see [Tri83, Sections 2.3.3 and 2.5.6], for instance), we conclude that for every $f \in W^{n,p}(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}^d$ with $|\alpha| = n$

$$D^\alpha T(f) = T D^\alpha(f) \quad (1.43)$$

and, thus, the operator T is bounded on $W^{n,p}(\mathbb{R}^d)$.

Definition 1.30. Let $K \in W_{loc}^{n,1}(\mathbb{R}^d \setminus \{0\})$ be the kernel of T and consider a function $f \in L^p$, a multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq n$ and $x \notin \text{supp}(f)$. We define

$$T^{(\alpha)} f(x) := \int D^\alpha K(x-y) f(y) dy.$$

Lemma 1.31. Let $f \in L^p$. Then Tf has weak derivatives up to order n in $\mathbb{R}^d \setminus \text{supp} f$. Moreover, for every multiindex $\alpha \in \mathbb{N}^d$ with $|\alpha| \leq n$ and $x \notin \text{supp} f$

$$D^\alpha T f(x) = T^{(\alpha)} f(x).$$

Proof. Take a compactly supported smooth function $\phi \in C_c^\infty(\mathbb{R}^d \setminus \text{supp} f)$. We can use Fubini's Theorem and get

$$\begin{aligned} \langle T^{(\alpha)} f, \phi \rangle &= \int_{\text{supp} \phi} \int_{\text{supp} f} D^\alpha K(x-y) f(y) dy \phi(x) dx \\ &= \int_{\text{supp} f} \int_{\text{supp} \phi} D^\alpha K(x-y) \phi(x) dx f(y) dy. \end{aligned}$$

Using the definition of distributional derivative, Tonelli's Theorem again and (1.40) we get

$$\begin{aligned} \langle T^{(\alpha)} f, \phi \rangle &= (-1)^{|\alpha|} \int_{\text{supp} f} \int_{\text{supp} \phi} K(x-y) D^\alpha \phi(x) dx f(y) dy \\ &= (-1)^{|\alpha|} \int_{\text{supp} \phi} \int_{\text{supp} f} K(x-y) f(y) dy D^\alpha \phi(x) dx = (-1)^{|\alpha|} \langle Tf, D^\alpha \phi \rangle. \end{aligned}$$

□

Proof of Lemma 1.28. Take a classical Whitney covering of Ω , \mathcal{W} , and for every $Q \in \mathcal{W}$, define a bump function $\varphi_Q \in C_c^\infty$ such that $\chi_{2Q} \leq \varphi_Q \leq \chi_{3Q}$. On the other hand, let $\{\psi_Q\}_{Q \in \mathcal{W}}$ be a partition of the unity associated to $\{\frac{3}{2}Q : Q \in \mathcal{W}\}$. Consider a multiindex α with $|\alpha| = n$. Then take $f_1^Q = \varphi_Q \cdot f$, and $f_2^Q = (f - f_1^Q)\chi_\Omega$. One can define

$$g(y) := \sum_{Q \in \mathcal{W}} \psi_Q(y) \left(T D^\alpha f_1^Q(y) + T^{(\alpha)} f_2^Q(y) \right).$$

This function is defined almost everywhere in Ω and is the weak derivative $D^\alpha T_\Omega f$.

Indeed, given a test function $\phi \in C_c^\infty(\Omega)$, then, since ϕ is compactly supported in Ω , its support intersects a finite number of Whitney double cubes and, thus, the following additions are finite:

$$\begin{aligned} \langle g, \phi \rangle &= \left\langle \sum_{Q \in \mathcal{W}} \psi_Q \cdot T D^\alpha f_1^Q + \psi_Q \cdot T^{(\alpha)} f_2^Q, \phi \right\rangle \\ &= \sum_{Q \in \mathcal{W}} \langle T D^\alpha f_1^Q, \phi_Q \rangle + \sum_{Q \in \mathcal{W}} \langle T^{(\alpha)} f_2^Q, \phi_Q \rangle, \end{aligned} \quad (1.44)$$

where $\phi_Q = \psi_Q \cdot \phi$. In the local part we can use (1.42), so

$$\langle TD^\alpha f_1^Q, \phi_Q \rangle = (-1)^{|\alpha|} \langle T f_1^Q, D^\alpha \phi_Q \rangle.$$

When it comes to the non-local part, bearing in mind that f_2^Q has support away from $2Q$ and $\phi_Q \in C_c^\infty(2Q)$, we can use the Lemma 1.31 and we get

$$\langle T^{(\alpha)} f_2^Q, \phi_Q \rangle = (-1)^{|\alpha|} \langle T f_2^Q, D^\alpha \phi_Q \rangle.$$

Back to (1.43) we have

$$\begin{aligned} \langle g, \phi \rangle &= \sum_{Q \in \mathcal{W}} (-1)^{|\alpha|} \langle T f_1^Q, D^\alpha \phi_Q \rangle + \sum_{Q \in \mathcal{W}} (-1)^{|\alpha|} \langle T f_2^Q, D^\alpha \phi_Q \rangle = \sum_{Q \in \mathcal{W}} (-1)^{|\alpha|} \langle T_\Omega f, D^\alpha \phi_Q \rangle \\ &= (-1)^{|\alpha|} \langle T_\Omega f, D^\alpha \phi \rangle, \end{aligned}$$

that is, $g = D^\alpha T_\Omega f$ in the weak sense. □

Chapter 2

T(P) theorems

Let $\Omega \subset \mathbb{R}^d$ be a domain and T a convolution Calderón-Zygmund operator. We are interested in conditions that allow us to infer that the restricted operator $T_\Omega = \chi_\Omega T \chi_\Omega$ is bounded on a certain Sobolev space $W^{s,p}(\Omega)$.

In this chapter we find some results for the supercritical case in terms of test functions, namely polynomials of degree strictly smaller than the considered smoothness s . This is in accordance with the pioneering results found in [CMO13], which deal with operators with even kernel satisfying some smoothness assumptions, and with spaces $W^{s,p}(\Omega)$ where $0 < s \leq 1$, $sp > d$ and Ω is a Lipschitz domain with parameterizations in $C^{1,\sigma}$. In that situation, the authors prove that T_Ω is bounded on $W^{s,p}(\Omega)$ if and only if $T_\Omega 1 \in W^{s,p}(\Omega)$ and an analogous result for $B_{p,p}^s(\Omega)$ with $s < 1$.

The reader will find two results in that spirit in this chapter. The first, Theorem 2.1, deals with $W^{s,p}(\Omega)$ with $s \in \mathbb{N}$ and $p > d$ (see the green segments in Figure 2.1), and the second, Theorem 2.8, deals with the supercritical case for $0 < s < 1$ (see the triangle in Figure 2.1). These results improve the previously known results in the sense that the class of operators considered

is wider, the range of indices s, p is larger and, moreover, the restrictions on the regularity of the domain are reduced to just asking the domain to be uniform. Moreover, in the case $0 < s < 1$, the result is valid for a greater family of Triebel-Lizorkin spaces.

The novelty in the approach exposed in this chapter is not only on the aforementioned improvements, but on the technique used to reach that results, which relies strongly on the properties of uniform domains and their hyperbolic metric. Furthermore, in both cases we make use of a Key Lemma that provides information even if the condition $sp > d$ is not satisfied. In Chapter 6 we will see how this lemmas can be used to provide other conditions in the critical and subcritical cases.

Sections 2.1 and 2.2 are devoted to Theorems 2.1 and 2.8 respectively. The rest of the chapter

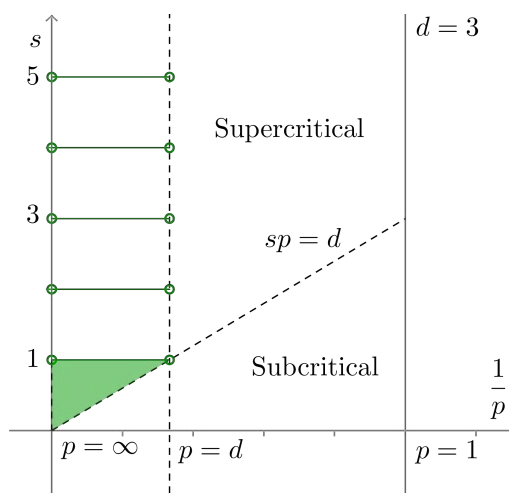


Figure 2.1: Indices studied regarding the $T(P)$ -theorems in this chapter.

serves to provide the tools that the author needs for Theorem 2.8 which he could not find in the literature.

2.1 Classic Sobolev spaces on uniform domains

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^d$ be a bounded ε -uniform domain, T an admissible convolution Calderón-Zygmund operator of order $n \in \mathbb{N}$ and $d < p < \infty$. Then the following statements are equivalent:*

- a) *The truncated operator T_Ω is bounded on $W^{n,p}(\Omega)$.*
- b) *For every polynomial P of degree at most $n - 1$, we have that $T_\Omega(P) \in W^{n,p}(\Omega)$.*

Moreover, if $x_0 \in \Omega$, writing $P_\lambda(x) := (x - x_0)^\lambda = \prod_{j=1}^d (x_j - x_{0,j})^{\lambda_j}$ for $\lambda \in \mathbb{N}^d$ we have that

$$\|T_\Omega\|_{W^{n,p}(\Omega) \rightarrow W^{n,p}(\Omega)} \lesssim_{\text{diam}(\Omega), \varepsilon, n, p, d} C_K + \|T\|_{L^p \rightarrow L^p} + \sum_{|\lambda| < n} \|\nabla^n(T_\Omega P_\lambda)\|_{L^p(\Omega)}. \quad (2.1)$$

To prove this theorem we need the following lemma, which says that it is equivalent to bound the transform of a function and its approximation by polynomials on Whitney cubes.

Key Lemma 2.2. *Let $\Omega \subset \mathbb{R}^d$ be an ε -uniform domain with Whitney covering \mathcal{W} (see Definition 1.15), T an admissible convolution Calderón-Zygmund operator of order $n \in \mathbb{N}$ and $1 < p < \infty$. Then the following statements are equivalent:*

- i) *For every $f \in W^{n,p}(\Omega)$ one has*

$$\|T_\Omega f\|_{W^{n,p}(\Omega)} \leq C \|f\|_{W^{n,p}(\Omega)},$$

where C depends only on ε, n, p, d and T .

- ii) *For every $f \in W^{n,p}(\Omega)$ one has*

$$\sum_{Q \in \mathcal{W}} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \leq C \|f\|_{W^{n,p}(\Omega)}^p,$$

where C depends only on ε, n, p, d and T .

Proof. Let Ω be an ε -uniform domain. Given a multiindex α with $|\alpha| = n$, we will bound the difference

$$\sum_{Q \in \mathcal{W}} \left\| D^\alpha T_\Omega \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \lesssim (C_K + \|T\|_{L^p \rightarrow L^p})^p \|\nabla^n f\|_{L^p(\Omega)}^p, \quad (2.2)$$

with constants depending on ε, n, p and d .

For each cube $Q \in \mathcal{W}$ we define a bump function $\varphi_Q \in C_c^\infty$ such that $\chi_{\frac{3}{2}Q} \leq \varphi_Q \leq \chi_{2Q}$ and $\|\nabla^j \varphi_Q\|_\infty \approx \ell(Q)^{-j}$ for every $j \leq n$. Then we can break (2.2) into local and non-local parts as follows:

$$\begin{aligned} \sum_{Q \in \mathcal{W}} \left\| D^\alpha T_\Omega \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p &\lesssim \sum_{Q \in \mathcal{W}} \left\| D^\alpha T \left(\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(Q)}^p \\ &+ \sum_{Q \in \mathcal{W}} \left\| D^\alpha T \left((\chi_\Omega - \varphi_Q) \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(Q)}^p = \textcircled{1} + \textcircled{2}. \end{aligned} \quad (2.3)$$

First of all we will show that the local term in (2.3) satisfies

$$\textcircled{1} = \sum_{Q \in \mathcal{W}} \left\| D^\alpha T \left(\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(Q)}^p \lesssim \|T\|_{L^p \rightarrow L^p}^p \|\nabla^n f\|_{L^p(\Omega)}^p. \quad (2.4)$$

To do so, note that $\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \in W^{n,p}(\mathbb{R}^d)$ and, by (1.42) and the boundedness of T in L^p ,

$$\begin{aligned} \left\| D^\alpha T \left(\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(Q)}^p &\lesssim \|T\|_{L^p \rightarrow L^p}^p \left\| D^\alpha \left(\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(\mathbb{R}^d)}^p \\ &= \|T\|_{L^p \rightarrow L^p}^p \left\| D^\alpha \left(\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(2Q)}^p. \end{aligned}$$

Using the Poincaré inequality (1.33), we get

$$\left\| D^\alpha T \left(\varphi_Q \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(Q)}^p \lesssim \|\nabla^n f\|_{L^p(3Q)}^p.$$

Summing over all Q we get (2.4).

For the non-local part in (2.3),

$$\textcircled{2} = \sum_{Q \in \mathcal{W}} \left\| D^\alpha T \left((\chi_\Omega - \varphi_Q) \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right) \right\|_{L^p(Q)}^p,$$

we will argue by duality. We can write

$$\textcircled{2}^{\frac{1}{p}} = \sup_{\|g\|_{L^{p'}} \leq 1} \sum_{Q \in \mathcal{W}} \int_Q \left| D^\alpha T \left[(\chi_\Omega - \varphi_Q) \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right] (x) \right| g(x) dx. \quad (2.5)$$

Note that given $x \in Q \in \mathcal{W}$, by Lemma 1.31 one has

$$D^\alpha T \left[(\chi_\Omega - \varphi_Q) \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right] (x) = \int_\Omega D^\alpha K(x-y) (1 - \varphi_Q(y)) \left(f(y) - \mathbf{P}_{3Q}^{n-1} f(y) \right) dy.$$

Taking absolute values and using the smoothness condition (1.41), we can bound

$$\begin{aligned} \left| D^\alpha T \left[(\chi_\Omega - \varphi_Q) \left(f - \mathbf{P}_{3Q}^{n-1} f \right) \right] (x) \right| &\leq C_K \int_{\Omega \setminus \frac{3}{2}Q} \frac{|f(y) - \mathbf{P}_{3Q}^{n-1} f(y)|}{|x-y|^{n+d}} dy \\ &\lesssim C_K \sum_{S \in \mathcal{W}} \frac{\|f - \mathbf{P}_{3Q}^{n-1} f\|_{L^1(S)}}{D(Q,S)^{n+d}} \end{aligned}$$

and, by (2.5), we have that

$$\textcircled{2}^{\frac{1}{p}} \lesssim C_K \sup_{\|g\|_{L^{p'}} \leq 1} \sum_{Q \in \mathcal{W}} \int_Q \sum_{S \in \mathcal{W}} \frac{\|f - \mathbf{P}_{3Q}^{n-1} f\|_{L^1(S)}}{D(Q,S)^{n+d}} g(x) dx. \quad (2.6)$$

By Definition 1.15, for every pair of Whitney cubes Q and S there exists an admissible chain $[S, Q]$ and, by (1.34), we have that

$$\|f - \mathbf{P}_{3Q}^{n-1} f\|_{L^1(S)} \lesssim \sum_{P \in [S, Q]} \frac{\ell(S)^d D(P, S)^{n-1}}{\ell(P)^{d-1}} \|\nabla^n f\|_{L^1(3P)}. \quad (2.7)$$

Thus, plugging (2.7) into (2.6), we get

$$\textcircled{2}^{\frac{1}{p}} \lesssim C_K \sup_{\|g\|_{p'} \leq 1} \sum_{Q \in \mathcal{W}} \int_Q g(x) dx \sum_{S \in \mathcal{W}} \sum_{P \in [S, Q]} \frac{\ell(S)^d D(P, S)^{n-1} \|\nabla^n f\|_{L^1(3P)}}{\ell(P)^{d-1} D(Q, S)^{n+d}}. \quad (2.8)$$

By (2.3), (2.4), (2.8) and Lemma 2.3 below, we have that (2.2) holds. \square

Lemma 2.3. *Consider a uniform domain Ω with Whitney covering \mathcal{W} , two functions $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ and $\rho \geq 1$. Then*

$$A_\rho(f, g) := \sum_{Q, S \in \mathcal{W}} \sum_{P \in [S, Q]} \frac{\ell(S)^d D(P, S)^{\rho-1} \|f\|_{L^1(50P)} \|g\|_{L^1(50Q)}}{\ell(P)^{d-1} D(Q, S)^{\rho+d}} \lesssim \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

Proof. Using that $P \in [S, Q]$ implies $D(P, S) \lesssim D(Q, S)$ (see Remark 1.14), we get

$$\begin{aligned} A_\rho(f, g) &\lesssim \sum_{Q, S \in \mathcal{W}} \sum_{P \in [S, S_Q]} \frac{\ell(S)^d \|f\|_{L^1(50P)} \|g\|_{L^1(50Q)}}{\ell(P)^{d-1} D(Q, S)^{d+1}} + \sum_{Q, S \in \mathcal{W}} \sum_{P \in [S_Q, Q]} \frac{\ell(S)^d \|f\|_{L^1(50P)} \|g\|_{L^1(50Q)}}{\ell(P)^{d-1} D(Q, S)^{d+1}} \\ &= A^{(1)}(f, g) + A^{(2)}(f, g). \end{aligned}$$

We consider first the term $A^{(1)}(f, g)$ where the sum is taken with respect to cubes $P \in [S, S_Q]$ and, thus, by (1.23) the long distance $D(Q, S) \approx D(P, Q)$. Moreover, we have $S \in \mathbf{SH}(P)$ by Definition 1.16. Thus, rearranging the sum,

$$A^{(1)}(f, g) \lesssim \sum_{P \in \mathcal{W}} \frac{\|f\|_{L^1(50P)}}{\ell(P)^{d-1}} \sum_{Q \in \mathcal{W}} \frac{\|g\|_{L^1(50Q)}}{D(Q, P)^{d+1}} \sum_{S \in \mathbf{SH}(P)} \ell(S)^d.$$

By Definition 1.16 again

$$\sum_{S \in \mathbf{SH}(P)} \ell(S)^d \approx \ell(P)^d$$

and, by (1.27) and the finite overlapping of the cubes $\{50Q\}_{Q \in \mathcal{W}}$, we get

$$\sum_{Q \in \mathcal{W}} \frac{\|g\|_{L^1(50Q)}}{D(Q, P)^{d+1}} \lesssim \frac{\inf_{x \in 50P} Mg(x)}{\ell(P)}.$$

Next we perform a similar argument with $A^{(2)}(f, g)$. Note that when $P \in [S_Q, Q]$, we have $D(Q, S) \approx D(P, S)$ and $Q \in \mathbf{SH}(P)$, leading to

$$A^{(2)}(f, g) \lesssim \sum_{P \in \mathcal{W}} \frac{\|f\|_{L^1(50P)}}{\ell(P)^{d-1}} \sum_{Q \in \mathbf{SH}(P)} \|g\|_{L^1(50Q)} \sum_{S \in \mathcal{W}} \frac{\ell(S)^d}{D(P, S)^{d+1}}.$$

By (1.28) we get

$$\sum_{Q \in \mathbf{SH}(P)} \|g\|_{L^1(50Q)} \lesssim \inf_{x \in 50P} Mg(x) \ell(P)^d$$

and, by (1.29),

$$\sum_{S \in \mathcal{W}} \frac{\ell(S)^d}{D(P, S)^{d+1}} \approx \frac{1}{\ell(P)}.$$

Thus,

$$A_\rho(f, g) \lesssim \sum_{P \in \mathcal{W}} \frac{\|f\|_{L^1(50P)} \inf_{50P} Mg}{\ell(P)^{d-1} \ell(P)} \ell(P)^d \lesssim \sum_{P \in \mathcal{W}} \|f \cdot Mg\|_{L^1(50P)}.$$

By Hölder inequality and the boundedness of the Hardy-Littlewood maximal operator in $L^{p'}$,

$$A_\rho(f, g) \lesssim \left(\sum_{P \in \mathcal{W}} \|f\|_{L^p(50P)}^p \right)^{1/p} \left(\sum_P \|Mg\|_{L^{p'}(50P)}^{p'} \right)^{1/p'} \lesssim \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

□

Proof of Theorem 2.1. The implication $a) \Rightarrow b)$ is trivial.

To see the converse, we assume by hypothesis that

$$\sum_{|\lambda| < n} \|T_\Omega(P_\lambda)\|_{W^{n,p}(\Omega)} \leq C. \quad (2.9)$$

Let $f \in W^{n,p}(\Omega)$. By the Key Lemma 2.2, we have to prove that

$$\sum_{Q \in \mathcal{W}} \|\nabla^n T_\Omega(\mathbf{P}_{3Q}^{n-1} f)\|_{L^p(Q)}^p \lesssim \|f\|_{W^{n,p}(\Omega)}^p.$$

We can write the polynomials

$$\mathbf{P}_{3Q}^{n-1} f(x) = \sum_{|\gamma| < n} m_{Q,\gamma} (x - x_Q)^\gamma,$$

where x_Q stands for the center of each cube Q . Taking the Taylor expansion in x_0 for each monomial, one has

$$\mathbf{P}_{3Q}^{n-1} f(x) = \sum_{|\gamma| < n} m_{Q,\gamma} \sum_{\vec{0} \leq \lambda \leq \gamma} \binom{\gamma}{\lambda} (x - x_0)^\lambda (x_0 - x_Q)^{\gamma - \lambda}.$$

Thus,

$$\nabla^n T_\Omega(\mathbf{P}_{3Q}^{n-1} f)(y) = \sum_{|\gamma| < n} m_{Q,\gamma} \sum_{\vec{0} \leq \lambda \leq \gamma} \binom{\gamma}{\lambda} (x_0 - x_Q)^{\gamma - \lambda} \nabla^n (T_\Omega P_\lambda)(y). \quad (2.10)$$

Recall the estimate (1.32) in Lemma 1.21, which implies that

$$|m_{Q,\gamma}| \leq C \sum_{j=|\gamma|}^{n-1} \|\nabla^j f\|_{L^\infty(3Q)} \ell(Q)^{j-|\gamma|} \lesssim \sum_{j=|\gamma|}^{n-1} \|\nabla^j f\|_{L^\infty(\Omega)} \text{diam} \Omega^{j-|\gamma|}. \quad (2.11)$$

Raising (2.10) to the power p , integrating in Q and using (2.11) we get

$$\left\| \nabla^n T_\Omega(\mathbf{P}_{3Q}^{n-1} f) \right\|_{L^p(Q)}^p \lesssim \sum_{j < n} \|\nabla^j f\|_{L^\infty(\Omega)}^p \sum_{|\lambda| < j} \text{diam} \Omega^{(j-|\lambda|)p} \|\nabla^n (T_\Omega P_\lambda)\|_{L^p(Q)}^p.$$

By the Sobolev Embedding Theorem, we know that $\|\nabla^j f\|_{L^\infty(\Omega)} \leq C \|\nabla^j f\|_{W^{1,p}(\Omega)}$ as long as $p > d$. If we add with respect to $Q \in \mathcal{W}$ and we use (2.9) we get

$$\sum_{Q \in \mathcal{W}} \left\| \nabla^n T_\Omega(\mathbf{P}_{3Q}^{n-1} f) \right\|_{L^p(Q)}^p \lesssim \sum_{j < n} \|\nabla^j f\|_{W^{1,p}(\Omega)}^p \sum_{|\lambda| < j} \|\nabla^n (T_\Omega P_\lambda)\|_{L^p(\Omega)}^p \lesssim \|f\|_{W^{n,p}(\Omega)}^p.$$

Note that the constants depend on the diameter of Ω , ε , n , p and d (the extension norm in [Jon81] and, as a consequence, the Sobolev embedding constant depend on all this parameters). This estimate, together with (2.2), implies (2.1). □

2.2 Fractional Sobolev spaces on uniform domains

In this section we study the counterpart of the $T(P)$ -theorem to the Sobolev spaces with fractional smoothness $0 < s < 1$, obtaining a $T(1)$ -theorem (Theorem 2.8). The methods we use are valid for any Triebel-Lizorkin space $F_{p,q}^s$ as long as the operator T is bounded on $F_{p,q}^s$. This boundedness is granted in some situations by the Calderón-Zygmund decomposition and interpolation, but in some cases it is not that easy. On the other hand, the smoothness hypothesis (1.41) assumed when $n \in \mathbb{N}$ can be relaxed. For these reasons we adapt the definition of admissible Calderón-Zygmund operator to this setting.

Definition 2.4. *Let $1 < p, q < \infty$. We say that an operator $T : \mathcal{S} \rightarrow \mathcal{S}'$ is a p, q -admissible convolution Calderón-Zygmund operator of order $s \in (0, 1)$ (or simply admissible when $q = 2$) with kernel K if T satisfies Definition 1.24, T also extends to a bounded operator in $F_{p,q}^0$ and K satisfies the smoothness condition*

$$|K(x+y) - K(x)| \leq \frac{C_K |y|^s}{|x|^{d+s}} \quad \text{for every } 0 < 2|y| \leq |x|. \quad (2.12)$$

Remark 2.5. *Note that the smoothness condition (2.12) implies the Hörmander condition (1.38). Moreover, being of order s implies being of order σ for every $\sigma < s$.*

The Fourier transform of a p, q -admissible convolution Calderón-Zygmund operator T is a Fourier multiplier for $F_{p,q}^0$, and for L^r with $1 < r < \infty$ as well by Remark 1.25. We refer the reader to [Tri83, Section 2.6] for a rigorous definition of multiplier and for the results on Fourier multipliers that we sum up next as well.

Being a Fourier multiplier for $F_{p,q}^0$ implies being a Fourier multiplier also for $F_{p,q}^s$ for every s , for $F_{p,p}^0$ and for $F_{p',q'}^0$, and the property is stable under interpolation (i.e., the set of indices $(\frac{1}{p}, \frac{1}{q})$ such that a T is bounded on $F_{p,q}^0$ is a convex set, see Figure 2.2).

Therefore, the fact of T being bounded on $F_{p,q}^0$ in Definition 2.4 is a consequence of being a Calderón-Zygmund operator when $0 < \frac{1}{2p} < \frac{1}{q} < \frac{1}{2p} + \frac{1}{2} < 1$ (see Figure 2.2).

Example 2.6. *The Beurling transform (1.1) is an admissible convolution Calderón-Zygmund operator of any order and, therefore, it is a p, q -admissible convolution Calderón-Zygmund operator for all indices p and q satisfying $0 < \frac{1}{2p} < \frac{1}{q} < \frac{1}{2p} + \frac{1}{2} < 1$.*

Definition 2.7. *Let $U \subset \mathbb{R}^d$ be a open set, $1 < p < \infty$, $1 < q < \infty$ and $0 < s < \infty$. Then for every measurable function $f : U \rightarrow \mathbb{C}$ we define*

$$\|f\|_{F_{p,q}^s(U)} := \inf_{g \in F_{p,q}^s(\mathbb{R}^d) : g|_U \equiv f} \|g\|_{F_{p,q}^s(\mathbb{R}^d)}.$$

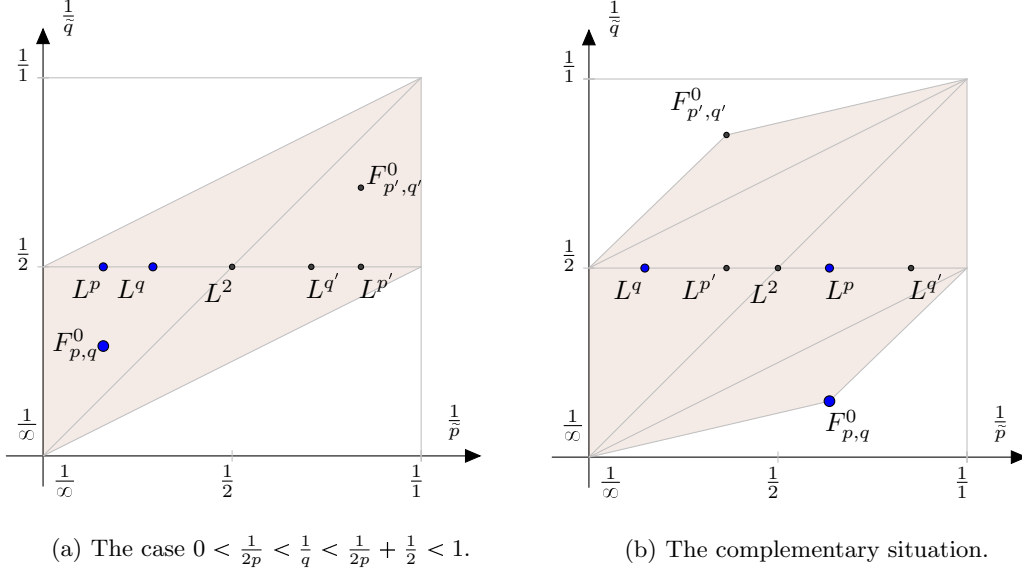
Theorem 2.8. *Let Ω be a bounded uniform domain, T a p, q -admissible convolution Calderón-Zygmund operator of order $0 < s < 1$, $1 < p < \infty$, $1 < q < \infty$ with $s > \frac{d}{p}$. Then*

$$\|T_\Omega 1\|_{F_{p,q}^s(\Omega)} < \infty \quad \iff \quad T_\Omega \text{ is bounded on } F_{p,q}^s(\Omega).$$

Furthermore,

$$\|T_\Omega\|_{F_{p,q}^s(\Omega) \rightarrow F_{p,q}^s(\Omega)} \lesssim C_K + \|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s} + \|T\|_{L^p \rightarrow L^p} + \|T\|_{L^q \rightarrow L^q} + \|T_\Omega 1\|_{F_{p,q}^s(\Omega)},$$

with constants independent of T .

Figure 2.2: Indices $\frac{1}{p}, \frac{1}{q}$ such that a p, q -admissible operator is bounded on $F_{p,\tilde{q}}^s$.


Similarly to Section 2.1, we will use a lemma which says that it is equivalent to bound the transform of a function and its approximation by constants on Whitney cubes. In the fractional case $0 < s < 1$, however, it is not clear what the ∇^s -gradient is.

Definition 2.9. Given a uniform domain Ω with Whitney covering \mathcal{W} and $f \in L^p(\Omega)$ for certain values $0 < s < 1$ and $1 < q < \infty$, the s -th (dyadic) fractional gradient of index q of f in a point $x \in Q \in \mathcal{W}$ is

$$\nabla_q^s f(x) := \left(\int_{\text{Sh}(Q)} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{1}{q}}.$$

In Sections 2.3 and 2.4 we will prove the following remarkable characterizations in terms of differences.

Theorem 2.10 (see Corollary 2.12, Corollary 2.17 and Lemma 2.18). Let $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then

$$F_{p,q}^s(\mathbb{R}^d) = \left\{ f \in L^{\max\{p,q\}} : \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}, \quad (2.13)$$

(with the usual modification for $q = \infty$), in the sense of equivalent norms.

Let Ω be a bounded uniform domain with an admissible Whitney covering \mathcal{W} . Given $1 < p < \infty$, $1 < q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$, we have that

$$\|f\|_{F_{p,q}^s(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_{\Omega} \left(\int_{\Omega} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \approx \|f\|_{L^p(\Omega)} + \|\nabla_q^s f\|_{L^p(\Omega)}. \quad (2.14)$$

Key Lemma 2.11. *Let Ω be a uniform domain with Whitney covering \mathcal{W} , let T be a p, q -admissible convolution Calderón-Zygmund operator of order $0 < s < 1$, $1 < p < \infty$ and $1 < q < \infty$ with $s > \frac{d}{p} - \frac{d}{q}$. The following statements are equivalent:*

i) *For every $f \in F_{p,q}^s(\Omega)$ one has*

$$\|T_\Omega f\|_{F_{p,q}^s(\Omega)} \leq C \|f\|_{F_{p,q}^s(\Omega)},$$

with C independent from f .

ii) *For every $f \in F_{p,q}^s(\Omega)$ one has*

$$\sum_{Q \in \mathcal{W}} |f_Q|^p \|\nabla_q^s T \chi_\Omega\|_{L^p(Q)}^p \leq C \|f\|_{F_{p,q}^s(\Omega)}^p,$$

with C independent from f .

Moreover,

$$\sum_{Q \in \mathcal{W}} \|\nabla_q^s T_\Omega (f - f_Q)\|_{L^p(Q)}^p \lesssim \left(C_K + \|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s} + \|T\|_{L^p \rightarrow L^p} + \|T\|_{L^q \rightarrow L^q} \right)^p \|f\|_{F_{p,q}^s(\Omega)}^p. \quad (2.15)$$

Proof. Let Ω be an ε -uniform domain. The core of the proof is showing that (2.15) holds. Once this is settled, since we have that

$$\sum_{Q \in \mathcal{W}} \|\nabla_q^s T_\Omega f\|_{L^p(Q)}^p \lesssim_p \sum_{Q \in \mathcal{W}} \|\nabla_q^s T_\Omega (f - f_Q)\|_{L^p(Q)}^p + \sum_{Q \in \mathcal{W}} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p,$$

and

$$\sum_{Q \in \mathcal{W}} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p \lesssim_p \sum_{Q \in \mathcal{W}} \|\nabla_q^s T_\Omega (f_Q - f)\|_{L^p(Q)}^p + \sum_{Q \in \mathcal{W}} \|\nabla_q^s T_\Omega f\|_{L^p(Q)}^p,$$

inequality (2.15) proves that

$$\sum_{Q \in \mathcal{W}} \|\nabla_q^s T_\Omega f\|_{L^p(Q)}^p \lesssim \|f\|_{F_{p,q}^s(\Omega)}^p \iff \sum_{Q \in \mathcal{W}} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p \lesssim \|f\|_{F_{p,q}^s(\Omega)}^p.$$

On the other hand, by assumption T is bounded on L^p and we have that $\|T_\Omega f\|_{L^p(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$. Since $\|T_\Omega f\|_{F_{p,q}^s(\Omega)}^p \approx \|T_\Omega f\|_{L^p(\Omega)}^p + \sum_{Q \in \mathcal{W}} \int_Q |\nabla_q^s T_\Omega f(x)|^p dx$ by (2.14), the lemma follows.

Again we use duality. That is, to prove (2.15) it suffices to prove that given a positive function $g \in L^{p'}(L^{q'}(\Omega))$ with $\|g\|_{L^{p'}(L^{q'}(\Omega))} = 1$, we have that

$$\sum_Q \int_Q \int_{\mathbf{Sh}(Q)} \frac{|T_\Omega(f - f_Q)(x) - T_\Omega(f - f_Q)(y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx \lesssim \|f\|_{F_{p,q}^s(\Omega)}.$$

Given a cube $Q \in \mathcal{W}$, we can define a bump function φ_Q such that $\chi_{6Q} \leq \varphi_Q \leq \chi_{7Q}$ and $\|\nabla \varphi_Q\|_{L^\infty} \leq C\ell(Q)^{-1}$. Given a cube $S \subset 5Q$ we define $\varphi_{QS} := \varphi_Q$. Otherwise, take $\varphi_{QS} := \varphi_S$. Note that in both situations, by (1.7) we have that $\text{supp} \varphi_{QS} \subset 23S$. Then, we can express the difference between $T_\Omega(f - f_Q)$ evaluated at $x \in Q$ and in $y \in S$ as

$$\begin{aligned} T_\Omega(f - f_Q)(x) - T_\Omega(f - f_Q)(y) &= T_\Omega[(f - f_Q)\varphi_Q](x) - T_\Omega[(f - f_Q)\varphi_{QS}](y) \\ &\quad + T_\Omega[(f - f_Q)(1 - \varphi_Q)](x) - T_\Omega[(f - f_Q)(1 - \varphi_{QS})](y). \end{aligned} \quad (2.16)$$

Note that the first two terms in the right-hand side of (2.16) are ‘local’ terms in the sense that the functions to which we apply the operator T_Ω are supported in a small neighborhood of the point of evaluation (and are globally $F_{p,q}^s$, as we will check later on) and the other two terms are ‘non-local’. What we will prove is that the local part

$$\boxed{1} := \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T_\Omega [(f - f_Q) \varphi_Q](x) - T_\Omega [(f - f_Q) \varphi_{QS}](y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx,$$

and the non-local part

$$\boxed{2} := \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T_\Omega [(f - f_Q)(1 - \varphi_Q)](x) - T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx,$$

are both bounded as

$$\boxed{1} + \boxed{2} \leq C \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.17)$$

We begin by the local part, that is, we want to prove that $\boxed{1} \lesssim \|f\|_{F_{p,q}^s(\Omega)}$. Note that for $x \in Q$ and $y \in S \in \mathbf{SH}(Q)$, if $y \in 3Q$ then $\varphi_{QS} = \varphi_Q$ and, otherwise $|x - y| \approx \ell(Q)$. Thus,

$$\begin{aligned} \boxed{1} &\leq \sum_Q \int_Q \int_{3Q} \frac{|T[(f - f_Q) \varphi_Q](x) - T[(f - f_Q) \varphi_Q](y)|}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx \\ &\quad + \sum_Q \int_Q \int_{\mathbf{SH}(Q)} \frac{|T[(f - f_Q) \varphi_Q](x)|}{\ell(Q)^{s + \frac{d}{q}}} g(x, y) dy dx \\ &\quad + \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T[(f - f_Q) \varphi_{QS}](y)|}{\ell(Q)^{s + \frac{d}{q}}} g(x, y) dy dx =: \boxed{1.1} + \boxed{1.2} + \boxed{1.3}. \end{aligned} \quad (2.18)$$

Of course, by Hölder’s inequality we have that

$$\boxed{1.1}^p \leq \sum_Q \int_Q \left(\int_{3Q} \frac{|T[(f - f_Q) \varphi_Q](x) - T[(f - f_Q) \varphi_Q](y)|^q}{|x - y|^{sq + d}} dy \right)^{\frac{p}{q}} dx \|g\|_{L^{p'}(L^{q'}(\Omega))}^p.$$

By (2.13) we get

$$\boxed{1.1}^p \lesssim \sum_Q \|T[(f - f_Q) \varphi_Q]\|_{F_{p,q}^s(\mathbb{R}^d)}^p.$$

Now, the operator T is bounded on $F_{p,q}^s$ by assumption (see Definition 2.4 and Remark 2.5). Thus,

$$\boxed{1.1}^p \lesssim \|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s}^p \sum_Q \|(f - f_Q) \varphi_Q\|_{F_{p,q}^s(\mathbb{R}^d)}^p.$$

Consider the characterization of the $F_{p,q}^s$ -norm given in (2.13). Since $\varphi_Q \leq \chi_{7Q}$, the first term $\sum_Q \|(f - f_Q) \varphi_Q\|_{L^p(\mathbb{R}^d)}^p$ is bounded by a constant times $\|f\|_{L^p}$ by the finite overlapping of the Whitney cubes and the Jensen inequality, and the second is

$$\sum_Q \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|(f(x) - f_Q) \varphi_Q(x) - (f(y) - f_Q) \varphi_Q(y)|^q}{|x - y|^{sq + d}} dy \right)^{\frac{p}{q}} dx,$$

where the integrand vanishes if both $x, y \notin 8Q$. Therefore, we can write

$$\begin{aligned}
\boxed{1.1}^p &\lesssim \|f\|_{L^p} + \sum_Q \int_{8Q} \left(\int_{8Q} \frac{|(f(x) - f_Q) \varphi_Q(x) - (f(y) - f_Q) \varphi_Q(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&+ \sum_Q \int_{\mathbb{R}^d \setminus 8Q} \left(\int_{7Q} \frac{|(f(y) - f_Q) \varphi_Q(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&+ \sum_Q \int_{7Q} \left(\int_{\mathbb{R}^d \setminus 8Q} \frac{|(f(x) - f_Q) \varphi_Q(x)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx =: \|f\|_{L^p} + \boxed{1.1.1} + \boxed{1.1.2} + \boxed{1.1.3},
\end{aligned} \tag{2.19}$$

where the constant depends linearly on the operator norm $\|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s}$.

Adding and subtracting $(f(x) - f_Q) \varphi_Q(y)$ in the numerator of the integral in $\boxed{1.1.1}$ we get that

$$\begin{aligned}
\boxed{1.1.1} &\lesssim \sum_Q \int_{8Q} \left(\int_{8Q} \frac{|f(x) - f_Q|^q |\varphi_Q(x) - \varphi_Q(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&+ \sum_Q \int_{8Q} \left(\int_{8Q} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx.
\end{aligned}$$

The second term is bounded by a constant times $\|f\|_{F_{p,q}^s(\Omega)}^p$, so

$$\boxed{1.1.1} \lesssim \sum_Q \int_{8Q} \left(\int_{8Q} \frac{\|\nabla \varphi_Q\|_{L^\infty}^q |x - y|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} |f(x) - f_Q|^p dx + \|f\|_{F_{p,q}^s(\Omega)}^p.$$

Using $\|\nabla \varphi_Q\|_{L^\infty} \lesssim \frac{1}{\ell(Q)}$ and the local inequality for the maximal operator (1.28) we get that

$$\begin{aligned}
\boxed{1.1.1} &\lesssim \sum_Q \int_{8Q} \ell(Q)^{(1-s)p} \frac{|f(x) - f_Q|^p}{\ell(Q)^p} dx + \|f\|_{F_{p,q}^s(\Omega)}^p \\
&\lesssim \sum_Q \int_{8Q} \left(\frac{\int_Q |f(x) - f(\xi)| d\xi}{\ell(Q)^{s+d}} \right)^p dx + \|f\|_{F_{p,q}^s(\Omega)}^p.
\end{aligned} \tag{2.20}$$

By Jensen's inequality $\frac{1}{\ell(Q)^d} \int_Q |f(x) - f(\xi)| d\xi \lesssim \left(\int_Q \frac{1}{\ell(Q)^d} |f(x) - f(\xi)|^q d\xi \right)^{\frac{1}{q}}$ and, therefore,

$$\boxed{1.1.1} \lesssim \|f\|_{F_{p,q}^s(\Omega)}^p. \tag{2.21}$$

Now we undertake the task of bounding $\boxed{1.1.2}$ in (2.19). Writing x_Q for the center of a given cube Q , we have that

$$\boxed{1.1.2} \lesssim \sum_Q \int_{\mathbb{R}^d \setminus 8Q} \frac{dx}{|x - x_Q|^{sp + \frac{dp}{q}}} \left(\int_{7Q} |f(y) - f_Q|^q dy \right)^{\frac{p}{q}}.$$

Since $s > \frac{d}{p} - \frac{d}{q}$ we have that $sp + \frac{dp}{q} > d$. Thus, by (1.29)

$$\boxed{1.1.2} \lesssim \sum_Q \frac{1}{\ell(Q)^{sp + \frac{dp}{q} - d}} \left(\int_{7Q} |f(y) - f_Q|^q dy \right)^{\frac{p}{q}} \leq \sum_Q \frac{\left(\int_{7Q} \left(\int_Q |f(y) - f(\xi)| d\xi \right)^q dy \right)^{\frac{p}{q}}}{\ell(Q)^{sp + \frac{dp}{q} - d + dp}}.$$

By Minkowski's inequality (1.9) we have that

$$\boxed{1.1.2} \lesssim \sum_Q \frac{\left(\int_Q \left(\int_{7Q} |f(y) - f(\xi)|^q dy \right)^{\frac{1}{q}} d\xi \right)^p}{\ell(Q)^{sp + \frac{dp}{q} + d(p-1)}},$$

and by Hölder's inequality, using that $p-1 = \frac{p}{p'}$ we get that

$$\boxed{1.1.2} \lesssim \sum_Q \frac{\int_Q \left(\int_{7Q} |f(y) - f(\xi)|^q dy \right)^{\frac{p}{q}} d\xi \ell(Q)^{\frac{dp}{p'}}}{\ell(Q)^{sp + \frac{dp}{q} + \frac{dp}{p'}}} \lesssim \sum_Q \int_Q \left(\int_{7Q} \frac{|f(y) - f(\xi)|^q dy}{|y - \xi|^{sq+d}} \right)^{\frac{p}{q}} d\xi$$

and

$$\boxed{1.1.2} \lesssim \|f\|_{F_{p,q}^s(\Omega)}^p. \quad (2.22)$$

Dealing with the last term in (2.19) is somewhat easier. Note that by (1.29) we have that

$$\boxed{1.1.3} \leq \sum_Q \int_{7Q} |f(x) - f_Q|^p \left(\int_{\mathbb{R}^d \setminus 8Q} \frac{1}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \leq \sum_Q \int_{7Q} \frac{|f(x) - f_Q|^p}{\ell(Q)^{sp}} dx$$

and, since this quantity is bounded by the right-hand side of (2.20), it follows that

$$\boxed{1.1.3} \lesssim \|f\|_{F_{p,q}^s(\Omega)}^p. \quad (2.23)$$

Summing up, by (2.19), (2.21), (2.22) and (2.23) we get

$$\boxed{1.1} \lesssim \|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s} \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.24)$$

Back to (2.18), it remains to bound $\boxed{1.2}$ and $\boxed{1.3}$. Recall that

$$\boxed{1.2} = \sum_Q \int_Q \frac{|T[(f - f_Q)\varphi_Q](x)|}{\ell(Q)^{s + \frac{d}{q}}} \int_{\mathbf{Sh}(Q)} g(x, y) dy dx.$$

Writing $G(x) = \|g(x, \cdot)\|_{L^{q'}(\Omega)}$ and using Hölder's inequality we get

$$\int_{\mathbf{Sh}(Q)} g(x, y) dy \leq \left(\int_{\mathbf{Sh}(Q)} g(x, y)^{q'} dy \right)^{\frac{1}{q'}} |\mathbf{Sh}(Q)|^{\frac{1}{q}} \lesssim_{\rho_\varepsilon, d} G(x) \ell(Q)^{\frac{d}{q}},$$

and using again Hölder's inequality it follows that

$$\boxed{1.2} \lesssim \left(\sum_Q \int_Q \frac{|T[(f - f_Q)\varphi_Q](x)|^p}{\ell(Q)^{sp}} dx \right)^{\frac{1}{p}} \|G\|_{L^{p'}(\Omega)}.$$

Of course, $\|G\|_{L^{p'}(\Omega)} \leq 1$. Now, by Definition 1.24 we can use the boundedness of T in L^p to find that

$$\boxed{1.2} \lesssim \|T\|_{L^p \rightarrow L^p} \left(\sum_Q \frac{\|(f - f_Q)\varphi_Q\|_{L^p(\mathbb{R}^d)}^p}{\ell(Q)^{sp}} \right)^{\frac{1}{p}} \lesssim \left(\sum_Q \frac{\|f - f_Q\|_{L^p(7Q)}^p}{\ell(Q)^{sp}} \right)^{\frac{1}{p}},$$

and we can argue again as in (2.20) to prove that

$$\boxed{1.2} \lesssim \|T\|_{L^p \rightarrow L^p} \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.25)$$

Finally, for the last term in (2.18), that is, for

$$\boxed{1.3} = \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T[(f - f_Q)\varphi_{QS}](y)|}{\ell(Q)^{s+\frac{d}{q}}} g(x, y) dy dx,$$

by Hölder's inequality we have that

$$\boxed{1.3} \leq \sum_Q \int_Q \left(\sum_{S \in \mathbf{SH}(Q)} \int_S \frac{|T[(f - f_Q)\varphi_{QS}](y)|^q}{\ell(Q)^{sq+d}} dy \right)^{\frac{1}{q}} G(x) dx.$$

The boundedness of T in L^q leads to

$$\boxed{1.3} \leq \|T\|_{L^q \rightarrow L^q} \sum_Q \left(\sum_{S \in \mathbf{SH}(Q)} \int_{\text{supp}(\varphi_{QS})} \frac{|(f(y) - f_Q)\varphi_{QS}(y)|^q}{\ell(Q)^{sq+d}} dy \right)^{\frac{1}{q}} \ell(Q)^d \inf_Q MG.$$

Given a cube Q , the finite overlapping of the family $\{50S\}_{S \in \mathcal{W}}$ (see Definition 1.5) implies the finite overlapping of the supports of the family $\{\varphi_{QS}\}$ (recall that $\text{supp}(\varphi_{QS}) \subset 23S$), so there is a certain ratio ρ_2 such that naming $\mathbf{Sh}^2(Q) := \mathbf{Sh}_{\rho_2}(Q)$ we have that

$$\begin{aligned} \boxed{1.3} &\lesssim \sum_Q \left(\int_{\mathbf{Sh}^2(Q)} \frac{|f(y) - f_Q|^q}{\ell(Q)^{sq+d-dq}} dy \right)^{\frac{1}{q}} \inf_Q MG \\ &\leq \sum_Q \left(\int_{\mathbf{Sh}^2(Q)} \left(\int_Q \frac{|f(y) - f(\xi)|}{\ell(Q)^{s+\frac{d}{q}-d+d}} d\xi \right)^q dy \right)^{\frac{1}{q}} \inf_Q MG. \end{aligned}$$

Finally, using Minkowski's inequality (1.9) and Hölder's inequality we get that

$$\boxed{1.3} \lesssim \sum_Q \int_Q \left(\int_{\mathbf{Sh}^2(Q)} \frac{|f(y) - f(\xi)|^q}{\ell(Q)^{sq+d}} dy \right)^{\frac{1}{q}} MG(\xi) d\xi \lesssim \left(\sum_Q \int_Q \left(\int_{\Omega} \frac{|f(y) - f(\xi)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{p}},$$

that is,

$$\boxed{1.3} \lesssim \|T\|_{L^q \rightarrow L^q} \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.26)$$

Now, by (2.18), (2.24), (2.25) and (2.26) we have that

$$\boxed{1} \lesssim \left(\|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s} + \|T\|_{L^p \rightarrow L^p} + \|T\|_{L^q \rightarrow L^q} \right) \|f\|_{F_{p,q}^s(\Omega)}, \quad (2.27)$$

and we have finished with the local part.

Now we bound the non-local part in (2.17). Consider $x \in Q \in \mathcal{W}$. By (1.40), since x is not in the support of $(f - f_Q)(1 - \varphi_Q)$, we have that

$$T_{\Omega}[(f - f_Q)(1 - \varphi_Q)](x) = \int_{\Omega} K(x - z)(f(z) - f_Q)(1 - \varphi_Q(z)) dm(z),$$

and by the same token for $y \in S \in \mathbf{SH}(Q)$

$$T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y) = \int_\Omega K(y - z)(f(z) - f_Q)(1 - \varphi_{QS}(z)) dm(z).$$

To shorten the notation, we will write

$$\lambda_{QS}(z_1, z_2) = K(z_1 - z_2)(f(z_2) - f_Q)(1 - \varphi_{QS}(z_2)),$$

for $z_1 \neq z_2$. Then we have that

$$\left| T_\Omega [(f - f_Q)(1 - \varphi_Q)](x) - T_\Omega [(f - f_Q)(1 - \varphi_{QS})](y) \right| = \left| \int_\Omega (\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)) dm(z) \right|,$$

that is,

$$\boxed{2} = \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \frac{\left| \int_\Omega (\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)) dz \right|}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx.$$

For ρ_3 big enough, $\mathbf{Sh}^3(Q) := \mathbf{Sh}_{\rho_3}(Q) \supset \bigcup_{S \in \mathbf{SH}(Q)} \mathbf{Sh}(S)$ (call $\mathbf{SH}^3(Q) := \mathbf{SH}_{\rho_3}(Q)$), we can decompose

$$\begin{aligned} \boxed{2} &\leq \sum_Q \int_Q \sum_{S \subset \mathbf{SH}(Q) \setminus 2Q} \int_S \frac{\int_{\mathbf{Sh}^3(Q)} |\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)| dz}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx & (2.28) \\ &+ \sum_Q \int_Q \sum_{S \subset \mathbf{SH}(Q) \setminus 2Q} \int_S \frac{\int_{\Omega \setminus \mathbf{Sh}^3(Q)} |\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)| dz}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx \\ &+ \sum_Q \int_Q \int_{5Q} \frac{\int_\Omega |\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)| dz}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx =: \boxed{A} + \boxed{B} + \boxed{C}. \end{aligned}$$

In the first term in the right-hand side of (2.28) the variable z is ‘close’ to either x or y , so smoothness does not help. Thus, we will take absolute values, giving rise to two terms separating λ_{QQ} and λ_{QS} . That is, we use that

$$\boxed{A} \leq \sum_Q \int_Q \sum_{S \subset \mathbf{SH}(Q) \setminus 2Q} \int_S \frac{\int_{\mathbf{Sh}^3(Q)} (|\lambda_{QQ}(x, z)| + |\lambda_{QS}(y, z)|) dz}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx.$$

Using the size condition (1.37),

$$|\lambda_{QQ}(x, z)| \leq C_K \frac{|f(z) - f_Q|}{|x - z|^d} |1 - \varphi_Q(z)|$$

and

$$|\lambda_{QS}(y, z)| \leq C_K \frac{|f(z) - f_Q|}{|y - z|^d} |1 - \varphi_{QS}(z)|.$$

Summing up,

$$\begin{aligned} \boxed{A} &\lesssim_{C_K} \sum_Q \int_Q \int_{\mathbf{SH}(Q) \setminus 2Q} \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f_Q| |1 - \varphi_Q(z)| dz}{|x - y|^{s + \frac{d}{q}} |x - z|^d} g(x, y) dy dx & (2.29) \\ &+ \sum_Q \int_Q \int_{\mathbf{SH}(Q) \setminus 2Q} \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f_Q| |1 - \varphi_{QS}(z)| dz}{|x - y|^{s + \frac{d}{q}} |y - z|^d} g(x, y) dy dx =: \boxed{2.1} + \boxed{2.2}, \end{aligned}$$

with constants depending linearly on the Calderón-Zygmund constant C_K .

We begin by the shorter part, that is

$$\boxed{2.1} = \sum_Q \int_Q \int_{\mathbf{Sh}(Q) \setminus 2Q} \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f_Q| |1 - \varphi_Q(z)| dz}{|x - y|^{s + \frac{d}{q}} |x - z|^d} g(x, y) dy dx.$$

Using the fact that $1 - \varphi_Q(z) = 0$ when z is close to the cube Q , we can say that

$$\boxed{2.1} \lesssim \sum_Q \frac{1}{\ell(Q)^{s + \frac{d}{q} + d}} \int_{\mathbf{Sh}^3(Q) \setminus 6Q} |f(z) - f_Q| \int_Q \int_{\mathbf{Sh}(Q) \setminus 2Q} g(x, y) dy dx dz.$$

Now, by the Hölder inequality we have that

$$\int_{\mathbf{Sh}(Q) \setminus 2Q} g(x, y) dy \lesssim_{\rho_\varepsilon, d} G(x) \ell(Q)^{\frac{d}{q}},$$

where $G(x) = \|g(x, \cdot)\|_{L^{q'}}$. Thus,

$$\boxed{2.1} \lesssim \sum_Q \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f_Q|}{\ell(Q)^{s+d}} \int_Q G(x) dx dz \lesssim \sum_Q \int_Q \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f(\xi)|}{\ell(Q)^{s+d}} MG(\xi) dz d\xi.$$

Finally, by Jensen's inequality and the boundedness of the maximal operator in $L^{p'}$ we have that

$$\begin{aligned} \sum_Q \int_Q \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f(\xi)|}{\ell(Q)^{s+d}} MG(\xi) dz d\xi &\lesssim \sum_Q \int_Q \left(\int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f(\xi)|^q}{\ell(Q)^{sq+d}} dz \right)^{\frac{1}{q}} MG(\xi) d\xi \quad (2.30) \\ &\lesssim \left(\int_\Omega \left(\int_\Omega \frac{|f(z) - f(\xi)|^q}{|z - \xi|^{sq+d}} dz \right)^{\frac{q}{p}} d\xi \right)^{\frac{1}{p}} \|MG\|_{L^{p'}}, \end{aligned}$$

that is,

$$\boxed{2.1} \lesssim \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.31)$$

The second term in (2.29) is the most delicate one. Given cubes Q , S and P and points $y \in S$ and $z \in P$ with $1 - \varphi_{QS}(z) \neq 0$, we have that $|z - y| \approx D(S, P)$. Therefore, we can write

$$\begin{aligned} \boxed{2.2} &= \sum_Q \int_Q \int_{\mathbf{Sh}(Q) \setminus 2Q} \int_{\mathbf{Sh}^3(Q)} \frac{|f(z) - f_Q| |1 - \varphi_{QS}(z)| dz}{|x - y|^{s + \frac{d}{q}} |y - z|^d} g(x, y) dy dx \\ &\lesssim \sum_Q \int_Q \sum_{S \in \mathbf{SH}(Q)} \int_S \sum_{P \in \mathbf{SH}^3(Q)} \int_P \frac{|f(z) - f_Q| dz}{\ell(Q)^{s + \frac{d}{q}} D(S, P)^d} g(x, y) dy dx. \end{aligned}$$

Next, we change the focus on the sum. Consider an admissible chain connecting two given cubes S and P both in $\mathbf{SH}^3(Q)$. Then $D(S, P) \approx \ell(S_P)$. Of course, using (1.21) and the fact that S and P are in $\mathbf{SH}^3(Q)$ we get

$$D(Q, S_P) \lesssim D(Q, S) + D(S, S_P) \approx D(Q, S) + D(S, P) \lesssim 2D(Q, S) + D(Q, P) \lesssim \ell(Q)$$

and, therefore, the cube S_P is contained in some $\mathbf{SH}_{\rho_4}(Q)$ for a certain constant ρ_4 depending on d and ε . For short, we write $L := S_P \in \mathbf{SH}^4(Q)$ and $\mathbf{Sh}^4(Q) := \mathbf{Sh}_{\rho_4}(Q)$. Then

$$\begin{aligned} \boxed{2.2} &\lesssim \sum_Q \int_Q \sum_{L \in \mathbf{SH}^4(Q)} \sum_{S \in \mathbf{SH}(L)} \int_S \sum_{P \in \mathbf{SH}(L)} \int_P \frac{|f(z) - f_Q| dz}{\ell(Q)^{s + \frac{d}{q}} \ell(L)^d} g(x, y) dy dx \\ &= \sum_Q \frac{1}{\ell(Q)^{s + \frac{d}{q}}} \int_Q \sum_{L \in \mathbf{SH}^4(Q)} \int_{\mathbf{Sh}(L)} |f(z) - f_Q| dz \frac{1}{\ell(L)^d} \int_{\mathbf{Sh}(L)} g(x, y) dy dx. \quad (2.32) \end{aligned}$$

If we write $g_x(y) = g(x, y)$, we have that for any cube L the integral

$$\int_{\mathbf{Sh}(L)} g(x, y) dy \leq \ell(L)^d \inf_L M g_x.$$

Arguing as before, for ρ_5 big enough we have that if $L \in SH^4(Q)$, then $\mathbf{Sh}(L) \subset \mathbf{Sh}_{\rho_5}(Q) =: \mathbf{Sh}^5(Q)$ and therefore

$$\int_{\mathbf{Sh}(L)} |f(z) - f_Q| dz = \int_{\mathbf{Sh}(L)} |f(z) - f_Q| \chi_{\mathbf{Sh}^5(Q)}(z) dz \lesssim \int_L M[(f - f_Q)\chi_{\mathbf{Sh}^5(Q)}](\xi) d\xi.$$

Back to (2.32) we have that

$$\begin{aligned} \boxed{2.2} &\lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \sum_{L \in \mathbf{SH}^4(Q)} \int_L M[(f - f_Q)\chi_{\mathbf{Sh}^5(Q)}](\xi) M g_x(\xi) d\xi dx \\ &= \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \int_{\mathbf{SH}^4(Q)} M[(f - f_Q)\chi_{\mathbf{Sh}^5(Q)}](\xi) M g_x(\xi) d\xi dx \end{aligned}$$

and, by Hölder's inequality and the boundedness of the maximal operator in L^q and $L^{q'}$, we have that

$$\begin{aligned} \boxed{2.2} &\lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \left(\int_{\mathbf{SH}^4(Q)} M[(f - f_Q)\chi_{\mathbf{Sh}^5(Q)}](\xi)^q d\xi \right)^{\frac{1}{q}} \left(\int_{\mathbf{SH}^4(Q)} M g_x(\xi)^{q'} d\xi \right)^{\frac{1}{q'}} dx \\ &\lesssim_q \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \left(\int_{\mathbf{Sh}^5(Q)} |f(\xi) - f_Q|^q d\xi \right)^{\frac{1}{q}} \left(\int_{\Omega} g(x, \xi)^{q'} d\xi \right)^{\frac{1}{q'}} dx. \end{aligned}$$

Again, we write $G(x) = \|g(x, \cdot)\|_{L^{q'}}$ and by Minkowski's integral inequality (1.9) we get that

$$\begin{aligned} \boxed{2.2} &\lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}+d}} \left(\int_{\mathbf{Sh}^5(Q)} \left(\int_Q |f(\xi) - f(\zeta)| d\zeta \right)^q d\xi \right)^{\frac{1}{q}} \int_Q G(x) dx \\ &\lesssim \sum_Q \frac{1}{\ell(Q)^{s+\frac{d}{q}}} \int_Q \left(\int_{\mathbf{Sh}^5(Q)} |f(\xi) - f(\zeta)|^q d\xi \right)^{\frac{1}{q}} M G(\zeta) d\zeta. \end{aligned}$$

Thus,

$$\boxed{2.2} \lesssim \left(\int_{\Omega} \left(\int_{\Omega} \frac{|f(\xi) - f(\zeta)|^q}{|\xi - \zeta|^{sq+d}} d\xi \right)^{\frac{2}{q}} d\zeta \right)^{\frac{1}{p}} \|MG\|_{L^{p'}} \lesssim \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.33)$$

Back to (2.28), it remains to bound $\boxed{\mathbb{B}}$ and $\boxed{\mathbb{C}}$. For the first one,

$$\boxed{\mathbb{B}} = \sum_Q \int_Q \sum_{S \subset \mathbf{Sh}(Q) \setminus 2Q} \int_S \frac{\int_{\Omega \setminus \mathbf{Sh}^3(Q)} |\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)| dz}{|x - y|^{s+\frac{d}{q}}} g(x, y) dy dx,$$

just note that if $x \in Q$, $y \in S \in \mathbf{SH}(Q)$ and $z \notin \mathbf{Sh}^3(Q)$ we have that $\varphi_{QQ}(z) = \varphi_{QS}(z) = 0$ and, if ρ_3 is big enough, $|x - z| > 2|x - y|$. Thus, we can use the smoothness condition (2.12), that is, $|\lambda_{QQ}(x, z) - \lambda_{QS}(y, z)| \leq |K(x - z) - K(y - z)| |f(z) - f_Q| \leq C_K \frac{|f(z) - f_Q| |x - y|^s}{|x - z|^{d+s}}$.

In the last term in (2.28),

$$\boxed{\mathbb{C}} = \sum_Q \int_Q \int_{5Q} \frac{\int_{\Omega} |\lambda_{QQ}(x, z) - \lambda_{QQ}(y, z)| dz}{|x - y|^{s + \frac{d}{q}}} g(x, y) dy dx,$$

we are integrating in the region where $x \in Q$, $y \in 5Q$ and $z \notin 6Q$ because otherwise $1 - \varphi_Q(z)$ would vanish. Also $|x - z| > C_d|x - y|$ and $|x - z| \approx |y - z|$. Thus, we have again that $|\lambda_{QQ}(x, z) - \lambda_{QQ}(y, z)| \leq |K(x - z) - K(y - z)| |f(z) - f_Q| \lesssim C_K \frac{|f(z) - f_Q| |x - y|^s}{|x - z|^{d+s}}$ by (2.12) and (1.37) (one may use the last one when $2|x - y| \geq |x - z| > C_d|x - y|$, that is $|x - y| \approx |x - z| \approx |y - z|$).

Summing up,

$$\boxed{\mathbb{B}} + \boxed{\mathbb{C}} \lesssim_{C_K} \sum_Q \int_Q \int_{\text{Sh}(Q)} \int_{\Omega \setminus 6Q} \frac{|f(z) - f_Q| |x - y|^s dz}{|x - y|^{s + \frac{d}{q}} |x - z|^{d+s}} g(x, y) dy dx =: \boxed{\mathbb{2.3}}. \quad (2.34)$$

with constants depending linearly on the Calderón-Zygmund constant C_K . Reordering,

$$\boxed{\mathbb{2.3}} = \sum_Q \int_Q \int_{\Omega \setminus 6Q} \frac{|f(z) - f_Q| dz}{|x - z|^{d+s}} \int_{\text{Sh}(Q)} \frac{g(x, y) dy}{|x - y|^{\frac{d}{q}}} dx.$$

The last integral above is easy to bound by the same techniques as before: Given $x \in Q \in \mathcal{W}$, since $\frac{d}{q} < d$, by (1.28), Hölder's Inequality and the boundedness of the maximal operator in $L^{q'}$ we have that

$$\int_{\text{Sh}(Q)} \frac{g(x, y) dy}{|x - y|^{\frac{d}{q}}} \lesssim \ell(Q)^{d - \frac{d}{q}} \inf_Q M g_x \leq \ell(Q)^{-\frac{d}{q}} \int_Q M g_x \leq \|M g_x\|_{L^{q'}} \lesssim_q G(x).$$

Thus,

$$\boxed{\mathbb{2.3}} \lesssim \sum_Q \int_Q \sum_P \int_P \frac{|f(z) - f_Q| dz}{D(P, Q)^{d+s}} G(x) dx.$$

For every pair of cubes $P, Q \in \mathcal{W}$, there exists an admissible chain $[P, Q]$ and, writing $[P, P_Q]$ for the subchain $[P, P_Q] \setminus \{P_Q\}$ and $[P_Q, Q]$ for $[P_Q, Q] \setminus \{Q\}$, we get

$$\begin{aligned} \boxed{\mathbb{2.3}} &\lesssim \sum_Q \int_Q \sum_P \int_P \frac{|f(z) - f_P| dz}{D(P, Q)^{d+s}} G(x) dx \\ &\quad + \sum_Q \int_Q \sum_P \sum_{L \in [P, P_Q]} \frac{|f_L - f_{N(L)}| \ell(P)^d}{D(P, Q)^{d+s}} G(x) dx \\ &\quad + \sum_Q \int_Q \sum_P \sum_{L \in [P_Q, Q]} \frac{|f_L - f_{N(L)}| \ell(P)^d}{D(P, Q)^{d+s}} G(x) dx =: \boxed{\mathbb{2.3.1}} + \boxed{\mathbb{2.3.2}} + \boxed{\mathbb{2.3.3}}. \end{aligned} \quad (2.35)$$

The first term in (2.35) can be bounded by reordering and using (1.27). Indeed, we have that

$$\boxed{\mathbb{2.3.1}} \leq \sum_P \int_P \int_P \frac{|f(z) - f(\xi)| d\xi dz}{\ell(P)^d} \sum_Q \int_Q \frac{G(x) dx}{D(P, Q)^{d+s}} \lesssim \sum_P \int_P \int_P \frac{|f(z) - f(\xi)| d\xi M G(z) dz}{\ell(P)^{d+s}},$$

and, by (2.30) we have that

$$\boxed{\mathbb{2.3.1}} \lesssim \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.36)$$

For the second term in (2.35) note that given cubes $L \in [P, P_Q]$ we have that $D(P, Q) \approx D(L, Q)$ by (1.23) and $P \in \mathbf{SH}(L)$ by Definition 1.16. Therefore, by (1.27) we have that

$$\begin{aligned} \boxed{2.3.2} &\lesssim \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)| d\zeta d\xi \sum_Q \frac{1}{D(L, Q)^{d+s}} \int_Q G(x) dx \sum_{P \in \mathbf{SH}(L)} \ell(P)^d \\ &\lesssim \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)| \frac{MG(\zeta)}{\ell(L)^s} d\zeta d\xi \ell(L)^d = \sum_L \int_L \int_{5L} \frac{|f(\xi) - f(\zeta)| MG(\zeta)}{\ell(L)^{d+s}} d\zeta d\xi, \end{aligned}$$

and, again by (2.30), we have that

$$\boxed{2.3.2} \lesssim \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.37)$$

Finally, the last term of (2.35) can be bounded analogously: Given cubes $L \in [P_Q, Q]$ we have that $D(Q, P) \approx D(L, P)$ by (1.23), and

$$\begin{aligned} \boxed{2.3.3} &\lesssim \sum_L \frac{1}{\ell(L)^{2d}} \int_L \int_{5L} |f(\xi) - f(\zeta)| d\zeta d\xi \sum_{Q \in \mathbf{SH}(L)} \int_Q G(x) dx \sum_P \frac{\ell(P)^d}{D(P, L)^{d+s}} \\ &\lesssim \sum_L \int_L \int_{5L} |f(\xi) - f(\zeta)| MG(\zeta) d\zeta d\xi \frac{\ell(L)^{d-s}}{\ell(L)^{2d}} = \sum_L \int_L \int_{5L} \frac{|f(\xi) - f(\zeta)| MG(\zeta)}{\ell(L)^{d+s}} d\zeta d\xi, \end{aligned}$$

and

$$\boxed{2.3.3} \lesssim \|f\|_{F_{p,q}^s(\Omega)}. \quad (2.38)$$

Now, putting together (2.28), (2.29), (2.34) and (2.35) we have that

$$\boxed{2} \lesssim_{C_K} \boxed{2.1} + \boxed{2.2} + \boxed{2.3.1} + \boxed{2.3.2} + \boxed{2.3.3},$$

and by (2.31), (2.33), (2.36), (2.37) and (2.38) we have that

$$\boxed{2} \lesssim C_K \|f\|_{F_{p,q}^s(\Omega)}, \quad (2.39)$$

with constants depending on ε , s , p , q and d . Estimates (2.27) and (2.39) prove (2.15). \square

Proof of Theorem 2.8. Let Ω be a bounded ε -uniform domain. Note that since $s > \frac{d}{p} > \frac{d}{p} - \frac{d}{q}$, we can use the Key Lemma 2.11, that is, we have that T_Ω is bounded if and only if for every $f \in F_{p,q}^s(\Omega)$ we have that

$$\sum_{Q \in \mathcal{W}} |f_Q|^p \|\nabla_q^s T \chi_\Omega\|_{L^p(Q)}^p \leq C \|f\|_{F_{p,q}^s(\Omega)}^p, \quad (2.40)$$

with C independent from f . Since $sp > d$, by Definition 2.7 and Proposition 1.11 we have the continuous embedding $F_{p,q}^s(\Omega) \subset L^\infty$. Therefore, given a cube Q we have that $|f_Q| \leq \|f\|_{L^\infty(\Omega)} \leq \|f\|_{F_{p,q}^s(\Omega)}$ and (2.40) holds as long as $T \chi_\Omega \in F_{p,q}^s(\Omega)$.

More precisely, putting together (2.15) and (2.40), we get

$$\|T_\Omega f\|_{F_{p,q}^s(\Omega)} \lesssim \left(C_K + \|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s} + \|T\|_{L^p \rightarrow L^p} + \|T\|_{L^q \rightarrow L^q} + \|T_\Omega 1\|_{F_{p,q}^s(\Omega)} \right) \|f\|_{F_{p,q}^s(\Omega)},$$

with C depending only on ε , s , p , q and d . \square

2.3 Characterization of $F_{p,q}^s$ via differences

In this section we prove part of Theorem 2.10. Namely, the estimate (2.13) is a consequence of Corollary 2.12 below, and the first estimate in (2.14) is proven in Corollary 2.17. The second estimate in (2.14) is left for Lemma 2.18 in Section 2.4.

For a function $f \in L_{loc}^1$, $M \in \mathbb{N}$, $0 < u \leq \infty$, $t > 0$ and $x \in \mathbb{R}^d$, we write

$$d_{t,u}^M f(x) := \left(t^{-d} \int_{|h| \leq t} |\Delta_h^M f(x)|^u dh \right)^{\frac{1}{u}},$$

with the usual modification for $u = \infty$. In [Tri06, Theorem 1.116] we find the following result.

Theorem (See [Tri06]). *Given $1 \leq r \leq \infty$, $0 < u \leq r$, $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 < s < M$ with $\frac{d}{\min\{p,q\}} - \frac{d}{r} < s$, we have that*

$$F_{p,q}^s(\mathbb{R}^d) = \left\{ f \in L^{\max\{p,r\}} : \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \left(\int_0^1 \frac{d_{t,u}^M f(x)^q}{t^{sq+1}} dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}$$

(with the usual modification for $q = \infty$), in the sense of equivalent quasinorms.

As an immediate consequence of this result, we get the following corollary.

Corollary 2.12. *Let $1 \leq p < \infty$, $1 \leq q < \infty$ and $0 < s < 1 \leq M$ with $s > \frac{d}{p} - \frac{d}{q}$. Then*

$$F_{p,q}^s(\mathbb{R}^d) = \left\{ f \in L^{\max\{p,q\}} \text{ s.t. } \|f\|_{A_{p,q}^s(\mathbb{R}^d)} := \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|\Delta_h^M f(x)|^q}{|h|^{sq+d}} dh \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty \right\}$$

(with the usual modification for $q = \infty$), in the sense of equivalent norms.

Proof. Let $f \in L^{\max\{p,q\}}$. Choosing $q = u = r$ all the conditions in the theorem above are satisfied. Therefore,

$$\|f\|_{F_{p,q}^s(\mathbb{R}^d)} \approx \|f\|_{L^p} + \left(\int_{\mathbb{R}^d} \left(\int_0^1 \frac{d_{t,q}^M f(x)^q}{t^{sq+1}} dt \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \quad (2.41)$$

Since $d_{t,q}^M f(x) = \left(t^{-d} \int_{|h| \leq t} |\Delta_h^M f(x)|^q dh \right)^{\frac{1}{q}}$ for $x \in \mathbb{R}^d$, we can change the order of integration to get that

$$\begin{aligned} \int_{\mathbb{R}^d} \left(\int_0^1 \frac{d_{t,q}^M f(x)^q}{t^{sq+1}} dt \right)^{\frac{p}{q}} dx &= \int_{\mathbb{R}^d} \left(\int_{|h| \leq 1} \int_{1 > t > |h|} \frac{dt}{t^{sq+1+d}} |\Delta_h^M f(x)|^q dh \right)^{\frac{p}{q}} dx \\ &= \int_{\mathbb{R}^d} \left(\int_{|h| \leq 1} \frac{|\Delta_h^M f(x)|^q}{sq+d} \left(\frac{1}{|h|^{sq+d}} - 1 \right) dh \right)^{\frac{p}{q}} dx. \end{aligned}$$

This shows that $\|f\|_{F_{p,q}^s(\mathbb{R}^d)} \lesssim \|f\|_{A_{p,q}^s(\mathbb{R}^d)}$ and also that

$$\int_{\mathbb{R}^d} \left(\int_{|h| < \frac{1}{2}} \frac{|\Delta_h^M f(x)|^q}{|h|^{sq+d}} dh \right)^{\frac{p}{q}} dx \lesssim \int_{\mathbb{R}^d} \left(\int_0^1 \frac{d_{t,q}^M f(x)^q}{t^{sq+1}} dt \right)^{\frac{p}{q}} dx \lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^d)}^p \quad (2.42)$$

by (2.41). It remains to see that $\int_{\mathbb{R}^d} \left(\int_{|h| > \frac{1}{2}} \frac{|\Delta_h^M f(x)|^q}{|h|^{sq+d}} dh \right)^{\frac{p}{q}} dx \lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^d)}^p$. Using appropriate changes of variables and the triangle inequality, it is enough to check that

$$\textcircled{\text{I}} := \int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|f(x+h)|^q}{(1+|h|)^{sq+d}} dh \right)^{\frac{p}{q}} dx \lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^d)}^p. \quad (2.43)$$

Let us assume first that $p \geq q$. Then, since the measure $(1+|h|)^{-(sq+d)} dh$ is finite, we may apply Jensen's inequality to the inner integral, and then Fubini to obtain

$$\textcircled{\text{I}} \lesssim \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \frac{|f(x+h)|^p}{(1+|h|)^{sp+d}} dh dx \lesssim \|f\|_{L^p}^p,$$

and (2.43) follows.

If, instead, $p < q$, cover \mathbb{R}^d with disjoint cubes $Q_{\vec{j}} = Q_0 + \ell \vec{j}$ for $\vec{j} \in \mathbb{Z}^d$. Fix the side-length ℓ of these cubes so that their diameter is $1/3$. By the subadditivity of $x \mapsto |x|^{\frac{p}{q}}$, we have that

$$\textcircled{\text{I}} \lesssim \sum_{\vec{k}} \int_{Q_{\vec{k}}} \sum_{\vec{j}} \left(\int_{Q_{\vec{j}}} \frac{|f(y)|^q}{(1+|x-y|)^{sq+d}} dy \right)^{\frac{p}{q}} dx \approx \sum_{\vec{j}} \left(\int_{Q_{\vec{j}}} |f(y)|^q dy \right)^{\frac{p}{q}} \sum_{\vec{k}} \frac{1}{(1+|\vec{j}-\vec{k}|)^{sp+\frac{dp}{q}}}.$$

Since $s + \frac{d}{q} > \frac{d}{p}$, the last sum is finite and does not depend on \vec{j} . By (2.42) we have that

$$\begin{aligned} \textcircled{\text{I}} &\lesssim \sum_{\vec{j}} \left(\int_{Q_{\vec{j}}} |f(y)|^q dy \right)^{\frac{p}{q}} \lesssim \sum_{\vec{j}} \int_{Q_{\vec{j}}} \left(\int_{Q_{\vec{j}}} |f(y) - f(x)|^q dy \right)^{\frac{p}{q}} dx + \sum_{\vec{j}} \int_{Q_{\vec{j}}} \left(\int_{Q_{\vec{j}}} |f(x)|^q dy \right)^{\frac{p}{q}} dx \\ &\lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^d)}^p. \end{aligned}$$

In the last step we have used that $\sum_{\vec{j}} \int_{Q_{\vec{j}}} \left(\int_{Q_{\vec{j}}} |f(x)|^q dy \right)^{\frac{p}{q}} dx \approx \|f\|_{L^p}^p$ because all the cubes have side-length comparable to 1. \square

Definition 2.13. Consider $1 \leq p < \infty$, $1 \leq q \leq \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Let U be an open set in \mathbb{R}^d . We say that a measurable function $f \in A_{p,q}^s(U)$ if

- The function $f \in L^p(U)$.
- The seminorm

$$\|f\|_{\dot{A}_{p,q}^s(U)} := \left(\int_U \left(\int_U \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \quad (2.44)$$

is finite.

We define the norm

$$\|f\|_{A_{p,q}^s(U)} := \|f\|_{L^p(U)} + \|f\|_{\dot{A}_{p,q}^s(U)}.$$

Remark 2.14. The condition $s > \frac{d}{p} - \frac{d}{q}$ ensures that the C_c^∞ -functions are in the class $A_{p,q}^s(\mathbb{R}^d)$.

Proof. Indeed, given a bump function $\varphi \in C_c^\infty(\mathbb{D})$,

$$\begin{aligned} \|\varphi\|_{A_{p,q}^s(\mathbb{R}^d)} &\geq \left(\int_{(2\mathbb{D})^c} \left(\int_{\mathbb{D}} \frac{|\varphi(x) - \varphi(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \\ &\approx \left(\int_{(2\mathbb{D})^c} \left(\int_{\mathbb{D}} |\varphi(y)|^q dy \right)^{\frac{p}{q}} \frac{1}{|x|^{sp+\frac{dp}{q}}} dx \right)^{\frac{1}{p}} \end{aligned}$$

which is finite if and only if $\frac{d}{p} < s + \frac{d}{q}$. The converse implication is an exercise. \square

In some situations, the classical Besov spaces $B_{p,p}^s(U) = A_{p,p}^s(U)$ and the fractional Sobolev spaces $W^{s,p}(U) = A_{p,2}^s(U)$. For instance, when Ω is a Lipschitz domain then $A_{p,2}^s(\Omega) = W^{s,p}(\Omega)$ (see [Str67]). We will see that this is a property of all the uniform domains.

Consider a given ε -uniform domain Ω . In [Jon81] Peter Jones defines an extension operator $\Lambda_0 : W^{1,p}(\Omega) \rightarrow W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$, that is, a bounded operator such that $\Lambda_0 f|_\Omega \equiv f|_\Omega$ for every $f \in W^{1,p}(\Omega)$. This extension operator is used to prove that the intrinsic characterization of $W^{1,p}(\Omega)$ and the infimum norms coincide, that is,

$$\inf_{g|_\Omega \equiv f} \|g\|_{W^{1,p}(\mathbb{R}^d)} \approx \|f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(\Omega)}.$$

Next we will see that the same operator is an extension operator for $A_{p,q}^s(\Omega)$ for $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. To define it we need a Whitney covering \mathcal{W}_1 of Ω (see Definition 1.5), a Whitney covering \mathcal{W}_2 of Ω^c and we define \mathcal{W}_3 to be the collection of cubes in \mathcal{W}_2 with side-lengths small enough, so that for any $Q \in \mathcal{W}_3$ there is a $S \in \mathcal{W}_1$ with $D(Q, S) \leq C\ell(Q)$ and $\ell(Q) = \ell(S)$ (see [Jon81, Lemma 2.4]). We define the symmetrized cube Q^* as one of the cubes satisfying these properties. Note that the number of possible choices for Q^* is uniformly bounded and, if Ω is an unbounded uniform domain, then

$$\mathcal{W}_2 = \mathcal{W}_3. \quad (2.45)$$

Lemma 2.15 (see [Jon81]). *For cubes $Q_1, Q_2 \in \mathcal{W}_3$ and $S \in \mathcal{W}_1$ we have that*

- *The symmetrized cubes have finite overlapping: there exists a constant C depending on the parameters ε and d such that $\#\{Q \in \mathcal{W}_3 : Q^* = S\} \leq C$.*
- *The long distance is invariant in the following sense:*

$$D(Q_1^*, Q_2^*) \approx D(Q_1, Q_2) \quad \text{and} \quad D(Q_1^*, S) \approx D(Q_1, S) \quad (2.46)$$

- *In particular, if $Q_1 \cap 2Q_2 \neq \emptyset$ (Q_1 and Q_2 are neighbors by (1.7)), then $D(Q_1^*, Q_2^*) \approx \ell(Q_1)$.*

We define the family of bump functions $\{\psi_Q\}_{Q \in \mathcal{W}_2}$ to be a partition of the unity associated to $\{\frac{11}{10}Q\}_{Q \in \mathcal{W}_2}$, that is, their sum is $\sum \psi_Q \equiv 1$, they satisfy the pointwise inequalities $0 \leq \psi_Q \leq \chi_{\frac{11}{10}Q}$ and $\|\nabla \psi_Q\|_\infty \lesssim \frac{1}{\ell(Q)}$. We can define the operator

$$\Lambda_0 f(x) = \sum_{Q \in \mathcal{W}_3} \psi_Q(x) f_{Q^*} \quad \text{for any } f \in L_{loc}^1(\Omega)$$

(recall that f_U stands for the mean of a function f in a set U).

Lemma 2.16. *Let Ω be a uniform domain, let $1 < p, q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then, $\Lambda_0 : A_{p,q}^s(\Omega) \rightarrow A_{p,q}^s(\mathbb{R}^d)$ is an extension operator. Furthermore, $\Lambda_0 f \in L^{\max\{p,q\}}$ for every $f \in A_{p,q}^s(\Omega)$.*

Proof. We have to check that

$$\|\Lambda_0 f\|_{A_{p,q}^s(\mathbb{R}^d)} = \|\Lambda_0 f\|_{L^p} + \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \lesssim \|f\|_{A_{p,q}^s(\Omega)}.$$

First, note that $\|\Lambda_0 f\|_{L^p} \leq \|f\|_{L^p(\Omega)} + \|\Lambda_0 f\|_{L^p(\Omega^c)}$. By Jensen's inequality, we have that

$$\|\Lambda_0 f\|_{L^p(\Omega^c)}^p \lesssim_p \sum_{Q \in \mathcal{W}_3} |f_{Q^*}|^p \|\psi_Q\|_{L^p}^p \leq \sum_{Q \in \mathcal{W}_3} \frac{1}{\ell(Q)^d} \|f\|_{L^p(Q^*)}^p \left(\frac{11}{10} \ell(Q) \right)^d.$$

By the finite overlapping of the symmetrized cubes,

$$\|\Lambda_0 f\|_{L^p(\Omega^c)}^p \lesssim \|f\|_{L^p(\Omega)}^p.$$

The same can be said about L^q when $q > p$. In that case, moreover, one can cover Ω with balls $\{B_j\}_{j \in J}$ with radius one such that $|B_j \cap \Omega| \approx 1$. Then, using the subadditivity of $x \mapsto |x|^{\frac{p}{q}}$ we get

$$\begin{aligned} \|f\|_{L^q(\Omega)}^p &\leq \left(\sum_j \int_{B_j \cap \Omega} |f(y)|^q dy \right)^{\frac{p}{q}} \\ &\lesssim_q \sum_j \left(\int_{B_j \cap \Omega} |f(y) - f(x)|^q dy \right)^{\frac{p}{q}} dx + \int_{B_j \cap \Omega} |f(x)|^q dy \Big)^{\frac{p}{q}} dx \\ &\lesssim \int_{\Omega} \left(\int_{\Omega} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx + \|f\|_{L^p(\Omega)}^p \approx \|f\|_{A_{p,q}^s(\Omega)}^p, \end{aligned} \quad (2.47)$$

by Definition 2.13.

It remains to check that

$$\|\Lambda_0 f\|_{A_{p,q}^s(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} \left(\int_{\mathbb{R}^d} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \lesssim \|f\|_{A_{p,q}^s(\Omega)}.$$

More precisely, we will prove that

$$\textcircled{a} + \textcircled{b} + \textcircled{c} \lesssim \|f\|_{A_{p,q}^s(\Omega)}^p,$$

where

$$\begin{aligned} \textcircled{a} &:= \int_{\Omega} \left(\int_{\Omega^c} \frac{|f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx, & \textcircled{b} &:= \int_{\Omega^c} \left(\int_{\Omega} \frac{|\Lambda_0 f(x) - f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \quad \text{and} \\ \textcircled{c} &:= \int_{\Omega^c} \left(\int_{\Omega^c} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx. \end{aligned}$$

Let us begin with

$$\textcircled{a} = \int_{\Omega} \left(\int_{\Omega^c} \frac{|f(x) - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S^*}|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx.$$

Call $\mathcal{W}_4 := \{S \in \mathcal{W}_3 : \text{all the neighbors of } S \text{ are in } \mathcal{W}_3\}$. Given $y \in \frac{11}{10}S$, where $S \in \mathcal{W}_4$, we have that $\sum_{P \in \mathcal{W}_3} \psi_P(y) \equiv 1$ and, otherwise $0 \leq 1 - \sum_{P \in \mathcal{W}_3} \psi_P(y) \leq 1$. Thus

$$\begin{aligned} \textcircled{a} &\lesssim \sum_{Q \in \mathcal{W}_1} \int_Q \left(\sum_{S \in \mathcal{W}_3} \frac{|f(x) - f_{S^*}|^q}{D(Q, S)^{sq+d}} \int_{\frac{11}{10}S} \psi_S(y) dy \right)^{\frac{p}{q}} dx \\ &\quad + \sum_{Q \in \mathcal{W}_1} \int_Q \left(\sum_{S \in \mathcal{W}_2 \setminus \mathcal{W}_4} \int_S \frac{|(1 - \sum_{P \in \mathcal{W}_3} \psi_P(y)) f(x)|^q}{D(Q, S)^{sq+d}} dy \right)^{\frac{p}{q}} dx =: \textcircled{a1} + \textcircled{a2}. \end{aligned}$$

In $\textcircled{a1}$ by the choice of the symmetrized cube we have that $\int_{\frac{11}{10}S} \psi_S(y) dy \approx \ell(S^*)^d$. Jensen's inequality implies that $|f(x) - f_{S^*}|^q \leq \frac{1}{\ell(S^*)^d} \int_{S^*} |f(x) - f(\xi)|^q d\xi$. By (2.46) and the finite overlapping of the symmetrized cubes, we get that

$$\textcircled{a1} \lesssim \sum_{Q \in \mathcal{W}_1} \int_Q \left(\sum_{S \in \mathcal{W}_3} \int_{S^*} \frac{|f(x) - f(\xi)|^q}{D(Q, S^*)^{sq+d}} d\xi \right)^{\frac{p}{q}} dx \lesssim \|f\|_{A_{p,q}^s(\Omega)}^p.$$

To bound $\textcircled{a2}$ just note that for $Q \in \mathcal{W}_1$ and $S \in \mathcal{W}_2 \setminus \mathcal{W}_4$, we have that S is far from the boundary, say $\ell(S) \geq \ell_0$, where ℓ_0 depends only on $\text{diam}(\Omega)$ and ε and, if Ω is unbounded, then $\ell_0 = \infty$ and $\textcircled{a2} = 0$ by (2.45). Thus, we have that

$$\textcircled{a2} \lesssim \sum_{Q \in \mathcal{W}_1} \int_Q \left(\sum_{S \in \mathcal{W}_2 \setminus \mathcal{W}_4} \int_{\frac{11}{10}S} \frac{|f(x)|^q}{D(Q, S)^{sq+d}} dy \right)^{\frac{p}{q}} dx \lesssim \left(\sum_{S \in \mathcal{W}_2 \setminus \mathcal{W}_4} \frac{\ell(S)^d}{D(\Omega, S)^{sq+d}} \right)^{\frac{p}{q}} \|f\|_{L^p}^p.$$

Recall that Whitney cubes have side-length equivalent to their distance to $\partial\Omega$. Moreover, the number of cubes of a given side-length bigger than ℓ_0 is uniformly bounded when Ω is bounded, so $\sum_{S \in \mathcal{W}_2 \setminus \mathcal{W}_4} \frac{\ell(S)^d}{\ell(S)^{sq+d}}$ is a geometric sum. Therefore,

$$\textcircled{a2} \lesssim \left(\sum_{S \in \mathcal{W}_2 \setminus \mathcal{W}_4} \frac{1}{\ell(S)^{sq}} \right)^{\frac{p}{q}} \|f\|_{L^p}^p \leq C_{\varepsilon, \text{diam}(\Omega)} \ell_0^{-sp} \|f\|_{L^p}^p.$$

Next, note that, using the same decomposition as above, we have that

$$\begin{aligned} \textcircled{b} &= \int_{\Omega^c} \left(\int_{\Omega} \frac{|\sum_{Q \in \mathcal{W}_3} \psi_Q(x) f_{Q^*} - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\ &\lesssim \sum_{Q \in \mathcal{W}_3} \int_{\frac{11}{10}Q} \psi_Q(x)^p dx \left(\sum_{S \in \mathcal{W}_1} \int_S \frac{|f_{Q^*} - f(y)|^q}{D(Q, S)^{sq+d}} dy \right)^{\frac{p}{q}} \\ &\quad + \sum_{P \in \mathcal{W}_2 \setminus \mathcal{W}_4} \int_P \left(1 - \sum_{Q \in \mathcal{W}_3} \psi_Q(x) \right)^p dx \left(\sum_{S \in \mathcal{W}_1} \int_S \frac{|f(y)|^q}{D(P, S)^{sq+d}} dy \right)^{\frac{p}{q}} =: \textcircled{b1} + \textcircled{b2}. \end{aligned}$$

We have that

$$\textcircled{b1} \lesssim \sum_{Q \in \mathcal{W}_3} \ell(Q)^d \left(\sum_{S \in \mathcal{W}_1} \int_S \frac{\left(\frac{1}{\ell(Q)^d} \int_{Q^*} |f(\xi) - f(y)| d\xi \right)^q}{D(Q^*, S)^{sq+d}} dy \right)^{\frac{p}{q}}$$

and, thus, by Minkowsky's integral inequality, we have that

$$\textcircled{\text{b1}} \lesssim \sum_{Q \in \mathcal{W}_3} \frac{\ell(Q)^d}{\ell(Q)^{dp}} \left(\int_{Q^*} \left(\sum_{S \in \mathcal{W}_1} \int_S \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{1}{q}} d\xi \right)^p.$$

By Hölder's inequality and the finite overlapping of symmetrized cubes, we get that

$$\textcircled{\text{b1}} \lesssim \sum_{Q \in \mathcal{W}_3} \frac{1}{\ell(Q)^{d(p-1)}} \int_{Q^*} \left(\int_{\Omega} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{p}{q}} d\xi \ell(Q)^{\frac{dp}{p'}} \lesssim \int_{\Omega} \left(\int_{\Omega} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{p}{q}} d\xi,$$

that is,

$$\textcircled{\text{b1}} \lesssim \|f\|_{A_{p,q}^s(\Omega)}^p.$$

To bound $\textcircled{\text{b2}}$, note that as before, if Ω is unbounded then $\textcircled{\text{b2}} = 0$ and, otherwise, we have that

$$\textcircled{\text{b2}} \approx \sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_4} \ell(Q)^d \left(\sum_{S \in \mathcal{W}_1} \int_S \frac{|f(y)|^q}{D(Q, \Omega)^{sq+d}} dy \right)^{\frac{p}{q}} \lesssim \|f\|_{L^q(\Omega)}^p \sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_4} \frac{\ell(Q)^d}{\text{dist}(Q, \Omega)^{sp + \frac{dp}{q}}}.$$

Now, since $s > \frac{d}{p} - \frac{d}{q}$ we have that $sp + \frac{dp}{q} > d$. Therefore,

$$\sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_4} \frac{\ell(Q)^d}{\text{dist}(Q, \Omega)^{sp + \frac{dp}{q}}} \approx \sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_4} \frac{1}{\ell(Q)^{sp + \frac{dp}{q} - d}} \leq C_{\varepsilon, \text{diam}(\Omega)} \ell_0^{d - sp - \frac{dp}{q}}.$$

On the other hand, if Ω is bounded and $q \leq p$ then $\|f\|_{L^q(\Omega)} \lesssim \|f\|_{L^p(\Omega)}$ by the Hölder inequality and, if $p < q$ then $\|f\|_{L^q(\Omega)} \lesssim \|f\|_{A_{p,q}^s(\Omega)}$ by (2.47).

Let us focus on $\textcircled{\text{c}}$. We have that

$$\textcircled{\text{c}} = \int_{\Omega^c} \left(\int_{\Omega^c} \frac{|\sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P^*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S^*}|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx.$$

Given $x \in \frac{11}{10}Q$ where $Q \in \mathcal{W}_4$ and $y \in \Omega^c \cap B(x, \frac{\ell_Q}{10})$, then neither x nor y are in the support of any bump function of a cube in $\mathcal{W}_2 \setminus \mathcal{W}_3$, so $\sum_{P \in \mathcal{W}_3} \psi_P(y) \equiv 1$ and $\sum_{P \in \mathcal{W}_3} \psi_P(x) \equiv 1$. Therefore

$$\sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P^*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S^*} = \sum_{P \cap 2Q \neq \emptyset} \sum_{S \in \mathcal{W}_3} \psi_P(x) \psi_S(y) (f_{P^*} - f_{S^*}).$$

If, moreover, $y \in B(x, \frac{1}{10}\ell(Q))$, since the points are 'close' to each other, we will use the Hölder regularity of the bump functions, so we write

$$\sum_{P \in \mathcal{W}_3} \psi_P(x) f_{P^*} - \sum_{S \in \mathcal{W}_3} \psi_S(y) f_{S^*} = \sum_{P \in \mathcal{W}_3} (\psi_P(x) - \psi_P(y)) f_{P^*}.$$

This decomposition is still valid if $Q \in \mathcal{W}_2 \setminus \mathcal{W}_4$ and $y \in B(x, \frac{1}{10}\ell(Q))$, that is, $y \in B(x, \frac{\ell_Q}{10})$, but we will treat this case apart since we lose the cancellation of the sums of bump functions but we gain a uniform lower bound on the side-lengths of the cubes involved. Finally, we will group the

remaining cases, when $x \in \Omega^c$ and $y \notin B(x, \frac{\ell(Q)}{10})$ in an error term. Considering all these facts we get

$$\begin{aligned}
\textcircled{c} &\lesssim \sum_{Q \in \mathcal{W}_4} \int_Q \left(\int_{\Omega^c \setminus B(x, \frac{1}{10}\ell(Q))} \sum_{P \cap 2Q \neq \emptyset} \sum_{S \in \mathcal{W}_3} |\psi_P(x)\psi_S(y)| \frac{|f_{P^*} - f_{S^*}|^q}{D(P^*, S^*)^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&+ \sum_{Q \in \mathcal{W}_4} \int_Q \left(\int_{B(x, \frac{1}{10}\ell(Q))} \frac{|\sum_{S \cap 2Q \neq \emptyset} (\psi_S(x) - \psi_S(y)) f_{S^*}|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&+ \sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_4} \int_Q \left(\int_{B(x, \frac{\ell(Q)}{10})} \frac{|\sum_{S \in \mathcal{W}_3: S \cap 2Q \neq \emptyset} (\psi_S(x) - \psi_S(y)) f_{S^*}|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&+ \int_{\Omega^c} \left(\int_{\Omega^c \setminus B(x, \frac{\ell(Q)}{10})} \frac{|\Lambda_0 f(x) - \Lambda_0 f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&=: \textcircled{c1} + \textcircled{c2} + \textcircled{c3} + \textcircled{c4},
\end{aligned}$$

where the last two terms are zero in case Ω is unbounded.

Using the same arguments as in $\textcircled{a1}$ and $\textcircled{b1}$ we have that

$$\textcircled{c1} \lesssim \|f\|_{A_{p,q}^s(\Omega)}^p.$$

Also combining the arguments used to bound $\textcircled{a2}$ and $\textcircled{b2}$ we get that if Ω is bounded, then

$$\textcircled{c4} \lesssim \left(\|f\|_{L^p(\Omega)} + \|f\|_{L^q(\Omega)} \right)^p,$$

and it vanishes otherwise.

The novelty comes from the fact that we are integrating in Ω^c both terms in \textcircled{c} , so the variables in the integrals $\textcircled{c2}$ and $\textcircled{c3}$ can get as close as one can imagine. Here we need to use the smoothness of the bump functions, but also the smoothness of f itself. The trick for $\textcircled{c2}$ is to use that $\{\psi_Q\}$ is a partition of the unity with ψ_Q supported in $\frac{11}{10}Q$, that is, $\sum_{S \in \mathcal{W}_3} \psi_S(x) = \sum_{S \cap 2Q \neq \emptyset} \psi_S(x) = 1$ if $x \in \frac{11}{10}Q$ with $Q \in \mathcal{W}_4$. Thus,

$$\textcircled{c2} = \sum_{Q \in \mathcal{W}_4} \int_Q \left(\int_{B(x, \frac{1}{10}\ell(Q))} \frac{|\sum_{S \cap 2Q \neq \emptyset} (\psi_S(x) - \psi_S(y)) (f_{S^*} - f_{Q^*})|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx,$$

and using the fact that $\|\nabla \psi_Q\|_\infty \lesssim \frac{1}{\ell(Q)}$ and (1.28), we have that

$$\begin{aligned}
\textcircled{c2} &\lesssim_q \sum_{Q \in \mathcal{W}_4} \int_Q \left(\sum_{S \cap 2Q \neq \emptyset} |f_{S^*} - f_{Q^*}|^q \int_{B(x, \frac{1}{10}\ell(Q))} \frac{|x-y|^q}{\ell(Q)^q} \frac{1}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\
&\lesssim_s \sum_{Q \in \mathcal{W}_4} \ell(Q)^d \left(\frac{\sum_{S \cap 2Q \neq \emptyset} |f_{S^*} - f_{Q^*}|^q}{\ell(Q)^{sq}} \right)^{\frac{p}{q}} \approx \sum_{Q \in \mathcal{W}_4} \ell(Q)^d \left(\sum_{S \cap 2Q \neq \emptyset} \frac{|f_{S^*} - f_{Q^*}|^q}{D(Q^*, S^*)^{sq}} \right)^{\frac{p}{q}},
\end{aligned}$$

which can be bounded as $\textcircled{c1}$.

Finally, we bound the error term $\textcircled{c3}$, assuming Ω to be a bounded domain. Here we cannot use the cancellation of the partition of the unity anymore. Instead, we will use the L^p norm of f , the Hölder regularity of the bump functions and the fact that all the cubes considered are roughly of the same size:

$$\begin{aligned} \textcircled{c3} &= \sum_{Q \in \mathcal{W}_2 \setminus \mathcal{W}_4} \int_Q \left(\int_{B(x, \frac{\ell_0}{10})} \frac{|\sum_{S \cap 2Q \neq \emptyset} (\psi_S(x) - \psi_S(y)) f_{S^*}|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \\ &\lesssim \sum_{\substack{Q \in \mathcal{W}_2 \\ \ell_0 \leq \ell(Q) \leq 2\ell_0}} \int_Q \sum_{\substack{S \in \mathcal{W}_3 \\ S \cap 2Q \neq \emptyset}} |f_{S^*}|^p \left(\int_{B(x, \frac{\ell_0}{10})} \frac{1}{\ell_0^q |x-y|^{(s-1)q+d}} dy \right)^{\frac{p}{q}} dx \\ &\lesssim_{\varepsilon, \ell_0, q, p} \sum_{\substack{S \in \mathcal{W}_3 \\ \frac{\ell_0}{2} \leq \ell(S) \leq \ell_0}} \|f\|_{L^p(S^*)}^p \lesssim \|f\|_{L^p(\Omega)}^p. \end{aligned}$$

□

Corollary 2.17. *Let Ω be a uniform domain with an admissible Whitney covering \mathcal{W} . Given $1 < p < \infty$, $1 < q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$, we have that $A_{p,q}^s(\Omega) = F_{p,q}^s(\Omega)$, and*

$$\|f\|_{F_{p,q}^s(\Omega)} \approx \|f\|_{A_{p,q}^s(\Omega)} \quad \text{for all } f \in F_{p,q}^s(\Omega).$$

Proof. By Corollary 2.12, given $f \in F_{p,q}^s(\Omega)$ we have that

$$\|f\|_{A_{p,q}^s(\Omega)} \leq \inf_{g \in L^{\max\{p,q\}}: g|_{\Omega} \equiv f} \|g\|_{A_{p,q}^s(\mathbb{R}^d)} \approx \inf_{g: g|_{\Omega} \equiv f} \|g\|_{F_{p,q}^s(\mathbb{R}^d)} = \|f\|_{F_{p,q}^s(\Omega)}.$$

By the Lemma 2.16 we have the converse: for every $f \in A_{p,q}^s(\Omega)$ we have that

$$\|f\|_{F_{p,q}^s(\Omega)} = \inf_{g: g|_{\Omega} \equiv f} \|g\|_{F_{p,q}^s(\mathbb{R}^d)} \leq \|\Lambda_0 f\|_{F_{p,q}^s(\mathbb{R}^d)} \approx \|\Lambda_0 f\|_{A_{p,q}^s(\mathbb{R}^d)} \leq C \|f\|_{A_{p,q}^s(\Omega)}.$$

□

2.4 Equivalent norms with reduction of the integration domain.

Next we present an equivalent norm for $F_{p,q}^s(\Omega)$ in terms of differences but reducing the domain of integration of the inner variable to the shadow of the outer variable in the seminorm $\|\cdot\|_{\tilde{A}_{p,q}^s(\Omega)}$ defined in (2.44).

Lemma 2.18. *Let Ω be a uniform domain with an admissible Whitney covering \mathcal{W} , let $1 < p, q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$. Then, $f \in F_{p,q}^s(\Omega)$ if and only if*

$$\|f\|_{\tilde{A}_{p,q}^s(\Omega)} = \|f\|_{L^p(\Omega)} + \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{\text{Sh}(Q)} \frac{|f(x) - f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty. \quad (2.48)$$

This quantity defines a norm which is equivalent to $\|f\|_{F_{p,q}^s(\Omega)}^p$ and, moreover, we have that $f \in L^{\max\{p,q\}}(\Omega)$.

Proof. Let Ω be an ε -uniform domain. Recall that in (2.44) we defined

$$\|f\|_{\dot{A}_{p,q}^s(\Omega)} = \left(\int_{\Omega} \left(\int_{\Omega} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}.$$

Trivially

$$\|f\|_{\dot{A}_{p,q}^s(\Omega)}^p \gtrsim \sum_{Q \in \mathcal{W}} \int_Q \left(\int_{\text{Sh}(Q)} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx.$$

Next, we will use the seminorm in the duality form

$$\|f\|_{\dot{A}_{p,q}^s(\Omega)} = \sup_{\|g\|_{L^{p'}(L^{q'}(\Omega))} \leq 1} \int_{\Omega} \int_{\Omega} \frac{|f(x) - f(y)|}{|x - y|^{s+\frac{d}{q}}} g(x, y) dy dx. \quad (2.49)$$

Let $g > 0$ be an L_{loc}^1 function with $\|g\|_{L^{p'}(L^{q'}(\Omega))} \leq 1$. Since the shadow of every cube Q contains $2Q$, we just use Hölder's inequality to find that

$$\sum_{Q \in \mathcal{W}} \int_Q \int_{2Q} \frac{|f(x) - f(y)|}{|x - y|^{s+\frac{d}{q}}} g(x, y) dy dx \leq \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{2Q} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \quad (2.50)$$

Therefore, we only need to prove the estimate

$$\sum_{Q,S} \int_Q \int_{S \setminus 2Q} \frac{|f(x) - f(y)|}{|x - y|^{s+\frac{d}{q}}} g(x, y) dy dx \lesssim \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{\text{Sh}(Q)} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}. \quad (2.51)$$

If $x \in Q$, $y \in S \setminus 2Q$, then $|x - y| \approx D(Q, S)$, so we can write

$$\sum_{Q,S} \int_Q \int_{S \setminus 2Q} \frac{|f(x) - f(y)|}{|x - y|^{s+\frac{d}{q}}} g(x, y) dy dx \lesssim \sum_{Q,S} \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q, S)^{s+\frac{d}{q}}} g(x, y) dy dx. \quad (2.52)$$

Since Ω is a uniform domain, for every pair of cubes Q and S in this sum, there exists an admissible chain $[Q, S]$ joining them. Thus, writing $f_Q = \int_Q f dm$ for the mean of f in Q , the right-hand side of (2.52) can be split as follows:

$$\begin{aligned} \sum_{Q,S} \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q, S)^{s+\frac{d}{q}}} g(x, y) dy dx &\leq \sum_{Q,S} \int_Q \int_S \frac{|f(x) - f_Q|}{D(Q, S)^{s+\frac{d}{q}}} g(x, y) dy dx \\ &\quad + \sum_{Q,S} \int_Q \int_S \frac{|f_Q - f_{Q_S}|}{D(Q, S)^{s+\frac{d}{q}}} g(x, y) dy dx \\ &\quad + \sum_{Q,S} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{D(Q, S)^{s+\frac{d}{q}}} g(x, y) dy dx \\ &=: \textcircled{1} + \textcircled{2} + \textcircled{3} \end{aligned} \quad (2.53)$$

The first term can be immediately bounded by the Cauchy-Schwarz inequality. Namely, writing $G(x) = \|g(x, \cdot)\|_{L^{q'}(\Omega)}$, by (1.29) we have that

$$\begin{aligned} \textcircled{1} &\leq \sum_{Q \in \mathcal{W}} \int_Q |f(x) - f_Q| \left(\sum_{S \in \mathcal{W}} \int_S g(x, y)^{q'} dy \right)^{\frac{1}{q'}} \left(\sum_{S \in \mathcal{W}} \frac{\ell(S)^d}{D(Q, S)^{sq+d}} \right)^{\frac{1}{q}} dx \\ &\leq \sum_{Q \in \mathcal{W}} \frac{\int_Q |f(x) - f_Q| G(x) dx}{\ell(Q)^s}. \end{aligned}$$

By Jensen's inequality, $|f(x) - f_Q| \leq \left(\frac{1}{\ell(Q)^d} \int_Q |f(x) - f(y)|^q dy \right)^{\frac{1}{q}}$ and thus, since $\ell(Q) \gtrsim_d |x - y|$ for $x, y \in Q$, we have that

$$\textcircled{1} \lesssim \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_Q \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \|G\|_{L^{p'}}. \quad (2.54)$$

Since $\|G\|_{L^{p'}} = \|g\|_{L^{p'}(L^{q'})} \leq 1$, this finishes this part.

For the second one, for all cubes Q and S we consider the subchain $[Q, Q_S] \subset [Q, S]$. Then

$$\textcircled{2} \leq \sum_{Q,S} \int_Q \int_S \frac{g(x,y)}{D(Q,S)^{s+\frac{d}{q}}} dy dx \sum_{P \in [Q, Q_S]} |f_P - f_{\mathcal{N}(P)}|.$$

Recall that all the cubes $P \in [Q, Q_S]$ contain Q in their shadow and the properties of the Whitney covering grant that $\mathcal{N}(P) \subset 5P$. Moreover, by (1.23) we have that $D(Q, S) \approx D(P, S)$. Thus,

$$\textcircled{2} \lesssim_d \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)| d\zeta d\xi \sum_{Q \in \mathbf{SH}(P)} \int_Q \sum_{S \in \mathcal{W}} \int_S \frac{g(x,y)}{D(P,S)^{s+\frac{d}{q}}} dy dx$$

and, using Hölder's inequality, and by (1.29), we have that

$$\begin{aligned} \textcircled{2} &\lesssim \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)| d\zeta d\xi \sum_{Q \in \mathbf{SH}(P)} \int_Q \left(\int_Q g(x,y)^{q'} dy \right)^{\frac{1}{q'}} \left(\sum_{S \in \mathcal{W}} \frac{\ell(S)^d}{D(P,S)^{sq+d}} \right)^{\frac{1}{q}} dx \\ &\lesssim_{d,s,q} \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)| d\zeta d\xi \sum_{Q \in \mathbf{SH}(P)} \int_Q G(x) dx \frac{1}{\ell(P)^s}. \end{aligned}$$

By (1.28) we have that $\int_{\mathbf{SH}(P)} G(x) dx \lesssim_{d,\varepsilon} \inf_{y \in P} MG(y) \ell(P)^d$, so

$$\begin{aligned} \textcircled{2} &\lesssim \sum_P \int_P \int_{5P} |f(\xi) - f(\zeta)| d\zeta d\xi \int_{5P} MG(\xi) d\xi \frac{\ell(P)^{d-s}}{\ell(P)^{2d}} \\ &\lesssim_{d,p} \sum_P \int_P \left(\int_{5P} |f(\xi) - f(\zeta)|^q d\zeta \right)^{\frac{1}{q}} \ell(P)^{\frac{d}{q}} MG(\xi) d\xi \frac{1}{\ell(P)^{d+s}}. \end{aligned}$$

Note that for $\xi, \zeta \in 5P$, we have that $|\xi - \zeta| \lesssim_d \ell(P)$. Thus, using Hölder's inequality again and the fact that $\|MG\|_{L^{p'}} \lesssim_p \|G\|_{L^{p'}} \leq 1$, we bound the second term by

$$\textcircled{2} \lesssim \sum_P \int_P \left(\int_{5P} \frac{|f(\xi) - f(\zeta)|^q}{|\xi - \zeta|^{sq+d}} d\zeta \right)^{\frac{1}{q}} MG(\xi) d\xi \lesssim \left(\sum_P \int_P \left(\int_{5P} \frac{|f(\xi) - f(\zeta)|^q}{|\xi - \zeta|^{sq+d}} d\zeta \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}. \quad (2.55)$$

Now we face the boundedness of

$$\textcircled{3} = \sum_{Q,S} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{D(Q,S)^{s+\frac{d}{q}}} g(x,y) dy dx.$$

Given two cubes Q and S , we have that for every admissible chain $[Q, S]$ the cubes $Q, S \in \mathbf{SH}(Q_S)$ by Definition 1.15 and $D(Q, S) \approx \ell(Q_S)$ by (1.21). Thus, we can reorder the sum, writing

$$\begin{aligned} \textcircled{3} &\lesssim \sum_R \sum_{Q \in \mathbf{SH}(R)} \sum_{S \in \mathbf{SH}(R)} \int_Q \int_S \frac{|f_R - f(y)|}{\ell(R)^{s+\frac{d}{q}}} g(x, y) dy dx \\ &\leq \sum_R \int_R \sum_{Q \in \mathbf{SH}(R)} \sum_{S \in \mathbf{SH}(R)} \int_Q \int_S \frac{|f(\xi) - f(y)|}{\ell(R)^{s+(1+\frac{1}{q})d}} g(x, y) dy dx d\xi. \end{aligned} \quad (2.56)$$

Using Hölder's inequality, Lemma 1.18 and the fact that for $S \in \mathbf{SH}(R)$ one has $\ell(R) \approx D(S, R)$, we get that

$$\begin{aligned} \textcircled{3} &\lesssim \sum_R \int_R \frac{1}{\ell(R)^{s+(1+\frac{1}{q})d}} \sum_{Q \in \mathbf{SH}(R)} \int_Q \sum_{S \in \mathbf{SH}(R)} \left(\int_S |f(\xi) - f(y)|^q dy \right)^{\frac{1}{q}} \left(\int_S g(x, y)^{q'} dy \right)^{\frac{1}{q'}} dx d\xi \\ &\leq \sum_R \int_R \frac{1}{\ell(R)^{s+(1+\frac{1}{q})d}} \left(\int_{\mathbf{SH}(R)} |f(\xi) - f(y)|^q dy \right)^{\frac{1}{q}} \sum_{Q \in \mathbf{SH}(R)} \int_Q G(x) dx d\xi \\ &\lesssim \sum_R \int_R \left(\int_{\mathbf{SH}(R)} \frac{|f(\xi) - f(y)|^q}{\ell(R)^{sq+d}} dy \right)^{\frac{1}{q}} \frac{1}{\ell(R)^d} MG(\xi) \ell(R)^d d\xi \end{aligned}$$

and, using the Hölder inequality again and the boundedness of the maximal operator in $L^{p'}$, we get

$$\begin{aligned} \textcircled{3} &\lesssim \left(\sum_R \int_R \left(\int_{\mathbf{SH}(R)} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}} \|MG\|_{L^{p'}} \\ &\lesssim \left(\sum_R \int_R \left(\int_{\mathbf{SH}(R)} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}. \end{aligned} \quad (2.57)$$

Thus, by (2.53), (2.54), (2.55) and (2.57), we have that

$$\sum_{Q, S} \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q, S)^{s+\frac{d}{q}}} g(x, y) dy dx \lesssim \left(\sum_R \int_R \left(\int_{\mathbf{SH}(R)} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}.$$

This fact, together with (2.52) proves (2.51) and thus, using (2.49) and (2.50), we get that

$$\|f\|_{A_{p,q}^s(\Omega)} \lesssim_{\varepsilon, s, p, q, d} \|f\|_{\tilde{A}_{p,q}^s(\Omega)}.$$

Finally, by (2.47) we have that $f \in L^{\max\{p, q\}}(\Omega)$. \square

Remark 2.19. Note that we have proven that the homogeneous seminorms are equivalent, that is,

$$\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{\mathbf{SH}(Q)} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \approx \|f\|_{A_{p,q}^s(\Omega)}^p,$$

which improves (2.48).

In some situations we can refine Lemma 2.18.

Lemma 2.20. *Let Ω be a uniform domain with an admissible Whitney covering \mathcal{W} , let $1 < q \leq p < \infty$ and $\max\left\{\frac{d}{p} - \frac{d}{q}, 0\right\} < s < 1$. Then, $f \in F_{p,q}^s(\Omega)$ if and only if*

$$\|f\|_{L^p(\Omega)} + \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{5Q} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} < \infty.$$

Furthermore, this quantity defines a norm which is equivalent to $\|f\|_{F_{p,q}^s(\Omega)}$.

Proof. Arguing as before by duality, we consider a function $g > 0$ with $\|g\|_{L^{p'}(L^q(\Omega))} \leq 1$. Combining (2.54) and (2.55) we know that

$$\sum_{Q,S} \int_Q \int_S \frac{|f(x) - f_{Q_S}|}{D(Q,S)^{s+\frac{d}{q}}} g(x,y) dy dx \lesssim \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{5Q} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}}$$

and, thus, we have

$$\sum_{Q,S} \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q,S)^{s+\frac{d}{q}}} g(x,y) dy dx \approx \left(\sum_{Q \in \mathcal{W}} \int_Q \left(\int_{5Q} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} + \textcircled{3}. \quad (2.58)$$

where

$$\textcircled{3} := \sum_{Q,S} \int_Q \int_S \frac{|f_{Q_S} - f(y)|}{D(Q,S)^{s+\frac{d}{q}}} g(x,y) dy dx \lesssim \sum_R \sum_{Q,S \in \mathbf{SH}(R)} \int_Q \int_S \frac{|f_R - f(y)|}{\ell(R)^{s+\frac{d}{q}}} g(x,y) dy dx$$

by (2.56).

Using Hölder's inequality and Lemma 1.18 we get that

$$\begin{aligned} \textcircled{3} &\lesssim \sum_R \frac{1}{\ell(R)^{s+\frac{d}{q}}} \left(\sum_{S \in \mathbf{SH}(R)} \int_S |f_R - f(y)|^q dy \right)^{\frac{1}{q}} \sum_{Q \in \mathbf{SH}(R)} \int_Q G(x) dx \\ &\lesssim \sum_R \left(\sum_{S \in \mathbf{SH}(R)} \int_S |f_R - f(y)|^q dy \right)^{\frac{1}{q}} \frac{\int_R MG(\xi) d\xi}{\ell(R)^{s+\frac{d}{q}}} \end{aligned}$$

and, using the Hölder inequality again, we get

$$\textcircled{3} \lesssim \left(\sum_R \left(\sum_{S \in \mathbf{SH}(R)} \int_S |f_R - f(y)|^q dy \right)^{\frac{p}{q}} \frac{\ell(R)^d}{\ell(R)^{sp+\frac{dp}{q}}} \right)^{\frac{1}{p}} \|MG\|_{L^{p'}}.$$

By the boundedness of the maximal operator in $L^{p'}$ we have that $\|MG\|_{L^{p'}} \lesssim 1$. Now, given $R, S \in \mathcal{W}$ there exists an admissible chain $[S, R]$, and we can decompose the previous expression as

$$\begin{aligned} \textcircled{3}^p &\lesssim \sum_R \left(\sum_{S \in \mathbf{SH}(R)} \left| \sum_{P \in [S,R]} (f_P - f_{\mathcal{N}(P)}) \frac{\ell(P)^{\frac{s}{q}}}{\ell(P)^{\frac{s}{q}}} \right|^q \ell(S)^d \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}} \quad (2.59) \\ &+ \sum_R \left(\sum_{S \in \mathbf{SH}(R)} \int_S |f_S - f(y)|^q dy \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}} =: \textcircled{3.1} + \textcircled{3.2}, \end{aligned}$$

where we wrote $[S, R] = [S, R] \setminus \{R\}$.

Using Hölder's inequality

$$(3.1) \lesssim \sum_R \left(\sum_{S \in \mathbf{SH}(R)} \sum_{P \in [S, R]} \frac{|f_P - f_{\mathcal{N}(P)}|^q}{\ell(P)^s} \left(\sum_{P \in [S, R]} \ell(P)^{\frac{sq'}{q}} \right)^{\frac{q}{q'}} \ell(S)^d \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}}.$$

But for $S \in \mathbf{SH}(R)$ by Remark 1.17 we have that $\sum_{P \in [S, R]} \ell(P)^{\frac{sq'}{q}} \lesssim \ell(R)^{\frac{sq'}{q}}$. Moreover, by (1.26) there exists a ratio ρ_τ such that for $P \in [S, R]$ we have that $S \in \mathbf{SH}^\tau(P) := \mathbf{SH}_{\rho_\tau}(P)$ and $P \in \mathbf{SH}^\tau(R)$. We also know that $\sum_{S \in \mathbf{SH}^\tau(P)} \ell(S)^d \lesssim \ell(P)^d$, so writing U_P for the union of the neighbors of P , we get

$$(3.1) \lesssim \sum_R \left(\sum_{P \in \mathbf{SH}^\tau(R)} \frac{\left(\int_{U_P} |f(\xi) - f_P| d\xi \right)^q \ell(P)^d}{\ell(P)^s} \right)^{\frac{p}{q}} \ell(R)^{d+\frac{sp}{q}-sp-\frac{dp}{q}}.$$

Recall that $p \geq q$ and, therefore, by Hölder's inequality and (1.24) we have that

$$(3.1) \lesssim \sum_R \sum_{P \in \mathbf{SH}^\tau(R)} \frac{\left(\int_{U_P} |f(\xi) - f_P| d\xi \right)^p \ell(P)^d}{\ell(P)^{\frac{sp}{q}}} \left(\sum_{P \in \mathbf{SH}^\tau(R)} \ell(P)^d \right)^{\left(1-\frac{q}{p}\right)\frac{p}{q}} \ell(R)^{d-\frac{sp}{q}-\frac{dp}{q}} \\ \lesssim \sum_P \frac{\left(\int_{U_P} |f(\xi) - f_P| d\xi \right)^p \ell(P)^d}{\ell(P)^{\frac{sp}{q}}} \sum_{R: P \in \mathbf{SH}^\tau(R)} \ell(R)^{-\frac{sp}{q}} \approx \sum_P \frac{\left(\int_{U_P} |f(\xi) - f_P| d\xi \right)^p \ell(P)^d}{\ell(P)^{sp}}$$

Using Jensen's inequality we get

$$(3.1) \lesssim \sum_P \int_{U_P} \frac{|f(\xi) - f_P|^p}{\ell(P)^{sp}} d\xi, \quad (2.60)$$

and Jensen's inequality again leads to

$$(3.1) \lesssim \sum_P \int_{U_P} \left(\frac{\int_P |f(\xi) - f(\zeta)|^q d\zeta}{\ell(P)^d} \right)^{\frac{p}{q}} \frac{1}{\ell(P)^{sp}} d\xi \lesssim \sum_P \int_P \left(\frac{\int_{5P} |f(\xi) - f(\zeta)|^q d\zeta}{|\xi - \zeta|^{sq+d}} \right)^{\frac{p}{q}} d\xi. \quad (2.61)$$

To bound (3.2) we follow the same scheme. Since $p \geq q$ we have that

$$(3.2) = \sum_R \left(\sum_{S \in \mathbf{SH}(R)} \int_S |f_S - f(y)|^q dy \frac{\ell(S)^{d(1-\frac{q}{p})}}{\ell(S)^{d(1-\frac{q}{p})}} \right)^{\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}} \\ \leq \sum_R \left(\sum_{S \in \mathbf{SH}(R)} \frac{\left(\int_S |f_S - f(y)|^q dy \right)^{\frac{p}{q}}}{\ell(S)^{d(\frac{p}{q}-1)}} \right)^{\frac{q}{p} \cdot \frac{p}{q}} \left(\sum_{S \in \mathbf{SH}(R)} \ell(S)^d \right)^{\left(1-\frac{q}{p}\right)\frac{p}{q}} \ell(R)^{d-sp-d\frac{p}{q}},$$

and, since $\sum_{S \in \mathbf{SH}(R)} \ell(S)^d \approx \ell(R)^d$, reordering and using (1.24) we get that

$$(3.2) \lesssim \sum_S \frac{\left(\int_S |f_S - f(y)|^q dy \right)^{\frac{p}{q}}}{\ell(S)^{d(\frac{p}{q}-1)}} \sum_{R: S \in \mathbf{SH}(R)} \ell(R)^{-sp} \lesssim \sum_S \left(\frac{\int_S |f_S - f(y)|^q dy}{\ell(S)^d} \right)^{\frac{p}{q}} \frac{\ell(S)^d}{\ell(S)^{sp}}.$$

Thus, by Jensen's inequality,

$$(3.2) \lesssim \sum_S \frac{\int_S |f_S - f(y)|^p dy}{\ell(S)^d} \frac{\ell(S)^d}{\ell(S)^{sp}}$$

and, arguing as in (2.60), we get that

$$(3.2) \lesssim \sum_S \int_S \left(\frac{\int_S |f(y) - f(\zeta)|^q d\zeta}{|y - \zeta|^{sq+d}} \right)^{\frac{p}{q}} dy. \quad (2.62)$$

Thus, by (2.58), (2.59), (2.61) and (2.62), we have that

$$\sum_{Q,S} \int_Q \int_S \frac{|f(x) - f(y)|}{D(Q,S)^{s+\frac{d}{q}}} g(x,y) dy dx \lesssim \left(\sum_S \int_S \left(\int_{5S} \frac{|f(\xi) - f(y)|^q}{|\xi - y|^{sq+d}} dy \right)^{\frac{p}{q}} d\xi \right)^{\frac{1}{p}}.$$

This fact, together with (2.49), (2.50) and (2.52) finishes the proof of Lemma 2.20. \square

Remark 2.21. An analogous result to Lemma 2.20 for Besov spaces $B_{p,p}^s$ can be found in [Dyd06, Proposition 5] where it is stated in the case of Lipschitz domains.

Corollary 2.22. Let Ω be a uniform domain. Let $\delta(x) := \text{dist}(x, \partial\Omega)$ for every $x \in \mathbb{C}$.

Given $1 < p < q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$, we have that $A_{p,q}^s(\Omega) = F_{p,q}^s(\Omega)$ and, moreover, for $\rho_1 > 1$ big enough, we have that

$$\|f\|_{F_{p,q}^s(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_{\Omega} \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \quad \text{for all } f \in F_{p,q}^s(\Omega).$$

Given $1 < q \leq p < \infty$ and $0 < s < 1$, we have that $A_{p,q}^s(\Omega) = F_{p,q}^s(\Omega)$ and, moreover, for $0 < \rho_0 < 1$ we have that

$$\|f\|_{F_{p,q}^s(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_{\Omega} \left(\int_{B_{\rho_0 \delta(x)}(x)} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{p}{q}} dx \right)^{\frac{1}{p}} \quad \text{for all } f \in F_{p,q}^s(\Omega).$$

Proof. This comes straight forward from Corollary 2.17, Lemma 2.18 and Lemma 2.20, taking smaller cubes in the Whitney covering if necessary when $\rho_0 \ll 1$. \square

Remark 2.23. In particular, for every $1 < p < \infty$ and $0 < s < 1$ we have that $A_{p,p}^s(\Omega) = B_{p,p}^s(\Omega)$, with

$$\|f\|_{B_{p,p}^s(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_{\Omega} \int_{B_{\rho_0 \delta(x)}(x)} \frac{|f(x) - f(y)|^p}{|x - y|^{sp+d}} dy dx \right)^{\frac{1}{p}} \quad \text{for all } f \in B_{p,p}^s(\Omega).$$

If in addition $s > \frac{d}{p} - \frac{d}{2}$, then $A_{p,2}^s(\Omega) = W^{s,p}(\Omega)$. If $p \geq 2$ we have that

$$\|f\|_{W^{s,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_{\Omega} \left(\int_{B_{\rho_0 \delta(x)}(x)} \frac{|f(x) - f(y)|^2}{|x - y|^{2s+d}} dy \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \quad \text{for all } f \in W^{s,p}(\Omega),$$

and, if $1 < p < 2$, we have that

$$\|f\|_{W^{s,p}(\Omega)} \approx \|f\|_{L^p(\Omega)} + \left(\int_{\Omega} \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} \frac{|f(x) - f(y)|^2}{|x - y|^{2s+d}} dy \right)^{\frac{p}{2}} dx \right)^{\frac{1}{p}} \quad \text{for all } f \in W^{s,p}(\Omega).$$

Chapter 3

Characteristic functions of planar domains

In this chapter we prove that for $p > 2$ the Beurling transform is bounded on $W^{n,p}(\Omega)$ when the boundary of the domain is regular enough. In particular, we will see that if $N \in B_{p,p}^{n-1/p}(\partial\Omega)$, then $\mathcal{B}\chi_\Omega \in W^{n,p}(\Omega)$, in the same spirit of [CT12]. Recall that this result is sharp for $n = 1$ and Lipschitz constant small enough by [Tol13].

Theorem 3.1. *Let $p > 2$, let $n \in \mathbb{N}$ and let Ω be a bounded Lipschitz domain with $N \in B_{p,p}^{n-1/p}(\partial\Omega)$. Then, for every $f \in W^{n,p}(\Omega)$ we have that*

$$\|\mathcal{B}(\chi_\Omega f)\|_{W^{n,p}(\Omega)} \leq C \|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)} \|f\|_{W^{n,p}(\Omega)},$$

where C depends on p , n , $\text{diam}(\Omega)$ and the Lipschitz character of the domain.

The proof will be slightly more tricky since we will need to approximate the boundary of the domain by polynomials instead of straight lines: the derivative of the Beurling transform of the characteristic function of a half-plane is zero (see [CT12]), but the derivative of the Beurling transform of the characteristic function of a domain bounded by a polynomial of degree greater than one is not zero anymore. Using the $T(P)$ -Theorem 2.1 this will suffice to see the boundedness of the Beurling transform in $W^{n,p}(\Omega)$.

Section 3.1 is devoted to present a family of convolution operators in the plane which include the Beurling transform as a particular case. The purpose of the notation that we will introduce at this point, which is not standard, is to simplify the proofs of Chapter 4. In Section 3.2 one finds the definition of some generalized β -coefficients related to Jones and David-Semmes' celebrated betas and an equivalent norm for the Besov spaces introduced in Section 1.2 in terms of the generalized β -coefficients is presented using a result by Dorrnsoro in [Dor85].

From that point, the proof of a quantitative version of Theorem 3.1 begins (see Theorem 3.28). The first step is to study the case of unbounded domains whose boundary can be expressed as the graph of a Lipschitz function. Section 3.3 contains the outline of the proof, reducing it to two lemmas. The first one studies the relation with the β -coefficients and is proven in Section 3.4. The second one, proven in Section 3.5, is about the case where the domain is bounded by the graph of a polynomial, and here one finds the exponential behavior of the bounds for the iterates of the Beurling transform, which entangles the more subtle details of the proof. Section 3.6 stops the flow for a while to relate the beta coefficients with the normal vector. Finally, in Sections 3.7 and 3.8 one finds the quantitative version (in the precise shape that we need in Chapter 4) of Theorem 3.1 for bounded Lipschitz domains using a localization principle and the $T(P)$ -theorem.

3.1 A family of convolution operators in the plane

Definition 3.2. Consider a function $K : \mathbb{C} \setminus \{0\} \rightarrow \mathbb{C}$. For any $f \in L^1_{loc}$ we define

$$T^K f(z) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{C} \setminus B_\varepsilon(z)} K(z-w)f(w) dm(w)$$

as long as the limit exists, for instance, when K is bounded away from 0, $f \in L^1$ and $y \notin \text{supp}(f)$ or when $f = \chi_U$ for an open set U with $y \in U$, $\int_{B_\varepsilon(0) \setminus B_{\varepsilon'}(0)} K dm = 0$ for every $\varepsilon > \varepsilon' > 0$ and K is integrable at infinity. We say that K is the kernel of T^K .

For any multiindex $\gamma \in \mathbb{Z}^2$, we will consider $K^\gamma(z) = z^\gamma = z^{\gamma_1} \bar{z}^{\gamma_2}$ and then we will put shortly $T^\gamma f := T^{K^\gamma} f$, that is,

$$T^\gamma f(z) = \lim_{\varepsilon \rightarrow 0} \int_{\mathbb{C} \setminus B_\varepsilon(z)} (z-w)^\gamma f(w) dm(w) \quad (3.1)$$

as long as the limit exists.

For any operator T and any domain Ω , we can consider $T_\Omega f = \chi_\Omega T(\chi_\Omega f)$.

Example 3.3. As the reader may have observed, the Beurling and the Cauchy transforms are in that family of operators. Namely, when $K(z) = z^{-2}$, that is, for $\gamma = (-2, 0)$, then $\frac{-1}{\pi} T^\gamma$ is the Beurling transform. The operator $\frac{1}{\pi} T^{(-1, 0)}$ coincides with the Cauchy transform.

Consider the iterates of the Beurling transform \mathcal{B}^m for $m > 0$. For every $f \in L^p$ and $z \in \mathbb{C}$ we have

$$\mathcal{B}^m f(z) = \frac{(-1)^m m}{\pi} \lim_{\varepsilon \rightarrow 0} \int_{|z-\tau| > \varepsilon} \frac{(\overline{z-\tau})^{m-1}}{(z-\tau)^{m+1}} f(\tau) dm(\tau) = \frac{(-1)^m m}{\pi} T^{(-m-1, m-1)} f(z). \quad (3.2)$$

That is, for $\gamma = (\gamma_1, \gamma_2)$ with $\gamma_1 + \gamma_2 = -2$ and $\gamma_1 \leq -2$, the operator T^γ is an iteration of the Beurling transform modulo constant (see [AIM09, Section 4.2]), and it maps $L^p(U)$ to itself for every open set U . If $\gamma_2 \leq -2$, then T^γ is an iterate of the conjugate Beurling transform and it is bounded on L^p as well. In both cases T^γ is an admissible convolution Calderón-Zygmund operator of order ∞ (see Definition 1.26).

3.2 Besov norm and beta coefficients

In [Dor85], Dorronsoro introduces a characterization of Besov spaces in terms of the mean oscillation of the functions on cubes, and he uses approximating polynomials to do so. If the polynomials are of degree one, that is straight lines, this definition can be written in terms of a certain sum of David-Semmes betas (see [CT12] for instance). Following the ideas of Dorronsoro in our case we will use higher degree polynomials to approximate the Besov function that we want to consider, giving rise to some generalized betas. The following proposition comes from [Dor85], where it is not explicitly proven. We give a short proof of it for the sake of completeness.

Proposition 3.4. Given a locally integrable function $f : \mathbb{R}^d \rightarrow \mathbb{R}$ and a cube $Q \subset \mathbb{R}^d$, there exists a unique polynomial $\mathbf{R}_Q^n f \in \mathcal{P}^n$ which we will call approximating polynomial of f on Q , such that given any multiindex γ with $|\gamma| \leq n$ one has that

$$\int_Q (\mathbf{R}_Q^n f - f) x^\gamma = 0. \quad (3.3)$$

Remark 3.5. *In case of existence, the approximating polynomial verifies*

$$\sup_{x \in Q} |\mathbf{R}_Q^n f(x)| \leq C_{n,d} \frac{1}{|Q|} \int_Q |f| dm.$$

Proof. By Remark 1.19, for any $P \in \mathcal{P}^n$ we have that

$$\|P\|_{L^\infty(Q)}^2 = \|P^2\|_{L^\infty(Q)} \approx \frac{1}{|Q|} \|P^2\|_{L^1(Q)} = \frac{1}{|Q|} \|P\|_{L^2(Q)}^2.$$

Using the linearity of the integral in (3.3), one has

$$\frac{1}{|Q|} \int_Q |\mathbf{R}_Q^n f|^2 dm = \frac{1}{|Q|} \int_Q \mathbf{R}_Q^n f \cdot f dm.$$

Combining both facts one gets

$$\|\mathbf{R}_Q^n f\|_{L^\infty(Q)}^2 \lesssim \frac{1}{|Q|} \|\mathbf{R}_Q^n f\|_{L^\infty(Q)} \|f\|_{L^1(Q)}.$$

□

Proof of Proposition 3.4. By the Hilbert Projection Theorem, $L^2(Q) = \mathcal{P}^n \oplus (\mathcal{P}^n)^\perp$. Thus, if $f \in L^2(Q)$, we can write $f|_Q = \mathbf{R}_Q^n f + (f|_Q - \mathbf{R}_Q^n f)$ satisfying (3.3).

For general $f \in L^1$, we can define a sequence of functions $\{f_j\}_{j \in \mathbb{N}} \subset L^2(Q)$ such that $|f_j| \leq |f|$ and $f_j \xrightarrow{a.e.} f$. By Remark 3.5 we have that the approximating polynomials $\mathbf{R}_Q^n f_j$ are uniformly bounded on Q by

$$\sup_{x \in Q} |\mathbf{R}_Q^n f_j(x)| \lesssim \frac{1}{|Q|} \int_Q |f_j| dm \leq \frac{1}{|Q|} \int_Q |f| dm.$$

Therefore there exists a convergent subsequence of $\{\mathbf{R}_Q^n f_j\}_j$ in L^1 (and in any other norm). We call $\mathbf{R}_Q^n f$ the limit of one such partial. By the Dominated Convergence Theorem we get (3.3).

To see uniqueness, we observe that if we find two polynomials P_1 and P_2 satisfying (3.3), then

$$\int_Q (P_1 - P_2)P = 0$$

for any $P \in \mathcal{P}^n$. In particular, if we take $P = P_1 - P_2$ we get that $\|P_1 - P_2\|_{L^2(Q)} = 0$. □

Remark 3.6. *Given $P \in \mathcal{P}^n$, a cube Q and $1 \leq p \leq \infty$ we have that*

$$\|f - \mathbf{R}_Q^n f\|_{L^p(Q)} \leq C_{d,n} \|f - P\|_{L^p(Q)}, \quad (3.4)$$

and given any cubes $Q \subset Q'$,

$$\|f - \mathbf{R}_Q^n f\|_{L^p(Q)} \leq C_{d,n} \|f - \mathbf{R}_{Q'}^n f\|_{L^p(Q')}. \quad (3.5)$$

Proof. By means of the triangle inequality and (3.3), we have that for any $P \in \mathcal{P}^n$

$$\|f - \mathbf{R}_Q^n f\|_{L^p(Q)} \leq \|f - P\|_{L^p(Q)} + \|P - \mathbf{R}_Q^n f\|_{L^p(Q)} = \|f - P\|_{L^p(Q)} + \|\mathbf{R}_Q^n(P - f)\|_{L^p(Q)}.$$

Therefore, we use twice Hölder's Inequality and Remark 3.5 to get

$$\begin{aligned} \|f - \mathbf{R}_Q^n f\|_{L^p(Q)} &\leq \|f - P\|_{L^p(Q)} + |Q|^{1/p} \|\mathbf{R}_Q^n(P - f)\|_{L^\infty(Q)} \\ &\lesssim_{n,d} \|f - P\|_{L^p(Q)} + \frac{|Q|^{1/p}}{|Q|} \|P - f\|_{L^1(Q)} \leq 2\|f - P\|_{L^p(Q)}. \end{aligned}$$

The inequality (3.5) is just a consequence of (3.4) replacing P by $\mathbf{R}_{Q'}^n f$. □

Remark 3.7. In the one dimensional case, if f is continuous and I is an interval one can easily see that $f - \mathbf{R}_I^n f$ has $n+1$ zeroes at least. Indeed, if it did not happen, one could find a polynomial $P \in \mathcal{P}^n$ with a simple zero at every point where $f - \mathbf{R}_I^n f$ changes its sign, and no more. Therefore, $(f - \mathbf{R}_I^n f) \cdot P$ would have constant sign and, thus, the integral in (3.3) would not vanish (see Figure 3.1).

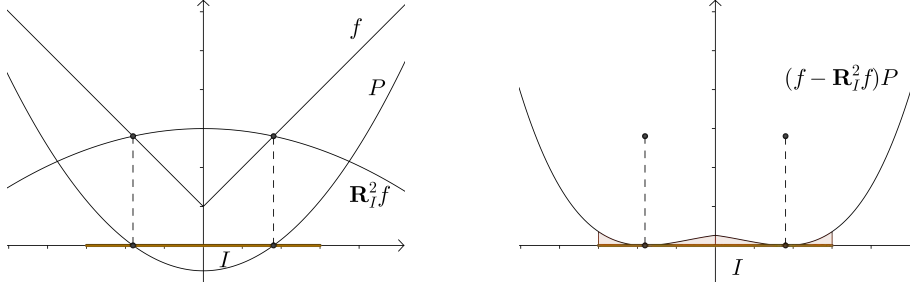


Figure 3.1: If $f - \mathbf{R}_I^2 f$ had only 2 zeroes, there would exist $P \in \mathcal{P}^2$ with $\int (f - \mathbf{R}_I^2 f)P \, dm > 0$.

Now we can define the generalized betas.

Definition 3.8. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be a locally integrable function and $Q \subset \mathbb{R}^d$ a cube. Then we define

$$\beta_{(n)}(f, Q) = \frac{1}{|Q|} \int_{3Q} \frac{|f(x) - \mathbf{R}_{3Q}^n f(x)|}{\ell(Q)} \, dm(x).$$

Remark 3.9. Taking into account (3.4), we can conclude that

$$\beta_{(n)}(f, Q) \approx \inf_{P \in \mathcal{P}^n} \frac{1}{|Q|} \int_{3Q} \frac{|f(x) - P(x)|}{\ell(Q)} \, dm(x).$$

This can be seen as a generalization of David and Semmes β_1 coefficient since $\beta_{(1)}$ and β_1 are comparable as long as some Lipschitz condition is assumed on f .

In [CT12] the authors point out that the seminorm of the homogeneous Besov space $\dot{B}_{p,q}^s$ for $0 < s < 1$ can be defined in terms of the approximating polynomials of degree 1 above. The same can be said for $s \geq 1$. Indeed, [Dor85, Theorem 1] says that for $f \in B_{p,q}^s$ and $n \geq [s]$ we have the equivalent seminorm

$$\|f\|_{\dot{B}_{p,q}^s} \approx \left(\int_0^\infty \frac{\left(\int_{\mathbb{R}^d} \left(\sup_{Q \ni x: |Q|=t^d} \int_Q |f(y) - \mathbf{R}_Q^n f(y)| \, dy \right)^p \, dx \right)^{\frac{q}{p}}}{t^{sq}} \, dt \right)^{\frac{1}{q}}$$

(see (1.17)). Note that, given $x \in \mathbb{R}^d$ and $t \in \mathbb{R}$, every cube $Q \ni x$ with $\ell(Q) = t$ satisfies that $Q \subset Q(x, t)$. Therefore, by (3.5) we have that $\sup_{Q \ni x: |Q|=t^d} \int_Q |f(y) - \mathbf{R}_Q^n f(y)| \, dy \lesssim_{d,n} t \beta_{(n)}(f, Q(x, t))$, and $t \beta_{(n)}(f, Q(x, t)) \lesssim_{d,n} \sup_{Q \ni x: |Q|=(6t)^d} \int_Q |f(y) - \mathbf{R}_Q^n f(y)| \, dy$. Thus,

$$\|f\|_{\dot{B}_{p,q}^s} \approx \left(\int_0^\infty \left(\frac{\|\beta_{(n)}(f, Q(\cdot, t))\|_{L^p}}{t^{s-1}} \right)^q \frac{dt}{t} \right)^{1/q}.$$

In the particular case when $p = q$, which is in fact the one we are interested on, via Fubini's Theorem and (3.5) one can conclude that for the canonical dyadic grid \mathcal{D} , for instance, we have that

$$\begin{aligned} \|f\|_{\dot{B}_{p,p}^s}^p &\approx \sum_{j=-\infty}^{\infty} \int_{2^j}^{2^{j+1}} \int_{\mathbb{R}^d} \left(\frac{\beta_{(n)}(f, Q(x, t))}{t^{s-1}} \right)^p dx \frac{dt}{t} \\ &\approx \sum_{j=-\infty}^{\infty} \sum_{Q \in \mathcal{D}: \ell(Q)=2^j} \left(\frac{\beta_{(n)}(f, Q)}{\ell(Q)^{s-1}} \right)^p |Q| \approx \sum_{Q \in \mathcal{D}} \left(\frac{\beta_{(n)}(f, Q)}{\ell(Q)^{s-1}} \right)^p |Q|. \end{aligned} \quad (3.6)$$

3.3 The case of unbounded domains $\Omega \subset \mathbb{C}$

Definition 3.10. Given $\delta > 0$ and $R > 0$, we say that $\Omega = \{x + iy \in \mathbb{C} : y > A(x)\}$ is a (δ, R, n, p) -admissible domain with defining function A if

- The defining function $A \in B_{p,p}^{n+1-1/p} \cap C^{n-1,1}$.
- We have $A(0) = 0$ and, if $n \geq 2$, $A'(0) = 0$.
- We have Lipschitz bounds on the function and its derivatives $\|A^{(j)}\|_{L^\infty} < \frac{\delta}{R^{j-1}}$ for $1 \leq j \leq n$.

We associate a Whitney covering \mathcal{W} with appropriate constants to Ω . The constants will be fixed along this section, depending on n and δ .

In this Section we will prove the next result for the operators T^γ defined in (3.1).

Theorem 3.11. Consider $\delta, R, \epsilon > 0$, $p > 1$ and a natural number $n \geq 1$. There exists a radius $\rho_\epsilon < R$ such that for every (δ, R, n, p) -admissible domain Ω and every multiindex $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n - 2$ and $\gamma_1 \cdot \gamma_2 \leq 0$, we have that $T^\gamma \chi_\Omega \in L^p(\Omega \cap B(0, \rho_\epsilon))$. In particular, if A is the defining function of Ω , we have that

$$\|T^\gamma \chi_\Omega\|_{L^p(\Omega \cap B(0, \rho_\epsilon))}^p \leq C \left(\|A\|_{\dot{B}_{p,p}^{n-1/p+1}}^p + \rho_\epsilon^{2-np} (1 + \epsilon)^{|\gamma|p} \right),$$

where C depends on p , n and the Lipschitz character of Ω .

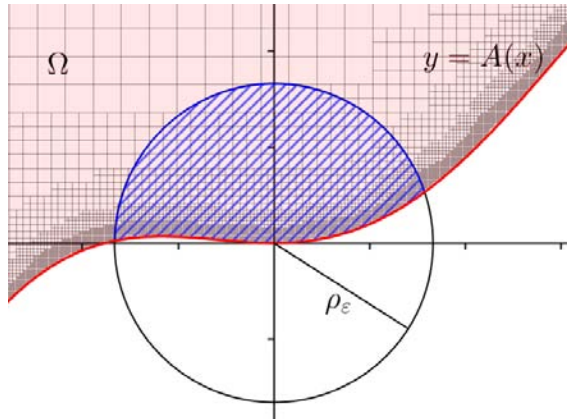


Figure 3.2: Disposition in Theorem 3.11.

Definition 3.12. Consider a given (δ, R, n, p) -admissible domain with defining function A . Then, for every interval I we have an approximating polynomial $\mathbf{R}_{3I}^n := \mathbf{R}_{3I}^n A$, and

$$\beta_{(n)}(I) := \beta_{(n)}(A, I) = \frac{1}{\ell(I)} \int_{3I} \frac{|A(x) - \mathbf{R}_{3I}^n(x)|}{\ell(I)} dx.$$

We call

$$\Omega_I^n := \{x + iy : y > \mathbf{R}_{3I}^n(x)\}.$$

Let $\pi : \mathbb{C} \rightarrow \mathbb{R}$ be the vertical projection (to the real axis) and Q a cube in \mathbb{C} . If $\pi(Q) = I$ we will write $\Omega_Q^n := \Omega_I^n$.

Remark 3.13. Note that π sends dyadic cubes of \mathbb{C} to dyadic intervals of \mathbb{R} and, in particular, any dyadic interval has a finite number of pre-images in the Whitney covering \mathcal{W} of Ω uniformly bounded by a constant depending on δ and the Whitney constants of \mathcal{W} .

Proof of Theorem 3.11. By (3.6) we have that

$$\sum_{I \in \mathcal{D}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-1/p}} \right)^p \ell(I) \approx \|A\|_{\dot{B}_{p,p}^{n-1/p+1}}^p,$$

so it is enough to prove that

$$\|T^\gamma \chi_\Omega\|_{L^p(\Omega \cap B(0, \rho_\epsilon))}^p \leq C \left(\sum_{I \in \mathcal{D}_\epsilon} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-1/p}} \right)^p \ell(I) + \rho_\epsilon^{2-np} (1 + \epsilon)^{|\gamma|p} \right), \quad (3.7)$$

where \mathcal{D}_ϵ stands for $\{I \in \mathcal{D} : \ell(I) \leq 2\rho_\epsilon \text{ and } D(\{0\}, I) \leq 3\rho_\epsilon\}$.

We begin the proof by some basic observations. Let $j_1, j_2 \in \mathbb{Z}$ such that $j_2 \neq j_1 + 1$. Then, the line integral

$$\int_{\partial \mathbb{D}} w^{j_1} \bar{w}^{j_2} dw = i \int_0^{2\pi} e^{i\theta(j_1 - j_2 + 1)} d\theta = 0 \quad (3.8)$$

so, as long as $j_2 > 0$, given $0 < \epsilon < 1$ Green's formula (1.12) says that

$$\int_{\mathbb{D} \setminus B(0, \epsilon)} w^{j_1} \bar{w}^{j_2 - 1} dm(w) = \frac{i}{2j_2} \int_{\partial \mathbb{D} \cup \partial B(0, \epsilon)} w^{j_1} \bar{w}^{j_2} dw = 0. \quad (3.9)$$

Consider a given $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n - 2$ and assume that $\gamma_2 \geq 0$ (the case $\gamma_1 \geq 0$ can be proven mutatis mutandis). Consider a Whitney cube Q and $z \in B(0, \rho_\epsilon) \cap Q$. Then by (3.9) we have that

$$\begin{aligned} |T^\gamma \chi_\Omega(z)| &= \left| \int_{|z-w| > \ell(Q)} (w-z)^\gamma \chi_\Omega(w) dm(w) \right| \\ &\leq \left| \int_{|z-w| > \ell(Q)} (w-z)^\gamma \chi_{\Omega_Q^n}(w) dm(w) \right| + \int_{|z-w| > \ell(Q)} \frac{|\chi_{\Omega_Q^n}(w) - \chi_\Omega(w)|}{|w-z|^{n+2}} dm(w). \end{aligned} \quad (3.10)$$

If we have taken appropriate Whitney constants, then we also have that $\ell(Q) < \text{dist}(Q, \partial \Omega_Q^n)$ (see Remark 3.5) and, thus, by (3.9) again, we have that

$$\int_{|z-w| > \ell(Q)} (w-z)^\gamma \chi_{\Omega_Q^n}(w) dm(w) = T^\gamma \chi_{\Omega_Q^n}(z). \quad (3.11)$$

We will see in Section 3.5 that the following claim holds.

Claim 3.14. *There exists a radius ρ_ϵ (depending on δ , R , n and ϵ) such that for every $z \in B(0, \rho_\epsilon)$ with $z \in Q \in \mathcal{W}$, we have that*

$$|T^\gamma \chi_{\Omega_Q^n}(z)| \lesssim_n \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n}. \quad (3.12)$$

The last term in (3.10) will bring the beta coefficients into play. Recall that we defined the symmetric difference of two sets A_1 and A_2 as $A_1 \Delta A_2 := (A_1 \cup A_2) \setminus (A_1 \cap A_2)$. For our choice of the Whitney constants we have that $2Q \subset \Omega_Q^n \cap \Omega$ so

$$\int_{|z-w|>\ell(Q)} \frac{|\chi_{\Omega_Q^n}(w) - \chi_\Omega(w)|}{|w-z|^{n+2}} dm(w) = \int_{\Omega_Q^n \Delta \Omega} \frac{1}{|w-z|^{n+2}} dm(w). \quad (3.13)$$

Next we split the domain of integration in vertical strips. Namely, if we call $S_j = \{w \in \mathbb{C} : |\operatorname{Re}(w-z)| \leq 2^j \ell(Q)\}$ for $j \geq 0$ and $S_{-1} = \emptyset$, we have that

$$\begin{aligned} \int_{\Omega_Q^n \Delta \Omega} \frac{1}{|w-z|^{n+2}} dm(w) &= \sum_{j \geq 0: 2^j \ell(Q) \leq \rho_\epsilon} \int_{(\Omega_Q^n \Delta \Omega) \cap S_j \setminus S_{j-1}} \frac{dm(w)}{|w-z|^{n+2}} + \int_{|w-z| > \rho_\epsilon/2} \frac{dm(w)}{|w-z|^{n+2}} \\ &\lesssim \sum_{j \geq 0: 2^j \ell(Q) \leq \rho_\epsilon} |(\Omega_Q^n \Delta \Omega) \cap S_j| \frac{1}{(2^{j-1} \ell(Q))^{n+2}} + \frac{1}{\rho_\epsilon^n}. \end{aligned} \quad (3.14)$$

We will see in Section 3.4 the following:

Claim 3.15. *We have that*

$$|(\Omega_Q^n \Delta \Omega) \cap S_j| \lesssim_n \sum_{\substack{I \in \mathcal{D} \\ \pi(Q) \subset I \subset 2^{j+1} \pi(Q)}} \frac{\beta_{(n)}(I)}{\ell(I)^{n-1}} (2^j \ell(Q))^{n+1}. \quad (3.15)$$

Summing up, plugging (3.11) and (3.12) in the first term of the right-hand side of (3.10) and plugging (3.13), (3.14) and (3.15) in the other term, we get

$$|T^\gamma \chi_\Omega(z)| \lesssim_n \sum_{j \geq 0} \sum_{\substack{I \in \mathcal{D} \\ 2^j \ell(Q) \leq \rho_\epsilon \\ \pi(Q) \subset I \subset 2^{j+1} \pi(Q)}} \frac{\beta_{(n)}(I)}{\ell(I)^{n-1}} (2^j \ell(Q))^{n+1} \frac{1}{(2^j \ell(Q))^{n+2}} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n}.$$

Note that the intervals I in the previous sum are in $\mathcal{D}_\epsilon = \{I \in \mathcal{D} : \ell(I) \leq 2\rho_\epsilon \text{ and } D(\{0\}, I) \leq 3\rho_\epsilon\}$. Reordering and computing,

$$|T^\gamma \chi_\Omega(z)| \lesssim_n \sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^{n-1}} \sum_{\substack{j \in \mathbb{N}_0 \\ I \subset 2^{j+1} \pi(Q)}} \frac{1}{2^j \ell(Q)} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n} \lesssim \sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n}.$$

Raising to power p , integrating in Q and adding we get that for ρ_ϵ small enough

$$\begin{aligned} \|T^\gamma \chi_\Omega\|_{L^p(\Omega \cap B(0, \rho_\epsilon))}^p &\lesssim_n \sum_{\substack{Q \in \mathcal{W} \\ Q \cap B(0, \rho_\epsilon) \neq \emptyset}} |Q| \left(\sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} + \frac{(1+\epsilon)^{|\gamma|}}{\rho_\epsilon^n} \right)^p \\ &\lesssim_p \sum_{\substack{Q \in \mathcal{W} \\ Q \cap B(0, \rho_\epsilon) \neq \emptyset}} |Q| \left(\sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} \right)^p + \rho_\epsilon^{2-np} (1+\epsilon)^{|\gamma|p}. \end{aligned} \quad (3.16)$$

Regarding the double sum, we use Hölder Inequality to find that

$$\begin{aligned}
\sum_{\substack{Q \in \mathcal{W} \\ Q \cap B(0, \rho_\epsilon) \neq \emptyset}} |Q| \left(\sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \frac{\beta_{(n)}(I)}{\ell(I)^n} \right)^p &\leq \sum_{Q \in \mathcal{W}} |Q| \sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{2p}}} \right)^p \left(\sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \frac{1}{\ell(I)^{\frac{p'}{2p}}} \right)^{\frac{p}{p'}} \\
&\lesssim_p \sum_{Q \in \mathcal{W}} \ell(Q)^2 \sum_{\substack{I \in \mathcal{D}_\epsilon \\ \pi(Q) \subset I}} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{2p}}} \right)^p \ell(Q)^{-\frac{1}{2}} \\
&\leq \sum_{I \in \mathcal{D}_\epsilon} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{2p}}} \right)^p \sum_{\substack{Q \in \mathcal{W} \\ \pi(Q) \subset I}} \ell(Q)^{\frac{3}{2}} \lesssim_{\mathcal{W}} \sum_{I \in \mathcal{D}_\epsilon} \left(\frac{\beta_{(n)}(I)}{\ell(I)^{n-\frac{1}{p}}} \right)^p \ell(I),
\end{aligned} \tag{3.17}$$

where the constant in the last inequality depends on the maximum number of Whitney cubes that can be projected to a given interval, depending only on δ and n .

Thus, by (3.16) and (3.17) we have proven (3.7) when $\gamma_2 \geq 0$. The case $\gamma_2 \leq 0$ can be proven analogously. \square

3.4 Beta-coefficients step in

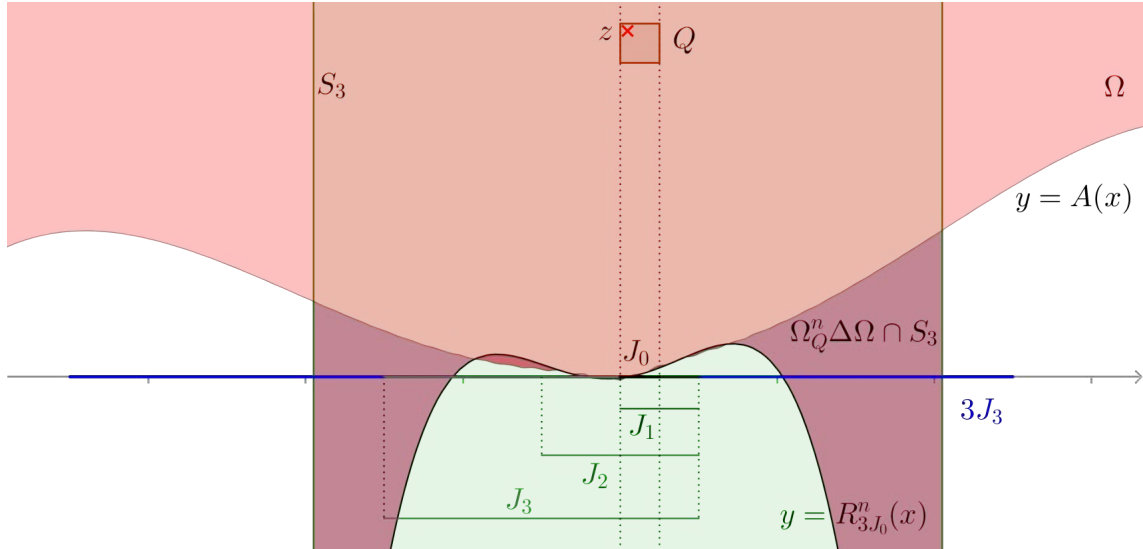


Figure 3.3: Disposition in the proof of Claim 3.15.

Proof of Claim 3.15. Consider $N \geq 0$. Recall that we have a point $z \in Q \in \mathcal{W}$, and a vertical strip $S_N = \{w \in \mathbb{C} : |\operatorname{Re}(w - z)| \leq 2^N \ell(Q)\}$. Let $J_0 = \pi(Q)$ and let J_N be the dyadic interval of length $2^N \ell(Q)$ containing J_0 . Then it is enough to see that

$$|(\Omega_Q^n \Delta \Omega) \cap S_N| \lesssim_n \sum_{\substack{I \in \mathcal{D} \\ J_0 \subset I \subset J_N}} \beta_{(n)}(I) \frac{\ell(J_N)^{n-1}}{\ell(I)^{n-1}} \ell(J_N)^2. \tag{3.18}$$

First note that

$$\begin{aligned} |(\Omega_Q^n \Delta \Omega) \cap S_N| &= \int_{\operatorname{Re}(z) - \ell(J_N)}^{\operatorname{Re}(z) + \ell(J_N)} |A - \mathbf{R}_{3J_0}^n| dm_1 \\ &\leq \int_{3J_N} |A - \mathbf{R}_{3J_N}^n| dm_1 + \int_{3J_N} |\mathbf{R}_{3J_N}^n - \mathbf{R}_{3J_0}^n| dm_1 = \boxed{\mathbb{1}} + \boxed{\mathbb{2}}. \end{aligned} \quad (3.19)$$

Trivially,

$$\boxed{\mathbb{1}} = \beta_{(n)}(J_N) \ell(J_N)^2. \quad (3.20)$$

To deal with the second term, we consider the chain of dyadic intervals

$$J_0 \subset \cdots \subset J_k \subset J_{k+1} \subset \cdots \subset J_N,$$

with $0 < k < N$ and $\ell(J_k) = 2^k \ell(J_0)$. We use the Triangle Inequality in the chain of intervals:

$$\boxed{\mathbb{2}} \leq \sum_{k=0}^{N-1} \int_{3J_N} |\mathbf{R}_{3J_{k+1}}^n - \mathbf{R}_{3J_k}^n| dm_1 = \sum_{k=0}^{N-1} \left\| \mathbf{R}_{3J_{k+1}}^n - \mathbf{R}_{3J_k}^n \right\|_{L^1(3J_N)}. \quad (3.21)$$

Fix $0 \leq k < N$. By Remark 1.19 we have that

$$\left\| \mathbf{R}_{3J_{k+1}}^n - \mathbf{R}_{3J_k}^n \right\|_{L^1(3J_N)} \lesssim_n \left\| \mathbf{R}_{3J_{k+1}}^n - \mathbf{R}_{3J_k}^n \right\|_{L^1(3J_k)} \frac{\ell(J_N)^{n+1}}{\ell(J_k)^{n+1}},$$

with constants depending only on n . Thus, by Remark 3.6

$$\begin{aligned} \left\| \mathbf{R}_{3J_{k+1}}^n - \mathbf{R}_{3J_k}^n \right\|_{L^1(3J_N)} &\lesssim_n \left(\left\| \mathbf{R}_{3J_{k+1}}^n - A \right\|_{L^1(3J_k)} + \left\| A - \mathbf{R}_{3J_k}^n \right\|_{L^1(3J_k)} \right) \frac{\ell(J_N)^{n+1}}{\ell(J_k)^{n+1}} \\ &\lesssim_n (\beta_{(n)}(J_{k+1}) + \beta_{(n)}(J_k)) \frac{\ell(J_N)^{n+1}}{\ell(J_k)^{n+1}} \ell(J_k)^2. \end{aligned} \quad (3.22)$$

Thus, combining (3.19), (3.20), (3.21) and (3.22) we get (3.18). \square

3.5 Domains which are bounded by the graph of a polynomial

We will consider only very “flat” polynomials. Let us see what we can say about their coefficients.

Lemma 3.16. *Let $n \geq 2$, $A \in C^{n-1,1}(\mathbb{R})$ with $A(0) = 0$, $A'(0) = 0$, $\|A^{(j)}\|_{L^\infty} < \frac{\delta}{R^{j-1}}$ for $j \leq n$ and consider two intervals J and I with $3J \subset I = [-R, R]$. Then we have the following bounds for the derivatives of the approximating polynomial $P = \mathbf{R}_J^n A$ in the interval I :*

$$\left\| P^{(j)} \right\|_{L^\infty(I)} \leq \frac{3^{n-j} \delta}{R^{j-1}} \quad \text{for } j \leq n.$$

Furthermore, if $\rho > 0$ and $3J \subset [-\rho, \rho]$, then

$$\|P\|_{L^\infty(-\rho, \rho)} \leq \frac{3^n \delta \rho^2}{R} \quad \text{and} \quad \|P'\|_{L^\infty(-\rho, \rho)} \leq \frac{3^{n-1} \delta \rho}{R}. \quad (3.23)$$

Proof. By Remark 3.7 we know that there are at least $n + 1$ common points $\tau_0^0, \dots, \tau_n^0 \in 3J$ for A and P , that is, $A(\tau_j^0) = P(\tau_j^0)$ for every j . By the Mean Value Theorem, there are n common points $\tau_0^1, \dots, \tau_{n-1}^1 \in 3J$ for their derivatives. By induction we find points $\tau_0^k \dots \tau_{n-k}^k \in 3J$ where the k -th derivatives coincide for $0 \leq k \leq n-1$, that is, $A^{(k)}(\tau_j^k) = P^{(k)}(\tau_j^k)$ for every $0 \leq j \leq n-k$.

Note that the polynomial derivative $P^{(n)}$, which is in fact a constant, coincides with the differential quotient of $P^{(n-1)}$ evaluated at any pair of points. In particular given $x \in \mathbb{R}$, for the points τ_0^{n-1} and τ_1^{n-1} we have that

$$\left| P^{(n)}(x) \right| = \left| \frac{P^{(n-1)}(\tau_0^{n-1}) - P^{(n-1)}(\tau_1^{n-1})}{\tau_0^{n-1} - \tau_1^{n-1}} \right| = \left| \frac{A^{(n-1)}(\tau_0^{n-1}) - A^{(n-1)}(\tau_1^{n-1})}{\tau_0^{n-1} - \tau_1^{n-1}} \right| \leq \frac{\delta}{R^{n-1}}.$$

Now we argue by induction again. Assume that $\|P^{(j+1)}\|_{L^\infty(I)} \leq 3^{n-j-1}\delta/R^j$ for a certain $j \leq n-1$. Consider $x \in I$ and, by the Mean Value Theorem, there exists a point ξ such that $|P^{(j)}(x) - P^{(j)}(\tau_0^j)| = |P^{(j+1)}(\xi)||x - \tau_0^j|$. Thus, since $P^{(j)}(\tau_0^j) = A^{(j)}(\tau_0^j)$ we have that

$$|P^{(j)}(x)| \leq |P^{(j+1)}(\xi)||x - \tau_0^j| + |A^{(j)}(\tau_0^j)| \leq \frac{3^{n-j-1}\delta}{R^j} 2R + \frac{\delta}{R^{j-1}} = \frac{3^{n-j}\delta}{R^{j-1}}.$$

We have not used yet the fact that $A'(0) = A(0) = 0$. Let us fix $\rho \leq R$ and assume that $3J \subset [-\rho, \rho]$. Then for every $x \in [-\rho, \rho]$, we can write $A'(x) = A'(x) - A'(0)$ so

$$|A'(x)| \leq \|A''\|_{L^\infty(I)} |x| \leq \frac{\delta}{R} \rho, \quad (3.24)$$

and we can also write $P'(x) = P'(x) - P'(\tau_0^1) + A'(\tau_0^1) - A'(0)$, so

$$|P'(x)| \leq \|P''\|_{L^\infty(I)} |x - \tau_0^1| + \|A''\|_{L^\infty(I)} |\tau_0^1| \leq \frac{3^{n-2}\delta}{R} 2\rho + \frac{\delta}{R} \rho \leq \frac{3^{n-1}\delta\rho}{R}.$$

By the same token, and using the estimate (3.24) on A' , we get

$$|P(x)| \leq \|P'\|_{L^\infty([-\rho, \rho])} |x - \tau_0^0| + \|A'\|_{L^\infty([-\rho, \rho])} |\tau_0^0| \leq \frac{3^{n-1}\delta\rho}{R} 2\rho + \frac{\delta\rho}{R} \rho \leq \frac{3^n\delta\rho^2}{R}.$$

□

Now we can prove Claim 3.14. Recall that we want to find a radius $\rho_{int} < R$ depending on ϵ such that every point z contained in a Whitney cube $Q \subset B(0, \frac{\rho_{int}}{2})$ satisfies (3.12), that is,

$$|T^\gamma \chi_{\Omega_Q^n}(z)| \lesssim_n \frac{(1 + \epsilon)^{|\gamma|}}{\rho_{int}^n},$$

where $\gamma \in \{(-j_1, j_2) : j_1, j_2 \in \mathbb{N}_0 \text{ and } j_1 - j_2 = n + 2\}$ (recall that we assumed that $\gamma_2 \geq 0$). According to the previous lemma, when $n \geq 2$ we are dealing with a domain Ω_Q^n whose boundary is the graph of a polynomial $P(x) = \sum_{j=0}^n a_j x^j$ such that

$$\begin{aligned} |a_0| &= |P(0)| \leq \frac{3^n \delta \rho_{int}^2}{R}, \\ |a_1| &= |P'(0)| \leq \frac{3^{n-1} \delta \rho_{int}}{R} \quad \text{and} \\ |a_j| &= \frac{|P^{(j)}(0)|}{j!} \leq \frac{3^{n-j} \delta}{j! R^{j-1}} \quad \text{for } 2 \leq j < n. \end{aligned} \quad (3.25)$$

We call $\Omega_P := \{x + iy : y > P(x)\}$ to such a domain. Note that (3.23) implies that for ρ_{int} small enough the polynomial P is “flat”, namely $|P(x)| < \frac{\rho_{int}}{4}$ for $|x| < \rho_{int}$.

One can think of the “exterior” radius ρ_{ext} below as a geometric version of ϵ , namely $\rho_{ext} = (\epsilon/16)^2$. Further, we can assume that $\rho_{ext} < R$.

Proposition 3.17. *Consider two real numbers $\delta, R > 0$ and $n \geq 2$. For ρ_{ext} small enough, there exists $0 < \rho_{int} < \rho_{ext}$ depending also on n, δ and R such that for all $j_1, j_2 \in \mathbb{N}_0$ with $j_1 - j_2 = n + 2$, all $P \in \mathcal{P}^n$ satisfying (3.25), all $z \in Q(0, \rho_{int}) \cap \Omega_P$ and $0 < \epsilon < \text{dist}(z, \partial\Omega_P)$ we have*

$$\left| \int_{\Omega_P \setminus B(z, \epsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w) \right| \leq \frac{C_n}{\rho_{int}^n} \left(1 + 16\rho_{ext}^{1/2}\right)^{j_2}, \quad (3.26)$$

with C_n depending only on n .

If $n = 1$ instead, then for all $j_1, j_2 \in \mathbb{N}_0$ with $j_1 - j_2 = 3$ and all $P \in \mathcal{P}^1$ we have that

$$\int_{\Omega_P \setminus B(z, \epsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w) = 0. \quad (3.27)$$

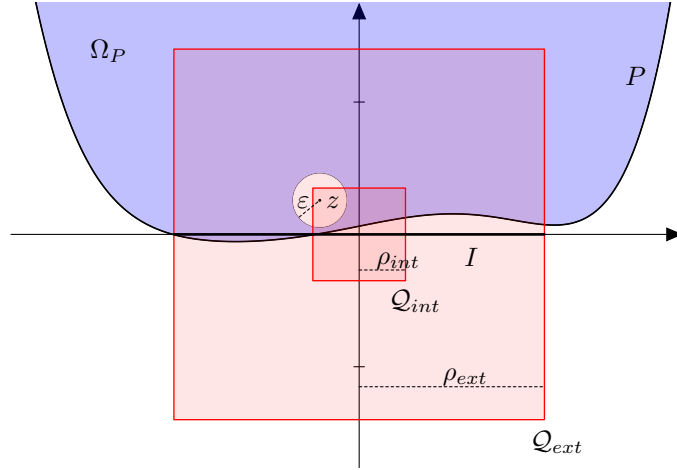


Figure 3.4: Disposition in Proposition 3.17.

Proof. First consider $n = 1$. In that case, Ω_P is a half plane. By rotation and dilation, we can assume $\Omega_P = \mathbb{R}_+^2 := \{w = x + iy : y > 0\}$. Note that $\frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}}$ is infinitely many times differentiable with respect to w in any ring centered in z . Then we can apply Green’s formula (1.12) and use the decay at infinity of the integrand and (3.8) to see that for $\epsilon > 0$ small enough

$$\begin{aligned} \int_{\mathbb{R}_+^2 \setminus B(z, \epsilon)} \frac{(\overline{z-w})^{j_1-3}}{(z-w)^{j_1}} dm(w) &= c_{j_1} \int_{\mathbb{R}} \frac{(\overline{z-w})^{j_1-3}}{(z-w)^{j_1-1}} d\bar{w} = c_{j_1} \int_{\mathbb{R}} \frac{(\overline{z-w})^{j_1-3}}{(z-w)^{j_1-1}} dw \\ &= c_{j_1} \int_{\mathbb{R}_+^2 \setminus B(z, \epsilon)} \frac{(\overline{z-w})^{j_1-4}}{(z-w)^{j_1-1}} dm(w). \end{aligned}$$

When $j_1 = 3$ the last constant is zero. By induction, all these integrals equal zero.

Now we assume that $n \geq 2$. Consider a given $\rho_{ext} > 0$. We define the interval $I := [-\rho_{ext}, \rho_{ext}]$, the exterior window $\mathcal{Q}_{ext} := Q(0, \rho_{ext})$, and the interior window $\mathcal{Q}_{int} := Q(0, \rho_{int})$. Note that

(3.25) implies that for ρ_{ext} small enough, the set $\{x + iP(x) : x \in I\} \subset \mathcal{Q}_{ext}$, that is, the boundary $\partial\Omega_P$, intersects the vertical sides of the window \mathcal{Q}_{ext} but does not intersect the horizontal ones. The same can be said for the sides of \mathcal{Q}_{int} (see Figure 3.4).

Fix $z \in \mathcal{Q}_{int}$ and $\varepsilon < \text{dist}(z, \partial\Omega)$. Splitting the domain of integration in two regions we get

$$\int_{\Omega_P \setminus B(z, \varepsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w) = \int_{\Omega_P \setminus \mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w) + \int_{\Omega_P \cap \mathcal{Q}_{ext} \setminus B(z, \varepsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w). \quad (3.28)$$

We bound the non-local part trivially by taking absolute values and using polar coordinates. Choosing $\rho_{int} < \rho_{ext}/2$, we have that

$$\int_{\Omega_P \setminus \mathcal{Q}_{ext}} \frac{1}{|z-w|^{j_1-j_2}} dm(w) \leq \int_{\frac{\rho_{ext}}{2}}^{\infty} \frac{1}{r^{j_1-j_2}} \int_0^1 dm_1 2\pi r dr = \frac{2\pi}{j_1-j_2-2} \frac{2^{j_1-j_2-2}}{(\rho_{ext})^{j_1-j_2-2}}, \quad (3.29)$$

where dm_1 stands for the Lebesgue length measure. Note that $j_1 - j_2 - 2 = n$.

To bound the local part, we can apply Green's Theorem again and we get

$$\begin{aligned} \frac{2(j_1-1)}{i} \int_{\Omega_P \cap \mathcal{Q}_{ext} \setminus B(z, \varepsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w) &= - \int_{|z-w|=\varepsilon} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\bar{w} \\ &\quad - \int_{\Omega_P \cap \partial\mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\bar{w} \\ &\quad + \int_{\partial\Omega_P \cap \mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\bar{w}. \end{aligned} \quad (3.30)$$

The first term in the right-hand side of (3.30) is zero arguing as in (3.8). For the second term we note that $z \in \mathcal{Q}_{int}$, and every w in the integration domain is in $\partial\mathcal{Q}_{ext}$, so $|z-w| > \rho_{ext} - \rho_{int}$. Thus,

$$\int_{\Omega_P \cap \partial\mathcal{Q}_{ext}} \frac{1}{|z-w|^{j_1-j_2-1}} d\bar{w} \leq \frac{1}{|\rho_{ext} - \rho_{int}|^{j_1-j_2-1}} 6\rho_{ext}. \quad (3.31)$$

Summing up, by (3.28), (3.29), (3.30) and (3.31), since $\rho_{int} < \frac{\rho_{ext}}{2}$, we get that

$$\left| \int_{\Omega_P \setminus B(z, \varepsilon)} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1}} dm(w) \right| \leq \left| \int_{\partial\Omega_P \cap \mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\bar{w} \right| + \frac{C_n}{\rho_{ext}^n}, \quad (3.32)$$

with C_n depending only on n .

It remains to bound the first term in the right-hand side of (3.32). We begin by using the change of coordinates $w = x + iP(x)$ to get a real variable integral:

$$\int_{\partial\Omega_P \cap \mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\bar{w} = \int_I \frac{(\bar{z} - (x - iP(x)))^{j_2}}{(z - (x + iP(x)))^{j_1-1}} (1 - iP'(x)) dx. \quad (3.33)$$

Note that the denominator on the right-hand side never vanishes because $z \notin \partial\Omega_P$. Now we take a closer look to the fraction in order to take as much advantage of cancellation as we can, namely

$$\begin{aligned} \frac{(\bar{z} - (x - iP(x)))^{j_2}}{(z - (x + iP(x)))^{j_1-1}} &= \frac{\left((\bar{z} - z + 2iP(x)) + (z - (x + iP(x))) \right)^{j_2}}{(z - (x + iP(x)))^{j_1-1}} \\ &= \sum_{j=0}^{j_2} \binom{j_2}{j} (\bar{z} - z + 2iP(x))^j (z - (x + iP(x)))^{j_2-j-j_1+1} \\ &= \sum_{j=0}^{j_2} \binom{j_2}{j} \frac{(-2i \text{Im}(z) + 2iP(x))^j}{(z - (x + iP(x)))^{n+1+j}}. \end{aligned} \quad (3.34)$$

Next, we complexify the right-hand side of (3.34) so that we have a holomorphic function in a certain neighborhood of I to be able to change the integration path. To do this change we need a key observation. If $\tau \in \mathcal{Q}_{ext}$, then $|\tau| < \sqrt{2}\rho_{ext}$ and by (3.25) writing $\tilde{\delta} = 3^n \delta$ we have that

$$|P'(\tau)| \leq |a_1| + 2|a_2||\tau| + \dots \leq \tilde{\delta} \left(\frac{\rho_{int}}{R} + \frac{2}{R} 2\rho_{ext} + \frac{3}{R^2} (2\rho_{ext})^2 + \dots \right) < 1/2 \quad (3.35)$$

if ρ_{ext} is small enough. Thus, we have that $\operatorname{Re}(1 + iP'(\tau)) > \frac{1}{2}$ in \mathcal{Q}_{ext} and, by the Complex Rolle Theorem 1.10, we can conclude that $\tau \mapsto \tau + iP(\tau)$ is injective in \mathcal{Q}_{ext} . In particular, $z - (\tau + iP(\tau))$ has one zero at most in \mathcal{Q}_{ext} , and this zero is not real because $z \notin \partial\Omega_P$. Therefore, since the real line divides \mathcal{Q}_{ext} in two congruent open rectangles, there is one of them whose closure has a neighborhood containing no zeros of this function. We call this open rectangle \mathcal{R} . Now, for any $j \geq 0$ we have that $\tau \mapsto \frac{(P(\tau) - \operatorname{Im}(z))^j}{(z - (\tau + iP(\tau)))^{n+1+j}} (1 - iP'(\tau))$ is holomorphic in \mathcal{R} , so we can change the path of integration and get

$$\int_I \frac{2^j (P(x) - \operatorname{Im}(z))^j}{(z - (x + iP(x)))^{n+1+j}} (1 - iP'(x)) dx = - \int_{\partial\mathcal{R} \setminus I} \frac{2^j (P(\tau) - \operatorname{Im}(z))^j}{(z - (\tau + iP(\tau)))^{n+1+j}} (1 - iP'(\tau)) d\tau. \quad (3.36)$$

On the other hand, if $|\tau| < \sqrt{2}\rho_{ext}$, then we have that

$$\begin{aligned} |P(\tau)| &\leq |a_0| + |a_1||\tau| + |a_2||\tau|^2 + |a_3||\tau|^3 + \dots \\ &\leq \tilde{\delta} \left(\frac{\rho_{int}^2}{R} + \frac{\rho_{int}}{R} 2\rho_{ext} + \frac{1}{R} (2\rho_{ext})^2 + \frac{1}{R^2} (2\rho_{ext})^3 + \dots \right) \leq \rho_{ext}^{3/2} \end{aligned} \quad (3.37)$$

for ρ_{ext} small enough. Then, taking absolute values inside the last integral in (3.36) and using (3.35) and (3.37) we get

$$\int_{\partial\mathcal{R} \setminus I} \frac{2^j |P(\tau) - \operatorname{Im}(z)|^j}{|z - (\tau + iP(\tau))|^{n+1+j}} |1 - iP'(\tau)| |d\tau| \leq \frac{3}{2} \int_{\partial\mathcal{R} \setminus I} \frac{2^j (\rho_{ext}^{3/2} + \rho_{int})^j}{|z - (\tau + iP(\tau))|^{n+1+j}} |d\tau|. \quad (3.38)$$

Finally, we have that for any $\tau \in \partial\mathcal{R} \setminus I \subset \partial\mathcal{Q}_{ext}$,

$$|z - (\tau + iP(\tau))| \geq |\tau| - |z| - |P(\tau)| \geq \rho_{ext} - \sqrt{2}\rho_{int} - \rho_{ext}^{3/2} \geq \frac{\rho_{ext}}{2} - 2\rho_{int}$$

for ρ_{ext} small enough. Using this fact we rewrite (3.38) as

$$\int_{\partial\mathcal{R} \setminus I} \frac{2^j |P(\tau) - \operatorname{Im}(z)|^j}{|z - (\tau + iP(\tau))|^{n+1+j}} |1 - iP'(\tau)| |d\tau| \leq \frac{3}{2} \frac{2^j (\rho_{ext}^{3/2} + \rho_{int})^j}{(\rho_{ext}/2 - 2\rho_{int})^{n+1+j}} \int_{\partial\mathcal{R} \setminus I} |d\tau|. \quad (3.39)$$

Putting together (3.33), (3.34), (3.36) and (3.39) we can write

$$\begin{aligned} \left| \int_{\partial\Omega_P \cap \mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\overline{w} \right| &\leq \frac{3}{2(\rho_{ext}/2 - 2\rho_{int})^{n+1}} \sum_{j=0}^{j_2} \left(2 \cdot \frac{\rho_{ext}^{3/2} + \rho_{int}}{\rho_{ext}/2 - 2\rho_{int}} \right)^j \binom{j_2}{j} 4\rho_{ext} \\ &= \frac{6\rho_{ext}}{(\rho_{ext}/2 - 2\rho_{int})^{n+1}} \left(1 + 2 \cdot \frac{\rho_{ext}^{3/2} + \rho_{int}}{\rho_{ext}/2 - 2\rho_{int}} \right)^{j_2}, \end{aligned}$$

and, choosing $\rho_{int} = \min\{\rho_{ext}/8, \rho_{ext}^{3/2}\}$,

$$\left| \int_{\partial\Omega_P \cap \mathcal{Q}_{ext}} \frac{(\overline{z-w})^{j_2}}{(z-w)^{j_1-1}} d\overline{w} \right| \leq \frac{C_n}{\rho_{ext}^n} \left(1 + 16\rho_{ext}^{1/2} \right)^{j_2}, \quad (3.40)$$

where the constant C_n depends only on n .

Now, (3.32) together with (3.40) prove (3.26). \square

Remark 3.18. Note that we have assumed $\gamma_2 \geq 0$ in the proof Theorem 3.11. When proving the case $\gamma_2 \leq 0$, we would have to prove Proposition 3.17 with $\gamma \in \{(j_1, -j_2) : j_1, j_2 \in \mathbb{N}_0 \text{ and } j_2 - j_1 = n + 2\}$. The proof is analogous to the one shown above with slight modifications, and it is left to the reader to complete the details.

3.6 The geometric condition

In this rather technical section we give equivalent expressions of the sufficient geometric condition for a $(\delta, R) - C^{n-1,1}$ domain Ω to satisfy that $\mathcal{B}_{\chi_\Omega} \in W^{n,p}(\Omega)$ (see Theorem 3.27).

Consider a window \mathcal{Q} (see Definition 1.4) of Ω , and its associated parameterization A . In particular, we assume that there is a rigid transformation F which sends the center of \mathcal{Q} to the origin and, in case $n > 1$ we also assume that the tangent to the curve $\partial\Omega$ at 0 is sent to the real axis so that

$$F(\mathcal{Q} \cap \partial\Omega) = \{(x, A(x)) : x \in I_R\},$$

where $I_R = \{-\frac{R}{2} < x < \frac{R}{2}\}$. In case $n = 1$, this assumption is too restrictive since the existence of a tangent to $\partial\Omega$ in the center of \mathcal{Q} is not granted and, therefore, $\|A\|_{L^\infty(I_R)}$ can reach $\delta R/2$ even if we take smaller radius R . Thus, we must use instead $I_R = \{-\frac{R}{2\delta} < x < \frac{R}{2\delta}\}$. We will use an auxiliary bump function φ_R such that $\varphi_R \equiv 1$ in $I_{\frac{R}{3}}$, $\varphi_R \equiv 0$ in I_R^c and

$$\|\varphi_R^{(j)}\|_{L^\infty} \leq C/R^j \quad \text{for every } j \leq n.$$

Thus, we have that $F(\frac{\mathcal{Q}}{3} \cap \partial\Omega)$ is parameterized by $\tilde{A} = \varphi_R A$ (with obvious modifications for $n = 1$).

Theorem 3.19. Let Ω be a bounded $(\delta, R) - C^{n-1,1}$ domain and let $\{\mathcal{Q}_k\}_{k=1}^M$ be a collection of R -windows such that $\left\{\frac{1}{\max\{20, 20\delta\}} \mathcal{Q}_k\right\}_{k=1}^M$ cover the boundary of Ω . Let $\{A_k\}_k$ be the parameterizations of the boundary associated to each window. Consider $N : \partial\Omega \rightarrow \mathbb{R}^2$ to be the unitary outward normal vector. Then, for any $1 < p < \infty$

$$\sum_{k=1}^M \sum_{I \in \mathcal{D}: I \subset I_{\frac{R}{6}}} \frac{\beta_{(n)}(A_k, I)^p}{\ell(I)^{n p - 2}} \lesssim \sum_{k=1}^M \|\varphi_R A_k\|_{\dot{B}_{p,p}^{n+1-1/p}}^p \lesssim \|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p. \quad (3.41)$$

with constants depending on n, p, δ , the length of the boundary $\mathcal{H}^1(\partial\Omega)$ and M . Moreover,

$$\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p \approx_{R, \mathcal{H}^1(\partial\Omega)} \sum_{k=1}^M \|\varphi_R A_k\|_{\dot{B}_{p,p}^{n+1-1/p}}^p + 1. \quad (3.42)$$

The proof of this lemma for $n = 1$ can be found in [CT12, Section 3].

We proceed to prove the general case using the same tools. The expert reader may skip this part. Note that M can be chosen to be $M \approx \frac{\mathcal{H}^1(\partial\Omega)}{R}$.

Proof. We only need to prove the case $n \geq 2$. By (3.6) the first estimate in (3.41) is immediate, while the second one is a consequence of Proposition 3.21 below and (3.42) is a consequence of all this facts and Corollary 3.22. \square

Lemma 3.20. Let $n \in \mathbb{N}$ with $n \geq 2$ and $s = n - 1 + \{s\}$ with $0 \leq \{s\} < 1$. With the notation introduced above, for $1 \leq k \leq M$ we have that

$$\|\varphi_R A_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim \int_{I_R} \int_{I_R} \frac{|A_k^{(n)}(x) - A_k^{(n)}(y)|^p}{|x - y|^{\{s\}p+1}} dy dx + 1.$$

Proof. Let us write fix a window \mathcal{Q}_k , and take $\tilde{A}_k = \varphi_R A_k$. First of all, note that, since $A_k^{(n-1)}$ is Lipschitz, for almost every $x \in I_R$

$$\tilde{A}_k^{(n)}(x) = \sum_{j=0}^n \binom{n}{j} A_k^{(j)}(x) \varphi_R^{(n-j)}(x).$$

Thus, using the lifting property for homogeneous Besov spaces (see [Tri83, Theorem 5.2.3/1]), since $s = n - 1 + \{s\}$ and $0 \leq \{s\} < 1$, we have that

$$\begin{aligned} \|\tilde{A}_k\|_{\dot{B}_{p,p}^{s+1}}^p &\approx \|\tilde{A}_k^{(n)}\|_{\dot{B}_{p,p}^{\{s\}}}^p \lesssim \sum_{j=0}^n \|A_k^{(j)} \varphi_R^{(n-j)}\|_{\dot{B}_{p,p}^{\{s\}}}^p \\ &\approx \sum_{j=0}^n \int_{\mathbb{R} \times \mathbb{R}} \frac{|A_k^{(j)}(y) \varphi^{(n-j)}(y) - A_k^{(j)}(x) \varphi^{(n-j)}(x)|^p}{|y-x|^{\{s\}p+1}} dy dx. \end{aligned}$$

But $\varphi_R \equiv 0$ in I_R^c , so we can reduce the integration domain and, using the symmetry between x and y , it is enough to consider the case $x \in I_R$, that is,

$$\|\tilde{A}_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim \sum_{j=0}^n \int_{I_R} \int_{\mathbb{R}} \frac{|\Delta_h (A_k^{(j)} \varphi^{(n-j)})(x)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dx$$

and, since

$$|\Delta_h(fg)(x)| \leq |g(x)| |\Delta_h f(x)| + |f(x+h)| |\Delta_h g(x)|, \quad (3.43)$$

we get

$$\begin{aligned} \|\tilde{A}_k\|_{\dot{B}_{p,p}^{s+1}}^p &\lesssim \sum_{j=0}^n \int_{I_R} \int_{\mathbb{R}} |A_k^{(n-j)}(x)| \frac{|\Delta_h \varphi^{(j)}(x)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dx + \int_{I_R} \int_{\mathbb{R}} |\varphi^{(n-j)}(x+h)| \frac{|\Delta_h A_k^{(j)}(x)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dx \\ &\lesssim \sum_{j=0}^n \frac{1}{R^{(n-j-1)p}} \int_{I_R} \int_{\mathbb{R}} \frac{|\Delta_h \varphi^{(j)}(x)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dx + \frac{1}{R^{(n-j)p}} \int_{I_R} \int_{I_R} \frac{|A_k^{(j)}(x) - A_k^{(j)}(y)|^p}{|x-y|^{\{s\}p+1}} dy dx \\ &\lesssim \sum_{j=0}^n \left(\textcircled{\text{I.j}} + \textcircled{\text{II.j}} \right). \end{aligned}$$

We take a closer look to the summands $\textcircled{\text{I.j}}$ and we separate the integral by the size of h ,

$$\textcircled{\text{I.j}} = \frac{1}{R^{(n-j-1)p}} \left(\int_{I_R} \int_{|h| < 3R} \frac{|\Delta_h \varphi^{(j)}(x)|^p}{|h|^{\{s\}p+1}} dh dx + \int_{I_R} \int_{|h| > 3R} \frac{|\Delta_h \varphi^{(j)}(x)|^p}{|h|^{\{s\}p+1}} dh dx \right).$$

Now we apply the Mean Value Theorem to the local part and we bound by the supremum in the non-local one to get

$$\begin{aligned} \textcircled{\text{I.j}} &\leq \frac{1}{R^{(n-j-1)p}} \left(\|\varphi^{(j+1)}\|_{L^\infty}^p \int_{|h| < 3R} |h|^{p(1-\{s\})-1} dh \int_{I_R} dx + \int_{I_R} \int_{|h| > 3R} \frac{|\varphi^{(j)}(x) - 0|^p}{|h|^{\{s\}p+1}} dh dx \right) \\ &\lesssim \frac{1}{R^{(n-j-1)p}} \left(\|\varphi^{(j+1)}\|_{L^\infty}^p R^{p-\{s\}p} R + \|\varphi^{(j)}\|_{L^\infty}^p R \frac{1}{R^{\{s\}p}} \right) \lesssim R^{-jp-\{s\}p+1-(n-j-1)p} = R^{1-sp}, \end{aligned}$$

that is, $\textcircled{\text{I.j}}$ are just *error* terms for $0 \leq j \leq n$.

When $j < n$, using the Lipschitz character of Ω , we have that

$$\begin{aligned} \textcircled{\text{II.j}} &:= \frac{1}{R^{(n-j)p}} \int_{I_R} \int_{I_R} \frac{|A_k^{(j)}(x) - A_k^{(j)}(y)|^p}{|x-y|^{\{s\}p+1}} dy dx \leq \frac{1}{R^{(n-j)p}} \int_{I_R} \int_{I_R} \frac{\delta^p}{R^{jp}} \frac{|x-y|^p}{|x-y|^{\{s\}p+1}} dy dx \\ &\lesssim \frac{1}{R^{np}} R^{p(1-\{s\})} \int_{I_R} dx \approx R^{1-sp}, \end{aligned}$$

that is, $\textcircled{\text{II.j}}$ are also *error* terms for $0 \leq j < n$. Summing up,

$$\|\tilde{A}_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim \textcircled{\text{II.n}} + R^{1-sp},$$

and the lemma follows. \square

Recall that we defined the arc parameter of the curve, $z : \mathbb{R} \rightarrow \partial\Omega$ with $|z'(t)| = 1$ and $z(t + \mathcal{H}^1(\partial\Omega)) = z(t)$ for every t and the auxiliary bump function $\varphi_\Omega : \mathbb{R} \rightarrow \mathbb{R}$ such that $\varphi_\Omega|_{2I} \equiv 1$ and $\varphi_\Omega|_{(4I)^c} \equiv 0$ where $I = (-\mathcal{H}^1(\partial\Omega)/2, \mathcal{H}^1(\partial\Omega)/2)$. Then, since $\|N\|_{L^p(\partial\Omega)}^p = \mathcal{H}^1(\partial\Omega)$, using (1.18), (1.16) and the lifting property, we get

$$\|N\|_{B_{p,p}^s(\partial\Omega)}^p \approx \|(N \circ z)\varphi_\Omega\|_{L^p}^p + \|(N \circ z)\varphi_\Omega\|_{\dot{B}_{p,p}^s}^p \approx \mathcal{H}^1(\partial\Omega) + \left\| [(N \circ z)\varphi_\Omega]^{(n-1)} \right\|_{\dot{B}_{p,p}^{\{s\}}}^p$$

and, by the Leibniz rule, Proposition 1.11 and (3.43), the reader can check with some effort that

$$\begin{aligned} \|N\|_{B_{p,p}^s(\partial\Omega)}^p &\approx \mathcal{H}^1(\partial\Omega) 1 + \sum_{j=0}^{n-1} \left\| (N \circ z)^{(j)} \varphi_\Omega^{(n-1-j)} \right\|_{\dot{B}_{p,p}^{\{s\}}}^p \approx \mathcal{H}^1(\partial\Omega) 1 + \left\| (N \circ z)^{(n-1)} \varphi_\Omega \right\|_{\dot{B}_{p,p}^{\{s\}}}^p \\ &\approx 1 + \int_{\mathbb{R}} \int_{\mathbb{R}} \frac{|\Delta_h((N \circ z)^{(n-1)}\varphi_\Omega)(t)|^p}{|h|^{\{s\}p}} dt \frac{dh}{|h|}. \end{aligned}$$

Moreover, using (3.43) again, Fubini and the periodicity of z , we get

$$\begin{aligned} \|N\|_{B_{p,p}^s(\partial\Omega)}^p &\lesssim 1 + \int_{\mathbb{R}} |(N \circ z)^{(n-1)}(t)|^p \int_{\mathbb{R}} \frac{|\Delta_h \varphi_\Omega(t)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dt + \int_{\mathbb{R}} \int_{4I-t} \frac{|\Delta_h(N \circ z)^{(n-1)}(t)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dt \\ &\lesssim 1 + \int_I \int_{2I} \frac{|\Delta_h(N \circ z)^{(n-1)}(t)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dt + \left\| (N \circ z)^{(n-1)} \right\|_{L^p(I)}^p \lesssim \|N\|_{B_{p,p}^s(\partial\Omega)}^p. \quad (3.44) \end{aligned}$$

Proposition 3.21. *With the notation introduced above,*

$$\sum_{k=1}^M \|\varphi_R A_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim \|N\|_{B_{p,p}^s(\partial\Omega)}^p.$$

Proof. Let us write fix a window \mathcal{Q}_k . By Lemma 3.20, it only remains to bound

$$\textcircled{\text{II.n}} = \int_{I_R} \int_{I_R} \frac{|A_k^{(n)}(x) - A_k^{(n)}(y)|^p}{|x-y|^{\{s\}p+1}} dy dx.$$

To do so, we need to relate $\Delta_h A_k^{(n)}(x)$ and $\Delta_h N_k^{(n-1)}(x)$, where N_k is the unit vector in $(x, A_k(x))$ normal to the graph of A_k defined as

$$N_k(x) = (N_{k,1}(x), N_{k,2}(x)) = g_k(x)(A_k'(x), -1),$$

with

$$\begin{aligned} g_k(x) &= \frac{1}{\sqrt{1 + A'_k(x)^2}} \text{ and, thus,} \\ g'_k(x) &= -\frac{A''_k(x)A'_k(x)}{\sqrt{1 + A'_k(x)^2}^3} = -A''_k(x)A'_k(x)g_k(x)^3, \\ &\dots \end{aligned}$$

First we note the trivial pointwise bounds of the derivatives of g_k . The first two bounds are obvious and the rest of them can be deduced by induction,

$$\begin{aligned} |g_k(x)| &= \left| \frac{1}{\sqrt{1 + A'_k(x)^2}} \right| \leq 1, \\ |g'_k(x)| &= |A''_k(x)A'_k(x)g_k(x)^3| \leq \frac{\delta^2}{R}, \\ &\dots, \\ |g_k^{(j)}(x)| &\leq \frac{C_\delta}{R^j} \text{ for all } j < n. \end{aligned}$$

Analogously, we have similar bounds for the multiplicative inverse of g_k , $\tilde{g}_k = \frac{1}{g_k}$,

$$\begin{aligned} |\tilde{g}_k(x)| &\leq \sqrt{1 + \delta^2}, \\ |\tilde{g}'_k(x)| &= |g_k(x)A'_k(x)A''_k(x)| \leq \frac{\delta^2}{R}, \\ &\dots, \\ |\tilde{g}_k^{(j)}(x)| &\leq \frac{C_\delta}{R^j} \text{ for every } j < n. \end{aligned}$$

Thus, for the k -th window normal vector

$$\begin{aligned} |N_{k,2}^{(j)}(x)| &= |g_k^{(j)}(x)| \leq \frac{C_\delta}{R^j} \text{ for all } j < n \text{ and} \\ |N_{k,1}^{(j)}(x)| &= \left| \sum_{i=0}^j \binom{j}{i} A_k^{(i+1)}(x)g_k^{(j-i)}(x) \right| \lesssim_{\delta,j} \sum_{i=0}^j \frac{1}{R^i} \frac{1}{R^{j-i}} \approx \frac{1}{R^j} \text{ for all } j < n. \end{aligned}$$

Summing up, we have that

$$\left\| A_k^{(j+1)} \right\|_{L^\infty}, \left\| g_k^{(j)} \right\|_{L^\infty}, \left\| \tilde{g}_k^{(j)} \right\|_{L^\infty}, \left\| N_k^{(j)} \right\|_{L^\infty} \lesssim_{\delta,n} \frac{1}{R^j} \text{ for } j < n. \quad (3.45)$$

Therefore, using the Mean Value Theorem one gets

$$|\Delta_h A_k^{(j)}(x)|, |\Delta_h g_k^{(j-1)}(x)|, |\Delta_h \tilde{g}_k^{(j-1)}(x)|, |\Delta_h N_k^{(j-1)}(x)| \lesssim \frac{|h|}{R^j} \text{ for } j < n. \quad (3.46)$$

Now we want to control $|\Delta_h A_k^{(n)}(x)|$ by an expression in terms of the differences of the derivatives of the normal vector, with $x, x+h \in I_R$. We have that

$$N_{k,1}^{(n-1)}(x) = \sum_{i=0}^{n-1} \binom{n-1}{i} A_k^{(i+1)}(x)g_k^{(n-1-i)}(x).$$

Thus, solving for $A_k^{(n)}(x)$ we get

$$A_k^{(n)}(x) = \frac{N_{k,1}^{(n-1)}(x) - \sum_{i=0}^{n-2} \binom{n-1}{i} A_k^{(i+1)}(x) g_k^{(n-1-i)}(x)}{g_k(x)},$$

and taking differences

$$|\Delta_h A_k^{(n)}(x)| \lesssim \left| \Delta_h \left(N_{k,1}^{(n-1)} \tilde{g}_k \right) (x) \right| + \sum_{i=0}^{n-2} \left| \Delta_h \left(A_k^{(i+1)} g_k^{(n-1-i)} \tilde{g}_k \right) (x) \right|. \quad (3.47)$$

On one hand, using (3.45) and (3.46) we have that

$$\begin{aligned} \left| \Delta_h \left(N_{k,1}^{(n-1)} \tilde{g}_k \right) (x) \right| &\leq \|\tilde{g}_k\|_{L^\infty} \left| \Delta_h N_{k,1}^{(n-1)}(x) \right| + \left\| N_{k,1}^{(n-1)} \right\|_{L^\infty} |\Delta_h \tilde{g}_k(x)| \\ &\lesssim \left| \Delta_h N_{k,1}^{(n-1)}(x) \right| + \frac{1}{R^{n-1}} \frac{|h|}{R}. \end{aligned}$$

On the other hand, if we consider $0 < i \leq n-2$, we obtain analogously

$$\begin{aligned} \left| \Delta_h \left(A_k^{(i+1)} g_k^{(n-1-i)} \tilde{g}_k \right) (x) \right| &\lesssim \frac{1}{R^{n-1-i}} \left| \Delta_h A_k^{(i+1)}(x) \right| + \frac{1}{R^i} \left| \Delta_h g_k^{(n-1-i)}(x) \right| + \frac{1}{R^{n-1}} |\Delta_h \tilde{g}_k(x)| \\ &\lesssim \frac{1}{R^{n-1-i}} \frac{|h|}{R^{i+1}} + \frac{1}{R^i} \frac{|h|}{R^{(n-i)}} + \frac{1}{R^{n-1}} \frac{|h|}{R}. \end{aligned}$$

When $i = 0$, instead, using that $N_{k,2}^{(n-1)}(x) = -g_k^{(n-1)}(x)$, we obtain that

$$\begin{aligned} \left| \Delta_h \left(A'_k g_k^{(n-1)} \tilde{g}_k \right) (x) \right| &\lesssim \frac{1}{R^{n-1}} \left| \Delta_h A'_k(x) \right| + \left| \Delta_h g_k^{(n-1)}(x) \right| + \frac{1}{R^{n-1}} |\Delta_h \tilde{g}_k(x)| \\ &\lesssim \frac{1}{R^{n-1}} \frac{|h|}{R} + \left| \Delta_h N_{k,2}^{(n-1)}(x) \right| + \frac{1}{R^{n-1}} \frac{|h|}{R}. \end{aligned}$$

Back to (3.47), we have deduced that

$$|\Delta_h A_k^{(n)}(x)| \lesssim \left| \Delta_h N_k^{(n-1)}(x) \right| + \frac{|h|}{R^n}.$$

Applying this result, we obtain that

$$\begin{aligned} \textcircled{\text{II.n}} &\lesssim \int_{I_R} \int_{I_R} \frac{\left| N_k^{(n-1)}(x) - N_k^{(n-1)}(y) \right|^p}{|x-y|^{\{s\}p+1}} dy dx + \frac{1}{R^{np}} \int_{-R}^R \frac{|h|^p}{|h|^{\{s\}p+1}} dh \int_{I_R} dx \\ &\lesssim \int_{I_R} \int_{I_R} \frac{\left| N_k^{(n-1)}(x) - N_k^{(n-1)}(y) \right|^p}{|x-y|^{\{s\}p+1}} dy dx + R^{1-sp}. \end{aligned} \quad (3.48)$$

Finally, note that $t = \tau_k(x) = \int_0^x \tilde{g}_k$ is the arc parameter of the curve, since

$$\frac{dx}{dt} = \frac{1}{\tilde{g}_k(x)} = \frac{1}{\sqrt{1 + A'_k(x)^2}}.$$

Thus, we have that $\tilde{N}_k(t) := N_k(\tau_k^{-1}(t))$ is the normal vector parameterized by the arc: according to our definitions, for $t \in \tau_k(I_R)$ we have that $\tilde{N}_k(t) = N \circ z(t + z^{-1}(z_k))$ where $z^{-1}(z_k)$ is assumed

to be chosen in I . That is, \tilde{N}_k is a translation of $N : \partial\Omega \rightarrow S^1$ parametrized by the arc $z : 2I \rightarrow \partial\Omega$ for values close to $z^{-1}(z_k)$. Of course, we have that $N_k(x) = \tilde{N}_k(\tau_k(x))$. Therefore,

$$N'_k(x) = \tilde{N}'_k(\tau_k(x))\tau'_k(x) = \tilde{N}'_k(\tau_k(x))\tilde{g}_k(x)$$

and, by induction, for $j \leq n-1$ we get

$$N_k^{(j)}(x) = \sum_{i=1}^j \tilde{N}_k^{(i)}(\tau_k(x)) \sum_{\substack{\alpha \in \mathbb{N}^i \\ |\alpha|=j-i}} C_\alpha \prod_{l=1}^i \tilde{g}_k^{(\alpha_l)}(x). \quad (3.49)$$

Solving this equation for $\tilde{N}_k^{(j)}$ and using (3.45), for $j \leq n-1$ we have that

$$\left\| \tilde{N}_k^{(j)} \right\|_{L^\infty(\tau_k(I_R))} \leq \frac{1}{R^j}. \quad (3.50)$$

In consequence, taking $t = \tau_k(x)$ and $\tilde{h} = \tau_k(y) - \tau_k(x)$, and applying (3.49), we get

$$\begin{aligned} |N_k^{(n-1)}(y) - N_k^{(n-1)}(x)| &\leq |\Delta_{\tilde{h}} \tilde{N}_k^{(n-1)}(t)| \|\tilde{g}_k\|_{L^\infty}^{n-1} \\ &\quad + \sum_{j=1}^{n-2} |\Delta_{\tilde{h}} \tilde{N}_k^{(j)}(t)| \sum_{\substack{\alpha \in \mathbb{N}^j \\ |\alpha|=n-1-j}} C_\alpha \prod_{i=1}^j \|\tilde{g}_k^{(\alpha_i)}\|_{L^\infty} \\ &\quad + \sum_{j=1}^{n-1} \|\tilde{N}_k^{(j)}\|_{L^\infty} \sum_{\substack{\alpha \in \mathbb{N}^j \\ |\alpha|=n-1-j}} C_\alpha \sum_{i=1}^j \prod_{l \neq i} \|\tilde{g}_k^{(\alpha_l)}(x) - \tilde{g}_k^{(\alpha_l)}(y)\| \|\tilde{g}_k^{(\alpha_i)}\|_{L^\infty}. \end{aligned}$$

Using (3.45), (3.46) and (3.50) we get

$$|\Delta_{\tilde{h}} \tilde{N}_k^{(n-1)}(t)| \|\tilde{g}_k\|_{L^\infty}^{n-1} \lesssim |\Delta_{\tilde{h}} \tilde{N}_k^{(n-1)}(t)|,$$

for all $j \leq n-2$ and $|\alpha| = n-1-j$ we get

$$|\Delta_{\tilde{h}} \tilde{N}_k^{(j)}(t)| \prod_{i=1}^j \|\tilde{g}_k^{(\alpha_i)}\|_{L^\infty} \lesssim |\tilde{h}| \|\tilde{N}_k^{(j+1)}\|_{L^\infty} \prod_{i=1}^j \frac{1}{R^{\alpha_i}} \lesssim \frac{|\tilde{h}|}{R^{j+1+|\alpha|}} = \frac{|\tilde{h}|}{R^n}$$

and, for all $j \leq n-1$, $|\alpha| = n-1-j$, we get

$$\|\tilde{N}_k^{(j)}\|_{L^\infty} \sum_{i=1}^j \prod_{l \neq i} \|\tilde{g}_k^{(\alpha_l)}(x) - \tilde{g}_k^{(\alpha_l)}(y)\| \|\tilde{g}_k^{(\alpha_i)}\|_{L^\infty} \lesssim \frac{1}{R^j} \frac{|x-y|}{R^{\alpha_i+1}} \frac{1}{R^{|\alpha|-\alpha_i}} \approx \frac{|\tilde{h}|}{R^n}.$$

Thus,

$$|N_k^{(n-1)}(x) - N_k^{(n-1)}(y)| \lesssim |\Delta_{\tilde{h}} \tilde{N}_k^{(n-1)}(t)| + \frac{|\tilde{h}|}{R^n}.$$

Therefore, using the bilipschitz change of variables $t = \tau_k(x)$ and $\tilde{h} = \tau_k(y) - \tau_k(x)$ in (3.48), we have that

$$\begin{aligned} \textcircled{\text{II.n}} &\lesssim \int_{I_R} \int_{I_R} \frac{|N_k^{(n-1)}(x) - N_k^{(n-1)}(y)|^p}{|x-y|^{\{s\}p+1}} dy dx + R^{1-sp} \\ &\lesssim \int_{\tau_k(I_R)} \int_{t+\tau_k(I_R)} \left(\frac{|\Delta_{\tilde{h}} \tilde{N}_k^{(n-1)}(t)|^p}{|\tilde{h}|^{\{s\}p+1}} + \frac{|\tilde{h}|^p}{R^{np} |\tilde{h}|^{\{s\}p+1}} \right) d\tilde{h} dt + R^{1-sp}. \quad (3.51) \end{aligned}$$

Taking sums on $1 \leq k \leq M$ and using Lemma 3.20, (3.48) and (3.51) we get

$$\sum_{k=1}^M \|\varphi_R A_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim \sum_{k=1}^M \left(\int_{\tau_k(I_R)} \int_{t+\tau_k(I_R)} \frac{|\Delta_{\tilde{h}} \tilde{N}_k^{(n-1)}(t)|^p}{|\tilde{h}|^{\{s\}p+1}} d\tilde{h} dt + R^{1-sp} \right) + R^{1-sp}.$$

Recall that $\tilde{N}_k(t)$ is a translation of the vector N parameterized by the arc. Namely, $\tilde{N}_k^{(n-1)}(t) = (N \circ z)^{(n-1)}(t + z^{-1}(z_k))$ and

$$\sum_{k=1}^M \|\varphi_R A_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim \sum_{k=1}^M \int_{\tau_k(I_R)} \int_{t+\tau_k(I_R)} \frac{|\Delta_{\tilde{h}}(N \circ z)^{(n-1)}(t + z^{-1}(z_k))|^p}{|\tilde{h}|^{\{s\}p+1}} d\tilde{h} dt + MR^{1-sp},$$

and changing variables,

$$\sum_{k=1}^M \|\varphi_R A_k\|_{\dot{B}_{p,p}^{s+1}}^p \lesssim 1 + \int_I \int_{2I} \frac{|\Delta_h(N \circ z)^{(n-1)}(t)|^p}{|h|^{\{s\}p}} \frac{dh}{|h|} dt \lesssim \|N\|_{\dot{B}_{p,p}^s(\partial\Omega)}^p.$$

□

Corollary 3.22. *We have that*

$$\|N\|_{\dot{B}_{p,p}^s(\partial\Omega)}^p \lesssim \sum_{k=1}^N \|\varphi_R A_k\|_{\dot{B}_{p,p}^{s+1}}^p + 1.$$

Sketch of the proof. Use (3.44), the overlapping of the windows $\frac{1}{20}Q_k$ (with the obvious modification for $n = 1$) and then argue as before for the small values of h . For the remaining part, use the L^p norm of $\nabla^{(n-1)}(N \circ z)$ and Proposition 1.11. □

By the same reasoning, one can also prove the following corollary.

Corollary 3.23. *Let Q_1, Q_2 be two windows of the same size of a Lipschitz domain and consider parameterizations $A_1 : I_1 \rightarrow \mathbb{R}$, $A_2 : I_2 \rightarrow \mathbb{R}$ such that there exist rigid transformations $F_1 : Q_1 \rightarrow \mathbb{C}$ and $F_2 : Q_2 \rightarrow \mathbb{C}$ such that*

$$F_1^{-1}(\{(x, A_1(x)) : x \in I_1\}) = F_2^{-1}(\{(x, A_2(x)) : x \in I_2\}).$$

Assume further that $\ell(I_1) \approx \ell(I_2) \approx R$ and take φ_1, φ_2 to be bump functions such that $\chi_{\frac{1}{3}I_j} \leq \varphi_j \leq \chi_{I_j}$. For $s < n + 1$ we have that

$$\|\varphi_1 A_1\|_{\dot{B}_{p,p}^s}^p + 1 \approx \|\varphi_2 A_2\|_{\dot{B}_{p,p}^s}^p + 1.$$

3.7 A localization principle: bounded smooth domains

We are going to follow a standard localization argument, so we will give a sketch, leaving some details for the reader.

Let us start with some remarks. First we make some general observations on admissible domains. In these first two remarks we assume $n \geq 2$ since the case $n = 1$ is simpler (there is no need for rotations).

Remark 3.24. If Ω is a (δ, R, n, p) -admissible domain with defining function A , then for every $\tau \in \partial\Omega$ one can perform a translation of the domain that sends τ to the origin and a rotation in the same spirit of Definition 1.4, so that $\partial\Omega$ coincides with the graph of a new function $\tilde{A} \in C^{n-1,1}(\mathbb{R})$ in a certain ball $B(0, \tilde{R})$ with fixed radius \tilde{R} (depending on δ and R) with $\tilde{A}(0) = 0$, $\tilde{A}'(0) = 0$, $\|\tilde{A}'\|_{L^\infty} \leq \tilde{\delta}$ and $\text{supp}(\tilde{A}) \subset [-2\tilde{R}, 2\tilde{R}]$. By Corollary 3.23, we have that $\|\tilde{A}\|_{\dot{B}_{p,p}^s} \lesssim \|A\|_{B_{p,p}^s} + 1$ for $s < n + 1$. Therefore \tilde{A} determines a $(\tilde{\delta}, \tilde{R}, n, p)$ -admissible domain $\tilde{\Omega}$ with compactly supported defining function (see Figure 3.5).

Consider $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n - 2$ and $\gamma_1 \cdot \gamma_2 \leq 0$. Note that $\chi_\Omega(z) = \chi_{\tilde{\Omega}}(z)$ for $z \in B(0, \tilde{R})$. For every $z \in \Omega \cap B(0, \frac{\tilde{R}}{2})$ we use the decomposition $T^\gamma \chi_\Omega(z) = T^\gamma \chi_{\tilde{\Omega}}(z) + T^\gamma(\chi_\Omega - \chi_{\tilde{\Omega}})(z)$:

$$|T^\gamma \chi_\Omega(z)| \leq |T^\gamma \chi_{\tilde{\Omega}}(z)| + \int_{|w| > \tilde{R}/2} \frac{|\chi_\Omega(w) - \chi_{\tilde{\Omega}}(w)|}{|w - z|^{n+2}} dm(w) \lesssim |T^\gamma \chi_{\tilde{\Omega}}(z)| + \frac{1}{\tilde{R}^n}. \quad (3.52)$$

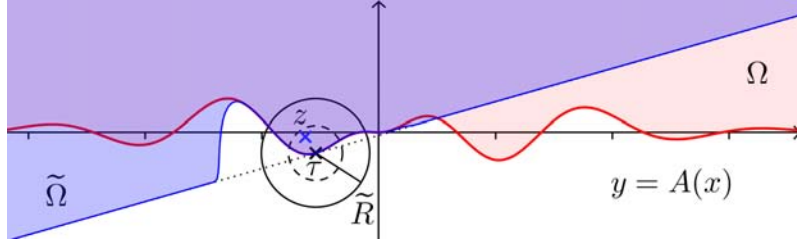


Figure 3.5: Disposition in Remark 3.24 before the rotation and the translation.

Next we take a look at admissible domains with compact support.

Remark 3.25. Let Ω be a (δ, R, n, p) -admissible domain with defining function A supported in $I = [-2R, 2R]$. For a given $\epsilon > 0$ small enough, take ρ to be the radius ρ_ϵ from Theorem 3.11 associated to the parameters $\tilde{\delta}, \tilde{R}, n, p$ of the previous remark. We assume $\rho < \tilde{R}/2$.

Since A is supported in I , we can cover the area close to the graph $\mathcal{G} = \{x + iA(x) : x \in I\}$ by a finite number of balls of radius ρ (see Figure 3.6). In each ball we can apply (3.52) for the corresponding domain $\tilde{\Omega}$. Thus, given $\gamma \in \mathbb{Z}^2$ with $\gamma_1 + \gamma_2 = -n - 2$ and $\gamma_1 \cdot \gamma_2 \leq 0$, writing $U_\rho \mathcal{G} = \bigcup_{z \in \mathcal{G}} B(z, \frac{\rho}{2})$ and using Theorem 3.11 we have that

$$\|T^\gamma \chi_\Omega\|_{L^p(\Omega \cap U_\rho \mathcal{G})}^p \leq C \left(\|A\|_{B_{p,p}^{n+1-1/p}(\partial\Omega)}^p + (1 + \epsilon)^{|\gamma|p} \right),$$

with C depending on n, p, δ, R and ϵ (but not on $|\gamma|$).

Finally, for $z \notin U_\rho \mathcal{G}$ we can use the same argument of (3.52) replacing the domain $\tilde{\Omega}$ by the half plane \mathbb{R}_+^2 . Namely,

$$|T^\gamma \chi_\Omega(z)| \leq |T^\gamma \chi_{\mathbb{R}_+^2}(z)| + \int_{\Omega \cap \Delta \mathbb{R}_+^2} \frac{1}{|w - z|^{n+2}} dm(w).$$

In that case, the first term is zero just by (3.27). Since A is compactly supported in $[-2R, 2R]$ and it is Lipschitz with constant δ , the domain of integration of the second term is contained in $Q(0, 2(1 + \delta)R)$. Thus, when $z \in \Omega \setminus Q(0, 4(1 + \delta)R)$ then $|T^\gamma \chi_\Omega(z)|$ is bounded by a constant times $\frac{R^2}{|z|^{n+2}}$. When $z \in \Omega \cap Q(0, 4(1 + \delta)R) \setminus U_\rho \mathcal{G}$ then $|T^\gamma \chi_\Omega(z)|$ is bounded by $\frac{C}{\rho^n}$. Summing up, we

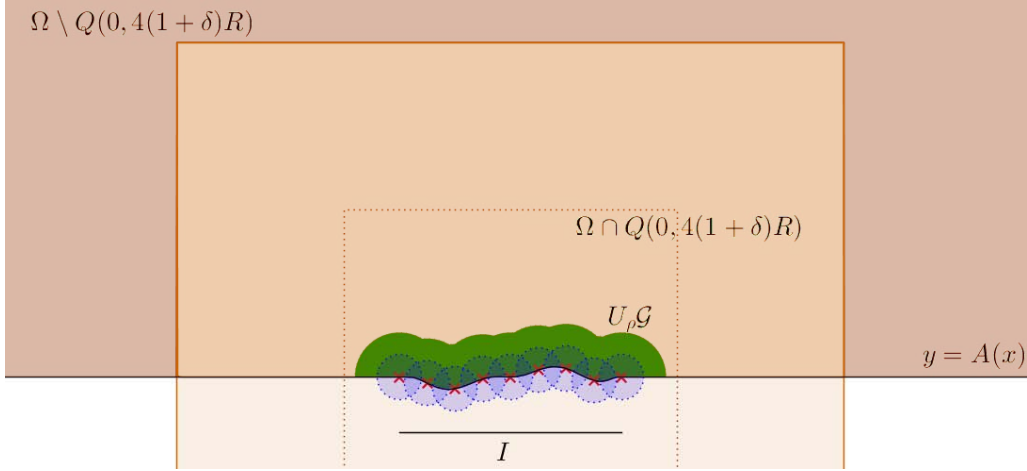


Figure 3.6: Decomposition of a (δ, R, n, p) -admissible domain Ω with defining function A supported in $I = [-2R, 2R]$ considered in Remark 3.25.

have a global bound

$$\|T^\gamma \chi_\Omega\|_{L^p(\Omega)}^p \leq C \left(\|A\|_{B_{p,p}^{n+1-1/p}}^p + (1+\epsilon)^{|\gamma|p} + \frac{R^2}{\rho^{np}} \right) \lesssim \|A\|_{B_{p,p}^{n+1-1/p}}^p + (1+\epsilon)^{|\gamma|p},$$

with constants depending on n, p, δ, R and ϵ .

Now we turn to the case of bounded domains. First we note how differentiation works for $T^\gamma \chi_\Omega$.

Remark 3.26. Consider a bounded $(\delta, R) - C^{(n-1,1)}$ domain Ω and let us fix $\gamma \in \mathbb{Z}^2$ with either $\gamma_1 \geq 0$ or $\gamma_2 \geq 0$, and $\alpha \in \mathbb{N}_0^2$ with modulus $|\alpha| = n$. Then for $z \in \Omega$ we have

$$D^\alpha T_\Omega^\gamma 1(z) = \begin{cases} C_n \chi_\Omega(z) & \text{if } \gamma = (n-1, -1) \text{ and } \alpha = (n, 0) \\ & \text{or } \gamma = (-1, n-1) \text{ and } \alpha = (0, n), \\ 0 & \text{if } \alpha_1 > \gamma_1 \geq 0 \text{ or } \alpha_2 > \gamma_2 \geq 0 \text{ except in the previous case,} \\ C_{\gamma, \alpha} T_\Omega^{\gamma-\alpha} 1(z) & \text{otherwise,} \end{cases}$$

where D^α stands for the weak derivative in Ω . The constants satisfy $|C_{\gamma, \alpha}| \lesssim (|\gamma| + n)^n$.

Proof. Let us assume that $\gamma_2 \geq 0$. If $\gamma_1 \geq 0$ as well, differentiating a polynomial under the integral sign makes the proof trivial, so we assume $\gamma_1 \leq -1$. Recall that we write $w^\gamma = w^{\gamma_1} \bar{w}^{\gamma_2}$. For every $z \in \Omega$ choose $\varepsilon_z := \text{dist}(z, \partial\Omega)/2$. By (3.9), Green's formula and (3.8) we get that

$$T^\gamma \chi_\Omega(z) = \int_{\Omega \setminus B(z, \varepsilon_z)} (z-w)^\gamma dm(w) = \frac{i}{2(\gamma_2+1)} \int_{\partial\Omega} (z-w)^{\gamma+(0,1)} dw, \quad (3.53)$$

and we can differentiate under the integral sign.

If $\gamma_2 \geq \alpha_2$, then we have

$$D^\alpha T^\gamma \chi_\Omega(z) = \frac{i}{2(\gamma_2+1)} (-1)^{\alpha_1} \frac{(\gamma_2+1)!}{(\gamma_2-\alpha_2+1)!} \frac{(-\gamma_1+\alpha_1-1)!}{(-\gamma_1-1)!} \int_{\partial\Omega} (z-w)^{\gamma-\alpha+(0,1)} dw.$$

Since $\gamma_2 - \alpha_2 \geq 0$ and $\gamma_1 - \alpha_1 < 0$, we can apply (3.53) to $\gamma - \alpha$ instead of γ and, thus,

$$D^\alpha T^\gamma \chi_\Omega(z) = (-1)^{\alpha_1} \frac{(\gamma_2)!}{(\gamma_2 - \alpha_2)!} \frac{(-\gamma_1 + \alpha_1 - 1)!}{(-\gamma_1 - 1)!} T^{\gamma - \alpha} \chi_\Omega(z).$$

If $\gamma_2 + 1 = \alpha_2$ we must pay special attention. In that case differentiating under the integral sign in (3.53) we get

$$\begin{aligned} D^\alpha T^\gamma \chi_\Omega(z) &= \frac{i}{2} (-1)^{\alpha_1} \frac{(\gamma_2)!}{(\gamma_2 - \alpha_2 + 1)!} \frac{(-\gamma_1 + \alpha_1 - 1)!}{(-\gamma_1 - 1)!} \int_{\partial\Omega} (z - w)^{\gamma - \alpha + (0,1)} dw \\ &= C_{\gamma, \alpha} \int_{\partial\Omega} \frac{1}{(z - w)^{-\gamma_1 + \alpha_1}} dw, \end{aligned}$$

where $|C_{\gamma, \alpha}| \lesssim (|\gamma| + n)^n$. If, moreover, $\gamma_1 - \alpha_1 \leq -2$, we can use (3.8) to write

$$D^\alpha T^\gamma \chi_\Omega(z) = C_{\gamma, \alpha} \int_{\partial\Omega \cup \partial B(0, \varepsilon_z)} \frac{1}{(z - w)^{-\gamma_1 + \alpha_1}} dw = C_{\gamma, \alpha} \int_{\Omega \setminus \partial B(0, \varepsilon_z)} 0 dm(w) = 0. \quad (3.54)$$

Otherwise, that is, if $\gamma_2 + 1 = \alpha_2$ and $\gamma_1 - \alpha_1 = -1$, then $\alpha = (0, n)$ and $\gamma = (-1, n - 1)$. This implies that

$$D^\alpha T^\gamma \chi_\Omega(z) = C_n \int_{\partial\Omega} \frac{1}{(z - w)} dw = C_n \chi_\Omega(z), \quad (3.55)$$

with $|C_n| \lesssim (n - 1)!$. Let us remark the fact that $\gamma = (-1, 0)$ together with $\alpha = (0, 1)$ is the case of the $\bar{\partial}$ -derivative of the Cauchy transform, which is the identity.

Finally, if $\gamma_2 < \alpha_2 - 1$, then differentiating (3.54) or (3.55) we get

$$D^\alpha T^\gamma \chi_\Omega(z) = 0.$$

One can argue analogously if $\gamma_1 \geq 0$. □

By the preceding remarks, Theorem 3.19 and other standard arguments, one gets the following theorem.

Theorem 3.27. *Let Ω be a bounded (δ, R) - $C^{n-1,1}$ domain with parameterizations in $B_{p,p}^{n+1-1/p}$. Then, for any $\gamma \in \mathbb{Z}^2 \setminus \{(-1, -1)\}$ with $\gamma_1 + \gamma_2 = -2$, we have that $T^\gamma \chi_\Omega \in W^{n,p}(\Omega)$ and, in particular, for any $\epsilon > 0$, we have that*

$$\|\nabla^n T^\gamma \chi_\Omega\|_{L^p(\Omega)}^p \lesssim C_\epsilon |\gamma|^{np} \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p + (1 + \epsilon)^{|\gamma|p} \right),$$

where C_ϵ depends on p, n, δ, R , the length of the boundary $\mathcal{H}^1(\partial\Omega)$ and ϵ but not on $|\gamma|$.

3.8 Bounded smooth domains, supercritical case

Using Theorems 3.27 and 2.1, we will prove the following theorem.

Theorem 3.28. *Consider $p > 2, n \geq 1$ and let Ω be a bounded Lipschitz domain with parameterizations in $B_{p,p}^{n+1-1/p}$. Then, for every $\epsilon > 0$ there exists a constant C_ϵ such that for every multiindex $\gamma \in \mathbb{Z}^2 \setminus \{(-1, -1)\}$ with $\gamma_1 + \gamma_2 \geq -2$, one has*

$$\|T^\gamma\|_{W^{n,p}(\Omega) \rightarrow W^{n+\gamma_1+\gamma_2+2,p}(\Omega)} \leq C_\epsilon |\gamma|^{n+\gamma_1+\gamma_2+2} \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)} + (1 + \epsilon)^{|\gamma|} \right) + \text{diam}(\Omega)^{\gamma_1+\gamma_2+2}. \quad (3.56)$$

In particular, for every $m \in \mathbb{N}$ we have that the iteration of the Beurling transform $(\mathcal{B}^m)_\Omega$ is bounded on $W^{n,p}(\Omega)$, with norm

$$\|(\mathcal{B}^m)_\Omega\|_{W^{n,p}(\Omega) \rightarrow W^{n,p}(\Omega)} \leq C_\epsilon m^{n+1} \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)} + (1+\epsilon)^m \right). \quad (3.57)$$

Proof. Note that by (1.15), we have that $B_{p,p}^{n+1-1/p} \subset B_{\infty,\infty}^{n+1-2/p}$ and, since $1 - 2/p > 0$, we also have that $B_{\infty,\infty}^{n+1-2/p} = C^{n,1-2/p}$ (see Remark 1.12) so Ω is in fact a (δ, R) - $C^{n-1,1}$ -domain, where δ and R depend on the size of the local parameterizations of the boundary and on $\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}$, and we can use Theorem 3.27.

First we study the case $\gamma_1 + \gamma_2 + 2 = 0$. Consider a given $\gamma \in \mathbb{Z}^2 \setminus \{(-1, -1)\}$ with $\gamma_1 + \gamma_2 = -2$. Recall that for $m \neq 0$, $\mathcal{B}^m = \frac{(-1)^m m}{\pi} T^{(-m-1, m-1)}$ by (3.2). The proof of the L^p boundedness of these operators with norm smaller than $C_p m^2$ can be found in [AIM09, Corollary 4.5.1]. Thus,

$$\|T^\gamma\|_{L^p \rightarrow L^p} = \frac{\pi}{m} \|\mathcal{B}^m\|_{L^p \rightarrow L^p} \lesssim m = \frac{|\gamma|}{2}. \quad (3.58)$$

On the other hand, a short computation shows that the constant C_K in (1.41) for these kernels is

$$C_{K_\gamma} = \sup_{z \neq 0} |\nabla^n K_\gamma(z)| |z|^{j+2} \lesssim |\gamma|^n, \quad (3.59)$$

with constant depending on n .

Following the notation of Theorem 2.1, given a multiindex $\lambda \in \mathbb{N}_0^2$, we write $P_\lambda(z) = z^{\lambda_1} \bar{z}^{\lambda_2}$, that is, $P_\lambda(z) = z^\lambda$. In order to use this theorem, it only remains to check the bounds for $\|D^\alpha T_\Omega^\gamma P_\lambda\|_{L^p(\Omega)}$ for all multiindices $\alpha, \lambda \in \mathbb{N}_0^2$ with $|\alpha| = n$ and $|\lambda| < n$. Using the binomial expansion $w^\lambda = \sum_{\nu \leq \lambda} (-1)^{|\nu|} \binom{\lambda}{\nu} (z-w)^\nu z^{\lambda-\nu}$, we can write

$$T_\Omega^\gamma P_\lambda(z) = \lim_{\epsilon \rightarrow 0} \int_{\mathbb{C} \setminus B_\epsilon(z)} (z-w)^\gamma w^\lambda dm(w) = \sum_{\bar{0} \leq \nu \leq \lambda} (-1)^{|\nu|} \binom{\lambda}{\nu} z^{\lambda-\nu} T^{\gamma+\nu} \chi_\Omega(z).$$

Differentiating (and assuming that $0 \in \Omega$) we find that

$$|\nabla^n T_\Omega^\gamma P_\lambda(z)| \lesssim 2^n \sum_{\bar{0} \leq \nu \leq \lambda} \sum_{j=0}^n (1 + \text{diam}(\Omega))^n |\nabla^j T^{\gamma+\nu} \chi_\Omega(z)|$$

and, thus, by the equivalence of norms in the Sobolev space (1.8), we have that

$$\|\nabla^n T_\Omega^\gamma P_\lambda\|_{L^p(\Omega)}^p \lesssim \sum_{\bar{0} \leq \nu \leq \lambda} \left(\|\nabla^{n+|\nu|} T^{\gamma+\nu} \chi_\Omega\|_{L^p(\Omega)}^p + \|T^{\gamma+\nu} \chi_\Omega\|_{L^p(\Omega)}^p \right),$$

with constants depending on n, p and the diameter and the Sobolev embedding constant of Ω . By Remark 3.26, Theorem 3.27 and (3.58), we have that

$$\|\nabla^n T_\Omega^\gamma P_\lambda\|_{L^p(\Omega)}^p \lesssim \sum_{\gamma \leq \nu \leq \gamma+\lambda} |\nu|^{np} \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p + (1+\epsilon)^{|\nu|p} \right) + \sum_{\gamma \leq \nu \leq \gamma+\lambda} \|T^\nu \chi_\Omega\|_{L^p(\Omega)}^p. \quad (3.60)$$

The Young Inequality (1.10) says that for all functions $f \in L^p$ and $g \in L^1$, $\|f * g\|_{L^p} \leq \|f\|_{L^p} \|g\|_{L^1}$. Thus, for $\gamma < \nu \leq \gamma + \lambda$ we have that

$$\|T_\Omega^\nu f\|_{L^p} \leq \text{diam}(\Omega)^{\nu_1 + \nu_2 + 2} \|f\|_{L^p}, \quad (3.61)$$

and taking $f = \chi_\Omega$, $\|T^\nu \chi_\Omega\|_{L^p}^p \lesssim 1 + \text{diam}(\Omega)^{(n-1)p+2}$. For $\nu = \gamma$, the same holds by the boundedness of the iterates of the Beurling transform with a slightly worse constant. Namely,

$$\|T_\Omega^\gamma f\|_{L^p} \leq C_p |\gamma| \|f\|_{L^p}. \quad (3.62)$$

Since $p > 2$, putting (2.1), (3.58), (3.59), (3.60), (3.61) and (3.62) together, we get

$$\begin{aligned} \|\nabla^n T_\Omega^\gamma\|_{W^{n,p}(\Omega) \rightarrow L^p(\Omega)} &\lesssim C_K + \|T^\gamma\|_{L^p \rightarrow L^p} + \sum_{|\lambda| < n} \|\nabla^n (T_\Omega P_\lambda)\|_{L^p(\Omega)} \\ &\lesssim |\gamma|^n + |\gamma| + |\gamma|^n \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)} + (1+\epsilon)^{|\gamma|} \right) \\ &\lesssim |\gamma|^n \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)} + (1+\epsilon)^{|\gamma|} \right), \end{aligned} \quad (3.63)$$

with constants depending on n, p, δ , the diameter of Ω , its Sobolev embedding constant and ϵ , but not on γ . This proves (3.56) when $\gamma_1 + \gamma_2 = -2$ and (3.57) for every $m > 0$.

It remains to study the operators of homogeneity greater than -2 . In that case we will see that we can differentiate under the integral sign to recover the previous situation. Fix $\gamma \in \mathbb{Z}^2$ such that $\gamma_1 + \gamma_2 + 2 > 0$. By (3.61) we have that $\|T_\Omega^\gamma f\|_{L^p} \leq \text{diam}(\Omega)^{\gamma_1 + \gamma_2 + 2} \|f\|_{L^p}$. Thus, to prove (3.56), it suffices to see that for $f \in W^{n,p}(\Omega)$ we have

$$\|\nabla^{n+\gamma_1+\gamma_2+2} T_\Omega^\gamma f\|_{L^p(\Omega)} \leq C_\epsilon |\gamma|^{(n+\gamma_1+\gamma_2+2)p} \left(\|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p + (1+\epsilon)^{|\gamma|p} \right) \|f\|_{W^{n,p}(\Omega)}.$$

By (3.63) it is enough to check that for any $\nu \in \mathbb{N}_0^2$ with $|\nu| = \gamma_1 + \gamma_2 + 2$, we have

$$D^\nu T_\Omega^\gamma f(z) = \begin{cases} C_n \chi_\Omega f(z) & \text{if } \gamma = (|\nu| - 1, -1) \text{ and } \nu = (|\nu|, 0) \\ & \text{or } \gamma = (-1, |\nu| - 1) \text{ and } \nu = (0, |\nu|), \\ 0 & \text{if } \nu_1 > \gamma_1 \geq 0 \text{ or } \nu_2 > \gamma_2 \geq 0 \text{ except in the previous case,} \\ C_{\nu,\gamma} T_\Omega^{\gamma-\nu} f(z) & \text{otherwise,} \end{cases} \quad (3.64)$$

with $|C_n|, |C_{\nu,\gamma}| \lesssim (|\nu| + |\gamma|)^{|\nu|}$.

To prove this statement, take $\alpha \leq \nu - (1, 0)$, and note that the partial derivative is

$$\begin{aligned} \partial T_\Omega^{\gamma-\alpha} f(z) &= \frac{\partial_x T_\Omega^{\gamma-\alpha} f(z) - i \partial_y T_\Omega^{\gamma-\alpha} f(z)}{2} \\ &= \lim_{h \rightarrow 0} \frac{T_\Omega^{\gamma-\alpha} (f - f(z))(z+h) - T_\Omega^{\gamma-\alpha} (f - f(z))(z)}{2h} \\ &\quad + \lim_{h \rightarrow 0} \frac{T_\Omega^{\gamma-\alpha} (f - f(z))(z+ih) - T_\Omega^{\gamma-\alpha} (f - f(z))(z)}{2ih} + \partial T_\Omega^{\gamma-\alpha} \chi_\Omega(z) f(z) \\ &=: \text{I} + \text{II} + \text{III}, \end{aligned}$$

where h is assumed to be real. Now, the principal value is not needed because $\gamma_1 - \alpha_1 + \gamma_2 - \alpha_2 > -2$, so

$$\text{I} = \lim_{h \rightarrow 0} \int_\Omega \frac{((z+h-w)^{\gamma-\alpha} - (z-w)^{\gamma-\alpha}) [f(w) - f(z)]}{2h} dm(w).$$

Moreover, since $f \in C^{0,\sigma}$ for a certain $\sigma > 0$ by the Sobolev Embedding Theorem, we get

$$\lim_{h \rightarrow 0} \int_{B(z,2|h|)} \frac{(|z+h-w|^{\gamma-\alpha} + |z-w|^{\gamma-\alpha}) |f(w) - f(z)|}{2h} dm(w) = 0.$$

On the other hand, using the Taylor expansion of order two of $(z - w + \cdot)^{\gamma - \alpha}$ around 0, there exists $\varepsilon = \varepsilon(h, w, z)$ with $|\varepsilon| < h$ such that

$$\boxed{\text{I}} = \lim_{h \rightarrow 0} \int_{\Omega \setminus B(z, 2|h|)} \left(\frac{\partial_x (z - w + \cdot)^{\gamma - \alpha}(0)}{2} + \frac{\partial_x^2 (z - w + \cdot)^{\gamma - \alpha}(\varepsilon)h}{2} \right) (f(w) - f(z)) dm(w).$$

Arguing analogously for $\boxed{\text{II}}$, we get that

$$\begin{aligned} \boxed{\text{I}} + \boxed{\text{II}} &= \lim_{h \rightarrow 0} \int_{\Omega \setminus B(z, 2|h|)} (\gamma_1 - \alpha_1)(z - w)^{\gamma - \alpha - (1-0)} f(w) dm(w) \\ &\quad - \lim_{h \rightarrow 0} \int_{\Omega \setminus B(z, 2|h|)} (\gamma_1 - \alpha_1)(z - w)^{\gamma - \alpha - (1-0)} dm(w) f(z) \end{aligned}$$

(when taking limits, the Taylor remainder vanishes by the Hölder continuity of f). If $\gamma_1 - \alpha_1 = 0$ then this part is null and $\boxed{\text{III}}$ will be also null unless $\gamma_2 - \alpha_2 = -1$ by Remark 3.26. Otherwise, the last term coincides with $\boxed{\text{III}}$ and they cancel out. By induction, we get (3.64). \square

Chapter 4

An application to quasiconformal mappings

Let $\mu \in L^\infty$ supported in a certain ball $B \subset \mathbb{C}$ with $k := \|\mu\|_{L^\infty} < 1$ and consider $K := \frac{1+k}{1-k}$. We say that f is a K -quasiregular solution to the Beltrami equation

$$\bar{\partial}f = \mu \partial f \tag{4.1}$$

with Beltrami coefficient μ if $f \in W_{loc}^{1,2}$, that is, if f and ∇f are square integrable functions in any compact subset of \mathbb{C} , and $\bar{\partial}f(z) = \mu(z)\partial f(z)$ for almost every $z \in \mathbb{C}$. Such a function f is said to be a K -quasiconformal mapping if it is a homeomorphism of the complex plane. If, moreover, $f(z) = z + \mathcal{O}(\frac{1}{z})$ as $z \rightarrow \infty$, then we say that f is the principal solution to (4.1).

Given a compactly supported Beltrami coefficient μ , the existence and uniqueness of the principal solution is granted by the measurable Riemann mapping Theorem (see [AIM09, Theorem 5.1.2], for instance). The principal solution can be given by means of the Cauchy and the Beurling transforms. If we call

$$h := (I - \mu\mathcal{B})^{-1}\mu,$$

then

$$f(z) = \mathcal{C}h(z) + z$$

is the principal solution of (4.1) because $\bar{\partial}f = h$ and $\partial f = \mathcal{B}h + 1$.

In this chapter we use the results of the previous one to improve the known results on the relation between the smoothness of a quasiconformal mapping and its Beltrami coefficient.

Theorem 4.1. *Let $n \in \mathbb{N}$, let Ω be a bounded Lipschitz domain with outward unit normal vector N in $B_{p,p}^{n-1/p}(\partial\Omega)$ for some $2 < p < \infty$ and let $\mu \in W^{n,p}(\Omega)$ with $\|\mu\|_{L^\infty} < 1$ and $\text{supp}(\mu) \subset \bar{\Omega}$. Then, the principal solution f to (4.1) is in the Sobolev space $W^{n+1,p}(\Omega)$.*

Section 4.1 is devoted to collecting some results that we will use in this chapter. Then in Section 4.2 we perform the main argument to present the outline of the proof of Theorem 4.1, reducing it to two lemmas, one on the compactness of a commutator which is proven in Section 4.3 and one on the compactness of $\chi_\Omega \mathcal{B}(\chi_\Omega \mathcal{B}^m(\chi_\Omega \cdot))$ which is studied in Section 4.5. Finally, Subsection 4.4 is devoted to establishing a generalization of some results in [MOV09] to be used in the last section.

4.1 Some tools

Let us sum up some properties of the Cauchy transform which will be useful in this section (see [AIM09, Theorems 4.3.10, 4.3.12 and 4.3.14]). We write $I_\Omega g := \chi_\Omega g$ for every $g \in L^1_{loc}$.

Theorem 4.2. *Let $1 < p < \infty$. Then*

- For every $f \in L^p$, we have that $\partial\mathcal{C}f = \mathcal{B}f$ and $\bar{\partial}\mathcal{C}f = f$.
- For every function $f \in L^1$ with compact support, if $p > 2$ or $\int f \, dm = 0$, then we have that

$$\|\mathcal{C}f\|_{L^p} \lesssim_p \text{diam}(\text{supp}(f)) \|f\|_{L^p}. \quad (4.2)$$

- Let Ω be a bounded open subset of \mathbb{C} . Then, we have that

$$I_\Omega \circ \mathcal{C} : L^p(\mathbb{C}) \rightarrow W^{1,p}(\Omega) \quad (4.3)$$

is bounded.

We will use some results from [RS96, Section 4.6.4].

Theorem 4.3. *Let $n \in \mathbb{N}$ and $\frac{d}{n} < p < \infty$. If Ω is a Lipschitz domain, then for every pair $f, g \in W^{n,p}(\Omega)$ we have that*

$$\|fg\|_{W^{n,p}(\Omega)} \leq C_{d,n,p,\Omega} \|f\|_{W^{n,p}(\Omega)} \|g\|_{W^{n,p}(\Omega)}.$$

Moreover, if the boundary of Ω has parameterizations in C^1 and $p > d$, then for $m \in \mathbb{N}$ we have that

$$\|f^m\|_{W^{n,p}(\Omega)} \leq C_{d,n,p,\Omega} m^n \|f\|_{L^\infty(\Omega)}^{m-n} \|f\|_{W^{n,p}(\Omega)}^n.$$

Proof. We have that $W^{n,p}(\mathbb{R}^d)$ is a multiplicative algebra (see [RS96, Section 4.6.4]), that is, if $f, g \in W^{n,p}(\mathbb{R}^d)$, then

$$\|fg\|_{W^{n,p}} \leq C_{n,p} \|f\|_{W^{n,p}} \|g\|_{W^{n,p}}.$$

Since Ω is an extension domain (see [Eva98, Section 5.4]), we have a bounded operator $E : W^{n,p}(\Omega) \rightarrow W^{n,p}(\mathbb{C})$ such that $Ef|_\Omega = f|_\Omega$ for every $f \in W^{n,p}(\Omega)$. The first property is a consequence of this fact.

To prove the second property, assume that $f \in C^\infty(\bar{\Omega})$. By (1.8) we only need to prove that $\|\partial_k^n(f^m)\|_{L^p(\Omega)} \leq C_{d,n,p,\Omega} m^n \left(\|f\|_{L^\infty(\Omega)}^{m-n} \|f\|_{W^{n,p}(\Omega)}^n \right)$ for $1 \leq k \leq d$. By the Leibniz' rule, it is an exercise to check that

$$\partial_k^n(f^m) = f^{m-n} \sum_{\substack{\vec{j} \in \mathbb{N}_0^n \\ j_i \geq j_{i+1} \text{ for } 1 \leq i < n \\ |\vec{j}|=n}} c_{\vec{j},m} \prod_{i=1}^n \partial_k^{j_i} f, \quad (4.4)$$

with $c_{\vec{j},m} > 0$ and $\sum_{\vec{j}} c_{\vec{j},m} = m^n$. Consider $\vec{j} = (n, 0, \dots, 0)$. Then, by (1.14), that is, the Sobolev embedding Theorem, we get

$$\left\| \prod_{i=1}^n \partial_k^{j_i} f \right\|_{L^p(\Omega)} = \|\partial_k^n f f^{n-1}\|_{L^p(\Omega)} \leq \|\partial_k^n f\|_{L^p(\Omega)} \|f\|_{L^\infty(\Omega)}^{n-1} \lesssim_{\Omega,p} \|f\|_{W^{n,p}(\Omega)}^n. \quad (4.5)$$

Otherwise, the indices $j_i < n$ for $1 \leq i \leq n$ and, since $p > d$, we use (1.14) again to state that

$$\left\| \prod_{i=1}^n \partial_k^{j_i} f \right\|_{L^p(\Omega)} \leq \prod_{i=1}^n \left\| \partial_k^{j_i} f \right\|_{L^\infty(\Omega)} |\Omega|^{\frac{1}{p}} \lesssim_{\Omega, p} \prod_{i=1}^n \left\| \partial_k^{j_i} f \right\|_{W^{1,p}(\Omega)} \leq \|f\|_{W^{n,p}(\Omega)}. \quad (4.6)$$

By (4.4), (4.5), (4.6) and the triangle inequality, this implies that

$$\left\| \partial_k^n (f^m) \right\|_{L^p(\Omega)} \leq \|f^{m-n}\|_{L^\infty(\Omega)} \sum_{\substack{\vec{j} \in \mathbb{N}_0^n \\ j_i \geq j_{i+1} \text{ for } 1 \leq i < n \\ |\vec{j}| = n}} c_{\vec{j}, m} \left\| \prod_{i=1}^n \partial_k^{j_i} f \right\|_{L^p(\Omega)} \lesssim m^n \|f\|_{L^\infty(\Omega)}^{m-n} \|f\|_{W^{n,p}(\Omega)}^n.$$

By an approximation procedure this property applies to every $f \in W^{n,p}(\Omega)$. \square

Finally, let us recall some results on compact operators and Fredholm theory (see [Sch02, Chapters 4 and 5], for instance). In the following definitions and results, X, Y, Z are Banach spaces, and all the operators are assumed to be bounded and linear.

Definition 4.4. An operator $K : X \rightarrow Y$ is called compact if every bounded sequence $\{x_k\}_k \subset X$ has a subsequence $\{x_{k_j}\}_j$ such that $\{K(x_{k_j})\}_j$ converges in Y .

Proposition 4.5. If $A : X \rightarrow Y$, $B : Z \rightarrow X$ are operators and $K : Y \rightarrow Z$ is compact, then $K \circ A$ and $B \circ K$ are compact.

Theorem 4.6. If $L : X \rightarrow Y$ and there is a sequence of compact operators $K_j : X \rightarrow Y$ such that $\|L - K_j\| \xrightarrow{j \rightarrow \infty} 0$, then L is compact.

Definition 4.7. An operator $A : X \rightarrow Y$ is said to be a Fredholm operator if it has closed range and the dimensions of the kernels $N(A)$ and $N(A')$ are finite. The Fredholm index of A is $i(A) := \dim N(A) - \dim N(A')$.

Theorem 4.8. Let $A : X \rightarrow Y$, $A_1, A_2 : Y \rightarrow X$, let $K_1, B_1 : X \rightarrow X$ and $K_2, B_2 : Y \rightarrow Y$ with B_j invertible and K_j compact and such that $A_1 \circ A = B_1 - K_1$ and $A \circ A_2 = B_2 - K_2$. Then A is Fredholm.

Theorem 4.9. The index is continuous with respect to the operator norm.

4.2 A Fredholm theory argument

Consider $m \in \mathbb{N}$. Recall that $(\mathcal{B}^m)_\Omega g = \chi_\Omega \mathcal{B}^m (\chi_\Omega g)$ for $g \in L^1_{loc}$ (see Definition 3.2) and $I_\Omega g = \chi_\Omega g$. Note that I_Ω is the identity in $W^{n,p}(\Omega)$. Let us define $P_m := I_\Omega + \mu \mathcal{B}_\Omega + (\mu \mathcal{B}_\Omega)^2 + \dots + (\mu \mathcal{B}_\Omega)^{m-1}$. Since $W^{n,p}(\Omega)$ is a multiplicative algebra (by Theorem 4.3), we have that P_m is bounded on $W^{n,p}(\Omega)$. It follows that

$$P_m \circ (I_\Omega - \mu \mathcal{B}_\Omega) = (I_\Omega - \mu \mathcal{B}_\Omega) \circ P_m = I_\Omega - (\mu \mathcal{B}_\Omega)^m, \quad (4.7)$$

and

$$\begin{aligned} I_\Omega - (\mu \mathcal{B}_\Omega)^m &= (I_\Omega - \mu^m (\mathcal{B}^m)_\Omega) + \mu^m ((\mathcal{B}^m)_\Omega - (\mathcal{B}_\Omega)^m) + (\mu^m (\mathcal{B}_\Omega)^m - (\mu \mathcal{B}_\Omega)^m) \\ &= A_m^{(1)} + \mu^m A_m^{(2)} + A_m^{(3)}. \end{aligned} \quad (4.8)$$

Note the difference between $(\mathcal{B}_\Omega)^m g = \chi_\Omega \mathcal{B}(\dots \chi_\Omega \mathcal{B}(\chi_\Omega \mathcal{B}(\chi_\Omega g)))$ and $(\mathcal{B}^m)_\Omega g = \chi_\Omega \mathcal{B}^m(\chi_\Omega g)$. Next we will see that for m large enough, the operator $I_\Omega - (\mu \mathcal{B}_\Omega)^m$ is the sum of an invertible operator and a compact one.

First we will study the compactness of $A_m^{(3)} = \mu^m (\mathcal{B}_\Omega)^m - (\mu \mathcal{B}_\Omega)^m$. To start, writing $[\mu, \mathcal{B}_\Omega](\cdot)$ for the commutator $\mu \mathcal{B}_\Omega(\cdot) - \mathcal{B}_\Omega(\mu \cdot)$ we have the telescopic sum

$$\begin{aligned} A_m^{(3)} &= \sum_{j=1}^{m-1} \mu^{m-j} [\mu, \mathcal{B}_\Omega] (\mu^{j-1} (\mathcal{B}_\Omega)^{m-1}) + (\mu \mathcal{B}_\Omega) (\mu^{m-1} (\mathcal{B}_\Omega)^{m-1} - (\mu \mathcal{B}_\Omega)^{m-1}) \\ &= \sum_{j=1}^{m-1} \mu^{m-j} [\mu, \mathcal{B}_\Omega] (\mu^{j-1} (\mathcal{B}_\Omega)^{m-1}) + (\mu \mathcal{B}_\Omega) A_{m-1}^{(3)}. \end{aligned}$$

Arguing by induction we can see that $A_m^{(3)}$ can be expressed as a sum of operators bounded on $W^{n,p}(\Omega)$ which have $[\mu, \mathcal{B}_\Omega]$ as a factor. It is well-known that the compactness of a factor implies the compactness of the operator (see Proposition 4.5). Thus, the following lemma, which we prove in Section 4.3 implies the compactness of $A_m^{(3)}$.

Lemma 4.10. *The commutator $[\mu, \mathcal{B}_\Omega]$ is compact in $W^{n,p}(\Omega)$.*

Consider now $A_m^{(2)} = (\mathcal{B}^m)_\Omega - (\mathcal{B}_\Omega)^m$. We define the operator $\mathcal{R}_m g := \chi_\Omega \mathcal{B}(\chi_{\Omega^c} \mathcal{B}^{m-1}(\chi_\Omega g))$ whenever it makes sense. This operator can be understood as a (regularizing) double reflection with respect to the boundary of Ω . For every $g \in W^{n,p}(\Omega)$ we have that

$$\begin{aligned} A_m^{(2)} g &= \chi_\Omega (\mathcal{B}((\chi_\Omega + \chi_{\Omega^c}) \mathcal{B}^{m-1}(\chi_\Omega g)) - \mathcal{B}(\chi_\Omega ((\mathcal{B}_\Omega)^{m-1} g))) \\ &= \chi_\Omega \mathcal{B}(\chi_{\Omega^c} \mathcal{B}^{m-1}(\chi_\Omega g)) + \chi_\Omega \mathcal{B}(\chi_\Omega (\mathcal{B}^{m-1}(\chi_\Omega \cdot) - (\mathcal{B}_\Omega(\cdot))^{m-1}) g) = \mathcal{R}_m g + \mathcal{B}_\Omega \circ A_{m-1}^{(2)} g. \end{aligned}$$

Note that by definition

$$\mathcal{R}_m = \left(A_m^{(2)} - \mathcal{B}_\Omega \circ A_{m-1}^{(2)} \right) \quad (4.9)$$

is bounded on $W^{n,p}(\Omega)$. In Section 4.5 we will prove the compactness of \mathcal{R}_m , which, by induction, will prove the compactness of $A_m^{(2)}$.

Lemma 4.11. *For every m , the operator \mathcal{R}_m is compact in $W^{n,p}(\Omega)$.*

Now, the following claim is the remaining ingredient for the proof of Theorem 4.1.

Claim 4.12. *For m large enough, $A_m^{(1)}$ is invertible.*

Proof. Since $p > 2$ we can use Theorem 4.3 to conclude that for every $g \in W^{n,p}(\Omega)$

$$\begin{aligned} \|\mu^m (\mathcal{B}^m)_\Omega g\|_{W^{n,p}(\Omega)} &\lesssim \|\mu^m\|_{W^{n,p}(\Omega)} \|(\mathcal{B}^m)_\Omega g\|_{W^{n,p}(\Omega)} \\ &\lesssim m^n \|\mu\|_{L^\infty}^{m-n} \|\mu\|_{W^{n,p}(\Omega)}^n \|(\mathcal{B}^m)_\Omega\|_{W^{n,p}(\Omega) \rightarrow W^{n,p}(\Omega)} \|g\|_{W^{n,p}(\Omega)}. \end{aligned}$$

By Theorem 3.28, for any $\epsilon > 0$ there are constants depending on the Lipschitz character of Ω (and other parameters) but not on m , such that

$$\|(\mathcal{B}^m)_\Omega\|_{W^{n,p}(\Omega) \rightarrow W^{n,p}(\Omega)}^p \lesssim m^{(n+1)p} \left((1 + \epsilon)^{mp} + \|N\|_{B_{p,p}^{n-1/p}(\partial\Omega)}^p \right).$$

In particular, if we choose $1 + \epsilon < \frac{1}{\|\mu\|_\infty}$, we get that for m large enough, the operator norm $\|\mu^m (\mathcal{B}^m)_\Omega\|_{W^{n,p}(\Omega) \rightarrow W^{n,p}(\Omega)} < 1$ and, thus, $A_m^{(1)}$ in (4.8) is invertible. \square

Proof of Theorem 4.1. Putting together Lemmas 4.10 and 4.11, Claim 4.12, and (4.8), we get that $I_\Omega - (\mu\mathcal{B}_\Omega)^m$ can be expressed as the sum of an invertible operator and a compact one for m big enough and, by (4.7), we can deduce that $I_\Omega - \mu\mathcal{B}_\Omega$ is a Fredholm operator (see Theorem 4.8). The same argument works with any other operator $I_\Omega - t\mu\mathcal{B}_\Omega$ for $0 < t < 1/\|\mu\|_\infty$. It is well known that the Fredholm index is continuous with respect to the operator norm on Fredholm operators (see Theorem 4.9), so the index of $I_\Omega - \mu\mathcal{B}_\Omega$ must be the same index of I_Ω , that is, 0.

It only remains to see that this operator is injective to prove that it is invertible. Since μ is continuous, we know from [Iwa92, Section 1] that the operator $I - \mu\mathcal{B}$ is injective in L^p . Thus, if $g \in W^{n,p}(\Omega)$, and $(I_\Omega - \mu\mathcal{B}_\Omega)g = 0$, we define $G(z) = g(z)$ if $z \in \Omega$ and $G(z) = 0$ otherwise, and then we have that

$$(I - \mu\mathcal{B})G = (I - \mu\chi_\Omega\mathcal{B})(\chi_\Omega G) = (I_\Omega - \mu\mathcal{B}_\Omega)g = 0.$$

By the injectivity of the former, we get that $G = 0$ and, thus, $g = 0$ as a function of $W^{n,p}(\Omega)$.

Now, remember that the principal solution of (4.1) is $f(z) = \mathcal{C}h(z) + z$ where

$$h = (I - \mu\mathcal{B})^{-1}\mu,$$

that is, $h + \mu\mathcal{B}(h) = \mu$, so $\text{supp}(h) \subset \text{supp}(\mu) \subset \bar{\Omega}$ and, thus, for almost every $z \in \Omega$ we have that $\chi_\Omega(z)h(z) + \mu(z)\mathcal{B}_\Omega(h)(z) = h(z) + \mu(z)\mathcal{B}(h)(z) = \mu(z)$, so

$$h|_\Omega = (I_\Omega - \mu\mathcal{B}_\Omega)^{-1}\mu,$$

proving that $h \in W^{n,p}(\Omega)$. By Theorem 4.2 we have that $\mathcal{C}h \in L^p(\mathbb{C})$. Since the derivatives of the principal solution, $\bar{\partial}f = h$ and $\partial f = \mathcal{B}h + 1 = \mathcal{B}_\Omega h + \chi_{\Omega^c}\mathcal{B}h + 1$, are in $W^{n,p}(\Omega)$, we have $f \in W^{n+1,p}(\Omega)$. \square

4.3 Compactness of the commutator

Proof of Lemma 4.10. We want to see that for any $\mu \in W^{n,p}(\Omega) \cap L^\infty$, the commutator $[\mu, \mathcal{B}_\Omega]$ is compact. The idea is to show that it has a regularizing kernel. In particular, we will prove that assuming some extra condition on the regularity of μ , then the commutator maps $W^{n,p}(\Omega)$ to $W^{n+1,p}(\Omega)$. This will imply the compactness of the commutator as a self-map of $W^{n,p}(\Omega)$ and, by a classical argument on approximation of operators, this will be extended to any given μ .

First we will see that we can assume μ to be $C_c^\infty(\mathbb{C})$ without loss of generality by an approximation procedure. Indeed, since Ω is an extension domain, for every $\mu \in W^{n,p}(\Omega)$, there is a function $E\mu$ with $\|E\mu\|_{W^{n,p}(\mathbb{C})} \leq C\|\mu\|_{W^{n,p}(\Omega)}$ such that $E\mu|_\Omega = \mu\chi_\Omega$. Now, $E\mu$ can be approximated by a sequence of functions $\{\mu_j\}_{j \in \mathbb{N}} \subset C_c^\infty(\mathbb{C})$ in $W^{n,p}(\mathbb{C})$ and one can define the operator $[\mu_j, \mathcal{B}_\Omega] : W^{n,p}(\Omega) \rightarrow W^{n,p}(\Omega)$. Since $W^{n,p}(\Omega)$ is a multiplicative algebra, one can check that $\{[\mu_j, \mathcal{B}_\Omega]\}_{j \in \mathbb{N}}$ is a sequence of operators converging to $[\mu, \mathcal{B}_\Omega]$ in the operator norm. Thus, it is enough to prove that the operators $[\mu_j, \mathcal{B}_\Omega]$ are compact in $W^{n,p}(\Omega)$ for all j by Theorem 4.6.

Let μ be a $C_c^\infty(\mathbb{C})$ function. We will prove that the commutator $[\mu, \mathcal{B}_\Omega]$ is a smoothing operator, mapping $W^{n,p}(\Omega)$ into $W^{n+1,p}(\Omega)$. Consider $f \in W^{n,p}(\Omega)$, a Whitney covering \mathcal{W} with appropriate constants and, for every $Q \in \mathcal{W}$, choose a bump function $\chi_{\frac{3}{2}Q} \leq \varphi_Q \leq \chi_{2Q}$ with $\|\nabla^j \varphi_Q\|_{L^\infty} \lesssim \frac{C_j}{\ell(Q)^j}$. Recall that in Section 1.5 we defined $\mathbf{P}_{3Q}^{n-1}f$ to be the approximating polynomial of f around $3Q$.

Then, we break the norm in three terms,

$$\begin{aligned} \|\nabla^{n+1}[\mu, \mathcal{B}_\Omega]f\|_{L^p(\Omega)}^p &\lesssim_p \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1}[\mu, \mathcal{B}_\Omega] \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right\|_{L^p(Q)}^p \\ &+ \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1}[\mu, \mathcal{B}_\Omega] \left((f - \mathbf{P}_{3Q}^{n-1}f) (\chi_\Omega - \varphi_Q) \right) \right\|_{L^p(Q)}^p \\ &+ \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1}[\mu, \mathcal{B}_\Omega] \left(\mathbf{P}_{3Q}^{n-1}f \right) \right\|_{L^p(Q)}^p =: \textcircled{1} + \textcircled{2} + \textcircled{3}. \end{aligned} \quad (4.10)$$

First we study $\textcircled{1}$. In this case, we can use the following classical trick for compactly supported functions. Given a ball $B \subset \mathbb{C}$, $\varphi \in C_c^\infty(B)$ and $g \in L^p$, then $\mathcal{C}g \in W^{1,p}(2B)$ by (4.3). Therefore, we can use Leibniz' rule (1.11) for the first order derivatives of $\varphi \cdot \mathcal{C}g$, and by Theorem 4.2 we get

$$\begin{aligned} \varphi \cdot \mathcal{B}(g) - \mathcal{B}(\varphi \cdot g) &= \varphi \cdot \partial \mathcal{C}g - \mathcal{B}(\varphi \cdot \bar{\partial} \mathcal{C}g) = -\partial \varphi \cdot \mathcal{C}g + \partial(\varphi \cdot \mathcal{C}g) - \bar{\partial} \mathcal{B}(\varphi \cdot \mathcal{C}g) + \mathcal{B}(\bar{\partial} \varphi \cdot \mathcal{C}g) \\ &= \mathcal{B}(\bar{\partial} \varphi \cdot \mathcal{C}g) - \partial \varphi \cdot \mathcal{C}g. \end{aligned} \quad (4.11)$$

Thus, for a fixed cube Q , since we assumed that $\mu \in C_c^\infty(\mathbb{C})$, we have that

$$[\mu, \mathcal{B}] \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) = \mathcal{B} \left(\bar{\partial} \mu \cdot \mathcal{C} \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right) - \partial \mu \cdot \mathcal{C} \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right).$$

Therefore, using the boundedness of the Beurling transform and the fact that it commutes with derivatives, we have that

$$\begin{aligned} \textcircled{1} &= \sum_Q \left\| \nabla^{n+1}[\mu, \mathcal{B}] \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right\|_{L^p(Q)}^p \\ &\lesssim_p \sum_Q \left\| \nabla^{n+1} \left(\bar{\partial} \mu \cdot \mathcal{C} \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right) \right\|_{L^p}^p + \sum_Q \left\| \nabla^{n+1} \left(\partial \mu \cdot \mathcal{C} \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right) \right\|_{L^p}^p \\ &\leq \sum_{j=0}^{n+1} \sum \|\mu\|_{W^{n+2,\infty}}^p \left\| \nabla^j \mathcal{C} \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right\|_{L^p}^p \end{aligned}$$

and, using the identities $\partial \mathcal{C} = \mathcal{B}$, $\bar{\partial} \mathcal{C} = Id$ (when $j > 0$ in the previous sum) together with (4.2) from Theorem 4.2 (when $j = 0$) we can estimate

$$\textcircled{1} \lesssim_p \|\mu\|_{W^{n+2,\infty}}^p \sum_Q \left(\sum_{j=1}^{n+1} \left\| \nabla^{j-1} \left((f - \mathbf{P}_{3Q}^{n-1}f) \varphi_Q \right) \right\|_{L^p(2Q)}^p + \ell(Q)^p \left\| f - \mathbf{P}_{3Q}^{n-1}f \right\|_{L^p(2Q)}^p \right)$$

and, by the Poincaré inequality (1.33) we get

$$\textcircled{1} \lesssim_{n,p} \|\mu\|_{W^{n+2,\infty}}^p \sum_Q \sum_{j=0}^{n+1} \ell(Q)^{(n+1-j)p} \left\| \nabla^j f \right\|_{L^p(2Q)}^p \lesssim_{n,\Omega} \|\mu\|_{W^{n+2,\infty}}^p \left\| \nabla^n f \right\|_{L^p(\Omega)}^p.$$

Second, we bound using duality

$$\begin{aligned} \textcircled{2} &= \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1}[\mu, \mathcal{B}_\Omega] \left((f - \mathbf{P}_{3Q}^{n-1}f) (\chi_\Omega - \varphi_Q) \right) \right\|_{L^p(Q)}^p \\ &= \left(\sup_{g \in L^{p'}: \|g\|_{L^{p'}} \leq 1} \sum_{Q \in \mathcal{W}} \int_Q \left| \nabla^{n+1}[\mu, \mathcal{B}_\Omega] \left((f - \mathbf{P}_{3Q}^{n-1}f) (\chi_\Omega - \varphi_Q) \right) (z) g(z) \right| dm(z) \right)^p. \end{aligned}$$

Let Q be a Whitney cube, let $z \in Q$ and let $\alpha \in \mathbb{N}^2$ with $|\alpha| = n + 1$. Then, if we call

$$K_\mu(z, w) = \frac{\mu(z) - \mu(w)}{(z - w)^2},$$

then, since z is not in the support of $(f - \mathbf{P}_{3Q}^{n-1}f)(\chi_\Omega - \varphi_Q)$, we have that

$$D^\alpha[\mu, \mathcal{B}_\Omega] \left((f - \mathbf{P}_{3Q}^{n-1}f)(\chi_\Omega - \varphi_Q) \right) (z) = \int_\Omega D_z^\alpha K_\mu(z, w) \left(f(w) - \mathbf{P}_{3Q}^{n-1}f(w) \right) (1 - \varphi_Q(w)) dm.$$

Note that

$$D_z^\alpha K_\mu(z, w) = (\mu(z) - \mu(w)) D_z^\alpha \frac{1}{(z - w)^2} + \sum_{\gamma < \alpha} \binom{\alpha}{\gamma} D^{\alpha - \gamma} \mu(z) D_z^\gamma \frac{1}{(z - w)^2},$$

so using $|\mu(z) - \mu(w)| \leq \|\nabla \mu\|_{L^\infty} |z - w|$ we get

$$|D_z^\alpha K_\mu(z, w)| \leq C_{n, \text{diam}(\Omega)} \|\mu\|_{W^{n+1, \infty}} \frac{1}{|z - w|^{n+2}}.$$

Note the similitude between this estimate and the size condition (1.41) (take smoothness n , dimension 2 and Calderón-Zygmund constant $C_{n, \text{diam}(\Omega)} \|\mu\|_{W^{n+1, \infty}}$). Using (1.34) and Lemma 2.3 we get

$$\begin{aligned} \textcircled{2}^{\frac{1}{p}} &\lesssim \|\mu\|_{W^{n+1, \infty}} \sup_{\|g\|_{L^{p'} \leq 1}} \sum_{Q, S} \frac{\int_S |f(w) - \mathbf{P}_{3Q}^{n-1}f(w)| dm(w)}{D(Q, S)^{n+2}} \int_Q |g(z)| dm(z) \\ &\lesssim \|\mu\|_{W^{n+1, \infty}} \sup_{\|g\|_{L^{p'} \leq 1}} \sum_{Q, S} \sum_{P \in [Q, S]} \frac{\|g\|_{L^1(Q)} \|\nabla^n f\|_{L^1(3P)} D(P, S)^{n-1} \ell(S)^2}{\ell(P) D(Q, S)^{n+2}} \\ &\lesssim_{n, \Omega} \|\mu\|_{W^{n+1, \infty}} \|\nabla^n f\|_{L^p(\Omega)}. \end{aligned}$$

Next we use a $T(1)$ argument reducing $\textcircled{3}$ to the boundedness of $[\mu, \mathcal{B}_\Omega](1)$. Consider the monomials $P_{Q, \gamma}(z) := (z - z_Q)^\gamma = (z - z_Q)^{\gamma_1} (z - z_Q)^{\gamma_2}$ for $\gamma \in \mathbb{N}_0^2$ where z_Q stands for the center of Q (see complex notation in Section 1.2). The Taylor expansion (1.31) of $\mathbf{P}_{3Q}^{n-1}f$ around z_Q can be written as $\mathbf{P}_{3Q}^{n-1}f(z) = \sum_{|\gamma| < n} m_{Q, \gamma} P_{Q, \gamma}(z)$. Thus, we have that

$$-\pi[\mu, \mathcal{B}_\Omega] \mathbf{P}_{3Q}^{n-1}f(z) = \left[\mu, T_\Omega^{(-2, 0)} \right] \mathbf{P}_{3Q}^{n-1}f(z) = \sum_{|\gamma| < n} m_{Q, \gamma} \left[\mu, T_\Omega^{(-2, 0)} \right] P_{Q, \gamma}(z),$$

and using the binomial expansion $(w - z_Q)^\gamma = \sum_{\lambda \leq \gamma} (-1)^\lambda \binom{\gamma}{\lambda} (z - w)^\lambda (z - z_Q)^{\gamma - \lambda}$ we have

$$-\pi[\mu, \mathcal{B}_\Omega] \mathbf{P}_{3Q}^{n-1}f(z) = \sum_{|\gamma| < n} m_{Q, \gamma} \sum_{\vec{0} \leq \lambda \leq \gamma} (-1)^\lambda \binom{\gamma}{\lambda} \left[\mu, T_\Omega^{(-2, 0) + \lambda} \right] (1)(z) P_{Q, \gamma - \lambda}(z), \quad (4.12)$$

that is,

$$\begin{aligned} \textcircled{3} &= \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} [\mu, \mathcal{B}_\Omega] \left(\mathbf{P}_{3Q}^{n-1}f \right) \right\|_{L^p(Q)}^p \\ &\lesssim \sum_{|\gamma| < n} \sum_{\vec{0} \leq \lambda \leq \gamma} \sum_{Q \in \mathcal{W}} |m_{Q, \gamma}|^p \left\| \nabla^{n+1} \left(\left[\mu, T_\Omega^{(-2, 0) + \lambda} \right] (1) \cdot P_{Q, \gamma - \lambda} \right) \right\|_{L^p(Q)}^p. \end{aligned}$$

But every coefficient $|m_{Q,\gamma}|$ is bounded by $C\|f\|_{W^{n-1,\infty}(Q)}$ by (1.32) and all the derivatives of $P_{Q,\gamma}$ are uniformly bounded on Ω . Therefore, we have that

$$\textcircled{3} \lesssim \|f\|_{W^{n-1,\infty}(\Omega)}^p \sum_{Q \in \mathcal{W}} \sum_{0 \leq |\lambda| < n} \left\| \left[\mu, T_{\Omega}^{(-2,0)+\lambda} \right] 1 \right\|_{W^{n+1,p}(Q)}^p.$$

Using the Sobolev Embedding Theorem, we get

$$\textcircled{3} \lesssim \|f\|_{W^{n,p}(\Omega)}^p \left(\sum_{0 < |\lambda| < n} \left\| \left[\mu, T_{\Omega}^{(-2,0)+\lambda} \right] 1 \right\|_{W^{n+1,p}(\Omega)}^p + \sum_{Q \in \mathcal{W}} \left\| \left[\mu, T_{\Omega}^{(-2,0)} \right] 1 \right\|_{W^{n+1,p}(Q)}^p \right).$$

Note that if $\lambda > \vec{0}$, then the operator $T_{\Omega}^{(-2,0)+\lambda}$ has homogeneity $-2 + \lambda_1 + \lambda_2 > -2$ and, therefore, by Theorem 3.28, $T_{\Omega}^{(-2,0)+\lambda} : W^{n,p}(\Omega) \rightarrow W^{n+1,p}(\Omega)$ is bounded and, since $p > 2$ and $W^{n+1,p}(\Omega)$ is a multiplicative algebra, we have that

$$\left\| \mu T_{\Omega}^{(-2,0)+\lambda} 1 \right\|_{W^{n+1,p}(\Omega)}^p + \left\| T_{\Omega}^{(-2,0)+\lambda} \mu \right\|_{W^{n+1,p}(\Omega)}^p \lesssim_{n,p,\Omega} \|\mu\|_{W^{n+1,p}(\Omega)}^p.$$

Therefore,

$$\textcircled{3} \lesssim \left(\|\mu\|_{W^{n+1,p}(\Omega)}^p + \|\mu, \mathcal{B}_{\Omega}\|_{W^{n+1,p}(\Omega)}^p \right) \|f\|_{W^{n,p}(\Omega)}^p,$$

so we have reduced the proof of Lemma 4.10 to the following claim.

Claim 4.13. *Let $2 < p < \infty$, $n \in \mathbb{N}$. Given a bounded Lipschitz domain Ω with parameterizations in $B_{p,p}^{n+1-1/p}$ and a function $\mu \in C_c^{\infty}(\mathbb{C})$, then $[\mu, \mathcal{B}_{\Omega}](1) \in W^{n+1,p}(\Omega)$.*

We know that $[\mu, \mathcal{B}_{\Omega}](1) = \mu \mathcal{B}_{\Omega}(1) - \mathcal{B}_{\Omega}(\mu) \in W^{n,p}(\Omega)$. We want to prove that $\nabla^{n+1}[\mu, \mathcal{B}_{\Omega}]1 \in L^p$. To do so, we split the norm in the same spirit of (4.10), but chopping μ instead of f :

$$\begin{aligned} \|\nabla^{n+1}[\mu, \mathcal{B}_{\Omega}](1)\|_{L^p(\Omega)}^p &\lesssim_p \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} \left[\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) \varphi_Q, \mathcal{B}_{\Omega} \right] (1) \right\|_{L^p(Q)}^p \\ &\quad + \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} \left[\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) (\chi_{\Omega} - \varphi_Q), \mathcal{B}_{\Omega} \right] (1) \right\|_{L^p(Q)}^p \\ &\quad + \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} \left[\mathbf{P}_{3Q}^{n+2} \mu, \mathcal{B}_{\Omega} \right] (1) \right\|_{L^p(Q)}^p =: \textcircled{4} + \textcircled{5} + \textcircled{6}. \end{aligned}$$

First we consider $\textcircled{4}$. Since $\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) \varphi_Q \in C_c^{\infty}$, by (4.11) we have that

$$\begin{aligned} \sum_Q \left\| \nabla^{n+1} \left[\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) \varphi_Q, \mathcal{B} \right] \chi_{\Omega} \right\|_{L^p(\mathbb{C})}^p &\lesssim_p \sum_Q \left\| \nabla^{n+1} \left(\partial \left(\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) \varphi_Q \right) \cdot \mathcal{C} \chi_{\Omega} \right) \right\|_{L^p(2Q)}^p \\ &\quad + \sum_Q \left\| \nabla^{n+1} \left(\bar{\partial} \left(\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) \varphi_Q \right) \cdot \mathcal{C} \chi_{\Omega} \right) \right\|_{L^p(2Q)}^p \end{aligned}$$

and, using Leibniz' rule (1.11), Hölder's inequality, and the finite overlapping of double Whitney cubes,

$$\textcircled{4} \lesssim_p \sum_{j=0}^{n+1} \left(\sup_{Q \in \mathcal{W}} \left\| \nabla^{j+1} \left(\left(\mu - \mathbf{P}_{3Q}^{n+2} \mu \right) \varphi_Q \right) \right\|_{L^{\infty}(2Q)}^p \right) \cdot \|\nabla^{n+1-j} \mathcal{C} \chi_{\Omega}\|_{L^p(\Omega)}^p. \quad (4.13)$$

To estimate ④ it remains to see that $\sup_{Q \in \mathcal{W}} \left\| \nabla^{j+1} \left((\mu - \mathbf{P}_{3Q}^{n+2} \mu) \varphi_Q \right) \right\|_{L^\infty(2Q)}^p < \infty$. But this is an immediate consequence of the Poincaré inequality (1.33), which shows that

$$\left\| \nabla^{j+1} \left((\mu - \mathbf{P}_{3Q}^{n+2} \mu) \varphi_Q \right) \right\|_{L^\infty(2Q)} \lesssim \ell(Q)^{n+2-j} \left\| \nabla^{n+3} \mu \right\|_{L^\infty(2Q)}. \quad (4.14)$$

Thus, the bounds (4.13) and (4.14) yield

$$\textcircled{4} \leq C_{p,n,\text{diam}\Omega} \left\| \nabla^{n+3} \mu \right\|_{L^\infty(\Omega)}^p \left\| \mathcal{C} \chi_\Omega \right\|_{W^{n+1,p}(\Omega)}^p,$$

which is finite by Theorem 3.28.

Next we face ⑤. Note that for a given Whitney cube Q , if $z \in Q$, then $\chi_\Omega(z) - \varphi_Q(z) = 0$, so

$$\textcircled{5} = \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} \mathcal{B} \left((\mu - \mathbf{P}_{3Q}^{n+2} \mu) (\chi_\Omega - \varphi_Q) \right) \right\|_{L^p(Q)}^p.$$

Moreover, for $z \in Q \in \mathcal{W}$, we have

$$\partial^{n+1} \mathcal{B} \left((\mu - \mathbf{P}_{3Q}^{n+2} \mu) (\chi_\Omega - \varphi_Q) \right) (z) = c_n \int_{\Omega \setminus \frac{3}{2}Q} \frac{(\mu(w) - \mathbf{P}_{3Q}^{n+2} \mu(w)) (1 - \varphi_Q(w))}{(z-w)^{3+n}} dm(w).$$

Since $\bar{\partial} \mathcal{B} \left((\mu - \mathbf{P}_{3Q}^{n+2} \mu) (\chi_\Omega - \varphi_Q) \right) (z) = 0$, only ∂^{n+1} is non zero in the $(n+1)$ -th gradient, so

$$\left| \nabla^{n+1} \mathcal{B} \left((\mu - \mathbf{P}_{3Q}^{n+2} \mu) (\chi_\Omega - \varphi_Q) \right) (z) \right| \lesssim \sum_{S \in \mathcal{W}} \frac{1}{\mathbb{D}(Q,S)^{3+n}} \left\| \mu - \mathbf{P}_{3Q}^{n+2} \mu \right\|_{L^1(S)}.$$

For every pair of Whitney cubes Q, S , consider an admissible chain $[S, Q]$. By (1.34) we have that

$$\left\| \mu - \mathbf{P}_{3Q}^{n+2} \mu \right\|_{L^1(S)} \lesssim \sum_{P \in [S, Q]} \frac{\ell(S)^2 \mathbb{D}(P, S)^{n+2}}{\ell(P)} \left\| \nabla^{n+3} \mu \right\|_{L^1(3P)}.$$

Combining all these facts with the expression of the norm by duality and Lemma 2.3, we get

$$\begin{aligned} \textcircled{5}^{\frac{1}{p}} &\lesssim \sup_{g \in L^{p'}(\Omega): \|g\|_{p'} \leq 1} \sum_Q \int_Q g dm \sum_{S \in \mathcal{W}} \frac{1}{\mathbb{D}(Q, S)^{3+n}} \sum_{P \in [S, Q]} \frac{\ell(S)^2 \mathbb{D}(P, S)^{n+2}}{\ell(P)} \left\| \nabla^{n+3} \mu \right\|_{L^1(3P)} \\ &\lesssim \text{diam}(\Omega)^2 \sup_{g \in L^{p'}(\Omega): \|g\|_{p'} \leq 1} \sum_Q \sum_S \sum_{P \in [S, Q]} \frac{\ell(S)^2 \left\| \nabla^{n+3} \mu \right\|_{L^1(3P)} \|g\|_{L^1(Q)}}{\ell(P) \mathbb{D}(Q, S)^3} \lesssim \left\| \nabla^{n+3} \mu \right\|_{L^p(\Omega)}. \end{aligned}$$

Finally we focus on

$$\textcircled{6} = \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} \left[\mathbf{P}_{3Q}^{n+2} \mu, \mathcal{B}_\Omega \right] (1) \right\|_{L^p(Q)}^p.$$

Consider first a monomial $P_{Q,\gamma}(z) = (z - z_Q)^\gamma$ for a multiindex $\gamma \in \mathbb{N}^2$. Then, as we did in (4.12), we use the binomial expression $P_{Q,\gamma}(w) = \sum_{\lambda \leq \gamma} (-1)^{|\lambda|} \binom{\gamma}{\lambda} (z - w)^\lambda (z - z_Q)^{\gamma - \lambda}$ to deduce that

$$-\pi \mathcal{B}_\Omega P_{Q,\gamma}(z) = T_\Omega^{(-2,0)} P_{Q,\gamma}(z) = \sum_{\vec{0} \leq \lambda \leq \gamma} (-1)^{|\lambda|} \binom{\gamma}{\lambda} T_\Omega^{(-2,0)+\lambda} (1)(z) (z - z_Q)^{\gamma - \lambda}.$$

Note that the term for $\lambda = \vec{0}$ in the right-hand side of this expression is $T_\Omega^{(-2,0)}(1)(z)P_{Q,\gamma}(z)$, so it cancels out in the commutator:

$$-\pi[P_{Q,\gamma}, \mathcal{B}_\Omega](1)(z) = \sum_{\vec{0} < \lambda \leq \gamma} (-1)^{|\lambda|} \binom{\gamma}{\lambda} T_\Omega^{(-2,0)+\lambda}(1)(z)P_{Q,\gamma-\lambda}(z). \quad (4.15)$$

Now, witting $\mathbf{P}_{3Q}^{n+2}\mu(z) = \sum_{|\gamma| \leq n+2} m_{Q,\gamma} P_{Q,\gamma}(z)$ we have that

$$\textcircled{6} = \sum_{Q \in \mathcal{W}} \left\| \nabla^{n+1} [\mathbf{P}_{3Q}^{n+2}\mu, \mathcal{B}_\Omega](1) \right\|_{L^p(Q)}^p \leq \sum_{Q \in \mathcal{W}} \sum_{\gamma \leq n+2} |m_{Q,\gamma}|^p \left\| \nabla^{n+1} [P_{Q,\gamma}, \mathcal{B}_\Omega](1) \right\|_{L^p(Q)}^p,$$

so using (1.32) and (4.15) together with Leibniz' rule (1.11), we get

$$\begin{aligned} \textcircled{6} &\lesssim \|\mu\|_{W^{n+2,\infty}}^p \sum_{Q \in \mathcal{W}} \sum_{\gamma \leq n+2} \sum_{\vec{0} < \lambda \leq \gamma} \sum_{j=0}^{n+1} \left\| \nabla^j T_\Omega^{(-2,0)+\lambda}(1) \right\|_{L^p(Q)}^p \left\| \nabla^{n+1-j} P_{Q,\gamma-\lambda} \right\|_{L^\infty(Q)}^p \\ &\leq C_{n,p,\Omega} \|\mu\|_{W^{n+2,\infty}}^p \sum_{\vec{0} < \lambda: |\lambda| \leq n+2} \left\| T_\Omega^{(-2,0)+\lambda}(1) \right\|_{W^{n+1,p}(\Omega)}^p. \end{aligned} \quad (4.16)$$

In the last sum we have that $T_\Omega^{(-2,0)+\lambda}(1) \in W^{n+1,p}(\Omega)$ for all $\lambda > \vec{0}$ by Theorem 3.28 because the operators $T^{(-2,0)+\lambda}$ have homogeneity bigger than -2 . Thus, the right-hand side of (4.16) is finite. \square

4.4 Some technical details

Given $\vec{m} = (m_1, m_2, m_3) \in \mathbb{N}^3$, let us define the line integral

$$K_{\vec{m}}(z, \xi) := \int_{\partial\Omega} \frac{(w - \xi)^{m_3}}{(z - w)^{m_1} (w - \xi)^{m_2}} dw \quad (4.17)$$

for all $z, \xi \in \Omega$, where the path integral is oriented counterclockwise.

Given a j times differentiable function f , we will write

$$P_z^j(f)(\xi) = \sum_{|\vec{i}| \leq j} \frac{D^{\vec{i}} f(z)}{i!} (\xi - z)^{\vec{i}}$$

for its j -th degree Taylor polynomial centered in the point z .

Mateu, Orobitg and Verdera study the kernel $K_{(2,m+1,m)}(z, \xi)$ for $m \in \mathbb{N}$ in [MOV09, Lemma 6] assuming the boundary of the domain Ω to be in $C^{1,\varepsilon}$ for $\varepsilon < 1$. They prove the size inequality

$$|K_{(2,m+1,m)}(z, \xi)| \lesssim \frac{1}{|z - \xi|^{2-\varepsilon}}$$

and a smoothness inequality in the same spirit. In [CMO13], when dealing with the compactness of the operator $\mathcal{R}_m f = \chi_\Omega \mathcal{B}(\chi_{\Omega^c} \mathcal{B}^{m-1}(\chi_\Omega f))$ on $W^{s,p}(\Omega)$ for $0 < s < 1$, this is used to prove that the Beltrami coefficient $\mu \in W^{s,p}(\Omega)$ implies the principal solution of $\bar{\partial} f = \mu \partial f$ being in $W^{s+1,p}(\Omega)$ only for $s < \varepsilon$. This bounds are not enough for us in this form and, moreover, we will consider $m_1 > 2$ (this comes from differentiating the kernel of \mathcal{R}_m , something that we have to do in order to study the classical Sobolev spaces). Nevertheless, their argument can be adapted to the case of the boundary being in the space $B_{p,p}^{n+1-1/p} \subset C^{n,1-2/p}$ to get Proposition 4.15 below, which will be used to prove Lemma 4.11. The proof follows the same pattern but it is more sophisticated and some combinatorial lemma will be handy.

We will use some auxiliary functions.

Definition 4.14. Let us define

$$H_{m_3, \xi}(w) := \frac{1}{2\pi i} \int_{\partial\Omega} \frac{(\overline{\tau - \xi})^{m_3}}{\tau - w} d\tau \quad \text{for every } w, \xi \notin \partial\Omega,$$

and

$$h_{m_3}(z) := \int_{\partial\Omega} \frac{(\overline{\tau - z})^{m_3}}{\tau - z} d\tau = 2\pi i H_{m_3, z}(z) \quad \text{for every } z \in \Omega. \quad (4.18)$$

Proposition 4.15. Let Ω be a Lipschitz domain, and let $\vec{m} = (m_1, m_2, m_3) \in \mathbb{N}^3$ with $m_1 \geq 3$, $m_2, m_3 \geq 1$ and $m_2 \leq m_1 + m_3 - 2$. Then, the weak derivatives of order m_3 of h_{m_3} are

$$\partial^j \bar{\partial}^{m_3-j} h_{m_3} = c_{m_3, j} \mathcal{B}^j \chi_\Omega, \quad \text{for } 0 \leq j \leq m_3. \quad (4.19)$$

Moreover, for every pair $z, \xi \in \Omega$ with $z \neq \xi$, we have that

$$K_{\vec{m}}(z, \xi) = c_{\vec{m}} \partial^{m_1-2} \mathcal{B} \chi_\Omega(z) \frac{(\overline{\xi - z})^{m_3-1}}{(\xi - z)^{m_2}} + \sum_{j \leq m_2-1} \frac{c_{\vec{m}, j} R_{m_1+m_3-3, j}^{m_3}(z, \xi)}{(\xi - z)^{m_2+m_1-1-j}}, \quad (4.20)$$

where

$$R_{M, j}^{m_3}(z, \xi) := \partial^j h_{m_3}(\xi) - P_z^{M-j}(\partial^j h_{m_3})(\xi) \quad (4.21)$$

is the Taylor error term of order $M - j$ for the function $\partial^j h_{m_3}$.

We begin by noting some remarkable properties of these functions.

Remark 4.16. Given $\xi \notin \partial\Omega$ and $w \in \partial\Omega$, if we write $H_{m_3, \xi}^-(w)$ for the interior non-tangential limit of $H_{m_3, \xi}(\zeta)$ when $\zeta \rightarrow w$ and $H_{m_3, \xi}^+(w)$ for the exterior one, we have the Plemelj formula

$$(\overline{w - \xi})^{m_3} = H_{m_3, \xi}^-(w) - H_{m_3, \xi}^+(w) \quad (4.22)$$

(see [Ver01, p. 143] for instance).

Remark 4.17. Given $\vec{j} = (j_1, j_2)$ with $j_2 \leq m_3$, taking partial derivatives in (4.18) we get

$$D^{\vec{j}} h_{m_3}(z) = \partial^{j_1} \bar{\partial}^{j_2} h_{m_3}(z) = \frac{m_3! j_1!}{(m_3 - j_2)!} (-1)^{j_2} \int_{\partial\Omega} \frac{(\overline{\tau - z})^{m_3 - j_2}}{(\tau - z)^{1+j_1}} d\tau \quad \text{for every } z \in \Omega$$

and, in particular, h_{m_3} is infinitely many times differentiable in Ω . Therefore, by Green's formula (1.12) and the cancellation of the integrand (3.8), for $j > 0$ we have

$$D^{(j, m_3-j)} h_{m_3}(z) = c_{m_3, j} \int_{\partial\Omega} \frac{(\overline{\tau - z})^j}{(\tau - z)^{1+j}} d\tau = c_{m_3, j} \int_{\Omega \setminus B(z, \varepsilon)} \frac{(\overline{w - z})^{j-1}}{(w - z)^{j+1}} dm(w) = c_{m_3, j} \mathcal{B}^j \chi_\Omega(z)$$

for $\varepsilon < \text{dist}(z, \partial\Omega)$ and, in case $j = 0$, by the Residue Theorem (1.13)

$$\bar{\partial}^{m_3} h_{m_3}(z) = c_{m_3} \int_{\partial\Omega} \frac{1}{\tau - z} d\tau = c_{m_3} 2\pi i \chi_\Omega(z),$$

proving (4.19).

Remark 4.18. We can also relate the derivatives of both $h_{m_3}(z)$ and $H_{m_3,\xi}(z)$ for any pair $z, \xi \in \Omega$. By Definition 4.14 and the previous remark, we have that

$$\begin{aligned} 2\pi i H_{m_3,\xi}(z) &= \sum_{l=0}^{m_3} \int_{\partial\Omega} \binom{m_3}{l} \frac{(\overline{\tau-z})^{m_3-l} (\overline{z-\xi})^l}{\tau-z} d\tau \\ &= \sum_{l=0}^{m_3} \frac{m_3!}{(m_3-l)!} \bar{\partial}^l h_{m_3}(z) \frac{(m_3-l)!}{m_3!} (-1)^l (\overline{\xi-z})^l (-1)^l, \end{aligned}$$

that is,

$$2\pi i \bar{\partial}^j H_{m_3,\xi}(z) = \sum_{l=0}^{m_3} \frac{1}{l!} D^{(j,l)} h_{m_3}(z) (\overline{\xi-z})^l. \quad (4.23)$$

Proof of Proposition 4.15. Consider $z, \xi \in \Omega$. Then $\frac{H_{m_3,\xi}(w)}{(z-w)^{m_1} (w-\xi)^{m_2}}$ decays at ∞ as $\frac{1}{|w|^{m_1+m_2+1}}$ and it is holomorphic in Ω^c and, thus, by Green's Theorem we have that

$$K_{\bar{m}}(z, \xi) = \int_{\partial\Omega} \frac{(\overline{w-\xi})^{m_3}}{(z-w)^{m_1} (w-\xi)^{m_2}} dw = \int_{\partial\Omega} \frac{(\overline{w-\xi})^{m_3} + H_{m_3,\xi}^+(w)}{(z-w)^{m_1} (w-\xi)^{m_2}} dw,$$

and using (4.22),

$$K_{\bar{m}}(z, \xi) = (-1)^{m_1} \int_{\partial\Omega} \frac{H_{m_3,\xi}^-(w)}{(w-z)^{m_1} (w-\xi)^{m_2}} dw.$$

Note that $H_{m_3,\xi}(w)$ is holomorphic in Ω , implying that the integrand above is meromorphic in Ω with poles in z and ξ . Moreover, $H_{m_3,\xi}^- \in L^2(\partial\Omega)$ by the boundedness of the Cauchy transform in $L^2(\Gamma)$ on a Lipschitz graph Γ (see [Ver01], for instance). Thus, combining then Dominated Convergence Theorem and the Residue Theorem, we get

$$(-1)^{m_1} K_{\bar{m}}(z, \xi) = 2\pi i \left\{ \frac{1}{(m_1-1)!} \bar{\partial}^{m_1-1} \left[\frac{H_{m_3,\xi}(\cdot)}{(\cdot-\xi)^{m_2}} \right] (z) + \frac{1}{(m_2-1)!} \bar{\partial}^{m_2-1} \left[\frac{H_{m_3,\xi}(\cdot)}{(\cdot-z)^{m_1}} \right] (\xi) \right\}.$$

Therefore,

$$\begin{aligned} \frac{(-1)^{m_1}}{2\pi i} K_{\bar{m}}(z, \xi) &= \frac{1}{(m_1-1)!} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1+j_2=m_1-1}} \frac{(m_1-1)!}{j_1! j_2!} \frac{\bar{\partial}^{j_2} H_{m_3,\xi}(z)}{(z-\xi)^{m_2+j_1}} (-1)^{j_1} \frac{(m_2+j_1-1)!}{(m_2-1)!} \\ &\quad + \frac{1}{(m_2-1)!} \sum_{\substack{j_1, j_2 \geq 0 \\ j_1+j_2=m_2-1}} \frac{(m_2-1)!}{j_1! j_2!} \frac{\bar{\partial}^{j_2} H_{m_3,\xi}(\xi)}{(\xi-z)^{m_1+j_1}} (-1)^{j_1} \frac{(m_1+j_1-1)!}{(m_1-1)!}. \end{aligned}$$

Simplifying and using (4.23) on the first sum of the right-hand side and (4.18) on the second one, we get

$$\begin{aligned} (-1)^{m_1+m_2} K_{\bar{m}}(z, \xi) &= \sum_{\substack{j_1, j_2 \geq 0 \\ j_1+j_2=m_1-1}} \binom{m_2+j_1-1}{m_2-1} \frac{1}{j_2!} \frac{1}{(\xi-z)^{m_2+j_1}} \sum_{l=0}^{m_3} \frac{1}{l!} D^{(j_2,l)} h_{m_3}(z) (\overline{\xi-z})^l \\ &\quad + \sum_{\substack{j_1, j_2 \geq 0 \\ j_1+j_2=m_2-1}} \binom{m_1+j_1-1}{m_1-1} \frac{1}{j_2!} \frac{\bar{\partial}^{j_2} h_{m_3}(\xi)}{(\xi-z)^{m_1+j_1}} (-1)^{j_2+1}. \end{aligned} \quad (4.24)$$

The key idea for the rest of the proof is that the first term in the right-hand side of (4.24) contains the Taylor expansion of the functions in the second one.

Let $m_2 - 1 \leq M \leq m_1 + m_3 - 2$ (we will consider $M = m_1 + m_3 - 3$). Using the Taylor approximating polynomial (4.21) of each $\partial^{j_2} h_{m_3}$ and multiplying by $(\xi - z)^{m_1 + m_2 - 1}$ we get

$$\begin{aligned} -K_{\bar{m}}(z, \xi)(z - \xi)^{m_1 + m_2 - 1} &= \sum_{j=0}^{m_1 - 1} \binom{m_2 + m_1 - 2 - j}{m_2 - 1} \frac{1}{j!} \sum_{l=0}^{m_3} \frac{1}{l!} D^{(j,l)} h_{m_3}(z) (\xi - z)^{(j,l)} \\ &\quad - \sum_{j=0}^{m_2 - 1} \binom{m_1 + m_2 - 2 - j}{m_1 - 1} \frac{(-1)^j}{j!} (\xi - z)^j R_{M,j}^{m_3}(z, \xi) \\ &\quad - \sum_{j=0}^{m_2 - 1} \binom{m_1 + m_2 - 2 - j}{m_1 - 1} \frac{(-1)^j}{j!} \sum_{|\vec{i}| \leq M - j} \frac{D^{\vec{i}} \partial^j h_{m_3}(z)}{\vec{i}!} (\xi - z)^{\vec{i} + (j,0)}. \end{aligned}$$

To simplify notation, let us define the error

$$E_M := -K_{\bar{m}}(z, \xi)(z - \xi)^{m_1 + m_2 - 1} + \sum_{j=0}^{m_2 - 1} \binom{m_1 + m_2 - 2 - j}{m_1 - 1} \frac{(-1)^j}{j!} (\xi - z)^j R_{M,j}^{m_3}(z, \xi). \quad (4.25)$$

Then,

$$\begin{aligned} E_M &= \sum_{\substack{\alpha \geq \vec{0} \\ \alpha \leq (m_1 - 1, m_3)}} \binom{m_1 + m_2 - 2 - \alpha_1}{m_2 - 1} \frac{D^\alpha h_{m_3}(z)}{\alpha!} (\xi - z)^\alpha \\ &\quad - \sum_{\substack{\alpha \geq \vec{0} \\ |\alpha| \leq M}} \sum_{0 \leq j \leq \min\{m_2 - 1, \alpha_1\}} \binom{m_1 + m_2 - 2 - j}{m_1 - 1} \frac{(-1)^j}{j!} \frac{D^\alpha h_{m_3}(z)}{(\alpha_1 - j)! \alpha_2!} (\xi - z)^\alpha. \end{aligned}$$

Note that if $\alpha_2 > m_3$, we have that $D^\alpha h_{m_3}(z) = 0$ (apply (4.19) with $j = 0$). The same happens for the case $\alpha = (\alpha_1, m_3)$ with $\alpha_1 > 0$. On the other hand, if $\alpha_1 > m_1 - 1$, then $\binom{m_1 + m_2 - 2 - \alpha_1}{m_2 - 1} = 0$. By the same token, if $j > m_2 - 1$, $\binom{m_1 + m_2 - 2 - j}{m_1 - 1} = 0$. Thus, we can write

$$\begin{aligned} E_M &= \sum_{|\alpha| \leq m_1 + m_3 - 2} \frac{D^\alpha h_{m_3}(z)}{\alpha!} (\xi - z)^\alpha \\ &\quad \cdot \left[\binom{m_1 + m_2 - 2 - \alpha_1}{m_2 - 1} - \chi_{|\alpha| \leq M} \sum_{j \leq \alpha_1} (-1)^j \binom{m_1 + m_2 - 2 - j}{m_1 - 1} \binom{\alpha_1}{j} \right]. \end{aligned}$$

Note that we have added many null terms in the previous expression, but now the proof of the proposition is reduced to Claim 4.19 below which implies that

$$E_M = \sum_{M < |\alpha| \leq m_1 + m_3 - 2} \binom{m_1 + m_2 - 2 - \alpha_1}{m_2 - 1} \frac{D^\alpha h_{m_3}(z)}{\alpha!} (\xi - z)^\alpha.$$

Taking $M = m_1 + m_3 - 3$ in this expression, only the terms with $|\alpha| = m_1 + m_3 - 2$ remain and, arguing as before, if $\alpha_1 > m_1 - 1$ then $\binom{m_1 + m_2 - 2 - \alpha_1}{m_2 - 1} = 0$ and

$$\text{if } \alpha_2 \geq m_3 \text{ then } D^\alpha h_{m_3}(z) = 0 \quad (4.26)$$

($|\alpha| > m_3$ because we assume that $m_1 \geq 3$). Summing up, by (4.19) we have that

$$E_{m_1+m_3-3} = \frac{D^{(m_1-1, m_3-1)} h_{m_3}(z)}{(m_1-1)!(m_3-1)!} (\xi-z)^{(m_1-1, m_3-1)} = c_{\vec{m}} \partial^{m_1-2} \mathcal{B}\chi_\Omega(z) (\xi-z)^{(m_1-1, m_3-1)}.$$

By (4.25) this implies (4.20). \square

Claim 4.19. For any natural numbers m_1, m_2 and α_1 we have that

$$\binom{m_1+m_2-2-\alpha_1}{m_2-1} = \sum_{j=0}^{\alpha_1} (-1)^j \binom{\alpha_1}{j} \binom{m_2+m_1-2-j}{m_1-1}.$$

Proof. We have the trivial identity

$$\binom{m_1+m_2-2-\alpha_1}{m_2-1} = \binom{m_1+m_2-2-\alpha_1}{m_1-1-\alpha_1} = \sum_{i=0}^0 (-1)^i \binom{0}{i} \binom{m_1+m_2-2-\alpha_1-i}{m_1-1-\alpha_1}.$$

Let $\kappa_1, \kappa_2, \kappa_3 \in \mathbb{Z}$ with $\kappa_1 \geq 0$. We have that

$$\begin{aligned} \sum_{i=0}^{\kappa_1} (-1)^i \binom{\kappa_1}{i} \binom{\kappa_3-i}{\kappa_2} &= \sum_{i=0}^{\kappa_1} (-1)^i \left[\binom{\kappa_1}{i} \binom{\kappa_3+1-i}{\kappa_2+1} - \binom{\kappa_1}{i} \binom{\kappa_3-i}{\kappa_2+1} \right] \\ &= \sum_{j=0}^{\kappa_1+1} (-1)^j \left[\binom{\kappa_1}{j} \binom{\kappa_3+1-j}{\kappa_2+1} + \binom{\kappa_1}{j-1} \binom{\kappa_3+1-j}{\kappa_2+1} \right] \\ &= \sum_{j=0}^{\kappa_1+1} (-1)^j \binom{\kappa_1+1}{j} \binom{\kappa_3+1-j}{\kappa_2+1}. \end{aligned}$$

Arguing by induction we get that

$$\sum_{i=0}^0 (-1)^i \binom{0}{i} \binom{m_1+m_2-2-\alpha_1-i}{m_1-1-\alpha_1} = \dots = \sum_{j=0}^{\alpha_1} (-1)^j \binom{\alpha_1}{j} \binom{m_2+m_1-2-j}{m_1-1}.$$

\square

Remark 4.20. In Proposition 4.15, if we assume that $m_1 = 2$, then for every pair $z, \xi \in \Omega$ with $z \neq \xi$ we have that

$$K_{\vec{m}}(z, \xi) = c_{\vec{m}} \mathcal{B}\chi_\Omega(z) \frac{(\overline{\xi-z})^{m_3-1}}{(\xi-z)^{m_2}} + c_{\vec{m}} \chi_\Omega(z) \frac{(\overline{\xi-z})^{m_3}}{(\xi-z)^{m_2+1}} + \sum_{j \leq m_2-1} \frac{c_{\vec{m},j} R_{m_3-1,j}^{m_3}(z, \xi)}{(\xi-z)^{m_2+1-j}}. \quad (4.27)$$

The proof is exactly the same, but (4.26) is only valid for $\alpha_2 > m_3$, leading to

$$E_{m_3-1} = \frac{D^{(1, m_3-1)} h_{m_3}(z)}{(m_3-1)!} (\xi-z)^{(1, m_3-1)} + \frac{m_2 D^{(0, m_3)} h_{m_3}(z)}{m_3!} (\xi-z)^{(0, m_3)},$$

leading to (4.27) by (4.19).

The thoughtful reader may wonder why we do not use $M = m_1 + m_3 - 2$ in (4.25) to get an analogous result to [MOV09, (27)], namely $E_{m_1+m_3-2} = 0$. This formula is still valid in our context (with $\vec{m} = (2, m+1, m)$), and taking $\vec{m} = (n+2, m+1, m)$ we could get a generalization with no extra terms, only Taylor remainders up to degree M . In the present dissertation, however, we deal with the situation $h_m \in W^{m+n,p}(\Omega)$ with $p > 2$ and, therefore, we will only use Taylor polynomials up to degree $m+n-1$ to use the Hölder estimates for Taylor remainders of Lemma 4.22, which is a consequence of the following lemma.

Lemma 4.21. *Let z, ξ be two points in an extension domain $\Omega \subset \mathbb{R}^d$ (open and connected), $M \geq 1$ a natural number, $0 < s \leq 1$, $p > d/s$ and $f \in W^{M+s,p}(\Omega)$. Then, writing $\sigma = \sigma_{d,s,p} = s - \frac{d}{p}$, the Taylor error term satisfies the estimate*

$$|f(\xi) - P_z^M f(\xi)| \leq C \|f\|_{W^{M+s,p}(\Omega)} |z - \xi|^{M+\sigma}.$$

Proof. Let us assume that $0 \in \Omega$. Using the extension $E : W^{M+s,p}(\Omega) \rightarrow W_0^{M+s,p}(B(0, 2 \operatorname{diam}(\Omega)))$ and the Sobolev Embedding Theorem, it suffices to prove the estimate for $f \in C^{M,\sigma}(\mathbb{R}^d)$. We will prove only the case $d = 1$ leaving to the reader the generalization. In that case, we define

$$F_t(u) := \frac{f(t) - P_u^M f(t)}{(t-u)^M}$$

for any $u \neq t \in \mathbb{R}$. We want to see that $|F_t(u)| \leq C \|f\|_{C^{M,\sigma}} |u-t|^\sigma$ for $t \neq u$. Note that the M -differentiability of f implies that $\lim_{\tau \rightarrow t} F_t(\tau) = 0$. Thus, decomposing $P_u^M f(t) = P_u^{M-1} f(t) + \frac{1}{M!} f^{(M)}(u)(t-u)^M$, we have that

$$\begin{aligned} F_t(u) &= \lim_{\tau \rightarrow t} F_t(u) - F_t(\tau) = \lim_{\tau \rightarrow t} \frac{(f(t) - P_u^{M-1} f(t)) - (f(t) - P_\tau^{M-1} f(t))}{(t-u)^M} \\ &\quad + \lim_{\tau \rightarrow t} (f(t) - P_\tau^{M-1} f(t)) \left(\frac{1}{(t-u)^M} - \frac{1}{(t-\tau)^M} \right) \\ &\quad + \lim_{\tau \rightarrow t} \frac{1}{M!} \left(-f^{(M)}(u) + f^{(M)}(\tau) \right) = \textcircled{\text{I}} + \textcircled{\text{II}} + \textcircled{\text{III}}. \end{aligned} \quad (4.28)$$

The first term in (4.28) is

$$\textcircled{\text{I}} = \frac{(f(t) - P_u^{M-1} f(t))}{(t-u)^M}$$

and, using the mean value form of the remainder term of the Taylor polynomial, there exists a point $c_1 \in (u, t)$ such that

$$\textcircled{\text{I}} = \frac{f^{(M)}(c_1)}{M!}.$$

The second term in (4.28) is

$$\begin{aligned} \textcircled{\text{II}} &= \lim_{\tau \rightarrow t} (f(t) - P_\tau^{M-1} f(t)) \left(\frac{(t-\tau)^M - (t-u)^M}{(t-u)^M (t-\tau)^M} \right) \\ &= \lim_{\tau \rightarrow t} (f(t) - P_\tau^{M-1} f(t)) (u-\tau) \left(\sum_{j=1}^M \frac{1}{(t-u)^j (t-\tau)^{M+1-j}} \right) \\ &= \lim_{\tau \rightarrow t} \frac{u-\tau}{t-u} \left(\sum_{j=1}^M \frac{f(t) - P_\tau^{M-1} f(t)}{(t-u)^{j-1} (t-\tau)^{M+1-j}} \right) = - \sum_{j=1}^M \lim_{\tau \rightarrow t} \frac{f(t) - P_\tau^{M-1} f(t)}{(t-u)^{j-1} (t-\tau)^{M+1-j}}. \end{aligned}$$

Applying the Taylor Theorem, only the term $j = 1$ has a non-null limit in the last sum, with

$$\textcircled{\text{II}} = - \frac{f^{(M)}(t)}{M!},$$

so

$$|F_t(u)| \leq \left| \frac{f^{(M)}(c_1)}{M!} - \frac{f^{(M)}(t)}{M!} \right| + \frac{1}{M!} \lim_{\tau \rightarrow t} |f^{(M)}(u) - f^{(M)}(\tau)| \leq \frac{2}{M!} \|f\|_{C^{M,\sigma}} |u-t|^\sigma.$$

□

Recall that in (4.21) we defined the Taylor error terms

$$R_{M,j}^{m_3}(z, \xi) := \partial^j h_{m_3}(\xi) - P_z^{M-j}(\partial^j h_{m_3})(\xi)$$

for $M, j, m_3 \in \mathbb{N}$ and $z, \xi \in \Omega$. Next we give bounds on the size and the smoothness of these terms.

Lemma 4.22. *Consider a real number $p > 2$ and naturals $n, m \in \mathbb{N}$ and let $\Omega \subset \mathbb{C}$ be a Lipschitz domain with parameterizations of the boundary in $B_{p,p}^{n+1-1/p}$. Writing $\sigma_p := 1 - \frac{2}{p}$, for $j \leq m$ we have that*

$$|R_{m+n,j}^{m+1}(z, \xi)| \leq C_{\Omega, n, m} |z - \xi|^{m+n-j+\sigma_p} \quad (4.29)$$

and, if $z_1, z_2, \xi \in \Omega$ with $|z_1 - \xi| > \frac{3}{2}|z_1 - z_2|$, then

$$|R_{m+n-1,j}^m(z_1, \xi) - R_{m+n-1,j}^m(z_2, \xi)| \leq C_{\Omega, n, m} |z_1 - z_2|^{\sigma_p} |z_1 - \xi|^{m+n-j-1}. \quad (4.30)$$

Proof. Recall that $\mathcal{B}^k \chi_\Omega \in W^{n,p}(\Omega)$ for every k by Theorem 3.28. Thus, by (4.19) we have that $\nabla^{m+1} h_{m+1} \in W^{n,p}(\Omega)$ and, since h_{m+1} is continuous and bounded in Ω as well (see (4.18)), we have that $\partial^j h_{m+1} \in W^{n+m+1-j,p}(\Omega)$ for $0 \leq j \leq m+n$. By Lemma 4.21, it follows that

$$|R_{m+n,j}^{m+1}(z, \xi)| \leq C \|\partial^j h_{m+1}\|_{W^{m+n-j+1,p}(\Omega)} |z - \xi|^{m+n-j+\sigma_p}.$$

The second inequality is obtained by the same procedure as [MOV09, Lemma 7]. We quote it here for the sake of completeness. Assume that $z_1, z_2, \xi \in \Omega$ with $|z_1 - \xi| > \frac{3}{2}|z_1 - z_2|$. Then

$$R_{m+n-1,j}^m(z_1, \xi) - R_{m+n-1,j}^m(z_2, \xi) = P_{z_1}^{m+n-1-j} \partial^j h_m(\xi) - P_{z_2}^{m+n-1-j} \partial^j h_m(\xi).$$

But for a natural number M and a function $f \in C^{M, \sigma_p}(\bar{\Omega})$ one has that

$$P_{z_1}^M f(\xi) - P_{z_2}^M f(\xi) = \sum_{|\vec{i}| \leq M} \frac{D^{\vec{i}} f(z_1)}{\vec{i}!} (\xi - z_1)^{\vec{i}} - \sum_{|\vec{j}| \leq M} \frac{D^{\vec{j}} f(z_2)}{\vec{j}!} (\xi - z_2)^{\vec{j}}.$$

Since $(\xi - z_2)^{\vec{j}} = \sum_{\vec{i} \leq \vec{j}} \binom{\vec{j}}{\vec{i}} (z_1 - z_2)^{\vec{j}-\vec{i}} (\xi - z_1)^{\vec{i}}$, one can write

$$\begin{aligned} P_{z_1}^M f(\xi) - P_{z_2}^M f(\xi) &= \sum_{|\vec{i}| \leq M} \frac{D^{\vec{i}} f(z_1)}{\vec{i}!} (\xi - z_1)^{\vec{i}} - \sum_{|\vec{j}| \leq M} \frac{D^{\vec{j}} f(z_2)}{\vec{j}!} \sum_{\vec{i} \leq \vec{j}} \binom{\vec{j}}{\vec{i}} (z_1 - z_2)^{\vec{j}-\vec{i}} (\xi - z_1)^{\vec{i}} \\ &= \sum_{|\vec{i}| \leq M} \frac{(\xi - z_1)^{\vec{i}}}{\vec{i}!} \left(D^{\vec{i}} f(z_1) - \sum_{\substack{|\vec{j}| \leq M \\ \vec{i} \leq \vec{j}}} \frac{D^{\vec{j}} f(z_2)}{\binom{\vec{j}}{\vec{i}}} (z_1 - z_2)^{\vec{j}-\vec{i}} \right) \\ &= \sum_{|\vec{i}| \leq M} \frac{(\xi - z_1)^{\vec{i}}}{\vec{i}!} \left(D^{\vec{i}} f(z_1) - P_{z_2}^{M-|\vec{i}|} D^{\vec{i}} f(z_1) \right). \end{aligned}$$

Therefore, arguing as before,

$$\begin{aligned} |P_{z_1}^M f(\xi) - P_{z_2}^M f(\xi)| &\lesssim \sum_{i \leq M} |\xi - z_1|^i \|f\|_{C^{M, \sigma_p}(\Omega)} |z_1 - z_2|^{M-i+\sigma_p} \\ &\lesssim |\xi - z_1|^M |z_1 - z_2|^{\sigma_p} \|f\|_{C^{M, \sigma_p}(\Omega)}. \end{aligned}$$

□

4.5 Compactness of the double reflection \mathcal{R}_m

Proof of Lemma 4.11. Recall that we want to prove that $\mathcal{R}_m : f \mapsto \chi_\Omega \mathcal{B}(\chi_{\Omega^c} \mathcal{B}^{m-1}(\chi_\Omega f))$ is a compact operator in $W^{n,p}(\Omega)$.

Since $\mathcal{R}_m f$ is analytic in Ω , it is enough to see that $\mathcal{T}_m := \partial^n \mathcal{R}_m : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$ is a compact operator.

Indeed, we have that \mathcal{R}_m is bounded on $W^{n,p}(\Omega)$ by (4.9) and, thus, since the inclusion $W^{n,p}(\Omega) \hookrightarrow W^{n-1,p}(\Omega)$ is compact for any extension domain (see [Tri83, 4.3.2/Remark 1]), we have that $\mathcal{R}_m : W^{n,p}(\Omega) \rightarrow W^{n-1,p}(\Omega)$ is compact. That is, given a bounded sequence $\{f_j\}_j \subset W^{n,p}(\Omega)$, there exists a subsequence $\{f_{j_k}\}_k$ and a function $g \in W^{n-1,p}(\Omega)$ such that $\mathcal{R}_m f_{j_k} \rightarrow g$ in $W^{n-1,p}(\Omega)$. If $\mathcal{T}_m : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$ was a compact operator, then there would be a subsubsequence $\{f_{j_{k_i}}\}_i$ and a function g_n such that $\mathcal{T}_m f_{j_{k_i}} \rightarrow g_n$ in $L^p(\Omega)$. It is immediate to see that g_n is the weak derivative $\partial^n g$ in Ω . Therefore, if \mathcal{T}_m is compact then \mathcal{R}_m is compact as well.

We will prove that \mathcal{T}_m is compact using an approximation argument. Let $f \in W^{n,p}(\Omega)$. For every cube Q , let f_Q be the mean of f in Q . Consider a partition of the unity $\{\psi_Q\}_{Q \in \mathcal{W}}$ such that $\text{supp } \psi_Q \subset \frac{1}{10}Q$ and $|\nabla^j \psi_Q| \lesssim \ell(Q)^{-j}$ for every Whitney cube Q .

For every $i \in \mathbb{N}$ we can define a finite partition of the unity $\{\psi_Q^i\}_{Q \in \mathcal{W}}$ such that

- If $\ell(Q) > 2^{-i}$ then $\psi_Q^i = \psi_Q$.
- If $\ell(Q) = 2^{-i}$ then $\text{supp } \psi_Q^i \subset \mathbf{Sh}(Q)$ (see Definition 1.16) and $|\nabla^j \psi_Q^i| \lesssim \ell(Q)^{-j}$.
- If $\ell(Q) < 2^{-i}$ then $\psi_Q^i \equiv 0$.

Then, writing $f_Q = \int_Q f \, dm$ for the mean of f in Q and $(\mathcal{T}_m(f - f_Q))_Q = \int_Q \mathcal{T}_m(f - f_Q) \, dm$, we can define

$$\mathcal{T}_m^i f(z) = \sum_{Q \in \mathcal{W}: \ell(Q) > 2^{-i}} \mathcal{T}_m(f)(z) \psi_Q(z) + \sum_{Q \in \mathcal{W}: \ell(Q) = 2^{-i}} (\mathcal{T}_m(f - f_Q))_Q \psi_Q^i(z).$$

We will prove the following two claims.

Claim 4.23. *For every $i \in \mathbb{N}$, the operator $\mathcal{T}_m^i : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$ is compact.*

Claim 4.24. *The norm of the error operator $\mathcal{E}^i := \mathcal{T}_m - \mathcal{T}_m^i : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$ tends to zero as i tends to infinity.*

Then the compactness of \mathcal{T}_m is a well-known consequence of the previous two claims (see Theorem 4.6). By all the exposed above, this proves Lemma 4.11. \square

Proof of Claim 4.23. We will prove that the operator $\mathcal{T}_m^i : W^{n,p}(\Omega) \rightarrow W^{1,p}(\Omega)$ is bounded. As before, since Ω is an extension domain, the embedding $W^{1,p}(\Omega) \hookrightarrow L^p(\Omega)$ is compact. Therefore we will deduce the compactness of $\mathcal{T}_m^i : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$. Note that the specific value of the operator norm $\|\mathcal{T}_m^i\|_{W^{n,p}(\Omega) \rightarrow W^{1,p}(\Omega)}$ is not important for our argument, since we only care about compactness.

Consider a fixed $i \in \mathbb{N}$ and $f \in W^{n,p}(\Omega)$. For every $z \in \Omega$ and every first order derivative D , since $\mathcal{T}_m f$ is analytic in Ω , we can use the Leibniz rule (1.11) to get

$$D\mathcal{T}_m^i f = \sum_{Q: \ell(Q) > 2^{-i}} D\mathcal{T}_m(f) \psi_Q + \sum_{Q: \ell(Q) > 2^{-i}} \mathcal{T}_m(f) D\psi_Q + \sum_{Q: \ell(Q) = 2^{-i}} (\mathcal{T}_m(f - f_Q))_Q D\psi_Q^i.$$

By Jensen's inequality $|\mathcal{T}_m(f - f_Q)|_Q \leq \|\mathcal{T}_m(f - f_Q)\|_{L^p(Q)} \ell(Q)^{-2/p}$, so

$$\begin{aligned} |\nabla \mathcal{T}_m^i f(z)| &\leq \sum_{Q:\ell(Q)>2^{-i}} \chi_{\frac{11}{10}Q}(z) |\nabla \mathcal{T}_m f(z)| + \sum_{Q:\ell(Q)>2^{-i}} |\nabla \psi_Q(z)| |\mathcal{T}_m f(z)| \\ &+ \sum_{Q:\ell(Q)=2^{-i}} |\nabla \psi_Q^i(z)| \|\mathcal{T}_m(f - f_Q)\|_{L^p(Q)} (2^{-i})^{-2/p}. \end{aligned} \quad (4.31)$$

Using the finite overlapping of the double Whitney cubes and the fact that $|\nabla \psi_Q^i(z)| \lesssim 2^i$ for every Whitney cube Q , writing Ω_i for $\bigcup_{Q:\ell(Q)>2^{-i}} \text{supp}(\psi_Q)$ we can conclude that

$$\|\nabla \mathcal{T}_m^i f\|_{L^p(\Omega)}^p \lesssim_{i,p} \|\nabla \mathcal{T}_m f\|_{L^p(\Omega_i)}^p + \|\mathcal{T}_m f\|_{L^p(\Omega_i)}^p + \sum_{Q:\ell(Q)=2^{-i}} \left(\|\mathcal{T}_m f\|_{L^p(Q)}^p + |f_Q|^p \|\mathcal{T}_m 1\|_{L^p(Q)}^p \right).$$

By the Sobolev Embedding Theorem

$$|f_Q| \leq \|f\|_{L^\infty(\Omega)} \lesssim_{\Omega,p} \|f\|_{W^{1,p}(\Omega)}. \quad (4.32)$$

Thus, since $\mathcal{T}_m : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$ is bounded, we have that

$$\|\nabla \mathcal{T}_m^i f\|_{L^p(\Omega)} \lesssim_{p,i,\Omega} \|\nabla \mathcal{T}_m f\|_{L^p(\Omega_i)} + \|f\|_{W^{n,p}(\Omega)}. \quad (4.33)$$

To see that $\|\nabla \mathcal{T}_m f\|_{L^p(\Omega_i)} \lesssim_i \|f\|_{W^{n,p}(\Omega)}$, note that $\nabla \mathcal{T}_m f = \nabla \partial^n \mathcal{B}(\chi_{\Omega^c} \mathcal{B}^{m-1}(\chi_\Omega f))$. We have that $\mathcal{B}^{m-1} : L^p(\Omega) \rightarrow L^p(\Omega^c)$ is bounded trivially, and for $z \in \Omega_i$ and $g \in L^p$ supported in Ω^c we have that

$$|\nabla \partial^n \mathcal{B}g(z)| \lesssim \int_{|z-w|>2^{-i}} \frac{1}{|z-w|^{n+3}} g(w) dm(w).$$

This is the convolution of g with an L^1 kernel, so Young's inequality (1.10) tells us that

$$\|\nabla \partial^n \mathcal{B}g\|_{L^p(\Omega_i)} \leq C_i \|g\|_{L^p},$$

proving that

$$\|\nabla \mathcal{T}_m f\|_{L^p(\Omega_i)} \lesssim_i \|\mathcal{B}^{m-1}(\chi_\Omega f)\|_{L^p(\Omega^c)} \lesssim \|f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{n,p}(\Omega)}. \quad (4.34)$$

Combining (4.33) and (4.34), we have seen that $\|\nabla \mathcal{T}_m^i f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{n,p}(\Omega)}$. The reader can use Jensen's inequality as in (4.31) to check that $\|\mathcal{T}_m^i f\|_{L^p(\Omega)} \lesssim \|f\|_{W^{n,p}(\Omega)}$ as well. This, proves that the operator $\mathcal{T}_m^i : W^{n,p}(\Omega) \rightarrow W^{1,p}(\Omega)$ is bounded and, therefore, composing with the compact inclusion, the operator $\mathcal{T}_m^i : W^{n,p}(\Omega) \rightarrow L^p(\Omega)$ is compact. \square

Proof of Claim 4.24. We want to see that the error operator

$$\mathcal{E}^i = \mathcal{T}_m - \mathcal{T}_m^i$$

satisfies that $\|\mathcal{E}^i\|_{W^{n,p}(\Omega) \rightarrow L^p(\Omega)}$ tends to zero as i tends to infinity.

Consider the set $\Omega_i = \bigcup_{Q:\ell(Q)>2^{-i}} \text{supp}(\psi_Q)$. We define the modified error operator \mathcal{E}_0^i acting in $f \in W^{n,p}(\Omega)$ as

$$\mathcal{E}_0^i f(z) := \chi_{\Omega \setminus \Omega_{i-1}}(z) \sum_{Q:\ell(Q)=2^{-i}} \sum_{\substack{S:\ell(S) \leq 2^{-i} \\ S \subset \text{Sh}(Q)}} \left| \mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_Q))_Q \right| \chi_{2S}(z)$$

for every $z \in \Omega$. The first step will be proving that

$$\|\mathcal{E}^i f\|_{L^p(\Omega)} \lesssim \|\mathcal{E}_0^i f\|_{L^p(\Omega)} + C_i \|f\|_{W^{1,p}(\Omega)}, \quad (4.35)$$

with $C_i \xrightarrow{i \rightarrow \infty} 0$.

Note that $\mathcal{T}_m 1 = \mathcal{T}_m \chi_\Omega$ because $\mathcal{T}_m \cdot = \partial^n \chi_\Omega \mathcal{B}(\chi_{\Omega^c} \mathcal{B}^{m-1}(\chi_\Omega \cdot))$. Let us write

$$\mathcal{T}_m f(z) = \sum_{S \in \mathcal{W}: \ell(S) > 2^{-i}} \mathcal{T}_m(f)(z) \psi_S(z) + \sum_{S \in \mathcal{W}: \ell(S) \leq 2^{-i}} (f_S \mathcal{T}_m(1)(z) + \mathcal{T}_m(f - f_S)(z)) \psi_S(z)$$

for $z \in \Omega$. Recall that

$$\mathcal{T}_m^i f(z) = \sum_{Q \in \mathcal{W}: \ell(Q) > 2^{-i}} \mathcal{T}_m(f)(z) \psi_Q(z) + \sum_{Q \in \mathcal{W}: \ell(Q) = 2^{-i}} (\mathcal{T}_m(f - f_Q))_Q \psi_Q^i(z).$$

Thus, for the error operator \mathcal{E}^i we have the expression

$$\begin{aligned} \mathcal{E}^i f(z) &= \mathcal{T}_m f(z) - \mathcal{T}_m^i f(z) = \sum_{S: \ell(S) \leq 2^{-i}} f_S \mathcal{T}_m(1)(z) \psi_S(z) \\ &\quad + \left(\sum_{S: \ell(S) \leq 2^{-i}} \mathcal{T}_m(f - f_S)(z) \psi_S(z) - \sum_{Q: \ell(Q) = 2^{-i}} (\mathcal{T}_m(f - f_Q))_Q \psi_Q^i(z) \right) \\ &= \mathcal{E}_1^i f(z) + \mathcal{E}_2^i f(z). \end{aligned} \quad (4.36)$$

The first part is easy to bound using again (4.32). Indeed, we have that

$$\|\mathcal{E}_1^i f\|_{L^p(\Omega)}^p \lesssim_p \sum_{S: \ell(S) \leq 2^{-i}} |f_S|^p \|\mathcal{T}_m(1)\|_{L^p(11/10S)}^p \lesssim_\Omega \|f\|_{W^{1,p}(\Omega)}^p \|\mathcal{T}_m(1)\|_{L^p(\Omega \setminus \Omega_{i-1})}^p, \quad (4.37)$$

where $\|\mathcal{T}_m(1)\|_{L^p(\Omega \setminus \Omega_i)}^p \xrightarrow{i \rightarrow \infty} 0$.

To control $\mathcal{E}_2^i f$ in (4.36), note that every $z \in \Omega$ satisfies

$$\sum_{S: \ell(S) \leq 2^{-i}} \psi_S(z) = \sum_{Q: \ell(Q) = 2^{-i}} \psi_Q^i(z) \leq 1, \quad (4.38)$$

with equality when $z \notin \bigcup_{\ell(Q) > 2^{-i}} \text{supp}(\psi_Q)$, that is, when $z \in \Omega \setminus \Omega_i$. Recall that

$$\mathcal{E}_2^i f(z) = \sum_{S: \ell(S) \leq 2^{-i}} \mathcal{T}_m(f - f_S)(z) \psi_S(z) - \sum_{Q: \ell(Q) = 2^{-i}} (\mathcal{T}_m(f - f_Q))_Q \psi_Q^i(z).$$

If $z \in \Omega \setminus \Omega_i$, we have equality in (4.38), i.e., $\sum_{S: \ell(S) \leq 2^{-i}} \psi_S(z) = \sum_{Q: \ell(Q) = 2^{-i}} \psi_Q^i(z) = 1$. Thus

$$\begin{aligned} \mathcal{E}_2^i f(z) &= \sum_{S: \ell(S) \leq 2^{-i}} \mathcal{T}_m(f - f_S)(z) \psi_S(z) - \sum_{Q: \ell(Q) = 2^{-i}} \psi_Q^i(z) \\ &\quad - \sum_{Q: \ell(Q) = 2^{-i}} (\mathcal{T}_m(f - f_Q))_Q \psi_Q^i(z) + \sum_{S: \ell(S) \leq 2^{-i}} \psi_S(z) \\ &= \sum_{Q: \ell(Q) = 2^{-i}} \sum_{S: \ell(S) \leq 2^{-i}} \left(\mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_Q))_Q \right) \psi_S(z) \psi_Q^i(z). \end{aligned} \quad (4.39)$$

If, instead, $z \in \Omega_i = \bigcup_{Q:\ell(Q)>2^{-i}} \text{supp}(\psi_Q)$ then there is a cube S_0 with $z \in \text{supp}(\psi_{S_0})$ and $\ell(S_0) \geq 2^{-i+1}$. Therefore, any other cube S with $\psi_S(z) \neq 0$ must have side-length $\ell(S) \geq 2^{-i}$ because any neighbor cube of S_0 has side-length at least $\frac{1}{2}\ell(S_0)$ (see Definition 1.5). Therefore,

$$\begin{aligned} \mathcal{E}_2^i f(z) &= \sum_{S:\ell(S)=2^{-i}} \mathcal{T}_m(f-f_S)(z)\psi_S(z) - \sum_{Q:\ell(Q)=2^{-i}} (\mathcal{T}_m(f-f_Q))_Q \psi_Q^i(z) \\ &= \sum_{Q:\ell(Q)=2^{-i}} \left(\mathcal{T}_m(f-f_Q)(z)\psi_Q(z) - (\mathcal{T}_m(f-f_Q))_Q \psi_Q^i(z) \right). \end{aligned}$$

Adding and subtracting $\mathcal{T}_m(f-f_Q)(z)\psi_Q^i(z)$ at each term of this sum, we get

$$\begin{aligned} \mathcal{E}_2^i f(z) &= \sum_{Q:\ell(Q)=2^{-i}} \mathcal{T}_m(f-f_Q)(z) (\psi_Q(z) - \psi_Q^i(z)) \\ &\quad + \sum_{Q:\ell(Q)=2^{-i}} \left(\mathcal{T}_m(f-f_Q)(z) - (\mathcal{T}_m(f-f_Q))_Q \right) \psi_Q^i(z). \end{aligned} \quad (4.40)$$

Summing up, by (4.39) and (4.40) we have that

$$\begin{aligned} \mathcal{E}_2^i f(z) &= \chi_{\Omega \setminus \Omega_i}(z) \sum_{Q:\ell(Q)=2^{-i}} \sum_{S:\ell(S) \leq 2^{-i}} \left(\mathcal{T}_m(f-f_S)(z) - (\mathcal{T}_m(f-f_Q))_Q \right) \psi_S(z)\psi_Q^i(z) \\ &\quad + \chi_{\Omega_i \setminus \Omega_{i-1}}(z) \sum_{Q:\ell(Q)=2^{-i}} \left(\mathcal{T}_m(f-f_Q)(z) - (\mathcal{T}_m(f-f_Q))_Q \right) \psi_Q^i(z) \\ &\quad + \chi_{\Omega_i \setminus \Omega_{i-1}}(z) \sum_{Q:\ell(Q)=2^{-i}} \mathcal{T}_m(f-f_Q)(z) (\psi_Q(z) - \psi_Q^i(z)). \end{aligned}$$

Every cube Q with $\ell(Q) = 2^{-i}$ satisfies that $\text{supp} \psi_Q^i \subset \mathbf{Sh}(Q)$, and, choosing conveniently ψ_Q^i we can assume that $\text{supp} \psi_Q^i \cap \Omega_i \subset 2Q$. Therefore, we get that

$$\begin{aligned} |\mathcal{E}_2^i f(z)| &\lesssim \chi_{\Omega \setminus \Omega_{i-1}}(z) \sum_{Q:\ell(Q)=2^{-i}} \sum_{\substack{S:\ell(S) \leq 2^{-i} \\ S \subset \mathbf{Sh}(Q)}} \left| \mathcal{T}_m(f-f_S)(z) - (\mathcal{T}_m(f-f_Q))_Q \right| \chi_{2S}(z) \\ &\quad + \chi_{\Omega_i \setminus \Omega_{i-1}}(z) \left| \sum_{Q:\ell(Q)=2^{-i}} \mathcal{T}_m(f-f_Q)(z) (\psi_Q(z) - \psi_Q^i(z)) \right|. \end{aligned} \quad (4.41)$$

For the last term, just note that for $z \in \Omega_i \setminus \Omega_{i-1}$, using the first equality in (4.38) we have that

$$\sum_{Q:\ell(Q)=2^{-i}} \mathcal{T}_m(f-f_Q)(z) (\psi_Q^i(z) - \psi_Q(z)) = \mathcal{T}_m(f-f_Q)(z) \left(\sum_{Q:\ell(Q)=2^{-i}} \psi_Q^i(z) - \sum_{Q:\ell(Q)=2^{-i}} \psi_Q(z) \right) \equiv 0.$$

Thus,

$$\sum_{Q:\ell(Q)=2^{-i}} \mathcal{T}_m(f-f_Q)(z) (\psi_Q^i(z) - \psi_Q(z)) = \sum_{Q:\ell(Q)=2^{-i}} -\mathcal{T}_m(f_Q)(z) (\psi_Q^i(z) - \psi_Q(z)),$$

which can be bounded as \mathcal{E}_1^i in (4.37). This fact, together with (4.36), (4.37) and (4.41) settles (4.35), that is,

$$\|\mathcal{E}^i f\|_{L^p(\Omega)} \lesssim \|\mathcal{E}_0^i f\|_{L^p(\Omega)} + C_{i,\Omega,n,p} \|f\|_{W^{1,p}(\Omega)},$$

with $C_{i,\Omega,n,p} \xrightarrow{i \rightarrow \infty} 0$.

Next we prove that for the modified error term,

$$\mathcal{E}_0^i f(z) = \chi_{\Omega \setminus \Omega_{i-1}}(z) \sum_{Q: \ell(Q)=2^{-i}} \sum_{\substack{S: \ell(S) \leq 2^{-i} \\ S \subset \mathbf{Sh}(Q)}} \left| \mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_Q))_Q \right| \chi_{2S}(z),$$

we have that $\|\mathcal{E}_0^i f\|_{L^p(\Omega)} \lesssim C_i \|f\|_{W^{1,p}(\Omega)}$ with $C_i \xrightarrow{i \rightarrow \infty} 0$.

Arguing by duality, we have that

$$\|\mathcal{E}_0^i f\|_{L^p} = \sup_{\substack{g \in L^{p'} \\ \|g\|_{p'}=1}} \int_{\Omega \setminus \Omega_{i-1}} \sum_{\substack{Q: \ell(Q)=2^{-i} \\ S: \ell(S) \leq 2^{-i} \\ S \subset \mathbf{Sh}(Q)}} \left| \mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_Q))_Q \right| \chi_{2S}(z) |g(z)| dm(z). \quad (4.42)$$

First note for every pair of Whitney cubes Q and S with $S \subset \mathbf{Sh}(Q)$ and every point z , using an admissible chain $[S, Q] = [S, Q] \setminus \{Q\}$ we get that

$$\begin{aligned} \mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_Q))_Q &= \mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_S))_S \\ &\quad + \sum_{P \in [S, Q]} (\mathcal{T}_m(f - f_P))_P - (\mathcal{T}_m(f - f_{\mathcal{N}(P)}))_{\mathcal{N}(P)}, \end{aligned}$$

where $\mathcal{N}(P)$ stands for the ‘‘next’’ cube in the chain $[S, Q]$ (see Definition 1.13). Note that the shadows of cubes of fixed side-length have finite overlapping since $|\mathbf{Sh}(Q)| \approx |Q|$ and, therefore, every Whitney cube S appears less than C times in the right-hand side of (4.42). Thus,

$$\begin{aligned} \|\mathcal{E}_0^i f\|_{L^p} &\lesssim \sup_{g: \|g\|_{p'}=1} \left(\sum_{S: \ell(S) \leq 2^{-i}} \int_{2S} |\mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_S))_S| |g(z)| dm(z) \right. \\ &\quad \left. + \sum_{\substack{Q: \ell(Q)=2^{-i} \\ S: \ell(S) \leq 2^{-i} \\ S \subset \mathbf{Sh}(Q)}} \sum_{P \in [S, Q]} \left| (\mathcal{T}_m(f - f_P))_P - (\mathcal{T}_m(f - f_{\mathcal{N}(P)}))_{\mathcal{N}(P)} \right| \int_{2S} |g(z)| dm(z) \right). \end{aligned} \quad (4.43)$$

All the cubes $P \in [S, Q]$ with $S \in \mathbf{Sh}(Q)$, satisfy that $\ell(P) \lesssim D(Q, S) \approx \ell(Q)$ by Definition 1.13. If we assume that $\ell(Q) = 2^{-i}$ this implies that $\ell(P) \leq C2^{-i}$. Moreover, we have that

$$\left| (\mathcal{T}_m(f - f_P))_P - (\mathcal{T}_m(f - f_{\mathcal{N}(P)}))_{\mathcal{N}(P)} \right| \leq \sum_{L \cap 2P \neq \emptyset} \int_P |\mathcal{T}_m(f - f_P)(z) - (\mathcal{T}_m(f - f_L))_L| dm(z). \quad (4.44)$$

Finally, we observe that $P \in [S, Q]$ with $S \subset \mathbf{Sh}(Q)$ imply that $D(P, S) \leq C\ell(P)$ (see (1.26)). Thus, for a fixed P with $\ell(P) \leq C2^{-i}$ and $g \in L^{p'}$, we have that

$$\sum_{\substack{Q: \ell(Q)=2^{-i} \\ S: S \subset \mathbf{Sh}(Q) \\ P \in [S, Q]}} \int_{2S} |g(z)| dm(z) \lesssim C \sum_{S: D(P, S) \leq C\ell(P)} \int_{2S} |g(z)| dm(z) \lesssim \ell(P)^d \inf_P M g. \quad (4.45)$$

Note that in the first step, as we did in (4.43), we have used that every cube S appears less than C times in the left-hand side. By (4.43), (4.44) and applying (4.45) after reordering, we get that

$$\|\mathcal{E}_0^i f\|_{L^p} \lesssim \sup_{\|g\|_{p'}=1} \sum_{\substack{S: \ell(S) \leq 2^{-i} \\ L \cap 2S \neq \emptyset}} \int_{2S} |(\mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_L))_L)| (|g(z)| + M g(z)) dm(z).$$

Since $\|Mg\|_{L^{p'}} \lesssim \|g\|_{L^{p'}} \leq 1$, we have that

$$\|\mathcal{E}_0^i f\|_{L^p} \lesssim \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} |\mathcal{T}_m(f - f_S)(z) - (\mathcal{T}_m(f - f_L))_L| |g(z)| dm(z),$$

where $\mathcal{W}_0 = \{(S, L) : \ell(S) \leq C2^{-i} \text{ and } 2S \cap L \neq \emptyset\}$.

For every cube Q , let φ_Q be a radial bump function with $\chi_{10Q} \leq \varphi_Q \leq \chi_{20Q}$ and the usual bounds in their derivatives. Now we use these bump functions to separate the local and the non-local parts. In the local part we can neglect the cancellation and use the triangle inequality (and the fact that $\int_{2S} |g| dm \lesssim \inf_{7S} Mg$), but in the non-local part the smoothness of a certain kernel will be crucial, so we write

$$\begin{aligned} \|\mathcal{E}_0^i f\|_{L^p} &\lesssim \sup_{\|g\|_{p'}=1} \sum_{S: \ell(S) \leq C2^{-i}} \int_{2S} |\mathcal{T}_m[(f - f_S)\varphi_S](z)| |g(z)| dm(z) \\ &\quad + \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2L} |\mathcal{T}_m[(f - f_L)\varphi_S](\xi)| Mg(\xi) dm(\xi) \\ &\quad + \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \left| \mathcal{T}_m[(f - f_S)(1 - \varphi_S)](z) - (\mathcal{T}_m[(f - f_L)(1 - \varphi_S)])_L \right| |g(z)| dm(z) \\ &= \boxed{7} + \boxed{7} + \boxed{8}. \end{aligned} \tag{4.46}$$

Note that the inequality $|g| \leq Mg$ (which is valid almost everywhere for g in L_{loc}^1) imply that $\boxed{7} \leq \boxed{7}$.

First we take a look at $\boxed{7}$. For any pair of neighbor Whitney cubes S and L and $z \in 2L$, using the definition of weak derivative and Fubini's Theorem we find that

$$\begin{aligned} \mathcal{T}_m[(f - f_L)\varphi_S](z) &= c_n \int_{\Omega^c} \frac{1}{(z - w)^{n+2}} \int_{20S} \frac{(\overline{w - \xi})^{m-1}}{(w - \xi)^{m+1}} (f(\xi) - f_L)\varphi_S(\xi) dm(\xi) dm(w) \\ &= c_{n,m} \int_{\Omega^c} \frac{1}{(z - w)^{n+2}} \int_{20S} \frac{(\overline{w - \xi})^m}{(w - \xi)^{m+1}} \bar{\partial}[(f - f_L)\varphi_S](\xi) dm(\xi) dm(w) \\ &= c_{n,m} \int_{20S} \left(\int_{\Omega^c} \frac{(\overline{w - \xi})^m}{(w - \xi)^{m+1} (z - w)^{n+2}} dm(w) \right) \bar{\partial}[(f - f_L)\varphi_S](\xi) dm(\xi). \end{aligned}$$

In the right-hand side above, we have that $\xi, z \in \Omega$. Therefore, we can use Green's Theorem in the integral on Ω^c and then (4.17) to get

$$\begin{aligned} \mathcal{T}_m[(f - f_L)\varphi_S](z) &= c_{n,m} \int_{20S} \left(\int_{\partial\Omega} \frac{(\overline{w - \xi})^{m+1}}{(w - \xi)^{m+1} (z - w)^{n+2}} dw \right) \bar{\partial}[(f - f_L)\varphi_S](\xi) dm(\xi) \\ &= c_{n,m} \int_{20S} K_{\vec{m}_0}(z, \xi) \bar{\partial}[(f - f_L)\varphi_S](\xi) dm(\xi), \end{aligned}$$

where $\vec{m}_0 := (2 + n, m + 1, m + 1)$.

Using Proposition 4.15 we have that

$$K_{\vec{m}_0}(z, \xi) = c_{m,n} \partial^n \mathcal{B}\chi_\Omega(z) \frac{(\overline{\xi - z})^m}{(\xi - z)^{m+1}} + \sum_{j \leq m} \frac{c_{m,n,j} R_{m+n,j}^{m+1}(z, \xi)}{(\xi - z)^{m+n+2-j}}.$$

The first part is $\partial^n \mathcal{B}\chi_\Omega(z)$ times the kernel of the operator $T^{(-m-1, m)}$. For the second part, we have that by Lemma 4.22

$$\frac{|R_{m+n,j}^{m+1}(z, \xi)|}{|\xi - z|^{m+n+2-j}} \lesssim \frac{1}{|\xi - z|^{2-\sigma_p}},$$

where $\sigma_p = 1 - \frac{2}{p}$. Thus,

$$\begin{aligned} \boxed{7} &= \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2L} |\mathcal{T}_m[(f - f_L)\varphi_S](z)| Mg(z) dm(z) \\ &\lesssim \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2L} \left| \partial^n \mathcal{B}\chi_\Omega(z) T^{(-m-1,m)} (\bar{\partial}[(f - f_L)\varphi_S])(z) \right| Mg(z) dm(z) \\ &\quad + \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2L} \int_{20S} \frac{|\bar{\partial}[(f - f_L)\varphi_S](\xi)|}{|\xi - z|^{2-\sigma_p}} dm(\xi) Mg(z) dm(z) = \boxed{7.1} + \boxed{7.2}. \end{aligned} \quad (4.47)$$

In the first sum we use that in $W^{1,p}(\mathbb{C})$ we have the identity $T^{(-m-1,m)} \circ \bar{\partial} = \bar{\partial} \circ T^{(-m-1,m)} = c_m \mathcal{B}^m$ and, therefore, $T^{(-m-1,m)} \bar{\partial}[(f - f_L)\varphi_S] = c_m \mathcal{B}^m[(f - f_L)\varphi_S] \in W^{1,p} \subset L^\infty$, so

$$\begin{aligned} \boxed{7.1} &\lesssim \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2L} |\partial^n \mathcal{B}\chi_\Omega(z)| Mg(z) dm(z) \|\mathcal{B}^m[(f - f_L)\varphi_S]\|_{L^\infty} \\ &\lesssim \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(2L)} \|Mg\|_{L^{p'}(2L)} \|\mathcal{B}^m[(f - f_L)\varphi_S]\|_{W^{1,p}(\mathbb{C})}. \end{aligned}$$

By the boundedness of \mathcal{B}^m in $W^{1,p}(\mathbb{C})$ we have that

$$\|\mathcal{B}^m[(f - f_L)\varphi_S]\|_{W^{1,p}(\mathbb{C})} \lesssim \|(f - f_L)\varphi_S\|_{W^{1,p}(20S)}.$$

Moreover, the Poincaré inequality (1.33) allows us to deduce that

$$\|(f - f_L)\varphi_S\|_{W^{1,p}(20S)} \lesssim \|\nabla f\|_{L^p(20S)}.$$

On the other hand, there is a certain i_0 such that for $\ell(S) \leq C2^{-i}$ and $L \cap 2S \neq \emptyset$, we have that $S, 2L \subset \Omega \setminus \Omega_{i-i_0}$, and

$$\|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(2L)} \leq \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})}.$$

Thus, by the Hölder inequality and the boundedness of the maximal operator in $L^{p'}$ we have that

$$\begin{aligned} \boxed{7.1} &\lesssim \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})} \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \|Mg\|_{L^{p'}(2L)} \|\nabla f\|_{L^p(20S)} \\ &\leq C_{\Omega,i} \|\nabla f\|_{L^p(\Omega)} \sup_{\|g\|_{p'}=1} \|Mg\|_{L^{p'}} \lesssim_p C_{\Omega,i} \|\nabla f\|_{L^p(\Omega)}, \end{aligned} \quad (4.48)$$

with $C_{\Omega,i} \xrightarrow{i \rightarrow \infty} 0$.

To bound the term $\boxed{7.2}$ in (4.47), note that given two neighbor cubes S and L and a point $z \in 2L$, by (1.28) we have that

$$\int_{20S} \frac{|\bar{\partial}[(f - f_L)\varphi_S](\xi)|}{|\xi - z|^{2-\sigma_p}} dm(\xi) \lesssim M(\bar{\partial}[(f - f_L)\varphi_S])(z) \ell(S)^{\sigma_p}.$$

Thus,

$$\begin{aligned} \boxed{7.2} &\lesssim \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2L} M(\bar{\partial}[(f - f_L)\varphi_S])(z) \ell(S)^{\sigma_p} Mg(z) dm(z) \\ &\lesssim 2^{-i\sigma_p} \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \|M(\bar{\partial}[(f - f_L)\varphi_S])\|_{L^p(\Omega)} \|Mg\|_{L^{p'}(2L)} \end{aligned}$$

and, by the boundedness of the maximal operator, (1.33) and the Hölder inequality, we get

$$\boxed{7.2} \lesssim 2^{-i\sigma_p} \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \|\bar{\partial}[(f - f_L)\varphi_S]\|_{L^p(20S)} \|Mg\|_{L^{p'}(2L)} \lesssim 2^{-i\sigma_p} \|\nabla f\|_{L^p(\Omega)}. \quad (4.49)$$

By (4.47), (4.48) and (4.49), we have that

$$\boxed{7} \lesssim C_{\Omega,i} \|\nabla f\|_{L^p(\Omega)}, \quad (4.50)$$

with $C_{\Omega,i} \xrightarrow{i \rightarrow \infty} 0$.

Back to (4.46) it remains to bound

$$\boxed{8} = \sup_{\|g\|_{p'}=1} \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} |\mathcal{T}_m[(f - f_S)(1 - \varphi_S)](z) - (\mathcal{T}_m[(f - f_L)(1 - \varphi_S)])_L| |g(z)| dm(z).$$

Fix $g \geq 0$ such that $\|g\|_{p'} = 1$. Then we will prove that

$$\boxed{8g} \leq C_{\Omega,i} \|f\|_{W^{1,p}(\Omega)},$$

with $C_{\Omega,i} \xrightarrow{i \rightarrow \infty} 0$, where

$$\boxed{8g} := \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \int_L |\mathcal{T}_m[(f - f_S)(1 - \varphi_S)](z) - \mathcal{T}_m[(f - f_L)(1 - \varphi_S)](\zeta)| dm(\zeta) g(z) dm(z).$$

First, we add and subtract $\mathcal{T}_m[(f - f_L)(1 - \varphi_S)](z)$ in each term of the last sum to get

$$\begin{aligned} \boxed{8g} &\leq \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} |\mathcal{T}_m[(f_L - f_S)(1 - \varphi_S)](z)| \int_L dm(\zeta) g(z) dm(z) \\ &\quad + \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \int_L |\mathcal{T}_m[(f - f_L)(1 - \varphi_S)](z) - \mathcal{T}_m[(f - f_L)(1 - \varphi_S)](\zeta)| dm(\zeta) g(z) dm(z). \end{aligned}$$

For a given $z \in \Omega$,

$$\int_{\Omega^c} \int_{\Omega} \frac{|f(\xi) - f_L|}{|z - w|^{n+2} |w - \xi|^2} dm(\xi) dm(w) \lesssim \|f\|_{L^\infty} \int_{\Omega^c} \frac{|\log(\text{dist}(w, \Omega))| + |\log(\text{diam}(\Omega))|}{|z - w|^{n+2}} dm(w),$$

which is finite since Ω is a Lipschitz domain (hint: compare the last integral above with the length of the boundary $\mathcal{H}^1(\partial\Omega)$ times the integral $\int_0^1 |\log(t)| dt$). Thus, we can use Fubini's Theorem and then Green's Theorem to state that

$$\begin{aligned} \mathcal{T}_m[(f - f_L)(1 - \varphi_S)](z) &= c_n \int_{\Omega^c} \frac{1}{(z - w)^{n+2}} \int_{\Omega} \frac{\overline{(w - \xi)}^{m-1}}{(w - \xi)^{m+1}} (f(\xi) - f_L)(1 - \varphi_S(\xi)) dm(\xi) dm(w) \\ &= c_{n,m} \int_{\Omega} \left(\int_{\partial\Omega} \frac{\overline{(w - \xi)}^m}{(w - \xi)^{m+1} (z - w)^{n+2}} dw \right) [(f - f_L)(1 - \varphi_S)](\xi) dm(\xi) \\ &= c_{n,m} \int_{\Omega} K_{\vec{m}_1}(z, \xi) [(f - f_L)(1 - \varphi_S)](\xi) dm(\xi), \end{aligned}$$

where $\vec{m}_1 := (2 + n, m + 1, m)$. Arguing analogously,

$$\mathcal{T}_m[(f_L - f_S)(1 - \varphi_S)](z) = c_{n,m} (f_L - f_S) \int_{\Omega \setminus 10S} K_{\vec{m}_1}(z, \xi) [(1 - \varphi_S)](\xi) dm(\xi).$$

Thus, we get that

$$\begin{aligned}
\boxed{\text{8g}} &\lesssim \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} |f_L - f_S| \left| \int_{\Omega \setminus 10S} K_{\bar{m}_1}(z, \xi) [(1 - \varphi_S)](\xi) dm(\xi) \right| g(z) dm(z) \\
&+ \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} f_L \left| \int_{\Omega} (K_{\bar{m}_1}(z, \xi) - K_{\bar{m}_1}(\zeta, \xi)) [(f - f_L)(1 - \varphi_S)](\xi) dm(\xi) \right| dm(\zeta) g(z) dm(z) \\
&= \boxed{\text{8.1}} + \boxed{\text{8.2}}.
\end{aligned} \tag{4.51}$$

Recall that Proposition 4.15 states that for $z \in 2S$ and $\xi \in \Omega$,

$$K_{\bar{m}_1}(z, \xi) = c_{m,n} \partial^n \mathcal{B}_{\chi\Omega}(z) \frac{(\overline{\xi - z})^{m-1}}{(\xi - z)^{m+1}} + \sum_{j \leq m} \frac{c_{m,n,j} R_{m+n-1,j}^m(z, \xi)}{(\xi - z)^{m+n+2-j}} \tag{4.52}$$

and, for any $z, \xi \in \Omega$, by (4.29) we have that

$$\left| \frac{R_{m+n-1,j}^m(z, \xi)}{(z - \xi)^{m+n+2-j}} \right| \leq C_{\Omega,n,m} \frac{1}{|z - \xi|^{3-\sigma_p}}, \tag{4.53}$$

where $\sigma_p = 1 - \frac{2}{p}$. Thus, by (4.30) and the identity $\frac{1}{a^j} - \frac{1}{b^j} = \frac{(b-a) \sum_{i=0}^{j-1} a^i b^{j-1-i}}{a^j b^j}$, when $z, \zeta \in 5S$ and $\xi \in \Omega \setminus 20S$ we have that

$$\begin{aligned}
&\left| \frac{R_{m+n-1,j}^m(z, \xi)}{(\xi - z)^{m+n+2-j}} - \frac{R_{m+n-1,j}^m(\zeta, \xi)}{(\xi - \zeta)^{m+n+2-j}} \right| \leq \left| R_{m+n-1,j}^m(z, \xi) \left(\frac{1}{(\xi - z)^{m+n+2-j}} - \frac{1}{(\xi - \zeta)^{m+n+2-j}} \right) \right| \\
&+ \left| \frac{R_{m+n-1,j}^m(z, \xi) - R_{m+n-1,j}^m(\zeta, \xi)}{(\xi - \zeta)^{m+n+2-j}} \right| \lesssim_{\Omega,n,m} \frac{|z - \zeta|}{|z - \xi|^{4-\sigma_p}} + \frac{|z - \zeta|^{\sigma_p}}{|z - \xi|^3} \lesssim \frac{|z - \zeta|^{\sigma_p}}{|z - \xi|^3}.
\end{aligned} \tag{4.54}$$

Then, using that $\text{dist}(2S, \text{supp}(1 - \varphi_S)) > 0$, we have that $\int_{\Omega} \frac{(\overline{\xi - z})^{m-1}}{(\xi - z)^{m+1}} [(1 - \varphi_S)](\xi) dm(\xi) = c_m \mathcal{B}_{\Omega}^m[(1 - \varphi_S)](z)$ for $z \in 2S$ and, by (4.51), (4.52) and (4.53) we get that

$$\begin{aligned}
\boxed{\text{8.1}} &\lesssim \sum_{(S,L) \in \mathcal{W}_0} |f_L - f_S| \int_{2S} |\partial^n \mathcal{B}_{\chi\Omega}(z) \mathcal{B}_{\Omega}^m[(1 - \varphi_S)](z)| g(z) dm(z) \\
&+ \sum_{(S,L) \in \mathcal{W}_0} |f_L - f_S| \int_{2S} \int_{\Omega \setminus 10S} \frac{1}{|z - \xi|^{3-\sigma_p}} dm(\xi) g(z) dm(z) \\
&= \boxed{\text{8.1.1}} + \boxed{\text{8.1.2}}.
\end{aligned} \tag{4.55}$$

By the same token, using (4.51), (4.52) and (4.54) we get

$$\begin{aligned}
\boxed{\text{8.2}} &\lesssim \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \int_L |\partial^n \mathcal{B}_{\chi\Omega}(z) \mathcal{B}_{\Omega}^m[(f - f_L)(1 - \varphi_S)](z)| dm(\zeta) g(z) dm(z) \\
&+ \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \int_L |\partial^n \mathcal{B}_{\chi\Omega}(\zeta) \mathcal{B}_{\Omega}^m[(f - f_L)(1 - \varphi_S)](\zeta)| dm(\zeta) g(z) dm(z) \\
&+ \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \int_L \int_{\Omega \setminus 10S} \frac{|z - \zeta|^{\sigma_p}}{|z - \xi|^3} |f(\xi) - f_L| dm(\xi) dm(\zeta) g(z) dm(z) \\
&= \boxed{\text{8.2.1}} + \boxed{\text{8.2.2}} + \boxed{\text{8.2.3}}.
\end{aligned} \tag{4.56}$$

We begin by the first term in the right-hand side of (4.55), that is,

$$\boxed{8.1.1} = \sum_{(S,L) \in \mathcal{W}_0} |f_L - f_S| \int_{2S} |\partial^n \mathcal{B}\chi_\Omega(z) \mathcal{B}_\Omega^m[(1 - \varphi_S)](z)| g(z) dm(z).$$

By the Poincaré and the Jensen inequalities, we have that

$$|f_L - f_S| \leq \frac{1}{\ell(L)^2} \int_L |f(\xi) - f_S| dm(\xi) \lesssim \frac{\ell(L)}{\ell(L)^2} \|\nabla f\|_{L^1(5S)} \lesssim \ell(S)^{1-\frac{2}{p}} \|\nabla f\|_{L^p(5S)}. \quad (4.57)$$

On the other hand, by Lemma 4.25 below we have that $\mathcal{B}^m \varphi_S(z) = 0$ for $z \in 2S$. Therefore, using (4.57) we have that

$$\begin{aligned} \boxed{8.1.1} &\lesssim \|\nabla f\|_{L^p(\Omega)} \sum_{S: \ell(S) \leq C2^{-i}} \ell(S)^{1-\frac{2}{p}} \int_{2S} |\partial^n \mathcal{B}\chi_\Omega(z) \mathcal{B}_\Omega^m \chi_\Omega(z)| g(z) dm(z) \\ &\lesssim 2^{-i(1-\frac{2}{p})} \|\nabla f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega)} \|\mathcal{B}_\Omega^m \chi_\Omega\|_{L^\infty(\Omega)} \lesssim_\Omega 2^{-i(1-\frac{2}{p})} \|\nabla f\|_{L^p(\Omega)}. \end{aligned} \quad (4.58)$$

Let us recall that the second term in the right-hand side of (4.55) is

$$\boxed{8.1.2} = \sum_{(S,L) \in \mathcal{W}_0} |f_L - f_S| \int_{2S} \int_{\Omega \setminus 10S} \frac{1}{|z - \xi|^{3-\sigma_p}} dm(\xi) g(z) dm(z)$$

and, by (4.57),

$$\begin{aligned} \boxed{8.1.2} &\lesssim \sum_{S: \ell(S) \leq C2^{-i}} \ell(S)^{1-\frac{2}{p}} \|\nabla f\|_{L^p(5S)} \frac{1}{\ell(S)^{1-\sigma_p}} \|g\|_{L^1(2S)} \\ &\lesssim \sum_{S: \ell(S) \leq C2^{-i}} \ell(S)^{\sigma_p - \frac{2}{p} + \frac{2}{p}} \|\nabla f\|_{L^p(5S)} \|g\|_{L^{p'}(2S)}. \end{aligned}$$

By Hölder's inequality,

$$\boxed{8.1.2} \lesssim 2^{-i\sigma_p} \|\nabla f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)} = 2^{-i\sigma_p} \|\nabla f\|_{L^p(\Omega)}.$$

Using this fact together with (4.55) and (4.58), we have that

$$\boxed{8.1} \lesssim C_{\Omega, i} \|\nabla f\|_{L^p(\Omega)}, \quad (4.59)$$

with $C_{\Omega, i} \xrightarrow{i \rightarrow \infty} 0$.

Let us focus now on the first term in the right-hand side of (4.56), that is,

$$\begin{aligned} \boxed{8.2.1} &= \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} |\partial^n \mathcal{B}\chi_\Omega(z)| |\mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)](z)| g(z) dm(z) \\ &\lesssim \sum_{(S,L) \in \mathcal{W}_0} \|g\|_{L^{p'}(2S)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(2S)} \|\mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)]\|_{L^\infty(2S)}. \end{aligned} \quad (4.60)$$

By the Sobolev Embedding Theorem and the boundedness of \mathcal{B}_Ω^m in $W^{1,p}(\Omega)$ (granted by Theorem 3.28) we have that

$$\|\mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)]\|_{L^\infty(\Omega)} \leq \|\mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)]\|_{W^{1,p}(\Omega)} \lesssim \|(f - f_L)(1 - \varphi_S)\|_{W^{1,p}(\Omega)}$$

and, using Leibniz' rule, Poincaré's inequality and the Sobolev embedding Theorem, we get

$$\begin{aligned} \|\mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)]\|_{L^\infty(\Omega)} &\leq \|\nabla f\|_{L^p(\Omega)} + \frac{1}{\ell(S)}\|f - f_L\|_{L^p(20S)} + \|f - f_L\|_{L^p(\Omega)} \\ &\lesssim_\Omega \|\nabla f\|_{L^p(\Omega)} + \|\nabla f\|_{L^p(20S)} + \|f\|_{L^p(\Omega)} + \|f\|_{L^\infty} \lesssim \|f\|_{W^{1,p}(\Omega)}. \end{aligned}$$

Thus, by Hölder's inequality we have that

$$\boxed{8.2.1} \lesssim \|f\|_{W^{1,p}(\Omega)} \|g\|_{L^{p'}(\Omega)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})} = \|f\|_{W^{1,p}(\Omega)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})}. \quad (4.61)$$

Note that $\|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})} \xrightarrow{i \rightarrow 0} 0$.

The second term in (4.56), that is,

$$\boxed{8.2.2} = \sum_{(S,L) \in \mathcal{W}_0} \frac{1}{\ell(L)^2} \int_L |\partial^n \mathcal{B}\chi_\Omega(\zeta) \mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)](\zeta)| dm(\zeta) \int_{2S} g(z) dm(z),$$

follows the same pattern. Since S and L in the sum above are neighbors, they have comparable side-length, and for $\zeta \in L$ we have that $\int_{2S} g(z) dm(z) \lesssim \ell(L)^2 Mg(\zeta)$. Therefore,

$$\begin{aligned} \boxed{8.2.2} &\lesssim \sum_{(S,L) \in \mathcal{W}_0} \int_L |\partial^n \mathcal{B}\chi_\Omega(\zeta) \mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)](\zeta)| Mg(\zeta) dm(\zeta) \\ &\lesssim \sum_{S: \ell(S) \leq C2^{-i}} \|Mg\|_{L^{p'}(5S)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(5S)} \|\mathcal{B}_\Omega^m[(f - f_L)(1 - \varphi_S)]\|_{L^\infty(5S)}. \end{aligned}$$

The last expression coincides with the right-hand side of (4.60) changing g by Mg and $2S$ by $5S$. Arguing analogously to that case, we get that

$$\boxed{8.2.2} \lesssim \|f\|_{W^{1,p}(\Omega)} \|Mg\|_{L^{p'}(\Omega)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})} \lesssim \|f\|_{W^{1,p}(\Omega)} \|\partial^n \mathcal{B}\chi_\Omega\|_{L^p(\Omega \setminus \Omega_{i-i_0})}. \quad (4.62)$$

Finally, we consider

$$\boxed{8.2.3} = \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} \int_L \int_{\Omega \setminus 10S} \frac{|z - \zeta|^{\sigma_p}}{|z - \xi|^3} |f(\xi) - f_L| dm(\xi) dm(\zeta) g(z) dm(z).$$

Note that for $z, \zeta \in 5S$ we have that $|z - \zeta| \lesssim \ell(S)$. Separating $\Omega \setminus 10S$ in Whitney cubes we get

$$\boxed{8.2.3} \lesssim \sum_{(S,L) \in \mathcal{W}_0} \int_{2S} g(z) dm(z) \sum_{P \in \mathcal{W}} \frac{\ell(S)^{\sigma_p}}{D(S,P)^3} \|f - f_L\|_{L^1(P)}.$$

Using the chain connecting two cubes P and L , by (1.34) we get that

$$\|f - f_L\|_{L^1(P)} \lesssim \sum_{Q \in [P,L]} \|\nabla f\|_{L^1(3Q)} \frac{\ell(P)^2}{\ell(Q)}.$$

Thus,

$$\boxed{8.2.3} \lesssim 2^{-i\sigma_p} \sum_{L,P \in \mathcal{W}} \sum_{Q \in [P,L]} \frac{\ell(P)^2 \|\nabla f\|_{L^1(3Q)} \|g\|_{L^1(7L)}}{\ell(Q) D(L,P)^3}.$$

By Lemma 2.3 (with $\rho = 1$), we get

$$\boxed{8.2.3} \lesssim 2^{-i\sigma_p} \|\nabla f\|_{L^p(\Omega)}, \quad (4.63)$$

and Claim 4.24 is proven. Indeed, by (4.56), (4.61), (4.62) and (4.63), we have that

$$\boxed{8.2} \lesssim C_{\Omega,i} \|\nabla f\|_{L^p(\Omega)}.$$

This fact combined with (4.51) and (4.59) prove that

$$\boxed{8} \leq \sup_{\|g\|_{p'}=1} \boxed{8g} \lesssim C_{\Omega,i} \|\nabla f\|_{L^p(\Omega)}$$

and, together with (4.35), (4.46) and (4.50), gives

$$\|\mathcal{E}^i f\|_{L^p(\Omega)} \lesssim C_{\Omega,i} \|f\|_{W^{1,p}(\Omega)},$$

with $C_{\Omega,i} \xrightarrow{i \rightarrow \infty} 0$. □

It remains to prove the following result.

Lemma 4.25. *Let φ be a radial function in L^2 such that $\varphi|_{\mathbb{D}} \equiv 0$. Then, for every $m \in \mathbb{N}$,*

$$\mathcal{B}^m \varphi(z) = 0 \quad \text{for } z \in \mathbb{D}.$$

Proof. Since $\mathcal{B}\varphi$ is in L^2 and it is radial by linearity, by induction, it is enough to prove that

$$\mathcal{B}\varphi(z) = 0 \quad \text{for } z \in \mathbb{D}.$$

Let $\varepsilon > 0$ and consider a simple radial function s such that $\|\varphi - s\|_{L^2} < \varepsilon$. Let $z \in \mathbb{D}$. Recall that $\mathcal{B}\chi_{\mathbb{D}}(z) = 0$ (see [AIM09, (4.24)]). Since s is a finite combination of characteristic functions of concentric disks $\{D_i\}_{i=1}^M$ with $z \in D^i$ for all i , then, $\mathcal{B}s(z) = 0$.

Therefore $\chi_{\mathbb{D}}\mathcal{B}\varphi = \chi_{\mathbb{D}}\mathcal{B}(\varphi - s)$ and, thus, we get $\|\chi_{\mathbb{D}}\mathcal{B}\varphi\|_{L^2} \leq \|\mathcal{B}(\varphi - s)\|_{L^2} < \varepsilon$. Since ε can be chosen as small as desired, $\chi_{\mathbb{D}}\mathcal{B}\varphi \equiv 0$. □

Chapter 5

Carleson measures on Lipschitz domains

Theorem 2.1 provides us with a useful tool to check if an operator is bounded on $W^{n,p}(\Omega)$ as long as $p > d$. Our concern for this chapter is to find a sufficient condition valid even if $p \leq d$. We want this condition to be related to some test functions (the polynomials of degree smaller than n seem the right choice) but somewhat more specific than the condition in the Key Lemma. In particular we seek for some Carleson condition in the spirit of the celebrated article [ARS02] by N. Arcozzi, R. Rochberg and E. Sawyer.

In Section 5.2 we define the *vertical* shadows $\mathbf{Sh}_v(x)$ and $\widetilde{\mathbf{Sh}}_v(x)$ for every point x in a Lipschitz domain Ω close enough to $\partial\Omega$ (see Definition 5.14 and Figure 5.2). This definition is similar to the shadow introduced in Section 1.4, but requires a local orientation. Therefore, in this chapter we will restrict ourselves to the study of Lipschitz domains, although there is hope that some of the techniques used here can be extrapolated to uniform domains (note that in the proof of the necessity of the Carleson condition, that is, Theorem 5.3 below, we use Lemma 5.8 which requires some restrictions in the dimension of the boundary). Those shadows, as before, can be understood as Carleson boxes of the domain. We say that a positive and finite Borel measure μ is an *s, p-Carleson measure* if for every $a \in \Omega$ and close enough to the boundary,

$$\int_{\widetilde{\mathbf{Sh}}_v(a)} \text{dist}(x, \partial\Omega)^{(d-sp)(1-p')} (\mu(\mathbf{Sh}_v(x) \cap \mathbf{Sh}_v(a)))^{p'} \frac{dx}{\text{dist}(x, \partial\Omega)^d} \leq C \mu(\mathbf{Sh}_v(a)).$$

In this chapter we study the relation between Carleson measures and Calderón-Zygmund operators. The first result we obtain is the following:

Theorem 5.1. *Let T be an admissible convolution Calderón-Zygmund operator of order n , and consider a bounded Lipschitz domain Ω and $1 < p \leq d$. If the measure $|\nabla^n T_\Omega P(x)|^p dx$ is a 1, p-Carleson measure for every polynomial P of degree at most $n - 1$, then T_Ω is a bounded operator on $W^{n,p}(\Omega)$.*

We also have a counterpart for Triebel-Lizorkin spaces with smoothness $0 < s < 1$:

Theorem 5.2. *Let $1 < p < \infty$, $1 < q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$, let T be a p, q-admissible convolution Calderón-Zygmund operator of order s , and consider a bounded Lipschitz domain Ω . If the measure $\mu(x) = |\nabla_q^s T_\Omega 1(x)|^p dx$ is an s, p-Carleson measure, then T_Ω is a bounded operator on $F_{p,q}^s(\Omega)$.*

The Carleson condition of Theorem 5.1 above is in fact necessary for $n = 1$:

Theorem 5.3. *Let T be an admissible convolution Calderón-Zygmund operator of order 1, and consider a bounded Lipschitz domain Ω and $1 < p < \infty$. The following statements are equivalent:*

1. T_Ω is a bounded operator on $W^{1,p}(\Omega)$.
2. The measure $|\nabla T\chi_\Omega(x)|^p dx$ is a p -Carleson measure for Ω .

Section 5.1 is devoted to modifying the definitions of Shadow and admissible chain introduced in Section 1.4. In Section 5.2 some results from [ARS02] which will be used in the subsequent sections are collected. Section 5.3 provides an adaptation of the Key Lemma 2.2 to proper orientations of the Whitney coverings and the proof of Theorem 5.1, that is, the sufficient condition in terms of Carleson measures for an operator to be bounded on $W^{n,p}(\Omega)$. After that, a counterpart to both results for $F_{p,q}^s(\Omega)$ with $\frac{d}{p} - \frac{d}{q} < s < 1$ (including Theorem 5.2 above) is given in Section 5.4. In Section 5.5 it is shown that when only the first derivatives are considered (that is, for $W^{1,p}(\Omega)$) this sufficient condition is in fact necessary, proving Theorem 5.3. Finally, Section 5.6 contains a different approach for the complex plane when the domain is more regular.

5.1 Oriented Whitney coverings

Along this section we consider Ω to be a fixed bounded (δ, R) -Lipschitz domain.

Recall that we say that \mathcal{Q} is an R -window of Ω if it is a cube centered in $\partial\Omega$, with side-length R inducing a Lipschitz parameterization of the boundary (see Definition 1.4). We can choose a number $N \approx \mathcal{H}^{d-1}(\partial\Omega)/R^{d-1}$ and a collection of windows $\{\mathcal{Q}_k\}_{k=1}^N$ such that

$$\partial\Omega \subset \bigcup_{k=1}^N \delta_1 \mathcal{Q}_k,$$

where $\delta_1 > 0$ is a value to fix later (in Remark 5.5). Each window \mathcal{Q}_k is associated to a parameterization A_k in the sense that, after a rotation,

$$\Omega \cap 2\mathcal{Q}_k = \{(y', y_d) \in (\mathbb{R}^{d-1} \times \mathbb{R}) \cap 2\mathcal{Q}_k : y_d > A_k(y')\}.$$

Thus, each \mathcal{Q}_k induces a *vertical* direction, given by the eventually rotated y_d axis. The number of Whitney cubes in \mathcal{Q}_k with the same side-length intersecting a given vertical line is bounded by a constant depending only on the Lipschitz character of Ω .

In the subsequent sections we will make use of a tree structure on the Whitney cubes compatible with admissible chains in a tubular neighborhood of the boundary. Therefore, we must modify the notions introduced in Section 1.4. We distinguish the cubes in the central region from those which are close to the boundary of the domain.

Definition 5.4. *We say that Q is central if $\text{dist}(Q, \partial\Omega) > \delta_2 R$, where $0 < \delta_2 < 1$ is a constant to fix in Remark 5.5. We denote this subcollection of cubes by \mathcal{W}_0 .*

We say that Q is peripheral if it is not central.

Given Whitney cubes $Q, S \subset \mathcal{Q}_k$, we will say that S is above Q if the “vertical” projection of the open cube Q intersects the vertical projection of S and the center of S has vertical coordinate greater or equal than the center of Q (the vertical direction is the one induced by the window again).

Consider $\delta_1 < \delta_0 < 1$ to be fixed. We call $\delta_0 \mathcal{Q}_k \cap \Omega$ the canvas of the window \mathcal{Q}_k , and we divide the peripheral cubes in collections $\mathcal{W}_k = \{Q \in \mathcal{W} \setminus \mathcal{W}_0 : Q \subset \delta_0 \mathcal{Q}_k \cap \Omega\}$.

Remark 5.5. *For Whitney constants big enough and for δ_0, δ_1 and δ_2 small enough we have that*

- 1) *The union of central cubes is a connected set.*

- 2) Every peripheral cube is contained in a window canvas. The subcollections \mathcal{W}_k are not disjoint and, if two peripheral cubes Q and S are not contained in any common \mathcal{W}_k , then $R \lesssim \text{dist}(Q, S) \lesssim \text{diam}(\Omega)$.
- 3) For each peripheral cube $Q \in \mathcal{W}_k$ there exists a cube $S \subset \mathcal{Q}_k$ above Q which is central.

Furthermore,

- 4) All the central cubes have comparable side-length.

Next we provide a tree-like structure to a particular family of cubes contained in a single window.

Definition 5.6. We say that a Whitney covering is properly oriented with respect to a window \mathcal{Q}_k if the cubes in the Whitney covering have sides parallel to the faces of \mathcal{Q}_k .

Given Whitney cubes $Q, S \subset \mathcal{Q}_k$, we will say that $Q \in \mathbf{SH}_v(S)$ if S is above Q . When the Whitney covering is properly oriented, this property is transitive and defines a partial order relation.

We want to have a somewhat rigid structure to gain some control on the chains we use, so we need to introduce a chain function.

Definition 5.7. Given two different cubes $Q, S \in \mathcal{W}_k$ (where \mathcal{W} is a properly oriented Whitney covering with respect to the k -th window), we define $[Q, S] := [Q, Q_S] \cup [S_Q, S]$ to be the chain such that

- the cubes in the subchains $[Q, Q_S] = (Q_1, \dots, Q_{M_1})$ and $[S_Q, S] = (S_{M_2}, \dots, S_1)$ satisfy that $Q_{j-1} \in \mathbf{SH}_v(Q_j)$ and $S_{j-1} \in \mathbf{SH}_v(S_j)$ for $j > 1$ with respect to the “vertical” direction induced by the window, and
- the only pair of cubes (Q_{j_1}, S_{j_2}) which has a vertical segment contained in the boundaries of both of them is (Q_{M_1}, S_{M_2}) , that is, (Q_S, S_Q) .

For δ_0 small enough, this definition makes sense, that is, the chain $[Q, S]$ exists and is unique. We define the collection of central cubes in a given window

$$\mathcal{W}_k^c = \{Q \in \mathcal{W}_0 : Q \subset \mathcal{Q}_k \cap \Omega \text{ and there exists } S \in \mathcal{W}_k \cap \mathbf{SH}_v(Q)\},$$

and we call the collection of central and peripheral cubes $\mathcal{W}_k^{cp} := \mathcal{W}_k \cup \mathcal{W}_k^c$. Then, $[\cdot, \cdot] : \mathcal{W}_k^{cp} \times \mathcal{W}_k^{cp} \rightarrow \bigcup_M (\mathcal{W}_k^{cp})^M$ satisfying the properties above is called (the) chain function.

In the next sections we will make use of the following technical results, specific for Lipschitz domains, which sharpen the results of Lemma 1.18 for g constant.

Lemma 5.8. Let $a > d - 1$ and $Q \in \mathcal{W}_k^{cp}$ a Whitney cube. Then

$$\sum_{S \in \mathbf{SH}_v(Q)} \ell(S)^a \approx \ell(Q)^a$$

with constants depending only on a and d .

Proof. Selecting the cubes by their side-length, we can write

$$\sum_{S \in \mathbf{SH}_v(Q)} \ell(S)^a = \sum_{j=0}^{\infty} \sum_{\substack{S \in \mathbf{SH}_v(Q) \\ \ell(S)=2^{-j}\ell(Q)}} (2^{-j}\ell(Q))^a = \ell(Q)^a \sum_{j=0}^{\infty} 2^{-ja} \#\{S \in \mathbf{SH}_v(Q) : \ell(S) = 2^{-j}\ell(Q)\}.$$

Since the domain is Lipschitz and the Whitney covering is oriented, the number of cubes with a given side-length intersecting a vertical line is uniformly bounded. Thus, we get that

$$\#\{S \in \mathbf{SH}_v(Q) : \ell(S) = 2^{-j}\ell(Q)\} \leq C2^{(d-1)j}$$

and therefore

$$\sum_{S \in \mathbf{SH}_v(Q)} \ell(S)^a \lesssim \ell(Q)^a \sum_{j=0}^{\infty} 2^{-j(a-(d-1))}.$$

This is bounded if $a > d - 1$. \square

Remark 5.9. *By the same token, given an R -window \mathcal{Q}_k , we get $\sum_{S \subset \mathcal{Q}_k} \ell(S)^a \lesssim R^a$ (see Remark 5.5, property 3)). Thus, the lemma is also valid for Q central writing*

$$\sum_{S \in \mathcal{W}} \ell(S)^a \approx \ell(Q)^a$$

by the last statement of Remark 5.5.

Lemma 5.10. *Let $b > a > d - 1$ and Q a Whitney cube. Then*

$$\sum_{S \in \mathcal{W}} \frac{\ell(S)^a}{D(Q, S)^b} \leq C\ell(Q)^{a-b},$$

with C depending only on a, b and d .

Proof. Let us assume that $Q \in \mathcal{W}_k$. First of all we consider the cubes contained in \mathcal{Q}_k and we classify those cubes by their side-length and their long distance to Q :

$$\begin{aligned} \sum_{S \subset \mathcal{Q}_k} \frac{\ell(S)^a}{D(Q, S)^b} &\leq \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} \sum_{\substack{S: \ell(S)=2^i\ell(Q) \\ 2^j\ell(Q) \leq D(S, Q) < 2^{j+1}\ell(Q)}} \frac{(2^i\ell(Q))^a}{(2^j\ell(Q))^b} \\ &\leq \ell(Q)^{a-b} \sum_{i=-\infty}^{\infty} \sum_{j=0}^{\infty} \frac{2^{ia}}{2^{jb}} \#\{S : \ell(S) = 2^i\ell(Q), D(S, Q) < 2^{j+1}\ell(Q)\}. \end{aligned}$$

Note that the value of j in the last sum must be greater or equal than i because, otherwise, the last cardinal would be zero.

Using again the fact that the number of cubes with a given side-length intersecting a vertical line is uniformly bounded, we can see that

$$\#\{S \in \mathcal{W}_k : \ell(S) = 2^i\ell(Q), D(S, Q) < 2^{j+1}\ell(Q)\} \leq C \left(\frac{(2^{j+1})\ell(Q)}{2^i\ell(Q)} \right)^{d-1} = C2^{(j-i)(d-1)}.$$

Thus,

$$\sum_{S \subset \mathcal{Q}_k} \frac{\ell(S)^a}{D(Q, S)^b} \lesssim \ell(Q)^{a-b} \sum_{j=0}^{\infty} \sum_{i=-\infty}^j 2^{i(a+1-d)-j(b+1-d)} \leq C_{a,b,d}\ell(Q)^{a-b}$$

as soon as $b > a > d - 1$.

On the other hand, when $S \not\subset \mathcal{Q}_k$ for every window containing Q . Then the long distance $D(Q, S)$ is always bounded from below by a constant times R (because $Q \subset \delta_0 \mathcal{Q}_k$), so separating \mathcal{W} in subcollections \mathcal{W}_k and using Remark 5.9,

$$\sum_{S \not\subset \mathcal{Q}_k} \frac{\ell(S)^a}{D(Q, S)^b} \lesssim \sum_{S \in \mathcal{W}_0} \frac{(\text{diam}\Omega)^a}{R^b} + \sum_{j \neq k} \sum_{S \in \mathcal{W}_j} \frac{\ell(S)^a}{R^b} \lesssim R^{a-b} \lesssim \ell(Q)^{a-b}. \quad (5.1)$$

To prove the lemma for a central cube $Q \in \mathcal{W}_0$, just apply an argument analogous to (5.1). \square

5.2 Carleson measures

Next we recall some useful results from [ARS02]. First we need to introduce some notation.

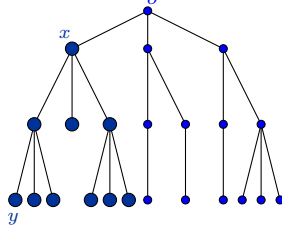


Figure 5.1: $y \in \mathbf{Sh}_{\mathcal{T}}(x)$.

Definition 5.11. We say that a connected, loopless graph \mathcal{T} is a tree, and we will fix a vertex $o \in \mathcal{T}$ and call it its root. This choice induces a partial order in \mathcal{T} , given by $x \leq y$ if $x \in [o, y]$ where $[o, y]$ stands for the geodesic path uniting those two vertices of the graph (see Figure 5.1). We call shadow of x in \mathcal{T} to the collection

$$\mathbf{Sh}_{\mathcal{T}}(x) = \{y \in \mathcal{T} : y \geq x\}.$$

We say that a function $\rho : \mathcal{T} \rightarrow \mathbb{R}$ is a weight if it takes positive values (by a function we mean a function defined in the vertices of the tree).

Definition 5.12. Given $h : \mathcal{T} \rightarrow \mathbb{R}$, we call the primitive $\mathcal{I}h$ the function

$$\mathcal{I}h(y) = \sum_{x \in [o, y]} h(x).$$

Theorem 5.13. [ARS02, Theorem 3] Let $1 < p < \infty$ and let ρ be a weight on \mathcal{T} . For a nonnegative measure μ on \mathcal{T} , the following statements are equivalent:

i) There exists a constant $C = C(\mu)$ such that

$$\|\mathcal{I}h\|_{L^p(\mu)} \leq C \|h\|_{L^p(\rho)}$$

ii) There exists a constant $C = C(\mu)$ such that for every $r \in \mathcal{T}$ one has

$$\sum_{x \in \mathbf{Sh}_{\mathcal{T}}(r)} \mu(\mathbf{Sh}_{\mathcal{T}}(x))^{p'} \rho(x)^{1-p'} \leq C \mu(\mathbf{Sh}_{\mathcal{T}}(r)).$$

For every $1 \leq p \leq \infty$, we say that a non-negative measure μ is a p -Carleson measure for (\mathcal{I}, ρ, p) if there exists a constant $C = C(\mu)$ such that the condition i) is satisfied.

To use the techniques on Carleson measures introduced in [ARS02] we need to have some tree structure coherent with the chain function from Definition 5.7. Note that this structure can be given in terms of the vertical shadow $\mathbf{SH}_{\mathbf{v}}$ as long as the Whitney covering is properly oriented. We also define the (almost) continuous version of the shadow:

Definition 5.14. Given an R -window \mathcal{Q} of a Lipschitz domain Ω with a properly oriented Whitney covering \mathcal{W} , for every $x \in \mathcal{Q}$, we write $x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and, if x is contained in a semi-open Whitney cube $Q \in \mathcal{W}$, we define the vertical shadow of x as

$$\mathbf{Sh}_{\mathbf{v}}(x) := \left\{ y \in \mathcal{Q} \cap \Omega : y_d < x_d \text{ and } \|x' - y'\|_{\infty} \leq \frac{1}{2} \ell(Q) \right\}.$$

Note that if x is the center of the upper $(d-1)$ -dimensional face of Q , the vertical projection of $\mathbf{Sh}_v(x)$ (which is a $(d-1)$ -dimensional square) coincides with the vertical projection of Q (see Figure 5.2). Finally, we define the vertical extension of $\mathbf{Sh}_v(x)$,

$$\widetilde{\mathbf{Sh}}_v(x) := \left\{ y \in \mathcal{Q} \cap \Omega : y_d < x_d + 2\ell(Q) \text{ and } \|x' - y'\|_\infty \leq \frac{1}{2}\ell(Q) \right\}.$$

More generally, given a set $U \subset \mathcal{Q}$ we call its shadow

$$\mathbf{Sh}_v(U) := \{y \in \mathcal{Q} \cap \Omega : \text{there exists } x \in U \text{ such that } y_d < x_d \text{ and } x' = y'\}.$$

Note that $\mathbf{Sh}_v(Q) = \bigcup_{P \in \mathbf{SH}_v(Q)} P$ for any Whitney cube $Q \subset \mathcal{Q}$ (see Figure 5.2).

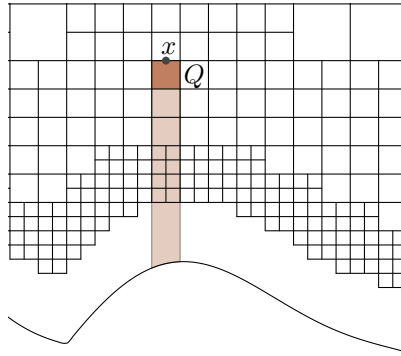


Figure 5.2: The shadows $\mathbf{Sh}_v(x)$ and $\mathbf{Sh}_v(Q)$ coincide when x is the center of the upper face of the cube. Furthermore, $P \subset \mathbf{Sh}_v(Q)$ if and only if $P \in \mathbf{SH}_v(Q) = \mathbf{SH}_v(Q)$.

Recall that we have a proper orientation in the Whitney covering. Thus, given a Whitney cube Q , we call the father of Q , $\mathcal{F}(Q)$ the neighbor Whitney cube which is immediately on top of Q with respect to the vertical direction. This parental relation is coherent with the order relation in Definition 5.6 ($P \in \mathbf{SH}_v(Q)$ if P is a descendant of Q). This would provide a tree structure to the Whitney covering \mathcal{W} if there was a common ancestor Q_0 for all the cubes. This does not happen, but we can add a “formal” cube Q_0 (root of the tree) and then we can write $\mathbf{SH}_v(Q_0) := \{Q \in \mathcal{W}_k^{cp} : Q \subset \mathcal{Q}\}$. If we call \mathcal{T} to the tree with the Whitney cubes as vertices complemented with Q_0 and the structure given by the aforementioned order relation, then for every Whitney cube $Q \subset \mathcal{Q}$, we have that $\mathbf{SH}_v(Q) = \mathbf{Sh}_{\mathcal{T}}(Q)$. Since we will only consider functions and measures supported in the window canvas $\delta_0 \mathcal{Q} \cap \Omega$, we can extend all of them formally in Q_0 as the null function.

Now, some minor modifications in the proof of [ARS02, Proposition 16] allow us to rewrite this proposition in the following way.

Proposition 5.15. *Given $1 < p < \infty$, a positive number $s \in \mathbb{R}$ and an R -window \mathcal{Q} of a Lipschitz domain Ω with a properly oriented Whitney covering \mathcal{W} , consider the weights $\rho(x) = \text{dist}(x, \partial\Omega)^{d-sp}$, $\rho_{\mathcal{W}}(Q) = \ell(Q)^{d-sp}$. For a positive Borel measure μ supported on $\delta_0 \mathcal{Q} \cap \Omega$, the following are equivalent:*

1. For every $a \in \delta_0 \mathcal{Q} \cap \Omega$ one has

$$\int_{\widetilde{\mathbf{Sh}}_v(a)} \rho(x)^{1-p'} (\mu(\mathbf{Sh}_v(x) \cap \mathbf{Sh}_v(a)))^{p'} \frac{dx}{\text{dist}(x, \partial\Omega)^d} \leq C\mu(\mathbf{Sh}_v(a)).$$

2. For every Whitney cube $P \subset Q$ one has

$$\sum_{Q \in \mathbf{SH}_v(P)} \mu(\mathbf{Sh}_v(Q))^{p'} \rho_{\mathcal{W}}(Q)^{1-p'} \leq C \mu(\mathbf{Sh}_v(P)). \quad (5.2)$$

In virtue of [ARS02, Theorem 1], when $d = 2$, $s = 1$ and the domain Ω is the unit disk in the plane, the first condition is equivalent to μ being a Carleson measure for the analytic Besov space $B_p(\rho)$, that is, for every analytic function defined on the unit disc \mathbb{D} ,

$$\|f\|_{L^p(\mu)}^p \lesssim \|f\|_{B_p(\rho)}^p := |f(0)|^p + \int_{\mathbb{D}} (1 - |z|^2)^p |f'(z)|^p \rho(z) \frac{dm(z)}{(1 - |z|^2)^2}.$$

Definition 5.16. We say that a measure satisfying the hypothesis of Proposition 5.15 is an s, p -Carleson measure for \mathcal{Q} (or simply p -Carleson measure if $s = 1$).

We say that a positive and finite Borel measure μ is an s, p -Carleson (or p -Carleson) measure for a Lipschitz domain Ω if it is an s, p -Carleson (resp. p -Carleson) measure for every R -window of the domain.

5.3 Integer smoothness: a sufficient condition for $p \leq d$

We will use local versions of the Key Lemmas in Chapter 2 in order to get rid of some technical difficulties:

Lemma 5.17. Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, T an admissible convolution Calderón-Zygmund operator of order $n \in \mathbb{N}$ and $1 < p < \infty$. Then the following statements are equivalent.

i) For every $f \in W^{n,p}(\Omega)$ one has

$$\|T_\Omega f\|_{W^{n,p}(\Omega)} \leq C \|f\|_{W^{n,p}(\Omega)}. \quad (5.3)$$

ii) For every window \mathcal{Q} and every $f \in W^{n,p}(\Omega)$ with $f|_{(\delta_0 \mathcal{Q})^c} \equiv 0$ one has

$$\sum_{Q \in \mathcal{W}_\mathcal{Q}} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \leq C \|f\|_{W^{n,p}(\Omega)}^p,$$

where the Whitney covering $\mathcal{W}_\mathcal{Q}$ is properly oriented with respect to \mathcal{Q} , that is, with the dyadic grid parallel to the local coordinates (see Definition 5.6).

Proof. To see that i) implies ii) just use the Key Lemma 2.2 with an appropriate dyadic grid.

To see the converse, choose a finite collection of windows $\{\mathcal{Q}_k\}_{k=1}^N$ with $N \approx \mathcal{H}^{d-1}(\partial\Omega)/R^{d-1}$ such that $\frac{\delta_0}{4} \mathcal{Q}_k$ is a covering of the boundary of Ω , call \mathcal{Q}_0 to the inner region $\Omega \setminus \bigcup_{k=1}^N \frac{\delta_0}{2} \mathcal{Q}_k$, and let $\{\psi_k\}_{k=0}^N \subset C^\infty$ be a partition of the unity related to the covering $\{\mathcal{Q}_0\} \cup \{\delta_0 \mathcal{Q}_k\}_{k=1}^N$. Consider a function $f \in W^{n,p}(\Omega)$. Note that hypothesis ii) does not give information about the inner region, but since ψ_0 is compactly supported in Ω , $\psi_0 f \in W^{n,p}(\mathbb{R}^d)$ and by Remark 1.29 also $T(\psi_0 f) \in W^{n,p}(\mathbb{R}^d)$, so

$$\|T_\Omega(\psi_0 f)\|_{W^{n,p}(\Omega)} = \|T(\psi_0 f)\|_{W^{n,p}(\Omega)} \leq \|T(\psi_0 f)\|_{W^{n,p}(\mathbb{R}^d)} \leq C \|\psi_0 f\|_{W^{n,p}(\Omega)}.$$

Now, replacing f by $\psi_k f$ and \mathcal{W} by a properly oriented Whitney covering \mathcal{W}_k with respect to \mathcal{Q}_k in (2.2) we get

$$\begin{aligned} \|\nabla^n T_\Omega(\psi_k f)\|_{L^p(\Omega)}^p &\leq \sum_{Q \in \mathcal{W}_k} \left\| \nabla^n T_\Omega \left(\psi_k f - \mathbf{P}_{3Q}^{n-1}(\psi_k f) \right) \right\|_{L^p(Q)}^p + \sum_{Q \in \mathcal{W}_k} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1}(\psi_k f) \right) \right\|_{L^p(Q)}^p \\ &\leq C \|\psi_k f\|_{W^{n,p}(\Omega)}^p. \end{aligned}$$

Thus, by the triangle inequality we have that

$$\|T_\Omega f\|_{W^{n,p}(\Omega)} \leq \sum_{k=0}^N \|T_\Omega(\psi_k f)\|_{W^{n,p}(\Omega)} \leq C \sum_{k=0}^N \|\psi_k f\|_{W^{n,p}(\Omega)}.$$

Choosing ψ_k as bump functions with the usual estimates on the derivatives $\|\nabla^j \psi_k\|_{L^\infty} \lesssim R^{-j}$, we get (5.3) using the Leibniz formula:

$$\|T_\Omega f\|_{W^{n,p}(\Omega)} \lesssim \sum_{k=0}^N \|\psi_k f\|_{W^{n,p}(\Omega)} \lesssim \sum_{k=0}^N \left(\sum_{j=0}^n \|\nabla^j f\|_{L^p(\delta_0 \mathcal{Q}_k)} \right) \lesssim \|f\|_{W^{n,p}(\Omega)}.$$

□

We are ready to prove Theorem 5.1. This proof is very much in the spirit of Theorem 2.1. Again we fix a point $x_0 \in \Omega$ and we use the polynomials $P_\lambda(x) = (x - x_0)^\lambda$ for every multiindex $|\lambda| < n$, but now the key point is to use the Poincaré inequality instead of the Sobolev Embedding Theorem. The hypothesis in Theorem 5.1 is equivalent to $d\mu_\lambda(x) = |\nabla^n T_\Omega P_\lambda(x)|^p dx$ being a p -Carleson measure for Ω for every $|\lambda| < n$.

Proof of Theorem 5.1. Consider a fixed R -window \mathcal{Q} and a properly oriented Whitney covering \mathcal{W} , that is, with dyadic grid parallel to the window faces. Making use of Lemma 5.17, we only need to show that

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \leq C \|f\|_{W^{n,p}(\Omega)}^p$$

for every $f \in W^{n,p}(\Omega)$ with $f|_{(\delta_0 \mathcal{Q})^c} \equiv 0$.

Fix such a function f . Using the expression (1.31) and expanding it as in (2.10) at a fixed point $x_0 \in \Omega$, we have

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \lesssim \sum_{|\gamma| < n} \sum_{\vec{0} \leq \lambda \leq \gamma} C_{\gamma, \lambda, \Omega} \sum_{Q \in \mathbf{SH}_v(Q_0)} |m_{Q, \gamma}|^p \|\nabla^n T_\Omega P_\lambda\|_{L^p(Q)}^p.$$

Moreover, by induction on (1.35), the coefficients are bounded by

$$|m_{Q, \gamma}| \lesssim \sum_{|\beta| < n: \beta \geq \gamma} \ell(Q)^{|\beta - \gamma|} C_{\beta, \gamma} \left| \int_{3Q} D^\beta f \, dm \right| \lesssim \sum_{|\beta| < n: \beta \geq \gamma} C_{\beta, \gamma, R} \left| \int_{3Q} D^\beta f \, dm \right|,$$

so

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \lesssim \sum_{\substack{|\beta| < n \\ \vec{0} \leq \lambda \leq \beta}} \sum_{Q \in \mathbf{SH}_v(Q_0)} \left| \int_{3Q} D^\beta f \, dm \right|^p \mu_\lambda(Q).$$

Taking into account that $f|_{(\delta_0 \mathcal{Q})^c} \equiv 0$, we have $\int_{3P} D^\beta f \, dm = 0$ for P close enough to the root Q_0 . Thus,

$$\int_{3Q} D^\beta f \, dm = \sum_{P \in [Q, Q_0)} \left(\int_{3P} D^\beta f \, dm - \int_{3\mathcal{F}(P)} D^\beta f \, dm \right).$$

With a slight modification of the proof of Poincaré inequality in [Eva98, Theorem 5.8.1/1], one can see that

$$\|f - f_{3Q}\|_{L^p(3Q \cap 3\mathcal{F}(Q))} \lesssim \ell(Q) \|\nabla f\|_{L^p(3Q)},$$

and, by the same token, $\|f - f_{3\mathcal{F}(Q)}\|_{L^p(3Q \cap 3\mathcal{F}(Q))} \lesssim \ell(Q)\|\nabla f\|_{L^p(3\mathcal{F}(Q))}$. Therefore,

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \lesssim \sum_{\substack{|\beta| < n \\ \bar{0} \leq \lambda \leq \beta}} \sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\sum_{P \in [Q, Q_0]} \ell(P) \int_{3P} |\nabla D^\beta f| dm \right)^p \mu_\lambda(Q). \quad (5.4)$$

By assumption, μ_λ is a p -Carleson measure for every $|\lambda| < n$, that is, it satisfies both conditions of Proposition 5.15. By Theorem 5.13, we have that, for every $h \in l^p(\rho_W)$,

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\sum_{P \in [Q, Q_0]} h(P) \right)^p \mu_\lambda(Q) \leq C \sum_{Q \in \mathbf{SH}_v(Q_0)} h(Q)^p \ell(Q)^{d-p}, \quad (5.5)$$

where $\rho_W(Q) = \ell(Q)^{d-p}$.

Let us fix β and λ momentarily and take $h(P) = \ell(P) \int_{3P} |\nabla D^\beta f| dm$ in (5.5). Using Jensen's inequality and the finite overlapping of the quintuple cubes, we have

$$\begin{aligned} \sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\sum_{P \in [Q, Q_0]} \ell(P) \int_{3P} |\nabla D^\beta f| dm \right)^p \mu_\lambda(Q) &\leq C \sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\int_{3Q} |\nabla D^\beta f| dm \right)^p \ell(Q)^d \\ &\lesssim \sum_{Q \in \mathbf{SH}_v(Q_0)} \int_{3Q} |\nabla D^\beta f|^p dm \ell(Q)^d \\ &\lesssim \int_\Omega |\nabla D^\beta f|^p dm. \end{aligned} \quad (5.6)$$

Plugging (5.6) into (5.4) for each β and λ , we get

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left\| \nabla^n T_\Omega \left(\mathbf{P}_{3Q}^{n-1} f \right) \right\|_{L^p(Q)}^p \leq C \|f\|_{W^{n,p}(\Omega)}^p.$$

□

5.4 Fractional smoothness: a sufficient condition for $sp \leq d$

Lemma 2.11 can be rewritten for fractional Triebel-Lizorkin spaces:

Lemma 5.18. *Let $1 < p < \infty$, $1 < q < \infty$ and $0 < s < 1$ with $s > \frac{d}{p} - \frac{d}{q}$, let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain, and let T be a p, q -admissible convolution Calderón-Zygmund operator of order s . Then the following statements are equivalent.*

i) For every $f \in F_{p,q}^s(\Omega)$ one has

$$\|T_\Omega f\|_{F_{p,q}^s(\Omega)} \leq C \|f\|_{F_{p,q}^s(\Omega)}.$$

ii) For every window \mathcal{Q} and every $f \in F_{p,q}^s(\Omega)$ with $f|_{(\delta_0 \mathcal{Q})^c} \equiv 0$ one has

$$\sum_{Q \in \mathcal{W}_\mathcal{Q}} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p \leq C \|f\|_{F_{p,q}^s(\Omega)}^p,$$

where the Whitney covering $\mathcal{W}_{\mathcal{Q}}$ is properly oriented with respect to \mathcal{Q} , that is, with the dyadic grid parallel to the local coordinates (see Definition 5.6), and the gradient $\nabla_q^s f$ is defined as

$$\nabla_q^s f(x) = \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{1}{q}},$$

with ρ_1 big enough (see Corollary 2.22).

Proof. It is trivial that *i*) implies *ii*) by the Key Lemma 2.11, increasing ρ_ε if necessary.

To see the converse, use the same partition of the unity $\{\psi_k\} \subset C^\infty$ as in the proof of Lemma 5.17. Again, for the central region, we have that

$$\|T_\Omega(\psi_0 f)\|_{F_{p,q}^s(\Omega)} = \|T(\psi_0 f)\|_{F_{p,q}^s(\Omega)} \leq \|T(\psi_0 f)\|_{F_{p,q}^s(\mathbb{R}^d)} \leq C \|\psi_0 f\|_{F_{p,q}^s(\mathbb{R}^d)}.$$

By means of (2.13), (2.14) and (2.47), it is immediate to see that $\|\psi_0 f\|_{F_{p,q}^s(\mathbb{R}^d)} \lesssim \|\psi_0 f\|_{F_{p,q}^s(\Omega)}$.

Let $1 \leq k \leq N$ and consider a properly oriented Whitney covering \mathcal{W}_k with respect to \mathcal{Q}_k . By (2.15), we have that

$$\sum_{Q \in \mathcal{W}_k} \|\nabla_q^s T_\Omega((\psi_k f) - (\psi_k f)_Q)\|_{L^p(Q)}^p \lesssim \|\psi_k f\|_{F_{p,q}^s(\Omega)}^p$$

and, using the hypothesis *ii*), we get

$$\sum_{Q \in \mathcal{W}_k} \|\nabla_q^s T_\Omega(\psi_k f)\|_{L^p(Q)}^p \lesssim_p \sum_{Q \in \mathcal{W}_k} |(\psi_k f)_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p + \|\psi_k f\|_{F_{p,q}^s(\Omega)}^p \lesssim \|\psi_k f\|_{F_{p,q}^s(\Omega)}^p.$$

Now we cannot use Leibniz' rule. Instead, for $x \in \Omega$ we have that

$$\begin{aligned} \nabla_q^s(f\psi_k)(x) &= \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} \frac{|f(x)\psi_k(x) - f(y)\psi_k(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{1}{q}} \\ &\leq |f(x)| \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} \frac{|\psi_k(x) - \psi_k(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{1}{q}} \\ &\quad + \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} |\psi_k(y)| \frac{|f(x) - f(y)|^q}{|x - y|^{sq+d}} dy \right)^{\frac{1}{q}} \lesssim \frac{R^{1-s}}{R} |f(x)| + \nabla_q^s(f)(x) \end{aligned}$$

and, by the triangle inequality, the previous statements and (2.14), we get

$$\begin{aligned} \|T_\Omega f\|_{F_{p,q}^s(\Omega)} &\leq \sum_{k=0}^N \|T_\Omega(\psi_k f)\|_{F_{p,q}^s(\Omega)} \lesssim \sum_{k=0}^N \|\psi_k f\|_{F_{p,q}^s(\Omega)} \\ &\approx \sum_{k=0}^N \left(\|\psi_k f\|_{L^p(\Omega)} + \|\nabla_q^s(\psi_k f)\|_{L^p(\Omega)} \right) \lesssim \sum_{k=0}^N \|f\|_{L^p(\Omega)} + \sum_{k=0}^N \|\nabla_q^s f\|_{L^p(\Omega)}. \end{aligned}$$

□

Proof of Theorem 5.2. As before, consider \mathcal{Q} and a properly oriented Whitney covering \mathcal{W} . By Lemma 5.18, we only need to show that

$$\sum_{Q \in \mathbf{SH}_v(\mathcal{Q}_0)} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p \leq C \|f\|_{F_{p,q}^s(\Omega)}^p$$

for every $f \in F_{p,q}^s(\Omega)$ with $f|_{(\delta_0 Q)^c} \equiv 0$.

Fix one such a function f . Then

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p = \sum_{Q \in \mathbf{SH}_v(Q_0)} |f_Q|^p \mu(Q).$$

Taking into account that $f|_{(\delta_0 Q)^c} \equiv 0$, we have $|f_P|^p = 0$ for P close enough to the root Q_0 . Thus, By Jensen's inequality

$$\begin{aligned} |f_Q| &= \left| \sum_{P \in [Q, Q_0]} (f_P - f_{\mathcal{F}(P)}) \right| \lesssim \sum_{P \in [Q, Q_0]} \frac{\ell(P)^s}{\ell(P)^d} \int_P \int_{5P} \frac{|f(x) - f(y)|}{\ell(P)^{d+s}} dy dx \\ &\lesssim \sum_{P \in [Q, Q_0]} \frac{\ell(P)^s}{\ell(P)^d} \int_P \left(\int_{5P} \frac{|f(x) - f(y)|^q}{\ell(P)^{d+sq}} dy \right)^{\frac{1}{q}} dx \leq \sum_{P \in [Q, Q_0]} \ell(P)^{s-d} \int_P \nabla_q^s f(x) dx, \end{aligned}$$

and, adding on Q , we get

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p \lesssim \sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\sum_{P \in [Q, Q_0]} \ell(P)^{s-d} \int_P \nabla_q^s f(x) dx \right)^p \mu(Q).$$

Since μ is an s, p -Carleson measure, it satisfies both conditions of Proposition 5.15. By Theorem 5.13, we have that, for every function $h : \mathcal{W} \rightarrow \mathbb{C}$,

$$\sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\sum_{P \in [Q, Q_0]} h(P) \right)^p \mu(Q) \leq C \sum_{Q \in \mathbf{SH}_v(Q_0)} h(Q)^p \ell(Q)^{d-sp}.$$

Take $h(P) = \ell(P)^{s-d} \int_P \nabla_q^s f(x) dx$. Using Jensen's inequality once again, we have

$$\begin{aligned} \sum_{Q \in \mathbf{SH}_v(Q_0)} |f_Q|^p \|\nabla_q^s T_\Omega 1\|_{L^p(Q)}^p &\lesssim \sum_{Q \in \mathbf{SH}_v(Q_0)} \left(\int_Q |\nabla_q^s f(x)| dx \right)^p \ell(Q)^{sp-dp+d-sp} \\ &\lesssim \sum_{Q \in \mathbf{SH}_v(Q_0)} \int_Q |\nabla_q^s f(x)|^p dx \lesssim \|\nabla_q^s f\|_{L^p(\Omega)}^p. \end{aligned}$$

□

5.5 Smoothness $n = 1$: a necessary condition

The implication 2. \implies 1. in Theorem 5.3 is a consequence of Theorem 5.1, so we only need to prove that 1. \implies 2.. To do so, we will solve a Neumann problem by means of the Newton potential: given an integrable function with compact support $g \in L_c^1(\mathbb{R}^d)$, its *Newton potential* is

$$Ng(x) = \int \frac{|x-y|^{2-d}}{(2-d)w_d} g(y) dy \quad \text{if } d > 2, \quad Ng(x) = \int \frac{\log|x-y|}{2\pi} g(y) dy \quad \text{if } d=2$$

for *a.e.* $x \in \mathbb{R}^d$, where w_d stands for the surface measure of the unit sphere in \mathbb{R}^d . Recall that the gradient of Ng is the $(d-1)$ -dimensional Riesz transform of g ,

$$\nabla Ng(x) = R^{(d-1)}g(x) = \int \frac{x-y}{w_d|x-y|^d} g(y) dy.$$

It is well known that $\Delta Ng(x) = g(x)$ for $x \in \mathbb{R}^d$ (see [Fol95, Theorem 2.21] for instance).

Remark 5.19. Given $g \in L^1_c(\overline{\mathbb{R}_+^d})$ and $d > 2$, consider the function

$$F(x) := N \left[2 \left(R_d^{(d-1)} g \right) d\sigma \right] (x) = \int_{\partial \mathbb{R}_+^d} \frac{2 \left(R_d^{(d-1)} g \right) (y)}{(2-d)w_d|x-y|^{d-2}} d\sigma(y) \quad \text{for } x \in \mathbb{R}_+^d, \quad (5.7)$$

where $R_d^{(d-1)}$ stands for the vertical component of $R^{(d-1)}$ and $d\sigma$ is the hypersurface measure in $\partial \mathbb{R}_+^d$. This function is well defined since

$$\begin{aligned} \left\| R_d^{(d-1)} g \right\|_{L^1(\sigma)} &\leq \int_{\partial \mathbb{R}_+^d} \int_{\mathbb{R}_+^d} \frac{z_d}{|y-z|^d} |g(z)| dz d\sigma(y) \\ &= \int_{\mathbb{R}_+^d} \left(\int_{\partial \mathbb{R}_+^d} \frac{z_d}{|y-z|^d} d\sigma(y) \right) |g(z)| dz \approx \|g\|_1 \end{aligned}$$

and, thus, the right-hand side of (5.7) is an absolutely convergent integral for each $x \in \mathbb{R}_+^d$, with $|F(x)| \leq \frac{2}{d-2} \frac{\|g\|_1}{|x_d|^{d-2}}$. By the same token, all the derivatives of F are well defined, F is $C^\infty(\mathbb{R}_+^d)$, harmonic and $\nabla F(x) = R^{(d-1)} \left[2 \left(R_d^{(d-1)} g \right) d\sigma \right] (x)$. When $d = 2$ we have to make the usual modifications.

Lemma 5.20. Consider a ball $B_1 \subset \mathbb{R}^d$ centered at the origin and a real number $\varepsilon > 0$. Let $g \in L^\infty$ supported in $(\mathbb{R}_+^d \cap \frac{1}{4}B_1) \setminus (\mathbb{R}^{d-1} \times (0, \varepsilon))$ and define

$$h(x) := N \left[2 \left(R_d^{(d-1)} g \right) d\sigma \right] (x) - Ng(x).$$

Then h has weak derivatives in \mathbb{R}_+^d and for every $\phi \in C_c^\infty(\overline{\mathbb{R}_+^d})$,

$$\int_{\mathbb{R}_+^d} \nabla \phi \cdot \nabla h dm = \int_{\mathbb{R}_+^d} \phi g dm. \quad (5.8)$$

Furthermore, if B_1 has radius r_1 then for every $x \in \mathbb{R}_+^d \setminus B_1$ we have

$$|h(x)| \lesssim \begin{cases} \frac{1}{|x|^{d-2}} \|g\|_1 & \text{if } d > 2, \\ \left(|\log|x|| + 1 + r_1 \frac{x_2 |\log|x_2||}{|x|^2} \right) \|g\|_1 & \text{if } d = 2, \end{cases} \quad (5.9)$$

and

$$|\nabla h(x)| \lesssim \frac{1}{|x|^{d-1}} \left(1 + \left| \log \frac{x_d}{|x|} \right| \right) \|g\|_1. \quad (5.10)$$

Remark 5.21. Note that h can be understood as a weak solution to the Neumann problem

$$\begin{cases} -\Delta h(x) = g(x) & \text{if } x \in \mathbb{R}_+^d, \\ \partial_a h(y) = 0 & \text{if } y \in \partial \mathbb{R}_+^d. \end{cases}$$

Proof of Lemma 5.20. Let us define F as in (5.7). Then,

$$\nabla F = R^{(d-1)} \left[2 \left(R_d^{(d-1)} g \right) d\sigma \right]$$

and $h = F - Ng$. Since $g \in L^\infty$, it follows that Ng is $C^1(\mathbb{R}^d)$ (in fact it is harmonic out of the support of g) and it satisfies that $\partial_d Ng = R_d^{(d-1)}g$. Moreover, we have that $\left(R_d^{(d-1)}g\right)|_{\partial\mathbb{R}_+^d}$ is in L^1 and it is C^∞ , so $\partial_j F = N\left[2\partial_j\left(R_d^{(d-1)}g\right)d\sigma\right] \in C^\infty$ up to the boundary as well. The remaining partial derivative of F satisfies $\partial_d F = R_d^{(d-1)}\left[2\left(R_d^{(d-1)}g\right)d\sigma\right]$. Note that the kernel of $R_d^{(d-1)}(2\cdot)$, when acting on functions defined on $\partial\mathbb{R}_+^d$, coincides with the Poisson kernel

$$\frac{2x_d}{w_d|x|^d}$$

and, therefore, it maps $L^\infty \cap C(\partial\mathbb{R}_+^d)$ functions to continuous functions in $\overline{\mathbb{R}_+^d}$, and it satisfies the pointwise identity $\lim_{x_d \rightarrow 0} R_d^{(d-1)}g(x', x_d) = g(x')$ for every $g \in L^\infty \cap C(\partial\mathbb{R}_+^d)$ and every $x' \in \mathbb{R}_+^d$ (see [Fol95, Theorem 2.44]). In particular, $\lim_{x_d \rightarrow 0} \partial_d F(x', x_d) = R_d^{(d-1)}g(x')$.

Consider $\phi \in C_c^\infty(\mathbb{R}^d)$. Using the Green identities, since F is harmonic in \mathbb{R}_+^d , we have

$$\int_{\mathbb{R}_+^d} \nabla\phi \cdot \nabla F dm - \int_{\mathbb{R}_+^d} \nabla\phi \cdot \nabla Ng dm = \int_{\partial\mathbb{R}_+^d} \phi \partial_d F d\sigma - \int_{\partial\mathbb{R}_+^d} \phi R_d^{(d-1)}g d\sigma + \int_{\mathbb{R}_+^d} \phi g dm = \int_{\mathbb{R}_+^d} \phi g,$$

proving (5.8).

To prove the pointwise bounds for ∇h , recall that

$$\nabla h(x) = R^{(d-1)}\left[2\left(R_d^{(d-1)}g\right)d\sigma\right](x) - R^{(d-1)}g(x).$$

Given $x \in \mathbb{R}_+^d \setminus B_1$, since $\text{supp}(g) \subset \frac{1}{4}B_1$,

$$|R^{(d-1)}g(x)| = c \left| \int_{B_1} \frac{g(z)(x-z)}{|x-z|^d} dz \right| \lesssim \frac{\|g\|_1}{|x|^{d-1}}. \quad (5.11)$$

On the other hand, consider $z \in \text{supp}(g) \subset \frac{1}{4}B_1$ and $x \notin B_1$. Then, for $y \in \partial\mathbb{R}_+^d \cap B(0, |x|/2)$ one has $|x-y| \approx |x|$, for $y \in \partial\mathbb{R}_+^d \cap B(0, 2|x|) \setminus B(0, |x|/2)$ one has $|y-z| \approx |x|$ and otherwise $|y-x| \approx |y-z| \approx |y|$. Thus,

$$\begin{aligned} \left| R^{(d-1)}\left[2\left(R_d^{(d-1)}g\right)d\sigma\right](x) \right| &= c \left| \int_{\partial\mathbb{R}_+^d} \left(\int_{B_1} \frac{g(z)z_d dz}{|y-z|^d} \right) \frac{(x-y)d\sigma(y)}{|x-y|^d} \right| \\ &\lesssim \int_{\partial\mathbb{R}_+^d \cap B(0, |x|/2)} \left(\int_{B_1} \frac{|g(z)|z_d dz}{|y-z|^d} \right) \frac{d\sigma(y)}{|x|^{d-1}} \\ &\quad + \int_{\partial\mathbb{R}_+^d \cap B(0, 2|x|) \setminus B(0, |x|/2)} \left(\int_{B_1} \frac{|g(z)|z_d dz}{|x|^d} \right) \frac{d\sigma(y)}{|x-y|^{d-1}} \\ &\quad + \int_{\partial\mathbb{R}_+^d \setminus B(0, 2|x|)} \left(\int_{B_1} |g(z)|z_d dz \right) \frac{d\sigma(y)}{|y|^{2d-1}}. \end{aligned} \quad (5.12)$$

The first term can be bounded by $C\frac{\|g\|_1}{|x|^{d-1}}$ because $\int_{\partial\mathbb{R}_+^d} \frac{d\sigma(y)}{|y-z|^d} = C\frac{1}{z_d}$. The second can be bounded by $C\frac{r_1\|g\|_1}{|x|^d} \left| \log \frac{|x_d|}{|x|} \right|$ using polar coordinates and the last one can be bounded by $C\frac{r_1\|g\|_1}{|x|^d}$ trivially. Thus,

$$\left| R^{(d-1)}\left[2\left(R_d^{(d-1)}g\right)d\sigma\right](x) \right| \lesssim \frac{\|g\|_1}{|x|^{d-1}} + \frac{r_1\|g\|_1}{|x|^d} \left| \log \frac{x_d}{|x|} \right| + \frac{r_1\|g\|_1}{|x|^d},$$

proving (5.10) since $r_1 \leq |x|$.

To prove the pointwise bounds for h , recall that

$$h(x) = N \left[2 \left(R_d^{(d-1)} g \right) d\sigma \right] (x) - Ng(x).$$

When $d > 2$ we use the same method as in (5.11) and (5.12) using Newton's potential instead of the vectorial $(d-1)$ -dimensional Riesz transform to get

$$|h(x)| \lesssim \frac{\|g\|_1}{|x|^{d-2}} + \frac{r_1 x_d \|g\|_1}{|x|^d} + \frac{r_1 \|g\|_1}{|x|^{d-1}}.$$

When $d = 2$ the Newton potential is logarithmic, but the spirit is the same. In this case, arguing as before,

$$|h(x)| \lesssim \log |x| \|g\|_1 + r_1 \|g\|_1 \frac{|x| + |x| \log |x| + x_2 \log x_2}{|x|^2}.$$

□

Proposition 5.22. *Let $1 < p < \infty$. Given a window \mathcal{Q} centered at the origin of a δ -special Lipschitz domain Ω (see Definition 1.4) with a Whitney covering \mathcal{W} and given $f \in W^{1,p}(\Omega)$, define the Whitney averaging function*

$$\mathcal{A}f(x) := \sum_{Q \in \mathcal{W}} \chi_Q(x) \int_{3Q} f(y) dy. \quad (5.13)$$

If μ is a finite positive Borel measure supported on $\delta_0 \mathcal{Q}$ with

$$\mu(\mathbf{Sh}_v(Q)) \leq C \ell(Q)^{d-p} \quad \text{for every Whitney cube } Q \subset \mathcal{Q}, \quad (5.14)$$

and $\mathcal{A} : W^{1,p}(\Omega) \rightarrow L^p(\mu)$ is bounded, then μ is a p -Carleson measure.

Proof. We will argue by duality. Let us assume that the window $\mathcal{Q} = Q(0, \frac{R}{2})$ is of side-length R and centered at the origin, which belongs to $\partial\Omega$. Note that the boundedness of \mathcal{A} is equivalent to the boundedness of its dual operator

$$\mathcal{A}^* : L^{p'}(\mu) \rightarrow (W^{1,p}(\Omega))^*.$$

We also assume that $\mu \equiv 0$ in a neighborhood of $\partial\Omega$. One can prove the general case by means of truncation and taking limits since the constants of the Carleson condition (5.2) and the the norm of the averaging operator will not get worse by this procedure.

Fix a cube P . Analogously to [ARS02, Theorem 3], we apply the boundedness of \mathcal{A}^* to the test function $g = \chi_{\mathbf{Sh}_v(P)}$ to get

$$\|\mathcal{A}^* g\|_{(W^{1,p}(\Omega))^*}^{p'} \lesssim \|g\|_{L^{p'}(\mu)}^{p'} = \mu(\mathbf{Sh}_v(P)).$$

Thus, it is enough to prove that

$$\sum_{Q \in \mathbf{SH}_v(P)} \mu(\mathbf{Sh}_v(Q))^{p'} \ell(Q)^{\frac{p-d}{p-1}} \lesssim \|\mathcal{A}^* g\|_{(W^{1,p}(\Omega))^*}^{p'} + \mu(\mathbf{Sh}_v(P)). \quad (5.15)$$

Given any $f \in W^{1,p}(\Omega)$, using (5.13) and Fubini's Theorem,

$$\langle \mathcal{A}^* g, f \rangle = \int g \mathcal{A}f d\mu = \int_{\Omega} f \left(\sum_{Q \in \mathcal{W}} \frac{\chi_{3Q}}{m(3Q)} \int_Q g d\mu \right) dm,$$

where we wrote $\langle \cdot, \cdot \rangle$ for the duality pairing. Consider

$$\tilde{g}(x) := \sum_{Q \in \mathcal{W}} \frac{\chi_{3Q}(x)}{m(3Q)} \int_Q g \, d\mu = \sum_{Q \in \mathbf{SH}_\nu(P)} \chi_{3Q}(x) \frac{\mu(Q)}{m(3Q)}. \quad (5.16)$$

Then,

$$\langle \mathcal{A}^* g, f \rangle = \int_{\Omega} f \tilde{g} \, dm.$$

Note that \tilde{g} is in L^∞ with norm depending on the distance from the support of μ to $\partial\Omega$ by (5.14), but the norm of \tilde{g} in L^1 is

$$\|\tilde{g}\|_{L^1} = \mu(\mathbf{SH}_\nu(P)).$$

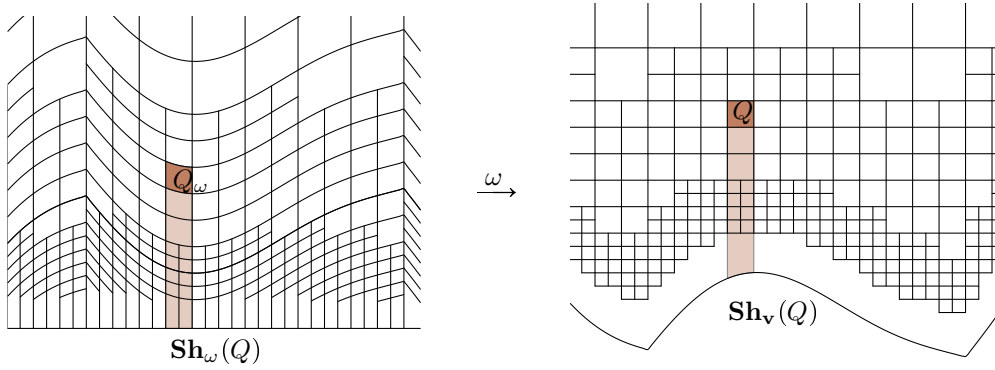


Figure 5.3: We divide \mathbb{R}_+^d in pre-images of Whitney cubes.

Consider also the change of variables $\omega : \mathbb{R}^d \rightarrow \mathbb{R}^d$, $\omega(x', x_d) = (x', x_d + A(x'))$ where A is the Lipschitz function whose graph coincides with $\partial\Omega$, and to every Whitney cube Q assign the set $Q_\omega = \omega^{-1}(Q)$ and its shadow $\mathbf{Sh}_\omega(Q) = \omega^{-1}(\mathbf{Sh}_\nu(Q))$ (see Figure 5.3). Then, for every $x \in \mathbb{R}^d$ we define

$$g_0(x) := \tilde{g}(\omega(x)) |\det(Dw(x))|, \quad (5.17)$$

where $\det(Dw(\cdot))$ stands for the determinant of the Jacobian matrix. Note that still $\|g_0\|_{L^1} = \|\tilde{g}\|_{L^1} = \mu(\mathbf{SH}_\nu(P))$, and, since $\omega(\mathbb{R}_+^d) = \Omega$,

$$\langle \mathcal{A}^* g, f \rangle = \int_{\Omega} f \tilde{g} \, dm = \int_{\mathbb{R}_+^d} f \circ \omega \cdot g_0 \, dm. \quad (5.18)$$

The key of the proof is using

$$h(x) := N \left[2 \left(R_d^{(d-1)} g_0 \right) d\sigma \right] (x) - N g_0(x), \quad (5.19)$$

which is the $W_{loc}^{1,1}(\mathbb{R}_+^d)$ solution of the Neumann problem

$$\int_{\mathbb{R}_+^d} \nabla \phi \cdot \nabla h \, dm = \int_{\mathbb{R}_+^d} \phi g_0 \, dm \quad \text{for every } \phi \in C_c^\infty(\overline{\mathbb{R}_+^d}), \quad (5.20)$$

provided by Lemma 5.20.

We divide the proof in four claims.

Claim 5.23. *If $\phi \in C_c^\infty(\overline{\mathbb{R}_+^d})$, then*

$$\langle \mathcal{A}^* g, \phi \circ \omega^{-1} \rangle = \int_{\mathbb{R}_+^d} \nabla \phi \cdot \nabla h \, dm.$$

Proof. Since ω is bilipschitz, the Sobolev $W^{1,p}$ norms before and after the change of variables ω are equivalent (see [Zie89, Theorem 2.2.2]). In particular, for $\phi \in C_c^\infty(\overline{\mathbb{R}_+^d})$, $\phi \circ \omega^{-1} \in W^{1,p}(\Omega)$ and we can use (5.18) and (5.20). \square

Now we look for bounds for $\|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))}$. The Hölder inequality together with a density argument would give us the bound

$$\|\mathcal{A}^* g\|_{(W^{1,p}(\Omega))^*} \lesssim \|\nabla h\|_{L^{p'}} + \mu(\mathbf{Sh}_\nu(P)),$$

with constants depending on the window size R , but we shall need a kind of converse.

Claim 5.24. *One has*

$$\|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))} \lesssim \|\mathcal{A}^* g\|_{(W^{1,p}(\Omega))^*} + \mu(\mathbf{Sh}_\nu(P)).$$

Proof. Take a ball B_1 containing $\omega^{-1}(4Q)$. The duality between L^p and $L^{p'}$ gives us the bound

$$\|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))} \lesssim \sup_{\substack{\phi \in C_c^\infty(B_1 \cap \mathbb{R}_+^d) \\ \|\phi\|_p \leq 1}} \left| \int \phi \partial_d h \, dm \right|.$$

To use the full potential of the Fourier transform, consider h^s to be the symmetric extension of h with respect to the hyperplane $x_d = 0$, $h^s(x', x_d) = h(x', |x_d|)$. It is immediate that h^s has global weak derivatives $\partial_j h^s = (\partial_j h)^s$ for $1 \leq j \leq d-1$ and $\partial_d h^s(x', x_d) = -\partial_d h(x', -x_d)$ for every $x_d < 0$. Thus,

$$\|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))} \lesssim \sup_{\substack{\phi \in C_c^\infty(B_1) \\ \|\phi\|_p \leq 1}} \left| \int \phi \partial_d h^s \, dm \right|. \quad (5.21)$$

Given $\phi \in C_c^\infty(B_1)$, consider the function $\tilde{\phi}(x) = \phi(x) - \phi(x - 2r_1 e_d)$, where e_d denotes the unit vector in the d -th direction and $r_1 = \frac{1}{2} \text{diam}(B_1)$, and take

$$I_\phi(x) = \int_{-\infty}^{x_d} \tilde{\phi}(x', t) dt. \quad (5.22)$$

Then, we have $I_\phi \in C_c^\infty(3B_1)$ with $\partial_d I_\phi \equiv \phi$ in the support of ϕ and $\|\partial_d I_\phi\|_p^p = 2\|\phi\|_p^p$. Thus,

$$\int \phi \partial_d h^s \, dm = \langle \partial_d I_\phi, \partial_d h^s \rangle - \int_{3B_1 \setminus B_1} \partial_d I_\phi \partial_d h^s \, dm, \quad (5.23)$$

where we use the brackets for the dual pairing of test functions and distributions. Using Hölder's inequality and the estimate (5.10) one can see that the error term in (5.23) is bounded by

$$\int_{3B_1 \setminus B_1} |\partial_d I_\phi \partial_d h^s| \, dm \leq \|\partial_d I_\phi\|_p \|\partial_d h^s\|_{L^{p'}(3B_1 \setminus B_1)} \leq C \|\phi\|_{L^p} \mu(\mathbf{Sh}_\nu(P)). \quad (5.24)$$

Note that C only depends on r_1 , which can be expressed as a function of the Lipschitz constant δ_0 and the window side-length R .

It is well known that the vectorial d -dimensional Riesz transform,

$$R^{(d)}f(x) = \frac{1}{2w_{d+1}} \text{p.v.} \int_{\mathbb{R}^d} \frac{x-y}{|x-y|^{d+1}} f(y) dy \text{ for every } f \in \mathcal{S}$$

is, in fact, a Calderón-Zygmund operator and, thus, it can be extended to a bounded operator in L^p . Writing $R_i^{(d)}$ for the i -th component of the transform and $R_{ij}^{(d)} := R_i^{(d)} \circ R_j^{(d)}$ for the double Riesz transform in the i -th and j -th directions, one has $\partial_{ii}I_\phi = R_{ii}^{(d)}\Delta I_\phi = \Delta R_{ii}^{(d)}I_\phi$ by a simple Fourier argument (see [Gra08, Section 4.1.4]). Thus, writing $f_\phi = R_{dd}^{(d)}I_\phi$, we have $\Delta f_\phi = \partial_{dd}I_\phi$, so

$$\langle \partial_d I_\phi, \partial_d h^s \rangle = -\langle \partial_{dd}I_\phi, h^s \rangle = -\langle \Delta f_\phi, h^s \rangle. \quad (5.25)$$

Let $f_r = \varphi_r f_\phi$ with φ_r a bump function in $C_c^\infty(B_{2r}(0))$ such that $\chi_{B_r(0)} \leq \varphi_r \leq \chi_{B_{2r}(0)}$, $|\nabla \varphi_r| \lesssim 1/r$ and $|\Delta \varphi_r| \lesssim 1/r^2$. We claim that

$$-\langle \Delta f_\phi, h^s \rangle = -\lim_{r \rightarrow \infty} \langle \Delta f_r, h^s \rangle = \lim_{r \rightarrow \infty} \langle \nabla f_r, \nabla h^s \rangle, \quad (5.26)$$

The advantage of f_r is that it is compactly supported, while only the Laplacian of f_ϕ is compactly supported. Recall that $\Delta f_\phi = \partial_{dd}I_\phi \in C_c^\infty(\mathbb{R}^d)$ so, by the hypoellipticity of the Laplacian operator, $f_\phi \in C^\infty(\mathbb{R}^d)$ itself (see [Fol95, Corollary (2.20)]). Thus, the second equality in (5.26) comes from the definition of distributional derivative. It remains to prove

$$\langle \Delta f_r - \Delta f_\phi, h^s \rangle \xrightarrow{r \rightarrow \infty} 0. \quad (5.27)$$

Since Δf_ϕ is compactly supported, taking r big enough we can assume that

$$\Delta[(\varphi_r - 1)f_\phi] = (\Delta \varphi_r)f_\phi + 2\nabla \varphi_r \cdot \nabla f_\phi,$$

so

$$|\langle \Delta f_r - \Delta f_\phi, h^s \rangle| \lesssim \int_{B_{2r}(0) \setminus B_r(0)} \left(\frac{|f_\phi||h^s|}{r^2} + \frac{|\nabla f_\phi||h^s|}{r} \right) dm.$$

If the dimension is $d = 2$ and $|x|$ large enough, equation (5.9) reads as

$$|h(x)| \lesssim |\log|x|| \|g_0\|_1.$$

If $d > 2$, we can use the same bound, since $|h(x)| \lesssim \frac{1}{|x|^{d-2}} \|g_0\|_1$. Using Hölder's inequality in (5.22) we have that $\|I_\phi\|_q \leq C\|\phi\|_q$ for any given $1 < q < \infty$. Now, $\partial_j f_\phi = \partial_j R_{dd}^{(d)}I_\phi = R_{dj}^{(d)}\partial_d I_\phi$, so using the boundedness of the d -dimensional Riesz transform in L^q , we get

$$\|f_\phi\|_{W^{1,q}} = \|f_\phi\|_{L^q} + \|\nabla f_\phi\|_{L^q} \leq C_q(\|I_\phi\|_q + \|\partial_d I_\phi\|_q) \leq C_q\|\phi\|_q. \quad (5.28)$$

Thus, for r large enough and choosing $q < d$ we get

$$\begin{aligned} \int_{B_{2r}(0) \setminus B_r(0)} \left(\frac{|f_\phi||h^s|}{r^2} + \frac{|\nabla f_\phi||h^s|}{r} \right) dm &\lesssim \frac{\log(r)}{r} \int_{B_{2r}(0) \setminus B_r(0)} (|f_\phi| + |\nabla f_\phi|) dm \|g_0\|_1 \\ &\lesssim \frac{\log(r)}{r} r^{\frac{d}{q}} \|f_\phi\|_{W^{1,q}(\mathbb{R}^d)} \|g_0\|_1 \xrightarrow{r \rightarrow \infty} 0, \end{aligned}$$

proving (5.27).

Back to (5.26), we can use $f_r^s(x', x_d) := f_r(x', -x_d)$ by a change of variables to obtain

$$\int \nabla f_r \cdot \nabla h^s dm = \int_{\mathbb{R}_+^d} \nabla f_r \cdot \nabla h dm + \int_{\mathbb{R}_+^d} \nabla f_r^s \cdot \nabla h dm = \langle \mathcal{A}^* g, (f_r + f_r^s) \circ \omega^{-1} \rangle \quad (5.29)$$

by means of Claim 5.23. Summing up, by (5.23), (5.24), (5.25), (5.26) and (5.29) and letting r tend to infinity, we get

$$\left| \int \phi \partial_d h^s dm \right| \lesssim |\langle \mathcal{A}^* g, (f_\phi + f_\phi^s) \circ \omega^{-1} \rangle| + \|\phi\|_{L^p} \mu(\mathbf{Sh}_\nu(P)). \quad (5.30)$$

Summing up, by (5.21), (5.30) and the estimate (5.28) with $q = p$, we have got that

$$\|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))} \lesssim \sup_{\|f\|_{W^{1,p}(\mathbb{R}^d)} \leq 1} |\langle \mathcal{A}^* g, f \circ \omega^{-1} \rangle| + \mu(\mathbf{Sh}_\nu(P)).$$

On the other hand, by [Zie89, Theorem 2.2.2] $\|f \circ \omega^{-1}\|_{W^{1,p}(\Omega)} \approx \|f\|_{W^{1,p}(\mathbb{R}_+^d)}$ for every f , so we have

$$\|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))} \lesssim \sup_{\|f\|_{W^{1,p}(\Omega)} \leq 1} |\langle \mathcal{A}^* g, f \rangle| + \mu(\mathbf{Sh}_\nu(P)) = \|\mathcal{A}^* g\|_{(W^{1,p}(\Omega))^*} + \mu(\mathbf{Sh}_\nu(P)),$$

that is Claim 5.24. □

Next we establish the relation between (5.15) and Claim 5.24.

Claim 5.25. *One has*

$$\begin{aligned} \sum_{Q \in \mathbf{SH}_\nu(P)} \mu(\mathbf{Sh}_\nu(Q))^{p'} \ell(Q)^{\frac{p-d}{p-1}} &\lesssim \|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))}^{p'} \\ &+ \sum_{Q \in \mathbf{SH}_\nu(P)} \int_{Q_\omega} \left(\int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz \right)^{p'} dx =: \textcircled{1} + \textcircled{2}. \end{aligned} \quad (5.31)$$

Proof. Note that in (5.19) we have defined h in such a way that

$$\begin{aligned} \partial_d h(x) &= R_d^{(d-1)} \left[2 \left(R_d^{(d-1)} g_0 \right) d\sigma \right] (x) - R_d^{(d-1)} g_0(x) \\ &= \frac{-1}{w_d} \int_{\mathbb{R}_+^d} \left(\frac{2x_d z_d}{w_d} \int_{\partial \mathbb{R}_+^d} \frac{d\sigma(y)}{|y - z|^d |x - y|^d} + \frac{x_d - z_d}{|x - z|^d} \right) g_0(z) dz. \end{aligned}$$

Given $x, z \in \mathbb{R}_+^d$, consider the kernel of $R_d^{(d-1)} \left[2 \left(R_d^{(d-1)}(\cdot) \right) d\sigma \right] - R_d^{(d-1)}(\cdot)$,

$$G(x, z) = \frac{2x_d z_d}{w_d} \int_{\partial \mathbb{R}_+^d} \frac{d\sigma(y)}{|y - z|^d |x - y|^d} + \frac{x_d - z_d}{|x - z|^d},$$

so that

$$\partial_d h(x) = \frac{-1}{w_d} \int_{\mathbb{R}_+^d} G(x, z) g_0(z) dz. \quad (5.32)$$

We have the trivial bound

$$G(x, z) + \frac{z_d - x_d}{|x - z|^d} \chi_{\{z_d > x_d\}}(z) \geq 0, \quad (5.33)$$

but given any Whitney cube $Q \in \mathbf{SH}_v(P)$, if $x \in Q_\omega$ and $z \in \mathbf{Sh}_\omega(Q)$ we can improve the estimate. In this case,

$$\int_{\partial\mathbb{R}_+^d \cap \overline{\mathbf{Sh}_\omega(Q)}} \frac{d\sigma(y)}{|y-z|^d} \gtrsim \int_{\partial\mathbb{R}_+^d \cap \omega^{-1}(\mathbf{Sh}_v(\omega(z)))} \frac{d\sigma(y)}{|y-z|^d} \approx \frac{1}{z_d}$$

and, thus,

$$\begin{aligned} G(x, z) + \frac{z_d - x_d}{|x - z|^d} \chi_{\{z_d > x_d\}}(z) &\geq \frac{2x_d z_d}{w_d} \int_{\partial\mathbb{R}_+^d} \frac{d\sigma(y)}{|y-z|^d |x-y|^d} \\ &\gtrsim \frac{\ell(Q)z_d}{\ell(Q)^d} \int_{\partial\mathbb{R}_+^d \cap \overline{\mathbf{Sh}_\omega(Q)}} \frac{d\sigma(y)}{|y-z|^d} \gtrsim \frac{\ell(Q)}{\ell(Q)^d}. \end{aligned} \quad (5.34)$$

By the Lipschitz character of Ω we know that $|\det D\omega(z)| \approx 1$ for every $z \in \mathbb{R}_+^d$. Thus, by (5.16) and (5.17), given $Q \in \mathbf{SH}_v(P)$ we have

$$\mu(\mathbf{Sh}_v(Q)) = \sum_{S \in \mathbf{SH}_v(Q)} \mu(S) \lesssim \int_{\mathbf{Sh}_v(Q)} \tilde{g}(w) dw \approx \int_{\mathbf{Sh}_\omega(Q)} g_0(z) dz.$$

For every $x \in Q_\omega$, using (5.34) first and then (5.33) we get

$$\begin{aligned} \mu(\mathbf{Sh}_v(Q)) &\lesssim \int_{\mathbf{Sh}_\omega(Q)} \left(G(x, z) + \frac{z_d - x_d}{|x - z|^d} \chi_{\{z_d > x_d\}}(z) \right) \frac{\ell(Q)^d}{\ell(Q)} g_0(z) dz \\ &\lesssim \left(\int_{\mathbb{R}_+^d} G(x, z) g_0(z) dz + \int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz \right) \ell(Q)^{d-1} \end{aligned}$$

and, by (5.32),

$$\mu(\mathbf{Sh}_v(Q)) \ell(Q)^{1-d} \lesssim |\partial_d h(x)| + \int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz.$$

Then, raising to the power p' , averaging with respect to $x \in Q_\omega$, we get that

$$\begin{aligned} \mu(\mathbf{Sh}_v(Q))^{p'} \ell(Q)^{p'-dp'} &\lesssim \int_{Q_\omega} |\partial_d h(x)|^{p'} dx + \int_{Q_\omega} \left(\int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz \right)^{p'} dx \\ &= \left(\|\partial_d h\|_{L^{p'}(Q_\omega)}^{p'} + \int_{Q_\omega} \left(\int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz \right)^{p'} dx \right) \ell(Q)^{-d} \end{aligned}$$

and, since $p' + d(1 - p') = \frac{p}{p-1} - \frac{d}{p-1}$, summing with respect to $Q \in \mathbf{SH}_v(P)$ we get the estimate (5.31), that is, Claim 5.25. \square

Finally, we bound the negative contribution of the $(d-1)$ -dimensional Riesz transform in (5.31), that is we bound $\textcircled{2}$.

Claim 5.26. *One has*

$$\textcircled{2} = \sum_{Q \in \mathbf{SH}_v(P)} \int_{Q_\omega} \left(\int_{\{z: z_d > x_d\}} \frac{z_d - x_d}{|x - z|^d} \tilde{g}(\omega(z)) dz \right)^{p'} dx \lesssim \mu(\mathbf{Sh}_v(P)). \quad (5.35)$$

Proof. Consider $x, z \in \mathbb{R}_+^d$ with $x_d < z_d$ and two Whitney cubes Q and S such that $x \in Q_\omega$ and $z \in \omega^{-1}(3S) \setminus \omega^{-1}(3Q)$. Then

$$\frac{z_d - x_d}{|x - z|^d} \lesssim \frac{\text{dist}(\omega(z), \partial\Omega)}{D(S, Q)^d} \approx \frac{\ell(S)}{D(S, Q)^d}.$$

On the other hand, when $3S \cap 3Q \neq \emptyset$,

$$\int_{\omega^{-1}(3Q)} \frac{|z_d - x_d|}{|x - z|^d} dz \lesssim \ell(Q) \approx \ell(S).$$

From the definition of \tilde{g} in (5.16) it follows that $\tilde{g}(\omega(z)) = \sum_{L \in \mathbf{SH}_v(P)} \chi_{3L}(\omega(z)) \frac{\mu(L)}{m(3L)}$. Bearing all these considerations in mind, one gets

$$\textcircled{2} \lesssim \sum_{Q \in \mathbf{SH}_v(P)} \ell(Q)^d \left(\sum_{S \in \mathbf{SH}_v(P)} \frac{\mu(S)\ell(S)}{D(S, Q)^d} \right)^{p'}.$$

Consider a fixed $\epsilon > 0$. One can apply first the Hölder inequality and then (5.14) to get

$$\begin{aligned} \textcircled{2} &\lesssim \sum_{Q \in \mathbf{SH}_v(P)} \ell(Q)^d \left(\sum_{S \in \mathbf{SH}_v(P)} \frac{\mu(S)\ell(S)^{1-\epsilon p'}}{D(S, Q)^d} \right) \left(\sum_{S \in \mathbf{SH}_v(P)} \frac{\mu(S)\ell(S)^{1+\epsilon p}}{D(S, Q)^d} \right)^{\frac{p'}{p}} \\ &\lesssim \sum_{Q \in \mathbf{SH}_v(P)} \ell(Q)^d \left(\sum_{S \in \mathbf{SH}_v(P)} \frac{\mu(S)\ell(S)^{1-\epsilon p'}}{D(S, Q)^d} \right) \left(\sum_{S \in \mathbf{SH}_v(P)} \frac{\ell(S)^{d-p+1+\epsilon p}}{D(S, Q)^d} \right)^{\frac{p'}{p}}. \end{aligned}$$

By Lemma 5.10, the last sum is bounded by $C\ell(Q)^{-p+1+\epsilon p}$ with C depending on ϵ as long as $d > d - p + 1 + \epsilon p > d - 1$, that is, when $\frac{p-2}{p} < \epsilon < \frac{p-1}{p}$. Thus,

$$\begin{aligned} \textcircled{2} &\lesssim \sum_{Q \in \mathbf{SH}_v(P)} \sum_{S \in \mathbf{SH}_v(P)} \frac{\mu(S)\ell(S)^{1-\epsilon p'} \ell(Q)^{d-p'+p'/p+\epsilon p'}}{D(S, Q)^d} \\ &= \sum_{S \in \mathbf{SH}_v(P)} \mu(S)\ell(S)^{1-\epsilon p'} \sum_{Q \in \mathbf{SH}_v(P)} \frac{\ell(Q)^{d-1+\epsilon p'}}{D(S, Q)^d}. \end{aligned}$$

Again by Lemma 5.10, the last sum does not exceed $C\ell(S)^{-1+\epsilon p'}$ with C depending on ϵ as long as $d > d - 1 + \epsilon p' > d - 1$, that is when $0 < \epsilon < \frac{1}{p'} = \frac{p-1}{p}$. Summing up, we need

$$\max \left\{ \frac{p-2}{p}, 0 \right\} < \epsilon < \frac{p-1}{p}.$$

Such a choice of ϵ is possible for every $p > 1$. Thus,

$$\textcircled{2} \lesssim \sum_{S \in \mathbf{SH}_v(P)} \mu(S) = \mu(\mathbf{Sh}_v(P)).$$

□

Now we can finish the proof of Proposition 5.22. The first term in the right-hand side of (5.31) is bounded due to Claim 5.24 by

$$\textcircled{1} = \|\partial_d h\|_{L^{p'}(\mathbf{Sh}_\omega(P))}^{p'} \lesssim \|\mathcal{A}^* g\|_{(W^{1,p}(\Omega))^*}^{p'} + \mu(\mathbf{Sh}_\mathbf{v}(P))^{p'}. \quad (5.36)$$

Being μ a finite measure, $\mu(\mathbf{Sh}_\mathbf{v}(P))^{p'} \leq \mu(\mathbf{Sh}_\mathbf{v}(P))\mu(\delta_0 \mathcal{Q})^{p'-1}$ and, thus, the bounds (5.35) and (5.36) combined with (5.31) prove (5.15), leading to

$$\sum_{Q \in \mathbf{SH}_\mathbf{v}(P)} \mu(\mathbf{Sh}_\mathbf{v}(Q))^{p'} \ell(Q)^{\frac{p-d}{p-1}} \lesssim \mu(\mathbf{Sh}_\mathbf{v}(P)).$$

□

For the sake of clarity, we restate Theorem 5.3 in terms of Carleson measures.

Theorem 5.27. *Given an admissible convolution Calderón-Zygmund operator of order 1, a Lipschitz domain Ω and $1 < p < \infty$, the following statements are equivalent:*

1. *Given any window \mathcal{Q} with a properly oriented Whitney covering, and given any Whitney cube $P \subset \delta_0 \mathcal{Q}$, one has*

$$\sum_{Q \in \mathbf{SH}_\mathbf{v}(P)} \left(\int_{\mathbf{Sh}_\mathbf{v}(Q)} |\nabla T_\Omega(\chi_\Omega)|^p dm \right)^{p'} \ell(Q)^{\frac{p-d}{p-1}} \leq C \int_{\mathbf{Sh}_\mathbf{v}(P)} |\nabla T_\Omega(\chi_\Omega)|^p dm.$$

2. *T_Ω is a bounded operator on $W^{1,p}(\Omega)$.*

Proof. The implication 1 \implies 2 is Theorem 5.1.

To prove that 2 \implies 1 we will use the previous proposition. Let us assume that we have a properly oriented Whitney covering \mathcal{W} associated to an R -window \mathcal{Q} of a Lipschitz domain Ω , where we assume that the window $\mathcal{Q} = Q(0, \frac{R}{2})$ is of side-length R and centered at the origin. Note that since T_Ω is bounded on $W^{1,p}(\Omega)$ then, by the Key Lemma 2.2,

$$\sum_{Q \in \mathcal{W}} |f_{3Q}|^p \int_Q |\nabla T_\Omega(\chi_\Omega)(x)|^p dx \lesssim \|f\|_{W^{1,p}(\Omega)}^p \quad \text{for } f \in W^{1,p}(\Omega). \quad (5.37)$$

Consider the Lipschitz function $A : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ whose graph coincides with the boundary of Ω in \mathcal{Q} . We say that $\tilde{\Omega}$ is the special Lipschitz domain defined by the graph of A that coincides with Ω in the window \mathcal{Q} . One can consider a Whitney covering $\tilde{\mathcal{W}}$ associated to $\tilde{\Omega}$ such that it coincides with \mathcal{W} in $\delta_0 \mathcal{Q}$. Consider the averaging operator

$$\mathcal{A}f(x) := \sum_{Q \in \tilde{\mathcal{W}}} \chi_Q(x) f_{3Q} \quad \text{for } f \in W^{1,p}(\tilde{\Omega}).$$

Writing $d\mu(x) := |\nabla T(\chi_\Omega)(x)|^p \chi_{\delta_0 \mathcal{Q}}(x) dx$ and taking a bump function $\chi_{\delta_0 \mathcal{Q}} \leq \psi_{\mathcal{Q}} \leq \chi_{\mathcal{Q}}$, we have that every $f \in W^{1,p}(\tilde{\Omega})$ satisfies that

$$\|\mathcal{A}f\|_{L^p(\mu)}^p = \sum_{Q \in \tilde{\mathcal{W}}} \mu(Q) f_{3Q}^p = \sum_{Q \in \tilde{\mathcal{W}}} \mu(Q) (f \psi_{\mathcal{Q}})_{3Q}^p \lesssim \|f \psi_{\mathcal{Q}}\|_{W^{1,p}(\Omega)}^p \lesssim \|f\|_{W^{1,p}(\tilde{\Omega})}^p.$$

by (5.37) and the Leibniz formula. That is, $\mathcal{A} : W^{1,p}(\tilde{\Omega}) \rightarrow L^p(\mu)$ is bounded.

In order to apply Proposition 5.22, we only need to show that $\mu(\mathbf{Sh}_v(Q)) \leq C\ell(Q)^{d-p}$ for every Whitney cube $Q \subset \mathcal{Q}$, which in particular implies that μ is finite. Fix a Whitney cube $Q \subset \mathcal{Q}$ and consider a bump function φ_Q such that $\chi_{\mathbf{Sh}_v(2Q)} \leq \varphi_Q \leq \chi_{\mathbf{Sh}_v(4Q)}$ with $|\nabla\varphi_Q| \lesssim \frac{1}{\ell(Q)}$.

Then,

$$\begin{aligned} \mu(\mathbf{Sh}_v(Q)) &= \int_{\mathbf{Sh}_v(Q) \cap \delta_0 \mathcal{Q}} |\nabla T \chi_\Omega(x)|^p dx \\ &\leq \int_{\mathbf{Sh}_v(Q)} |\nabla T(\chi_\Omega - \varphi_Q)(x)|^p dx + \int_\Omega |\nabla T \varphi_Q(x)|^p dx. \end{aligned} \quad (5.38)$$

With respect to the first term, notice that given $x \in \mathbf{Sh}_v(Q)$, $\text{dist}(x, \text{supp}(\chi_\Omega - \varphi_Q)) > \frac{1}{2}\ell(Q)$ so Lemma 1.31 together with (1.41) allows us to write

$$|\nabla T(\chi_\Omega - \varphi_Q)(x)| \lesssim \int_{\Omega \setminus \mathbf{Sh}_v(2Q)} \frac{1}{|y-x|^{d+1}} dy \lesssim \frac{1}{\ell(Q)}.$$

Being Ω a Lipschitz domain, $m(\mathbf{Sh}_v(Q)) \approx \ell(Q)^d$, so

$$\int_{\mathbf{Sh}_v(Q)} |\nabla T(\chi_\Omega - \varphi_Q)(x)|^p dx \lesssim \ell(Q)^{d-p}.$$

The second term in the right-hand side of (5.38) is bounded by hypothesis by a constant times $\|\varphi_Q\|_{W^{1,p}(\Omega)}^p$, and

$$\|\varphi_Q\|_{W^{1,p}(\Omega)}^p \approx \|\varphi_Q\|_{L^p(\Omega)}^p + \|\nabla\varphi_Q\|_{L^p(\Omega)}^p \lesssim \ell(Q)^d + \ell(Q)^{d-p} \lesssim (R^p + 1)\ell(Q)^{d-p},$$

where R is the side-length of the R -window \mathcal{Q} , proving that μ satisfies (5.14). \square

5.6 On the complex plane

Remark 5.28. *The article of Arcozzi, Rochberg and Sawyer [ARS02] has been the cornerstone in our quest for necessary conditions related to Carleson measures. In fact their article provides a quick shortcut for the proof of Theorem 5.27 (avoiding Proposition 5.22) for simply connected domains of class C^1 in the complex plane, and we believe it is worth to give a hint of the reasoning.*

Sketch of the proof. In the case of the unit disk, we found in the Key Lemma 2.2 that if T is an admissible convolution Calderón-Zygmund operator of order 1 bounded on $W^{1,p}(\mathbb{D})$, then

$$\sum_{Q \in \mathcal{W}} \left| \int_{3Q} f dm \right|^p \int_Q |\nabla T \chi_{\mathbb{D}}(z)|^p dm(z) \lesssim \|f\|_{W^{1,p}(\mathbb{D})}^p \quad (5.39)$$

for all $f \in W^{1,p}(\mathbb{D})$. If one considers $d\mu(z) = |\nabla T \chi_{\mathbb{D}}(z)|^p dm(z)$ and $\rho(z) = (1 - |z|^2)^{2-p}$, then, when f is in the Besov space of analytic functions on the unit disk $B_p(\rho)$, we have that

$$\|f\|_{B_p(\rho)}^p := |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1 - |z|^2)^p \rho(z) \frac{dm(z)}{(1 - |z|^2)^2} \approx \|f\|_{W^{1,p}(\mathbb{D})}^p.$$

Using the mean value property (and (5.14) for the error terms), one can see that if T is bounded, then for every holomorphic function f the bound in (5.39) is equivalent to

$$\int_{\mathbb{D}} |f(z)|^p |\nabla T \chi_{\mathbb{D}}(z)|^p dm(z) \lesssim \|f\|_{B_p(\rho)}^p,$$

i.e., $\|f\|_{L^p(\mu)} \lesssim \|f\|_{B_p(\rho)}$. Following the notation in [ARS02], the measure μ is a Carleson measure for $(B_p(\rho), p)$, establishing Theorem 5.27 for the unit disk by means of Theorem 1 in that article.

For $\Omega \subset \mathbb{C}$ Lipschitz and f analytic in Ω , we also have that, if $T : W^{1,p}(\Omega) \rightarrow W^{1,p}(\Omega)$ is bounded, then

$$\int_{\Omega} |f(z)|^p |\nabla T \chi_{\Omega}(z)|^p dm(z) \lesssim \|f\|_{W^{1,p}(\Omega)}^p.$$

If Ω is simply connected, considering a Riemann mapping $F : \mathbb{D} \rightarrow \Omega$, and using it as a change of variables, one can rewrite the previous inequality as

$$\int_{\mathbb{D}} |f \circ F(\omega)|^p \mu(F(\omega)) |F'(\omega)|^2 dm(\omega) \lesssim |f(F(0))|^p + \int_{\mathbb{D}} |(f \circ F)'(\omega)|^p |F'(\omega)|^{2-p} dm(\omega).$$

Writing $d\tilde{\mu}(\omega) = \mu(F(\omega)) |F'(\omega)|^2 dm(\omega)$, and $\rho(\omega) = |F'(\omega)(1 - |\omega|^2)|^{2-p}$, one has that given any g analytic on \mathbb{D} ,

$$\|g\|_{L^p(\tilde{\mu})} \lesssim \|g\|_{B_p(\rho)}.$$

So far so good, we have seen that $\tilde{\mu}$ is a Carleson measure for $(B_p(\rho), p)$, but we only can use [ARS02, Theorem 1] if two conditions on ρ are satisfied. The first condition is that the weight ρ is “almost constant” in Whitney squares, that is

$$\text{for } x_1, x_2 \in Q \in \mathcal{W} \implies \rho(x_1) \approx \rho(x_2),$$

and this is a consequence of Koebe distortion theorem, which asserts that for every $w \in \mathbb{D}$ we have

$$|F'(w)|(1 - |w|^2) \approx \text{dist}(F(w), \partial\Omega)$$

(see [AIM09, Theorems 2.10.6 and 2.10.8], for instance). The second condition is the Bekollé-Bonami condition, which is

$$\int_Q (1 - |z|^2)^{p-2} \rho(z) dm(z) \left(\int_Q ((1 - |z|^2)^{p-2} \rho(z))^{1-p'} dm(z) \right)^{p-1} \lesssim m(Q)^p.$$

If the domain Ω is Lipschitz with small constant depending on p (in particular if it is C^1), then this condition is satisfied (see [Bék86, Theorem 2.1]). \square

Conclusions

In this dissertation we have found answers to different problems related to the boundedness of convolution Calderón-Zygmund operators on Sobolev (and Triebel-Lizorkin) spaces on domains. It is worthy to finish pointing out possible further steps in this research.

Chapters 2 and 5:

Putting together Theorems 2.1 and 2.8, we have obtained the following $T(P)$ -theorem:

Theorem. *Let $\Omega \subset \mathbb{R}^d$ be a bounded uniform domain, T an admissible convolution Calderón-Zygmund operator of order $s \in \mathbb{N}$ or $0 < s < 1$, $d < p < \infty$ with $s > \frac{d}{p}$ and let $\mathcal{P}^{[s]-1}$ stand for the polynomials of degree smaller than s . Then*

$$\|T_\Omega P\|_{W^{s,p}(\Omega)} < \infty \text{ for every } P \in \mathcal{P}^{[s]-1}$$

if and only if

$$T_\Omega \text{ is bounded on } W^{s,p}(\Omega).$$

On the other hand, putting together Theorems 5.1 and 5.2 (writing ∇^s for ∇_2^s) we have shown the following theorem (see Figure 5.4):

Theorem. *Let Ω be a bounded Lipschitz domain, let $1 < p \leq d$ and let T be an admissible convolution Calderón-Zygmund operator of order $s \in \mathbb{N}$ or $0 < s < 1$ (with $s > \frac{d}{p} - \frac{d}{2}$ in the fractional case). If the measure $|\nabla^s T_\Omega P(x)|^p dx$ is a p -Carleson measure for every polynomial P of degree smaller than s , then T_Ω is a bounded operator on $W^{s,p}(\Omega)$.*

I expect that, combining the techniques of the integer orders of smoothness and the fractional ones, the first theorem above can be extended to the green sawtooth region (see Figure 5.5a) and the second theorem above to the red one, using

$$\nabla_q^s f(x) = \left(\int_{B_{\rho_1 \delta(x)}(x) \cap \Omega} \frac{|\nabla^{n-1} f(x) - \nabla^{n-1} f(y)|^q}{|x-y|^{sq+d}} dy \right)^{\frac{1}{q}}.$$

Moreover, I believe that the restriction $s > \frac{d}{p} - \frac{d}{2}$ is rather unnatural. I expect that using other expressions for the gradient in terms of means on balls, for instance, this restriction can be avoided (see Figure 5.5b).

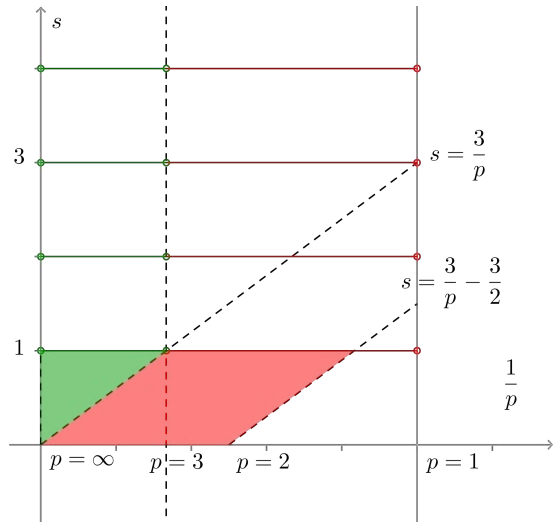


Figure 5.4: Indices for which the $T(P)$ -theorem (in green) and the Carleson theorem (in red) are valid in $W^{s,p}(\Omega)$ for uniform domains in \mathbb{R}^3 .

In fact, there is hope that, using higher order differences, a $T(P)$ -theorem can be obtained for all the supercritical range, and for $F_{p,q}^s(\Omega)$ in general.

Even more challenging is the question of whether Theorem 5.3 has a counterpart for other orders of smoothness or not, that is, if there is a necessary Carleson condition when $s \neq 1$.

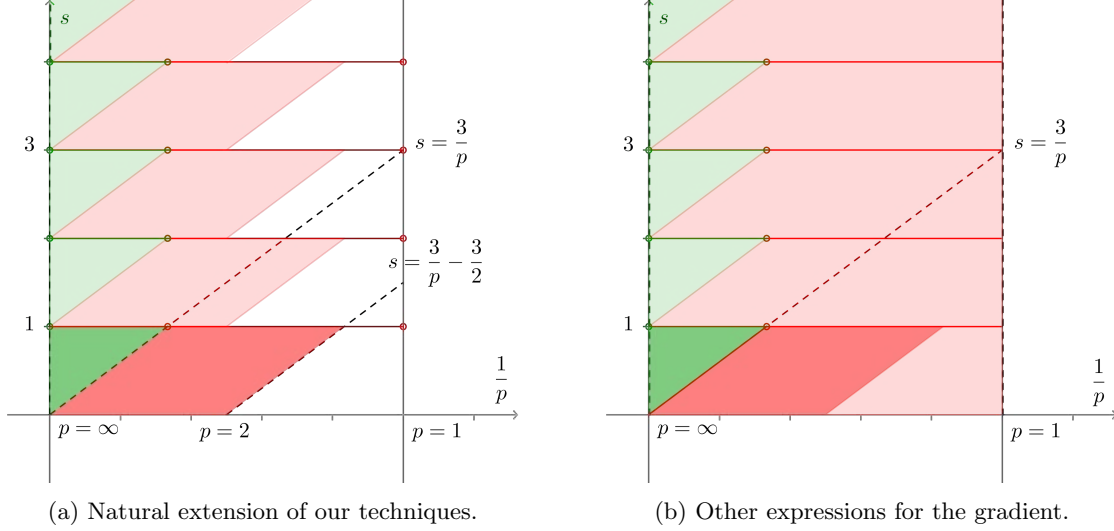


Figure 5.5: Conjectures on the indices where our results are valid for higher orders of fractional smoothness, in \mathbb{R}^3 .

Remark 5.29. For $1 < p, q < \infty$ and $0 < s < \frac{1}{p}$, we have that the multiplication by the characteristic function of a half plane is bounded in $F_{p,q}^s(\mathbb{R}^d)$. This implies that for domains Ω whose boundary consists of a finite number of polygonal boundaries, the pointwise multiplication with χ_Ω is also bounded in this space and, using characterizations by differences, this property can be seen to be stable under bi-Lipschitz changes of coordinates. Summing up, given a Lipschitz domain Ω and a function $f \in F_{p,q}^s(\mathbb{R}^d)$, we have that

$$\|\chi_\Omega f\|_{F_{p,q}^s(\mathbb{R}^d)} \lesssim \|f\|_{F_{p,q}^s(\mathbb{R}^d)}.$$

Moreover, if T is an operator bounded in $F_{p,q}^s$, using the extension $\mathcal{E} : F_{p,q}^s(\Omega) \rightarrow F_{p,q}^s(\mathbb{R}^d)$ (see [Ryc99], for instance), for every $f \in F_{p,q}^s(\Omega)$ we have that

$$\begin{aligned} \|T_\Omega f\|_{F_{p,q}^s(\Omega)} &= \|T(\chi_\Omega \mathcal{E}f)\|_{F_{p,q}^s(\Omega)} \leq \|T(\chi_\Omega \mathcal{E}f)\|_{F_{p,q}^s} \leq \|T\|_{F_{p,q}^s \rightarrow F_{p,q}^s} \|\chi_\Omega \mathcal{E}f\|_{F_{p,q}^s} \lesssim \|\mathcal{E}f\|_{F_{p,q}^s} \\ &\lesssim \|f\|_{F_{p,q}^s(\Omega)}. \end{aligned}$$

That is, given a p, q -admissible convolution Calderón-Zygmund operator T and a Lipschitz domain Ω we have that T_Ω is bounded in $F_{p,q}^s(\Omega)$ for any $0 < s < \frac{1}{p}$.

Chapter 3: Characteristic functions of planar domains

In this chapter we have studied when $\mathcal{B}\chi_\Omega \in W^{s,p}(\Omega)$. By the previous remark, when $sp < 1$ and Ω is a Lipschitz domain $\mathcal{B}\chi_\Omega \in W^{s,p}(\Omega)$ regardless of any other consideration in the smoothness of the boundary (see Figure 5.6).

Conjecture 5.30. *Let $0 < s < \infty$, let $1 < p < \infty$ and let Ω be a bounded (δ, R) - $C^{\lfloor s \rfloor - 1, 1}$ domain with parameterizations in $B_{p,p}^{s+1-1/p}$. Then, we have that $\mathcal{B}\chi_\Omega \in W^{s,p}(\Omega)$ and, if $s > \frac{1}{p}$, then*

$$\|\nabla^s \mathcal{B}\chi_\Omega\|_{L^p(\Omega)}^p \lesssim \|N\|_{B_{p,p}^{s-1/p}(\partial\Omega)}^p.$$

Theorem 3.27, proves the case $s \in \mathbb{N}$, and the result in [CT12] proves the case $\frac{1}{p} < s < 1$. I expect the cases $1 < s < \infty$ with $s \notin \mathbb{N}$ to be proven without much effort using the techniques of both. The case $sp = 1$ may also be of some interest. There is no need to say that further improvements in the $T(P)$ -theorems discussed above could help to deduce if, given a domain Ω , the truncated Beurling transform \mathcal{B}_Ω is bounded on $W^{s,p}(\Omega)$.

I conclude the discussion on the results of this chapter with two open questions.

Remark 5.31. *In [Tol13] it is seen that this result is sharp for $s = 1$. It remains to see if this is the case for $\frac{1}{p} < s < 1$ and for $1 < s < \infty$.*

At some point in my PhD studies I got interested in finding a counterpart of this theorem in higher dimensions. The main obstruction is Proposition 3.17, which depends strongly on complex analysis. It remains to see if this technique can be bypassed using real analysis tools for any operator in higher dimensions.

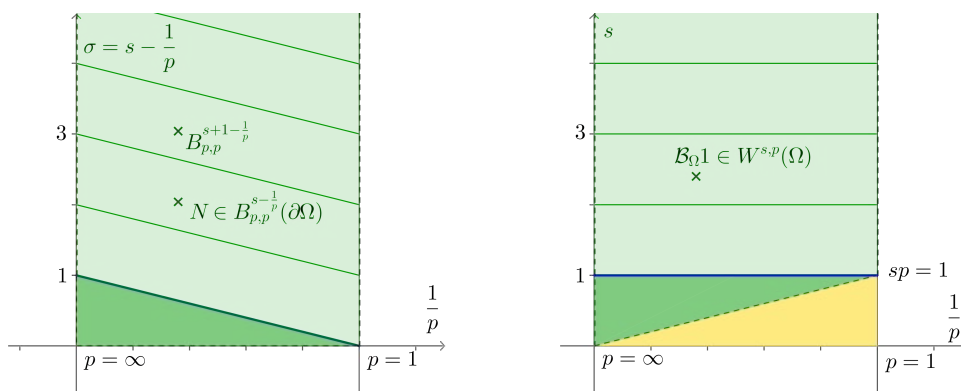


Figure 5.6: Indices considered on Conjecture 5.30. For $s = 1$ the result is sharp (blue segments). For $sp < 1$ (in yellow), every Lipschitz domain satisfies that $\mathcal{B}\chi_\Omega \in W^{s,p}(\Omega)$ as a consequence of Remark 5.29.

Chapter 4: An application to quasiconformal mappings

In relation to quasiconformal mappings, I expect the following conjecture to be true.

Conjecture 5.32. *Let $n \in \mathbb{N}$, and let $0 < \{s\} \leq 1$ and $s = n - 1 + \{s\}$. Let Ω be a bounded Lipschitz domain with outward unit normal vector N in $B_{p,p}^{s-1/p}(\partial\Omega)$ for some $p < \infty$ with $\{s\}p > 2$ and let $\mu \in W^{s,p}(\Omega)$ with $\|\mu\|_{L^\infty} < 1$ and $\text{supp}(\mu) \subset \bar{\Omega}$. Then, the principal solution f to the Beltrami equation is in the Sobolev space $W^{s+1,p}(\Omega)$ (see Figure 5.7).*

This conjecture for $s \in \mathbb{N}$ is exactly Theorem 4.1. I expect this result to be proven straight ahead using the techniques exposed in this chapter when $0 < s < 1$ (Conjecture 5.30 holds by [CT12]). The main difficulties will be to prove the compactness of $[\mu, \mathcal{B}]$ and \mathcal{R}^m . For higher fractional orders of smoothness I expect some complications but the core of the proof should remain the same. However, to study this case, we will need a quantitative version of Conjecture 5.30 to be proven for the corresponding indices.

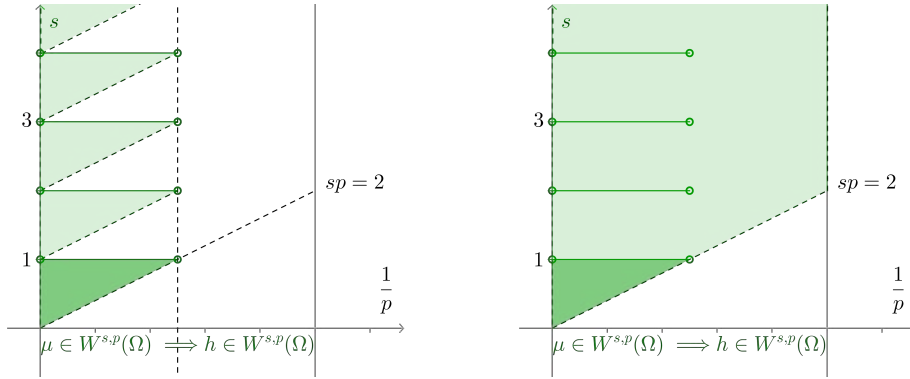


Figure 5.7: Conjectures on the indices where the application to quasiconformal maps holds.

Again, in case that further improvements in the $T(P)$ -theorems discussed above lead to a wider range of indices than expected (say to the whole supercritical region), this conjecture could be extended in the same spirit, probably using higher order differences for the homogeneous Sobolev seminorms. I do not expect to have such a result for the critical and the subcritical case, where a decay in the integrability or even in smoothness in $\mu \mapsto h$ seems to be unavoidable when $\|\mu\|_{L^\infty} > 0$, in view of the examples in [CFM⁺09, pages 205-206].

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