Chapter 4

Sequential Formation of Coalitions through Bilateral Agreements

This Chapter is a joint work with Inés Macho Stadler and David Pérez Castrillo.

4.1 Introduction

The incentives of firms to merge have recently been studied in non cooperative games of endogenous coalition formation. In these games, any set of players can decide to form a coalition. However, negotiations by large number of agents may be difficult or costly. In these cases, a large set of players may not be able to agree on forming a coalition, that is, players only have the possibility of forming coalitions that involve a small number of participants. This does not mean that once some coalitions are formed, they cannot decide to continue with the process of forming bigger entities.

The sector of firms providing professional services (accounting, consulting, etc.)

offers a relevant set of examples of such a sequential process of mergers. Some of the major firms in this sector, for instance, Ernst & Young, KPGM and PricewaterhouseCoopers, are the outcome of a sequential process of mergers with a small number of parties involved. In particular, since Arthur Young opened an accounting firm in Chicago (1894), and the brothers Alvin and Theodore Ernst settled their firm in Cleveland (1903) until the present structure of Ernst and Young, at least four bilateral mergers have taken place.

The banking sector provides other examples. In Spain, the now called SCH is the outcome of the merger of the Banco de Santander with the Banco Central Hispano, which in turn was issued of the merger of banks Central and Hispano. Similarly, banks Bilbao and Vizcaya first merged to form the BBV and then this new entity merged with the Banco Argentaria to form the BBVA.

We model the formation of coalitions as a sequential process in which, at each moment in time, only two existing coalitions can decide to merge. We study the subgame perfect equilibria of such a game. The sequential process of coalition formation that we propose can be useful to analyze sequential formation of bilateral agreements in several economic environments where groups of agents interact, including mergers, environmental cartels, and networks.

In this paper, we consider a market where identical firms with constant returns facing linear demand compete à la Cournot. At each period, firms take decisions on quantity. To concentrate our analysis on the incentives to form coalitions, we assume that production is a short-term decision. Also, at each period, two randomly chosen coalitions in the existing partition can merge. A merger means forming a cartel where the decision on the total level of production by the partners is made jointly. The decision on the merger is made taking into account the long-term profits.

As Salant, Switzer, and Reynolds (1983) pointed out, two firms (or coalitions) will not be interested in merging if they only consider the present period profits and there are at least three firms (coalitions) in the industry. Their result extends easily to our model: If the firms' discount rate is low enough, they will not merge at any period in the unique subgame perfect equilibrium of the game. Hence, the outcome is that all firms stay as singletons.

The situation when firms are forward looking is more interesting. In this case, firms may want to merge even if they lose profits in the short run. In fact, we show that when firms are patient enough, and there are enough firms in the industry, the final outcome of any subgame perfect equilibria is the grand coalition.[1] The firms form coalitions sequentially, growing gradually, so that they end up all together. We characterize the sequences of mergers that the firms will undertake at equilibrium. In those sequences, firms will accept some of the mergers and will reject others.

The fact that, in a linear Cournot model, the grand coalition can result as the equilibrium of a game of coalition formation is in contrast with other results in the literature on mergers. Interestingly, it is the fact that *only* a pair of coalitions can merge at each period (so bigger coalitions cannot form immediately) which allows reaching the grand coalition. Even if this restriction to bilateral agreements is ex-ante damaging for the formation of coalitions, since it reduces the choice set of the firms, it ex-post results in an impulse to the process of merging. The reason is that the bilateral nature of the mergers induces the coalitions to be formed gradually, in such a way that the incentives of all the players to free

[1] This intertemporal effect is also present in Pesendorfer (2000).

ride are always outweighed by the future benefits from merging.

Bloch (1996) and Ray and Vohra (1999) also analyze an infinite-horizon sequential game. In their model, payoffs are realized only once the coalitions are formed. In the coalition formation game previous to the production, the first agent, according to a rule of order, makes an offer to other agents to join him in a coalition. If all members accept the offer, the partnership is formed and partners in the coalition leave the game. Then, the first agent in the set of remaining players makes a partnership proposal, and the game continues following the same rule until all players have left the game. If someone rejects, he will do the next proposal. This model applies to general games. For the linear Cournot game, Bloch (1996) proves that, when players are ex-ante symmetric and the discount rate is high enough, the coalition structures that result in the Markov symmetric perfect equilibria in pure strategies contain a coalition whose size is about 80% of market, while the other firms remain isolated. Hence, the grand coalition is not formed because some individual firms are able to appropriate the positive externalities generated by the large coalition, while firms in the large coalitions are better off inside than outside the coalition.

The three main differences between the game by Bloch (1996) and Ray and Vohra (1999) and our proposal are that, in their game, first, a player can make an offer to any set of partners, second, if the offer is accepted the coalition leaves the game and third, production takes place only once the coalitions are formed. The third difference is however not relevant. In the last section of the paper, we propose a variant of our game in which firms only produce after the coalition formation game has been played. We check that our results also hold in this environment. In that section, we also show that our analysis can be easily adapted to cope with situations where the identity of the firms or coalitions that can merge at a given period is not random but it follows a deterministic protocol.

Several authors have addressed the question of the coalition structures that would prevail in Cournot games with homogenous goods and linear demand by analyzing the stability of the coalition structures.[2] This literature suggests that we would observe a large coalition and some players as singletons. Our game has never these intermediate results: If there is a small number of players or the discount rate is low, all players remain as singletons, while the grand coalition is the only final outcome when both the set of players and the discount rate are large enough. In fact, all singleton and the grand coalition are the only possible subgame perfect equilibrium outcomes of our game.

In addition to the literature that analyzes the formation or stability of coalition structures in Cournot games, our work is also related to Gul (1989). This author analyzes a transferable utility economy where random meeting between two agents occur. A each meeting, one of the agents makes a proposal to the other that this last agent can either accept or reject. If the proposal is accepted, then resources of both agents are in the hands of the proposer from this moment on, otherwise both players are staying at the game. Gul (1989) shows that, under some conditions, all the players will eventually end up together and the expected payoff of each player at an efficient Markov perfect equilibrium is his Shapley value.[3]

[2] We can refer to four stability concepts (Aumann (1967) and Hart and Kurtz (1983)). A coalition structure is α-stable if no group of firms can guarantee an improvement, independently of what the others do. A partition is β-stable if no group of firms has, for any possible reaction of the external players, a strategy that can improve its situation. A coalition structure is γ-stable (respectively, δ-stable) if no set of players has incentives to deviate when the players of their original coalitions split apart (respectively, they still form a coalition). In the linear Cournot game, α-stable, β-stable, and γ-stable outcomes have always the form $\{s, 1, ..., 1\}$ with s higher or equal than 80% of the market. On the other hand, the set of δ-stable outcomes is empty.

[3] Other papers analyzing gradual coalition formation in games without externalities accross coalitions are

In the next section, we present the coalition formation game. In Section 4.3, we analyze the outcomes of the game when firms are myopic, while in Section 4.4 we do the analysis when firms are forward looking. In Section 4.5, we show how our results extend to several variants of our game.

4.2 The Coalition Formation Game

We study the sequential formation of coalitions between firms competing à la Cournot in a framework where only *bilateral agreements* are allowed. We assume that, at each moment in time only two of the existing coalitions can decide to merge.

At the beginning of the game, there are n identical firms, with $n \geq 2$. We denote by $N = \{1, ..., n\}$ the set of firms. Firms can form coalitions following a certain protocol, that will be described later. Hence, at any point in time, these n firms form a partition of N, that is, they constitute a *coalition structure.*

Let $\prod$ denote the set of coalition structures over N. Denote $\pi \in \prod$ an element of this set, that is, $\pi = \{S_1, ..., S_r\}$, with $S_a \subset N$ for all $a = 1, ..., r$ and $S_a \cap S_b = \emptyset$ for all $S_a, S_b \in \pi$, with $S_a \neq S_b$. We denote by s_a the size of coalition S_a. A particular coalition structure is the one where all the agents are alone, that is, all the coalitions are singletons. We denote by π^n such a partition and by $\pi^1 \equiv N$ the grand coalition, that is, the coalition structure with only one element. We denote by $(\pi \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$, the coalition structure that results when we replace two elements of π, S_a and S_b, by their union. Therefore, if π is formed by r coalitions, $(\pi \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ consists of $r - 1$

Chatterjee *et al.* (1993), Okada (1996), and Seidmann and Winter (1998).

coalitions.

Firms take decisions at any time $t = 0, 1, 2, ...$ At time t, the present profits of a firm depend on the whole coalition structure that is formed at that time. We assume for simplicity that firms face a linear demand function and bear equal constant average costs. That is, the inverse demand function is:

$$P(\sum_{j=1}^{n} q_j) = \alpha - \beta \sum_{j=1}^{n} q_j.$$

The costs of production of firm i are given by:

$$C_i(q_i) = cq_i.$$

When firms merge, they form a cartel. That is, merging allows firms to coordinate their quantity decisions. We calculate firms' profit at any point in time given a cartel structure (i.e., a coalition structure) $\pi = \{S_1, ..., S_r\}$. We assume that production is a short-term decision, being taken by short-term managers.[4] Given that there are r cartels in this structure and that marginal costs are equal for all firms in a cartel, cartel S_a chooses the total level of production q_a of its firms by solving the following maximization program:

$$\max_{q_a} \left\{ \left(\alpha - \beta \sum_{b=1}^{r} q_b \right) q_a - cq_a \right\}. \tag{4.1}$$

From here we find that the equilibrium quantities are equal for all the cartels and they are equal to: $q^r = \frac{\alpha - c}{\beta(r+1)}$. Hence, the Cournot profits per-cartel V^r in a coalition structure with

[4]It is well known that, in an infinite game like ours, there are strategies under which firms may reach implicit collusion in production if the discount rate is high enough (notice however that the set of equilibrium outcomes is usually very large). Our objective in this paper is the analysis of the incentives for coalition formation, so we will abstract from the possibility of collusion by assuming that production is a short-term decision. An equivalent assumption is that firms use Markov, or "state-space" strategies when they decide their production level. In Section 5, we analyze a simpler game where this assumption is not necessary because production only takes place once and where all our results stil hold.

r cartels are:

$$V^r = \frac{(\alpha - c)^2}{\beta(r+1)^2}.$$

We normalize $\frac{(\alpha-c)^2}{\beta} = 1$, so:

$$V^r = \frac{1}{(r+1)^2}.$$

It can be easily verified that the efficient outcome is reached when all the firms merge, and the grand coalition is formed.

We assume that the sharing of the profits among the firms that form the cartel is exogenously fixed and egalitarian. Therefore, the individual profits $V_i(\pi)$ of any firm i belonging to the cartel $S_a \in \pi$, with size s_a, when there are r cartels in the coalition structure π, are:

$$V_i(\pi) = \frac{1}{(r+1)^2 s_a}. \tag{4.2}$$

Firms value future payoffs with a homogeneous discount factor $\delta \in [0, 1)$. Therefore, if π_t is the coalition structure existing at time t, for $t \geq t^\circ$, the discounted payoff of firm i at time t° is $\sum_{t=t^\circ}^{\infty} \delta^{(t-t^\circ)} V_i(\pi_t)$.[5] We will also discuss at some point the particular case when the players are perfectly patient ($\delta = 1$) and evaluate future profits with the "time-average criterion". That is, firm i maximizes:

$$\lim_{T\to\infty} \inf \frac{\sum_{t=0}^{T} V_i(\pi_t)}{T}.$$

We study the outcome of a *process of sequential coalition formation.* This infinite-horizon process is undertaken according to the following protocol. At each period t, first

[5] When $\delta = 0$, the discounted payoff of player i at time t^o is $V_i(\pi_{t^o})$.

(phases $t.1$ and $t.2$) there is the decision about merging and second (phase $t.3$), there is the production stage. We have already described the result of the production stage, summarized by the profit function $V_i(\pi_t)$. More precisely:

At $t = 0$:

0.1 Two different firms i and j are randomly selected. All the firms have identical probability of being selected.

0.2 Firms i and j sequentially decide whether or not to merge. The merger occurs if both players agree.

The coalition structure at time $t = 0$ is then either $\pi_0 = (\pi^n \backslash \{\{i\},\{j\}\}) \cup \{i,j\}$ if firms i and j merged or $\pi_0 = \pi^n$ if they did not.

0.3 Each firm $k \in N$ obtains, at $t = 0$, profits $V_k(\pi_0)$.

Consider now any time $t \geq 1$. The coalition structure existing at $t-1$ was π_{t-1}. If $\pi_{t-1} = N$, then $\pi_t = N$. Otherwise:

t.1 Two coalitions S_a and S_b in π_{t-1} are randomly selected. All the coalitions have identical probability of being selected.

t.2 Firms in coalitions S_a and S_b sequentially decide whether to merge. The merger happens if all the firms in coalitions S_a and S_b agree on it.

The coalition structure at time t is either $\pi_t = (\pi_{t-1} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ if coalitions S_a and S_b merged or $\pi_t = \pi_{t-1}$ if they did not.

t.3 Each player $k \in N$ obtains profits $V_k(\pi_t)$ at time t.

The solution concept that we consider is *Subgame Perfect Equilibrium* and we concentrate in *pure strategies*. We will denote by SPE the set of Subgame Perfect Equilibria in pure strategies.

Let us remark that the proposed process of formation of coalitions is irreversible in the sense that the players cannot undo a merger once it is formed. Allowing for mergers to split apart considerably enlarges the set of possible SPE.

Given the irreversibility of the process of coalition formation, with probability one the game will reach a situation where the existing coalition structure at that period will remain forever. We will refer to such a coalition structure as a *final coalition structure* or a *final outcome*. If there are SPE strategies that lead to a particular final outcome, then we say that it is a *SPE final outcome*.

4.3 Myopic Firms

The objective of the paper is to look at the SPE of the game of sequential formation of coalitions. The easiest analysis is done in the simple benchmark where $\delta = 0$, that is, players have completely *myopic behavior*.

If the players are myopic, the firms in two coalitions S_a and S_b in partition π will decide to merge (if they are chosen by the protocol) at any period if and only if:[6]

$$V_i(\pi) < V_i\left((\pi \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}\right) \text{ for all } i \in S_a \cup S_b.$$

Suppose that the coalition structure π is formed by $r \geq 2$ coalitions. Then, the

[6] For convention, we make the implicit assumption that a player will only be willing to join a coalition if he gains strictly more by doing it.

firms of S_a and S_b will want to merge and move to a structure with $r-1$ coalitions if:

$$\max\left\{\frac{1}{(r+1)^2 s_a}, \frac{1}{(r+1)^2 s_b}\right\} < \frac{1}{r^2(s_a+s_b)}.$$

Let us assume without loss of generality that $s_a \leq s_b$, then the condition becomes:

$$\frac{1}{(r+1)^2 s_a} < \frac{1}{r^2(s_a+s_b)},$$

that is,

$$s_a > \frac{r^2}{2r+1} s_b.$$

Remark that the previous equation implies $s_a > s_b$ as long as $r \geq 3$, which would be in contradiction with our hypothesis that $s_a \leq s_b$. Therefore, two coalitions of firms will never be interested in merging if they only care about this period profits and there are at least three existing coalitions in the industry. This is a well-known result in static games that goes back to Salant, Switcher, and Reynolds (1983).

The previous remark implies that *if there are at least three firms* in the market, *the only myopic final outcome of the game of coalition formation is all singleton.* That is, when $\delta = 0$ no merger will occur.

For low enough discount rates, a firm is not interested in compensating short-term loses with long-term gains. Therefore, the myopic final outcome will also be the SPE final outcome when the discount parameter δ is low enough. We state this result formally in the following proposition:

Proposition 15 *If $n \geq 3$ and the discount rate δ is low enough, then the only SPE final outcome of the process of sequential coalition formation in the linear Cournot setting is that all firms remain as singleton.*

Proof. Immediate after the discussion for the case $\delta = 0$. ■

4.4 Forward-Looking Firms

When firms are forward looking, they may be interested in merging even if they lose profits in the short run when they anticipate higher profits in the future. A (non-profitable) merger by two firms or two coalitions may further other mergers. Hence, although the initial merging firms (or coalitions) lose profits because of the first merger, they may improve their situation later on if other mergers happen.

Next proposition restricts the set of potential SPE final outcomes of the sequential game, for any discount rate. It shows that, at equilibrium, firms will surely not start merging to end up in a coalition structure with more than one coalition.

Proposition 16 *The SPE final outcome of the game of coalition formation in a Cournot competition model is either monopoly or all singletons.*

Proof. We do the proof by contradiction. Suppose that the final outcome is a coalition structure π formed by r coalitions, with $2 \leq r \leq n - 1$. Denote by S_a and S_b the last two coalitions that merged, say at period t^0, with $s_a \leq s_b$. We already saw that, for a firm $i \in S_a$, $V_i(\pi) > V_i((\pi \backslash \{S_a \cup S_b\}) \cup \{S_a, S_b\})$. In addition, firms in S_a would even get strictly higher profits if, at some periods after t^0, other mergers not involving S_a happen. Therefore, for firms in S_a, the strategy of merging with S_b at t^0 (leading to the final outcome $\pi \neq \pi^1$) is strictly dominated by the strategy consisting in not accepting any merger from t^0 on. Therefore, the firms in S_a have a profitable deviation. Hence, no SPE strategy can lead to a final outcome with r coalitions, for $2 \leq r \leq n - 1$. ■

In Proposition 16, we have shown that the process of coalition formation in a linear Cournot model will only start if it leads to full integration (monopoly), otherwise all the firms will remain as singletons. The reason for this result is that no couple of coalitions wants to be the last to merge (unless the merger leads to a monopoly). Therefore, at equilibrium, a merger can only happens if the firms involved anticipate that it will be followed by another one, until the grand coalition is formed.

We have seen (Proposition 15) that, if the discount rate δ is low enough, no merger will take place. We now look at the conditions under which the firms can or will end up all together. That is, we investigate when the SPE final outcome of the game of coalition formation is a monopoly. We first look for necessary conditions for a monopoly to emerge.

We know that two coalitions may decide to merge only it they anticipate that after their action, some other mergers will happen that compensate for the immediate loses suffered. Given Proposition 16, the previous argument implies that two coalitions will never merge if there is not a sequence of unions leading to full integration. Therefore, a necessary condition for the process of merging to arise is that for every value of r, for $2 \leq r \leq n$, there exists a coalition structure with r coalitions such that at least two of them obtain smaller profits in this structure than under monopoly (hence, they might be willing to merge).

The profits of the members of a coalition of size s in a coalition structure with r cartels are strictly smaller than their profits under monopoly if:

$$\frac{1}{(r+1)^2 s} < \frac{1}{4n}, \text{ i.e., } s > \frac{4n}{(r+1)^2}.$$

Given that s is a natural number, the condition can be rewritten as:[7]

$$s \geq \underline{s}^r \equiv int\left\{\frac{4n}{(r+1)^2}\right\} + 1.$$

Hence, in a partition with r coalitions, a necessary condition for two coalitions to merge is that the size of each them is at least $\underline{s}^r$. This implies that, starting with a group of n firms, for a sequence of mergers that may lead to full integration to exist, it is necessary that, if a situation with $r-1$ coalitions has been attained, the size of the biggest coalition is at least $2\underline{s}^r$. We state this condition formally.

Let us consider a sequence of coalition structures $M = \{\pi^r\}_{r=1}^n$ with the property that $\pi^1 = N$ and $\pi^{r-1} = (\pi^r \backslash \{S_a, S_b\}) \cup \{S_a \cup S_v\}$, for some S_a and S_b in π^r, and for all $r > 1$. If the final outcome of the game of coalition formation leads to full integration, it needs to be through such a sequence M. Moreover, M must satisfy that, for every $\pi^{r-1} \in M$, and for all $2 \leq r \leq n$,

$$\max_{S_j \in \pi^{r-1}} \{s_j\} \geq 2\underline{s}^r. \tag{C(r)}$$

We denote by $\mathcal{M}$ the set of such sequences of coalition structures that fulfill condition C(r) for all $2 \leq r \leq n$. Also, denote by $\mathcal{N}$ the set of natural numbers for which there exists a sequence $M \in \mathcal{M}$. To prove that $\mathcal{N} \neq \mathbb{N}$, it suffices to check that $\mathcal{N}$ does not include numbers as 3 or 4. Next lemma provides a sufficient condition for n to belong to $\mathcal{N}$.

Lemma 3 *If $n \geq 37$, then $n \in \mathcal{N}$.*

Proof. The proof proceeds as follows. First, we construct a sequence $M = \{\pi^r\}_{r=1}^n$ by departing from the grand coalition, $\pi^1 = N$, and splitting one coalition each time. Second, we prove that the sequence M belongs to the set $\mathcal{M}$.

[7] We use $int\{m\}$ to denote the integer of $m \in \mathbb{R}$.

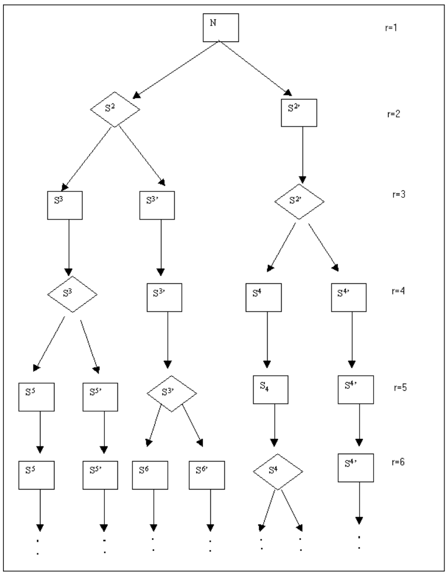

Figure 4.1: Outline of the Sequence of Moves

a) We denote by S^r and $S^{r'}$ the two coalitions that are split from π^{r-1} (the interpretation is that S^r and $S^{r'}$ are the "candidates" to merge if the coalition structure π^r emerges). That is, $\pi^r = (\pi^{r-1} \backslash \{S^r \cup S^{r'}\}) \cup \{S^r, S^{r'}\}$.

We divide $S^1 = N$ into S^2 and $S^{2'}$ with $s^2 = n - \underline{s}^2$ and $s^{2'} = \underline{s}^2$. From this point on, we divide the selected coalition into two of equal size, or as equal as possible. For $r = 3$, S^3 and $S^{3'}$ are obtained by dividing S^2 in such a way that $s^3 = s^{3'} = \frac{n-s^2}{2}$ if s^2 is even and $s^3 = s^{3'} + 1 = \frac{n-s^2+1}{2}$ if s^2 is odd. For $r \geq 4$, we split the biggest coalition in π^{r-1}, that corresponds to the biggest coalition out of the coalitions with the smallest index in π^{r-1}. Formally, we divide $S^{\frac{r}{2}'}$ if r is even, and $S^{\frac{r+1}{2}}$ if r is odd (see Figure 4.1).

b) We now prove that the sequence M constructed before belongs to $\mathcal{M}$ when n is high enough. We do the proof by induction. At each step, we provide conditions over n under which the sequence proposed satisfies condition $C(r)$ for all $r \geq 2$. Note that, since the minimum size of a coalition is 1, when $\underline{s}^r = 1$, condition C(r) imposes no restriction on the size of the coalitions. This is the case if

$$int\left\{\frac{4n}{(r+1)^2}\right\} = 0, \text{ i.e., } r > r_{\max}(n) \equiv \sqrt{4n} - 1.$$

Therefore, we will concentrate in $r \in [2, r_{\max}(n)]$.

($r = 2$) Condition C($r = 2$) is satisfied if $s^2 \geq \underline{s}^2$, that is $s^1 = n \geq 2\underline{s}^2$, i.e.,

$$n \geq 2\left(int\left\{\frac{4n}{9}\right\} + 1\right).$$

A sufficient condition for the above inequality to hold is $n \geq 18$.

($r = 3$) Condition C($r = 3$) is:

$$\max\{s^2, s^{2'}\} = s^2 \geq 2\underline{s}^3, \text{ i.e., } n - \left(int\left\{\frac{4n}{9}\right\} + 1\right) \geq 2\left(int\left\{\frac{n}{4}\right\} + 1\right).$$

It can be checked that the above inequality holds for any $n \geq 37$.

We make the induction hypothesis that $s^l \geq s^{l'} \geq \underline{s}^l$ for all $l < r$. This property implies that $C(l)$ holds for all $l < r$, since $\underset{s_j \in \pi^{l-1}}{max}\{s_j\} \geq s^l + s^{l'} \geq 2\underline{s}^l$, given that s^l and $s^{l'}$ are the sizes of the two coalitions that are constructed by spliting one of the coalitions in π^{l-1}.

We prove the induction property for r.

(a) Suppose r is even, $s^{r'} \geq \underline{s}^r$ provided $s^{\frac{r}{2}'} \geq 2\underline{s}^r$. This is implied by:

$$\underline{s}^{\frac{r}{2}} \geq 2\underline{s}^r, \text{ i.e., } int\left\{\frac{4n}{\left(\frac{r}{2}+1\right)^2}\right\} + 1 \geq 2\left(int\left\{\frac{4n}{(r+1)^2}\right\} + 1\right).$$

This condition is weaker than:

$$\frac{r^2 - 2}{(r+1)^2(r+2)^2} \geq \frac{1}{8n}.$$

Denoting $f_e(r) = \frac{r^2-2}{(r+1)^2(r+2)^2}$, it can be shown that $\forall r \geq 4$, $f_e'(r) < 0$. Hence this condition is more demanding the higher is r. Evaluating in the extreme of the domain ($r_{\max}(n)$), the condition becomes:

$$n \geq \left(\frac{3+2\sqrt{3}}{2}\right)^2 \simeq 10.446.$$

Therefore, for any even value of r greater or equal than 4, the condition is satisfied provided $n \geq 11$.

(b) When r is odd, $s^{r'} \geq \underline{s}^r$ provided:

$$\underline{s}^{\frac{r+1}{2}} \geq 2\underline{s}^r, \text{ i.e., } int\left\{\frac{4n}{\left(\frac{r+1}{2}+1\right)^2}\right\}+1 \geq 2\left(int\left\{\frac{4n}{(r+1)^2}\right\}+1\right).$$

This condition is weaker than:

$$\frac{r^2-2r-7}{(r+1)^2\,(r+3)^2} \geq \frac{1}{8n}.$$

Denoting $f_o(r) = \frac{r^2-2r-7}{(r+1)^2(r+3)^2}$, it can be shown that $f_o'(r) < 0$ if and only if $r \geq 7$. Hence we need to check the condition in the two extreme values of r, $r = 5$ and $r = r_{\max}(n)$. Evaluating it in $r = 5$, the condition becomes $n \geq 36$. Proceeding analogously for $r = r_{\max}(n)$, we find: $n - 6\sqrt{n} - 3 \geq 0$, and this is fulfilled for any value of $n \geq 21 + 12\sqrt{3} \simeq 41.785$.

Given the different requirements imposed on n, it is immediate that if $n \geq 42$, all the conditions C(r) hold. Once we have provided the way to construct the sequence of coalition structures π^r from $r = 1$ to $r = r_{\max}(n)$, notice that any split of any coalition satisfies condition C(r), for $r \geq r_{\max}(n)$. Therefore, for all $n \geq 42$ there exists a sequence $M \in \mathcal{M}$. Finally, it is easy to check numerically that $\mathcal{N}$ also includes the numbers from 37 to 41. ■

Can monopoly be a SPE final outcome? A necessary condition is that the number of firms, n, is such that $n \in \mathcal{N}$, for example (according to Lemma 3) because $n \geq 37$.[8] However, the condition is not sufficient. Indeed, we have seen in Proposition 1 that if δ is small enough, and $n \geq 3$, the only SPE final outcome of the game of coalition formation is all singleton.

Proposition 17 shows that the reverse happens if $n \in \mathcal{N}$ and δ is large enough. In this case, the firms will enter into a sequential process of forming coalitions that will end up in the creation of a monopoly.

Proposition 17 *If $n \in \mathcal{N}$, there exists a $\bar{\delta} < 1$, such that $\forall \delta \geq \bar{\delta}$, the final outcome of any SPE of the process of sequential coalition formation is the grand coalition.*

Proof. We provide characteristics of the SPE strategies of the game of coalition formation when $n \in \mathcal{N}$ and δ is very close to 1. We then show that, given these characteristics, the only possible final outcome of any SPE is monopoly. We use induction arguments. ($r = 2$) Take any subgame where only two coalitions S_a and S_b are left, $S_a \cup S_b = N$. Consider the case where S_a and S_b fulfil condition C($r = 2$). Then, the best response of firms in both coalitions is to merge (all the firms in S_a and S_b will obtain higher profits from this moment on). That is, every SPE strategy will lead to full integration if such a node is reached. On the other hand, using a similar argument, every SPE should lead S_a and S_b to remain separated if condition C($r = 2$) does not hold since any firm in the smallest coalition between S_a and S_b would rather reject the merger than accept that the merger would go on.

[8] One can also check that $\mathcal{N}$ also includes 15, 22, 23, 26, 29 to 32, 34, and 35.

Our induction hypothesis is that, for any $r' < r^\circ$, the following characteristic, denoted by B(r'), is true for any SPE strategy:

B(r'): If the coalition structure $\pi^{r'}$ with r' coalition belongs to some $M \in \mathcal{M}$, then the unique final outcome is monopoly. Otherwise, the unique final outcome is $\pi^{r'}$.

We now show that, if B(r') is true for any $r' < r^\circ$, then it is also true for r° at any node of the game where r° coalitions exist. We first prove that S_a and S_b in π^{r° will not merge if $(\pi^{r^\circ} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ does not belong to any $M \in \mathcal{M}$. Indeed, if S_a and S_b decide to merge then, by induction hypothesis, the final structure will be $(\pi^{r^\circ} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$. On the other hand, if they chose a strategy of never merging (this is not necessarily the optimal strategy, but it is one possibility), they obtain from this moment on at least the benefits that they have under the structure π^{r°. Given that $r^\circ > 2$, $V_i(\pi^{r^\circ}) > V_i(\pi^{r^\circ} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ for either every firm in S_a or every firm in S_b. Therefore, merging is not an optimal strategy for firms either in S_a or in S_b.

Given the previous property, if π^{r° does not belong to any $M \in \mathcal{M}$ (i.e., if $(\pi^{r^\circ} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ does not belong to any $M \in \mathcal{M}$, for all S_a, $S_b \in \pi^{r^\circ}$), then no merger will happen. Hence, the final outcome is π^{r°.

We now prove that if π^{r° belongs to some $M \in \mathcal{M}$ then the strategies of members of (at least) two coalitions S_a and S_b in π^{r° such that $(\pi^{r^\circ} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ belongs to some $M \in \mathcal{M}$, will consist in accepting the merger if they are selected by the mechanism. We do the proof by contradiction. If the property does not hold, then take any pair of coalitions S_a and S_b in π^{r° such that $(\pi^{r^\circ} \backslash \{S_a, S_b\}) \cup \{S_a \cup S_b\}$ belongs to some $M \in \mathcal{M}$. If δ is high enough, the members of S_a and S_b strictly prefer arriving at the monopoly situation after some periods than staying at S_a and S_b forever. Therefore, they will have incentives to change their strategy and accept the merger.[9]

Because of the last property and the induction argument, the unique final outcome is monopoly if we start from a coalition structure π^{r° that belongs to some $M \in \mathcal{M}$.

Using the induction argument, we arrive until $r^\circ = n$. Since $n \in \mathcal{N}$, the existence of a sequence $M \in \mathcal{M}$ is guaranteed. That is, we are sure that the process of coalition formation will eventually start and will lead to a monopoly. Therefore, we have shown that if δ is large enough, and $n \in \mathcal{N}$, all the SPE strategies lead to a sequence of coalition structures that satisfy C(r) for all $r \geq 2$ and whose final outcome is monopoly. ■

Proposition 16 showed that in our coalition formation game, only the extreme coalition structures, all singletons or the grand coalition, may be equilibrium outcomes. Proposition 17 shows that, when the number of initial players is high enough and these players are patient enough, the efficient outcome is the only equilibrium outcome. That is, under these two conditions, the possibility of establishing bilateral agreements sequentially makes the firms to merge in such a way that they end up as a monopoly. This result is in contrast with previous results in merger games. Indeed, the grand coalition is often not an equilibrium (or stable) outcome or, when it is, it is not the only one.

The proof of Proposition 17 provides the two main characteristics of the SPE strategies when the discount factor δ is high enough and $n \in \mathcal{N}$. First, in a SPE strategy profile, the members of two randomly chosen coalitions will only decide to merge if the

[9] Let us make two remarks. First, the members in S_a and S_b do not face a coordination problem because they chose sequentially. Therefore, if merging is optimal for all of them, they will sequentially chose merging. Second, in order to arrive to monopoly, there is at least a pair of coalitions at each moment whose strategy leads to merging. Then, there is a maximum expected time at which the monopoly can be reached. This is why, waiting for the monopoly is profitable for the members of the previous coalitions S_a and S_b as soon as δ is high enough.

resulting coalition structure after the merger belongs to some sequence $M \in \mathcal{M}$. Second, when it is possible to keep the chain of coalitions in a sequence satisfying C(r), then at least a pair of coalitions will decide to merge. The two properties together imply that, in any SPE strategy profile, at any period, the firms will form partitions that satisfy the property $C(r)$, and they will end up all together.

To make sure that a sequence satisfying $C(r)$ for all r exists, the number of initial players is crucial. To understand why, remember that, in order to be willing to merge, two coalitions may not be very different in size. But this is a characteristic that need to be fulfilled along the whole sequence of mergers. If, at one stage, all the coalitions are too similar, when two of them merge they will create a coalition big as compared to the others and the small ones may stop the process to free ride on the big one. With many players, there is way to have coalitions whose sizes are balanced enough at every stage.

To highlight the previous argument, consider the case with three firms. In order to reach the grand coalition, a firm of size two has to merge with the a firm of size one. However, the process will not be completed because they are very asymmetric in size and the firm which is alone receives higher profits in the duopoly than if it obtains a third of the profits of the monopoly. Consider now the case $n = 39$. For the same reason as before, a sequence of mergers that leads to a duopoly with a firm of size 26 and another of size 13 will never get to the grand coalition. However, a path reaching a duopoly with two firms of sizes 21 and 18 will end up as a monopoly.

It is difficult to give a complete characterization of the set of SPE strategies. The reason is that the members of two coalitions may have incentives to wait to merge (even if they keep the coalition structure in a "good path") in order to obtain short-term profits when they know that some other coalitions will eventually start up merging to lead to monopoly. Next proposition specifies some SPE strategies for the coalition formation game in the particular case when the players are perfectly patient ($\delta = 1$).

Proposition 18 *If firms evaluate future profits according to the "time-average criterion", then the following strategy profile is a SPE profile in the game of coalition formation.*

At any period at which the members of the coalitions S_a and S_b have to chose whether to merge, they will merge if and only if the resulting coalition structure belongs to some $M \in \mathcal{M}$.

Proof. Straightforward after the proof of Proposition 17. ■

The above stated equilibrium strategy is symmetric. Moreover, it is stationary.

4.5 Comments and Extensions

In this paper, we have shown that when the initial number of firms (players) is large enough and firms are forward looking, then a sequential process of bilateral agreements will lead to the creation of a monopoly (the grand coalition). In this section, we discuss the main ingredients of our model by proposing several processes of gradual agreements. We introduce modifications that affect the timing of the coalition formation and the production stages, the protocol that chooses the candidates that can merge, the exogenous sharing rule, and the bilateral nature of the agreements.

4.5.1 Timing of the Production Stage

Consider the case where production takes place and profits are realized only once the whole process of coalition formation has ended. This is the framework that most models in the literature have considered.[10] The difference between this game and the one described in Section 4.2 is that in the later production takes place at every period while in the former it is only undertaken once the coalition formation stage has finished. In fact, this variant makes the analysis simpler.

To adapt our model to this framework, we assume the same protocol for coalition formation as before. However, players should be able to move to the production stage if they wish. Hence, we have to be specific about how the players can decide to end the coalition formation stage. We consider the following natural rule: *"at the beginning of each round, firms are asked sequentially whether to move to the production stage, or to continue with the process of coalition formation. The formation of coalitions stage continues only if all the firms agree on it."*

In this set up, all our results still hold. In fact, the process of coalition formation is fostered in this framework since the absence of intermediate payoffs reduces the possibility of free-riding. Moreover, the equilibrium players' strategies are easier to characterize. For example, in Proposition 17, firms will all decide to merge if and only if this decision minimizes the expected loses due to the discounting. Notice that, in this case, all the firms share the same objective function when they decide whether to merge or not. Finally, players' strategy when the existing coalition structure is π^r specifies that they will decide to move

[10] See Bloch (1996), Ray and Vohra (1999), and Montero (1999).

to the production stage if and only if π^r does not belong to any $M \in \mathcal{M}$.

4.5.2 Protocol

A second feature of our coalition formation game that is not crucial for obtaining the results is the random choice of the coalitions that meet at a given period. If there is a deterministic protocol to determine the identity of the two coalitions that can merge, the results still hold provided the protocol is exhaustive in the set of possible couples for each coalition structure (i.e., all the possible pairs of coalitions in any coalition structure are called by the protocol at some moment).

Similarly, the analysis is robust to frameworks where the protocol selects (either randomly or deterministically) one of the coalitions which has the possibility to offer a merger to any other coalition. Finally, we can also consider situations in which a particular player is in charge of naming, at each period, the two coalitions that may merge. In this case, at the SPE of the game with high patience rate, the grand coalition will be attained through the sequence of mergers that is most favorable to this player. The same happens if, for example, the coalitions that decide whether to merge or not at time t are chosen by some player belonging to one of the coalitions selected at time $t-1$.

4.5.3 Endogenous Sharing Rule

We have chosen to study the outcomes of a coalition formation procedure when the payoffs of the players at any moment only depend on the coalition structure prevailing at that moment. Indeed, we have assumed an exogenous equal sharing rule independent of the history. We could also study the outcomes of a similar procedure allowing for endogenous sharing rules that would depend on the bargaining power of the coalitions at the moment at which they have to decide whether to merge. Although it may at first sight seem that allowing for endogenous sharing rules should help the formation of coalitions, since it allows compensating players in any way, this possibility makes forming coalitions more difficult. The reason is that merging at an early stage lowers the bargaining power of the players in the continuation of the game. Hence, although the final mergers are easier to implement, players have no incentives to start the process.

The SPE outcome of the linear Cournot game with endogenous sharing rule is that all the players remain as singleton. To illustrate the result, consider a variant of our coalition formation game in which, out of the two coalitions that have to decide whether to merge, one of them must make a proposal to the other concerning the sharing of the surplus. In expected terms, the possible surplus will be shared equally among the two coalitions (not necessarily equally among the firms, since the coalitions can be of different sizes). Imagine a situation where all the players have been merging until the structure in the market consists in three coalitions. The sum of the payoffs of the firms in each of the coalitions is 1/16. If a duopoly is formed, the two coalitions will have incentives to merge, each obtaining at the end an expected payoff of 1/8, since they will share the benefits of the monopoly, i.e., 1/4. But this implies that no two coalitions in a triopoly will have incentives to merge, going through a duopoly structure, in which their join profits decrease, to end up obtaining the same. Notice that the previous argument is independent of the size of the existing coalitions, which is not true in the game proposed in our paper where the two smallest coalitions in a triopoly may have incentives to merge.

4.5.4 Multilateral Agreements

The bilateral nature of the agreements is a key feature of our analysis. The results obtained in this paper do not extend if players have the possibility of forming coalitions of any size in a single round.

Consider the following variant of our coalition formation game. At each period t, one of the existing coalitions S_a in π_t is randomly chosen. The firms in S_a choose any set of potential partners, that is, they select any subset Z of π_t such that $S_a \in Z$. All the firms in Z sequentially decide whether to accept the offer. The merger is formed if all the firms agree on it, otherwise the same coalition structure remains until period $t+1$. Other than this modification, the coalition formation and production game is undertaken under the rules described in Section 4.2. Notice that this is the game proposed in Bloch (1996) with the differences that production takes place at each and every period (as discussed previously, this difference is innocuous) and that once a coalition is formed it does not leave the game, so it can be part of another future merger. This last difference is shown to be crucial.

In this framework, the set of SPE outcomes is very large. In particular, monopoly can be sustained as a SPE outcome. Indeed, consider any SPE strategy profile for those subgames where at least one merger has already taken place.[11] Suppose that, when π_t is all singletons, each player's strategy prescribes accepting an offer if and only if $Z = \pi_t$ and proposing $Z = \pi_t$ if he has been chosen by the protocol. One can check, using the one-stage deviation principle,[12] that the previous profile constitutes a SPE whose outcome is monopoly, independently of the discount rate. However, many other coalition structures

[11] In our simple game, it is not difficult to show that such SPE profiles always exist.

[12] For a formal statement of this principle see, for example, Fudenberg and Tirole (1991).

are also SPE outcomes for every discount rate. For instance, take $S \subset N$ such that $s \geq s^*$, where s^* is the minimum size of any profitable coalition, defined by Salant, Switzer, and Reynolds (1983), and represents about 80% of the market. Consider strategies where, when π_t is all singletons, players in S accept any offer involving at least s players, propose $Z = S$ if they are selected by the protocol, while players outside S never accept nor offer any merger (and take any SPE profile for the remaining subgames). This profile also constitutes a SPE that yields a coalition structure with a coalition of s players and $(n - s)$ singletons. When $s = s^*$, this is precisely the outcome of the game proposed by Bloch (1996).

Many equilibrium profiles can also be devised that induce more than one merger. In particular, any duopoly in which the size of the smallest coalition is less than $(4/9)n$ (so that the members of this coalition are better off in duopoly than in monopoly) can be sustained by SPE strategies if δ is high enough. The SPE profile will be such that the small coalition is formed first, then the big coalition forms. Clearly, similar arguments allow to sustain more complex coalition structures.

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