Appendix A

Appendix to chapter 2

1. Land rents equilibrium values with two controlling cities

1.1 Total land rents with competition (TR_{comp})

The expression of land rents in 2.4 can be simplified to:

$$TR_1 = -\frac{t}{18k_2}[\alpha^2(-12+5k)N_2^A + \alpha N^A((-6+k)N^A - 18k_2 r_1) + 9k_2 r_1(-2N^B + 3kr_1 + 2kr_2)], \quad (1.1)$$

by properly considering the expression of z^A and z^B in function of r_1 and r_2, and the best response function of 2, Substituting r_1 and r_2 with their competitive values

$$r_{comp} = r_1 = r_2 = \frac{1}{4k}\left[\alpha N^A + N^B\right],$$

it is obtained:

$$TR_{comp} = \frac{t}{288k}\left[11\alpha(N^A)^2 + 54\alpha N^A N^B + 27(N^B)^2 + 16\alpha N^A\right].$$

1.2 Total land rents with cooperation (TR_{coop})

In this instance, the value of r_1 corresponds to the cooperative city size, i.e.

$$r_{coop} = \frac{1}{5k}\left[\alpha N^A + N^B\right].$$

Substituting in the expression of total land rents above, it is obtained

$$TR_{coop} = \frac{t}{450k}\left[20\alpha^2(N^A)^2 + 90\alpha N^A N^B + 45(N^B)^2 + 25\alpha(N^A)^2\right].$$

1.3 Total land rents when one city deviates while the other one cooperates (TR_{dev} and $TR_{coop'}$, respectively)

The respective city sizes will be

$$\overline{r}_{dev} = \frac{4}{15k}\left[\alpha N^A + N^B\right] \quad \text{and}$$

$$r_{coop} = \frac{1}{5k}\left[\alpha N^A + N^B\right].$$

Substituting this values into the expression of land rents in 1.1, we have

$$TR_{dev} = \frac{t}{450k}\left[23\alpha^2(N^A)^2 + 96\alpha N^A N^B + 48(N^B)^2 + 25\alpha(N^A)^2\right] \quad \text{and}$$

$$TR_{coop'} = \frac{t}{450k}\left[14\alpha(N^A)^2 + 78\alpha N^A N^B + 39(N^B)^2 + 25\alpha(N^A)^2\right].$$

2. Present values of total land rents ($PVTR$) in an infinite-horizon game

2.1 PVTR with competition

The present value of total rents are

$$PVTR_{comp} = \sum_{t=0}^{\infty} TR^t_{comp} = TR^0_{comp} + \sum_{t=1}^{\infty} TR^t_{comp},$$

with the superscript denoting the time period and assuming that land rents when cities compete are the same in each period. Then:

$$PVTR_{comp} = TR_{comp} + \frac{TR_{comp}}{r(1+r)},$$

with r representing the interest rate. Substituting the value of TR_{comp} in 2.15 in page 21,

$$PVTR_{comp} = \frac{t}{288k}\left[11\alpha^2(N^A)^2 + 54\alpha N^A N^B + 27(N^B)^2 + 16\alpha(N^A)^2\right] + \frac{t}{288r(1+r)k}\left[11\alpha^2(N^A)^2 + 54\alpha N^A N^B + 27(N^B)^2 + 16\alpha(N^A)^2\right].$$

2.2 PVTR with cooperation

$$PVTR_{coop} = TR^0_{coop} + \sum_{t=1}^{\infty} TR^t_{coop},$$

again assuming that land rents from cooperation are the same in each period. Then,

$$PVTR_{coop} = TR_{coop} + \frac{TR_{coop}}{r(1+r)},$$

that can be rewritten using 2.20 in page 22 in chapter 2:

$$PVTR_{coop} = \frac{t}{450k}\left[20\alpha^2(N^A)^2 + 90\alpha N^A N^B + 45(N^B)2 + 25\alpha(N^A)^2\right] + \frac{t}{450r(1+r)k}\left[20\alpha^2(N^A)^2 + 90\alpha N^A N^B + 45(N^B)^2 + 25\alpha(N^A)^2\right]$$

.

2.3 PVTR when one city deviates and the other one cooperates

For the city that deviates from cooperation,

$$PVTR_{dev} = TR^0_{dev} + \sum_{t=1}^{\infty} TR^t_{comp},$$

or

$$PVTR_{dev} = TR_{dev} + \frac{TR_{comp}}{r(1+r)}.$$

Using the expressions of TR_{dev} and TR_{comp} in equations 2.23 and 2.15, it is obtained:

$$PVTR_{dev} = \frac{t}{450k}\left[23\alpha^2(N^A)^2 + 96\alpha N^A N^B + 48(N^B)^2 + 25\alpha(N^A)^2\right] + \frac{t}{288r(1+r)k}\left[11\alpha^2(N^A)^2 + 54\alpha N^A N^B + 27(N^B)^2 + 16\alpha(N^A)^2\right].$$

For the city choosing the cooperative population control while the otherone deviates,

$$PVTR_{coop'} = TR^0_{coop'} + \sum_{t=1}^{\infty} TR^t_{comp}.$$

Then,

$$PVTR_{coop'} = TR_{coop'} + \frac{TR_{comp}}{r(1+r)},$$

and substituting with the expressions in 2.24 and 2.15 respectively:

$$\begin{aligned} PVTR_{coop'} = & \frac{t}{450k}\left[14\alpha^2(N^A)^2 + 78\alpha N^A N^B + 39(N^B)^2 + 25\alpha(N^A)^2\right] \\ & + \frac{t}{288r(1+r)k}\left[11\alpha(N^A)^2 + 54\alpha N^A N^B + 27(N^B)^2 + 16\alpha(N^A)^2\right]. \end{aligned}$$

2.4 Comparison of the values of PVTR

- $PVTR_{comp}$ and $PVTR_{coop'}$

  The only difference between the two expressions above are land rents from the first period. Since $TR_{comp} > TR_{coop'}$, then $PVTR_{comp} > PVTR_{coop'}$.

- $PVTR_{dev}$ and $PVTR_{coop}$

  It was already shown in subsection 3.3 in page 25 that cooperation leads to a higher level of total land rents as long as the rate interest is $r < 0.589$.

3. Optimal housing tax with cooperation

The general objective function for local authorities in city A when maximizing tax revenues was:

$$\max_{h^1} \; Rh_1 = h_1 k \, \overline{r^1} = h_{coop}\, k \left[\frac{\alpha N^A + N^B}{3k} - \frac{h_{coop}}{3t}\right],$$

once assumed that the two active cities would implement identical tax levels.

For every single city, the value of h_{coop} is obtained when solving for h_{coop} in the following expression:

$$\frac{\partial Rh_{coop}}{\partial h_{coop}} = k\left[\frac{\alpha N^A + N^B}{3k} - \frac{h_{coop}}{3t}\right] - \frac{k}{3t}h_{coop} = 0,$$

what yields the value of

$$h_{coop} = \frac{t}{2k}\left[\alpha N^A + N^B\right].$$

4. Housing tax revenues in a single period game

4.1 Revenues with competition

In this instance the expression of revenues is obtained from the appropriate tax under competition, h_{comp}, and the general expression for $Rh(h_1, h_2)$:

$$\begin{aligned} Rh_{comp} &= h_{comp}\, k\left[\frac{\alpha N^A + N^B}{3k} - \frac{h_{comp}}{3t}\right] \\ &= \frac{t}{3k}\left[\alpha N^A + N^B\right] k\left[\frac{\alpha N^A + N^B}{3k} - \frac{1}{9k}(\alpha N^A + N^B)\right] \\ &= \frac{2t}{27k}\left[\alpha N^A + N^B\right]^2. \end{aligned}$$

4.2 Revenues with cooperation

With cooperation, revenues are calculated combining the general expression for tax revenues and the arising tax under competition, h_{coop}:

$$\begin{aligned} Rh_{coop} &= h_{coop}\, k\left[\frac{\alpha N^A + N^B}{3k} - \frac{h_{coop}}{3t}\right] \\ &= \frac{t}{2k}\left[\alpha N^A + N^B\right] k\left[\frac{\alpha N^A + N^B}{3k} - \frac{1}{6k}(\alpha N^A + N^B)\right] \\ &= \frac{t}{12k}\left[\alpha N^A + N^B\right]^2. \end{aligned}$$

4.3 Revenues when one city cooperates and the other one deviates

Let h_{coop} denote the tax implemented by city that cooperates, and h_{dev} the tax chosen by the city that deviates from the cooperation agreement. Tax revenues for the first one are

$$Rh_{coop'}$$

, and can be expressed as:

$$Rh_{coop'} = h_{coop}\, k \left[\frac{\alpha N^A + N^B}{3k} + \frac{1}{3t}(h_{dev} - 2h_{coop}) \right].$$

From the values of $h_{coop} = \frac{t}{2k}\left[\alpha N^A + N^B\right]$ and $h_{dev} = \frac{3t}{8k}\left[\alpha N^A + N^B\right]$, it results

$$Rh_{coop'} = \frac{t}{16k}\left[\alpha N^A + N^B\right]^2.$$

As for the city deviating, revenues are

$$Rh_{dev} = h_{dev}\, k \left[\frac{\alpha N^A + N^B}{3k} + \frac{1}{3t}(h_{coop} - 2h_{dev}) \right],$$

expression that can be simplified to

$$Rh_{dev} = \frac{3t}{32k}\left[\alpha N^A + N^B\right]^2.$$

4.4 Comparison of tax revenues

If we compare the expressions of tax revenues under the different scenarios above considered, it can be easily seen that

$$Rh_{dev} > Rh_{coop} > Rh_{comp} > Rh_{coop'}.$$

5. Comparison of tax revenues in the cooperative and deviation scenarios in the infinite horizon game

Cooperating becomes the equilibrium solution as long as $PVRh_{coop} > PVRh_{dev}$, that is

$$\frac{t}{12k}(\alpha N^A + N^B)^2 + \frac{t(\alpha N^A + N^B)^2}{12kr(1+r)} > \frac{3t}{32k}(\alpha N^A + N^B)^2 + \frac{2t(\alpha N^A + N^B)^2}{27kr(1+r)};$$

$$\frac{1}{12} + \frac{1}{12r(1+r)} > \frac{3}{32} + \frac{2}{27r(1+r)};$$

$$r_2 + r - \frac{8}{9} < 0;$$

Thus, cooperating will be the preferred strategy as long as the interest rate accomplishes

$$r < r^* \approx 0,567.$$

6. Comparison of revenues with population controls and taxes

6.1 Relationship between the housing tax, city size and tax revenues when two cities impose identical controls

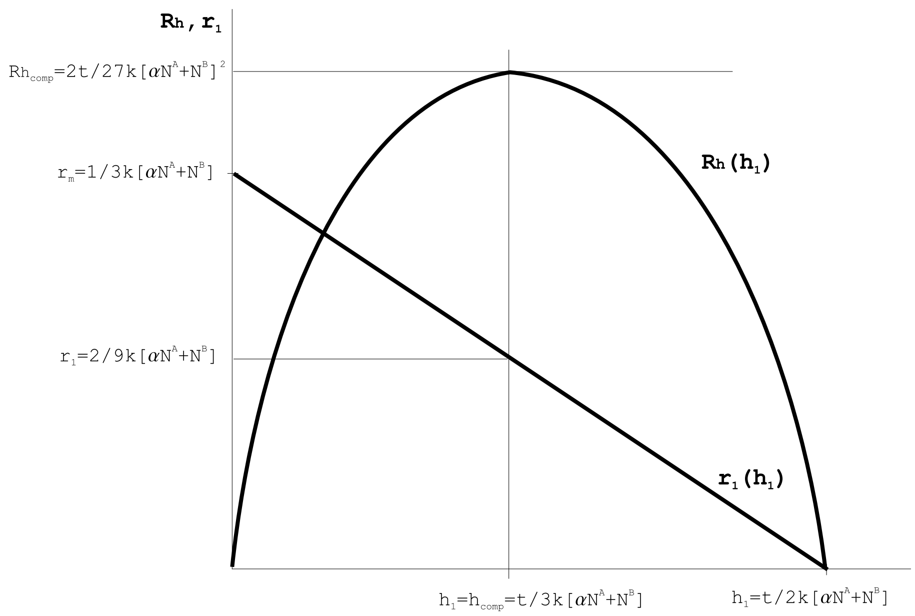

Figure A.1. The effect of the housing tax on city size and revenues