UNIVERSIDAD DE CANTABRIA
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Enumeración y Anchura de Politopos Reticulares por su Número de Puntos Reticulares

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Enumeration and Width of Lattice Polytopes by their Number of Lattice Points

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A mis padres

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Índice general

Capítulo I

Introducción

Esta tesis es trabajo conjunto con mi director Francisco Santos y (parcialmente) con Christian Haase y Jan Hofmann, de la Universidad Libre de Berlín. Los resultados que hemos obtenido están contenidos en los artículos [BHHS16] y [BS16c]. Se trata de una continuación de la investigación presentada en mi Tesis de Máster, que ha sido posteriormente publicada en la revista SIAM Journal on Discrete Mathematics en los artículos [BS16a] y [BS16b].

Este capítulo introductorio está organizado como sigue:

La Sección I.1 recopila conceptos básicos y parámetros de politopos reticulares, principal objeto de nuestro estudio. Todas las definiciones y notación generales aparecen en esta sección.

La Sección I.2 es un resumen de resultados de politopos reticulares y otros enfoques sobre la clasificación de los mismos. Trataremos con más detalle los resultados que son necesarios para nuestra investigación.

La Sección I.3 describe cómo enfocamos nosotros la clasificación de los 3-politopos reticulares. Este trabajo ya lo comenzamos en los artículos [BS16a] y [BS16b], y en esta sección resumimos los resultados principales obtenidos en ellos. Aquellos resultados que utilicemos en esta tesis serán presentados con más detalle.

Finalmente, en la Sección I.4 daremos una idea general del contenido del resto de la tesis, incluyendo los principales resultados y conclusiones de la misma.

I.1 Politopos. Politopos reticulares. Equivalencia unimodular. Anchura

Dos libros básicos de referencia sobre politopos y convexidad son [Gru07, Zie95].

Definición I.1 (Politopo)**.** Un *politopo* $P \subset \mathbb{R}^d$ está definido de dos maneras equivalentes:

- Un poliedro acotado.
- La envolvente convexa de un número finito de puntos.

Un *poliedro* es la intersección de un número finito de semi-espacios afines. Recuérdese que un *semi-espacio afín* es la región $\{x \in \mathbb{R}^d \mid f(x) \geq 0\} \subset \mathbb{R}^d$, donde $f : \mathbb{R}^d \to \mathbb{R}$ es un funcional afín:

$$f(x_1, \ldots, x_d) = a_0 + a_1x_1 + \ldots a_dx_d, \text{ con } a_i \in \mathbb{R}$$

Por otro lado, la *envolvente convexa* de un conjunto $K \subseteq \mathbb{R}^d$ es el conjunto convexo más pequeño que contiene a K, y se denota por $\operatorname{conv}(K)$:

$$\operatorname{conv}(K) := \bigcap_{C \text{ convexo},\ K \subseteq C} C$$

Equivalentemente, la envolvente convexa de K es el conjunto de todas las *combinaciones convexas* de puntos de K:

$$\operatorname{conv}(K) = \left\{ \lambda_1 p_1 + \cdots + \lambda_n p_n \;\middle|\; \lambda_i \geq 0,\ \sum_{i=1,\dots,n} \lambda_i = 1,\ p_i \in K,\ n \in \mathbb{N} \right\}.$$

Por ejemplo, la envolvente convexa de dos puntos x e y es $\operatorname{conv}\{x, y\} = \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$ (el segmento de recta que une x con y). Y la envolvente convexa de tres puntos (no alineados) es el triángulo que forman. Dada una lista finita de puntos $p_1, \dots, p_d \in \mathbb{R}^d$, algunas veces denotaremos su envolvente convexa como $p_1 p_2 \dots p_d := \operatorname{conv}\{p_1, \dots, p_d\}$.

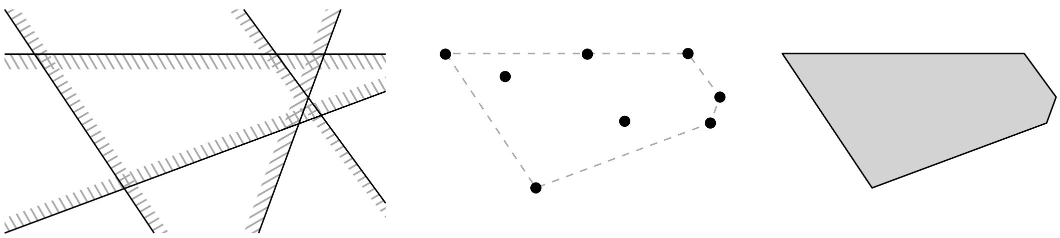


Figura I.1: Un conjunto de semi-espacios y un conjunto de puntos que definen el mismo politopo.

Decimos que $K \subset \mathbb{R}^d$ es un *cuerpo convexo* si es un conjunto compacto y tal que $K = \operatorname{conv}(K)$.

La *dimensión* de un politopo $P \subset \mathbb{R}^d$ (o de cualquier otro cuerpo convexo) es la dimensión del espacio afín que genera:

$$\operatorname{aff}(P) := \left\{ \lambda_1 p_1 + \cdots + \lambda_n p_n \;\middle|\; \sum_{i=1,\dots,n} \lambda_i = 1,\ p_i \in P,\ n \in \mathbb{N} \right\}$$

Llamaremos *d-politopo* a un politopo P de dimensión d, aunque P puede estar definido en un espacio de dimensión mayor. Los politopos de dimensión uno se llaman *segmentos* y los de dimensión dos *polígonos*.

Una *cara* de un d-politopo $P \subset \mathbb{R}^d$ es la intersección $P_f := P \cap \{x \in \mathbb{R}^d \mid f(x) = 0\}$ de P con un hiperplano válido. Decimos que un hiperplano $\{x \in \mathbb{R}^d \mid f(x) = 0\}$, donde $f : \mathbb{R}^d \to \mathbb{R}$ es un funcional afin no constante, es un *hiperplano válido* de un politopo P si $f(P) \subset [0, \infty)$. Un hiperplano *soporte* es un hiperplano válido tal que $0 \in f(P)$. Nótese que una cara de un politopo es a su vez un politopo. Se llama *vértice* a una cara 0-dimensional, y *arista* a una cara 1-dimensional. Llamamos *facetas* a las caras de dimensión $d-1$, y *crestas* a las de dimensión $d-2$. El vacío es siempre una cara de P, obtenida escogiendo cualquier hiperplano que no interseca a P, y, por convenio, el propio P se considera una cara, escogiendo el funcional $f(x) = 0,\ \forall x \in \mathbb{R}^d$. Una *cara propia* es una cara $F \subsetneq P$.

La *frontera* o *borde* de un politopo P es el conjunto ∂P de los puntos $x \in P$ que pertenecen a alguna cara propia. El *interior* de P es su interior topológico, y lo denotamos $\mathrm{int}(P)$. Llamamos *interior relativo* de un politopo $P \subset \mathbb{R}^d$, y lo denotamos $\mathrm{relint}(P)$, al interior topológico de P en $\mathrm{aff}(P)$. Si P es de dimensión máxima, el interior y el interior relativo coinciden. Cuando $P \subset \mathbb{R}^d$ es de dimensión menor, su interior es el conjunto vacío. Decimos que $x \in P$ es un *punto interior* si $x \in \mathrm{int}(P)$ en el caso de que P sea de dimensión máxima, o por el contrario, si $x \in \mathrm{relint}(P)$. Del mismo modo, x es un *punto del borde* si está en $\mathrm{int}(P)$ o ∂P, respectivamente.

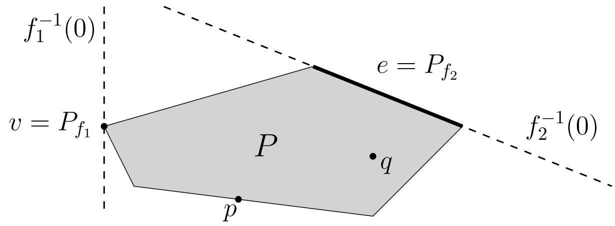


Figura I.2: Un polígono P y dos de sus caras: v es un vértice (cara 1-dimensional) y e es una arista y faceta al mismo tiempo (cara 1 y $(d-1)$-dimensional). El punto p está en el borde de P y q es un punto interior.

Un *símplice* o *d-símplice* es un politopo d-dimensional con $d+1$ vértices. Todo politopo 1-dimensional es un símplice. Un 2-símplice es un triángulo, y un 3-símplice es un tetraedro.

El objeto de estudio en esta tesis son los politopos reticulares. Un *retículo lineal* $\Lambda \subset \mathbb{R}^d$ es cualquier subgrupo aditivo discreto. En particular, es isomorfo a $\mathbb{Z}^k$, donde k es la *dimensión* del retículo. Dicho de otra manera, para k vectores linealmente independientes en $\mathbb{R}^d$, el conjunto de las combinaciones enteras lineales de dichos vectores es un retículo lineal de dimensión k. Una traslación de un retículo lineal es un *retículo afín*, o *retículo* sin más. Salvo que se indique lo contrario, por defecto trabajaremos en el *retículo estándar* $\mathbb{Z}^d$.

Definición I.2 (Politopo reticular)**.** Un *politopo reticular* es un politopo con vértices en $\mathbb{Z}^d$ (o en cualquier otro retículo dado Λ). Es decir, es la envolvente convexa de un número finito de puntos reticulares.

Un *punto reticular* o *punto entero* es un punto $p \in \mathbb{Z}^d$. El conjunto de los puntos reticulares de un politopo P (o de cualquier otro cuerpo convexo) es $P \cap \mathbb{Z}^d$ (y es finito). Sea $P \subset \mathbb{R}^d$ un politopo reticular, llamaremos *tamaño* de P al número natural $\#(P \cap \mathbb{Z}^d)$.

Nótese que, si P es un politopo reticular, entonces es la intersección de un número finito de semi-espacios afines enteros (el funcional afín correspondiente tiene coeficientes enteros), pero el recíproco no es cierto ya que la intersección de hiperplanos enteros no tiene por qué ser entera.

Un *símplice unimodular* es un símplice cuyos vértices son una base entera afín del retículo (véase la Figura I.3). El *símplice estándar unimodular* es $\Delta_d := \mathrm{conv}\{0, e_1, \dots, e_d\} \subset \mathbb{R}^d$, donde 0 es el origen y e_i es el i-ésimo vector de la base estándar.

Salvo que se especifique lo contrario, por *volumen* de un d-politopo reticular haremos referencia a su volumen euclídeo multiplicado por un factor $d!$ (o, para un retículo genérico Λ, por el factor que hace que los símplices unimodulares de Λ tengan volumen uno). A este

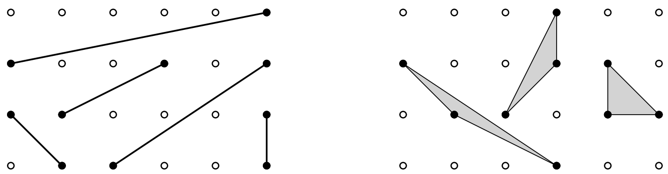


Figura I.3: La figura de la izquierda muestra segmentos reticulares unimodulares en el espacio ambiente $\mathbb{R}^2$. Su volumen normalizado es uno en todos ellos, a pesar de que su volumen 2-dimensional es 0. La figura de la derecha muestra triángulos reticulares unimodulares.

concepto de volumen se le suele llamar *volumen normalizado*. Esta normalización hace que el volumen de todo politopo reticular sea un número entero.

Obsérvese que, para $P \subset \mathbb{R}^d$ un politopo reticular que no es de dimensión máxima, la forma natural de definir su volumen es con respecto al retículo $\Lambda := \mathrm{aff}(P) \cap \mathbb{Z}^d$ (en vez de su volumen "d-dimensional", que sería cero). Nótese que para politopos reticulares de dimensión no máxima, la constante de proporcionalidad entre el volumen euclídeo y el normalizado depende del espacio $\mathrm{aff}(P)$. Véase por ejemplo la Figura I.3, donde los segmentos en la figura de la izquierda son todos de volumen uno, mientras que su longitud euclídea es diferente en cada caso.

A veces se llama *determinante* de un símplice reticular $T = \mathrm{conv}\{p_1, \ldots, p_{d+1}\} \subset \mathbb{R}^d$, con $p_i \in \mathbb{Z}^d$, a su volumen:

$$\mathrm{vol}(T) = \left| \det \begin{pmatrix} 1 & \ldots & 1 \\ p_{i_1} & \ldots & p_{i_{d+1}} \end{pmatrix} \right|$$

La noción más natural de equivalencia afín entre politopos reticulares es la siguiente:

Definición I.3 (Equivalencia unimodular)**.** Una *equivalencia unimodular*, o simplemente *equivalencia*, es una transformación afín $t : \mathbb{R}^d \to \mathbb{R}^d$ que conserva el retículo.

Dos politopos reticulares P y Q son *equivalentes unimodularmente*, o simplemente *equivalentes*, si existe una equivalencia unimodular que envía uno al otro. En este caso se denota $P \cong Q$.

En el caso del retículo estándar $\mathbb{Z}^d$, una equivalencia unimodular es cualquier aplicación de la forma $t(x) = Mx + b$, donde $M \in \mathbb{Z}^{d\times d}$ es una matriz entera con $\det(M) = \pm 1$, y $b \in \mathbb{Z}^d$ es un vector de traslación.

Cuando decimos que sólo existe un número finito de politopos reticulares que satisfacen unas ciertas propiedades, queremos decir módulo equivalencia unimodular. Véase el Algoritmo A.7 para nuestra implementación en MATLAB de un test que evalúa la equivalencia unimodular entre dos 3-politopos reticulares. Este algoritmo se puede extender a dimensión arbitraria.

En la siguiente definición, K es un cuerpo convexo y consideramos el retículo estándar $\mathbb{Z}^d$.

Definición I.4 (Anchura reticular)**.** La *anchura* de un cuerpo convexo $K \subset \mathbb{R}^d$ *con respecto a un funcional afín* $f : \mathbb{R}^d \to \mathbb{R}$ es

$$\mathrm{width}_f(K) := \max_{p,\, q \in K} |f(p) - f(q)| \,.$$

Obsérvese que es igual a $\text{máx}_{p,\, q\in K} |\hat{f}(p) - \hat{f}(q)|$, donde $\hat{f}$ es el funcional lineal paralelo a f. La *anchura reticular*, o simplemente *anchura* de K, denotada por $\text{width}(K)$, es el mínimo $\text{width}_f(K)$ de entre todos los funcionales f enteros no constantes. Es decir:

$$\text{width}(K) := \min_{f\in(\mathbb{Z}^d)^*\setminus\{0\}} \text{width}_f(K).$$

Obsérvese que la anchura de $K \subset \mathbb{R}^d$ es cero si, y sólo si, K no es de dimensión máxima. Si ese es el caso, se especificará si en algún momento nos referimos a la anchura de K con respecto al retículo $\text{aff}(K) \cap \mathbb{Z}^d$.

En el caso de politopos reticulares, la anchura es un número entero. El Algoritmo A.9 calcula la anchura de un politopo reticular.

En el caso de un retículo general $\Lambda \subset \mathbb{R}^d$, el papel de los funcionales enteros lineales lo juegan los elementos del *retículo dual*:

$$\Lambda^* := \left\{ f \in (\mathbb{R}^d)^* \mid f(\Lambda) \subseteq \mathbb{Z} \right\} \subset (\mathbb{R}^d)^*,$$

donde $(\mathbb{R}^d)^* := \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ funcional lineal } \right\} \simeq \mathbb{R}^d$ es el espacio real dual usual. Obsérvese que por el isomorfismo canónico $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$ también tenemos $(\mathbb{Z}^d)^* \simeq \mathbb{Z}^d$. Por la existencia de estos isomorfismos, identificamos un funcional lineal $f \in (\mathbb{R}^d)^*$ con un vector, y la imagen por f de un punto $x \in \mathbb{R}^d$ la denotaremos indistintamente por $f(x)$ o $f \cdot x$.

Recuérdese que el *polar* de un cuerpo convexo $K \subset \mathbb{R}^d$ se define como:

$$K^\vee := \{f \in (\mathbb{R}^d)^* \mid f \cdot x \le 1\ \forall x \in K\} \ \subset (\mathbb{R}^d)^*$$

que es a su vez convexo. Si el origen está en el interior de K (lo que en particular implica que K is de dimensión máxima), entonces $K^\vee$ está acotado y $(K^\vee)^\vee = K$. El polar de un politopo $P \subset \mathbb{R}^d$ es un poliedro y, si el origen está en el interior de P, $P^\vee$ es también un politopo y $(P^\vee)^\vee = P$. Por convexidad de P, también se tiene que $P^\vee = \{f \in (\mathbb{R}^d)^* \mid f \cdot v \le 1\ \forall v \in \text{vert}(P)\}$. Es decir, $(v_1, \ldots, v_d) \in \mathbb{R}^d$ es vértice de P si, y sólo si, $v_1x_1 + \cdots + v_dx_d \le 1$ es la ecuación de una faceta de $P^\vee \subset \mathbb{R}^d$. Puesto que $(P^\vee)^\vee = P$, la dualidad establece una biyección entre los vértices de P y las facetas de $P^\vee$, y entre las facetas de P y los vértices de $P^\vee$.

En la literatura, el *cuerpo polar* a veces se llama *dual*, *polar recíproco*, o *polar dual*. Además, el polar se define a veces como $\{f \in (\mathbb{R}^d)^* \mid f \cdot x \ge -1\ \forall x \in K\}$, que es la misma definición módulo una simetría respecto del origen.

En términos de dualidad, la anchura se puede interpretar de la siguiente manera. En el enunciado, λQ, donde $Q \subset \mathbb{R}^d$ es un cuerpo convexo, denota la *dilatación* de Q for el factor λ, que es el conjunto $\lambda Q := \{\lambda x \mid x \in Q\}$. Para $i \in \mathbb{N}$, llamamos a iQ la *i-ésima dilatación* de Q.

Proposición I.5. *La anchura de un cuerpo convexo $K \subset \mathbb{R}^d$ con respecto a un retículo $\Lambda \subset \mathbb{R}^d$ es el mínimo valor $\lambda > 0$ tal que $\lambda(K - K)^\vee$ contiene un punto reticular no nulo $f \in \Lambda^*$.*

Demostración. Sea $f \in \Lambda^* \setminus \{0\}$, la anchura de K con respecto a f es

$$\text{width}_f(K) = \max_{x,y\in K} |f(x) - f(y)| = \max_{x,y\in K} |f(x - y)| = \max_{z\in K-K} |f(z)|$$

Sea $\lambda > 0$. Entonces

$$\mathrm{width}_f(K) \leq \lambda \iff |f(z)| \leq \lambda,\ \forall z \in K - K \iff f \in \lambda \cdot (K-K)^\vee$$

donde $\lambda(K-K)^\vee := \{f \in (\mathbb{R}^3)^* \mid f(z) \leq \lambda, \forall z \in K - K\} \subset (\mathbb{R}^3)^*$. Es decir, la anchura de K con respecto a f es el mínimo valor $\lambda > 0$ para el cual $f \in \lambda(K-K)^\vee$. El resultado se concluye de que la anchura de K es el mínimo de $\mathrm{width}_f(K)$, para $f \in \Lambda^* \setminus \{0\}$. $\square$

Dicho de otra manera, la proposición dice que la anchura de K es igual al primer mínimo sucesivo de $(K-K)^\vee$. (Recuérdese que el *primer mínimo sucesivo* de un cuerpo convexo y simétrico respecto del origen C es el mínimo λ tal que $\lambda C \cap (\Lambda^* \setminus \{0\}) \neq \emptyset$ [Gru07]).

El vector de coeficientes $(a_1, \ldots, a_d) \in \mathbb{R}^d$ de un funcional lineal es ortogonal al hiperplano $\{(x_1, \ldots, x_d) \in \mathbb{R}^d \mid a_1 x_1 + \cdots a_d x_d = 0\}$. En el caso de un hiperplano reticular, el vector ortogonal es entero y es posible escogerlo primitivo, donde un vector o punto $x \in \mathbb{Z}^d$ es *primitivo* si el máximo común divisor de sus coordenadas es uno. De la misma manera, un funcional lineal $f \in (\mathbb{Z}^d)^*$ es primitivo si su vector de coeficientes es primitivo, y un funcional afín es primitivo si su correspondiente funcional lineal $\hat{f}$ es primitivo. Como observación, un funcional f es primitivo si, y sólo si, $f(\mathbb{Z}^d) = \mathbb{Z}$.

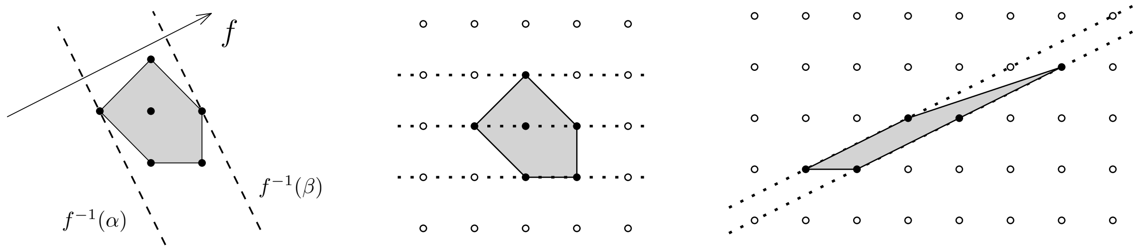


Figura I.4: La figura de la izquierda muestra la anchura $|\beta - \alpha|$ de un polígono reticular con respecto a un funcional f. Las otras dos figuras muestran polígonos reticulares delimitados por hiperplanos reticulares.

Un *subespacio reticular* es un subespacio afín S tal que $\dim(S) = \dim(S \cap \mathbb{Z}^d)$. Equivalentemente, tal que $S \cap \mathbb{Z}^d$ es un retículo de dimensión $\dim(S)$. Obsérvese que un *hiperplano reticular* es un hiperplano dado por una ecuación afín entera, pero que un subespacio afín de menor dimensión determinado por ecuaciones enteras no es necesariamente un subespacio reticular.

Llamamos *distancia reticular*, o simplemente *distancia*, entre un punto reticular $x \in \mathbb{Z}^d$ y un subespacio afín $S \subset \mathbb{R}^d$, a la anchura de $\mathrm{conv}(S \cup \{x\})$ como cuerpo de dimensión máxima en $\mathrm{aff}(S \cup \{x\})$. La distancia entre $x \in \mathbb{Z}^d$ y un politopo reticular $P \subset \mathbb{R}^d$ es la distancia entre x y $\mathrm{aff}(P)$. La distancia entre dos hiperplanos reticulares paralelos H_1 y H_2 es la distancia desde un punto $x \in H_1$ a H_2, la cual es independiente de la elección de x. En dimensión 3, la distancia entre dos segmentos (o dos rectas) reticulares no coplanares es la distancia entre el par único de hiperplanos reticulares paralelos que contienen a cada uno de los segmentos (o rectas). Obsérvese que la anchura de un politopo reticular P es la mínima distancia entre planos reticulares paralelos que delimitan a P (véase la Figura I.4).

Obsérvese también que el volumen de un símplice es el volumen de una de sus facetas F multiplicado por la distancia de F al vértice opuesto.

Llamamos *longitud reticular*, o simplemente *longitud*, de un segmento reticular $\mathrm{conv}\{x, y\}$ a la distancia entre los puntos x e y, medida en el subespacio afín 1-dimensional que generan. La longitud reticular de un segmento es igual a su volumen y también a su anchura.

Observación I.6. Todos estos parámetros de politopos reticulares (tamaño, volumen, dimensión, anchura, etc) son invariantes por equivalencia unimodular.

I.2 Clasificaciones de politopos reticulares

Los politopos reticulares aparecen en diversos campos de las matemáticas, como la geometría, la combinatoria, la optimización y la geometría algebraica.

Los politopos reticulares aparecen como la envolvente convexa de las soluciones enteras de problemas de optimización. Una cuestión importante en optimización es el poder decidir si un cuerpo convexo tiene o no algún punto reticular (es decir, si existe o no una solución entera al problema de optimización). E incluso contar cuántos puntos reticulares (cuántas soluciones enteras) hay en dicho cuerpo convexo. En este sentido es también importante el concepto de anchura, ya que los cuerpos convexos sin puntos reticulares son en cierta manera "planos".

El número de puntos reticulares en cada dilatación natural de un cierto politopo reticular coincide con un polinomio en el factor de dilatación (el polinomio de Ehrhart), y a través de la secuencia de estos valores y de su función generatriz, se encuentran interesantes relaciones entre objetos combinatorios y la geometría de los politopos reticulares.

En geometría algebraica, los politopos reticulares se corresponden con variedades tóricas, y conceptos combinatorios y geométricos sobre politopos reticulares tienen sus parejos en conceptos algebraicos.

En esta sección vamos a repasar algunas de las clasificaciones de ciertas clases de politopos reticulares. Más concretamente, las clasificaciones que aquí aparecen están muy relacionadas con la investigación que aparece en esta tesis, que es el estudio y clasificación de politopos reticulares de acuerdo a su número de puntos reticulares y su anchura.

I.2.1 Politopos huecos

Un *politopo hueco* es un politopo reticular sin puntos reticulares en el interior.

La anchura de cuerpos convexos sin puntos reticulares interiores está acotada por el llamado "flatness theorem", que se remonta a Khinchine (1948); véase, por ejemplo, [KL88]. La mejor cota superior es:

Teorema I.7 (Banaszczyk et. al, [BLPS99, Theorem. 2.4])**.** *La anchura reticular de un cuerpo convexo sin puntos reticulares en el interior es* $\leq O(d^{\frac{3}{2}})$.

Existen d-politopos huecos de anchura d (por ejemplo la d-ésima dilatación del símplice estándar unimodular). Es decir, si $w_H(d)$ denota la máxima anchura de entre todos los d-politopos huecos, entonces

$$d \leq w_H(d) \leq O(d^{\frac{3}{2}}). \tag{I.1}$$

En dimensión uno, el único segmento hueco es el de longitud uno (los únicos puntos reticulares son los extremos del segmento). Para dimensión arbitraria, todo politopo reticular de anchura uno es hueco, ya que no hay puntos reticulares estrictamente entre los dos hiperplanos reticulares a distancia uno. En general, cualquier d-politopo reticular que admite una proyección reticular a un k-politopo hueco, es a su vez hueco. Llamamos *proyección reticular* a una aplicación afín $\pi : \mathbb{R}^d \to \mathbb{R}^k, \pi(p) = A \cdot p + b$, con $\pi(\mathbb{Z}^d) = \mathbb{Z}^k$.

Teorema I.8 (Clasificación de polígonos huecos, véase por ejemplo [Tre08] o [Ark80]). *Un polígono hueco o es de anchura uno o es equivalente a* $2\Delta_2$*, la segunda dilatación del triángulo estándar unimodular.*

En dimensión 3, la clasificación de los 3-politopos huecos está fuertemente relacionada con la clasificación de polígonos huecos.

Teorema I.9 (Treutlein, [Tre08, Theorem 1.3]). *Un* 3*-politopo hueco está exactamente en una de las siguientes categorías:*

(1) Tiene anchura 1*.*

(2) Tiene anchura 2 *y admite una proyección a* $2\Delta_2$*.*

(3) Tiene anchura ≥ 2*, y no admite una proyección a* $2\Delta_2$*. Existe sólo un número finito de politopos con estas características, y cada uno de ellos está contenido en un* 3*-politopo hueco-maximal.*

En el anterior teorema, un d-politopo *hueco-maximal* es un d-politopo hueco que no está propiamente contenido en otro d-politopo hueco.

Observación I.10. *Existen politopos huecos que no están contenidos en un politopo hueco-maximal: el rectángulo* $\mathrm{conv}\{(0,0),(0,1),(2,0),(2,1)\}$ *de anchura uno no está contenido en un polígono hueco-maximal, ya que todo polígono que lo contiene está a su vez contenido en la región hueca no acotada* $\{0 \leq y \leq 1\}$*.*

Nill–Ziegler extienden a dimensión arbitraria la relación entre d-politopos huecos y $(d-1)$-politopos huecos.

Teorema I.11 (Nill-Ziegler, [NZ11, Theorem 1.2, Corollary 1.3]).
Excepto una cantidad finita, todos los d*-politopos huecos admiten una proyección a un* $(d-1)$*-politopo hueco. Además, si un* d*-politopo hueco no admite tal proyección, entonces está contenido en un* d*-politopo hueco-maximal.*

Como consecuencia tenemos que sólo existe un número finito de d-politopos huecos con anchura $> w_H(d-1)$.

Los 3-politopos hueco-maximales mencionados en la parte 3 del Teorema I.9 fueron clasificados en [AWW11, AKW15]. Más concretamente, Averkov, Wagner y Weismantel [AWW11] clasificaron los 3-politopos huecos que no están propiamente contenidos en otro cuerpo convexo sin puntos reticulares interiores. Más adelante, Averkov, Krümpelmann y Weltge [AKW15] demostraron que los 3-politopos huecos maximales en este sentido (que ellos llaman $\mathbb{R}$-maximales) coinciden con los 3-politopos hueco-maximales en nuestro sentido (a lo que ellos llaman $\mathbb{Z}$-maximales). Se sabe que las dos nociones de maximalidad de politopos huecos no coinciden de dimensión cuatro en adelante [NZ11, Theorem 1.4].

Teorema I.12 ([AWW11, Theorem 2.2] y [AKW15, Theorem 1]). *Estos* 12 *politopos son todos los* 3*-politopos hueco-maximales:*

$$\mathcal{M}_1 \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \quad \mathcal{M}_2 \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \mathcal{M}_3 \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_4 \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \mathcal{M}_5 \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \quad \mathcal{M}_6 \begin{pmatrix} 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_7 \begin{pmatrix} 0 & 4 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \mathcal{M}_8 \begin{pmatrix} 2 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \mathcal{M}_9 \begin{pmatrix} -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_{10} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix} \quad \mathcal{M}_{11} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

$$\mathcal{M}_{12} \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$

Todos tienen anchura dos, excepto $\mathcal{M}_3$*,* $\mathcal{M}_5$*,* $\mathcal{M}_6$*,* $\mathcal{M}_9$ *y* $\mathcal{M}_{10}$ *que tienen anchura* 3.

En el Capítulo 2 necesitamos conocer los 3-politopos huecos de anchura mayor que dos. Por el Teorema I.11 (Nill–Ziegler), un tal politopo tiene que ser subpolitopo (no necesariamente propio) de uno de los cinco 3-politopos hueco-maximales de anchura 3 en el Teorema I.12. Estos cinco politopos aparecen en la Figura I.5 (esta figura ha sido tomada de [AKW15]), y el siguiente resultado demuestra que en realidad son los únicos de anchura 3:

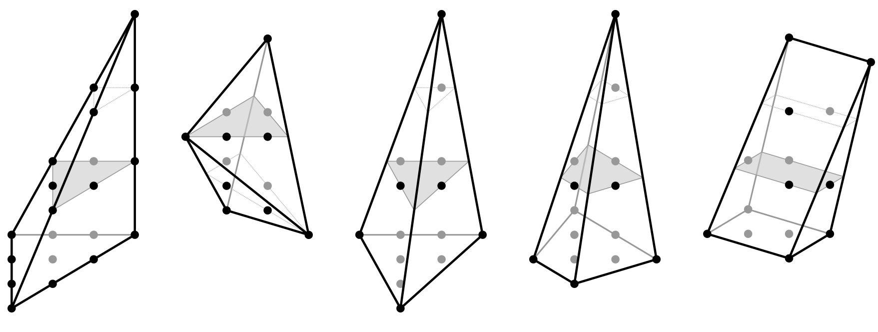


Figura I.5: Los cinco 3-politopos hueco-maximales de anchura 3. Esta figura aparece en Averkov et al [AKW15].

Corolario I.13. *Los únicos* 3*-politopos huecos de anchura* > 2 *son* $\mathcal{M}_3$*,* $\mathcal{M}_5$*,* $\mathcal{M}_6$*,* $\mathcal{M}_9$ *y* $\mathcal{M}_{10}$*, y todos tienen anchura* 3.

Demostración. Es suficiente comprobar que todos los subpolitopos propios de $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ y $\mathcal{M}_{10}$ que se obtienen al quitar uno de los vértices tienen anchura dos (o menor). En algunos de los casos, esto es evidente en la Figura 2.10. En los dos casos que nos parece menos claro, es decir $\mathcal{M}_5$ y $\mathcal{M}_{10}$, la Figura I.6 muestra proyecciones de ambos para las cuales los funcionales paralelos a los ejes de coordenadas dan anchura 2 a los subpolitopos (una única proyección para $\mathcal{M}_5$; tres proyecciones $\mathcal{M}_{10}$, cada una de ellas mostrando el resultado para dos de los vértices). $\square$

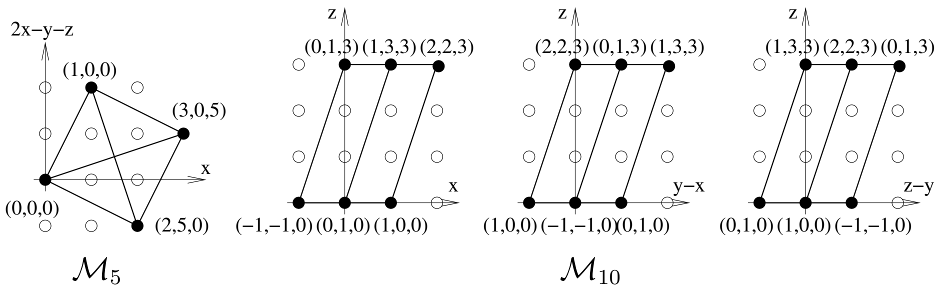


Figura I.6: Proyecciones que muestran que todos los subpolitopos propios de $\mathcal{M}_5$ y $\mathcal{M}_{10}$ tienen anchura a lo sumo dos.

Corollary I.1. *$w_H(d) \geq d$ para todo d, con igualdad para $d = 1, 2, 3$.*

I.2.2 Politopos vacíos

Un *politopo vacío* es un politopo reticular que no tiene otros puntos reticulares aparte de los vértices. Son un caso especial de politopos huecos.

En dimensión 1, el único segmento vacío es el segmento unimodular. En dimensión 2, si P es un polígono vacío, entonces está contenido en un paralelogramo de volumen euclídeo uno (que es equivalente al cuadrado unidad). Este resultado, así como el de dimensión 3, está dado en términos de maximalidad. Un d-politopo *vacío-maximal* es un d-politopo vacío que no está propiamente contenido en otro d-politopo vacío.

Los politopos vacíos en dimensión d tienen a lo sumo tamaño 2^d, puesto que si no contienen tres puntos alineados y entonces no son vacíos. En consecuencia, todo 3-politopo vacío está contenido en un d-politopo vacío-maximal (no existen regiones vacías no acotadas; véase Observación I.10). En dimensión 3, todo 3-politopo vacío-maximal tiene ocho vértices:

Teorema I.14 (Teorema de Howe [Sca85])**.** *Si P es un 3-politopo vacío, entonces P está contenido en un politopo reticular de anchura uno que tiene un paralelogramo de volumen euclídeo uno en cada uno de los planos.*

Los símplices vacíos son las *piezas fundamentales* de los politopos reticulares, en el sentido de que un símplice vacío no contiene ningún subpolitopo propio de la misma dimensión, y que todo politopo reticular se puede subdividir en símplices vacíos.

En dimensiones 1 y 2, el único símplice vacío es el símplice unimodular. En dimensiones 3 y más, esto ya no se cumple:

Example I.2 (Tetraedros de Reeve, [Ree57])**.** *La familia $\{T_r\}_{r\in\mathbb{Z}_{>0}}$, donde*

$$T_r := \operatorname{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,r)\},$$

es una familia infinita de tetraedros vacíos, y el volumen de cada T_r es r.

White realizó la clasificación completa:

Teorema I.15 (Clasificación de tetraedros vacíos, White 1964 [Whi64])**.** *Todo tetraedro vacío de volumen $q \in \mathbb{N}$ es equivalente a*

$$T(p,q) := \mathrm{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\},$$

para algún $p \in \mathbb{Z}$ con $\gcd(p,q) = 1$. Además, $T(p,q)$ es equivalente a $T(p',q)$ si, y sólo si, $p' = \pm p^{\pm 1} \pmod{q}$.

Todos los tetraedros vacíos tienen anchura uno con respecto a un par de aristas opuestas (la distancia entre las aristas es uno). Sólo los tetraedros vacíos de volúmenes 1 o 2 ($q \in \{1,2\}$) tienen anchura uno con respecto a los tres pares de aristas opuestas, y una familia infinita de tetraedros vacíos (con $p \in \{1, q-1\}$) tiene anchura uno con respecto a dos pares de aristas opuestas.

En dimensión 4, de nuevo sólo existe un número infinito de símplices vacíos, pero su anchura no está acotada por uno:

Example I.3 (Haase–Ziegler, [HZ00, Proposition 6])**.** *Para todo $D \geq 8$ con $\gcd(D,6) = 1$, el 4-símplice reticular $\mathrm{conv}\{e_1, e_2, e_3, e_4, (2,2,3,D-6)\}$ es vacío y tiene anchura 2.*

Como consecuencia de nuestro trabajo en el Capítulo 2 demostramos que, sin embargo, sólo existe un número finito de 4-símplices vacíos de anchura mayor que dos. La enumeración completa de todos ellos ha sido realizada por Iglesias–Santos [IS17]. (Esta lista ya era conocida por Haase-Ziegler [HZ00], pero se desconocía si era completa.)

Recapitulando, si llamamos $w_E(d)$ a la anchura máxima de entre todos los d-politopos vacíos, sabemos que $w_E(1) = w_E(2) = w_E(3) = 1$ y $w_E(4) \geq 4$. En general, una cota inferior es $2\lfloor d/2 \rfloor - 1 \leq w_E(d)$. Un d-símplice vacío de esta anchura aparece en Sebő [Seb99, Equations 1.1 and 1.2]. Como cota superior no conocemos otra mejor que la cota superior para politopos huecos (véase la Sección I.2.1).

I.2.3 Politopos reticulares con un punto reticular en el interior

Para cada d y $k > 0$, el número de d-politopos reticulares con k puntos reticulares en el interior es finito. Este hecho se deduce de combinar los siguientes dos teoremas:

Teorema I.16 (Hensley, [Hen83, Theorem. 3.6])**.** *Para un d fijo y para cada $k > 0$, existe una cota al volumen de los d-politopos reticulares con k puntos reticulares en el interior.*

Teorema I.17 (Lagarias–Ziegler, [LZ91, Theorem. 2])**.** *Para d y V fijos, el número de clases de equivalencia de d-politopos reticulares cuyo volumen está acotado por V es finito.*

En dimensión 1, el único segmento reticular con exactamente k puntos reticulares interiores es un segmento de longitud $k+1$. En dimensión 2, la Figura I.7 muestra la lista completa de polígonos reticulares con exactamente un punto reticular interior (hay 16 tales polígonos).

Llamamos *politopo canónico* a un politopo reticular con exactamente un punto reticular en el interior. Son de gran importancia en el ámbito de la geometría algebraica, donde el punto interior se supone el origen. Se llama *politopo terminal* a un politopo canónico en el que todos los puntos reticulares del borde son vértices.

Los politopos canónicos 3-dimensionales han sido completamente clasificados por Kasprzyk [Kas10] (hay 674 688; la información sobre ellos y más cosas pueden encontrarse en la "graded ring database" www.grdb.co.uk). Más recientemente, Balletti y Kasprzyk clasificaron todos los 3-politopos reticulares con exactamente dos puntos reticulares en el interior (hay 22 673 449; véase [BK16]).

I.2.4 Politopos reflexivos

Un d-politopo *reflexivo* es un d-politopo reticular P que tiene un punto reticular interior $p \in \operatorname{int}(P) \cap \mathbb{Z}^d$ y tal que la distancia desde p a cada faceta es 1. Por definición, este punto p es el único punto en el interior, y por norma general suponemos que es el origen. Este concepto se introdujo por primera vez en [Bat94].

En dimensiones 1 y 2, resulta que un politopo reticular es reflexivo si, y sólo si, tiene exactamente un punto reticular en el interior. Sin embargo, para $d \geq 3$, un d-politopo reticular con un único punto reticular interior no es necesariamente reflexivo. La explicación es relativamente sencilla. Sea P un politopo reticular con un único punto reticular interior p. Entonces todos los subpolitopos $\operatorname{conv}(F \cup \{p\})$, con F una faceta de P, tienen todos los puntos reticulares excepto p en F. La única manera de que esto ocurra en dimensiones 1 y 2 es que la distancia entre p y F sea uno (trivial en dimensión 1; véase el Lema I.21 para el caso de dimensión 2. Pero en dimensiones mayores, existen tales pirámides con la distancia mayor que uno. Los tetraedros de Reeve son un ejemplo para dimensión 3 (véase el Ejemplo I.2).

Una propiedad interesante de los politopos reflexivos tiene que ver con la dualidad: *un politopo reticular P es reflexivo si, y sólo si, su polar $P^\vee$ es un politopo reticular.* Este hecho se deduce trivialmente de la biyección entre facetas de P y vértices de $P^\vee$. Sea P con el origen en el interior, y sea $v \in \mathbb{R}^d$ un vértice de $P^\vee$. Entonces v es de la forma $v = \frac{1}{a_0}(a_1, \dots, a_d)$, donde $x_1a_1 + \cdots + x_da_d = a_0$, con $a_0 > 0$, es la ecuación del hiperplano soporte de una faceta F de P. Si P es un politopo reticular, los coeficientes de la ecuación son enteros y, sin pérdida de generalidad, podemos escoger $a_0, a_1, \dots, a_d \in \mathbb{Z}$ tales que $\gcd(a_1, \dots, a_d) = 1$ (obsérvese que esto no afecta a v). Ahora, si P es reflexivo, entonces la distancia del origen a la faceta F es uno. Es decir, el funcional $f(x_1, \dots, x_d) = x_1a_1 + \cdots + x_da_d$ (que es primitivo) toma el valor 0 en el origen y 1 en la faceta F. Entonces $a_0 = 1$ y $v \in \mathbb{Z}^d$.

En general, el polar de un politopo reticular no tiene por qué ser reticular. Por otro lado, puesto que $(P^\vee)^\vee = P$, entonces P es reflexivo si, y sólo si, $P^\vee$ es reflexivo. Es decir, los politopos reflexivos están emparejados, y sólo hay un número finito de ellos en cada dimensión (tienen exactamente un punto reticular interior, y el número de d-politopos reticulares con un número fijo y positivo de puntos reticulares interiores es finito, tal y como se ha explicado en la Sección I.2.3).

Existen exactamente 1, 16, 4319 y 473 800 776 d-politopos reflexivos, para dimensiones 1 a 4, respectivamente. Los de dimensión 3 y 4 fueron clasificados por Kreuze–Skarke in [KS98] and [KS00], respectivamente. La Figura I.7 muestra los polígonos reflexivos.

Dado que el número de politopos reflexivos crece dramáticamente con la dimensión, los trabajos de clasificación se han concentrado en una subclase de los reflexivos, los llamados politopos reflexivos *suaves*.

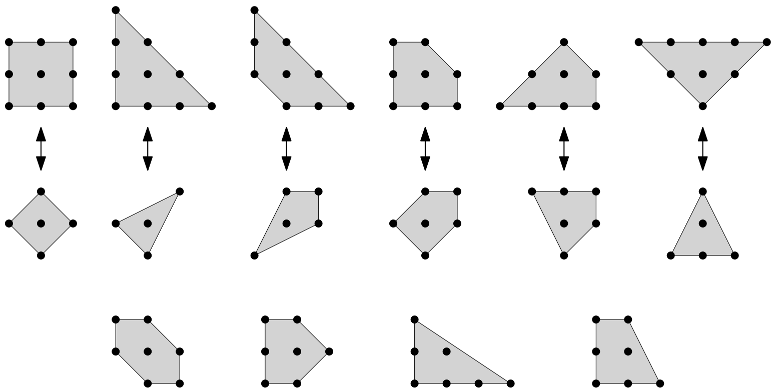


Figura I.7: Clasificación de los polígonos reflexivos, que coinciden con los polígonos reticulares con exactamente un punto reticular interior. Hay seis pares de polares reflexivos y cuatro polígonos reflexivos autoduales.

Un politopo *suave* (del inglés *smooth*) es un politopo reticular que es *simple* (cada vértice está contenido en exactamente d facetas) y tal que el cono en cada vértice es unimodular. Sin entrar en detalle de lo que es un cono en un vértice, lo que que queremos decir es lo siguiente: un vértice v de un d-politopo simple P pertenece exactamente a d aristas. Sea S_v el símplice cuyos vértices son v y los primeros puntos reticulares en cada una de las d aristas que salen de v. Decimos que el cono de P en v es unimodular si S_v es un símplice unimodular.

En dimensión 1, el único segmento reflexivo es suave. En dimensión 2, los cuatro primeros polígonos (empezando por la izquierda) en la primera fila de la Figura I.7 son suaves, así como el hexágono autodual (primer polígono de la última fila).

Los politopos reflexivos suaves fueron clasificados en [Øbr07] (Øbro) hasta dimensión 8. Lorenz y Paffenholz [LP08] clasificaron los de dimensión 9.

Hay exactamente los siguientes números de d-politopos reflexivos suaves, para $d \in \{3, \dots, 9\}$: 18, 124, 866, 7 622, 72 256, 749 892 y 8 229 721. Estas clasificaciones han dado lugar a resultados sobre politopos reflexivos suaves en dimensión arbitraria y a resolver problemas que llevaban abiertos bastante tiempo [AJP14, LN15, NP11, OSY12].

I.2.5 Politopos dps, suma de Minkowski

Un *politopo dps* (del inglés *distinct pair-sums*: sumas de pares distintas) es un politopo reticular $P \subset \mathbb{R}^d$ para el cual las sumas de pares $\{a+b : a, b \in P \cap \mathbb{Z}^d\}$ son todas distintas. Equivalentemente, P es dps si

$$\#\Big(P \cap \mathbb{Z}^d \; + \; P \cap \mathbb{Z}^d\Big) \; = \; \binom{\#(P \cap \mathbb{Z}^d)}{2} \; + \; \#(P \cap \mathbb{Z}^d).$$

donde la suma de dos conjuntos $A, B \subset \mathbb{R}^d$ es $A + B := \{a + b \mid a \in A, \; b \in B\}$, y se llama *suma de Minkowski*.

Fueron introducidos por primera vez en [CLR02], donde también se observó que estos politopos coinciden con aquellos que no ni contienen tres puntos alineados ni los vértices de un paralelogramo reticular [CLR02, Lemma 1].

También coinciden con los politopos reticulares de *longitud de Minkowski* igual a uno [BGSW12], donde la *longitud de Minkowski* de un politopo reticular P es el mayor número de vectores primitivos no nulos cuya suma de Minkowski está contenida en P. Por esta razón, también se les llama *fuertemente indescomponibles* en [SS09].

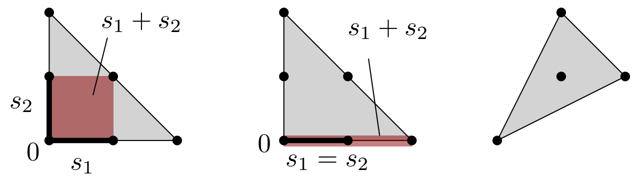


Figura I.8: El polígono $2\Delta_2$ de longitud de Minkowski 2 (con dos elecciones diferentes de segmentos primitivos s_1 y s_2 cuya suma de Minkowski $s_1 + s_2 \subseteq 2\Delta_2$). Y un triángulo terminal de longitud de Minkowski uno, es decir, dps, es decir, fuertemente indescomponibles.

Los politopos dps d-dimensionales no pueden tener más de 2^d puntos reticulares, ya que si no dos de sus puntos reticulares estarían en la misma clase módulo 2, y el politopo contendría al menos tres puntos alineados. Por esta razón, una manera de clasificarlos es como subproducto de clasificar todos los politopos reticulares de tamaño a lo sumo 2^d.

En dimensión 1, sólo el segmento unidad es dps. En dimensión 2, los únicos polígonos dps son el símplice unimodular y el único triángulo terminal (polígono de más a la derecha en la Figura I.8). En [Cur12] aparecen clasificaciones parciales de los 3-politopos dps.

Observación I.18. La clasificación de los 3-politopos dps fue en parte nuestra motivación inicial cuando comenzamos la clasificación de los 3-politopos reticulares de tamaño pequeño ([BS16a], [BS16b] y el Capítulo 3 en esta tesis), puesto que los 3-politopos dps tienen a lo sumo tamaño 8.

Después de haber terminado este trabajo, hemos sabido que la clasificación completa de los 3-politopos *fuertemente indescomponibles* aparece en la tesis no publicada de J. Whitney [Whi10]. Esta clasificación coincide con la nuestra.

En dimensión 3, existen infinitos 3-politopos dps de anchura uno, y 108 de anchura mayor que uno (véase la Sección 3.4.6).

I.3 Trabajo previo: clasificación de los 3-politopos reticulares de tamaños 5 y 6

En gran parte de los enfoques de clasificación explicados en la sección anterior, la clasificación de símplices vacíos juega un importante papel, ya que son los subpolitopos reticulares más pequeños que podemos encontrar. Un profundo conocimiento de los d-símplices vacíos proporciona herramientas e intuición para entender y descubrir lo que ocurre para otros d-politopos reticulares. En particular, es importante observar que estos símplices son politopos de un tamaño fijo, en concreto $d + 1$. Nos parece natural clasificar, o enumerar, *todos* los politopos reticulares de una dimensión dada y con un cierto número de puntos reticulares.

Proposición I.19. *Para $d \leq 2$, y para cada $n \geq d+1$, el número de d-politopos reticulares de tamaño n es finito.*

Demostración. En dimensión 1, un segmento reticular de tamaño n es equivalente a $[1, n]$, de longitud $n-1$. En dimensión 2, el Teorema de Pick (véase [BR07]) relaciona el volumen de un polígono reticular con el número de puntos reticulares interiores (i) y en el borde (b):

$$\operatorname{vol}(P) = 2i + b - 2 = n + i - 2$$

lo cual, dado que un polígono reticular de tamaño n tiene a lo sumo $n-3$ puntos reticulares interiores, implica que su volumen está acotado por $2n-5$. El Teorema de Hensley I.16 demuestra que sólo existe un número finito de clases de polígonos de cada tamaño. □

Una idea de una demostración constructiva es la siguiente: cualquier d-politopo reticular P de tamaño n se puede construir como la envolvente convexa de un polígono reticular Q (de dimensión d o $d-1$) de tamaño $n-1$, y un punto reticular p que *extienda a Q por un único punto.*

Observación I.20 (Extendiendo politopos con un único punto)**.** Sea P un d-politopo reticular de tamaño n y sea v uno de sus vértices, denotamos por $P^v := \operatorname{conv}(P \cap \mathbb{Z}^d \setminus \{v\})$ a la envolvente convexa de todos los puntos reticulares de P con la excepción de v. Obsérvese que P^v es de tamaño $n-1$, pero su dimensión puede ser d o $d-1$. Simplificamos $P^{u,v} := (P^u)^v = (P^v)^u$.

Recíprocamente, sea $Q \subset \mathbb{R}^d$ un politopo reticular d o $(d-1)$-dimensional y sea $v \in \mathbb{Z}^d \setminus Q$. Decimos que v *extiende a Q con un único punto* si el politopo reticular $P := \operatorname{conv}(Q \cup \{v\})$ es d-dimensional, y $P \cap \mathbb{Z}^d = (Q \cap \mathbb{Z}^d) \cup \{v\}$. Es decir, si $Q = P^v$.

En dimensión uno, la única manera de extender a un segmento con un único punto es incrementando la longitud por uno. Es decir, estamos añadiendo un punto reticular a distancia uno de una faceta. En dimensión 2 ocurre lo mismo:

Lema I.21. *Sea $T := \operatorname{conv}\{v_1, v_2, v_3\}$ un triángulo reticular, con v_3 a distancia mayor que uno del segmento v_1v_2. Entonces existe un punto reticular $p \in T$ a menor distancia (pero no nula) de v_1v_2 que v_3.*

Demostración. Sin pérdida de generalidad, sea $T := \operatorname{conv}\{(0,0), (0,k), (a,b)\}$ con $k \geq 1$ y $a \geq 2$. Consideramos el triángulo $T' = \operatorname{conv}\{(0,0), (0,1), (a,b)\} \subseteq T$. Puesto que T' no es unimodular, T' contiene algún punto reticular extra (c,d). Claramente $0 < c < a$. □

El lema anterior dice que un punto v extiende un segmento reticular $S \subset \mathbb{R}^2$ con un único punto si, y sólo si, la distancia de v a S es uno. Sea $Q \subset \mathbb{R}^2$ un polígono reticular de tamaño n, y sea $v \in \mathbb{Z}^2 \setminus Q$. Utilizando el Lema I.21 es fácil ver que la única manera de que v extienda a Q con un único punto es si v está a distancia uno de todas las facetas de Q que son visibles desde v. Por faceta *visible* queremos decir las facetas de Q cuyo hiperplano soporte deja v y Q en lados opuestos. Véase la Figura I.9. Esto nos deja con un número finito de posibilidades para v, para cualquier dado Q. Por inducción podemos construir los (finitos) polígonos reticulares de cada tamaño, empezando por el único polígono reticular de tamaño 3, es decir, el triángulo estándar unimodular Δ_1.

Sin embargo, en dimensiones 3 y más, el número de d-politopos reticulares de un tamaño fijo es infinito, como ya se argumentó en [LZ11]. Un ejemplo son los tetraedros de Reeve (Ejemplo I.2), que son infinitos 3-politopos reticulares de tamaño 4. Todos los de este tamaño fueron completamente clasificados por White (véase el Teorema I.15), y todos ellos tienen anchura uno.

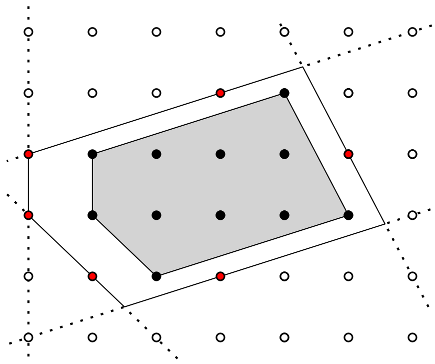


Figura I.9: Un polígono y el conjunto finito de puntos (en rojo) que lo extienden con un único punto.

Observación I.22. En general, hay infinitos d-politopos reticulares de anchura uno para cada tamaño dado $n \geq d+1$: consisten en dos politopos reticulares paralelos de dimensión $\leq d-1$ y de tamaños $n_1 \leq n$ y $n-n_1$ a distancia uno. Las infinitas opciones corresponden con las infinitas rotaciones $GL(\mathbb{Z}, d-1)$ de un politopo con respecto al otro.

Resulta que la anchura juega un papel importante en la existencia o no de infinitos politopos reticulares de un cierto tamaño. Por ejemplo, en dimensión tres el siguiente resultado es cierto:

Teorema I.23 ([BS16a, Corollary 22])**.** *Para cada $n \geq 4$ sólo existe un número finito de 3-politopos reticulares de tamaño n y anchura mayor que uno.*

Véanse en el Capítulo 2 resultados similares en dimensión arbitraria.

En el resto de esta sección revisaremos las listas completas de 3-politopos reticulares de tamaños 5 y 6 que calculamos en [BS16a, BS16b]. Los métodos que utilizamos eran bastante específicos y se basaban en clasificar primero las posibles matroides orientadas (véase [DLRS10]) de los cinco o seis puntos reticulares, y luego estudiarlos caso por caso. Los argumentos utilizados en esos artículos hacían una gran distinción entre politopos de anchura uno y politopos de anchura mayor que uno.

Explicaremos algunos resultados y definiciones que aparecen en esos artículos y que son necesarias para comprender la información que se proporciona sobre cada politopo (Tablas I.1, I.2 y I.3). También enunciaremos un par de lemas específicos que se deducen de esta clasificación y que son necesarios en el Capítulo 3.

Como herramienta útil para evaluar la equivalencia entre los politopos de tamaños 5 y 6, definimos el siguiente invariante (casi completo) de la equivalencia unimodular.

Definition I.4 (Vector de volúmenes)**.** *Sea $A = \{p_1, p_2, \ldots, p_n\}$, con $n \geq d+1$, un conjunto finito de puntos en $\mathbb{Z}^d$. El vector de volúmenes de A es el vector*

$$w = (w_{i_1 \ldots i_{d+1}})_{1 \leq i_1 < \ldots < i_{d+1} \leq n} \in \mathbb{Z}^{\binom{n}{d+1}}$$

donde $w_{i_1 \ldots i_{d+1}}$ es el determinante de los puntos $p_{i_1}, \ldots, p_{i_{d+1}}$, en ese orden.

El vector de volúmenes es claramente un invariante de la equivalencia unimodular: para $A, B \subset \mathbb{Z}^d$ configuraciones con el mismo número de puntos reticulares, si $\mathrm{conv}(A) \cong \mathrm{conv}(B)$, entonces existe una permutación de los puntos de manera que los vectores de volúmenes son iguales, módulo una orientación global. El recíproco es cierto sólo si los vectores de volúmenes son primitivos [BS16a, Proposition 5].

Véase el Algoritmo A.6 para una función implementada de MATLAB que calcula el vector de volúmenes, así como la versión ordenada de los valores absolutos (por la cual nos referimos al vector de los valores absolutos de las coordenadas, ordenados de menor a mayor). El vector de volúmenes del conjunto de vértices de un politopo se utilizará a la hora de evaluar equivalencia unimodular.

La clasificación de 3-politopos reticulares de tamaño 5 se muestra en la Tabla I.1. En dicha tabla está incluida la siguiente información de cada clase de equivalencia: un representante de la clase, la anchura, la *signatura*, y el vector de volúmenes con una cierta modificación de manera que refleje la signatura.

Sign.	**Vector de volúmenes**	**Anchura**	**Representante**
$(2,2)$	$(-1,1,1,-1,0)$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$,$(0,0,1)$
$(2,1)$	$(-2q,q,0,q,0)$ $0 \le p \le \frac{q}{2}$, $\gcd(p,q)=1$	1	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(-1,0,0)$,$(p,q,1)$
$(3,2)^*$	$(-a-b,a,b,1,-1)$ $0 < a \le b$, $\gcd(a,b)=1$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$,$(a,b,1)$
$(3,1)^*$	$(-3,1,1,1,0)$ $(-9,3,3,3,0)$	1 2	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$,$(0,0,1)$ $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$,$(1,2,3)$
$(4,1)^*$	$(-4,1,1,1,1)$ $(-5,1,1,1,2)$ $(-7,1,1,2,3)$ $(-11,1,3,2,5)$ $(-13,3,4,1,5)$ $(-17,3,5,2,7)$ $(-19,5,4,3,7)$ $(-20,5,5,5,5)$	2 2 2 2 2 2 2 2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,1,1)$,$(-2,-1,-2)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,2,1)$,$(-1,-1,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,3,1)$,$(-1,-2,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-1,-2,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-1,-1,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,7,1)$,$(-1,-2,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(3,7,1)$,$(-2,-3,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-3,-5,-2)$

Tabla I.1: Clasificación de 3-politopos reticulares de tamaño 5. Los que están marcados con un * son dps.

Recuérdese que $d+2$ puntos en dimensión d tienen una única dependencia afín, módulo un escalar. La *signatura* de la configuración de puntos es el par (c^+, c^-), donde c^+ y c^- son el número de coeficientes positivos y negativos de la dependencia, respectivamente. El (único) *circuito* de esta configuración está formado por los $c^+ + c^-$ puntos con coeficientes no nulos en la dependencia, y su signatura es también (c^+, c^-). El vector de volúmenes de $d+2$ puntos en dimensión d tiene un cambio de signo en alguna de sus coordenadas, de manera que las coordenadas del vector son en sí los coeficientes de la dependencia. Por otro lado, la matroide orientada de $d+2$ puntos en dimensión d está completamente determinada por su signatura (véase la Figura I.10 para el caso de 5 puntos en dimensión 3).

La clasificación de 3-politopos reticulares de tamaño 6 y anchura mayor que uno se muestra en las Tablas I.2 y I.3, donde se da, para cada uno de ellos, la matroide orientada,

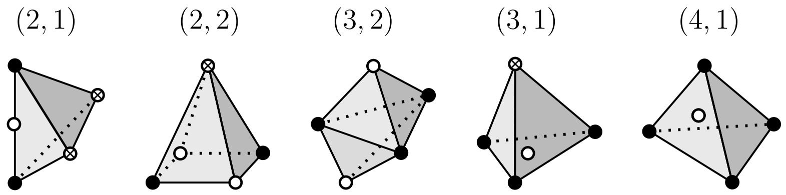


Figura I.10: Las cinco posibles signaturas de configuraciones de cinco puntos en dimensión 3. Puntos negros y blancos representan los puntos con coeficientes positivos y negativos en la dependencia, respectivamente. Las cruces representan puntos con coeficiente nulo.

un ID de referencia, el vector de volúmenes, la anchura, y un funcional que da dicha anchura. El número que aparece en la columna "MO" de las tablas es el identificador de la matroide orientada tal y como aparece en [BS16b, Figure 1].

En lo que sigue también damos una matriz entera 3×6 para cada uno de los politopos (con el ID de referencia de las tablas), cuyas columnas son los seis puntos reticulares de un representante de la clase. El vector de volúmenes dado en las tablas corresponde al orden de columnas de las matrices.

Observación I.24. En las Tablas I.2 y I.3, una (a, b)-*coplanaridad* hace referencia a $a + b \geq 4$ puntos reticulares que son coplanares y forman un circuito de signatura (a, b). Por otro lado, una $(2, 1)$-*colinealidad* hace referencia a tres puntos reticulares alineados.

$$A.1 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad B.6 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.13 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix} \quad C.5 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$A.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad B.7 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.14 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad C.6 \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$B.1 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.8 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.15 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad D.1 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$B.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.9 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad C.1 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad D.2 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$B.3 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.10 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad C.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad E.1 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix}$$

$$B.4 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.11 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix} \quad C.3 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 6 \end{pmatrix} \quad E.2 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$B.5 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.12 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix} \quad C.4 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \quad F.1 \begin{pmatrix} 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 \end{pmatrix}$$

$$F.2\begin{pmatrix}0&1&0&-1&1&2\\0&2&0&-1&0&0\\0&1&1&-1&0&0\end{pmatrix}\quad F.5\begin{pmatrix}0&0&1&1&-1&-2\\0&0&0&3&-2&-4\\0&1&0&1&-1&-2\end{pmatrix}\quad F.8\begin{pmatrix}0&1&0&-1&1&-1\\0&2&0&-1&0&0\\0&1&1&-1&0&0\end{pmatrix}\quad F.11\begin{pmatrix}0&0&1&1&-1&1\\0&0&0&3&-2&2\\0&1&0&1&-1&1\end{pmatrix}$$

$$F.3\begin{pmatrix}0&0&1&1&-1&-2\\0&0&0&2&-1&-2\\0&1&0&1&-1&-2\end{pmatrix}\quad F.6\begin{pmatrix}0&1&2&0&-1&-2\\0&0&5&0&-2&-4\\0&0&1&1&-1&-2\end{pmatrix}\quad F.9\begin{pmatrix}0&0&1&1&-1&1\\0&0&0&2&-1&1\\0&1&0&1&-1&1\end{pmatrix}\quad F.12\begin{pmatrix}0&1&2&0&-1&1\\0&0&5&0&-2&2\\0&0&1&1&-1&1\end{pmatrix}$$

$$F.4\begin{pmatrix}0&0&1&-1&1&2\\0&0&3&-2&0&0\\0&1&1&-1&0&0\end{pmatrix}\quad F.7\begin{pmatrix}0&1&0&-2&1&-1\\0&1&0&-1&0&0\\0&1&1&-2&0&0\end{pmatrix}\quad F.10\begin{pmatrix}0&0&1&-1&1&-1\\0&0&3&-2&0&0\\0&1&1&-1&0&0\end{pmatrix}\quad F.13\begin{pmatrix}0&1&0&-2&1&4\\0&1&0&-1&0&1\\0&1&1&-2&0&2\end{pmatrix}$$

$$F.14\begin{pmatrix}0&0&-1&1&1&1\\0&0&-1&2&0&-2\\0&1&-1&1&0&-1\end{pmatrix}\quad G.6\begin{pmatrix}0&0&1&1&-1&-3\\0&0&2&0&-1&-5\\0&1&1&0&-1&-4\end{pmatrix}\quad G.15\begin{pmatrix}0&0&1&2&-1&-1\\0&0&0&5&-2&-1\\0&1&0&1&-1&-1\end{pmatrix}\quad H.4\begin{pmatrix}0&0&1&-1&1&-8\\0&0&2&-1&0&-9\\0&1&1&-1&0&-10\end{pmatrix}$$

$$F.15\begin{pmatrix}0&0&1&1&-1&-3\\0&0&0&2&-1&-4\\0&1&0&1&-1&-3\end{pmatrix}\quad G.7\begin{pmatrix}0&0&1&1&-1&-5\\0&0&2&0&-1&-7\\0&1&1&0&-1&-6\end{pmatrix}\quad G.16\begin{pmatrix}0&0&1&2&-1&-2\\0&0&0&5&-2&-3\\0&1&0&1&-1&-2\end{pmatrix}\quad H.5\begin{pmatrix}0&0&1&1&-1&3\\0&0&0&2&-1&7\\0&1&0&1&-1&2\end{pmatrix}$$

$$F.16\begin{pmatrix}0&1&-1&0&1&2\\0&0&-2&0&3&6\\0&0&-1&1&1&1\end{pmatrix}\quad G.8\begin{pmatrix}0&-1&0&1&1&2\\0&-2&0&3&0&1\\0&-1&1&1&0&0\end{pmatrix}\quad G.17\begin{pmatrix}0&2&1&0&-1&-3\\0&5&0&0&-2&-5\\0&1&0&1&-1&-2\end{pmatrix}\quad H.6\begin{pmatrix}0&1&0&1&-1&4\\0&0&0&2&-1&11\\0&0&1&1&-1&5\end{pmatrix}$$

$$F.17\begin{pmatrix}0&1&1&0&-1&-2\\0&0&3&0&-2&-4\\0&0&1&1&-1&-3\end{pmatrix}\quad G.9\begin{pmatrix}0&-1&0&1&1&2\\0&-2&0&0&3&5\\0&-1&1&0&1&1\end{pmatrix}\quad G.18\begin{pmatrix}0&-1&2&0&1&1\\0&-1&5&0&0&-1\\0&-1&1&1&0&0\end{pmatrix}\quad H.7\begin{pmatrix}0&0&1&-1&1&-4\\0&0&0&-2&3&-7\\0&1&0&-1&1&-5\end{pmatrix}$$

$$G.1\begin{pmatrix}0&1&-2&0&1&4\\0&1&-1&0&0&1\\0&1&-2&1&0&3\end{pmatrix}\quad G.10\begin{pmatrix}0&1&0&-1&1&2\\0&0&0&-2&3&7\\0&0&1&-1&1&1\end{pmatrix}\quad G.19\begin{pmatrix}0&1&2&-1&0&-1\\0&0&5&-1&0&-2\\0&0&1&-1&1&1\end{pmatrix}\quad H.8\begin{pmatrix}0&1&0&-1&1&-7\\0&0&0&-2&3&-11\\0&0&1&-1&1&-6\end{pmatrix}$$

$$G.2\begin{pmatrix}0&1&-2&0&1&6\\0&1&-1&0&0&1\\0&1&-2&1&0&3\end{pmatrix}\quad G.11\begin{pmatrix}0&0&1&1&-1&-2\\0&0&0&3&-2&-1\\0&1&0&1&-1&-1\end{pmatrix}\quad G.20\begin{pmatrix}0&0&2&1&-1&-2\\0&0&5&0&-1&-3\\0&1&1&0&-1&-2\end{pmatrix}\quad H.9\begin{pmatrix}0&0&1&1&-1&2\\0&0&3&0&-2&-1\\0&1&1&0&-1&-1\end{pmatrix}$$

$$G.3\begin{pmatrix}0&-1&1&0&1&1\\0&-1&2&0&0&-1\\0&-1&1&1&0&0\end{pmatrix}\quad G.12\begin{pmatrix}0&1&1&0&-1&-3\\0&3&0&0&-2&-5\\0&1&0&1&-1&-2\end{pmatrix}\quad H.1\begin{pmatrix}0&1&0&-2&1&-12\\0&1&0&-1&0&-7\\0&1&1&-2&0&-13\end{pmatrix}\quad H.10\begin{pmatrix}0&0&1&1&-1&4\\0&0&0&3&-2&13\\0&1&0&1&-1&3\end{pmatrix}$$

$$G.4\begin{pmatrix}0&-1&0&1&1&3\\0&-1&0&2&0&1\\0&-1&1&1&0&0\end{pmatrix}\quad G.13\begin{pmatrix}0&1&1&0&-1&-4\\0&3&0&0&-2&-7\\0&1&0&1&-1&-3\end{pmatrix}\quad H.2\begin{pmatrix}0&0&1&-2&1&-15\\0&0&1&-1&0&-8\\0&1&1&-2&0&-17\end{pmatrix}\quad H.11\begin{pmatrix}0&1&2&-1&0&-5\\0&0&5&-2&0&-9\\0&0&1&-1&1&-4\end{pmatrix}$$

$$G.5\begin{pmatrix}0&0&1&-1&1&1\\0&0&2&-1&0&-3\\0&1&1&-1&0&-2\end{pmatrix}\quad G.14\begin{pmatrix}0&0&1&1&-1&-4\\0&0&0&3&-2&-5\\0&1&0&1&-1&-3\end{pmatrix}\quad H.3\begin{pmatrix}0&1&0&-1&1&-7\\0&2&0&-1&0&-9\\0&1&1&-1&0&-8\end{pmatrix}\quad H.12\begin{pmatrix}0&1&0&2&-1&4\\0&0&0&5&-2&11\\0&0&1&1&-1&2\end{pmatrix}$$

En el Capítulo 3 necesitamos dos resultados que se deducen de estas clasificaciones de 3-politopos reticulares de tamaños 5 y 6. El primer lema recopila información sobre qué puntos en $\mathbb{Z}^3$ pueden extender un cierto polígono reticular con un único punto. En él, un *paralelogramo unimodular* es un paralelogramo vacío o, equivalentemente, un paralelogramo reticular de volumen euclídeo uno.

Lema I.25 ([BS16a] y [BS16b, Section 5])**.** *Sea P un 3-politopo reticular con un vértice v tal que P^v es 2-dimensional (suponemos que está en el plano $\{z = 0\}$). Entonces:*

(1) Si P^v contiene el siguiente conjunto de cuatro puntos, entonces bien v está a distancia uno de P^v o $v = (a, b, \pm 3)$ con $a \equiv 1 \equiv -b \mod 3$:

$$\{(-1, -1), (0, 0), (1, 0), (0, 1)\}$$

(2) Si P^v contiene el siguiente conjunto de cuatro puntos, entonces la arista $\operatorname{conv}\{v, (0, 1)\}$ está a distancia uno de la arista $\operatorname{conv}\{(-1, 0), (1, 0)\}$:

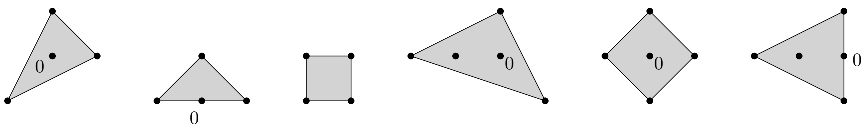
$$\{(-1, 0), (0, 0), (1, 0), (0, 1)\}$$

(3) Si P^v contiene un paralelogramo unimodular o los siguientes cinco puntos, entonces v está a distancia uno de P^v:

$$\{(0, 0), (-1, 0), (-2, 0), (0, 1), (1, -1)\}.$$

(4) Si P^v contiene alguno de los siguientes conjuntos de cinco puntos, entonces o bien v está a distancia uno de P^v o bien $v = (a, b, \pm 2)$ con $a \equiv 1 \equiv b \mod 2$:

$$\{(-1, 0), (0, 0), (1, 0), (0, 1), (0, -1)\}, \qquad \{(0, 0), (-1, 0), (-2, 0), (0, 1), (0, -1)\}.$$



Demostración. Para los casos en que P^v contiene uno de los polígonos reticulares de tamaño 4, mirar las filas de las signaturas $(3, 1)$, $(2, 1)$ y $(2, 2)$ en la Tabla I.1, respectivamente.

Para los polígonos reticulares de tamaño 5, buscar en la Tabla I.2 las filas correspondientes a "Politopos que contienen 5 puntos coplanares" y tener en cuenta que si v está a distancia uno de P^v, entonces siempre extiende a P^v con un único punto, pero que en ese caso P tiene anchura uno. $\square$

Lema I.26. *Las siguientes configuraciones (B.14 y B.15 en la Tabla I.2) son los únicos 3-politopos reticulares de tamaño 6 con dos vértices u, v tales que $P^{u,v} = \operatorname{conv}\{(-1, -1, 0), (1, 0, 0), (0, 1, 0)\}$ y tales que u y v están en lados opuestos y a distancia mayor que uno del plano que contiene a $P^{u,v}$:*

$$B.14 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad B.15 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}$$

Demostración. El subpolitopo $P^{u,v}$ es una $(3, 1)$-coplanaridad (véase la Observación I.24) en el plano $\{z = 0\}$. Mirando a la Tabla I.2, uno puede observar que los 3-politopos reticulares de tamaño 6, con una $(3, 1)$-coplanaridad y dos puntos a lados opuestos de dicha coplanaridad tienen identificadores $B.*$. Mirando a la lista de matrices, se puede ver que los únicos politopos con los dos puntos fuera de este plano a distancia mayor que uno del mismo son $B.14$ y $B.15$. $\square$

MO	Id.	Vector de volúmenes															Anchura, funcional	
Politopos que contienen 5 puntos coplanares																		
3.2	A.1	0	0	2	0	0	4	0	2	0	−4	0	4	−2	−8	−2	2	z
3.3	A.2	0	0	2	0	4	4	0	2	0	−4	0	0	−2	−4	−2	2	z
Politopos que contienen una $(3,1)$-coplanaridad, pero no 5 puntos coplanares																		
Una $(3,1)$-coplanaridad deja los otros dos puntos en lados opuestos																		
3.8	B.7	0	1	−1	−1	1	−2	1	−1	0	2	3	−3	0	6	0	2	z
3.9	B.9	0	1	−1	−1	1	−1	1	−1	1	0	3	−3	0	3	−3	2	z
3.13	B.10	0	1	−1	−1	1	0	1	−1	0	0	3	−3	−2	2	−2	2	z
	B.15	0	3	−3	−3	3	0	3	−3	0	0	9	−9	−6	6	−6	2	x
4.13*	B.1	0	1	−1	−1	1	−4	1	−1	1	3	3	−3	3	9	0	2	z
4.17*	B.2	0	1	−1	−1	1	−3	1	−1	0	3	3	−3	1	8	1	2	z
	B.14	0	3	−3	−3	3	−9	3	−3	0	9	9	−9	3	24	3	2	x
4.18	B.8	0	1	−1	−1	1	−1	1	−1	0	1	3	−3	−1	4	−1	2	z
5.10*	B.5	0	1	−1	−1	1	−5	1	−1	1	4	3	−3	4	11	1	2	z
	B.6	0	1	−1	−1	1	−6	1	−1	1	5	3	−3	5	13	2	2	z
	B.11	0	1	−3	−1	3	−8	1	−3	1	7	3	−9	5	19	2	2	x
5.11*	B.12	0	1	−3	−1	3	−2	1	−3	1	1	3	−9	−1	7	−4	2	x
5.12*	B.3	0	1	−1	−1	1	−2	1	−1	1	1	3	−3	1	5	−2	2	z
	B.4	0	1	−1	−1	1	−3	1	−1	1	2	3	−3	2	7	−1	2	z
	B.13	0	1	−3	−1	3	−5	1	−3	1	4	3	−9	2	13	−1	2	x
Todas las $(3,1)$-coplanaridades dejan los otros dos puntos en el mismo lado																		
3.6	C.1	0	1	2	−1	−2	0	1	2	−1	1	3	6	0	0	3	2	x
3.11	C.2	0	1	2	−1	−2	0	1	2	0	0	3	6	1	−1	1	2	x
	C.3	0	3	6	−3	−6	0	3	6	0	0	9	18	3	−3	3	3	x
5.4*	C.4	0	1	5	−1	−5	1	1	5	−2	1	3	15	1	−4	7	2	y
	C.5	0	1	7	−1	−7	1	1	7	−2	1	3	21	3	−6	9	2	y
5.6*	C.6	0	1	3	−1	−3	−2	1	3	1	1	3	9	5	1	2	2	x
Politopos que contienen una $(2,2)$-coplanaridad, pero ninguna de las anteriores																		
Una $(2,2)$-coplanaridad deja los otros dos puntos en lados opuestos																		
5.13	D.1	0	1	−1	1	−1	−4	−1	1	3	−1	−1	1	5	1	−2	2	z
	D.2	0	1	−1	1	−1	−5	−1	1	4	−1	−1	1	7	2	−3	2	z
Todas las $(2,2)$-coplanaridades dejan los otros dos puntos en el mismo lado																		
5.5	E.1	0	1	5	1	5	1	−1	−5	−2	−1	−1	−5	1	2	−3	2	y
	E.2	0	1	7	1	7	2	−1	−7	−3	−1	−1	−7	1	3	−4	2	$x-y$
Politopos que contienen una $(2,1)$-colinealidad, pero ninguna otra coplanaridad																		
4.21	F.1	1	−1	−2	1	2	0	−1	−2	0	0	−4	−7	−1	1	−1	2	y
	F.2	1	−2	−4	1	2	0	−1	−2	0	0	−5	−9	−2	1	−1	2	z
	F.3	2	−1	−2	1	2	0	−1	−2	0	0	−5	−8	−1	1	−1	2	$x-z$
	F.4	1	−3	−6	2	4	0	−1	−2	0	0	−7	−13	−3	2	−1	2	z
	F.5	3	−2	−4	1	2	0	−1	−2	0	0	−7	−11	−2	1	−1	2	$x-z$
	F.6	5	−3	−6	2	4	0	−1	−2	0	0	−11	−17	−3	2	−1	2	$x-z$
4.22	F.7	1	−1	1	1	−1	0	−1	1	0	0	−4	2	2	−2	2	2	y
	F.8	1	−2	2	1	−1	0	−1	1	0	0	−5	3	4	−2	2	2	z
	F.9	2	−1	1	1	−1	0	−1	1	0	0	−5	1	2	−2	2	2	z
	F.10	1	−3	3	2	−2	0	−1	1	0	0	−7	5	6	−4	2	2	z
	F.11	3	−2	2	1	−1	0	−1	1	0	0	−7	1	4	−2	2	2	z
	F.12	5	−3	3	2	−2	0	−1	1	0	0	−11	1	6	−4	2	2	z
4.11	F.13	1	−1	−3	1	2	1	−1	−2	−1	0	−4	−8	−4	0	0	2	y
	F.14	1	−1	−3	2	4	2	−1	−2	−1	0	−5	−10	−5	0	0	2	z
	F.15	2	−1	−4	1	2	1	−1	−2	−1	0	−5	−10	−5	0	0	2	$x-z$
	F.16	2	−1	−4	3	6	3	−1	−2	−1	0	−7	−14	−7	0	0	2	$x-z$
	F.17	3	−1	−5	2	4	2	−1	−2	−1	0	−7	−14	−7	0	0	2	$x-z$

Tabla I.2: Politopos reticulares 3-dimensionales de tamaño 6 y anchura > 1 que contienen alguna coplanaridad. Los que son dps están marcados con un * en la primera columna.

MO	Id.	Vector de volúmenes															Anchura,	funcional
Politopos sin coplanaridades y con 1 punto reticular interior																		
	G.1	1	−1	−1	1	3	−2	−1	−2	1	1	−4	−7	3	5	−1	2	y
	G.2	1	−1	−3	1	5	−2	−1	−4	1	1	−4	−13	1	7	−3	2	y
	G.3	1	−1	−1	1	2	−1	−2	−3	1	1	−5	−7	2	3	−1	2	z
	G.4	1	−1	−2	1	3	−1	−2	−5	1	1	−5	−11	1	4	−3	2	z
	G.5	1	−2	−5	1	4	−3	−1	−3	1	1	−5	−13	1	7	−2	2	x
	G.6	2	−1	−1	1	5	−2	−1	−3	1	1	−5	−11	3	7	−2	2	$x-z$
	G.7	2	−1	−3	1	7	−2	−1	−5	1	1	−5	−17	1	9	−4	2	$x-z$
	G.8	1	−2	−3	1	2	−1	−3	−5	1	1	−7	−11	1	3	−2	2	z
	G.9	2	−1	−1	1	3	−1	−3	−5	1	2	−7	−11	2	5	−1	2	z
6.2*	G.10	2	−3	−7	1	5	−4	−1	−3	1	1	−7	−17	1	9	−2	2	z
	G.11	3	−2	−1	1	5	−3	−1	−2	1	1	−7	−11	5	8	−1	2	z
	G.12	3	−1	−1	1	4	−1	−2	−5	1	1	−7	−13	2	5	−3	2	$x-z$
	G.13	3	−1	−2	1	5	−1	−2	−7	1	1	−7	−17	1	6	−5	2	$x-z$
	G.14	3	−2	−5	1	7	−3	−1	−4	1	1	−7	−19	1	10	−3	2	$x-z$
	G.15	5	−2	−1	1	3	−1	−3	−4	1	1	−11	−13	3	4	−1	2	z
	G.16	5	−2	−3	1	4	−1	−3	−7	1	1	−11	−19	1	5	−4	2	$x-z$
	G.17	5	−3	−5	1	5	−2	−2	−5	1	1	−11	−20	1	7	−3	2	$x-z$
	G.18	3	−4	−5	1	2	−1	−5	−7	1	1	−13	−17	1	3	−2	2	z
	G.19	4	−5	−7	1	3	−2	−3	−5	1	1	−13	−19	1	5	−2	2	z
	G.20	5	−3	−4	1	3	−1	−4	−7	1	1	−13	−19	1	4	−3	2	$x-z$
Politopos sin coplanaridades y con 2 puntos reticulares interiores																		
	H.1	1	−1	5	1	1	−6	−1	−2	7	−1	−4	1	19	5	−9	2	$x-z$
	H.2	1	−1	7	1	1	−8	−1	−2	9	−1	−4	3	25	7	−11	2	$x-2y$
	H.3	1	−2	5	1	1	−7	−1	−2	9	−1	−5	1	23	6	−11	2	$x-z$
	H.4	1	−2	7	1	1	−9	−1	−2	11	−1	−5	3	29	8	−13	2	$x-y$
	H.5	2	−1	7	1	1	−4	−1	−3	5	−1	−5	1	17	3	−8	2	$x-z$
6.1*	H.6	2	−1	11	1	1	−6	−1	−3	7	−1	−5	5	25	5	−10	2	$x-z$
	H.7	2	−3	7	1	1	−5	−1	−3	8	−1	−7	1	23	4	−11	2	$x-z$
	H.8	2	−3	11	1	1	−7	−1	−3	10	−1	−7	5	31	6	−13	2	$x-z$
	H.9	3	−1	7	2	1	−5	−1	−2	3	−1	−7	1	16	3	−5	2	z
	H.10	3	−2	13	1	1	−5	−1	−4	7	−1	−7	5	27	4	−11	2	$x-z$
	H.11	3	−5	11	2	1	−9	−1	−2	7	−1	−11	5	32	7	−9	2	$x-z$
	H.12	5	−2	11	3	1	−7	−1	−2	3	−1	−11	3	23	4	−5	3	$x-z$

Tabla I.3: Politopos reticulares 3-dimensionales de tamaño 6 y anchura > 1 sin coplanaridades. Todos son dps.

I.4 Resumen de la tesis y resultados principales

Como continuación de los resultados obtenidos en 3-politopos reticulares de tamaños 5 y 6 surgieron dos líneas de investigación. Por un lado, la demostración del Teorema I.23 que aparece en [BS16a] utilizaba resultados que son ciertos en dimensión arbitraria, con la excepción del hecho de que el único polígono hueco de anchura mayor que uno es $2\Delta_2$ (véase el Teorema I.8). En colaboración con Christian Haase, Jan Hofmann y Francisco Santos, estudiamos este problema en dimensión arbitraria. El Capítulo 2 recoge esta línea de investigación, que aparece en el artículo [BHHS16].

En el Capítulo 3 continuamos la clasificación de (el número finito de) 3-politopos reticulares de anchura mayor que uno y de un cierto tamaño. Los métodos generales utilizados en [BS16a, BS16b] para la clasificación de dichos politopos de tamaños 5 y 6 no son aplicables para tamaño 7 o más. Sin embargo, en partes específicas de la clasificación de 3-politopos de tamaño 6 [BS16b] utilizábamos implícitamente una idea, un método que hemos desarrollado posteriormente para elaborar un algoritmo que enumera todos los 3-politopos reticulares de anchura mayor que uno y hasta un cierto tamaño dado. La idea es utilizar inducción en el tamaño del politopo y hacer uso del hecho de que, si un 3-politopo reticular de anchura mayor que uno y tamaño N contiene un subpolitopo de tamaño $N-1$

y anchura aún mayor que uno, entonces este subpolitopo pertenece a una lista finita, que suponemos ya está calculada. Esta investigación aparece en el artículo [BS16c].

El Apéndice A de esta tesis contiene los programas de ordenador utilizados para los cálculos del Capítulo 3.

A continuación presentamos un resumen de los resultados principales de esta tesis.

Anchura umbral de finitud de politopos reticulares (Capítulo 2).

Teorema (Definición 2.1 y Teorema 2.4(3))**.** *Para cada dimensión d existe una constante $W \in \mathbb{N}$ que depende únicamente de d y tal que, para cada $n \geq d+1$, todos excepto un número finito de los d-politopos de tamaño n tienen anchura $\leq W$.*

Definimos la *anchura umbral de finitud*, y la denotamos por $w^\infty(d)$, como la mínima de dichas constantes W. Por ejemplo, en dimensiones 1 y 2, la anchura umbral de finitud es 0, ya que sólo existe un número finito de d-politopos reticulares de cada tamaño $n \geq d+1$ (véase el Teorema I.19). De la misma manera, el Teorema I.23 dice que $w^\infty(3) = 1$ (Blanco–Santos).

El Teorema I.11 (Nill–Ziegler) nos dice que, para encontrar una familia infinita de d-politopos reticulares y un cierto tamaño, podemos centrarnos en familias de d-politopos huecos que admiten proyecciones a $(d-1)$-politopos huecos. Ese mismo teorema es el que nos permite, junto con los Teoremas I.16 (Hensley) y I.17 (Lagarias–Ziegler), demostrar que $w^\infty(d) \leq w_H(d-1)$, y $w_H(d-1)$ está acotada para cada d por el Flatness Theorem (véase el Teorema I.7).

Estudiamos entonces *levantamientos* de (politopos reticulares que se proyectan sobre) un politopo reticular (Definición 2.14), y estudiamos la relación entre la anchura y la dimensión de un politopo con las de sus levantamientos. También definimos *levantamientos ajustados*, que son levantamientos minimales por inclusión (Definición 2.19). Estas definiciones y resultados (Sección 2.1) nos permiten simplificar considerablemente el trabajo a la hora de decidir si un politopo tiene o no infinitos levantamientos de tamaño acotado.

Es fácil ver que los símplices reticulares y los politopos reticulares no huecos tienen sólo un número finito de levantamientos de tamaño acotado (Sección 2.2). Por otro lado encontramos politopos huecos y vacíos de anchuras máximas (en su dimensión), que nos dan las cotas inferiores $w^\infty(d) \geq w_H(d-2)$ y $w^\infty(d) \geq w_E(d-1)$ (Sección 2.3).

En resumen:

Teorema (Corolario 2.6)**.** *Para cada $d \geq 3$,*

$$d-2 \;\leq\; \text{máx}\{w_H(d-2), w_E(d-1)\} \;\leq\; w^\infty(d) \;\leq\; w_H(d-1) \;\leq\; O\left(d^{3/2}\right).$$

Además, demostramos que el valor *exacto* de $w^\infty(d)$ puede obtenerse a partir de cierta información sobre $(d-1)$-politopos huecos como sigue:

Teorema (Teorema 2.7)**.** *Para cada $d \geq 3$, $w^\infty(d)$ es igual a la máxima anchura de un $(d-1)$-politopo hueco para el cual existen infinitos d-politopos que se proyectan sobre Q y con un cierto tamaño fijo $n \in \mathbb{N}$. Además, todo Q en esas condiciones ha de ser hueco y no puede ser un símplice.*

Utilizando la clasificación de 3-politopos huecos detallada en la Sección I.2.1, hemos obtenido el valor exacto de $w^\infty(d)$ para $d = 4$:

Teorema (Teorema 2.2)**.** $w^\infty(4) = 2$. *Es decir, para cada $n > 4$ sólo existe un número finito de 4-politopos reticulares de tamaño n y anchura mayor que 2.*

En dimensión 5, de la cota $w_E(d-1) \leq w^\infty(d)$ es trivial deducir que $w^\infty(5) \geq 4$, dado que $w_E(4) \geq 4$ por la existencia de un 4-símplice vacío de anchura 4 (véase la Sección I.2.2).

Observación I.27. Una consecuencia de $w^\infty(4) = 2$ es que el número de 4-símplices vacíos de anchura > 2 es finito (Corolario 2.3). Este resultado aparece en Barile et al. [BBBK11], pero la demostración que allí aparece es incompleta. Más específicamente, los autores utilizan una clasificación de familias infinitas de 4-símplices vacíos de anchura > 1 que se conjeturó completa en Mori et al. [MMM88]. Fue demostrada más tarde por Sankaran [San90] y Bover [Bob09], pero sólo para símplices vacíos con determinante (volumen) primo. Cuando el determinante no es un número primo, aparecen otras familias infinitas como la siguiente, no considerada por Barile et al: los 4-símplices vacíos con vértices e_1, e_2, e_3, e_4 y $(2, N/2-1, a, N/2-a)$, donde el determinante es un múltiplo de 4 y coprimo con a.

En resumen, la demostración del Corolario 2.3 que aparece en in [BBBK11] es válida para símplices de determinante primo. Agradecemos a O. Iglesias los cálculos que proporcionan ésta (y otras) familias y a los autores de [BBBK11] sus comentarios acerca del alcance del error.

Para mayores valores de d, los resultados de este capítulo servirían para obtener el valor exacto de $w^\infty(d)$, en el caso de que se conozcan nuevas clasificaciones de $(d-1)$-politopos huecos.

En vista de los resultados obtenidos, surgen las siguientes preguntas. En ellas, $w^\infty(d,n)$ es la mínima anchura $W \geq 0$ para la cual sólo existe un número finito de d-politopos reticulares de tamaño n y anchura W.

Pregunta I.28. *El Teorema 2.4 demuestra que $w^\infty(d,n) \leq w^\infty(d,n+1)$ (1) y $w^\infty(d) \leq w^\infty(d+1)$ (2), para todo d, y para todo $n \geq d+1$.*

¿Es también cierto que $w^\infty(d,n) \leq w^\infty(d+1,n+1)$? En el caso de símplices vacíos sí se verifica: $w^\infty(d,d+1) \leq w^\infty(d+1,d+2)$ es consecuencia de que todo d-símplice vacío es una faceta de un número infinito de $(d+1)$-símplices vacíos de al menos la misma anchura *([HZ00, Proposition 1]).*

Pregunta I.29. *Para todos los valores conocidos ($d \leq 4$) tenemos que $w^\infty(d) = w^\infty(d,d+1)$. Es decir, que la anchura umbral de finitud para los d-politopos reticulares está determinada por los d-símplices vacíos. ¿Es esto cierto en cualquier dimensión d?*

Enumeración de 3-politopos reticulares (Capítulo 3).

El objetivo de este capítulo es clasificar todos los 3-politopos reticulares de anchura mayor que uno y tamaño fijo N. Suponemos que $N \geq 7$ y demostramos que cada uno de estos politopos pertenece a uno de tres grupos:

Teorema (Teoremas 3.13 and 3.15)**.** *Sea $P \subset \mathbb{R}^3$ un 3-politopo reticular de anchura mayor que uno y tamaño al menos siete. Entonces P está en al menos una de las siguientes condiciones:*

(1) P proyecta de una manera muy concreta a uno de entre siete polígonos reticulares dados. (Los llamamos politopos espinados*; en inglés* spiked*)*

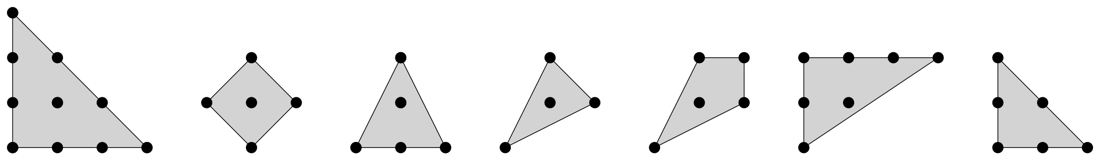
(2) Todos excepto tres de los puntos reticulares de P están en un paralelepípedo racional de anchura uno con respecto a cada par de facetas opuestas. (Politopos encajonados*; en inglés* boxed*).*

(3) P tiene (al menos) dos vértices u y v tales que tanto P^u como P^v tienen anchura mayor que uno, y tales que $P^{u,v}$ es aún 3-dimensional (Politopos fusionados*; en inglés* merged*).*

Debemos advertir que las descripciones de politopos tanto espinado como encajonado en el teorema anterior no son del todo precisas (para más detalles, véase la Sección 3.1.1). Para la parte de politopos fusionados, recuérdese que $P^u := \mathrm{conv}(P \cap \mathbb{Z}^3 \setminus \{u\})$, y que $P^{u,v} = (P^u)^v$ (véase la Observación I.20).

Dado ese resultado, procedemos a la clasificación de cada uno de estos grupos de manera distinta. Para politopos espinados obtenemos lo siguiente:

Teorema (Corolario 3.17)**.** *Sea P un 3-politopo espinado de tamaño al menos siete. Entonces P admite una proyección a uno de los siguientes polígonos reticulares, de tal manera que cada vértice en la proyección tiene una única anti-imagen en P.*



Recuérdese que, por el Teorema I.23, hay sólo un número finito de 3-politopos espinados con un cierto tamaño. Fijado Q uno de esos siete polígonos reticulares, el hecho de que sólo un vértice de P puede proyectar a cada vértice de Q, y el requerimiento extra (que no se especifica en el teorema) de que a lo sumo P puede tener otro vértice más (proyectando a un punto que no es vértice de Q), el número de posibilidades es muy limitado y hace posible una descripción explícita de cada posible P, para cada Q y N dados (Teoremas 3.18 y 3.19).

En el caso de 3-politopos encajonados, una descripción más precisa es la siguiente: un 3-politopo reticular P de anchura mayor que uno es encajonado si $P \cap \mathbb{Z}^3 = A \cup \{v_1, v_2, v_3\}$ y existen funcionales primitivos afines f_1, f_2, f_3 tales que $f_i(A) \subseteq \{0,1\}$ y $f_i(v_j) \notin \{0,1\}$ si, y sólo si, $i = j$. El paralelepípedo mencionado en la parte 2 del Teorema I.4 es $\bigcap_i f_i^{-1}([0,1])$.

En particular, los 3-politopos encajonados tienen tamaño a lo sumo 11 (3 vértices más posiblemente los 2^3 vértices del paralelepípedo). Por el Teorema I.23, sólo existe un número finito para cada tamaño, y por tanto sólo un número finito de ellos en total. Primero estudiamos las posibilidades para los f_i, y luego, una vez fijados estos funcionales, acotamos el valor de $f_i(v_i)$, lo que nos deja sólo un número finito de posibilidades para cada v_i:

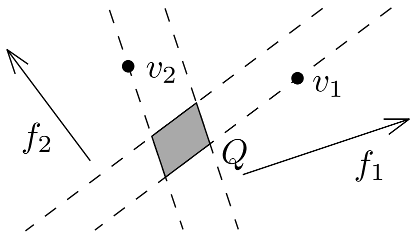


Figura I.11: Un polígono encajonado. Q es el paralelepípedo $f_1^{-1}([0,1]) \cap f_2^{-1}([0,1])$.

Teorema (Lema 3.22 y Teorema 3.26). *Si P es un 3-politopo encajonado, entonces sin pérdida de generalidad $f_1 = x$, $f_2 = y$, $f_3 = z$, ó $f_1 = y + z$, $f_2 = x + z$, $f_3 = x + y$.*

Además, $f_i(v_i) \in [-6, 7]$ para cada i.

Obsérvese que la primera elección de los f_i nos da como paralelepípedo el cubo unidad $[0,1]^3$, y la segunda elección da un paralelepípedo cuyos únicos puntos reticulares son los del tetraedro estándar unimodular Δ_3. Este resultado permite enumerar todos los 3-politopos encajonados a través de funciones implementadas en MATLAB (véase la Sección 3.3.3).

Finalmente, todo 3-politopo fusionado de tamaño N se puede obtener como la unión o fusión (*merging*) de los dos politopos P^u y P^v (de tamaño $N-1$ y anchura mayor que uno). El hecho de que $P^{u,v}$ (de tamaño $N-2$) sea 3-dimensional implica que sólo existe un número finito de maneras de fusionar dichos politopos:

Algoritmo I.30 (Algoritmo de fusionado en dimensión 3, Algoritmo 3.2).
INPUT: dos 3-politopos reticulares P_1 y P_2 de tamaño $N-1$ y anchura > 1.
OUTPUT: todos los 3-politopos reticulares de tamaño N que se obtienen fusionando P_1 y P_2.
Para cada vértice v_1 de P_1 y v_2 of P_2:

(1) Sean $P_1' = (P_1)^{v_1} \subset P_1$ y $P_2' = (P_2)^{v_2} \subset P_2$.

(2) Comprobar que tanto P_1' como P_2' son 3-dimensionales.

(3) Para cada equivalencia unimodular $t : \mathbb{R}^3 \to \mathbb{R}^3$ con $t(P_1') = P_2'$, si el tamaño de $P := \mathrm{conv}(t(P_1) \cup \{v_2\}) = \mathrm{conv}(\{t(v_1)\} \cup P_2)$ es N, añadir P al output. (Obsérvese que t puede no ser única, pero que sólo hay un número finito de posibilidades)

Véase la Figura I.12 para un esquema del fusionado en dimensión 2.

Es decir, que para obtener todos los 3-politopos fusionados de tamaño N, cogemos la lista (finita) de 3-politopos reticulares de anchura mayor que uno y tamaño $N-1$, que suponemos ya conocida, y para cada dos politopos de esa lista, enumeramos todos sus posibles fusionados.

Juntando estos tres grupos, hemos conseguido enumerar todos los 3-politopos reticulares de anchura mayor que uno y tamaño hasta 11 (hay 216 453; véase el Teorema 3.6).

Observación I.31. Pudiera parecer que los resultados de este capítulo hacen obsoletos los resultados de [BS16a, BS16b], pero no es el caso. Más bien lo contrario, necesitamos las clasificaciones de tamaños 5 y 6 como punto de partida del algoritmo, ya que, por diversas

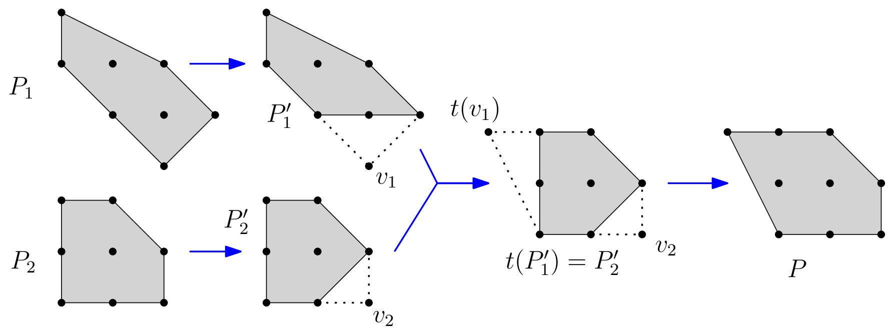


Figura I.12: Un polígono reticular P de tamaño 9 obtenido fusionando dos de tamaño 8.

razones, las técnicas utilizadas en el Capítulo 3 sólo sirven para tamaño al menos siete. Una de ellas es el hecho de que existe un único 3-politopo de anchura mayor que uno (y tamaño 6) que no pertenece a ninguno de los tres grupos antes descritos (véase el Teorema 3.15). Más importante es el hecho de que, el suponer que el tamaño sea al menos siete, simplifica considerablemente las clasificaciones de 3-politopos espinados y encajonados (Teorema 3.16 y Lema 3.22). En particular, clasificar los 3-politopos espinados y encajonados de tamaño 6 no habría sido más sencillo que repetir el trabajo realizado en [BS16b].

En la Sección 3.4 se da información más detallada sobre el resultado del algoritmo (las listas de 3-politopos reticulares de anchura mayor que uno y tamaño hasta 11) y algunos comentarios y observaciones que surgen de su estudio. A continuación hacemos un resumen de dicha sección.

Las Tablas 3.2 y 3.9 muestran los números de 3-politopos reticulares de cada tamaño según su número de vértices y de puntos reticulares interiores. Nuestros resultados coinciden con las clasificaciones de 3-politopos reticulares con 1 y 2 puntos reticulares interiores mencionadas en la Sección I.2.3. La Tabla 3.3 muestra los números de 3-politopos canónicos y terminales de tamaño hasta 11.

La Tabla 3.4 muestra la clasificación de acuerdo a la anchura de los politopos. La máxima anchura que alcanzan los politopos de cada tamaño es: 2 para tamaño 5, 3 para tamaños 6 a 9, y 4 para tamaños 10 y 11.

En la Sección 3.4.3 observamos los volúmenes que aparecen en politopos de cada tamaño. Experimentalmente observamos que, para tamaños $n = 5, \dots, 11$, existe siempre un único 3-politopo de anchura mayor que uno que maximiza el volumen, y es un tetraedro limpio de volumen $12(n-4)+8$ (un politopo reticular es *limpio* si los únicos puntos del borde son los vértices). Este tetraedro se puede generalizar a tamaño arbitrario (Proposición 3.35) y conjeturamos que es el único 3-politopo de anchura mayor que uno que maximiza el volumen para cada tamaño (Conjetura 3.36).

La mayoría de los politopos en nuestras listas son *primitivos*, con lo que nos referimos a politopos cuyos puntos reticulares generan afínmente el retículo. Cuando éste no es el caso, llamamos *índice subreticular* de un d-politopo P al índice del retículo generado por $P \cap \mathbb{Z}^d$ como subretículo de $\mathbb{Z}^d$. Experimentalmente observamos que existe un único 3-politopo reticular (de anchura mayor que uno y tamaño ≤ 11) con índice subreticular 5 (un tetraedro termina) y que los únicos otros índices subreticulares que aparecen son 2 y 3 (véase la Tabla 3.5).

Observación I.32. En un próximo artículo de Blanco–Santos [BS17] demostramos que lo mismo ocurre para todos los tamaños, y caracterizamos los 3-politopos reticulares de anchura mayor que uno que no son primitivos: hay un número lineal en n de índice 3 y un número cuadrático en n de índice 2, para cada tamaño n. Véase la Sección 3.4.4 para más detalles.

Es importante comentar que la principal herramienta utilizada para los resultados de [BS17] es la separación de los 3-politopos reticulares de anchura mayor que uno y tamaño al menos siete en los tres grupos del Teorema I.4. Para un tamaño fijo N, el índice subreticular de 3-politopos espinados se estudia a partir de la descripción explícita que se da de los mismos. En el caso de los 3-politopos encajonados, se comprueba la lista completa que tenemos de ellos. Finalmente, para los 3-politopos fusionados, utilizamos la información sobre el índice subreticular de los 3-politopos reticulares de anchura mayor que uno y tamaño hasta 11, y luego utilizamos inducción para estudiar el índice subreticular de los de tamaño arbitrario. Los politopos de anchura uno los estudiamos separadamente.

Creemos que esta idea tiene un gran potencial a la hora de estudiar ciertas propiedades o parámetros de los 3-politopos reticulares, siempre y cuando sea posible estudiar dicha propiedad en función de aquella de sus subpolitopos.

En la Sección 3.4.5 observamos también cuáles de nuestra lista de politopos son *normales* (un 3-politopo reticular P es normal si todo punto reticular $p \in 2P \cap \mathbb{Z}^3$ se puede escribir como suma de dos puntos reticulares de P). Experimentalmente parece que el porcentaje de politopos de cada tamaño que son normales no cambia mucho con el tamaño y se acerca al 13 %. No sabemos si esto sigue ocurriendo para tamaños mayores. También hemos comprobado (hasta tamaño 11) que todo 3-politopo normal de anchura mayor que uno tiene un vértice v tal que P^v es también normal (en [BGM16] se pregunta si este es siempre el caso).

Puesto que los 3-politopos dps tienen tamaño a lo sumo 8 (véase la Sección I.2.5), los resultados del Capítulo 3 completan la clasificación de 3-politopos dps. En la Sección 3.4.6 damos estos resultados. La Tabla 3.6 da los números de 3-politopos de anchura uno, para cada tamaño y cada número de vértices. En particular podemos contestar en dimensión 3 algunas de las preguntas de Reznick [BNR+08] sobre politopos dps. También observamos que los d-politopos dps para $d = 2, 3$ tienen a lo sumo $3 \cdot 2^{d-2}$ vértices y nos preguntamos si éste es el caso para cualquier dimensión (véase la Pregunta 3.38).

Programas de MATLAB (Apéndice A).

Finalmente, en el Apéndice A aparecen las implementaciones de los algoritmos utilizadas para los cálculos del Capítulo 3. Estos cálculos han sido hechos con el software MATLAB, y todos los algoritmos han sido implementados por nosotros. Cada uno de ellos está precedido de una descripción de la teoría que interviene en el algoritmo. El apéndice está organizado en tres partes:

- La Sección A.1 recopila los algoritmos que calculan información básica sobre los 3-politopos (envolvente convexa, vértices, puntos reticulares, etc) y los algoritmos que necesitamos para trabajar con listas de 3-politopos reticulares de anchura mayor que uno (equivalencia unimodular y anchura).

- La Sección A.2 se centra en los algoritmos específicos utilizados en la clasificación de 3-politopos reticulares de anchura mayor que uno y un tamaño dado $N \geq 7$. Esto incluye los algoritmos que enumeran los 3-politopos espinados y encajonados, y el algoritmo de fusionado que enumera los 3-politopos fusionados.

- La Sección A.3 se centra en el cálculo de otras propiedades o parámetros de los politopos que hemos clasificado y que nos parecen de interés. Mostramos la lista de propiedades calculadas para cada politopo, así como algunos ejemplos de cómo se almacena (y se lee) la información sobre un politopo concreto. Las propiedades que aquí se calculan son las que dan lugar a los comentarios y tablas en la Sección 3.4 del Capítulo 3.

En la página 186 aparece un listado de los algoritmos de esta sección.

Chapter 1

Introduction

The work on this PhD thesis is joint work with my supervisor Francisco Santos and (partially) with Christian Haase and Jan Hofmann, from the Free University of Berlin. The obtained results are contained in the papers [BHHS16] and [BS16c]. It is a continuation of the research presented in my Master's Thesis, later published at SIAM Journal on Discrete Mathematics as [BS16a] and [BS16b].

This introductory chapter is organized as follows:

Section 1.1 introduces basic concepts and parameters of lattice polytopes, main object of our study. All the general definitions and notations of the thesis appear in this section.

Section 1.2 is a summary of results on lattice polytopes and other classification approaches. We give in more detail those results needed in our research.

Section 1.3 describes our approach on the classification of lattice 3-polytopes. We already started this work in the papers [BS16a] and [BS16b], and in this section we summarize the main results there obtained. Those result that we use in this thesis will be presented with more detail.

Finally, Section 1.4 will be an outline of the rest of the thesis, with the main results and conclusions.

1.1 Polytopes. Lattice polytopes. Unimodular equivalence. Width

Two basic references on polytopes and convexity are [Gru07, Zie95].

Definition 1.1 (Polytope)**.** A *polytope* $P \subset \mathbb{R}^d$ can be defined in two equivalent ways:

- A bounded polyhedron.
- The convex hull of finitely many points.

A *polyhedron* is the intersection of finitely many affine halfspaces. Remember that an *affine halfspace* is defined as the region $\{x \in \mathbb{R}^d \mid f(x) \geq 0\} \subset \mathbb{R}^d$, where $f : \mathbb{R}^d \to \mathbb{R}$ is an affine functional

$$f(x_1, \ldots, x_d) = a_0 + a_1x_1 + \ldots a_dx_d, \text{ for } a_i \in \mathbb{R}$$

On the other hand, the *convex hull* of a set $K \subseteq \mathbb{R}^d$ is the smallest convex set that contains K, and is denoted by $\operatorname{conv}(K)$:

$$\operatorname{conv}(K) := \bigcap_{C \text{ convex}, \ K \subseteq C} C$$

Equivalently, the convex hull of K is also the set of all *convex combinations* of points of K:

$$\operatorname{conv}(K) = \left\{ \lambda_1 p_1 + \cdots + \lambda_n p_n \;\middle|\; \lambda_i \geq 0, \ \sum_{i=1,\ldots,n} \lambda_i = 1, \ p_i \in K, \ n \in \mathbb{N} \right\}.$$

For example, the convex hull of two points x and y is $\operatorname{conv}\{x,y\} = \{\lambda x + (1-\lambda)y \mid \lambda \in [0,1]\}$ (the straight segment joining x and y). And the convex hull of three (not aligned) points is the triangle they form. For a given (short) list of points $p_1, \ldots, p_d \in \mathbb{R}^d$, we may sometimes denote their convex hull as $p_1 p_2 \ldots p_d := \operatorname{conv}\{p_1, \ldots, p_d\}$.

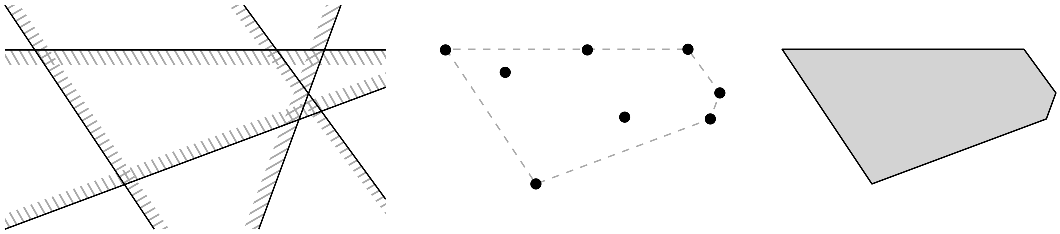


Figure 1.1: A set of halfspaces and a set of points that define the same polytope.

We say that $K \subset \mathbb{R}^d$ is a *convex body* if it is a compact set such that $K = \operatorname{conv}(K)$.

The *dimension* of a polytope $P \subset \mathbb{R}^d$ (or of any convex body) is the dimension of the affine space it spans:

$$\operatorname{aff}(P) := \left\{ \lambda_1 p_1 + \cdots + \lambda_n p_n \;\middle|\; \sum_{i=1,\ldots,n} \lambda_i = 1, \ p_i \in P, \ n \in \mathbb{N} \right\}$$

We will say d-polytope whenever P is of dimension d, although P may be embedded in a larger dimensional space. Polytopes of dimension one are called *segments* and those of dimension two are *polygons*.

A *face* of a d-polytope $P \subset \mathbb{R}^d$ is the intersection $P_f := P \cap \{x \in \mathbb{R}^d \mid f(x) = 0\}$ of P with a valid hyperplane. We say that a hyperplane $\{x \in \mathbb{R}^d \mid f(x) = 0\}$, where $f : \mathbb{R}^d \to \mathbb{R}$ is a non-constant affine functional, is a *valid hyperplane* of a polytope P if $f(P) \subset [0, \infty)$. A *supporting hyperplane* is a valid hyperplane with $0 \in f(P)$. Notice that a face of a polytope is a polytope itself. A 0-dimensional face is called a *vertex* and a 1-dimensional face is an *edge*. The faces of dimension $d-1$ are called *facets* and those of dimension $d-2$ are *ridges*. The empty set is always a face of P, by choosing any hyperplane that does not intersect P, and by convention P itself is a face for the affine functional $f(x) = 0$, $\forall x \in \mathbb{R}^d$. A *proper face* is any face $F \subsetneq P$.

The *boundary* of a polytope P is the set ∂P of all points $x \in P$ that belong to some proper face. The *interior* of P is its topological interior, and is denoted $\operatorname{int}(P)$. We call *relative interior* of $P \subset \mathbb{R}^d$, and denote it by $\operatorname{relint}(P)$, the topological interior of P in $\operatorname{aff}(P)$. If P is full-dimensional, the interior and the relative interior coincide. When

$P \subset \mathbb{R}^d$ a lower dimensional polytope, its interior is the empty set. We say that $x \in P$ is an *interior point* if $x \in \text{int}(P)$ in the case of P full-dimensional, or if $x \in \text{relint}(P)$ otherwise. The same way, x is a *boundary point* if x is in $\text{int}(P)$ or ∂P, respectively.

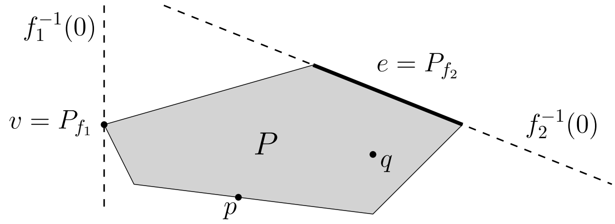


Figure 1.2: A polygon P and two of its faces: v is a vertex (1-dimensional face) and e is an edge and a facet (1 and $(d-1)$-dimensional face). Point p is in the boundary of P and q is an interior point.

A *simplex* or *d-simplex* is a d-dimensional polytope with $d+1$ vertices. All 1-polytopes are simplices. A 2-simplex is a triangle, and a 3-simplex is a tetrahedron.

The objects of study in this thesis are lattice polytopes. A *linear lattice* $\Lambda \subset \mathbb{R}^d$ is any discrete additive subgroup. In particular, it is isomorphic to $\mathbb{Z}^k$, where k the *dimension* of Λ. That is, for k linearly independent vectors in $\mathbb{R}^d$, the set of all linear integer combinations of those vectors is a linear lattice of dimension k. Any translation of a linear lattice is an *affine lattice*, or simply a *lattice*. Unless otherwise specified we work in the *standard lattice* $\mathbb{Z}^d$.

Definition 1.2 (Lattice polytope)**.** A *lattice polytope* is a polytope with vertices in $\mathbb{Z}^d$ (or in a specified lattice Λ). That is, it is the convex hull of finitely many lattice points.

A *lattice* or *integer point* is a point $p \in \mathbb{Z}^d$. The set of lattice points of a lattice polytope P (or any other body) is $P \cap \mathbb{Z}^d$ (and it is finite). For $P \subset \mathbb{R}^d$ a lattice polytope, we will call *size* of P the natural number $\#(P \cap \mathbb{Z}^d)$.

Notice that if P is a lattice polytope, then it is the intersection of finitely many affine integer halfspaces (the affine functional has integer coefficients), but the converse is not true since integer hyperplanes may intersect in non-integer points.

A *unimodular* simplex is a simplex whose vertices are an affine integer basis for the lattice (see Figure 1.3). The *standard unimodular simplex* is $\Delta_d := \text{conv}\{o, e_1, \ldots, e_d\} \subset \mathbb{R}^d$, where o is the origin, and e_i is the i-th standard basis vector.

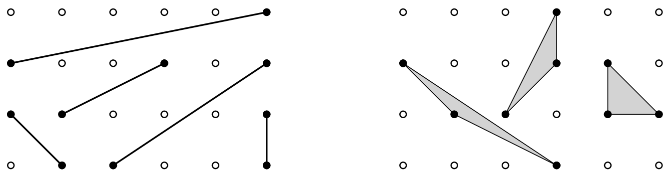


Figure 1.3: The figure in the left shows unimodular lattice segments embedded in $\mathbb{R}^2$. Their normalized volume is one in all of them, even though their 2-dimensional volume is 0. The figure in the right shows unimodular lattice triangles.

Unless otherwise specified, by *volume* of a lattice d-polytope $P \subset \mathbb{R}^d$ we mean its Euclidean volume multiplied by a factor $d!$ (or, for a lattice Λ, by the factor that makes unimodular simplices of Λ have volume 1). This is commonly referred to as *normalized volume*. This normalization makes the volume of every lattice polytope to be an integer.

Observe that for a lower dimensional lattice polytope $P \subset \mathbb{R}^d$ the natural way to define its volume is with respect to the lattice $\Lambda := \operatorname{aff}(P) \cap \mathbb{Z}^d$ (as opposed to its "d-dimensional" volume, which would be zero). Notice that for lower dimensional lattice polytopes the proportionality constant between Euclidean and normalized volume depends on $\operatorname{aff}(P)$. See for example Figure 1.3, where the segments in the left figure are all of volume one, whereas their Euclidean length is different in each case.

The volume of a lattice simplex $T = \operatorname{conv}\{p_1, \dots, p_{d+1}\} \subset \mathbb{R}^d$, for $p_i \in \mathbb{Z}^d$, is sometimes called its *determinant*, since we have:

$$\operatorname{vol}(T) = \left| \det \begin{pmatrix} 1 & \dots & 1 \\ p_{i_1} & \dots & p_{i_{d+1}} \end{pmatrix} \right|$$

The following is the most natural notion of affine equivalence between lattice polytopes:

Definition 1.3 (Unimodular equivalence)**.** A *unimodular equivalence*, or just *equivalence*, is an affine transformation $t : \mathbb{R}^d \to \mathbb{R}^d$ that preserves the lattice.

Two lattice polytopes P and Q are called *unimodularly equivalent* or just *equivalent* if there exists a unimodular equivalence that maps one polytope to the other. In this case we write $P \cong Q$.

For the standard lattice $\mathbb{Z}^d$, a unimodular equivalence is any map of the form $t(x) = Mx + b$ where $M \in \mathbb{Z}^{d \times d}$ is an integer matrix of $\det(M) = \pm 1$, and $b \in \mathbb{Z}^d$ is a translation vector. When we say that there are only finitely many lattice polytopes satisfying certain properties, we will always mean modulo unimodular equivalence. See Algorithm A.7 for our MATLAB routine that computes unimodular equivalence between two lattice 3-polytopes. The algorithm used can be extended to arbitrary dimension.

In the next definition K is a convex body and we consider the standard lattice $\mathbb{Z}^d$.

Definition 1.4 (Lattice width)**.** The *width* of a convex body $K \subset \mathbb{R}^d$ *with respect to an affine functional* $f : \mathbb{R}^d \to \mathbb{R}$ is defined as

$$\operatorname{width}_f(K) := \max_{p,\, q \in K} |f(p) - f(q)| \,.$$

Observe that this equals $\max_{p,\, q \in K} |\hat{f}(p) - \hat{f}(q)|$, where $\hat{f}$ is the linear functional parallel to f. The *lattice width* (or just *width*) of K, denoted $\operatorname{width}(K)$, is the minimum such $\operatorname{width}_f(K)$ where f ranges over all non-constant integer functionals. That is:

$$\operatorname{width}(K) := \min_{f \in (\mathbb{Z}^d)^* \setminus \{0\}} \operatorname{width}_f(K).$$

Observe that the width of $K \subset \mathbb{R}^d$ is zero if, and only if, K is lower dimensional. If we want to refer to the width of such a K with respect to the lattice $\operatorname{aff}(K) \cap \mathbb{Z}^d$ we will explicitly say so.

In the case of lattice polytopes, the width is an integer. See Algorithm A.9 for the computer routine that computes the width of a lattice polytope.

For the case of a general lattice $\Lambda \subset \mathbb{R}^d$ the role of linear integer functionals is played by the elements of the *dual lattice*

$$\Lambda^* := \left\{ f \in (\mathbb{R}^d)^* \mid f(\Lambda) \subseteq \mathbb{Z} \right\} \subset (\mathbb{R}^d)^*,$$

where, as usual, $(\mathbb{R}^d)^* := \left\{ f : \mathbb{R}^d \to \mathbb{R} \text{ linear functional } \right\} \simeq \mathbb{R}^d$. Observe that under the canonical isomorphism $(\mathbb{R}^d)^* \simeq \mathbb{R}^d$ we have $(\mathbb{Z}^d)^* \simeq \mathbb{Z}^d$. Via these isomorphisms, the image of a point $x \in \mathbb{R}^d$ by a linear functional $f \in (\mathbb{R}^d)^*$ will indistinctly be written $f(x)$ or $f \cdot x$.

Remember that the *polar of a convex body* $K \subset \mathbb{R}^d$ is:

$$K^\vee := \{f \in (\mathbb{R}^d)^* \mid f \cdot x \le 1 \; \forall x \in K\} \; \subset (\mathbb{R}^d)^*$$

which is itself convex. If the origin is in the interior of K (which in particular implies that K is full-dimensional), then $K^\vee$ is bounded and $(K^\vee)^\vee = K$. The *polar of a polytope* $P \subset \mathbb{R}^d$ is a polyhedron, and if the origin is in the interior of P, then $P^\vee$ is also a polytope and $(P^\vee)^\vee = P$. By convexity, we also have that $P^\vee = \{f \in (\mathbb{R}^d)^* \mid f \cdot v \le 1 \; \forall v \in \operatorname{vert}(P)\}$. That is, $(v_1, \dots, v_d) \in \mathbb{R}^d$ is a vertex of P if, and only if, $v_1 x_1 + \cdots + v_d x_d \le 1$ is the equation of a facet of $P^\vee \subset \mathbb{R}^d$. Since $(P^\vee)^\vee = P$, duality establishes a bijection between vertices of P and facets of $P^\vee$ and between vertices of $P^\vee$ and facets of P.

In the literature, the *polar body* is sometimes called *dual*, *polar reciprocal*, or *polar dual*. Also, the polar is sometimes defined to be $\{f \in (\mathbb{R}^d)^* \mid f \cdot x \ge -1 \; \forall x \in K\}$, which is the same definition up to central symmetry from the origin.

In terms of duality, the width has the following interpretation. In the statement, λQ, for $Q \subset \mathbb{R}^d$ a convex body, denotes the *dilation* of Q by the factor λ, which is the set $\lambda Q := \{\lambda x \mid x \in Q\}$. For $i \in \mathbb{N}$, we call iQ the *i-th dilation* of Q.

Proposition 1.5. *The width of a convex body $K \subset \mathbb{R}^d$ with respect to a lattice $\Lambda \subset \mathbb{R}^d$ equals the minimum $\lambda > 0$ such that $\lambda(K-K)^\vee$ contains a non-zero lattice point $f \in \Lambda^*$.*

Proof. Let $f \in \Lambda^* \setminus \{0\}$, the width of K with respect to f is

$$\operatorname{width}_f(K) = \max_{x,y \in K} |f(x) - f(y)| = \max_{x,y \in K} |f(x-y)| = \max_{z \in K-K} |f(z)|$$

Let $\lambda > 0$. Then

$$\operatorname{width}_f(K) \le \lambda \quad \Longleftrightarrow \quad |f(z)| \le \lambda, \; \forall z \in K-K \quad \Longleftrightarrow \quad f \in \lambda \cdot (K-K)^\vee$$

where $\lambda(K-K)^\vee := \{f \in (\mathbb{R}^3)^* \mid f(z) \le \lambda, \forall z \in K-K\} \subset (\mathbb{R}^3)^*$. That is, the width of K with respect to f is the minimum $\lambda > 0$ for which $f \in \lambda(K-K)^\vee$. The statement follows since the width of K is the minimum of those, over $f \in \Lambda^* \setminus \{0\}$. $\square$

Put differently, the proposition says that the width of K, equals the first successive minimum of $(K-K)^\vee$. (Remember that the *first successive minimum* of a centrally symmetric convex body C is precisely the minimum λ such that $\lambda C \cap (\Lambda^* \setminus \{0\}) \neq \emptyset$ [Gru07]).

Remember that the vector of coefficients $(a_1, \dots, a_d) \in \mathbb{R}^d$ of a linear functional is orthogonal to the hyperplane $\{(x_1, \dots, x_d) \in \mathbb{R}^d \mid a_1 x_1 + \cdots a_d x_d = 0\}$. In the case of a lattice hyperplane, the orthogonal vector is integer and can be chosen primitive, where a vector or point $x \in \mathbb{Z}^d$ is *primitive* if the gcd of its coordinates is one. In turn, we say that a linear functional $f \in (\mathbb{Z}^d)^*$ is *primitive* if its vector of coefficients is primitive and that an affine functional f is primitive if its corresponding linear functional $\hat{f}$ is primitive. Observe that a functional f is primitive if, and only if, $f(\mathbb{Z}^d) = \mathbb{Z}$.

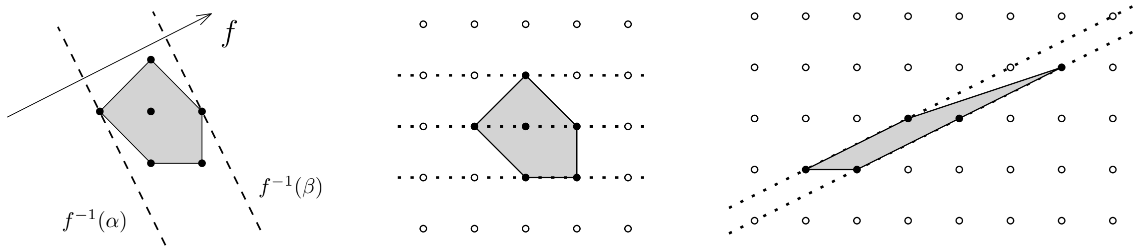


Figure 1.4: The figure in the left shows the width $|\beta - \alpha|$ of a lattice polygon with respect to some functional f. The other two figures are lattice polygons enclosed between lattice hyperplanes.

A *lattice subspace* is an affine subspace S such that $\dim(S) = \dim(S \cap \mathbb{Z}^d)$. Equivalently, such that $S \cap \mathbb{Z}^d$ is a lattice of dimension $\dim(S)$. Observe that a *lattice hyperplane* is a hyperplane defined by an integer affine equation, but a lower dimensional affine space defined by integer equations may not be a lattice subspace.

We define as *lattice distance*, or simply *distance*, between a lattice point $x \in \mathbb{Z}^d$ and a lattice subspace $S \subset \mathbb{R}^d$, the width of $\operatorname{conv}(S \cup \{x\})$ measured in the affine subspace $\operatorname{aff}(S \cup \{x\})$. By lattice distance from $x \in \mathbb{Z}^d$ to a lattice polytope $P \subset \mathbb{R}^d$ we mean the distance between x and $\operatorname{aff}(P)$. By distance between two parallel lattice hyperplanes H_1 and H_2 we mean the distance from a point $x \in H_1$ to H_2, which is independent of x. In dimension 3, by distance between two non coplanar lattice segments (or lines) we mean the distance between the unique pair of parallel lattice hyperplanes containing each of the segments (or lines). Notice that the width of a lattice polytope P equals the minimum distance between parallel lattice hyperplanes enclosing the polytope (see Figure 1.4). Observe also that the volume of a simplex equals the volume of any facet F of it times the distance from F to the opposite vertex.

We call *lattice length*, or simply *length* of a lattice segment $\operatorname{conv}\{x, y\}$ the lattice distance between the points x e y, measured in the 1-dimensional affine space they span. Notice that the lattice length of a segment equals its volume, and also its width.

Remark 1.6. All these properties of lattice polytopes (size, volume, dimension, width, etc) are invariant under unimodular equivalence.

1.2 Classifications of lattice polytopes

Lattice polytopes appear in several fields of mathematics, like for example geometry, combinatorics, optimization and algebraic geometry.

Lattice polytopes appear as the convex hull of the integer solutions in optimization problems. An important matter in optimization is the problem of being able to decide whether a convex body has or not lattice points (that is, if there exists an integer solution to the optimization problem). Or even to be able to count how many lattice points (how many integer solutions) are there in said convex body. The concept of width is also of importance here, since convex bodies without lattice points are somehow “flat”.

The number of lattice points in each of the natural dilations of some lattice polytope agrees with a polynomial on the dilation factor (the Ehrhart polynomial), and through the sequence of these values and its generating function, interesting relations are found between combinatorial objects and the geometry of lattice polytopes.

In algebraic geometry, lattice polytopes correspond to toric varieties, and combinatorial and geometric concepts of lattice polytopes have their algebraic equals.

In this section we are going to review some of the classifications of certain classes of lattice polytopes. More specifically, the classifications listed here are strongly related with the research in this thesis, which is the study and classification of lattice polytopes according to their number of lattice points and their width.

1.2.1 Hollow polytopes

A convex body with no lattice points in its interior is called *lattice-free*. A *hollow* polytope is a lattice-free lattice polytope.

The width of lattice-free convex bodies is bounded by the so-called "flatness theorem", dating back to Khinchine (1948); see, e.g., [KL88]. The best known upper bound is:

Theorem 1.7 (Banaszczyk et. al, [BLPS99, Theorem. 2.4])**.**
The lattice width of lattice-free convex bodies is $\leq O(d^{\frac{3}{2}})$.

There exist hollow d-polytopes of width d (take the d-th dilation of the standard unimodular simplex). That is, if $w_H(d)$ denotes the maximum width among that of a hollow d-polytopes, then

$$d \leq w_H(d) \leq O(d^{\frac{3}{2}}). \tag{1.1}$$

In dimension one, the unique hollow segment is that of length one (the only lattice points are the endpoints of the segment). In every dimension, every lattice polytope of width one is hollow, since there are no lattice points strictly between the two hyperplanes at distance one. More generally, any lattice d-polytope admitting a lattice projection to a hollow k-polytope, is hollow itself. Here, a *lattice projection* is an affine map $\pi : \mathbb{R}^d \to \mathbb{R}^k, \pi(p) = A \cdot p + b$, with $\pi(\mathbb{Z}^d) = \mathbb{Z}^k$.

Theorem 1.8 (Classification of hollow polygons, see e.g. [Tre08] or [Ark80])**.** *A hollow polygon is either of width one, or is equivalent to* $2\Delta_2$*, the second dilation of the standard unimodular triangle.*

In dimension 3, the behavior of hollow 3-polytopes is heavily related to the classification of hollow polygons.

Theorem 1.9 (Treutlein, [Tre08, Theorem 1.3])**.** *A hollow* 3*-polytope falls exactly under one of the following categories:*

(1) It has width 1*.*

(2) It has width 2 *and admits a projection onto the polygon* $2\Delta_2$*.*

(3) It has width ≥ 2*, and does not admit a projection to* $2\Delta_2$*. There are only finitely many of them and they are all contained in hollow-maximal* 3*-polytopes.*

In the previous theorem, a *hollow-maximal* d-polytope is a hollow d-polytope not properly contained in another hollow d-polytope.

Remark 1.10. *Notice that there exist hollow polytopes not contained in any hollow-maximal polytope: the rectangle* $\operatorname{conv}\{(0,0),(0,1),(2,0),(2,1)\}$ *of width one is not contained in a hollow-maximal polygon, since every hollow polygon containing it is, in turn, contained in the hollow unbounded region* $\{0 \le y \le 1\}$*.*

Nill–Ziegler extend to arbitrary dimension the relation between hollow d-polytopes and hollow $(d-1)$-polytopes.

Theorem 1.11 (Nill-Ziegler, [NZ11, Theorem 1.2, Corollary 1.3])**.**
All but finitely many hollow d-polytopes admit a lattice projection to a hollow $(d-1)$-polytope. Moreover, if a hollow d-polytope does not admit such a projection, then it is contained in a hollow-maximal d-polytope.

As a consequence, only finitely many hollow d-polytopes have width $> w_H(d-1)$.

The hollow-maximal 3-polytopes referred to in part 1.9 of Theorem 1.9 have been classified in [AWW11, AKW15]. More precisely, Averkov, Wagner and Weismantel [AWW11] classified the hollow 3-polytopes that are not properly contained in any other lattice-free convex body. Then Averkov, Krümpelmann and Weltge [AKW15] showed that the maximal lattice 3-polytopes in this sense (which they call $\mathbb{R}$-maximal) coincide with the hollow-maximal 3-polytopes in our sense (which they call $\mathbb{Z}$-maximal). It is known that the two notions of maximal hollow polytopes do not coincide in dimensions four and higher [NZ11, Theorem 1.4].

Theorem 1.12 ([AWW11, Theorem 2.2] and [AKW15, Theorem 1])**.** *There are the following* 12 *hollow-maximal* 3*-polytopes:*

$$\mathcal{M}_1 \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 6 \end{pmatrix} \qquad \mathcal{M}_2 \begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \mathcal{M}_3 \begin{pmatrix} 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_4 \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \qquad \mathcal{M}_5 \begin{pmatrix} 0 & 1 & 2 & 3 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 5 \end{pmatrix} \qquad \mathcal{M}_6 \begin{pmatrix} 0 & 3 & 1 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_7 \begin{pmatrix} 0 & 4 & 1 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix} \quad \mathcal{M}_8 \begin{pmatrix} 2 & -2 & 0 & 0 & 1 \\ 0 & 0 & 2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad \mathcal{M}_9 \begin{pmatrix} -1 & 2 & 0 & 0 & 1 \\ 0 & 0 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 \end{pmatrix}$$

$$\mathcal{M}_{10} \begin{pmatrix} 1 & 0 & -1 & 2 & 1 & 0 \\ 0 & 1 & -1 & 2 & 3 & 1 \\ 0 & 0 & 0 & 3 & 3 & 3 \end{pmatrix} \qquad \mathcal{M}_{11} \begin{pmatrix} 1 & -1 & 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 2 & 2 \end{pmatrix}$$

$$\mathcal{M}_{12} \begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 0 & 2 & 1 \\ 0 & 1 & 1 & 2 & 1 & 2 & 2 & 3 \\ 0 & 0 & 0 & 0 & 2 & 2 & 2 & 2 \end{pmatrix}$$

They all have width two except $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$, of width three.

In Chapter 2 we will be particularly interested in hollow 3-polytopes of width larger than two. By Theorem 1.11 (Nill–Ziegler), any such polytope has to be a (non necessarily proper) subpolytope of one of the five hollow-maximal 3-polytopes of width 3 in

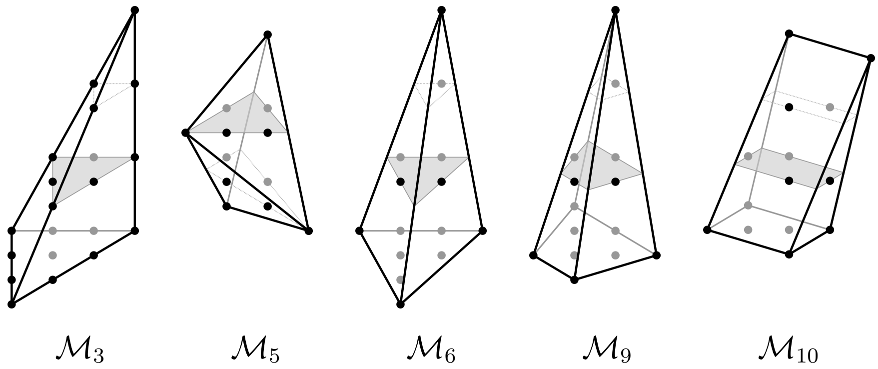


Figure 1.5: The five hollow 3-polytopes of width three. This picture has been taken from Averkov et al [AKW15].

Theorem 1.12. These five polytopes appear in Figure 1.5 (this picture has been taken from [AKW15]), and the following proves that they are actually the only ones of width 3:

Corollary 1.13. *The only hollow* 3*-polytopes of width* > 2 *are* $\mathcal{M}_3$*,* $\mathcal{M}_5$*,* $\mathcal{M}_6$*,* $\mathcal{M}_9$ *and* $\mathcal{M}_{10}$*, and they have width three.*

Proof. It suffices to check that all the proper subpolytopes of $\mathcal{M}_3$, $\mathcal{M}_5$, $\mathcal{M}_6$, $\mathcal{M}_9$ and $\mathcal{M}_{10}$ obtained removing a single vertex have width two (or lower). In some of them this is evident from Figure 2.10. In the two that seem less clear to us, namely $\mathcal{M}_5$ and $\mathcal{M}_{10}$, Figure 1.6 shows projections of them for which the coordinate functionals prove this fact (a single projection for $\mathcal{M}_5$; three projections for $\mathcal{M}_{10}$, each showing it for two of the vertices). $\square$

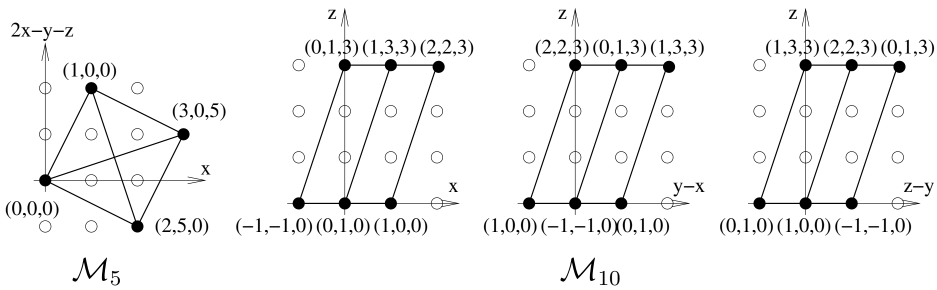


Figure 1.6: Projections showing that all proper subpolytopes of $\mathcal{M}_5$ and $\mathcal{M}_{10}$ have width at most two.

Corollary 1.14. $w_H(d) \geq d$ *for all* d*, with equality for* $d = 1, 2, 3$.

1.2.2 Empty polytopes

An *empty* polytope is a lattice polytope whose only lattice points are its vertices. They are a especial case of hollow polytopes.

In dimension 1, the only empty segment is the unimodular segment. In dimension 2, if P is an empty polygon, then it is contained in a parallelogram of Euclidean volume one (which is equivalent to the unit square). This result, and the one in dimension 3, is given in terms of maximality. An *empty-maximal* d-polytope is an empty d-polytope that is not properly contained in another empty d-polytope.

Empty d-polytopes have size at most 2^d, since otherwise they contain three aligned points and hence are not empty. As a consequence, every empty d-polytope is contained in an empty-maximal d-polytope (unbounded empty regions cannot exist; see Remark 1.10). In dimension 3 every empty-maximal 3-polytope has eight vertices:

Theorem 1.15 (Howe's Theorem [Sca85])**.** *If P is an empty 3-polytope, then P is contained in a lattice polytope of width one having a parallelogram of Euclidean volume one in each of the hyperplanes.*

Empty simplices are the "building blocks" of lattice polytopes, in the sense that they do not properly contain other full-dimensional lattice polytopes, and every lattice polytope can be subdivided into empty simplices.

In dimensions 1 and 2, the unique empty simplex is the unimodular simplex. In dimensions 3 and higher, this is no longer true:

Example 1.16 (Reeve tetrahedra, [Ree57])**.** *The family $\{T_r\}_{r\in\mathbb{Z}_{>0}}$, where*

$$T_r := \operatorname{conv}\{(0,0,0),(1,0,0),(0,1,0),(1,1,r)\},$$

is an infinite family of empty tetrahedra, and the volume of each T_r is r.

White worked out the full classification:

Theorem 1.17 (Classification of empty tetrahedra, White 1964 [Whi64])**.** *Every empty tetrahedron of volume $q \in \mathbb{N}$ is unimodularly equivalent to*

$$T(p,q) := \operatorname{conv}\{(0,0,0),(1,0,0),(0,0,1),(p,q,1)\},$$

for some $p \in \mathbb{Z}$ with $\gcd(p,q)=1$. Moreover, $T(p,q)$ is equivalent to $T(p',q)$ if, and only if, $p' = \pm p^{\pm 1} \pmod q$.

All empty tetrahedra have width one with respect to a pair of opposite edges (the distance between the edges is one). Only the empty tetrahedra of volumes 1 or 2 ($q \in \{1,2\}$) have width one width respect to all three pairs of opposite edges, and an infinite family of empty tetrahedra (for $p \in \{1, q-1\}$) has width one with respect to two pairs of opposite edges.

In dimension 4 there exist again infinitely many empty simplices, but they no longer have width bounded by one:

Example 1.18 (Haase–Ziegler, [HZ00, Proposition 6])**.** *For every $D \geq 8$ with $\gcd(D,6) = 1$, the lattice 4-simplex $\operatorname{conv}\{e_1,e_2,e_3,e_4,(2,2,3,D-6)\}$ is empty and has width 2.*

As a byproduct of our work in Chapter 2 we prove that, in contrast, there are only finitely many empty 4-simplices of width larger than two. The full classification of them has later been proven by Iglesias–Santos [IS17]. (This list was already known by Haase-Ziegler [HZ00], but it was not known to be complete.)

Summing up, let $w_E(d)$ denote the maximum width among that of empty d-polytopes, then $w_E(1) = w_E(2) = w_E(3) = 1$ and $w_E(4) \geq 4$. For general d, a lower bound is $2\lfloor d/2 \rfloor - 1 \leq w_E(d)$. An empty simplex of this width was described by Sebő [Seb99, Equations 1.1 and 1.2]. An upper bound is for example the upper bound for hollow polytopes (see Section 1.2.1 above).

1.2.3 Lattice polytopes with one interior lattice point

For every d and $k > 0$, the number of lattice d-polytopes with k interior lattice points is finite. This fact follows from the combination of these two theorems:

Theorem 1.19 (Hensley, [Hen83, Theorem. 3.6])**.** *For fixed d and $k > 0$, there is a bound on the volume of lattice d-polytopes with k interior lattice points.*

Theorem 1.20 (Lagarias–Ziegler, [LZ91, Theorem. 2])**.** *For fixed d and V, the number of equivalence classes of lattice d-polytopes with volume bounded by V is finite.*

In dimension 1, the unique lattice segment with exactly k interior lattice points is a segment of length $k + 1$. In dimension 2, Figure 1.7 shows the classification of all lattice polygons with exactly one interior point (there are 16 of them).

Lattice polytopes with exactly one interior lattice point are called *canonical*, and they are of great importance in algebraic geometry (where the interior point is generally assumed to be the origin). Canonical polytopes all of whose boundary lattice points are vertices are called *terminal*.

Canonical 3-polytopes were fully enumerated by Kasprzyk [Kas10] (there are 674 688 of them; the data for this and a lot more can be found in the graded ring database www.grdb.co.uk). In more recent work, Balletti and Kasprzyk classified all lattice 3-polytopes with 2 interior points (there are 22 673 449 of them; see [BK16]).

1.2.4 Reflexive polytopes

A *reflexive* d-polytope is a lattice d-polytope P for which there exist an interior lattice point $p \in \operatorname{int}(P) \cap \mathbb{Z}^d$ such that the lattice distance from p to every facet is 1. By definition, this interior point has to be unique, and is generally assumed to be the origin. This concept was first introduced in [Bat94].

In dimensions 1 and 2, it so happens that a lattice polytope is reflexive if, and only if, it has exactly one interior point. However, in dimensions 3 and and higher, lattice polytopes with exactly one interior point need not be reflexive. The explanation for this is fairly simple. Let P be a lattice polytope with exactly one interior point p. Then, all the pieces $\operatorname{conv}(F \cup \{p\})$, for F a facet of P, have all lattice points except p in F. The only way this can happen in dimensions 1 and 2 is if the distance from p to F is one (trivial in dimension 1; see Lemma 1.24 for the case in dimension 2), whereas in dimensions larger than two there exist such pyramids with this distance being larger than one. The Reeve tetrahedra are examples for that in dimension 3 (see Example 1.16).

An interesting feature about reflexive polytopes comes through duality: *a lattice polytope P is reflexive if, and only if, its polar $P^\vee$ is a lattice polytope.* This is a trivial fact from the bijection between the facets of P and the vertices of $P^\vee$. Let P have the origin in its interior and $v \in \mathbb{R}^d$ a vertex of $P^\vee$. Then v is of the form $v = \frac{1}{a_0}(a_1, \dots, a_d)$, where $x_1 a_1 + \dots + x_d a_d = a_0$, with $a_0 > 0$, is the equation of the supporting hyperplane of a facet F of P. If P is a lattice polytope, the coefficients of the equation are integer and, without loss of generality, we can choose $a_0, a_1, \dots, a_d \in \mathbb{Z}$ such that $\gcd(a_1, \dots, a_d) = 1$ (notice that this does not change v). Now, if P is reflexive then the distance from the origin to the facet F is one. That is, the functional $f(x_1, \dots, x_d) = x_1 a_1 + \dots + x_d a_d$ (which is primitive) takes value 0 in the origin and 1 in the facet F. Hence $a_0 = 1$ and $v \in \mathbb{Z}^d$.

In general, the polar of any lattice polytope needs not be lattice. Notice also that, since $(P^\vee)^\vee = P$, then P is reflexive if, and only if, $P^\vee$ is reflexive. That is, reflexive polytopes come in pairs, and there are only finitely many such pairs in each dimension (they have exactly one interior point, and lattice d-polytopes with a fixed positive number of interior points are only finitely many, as explained in Section 1.2.3).

There exist exactly 1, 16, 4319 and 473 800 776 reflexive d-polytopes, for dimensions 1 to 4, respectively. Those of dimensions 3 and 4 where classified by Kreuzer–Skarke in [KS98] and [KS00], respectively. Figure 1.7 shows the reflexive polygons.

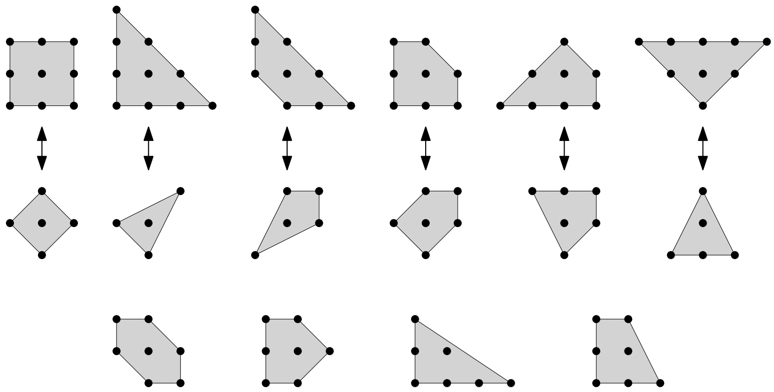


Figure 1.7: Classification of reflexive polygons, which coincide with lattice polygons with exactly one interior lattice point. There are six reflexive polar pairs plus four self-dual reflexive polygons.

Because the number of reflexive polytopes dramatically increases with the dimension, efforts have been directed into classifying a subclass of those, namely the smooth reflexive polytopes. A *smooth* polytope is a lattice polytope that is *simple* (each vertex is contained in exactly d facets) and such that the cone at each vertex is unimodular. Without going into detail of what a cone at a vertex is, let us explain what we mean. Any vertex v of a simple d-polytope P belongs to exactly d edges. Take the simplex S_v whose vertices are v and the first lattice point in each of those d edges starting at v. We say that the cone of P at v is unimodular if S_v is a unimodular simplex.

In dimension 1, the unique reflexive segment is smooth. In dimension 2, the first four polygons (starting from the left) in the top row of Figure 1.7 are smooth, and also the self-dual hexagon (left-most polygon in the last row).

Reflexive smooth polytopes where classified in [Øbr07] (Øbro) up to dimension 8. Lorenz and Paffenholz [LP08]) classified them in dimension 9.

There are the following numbers of smooth reflexive d-polytopes, for $d \in \{3, \dots, 9\}$: 18, 124, 866, 7 622, 72 256, 749 892 and 8 229 721. This classification led to new discoveries about smooth reflexive polytopes in arbitrary dimension and to solve long-open problems [AJP14, LN15, NP11, OSY12].

1.2.5 Dps polytopes, Minkowski sum

A *distinct pair-sums polytope* (or *dps* for short), is a lattice polytope $P \subset \mathbb{R}^d$ in which all the pairwise sums $\{a + b : a, b \in P \cap \mathbb{Z}^d\}$ are distinct. Equivalently, P is dps if

$$\#\big(P \cap \mathbb{Z}^d \ + \ P \cap \mathbb{Z}^d\big) \ = \ \binom{\#(P \cap \mathbb{Z}^d)}{2} \ + \ \#(P \cap \mathbb{Z}^d).$$

where the sum between two sets $A, B \subset \mathbb{R}^d$ is $A + B := \{a + b \mid a \in A,\ b \in B\}$, and is called the *Minkowski sum*.

They were first introduced in [CLR02], where it was also observed that they coincide with lattice polytopes containing neither three collinear lattice points nor the vertices of a lattice parallelogram [CLR02, Lemma 1].

They also coincide with the polytopes of *Minkowski length* equal to one [BGSW12], where the *Minkowski length* of a polytope P is the largest number of non-trivial primitive segments whose Minkowski sum lies in P. For this reason, dps polytopes are also called *strongly indecomposable* in [SS09].

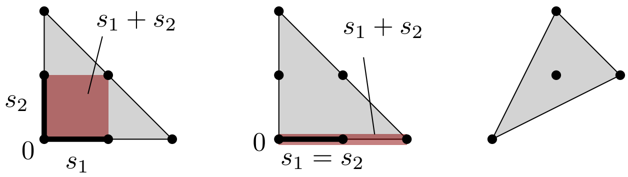


Figure 1.8: The polygon $2\Delta_2$ of Minkowski length 2 (with two different choices of primitive segments s_1 and s_2 whose Minkowski sum $s_1 + s_2 \subseteq 2\Delta_2$). And a terminal triangle of Minkowski length one, i.e. dps, i.e strongly decomposable.

Dps d-polytopes cannot have more than 2^d lattice points, or otherwise two of the lattice points would be in the same class modulo 2, and the polytope would contain at least three aligned points. Because of this, a way of classifying them is as a by-product of a classification of all lattice polytopes of at most that size.

In dimension 1, only the unit segment is dps. In dimension 2, the only dps polygons are the unimodular simplex and the unique terminal triangle (right-most polygon in Figure 1.8). Partial classifications of dps 3-polytopes appear in [Cur12].

Remark 1.21. Classifying dps 3-polytopes was partially our motivation when we started the classification of lattice 3-polytopes of small sizes ([BS16a], [BS16b] and Chapter 3 of this thesis), since dps 3-polytopes have size at most 8.

After this work was completed we have learned that a full classification of strongly indecomposable 3-polytopes is contained in the unpublished PhD thesis of J. Whitney [Whi10]. This classification agrees with ours.

In dimension 3, there are infinitely many dps 3-polytopes of width one, and 108 of them of width larger than one (see Section 3.4.6).

1.3 Previous work: classification of lattice 3-polytopes of sizes 5 and 6

In most of the classification approaches explained in the previous section, the classification of empty simplices plays an important role, as they are the smallest lattice subpolytopes we can have. A deep understanding of empty d-simplices gives tools and intuition to derive what will happen for other lattice d-polytopes. Notice that these simplices are, in particular, polytopes of a fixed size, namely $d + 1$. It seems natural to us to classify, or enumerate, *all* lattice polytopes of a given dimension and with a certain number of lattice points.

Proposition 1.22. *For $d \leq 2$, and for each $n \geq d+1$, there exist only finitely many lattice d-polytopes of size n.*

Proof. In dimension 1, a lattice segment of size n is equivalent to $[1,n]$, of length $n-1$. In dimension 2, Pick's Theorem (see [BR07]) relates the volume of a lattice polygon with the number of interior (i) and boundary (b) lattice points:

$$\mathrm{vol}(P) = 2i + b - 2 = n + i - 2$$

which, since a polygon of size n has at most $n-3$ interior lattice points, implies that its volume is bounded by $2n-5$. Hensley's Theorem 1.19 then proves that there are only finitely many lattice polygons for each size. □

An idea of a constructive proof is the following: any lattice d-polytope P of size n can be constructed by taking the convex hull of a lattice polytope Q (of dimension d or $d-1$) of size $n-1$ and a lattice point p that *extends Q by a single point.*

Remark 1.23 (Extending polytopes by a single point)**.** Let P be a lattice d-polytope of size n and let v be one of its vertices, we denote $P^v := \mathrm{conv}(P \cap \mathbb{Z}^d \setminus \{v\})$ the convex hull of all the lattice points of P with the exception of v. Notice that P^v is of size $n-1$, but its dimension can be d or $d-1$. We simplify $P^{u,v} := (P^u)^v = (P^v)^u$.

Conversely, let $Q \subset \mathbb{R}^d$ be a d or $(d-1)$-dimensional lattice polytope and let $v \in \mathbb{Z}^d \setminus Q$. We say that v *extends Q by a single point* if the polytope $P := \mathrm{conv}(Q \cup \{v\})$ is d-dimensional, and $P \cap \mathbb{Z}^d = (Q \cap \mathbb{Z}^d) \cup \{v\}$. That is, if $Q = P^v$.

In dimension one, the only way of extending a segment by a single point is by increasing its length by one. That is, we are taking a lattice point at distance one from a facet. In dimension 2, the same happens:

Lemma 1.24. *Let $T := \mathrm{conv}\{v_1, v_2, v_3\}$ be a lattice triangle, with v_3 at distance more than one from the segment v_1v_2. Then, there exists a lattice point $p \in T$ at smaller, but non-zero, distance to v_1v_2 than v_3.*

Proof. Without loss of generality let $T := \mathrm{conv}\{(0,0),(0,k),(a,b)\}$ with $k \geq 1$ and $a \geq 2$. Consider the triangle $T' = \mathrm{conv}\{(0,0),(0,1),(a,b)\} \subseteq T$. Since T' is not unimodular, it contains some extra lattice point (c,d). It is clear that $0 < c < a$. □

The previous lemma states that a point v extends a lattice segment $S \subset \mathbb{R}^2$ by a single point if, and only if, the distance from v to S is one. Let $Q \subset \mathbb{R}^2$ be a lattice polygon of size n, and let $v \in \mathbb{Z}^2 \setminus Q$. Using Lemma 1.24 it is easy to see that the only way v extends Q by a single point is if v is at distance one from all the facets of Q that are visible from v. By *visible* we mean facets of Q whose supporting hyperplane leaves v and Q in opposite sides. See Figure 1.9. This leaves only finitely many possibilities for v, for any fixed Q. Inductively, we can construct the finitely many lattice polygons of any given size, starting with the unique lattice polygon of size 3, namely the standard unimodular triangle Δ_1.

However, in dimensions 3 and higher, the number of lattice d-polytopes of a fixed size is infinite, as argued in [LZ11]. An example are the Reeve tetrahedra (Example 1.16), which are infinitely many lattice 3-polytopes of size 4. These were completely classified by White (see Theorem 1.17), and they all have width one.

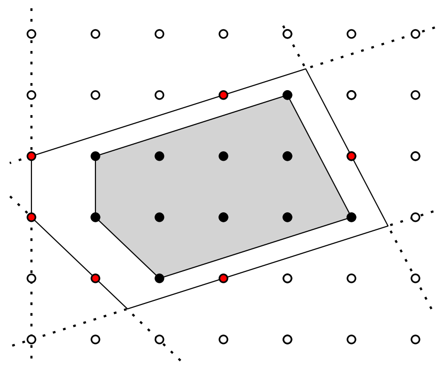


Figure 1.9: A polygon and the finitely many lattice points (in red) that extend it by a single point.

Remark 1.25. In general, lattice d-polytopes of width one are infinitely many for each fixed size $n \geq d+1$: they consist of two parallel lattice polytopes of dimension $\leq d-1$ of sizes $n_1 \leq n$ and $n - n_1$ at distance one. The infinitely many options correspond to the infinitely many possible $GL(\mathbb{Z}, d-1)$-rotations of one polytope with respect to the other.

It so happens that the width plays a role in the existence or not of infinitely many polytopes of a given size. For example, in dimension three, the following is true:

Theorem 1.26 ([BS16a, Corollary 22]). *For each $n \geq 4$, there exist only finitely many lattice 3-polytopes of size n and width larger than one.*

See Chapter 2 for similar results in arbitrary dimension.

In the rest of this section we review the complete lists of lattice 3-polytopes of sizes 5 and 6, that we computed in [BS16a, BS16b]. The methods used were quite ad-hoc and based on first classifying the possible oriented matroids (see [DLRS10]) of the five or six lattice points and then doing a detailed case study. The arguments used in those papers make a great distinction between polytopes of width one and polytopes of width larger than one.

We will explain some results and definitions appearing in these papers that are needed to understand the information given of the polytopes (Tables 1.1, 1.2 and 1.3). We will also state here some specific lemmas that can be deduce from these classifications and that are later needed on Chapter 3.

As a means to evaluate equivalences between different polytopes of sizes 5 and 6, we defined the following (almost complete) invariant of unimodular equivalence.

Definition 1.27 (Volume vectors). *Let $A = \{p_1, p_2, \ldots, p_n\}$, with $n \geq d+1$, be a finite set of lattice points in $\mathbb{Z}^d$. The volume vector of A is the vector*

$$w = (w_{i_1 \ldots i_{d+1}})_{1 \leq i_1 < \ldots < i_{d+1} \leq n} \in \mathbb{Z}^{\binom{n}{d+1}}$$

where $w_{i_1 \ldots i_{d+1}}$ is the determinant of the points $p_{i_1}, \ldots, p_{i_{d+1}}$, in that order.

The volume vector is clearly invariant under unimodular equivalence: for $A, B \subset \mathbb{Z}^d$ configurations with the same number of lattice points, if $\mathrm{conv}(A) \cong \mathrm{conv}(B)$, then there exists a permutation of the points such that the volume vectors are the same, up to a global orientation. The converse is only true if the volume vectors are primitive [BS16a, Proposition 5].

See Algorithm A.6 for the MATLAB routine that computes the volume vector, and also its ordered unsigned version (by which we mean the vector of absolute values of entries, ordered from smallest to greatest). The volume vector of the set of vertices of a polytope will be used in determining unimodular equivalence.

The classification of lattice 3-polytopes of size 5 is shown in Table 1.1. In this table, the following information of the different classes of polytopes is stored: a representative for the class, its width, the *signature*, and the volume vector slightly modified to reflect the signature of the configuration.

Sign.	**Volume vector**	**Width**	**Representative**
$(2,2)$	$(-1,1,1,-1,0)$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(1,1,0)$,$(0,0,1)$
$(2,1)$	$(-2q,q,0,q,0)$ $0 \le p \le \frac{q}{2}$, $\gcd(p,q)=1$	1	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(-1,0,0)$,$(p,q,1)$
$(3,2)^*$	$(-a-b,a,b,1,-1)$ $0 < a \le b$, $\gcd(a,b)=1$	1	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(0,0,1)$,$(a,b,1)$
$(3,1)^*$	$(-3,1,1,1,0)$ $(-9,3,3,3,0)$	1 2	$(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$,$(0,0,1)$ $(0,0,0)$, $(1,0,0)$, $(0,1,0)$, $(-1,-1,0)$,$(1,2,3)$
$(4,1)^*$	$(-4,1,1,1,1)$ $(-5,1,1,1,2)$ $(-7,1,1,2,3)$ $(-11,1,3,2,5)$ $(-13,3,4,1,5)$ $(-17,3,5,2,7)$ $(-19,5,4,3,7)$ $(-20,5,5,5,5)$	2 2 2 2 2 2 2 2	$(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,1,1)$,$(-2,-1,-2)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,2,1)$,$(-1,-1,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(1,3,1)$,$(-1,-2,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-1,-2,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-1,-1,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,7,1)$,$(-1,-2,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(3,7,1)$,$(-2,-3,-1)$ $(0,0,0)$, $(1,0,0)$, $(0,0,1)$, $(2,5,1)$,$(-3,-5,-2)$

Table 1.1: Complete classification of lattice 3-polytopes of size 5. Those marked with an * are dps.

Remember that $d+2$ lattice points in dimension d have a unique (modulo a scalar factor) affine dependence. The *signature* of the configuration of points is the pair (c^+,c^-), where c^+ and c^- are the number of positive and negative coefficients in this dependence, respectively. The (unique) *circuit* of this configuration of points is the subset of $c^+ + c^-$ points with non-zero coefficients, and its signature is as well (c^+,c^-). The volume vector for $d+2$ points in dimension d has a sign change in some of its coordinates, so that the volume vector itself is a choice of coefficients of this dependence. Also, in dimension 3, the signature of five points completely determines the oriented matroid of the configuration (see Figure 1.10).

The classification of lattice 3-polytopes of size 6 and width larger than one is shown in Tables 1.2 and 1.3, where we give, for each of them, its oriented matroid, a reference ID, its volume vector, its width, and a functional that achieves this width. The number in the "OM" column of those tables is the identifier of the oriented matroid as it appears in [BS16b, Figure 1].

In what follows we also give a 3×6 integer matrix for each of them (with the reference ID of the tables), whose columns are the six lattice points in a representative of the class.

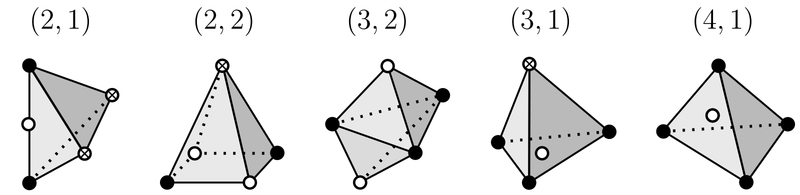


Figure 1.10: The five possible signatures of configurations of five points in dimension 3. Black and white points are those with positive and negative coefficients in the dependence, respectively. Crossed points are those with a zero coefficient.

The volume vector in the tables is always given with respect to the order of columns in the corresponding matrix.

Remark 1.28. In Tables 1.2 and 1.3, an (a, b)-*coplanarity* refers to $a+b \geq 4$ lattice points that are coplanar and form a circuit of signature (a, b). In turn, a $(2, 1)$-*collinearity* refers to three aligned lattice points.

$$A.1 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix} \quad A.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

$$B.1 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.3 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$B.4 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 3 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.5 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.6 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 6 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$B.7 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.8 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.9 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$B.10 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad B.11 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 8 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix} \quad B.12 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix}$$

$$B.13 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -3 \end{pmatrix} \quad B.14 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad B.15 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}$$

$$C.1 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad C.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{pmatrix} \quad C.3 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 1 & -1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 3 & 6 \end{pmatrix}$$

$$C.4 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \quad C.5 \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix} \quad C.6 \begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \end{pmatrix}$$

$$D.1 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix} \quad D.2 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 & 0 & 5 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$E.1 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & 5 \end{pmatrix} \quad E.2 \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{pmatrix}$$

$$F.1 \begin{pmatrix} 0 & 1 & 0 & -2 & 1 & 2 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 \end{pmatrix} \quad F.2 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \quad F.3 \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 2 & -1 & -2 \\ 0 & 1 & 0 & 1 & -1 & -2 \end{pmatrix}$$

$$F.4 \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 2 \\ 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \quad F.5 \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 3 & -2 & -4 \\ 0 & 1 & 0 & 1 & -1 & -2 \end{pmatrix} \quad F.6 \begin{pmatrix} 0 & 1 & 2 & 0 & -1 & -2 \\ 0 & 0 & 5 & 0 & -2 & -4 \\ 0 & 0 & 1 & 1 & -1 & -2 \end{pmatrix}$$

$$F.7 \begin{pmatrix} 0 & 1 & 0 & -2 & 1 & -1 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 0 & 0 \end{pmatrix} \quad F.8 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \quad F.9 \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 2 & -1 & 1 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{pmatrix}$$

$$F.10 \begin{pmatrix} 0 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 3 & -2 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 0 \end{pmatrix} \quad F.11 \begin{pmatrix} 0 & 0 & 1 & 1 & -1 & 1 \\ 0 & 0 & 0 & 3 & -2 & 2 \\ 0 & 1 & 0 & 1 & -1 & 1 \end{pmatrix} \quad F.12 \begin{pmatrix} 0 & 1 & 2 & 0 & -1 & 1 \\ 0 & 0 & 5 & 0 & -2 & 2 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{pmatrix}$$

$$F.13 \begin{pmatrix} 0 & 1 & 0 & -2 & 1 & 4 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 & 0 & 2 \end{pmatrix}$$

$$F.14\begin{pmatrix} 0 & 0 & -1 & 1 & 1 & 1 \\ 0 & 0 & -1 & 2 & 0 & -2 \\ 0 & 1 & -1 & 1 & 0 & -1 \end{pmatrix} \quad G.6\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & 2 & 0 & -1 & -5 \\ 0 & 1 & 1 & 0 & -1 & -4 \end{pmatrix} \quad G.15\begin{pmatrix} 0 & 0 & 1 & 2 & -1 & -1 \\ 0 & 0 & 0 & 5 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix} \quad H.4\begin{pmatrix} 0 & 0 & 1 & -1 & 1 & -8 \\ 0 & 0 & 2 & -1 & 0 & -9 \\ 0 & 1 & 1 & -1 & 0 & -10 \end{pmatrix}$$

$$F.15\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -3 \\ 0 & 0 & 0 & 2 & -1 & -4 \\ 0 & 1 & 0 & 1 & -1 & -3 \end{pmatrix} \quad G.7\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -5 \\ 0 & 0 & 2 & 0 & -1 & -7 \\ 0 & 1 & 1 & 0 & -1 & -6 \end{pmatrix} \quad G.16\begin{pmatrix} 0 & 0 & 1 & 2 & -1 & -2 \\ 0 & 0 & 0 & 5 & -2 & -3 \\ 0 & 1 & 0 & 1 & -1 & -2 \end{pmatrix} \quad H.5\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & 3 \\ 0 & 0 & 0 & 2 & -1 & 7 \\ 0 & 1 & 0 & 1 & -1 & 2 \end{pmatrix}$$

$$F.16\begin{pmatrix} 0 & 1 & -1 & 0 & 1 & 2 \\ 0 & 0 & -2 & 0 & 3 & 6 \\ 0 & 0 & -1 & 1 & 1 & 1 \end{pmatrix} \quad G.8\begin{pmatrix} 0 & -1 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 3 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad G.17\begin{pmatrix} 0 & 2 & 1 & 0 & -1 & -3 \\ 0 & 5 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 1 & -1 & -2 \end{pmatrix} \quad H.6\begin{pmatrix} 0 & 1 & 0 & 1 & -1 & 4 \\ 0 & 0 & 0 & 2 & -1 & 11 \\ 0 & 0 & 1 & 1 & -1 & 5 \end{pmatrix}$$

$$F.17\begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -2 \\ 0 & 0 & 3 & 0 & -2 & -4 \\ 0 & 0 & 1 & 1 & -1 & -3 \end{pmatrix} \quad G.9\begin{pmatrix} 0 & -1 & 0 & 1 & 1 & 2 \\ 0 & -2 & 0 & 0 & 3 & 5 \\ 0 & -1 & 1 & 0 & 1 & 1 \end{pmatrix} \quad G.18\begin{pmatrix} 0 & -1 & 2 & 0 & 1 & 1 \\ 0 & -1 & 5 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad H.7\begin{pmatrix} 0 & 0 & 1 & -1 & 1 & -4 \\ 0 & 0 & 0 & -2 & 3 & -7 \\ 0 & 1 & 0 & -1 & 1 & -5 \end{pmatrix}$$

$$G.1\begin{pmatrix} 0 & 1 & -2 & 0 & 1 & 4 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 3 \end{pmatrix} \quad G.10\begin{pmatrix} 0 & 1 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & -2 & 3 & 7 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \quad G.19\begin{pmatrix} 0 & 1 & 2 & -1 & 0 & -1 \\ 0 & 0 & 5 & -1 & 0 & -2 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{pmatrix} \quad H.8\begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -7 \\ 0 & 0 & 0 & -2 & 3 & -11 \\ 0 & 0 & 1 & -1 & 1 & -6 \end{pmatrix}$$

$$G.2\begin{pmatrix} 0 & 1 & -2 & 0 & 1 & 6 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 & 3 \end{pmatrix} \quad G.11\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -2 \\ 0 & 0 & 0 & 3 & -2 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \end{pmatrix} \quad G.20\begin{pmatrix} 0 & 0 & 2 & 1 & -1 & -2 \\ 0 & 0 & 5 & 0 & -1 & -3 \\ 0 & 1 & 1 & 0 & -1 & -2 \end{pmatrix} \quad H.9\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 3 & 0 & -2 & -1 \\ 0 & 1 & 1 & 0 & -1 & -1 \end{pmatrix}$$

$$G.3\begin{pmatrix} 0 & -1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 2 & 0 & 0 & -1 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad G.12\begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -3 \\ 0 & 3 & 0 & 0 & -2 & -5 \\ 0 & 1 & 0 & 1 & -1 & -2 \end{pmatrix} \quad H.1\begin{pmatrix} 0 & 1 & 0 & -2 & 1 & -12 \\ 0 & 1 & 0 & -1 & 0 & -7 \\ 0 & 1 & 1 & -2 & 0 & -13 \end{pmatrix} \quad H.10\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & 4 \\ 0 & 0 & 0 & 3 & -2 & 13 \\ 0 & 1 & 0 & 1 & -1 & 3 \end{pmatrix}$$

$$G.4\begin{pmatrix} 0 & -1 & 0 & 1 & 1 & 3 \\ 0 & -1 & 0 & 2 & 0 & 1 \\ 0 & -1 & 1 & 1 & 0 & 0 \end{pmatrix} \quad G.13\begin{pmatrix} 0 & 1 & 1 & 0 & -1 & -4 \\ 0 & 3 & 0 & 0 & -2 & -7 \\ 0 & 1 & 0 & 1 & -1 & -3 \end{pmatrix} \quad H.2\begin{pmatrix} 0 & 0 & 1 & -2 & 1 & -15 \\ 0 & 0 & 1 & -1 & 0 & -8 \\ 0 & 1 & 1 & -2 & 0 & -17 \end{pmatrix} \quad H.11\begin{pmatrix} 0 & 1 & 2 & -1 & 0 & -5 \\ 0 & 0 & 5 & -2 & 0 & -9 \\ 0 & 0 & 1 & -1 & 1 & -4 \end{pmatrix}$$

$$G.5\begin{pmatrix} 0 & 0 & 1 & -1 & 1 & 1 \\ 0 & 0 & 2 & -1 & 0 & -3 \\ 0 & 1 & 1 & -1 & 0 & -2 \end{pmatrix} \quad G.14\begin{pmatrix} 0 & 0 & 1 & 1 & -1 & -4 \\ 0 & 0 & 0 & 3 & -2 & -5 \\ 0 & 1 & 0 & 1 & -1 & -3 \end{pmatrix} \quad H.3\begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -7 \\ 0 & 2 & 0 & -1 & 0 & -9 \\ 0 & 1 & 1 & -1 & 0 & -8 \end{pmatrix} \quad H.12\begin{pmatrix} 0 & 1 & 0 & 2 & -1 & 4 \\ 0 & 0 & 0 & 5 & -2 & 11 \\ 0 & 0 & 1 & 1 & -1 & 2 \end{pmatrix}$$

In Chapter 3 we need two results that follow from these classifications of lattice 3-polytopes of size 5 and 6. The first lemma gathers information about which points in $\mathbb{Z}^3$ can extend a given lattice polygon by a single point. In it, a *unimodular parallelogram* is an empty parallelogram or, equivalently, a lattice parallelogram of Euclidean volume one.

Lemma 1.29 ([BS16a] and [BS16b, Section 5])**.** *Let P be a lattice 3-polytope with a vertex v such that P^v is two-dimensional (we assume it to be in the horizontal plane $\{z = 0\}S$). Then:*

(1) If P^v contains the following four-point set, then either v is at distance one from P^v or $v = (a, b, \pm 3)$ with $a \equiv 1 \equiv -b \mod 3$:

$$\{(-1, -1), (0, 0), (1, 0), (0, 1)\}$$

OM	Id.	Volume vector															Width, functional	
Polytopes containing 5 coplanar lattice points																		
3.2	A.1	0	0	2	0	0	4	0	2	0	−4	0	4	−2	−8	−2	2	z
3.3	A.2	0	0	2	0	4	4	0	2	0	−4	0	0	−2	−4	−2	2	z
Polytopes containing a $(3,1)$-coplanarity, but no 5 coplanar lattice points																		
One $(3,1)$-coplanarity leaves the other two lattice points on opposite sides																		
3.8	B.7	0	1	−1	−1	1	−2	1	−1	0	2	3	−3	0	6	0	2	z
3.9	B.9	0	1	−1	−1	1	−1	1	−1	1	0	3	−3	0	3	−3	2	z
3.13	B.10	0	1	−1	−1	1	0	1	−1	0	0	3	−3	−2	2	−2	2	z
	B.15	0	3	−3	−3	3	0	3	−3	0	0	9	−9	−6	6	−6	2	x
4.13*	B.1	0	1	−1	−1	1	−4	1	−1	1	3	3	−3	3	9	0	2	z
4.17*	B.2	0	1	−1	−1	1	−3	1	−1	0	3	3	−3	1	8	1	2	z
	B.14	0	3	−3	−3	3	−9	3	−3	0	9	9	−9	3	24	3	2	x
4.18	B.8	0	1	−1	−1	1	−1	1	−1	0	1	3	−3	−1	4	−1	2	z
5.10*	B.5	0	1	−1	−1	1	−5	1	−1	1	4	3	−3	4	11	1	2	z
	B.6	0	1	−1	−1	1	−6	1	−1	1	5	3	−3	5	13	2	2	z
	B.11	0	1	−3	−1	3	−8	1	−3	1	7	3	−9	5	19	2	2	x
5.11*	B.12	0	1	−3	−1	3	−2	1	−3	1	1	3	−9	−1	7	−4	2	x
5.12*	B.3	0	1	−1	−1	1	−2	1	−1	1	1	3	−3	1	5	−2	2	z
	B.4	0	1	−1	−1	1	−3	1	−1	1	2	3	−3	2	7	−1	2	z
	B.13	0	1	−3	−1	3	−5	1	−3	1	4	3	−9	2	13	−1	2	x
All $(3,1)$-coplanarities leave the other two lattice points on the same side																		
3.6	C.1	0	1	2	−1	−2	0	1	2	−1	1	3	6	0	0	3	2	x
3.11	C.2	0	1	2	−1	−2	0	1	2	0	0	3	6	1	−1	1	2	x
	C.3	0	3	6	−3	−6	0	3	6	0	0	9	18	3	−3	3	3	x
5.4*	C.4	0	1	5	−1	−5	1	1	5	−2	1	3	15	1	−4	7	2	y
	C.5	0	1	7	−1	−7	1	1	7	−2	1	3	21	3	−6	9	2	y
5.6*	C.6	0	1	3	−1	−3	−2	1	3	1	1	3	9	5	1	2	2	x
Polytopes containing a $(2,2)$-coplanarity, but none of the above																		
One $(2,2)$-coplanarity leaves the other two lattice points on opposite sides																		
5.13	D.1	0	1	−1	1	−1	−4	−1	1	3	−1	−1	1	5	1	−2	2	z
	D.2	0	1	−1	1	−1	−5	−1	1	4	−1	−1	1	7	2	−3	2	z
All $(2,2)$-coplanarities leave the other two lattice points on the same side																		
5.5	E.1	0	1	5	1	5	1	−1	−5	−2	−1	−1	−5	1	2	−3	2	y
	E.2	0	1	7	1	7	2	−1	−7	−3	−1	−1	−7	1	3	−4	2	$x-y$
Polytopes containing a $(2,1)$-collinearity, but no other coplanarity																		
4.21	F.1	1	−1	−2	1	2	0	−1	−2	0	0	−4	−7	−1	1	−1	2	y
	F.2	1	−2	−4	1	2	0	−1	−2	0	0	−5	−9	−2	1	−1	2	z
	F.3	2	−1	−2	1	2	0	−1	−2	0	0	−5	−8	−1	1	−1	2	$x-z$
	F.4	1	−3	−6	2	4	0	−1	−2	0	0	−7	−13	−3	2	−1	2	z
	F.5	3	−2	−4	1	2	0	−1	−2	0	0	−7	−11	−2	1	−1	2	$x-z$
	F.6	5	−3	−6	2	4	0	−1	−2	0	0	−11	−17	−3	2	−1	2	$x-z$
4.22	F.7	1	−1	1	1	−1	0	−1	1	0	0	−4	2	2	−2	2	2	y
	F.8	1	−2	2	1	−1	0	−1	1	0	0	−5	3	4	−2	2	2	z
	F.9	2	−1	1	1	−1	0	−1	1	0	0	−5	1	2	−2	2	2	z
	F.10	1	−3	3	2	−2	0	−1	1	0	0	−7	5	6	−4	2	2	z
	F.11	3	−2	2	1	−1	0	−1	1	0	0	−7	1	4	−2	2	2	z
	F.12	5	−3	3	2	−2	0	−1	1	0	0	−11	1	6	−4	2	2	z
4.11	F.13	1	−1	−3	1	2	1	−1	−2	−1	0	−4	−8	−4	0	0	2	y
	F.14	1	−1	−3	2	4	2	−1	−2	−1	0	−5	−10	−5	0	0	2	z
	F.15	2	−1	−4	1	2	1	−1	−2	−1	0	−5	−10	−5	0	0	2	$x-z$
	F.16	2	−1	−4	3	6	3	−1	−2	−1	0	−7	−14	−7	0	0	2	$x-z$
	F.17	3	−1	−5	2	4	2	−1	−2	−1	0	−7	−14	−7	0	0	2	$x-z$

Table 1.2: Lattice 3-polytopes of size 6 and width >1 with some coplanarity. Dps ones are marked with an * in the first column.

OM	Id.	Volume vector															Width, functional	
Polytopes with no coplanarities and 1 interior lattice point																		
6.2*	G.1	1	−1	−1	1	3	−2	−1	−2	1	1	−4	−7	3	5	−1	2	y
	G.2	1	−1	−3	1	5	−2	−1	−4	1	1	−4	−13	1	7	−3	2	y
	G.3	1	−1	−1	1	2	−1	−2	−3	1	1	−5	−7	2	3	−1	2	z
	G.4	1	−1	−2	1	3	−1	−2	−5	1	1	−5	−11	1	4	−3	2	z
	G.5	1	−2	−5	1	4	−3	−1	−3	1	1	−5	−13	1	7	−2	2	x
	G.6	2	−1	−1	1	5	−2	−1	−3	1	1	−5	−11	3	7	−2	2	$x-z$
	G.7	2	−1	−3	1	7	−2	−1	−5	1	1	−5	−17	1	9	−4	2	$x-z$
	G.8	1	−2	−3	1	2	−1	−3	−5	1	1	−7	−11	1	3	−2	2	z
	G.9	2	−1	−1	1	3	−1	−3	−5	1	2	−7	−11	2	5	−1	2	z
	G.10	2	−3	−7	1	5	−4	−1	−3	1	1	−7	−17	1	9	−2	2	z
	G.11	3	−2	−1	1	5	−3	−1	−2	1	1	−7	−11	5	8	−1	2	z
	G.12	3	−1	−1	1	4	−1	−2	−5	1	1	−7	−13	2	5	−3	2	$x-z$
	G.13	3	−1	−2	1	5	−1	−2	−7	1	1	−7	−17	1	6	−5	2	$x-z$
	G.14	3	−2	−5	1	7	−3	−1	−4	1	1	−7	−19	1	10	−3	2	$x-z$
	G.15	5	−2	−1	1	3	−1	−3	−4	1	1	−11	−13	3	4	−1	2	z
	G.16	5	−2	−3	1	4	−1	−3	−7	1	1	−11	−19	1	5	−4	2	$x-z$
	G.17	5	−3	−5	1	5	−2	−2	−5	1	1	−11	−20	1	7	−3	2	$x-z$
	G.18	3	−4	−5	1	2	−1	−5	−7	1	1	−13	−17	1	3	−2	2	z
	G.19	4	−5	−7	1	3	−2	−3	−5	1	1	−13	−19	1	5	−2	2	z
	G.20	5	−3	−4	1	3	−1	−4	−7	1	1	−13	−19	1	4	−3	2	$x-z$
Polytopes with no coplanarities and 2 interior lattice points																		
6.1*	H.1	1	−1	5	1	1	−6	−1	−2	7	−1	−4	1	19	5	−9	2	$x-z$
	H.2	1	−1	7	1	1	−8	−1	−2	9	−1	−4	3	25	7	−11	2	$x-2y$
	H.3	1	−2	5	1	1	−7	−1	−2	9	−1	−5	1	23	6	−11	2	$x-z$
	H.4	1	−2	7	1	1	−9	−1	−2	11	−1	−5	3	29	8	−13	2	$x-y$
	H.5	2	−1	7	1	1	−4	−1	−3	5	−1	−5	1	17	3	−8	2	$x-z$
	H.6	2	−1	11	1	1	−6	−1	−3	7	−1	−5	5	25	5	−10	2	$x-z$
	H.7	2	−3	7	1	1	−5	−1	−3	8	−1	−7	1	23	4	−11	2	$x-z$
	H.8	2	−3	11	1	1	−7	−1	−3	10	−1	−7	5	31	6	−13	2	$x-z$
	H.9	3	−1	7	2	1	−5	−1	−2	3	−1	−7	1	16	3	−5	2	z
	H.10	3	−2	13	1	1	−5	−1	−4	7	−1	−7	5	27	4	−11	2	$x-z$
	H.11	3	−5	11	2	1	−9	−1	−2	7	−1	−11	5	32	7	−9	2	$x-z$
	H.12	5	−2	11	3	1	−7	−1	−2	3	−1	−11	3	23	4	−5	3	$x-z$

Table 1.3: Lattice 3-polytopes of size 6 and width >1 with no coplanarities. All dps.

(2) *If P^v contains the following four-point set, then the edge* $\operatorname{conv}\{v,(0,1)\}$ *is at distance one from the edge* $\operatorname{conv}\{(-1,0),(1,0)\}$*:*

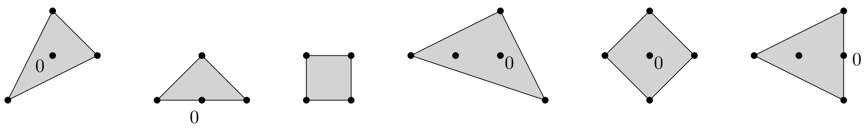
$$\{(-1,0),(0,0),(1,0),(0,1)\}$$

(3) *If P^v contains either a unimodular parallelogram or the following five points, then v is at distance one from P^v:*

$$\{(0,0),(-1,0),(-2,0),(0,1),(1,-1)\}.$$

(4) *If P^v contains one of the following five-point sets, then either v is at distance one from P^v or $v=(a,b,\pm 2)$ with $a\equiv 1\equiv b \mod 2$:*

$$\{(-1,0),(0,0),(1,0),(0,1),(0,-1)\},\qquad \{(0,0),(-1,0),(-2,0),(0,1),(0,-1)\}.$$



Proof. For the cases where P^v contains one of the polygons of size four, look at the rows of signatures $(3,1)$, $(2,1)$ and $(2,2)$ in Table 1.1, respectively.

For the five-point configurations, look in Table 1.2 for the rows corresponding to "Polytopes containing 5 coplanar points" and take into account that v at distance one always extends P^v by a single point, but then P has width one. □

Lemma 1.30. *The following two configurations (B.14 and B.15 from Table 1.2) are the only lattice* 3*-polytopes of size* 6 *having two vertices* u, v *such that* $P^{u,v} = \operatorname{conv}\{(-1,-1,0), (1,0,0), (0,1,0)\}$ *and such that* u *and* v *lie in opposite sides and at distance more than one from the plane containing* $P^{u,v}$*:*

$$B.14 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad B.15 \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}$$

Proof. The subpolytope $P^{u,v}$ is a $(3,1)$-coplanarity (see Remark 1.28) in the plane $\{z = 0\}$. By looking at Table 1.2, one can see that lattice 3-polytopes of size 6 with a $(3,1)$-coplanarity plus two other vertices in opposite sides of the coplanarity have identifiers $B.*$. By looking at the list of matrices one can find that the only polytopes with both points outside of that plane at distance more than one are $B.14$ and $B.15$. □

1.4 Summary of the thesis and main results

Following the results on lattice 3-polytopes of sizes 5 and 6, two different lines of investigation arose. One one hand the proof of Theorem 1.26 that we derived in [BS16a] relied on results that were valid in arbitrary dimension, with the exception of the fact that $2\Delta_2$ was the unique hollow polygon of width larger than one (see Theorem 1.8). In joint work with Christian Haase, Jan Hofmann and Francisco Santos, we studied the same problem in arbitrary dimension. Chapter 2 is devoted to this research, which appears as well in the paper [BHHS16].

In Chapter 3 we continue the classification of the (finitely many) lattice 3-polytopes of width larger than one and any given size. The general methods used in [BS16a, BS16b] for the classification of such polytopes of sizes 5 and 6 did not seem feasible for size seven and beyond. However, in specific parts of the classification of lattice 3-polytopes of size 6 [BS16b] we were implicitly using an idea and method that we have later developed into an algorithm that enumerates all lattice 3-polytopes of width larger than one up to any given size. The idea is to use induction on the size of a polytope, and make use of the fact that, if a lattice 3-polytope of width larger than one and a certain size N contains a subpolytope of size $N-1$ and still of width larger than one, then this subpolytope belongs to a finite list, that is assumed precomputed. This research appears on the paper [BS16c].

Appendix A of this thesis contains the routines used for computations in Chapter 3.

In what follows we present a summary of the main results of this thesis.

The finiteness threshold width of lattice polytopes (Chapter 2).

Theorem (Definition 2.1 and Theorem 2.4(3))**.** *For every dimension* d*, there exists a constant* $W \in \mathbb{N}$*, depending solely on* d*, such that, for each* $n \geq d+1$*, all but finitely many lattice* d*-polytopes of size* n *have width* $\leq W$*.*

We define the *finiteness threshold width*, and denote it $w^\infty(d)$, to be the minimal such constant W. For example, in dimensions 1 and 2 the finiteness threshold width is 0, since there exist only finitely many lattice d-polytopes of each size $n \geq d+1$ (see Theorem 1.22). Similarly, Theorem 1.26 states that $w^\infty(3) = 1$ (Blanco–Santos).

Theorem 1.11 (Nill–Ziegler) tells us that, in order to find an infinite family of lattice d-polytopes of a certain size, we can restrict ourselves to families of hollow d-polytopes that admit projections onto hollow $(d-1)$-polytopes. That theorem is the one that allows us, together with Theorems 1.19 (Hensley) and 1.20 (Lagarias–Ziegler), to prove that $w^\infty(d) \leq w_H(d-1)$, and $w_H(d-1)$ is bounded for each d by the Flatness Theorem (see Theorem 1.7).

We then study *lifts* of (lattice polytopes projecting onto) some lattice polytope (Definition 2.14), and we study the relation between the width and the dimension of a polytope and those of its lifts. We also define *tight lifts*, that are inclusion-minimal lifts (Definition 2.19). These definitions and results (Section 2.1) allow us to considerably simplify the work we have to do in order to decide whether a polytope has infinitely or only finitely many lifts of bounded size.

It is easy to see that lattice simplices and non-hollow lattice polytopes have only finitely many lifts of bounded size (Section 2.2). On the other hand, we find hollow and empty polytopes of maximum widths (in their dimensions), that give us the lower bounds $w^\infty(d) \geq w_H(d-2)$ and $w^\infty(d) \geq w_E(d-1)$ (Section 2.3).

Summing up:

Theorem (Corollary 2.6)**.** *For every $d \geq 3$,*

$$d-2 \;\leq\; \max\{w_H(d-2), w_E(d-1)\} \;\leq\; w^\infty(d) \;\leq\; w_H(d-1) \;\leq\; O\left(d^{3/2}\right).$$

Moreover, we find that the *exact* value of $w^\infty(d)$ can be derived from some information on hollow $(d-1)$-polytopes as follows:

Theorem (Theorem 2.7)**.** *For all $d \geq 3$, $w^\infty(d)$ equals the maximum width of a lattice $(d-1)$-polytope Q for which there are infinitely many lattice d-polytopes projecting to Q of some fixed size $n \in \mathbb{N}$. Moreover, every such Q is hollow it cannot be a simplex.*

Using the classification of hollow 3-polytopes detailed in Section 1.2.1, we were able to derive the value for $w^\infty(d)$ for $d = 4$:

Theorem (Theorem 2.2)**.** $w^\infty(4) = 2$*. That is, for each $n > 4$ there are only finitely many lattice 4-polytopes of size n and width greater than 2.*

In dimension 5, it is trivial from $w_E(d-1) \leq w^\infty(d)$ that $w^\infty(5) \geq 4$, since $w_E(4) \geq 4$ by the existence of an empty 4-simplex of width 4 (see Section 1.2.2).

Remark 1.31. As a consequence of $w^\infty(4) = 2$, we have that there exist only finitely many empty 4-simplices of width > 2 (Corollary 2.3). This result was claimed by Barile et al. [BBBK11], but the proof given there is incomplete. More precisely, the authors use a classification of infinite families of empty 4-simplices of width > 1 that had been conjectured to be complete by Mori et al. [MMM88] for simplices whose determinant (i.e, their *normalized volume*) is a prime number, and proved by Sankaran [San90] and

Bover [Bob09]. But when the determinant is not prime other infinite families do arise, such as the following explicit example, not considered by Barile et al: the empty 4-simplices with vertices e_1, e_2, e_3, e_4 and $(2, N/2-1, a, N/2-a)$, where the determinant N is a multiple of 4 and coprime with a.

As a conclusion, the proof of Corollary 2.3 given in [BBBK11] is valid only for simplices of prime determinant. We thank O. Iglesias for the computations leading to finding this (and other) families and the authors of [BBBK11] for helpful discussions about the extent of this mistake.

For larger values of d, the results in this chapter would prove useful in finding the exact value of $w^\infty(d)$, provided that new information on hollow $(d-1)$-polytopes is known.

In light of these results, we ask the following questions. In them, $w^\infty(d, n)$ is the minimal width $W \geq 0$ for which there exist only finitely many lattice d-polytopes of size n and width W.

Question 1.32. *Theorem 2.4 states that $w^\infty(d, n) \leq w^\infty(d, n+1)$ (1) and $w^\infty(d) \leq w^\infty(d+1)$ (2), for all d, and for all $n \geq d+1$.*

Is it also true that $w^\infty(d, n) \leq w^\infty(d+1, n+1)$? It holds in the case of empty simplices: $w^\infty(d, d+1) \leq w^\infty(d+1, d+2)$ follows from every empty d-simplex is a facet of infinitely many empty $(d+1)$-simplices of at least the same width *([HZ00, Proposition 1]).*

Question 1.33. *For all known values ($d \leq 4$) we have $w^\infty(d) = w^\infty(d, d+1)$. That is, the finiteness threshold width for all d-polytopes is determined by empty d-simplices. Does this hold for arbitrary d?*

Enumeration of lattice 3-polytopes (Chapter 3).

In this chapter we aim to classify all lattice 3-polytopes of width larger than one and any fixed size N. We assume the size to be at least seven, and first prove that any such polytope has to be in one of three different groups:

Theorem (Theorems 3.13 and 3.15)**.** *Let $P \subset \mathbb{R}^3$ be a lattice 3-polytope of width larger than one and size at least seven. Then one of the following happens:*

(1) P projects in a very specific manner to one of a list of seven particular lattice polygons. (We call them spiked *polytopes)*

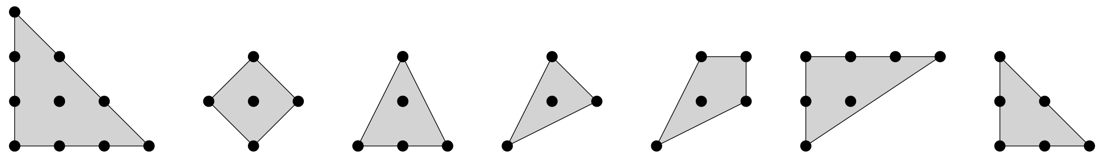
*(2) All except three of the lattice points in P lie in a rational parallelepiped of width one with respect to every facet. (*Boxed *polytopes).*

*(3) P has (at least) two vertices u and v such that both P^u and P^v still have width larger than one, and such that $P^{u,v}$ is still 3-dimensional (*Merged *polytopes).*

Please beware that the descriptions of both spiked and boxed 3-polytopes in the previous theorem are not totally precise (for more details, see Section 3.1.1). For the part of merged polytopes, remember that $P^u := \operatorname{conv}(P \cap \mathbb{Z}^3 \setminus \{u\})$, and that $P^{u,v} = (P^u)^v$ (see Remark 1.23).

Given that result, we approach the classification of each of those groups in a different way. For spiked polytopes, we obtain the following:

Theorem (Corollary 3.17)**.** *Let P be a spiked 3-polytope of size at least seven. Then P admits a lattice projection to one of the following polygons, in such a way that every vertex in the projection has a unique preimage in P.*



Remember that, by Theorem 1.26, there exist only finitely many spiked 3-polytopes of a given size. Fixed Q one of those seven lattice polygons, the fact that only one vertex of P can project to each of the vertices of Q, and the extra requirement (not mentioned above) that at most one other vertex of P (projecting to a non-vertex of Q) can appear, the number of possibilities is very limited, and an explicit description of each possible P, for any given Q and size N, is possible (Theorems 3.18 and 3.19).

For boxed 3-polytopes, a more precise description is the following: a lattice 3-polytope P of width larger than one is boxed if $P \cap \mathbb{Z}^3 = A \cup \{v_1, v_2, v_3\}$ and there exist affine primitive functionals f_1, f_2, f_3 such that $f_i(A) \subseteq \{0, 1\}$ and $f_i(v_j) \notin \{0, 1\}$ if, and only if, $i = j$. The parallelepiped mentioned in part 2 of Theorem 1.4 is $\bigcap_i f_i^{-1}([0, 1])$.

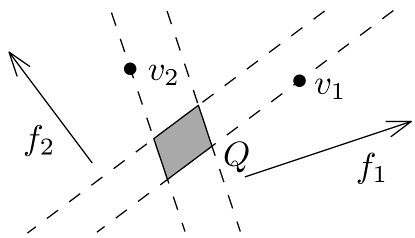


Figure 1.11: A boxed polygon. Q is the parallelepiped $f_1^{-1}([0, 1]) \cap f_2^{-1}([0, 1])$.

In particular, boxed 3-polytopes have size at most 11 (3 vertices plus possibly the 2^3 vertices of the parallelepiped). By Theorem 1.26, they are finitely many for each size, hence finitely many in total. We first study the options for the f_i, and with these functionals fixed, we bound the value $f_i(v_i)$, which gives only finitely many possibilities for each v_i:

Theorem (Lemma 3.22 and Theorem 3.26)**.** *If P is a boxed 3-polytope, then without loss of generality $f_1 = x$, $f_2 = y$, $f_3 = z$, or $f_1 = y + z$, $f_2 = x + z$, $f_3 = x + y$.*

Moreover, $f_i(v_i) \in [-6, 7]$ for all i.

Notice that the first choice of f_i gives as parallelepiped the unit cube $[0, 1]^3$, and the second choice gives a parallelepiped with only lattice points the vertices of the standard unimodular tetrahedron. That result allows for a full enumeration of boxed 3-polytopes via computer search (see Section 3.3.3).

Finally, any merged 3-polytope P of size N is obtained as a union (*merging*) of the two polytopes P^u and P^v (of size $N - 1$ and width larger than one). The fact that $P^{u,v}$ (of size $N - 2$) is full-dimensional means that there is only finitely many ways of merging them:

Algorithm 1.34 (Merging algorithm in dimension 3, Algorithm 3.2)**.**
INPUT: two lattice 3-polytopes P_1 and P_2 of size $N - 1$ and width > 1.

OUTPUT: all the lattice 3-polytopes of size N obtained merging P_1 and P_2.
For each vertex v_1 of P_1 and v_2 of P_2:

(1) Let $P_1' = (P_1)^{v_1} \subset P_1$ and $P_2' = (P_2)^{v_2} \subset P_2$.

(2) Check that P_1' and P_2' are 3-dimensional.

(3) For each equivalence $t : \mathbb{R}^3 \to \mathbb{R}^3$ with $t(P_1') = P_2'$, if the size of $P := \operatorname{conv}(t(P_1) \cup \{v_2\}) = \operatorname{conv}(\{t(v_1)\} \cup P_2)$ equals N, add P to the output list. (Observe that t may be not unique, but there are only finitely many possibilities for it).

See Figure 1.12 for an idea of the "merging" in dimension 2.

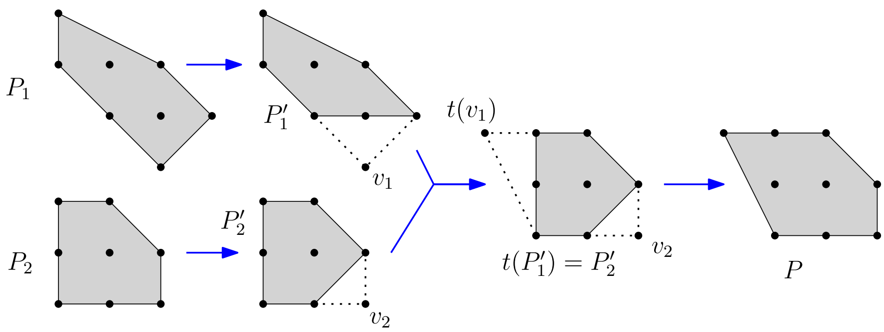


Figure 1.12: A lattice polygon P of size 9 constructed by merging two polygons of size 8.

Hence, to obtain all merged 3-polytopes of size N, we take the finite list of lattice 3-polytopes of size $N-1$ and width larger than one, which we assume precomputed, and for each two of these polytopes, try all the (finitely many) mergings.

Putting together these three groups, we have been able to enumerate all lattice 3-polytopes of width larger than one and size up to 11 (there are 216 453 of them; see Theorem 3.6).

Remark 1.35. It may seem that the results of this chapter make the results of [BS16a, BS16b] useless, but that is not the case. Quite the opposite, we need these classification of sizes 5 and 6 as the starting point for our algorithm, since there are several issues that make the techniques in Chapter 3 only applicable for size at least seven. One of them is the fact that there exists a (unique) polytope of size 6 that does not belong to any of the three groups described above (see Theorem 3.15). But most importantly, the assumption of the size being at least seven considerably simplifies the cases to be considered in the classification of spiked and boxed 3-polytopes (Theorem 3.16 and Lemma 3.22). In particular, classifying spiked and boxed 3-polytopes of size six might have been not significantly simpler than repeating the work done in [BS16b].

More detailed information about the output of the algorithm (the lists of lattice 3-polytopes of width larger than one and size up to 11) and some remarks that can be derived from it are contained in Section 3.4. We here include a summary of that section.

Tables 3.2 and 3.9 show the numbers of 3-polytopes of each size in terms of their number of vertices and of interior points. Our results agree with the classifications of lattice 3-polytopes with 1 and 2 interior points mentioned in Section 1.2.3. Table 3.3 shows the numbers of canonical and terminal 3-polytopes up to size 11.

Table 3.4 shows the classification according to the width. The maximum widths obtained by the polytopes of each size are: 2 for size 5, 3 for sizes 6 to 9, and 4 for sizes 10 and 11.

In Section 3.4.3 we look at the volumes that arise for each size. Experimentally, we see that in sizes $n = 5, \dots, 11$ there is always a unique 3-polytope that maximizes volume, and it is a clean tetrahedron of volume $12(n-4)+8$ (a lattice polytope is *clean* if the only boundary points are the vertices). This tetrahedron can be generalized to arbitrary size (Proposition 3.35) and we conjecture it to be the unique maximizer of volume for every size (Conjecture 3.36).

Most of the polytopes in our output are *primitive*, by which we mean that their integer points affinely span the whole integer lattice. When this is not the case, we call *sublattice index* of P the index of the lattice spanned by $P \cap \mathbb{Z}^d$ as a sublattice of $\mathbb{Z}^d$. Experimentally we see that there is a unique lattice 3-polytope (of width larger than one and size ≤ 11) of sublattice index 5 (a terminal tetrahedron) and that the only other sublattice indices that arise are 2 and 3 (see Table 3.5).

Remark 1.36. In an upcoming paper of Blanco–Santos [BS17] we show that this is actually the case for all sizes and characterize the non-primitive 3-polytopes of width larger than one: there are linearly many on n of index 3 and quadratically many on n of index 2, for each size n. See Section 3.4.4 for more details.

It is worth mentioning that the main tool used for the results in [BS17] is the separation of lattice 3-polytopes of size N and width larger than one in the three groups of Theorem 1.4. For any fixed size N, the sublattice index of spiked 3-polytopes was derived from the explicit description of them. For boxed 3-polytopes, we checked the full list of them (which we have computed). Finally, for merged 3-polytopes, we used the information on the sublattice index of lattice 3-polytopes of smaller sizes, and then used induction in order to study the sublattice index of merged 3-polytopes of arbitrary large size. Polytopes of width one where studied independently.

We believe that this idea has a lot of potential in studying some specific property or parameter of lattice 3-polytopes, as long as it is possible to study this property of a polytope, with relation to that of its subpolytopes.

In Section 3.4.5 we look at how many of our polytopes are *normal* (a lattice 3-polytope is normal if every lattice point $p \in 2P \cap \mathbb{Z}^3$ is the sum of two lattice points in P). Experimentally it seems that the fraction of polytopes that are normal does not vary much with size and stays close to a 13%. We do not know whether the same keeps happening for higher sizes. We also check that up to size 11, all normal 3-polytopes have a vertex that can be removed and still leave a normal polytope (whether this is always the case is a question from [BGM16]).

Since dps 3-polytopes have size at most 8 (see Section 1.2.5), the results of Chapter 3 complete the classification of dps 3-polytopes. We devote Section 3.4.6 to them. Table 3.6 gives the number of those of width larger than one, for each size and number of vertices. In particular we can answer in dimension 3 the several questions posed by Reznick [BNR$^+$08] regarding dps polytopes. We also observe that dps d-polytopes for $d = 2, 3$ have at most $3 \cdot 2^{d-2}$ vertices and ask whether the same happens in higher dimensions (see Question 3.38).

MATLAB routines (Appendix A).

Finally, in Appendix A we collect the implementation of the algorithms used for all the computations needed in Chapter 3. These computations have been done with the software MATLAB, and all routines were implemented by us. Each of them is preceded by a description of the theory involved in the computations. The appendix is organized in three parts:

- Section A.1 collects the routines used to compute the basic information we need about a polytope (convex hull, vertices, lattice points, etc), and the algorithms we need in order to work with lists of lattice 3-polytopes of width larger than one (unimodular equivalence and width).
- Section A.2 is centered on the specific routines of the classification of lattice 3-polytopes of width larger than one and some given size $N \geq 7$. This includes the classifications of boxed and spiked 3-polytopes mentioned above, and the *merging* algorithm for computing merged 3-polytopes.
- Section A.3 is concentrated on the computation of further properties of lattice 3-polytopes that seem of interest, and it shows a few examples of how the information of any of the classified polytopes is stored and read. The properties here computed lead to the comments and classification tables in Section 3.4 in Chapter 3.

The algorithms in this section are listed in page 186.

Chapter 2

The finiteness threshold width of lattice polytopes

In this chapter we study, in general dimension, the width of lattice d-polytopes of a certain fixed size n. For this we introduce the following constants:

Definition 2.1 (Finiteness Threshold Width). *For each d and each $n \geq d+1$, denote by $w^\infty(d,n) \in \mathbb{N} \cup \{\infty\}$ the minimal width $W \geq 0$ such that there exist only finitely many lattice d-polytopes of size n and width $> W$. We call $w^\infty(d) := \sup_{n\in\mathbb{N}} w^\infty(d,n)$ the finiteness threshold width.*

For instance, $w^\infty(1) = w^\infty(2) = 0$, since there are only finitely many lattice segments or polygons for any fixed size (see Section 1.3). In dimension 3, Reeve tetrahedra are infinitely many empty simplices of width one (Example 1.16). That is, $w^\infty(3) \geq w^\infty(3,4) \geq 1$. In dimension 4, Example 1.18 (Haase–Ziegler) is an infinite list of empty simplices of width 2. That is, $w^\infty(4) \geq w^\infty(4,5) \geq 2$.

As one of the main results in this chapter, we determine the finiteness threshold width in dimension 4. The result for dimension 3 is Theorem 1.26, but we include a proof here as well.

Theorem 2.2.

- *$w^\infty(3) = 1$. That is, for each $n > 3$ there are only finitely many lattice 3-polytopes of size n and width greater than 1.*
- *$w^\infty(4) = 2$. That is, for each $n > 4$ there are only finitely many lattice 4-polytopes of size n and width greater than 2.*

Observe that this implies the following result (see Remark 1.31):

Corollary 2.3. *There are only finitely many empty 4-simplices of width larger than two.*

Along the way, we prove the following properties of the parameters $w^\infty(d)$ and $w^\infty(d,n)$.

Theorem 2.4.

(1) $w^\infty(d,n) \leq w^\infty(d,n+1)$ for all d,n. *(Proposition 2.9)*

(2) $w^\infty(d) \leq w^\infty(d+1)$ for all d. *(Proposition 2.10)*

(3) $w^\infty(d) \leq w_H(d-1) < \infty$. *(Corollary 2.12)*

(4) $w_E(d-1) \leq w^\infty(d)$ for $d \geq 3$. *(Corollary 2.29)*

(5) $w_H(d-2) \le w^\infty(d)$. *(Corollary 2.31)*

Remember that the constants $w_E(d)$ and $w_H(d)$ are the maximum width achieved by an empty or a hollow d-polytope, respectively.

Remark 2.5. *None of the inequalities* $w_H(d-2) \le w^\infty(d) \le w_H(d-1)$ *or* $w_E(d-1) \le w^\infty(d)$ *(for* $d \ge 3$*) is sharp, as the following table of known values shows.*

d	$w_E(d-1)$	$w_H(d-2)$	$w^\infty(d)$	$w_H(d-1)$
1	$-$	$-$	0	$-$
2	1	$-$	0	1
3	1	1	1	2
4	1	2	2	3
5	≥ 4	3	≥ 4	≥ 4

The values of $w^\infty(d)$*,* $d = 1, 2, 3, 4$*, have been discussed above. In the case of* $d = 5$*, its lower bound follows from* $w^\infty(5) \ge w_E(4) \ge 4$ *(Theorem 2.4(4)). For the values of* w_H *and* w_E*, see references in Sections 1.2.1 and 1.2.2.*

Finiteness of $w_H(d)$ follows from the “flatness theorem". Best lower and upper bounds on this constant are in Section 1.2.1. Combined with the bounds $w_H(d-2) \le w^\infty(d) \le w_H(d-1)$ (parts (3) and (5) in Theorem 2.4), we get:

Corollary 2.6. *For all* $d \ge 3$*,*

$$d-2 \quad \le \quad w^\infty(d) \quad \le \quad O\left(d^{3/2}\right). \tag{2.1}$$

Let us now summarize the contents and structure of the chapter. Section 2.1 is organized as follows:

- The monotonicity properties stated in parts (1) and (2) of Theorem 2.4 are almost straightforward, and proved in Section 2.1.1.
- We then prove in Section 2.1.2 that all but finitely many lattice d-polytopes of bounded size are hollow and project to hollow $(d-1)$-polytopes (see Lemma 2.11). This gives the upper bound $w^\infty(d) \le w_H(d-1)$ (Corollary 2.12), which in particular implies that *the finiteness threshold width is finite for all d.*

  That fact also implies that to search for an infinite family of lattice d-polytopes of bounded size we can focus on those projecting to hollow $(d-1)$-polytopes.
- In Section 2.1.3, we look at *lifts* (see Definition 2.14) of polytopes, and prove that all but finitely many of the lifts of a $(d-1)$-polytope Q of width W have dimension d and width W.
- In Section 2.1.4 we introduce the concept of *tight* lifts (see Definition 2.19), which are inclusion-minimal lifts of a polytope. We prove that, if the size of the lifts is consider bounded, it is enough to look at tight lifts.

This last result, and the conclusions of the entire Section 2.1, are collected in Corollary 2.21 which, in summary, states the following: *there exist infinitely many lattice d-polytopes projecting to a lattice $(d-1)$-polytope Q, of bounded size and of the same width as Q if, and only if, Q has infinitely many tight lifts of bounded size.* This proves extremely useful in the rest of the chapter, since we no longer need to care about the width or the dimension of the lifts, and we can concentrate in the very-easy-to-understand tight lifts and their size.

In Section 2.2 we prove certain lattice polytopes to have only finitely many lifts of bounded size: lattice simplices (Lemma 2.22), lattice pyramids whose basis have only finitely many lifts of bounded size (Lemma 2.23), and lattice polytopes with interior points (Corollary 2.25).

In Section 2.3 we prove sufficient conditions for lattice polytopes to have infinitely many lifts of bounded size (Lemma 2.26). We then prove the existence of hollow $(d-1)$-polytopes with such properties and of widths $w_E(d-1)$ and $w_H(d-2)$, which provides the lower bounds $w_E(d-1) \leq w^\infty(d)$ (Corollary 2.29) and $w_H(d-2) \leq w^\infty(d)$ (Corollary 2.31). Moreover, these constructions allow us to prove the following characterization of the finiteness threshold width:

Theorem 2.7 (Theorem 2.32). *For all $d \geq 3$, $w^\infty(d)$ equals the maximum width of a lattice $(d-1)$-polytope Q for which there are infinitely many lattice d-polytopes projecting to Q of some fixed size $n \in \mathbb{N}$. Moreover, every such Q is hollow.*

One direction of the theorem is trivial, but the other is not since a priori there could exist infinitely many hollow $(d-1)$-polytopes with lifts of size n, but each of them with only finitely many such lifts.

Example 2.8. *In dimension 3, the infinite family of Reeve tetrahedra (Example 1.16) are lifts of size 4 of a unit square, which is a hollow polygon of width one. Hence $w^\infty(3) \geq 1$.*

On the other hand, $w^\infty(4) \geq 2$ follows from the fact that the following hollow 3-polytope of width two can be lifted to infinitely many empty simplices (Example 1.18, Haase–Ziegler):

$$Q = \operatorname{conv}\left\{ \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 2 \\ 3 \end{pmatrix} \right\}.$$

Hence, to prove the exact values $w^\infty(d)$, for $d = 3, 4$ (Theorem 2.2), it suffices to see whether hollow $(d-1)$-polytopes of width > 1 or > 2, respectively, have infinitely or only finitely many lifts of bounded size. This is done in Section 2.4, using the classification of hollow $(d-1)$-polytopes detailed in Section 1.2.1.

2.1 Finiteness threshold width and lifts of hollow polytopes

2.1.1 Monotonicity of the finiteness threshold widths

Parts (1) and (2) of Theorem 2.4 have the following proofs:

Proposition 2.9. $w^\infty(d, n) \leq w^\infty(d, n+1)$ *for all $n \geq d+1$.*

Proof. Let $W \in \mathbb{N}$ be such that there exists an infinite family $\{P_i\}_{i\in\mathbb{N}}$ of d-polytopes of size n and width W. We are going to show that for each P_i there is a P'_i of size $n+1$ and width W containing P_i. To prove this, let ℓ_i be an integer functional giving width W to P_i, and assume without loss of generality that $\ell_i(P_i) = [0, W]$. Taking any point $q_i \in \mathbb{Z}^d \cap \ell_i^{-1}[0, W] \setminus P_i$ we easily get a $Q_i = \text{conv}(P_i \cup \{q_i\})$ of width W and properly containing P_i. If $Q_i \setminus P_i$ has more than one lattice point, remove them one by one until only one remains (which can always be done; simply choose a vertex v of Q_i not in P_i and replace Q_i to the convex hull of $(Q_i \cap \mathbb{Z}^d) \setminus \{v\}$; then iterate).

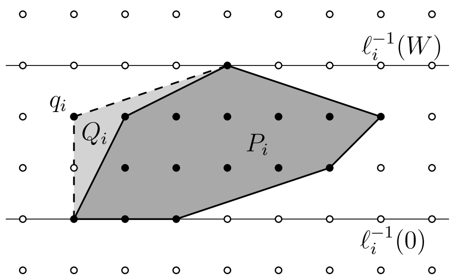


Figure 2.1: The setting of the proof of Proposition 2.9.

That implies the lemma except for the fact that different polytopes P_i and P_j may produce isomorphic P'_i and P'_j, so it is not obvious that $\{P'_i\}_{i\in\mathbb{N}}$ is an infinite family. But each element of $\{P'_i\}_{i\in\mathbb{N}}$ can only correspond to at most $n+1$ elements from $\{P_i\}_{i\in\mathbb{N}}$ (because P_i is recovered from P'_i by removing one of its $n+1$ lattice points), so the proof is complete. □

Proposition 2.10. *$w^\infty(d) \leq w^\infty(d+1)$, for all d.*

Proof. Let $W \in \mathbb{N}$ be such that, for some $n \in \mathbb{N}$, there is an infinite family $\{P_i\}_{i\in\mathbb{N}}$ of d-polytopes of size n and width W. Then, $\{P_i \times [0, W]\}_{i\in\mathbb{N}}$ is a sequence of $(d+1)$-polytopes of size $n(W+1)$ and width W. The same argument of the previous lemma shows that $\{P_i \times [0, W]\}_{i\in\mathbb{N}}$ is infinite. Hence $w^\infty(d+1) \geq W$. □

2.1.2 The finiteness threshold width is finite

The following lemma provides us with a bound for the finiteness threshold width:

Lemma 2.11. *Let $d < n \in \mathbb{N}$. All but finitely many lattice d-polytopes of size bounded by n are hollow. Furthermore, all but finitely many of the hollow d-polytopes admit a projection to some hollow $(d-1)$-polytope.*

Proof. By Theorem 1.19 (Hensley), all lattice d-polytopes of size $\leq n$ and with interior points have a bound for the volume. In turn, by Theorem 1.20 (Lagarias–Ziegler) there are only finitely many of them. Hence, all but finitely many lattice d-polytopes of size bounded by n are hollow.

Finally, by Theorem 1.11 (Nill–Ziegler) all but finitely many hollow d-polytopes admit a projection to a hollow $(d-1)$-polytope. □

Corollary 2.12. *$w^\infty(d) \leq w_H(d-1)$ for all d. In particular, $w^\infty(d) \in O(d^{3/2})$.*

Proof. The previous lemma implies $w^\infty(d,n) \le w_H(d-1)$ for each n, so $w^\infty(d) \le w_H(d-1)$. $w_H(d-1) \in O(d^{3/2})$ follows from Theorem 1.7 (Banaszczyk et. al). $\square$

Another consequence of Lemma 2.11 is that, for fixed d and n, the width of lattice d-polytopes of size n is bounded. With this we can reformulate the definition of $w^\infty(d)$ in a way that allows us to better obtain information on it:

Corollary 2.13. *For each d and each $n \ge d+1$, $w^\infty(d,n)$ equals zero if there are only finitely many lattice d-polytopes of size n, and it otherwise equals the maximal width W for which there are infinitely many lattice d-polytopes of size n and width W.*

Proof. The reason why this statement might not be equivalent to the definition of $w^\infty(d,n)$ is that there could be d-polytopes of size n and arbitrarily large width, but only finitely many for each width. Lemma 2.11 prevents this. All but finitely many lattice d-polytopes of size n have width bounded by $w_H(d-1)$, which is finite. $\square$

2.1.3 Finiteness threshold width via polytopes with infinitely many lifts of bounded size

Definition 2.14. *Let Q be a lattice $(d-1)$-polytope. A lift of Q is a lattice polytope $P \subset \mathbb{R}^d$ such that $\pi(P) = Q$, for $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ a lattice projection. Without loss of generality, we will typically assume π to be the map that forgets the last coordinate.*

Two lifts $\pi_1 : P_1 \to Q$ and $\pi_2 : P_2 \to Q$ are equivalent if there is a unimodular equivalence $f : P_1 \to P_2$ with $\pi_2 \circ f = \pi_1$. That is, if for each $p \in \mathbb{Z}^d$, $f(p) \in \pi_2^{-1}(\pi_1(p))$ (the equivalence maps a point in the fiber of p under π_1, to a point in the fiber of p under π_2).

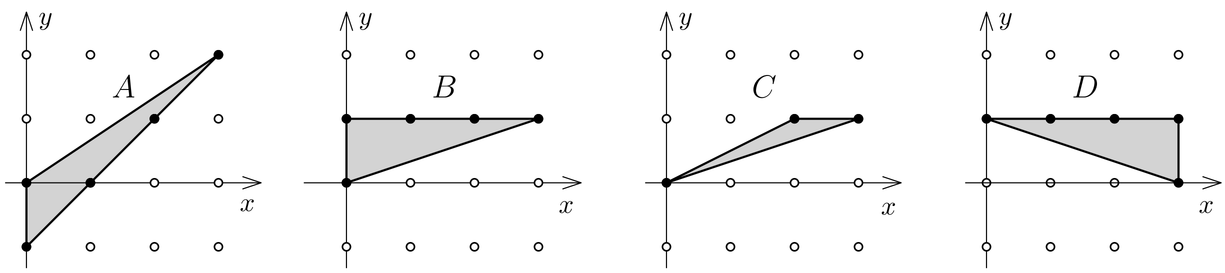


Figure 2.2: Polytopes $A, B, C, D \subset \mathbb{R}^2$ are lifts of $[0,3] \subset \mathbb{R}$ under projection $\pi(x,y) = x$. Only A and B are equivalent lifts. D is equivalent to A and B as a lattice polytope, but not as a lift of $[0,3]$ under π.

Remark 2.15. Notice that two non-equivalent lifts $\pi_1 : P_1 \to Q$ and $\pi_2 : P_2 \to Q$, with P_1 and P_2 equivalent polytopes, are equivalent lifts modulo an automorphism of Q.

That is, there exists a unimodular automorphism $t : Q \to Q$ such that $\pi_2 \circ f = t \circ \pi_1$, for some unimodular equivalence $f : P_1 \to P_2$.

In the rest of the chapter we often use the expression "Q has only finitely many lifts of bounded size". What we precisely mean is: for every $n \in \mathbb{N}$ there are only finitely many equivalence classes of lifts of Q with size n. Accordingly, "Q has infinitely many lifts of bounded size" means that there is a size $n \in \mathbb{N}$ for which there are infinitely many equivalence classes of lifts of Q.

Theorem 2.18 relates the width of a polytope with the width of its lifts. For its proof, we need a couple of technical lemmas. In the first one we only assume Q to be a convex body.

Lemma 2.16. *Let $Q \subset \mathbb{R}^d$ be a full-dimensional convex body, and $W \in \mathbb{N}$. Then, there is only a finite number of functionals $\ell \in (\mathbb{Z}^d)^*$ such that width$_\ell(Q) \leq W$.*

Proof. As detailed in the proof of Proposition 1.5, $\mathrm{width}_\ell(Q) \leq W$ if, and only if, $\ell \in W \cdot (Q-Q)^\vee$. Since Q is full-dimensional, $Q-Q$ has the origin in its interior and $(Q-Q)^\vee$ is bounded. Thus the number of lattice points in $W \cdot (Q-Q)^\vee$ is finite. □

A lift of a lattice polytope Q may have the same dimension as Q and still not be unimodularly equivalent to it. For example, the segment $[0,k]$ in $\mathbb{R}^1$ can be lifted to the primitive segment $\mathrm{conv}\{(0,0),(k,1)\}$. However, the number of different such lifts of Q is finite, modulo the equivalence relation in Definition 2.14:

Lemma 2.17. *A lattice $(d-1)$-polytope Q has only finitely many $(d-1)$-dimensional lifts.*

Proof. Every $(d-1)$-dimensional lift P of Q can be described as follows: there is an affine map $f : \mathbb{R}^{d-1} \to \mathbb{R}$ with

$$P = \mathrm{conv}\{(v, f(v)) : v \text{ is a vertex of } Q\},$$

and such that f is integer in all vertices of Q. Assuming, without loss of generality, that f is linear and the origin is a vertex of Q, this implies $f \in \Lambda(Q)^*$, where $\Lambda(Q) \subseteq \mathbb{Z}^{d-1}$ is the lattice spanned by the vertices of Q. Two such functionals give equivalent lifts if, and only if, they are in the same class modulo $(\mathbb{Z}^{d-1})^*$. Thus, the number of different lifts equals the index of $(\mathbb{Z}^{d-1})^*$ in $\Lambda(Q)^*$. □

Theorem 2.18. *Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope of width W. Then all lifts $P \subset \mathbb{R}^d$ of Q have width $\leq W$. All but finitely many of them have width $= W$.*

Proof. As a convenient notation, for a given vector $v = (v_1, \dots, v_d)$, we set $\tilde{v} = (v_1, \dots, v_{d-1})$. As projections do not decrease the width, we only need to prove that all but finitely many lifts of Q have width $\geq W$. For this, let $\ell \in (\mathbb{Z}^d)^*$ be a functional and P be some lift of Q.

If $\ell_d = 0$, then $\mathrm{width}_\ell(P) = \mathrm{width}_{\tilde{\ell}}(Q) \geq W$, so for the rest of the proof we assume that $\ell_d \neq 0$ for all our functionals.

Let $T \subseteq Q$ be a $(d-1)$-simplex with $\mathrm{vert}(T) \subseteq \mathrm{vert}(Q)$ and S be one of the finitely many $(d-1)$-dimensional lifts of T (see Lemma 2.17). Every lift P of Q has to contain one of these S, so it suffices to show that there are only finitely many such P having width $< W$. Furthermore we can assume that both P and S are contained in $Q \times \mathbb{R}_{\geq 0}$. Let H be the hyperplane containing S, and $L_1 := \max\{x_d \,:\, x \in H \cap (Q \times \mathbb{R})\} + 1$ (the value L_1 is such that $Q \times \{L_1\}$ lies strictly over $H \cap (Q \times \mathbb{R})$; see Figure 2.3). Then $R := \bigcap_{x \in Q \times \{L_1\}} \mathrm{conv}(S \cup \{x\})$ is a full-dimensional convex body, which following Lemma 2.16 implies that there are only finitely many integer functionals such that $\mathrm{width}_\ell(R) \leq W$. Let $\ell^1, \dots, \ell^s$ be those functionals and set $D := \max_{x \in Q,\, i \in \{1,\dots,s\}} |\tilde{\ell^i} \cdot x|$ and $L := \max\{L_1, 2D + W\}$.

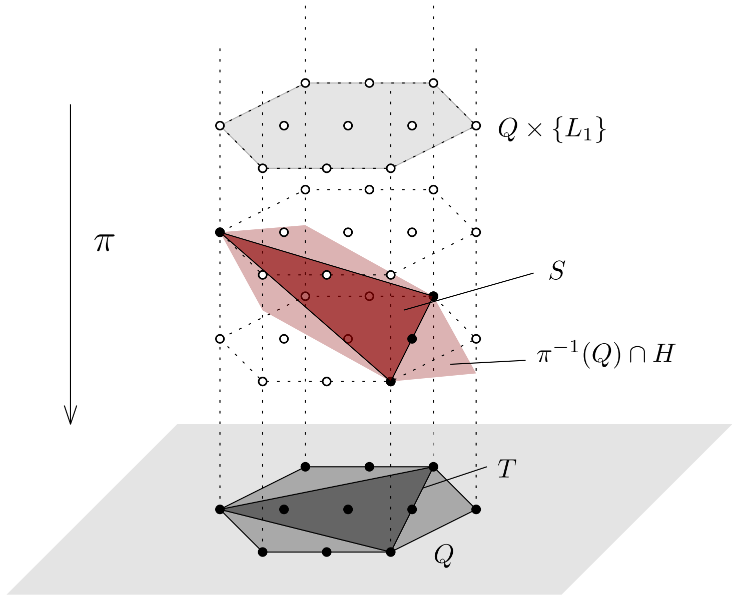


Figure 2.3: The setting of the proof of Theorem 2.18.

Now assume that there exists a point $p \in P$ with $p_d \geq 2L$ and let $\hat{S} := \operatorname{conv}(S \cup \{p\}) \subset P$. Then thanks to $p_d \geq L_1$, we have $R \subseteq \hat{S}$. Suppose ℓ is such that $\operatorname{width}_\ell(P) \leq W$, then $\operatorname{width}_\ell(R) \leq \operatorname{width}_\ell(\hat{S}) \leq \operatorname{width}_\ell(P) \leq W$ and hence $\ell = \ell^i$ is one of the finitely many functionals giving width $\leq W$ to R. But if ℓ is one of those functionals, and for any $r \in R$, the following happens:

$$\begin{aligned}
\operatorname{width}_\ell(P) &\geq \operatorname{width}_\ell(\hat{S}) = \max_{x,y,\in\hat{S}} |\ell \cdot (x-y)| \underset{p,r\in\hat{S}}{\geq} |\ell \cdot (p-r)| = \\
&= |\ell_d(p_d - r_d) + \tilde{\ell} \cdot (\tilde{p} - \tilde{r})| \geq |\ell_d(p_d - r_d)| - |-\tilde{\ell} \cdot (\tilde{p} - \tilde{r})| = \\
&= |\ell_d||p_d - r_d| - |-\tilde{\ell} \cdot \tilde{p} + \tilde{\ell} \cdot \tilde{r}| \underset{r_d \leq L_1 \leq L,\ p_d \geq 2L}{\geq} \\
&\geq |\ell_d|L - |-\tilde{\ell} \cdot \tilde{p} + \tilde{\ell} \cdot \tilde{r}| \underset{|l_d|\geq 1}{\geq} L - |-\tilde{\ell} \cdot \tilde{p} + \tilde{\ell} \cdot \tilde{r}| \geq \\
&\geq L - \left(|\tilde{\ell} \cdot \tilde{p}| + |\tilde{\ell} \cdot \tilde{r}|\right) \underset{\tilde{p},\tilde{r}\in Q}{\geq} L - 2D \geq W.
\end{aligned}$$

That is, P has width exactly W.

Hence all lifts P that extend S and contain a point p with $p_d \geq 2L$ have width W. That leaves us with those P that are contained in $Q \times [0, 2L]$, but there are only finitely many lattice subpolytopes of $Q \times [0, 2L]$ and hence only finitely many of width $< W$. $\square$

2.1.4 Tight lifts

We finish this section showing that in order to decide whether a given Q has infinitely many lifts it is enough to look at *tight* lifts. This will simplify the work in Sections 2.3 and 2.2:

Definition 2.19. *Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope. We say that a lift $P \subset \mathbb{R}^d$ of Q is* tight *if the projection sending P to Q bijects their sets of vertices. That is, if $P = \operatorname{conv}\{(v, h_v) : v \in \operatorname{vert}(Q)\}$ for some $\mathbf{h} \in \mathbb{Z}^{\operatorname{vert}(Q)}$.*

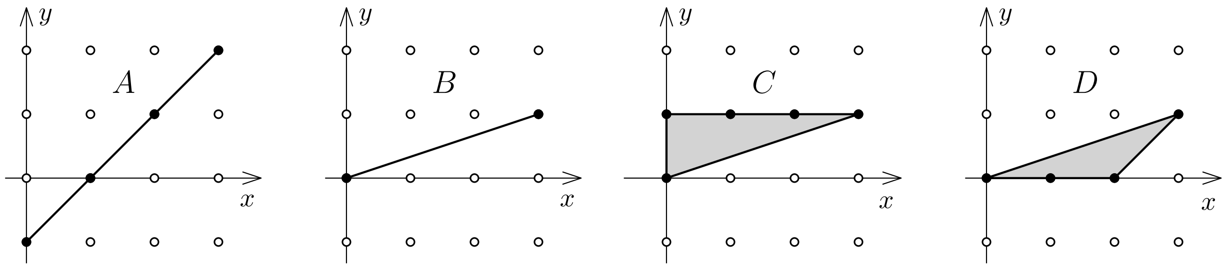


Figure 2.4: Polytopes $A, B, C, D \subset \mathbb{R}^2$ are lifts of $[0, 3] \subset \mathbb{R}$ under projection $\pi(x, y) = x$. A and B are tight lifts; C and D are not.

Notice that a tight lift is not necessarily full-dimensional and that every lift of Q contains a tight lift.

Lemma 2.20. *Let $P \subset \mathbb{R}^d$ be a (not necessarily full-dimensional) lift of a lattice $(d-1)$-polytope Q. Then, there are only finitely many lifts of Q of bounded size and that contain P.*

Proof. For each $q \in Q \cap \mathbb{Z}^{d-1}$, let $h_q \in \mathbb{R}$ such that $p_q = (q, h_q) \in P$, which exists since P projects to Q.

Let $P' \subset \mathbb{R}^d$ be a lift of Q that contains P. Let $p' \in P' \cap \mathbb{Z}^d$, then $p' = (q, h')$ for $q \in Q \cap \mathbb{Z}^{d-1}$ and $h' \in \mathbb{Z}$. Without loss of generality assume that $h' \geq h_q$ (the other case is symmetric). Then P' contains the segment $\operatorname{conv}\{p_q, p'\} = \{q\} \times [h_q, h'] \subset P'$, which contains already $h' - \lceil h_q \rceil + 1$ lattice points. Since the size of P' is bounded, there are only finitely many possibilities for h', hence for all the points of P'.

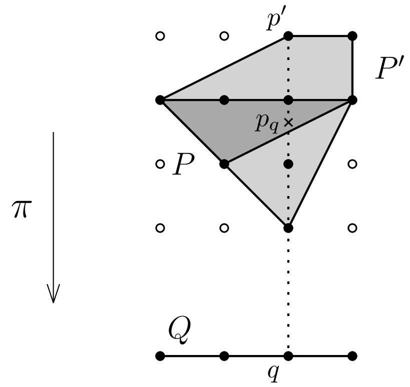


Figure 2.5: The setting of proof of Lemma 2.20.

□

The results in this section lead to the following:

Corollary 2.21. *Let $Q \subset \mathbb{R}^{d-1}$ be a lattice $(d-1)$-polytope. The following are equivalent:*

(1) There exist infinitely many lattice d-polytopes projecting to Q of bounded size and the same width as Q.

(2) Q has infinitely many d-dimensional lifts of bounded size and the same width as Q.

(3) Q has infinitely many lifts of bounded size.

(4) Q has infinitely many tight lifts of bounded size.

In any of those cases, the width of Q is a lower bound for $w^\infty(d)$.

Proof. Since there are only finitely many automorphisms of Q, (1) $\iff$ (2) comes from Remark 2.15. (2) $\iff$ (3) follows from the fact that there exist only finitely many $(d-1)$-dimensional lifts of Q (Lemma 2.17) and that all but finitely many lifts of Q have width $\mathrm{width}(Q)$ (Lemma 2.16). Finally, (4) $\implies$ (3) is trivial, and (3) $\implies$ (4) comes from Lemma 2.20 and the fact that any lift of a polytope contains a tight lift.

That $w^\infty(d) \geq \mathrm{width}(Q)$ is by definition of the finiteness threshold width. □

2.2 Polytopes with only finitely many lifts of bounded size

In this section we look at polytopes with only finitely many lifts of bounded size. It is fairly easy to see that simplices and some pyramids have this property:

Lemma 2.22. *Lattice simplices have only finitely many lifts of bounded size.*

Proof. A tight lift $P \subset \mathbb{R}^d$ of a lattice $(d-1)$-dimensional simplex is the convex hull of d points in $\mathbb{R}^d$. That is, P is $(d-1)$-dimensional, and by Lemma 2.17 there are only finitely many of those. Hence a lattice simplex has only finitely many tight lifts, regardless of the size. Corollary 2.21 implies the statement. □

For the rest of the chapter, we will call a *lattice pyramid* a lattice polytope such that all but one of the vertices lie in the same facet. This facet is the *basis* of the pyramid, and the vertex that is not in it is the *apex*.

Lemma 2.23. *Let Q be a lattice pyramid with basis B and apex v. If B has only finitely many lifts of bounded size, then so does Q.*

Proof. Let Q be $(d-1)$-dimensional. Any tight lift of Q is of the form $P(\tilde{B}, h) := \mathrm{conv}(\tilde{B} \cup \{\tilde{v}\})$, where $\tilde{B}$ is a tight lift of B and $\tilde{v} = (v, h)$ is a point in the fiber of v. Since $\tilde{B}$ is contained in some hyperplane H orthogonal to $\{x_d = 0\}$ and containing $B \times \{0\}$, $P(\tilde{B}, h)$ is a pyramid with basis $\tilde{B}$ and apex v (see Figure 2.6).

Let m be the distance from v to B, $P(\tilde{B}, h)$ is equivalent to $P(\tilde{B}, h+m)$ for all $h \in \mathbb{Z}$ (take the unimodular transformation that fixes the hyperplane H and shifts the parallel hyperplanes vertically by their distance to H). That is, there are at most $m-1$ values of h that give non-equivalent tight lifts $P(\tilde{B}, h)$, for any fixed $\tilde{B}$.

Then, if B has only finitely many tight lifts $\tilde{B}$ of bounded size, then there are only finitely many tight lifts of Q of bounded size. Corollary 2.21 implies the statement. □

Notice Lemma 2.22 follows also from this last result, by induction on the dimension and using the fact that a single lattice point has only finitely many lifts of bounded size.

Another property that affects the finiteness or not of the number of lifts of a polytope, is the existence of interior points. We want to show that non-hollow lattice polytopes have only finitely many lifts of bounded size. The following geometric lemma (in which Q need not be a lattice polytope) will be helpful:

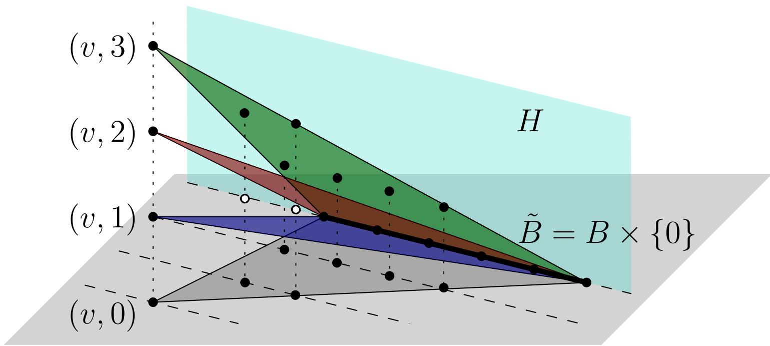


Figure 2.6: The setting of the proof of Lemma 2.23. In the figure, the case when $\tilde{B}$ is the tight lift $B \times \{0\}$ is represented. The apex v is at distance 3, hence $(v, 3)$ yields an equivalent lift as $(v, 0)$, while $(v, 1)$ and $(v, 2)$ do not.

Lemma 2.24. *Let $\pi : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be the standard projection that forgets the last coordinate. Let q be a point in the interior of a $(d-1)$-polytope Q. Then, there is a $c \in \mathbb{R}$ such that for every d-polytope $P \in \mathbb{R}^d$ with $\pi(P) = Q$ we have*

$$\mathrm{vol}(P) \leq c \cdot \mathrm{length}(P \cap \pi^{-1}(q)).$$

Proof. Assume without loss of generality that q is the origin and that the vertical segment $P \cap \pi^{-1}(q)$ goes from $(q, 0)$ to $(q, 1)$. This is no loss of generality since the parameter $\mathrm{length}(P \cap \pi^{-1}(q))/\mathrm{vol}(P)$ does not change by translations or vertical dilation/contraction of P. Observe that in this statement we do not assume P to be a lattice polytope. Under these assumptions what we want to show that there is a global upper bound c for the volume of P.

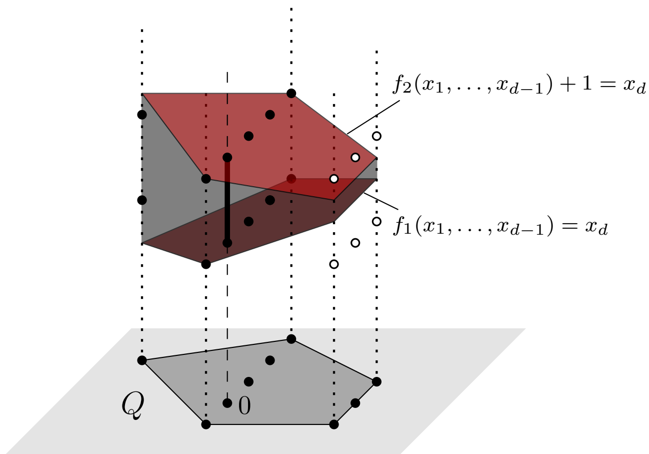


Figure 2.7: The setting of the proof of Lemma 2.24. The figure shows the d-dimensional polytope $\pi^{-1}(Q) \cap \{f_1(x_1, \dots, x_{d-1}) \leq x_d \leq f_2(x_1, \dots, x_{d-1}) + 1\}$.

By considering respective supporting hyperplanes of P at $(q, 0)$ and $(q, 1)$ we see that P is contained in the region $f_1(x_1, \dots, x_{d-1}) \leq x_d \leq f_2(x_1, \dots, x_{d-1}) + 1$, for some linear functionals $f_1, f_2 \in (\mathbb{R}^{d-1})^*$, and there is no loss of generality in assuming that P actually equals the intersection of $\pi^{-1}(Q)$ with that region (see Figure 2.7). Now, for $\pi(P)$ to equal

Q we need $f_1 \leq f_2 + 1$ on Q, which is equivalent to saying that $f_1 - f_2$ is in the polar $Q^\vee$ of Q. The volume of P is a continuous function of the functional $f_1 - f_2$. (In fact, it equals the integral in Q of the function $f_2 + 1 - f_1$. Since the origin is in the interior of Q, $Q^\vee$ is compact, and there is a global bound on the volume of P. ☐

Corollary 2.25. *A non-hollow lattice polytope has only finitely many lifts of bounded size.*

Proof. Let q be a point in the interior of $Q \subset \mathbb{R}^{d-1}$. A bound n for the size of a lift P of Q implies a bound $n+1$ for the length of $\pi^{-1}(q) \cap P$. By Lemma 2.24, this gives a bound for the volume of P. Since there are only finitely many lattice d-polytopes with bounded volume (see Hensley 1.19), the result follows. ☐

2.3 Hollow polytopes with infinitely many lifts of bounded size

In light of the results of the previous section, in order to find polytopes with infinitely many lifts of bounded size, we need to look for polytopes that are neither simplices, nor pyramids, and that have no interior points. The following result is the least restrictive we could find a proof for, in terms of sufficient properties of a polytope having infinitely many lifts of bounded size.

Lemma 2.26. *Let Q be a hollow $(d-1)$-polytope and let $v \in \operatorname{vert}(Q)$ be such that $Q' := \operatorname{conv}(\operatorname{vert}(Q) \setminus \{v\})$ is $(d-1)$-dimensional (that is, Q is not a pyramid with apex at v). Suppose that every proper face F of Q with $v \in F$ is either hollow or a pyramid with apex v. Then, for every $h \in \mathbb{Z}_{>0}$ the d-dimensional tight lift*

$$P(h) := \operatorname{conv}(Q' \times \{0\}) \cup \{(v, h)\}$$

of Q has size$(P(h)) \leq$ size(Q), with equality for infinitely many values of h, and different choices of h yield non-equivalent lifts of Q.

Proof. Let $q \in Q \cap \mathbb{Z}^{d-1}$, which has to be a point in the boundary. We claim that the fiber $\pi^{-1}(q)$ has at most one lattice point in P, with equality in many cases. For this, let F be the carrier face of q in Q; that is, the unique face of Q with $q \in \operatorname{relint}(F)$. Since Q is hollow, F is a proper face. Notice that a proper face that does not contain v is entirely lifted at height 0, and any face containing v has this vertex lifted to (v, h) and every other vertex at height 0.

By assumption, there are three possibilities for F:

- *F does not contain v.* Then $\pi^{-1}(F) \cap P = F \times \{0\}$. In particular, $(q, 0)$ is the only lattice point of P in the fiber $\pi^{-1}(q)$.
- *$v \in F$ and F is hollow.* Since $q \in \operatorname{relint}(F)$, we must have $F = \{q\} = \{v\}$. In particular, (v, h) is the only lattice point of P in the fiber $\pi^{-1}(q)$.
- *F is a pyramid with apex at v.* Let B be the base of the pyramid. The face $\pi^{-1}(F) \cap P$ of P equals the affine image of F under the map $x \mapsto (x, h \cdot \operatorname{dist}(B, x)/\operatorname{dist}(B, v))$, where $\operatorname{dist}(B, x)$ denotes the lattice distance from x to B as defined in Section 1.1. Thus, $(q, h \cdot \operatorname{dist}(B, q)/\operatorname{dist}(B, v))$ is the only point of P in the fiber $\pi^{-1}(q)$. That point will be a lattice point if (but perhaps not only if) h is an integer multiple of $\operatorname{dist}(B, v)$.

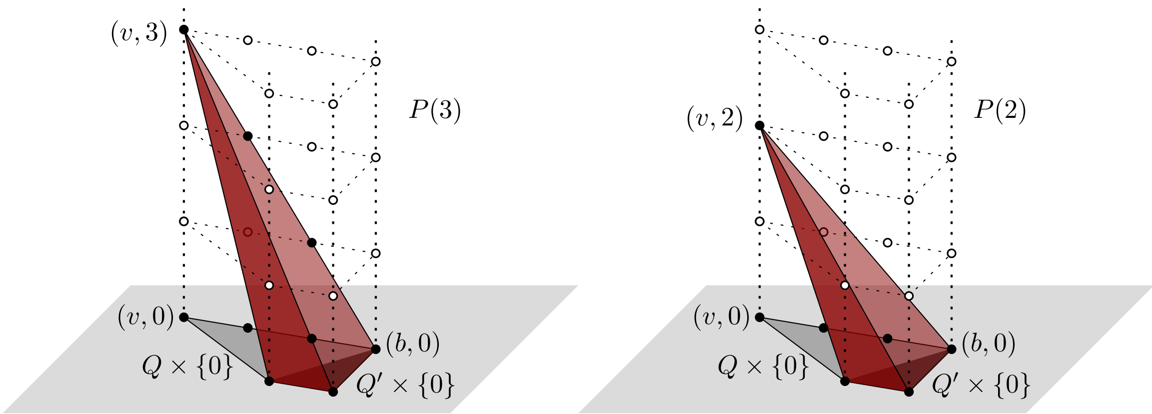


Figure 2.8: The setting of Lemma 2.26. The figure shows the hollow polygon Q and two of its tight lifts $P(h)$. One of the proper faces of Q containing v is empty, and the other is a 1-dimensional pyramid over a point b, with the distance from v to b being 3. This implies that $h = 3$ yields $P(3)$ with as many lattice points as Q, and $h = 2$ yields $P(2)$ with strictly less lattice points.

In particular, we have $\mathrm{size}(P(h)) = \mathrm{size}(Q)$ for any h that is an integer multiple of $\mathrm{lcm}\{\mathrm{dist}(B, v) : F \text{ face of } Q \text{ that is a pyramid with base } B \text{ and apex } v\}$.

Finally, the polytopes $P(h)$ are unimodularly non-isomorphic for different values of $h \in \mathbb{Z}_{>0}$, since their volume is proportional to $|h|$.

□

The following weaker version of the previous result will be used later in this chapter.

Corollary 2.27. *Let Q be hollow and not a simplex. If Q is either empty or simplicial then it has infinitely many lifts of bounded size.*

Proof. Since Q is not a simplex, there is a vertex v such that Q is not a pyramid with apex at v. Being empty or simplicial guarantees that the conditions of Lemma 2.26 for v are met. □

This result gives us a way to lower-bound $w^\infty(d)$; if a $Q \subset \mathbb{R}^{d-1}$ is in the conditions of the corollary, then $w^\infty(d) \geq \mathrm{width}(Q)$.

Lemma 2.28. *Let Q be a hollow-maximal or empty-maximal d-polytope, for $d \geq 2$. Then, for every vertex v of Q there is a lattice point $u \in Q$ that is not contained in any facet containing v.*

Proof. Let v be a vertex of Q and suppose that every lattice point of Q is in a facet containing v. We claim that this contradicts Q being hollow-maximal or empty-maximal. For this, let $C_v = v + \mathbb{R}^+(Q - v)$ be the cone of Q at v, then all the lattice points of Q lie in the boundary of the cone. Let $u \in \mathrm{int}(C_v) \cap \mathbb{Z}^d$ be such that u is the only lattice point of $Q' := \mathrm{conv}(Q, u)$ in the interior of C_v. (Such a u can be found, for example, minimizing in $\mathrm{int}(C_v) \cap \mathbb{Z}^d$ any supporting linear functional of C_v). Then Q' strictly contains Q and it is still empty or hollow if Q was empty or hollow, respectively (see Figure 2.9).

□

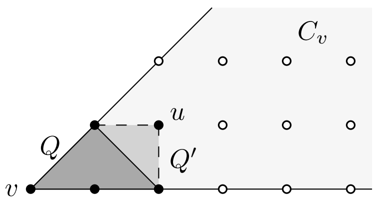


Figure 2.9: The setting of the proof of Lemma 2.28.

With this we can now prove that $w^\infty(d)$ is at least $\max\{w_E(d-1), w_H(d-2)\}$.

Corollary 2.29. *For every $d \geq 3$ there exists an empty $(d-1)$-polytope of width $w_E(d-1)$ with infinitely many lifts of bounded size. In particular, $w^\infty(d) \geq w_E(d-1)$.*

Proof. Lemma 2.28 implies that $w_E(d-1)$ is achieved by a non-simplex Q, and then Lemma 2.27 shows Q has infinitely many lifts of bounded size. Corollary 2.21 proves that $w^\infty(d) \geq \operatorname{width}(Q) = w_E(d-1)$. □

We call *bipyramid* a polytope with two vertices u and v such that every facet is a pyramid with apex at u or v, and no facet contains both. Notice that we do not require the rest of the vertices to be coplanar.

Lemma 2.30. *For every $d \geq 2$ there exists a hollow bipyramid of dimension d and width $w_H(d-1)$.*

Proof. By induction on d. For $d = 2$, the unit square is a hollow bipyramid of width $1 = w_H(1)$. For higher d, let us first see that there exists a hollow $(d-1)$-polytope Q of width $w_H(d-1)$ and having two lattice points u and v not sharing any facet.

- If $w_H(d-1) = w_H(d-2)$ then let Q be a hollow bipyramid of dimension $d-1$ and width $w_H(d-2)$, which exists by induction hypothesis.
- If $w_H(d-1) > w_H(d-2)$ then there are only finitely many hollow $(d-1)$-polytopes of width $w_H(d-1)$, hence there is one such Q that is maximal. By Lemma 2.28, there are lattice points u and v in Q not contained in the same facet.

Consider now, for $h \in \mathbb{Z}_{>0}$, the polytope

$$\operatorname{conv}\Big((Q \times \{0\}) \cup \{(u,h),(v,-h)\}\Big).$$

This is a hollow d-dimensional bipyramid and, for sufficiently large h, it has the same width of Q (by Theorem 2.18, since different values of h yield non-equivalent lifts of Q). □

Corollary 2.31. *For every $d \geq 3$ there exists a hollow $(d-1)$-polytope of width $w_H(d-2)$ with infinitely many lifts of bounded size. In particular, $w^\infty(d) \geq w_H(d-2)$.*

Proof. Let Q be a hollow $(d-1)$-dimensional bipyramid of width $w_H(d-2)$, which exists by Lemma 2.30. Hollow bipyramids are clearly in the conditions of Lemma 2.26, hence Q has infinitely many lifts of bounded size. Corollary 2.21 proves that $w^\infty(d) \geq \operatorname{width}(Q) = w_H(d-2)$. □

This finally allows us to prove that:

Theorem 2.32. *For all $d \geq 3$, $w^\infty(d)$ equals the maximum width of a lattice $(d-1)$-polytope Q for which there are infinitely many lattice d-polytopes projecting to Q of some fixed size $n \in \mathbb{N}$. Moreover, every such Q is hollow and not a simplex.*

Proof. That $w^\infty(d)$ is at least the width of any $(d-1)$-polytope with infinitely many lifts of bounded size follows from Corollary 2.21.

For the other inequality, Corollary 2.31 proves the statement in the case when $w^\infty(d) = w_H(d-2)$ (it proves as well that $w^\infty(d) \geq w_H(d-2)$).

The only remaining case is then when $w^\infty(d) > w_H(d-2)$. First of all, since $w^\infty(3) \geq 1$, by Proposition 2.10 we have that $w^\infty(d) > 0$ for all $d \geq 3$ (this guarantees the existence of infinitely many lattice d-polytopes of some fixed size). Let n be such that $W := w^\infty(d) = w^\infty(d, n)$. That is, there is an infinite family $\{P_i\}_{i\in\mathbb{N}}$ of lattice d-polytopes of size n and width W. Without loss of generality (Lemma 2.11) assume all P_i's are hollow and have a hollow projection Q_i. Since projecting does not decrease the width, every Q_i has width at least W, and since $W = w^\infty(d) > w_H(d-2)$ no Q_i admits a hollow projection to dimension $d-2$. By Nill–Ziegler 1.11, this implies the family $\{Q_i\}_{i\in\mathbb{N}}$ to be finite, so one of them, call it Q, lifts to infinitely many members of the family $\{P_i\}_{i\in\mathbb{N}}$. Theorem 2.18 implies then that Q has width *exactly* W.

The first part of the statement then follows from Corollary 2.21. That Q must be hollow and not a simplex follows from the fact that simplices and lattice polytopes with interior points have only finitely many lifts of bounded size (Lemma 2.22 and Corollary 2.25). ☐

2.4 The finiteness threshold width in dimensions 3 and 4

According to Theorem 2.32, in order to derive the exact value of $w^\infty(d)$ we need to look at the largest width of a hollow $(d-1)$-polytope with infinitely many lifts of bounded size. Since the classification of hollow polytopes is complete up to dimension 3, we will be able to determine $w^\infty(3)$ and $w^\infty(4)$.

In the introduction to this chapter, it was already established that $w^\infty(3) \geq 1$ and $w^\infty(4) \geq 2$ (see Example 2.8) by showing explicit hollow $(d-1)$-polytopes of those widths with infinitely many lifts of bounded size. We now explore hollow $(d-1)$-polytopes of widths larger than those (in their respective cases).

Theorem 2.33 (Finiteness Threshold Width in dimension 3, Blanco–Santos [BS16a, Corollary 22])**.** $w^\infty(3) = 1$. *That is, for each $n \geq 4$, there exist only finitely many lattice 3-polytopes of size n and width larger than one.*

Proof. By Theorem 1.8, the only hollow polygon of width larger than one is the second dilation of the unimodular triangle, which is a simplex and hence has only finitely many lifts of bounded size by Lemma 2.22. Theorem 2.32 then implies that $w^\infty(3) \leq 1$. That $w^\infty(3) \geq 1$ follows from Example 2.8. ☐

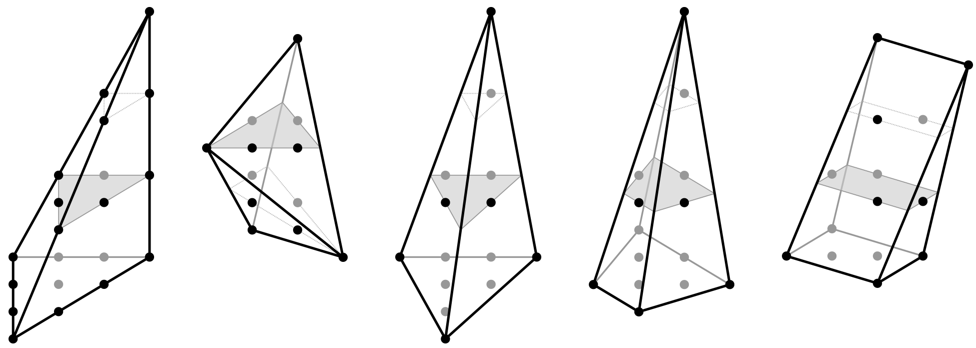
We now look at hollow 3-polytopes of width at least 3 and show that they all have only finitely many lifts of bounded size.

Corollary 2.34 (Finiteness Threshold Width in dimension 4)**.** $w^\infty(4) = 2$. *That is, for each $n \geq 5$, there exist only finitely many lattice 4-polytopes of size n and width larger than two.*

Proof. By Corollary 1.13, we know that 3 is the maximum width of hollow 3-polytopes, and that there exist only five hollow 3-polytopes of width 3, depicted in Figure 2.10.

$\mathcal{M}_3$, $\mathcal{M}_5$ and $\mathcal{M}_6$ are simplices and hence have only finitely many lifts of bounded size by Corollary 2.22. $\mathcal{M}_9$ and $\mathcal{M}_{10}$ have only finitely many lifts of bounded size by Propositions 2.35 and 2.36 below, respectively.

Theorem 2.32 then implies that $w^\infty(4) \leq 2$. That $w^\infty(4) \geq 2$ follows from Example 2.8. □



$\mathcal{M}_3$ $\mathcal{M}_5$ $\mathcal{M}_6$ $\mathcal{M}_9$ $\mathcal{M}_{10}$

Figure 2.10: The five hollow 3-polytopes of width three. This picture has been taken from Averkov et al [AKW15].

Proposition 2.35. *The pyramid $\mathcal{M}_9$ has only finitely many lifts of bounded size.*

Proof. The basis of the pyramid is a lattice polygon with three interior lattice points, which, by Corollary 2.25, has only finitely many lifts of bounded size. By Lemma 2.23, $\mathcal{M}_9$ also has only finitely many lifts of bounded size. □

Proposition 2.36. *The prism $\mathcal{M}_{10}$ has only finitely many lifts of bounded size.*

Proof. Let $u, v, w, u', v'w'$ be the vertices of the prism, where uu', vv', ww' are edges. Let $Q := \text{conv}\{u, v, w, u', v'\} \subset \mathcal{M}_{10}$, which is a quadrangular pyramid over a polygon with interior lattice points. Any tight lift of $\mathcal{M}_{10}$ will be of the form $P(\tilde{Q}, \tilde{w}')$, where $\tilde{Q}$ is a tight lift of Q and $\tilde{w}'$ is a point in the fiber of w'. By Lemma 2.23 and Corollary 2.25, there are only finitely many such $\tilde{Q}$ of bounded size. Fix one, and let us see that there are only finitely many possibilities for $\tilde{w}'$.

Each lift $\tilde{w}'$ (together with the fixed tight lift $\tilde{Q}$) induces a tight lift of the quadrilateral $R := \text{conv}\{u, w, u', w'\} \subset \mathcal{M}_{10}$. We claim that at most two choices of $\tilde{w}'$ correspond to equivalent lifts of R.

By fixing $\tilde{Q}$ we already have fixed a lift of the three vertices u, w, u'. These three lifts are contained in a plane Π. On the other hand, the possible lifts of the point w' are in the line $\pi^{-1}(w')$. This line is not contained in Π, so these tight lifts of R are all 3-dimensional (except for at most one lift of $\tilde{w}'$), and their volume is proportional to the distance between

$\tilde{w}'$ and Π. That is, each of the possibilities for $\tilde{w}'$ induces non-equivalent tight lifts of the quadrilaterals, up to (perhaps) reflection with respect to the plane Π.

Now, as the quadrilateral R contains interior points, Corollary 2.25 implies that it has finitely many lifts of bounded size. Infinitely many choices of $\tilde{w}'$ would then have unbounded size, and so would happen for $P(\tilde{Q}, \tilde{w}')$. That is, $\mathcal{M}_{10}$ has only finitely many tight lifts of bounded size. Corollary 2.21 implies the statement. $\square$

Chapter 3

Enumeration of lattice 3-polytopes of width larger than one

In this chapter we describe an algorithm to classify all lattice 3-polytopes of width larger than one and up to any given size. We have implemented the algorithm and run it up to size eleven. Running it for larger sizes requires either more efficient implementations or more computer power (or both).

Our starting point is the fact that there are only finitely many equivalence classes of lattice 3-polytopes of a given size n and width larger than one (Theorem 2.33).

Suppose that we know already the list of lattice 3-polytopes of size $n-1$ and width > 1. One can expect that *most* of the polytopes of size n can be obtained by "merging" two polytopes of size $n-1$ in the sense of the following definition. In it, remember that $P^v := \operatorname{conv}(\mathbb{Z}^d \cap P \setminus \{v\})$ for a lattice d-polytope P and a vertex v of it, and that $P^{v,w} := (P^v)^w$ (see Remark 1.23). Observe that if P has size n, P^v has size $n-1$ and P^{vw} has size $n-2$:

Definition 3.1. *We say that a lattice d-polytope P is a* merging of P_1 and P_2 *if P_1 and P_2 are both of width larger than one and there are vertices $v, w \in P$ such that $P_1 \cong P^v$, $P_2 \cong P^w$ and P^{vw} is d-dimensional.*

For given P_1 and P_2, the following algorithm computes all the polytopes P that can be obtained merging them:

Algorithm 3.2 (Merging)**.**
INPUT: two lattice d-polytopes P_1 and P_2 of size $n-1$ and width > 1.
OUTPUT: all the lattice d-polytopes of size n obtained merging P_1 and P_2.
For each vertex v_1 of P_1 and v_2 of P_2:

(1) Let $P_1' = (P_1)^{v_1} \subset P_1$ and $P_2' = (P_2)^{v_2} \subset P_2$.

(2) Check that P_1' and P_2' are d-dimensional.

(3) For each equivalence $t : \mathbb{R}^d \to \mathbb{R}^d$ with $t(P_1') = P_2'$, if the size of $P := \operatorname{conv}(t(P_1) \cup \{v_2\}) = \operatorname{conv}(\{t(v_1)\} \cup P_2)$ equals N, add P to the output list. (Observe that t may be not unique, but there are only finitely many possibilities for it).

In dimension 2, Figure 3.1 shows a merging of two polygons.

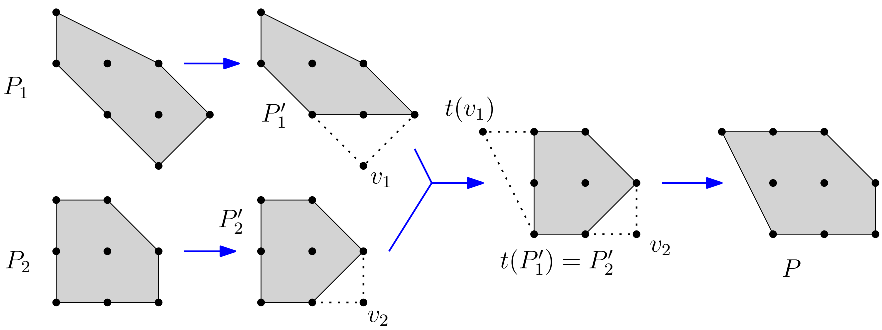


Figure 3.1: A lattice polygon P of size 9 constructed by merging two polygons of size 8.

Thanks to Algorithm 3.2, to completely enumerate lattice 3-polytopes of a given size and width > 1 we only need to understand (and enumerate) the lattice 3-polytopes that are *not* obtained by merging smaller polytopes. For this we introduce the following definitions:

Definition 3.3 (Quasi-minimal and merged polytopes)**.** *Let P be a lattice d-polytope of width > 1. We say that P is*

- Quasi-minimal *if it has at most one vertex v such that P^v is still of width larger than one.*
- Merged *if there exist some polytopes P_1 and P_2 such that P is obtained merging P_1 and P_2. Equivalently, if P has two vertices v, w such that P^v and P^w have width larger than one and $P^{v,w}$ is 3-dimensional.*

See Figures 3.2 and 3.3 for examples of a merged and a quasi-minimal 3-polytope, respectively.

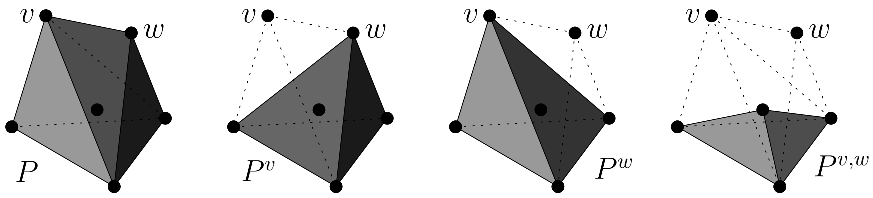


Figure 3.2: A merged 3-polytope P. The interior point of P is in the interior of both P^v and P^w, hence they have width larger than one. $P^{v,w}$ is 3-dimensional.

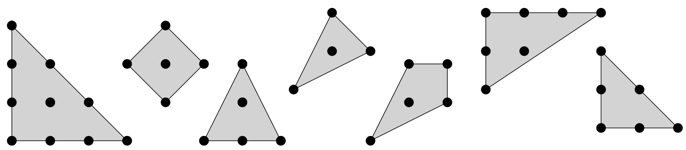
The MATLAB routine `is_quasiminimal` in page 154 computes whether a polytope is quasi-minimal or not.

Observe that a polytope P of width > 1 may have two vertices v_1 and v_2 with P^{v_1} and P^{v_2} of width larger than one and not be merged, because P^{v_1,v_2} can be 2-dimensional. (This is excluded in the definition of merging, because merging over a 2-dimensional intersection makes the set of transformations t to be tested in Algorithm 3.2 infinite). Section 3.1 is aimed at proving that there is no lattice 3-polytope of size greater than six in which this is a problem:

Theorem 3.4. *All lattice 3-polytopes P of size $n \geq 7$ and width larger than one are either quasi-minimal, or merged.*

Once this is established, the algorithm for classifying all lattice 3-polytopes of size $n > 6$ and width larger than one consists simply in computing all mergings of lattice 3-polytopes of size $n-1$ and width > 1 (which are assumed recursively precomputed), and adding to those the quasi-minimal ones. Computing mergings is done via Algorithm 3.2 (see Algorithm A.11 for the MATLAB routine implementing this), but the quasi-minimal polytopes need to be classified. The main tool for this is the following result:

Theorem 3.5. *Let P be a quasi-minimal 3-polytope with more than 11 lattice points. Then P projects to one of the following lattice polygons in such a way that all of the vertices in the projection have a unique element in the preimage.*



More precise versions of Theorems 3.4 and 3.5 are proved as Theorem 3.15 and Corollary 3.17 in Sections 3.1 and 3.2. There, we divide the quasi-minimal 3-polytopes into two types. Those that project as stated in Theorem 3.5 (or, more precisely, as in Corollary 3.17) are called *spiked* (see Definition 3.10) because *most of its lattice points lie in a lattice segment*. There are infinitely many spiked polytopes in total but only finitely many for each size, and they are very explicitly described in Section 3.2 (Theorems 3.18 and 3.19).

The rest, apart of having at most 11 lattice points, are called *boxed* (see Definition 3.11) because *most of its lattice points are vertices of a parallelepiped*. Boxed 3-polytopes are only finitely many, but they also have less structure so their enumeration is in fact more complicated than that of spiked ones. Their defining property is that P is boxed if there is a rational parallelepiped Q of width one with respect to every facet and such that at most three lattice points v_1, v_2 and v_3 of P do not lie in Q. Two problems arise, that we solve in Section 3.3:

(1) A priori there are many possibilities for Q. Lemma 3.22 shows that if P has size at least seven then there are only two: the unit cube and a certain parallelepiped with four integer and four non-integer vertices.

(2) A priori there are infinitely many possibilities to check for the v_i's. We solve this in Theorem 3.26, by showing that the v_i's must be at "distance" at most six from Q, which reduces the possibilities to only finitely many. (Observe that here, this distance is not refering to the lattice distance defined in the introduction, but to a more specific parameter that will be later properly defined.)

Once this is proved, a complete enumeration of boxed 3-polytopes is possible via a computer exhaustive search, as explained in Section 3.3.3. Table 3.1 in Section 3.4 shows the numbers of boxed and spiked 3-polytopes for each size and number of vertices.

After we have a classification of quasi-minimal polytopes we can run the algorithm. Our results are summarized as follows:

Theorem 3.6. *There are exactly 9, 76, 496, 2675, 11698, 45035 and 156464 lattice 3-polytopes of width larger than one and sizes 5, 6, 7, 8, 9, 10 and 11, respectively.*

Section 3.4 studies the output according to different properties of the polytopes.

3.1 Quasi-minimal polytopes: spiked vs. boxed

Let $A \subset \mathbb{Z}^d$ be a finite set of lattice points. In the rest of this section we assume the corresponding lattice polytope $\operatorname{conv}(A)$ to have width greater than one. For each vertex v of $\operatorname{conv}(A)$ we denote $A^v := A \setminus \{v\}$.

Definition 3.7. *We say that a vertex v of $\operatorname{conv}(A)$ is essential if $\operatorname{conv}(A^v)$ has width at most one. That is, if $\operatorname{conv}(A^v)$ either has width one or is $(d-1)$-dimensional.*

Let $\operatorname{vert}(A)$ be the set of all vertices of $\operatorname{conv}(A)$ and $\operatorname{vert}^*(A) \subseteq \operatorname{vert}(A)$ the set of essential vertices. We are primarily interested in the case $A = P \cap \mathbb{Z}^d$, for a lattice d-polytope P, in which case we use $\operatorname{vert}(P)$, $\operatorname{vert}^*(P)$ and P^v for $\operatorname{vert}(A)$, $\operatorname{vert}^*(A)$ and $\operatorname{conv}(A^v)$, but we also need to consider more general cases.

Definition 3.8. *We say that A is a minimal configuration if all its vertices are essential, and that it is a quasi-minimal configuration if at most one vertex is not essential. That is, if $\operatorname{vert}^*(A) = \operatorname{vert}(A)$ and $|\operatorname{vert}^*(A)| \geq |\operatorname{vert}(A)| - 1$, respectively. We call a d-polytope P minimal or quasi-minimal if $P \cap \mathbb{Z}^d$ is a minimal or quasi-minimal configuration, respectively.*

Observe that Definition 3.3 agrees with this. See Figure 3.3 for an example showing a quasi-minimal 3-polytope and its essential vertices.

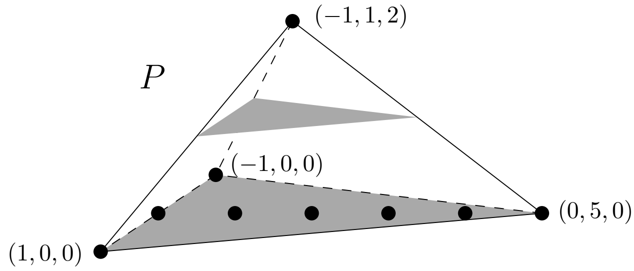


Figure 3.3: A quasi-minimal 3-polytope. Black dots are lattice points. The gray triangles represent the intersection of P with the planes $\{z = 0, 1\}$. Vertices $(-1,1,2)$, $(1,0,0)$ and $(-1,0,0)$ are essential ($P^{(-1,1,2)}$ is 2-dimensional; $P^{(1,0,0)}$ and $P^{(-1,0,0)}$ have width one). Vertex $(0,5,0)$ is the only non-essential vertex.

One of the main results in this chapter is the complete classification of quasi-minimal 3-polytopes. As a warm-up let us show that there are infinitely many of them in any dimension:

Proposition 3.9. *For every $d \geq 2$ and $k \geq 2$ the following lattice d-polytope with $2^{d-1}+1$ vertices and $2^{d-1}+k+1$ lattice points is quasi-minimal:*

$$\operatorname{conv}\{\pm e_1, \ldots, \pm e_{d-1}, k e_d\}.$$

Proof. It has width at least two since e_d is an interior lattice point in it. For every $i = 1, \ldots, d-1$, removing e_i or $-e_i$ gives width one with respect to the i-th coordinate. $\square$

A slight modification of this construction shows that in $d \geq 3$ there are infinitely many *minimal* polytopes:

$$\operatorname{conv}\{\pm e_1, \dots, \pm e_{d-2}, -e_{d-1}, e_{d-1} + ke_d\}.$$

In dimension 2, however, there are only four minimal polygons, as follows from the complete classification of quasi-minimal polygons that we work out in Lemma 3.14 below (see also Figure 3.4).

3.1.1 A dichotomy

Let us introduce the following two types of configurations:

Definition 3.10 (Spiked configuration). *Let $A \subset \mathbb{Z}^d$ be a quasi-minimal configuration and let $A' := \pi(A)$, where $\pi : \mathbb{Z}^d \to \mathbb{Z}^{d-1}$ is a lattice projection such that:*

- *Every vertex of $\operatorname{conv}(A')$ has a unique preimage in A.*
- *The projection bijects essential vertices of A and A'.*

Then, we say that A is spiked *with respect to A'.*

Definition 3.11 (Boxed configuration). *Let $A \subset \mathbb{Z}^d$ be a configuration of width larger than one, and let $Q \subset \operatorname{conv}(A)$ be a rational d-dimensional parallelepiped such that:*

- *The facets of Q are defined by lattice hyperplanes, with opposite facets at distance one. That is,*
$$Q = \bigcap_{i=1}^{d} f_i^{-1}([0, 1]),$$
where the f_i are affinely independent primitive functionals. In particular, only the vertices of Q can be integer points.
- *$A \setminus Q = \{v_1, \dots, v_d\}$, with $f_i(v_j) \notin \{0, 1\}$ if, and only if, $i = j$. In particular, each v_i is an essential vertex of A.*

Then, we say that A is boxed *with respect to Q.*

We say that a quasi-minimal d-polytope is *spiked* (resp. *boxed*) if its set of lattice points is a spiked (resp. boxed) configuration.

Remark 3.12. *The definition of spiked assumes A to be quasi-minimal, but the definition of boxed does not. Observe also the following immediate consequences of the definitions:*

- *The $(d-1)$-dimensional configuration A' in the definition of spiked is automatically quasi-minimal. More precisely, it has at most as many non-essential vertices as A.*
- *Boxed configurations have at most $d + 2^d$ lattice points: apart from $v_1, \dots, v_d$, only the 2^d vertices of Q can be lattice points of A.*

The reason for these seemingly strange definitions is the following result:

Theorem 3.13. *Every quasi-minimal configuration is spiked or boxed.*

Proof. Let $A \subset \mathbb{Z}^d$ be a quasi-minimal configuration. For each $v_i \in \mathrm{vert}^*(A)$ let f_i be an affine primitive functional with $A^{v_i} \subset f_i^{-1}(\{0,1\})$. Let $\hat{f}_i$ be the corresponding linear functional. Two things can happen:

(1) If the set $\{\hat{f}_i : v_i \in \mathrm{vert}^*(A)\}$ linearly spans $(\mathbb{R}^d)^*$, then assume without loss of generality that $\hat{f}_1, \dots, \hat{f}_d$ are independent. By construction A is boxed with respect to the parallelepiped

$$\bigcap_{i=1}^{d} f_i^{-1}([0,1]).$$

(2) If the set $\{\hat{f}_i : v_i \in \mathrm{vert}^*(A)\}$ does not linearly span $(\mathbb{R}^d)^*$, then there is a line where all the $\hat{f}_i$ are constant. That is, there exists an element $r \in \mathbb{Z}^d$ such that $\hat{f}_i(r) = 0$ for all i. Let $\pi : \mathbb{Z}^d \to L := \mathbb{Z}^d/\langle r\rangle \cong \mathbb{Z}^{d-1}$, and let f_i' be the functional in L defined by $f_i(x) = f_i'(\pi(x))$ for each $x \in \mathbb{Z}^d$. Since A has width greater than one, so does $A' := \pi(A)$ with respect to L. We claim that A is spiked with respect to A'.

Let us first see that each vertex of $\mathrm{conv}(A')$ has a unique preimage in A. For this, let v' be a vertex of $\mathrm{conv}(A')$ and suppose u, v are two different vertices of A projecting to v'. Then we would have $\pi(A^u) = \pi(A^v) = \pi(A) = A'$. But at least one of u and v must be an essential vertex of A. Say $v = v_i \in \mathrm{vert}^*(A)$, then A^v has width one with respect to f_i, which is constant in r. This would imply $\pi(A^v) = A'$ to have width one in L with respect to f_i', which is a contradiction.

Finally, let us prove that π bijects essential vertices of A and A'. Let v' be a vertex of $\mathrm{conv}(A')$ and let v be the unique vertex of $\mathrm{conv}(A)$ with $\pi(v) = v'$. Then $v \in \mathrm{vert}^*(A)$ if, and only if, A^v has width one with respect to a functional f with $\hat{f}(r) = 0$. This happens if, and only if, the corresponding functional f' gives width one to $A' \setminus \{v'\}$ in L, which in turn is equivalent to $v' \in \mathrm{vert}^*(A')$.

□

See Algorithm A.21 for the MATLAB routine that computes whether a polytope is boxed and/or spiked, following the idea in the proof of Theorem 3.13.

In dimension one the only minimal configurations are $\{0,1,2\}$ and $\{0,k\}$ for any $k \geq 2$. The only quasi-minimal ones that are not minimal are $\{0,1,k\}$, for any $k \geq 3$. In dimension two, classifying quasi-minimal configurations is not that easy, but Theorem 3.13 allows us to classify quasi-minimal polygons:

Lemma 3.14. *Every quasi-minimal polygon is unimodularly equivalent to one in Figure 3.4.*

Proof. An exhaustive search proves the list for lattice polygons with up to 6 points. So, for the rest of the proof let P be a quasi-minimal polygon of size $\geq 7 > 2^2 + 2$. Since it has to be spiked, Theorem 3.13 implies that P projects to $\{0,1,k\}$ or $\{0,k\}$ ($k \geq 2$) with a single point in the fibers of 0 and k. Hence it can only project to $\{0,1,k\}$ and have at least five points in the fiber of 1.

If $k = 2$, then P is a quasi-minimal triangle like one of the two in the statement with an arbitrary number of interior points. If $k \geq 3$, the fact that there are (at least) five points in the fiber of 1 implies that there must be some other lattice points of P in the fiber of 2. Indeed, the fiber of 1 is a segment of length at least 4, hence the fiber of 2 has length at least $4(k-2)/(k-1) \geq 2$. This cannot happen. □

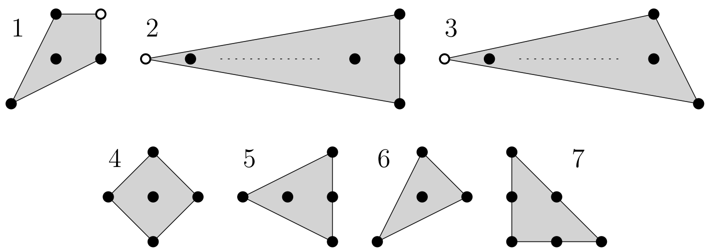


Figure 3.4: The quasi-minimal polygons. In the non-minimal ones (top row) the white dot is the non-essential vertex. Labels 1 to 7 are used in the proof of Theorem 3.15.

Observe that a polytope can be both spiked and boxed. An example is

$$P = \operatorname{conv}\{(1,0,0),(0,1,0),(-1,0,0),(0,-1,2)\},$$

whose set of lattice points is $A = \{(1,0,0),\ (0,1,0),(-1,0,0),(0,-1,2),(0,0,0),(0,0,1)\}$.

3.1.2 An exception

We said in the introduction that all lattice 3-polytopes of width > 1 and size at least 7 that are not quasi-minimal can be merged. Here we prove a more explicit result:

Theorem 3.15. *For every lattice 3-polytope P of width larger than one exactly one of the following happens:*

(1) P is quasi-minimal.

(2) P is merged.

(3) P has size 6 and is equivalent to

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}.$$

Proof. We first show that the three cases are mutually exclusive. Cases (1) and (2) are by definition, so we only need to check that the polytope in (3) is neither quasi-minimal nor merged. Geometrically, it is a triangular bipyramid with the origin as the common barycenter of the triangle $\operatorname{conv}\{(1,0,0),(0,1,0),(-1,-1,0)\}$ and the segment $\operatorname{conv}\{(1,2,3),(-1,-2,-3))\}$. The vertices in the triangle are essential (consider the functionals $-2x+y+z$, $x-2y+z$ and $x+y-z$) but the vertices $u=(1,2,3)$ and $v=(-1,-2,-3)$ in the segment are not (see the configuration of signature $(3,1)$ and width 2 in Table 1.1). This means P is not quasi-minimal (it has two non-essential vertices) but it is not merged either, since $P^{u,v}$ is two-dimensional.

So, to finish the proof it only remains to show that if P is not in the conditions of (1) or (2) then it is unimodularly equivalent to the configuration in part (3). Observe that P must have size at least six: every lattice 3-polytope of size four has width one, which implies that every lattice 3-polytope of size five and width larger than one is minimal.

Since P is neither quasi-minimal nor merged, it has at least two non-essential vertices v_1 and v_2 and $Q := P^{v_1,v_2}$ is lower dimensional. We assume without loss of generality that $Q \subset \mathbb{Z}^2 \times \{0\}$. On the other hand, notice that $P^{v_i} = \mathrm{conv}(Q \cup \{v_i\})$ has to be 3-dimensional and of width > 1, since v_i is not essential. We are going to conclude that the only possibility is the configuration in (3), by proving some properties of P and Q:

(a) *Q is 2-dimensional*, because if Q was contained in a line then P^{v_i} would be at most 2-dimensional.

(b) *v_1 and v_2 both lie at distance greater than one from Q*, since P^{v_1} or P^{v_2} have width greater than one.

(c) *Q does not contain a unimodular parallelogram*, by Lemma 1.29(3).

(d) *Q has width larger than one.*

Suppose not, so that the lattice points of Q lie in two consecutive parallel lattice lines in the plane $\mathrm{aff}(Q)$ that it spans. If both lines have at least two points, then Q contains a unimodular parallelogram, which contradicts (c). If one of the lines contains only one point, call it v, then $P^{v_1 v_2 v}$ is a lattice segment. Let a, b and c be three consecutive lattice points in $P^{v_1 v_2 v}$ (which exist since P has size at least six). By Lemma 1.29(2) we know that $\mathrm{conv}\{a, b, c, v, v_1\}$ of size 5 and with three collinear points a, b and c, must have width one with respect to a functional that is constant in those three points. Since that functional must then be constant in the whole segment $P^{v_1 v_2 v}$, P^{v_2} has width one as well.

(e) *None of the non-essential vertices of Q (if it has any) are vertices of P.*

Suppose otherwise, and let w be a non-essential vertex of Q that is a vertex of P. Since Q^w has width larger than one and both $v_i \in P^w$ are at distance greater than one from Q^w, P^w has width larger than one as well. Thus, w is also a non-essential vertex of P and P is merged from P^w and P^{v_i}, for any choice of i, since the polytope $P^{w,v_i} = \mathrm{conv}(Q^w \cup \{v_j\})$, for $\{i, j\} = \{1, 2\}$, is 3-dimensional.

(f) *Q is quasi-minimal (as a polygon in $\mathbb{Z}^2 \times \{0\} \cong \mathbb{Z}^2$).*

Suppose not. Then Q has at least two non-essential vertices which, by part (e) are not vertices of P. This implies that the segment $v_1 v_2$ intersects $\mathrm{aff}(Q)$ in a point $v_0 \in \mathbb{R}^2 \times \{0\}$ outside Q and that the two non essential vertices are in $Q_0 := \mathrm{conv}(Q \cup \{v_0\})$, but are not vertices of it. (Notice that if $v_1 v_2$ intersects $\mathrm{aff}(Q)$ in a point of Q then at most one vertex of Q is not a vertex of P). In particular, we can find two consecutive vertices w and w' of Q such that the line containing the edge ww' separates Q from v_0 and such that w and w' are not vertices of Q_0. Observe that the triangle $v_0 w w' \subset Q_0$ cannot contain lattice points outside the segment ww', because they would be lattice points of $Q = P^{v_1,v_2}$.

To fix ideas, without loss of generality let $w = (0, 0, 0)$, $w' = (k, 0, 0)$, and $v_0 = (a, b, 0)$ with $k, b > 0$ and $k \in \mathbb{Z}$. Consider the first lattice point w'' in the segment from w to w' (which could equal w'); that is, let $w'' = (1, 0, 0)$. Since the triangle $w w'' v_0$ does not contain lattice points other than w and w'', we can assume by an affine transformation in the plane $z = 0$ that $0 \le a \le 1$. See Figure 3.5.

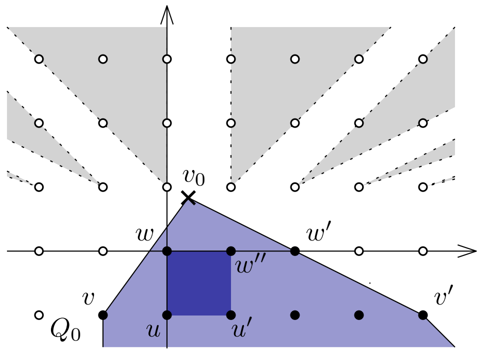


Figure 3.5: The setting of case (f) in the proof of Theorem 3.15. Black dots are lattice points in $Q \subset Q_0$, white dots represent other lattice points. The cross represents the intersection v_0 of the edge v_1v_2 with the plane $\{z = 0\}$.

Let v and v' be the vertices of $Q_0 = \mathrm{conv}(Q^{w,w'} \cup \{v_0\})$ adjacent to v_0 on the sides of w and w', respectively. Notice that both v and v' are also vertices of Q. The wedge formed by the rays from v_0 to v and v' contains the points $u = (0, -1, 0)$ and $u' = (1, -1, 0)$. Since the y coordinate of v and v' is clearly ≤ -1 (or otherwise w or w' would not be vertices of Q), those two points are actually in $\mathrm{conv}\{w, w', v, v'\} \subseteq Q$. That is impossible because then Q contains the unimodular parallelogram $ww''u'u$.

(g) *Q is (equivalent to) the triangle* $\mathrm{conv}\{(-1, -1), (1, 0), (0, 1)\}$.
The full list of quasi-minimal polygons is worked out in Lemma 3.14 above and is shown in Figure 3.4. Configurations 1, 3 and 7 are excluded by Lemma 1.29(3). Let us see how to exclude 2, 4 and 5.

Observe that in the three of them, Q together with any of v_1 or v_2 is in the conditions of Lemma 1.29(4), which means that v_i is of the form $(a_i, b_i, \pm 2)$ with $a_i \equiv 1 \equiv b_i \mod 2$, for both $i = 1, 2$. v_1 and v_2 cannot be both at the same side of Q, for then their mid-point is another lattice point in P apart from those of Q, v_1 and v_2. So, v_1 and v_2 are on opposite sides and, in particular, their mid-point has to be one of the lattice points in Q.

Consider first configuration 2 and without loss of generality assume that the polygon Q is $Q = \mathrm{conv}\{(-k, 0, 0), (0, 1, 0), (0, -1, 0)\}$ and $v_1 = (1, 1, 2)$. Its non-essential vertex $(-k, 0, 0)$ must not be a vertex in P, by part (e). Hence, that non-essential vertex is the mid-point of v_1 and v_2, which means $v_2 = (-2k - 1, -1, -2)$. But then let $v_3 = (0, 1, 0)$ and let us show that the polytope P^{v_3} has width at least two. A linear functional f giving width one to P^{v_3} must be constant in the collinearities $\{(0, 0, 0), (-1, 0, 0), (-2, 0, 0)\}$ and $\{(1, 1, 2), (-k, 0, 0), (-2k-1, -1, -2)\}$ which means that $f(x, y, z) = 2y - z$, which is primitive. But then $f(0, 0, 0) = 0$ and $f(0, -1, 0) = -2$, which means P^{v_3} has width two. Then P is obtained merging P^{v_3} and P^{v_i}, for any choice of i, since the polytope P^{v_3,v_i}, is 3-dimensional.

In configurations 4 and 5 the argument is the same, except now the mid-point of v_1 and v_2 can be any of the five lattice points in Q. But in all cases there is a vertex $v_3 \in Q$ of P such that P^{v_3} has width at least two and P is merged from P^{v_3} and P^{v_i}. We omit the details of these two cases.

Once we know that $Q = \text{conv}\{(1,0,0),(0,1,0),(-1,-1,0)\}$ (modulo unimodular equivalence), then P has six lattice points. By Lemma 1.30, the only lattice 3-polytopes of size 6 in which the two extra points outside of Q are in opposite sides of $\text{aff}(Q)$ and at distance larger than one from Q are:

$$\begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 0 & -1 & 1 & -1 \\ 0 & 0 & 1 & -1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 3 & -3 \end{pmatrix}$$

The first one is the configuration in part (3). The second one is merged from P^{v_1} and P^{v_3}, for $v_3 = (1,0,0)$, since P^{v_3} has width larger than one (it is equivalent to the configuration of signature $(3,1)$ and width 2 in Table 1.1) and the polytope

$$P^{v_1,v_3} = \text{conv}\{(0,0,0),(0,1,0),(-1,-1,0),(-1,1,-3)\}$$

is 3-dimensional. □

3.2 The classification of spiked 3-polytopes

The definition of *spiked* translates in dimension 3 to Theorem 3.16 below.

Theorem 3.16. *Let P be a lattice 3-polytope of size at least seven, spiked with respect to a certain 2-dimensional configuration A'. Let $P' = \text{conv}(A')$. Then one of the following three things happens:*

(1) P' is the second dilation of a unimodular triangle.

(2) P' is a reflexive triangle and A' consists of the three vertices of it plus its unique interior point.

(3) $A' = P' \cap \mathbb{Z}^2$ and P' has exactly four lattice points in the boundary and one in the interior (That is, P' is one of the three reflexive polygons with four boundary points; see Figure 1.7).

Proof. Let $\pi : \mathbb{Z}^3 \to \mathbb{Z}^2$ be the projection that makes P spiked and let $A = P \cap \mathbb{Z}^3$, so that $A' = \pi(A) \subset \mathbb{Z}^2$. Remember that A' has width at least two, each vertex of $\text{conv}(A')$ has a unique preimage in A, at most one vertex of A' is not essential, and π bijects the essential vertices of A to the essential vertices of A'.

Since the only lattice polygon of width greater than one and without interior lattice points is the second dilation of a unimodular triangle (case (1)), for the rest of the proof we assume that $\text{conv}(A')$ has interior lattice points. We are going to prove that $A' \cap \partial P'$ has three or four points, and that A' is, respectively, as in case (2) or case (3).

- If A' has only three points v'_1, v'_2 and v'_3 in the boundary of P', these three points must be essential vertices, because at least three vertices of A are essential and π bijects essential vertices. That is, $\text{conv}(A')$ is a triangle and A' is a minimal configuration. A' must have at least one additional lattice point p', since $\pi^{-1}(v'_i)$ have a single point in A and A has size at least seven. We then claim that:

- p' *is the only point of* A' *in the interior of* $\mathrm{conv}(A')$*. That is,* $A' = \{v'_1, v'_2, v'_3, p'\}$. Indeed, if A' has a second interior lattice point p'' then let v'_i and v'_j be vertices of $\mathrm{conv}(A')$ on opposite (open) sides of the line containing p' and p''. The contradiction is that $\mathrm{conv}\{p', p'', v'_i, v'_j\} \subseteq \mathrm{conv}(A' \setminus \{v'_k\})$ cannot have width one, since one of p' or p'' is in its interior.
- p' *is the only lattice point in the interior of* $\mathrm{conv}(A')$*. That is,* $\mathrm{conv}(A')$ *is reflexive with respect to* p' (see Section 1.2.4). Suppose p'' was a second interior lattice point. p'' cannot be in the interior of a triangle $\mathrm{conv}\{v'_i, v'_j, p'\}$, since these triangles have width one. Thus, p'' lies along a segment $\mathrm{conv}\{v'_i, p'\}$. We now apply Lemma 1.24 to the triangle $\mathrm{conv}\{v_i, p_1, p_2\}$, where $v_i, p_1, p_2 \in A$ are, respectively, the unique point in $\pi^{-1}(v'_i)$ and two points in $\pi^{-1}(p')$ (the latter exist because A has at least seven points, and only one projects to each v_i). Existence of p'' implies that v_i is at distance at least two from the segment $\mathrm{conv}\{p_1, p_2\}$, so Lemma 1.24 says the triangle $\mathrm{conv}\{v_i, p_1, p_2\}$ contains a lattice point q closer to $\mathrm{conv}\{p_1, p_2\}$ than v_i. The point $q' = \pi(q)$ is then in A', in contradiction to the fact that $A' = \{v'_1, v'_2, v'_3, p'\}$.

* If A' has at least four points in the boundary of P', let $v'_1, ..., v'_3$ be essential vertices and let v'_4 be another boundary lattice point, which may or may not be a vertex. We assume $v'_1, ..., v'_4$ to be cyclically ordered along the boundary.

  Observe that the segment $v'_2v'_4$ decomposes $\mathrm{conv}(A')$ as the union of two polygons P'_1 and P'_3 contained respectively in $\mathrm{conv}(A' \setminus \{v'_1\})$ and $\mathrm{conv}(A' \setminus \{v'_3\})$, with the point p' lying either in the segment $v'_2v'_4$ or in the interior of one of the two subpolygons. Since $\mathrm{conv}(A' \setminus \{v'_1\})$ and $\mathrm{conv}(A' \setminus \{v'_3\})$ have width one, the latter is impossible and v'_1 and v'_3 are at lattice distance one from the segment. We also claim that v'_1 and v'_3 are the only lattice points of $\mathrm{conv}(A')$ outside the segment $v'_2v'_4$. If not, let v' be an additional one, say on the side of v'_1. Then $A'^{v'_1}$ cannot have width one since it contains three collinear points (v'_2, v'_4 and p') plus points v' and v'_3 on opposite sides of the line containing them.

  So, v'_1, v'_2, v'_3, and v'_4 are the only boundary lattice points in $\mathrm{conv}(A')$, and v'_4 is either a vertex (in which case $\mathrm{conv}(A')$ is a quadrilateral) or it lies in the segment $v'_1v'_3$ (and $\mathrm{conv}(A')$ is a triangle).

  Let us see that A' must have some interior lattice point (a priori, p' may not be in A'). Since the preimages of vertices of A' in A consist of a single point and A has at least size seven, if v'_4 is a vertex then A' must have some other lattice point, hence an interior point. In case v'_4 is not a vertex, if there were no other points in A' then the fiber of v'_4 would have at least four lattice points. Applying Lemma 1.24 to the triangle formed by the fiber of v'_4 (a segment) and v'_2 (a point) we conclude that A has some lattice point projecting to the relative interior of the segment $v'_4v'_2$, a contradiction.

  On the other hand, A' cannot have *two* interior lattice points: if it does, they are both in the segment $v'_2v'_4$, and call q' the closest to v'_4. Then q' is in the interior of $\mathrm{conv}(A' \setminus \{v'_2\})$, which is a contradiction since v'_2 is an essential vertex of A'.

  That is, we can assume that p' is the only point of A' in the interior of $\mathrm{conv}(A')$. We claim that there is no other lattice point in the interior of $\mathrm{conv}(A')$ for which we only need to check that $p'v'_2$ and $p'v'_4$ are primitive. If the fiber of p' in A has at least two points, then Lemma 1.24 applied to the triangle formed by these two points plus v_2 (resp. v_4) implies that $p'v_2'$ (resp. $v_4'p'$) is primitive: otherwise, A must have lattice points projecting to their relative interiors. If the fiber of p' in A has a single point then the fiber of v_4' must have at least three and the same argument shows that $v_4'p'$ is primitive, but we need an extra argument for $p'v_2'$.

So, suppose that p' has a single point p in its fiber, which implies v_4' has at least three. Call v_4^+, v_4^0 and v_4^- three consecutive points in the fiber of v_4', and call $p^+ = p+v_4^+-v_4^0$ and $p^- = p+v_4^- - v_4^0$. That is, p^+, p and p^- are consecutive lattice points projecting to p', in the same order as v_4^+, v_4^0 and v_4^-. Let v_2 be the point in the fiber of v_2'. See Figure 3.6. Since the triangle $v_4^+v_4^-v_2$ leaves p^+ and p^- outside, v_2 must lie in the ray from v_4^0 through p. Then the lattice points in the segment pv_2 are all in A, but no such point can arise other than p and v_2 because it would project to a point of A' in the relative interior of the segment $p'v_2'$, which does not exist. Hence, pv_2 is primitive, and so is $p'v_2'$.

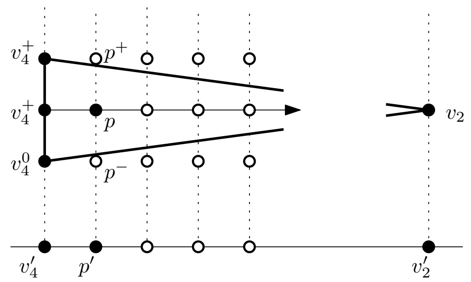


Figure 3.6: The situation in the final part of the proof of Theorem 3.16. White dots represent lattice points. Black dots are the lattice points in A and A'.

□

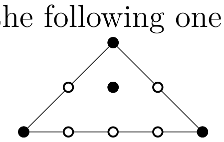
Corollary 3.17. *A spiked 3-polytope is spiked with respect to one of the ten quasi-minimal configurations of Figure 3.7.*

Proof. Let P be our spiked 3-polytope and let $A' = \pi(P \cap \mathbb{Z}^3) \subset \mathbb{Z}^2$ be the quasi-minimal configuration for which P is spiked. Without loss of generality assume that π is the projection that forgets the third coordinate.

Let us look at the three cases allowed by Theorem 3.16 for $P' := \operatorname{conv}(A')$:

(1) If P' is the second dilation of the unimodular triangle, all three vertices of P' are in A' but the mid-points of edges may or may not be in A'. The statement simply says that at least one of them is in A'. (This is the only case missing from the top row of Figure 3.7). This must be so because A has at least seven points and only three of them project to the vertices of A'.

(2) Suppose P' is a reflexive triangle and the unique points of A' are the three vertices and the unique interior point of P'. There are five reflexive triangles; the four in the middle row of Figure 3.7 plus the following one:



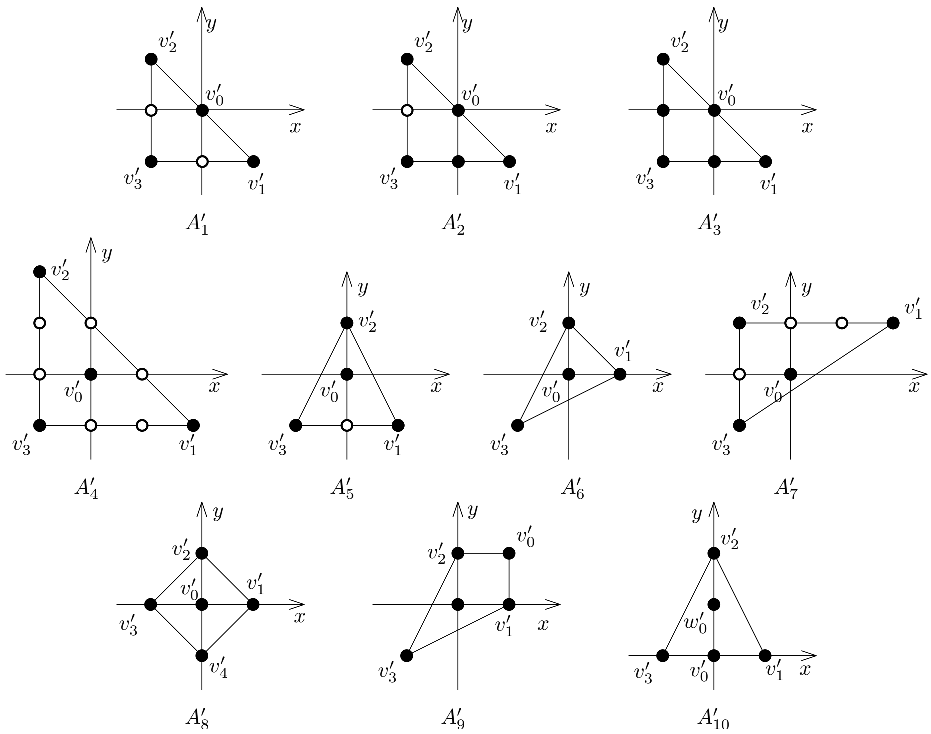


Figure 3.7: The ten quasi-minimal configurations A'_i that can arise as the projection of a spiked 3-polytope. Black dots are lattice points of A'_i, and white dots are lattice points in $\operatorname{conv}(A'_i) \setminus A'_i$. Labels v'_i of certain lattice points are there for reference in the proof of Theorem 3.19.

But, if P' was this triangle then two of the essential vertices of P should be at heights of the same parity and their mid-point would be a lattice point in A. In particular, one of the midpoints of edges of P' would by in A', a contradiction.

(3) In case (3) of the theorem, P' is a reflexive polygon with four boundary points and $A' = P' \cap \mathbb{Z}^2$. All reflexive polygons are depicted in Figure 1.7, and the last row of Figure 3.7 shows the three of them with exactly four lattice points in the boundary.

□

Theorem 3.18 (Classification of spiked minimal 3-polytopes)**.** *Let P be a spiked minimal 3-polytope of size at least seven. Then P is equivalent to*

$$\operatorname{conv}\{(1,0,0),(0,1,0),(-1,0,-a),(0,-1,2k+b)\}$$

for some $(a,b) \in \{(0,0),(0,1),(1,1)\}$ and an integer $k \geq 2$. It has size $k+5$.

Proof. By Corollary 3.17, P is spiked with respect to one of the configurations A'_i in Figure 3.7. Since P has at least four essential vertices, so does A'_i, which leaves only the possibility A'_8. We use the coordinates and labels from Figure 3.7, and assume that the projection is the one that forgets the z coordinate.

By definition of spiked the lattice points in P are its four essential vertices v_i that project to each v'_i, plus the lattice points projecting to $(0,0)$, none of which are vertices. We assume that there are $k+1$ such points and they form the segment $S_k =$

$\{(0,0,0),(0,0,1),\dots,(0,0,k)\}$ for some k (with $k \geq 2$ or otherwise P has size less than seven). Since the triangle $v_0'v_1'v_2'$ is unimodular, we can arbitrarily change the heights of v_1 and v_2 keeping the choices so far, so we assume $v_1 = (1,0,0)$ and $v_2 = (0,1,0)$. Let l be the vertical line $\{x = y = 0\}$. In order for the fiber of v_0' in $P \cap \mathbb{Z}^3$ to equal S_k, one of the segments v_1v_3 and v_2v_4 must cut the vertical line $\{x = y = 0\}$ at height in $(-1,0]$, and the other at height in $[k, k+1)$. That is, without loss of generality, $v_3 = (-1,0,-a)$ and $v_4 = (0,-1,2k+b)$ for $a, b \in \{0,1\}$. Furthermore, by the symmetry of the configuration of points, we can assume that $a \leq b$. □

Theorem 3.19 (Classification of spiked quasi-minimal 3-polytopes)**.** *Let P be a spiked quasi-minimal, but not minimal, 3-polytope of size at least seven. Then P is equivalent to one of the following. In all cases, $k \geq 2$ is an integer and the point in boldface is the non essential vertex of P.*

(1) $\operatorname{conv}\{(1,-1,-1),(-1,1,1),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\}$, *of size* $k+4$.

(2) $\operatorname{conv}\{(1,-1,0),(-1,1,-1),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\}$, *of size* $k+5$.

(3) $\operatorname{conv}\{(1,-1,0),(-1,1,0),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\}$, *of size* $k+6$.

(4) $\operatorname{conv}\{(2,-1,-1),(-1,2,1),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\}$, *of size* $k+4$.

(5) $\operatorname{conv}\{(1,-1,-1),(0,1,a),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\}, a \in \{-1,0\}$, *of size* $k+4$.

(6) $\operatorname{conv}\{(1,0,0),(0,1,a),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\},\ a \in \{-2,-1,0\}$, *of size* $k+4$.

(7) $\operatorname{conv}\{(2,1,0),(-1,1,a),(-1,-1,0),(\mathbf{0},\mathbf{0},\mathbf{k})\},\ a \in \{-5,-1\}$, *of size* $k+4$.

(8) $\operatorname{conv}\{(1,0,0),(0,1,0),(-1,0,a),(0,-1,b),(\mathbf{0},\mathbf{0},\mathbf{k})\},\ a \in \{-1,0\}, a \leq b < 2k$, *of size* $k+5$.

(9) $\operatorname{conv}\{(1,0,0),(0,1,0),(-1,-1,a),(\mathbf{1},\mathbf{1},\mathbf{2k}-\mathbf{a}+\mathbf{b})\},\ a \in \{-2,-1,0\}, b \in \{0,1\}$, *of size* $k+5$.

(10a) $\operatorname{conv}\{(1,0,a),(0,2,b),(-1,0,0),(\mathbf{0},\mathbf{0},\mathbf{k})\},\ a,b \in \{-1,0\}$, *of size* $\lfloor(3k+b)/2\rfloor + 5$.

(10b) $\operatorname{conv}\{(1,0,0),(0,2,a),(-1,0,0),(\mathbf{0},\mathbf{1},\mathbf{k})\},\ a \in \{-1,0\}$, *of size* $k+5$.

Proof. By Corollary 3.17, P is spiked with respect to one of the ten configurations A_i' in Figure 3.7. This will correspond to the ten cases in the statement, except that case (10) subdivides into two subcases as we show below. Without loss of generality we can assume that the projection is the one that forgets the z coordinate and we take in $\mathbb{Z}^2$ the system of coordinates of Figure 3.7.

Let us first concentrate in cases (1) to (7), in which A_i' is a minimal configuration and $\operatorname{conv}(A_i')$ a triangle (with vertices v_1', v_2' and v_3' as labeled in the figure). This implies that P is a tetrahedron with three essential vertices v_1, v_2, v_3 projecting to the three vertices of $\operatorname{conv}(A_i')$, plus a non-essential vertex v_0 projecting to a non-vertex lattice point v_0' of A_i'. In all except A_2' and A_3' there is only one choice for v_0'. In cases A_2' and A_3' there are several possibilities for v_0' but they are equivalent to one another. This allows as to assume

v_0' is as shown in the figure in all cases. Hence, to finish the proof for these seven cases we only need to derive the possible third coordinates (the heights) for the four vertices v_i, in each case. We denote these heights h_0, h_1, h_2 and h_3, and let us check that without loss of generality they are as in the statement:

- Let $k+1$ be the number of lattice points in the fiber of v_0' in $P \cap \mathbb{Z}^3$. We take without loss of generality $v_0 = (0,0,k)$ (that is, $h_0 = k$), so that $(0,0,0)$ is the bottom-most point in the fiber. Observe that v_0' is the only point of A_i' whose fiber in $P \cap \mathbb{Z}^3$ has more than a single point. For all except A_2' and A_3' this is obvious, since v_0' is the only non-vertex. For A_2' and A_3', the fibers of $(0,-1)$ (in both) and $(-1,0)$ (in A_3') must be single points or otherwise they produce additional vertices in P, which do not exist.

- Since the segment $v_0'v_3'$ is primitive, there is no loss of generality in taking height zero for v_3. That is to say, $h_3 = 0$ and $v_3 = (-1,-1,0)$, in all seven cases.

- Let $d_i \geq 1$ be the lattice distance from v_1' to the segment $v_0'v_3'$. (That is, $d_i = 2,2,2,3,2,1$ and 1, respectively, in cases (1) to (7)). Since the unimodular transformation $(x,y,z) \to (x,y,z \pm (x-y))$ fixes the plane containing all the choices so far (points projecting to v_0' and v_3') and changes the height of v_1 by d_i units, without loss of generality we choose the height of v_1 to be in $\{0,-1,\dots,-d_i+1\}$. Moreover, this height must be even in cases (2) and (3), in order for the midpoint of v_1v_3 to be a lattice point, and it must be prime with d_i in all other cases, in order for the segment v_1v_3 to be primitive. Summing up, the height h_1 of v_1 equals: 0 in (2), (3), (6), and (7); -1 in (1) and (5); and -1 or -2 in (4).

- In order to study h_2, let h be the height at which the triangle $v_1v_2v_3$ intersects the vertical line projecting to v_0'. Since our choice is that $(0,0,0)$ is the bottom-most lattice point in the fiber of v_0', we must have $h \in (-1,0]$. This in turn implies a bounded interval for the height h_2 in each case, namely:

  (1) $h_2 \in (-1,1]$.
  (2) $h_2 \in (-2,0]$.
  (3) $h_2 \in (-2,0]$.
  (4) $h_2 \in (-2,1]$ if $h_1 = -1$ and $h_2 \in (-1,2]$ if $h_1 = -2$.
  (5) $h_2 \in (-3/2,1/2]$.
  (6) $h_2 \in (-3,0]$.
  (7) $h_2 \in (-6,0]$.

- This already gives a finite list of possibilities for all heights, but there are the following additional considerations:

  - In (3), h_2 must be even in order for the midpoint of v_2v_3 to be integer.
  - In (1), (2) and (7), h_2 must be odd for the segment v_2v_3 to be primitive.
  - In (4), $h_2 \neq 0 \pmod 3$ for the segment v_2v_3 to be primitive.
  - In (4) and (7), $h_2 \neq h_1 \pmod 3$ for the segment v_1v_2 to be primitive.

  Together with the intervals stated above, this fixes h_2 to be 1, -1, and 0, in cases (1), (2) and (3), respectively. In case (4) we have two possibilities for (h_1,h_2), namely

$(-1,1)$ and $(-2,2)$, but they produce equivalent configurations via the transformation $(x,y,z) \mapsto (y,x,x-y+z)$), so we take the first one. In cases (5), (6) and (7) we have $h_2 \in \{-1,0\}$, $h_2 \in \{-2,-1,0\}$, and $h_2 \in \{-5,-1\}$, respectively. This finishes the proofs of these seven cases.

We now look at the three remaining cases, A'_8, A'_9, and A'_{10}. As before, we will denote by h_i the height of the vertex v_i of P projecting to a point $v'_i \in A'_i$. The ideas are essentially the same, with slight modifications:

(8) A'_8 is minimal, hence in this case P has four essential vertices v_1, v_2, v_3, v_4 projecting to the four vertices of A'_8 plus a fifth non-essential vertex v_0 projecting to $v'_0 = (0,0)$. Again, we choose $v_0 = (0,0,k)$ where $k+1$ is the number of lattice points in the fiber of v'_0, and as in Theorem 3.18, we can take without loss of generality, the heights of v_1 and v_2 to be zero. By symmetry, we can also assume $h_3 \le h_4$ which implies, in order for the bottom-most point in the fiber of v'_0 to be $(0,0,0)$, that $h_3 \in \{-1,0\}$. Finally, in order for v_0 to be above the segment v_2v_4 we need $h_4 < 2k$.

(9) A'_9 is not minimal, so the vertices of P biject to those of A'_9 and all other lattice points in P project to the unique non-vertex point $p'_0 = (0,0)$. We let the fiber of p'_0 consist of $(0,0,0), \ldots, (0,0,k)$, as in previous cases. Since the triangle $p'_0v'_1v'_2$ is unimodular we can choose the heights of v_1 and v_2 to be zero. Also, we make the choice that the triangle $v_1v_2v_3$ lies below (perhaps not strictly) $(0,0,0)$ and the segment v_0v_3 lies above. The opposite choice would lead to equivalent configurations.

    Then, for the triangle $v_1v_2v_3$ to cut the line $\{x=y=0\}$ at height in $(-1,0]$ we need $h_3 \in (-3,0]$. And for the segment v_0v_3 to cut that line at height in $[k,k+1)$ we need $(h_0+h_3)/2 \in [k,k+1)$. Hence $h_0 \in \{2k-h_3, 2k-h_3+1\}$ and $k = \lfloor (h_0+h_3)/2 \rfloor$.

(10) A'_{10} is minimal, which implies P to have three essential vertices v_1, v_2, v_3 projecting to the three vertices of A'_{10}, plus a fourth non-essential vertex projecting to one of the other two lattice points, $v'_0 = (0,0)$ and $w'_0 = (0,1)$. We consider the two cases separately:

    (10a) If the non-essential vertex projects to v'_0, call it v_0. By the same arguments as used for configurations (1) to (7), we can assume that $v_0 = (0,0,k)$, $(0,0,0)$ is the bottom-most point in the fiber of v'_0, $h_3 = 0$ and $h_1 \in \{-1,0\}$. Once these are fixed, unimodular transformations can change the height of v_2 by arbitrary even numbers, so we can take the height of v_2 in $\{0,-1\}$ as well. Observe that the fiber of point w'_0 in P is the segment going from $(0,1,h')$, with $h' = (h_1+2h_2)/4 \in (-1,0]$, to $(0,1,(h_2+k)/2)$. It then contains the $\lfloor (h_2+k)/2 \rfloor + 1$ lattice points from $(0,1,0)$ to $(0,1,\lfloor (h_2+k)/2 \rfloor)$.

    (10b) If the non-essential vertex projects to w'_0, call it w_0. By the same arguments as before, we can assume that $w_0 = (0,1,k)$, $(0,1,0)$ is the bottom-most point in the fiber of w'_0, $h_3 = 0$ and $h_1 \in \{-1,0\}$. In this case, the fiber of v'_0 consists of a single point, the middle point of segment v_1v_3. In order for this point to be a lattice point, h_1 has to be even, hence $h_1 = 0$. Then, in order for the triangle $v_1v_2v_3$ to cut the fiber of w'_0 at a height in $(-1,0]$, we need $h_2 \in \{-1,0\}$.

In all cases k can be assumed at least two: In case A'_3 because otherwise P has width one with respect to the vertical direction. In all other cases because otherwise P has size at most 6. □

See Algorithm A.12 for the MATLAB routine that enumerates all the spiked 3-polytopes of any given size $N \geq 7$.

Remark 3.20. For $k \geq 3$ all the polytopes described in Theorem 3.19 are spiked, quasi-minimal, not minimal, non-equivalent to one another, and have size ≥ 7. But for $k = 2$ the following happens:

- Cases (1), (4), (5), (6), and (7) produce size six.
- In some cases (sometimes depending also on the values of a and b) the vertex that should be non-essential (the vertex v_0 or w_0 in the proof, projecting to v_0' or w_0' in Figure 3.7) turns out to be essential. In this case the polytope obtained is minimal, and it is not spiked: the projection does not biject essential vertices to essential vertices.

This means that for each size $n \geq 9$ there are *exactly* the following non-equivalent spiked 3-polytopes: 3 spiked minimal tetrahedra; 23 if $n = 0 \pmod 3$ and 21 if $n \neq 0 \pmod 3$ spiked quasi-minimal, not minimal tetrahedra; and $4n - 19$ spiked quasi-minimal, not minimal 3-polytopes with 5 vertices. For $n = 7$ and 8 the global counts are decreased by two. See exact numbers in Table 3.1.

Remark 3.21. Observe as well that no quasi-minimal 3-polytope of size at least seven can be both spiked and boxed. Indeed, with $k \geq 2$ in Theorems 3.18 and 3.19 the only way to get a boxed and spiked polytope is if there exists an essential vertex v such that P^v has width one with respect to a functional that is not constant on the fibers of the projection. This implies that each fiber can contain at most two lattice points of P^v. Since v is an essential vertex, it does not belong to the fiber of S_k, and the previous implies that $k = 1$.

3.3 The classification of boxed 3-polytopes

Let P be a boxed 3-polytope of size at least seven. That is to say, there are three affine primitive functionals $f_1, f_2, f_3 : \mathbb{R}^3 \to \mathbb{R}$ such that the lattice points in P are:

- Some or all of the vertices of the rational parallelepiped $Q := \bigcap_{i=1}^3 f_i^{-1}[0,1]$.
- Three additional points v_1, v_2, v_3 (essential vertices of P) with $f_i(v_j) \notin \{0,1\}$ if, and only if, $i = j$.

Without loss of generality we assume the origin to be a vertex of Q, so that the f_i's can be taken primitive *linear* functionals.

For each $i \in \{1,2,3\}$, let

$$C_i^+ = \Big(\bigcap_{j \neq i} f_j^{-1}[0,1]\Big) \cap f_i^{-1}(1,\infty)$$

and

$$C_i^- = \Big(\bigcap_{j \neq i} f_j^{-1}[0,1]\Big) \cap f_i^{-1}(-\infty,0),$$

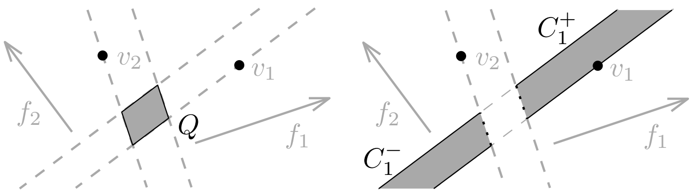


Figure 3.8: The chimneys of a boxed polytope

and let $C_i = C_i^+ \cup C_i^-$. We call the C_i's *chimneys* of Q and refer to C_i^+ and C_i^- as *half-chimneys*. With this notation, $v_i \in C_i$ for each i. Observe that each half-chimney C_i^* contains at most 4 lattice rays. See Figure 3.8.

In order to classify boxed 3-polytopes, in this section we do the following: in Section 3.3.1 we look at the possibilities for Q and prove that all boxed 3-polytopes of size at least seven are boxed with respect to the unit cube or one specific rational parallelepiped Q_0. Once we know that Q is one of these two parallelepipeds, in Section 3.3.2 we use their coordinates to bound the possibilities for vertices v_i, which a priori are infinitely many. Finally, in Section 3.3.3 we explain how we use the theoretical results to actually implement computer algorithms that enumerate all boxed 3-polytopes.

3.3.1 Possibilities for the parallelepiped

The Euclidean volume of the parallelepiped Q equals the inverse of the determinant of (f_1, f_2, f_3), which is an integer. In particular, the volume of Q is exactly one if, and only if, $Q \cong [0,1]^3$, and is at most $1/2$ otherwise. The following lemma shows that, if we restrict ourselves to boxed 3-polytopes of size at least seven, there is only one other possibility for Q.

Lemma 3.22. *Let P be a boxed 3-polytope with size at least seven and suppose P is not boxed with respect to a parallelepiped unimodularly equivalent to the standard cube. Then, modulo unimodular equivalence, we can assume that $f_1 = y+z$, $f_2 = x+z$, $f_3 = x+y$ so that P is boxed with respect to the parallelepiped*

$$\begin{aligned} Q_0 \quad := \quad \operatorname{conv}\Bigg\{ & (0,0,0), \left(\frac{-1}{2}, \frac{1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{-1}{2}, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}, \frac{-1}{2}\right), \\ & (1,0,0), (0,1,0), (0,0,1), \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) \Bigg\}. \end{aligned}$$

Proof. Let P be of size at least seven and boxed with respect to a parallelepiped Q not unimodularly equivalent to the unit cube. As usual, let $P \cap \mathbb{Z}^3 = A \cup \{v_1, v_2, v_3\}$ where $A \subseteq Q \cap \mathbb{Z}^3$ has size at least four.

If $T \subseteq A$ consists of four non-coplanar lattice points, the convex hull of them is a lattice tetrahedron (of Euclidean volume at least $1/6$) whose vertices are vertices of the parallelepiped Q of Euclidean volume at most $1/2$. The only way this can happen is that this tetrahedron T consists of alternating vertices of Q and it has Euclidean volume exactly $1/6$ (that is, it is *unimodular*). Since all unimodular tetrahedra are equivalent, there is no loss of generality in assuming $T = \{(0,0,0), (1,0,0), (0,1,0), (0,0,1)\}$. The three linear primitive functionals with values 0 and 1 on the edges of $\operatorname{conv}(T)$ are $x+y$, $y+z$ and $x+z$, as in the statement. Hence $Q = Q_0$ and $A = T$.

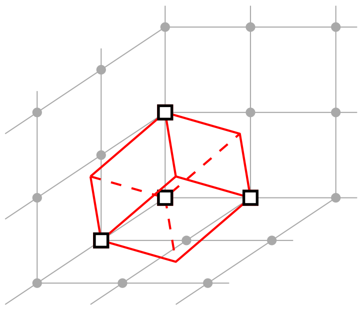


Figure 3.9: The parallelepiped Q_0. Grey dots are lattice points. White squares are the lattice points of Q_0.

So, for the rest of the proof we assume that A is contained in a plane (in particular, it has exactly four points) and try to get a contradiction. The two possibilities are that the points in A form a facet of Q or they span two opposite parallel edges:

- If they span a facet F, then let H be the plane containing F, and let F' be the opposite facet and H' the plane containing it. H and H' are both lattice planes by definition of boxed, and the lattice $H' \cap \mathbb{Z}^3$ is an integer translation of the lattice $H \cap \mathbb{Z}^3$. Since F is an empty parallelogram (in $H \cap \mathbb{Z}^3$), F' is also empty parallelogram (in the lattice $H' \cap \mathbb{Z}^3$). This is a contradiction with the fact that F' contains no lattice points.

- If they span two opposite parallel edges, then the four points still form an empty parallelogram. Assume without loss of generality that

$$A = \mathrm{conv}\{(0,0,0), (1,0,0), (0,1,0), (1,1,0)\} \subset Q$$

  where the lines $\{x = z = 0\}$ and $\{x = 1, z = 0\}$ contain opposite edges of Q. Since Q has no other lattice points, and the points in A are in opposite parallel edges, one of the chimneys, say C_1, contains these two lattice lines and no other. By symmetry of the conditions so far, we can assume that $v_1 = (0, b_1, 0)$, for $b_1 \geq 2$. But for any choice of b_1 the point $(0,2,0)$ is in P, so we must actually have $v_1 = (0,2,0)$.

  Now, vertices v_2 and v_3 must satisfy that $\mathrm{conv}(A \cup \{v_i\})$ $(i = 2, 3)$ does not have any extra lattice points. Since A contains a unimodular parallelogram, v_2 and v_3 must be at distance at most one from A (Lemma 1.29(3)). Moreover, they must be in opposite sides and at distance one of A or otherwise P has width one. That is, without loss of generality we can assume $v_2 = (0,0,1)$, and $v_3 = (a, b, -1)$, for some $a, b \in \mathbb{Z}$.

  The functional f_1 has to be equal to zero on the segment $(0,0,0)(1,0,0)$, and to 1 in the segment $(0,1,0), (1,1,0)$, so it has the form $f_1(x,y,z) = y + cz$ for some $c \in \mathbb{Z}$. By definition of boxed, we need to have that $v_2, v_3 \in f_1^{-1}(\{0,1\})$, which implies that $f_1(v_2) = f_1(0,0,1) = c \in \{0,1\}$ and $f_1(v_3) = f_1(a,b,-1) = b - c \in \{0,1\}$. In particular, $b \in \{c, c+1\} \subset \{0,1,2\}$. But then P is boxed also with respect to the parallelepiped

$$Q' := \bigcap_{i=1}^{3} f_i'^{-1}([0,1]),$$

for $f'_2 = -z$, $f'_3 = x$ and f'_1 equal to y if $b \in \{0,1\}$ and to $y+z$ if $b = 2$. This contradicts the assumption that P is not boxed to a parallelepiped unimodularly equivalent to the unit cube. Observe that to check the boxed property we need $f'_3(v_3) = a \notin \{0,1\}$, which happens because $a \in \{0,1\}$ gives P width one with respect to x.

□

3.3.2 Possibilities for the vertices v_i

A priori, v_i can be any of the (infinitely many) lattice points in the chimney C_i. In this section we give bounds on how far v_i can be from Q, which reduces the infinite possibilities to only finitely many.

For each $i \in \{1,2,3\}$ denote by r_i the (unique) line that contains v_i and an edge of Q, and let $s_i = r_i \cap Q$ be such edge. In case s_i contains a lattice point of P, bounding the possible positions of v_i is quite straightforward. (We assume $d = 3$, but Lemma 3.23 and Corollary 3.24 are valid in arbitrary dimension):

Lemma 3.23. *Let P be boxed with respect to a parallelepiped Q, and let v_i be one of the three lattice points in $P \setminus Q$. If s_i contains a lattice point of P, then there is no lattice point $p \in r_i$ strictly between Q and v_i.*

Proof. Let q be a lattice point in $s_i \cap P$. If there was a $p \in r_i \cap \mathbb{Z}^3$ strictly between Q and v_i then $p \in P$ would neither be in Q nor be a vertex of P (since it lies in the segment from q to v_i). This is a contradiction with the definition of boxed. □

Corollary 3.24. *Let P be boxed with respect to a parallelepiped Q. If all edges of Q contain lattice points of P then each v_i is the first lattice point in one of the eight rays in the corresponding chimney.*

This allows us to fully understand boxed 3-polytopes with respect to the parallelepiped Q_0 of Lemma 3.22. Since Q_0 contains only four lattice points and we assume P has size at least seven, we conclude that P has size exactly seven and consists of those four lattice points plus v_1, v_2 and v_3. Moreover, since those four lattice points are alternate vertices of Q_0, P contains lattice points in all edges of Q_0 and Corollary 3.24 implies:

Corollary 3.25. *If P is a boxed 3-polytope of size at least seven, and suppose that it is not boxed with respect to a parallelepiped unimodularly equivalent to $[0,1]^3$. Then $P \cap \mathbb{Z}^3 \cong \{(0,0,0), (1,0,0), (0,1,0), (0,0,1), v_1, v_2, v_3\}$ with*

$$v_1 \in \left\{ \begin{matrix} (-1, & 1, & 1) \\ (-1, & 1, & 2) \\ (-1, & 2, & 1) \\ (\ 0, & 1, & 1) \\ (\ 1, & -1, & -1) \\ (\ 1, & -1, & 0) \\ (\ 1, & 0, & -1) \\ (\ 2, & -1, & -1) \end{matrix} \right\}, \quad v_2 \in \left\{ \begin{matrix} (\ 1, & -1, & 1) \\ (\ 1, & -1, & 2) \\ (\ 2, & -1, & 1) \\ (\ 1, & 0, & 1) \\ (-1, & 1, & -1) \\ (-1, & 1, & 0) \\ (\ 0, & 1, & -1) \\ (-1, & 2, & -1) \end{matrix} \right\}, \quad v_3 \in \left\{ \begin{matrix} (\ 1, & 1, & -1) \\ (\ 1, & 2, & -1) \\ (\ 2, & 1, & -1) \\ (\ 1, & 1, & 0) \\ (-1, & -1, & 1) \\ (-1, & 0, & 1) \\ (\ 0, & -1, & 1) \\ (-1, & -1, & 1) \end{matrix} \right\}.$$

So, we now assume that P is boxed with respect to $Q = [0,1]^3$, so that $f_1 = x$, $f_2 = y$ and $f_3 = z$. In particular,

$$v_1 = (a_1, \lambda_y^1, \lambda_z^1), v_2 = (\lambda_x^2, a_2, \lambda_z^2), v_3 = (\lambda_x^3, \lambda_y^3, a_3),$$

where $\lambda_*^i \in \{0,1\}$ for all $i \in \{1,2,3\}$, and $a_i \in \mathbb{Z} \setminus \{0,1\}$.

Our final result in this section says that in these conditions each a_i lies within $[-6,7]$. It relies on Lemmas 3.30 and 3.32, whose proofs are quite technical and are postponed to Section 3.3.4 in order not to interrupt the flow of reading:

Theorem 3.26. *Let P be a boxed 3-polytope, boxed with respect to the unit cube and of size at least seven. Then, with the notations above, $a_i \in \{-6,-5,-4,-3,-2,-1,2,3,4,5,6,7\}$.*

Proof. For each i, if the edge s_i of $[0,1]^3$ in the line r_i through v_i contains some lattice point of P, then Lemma 3.23 implies that $a_i \in \{-1,2\}$. So, assume this is not the case.

Without loss of generality let $v_3 = (1,1,a_3)$ and assume that neither $(1,1,0)$ nor $(1,1,1)$ are in P. Under these conditions, if $P \cap Q \cap \mathbb{Z}^3$ is a facet of Q then Lemma 3.30 shows that $a_3 \in [-6,7]$. If $P \cap Q \cap \mathbb{Z}^3$ is not a facet of Q then Lemma 3.32 shows that $a_3 \in [-4,5]$. $\square$

3.3.3 Enumeration of boxed 3-polytopes

We here explain how we combine the results from Sections 3.3.1 and 3.3.2 to computationally enumerate boxed 3-polytopes of size at least seven. See Algorithm A.13 for the MATLAB routines that implement this.

Let P be a boxed 3-polytope of size at least 7, so that $P \cap \mathbb{Z}^3 = A \cup \{v_1, v_2, v_3\}$ and A, of size at least four, is a subset of vertices of a rational parallelepiped Q.

(1) *If Q is not the unit cube*, then by Corollary 3.25, there are at most $8 \times 8 \times 8 = 512$ possibilities to check for P. Doing so we find that:

   Proposition 3.27. *All boxed 3-polytopes are boxed with respect to the unit cube.*

   Proof. Checking the 512 possibilities of Corollary 3.25 we find that there are only five non-isomorphic 3-polytopes of size seven that are boxed with respect to Q_0 and that the five of them are also boxed with respect to the unit cube. For size less than seven we use the full list of lattice 3-polytopes of width larger than one and sizes five or six, detailed in Section 1.3. Again, it turns out that all that are boxed are boxed with respect to the unit cube. $\square$

(2) *If Q is the unit cube and A meets every edge of it* then we know that all v_i's are at distance one from the cube. We could enumerate all possibilities and check boxedness one by one, but the following lemma allows us to do better:

   Lemma 3.28. *Let $P \subset \mathbb{R}^d$ be boxed with respect to the unit cube $Q = [0,1]^d$ and such that all edges of Q contain at least a lattice point of P. Suppose that the size of P is not $2^d + d$ (that is, $Q \not\subset P$). Then, for any $u \in \{0,1\}^d \setminus P$, $\mathrm{conv}(P \cup \{u\})$ is also boxed with respect to Q and it has size one more than P (that is, u is the only new lattice point).*

Proof. By Corollary 3.24, P is contained in the following polytope, which can also be expressed as the Minkowski sum of Q and the unit cross-polytope:

$$D := \mathrm{conv}\{(x_1, \dots, x_d) \in \mathbb{Z}^d : \exists i \text{ with } x_i \in \{-1, 2\} \text{ and } x_j \in \{0, 1\} \forall j \neq i\}.$$

All lattice points in D except the 2^d vertices of Q are vertices of D. In particular, $\mathrm{conv}(P \cup \{u\})$ cannot add any lattice point that is not a vertex of Q. But it cannot add any vertex of Q either, because the fact that P meets all edges of Q implies that the d vertices of Q^u visible from u (the neighbors of u in the graph of Q) are already in P. That is, $\mathrm{conv}((P \cap Q) \cup \{u\})$ equals $P \cap Q$ together with the d-simplex formed by u and its neighbors. $\square$

Thus, in order to enumerate boxed 3-polytopes of this type we can:

- Start with the maximal ones, in which A has size eight and we have a priori $8^3 = 512$ possibilities for v_1, v_2 and v_3 by Corollary 3.24. Discard those that have extra lattice points and eliminate redundancies.
- Remove vertices of P that belong to Q one by one, in all possible manners. Discard those polytopes that have extra lattice points and eliminate redundancies.

This procedure gives us the following numbers of boxed 3-polytopes:

# **vertices**	4	5	6	7	8	9	10	Total
size 7	1	21	28	0	-	-	-	50
size 8	2	11	48	30	0	-	-	91
size 9	0	5	24	45	16	0	-	90
size 10	1	0	7	21	20	6	0	55
size 11	0	1	0	4	6	4	1	16

(3) *If Q is the unit cube and A does not meet every edge of it*, there are eight possibilities (modulo symmetry) for A: one of size six (vertices of a triangular prism), two of size five (vertices of a square pyramid, with base on a facet or with base one a halving quadrilateral), and five of size four (two coplanarities plus the three types of unimodular tetrahedra in the unit cube).

We then exhaust all the possible coordinates for the vertices v_i which are, according to Lemmas 3.30 and 3.32, less than $(2 \times 2 \times 12)^3$ since the i-th coordinate of v_i is in $\{-6, \dots, -1, 2, \dots, 7\}$ and the other two coordinates are in $\{0, 1\}$. (This is a huge overcount, since the twelve possibilities have to be considered only when A does not meet the particular edge contained in the same line as v_i, which happens quite rarely). This results in the following counts of boxed 3-polytopes:

# **vertices**	4	5	6	7	8	Total
size 7	4	51	47	0	-	102
size 8	2	19	72	31	0	124
size 9	0	3	20	35	8	66

Cases (1), (2) and (3) contain some redundancy, since the same configuration can be boxed in more than one way. The following is the irredundant classification of boxed 3-polytopes by size and number of vertices:

# vertices	4	5	6	7	8	9	10	Total
size 7	4	51	49	0	-			104
size 8	2	19	77	38	0			136
size 9	0	5	30	56	18			279
size 10	1	0	7	21	20	6		55
size 11	0	1	0	4	6	4	1	16

Only 32 of these 590 boxed 3-polytopes are quasi-minimal. These are the numbers of them, in terms of their size and number of vertices:

# vertices	4	5	6	Total
size 7	4	15	4	23
size 8	2	5	0	7
size 9	0	1		1
size 10	1			1

The following matrices, with columns corresponding to vertices, are representatives for them.

Size 7

$$\begin{pmatrix} 0 & 1 & 1 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 2 & 0 & 1 & 0 \\ 1 & 0 & 0 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 1 & 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 1 & 2 & 0 & 0 \\ 1 & 0 & 2 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 2 \\ -1 & 1 & 1 & 0 \\ 0 & 0 & 5 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 0 & 0 & 1 & 1 \\ 2 & 0 & 0 & -1 & 1 \\ 1 & 0 & 1 & 1 & -1 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 1 & 2 & 0 & 0 & 1 \\ 0 & 1 & 1 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 1 & 1 & 2 \\ 0 & 2 & 1 & 1 & 1 \\ 1 & 1 & 0 & 2 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 2 \\ 1 & 0 & 1 & 1 & 2 & 1 \\ 1 & 1 & 0 & 2 & 1 & 1 \end{pmatrix}$$

Size 8

$$\begin{pmatrix} 0 & 2 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 1 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 1 & 1 & 1 & 2 \\ 1 & 0 & 0 & 2 & 1 \\ 0 & 0 & 2 & 0 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 & 1 & 2 \\ 0 & 1 & 2 & 1 & 1 \\ 1 & 0 & 1 & 2 & 0 \end{pmatrix}$$

Size 9

$$\begin{pmatrix} 0 & 2 & 0 & 1 & 1 \\ 0 & 0 & 2 & 0 & 1 \\ 0 & 0 & 0 & 2 & 1 \end{pmatrix}$$

Size 10

$$\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$$

3.3.4 Technical lemmas for the proof of Theorem 3.26

We here prove the lemmas that lead to the bound on the a_i's stated in Theorem 3.26. For the sake of symmetry, rather than looking at a_i we look at the *distance from v_i to the unit cube*, which equals $d_i := \max\{a_i - 1, -a_i\} = |a_i - 1/2| - 1/2$.

Remark 3.29 (Parallel-planes method)**.** In Lemma 3.30 we use a technique that we call the *parallel planes method*, and that is used in [BS16b].

Let $P \subset \mathbb{R}^3$ be a lattice 3-polytope, and let $L := P \cap \mathbb{Z}^3$. We use the *parallel-planes method* when we can guarantee that L is contained in three parallel planes H_1, H_2 and H_3 and we know (or pose without loss of generality) the coordinates of all points but one. In this case we look at what conditions must the coordinates of the last point satisfy for P not to have extra lattice points in the intermediate plane H_2. This is a 2-dimensional problem that can be solved as follows.

Denote by $L_i := L \cap H_i$. Observe that $P \cap H_2 = \operatorname{conv}(L_2 \cup I)$, where

$$I := \{\operatorname{conv}\{p, q\} \cap H_2 \mid p \in L_1, q \in L_3\}$$

That is $P \cap H_2$ is the convex hull of the points of L_2 and the intersection points of the edges $\operatorname{conv}\{p, q\}$ with H_2, for $p \in L_1$ and $q \in L_3$. Since we are assuming that all the points in L have fixed coordinates except for at most one, there are some points of $L_2 \cup I$ that are unknown.

In most cases, we have the situation where there is a single unknown point $v_0 \in L_2 \cup I$. Let $Q := \operatorname{conv}(L \cup I \setminus \{v_0\})$, which is a polygon in the plane H_2. Then, for each lattice point $x \in H_2 \cap \mathbb{Z}^3$, the point v_0 has to be a point outside of the cone $x - \mathbb{R}_{\geq 0}(Q - x)$, except for maybe $v_0 = x$.

It is easy to see that removing the cones corresponding to lattice points "close" to Q soon leaves us with a semi-open, *star-shaped* area where the point v_0 must lie (see Figures 3.10 and 3.11). Fixing a choice of v_0 in the allowed region fixes the unknown point of L, and hence P. Notice that we still need to check each of those choices because they could produce extra lattice points in other parallel planes between H_1 and H_3 (we do not assume H_1, H_2 and H_3 to be consecutive).

Lemma 3.30. *Let P be a lattice 3-polytope, boxed with respect to the unit cube $Q = [0, 1]^3$, such that $P \cap Q \cap \mathbb{Z}^3$ is a facet of Q. Then $d_i \leq 6$ for all i.*

Proof. As usual, let $(P \setminus [0, 1]^3) \cap \mathbb{Z}^3 = \{v_1, v_2, v_3\}$. Without loss of generality we can assume that

$$A_0 := P \cap \{0, 1\}^3 = \{(0, 0, 0), (1, 0, 0), (0, 1, 0), (1, 1, 0)\},$$

and that $v_3 = (0, 0, a_3)$. By Lemma 3.23 a_3 must be either -1 or 2, but if v_3 were $(0, 0, 2)$ then the point $(0, 0, 1)$ would be in P, which contradicts the assumptions. Thus, we assume that $v_3 = (0, 0, -1)$ for the rest of the proof.

Vertices v_1 and v_2 have the third coordinate in $\{0, 1\}$. In order for P not to have width one with respect to the functional z, at least one of v_1 and v_2 must lie in the plane $\{z = 1\}$.

Two things can happen: either the two vertices are in the plane $\{z=1\}$, or there is one in $\{z=1\}$ and another in $\{z=0\}$.

Let us see, in both cases, what conditions on v_1 and v_2 are necessary for no extra lattice points to arise in the plane $\{z=0\}$. For this, we use the *parallel planes method* explained in Remark 3.29. In this case, the planes are $\{z=-1\}$, $\{z=0\}$ and $\{z=1\}$: the intersection of P with $\{z=-1\}$ is the point v_3; the intersection of P with $\{z=1\}$ is a point or an edge; and the intersection of P with $\{z=0\}$ equals the convex hull of A_0 together with the mid-points of edges joining the vertices of P in the other two planes.

(a) One point in each plane. Without loss of generality, since the conditions on v_1 and v_2 are symmetric under the exchange of x and y:

$$v_1 = (a_1, \lambda_y^1, 1), \qquad v_2 = (\lambda_x^2, a_2, 0),$$

with $a_1 \in \mathbb{Z}$, $a_2 \in \{-1, 2\}$ (by Lemma 3.23) and $\lambda_*^i \in \{0,1\}$. Since the conditions so far are symmetric under $(x,y,z) \mapsto (1-x+z, y, z)$ and this symmetry exchanges the two possible values of λ_x^2, we can further assume that $\lambda_x^2 = 0$ and hence

$$v_2 \in \{(0,-1,0), (0,2,0)\}.$$

Let us see which values are allowed for the coordinates of v_1 so that P has no extra lattice point. Observe that in this case, the intersection of P with the plane $\{z=0\}$ equals the convex hull of $A_0 \cup \{v_2, v_1'\}$, where v_1' is the intersection point of the edge v_1v_3 with that plane (see Remark 3.29). This intersection point is

$$v_1' = \left(\frac{a_1}{2}, \frac{\lambda_y^1}{2}, 0\right) \in \frac{1}{2}\mathbb{Z} \times \left\{0, \frac{1}{2}\right\} \times \{0\}.$$

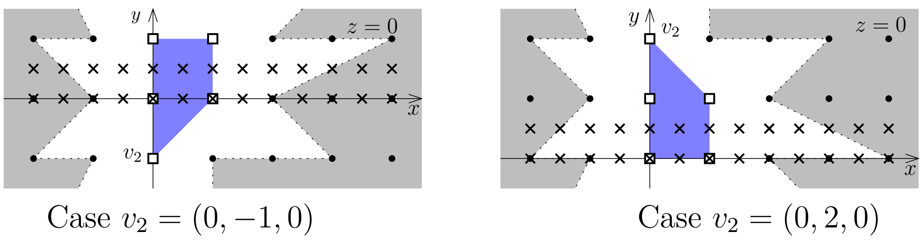


Figure 3.10: The possible positions for v_1' (hence for v_1) in case (a) in the proof of Lemma 3.30. Crosses mark the positions for v_1' corresponding to v_1 lying in its chimney. The white (open) region are the positions where a point v_1' can be placed with the property that $\mathrm{conv}(A_0 \cup \{v_2, v_1'\})$ does not have extra lattice points. The intersection of both gives the valid positions for v_1'. The blue area is $\mathrm{conv}(A_0 \cup \{v_2\})$.

Figure 3.10 shows that in order for no extra lattice points to arise we must have $a_1/2 \in [-1, 5/2]$ so that $a_1 \in [-2, 5]$. Via the symmetry $(x,y,z) \mapsto (1-x+z, y, z)$, the case $\lambda_x^2 = 1$ would give $a_1 \in [-3, 4]$, so all in all $a_1 \in [-3, 5]$.

(b) Both points in the plane $\{z = 1\}$:

$$v_1 = (a_1, \lambda_y^1, 1), \qquad v_2 = (\lambda_x^2, a_2, 1)$$

with $a_i \in \mathbb{Z}$ and $\lambda_*^i \in \{0, 1\}$.

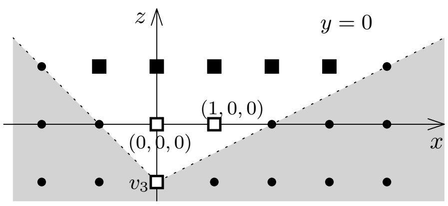
Let us see which values of a_i and λ_*^i are allowed so that no extra lattice point is added when considering the whole polytope. In this case, the intersection of P with $\{z = 0\}$ equals $\operatorname{conv}(A_0 \cup \{v_1', v_2'\})$ where v_1' and v_2' are the intersection points of the edges v_1v_3 and v_2v_3 with that plane. Namely:

$$v_1' = \left(\frac{a_1}{2}, \frac{\lambda_y^1}{2}, 0\right) \in \frac{1}{2}\mathbb{Z} \times \left\{0, \frac{1}{2}\right\} \times \{0\}$$

and

$$v_2' = \left(\frac{\lambda_x^2}{2}, \frac{a_2}{2}, 0\right) \in \left\{0, \frac{1}{2}\right\} \times \frac{1}{2}\mathbb{Z} \times \{0\}.$$

We are going to do a case-study based on the four possibilities for λ_y^1 and λ_x^2. Let us first see that, if $\lambda_*^i = 0$, then $a_i \in \{-1, 2, 3\}$. Suppose $\lambda_y^1 = 0$. Then the intersection of P with the plane $y = 0$ is the convex hull of the points $(0, 0, 0)$, $(1, 0, 0)$, $v_3 = (0, 0, -1)$ and $v_1 = (a_1, 0, 1)$. In order for $(2, 0, 0)$ and $(-1, 0, 0)$ not to lie in $P \cap \{y = 0\}$, the value of a_1 is restricted to $(-2, 4)$. See the following figure, where the black squares are the possibilities for v_1:



Since $a_{\in}\mathbb{Z} \setminus \{0, 1\}$, then $a_i \in \{-1, 2, 3\}$. By symmetry under the exchange of x and y, the same happens for a_2 if $\lambda_x^2 = 0$. Then:

- If $\lambda_y^1 = \lambda_x^2 = 0$, then $a_1, a_2 \in \{-1, 2, 3\}$, so the statement holds.
- If $\lambda_y^1 = 1$ and $\lambda_x^2 = 0$, then $a_2 \in \{-1, 2, 3\}$ and we need to look at possible values of a_1. We can forget the case $a_2 = 2$, because then P has width one with respect to the functional $y - z$. Since the conditions so far are symmetric under $(x, y, z) \mapsto (x, 1 - y + z, z)$ and this symmetry exchanges $(0, -1, 1)$ and $(0, 3, 1)$, we can assume that $v_2 = (0, 3, 1)$ (hence $v_2' = (0, \frac{3}{2}, 0)$).

  The admissible positions of v_1 (or, rather, of v_1') are drawn in Figure 3.11. As seen in the figure, the valid positions of v_1' have first coordinate $a_1/2 \in [-3/2, 7/2]$ so that $a_1 \in [-3, 7]$ (notice that the symmetry $(x, y, z) \mapsto (x, 1 - y + z, z)$ fixes $v_1 = (a_1, 1, 1)$).
- The case $\lambda_y^1 = 0$ and $\lambda_x^2 = 1$ is symmetric to the previous one, so it leads to $a_1 \in \{-1, 2, 3\}$ and $a_2 \in [-3, 7]$.

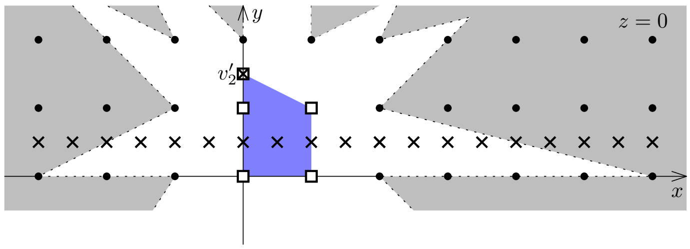


Figure 3.11: The possible positions for v_1' (hence for v_1) in case (b), $\lambda_y^1 = 1$ and $\lambda_x^2 = 0$, in the proof of Lemma 3.30. Crosses mark the positions for v_1' corresponding to v_1 lying in its chimney. The white (open) region are the positions where a point v' has the property that $A_0 \cup \{v_2', v'\}$ does not add extra lattice points. The intersection of both gives the valid positions for v_1'. The blue area is $\operatorname{conv}(A_0 \cup \{v_2'\})$.

- If $\lambda_y^1 = \lambda_x^2 = 1$ we have $v_1 = (a_1, 1, 1)$ and $v_2 = (1, a_2, 1)$ with $a_i \in \mathbb{Z} \setminus \{0, 1\}$. Notice that we can also assume that $a_i \neq 2$ (if $a_1 = 2$ P has width one with respect to $x - z$ and if $a_2 = 2$ it has width one with respect to $y - z$). In this case, the conditions so far on the configuration are symmetric under both $(x, y, z) \mapsto (1 - x + z, y, z)$ and $(x, y, z) \mapsto (x, 1 - y + z, z)$, which reflect v_1 and v_2 within their respective chimneys. Hence we can assume that both v_1 and v_2 lie in their respective positive half-chimneys, that is, $a_1, a_2 > 2$.

  In the plane $z = 0$, we have now that

$$v_1' = \left(a_1', \frac{1}{2}, 0\right), \quad v_2' = \left(\frac{1}{2}, a_2', 0\right),$$

  where $a_i' = a_i/2 > 1$.

  The crosses in Figure 3.12 show the possible positions for v_1' and v_2'.

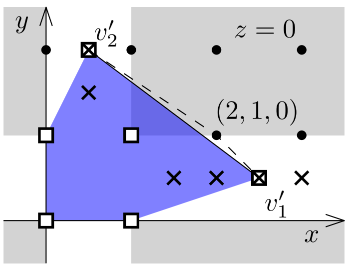


Figure 3.12: The possible positions for v_i' (hence for v_i) in case (b), $\lambda_y^1 = 1 = \lambda_x^2$, in the proof of Lemma 3.30. Crosses mark the positions for v_i' corresponding to v_i lying in their chimneys (the white regions). The $v_1'v_2'$ must separate $(2, 1, 0)$ from Q. The blue area is $P \cap \{z = 0\}$.

  Then, in order for the point $(2, 1, 0)$ not to be in P we need the triangle $\operatorname{conv}\{(2, 1), (a_1', 1/2), (1/2, a_2')\}$ to be negatively oriented, which amounts to:

$$\begin{vmatrix} 1 & 2 & 1 \\ 1 & a_1' & 1/2 \\ 1 & 1/2 & a_2' \end{vmatrix} = a_1'a_2' + \frac{5}{4} - a_1' - 2a_2' < 0.$$

Equivalently,

$$a_1'(a_2'-1) < 2a_2' - \frac{5}{4}.$$

Since $a_2' \geq 3/2$, this is the same as

$$a_1' < \frac{2a_2' - \frac{5}{4}}{a_2'-1} = 2 + \frac{\frac{3}{4}}{a_2'-1} \leq \frac{7}{2}.$$

The same arguments using the point $(1,2,0)$ and that $a_1' \geq 3/2$ (or simply the symmetry $x \leftrightarrow y$) give $a_2' < \frac{7}{2}$. Hence $a_1, a_2 \leq 6$. By the symmetries $(x,y,z) \mapsto (1-x+z, y, z)$ and $(x,y,z) \mapsto (x, 1-y+z, z)$, we get $a_1, a_2 \geq -4$. All in all, $a_1, a_2 \in [-4, 6]$.

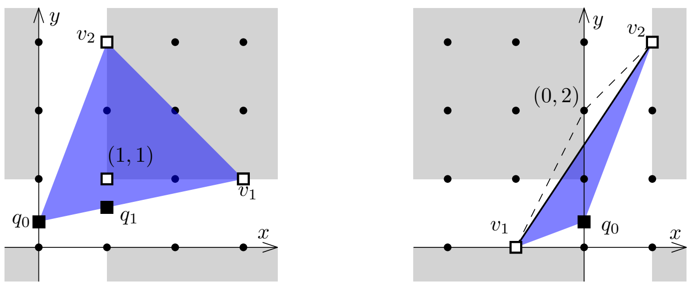
Summarizing, an upper bound for the distance in this case is $d_i \leq 6$ for all i. $\square$

For the remaining case we first study a similar question for boxed 2-polytopes.

Lemma 3.31. *Let P be a lattice polygon, boxed with respect to the unit square $Q = [0,1]^2$. Suppose that P intersects the edge $\{x = 0\}$ of Q and does not contain the vertices $(1,0)$ and $(1,1)$. Assume further that $v_2 = (1, a_2)$. Then, $a_2 \in \{-2, -1, 2, 3\}$.*

Proof. We assume without loss of generality that $a_2 > 1$ and want to show that $a_2 < 4$ (the case $a_2 < 0$ is symmetric with respect to the line $y = 1/2$). Let $q_0 = (0, y_0) \in P$, with $y_0 \in [0,1]$, be the point guaranteed by the hypotheses. We distinguish according to the possible positions of $v_1 = (a_1, \lambda_y^1)$. Remember that $\lambda_y^1 \in \{0,1\}$ and $a_1 \in \mathbb{Z} \setminus \{0,1\}$:

- If $a_1 > 1$, then P contains a point $q_1 = (1, y_1) \in \operatorname{conv}\{v_1, q_0\}$ with $y_1 \in [0,1]$. But then, the point $(1,1)$ must be in P (a contradiction) since it lies in the segment $q_1 v_2$ (see Figure 3.13).
- If $a_1 < 0$, consider the segment $v_1 v_2$. We need it to intersect the line $\{x = 0\}$ at height smaller than 2, or otherwise the point $(0,2)$ is in P (see Figure 3.13).



Case $a_1 > 1$ | Case $a_1 < 0$

Figure 3.13: The analysis of the cases where v_1 lies in $x > 1$ or $x < 0$ in the proof of Lemma 3.31. Black dots represent lattice points. Black squares represent the (possibly non-integer) points q_0 and q_1 that lie in P. White squares represent the vertices v_1 and v_2 of P and the lattice point $(1,1)$ when it lies in P. Each of the v_i lie in their corresponding chimney (the white regions). The blue area is a (maybe rational) subpolytope of P.

That is, we want the following determinant to be positive:

$$\begin{vmatrix} 1 & 1 & 1 \\ a_1 & 1 & 0 \\ \lambda_y^1 & a_2 & 2 \end{vmatrix} > 0.$$

Equivalently,

$$a_1a_2 + 2 - \lambda_y^1 - 2a_1 > 0,$$

or (since $a_1 \leq -1$ and $\lambda_y^1 \in \{0, 1\}$)

$$a_2 < 2 + \frac{2 - \lambda_y^1}{|a_1|} \leq 4 - \lambda_y^1 \leq 4.$$

□

Lemma 3.32. *Let P be a lattice 3-polytope, boxed with respect to the unit cube $Q = [0, 1]^3$ and of size at least seven. Suppose that:*

- *$v_3 = (1, 1, a_3)$, but neither $(1, 1, 0)$ nor $(1, 1, 1)$ are in P.*
- *$P \cap Q \cap \mathbb{Z}^3$ is not a facet of Q.*

Then, $d_i \leq 4$ for all i.

Proof. Since $P \cap Q \cap \mathbb{Z}^3$ has at least four points but is not a facet of Q, and since $(1, 1, 0)$ nor $(1, 1, 1)$ are not in P, $P \cap Q \cap \mathbb{Z}^3$ contains at least one lattice point from each of $\{(1, 0, 0), (1, 0, 1)\}$ and $\{(0, 1, 0), (0, 1, 1)\}$. That is, let $q_{10} = (1, 0, z_{10}), q_{01} = (0, 1, z_{01}) \in P$ for some $z_{10}, z_{01} \in \{0, 1\}$. We can also assume, without loss of generality, that $a_3 > 1$ (the case $a_3 < 0$ is symmetric with respect to the plane $\{y = 1/2\}$.

We distinguish according to the possible positions of

$$v_1 = (a_1, \lambda_y^1, \lambda_z^1),\ v_2 = (\lambda_x^2, a_2, \lambda_z^2).$$

Remember that $\lambda_*^i \in \{0, 1\}$ and $a_i \in \mathbb{Z} \setminus \{0, 1\}$, for $i = 1, 2$.

First, suppose $\lambda_x^2 = 1$. Then $v_2, v_3, q_{10} \in P$ are in the plane $\{x = 1\}$. Then $P \cap \{x = 1\}$ is as in the hypothesis of Lemma 3.31, hence $a_3 \in \{2, 3\}$. By symmetry of coordinates x and y, the same happens if $\lambda_y^1 = 1$, using the point q_{01}.

So now we have the case where $\lambda_y^1 = 0 = \lambda_x^2$:

$$v_1 = (a_1, 0, \lambda_z^1),\ v_2 = (0, a_2, \lambda_z^2)$$

We will now separate the cases where a_1, a_2 are positive or negative (see Figure 3.14):

- If $a_1, a_2 > 1$, then P contains a point $q_{11} = (1, 1, z_{11}) \in \operatorname{conv}\{v_1, v_2, q_{10}\}$ with $z_1 \in [0, 1]$. But then, the point $(1, 1, 1)$ must be in P (a contradiction) since it lies in the segment $q_{11}v_3$.

- If $a_1, a_2 < 0$, then P contains a point $q_{00} = (0,0,z_{00}) \in \operatorname{conv}\{v_1, v_2, q_{10}\}$, with $z_{00} \in [0,1]$. Consider now the triangle $v_1v_2v_3$. We need it to intersect the line $\{x = 0 = y\}$ at a height smaller than 2, or otherwise, since $q_{00} \in P$, the point $(0,0,2)$ must be in P (a contradiction). That is, we want the following determinant to be positive:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ a_1 & 0 & 1 & 0 \\ 0 & a_2 & 1 & 0 \\ \lambda_z^1 & \lambda_z^2 & a_3 & 2 \end{vmatrix} > 0.$$

That is, $-a_1a_2a_3 + \lambda_z^1 a_2 + \lambda_z^2 a_1 + 2(a_1a_2 - a_1 - a_2) > 0$ or, (since $a_1, a_2 \leq -1$ and $\lambda_z^1, \lambda_z^2 \in \{0,1\}$)

$$\begin{aligned} a_3 &< 2\left(\frac{(-a_1)(-a_2) + (-a_1)(1-\lambda_z^2/2) + (-a_2)(1-\lambda_z^1/2)}{(-a_1)(-a_2)}\right) = \\ &= 2\left(1 + \frac{1-\lambda_z^2/2}{-a_2} + \frac{1-\lambda_z^1/2}{-a_1}\right) \leq 2\left(1 + \frac{1}{-a_2} + \frac{1}{-a_1}\right) \leq 6 \end{aligned}$$

- One is positive and one negative: suppose $a_1 > 1$ and $a_2 < 0$. In this case, the triangle with vertices $v_1, v_2, v_3 \in P$ must intersect the line $\{x = 1, y = 0\}$ at height smaller than 2 or otherwise, since $q_{10} \in P$, the point $(1,0,2)$ is in P. This is equivalent to the following determinant being positive:

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & a_1 & 1 & 1 \\ a_2 & 0 & 1 & 0 \\ \lambda_z^2 & \lambda_z^1 & a_3 & 2 \end{vmatrix} > 0.$$

That is, $2a_2 + a_1a_2a_3 - 2a_1a_2 - a_2a_3 + 2a_1 + \lambda_z^2 - a_1\lambda_z^2 - \lambda_z^1 > 0$ or (since $a_1 \geq 2$, $a_2 \leq -1$ and $\lambda_z^1, \lambda_z^2 \in \{0,1\}$)

$$\begin{aligned} a_3 &< 2\left(\frac{(-a_2)(a_1-1) + a_1 - \lambda_z^1/2 - \lambda_z^2(a_1-1)/2}{(-a_2)(a_1-1)}\right) \leq \\ &\leq 2\left(\frac{(-a_2)(a_1-1) + a_1}{(-a_2)(a_1-1)}\right) = \\ &= 2\left(1 + \frac{1}{-a_2} + \frac{1}{(-a_2)(a_1-1)}\right) \leq 6 \end{aligned}$$

By symmetry $x \leftrightarrow y$, the same happens if $a_1 < 0$ and $a_2 > 1$.

Summarizing, an upper bound for the distance in this case is $d_i \leq 4$ for all i. $\square$

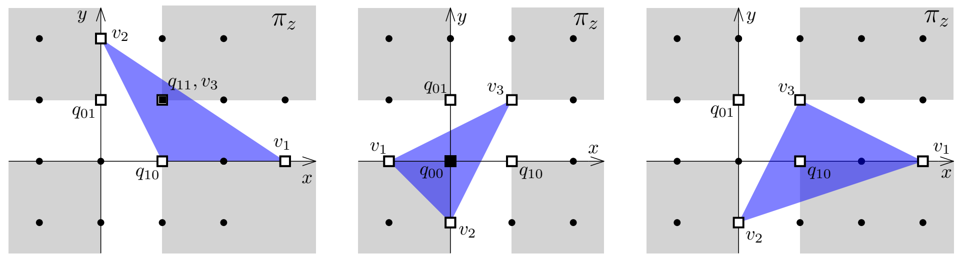


Figure 3.14: The analysis of the cases where v_i lies in C_i^+ or in C_i^-, for $i = 1, 2$, in the proof of Lemma 3.32. The figures represent the projection in the direction of the third coordinate. Black dots represent lattice points. Black squares represent the (possibly non-integer) points q_{ij} that lie in P. White squares represent the vertices v_i and lattice points q_{ij} of P. Each of the vertices v_i lies in its corresponding chimney (the white regions). The blue area is a (maybe rational) subpolytope of P.

3.4 Results of the enumeration

The results of Sections 3.2 and 3.3 allow us to completely enumerate quasi-minimal 3-polytopes. The counts of them are given in Table 3.1. Boxed ones are only finitely many and of size at most ten. They are enumerated by computer as explained in Section 3.3.3. For spiked ones, the number was computed in Section 3.2 (see Remark 3.20). These two counts are shown in the left and center parts of the table, and the right part contains the union of the two sets. For sizes 5 and 6 a polytope can be boxed and spiked at the same time (see Remarks 3.20 and 3.21), so we do not give the separate numbers. In fact, the numbers of quasi-minimal 3-polytopes of these sizes were not computed with the methods of this chapter, but directly extracted from the classifications of sizes 5 and 6 detailed in Section 1.3.

	boxed				spiked			all			
# vertices	4	5	6	total	4	5	total	4	5	6	total
size 5								9	0	–	9
size 6								22	13	0	35
size 7	4	15	4	23	21	6	27	25	21	4	50
size 8	2	5	0	7	22	13	35	24	18	0	42
size 9	0	1	0	1	26	17	43	26	18	0	44
size 10	1	0	0	1	24	21	45	25	21	0	46
size 11	0	0	0	0	24	25	49	24	25	0	49
size > 11	0	0	0	0	∞	∞	∞	∞	∞	0	∞
Total	7	21	4	32	∞	∞	∞	∞	∞	4	∞

Table 3.1: Quasi-minimal 3-polytopes, classified according to their size (row) and number of vertices (column). For size $n > 11$ there exist $4n-19$ spiked polytopes with 5 vertices and 24 (or 26 if $n \equiv 0 \pmod 3$) with 4 vertices.

Once we have quasi-minimal 3-polytopes completely classified, we can run the enumeration algorithm described in the introduction of this chapter, taking as input the list of lattice 3-polytopes of size 6 and width larger than one, detailed in Section 1.3. In the following sections we show the results of this enumeration, that we carried out up to size 11.

3.4.1 Classification by number of vertices and/or interior points

The summary of our enumeration of lattice 3-polytopes is given in Table 3.2. Observe that the zeros in the diagonal "size=vertices" follow from Howe's Theorem 1.15. We also show the approximate computation times.

# vertices	4	5	6	7	8	9	10	total	time
size 5	9	0	–	–	–	–	–	9	from [BS16a]
size 6	36	40	0	–	–	–	–	76	from [BS16b]
size 7	103	296	97	0	–	–	–	496	14 mins.
size 8	193	1195	1140	147	0	–	–	2675	70 mins.
size 9	282	2853	5920	2491	152	0	–	11698	7 hours
size 10	478	5985	18505	16384	3575	108	0	45035	48 hours
size 11	619	11432	48103	64256	28570	3425	59	156464	20 days

Table 3.2: Lattice 3-polytopes of width larger than one and size ≤ 11, classified according to their size (row) and number of vertices (column).

Remark 3.33. The total number of lattice 3-polytopes of width larger than one seems experimentally to grow more slowly than a single exponential. But the only theoretical upper bound that we can derive from the merging algorithm is doubly exponential, which follows from the following recurrence: let $S(n)$ be the number of lattice 3-polytopes of width larger than one and size n, then

$$S(n+1) \leq 24\binom{n}{4}S(n)^2 + 4n + 11.$$

(The term $4n + 11$ is an upper bound for the number of quasi-minimal 3-polytopes of size $n + 1 \geq 11$).

Table 3.9 shows a finer classification, in which the number of interior lattice points is also considered.

The numbers of canonical and terminal polytopes (see Section 1.2.3), for each size can be extracted from Table 3.9 and are shown in Table 3.3. Our results agree with both the classifications of canonical 3-polytopes [Kas06] and of lattice 3-polytopes with exactly 2 interior lattice points [BK16] (the latter we have compared through personal communication with the authors).

Size	5	6	7	8	9	10	11
Canonical	8	49	218	723	1990	4587	9376
Terminal	8	38	95	144	151	107	59

Table 3.3: Canonical and terminal 3-polytopes of size ≤ 11.

3.4.2 Classification by width

In Table 3.4 we present the classification of lattice 3-polytopes by size and width larger than one (remember that those of width one are infinitely many for each size; see Remark 1.25). There is a remarkable gap between 3-polytopes of width three, that exist already with six lattice points, and of width four, which need ten lattice points at the least. It is also worth noting that in every size the maximum width is achieved (perhaps not uniquely) at some clean tetrahedron.

Size	4	5	6	7	8	9	10	11
width 2	0	9	74	477	2524	10862	40885	137803
width 3		0	2	19	151	836	4148	18635
width 4			0	0	0	0	2	26
width 5							0	0

Table 3.4: Lattice 3-polytopes of size ≤ 11, classified by width.

3.4.3 Volumes of 3-polytopes

Figure 3.15 shows the normalized volumes that arise among lattice 3-polytopes of width larger than one and sizes 5 to 11.

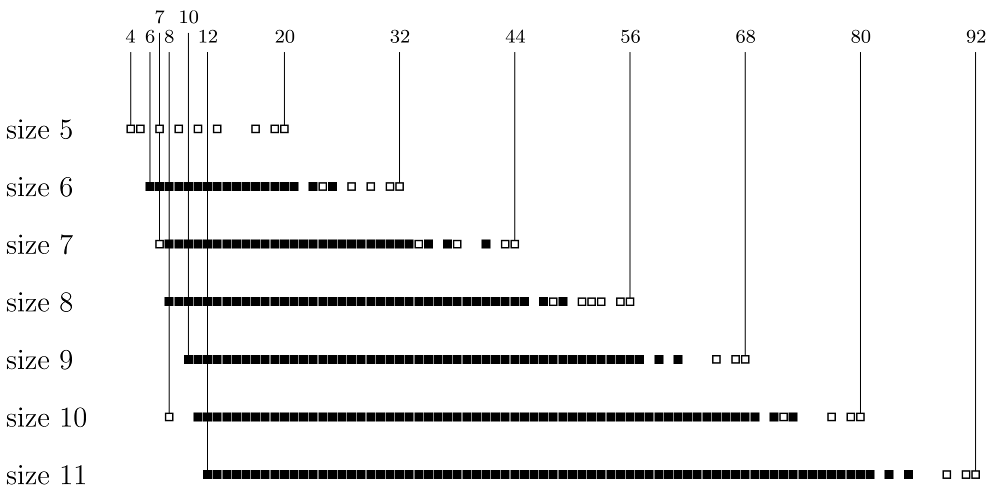


Figure 3.15: A square in the row n and column v means that there is some lattice 3-polytope of width > 1 and size n of (normalized) volume v. A white square means that there is exactly one such polytope. The values for v displayed in the top row are the minimum and maximum values achieved for each size.

Observe that the minimum volume is not a monotone function of size. There is a (unique) polytope of size 10 and volume 8, while the minimum volume in size 9 is 10. This may seem contradictory, since for any polytope P of size n, the volume of P^v, which has size $n-1$, is strictly smaller than that of P. However, remember that our table *does not* show the volumes of polytopes of width one (which go from $n-3$ to ∞ for each size

$n \geq 4$). The polytope of size 10 and volume 8 is the second dilation of the unimodular tetrahedron, which is minimal (every proper subpolytope of size one less has width one).

In turn, the maximum volume achieved for each size is very consistent:

Theorem 3.34. *For each $n \in \{5, \dots, 11\}$ the maximum volume of a lattice 3-polytope of size n and width larger than one is $12(n-4)+8$ and is achieved by a unique polytope.*

Looking closer at these unique polytopes, we can see that there is a very simple description, with only one coordinate of one of the vertices being dependent of the size $n \in \{5, 11\}$. It turns out that this polytope has the same properties for any $n \geq 5$:

Proposition 3.35. *The following lattice 3-polytope*

$$R_n := \operatorname{conv}\{(-1,-1,1), (-1,1,-2), (0,1,2n-9), (2,-1,0)\}$$

is a clean tetrahedron of size n, width 2 and normalized volume $12(n-4)+8$, for all $n \geq 5$.

Proof. The width 2 is achieved with functional y. Let us see that, besides the four vertices, the only other lattice points of R_n are $n-4$ aligned interior points. The following is the projection of R_n in the direction of the z coordinate:

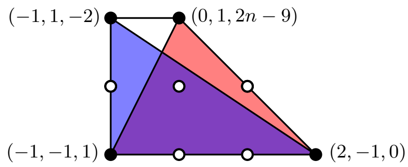


Figure 3.16: The projection of R_n in the direction of the z coordinate. Black dots represent the projection of vertices, white dots represent other lattice points in the convex hull of the projection, black lines represent edges and the blue and red triangles are the two facets of R_n that intersect the line $\{x = 0 = y\}$.

As the image shows, the only possible lattice points of R_n, besides its vertices, could appear as points in the edges $\{(-1,-1,1), (-1,1,-2)\}$, $\{(-1,-1,1), (2,-1,0)\}$ and $\{(0,1,2n-9), (2,-1,0)\}$, or points in the line $\ell := \{x = 0 = y\}$. Since those three edges are primitive, R_n can only have more lattice points in ℓ. The figure shows the only two facets that cut this line. The plane passing through $(-1,-1,1)$, $(-1,1,-2)$ and $(2,-1,0)$ cuts ℓ at $z = -\frac{5}{6} \in (-1, 0]$ and the plane passing through $(-1,-1,1)$, $(2,-1,0)$ and $(0,1,2n-9)$ cuts ℓ at $z = n-5+\frac{5}{6} \in [n-5, n-4)$, for $n-5 \geq 0$. Hence the only lattice points of R_n other than its vertices are the points $(0,0,0)$ to $(0,0,n-5)$. $\square$

Given this, it is quite natural to conjecture the following:

Conjecture 3.36. *For each $n \geq 5$ the maximum volume of a lattice 3-polytope of size n and width larger than one is $12(n-4)+8$, and this volume is achieved only by R_n.*

Remark 3.37. Han Duong conjectured that the maximum volume of a *clean tetrahedron with exactly k interior points* is $12k+8$, and that the unique clean tetrahedron achieving this bound was $\operatorname{conv}\{(0,0,0), (1,0,0), (0,1,0), (2k+1, 4k+3, 12k+8)\}$ (Conjecture 2 in [Duo08]). Since that polytope is equivalent to R_{k+4} under the unimodular transformation $(x,y,z) \to (3y-z-1, -2x-2y+z+1, 3x+2y-z-2)$, our conjecture is in fact stronger than his: we conjecture that this polytope maximizes volume not only among clean tetrahedra, but actually among all 3-polytopes of a given size (and width larger than one).

3.4.4 Nonprimitive 3-polytopes

We call a lattice d-polytope P *primitive* if $P \cap \mathbb{Z}^d$ generates with affine integer combinations the whole lattice $\mathbb{Z}^d$. More generally, we call *sublattice index* of P the index, as a sublattice of $\mathbb{Z}^d$, of the affine lattice generated by $P \cap \mathbb{Z}^d$.

It is easy to prove (see [BS17]) that this index coincides with the gcd of all determinants of $(d+1)$-tuples of lattice points in P. Using this fact we have computed the index of all 3-polytopes of width larger than one and size up to 11. Table 3.5 shows that most polytopes are primitive and that only indices 2, 3 and 5 appear.

size	**5**	**6**	**7**	**8**	**9**	**10**	**11**
index 1	7	71	486	2658	11680	45012	156436
index 2	-	2	8	14	15	19	24
index 3	1	3	2	3	3	4	4
index 5	1	-	-	-	-	-	-

Table 3.5: Lattice 3-polytopes of width greater than one and size ≤ 11, classified by sublattice index.

In an upcoming paper of Blanco–Santos [BS17] we study the sublattice index of lattice 3-polytopes and show that, as hinted by Table 3.5:

- All 3-polytopes of width greater than one have indices 1, 2, 3 or 5.
- There is only one of index 5, a terminal tetrahedron of normalized volume 20.
- In size $n \geq 7$ there are exactly $\lfloor n/2 \rfloor - 1$ lattice 3-polytopes of index three, all closely related to the spiked 3-polytopes of type (4) from Theorem 3.19.
- In size $n \geq 9$ there are exactly $\lceil n(n-2)/4 \rceil - 1$ lattice 3-polytopes of index two, all closely related to the spiked 3-polytopes of type (1) from Theorem 3.19.

3.4.5 Normality in dimension 3

Following [BGM16], we say that a lattice d-polytope P is *normal* if, for all $k \in \mathbb{N}$, every point in $kP \cap \mathbb{Z}^d$ can be written as the sum of k points in $P \cap \mathbb{Z}^d$. That is, if

$$kP \cap \mathbb{Z}^d = \left\{ p_1 + \dots p_k \mid p_1, \dots, p_k \in P \cap \mathbb{Z}^d \right\}.$$

Please observe that with this definition every normal polytope is primitive. Sometimes a weaker definition of normality is used, and the concept defined here, which becomes equivalent to "normal and primitive", is called *integrally closed* (see, e.g., [BG09]).

It is easy to prove that for a lattice d-polytope to be normal it is enough that it satisfies the definition for $k \in \{2, \dots, d-1\}$. In particular, a lattice 3-polytope P is normal if, and only if,

$$\#\Big(2P \cap \mathbb{Z}^3\Big) = \#\Big(P \cap \mathbb{Z}^3 + P \cap \mathbb{Z}^3\Big).$$

Via this characterization, we have checked normality in all lattice 3-polytopes from our database. See Algorithm A.22 for the MATLAB routine computing normality of a polytope. The resulting numbers are given in the following table, where we also show what

size	5	6	7	8	9	10	11
normal	1	10	61	325	1532	6661	25749
fraction	0.111	0.132	0.123	0.121	0.131	0.148	0.165

fraction of the total are normal, for each size. It is not clear with this data what the asymptotic behavior of this fraction is.

An interesting question about normality that arises in the work of Bruns et al. [BGM16, Question 7.2(a)] is whether, apart of the unimodular tetrahedron, there is a lattice 3-polytope P that is normal but in which P^v (if it is 3-dimensional) is not normal for any vertex v of P. We can guarantee that, among the 34339 normal 3-polytopes of width larger than one and size ≤ 11 there is none.

3.4.6 Results on dps 3-polytopes

Finally, from the classification it is also easy to extract the full list of dps 3-polytopes of width greater than one, as shown in Table 3.6.

# vertices	4	5	6	7	total
size 5	9	0	–	–	9
size 6	20	25	0	–	45
size 7	5	31	12	0	48
size 8	3	2	1	0	6
total	37	58	13	0	108

Table 3.6: The number of dps 3-polytopes of width larger than one, classified according to their size (row) and number of vertices (column).

Dps 3-polytopes of width one are infinitely many for each given size (see Remark 1.25). In [BS16a] (see Table 1.1) and [BS16b, Section 4] we worked up a full classification of lattice 3-polytopes of width one and sizes 5 and 6, respectively, and singled out the dps ones. Dps 3-polytopes of sizes 7 and 8 and width one are also easy to describe: they consist of two polygons in parallel consecutive hyperplanes, a unimodular triangle and a terminal triangle in the case of size 7, and two terminal triangles in the case of size 8.

After this full classification, we can answer in dimension 3 the several questions posed by Reznick [BNR+08] regarding dps polytopes:

- What is the range for the volume of dps d-polytopes of size 2^d?
- Is every dps d-polytope a subset of one of size 2^d?
- How many "inequivalent" dps d-polytopes of size 2^d are there?

In dimension 2 there are only two dps polygons: a unimodular triangle, and a terminal triangle (of volume 3), and only the second one is maximal. In dimension 3, by looking at the list of dps 3-polytopes of size 8:

- There are six dps 3-polytopes of width larger than one and of maximal size 8. Specific coordinates for each of them, together with some other properties, are displayed in Table 3.8. Five of them (first five rows in the table) were found by Curcic, (unpublished PhD thesis; see [Cur12]), who asked whether his list was complete.

- The one of minimum volume is a clean 3-polytope of width 3 and normalized volume 25, with 6 vertices and 2 interior lattice points. That of maximal volume is a clean tetrahedron of volume 51.

- Not every dps 3-polytope is a subset of one of size 8. Specifically, there are exactly 33 dps 3-polytopes that have no extension to size 8. They all are of size 7 and of width larger than one. Table 3.7 shows the numbers of them, organized according to number of lattice points and vertices:

# vertices	4	5	6	7	total
size 7	3	21	9	0	33
size 8	3	2	1	0	6

Table 3.7: Dps 3-polytopes of width larger than one that are *dps-maximal* (those dps polytopes that are not contained in another dps polytope) are counted.

Coordinates	**Vertices**	**Interior points**	**Volume**	**Width**
$\begin{pmatrix} -3 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -3 & -1 & -1 & 0 & 0 & 1 & 0 & 4 \\ -1 & 0 & 3 & 0 & 1 & 0 & 0 & -1 \end{pmatrix}$	4	4	51	3
$\begin{pmatrix} -3 & -1 & 0 & 0 & 0 & 0 & 1 & 1 \\ -5 & -1 & 0 & 0 & 1 & 3 & 0 & 8 \\ 1 & 0 & 0 & 1 & 0 & -1 & 0 & -3 \end{pmatrix}$	4	4	39	3
$\begin{pmatrix} -1 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ -1 & 0 & 0 & 1 & -1 & 0 & 2 & -3 \\ -1 & 0 & 1 & 3 & 0 & 0 & 1 & -2 \end{pmatrix}$	4	4	35	3
$\begin{pmatrix} -2 & -1 & 0 & 0 & 0 & 1 & 1 & 3 \\ 1 & -1 & 0 & 0 & 1 & 0 & 2 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & 1 \end{pmatrix}$	5	2	28	2
$\begin{pmatrix} -2 & -1 & 0 & 0 & 0 & 1 & 1 & 5 \\ 1 & -1 & 0 & 0 & 1 & 0 & 2 & -1 \\ 1 & 0 & 0 & 1 & 0 & 0 & -1 & -1 \end{pmatrix}$	5	3	36	2
$\begin{pmatrix} -1 & -1 & 0 & 0 & 0 & 1 & 1 & 2 \\ -2 & -1 & 0 & 0 & 1 & 0 & 3 & 1 \\ -1 & 2 & 0 & 1 & 1 & 0 & 1 & 0 \end{pmatrix}$	6	2	25	3

Table 3.8: Some properties of dps 3-polytopes of size 8.

We have also looked at the number of vertices of dps 3-polytopes. In dimension 2, the maximum number of vertices is 3. In dimension 3, and for polytopes of width larger than one, Table 3.6 shows that the maximum number of vertices is 6. The same happens for those of width one, since the lattice dps polygons in each of the parallel planes can only have 3 vertices each.

Question 3.38. *Is the maximum number of vertices of a dps d-polytope* $3 \cdot 2^{d-2}$*?*

Dps polytopes with this number of vertices are easy to construct by induction on the dimension. For $d = 2$ the two dps polygons have $3 = 3 \cdot 2^0$ vertices. For $d > 2$ take two dps $(d-1)$-polytopes with $3 \cdot 2^{(d-1)-2} = 3 \cdot 2^{d-3}$ vertices and place them in consecutive parallel hyperplanes in a way that no edge in one of the polytopes is parallel to an edge in the other (there are infinitely many possibilities that have this property). Then the resulting polytope is still dps and has $2(3 \cdot 2^{d-3}) = 3 \cdot 2^{d-2}$ vertices.

	Size 5		
# vertices	**4**	**total**	
0 int. pts.	1	1	
1 int. pts.	8	8	
2 int. pts.	–	–	
3 int. pts.	–	–	
total	9	9	

	Size 6		
# vertices	**4**	**5**	**total**
0 int. pts.	2	2	4
1 int. pts.	11	38	49
2 int. pts.	23	–	23
3 int. pts.	–	–	–
total	36	40	76

	Size 7			
# vertices	**4**	**5**	**6**	**total**
0 int. pts.	5	10	2	17
1 int. pts.	17	106	95	218
2 int. pts.	30	180	–	210
3 int. pts.	51	–	–	51
total	103	296	97	496

	Size 8				
# vertices	4	5	6	7	**total**
0 int. pts.	5	27	24	3	59
1 int. pts.	10	176	393	144	723
2 int. pts.	31	429	723	–	1183
3 int. pts.	57	563	–	–	620
4 int. pts.	90	–	–	–	90
5 int. pts.	–	–	–	–	–
total	193	1195	1140	147	2675

	Size 9					
# vertices	4	5	6	7	8	**total**
0 int. pts.	4	43	69	26	1	143
1 int. pts.	19	195	833	792	151	1990
2 int. pts.	15	524	2303	1673	–	4515
3 int. pts.	50	1075	2715	–	–	3840
4 int. pts.	92	1016	–	–	–	1108
5 int. pts.	102	–	–	–	–	102
total	282	2853	5920	2491	152	11698

	Size 10						
# vertices	4	5	6	7	8	9	**total**
0 int. pts.	8	56	156	109	16	1	346
1 int. pts.	15	300	1235	1975	955	107	4587
2 int. pts.	21	554	3822	6774	2604	–	13775
3 int. pts.	37	1304	7504	7526	–	–	16371
4 int. pts.	92	2029	5788	–	–	–	7909
5 int. pts.	119	1742	–	–	–	–	1861
6 int. pts.	186	–	–	–	–	–	186
total	478	5985	18505	16384	3575	108	45035

	Size 11							
# vertices	4	5	6	7	8	9	10	**total**
0 int. pts.	6	59	235	267	81	5	–	653
1 int. pts.	19	302	1809	3658	2781	748	59	9376
2 int. pts.	23	661	5208	13859	12234	2672	–	34657
3 int. pts.	32	1326	11892	27467	13474	–	–	54191
4 int. pts.	46	2421	16239	19005	–	–	–	37711
5 int. pts.	99	3307	12720	–	–	–	–	16126
6 int. pts.	185	3356	–	–	–	–	–	3541
7 int. pts.	209	–	–	–	–	–	–	209
total	619	11432	48103	64256	28570	3425	59	156464

Table 3.9: The total number of lattice 3-polytopes of width larger than one and sizes 5 to 11, classified according to their numbers of interior points (row) and vertices (column).

Appendix A

MATLAB routines

All the MATLAB routines presented here work for lattice polytopes in $\mathbb{R}^3$.

A.1 Basic properties of a polytope

A.1.1 Convex hull and lattice points

Let $A = \{p_1, \dots, p_k\} \subset \mathbb{Z}^3$ be a configuration of lattice points. The polytope $P = \operatorname{conv}(A) \subset \mathbb{R}^3$ is completely determined by either of these pieces of information: the list of equations HP of the facet-defining halfspaces, and the list of vertices VP. We also often have, or want to have, the complete list of lattice points LP, since we are interested in classifying polytopes via their (number of) lattice points.

Algorithm A.1. Facet-defining hyperplanes. The facet-defining hyperplanes come uniquely determined by primitive integer affine functionals $f : \mathbb{R}^3 \to \mathbb{R}$, $f(x, y, z) = ax + by + cz + d$, for $a, b, c, d \in \mathbb{Z}$, that are zero in the points of on some facet, and non-negative on A.

For each 3-tuple of points p_i, p_j, p_k, we compute the equation of the hyperplane passing through those three points:

$$ax + by + cz + d = \det \begin{pmatrix} x - p_1(1) & y - p_1(2) & z - p_1(3) \\ p_2(1) - p_1(1) & p_2(2) - p_1(2) & p_2(3) - p_1(3) \\ p_3(1) - p_1(1) & p_3(2) - p_1(2) & p_3(3) - p_1(3) \end{pmatrix} = 0$$

$\big(p_i(s)$ denotes the s-th coordinate of $p_i\big)$

The affine functional that is zero in the hyperplane is $f(x, y, z) = ax + by + cz + d$. If $f(p) \geq 0$ for all $p \in A$, or $f(p) \leq 0$ for all $p \in A$, then f is a facet-defining hyperplane. For uniqueness, we choose f or $-f$, so that the functional is non-negative on A, and we choose it primitive ($\gcd(a, b, c) = 1$).

The list of facet-defining hyperplanes is stored as `hyperplanes` or `HP`, and it is an $m \times 4$ matrix where each row contains the coefficients $[a, b, c, d]$ of a functional.

There is an optional variable `rat` as input, which is equal to 0 by default, and 1 if we want to allow for non-integer points. We use this feature in the computation of width (see Algorithm A.9). In this case, since HP is not necessarily integer and MATLAB stores its entries in floating point, different rows may define the same facet.

Remark A.2. The convex hull of n points in $\mathbb{R}^3$ can be computed in $O(n \log n)$ time (see [PH77]). However, in our case, we are generally computing the convex hull of very few points and we generally only compute the convex hull of each polytope once, hence we use a brute force $O(n^3)$ algorithm.

```
function HP=convex_hull(A,rat)

%A is a set of lattice points: a 3 X n matrix where each
%column is a point

%HP is an m x 4 matrix, where each row is primitive and represents a
%facet of P=conv(A): HP(i,:)=(a,b,c,d)==> ax+by+cz+d >= 0 for all
%(x,y,z) in P

%rat=1 if the points are possibly rational. In this case THERE MAY BE
%REPETITION of facets, as rational coefficients are more difficult
%to treat
if nargin<2
    rat=0;
end

n=size(A,2);

HP=[];
s=0;

%We consider each 3-tuple of points
for i=1:n-2
    for j=i+1:n-1
        for k=j+1:n
            M=A(:,[i j k]);
            M=M'; %Each row is a point

            %We want to eliminate the cases when the 3 points are
            %collinear. For this, we compute the vectors that join one
            %to the other two. The points will be collinear if the rank
            %of this vectors is 1.
            O=zeros(2,3);
            O(1,:)=M(2,:)-M(1,:);
            O(2,:)=M(3,:)-M(1,:);
            if rank(O)>1
                %The equation of the plane that passes through those
                %three points is:
                %     (x-M11  y-M12  z-M13)
                % det(O21     O22    O23   )=0=ax+by+cz+d
                %     (O31    O32    O33   )

                detx=det(O(:,[2,3]));
                dety=det(O(:,[1,3]));
                detz=det(O(:,[1,2]));
                %vec=(a,b,c,d);
                vec=[detx,-dety,detz];
                vec=[vec,-detx*M(1,1)+dety*M(1,2)-detz*M(1,3)];
                if rat==0
                    vec=round(vec);
```

```
                    %vec is the coefficients of the affine functional
                    %that is constant and zero in the plane that goes
                    %through those three points.
                    %We want to make it primitive. By construction, the
                    %gcd of the first three coordinates (the linear part
                    %of the functional) divides the independent
                    %coefficient
                    d=gcd(vec(1),gcd(vec(2),vec(3)));
                    vec=vec./d;
                end
                %In vecA we will store the values the functional takes
                %in all the points of A
                vecA=vec*[A;ones(1,n)];

                if rat==0
                    vecA=round(vecA); %The values are integer
                end

                %If all lattice points are in this plane, we want to
                %only have one equation so that we can later interprate
                %that the configuration is 2-dimensional
                %If HP=[], the configuration would be 1-dimensional.
                if isequal(vecA,zeros(1,4))==1
                    s=s+1;
                    HP(s,:)=vec;
                    return
                end

                %we now only consider the signs.
                sign_vecA=sign(vecA);
                %For uniqueness, we want as many pluses as minuses
                if sum(sign_vecA)<0
                    vec=-vec;
                    sign_vecA=-sign_vecA;
                end

                %If there is a -1 (there is at least a 1), then the
                %functional does not define a facet
                if ismember(-1,sign_vecA)==0
                    %If there is no -1, then the functional is valid,
                    %but we want to check that it has not been stored in
                    %HP yet.
                    if  s>0
                        if ismember(vec,HP,'rows')==0
                            s=s+1;
                            HP(s,:)=vec;
                        end
                    else
                        HP=vec;
                        s=1;
                    end
                end
            end
        end
    end
end
```

Algorithm A.3. Lattice points in a polytope.

We assume that our input is A a set of lattice points plus the list HP of facet-defining hyperplanes of $P = \operatorname{conv}(A)$. In the case that HP has not been computed yet, we do so.

Since all lattice points in P are convex combinations of points in A, the i-th coordinate of any $p \in P$ is going to verify $p(i) \in [\min_{q\in A} q(i), \max_{q\in A} q(i)]$. We simply take all points $p \in \mathbb{Z}^3$ in the cartesian product

$$\prod_{i=1,2,3} \left[\min_{q\in A} q(i), \max_{q\in A} q(i)\right]$$

and check whether $f(p) \geq 0$ for all $f \in HP$. If this holds, p is a lattice point of P.

The list of lattice points is stored as `lattice_pts` or `LP`, and it is a $3 \times N$ matrix where each column corresponds to a point.

As in the previous case, `rat` is an optional input variable, which equals 0 by default, and 1 if we want to allow for the polytope to have non-integer vertices.

Remark A.4. Because of the *floating point* treatment of non-integer values in MATLAB, there is some margin of error in the equations of the convex hull. The only case this routine is used with `rat=1` is in the width Algorithm A.9. In this algorithm, we need the call to the function `lattice_points` to compute all the lattice points in the convex hull of a non-lattice polytope, but it is not a problem if the function gives us additional lattice points. Since there may be rounding errors, we slightly shift the facet-defining hyperplanes in the outward normal direction (adding a positive small constant to the independent term).

```
function LP=lattice_points(A,HP,rat)

%INPUTS
%A is a 3-dimensional set of lattice points given by a 3 x N matrix

%HP is an optional variable. If it is not input, we have to compute it.
%It is an m x 4 integer matrix, where each row is primitive and
%represents a facet of P=conv(A): HP(i,:)=(a,b,c,d)===>  ax+by+cz+d >=0
%for all (x,y,z) in P
if nargin < 2
    HP=convex_hull(A);
end

%rat is an optional variable which will be 0 by default and 1 if we are
%looking at a rational polytope. In this case, we will assume that HP
%has been inputed, and is a matrix with rational coefficients
if nargin < 3
    rat=0;
end

%OUTPUTS
%LP: total set of lattice points of the convex hull

LP=[];
sLP=0;

%We take the ceiling of the minimum of each coordinate as a lower bound,
%and the floor of the maximum.
```

```
%If they are integer, the value does not change.
x1=ceil(min(A(1,:)));
y1=ceil(min(A(2,:)));
z1=ceil(min(A(3,:)));

x2=floor(max(A(1,:)));
y2=floor(max(A(2,:)));
z2=floor(max(A(3,:)));

if rat==1
    %In the case of looking at rational polytope, we are given the facet
    %equations by rational functionals, and there is some margin of
    %error in computing whether a point lies in one side or the other of
    %the hyperplane. To ensure that we consider all the points of the
    %polytope we will shift slightly the hyperplanes in the outward
    %normal direction. For this, we simply add +0.2 to the independent
    %term of each functional.
    %We will count with the possibility that points outside of P are
    %considered as points of P.
    HP(:,4)=HP(:,4)+1/5*ones(size(HP,1),1);
end

for x=x1:x2
    for y=y1:y2
        for z=z1:z2

            % X is a lattice point
            X=[x;y;z];

            % val is a vector m x 1, where the i-th value is the value
            % that hyperplane i takes in point X
            val=HP*[X;1];

            %If rat=0, the coordinates in val are integers.
            if rat==0
                val=round(val);
            end

            %sign(val) records the signs of vector val
            % The i-th coordinate is -1 if point X is in the wrong side
            % of hyperplane i
            if ismember(-1,sign(val))==0
                sLP=sLP+1;
                LP(:,sLP)=X;
            end
        end
    end
end
```

Algorithm A.5. Interior and boundary points, vertices. Let $A \subset \mathbb{Z}^3$ with $P = \text{conv}(A)$ (any set with $\text{vert}(P) \subseteq A \subseteq LP$), we want to decide whether each of those lattice points is in the boundary or in the interior, and of those of the boundary, decide whether they are vertices or not.

For each $p \in A$, and each $f \in HP$, we check whether $f(p)$ is zero. If $f(p) = 0$, it will mean that p belongs to the facet $P^f := P \cap f^{-1}(0)$. If $f(p) > 0$ for all $f \in HP$, then p is an interior point. Otherwise, it is a boundary point. On the other hand, a boundary point is a vertex if it belongs to at least three different facets: in dimension 3, if three different facets intersect at a lattice point of the polytope, then this lattice point must be a vertex.

The list of vertices is stored in `vertices` or `VP`. The information on whether each lattice point of A is a vertex, a boundary point, or an interior point, is stored in three separate boolean vectors where the i-th coordinate is 1 if the point verifies each property, or 0 otherwise. We will typically input $A = LP$.

```
function [VP,V_ind,I_ind,B_ind]=vertex_int_boun_pt(HP,A)

n=size(A,2);     %number of lattice points
f=size(HP,1);    %number of facets

SHP=sign(HP*[A;ones(1,n)]);
%SHP(j,i) is 0 if point i is in facet j, and 1 otherwise

VP=[];
V_ind=zeros(1,n);
I_ind=zeros(1,n);
B_ind=zeros(1,n);

for i=1:n
    %S is the number of facets the point A(:,i) DOES NOT belong to
    S=sum(SHP(:,i));

    %if S is equal to the the number of facets, then it is an interior
    %point
    if S==f %interior point
        I_ind(i)=1;

    %otherwise it is a boundary point
    else
        B_ind(i)=1;

        %If the point belongs to at least 3 facets, it is a vertex
        if S<f-2
            VP=[VP,A(:,i)];
            V_ind(i)=1;
        end
    end
end
```

A.1.2 Unimodular equivalence and volume vectors

To prove whether two lattice polytopes P and Q are unimodularly equivalent or not, we need to check the existence of an affine integer unimodular transformation that maps one polytope to the other. That is, we need to find a matrix $M \in \mathbb{Z}^{3\times 3}$ with $\det(M) = \pm 1$ and a vector $b \in \mathbb{Z}^3$ such that the transformation $t : \mathbb{R}^3 \to \mathbb{R}^3$ given by $t(x, y, z) = M \cdot (x, y, z)^T + b$ maps one polytope to the other.

The *unimodular equivalence* Algorithm A.7 we use needs as input the volume vector (and its ordered unsigned version) of the sets of vertices VP and VQ.

Algorithm A.6. Volume vectors of a point configuration. Given A a point configuration, w is an integer vector with $\binom{\#A}{4}$ coordinates. The coordinates record the determinants of each 4-tuples of points of A in lexicographic order. For this, we require A to be an *ordered* list, so that each element of the list can be identified with its position in said list. We then store in w_0 the absolute values of the coordinates of w, in order of increasing value. See routine `volumevectors` in page 121.

We also need to be able to recover the 4 points whose determinant corresponds to a given entry of the volume vector (routine `correspondance` in page 122), and the non-zero value that is the least repeated in the absolute value vector (routine `least_freq_vol` in page 122).

```
function [w,w0] = volumevectors(A)
% A is an integer 3 by n matrix, where column A(:,i) represents a
%lattice point.
% We compute the volume vector
% w = ( w_{i,j,k,l}), with 1 =< i < j < k < l =< n and where the order
% of the indexes is lexicographical. w_{i,j,k,l} is the determinant of
% the lattice points in positions i,j,k,l, in that order.

% In w0 we will store the coordinates of w in its absolute value.

n=size(A,2); %number of lattice points.
w=zeros(1,nchoosek(n,4)); %empty volume vector
w0=zeros(1,nchoosek(n,4)); %emtpy absolute volumes vector
r=0;
for i=1:(n-3)
    for j=(i+1):(n-2)
        for k=(j+1):(n-1)
            for l=(k+1):n
                r=r+1;
                %Anew is a 4 by 4 matrix with a first row of ones and
                %then the four other points each one completing a
                %column
                Anew=ones(4);
                Anew(2:4,1)=A(:,i);
                Anew(2:4,2)=A(:,j);
                Anew(2:4,3)=A(:,k);
                Anew(2:4,4)=A(:,l);
                M=round(det(Anew));
                w(r)=M;
                w0(r)=abs(M);
            end
        end
    end
end

%Now we sort w0 in increasing order.
w0=sort(w0);

end
```

```
function [CO]=correspondance(N)

CO=zeros(nchoosek(N,4),4);
%CO will be a \binom{N}{4} x 4 matrix, where row r is
% CO(r,:)=[i,j,k,l] if the r-th coordinate of a volume
% vector of a point configuration of N points is the
%determinant of points i,j,k,l

r=0;

for i=1:(N-3)
    for j=(i+1):(N-2)
        for k=(j+1):(N-1)
            for l=(k+1):N
                r=r+1;
                CO(r,:)=[i,j,k,l];
            end
        end
    end
end
```

```
function [q0,pos]=least_freq_vol(wA,w0A)

%wA is a vector of integers, and w0A is the vector with the absolute
%values ordered increasingly

%q0 will be the least frequent non-zero volume in wA, and pos the first
%position in which it appears in wA (in absolute value)

k=size(w0A,2);
q0=0;
frec0=k+1; %This is a strict upper bound on the frequency of a value

C=unique(w0A);  % C stores the different values appearing in W0A, with
                % no repetition
for i=1:size(C,2);
    %For each non-zero value q, count how many times it appears on the
    %vector. If the number of appearances is less than frec0 (frequency
    %of q0, then update q0=q and frec0=frec
    q=C(i);
    if q ~=0
        frec=0;
        for j=1:k
            if w0A(j)==q
                frec=frec+1;
            end
        end
        if frec<frec0
            q0=q;
            frec0=frec;
        end
    end
end
```

```
%Run through the vector until you find \pm q0 in the vector
for i=1:k
    if abs(wA(i))==q0
        pos=i;
        return
    end
end
```

Algorithm A.7. Unimodular equivalence between two lattice 3-polytopes. The information of $t(P)$ and Q, for t a unimodular transformation, is usually stored in matrices. We need to compare these matrices modulo a permutation of the columns (which correspond to lattice points). For that we use the auxiliary routine `compare_cols` in page 124.

To check unimodular equivalence, we compute all possible unimodular transformations that could map one polytope to the other. If we fail to find any, then the polytopes are not equivalent. We have an extra input variable called `fast` that indicates whether we want all unimodular transformations (0) or whether we stop when we find one (1).

We treat separately the case when the configurations are tetrahedra.

- **Equivalence among tetrahedra.** Routine `maps_simplex` in page 125.

  Let P and Q be lattice tetrahedra. A necessary condition for them to be equivalent is that their normalized volumes (the determinant, in this case) have to be equal. The inputs of the function are the volume q, the sets of vertices $VP = \{p_1, p_2, p_3, p_4\}$ and $VQ = \{q_1, q_2, q_3, q_4\}$, and the aforementioned variable `fast`.

  First and foremost, if VP and VQ contain the same lists of four points, then the configurations are automatically equivalent under the identity. This is useful in the case that `fast=1`, in which case we have finished already.

  We next want to compute all possible unimodular transformations t that map one tetrahedron to the other. There are 24 possible permutations σ of the set $\{1, 2, 3, 4\}$ so that $t(p_i) = q_{\sigma(i)}$, and we need to check them all.

  Fixed said permutation, and for 3-dimensional configurations of four points in $\mathbb{R}^3$, there is a unique affine transformation t mapping one to the other in that specific order, and it is defined by the unique matrix M_0 that verifies:

  $$M_0 \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_{\sigma(1)} & q_{\sigma(2)} & q_{\sigma(3)} & q_{\sigma(4)} \end{pmatrix}$$

  The matrix M_0 is of the form

  $$M_0 = \begin{pmatrix} 1 & \mathbf{0} \\ b & M \end{pmatrix} \in \mathbb{R}^{4\times 4}$$

  where the affine transformation is given by $t(x, y, z) = M \cdot (x, y, z)^T + b$.

  Now, since the determinant of both P and Q is $\pm q$, the determinant of M_0 is then ± 1, and so is that of M.

Finally, being M_0 defined as a product of an integer matrix with the inverse of another integer matrix of determinant $\pm q$, all the coefficients of M_0 are in $\frac{1}{q}\mathbb{Z}$.

$$M_0 = \begin{pmatrix} 1 & 1 & 1 & 1 \\ q_{\sigma(1)} & q_{\sigma(2)} & q_{\sigma(3)} & q_{\sigma(4)} \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 & 1 \\ p_1 & p_2 & p_3 & p_4 \end{pmatrix}^{-1} \in \frac{1}{q}\mathbb{Z}^{4\times 4}$$

This transformation is a unimodular equivalence between lattice polytopes if, and only if, all the coefficients of M_0 are integers. Notice that it is enough to check that M is integer, because then $b = q_{\sigma(i)} - M \cdot p_i \in \mathbb{Z}^3$.

- **Equivalence among polytopes with any number of vertices.** Routine `maps` in page 127.

  In this case, VP and VQ are again inputs, and this time we also need the volume vectors wP and wQ of each configuration, as well as the ordered unsigned volume vectors $w0P$ and $w0Q$. Notice that in the case of equivalence between tetrahedra, the volume q would correspond to the information stored in these four vectors, except perhaps for the orientation of the points.

  Necessary conditions for the polytopes to be equivalent are that they have the same number of vertices and that the ordered unisgned volume vectors $w0P$ and $w0Q$ are equal (unimodular transformations preserve volumes).

  Now, in the case that `fast=1` and VP and VQ contain the same lists of lattice points, we have finished. And in the case that VP and VQ are tetrahedra of the same volume, we use the routine `maps_simplex` (see page 125).

  Once all those things are checked, we move on to compute all possible unimodular affine transformations that map VP to VQ. For this we fix a tetrahedron (4-tuple of points) $T_P \subset VP$, and for each tetrahedron $T_Q \subset VQ$ of the same volume as tetrahedron T_P, we check whether one of the 24 affine maps t with $t(T_P) = T_Q$ is integer and satisfies $t(VP) = VQ$.

  To reduce the number of computations as much as possible, we choose a tetrahedron T_P whose volume is the least repeated among tetrahedra in VP (and VQ). This means that, for fixed T_P, we have less possible tetrahedra T_Q to check.

```
function bool=compare_cols(A,B)

%A and B are matrices with the same number of rows
%bool=1 if the number of columns is the same AND B is a column
%permutation of A.

%If the number of columns is different, then finish
if size(A,2) ~= size(B,2)
    bool=0; return
end

%Now A and B are matrices m x n.

n=size(A,2);
```

```
% c will be a vector with c(i)=j if column A(:,i) is equal to column
% B(:,j). It has to be an injective map
c=zeros(n,1);
for i=1:n
    for j=1:n
        if isequal(A(:,i),B(,j))==1 && ismember(j,c)==0 && c(i)==0
            c(i)=j;
        end
    end
end

%If some coordinate is 0, it means that a column in A is not in B
%(or it appears with less repetitions)
if ismember(0,c)==1
    bool=0;
else
    bool=1;
end
```

```
function [bool,Mx,bx,sx]= maps_simplex(q,VP,VQ,fast)

%VP and VQ are 3x4 matrices representing tetrahedra of volume q, where
%each column represents a vertex
% maps_simplex returns a boolean bool that will be bool=1 if the
%configurations are equivalent, and bool=0 if not.

if nargin < 4
    fast=0;
end
%fast=1 if we just want one equivalence.
%fast=0 if we want all equivalences.

Mx=[];
bx=[];
sx=0;

%If bool=1, each linear transformation Mx(:,:,i) together with the
%translation bx(:,:,i) correspond to an affine unimodular transformation
%that maps VP to VQ, up to a permutation of the vertices

%If VP and VQ contain the exact same set of columns, then they are
% equivalent by the identity transformation
%If fast=1, we can already exit the program
if compare_cols(VP,VQ)==1 && fast==1
    bool=1;
    sx=1;
    Mx(:,:,sx)=eye(3);
    bx(:,:,sx)=zeros(3,1);
    return
end


%We now need to check maps from VP to VQ, in all the possible orderings
%of the four vertices of one configuration with respect to the other.
%That is, we fix VP, and we run through all the permutations of the
```

```
%vertices of VQ

per=perms([4,3,2,1]);
%per is a 24 x 4 matrix, where each row represents a permutation of the
%elements 1,2,3,4

Px=[ones(1,4);VP];

bool=0;

for i=1:24
%We run through all the permutations
    %we permute the vertices of VQ under permutation per(i,:)
    Qx=zeros(4,4);
    for j=1:4
        Qx(2:4,j)=VQ(:,per(i,j));
    end
    Qx(1,:)=ones(1,4);
    %Px(2:4,:) and Qx(2:4,:) have as columns pj=VP(:,j),
    %qj=VQ(:,per(i,j))
    %We want to compute the unique affine unimodular map that sends
    %pi-->qi for each i=1,2,3,4

    %The determinant of Px is non-zero, hence it has an inverse, that is
    %a rational matrix with coefficients in <1/det(Px)>. Then the matrix
    %M0 is uniquely defined and of determinant one, since det(Px)=det(Qx)
    %Hence the system has the following structure:
    %
    %         / 1  | 0 0 0 \       / 1  1  1  1 \     /  / 1  1  1  1 \
    %        |     |        |     |              |   /  |              |
    %   M0 = |  b  |  M     |  =  | q1 q2 q3 q4  |  /   | p1 p2 p3 p4  |
    %         \    |       /       \            /  /     \            /
    %
    %Where M is the linear transformation (of determinant one) and b is
    %the translation vector:  M(p_i) + b = q_i for all i

    M0=Qx/Px;

    b=M0(2:4,1);
    M=M0(2:4,2:4);

    %M0 has integer coefficients iff M does
    %M*q has integer coefficients. We want to know whether M does too.

    Maux=round(q*M); %We are now considering integer numbers
    d=gcd(Maux(1,:),gcd(Maux(2,:),Maux(3,:)));
    %d is a vector that computes gcd of columns
    d=gcd(d(1),gcd(d(2),d(3)));
    % d is now the gcd of the entries of Maux
    %rem(d,q)
    %M is integer if and only if q divides d
    if rem(d,q)==0
        bool=1;
        M=round(M);
        b=round(b);
        sx=sx+1; %add element to lists Mx and bx
        Mx(:,:,sx)=M;
```

```
        bx(:,:,sx)=b;
    end
end
```

```
function [bool,Mx,bx,sx] = maps(VP,wP,w0P,VQ,wQ,w0Q,fast)

%VP and VQ are the vertices of the polytopes that are being compared.
%They are given as matrices 3 x n, where n is the number of vertices

%wP and wQ are the volume vectors of the set of vertices VP and VQ
%w0P and w0Q are the respective absolute volume vectores

if nargin < 7
    fast=0;
end
%fast=1 if we just want one equivalence.
%fast=0 if we want all equivalences.

%bool=0 if they are not equivalent. Then Mx, bx and sx are:
Mx=[];
bx=[];
sx=0;

%bool=1 if they are equivalent. In this case, each linear transformation
% Mx(:,:,i) together with the translation bx(:,:,i) correspond to an
% affine unimodular transformation that maps VP to VQ, up to permutation
% of points

%VP and VQ have to have the same number of vertices:
if size(VP,2)~=size(VQ,2)
    bool=0;
    return
end

%The absolute volume vectors have to be equal to have equivalence
if isequal(w0P,w0Q)==0
    bool=0;
    return
end

n=size(VP,2);

%If VP and VQ contain the exact same set of columns, then they are
%equivalent by the identity transformation
%If fast=1, we can already exit the program
if compare_cols(VP,VQ)==1 && fast==1
    bool=1;
    sx=1;
    Mx(:,:,sx)=eye(3);
    bx(:,:,sx)=zeros(3,1);
    return
end

%In the case of a simplex we do it with its specific program
if n==4
```

```
    [bool,Mx,bx,sx]=maps_simplex(w0P,VP,VQ,fast);
    return
end

%q0 is the least frequent non-zero volume in wP, and pos is minimal such
%that abs(wP(pos))=q0
[q0,pos]=least_freq_vol(wP,w0P);

%postuple=[i,j,k,l], where q0 is the absolute value of the determinant
%of points i,j,k,l
CO=correspondance(n);
postuple=CO(pos,:);

Pnew=zeros(3,4);
Pnew(:,1)=VP(:,postuple(1));
Pnew(:,2)=VP(:,postuple(2));
Pnew(:,3)=VP(:,postuple(3));
Pnew(:,4)=VP(:,postuple(4));

bool=0;
for i=1:nchoosek(n,4)
    if abs(wQ(i))==q0

        %indices of the four points of VQ
        v=CO(i,:);
        Qnew=zeros(3,4);
        Qnew(:,1)=VQ(:,v(1));
        Qnew(:,2)=VQ(:,v(2));
        Qnew(:,3)=VQ(:,v(3));
        Qnew(:,4)=VQ(:,v(4));

        %Pnew and Qnew now have (as columns) the vertices of two
        %tetrahedra of the same volume. We now compute all the
        %possible unimodular transformation that map one set of
        %points to the other
        [k,M,b,s]= maps_simplex(q0,Pnew,Qnew);

        %If they are equivalent, check for each of the transformation
        %whether they also map all the points in VP to those in VQ
        if k==1
            for j=1:s
                P2=M(:,:,j)*VP+b(:,:,j)*ones(1,n);
                if compare_cols(P2,VQ)==1
                    bool=1;
                    sx=sx+1;
                    Mx(:,:,sx)=M(:,:,j);
                    bx(:,:,sx)=b(:,:,j);
                    if fast==1
                       return
                    end
                end
            end
        end
    end
end
```

Algorithm A.8. Equivalence classes in a list.

When dealing with lists of polytopes, we often want to keep one representative for each class. In order to do this, we have the following information of each polytope: the set of lattice points, the set of vertices, the volume vector, and the ordered unsigned volume vector. These are stored, respectively, in L, V, W and $W0$.

Also, we assume that all the polytopes in the lists have the same number N of lattice points and the same number $j0$ of vertices.

```
function [L1,V1,W1,W01,C1]=classes(L,V,W,W0)

C=size(L,3);
%N=size(L,2);
%j0=size(V,2);
%clases returns a list of representatives of all the equivalence classes
%present in the input. The input is as follows:
%L is a 3 x N x C matrix, where each L(:,:,i) is the list of a lattice
%points
%V is a 3 x j0 x C matrix, where each V(:,:,i) is the list of vertices
%W is a 1 x nchoosek(j0,4) x C matrix, where each W(:,:,i) is the volume
%vector
%W0 is a 1 x nchoosek(j0,4) x C matrix, where each W0(:,:,i) is the
%absolute volume vector

%WE ASSUME ALL POLYTOPES IN THE LIST HAVE THE SAME NUMBER OF LATTICE
%POINTS AND THE SAME NUMBER OF VERTICES.

%The output will be the list of representatives
L1=[];
V1=[];
W1=[];
W01=[];

C1=0;
%C1 will count how many elements have been added to L1, V1

%List CC is a list of all ones with as many entries as polytopes in the
%list L
%Whenever one configuration is compared, and equal, to another in
%inferior position (lower index in L), the corresponding entry becomes
%0, so we do not consider that configuration again.
%In the end, CC will be a list of 1's and 0's, where the position of the
%1's will represent the first (to appear) configuration of each
%equivalence class.
CC=ones(1,C);

for i=1:C-1
    if CC(i)==1
        %If CC(i)=1, then it means that a polytope in the same class of
        %equivalence has not yet been added. We add it
        C1=C1+1;
        L1(:,:,C1)=L(:,:,i);
        V1(:,:,C1)=V(:,:,i);
        W1(:,:,C1)=W(:,:,i);
        W01(:,:,C1)=W0(:,:,i);
```

```
            %We now compare it with all configurations i+1,...,C
            for j=(i+1):C
                if CC(j)==1
                    A=V(:,:,i);
                    B=V(:,:,j);

                    %If configuration j is equivalent to i, then we put
                    %CC(j)=1
                    bool=maps(A,W(:,:,i),W0(:,:,i),B,W(:,:,j),W0(:,:,j),1);
                    if bool==1
                        CC(j)=0;
                    end
                end
            end
        end
end

%If CC(C)=1, the last configuration was not equivalent to any of the
%others, and we have to add it
if CC(C)==1
    C1=C1+1;
    L1(:,:,C1)=L(:,:,C);
    V1(:,:,C1)=V(:,:,C);
    W1(:,:,C1)=W(:,:,C);
    W01(:,:,C1)=W0(:,:,C);
end
```

A.1.3 Width of lattice 3-polytopes

To compute the (lattice) width of a polytope we use the ideas in Proposition 1.5 and Lemma 2.16. Suppose that we already have an upper bound W for the width of P (for example, but not necessarily, the width of P with respect to a particular functional). Then every $f \in (\mathbb{Z}^3)^*$ such that $\mathrm{width}_f(P) \leq W$ is a lattice point in $W(P-P)^\vee$. That is, we simply need to compute the lattice points in the W-th dilation of the polar of $P-P$ and check which one gives the minimum width.

Notice that if W is smaller than the width, the same algorithm certifies it, by showing that $W(P-P)^\vee$ has no nonzero lattice points.

Algorithm A.9. Width of a polytope. Routine `width_p` in page 131. To compute the width of a polytope P we only need to input the set of vertices VP.

- For computing $Q = VP - VP$ the Minkoswi sum of P and $-P$, we use routine `minkowski_sum` in page 131.

- We then need to compute the polar of Q. Since $\mathbf{0} \in \mathrm{int}(Q)$, then $S = Q^\vee$ is bounded and $Q = (Q^\vee)^\vee = S^\vee$. Let $f \in HQ$ be a facet-defining hyperplane of Q, with $f(x,y,z) = ax + by + cz + d$. The equation $ax + by + cz + d \geq 0$ is equivalent to $-\frac{a}{d}x - \frac{b}{d}y - \frac{c}{d}z \leq 1$, and by the bijection between facets of Q and vertices of S established by duality, we have that $(-\frac{a}{d}, -\frac{b}{d}, -\frac{c}{d}) \in \mathrm{vert}(S)$. This way we compute the (maybe rational) vertices of S.

- The next step is to take the w-th dilate of S, and to compute the list LS of lattice points of it. Since wS is a rational polytope, we use the routines `convex_hull` and `lattice_points` with the variable `rat=1` as input (see Algorithms A.1 and A.3).

- We then check all the lattice points of LS, considered now as coefficient vectors of linear functionals, and check the width of P with respect to each of those functionals.

- There is an optional input variable `wfix`. If this variable is input, it is a natural number, and the algorithm returns all linear functionals with respect to which P has that specified width. If it is not inputed, the algorithm returns the width of the polytope and all the functionals that achieve it.

```
function A=minkowski_sum(A1,A2)

%A1 and A2 are lists of points, and A is the minkowski sum.

A=[];
c=0;
for i=1:size(A1,2)
    for j=1:size(A2,2)
        sum=A1(:,i)+A2(:,j);
        %If the point sum is not yet in A, add it
        if size(A,2)>0
            if ismember(sum',A','rows')==0
                c=c+1;
                A(:,c)=sum;
            end
        else
            c=c+1;
            A(:,c)=sum;
        end
    end
end
```

```
function  [wi,fu]=width_p(VP,wfix)
% VP is a the list of vertices of a lattice polytope
% It can also just be any list of points

% If wfix is not inputed:
% wi is the lattice width of the polytope conv(VP)
% fu is a list of all the integer linear functionals with respect to
% which such width is achieved

%If w is inputed:
% wi is the given width wfix
% fu is a (maybe empty) list of all the integer linear functionals with
% respect to which width wfix is achieved

%Compute the polytope P + (-P)
Q=minkowski_sum(VP,-VP);

%Compute S the polar of Q.
```

```
HQ=convex_hull(Q);
S=zeros(size(HQ,1),3);
for i=1:size(HQ,1)
    S(i,:)=-HQ(i,1:3)/HQ(i,4);
end
%We want the vertices of S to be columns:
S=S';

%Compute the w-th dilate of S:
if nargin < 2
    % w is the width of P in direction of (1,0,0). It is an upper bound
    %on the width
    fix=0;
    w=max(VP(1,:))-min(VP(1,:));
else
    fix=1; %if the value w is inputed, the width is fixed
    w=wfix;
end
S=S*w;

%Compute the list of lattice points in the (rational) polytope S
HS=convex_hull(S,1);
LS=lattice_points(S,HS,1);

%We now want to remove, from this list of points, the all-zeros,
%opposites, and multiples. LSx will be an auxiliary list where we store
%the non-zero and non-proportional ones, and cx keeps count of how many
%of them are.
LSx=[]; cx=0;
for i=1:size(LS,2)  %number of lattice points in LS
    vec=LS(:,i);    %take one of them
    if isequal(vec,zeros(3,1))==0  %if it is not the all-zero vector
        d=gcd(vec(1),gcd(vec(2),vec(3))); %take d the gcd of its entries
        vec=round(vec./d);      %make the vector primitive
        %we now want the first non-zero coordinate to be positive
        r=find(vec);
        if vec(r(1))<0
            vec=-vec;
        end

        %if vec is not yet in LSx, add it
        if size(LSx,1)>0
            if ismember(vec',LSx','rows')==0
                cx=cx+1;
                LSx(:,cx)=vec;
            end
        else
            cx=cx+1;
            LSx(:,cx)=vec;
        end
    end
end

%We have now LS to be the list of all non-proportional, primitive, not
%zero lattice points in S
LS=LSx;
clear LSx
```

```
%The minimum width of the initial polytope will be achieved by an
%element in LS (as a linear functional). We take transpose:
LS=LS';


if fix==0
    %We need to compute the width and all the functionals that achieve
    %it
    %We first put the upper bound:
    wi=w; %minimum width (so far)
    fu=[1,0,0]; %functional achieving width wi

    %We then check if the functionals in LS give smaller width or not
    for i=1:size(LS,1)
        vec=LS(i,:)*VP;
        waux=max(vec)-min(vec);
        if waux==wi %if the width is the minimum, add the functional
            fu=[fu;LS(i,:)];
        elseif waux<wi %if the width is smaller, change wi and fu
            wi=waux;
            fu=LS(i,:);
        end
    end
else
    %We need to add to fu all the functionals giving width wfix to VP
    fu=[];
    nofu=0;

    for i=1:size(LS,1)
        vec=LS(i,:)*VP;
        widthi=max(vec)-min(vec);
        if widthi==wfix
            nofu=nofu+1;
            fu(nofu,:)=LS(i,:);
        end
    end
    wi=wfix;
end
```

Algorithm A.10. Width one evaluation. Routine `width_one` in page 134.

We are sometimes only interested in knowing whether a polytope has width one or not. This is for example the case when we are computing boxed 3-polytopes, where we reject any configuration of points that is of width one.

We make a separate algorithm for this because there is a much easier way of computing the complete list of functionals that could possibly give width one to a polytope: let P be a lattice 3-polytope of width one, and let f be a functional such that $f(P) = [0, 1]$. Then one of the following happens:

- f corresponds to a facet-defining functional.
- P is a tetrahedron and f is the functional that is orthogonal to a pair of opposite edges.

The algorithm has an optional variable `fast` that is 0 by default. If `fast=0` the algorithm returns the complete (maybe empty) list of functionals that give width one to P, and if `fast=1`, the algorithm finishes as soon as it finds one.

```
function  [w1,func]=width_one(VP,HP,fast)

if nargin < 3
    fast=0;
end
% VP is the set of vertices of P and HP are the facet-defining
%functionals

% w1 will be 1 if the width of P is one, and 0 otherwise.
% func will contain the list of functionals that give width one, if any
n=size(VP,2);
func=[];
s=0; %number of functionals

%fast=1 if we just want to check that the width is one.
%fast=0 if we want to have all functionals that give width one to P

%HPVP is the set of distances from the points to the facet defining
%hyperplanes.
HPVP=HP*[VP;ones(1,n)];

%We set w1=0 and will change it to 1 if we find a width one functional
w1=0;



%We first check width one with respect to facet-defining functionals:
for i=1:size(HP,1)
    if max(HPVP(i,:))==1 % By construction of HP, the minimum is 0
        w1=1;
        s=s+1;
        func(s,:)=HP(i,:);

        %If fast==1, we have finished already
        if fast==1
            return
        end
    end
end

%We now check the possibility of P having width one with respect to a
%functional that is not facet defining. In this case, the only
%possibility is that P is a tetrahedron and the width is achieved wrt a
%pair of opposite edges. If VP is not a tetrahedron, this cannot happen
%and we have finished.
if size(VP,2)>4
    return
end

%We have three possible pairs of opposite edges: 12/34, 13/24 and 14/23.
%We construct an auxiliar matrix that will give us these choices as rows.
```

```
M=[1,2,3,4;
   1,3,2,4;
   1,4,2,3];

for i=1:3
    %v1 and v2 will be the two vectors representing the opposite edges
    v1=VP(:,M(i,1))-VP(:,M(i,2));
    v2=VP(:,M(i,3))-VP(:,M(i,4));
    v=round(cross(v1,v2)); %This is the orthogonal vector (integer)
    d=gcd(v(1),gcd(v(2),v(3))); %Take the gcd of the coordinates of v
    v=round(v./d); %The vector is now primitive
    v=v'; %Row vector

    %Now, since the functional with coefficient vector v is constant in
    %each edge, it suffices to check that the difference of values is 1
    diff=v*VP(:,M(i,1))-v*VP(:,M(i,3));
    if abs(diff)==1
        w1=1;
        s=s+1;
        %We want the functional to take minimum value 0 in the polytope
        func(s,:)=[v,round(-min(v*VP(:,M(i,1)),v*VP(:,M(i,3))))];
        if fast==1
            return
        end
    end
end
```

A.2 Enumerating lattice 3-polytopes of a fixed size N

The MATLAB functions in this section correspond to the classification work on Chapter 3 of lattice 3-polytopes of a given size.

Algorithm A.11. Merged 3-polytopes of size N. We are going to apply the merging algorithm (see Algorithm 3.2) to the list of lattice 3-polytopes of size $N-1$ and width larger than one, which we assume precomputed.

- Routine `merging_pairs` in page 136.

  As a first step to the merging algorithm we compute the pairs (P^v, v) for P of size $N-1$ and of width larger than one, and v a vertex such that P^v is full dimensional. Since the "merging" (the equivalences) are checked between different P^v, we store the list of lattice points of P^v in $L1$, the list of vertices in $V1$, the volume vector and the ordered unsigned volume vector in $W1$ and $W01$, respectively, and the removed point v in $V0$.

  There is an extra variable `prop` as input, that is 0 by default and 1 if we have also computed several extra properties of the polytopes of size $N-1$ (see Section A.3 for more details on the computed properties). Computing these properties is not necessary for the merging algorithm but it may make computations a bit faster. More specifically, among these properties we have which vertices v are such that P^v is full dimensional, and we have the automorphisms of every polytope: given P, $v \in \text{vert}(P)$ and t an automorphism of P, the pair (P^v, v) is equivalent (for our purposes) to the pair $(P^{t(v)}, t(v)) = (t(P^v), t(v))$, and we need not consider it. This makes the list of pairs shorter.

- Routine merged_3polytopesofsizeN in page 138.

  Once we have the pairs computed, the remaining part of the merging algorithm consists on the following: for each two pairs $(P_1^{v_1}, v_1)$ and $(P_2^{v_2}, v_2)$, and for each t unimodular transformation with $t(P_1^{v_1}) = P_2^{v_2}$, take the polytope $P = \text{conv}(t(P_1) \cup \{v_2\}) = \text{conv}(\{t(v_1)\} \cup P_2)$. If P has size N, add it to the list.

  In this case we have as input the (optional) variable fast, which is 0 by default and 1 if we know that the short list of mergings has been computed (prop=1 in merging_pairs).

```
function [L1,V1,V0,W1,W01,C1]=merging_pairs(N,prop)

T=cputime;
if nargin<2
    prop=0;
end

%this program returns all the pairs (P^v,v), for P a lattice 3-polytope
%of size N and v a vertex such that P^v is full-dimensional.

%We will store the pairs as follows:
%C1 is a 1 x N-1 matrix:
    %C1(K) is the number of pairs with P^v having K vertices
%L1 is a 1 x N array:
    %L1{K} is a 3 x N-1 x C1(K) containing the list of lattice points
%V1 is a 1 x N array:
    %V1{K} is a 3 x K x C1(K) containing the list of vertices
%V0 is a 1 x N array:
    %V0{K} is a 3 x 1 x C1(K) containing the removed point
%W1 is a 1 x N array: W1{K} is a 1 x binom{K}{4} x C1(K) containing the
    %volume vectors of the set of vertices
%W01 is a 1 x N array: W01{K} is a 1 x binom{K}{4} x C1(K) containing
    %the absolute volume vectors of the set of vertices
L1=cell(1,N-1);
V1=cell(1,N-1);
V0=cell(1,N-1);
W1=cell(1,N-1);
W01=cell(1,N-1);
C1=zeros(1,N-1);

%We take the full list of lattice 3-polytopes of size N and width >1

%If prop=0, it takes this list from the file PP_3polytopesofsizeN.mat,
%if it exists, or computes them otherwise.
if prop==0
    name=sprintf('PP_3polytopesofsize%d.mat',N);
    if exist(name,'file')==2
        load(name,'L','V','C')
    else
        [L,V,~,~,C]=PP_3polytopesofsizeN(N);
    end
```

```
%C is a 1 x N matrix:
    %C(k) is the number of polytopes of size N and with K vertices
%L is a 1 x N array:
    %L{K} is a 3 x N x C(K) containing the list of lattice points
%V is a 1 x N array:
    %V{K} is a 3 x K x C(K) containing the list of vertices

    for K=4:N   %Configurations of size N and K vertices
        if C(K)~=0
            for c=1:C(K) %c is number of configuration
                for j=1:K   %run through the vertices
                    LP=L{K}(:,:,c);
                    %Copy the list of lattice points, remove vertex j
                    for s=1:N
                        if isequal(V{K}(:,j,c),L{K}(:,s,c))==1
                            LP(:,s)=[];
                        end
                    end
                    HP=convex_hull(LP); %affine hyperplane defining
                                        %facets

                    if size(HP,1)>3  %that is, if the remaining N-1
                                     %points are not lower dimensional
                        VP=vertex_int_boun_pt(HP,LP);
                        j1=size(VP,2);
                        [w,w0] = volumevectors(VP);
                        C1(j1)=C1(j1)+1;
                        L1{j1}(:,:,C1(j1))=LP;
                        V1{j1}(:,:,C1(j1))=VP;
                        V0{j1}(:,:,C1(j1))=V{K}(:,j,c);
                        W1{j1}(:,:,C1(j1))=w;
                        W01{j1}(:,:,C1(j1))=w0;
                    end
                end
            end
        end
    end
    name1=sprintf('PP_merging_pairs_%d.mat',N);
    save(name1,'L1','V1','V0','W1','W01','C1')

%If prop=1, it means that the properties of polytopes of size N have
%been computed, hence it is easier to compute the merging pairs.

elseif prop==1
    readfolder=sprintf('Size%d_configurations',N);
    Files=dir(fullfile(readfolder,'*.mat'));
    len=length(Files); clear Files
    for c= 1:len
        name=sprintf('%d_%d.mat',[N,c]);

        fileName = fullfile(readfolder,name);
        load(fileName,'lattice_pts','fulldim_indices','vertices')
        load(fileName,'automorphisms')

        good_vert=vertices;
        i=1;    %number of lattice point
```

```
        %in good_vertices we will keep the vertices that we still need
        %to remove for a new pair.
        while size(good_vert,2)>0 && i<N+1
            %If point i is a vertex such that P^v is full-dimensional:
            aux=ismember(lattice_pts(:,i)',good_vert','rows');
            if fulldim_indices(i)==1 && aux==1
                LP=lattice_pts;
                LP(:,i)=[]; %Polytope P^i
                rv=lattice_pts(:,i); %removed vertex

                HP=convex_hull(LP);
                VP=vertex_int_boun_pt(HP,LP);
                j1=size(VP,2);
                [w,w0] = volumevectors(VP);
                C1(j1)=C1(j1)+1;
                L1{j1}(:,:,C1(j1))=LP;
                V1{j1}(:,:,C1(j1))=VP;
                V0{j1}(:,:,C1(j1))=rv;
                W1{j1}(:,:,C1(j1))=w;
                W01{j1}(:,:,C1(j1))=w0;

%Let us now remove vertex rv and all the ones that are in the same orbit
%from good_vert.
                %For each automorphism
                for s=1:size(automorphisms,3)
                    %Compute the image of rv by this automorphism
                    M=automorphisms(:,:,s)*[rv;1];
                    m=1;
                    while size(good_vert,2)>0 && m<size(good_vert,2)+1
                        if isequal(good_vert(:,m),M)==1
                            good_vert(:,m)=[];
                        else
                            m=m+1;
                        end
                    end
                end
            end

            i=i+1;
        end
    end
    %Save the merging pairs
    name1=sprintf('PP_merging_pairs_%d_short.mat',N);
    save(name1,'L1','V1','V0','W1','W01','C1')

end
cputime-T
```

```
function [L,V,W,W0,C]=merged_3polytopesofsizeN(N,fast)
T=cputime;
if nargin<2
    fast=0;
end

%Take the list of merging pairs of size N-1. Take the short one if it
```

```
%has been computed (fast=1)
if fast==0
    name=sprintf('PP_merging_pairs_%d.mat',N-1);
    if exist(name,'file')==2
        load(name)
    else
        [~,V1,V0,W1,W01,C1]=merging_pairs(N-1);
    end
elseif fast==1
    name=sprintf('PP_merging_pairs_%d_short.mat',N-1);
    if exist(name,'file')==2
        load(name)
    else
        [~,V1,V0,W1,W01,C1]=merging_pairs(N-1,1);
    end
end

L=cell(1,N);
V=cell(1,N);
W=cell(1,N);
W0=cell(1,N);
C=zeros(1,N);

for K=1:N-2 %K is the number of vertices of the P^v
    if C1(K)~=0
        for i=1:C1(K)
            for j=i:C1(K)
                %For each two pairs (P^i,i) and (P^j,j) we check whether
                %P^i and P^j are equivalent
                Pi=V1{K}(:,:,i);
                wi=W1{K}(:,:,i);
                w0i=W01{K}(:,:,i);

                Pj=V1{K}(:,:,j);
                wj=W1{K}(:,:,j);
                w0j=W01{K}(:,:,j);

                [bool,Mx,bx,sx]=maps(Pi,wi,w0i,Pj,wj,w0j);
                if bool==1  %They are equivalent
                    %for each of the equivalences t
                    for r=1:sx
                        A=zeros(3,K+2);
                        A(:,1:K)=Pj;
                        A(:,K+1)=V0{K}(:,:,j);
                        A(:,K+2)=Mx(:,:,r)*V0{K}(:,:,i)+bx(:,:,r);
                        %A is t(P^i) union P^j
                        HP=convex_hull(A);
                        LP=lattice_points(A,HP);
                        %If conv(A) has exactly N lattice points, then
                        %it is merged
                        if size(LP,2)==N
                            VP=vertex_int_boun_pt(HP,LP);
                            j0=size(VP,2);
                            [w,w0] = volumevectors(VP);
                            C(j0)=C(j0)+1;
                            L{j0}(:,:,C(j0))=LP;
                            V{j0}(:,:,C(j0))=VP;
```

```
                        W{j0}(:,:,C(j0))=w;
                        W0{j0}(:,:,C(j0))=w0;
                    end
                end
            end
        end
    end
end
end

%Remove redundancies
for j=4:N
    if C(j)~=0
        [L{j},V{j},W{j},W0{j},C(j)]=classes(L{j},V{j},W{j},W0{j});
    end
end

%Save the results
if fast==0
    name=sprintf('PP_merged_3polytopesofsize%d',N);
elseif fast==1
    name=sprintf('PP_merged_3polytopesofsize%d_fast',N);
end
save(name,'L','V','W','W0','C')

cputime-T
```

Algorithm A.12. Spiked 3-polytopes of size N. Routine `spiked_3polytopesofsizeN` in page 141.

Spiked 3-polytopes of each size (at least seven) are explicitly given with vertices in Theorems 3.18 and 3.19. They are polytopes projecting to the lattice point configurations A'_i of dimension 2 appearing in Figure 3.7. There are twelve different *types* represented in the MATLAB routines as $0, 1, \ldots, 11$. They correspond to: type 0 are the minimal polytopes appearing in Theorem 3.18, projecting to configuration A'_8; types $i \in \{1, \ldots, 9\}$ are the polytopes (i) in Theorem 3.19, projecting to configuration A'_i; and types 10 and 11 correspond, respectively, to polytopes (10a) and (10b) in Theorem 3.19, both projecting to configuration A'_{10}.

Let P be such a spiked 3-polytope, of size N. The coordinates of P are given in terms of a parameter k which equals the length of the largest lattice segment contained in one of the lattice fibers in P. For all types, also the complete list of lattice points in P are known:

- In each of the types except 10 and 11, the point $(0, 0)$ is the only one that has more than one lattice point of the polytope projecting to it, and the lattice points projecting to $(0, 0)$ are precisely the $k+1$ points in the segment $(0, 0, 0)(0, 0, k)$. The rest of lattice points are the vertices of P plus some extra points in the following cases:

  Type 2: Point $(0, -1, 0)$ in the edge $(-1, -1, 0)(1, -1, 0)$.

  Type 3: Points $(0, -1, 0)$ and $(-1, 0, 0)$ in the edges $(-1, -1, 0)(1, -1, 0)$ and $(-1, -1, 0)$ $(-1, 1, 0)$, respectively.

In particular, this makes N equal to k plus a fixed number depending only on the type of P.

- In type 10, the number of lattice points is: the three vertices v_1, v_2 and v_3, $k+1$ points projecting to $(0,0)$, and $\left\lfloor \frac{k+b}{2} \right\rfloor + 1$ points projecting to $(0,1)$:

$$N = 3 + (k+1) + \left\lfloor \frac{k+b}{2} \right\rfloor + 1 = \left\lfloor \frac{3k+b}{2} \right\rfloor + 5$$

  This makes the exact relation between k and N depend on the parities of b and k and the value of $N \pmod 3$.

- Finally, in type 11, the only point with more than one lattice point projecting to it is $(0,1)$, and the lattice points projecting to it are those of the segment $(0,1,0)(0,1,k)$. The rest of the lattice points are $(0,0,0)$ and the rest of the vertices.

The last thing to be careful with is that sometimes, for small values of k, the resulting polytope is of width one, or it is not spiked (the vertex that is supposed to be non-essential turns out to be essential). For $k \geq 3$ the polytopes are of width > 1 and any subpolytope of size $N-1$ contains at least three aligned points and is hence of width > 1 as well. For $k \geq 2$, the polytopes are of width > 1 since they contain three aligned points. The only cases that remain to check are those when $k \leq 2$: if $k = 1$ we check that P and P^v, where v is the non-essential vertex, are both of width > 1; if $k = 2$, we only need to check that P^v does.

```
function [L,V,W,W0,C]=spiked_3polytopesofsizeN(N)

%We assume N > 6
if N<7
    disp('This algorithm is not correct for N<7')
    return
end

K=[]; %In K we will store the k that corresponds to each configuration
I=[]; %In I we will store the type
VNE=[]; %In VNE we store the position of the non-essential vertex
        %in the list of lattice points

%We first input the lists of vertices:
Vx=[];
cx=0;

%TYPE 0: M(a,b) minimal spiked polytope
ABaux=[0,0;
       0,1;
       1,1];
k=N-5;
for i=1:size(ABaux,1)
    a=ABaux(i,1);
    b=ABaux(i,2);
    cx=cx+1; K(cx)=k; I(cx)=0; VNE(cx)=0;
    Vx{cx}=[-1,    0,0,1;
             0,   -1,1,0;
```

```
             -a,2*k+b,0,0];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 1: P_1(k)
k=N-4;
cx=cx+1; K(cx)=k; I(cx)=1; VNE(cx)=3;
Vx{cx}=[-1,-1,0, 1;
        -1, 1,0,-1;
         0, 1,k,-1];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 2: P_2(k)
k=N-5;
cx=cx+1; K(cx)=k; I(cx)=2; VNE(cx)=3;
Vx{cx}=[-1,-1,0, 1;
        -1, 1,0,-1;
         0,-1,k, 0];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 3: P_3(k)
k=N-6;
cx=cx+1; K(cx)=k; I(cx)=3; VNE(cx)=3;
Vx{cx}=[-1,-1,0, 1;
        -1, 1,0,-1;
         0, 0,k, 0];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 4: P_4(k)
k=N-4;
cx=cx+1; K(cx)=k; I(cx)=4; VNE(cx)=3;
Vx{cx}=[-1,-1,0, 2;
        -1, 2,0,-1;
         0, 1,k,-1];
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 5: P_5(a,k)
Aaux=[-1,0];
k=N-4;
for i=1:length(Aaux)
    a=Aaux(i);
    cx=cx+1; K(cx)=k; I(cx)=5; VNE(cx)=2;
    Vx{cx}=[-1,0,0, 1;
            -1,0,1,-1;
             0,k,a,-1];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 6: P_6(a,k)
Aaux=[-2,-1,0];
k=N-4;
for i=1:length(Aaux)
    a=Aaux(i);
    cx=cx+1; K(cx)=k; I(cx)=6; VNE(cx)=2;
    Vx{cx}=[-1,0,0,1;
            -1,0,1,0;
             0,k,a,0];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 7: P_7(a,b,k)
Aaux=[-5,-1];
k=N-4;
for i=1:length(Aaux)
```

```
    a=Aaux(i);
    cx=cx+1; K(cx)=k; I(cx)=7; VNE(cx)=3;
    Vx{cx}=[-1,-1,0,2;
            -1, 1,0,1;
             0, a,k,0];
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 8: P_8(a,b,k)
Aaux=[-1,0];
k=N-5;
for i=1:length(Aaux)
    a=Aaux(i);
    Baux=(a:1:2*k-1);
    for j=1:length(Baux)
        b=Baux(j);
        cx=cx+1; K(cx)=k; I(cx)=8; VNE(cx)=4;
        Vx{cx}=[-1, 0,0,0,1;
                 0,-1,1,0,0;
                 a, b,0,k,0];
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 9: P_9(a,b)
ABaux=[-2,0;
       -1,0;
        0,0;
       -2,1;
       -1,1;
        0,1];
k=N-5;
for i=1:size(ABaux,1)
    a=ABaux(i,1);
    b=ABaux(i,2);
    cx=cx+1; K(cx)=k; I(cx)=9; VNE(cx)=4;
    Vx{cx}=[-1,0,1,      1;
            -1,1,0,      1;
             a,0,0,2*k-a+b];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 10: P_10A(a,b,k)
ABaux=[-1,-1;
       -1, 0;
        0,-1;
        0, 0];
for i=1:size(ABaux,1)
    a=ABaux(i,1);
    b=ABaux(i,2);
    k=0;
    if b==-1 && rem(N,3)==0
        k=(2*N-9)/3;
    elseif b==-1 && rem(N-1,3)==0
        k=(2*N-8)/3;
    elseif b==0 && rem(N,3)==0
        k=(2*N-9)/3;
    elseif b==0 && rem(N-2,3)==0
```

```
        k=(2*N-10)/3;
    end
    if k>0
        cx=cx+1; K(cx)=k; I(cx)=10; VNE(cx)=2;
        Vx{cx}=[-1,0,0,1;
                 0,0,2,0;
                 0,k,b,a];
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%TYPE 11: P_10B(a,k)
Aaux=[-1,0];
k=N-5;
for i=1:length(Aaux)
    a=Aaux(i);
    cx=cx+1; K(cx)=k; I(cx)=11; VNE(cx)=2;
    Vx{cx}=[ -1,0,0,1;
              0,1,2,0;
              0,k,a,0];
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

L=cell(1,N);
V=cell(1,N);
W=cell(1,N);
W0=cell(1,N);
C=zeros(1,N);


%We now add the lattice points
for i=1:cx
    VP=Vx{i};
    LP=VP;
    %First add the points in the LONG segment
    for s=0:K(i)
        %Segment (0,0,0)(0,0,k) for types 0 to 10
        if I(i)<11
            if ismember([0,0,s],LP','rows')==0
                LP=[LP,[0;0;s]];
            end
        %Segment (0,1,0)(0,1,k) for type 11
        elseif I(i)==11
            if ismember([0,1,s],LP','rows')==0
                LP=[LP,[0;1;s]];
            end
        end
    end

    %Add the extra points in the cases needed
    if I(i)==2  %Type 2
        LP=[LP,[0;-1;0]];
    elseif I(i)==3  %Type 3
        LP=[LP,[0;-1;0],[-1;0;0]];
    elseif I(i)==10 %Type 10
        if ismember([0,2,0],LP','rows')==1
            b=0;
```

```
        elseif ismember([0,2,-1],LP','rows')==1
            b=-1;
        end
        kw=floor((b+K(i))/2); %kw will be the top most point (0,1,_)
        for s=0:kw
            LP=[LP,[0;1;s]];
        end
    elseif I(i)==11 %Type 11
        LP=[LP,[0;0;0]];
    end

    %Take care of the k<3 problem
    bool1=0;
    if K(i)==1
        HP=convex_hull(VP);
        bool1=width_one(VP,HP,1);
    end
    bool2=0;
    if K(i)==2 && VNE(i)>0  %If is the minimal case, VNE(i)=0
        LP0=LP; LP0(:,VNE(i))=[];
        HP0=convex_hull(LP0);
        VP0=vertex_int_boun_pt(HP0,LP0);
        bool2=width_one(VP0,HP0,1);
    end

    %If neither P nor P^v have width one, add polytope to the list
    if bool1==0 && bool2==0
        [w,w0] = volumevectors(VP);
        j=size(VP,2);
        C(j)=C(j)+1;
        V{j}(:,:,C(j))=VP;
        L{j}(:,:,C(j))=LP;
        W{j}(:,:,C(j))=w;
        W0{j}(:,:,C(j))=w0;
    end
end

%Remove redundancies
[L{4},V{4},W{4},W0{4},C(4)]=classes(L{4},V{4},W{4},W0{4});
[L{5},V{5},W{5},W0{5},C(5)]=classes(L{5},V{5},W{5},W0{5});

%Save the results
name=sprintf('PP_spiked_3polytopesofsize%d',N);
save(name,'L','V','W','W0','C')
```

Algorithm A.13. Boxed quasi-minimal 3-polytopes. The enumeration of boxed 3-polytopes (not necessarily quasi-minimal) is described in Section 3.3.3. The first step is to classify all boxed 3-polytopes that correspond to cases (1), (2) and (3) in the description, and then to discard those that are not quasi-minimal.

For this, the starting point is always fixing the set of lattice points LP of our polytope: there is a small number of possibilities for the configuration $A_0 := LP \cap Q$ (lattice points of P that are in the parallelepiped Q), and then several possibilities for each of the three vertices v_i of P that are not in Q. For each choice of such vertices, we first check whether the convex hull $\operatorname{conv}(A_0 \cup \{v_1, v_2, v_3\})$ contains lattice points other than $LP = A_0 \cup \{v_1, v_2, v_3\}$, in which case we discard it. The last step is to check that the polytope has width larger than one (there may be some functional, not corresponding to the parallelepiped functionals, that gives width one to the polytope).

- Routine `boxed_3polytopes_case_1` in page 146. In case (1), the choice for A_0 and the vertices v_i is as in Corollary 3.25.
- Routine `boxed_3polytopes_case_2` in page 148. In case (2), $A_0 = \{0,1\}^3$, and v_i has the i-th coordinate in $\{-1,2\}$ and the other two in $\{0,1\}$.

  In this case there is also a second part for the algorithm. The first step computes all boxed 3-polytopes of size 11. But then we also need to compute those of sizes 7 to 10, which are subpolytopes of the ones of size 11, where the points removed are vertices of the polytope that are in the parallelepiped Q.
- Routine `boxed_3polytopes_case_3` in page 150. In case (3), A_0 is one of the eight (modulo symmetries of the cube) configurations of vertices of the unit cube such that $\#A_0 \geq 4$ and the two points in one of the edges of the cube are not in A_0. For consistency, the coordinates of the eight configurations have been chosen so that in all cases $(1,1,0),(1,1,1) \notin A_0$.

  The options for the vertices depend on the line they are contained in: for each v_i let s_i be the edge of $[0,1]^3$ such that s_i and v_i are contained in a line. If $A_0 \cap s_i \neq \emptyset$, then the i-th coordinate must be in $\{-1,2\}$, whereas if $A_0 \cap s_i = \emptyset$ the i-th coordinate can be in $\{-6,-5,-4,-3,-2,-1,2,3,4,5,6,7\}$. The other two coordinates must always be in $\{0,1\}$.

Routine `boxed_3polytopes` in page 153 puts together all the boxed 3-polytopes of size at least seven and removes all possible redundancies.

Routine `boxed_quasiminimal_3polytopes` in page 155 removes the configurations that are not quasi-minimal. This is not necessary in order to get all lattice 3-polytopes of width larger than one and a certain size, but removing the non quasi-minimal ones guarantees that the sets of polytopes are disjoint, and we need not remove redundancies afterwards.

Routine `is_quasiminimal` in page 154 computes whether a polytope P is quasi-minimal or not: for each vertex $v \in VP$, we evaluate whether P^v is lower dimensional (only one hyperplane equation is stored in HP) or has width one (we use `width_one` with `fast=1`, see Algorithm A.10). If that is the case, vertex v is essential. The algorithm decides as soon as it finds two non-essential vertices, in which case it is not quasi-minimal, or it runs through all the vertices finding at most one, in which case it is quasi-minimal.

```
function [L,V,W,W0,C]=boxed_3polytopes_case_1
T=cputime;

%Parallelepiped of the standard unimodular tetrahedron

%Options for vertex v1
V1=[-1,-1,-1, 0, 1, 1, 1, 2;
     1, 1, 2, 1,-1,-1, 0,-1;
     1, 2, 1, 1,-1, 0,-1,-1];
```

```
%Options for vertex v2
V2=[V1(2,:);
    V1(1,:);
    V1(3,:)];

%Options for vertex v3
V3=[V1(2,:);
    V1(3,:);
    V1(1,:)];

%A is the matrix that stores the 7 lattice points that can only be in
%the polytope
A=zeros(3,7);
%The first four points are the origin and the 3 unit vectors
A(1,2)=1; A(2,3)=1; A(3,4)=1;

L=cell(1,7);
V=cell(1,7);
W=cell(1,7);
W0=cell(1,7);
C=zeros(1,7);

for i=1:size(V1,2)
    for j=1:size(V2,2)
        for k=1:size(V3,2)

            %For each choice of vertices v1, v2, v3, take the convex
            %hull of them with the unimodular tetrahedron:
            A(:,5)=V1(:,i);
            A(:,6)=V2(:,j);
            A(:,7)=V3(:,k);

            HP=convex_hull(A);
            LP=lattice_points(A,HP);
            %If the convex hull does not contain any more points...
            if size(LP,2)==7
                VP=vertex_int_boun_pt(HP,LP);
                bool=width_one(VP,HP,1);
                %...and it has width larger than one, then add it to the
                %list
                if bool==0
                    j0=size(VP,2);
                    [w,w0]=volumevectors(VP);
                    C(j0)=C(j0)+1;
                    L{j0}(:,:,C(j0))=LP;
                    V{j0}(:,:,C(j0))=VP;
                    W{j0}(:,:,C(j0))=w;
                    W0{j0}(:,:,C(j0))=w0;
                end
            end
        end
    end
end

%Remove redundancies
for i=4:7
```

```
    if C(i) ~=0
        [L{i},V{i},W{i},W0{i},C(i)]=classes(L{i},V{i},W{i},W0{i});
    end
end

%Save the results
save('PP_boxed_3polytopes_case_1.mat','L','V','W','W0','C')

cputime-T
```

```
function [L,V,W,W0,C]=boxed_3polytopes_case_2

T=cputime;
%Parallelepiped of the unit cube

%Options for vertex v1
[p,q,r] = meshgrid([-1,2],[0,1],[0,1]);
V1 = [p(:) q(:) r(:)]';

%Options for vertex v2
[p,q,r] = meshgrid([0,1],[-1,2],[0,1]);
V2 = [p(:) q(:) r(:)]';

%Options for vertex v3
[p,q,r] = meshgrid([0,1],[0,1],[-1,2]);
V3 = [p(:) q(:) r(:)]';

%C3 is the 3-dimensional unit cube
C3=[0,0,0,0,1,1,1,1;
    0,0,1,1,0,0,1,1;
    0,1,0,1,0,1,0,1];

L=cell(11,11);
V=cell(11,11);
W=cell(11,11);
W0=cell(11,11);
C=zeros(11,11);

%Compute all boxed 3-polytopes with 11 lattice points: 8 points of the
%unit cube and 3 vertices at distance one in each chimney
for i=1:size(V1,2)
    for j=1:size(V2,2)
        for k=1:size(V3,2)
            %For each choice of vertices v1, v2, v3, take the convex
            %hull of them with the unit cube:
            A=C3;
            A(:,9)=V1(:,i);
            A(:,10)=V2(:,j);
            A(:,11)=V3(:,k);
            HP=convex_hull(A);
            LP=lattice_points(A,HP);
            %If the convex hull does not contain any more points...
            if size(LP,2)==11
                VP=vertex_int_boun_pt(HP,LP);
                bool=width_one(VP,HP,1);
```

```
                %...and it has width larger than one, then add it to
                %the list
                if bool==0
                    j0=size(VP,2);
                    [w,w0]=volumevectors(VP);
                    C(11,j0)=C(11,j0)+1;
                    L{11,j0}(:,:,C(11,j0))=LP;
                    V{11,j0}(:,:,C(11,j0))=VP;
                    W{11,j0}(:,:,C(11,j0))=w;
                    W0{11,j0}(:,:,C(11,j0))=w0;
                end
            end
        end
    end
end

%Remove redundancies
for i=4:11
    if C(11,i) ~=0
        Li=L{11,i};
        Vi=V{11,i};
        Wi=W{11,i};
        W0i=W0{11,i};
        [L{11,i},V{11,i},W{11,i},W0{11,i},C(11,i)]=classes(Li,Vi,Wi,W0i);
    end
end


%We now proceed to remove, one by one, the vertices of P that are in C3

for N=11:-1:8        %N is the size
    for j=4:N        %j is the number of vertices

        %in lists V{N,j} and L{N,j} we have the information VP
        %(vertices) and LP (lattice points) of the polytopes
        for k=1:C(N,j)
            %For each of the configurations of size N and with j
            %vertices
            for v=1:j
                %For each of the vertices in configuration k
                ver=V{N,j}(:,v,k);
                %If said vertex is a point of the cube
                if ismember(ver',C3','rows')==1
                    LP=L{N,j}(:,:,k);
                    %Copy the list of lattice points, remove vertex v
                    s=1;
                    while s<N+1
                        if isequal(ver,LP(:,s))==1
                            LP(:,s)=[];
                            s=N+1;
                        else
                            s=s+1;
                        end
                    end
                    HP=convex_hull(LP);
                    VP=vertex_int_boun_pt(HP,LP);
                    bool=width_one(VP,HP,1);
```

```
                        %If the polytope is of width >1, add it to the list
                        if bool==0
                            [w,w0]=volumevectors(VP);
                            j0=size(VP,2);
                            C(N-1,j0)=C(N-1,j0)+1;
                            L{N-1,j0}(:,:,C(N-1,j0))=LP;
                            V{N-1,j0}(:,:,C(N-1,j0))=VP;
                            W{N-1,j0}(:,:,C(N-1,j0))=w;
                            W0{N-1,j0}(:,:,C(N-1,j0))=w0;
                        end
                    end
                end
            end
        end

        %Remove redundancies
        for j=4:(N-1)
            if C(N-1,j)~=0
                Lj=L{N-1,j}; Vj=V{N-1,j}; Wj=W{N-1,j}; W0j=W0{N-1,j};

                [Lj,Vj,Wj,W0j,C(N-1,j)]=classes(Lj,Vj,Wj,W0j);

                L{N-1,j}=Lj; V{N-1,j}=Vj; W{N-1,j}=Wj; W0{N-1,j}=W0j;
            end
        end
end

%Save the results
save('PP_boxed_3polytopes_case_2.mat','L','V','W','W0','C')

cputime-T
```

```
function [L,V,W,W0,C]=boxed_3polytopes_case_3
T=cputime;
%Parallelepiped of the unit cube

%In A0 we will store the set points of the unit cube that are in P
A0=cell(8,1);

A0{1}=[0,1,0,0,1,0;
       0,0,1,0,0,1;
       0,0,0,1,1,1];

A0{2}=[0,1,0,0,1;
       0,0,1,0,0;
       0,0,0,1,1];

A0{3}=[0,1,0,1,0;
       0,0,1,0,1;
       0,0,0,1,1];

A0{4}=[0,1,0,0;
       0,0,1,0;
       0,0,0,1];
```

```
A0{5}=[0,1,0,1;
       0,0,1,0;
       0,0,0,1];

A0{6}=[1,0,0,1;
       0,1,0,0;
       0,0,1,1];

A0{7}=[1,0,1,0;
       0,1,0,1;
       0,0,1,1];

A0{8}=[0,1,0,1;
       0,0,0,0;
       0,0,1,1];

%Options for vertex v1
[p,q,r] = meshgrid([-1,2],[0,1],[0,1]);
V1 = [p(:) q(:) r(:)]';

%Options for vertex v2
[p,q,r] = meshgrid([0,1],[-1,2],[0,1]);
V2 = [p(:) q(:) r(:)]';

%Options for vertex v3
[p,q,r] = meshgrid([0,1],[0,1],[-1,2]);
V3 = [p(:) q(:) r(:)]';

%When some edges of the cube does not contain points of P, we have to
%allow the vertices vi to go further in their chimney:

%Edge x=1=y is never in P:
[p,q,r] = meshgrid(1,1,[-6,-5,-4,-3,-2,3,4,5,6,7]);
V3=[V3,[p(:) q(:) r(:)]'];
%Edge x=0=y:
[p,q,r] = meshgrid(0,0,[-6,-5,-4,-3,-2,3,4,5,6,7]);
x0y0=[p(:) q(:) r(:)]';
%Edge x=0, y=1:
[p,q,r] = meshgrid(0,1,[-6,-5,-4,-3,-2,3,4,5,6,7]);
x0y1=[p(:) q(:) r(:)]';

%Edge x=1=z:
[p,q,r] = meshgrid(1,[-6,-5,-4,-3,-2,3,4,5,6,7],1);
x1z1=[p(:) q(:) r(:)]';
%Edge x=0,z=1:
[p,q,r] = meshgrid(0,[-6,-5,-4,-3,-2,3,4,5,6,7],1);
x0z1=[p(:) q(:) r(:)]';

%Edge y=1=z:
[p,q,r] = meshgrid([-6,-5,-4,-3,-2,3,4,5,6,7],1,1);
y1z1=[p(:) q(:) r(:)]';
%Edge y=1,z=0:
[p,q,r] = meshgrid([-6,-5,-4,-3,-2,3,4,5,6,7],1,0);
y1z0=[p(:) q(:) r(:)]';


L=cell(9,9);
```

```
V=cell(9,9);
W=cell(9,9);
W0=cell(9,9);
C=zeros(9,9);


for b=1:8 %configuration of the cube
    V1b=V1; V2b=V2; V3b=V3;
    %We will add possibilities depending which edges P does not touch
    %for each configuration
    if b==2 || b==4 || b==5 || b==6 || b==8
        V1b=[V1b,y1z1];
    end
    if b==4
        V2b=[V2b,x1z1];
    end
    if b==5
        V2b=[V2b,x0z1];
    end
    if b==7
        V3b=[V3b,x0y0];
    end
    if b==8
        V3b=[V3b,x0y1];
        V1b=[V1b,y1z0];
    end
    n=size(A0{b},2);
    %Now, for each choice of vertices v1, v2, v3, take the convex hull
    %of them with A0{b}
    for i=1:size(V1b,2)
        for j=1:size(V2b,2)
            for k=1:size(V3b,2)
                A=[A0{b},V1b(:,i),V2b(:,j),V3b(:,k)];
                HP=convex_hull(A);
                LP=lattice_points(A,HP);
                %If the convex hull does not contain any more points...
                if size(LP,2)==n+3
                    VP=vertex_int_boun_pt(HP,LP);
                    bool=width_one(VP,HP,1);
                    %...and it has width larger than one, then add it to
                    %the list
                    if bool==0
                        N=n+3;
                        j0=size(VP,2);
                        [w,w0]=volumevectors(VP);
                        C(N,j0)=C(N,j0)+1;
                        L{N,j0}(:,:,C(N,j0))=LP;
                        V{N,j0}(:,:,C(N,j0))=VP;
                        W{N,j0}(:,:,C(N,j0))=w;
                        W0{N,j0}(:,:,C(N,j0))=w0;
                    end
                 end
            end
        end
    end
end
```

```
%Remove redundancies
for N=7:9
    for j=4:N
        if C(N,j)~=0
            Lj=L{N,j};
            Vj=V{N,j};
            Wj=W{N,j};
            W0j=W0{N,j};
            [L{N,j},V{N,j},W{N,j},W0{N,j},C(N,j)]=classes(Lj,Vj,Wj,W0j);
        end

    end
end

%Save the results
save('PP_boxed_3polytopes_case_3.mat','L','V','W','W0','C')

cputime-T
```

```
function [L,V,W,W0,C]=boxed_3polytopes

%Load the boxed polytopes for each of the cases:

%Case 1
if exist('PP_boxed_3polytopes_case_1.mat','file')==2
    load('PP_boxed_3polytopes_case_1.mat')
    L1=L; V1=V; W1=W; W01=W0; C1=C;
else
    [L1,V1,W1,W01,C1]=boxed_3polytopes_case_1;
end

%Case 2
if exist('PP_boxed_3polytopes_case_2.mat','file')==2
    load('PP_boxed_3polytopes_case_2.mat')
    L2=L; V2=V; W2=W; W02=W0; C2=C;
else
    [L2,V2,W2,W02,C2]=boxed_3polytopes_case_2;
end

%Case 3
if exist('PP_boxed_3polytopes_case_3.mat','file')==2
    load('PP_boxed_3polytopes_case_3.mat')
    L3=L; V3=V; W3=W; W03=W0; C3=C;
else
    [L3,V3,W3,W03,C3]=boxed_3polytopes_case_3;
end

%L1 has dimensions 1 x 7
%L2 has dimensions 11 x 11
%L3 has dimensions 9 x 9
L=cell(11,11);
V=cell(11,11);
W=cell(11,11);
W0=cell(11,11);
C=zeros(11,11);
```

```
%Add the boxed case 1
for j=4:7
    L{7,j}=L1{j}; V{7,j}=V1{j}; W{7,j}=W1{j}; W0{7,j}=W01{j};
    C(7,j)=C1(j);
end

%Add the boxed case 2
for N=7:11
    for j=4:N
        L{N,j}=cat(3,L{N,j},L2{N,j}); V{N,j}=cat(3,V{N,j},V2{N,j});
        W{N,j}=cat(3,W{N,j},W2{N,j}); W0{N,j}=cat(3,W0{N,j},W02{N,j});
        C(N,j)=C(N,j)+C2(N,j);
    end
end

%Add the boxed case 3
for N=7:9
    for j=4:N
        L{N,j}=cat(3,L{N,j},L3{N,j}); V{N,j}=cat(3,V{N,j},V3{N,j});
        W{N,j}=cat(3,W{N,j},W3{N,j}); W0{N,j}=cat(3,W0{N,j},W03{N,j});
        C(N,j)=C(N,j)+C3(N,j);
    end
end

%Remove redundancies
for N=7:11
    for j=4:N
        if C(N,j)~=0
            Lj=L{N,j};
            Vj=V{N,j};
            Wj=W{N,j};
            W0j=W0{N,j};
            [L{N,j},V{N,j},W{N,j},W0{N,j},C(N,j)]=classes(Lj,Vj,Wj,W0j);
        end

    end
end

%Save the results
save('PP_boxed_3polytopes.mat','L','V','W','W0','C')
```

```
function bool=is_quasiminimal(VP,LP)

%VP are the vertices of the polytope
%LP are the lattice points
j=size(VP,2);
N=size(LP,2);

%bool=1 if the polytope is quasi-minimal, 0 otherwise

%In nev we will keep count of the non-essential vertices in VP
nev=0;
v=1;
%As long as we have found less than two non-essential vertices...
```

```
while nev <2  && v<j+1
    %Compute whether P^v has width one or not

    %Take the list of lattice points and remove vertex v
    LP0=LP;
    s=1;
    while s<N+1
        if isequal(VP(:,v),LP0(:,s))==1
            LP0(:,s)=[];
            s=N+1;
        else
            s=s+1;
        end
    end

    HP0=convex_hull(LP0);
    if size(HP0,1)>1     %If P^v is lower dimensional, v is essential
        VP0=vertex_int_boun_pt(HP0,LP0);
        bool=width_one(VP0,HP0,1);
        %If P^v does not have width one, then v is not essential
        if bool==0
            nev=nev+1;
        end
    end
    v=v+1;
end

%The previous loop while finishes as soon as it finds two non-essential
%vertices, or if there are no more vertices to check.
if nev<2
    bool=1;
else
    bool=0;
end
```

```
function [L,V,W,W0,C]=boxed_quasiminimal_3polytopes

%Load the boxed polytopes (or compute them):
if exist('PP_boxed_3polytopes.mat','file')==2
    load('PP_boxed_3polytopes.mat')
    Lb=L; Vb=V; Wb=W; W0b=W0; Cb=C;
else
    [Lb,Vb,Wb,W0b,Cb]=boxed_3polytopes;
end


%Lb has dimensions 11 x 11
L=cell(11,11);
V=cell(11,11);
W=cell(11,11);
W0=cell(11,11);
C=zeros(11,11);

%For each polytope, keep only the quasi-minimal ones:
for N=7:11  %N number of lattice points
```

```
    for j=4:N   %j number of vertices
        for k=1:Cb(N,j)
            if is_quasiminimal(Vb{N,j}(:,:,k),Lb{N,j}(:,:,k))==1
                C(N,j)=C(N,j)+1;
                L{N,j}(:,:,C(N,j))=Lb{N,j}(:,:,k);
                V{N,j}(:,:,C(N,j))=Vb{N,j}(:,:,k);
                W{N,j}(:,:,C(N,j))=Wb{N,j}(:,:,k);
                W0{N,j}(:,:,C(N,j))=W0b{N,j}(:,:,k);
            end
        end
    end
end


save('PP_boxed_quasiminimal_3polytopes.mat','L','V','W','W0','C')
```

Algorithm A.14. Quasi-minimal 3-polytopes of size N.

Putting together spiked and boxed quasi-minimal 3-polytopes of a certain size (at least seven) does not require removing redundancies since, by Remark 3.21, no polytope of size at least seven can be both spiked and boxed.

```
function [L,V,W,W0,C]=quasiminimal_3polytopesofsizeN(N)

%If N>11, we only need to take (or compute) the spiked polytopes
name=sprintf('PP_spiked_3polytopesofsize%d.mat',N);
if N>11
    if exist(name,'file')==2
        load(name)
    else
        [L,V,W,W0,C]=spiked_3polytopesofsizeN(N);
    end
    return
end


%If N<12, we have to put together spiked and boxed polytopes
%Load the spiked 3-polytopes of size N (or compute them)
if exist(name,'file')==2
    load(name)
    Ls=L;  Vs=V; Ws=W; W0s=W0; Cs=C;
elseif exist(name,'file')==0
    [Ls,Vs,Ws,W0s,Cs]=spiked_3polytopesofsizeN(N);
end

%Load the boxed quasi-minimal 3-polytopes (or compute them)
name2='PP_boxed_quasiminimal_3polytopes.mat';
if exist(name2,'file')==2
    load(name2)
    Lb=L(N,1:N); Vb=V(N,1:N); Wb=W(N,1:N); W0b=W0(N,1:N); Cb=C(N,1:N);
else
    [L,V,W,W0,C]=boxed_quasiminimal_3polytopes;
    Lb=L(N,1:N); Vb=V(N,1:N); Wb=W(N,1:N); W0b=W0(N,1:N); Cb=C(N,1:N);
end
```

```
L=cell(1,N);
V=cell(1,N);
W=cell(1,N);
W0=cell(1,N);
C=zeros(1,N);

%Put them together in the same list
for j=4:N
    C(j)=Cs(j)+Cb(j);
    if C(j)~=0
        L{j}=cat(3,Ls{j},Lb{j});
        V{j}=cat(3,Vs{j},Vb{j});
        W{j}=cat(3,Ws{j},Wb{j});
        W0{j}=cat(3,W0s{j},W0b{j});
    end
end

%Save the results
name3=sprintf('PP_quasiminimal_3polytopesofsize%d.mat',N);
save(name3,'L','V','W','W0','C')
```

Algorithm A.15. Lattice 3-polytopes of size N.

The list of all lattice 3-polytopes of a certain size (at least seven) is putting together the lists of merged and quasi-minimal polytopes, which are disjoint sets.

```
function [L,V,W,W0,C]=all_3polytopesofsizeN(N,fast)

if nargin < 2
    fast=0;
end

%Take all quasi-minimal 3-polytopes of size N
name=sprintf('PP_quasiminimal_3polytopesofsize%d.mat',N);
if exist(name,'file')==2
    load(name)
    Lq=L;  Vq=V; Wq=W; W0q=W0; Cq=C;
elseif exist(name,'file')==0
    [Lq,Vq,Wq,W0q,Cq]=quasiminimal_3polytopesofsizeN(N);
end

%Take all merged 3-polytopes of size N. We can take this list from the
%short lists of merging, if it was computed, or from the normal one.
if fast==0
    name2=sprintf('PP_merged_3polytopesofsize%d.mat',N);
elseif fast==1
    name2=sprintf('PP_merged_3polytopesofsize%d_fast.mat',N);
end
if exist(name2,'file')==2
    load(name2)
    Lm=L; Vm=V; Wm=W; W0m=W0; Cm=C;
elseif fast==0
    [Lm,Vm,Wm,W0m,Cm]=merged_3polytopesofsizeN(N);
```

```
elseif fast==1
    [Lm,Vm,Wm,W0m,Cm]=merged_3polytopesofsizeN(N,1);
end

L=cell(1,N);
V=cell(1,N);
W=cell(1,N);
W0=cell(1,N);
C=zeros(1,N);

%Take the union of these (disjoint) lists
for j=4:N
    C(j)=Cq(j)+Cm(j);
    if C(j)~=0
        L{j}=cat(3,Lq{j},Lm{j});
        V{j}=cat(3,Vq{j},Vm{j});
        W{j}=cat(3,Wq{j},Wm{j});
        W0{j}=cat(3,W0q{j},W0m{j});
    end
end

%Save the results. Both fast=0 and fast=1 should yield the same lists of
%polytopes (modulo ordering of points and polytopes), but we will store
%it differently for the sake of comparison.
if fast==0
    name3=sprintf('PP_3polytopesofsize%d.mat',N);
elseif fast==1
    name3=sprintf('PP_3polytopesofsize%d_fast.mat',N);
end
save(name3,'L','V','W','W0','C')
```

A.3 Further properties of a polytope

Once we have enumerated all lattice 3-polytopes of a given size, we output for each of them certain properties we are interested in. These properties are listed in Table A.16, and the rest of this section gives details of the algorithms used to compute them.

Table A.16. *List of computed properties.* The following table is a guide to the list of properties of a polytope we compute. The first column contains the name of the variable, the middle column explains what kind of mathematical object is stored, and the right column explains what information is stored.

automorphisms	matrix $3 \times 4 \times t$ in $\mathbb{Z}$	each 3×4 matrix in `(:,:,i)` represents an automorphism of the polytope (last column is translation)
boxed_parallelepiped	cell of matrices 3×4 in $\mathbb{Z}$, paired with natural number	each matrix contains the three affine functionals that define the parallelepiped the polytope is boxed with respect to, the natural number is the inverse of the volume
ehrhart_coeff	vector 1×4 in $\mathbb{Z}$	$6 \cdot [e_0, e_1, e_2, e_3]$, with e_i the i-th coefficient of the Ehrhart polynomial
fulldim_subpolytopes	cell array $1 \times N$	each cell `{i}` is a matrix $3 \times (N-1)$ in $\mathbb{Z}$ with the list of lattice points in P^u, if it is 3-dimensional
func_min_width	matrix $r \times 3$ in $\mathbb{Z}$	each row `(i,:)` is a linear functional that gives width "width" to the polytope

fvector	vector 1×3 in $\mathbb{Z}_{\geq 0}$	$[f_0, f_1, f_2]$, where f_i is the number of faces of dimension i
hstar_vector	vector 1×4 in $\mathbb{Z}_{\geq 0}$	$[h_0^*, h_1^*, h_2^*, h_3^*]$, the h^*-vector
hyperplanes	matrix $m \times 4$ in $\mathbb{Z}$	m is the number of facets, and each row (a, b, c, d) represents a facet $ax + by + cz + d \geq 0$
hyperplanes_values	matrix $m \times N$ in $\mathbb{Z}_{\geq 0}$	the entry `(i,j)` is the value that takes hyperplane i in point j
lattice_pts	matrix $3 \times N$ in $\mathbb{Z}$	each column `(:,i)` is a lattice point
spiked_directions	matrix $1 \times 3 \times s$ in $\mathbb{Z}$	each vector `(:,:,i)` contains a direction in which the polytope is spiked
sublattice_index	natural number	gcd of the volume_vector entries
vertices	matrix $3 \times v$in $\mathbb{Z}$	each column `(:,i)` is a vertex
volume	natural number	normalized volume
volume_vector	vector $1 \times \binom{N}{4}$in $\mathbb{Z}$	determinant of all the 4-tuples of lattice points
volume_vector_abs	vector $1 \times \binom{N}{4}$ in $\mathbb{Z}_{\geq 0}$	ordered absolute values in volume_vector
width	natural number	lattice width
n_boundary_pts	natural number	number of boundary points (k)
n_edges	natural number	number of edges (e)
n_facets	natural number	number of facets (m)
n_interior_pts	natural number	number of interior points (k)
n_lattice_pts	natural number	number of lattice points (N)
n_vertices	natural number	number of vertices (v)
TF_boxed	boolean	1 if the polytope is boxed, 0 otherwise
TF_canonical	boolean	1 if the polytope is canonical, 0 otherwise
TF_clean	boolean	1 if the polytope is clean, 0 otherwise
TF_dps	boolean	1 if the polytope is dps, 0 otherwise
TF_empty	boolean	1 if the polytope is empty, 0 otherwise
TF_hollow	boolean	1 if the polytope is hollow, 0 otherwise
TF_normal	boolean	1 if the polytope is normal, 0 otherwise
TF_normal_minimal	boolean	1 if the polytope is normal minimal, 0 otherwise
TF_primitive	boolean	1 if the polytope is primitive, 0 otherwise
TF_quasiminimal	boolean	1 if the polytope is quasi-minimal, 0 otherwise
TF_spiked	boolean	1 if the polytope is spiked, 0 otherwise
TF_terminal	boolean	1 if the polytope is terminal, 0 otherwise
boundary_pts_indices	boolean vector $1 \times N$	the i-th value is 1 if the point is a boundary point or 0 otherwise
edges_indices	boolean matrix $e \times N$	entry `(i,j)` is 1 if the j-th point belongs to the i-th edge
essential_indices	boolean vector $1 \times N$	the i-th value is 1 if the vertex i-th lattice point is an essential vertex
facets_indices	boolean matrix $m \times N$	entry `(i,j)` is 1 if the j-th point belongs to the i-th facet
fulldim_indices	boolean vector $1 \times N$	the i-th value is 1 if P^u, for u the i-th lattice point and a vertex, is full-dimensional
interior_pts_indices	boolean vector $1 \times N$	the i-th value is 1 if the point is an interior point or 0 otherwise
normal_indices	boolean vector $1 \times N$	the i-th value is 1 if P^u, for u the i-th lattice point and a vertex, is fulldimensional and normal
vertices_indices	boolean vector $1 \times N$	the i-th value is 1 if the point is a vertex or 0 otherwise

A.3.1 Algorithms

The properties of each polytope that remain to be computed are the following:

Algorithm A.17. Computing if a polytope is dps. As stated by [CLR02], a lattice polytope is dps if, and only if, it does not contain three aligned points, nor four points forming an empty parallelogram.

```
function [bool] = is_dps(LP)

% LP is a matrix 3 x N where the columns are the list of lattice points
% of a polytope
% bool=1 if LP is dps or 0 otherwise

N=size(LP,2);
bool=1;

% A lattice polytope is dps if none of these two things happen:

% 1- three points pi,pj,pk are consecutive points in a segment
%(pi+pj=2*pk for a permutation of them)
for i=1:(N-2)
    for j=(i+1):(N-1)
        for k=(j+1):N
            if isequal(LP(:,i)+LP(:,j),2*LP(:,k))==1
                bool=0;
                return
            end
            if isequal(LP(:,i)+LP(:,k),2*LP(:,j))==1
                bool=0;
                return
            end
            if isequal(LP(:,j)+LP(:,k),2*LP(:,i))==1
                bool=0;
                return
            end
        end
    end
end


% 2- four points pi,pj,pk,pl form a parallelepiped (pi+pj=pk+pl for a
%permutation of them)
for i=1:(N-3)
    for j=(i+1):(N-2)
        for k=(j+1):(N-1)
            for l=(k+1):N
                if isequal(LP(:,i)+LP(:,j),LP(:,k)+LP(:,l))==1
                    bool=0;
                    return
                end
                if isequal(LP(:,i)+LP(:,k),LP(:,j)+LP(:,l))==1
                    bool=0;
                    return
                end
                if isequal(LP(:,i)+LP(:,l),LP(:,k)+LP(:,j))==1
                    bool=0;
                    return
                end
            end
        end
    end
end
```

Algorithm A.18. Edges of a polytope. Having the information in `hyperplanes` and `lattice_pts` it is easy to deduce which points land in a specific edge: if a lattice point of the polytope belongs to at least two facets, then the facets form an edge (if more than one point of LP is in the intersection) or contain a vertex (if the point is unique).

```
function EE=edges(FF)

%FF is a m x N matrix, where m is the number of facets, N is the number
%of lattice points in the polytope, and FF(i,j) is 1 if point j belongs
%to facet i

%EE is a e x N matrix, where e is the number of edges, N is the number
%of lattice points in the polytope, and EE(i,j) is 1 if point j belongs
%to edge i

%Each edge is the intersection of two facets, and any intersection of
%two facets forms and edge if at least two different lattice points
%belong to both of them

N=size(FF,2);

EE=[];
e=0;

for i=1:size(FF,1)-1                    %facet i
    for j=(i+1):size(FF,1)              %facet j
        vec=FF(i,:)+FF(j,:);            %sum of incidencies of facets i,j
        vecsign=zeros(1,N);
        for k=1:N                       %for each lattice point k
            if vec(k)==2                %if point k is in both facets
                vecsign(k)=1;           %vecsign(k)=1
            end
        end
        if sum(vecsign)>1           %If at least two lattice points of the
                                 %polytope are in both facets, they form an edge
            e=e+1;
            EE=[EE;vecsign];
        end
    end
end
```

Algorithm A.19. Ehrhart and h^*-polynomial, volume. The Ehrhart polynomial of a lattice 3-polytope P is $E_P(t) := e_0 + e_1 t + e_2 t^2 + e_3 t^3$ with coefficients in $\frac{1}{6}\mathbb{Z}$, and it verifies: $E_P(k) = \#(kP \cap \mathbb{Z}^3)$ the number of lattice points in the k-th dilation of P. Moreover, $E_P(0) = 1$ and $E_P(-1) = \#(\operatorname{int}(P) \cap \mathbb{Z}^3)$ the number of lattice points in the interior of P. The coefficients of the polynomial can be interpolated taking the values at $t = -1, 0, 1, 2$.

The normalized volume of the polytope is $6e_3$.

The h^*-vector is $(h_0^*, h_1^*, h_2^*, h_3^*) \in \mathbb{Z}_{\geq 0}^4$ (see [Sta80]), where the entries are the coefficients of the h^*-polynomial $h_0^* + h_1^* z + h_2^* z^2 + h_3^* z^3$, which is the unique polynomial such that:

$$\sum_{t \geq 0} E_P(t) z^t = \frac{h_0^* + h_1^* z + h_2^* z^2 + h_3^* z^3}{(1-z)^4}$$

From there we derive:

$$
\begin{aligned}
h_0^* &= e_0 = 1 \\
h_1^* &= -3e_0 + e_1 + e_2 + e_3 \\
h_2^* &= 3e_0 - 2e_1 + 4e_3 \\
h_3^* &= -e_0 + e_1 - e_2 + e_3
\end{aligned}
$$

See [BR07] for general Ehrhart Theory references.

```
function [E,hvec]=ehrhart(N,VP,int,HP)

% VP is a matrix 3 x j, where each column is a vertex
% int is the number of interior points of the polytope
% HP is a matrix m x N, the facet-defining hyperplanes

% E is going to be a vector 6*[e0 e1 e2 e3] where ei is the coefficient
% of degree i in the Ehrhart polynomial.
% In particular, e0=1.


% The Ehrhart polynomail E(t)= e0 + e1*t + e2*t^2 + e3*t^3 is the unique
% polynomial of degree 3 that satisfies the following:
%E( 0)=1          -->e0             =1
%E(-1)=-|int(P)|-->e0- e1+ e2- e3=-|int(P)|-->-e1+ e2- e3 =-|int(P)|-1
%E( 1)=|P|       -->e0+ e1+ e2+ e3=|P|       --> e1+ e2+ e3 =|P|-1
%E( 2)=|2P|      -->e0+2e1+4e2+8e3=|2P|      -->2e1+4e2+8e3 =|2P|-1

b1=-int-1;
b2=N-1;

HH=[HP(:,1:3),2*HP(:,4)];    %HH is the convex hull of 2P
VV=2*VP;                %VV are the vertices of 2P

LL=lattice_points(VV,HH);    %PP are the lattice points of 2P

b3=size(LL,2)-1;

A = [ -1, 1,-1;
       1, 1, 1;
       2, 4, 8];

B=[b1;
   b2;
   b3];

%We solve Ax=B, and the solution will be x=[e1,e2,e3] the last three
%coefficients
E=[1,(A\B)'];

%We know that E is a rational vector, but that 6*E is integral, so we
%store 6*E
E=round(6*E);

% The h*-vector [h0*,h1*,h2*,h3*] verifies the following equations,
% where [e0,e1,e2,e3] are the coefficients of the Ehrhart polynomial:
```

```
%     h0* = e0 = 1
%     h1* = - 3 e0 +    e1 + e2 +    e3
%     h2* =    3 e0 - 2 e1 +       4 e3
%     h3* = -    e0 +    e1 - e2 +    e3

h=zeros(1,4);
h(1)=E(0);
h(2)=-3*E(0)+E(1)+E(2)+E(3);
h(3)=3*E(0)-2*E(1)+4*E(3);
h(4)=-E(0)+E(1)-E(2)+E(3);

%Remember that E had the ehrhart coefficients multiplied by 6. We have
%to divide:
hvec=round(h./6);
```

Algorithm A.20. Full-dimensional subpolytopes of size $N-1$. The computation of which subpolytopes of size $N-1$ (of the form P^v, for v a vertex) are full-dimensional or not is used in two other functions: computing whether a polytope is normal minimal (Algorithm A.22), and computing the essential vertices (Algorithm A.21).

An extra variable (not listed as a property) is output: a vector array `F2dim`, $1 \times N$, where each `F2dim{i}` stores the equation of the hyperplane containing P^i, in the case that i is a vertex and P^i is 2-dimensional, and is empty otherwise. This variable is used when evaluating if a polytope is spiked and/or boxed (Algorithm A.21).

```
function [SS,V_full,F2dim]=subpolytopes(LP,VI,HP)

%N number of lattice points
N=size(LP,2);

%LP is a 3 x N matrix. LP(:,i) are the lattice points
%VI is a 1 x N vector where VI(i)=1 if LP(:,i) is a vertex
%HP is an m x 4 matrix where HP(i,:) is a facet-defining functional

%SHP is the sign matrix of hyperplanes_values
SHP=sign(HP*[LP;ones(1,N)]);

%SS will be a 1 x N cell, where SS{v}=[] if the polytope P^v is lower
%dimensional or if v is not a vertex.
%It will be the list of lattice points of P^v otherwise (a matrix
% 3 x N-1)

%V_full is a 1 x N vector and VVfull(i)=1 if P^v is full-dimensional,
% for v=LP(:,i) a vertex

%F2dim is a 1 x N cell where F2dim{v}=[a,b,c,d] an affine functional
%defining the hyperplane containing P^v, if this is lower-dimensional,
%or is [] otherwise.

SS=cell(1,N);
V_full=zeros(1,N);
F2dim=cell(1,N);

for i=1:N
    if VI(i)==1 %If LP(:,i) is a vertex
```

```
        %vaux will be the unit vector in the direction of the i-th
        %coordinate
        vaux=zeros(1,N);
        vaux(i)=1;

        %vaux will be a row in SHP if and only if all the points of P
        % except for LP(:,i) are in one facet

        if ismember(vaux,SHP,'rows')==1
            %We want to store in F2dim{i} the row s such that
            %vaux=SHP(s,:)
            s=1;
            while s<size(SHP,1)+1
                if isequal(vaux,SHP(s,:))==1
                    F2dim{i}=HP(s,:);
                    s=size(SHP,1)+1;
                end
                s=s+1;
            end
        else
            LPi=LP;
            LPi(:,i)=[];
            SS{i}=LPi;
            V_full(i)=1;
        end
    end
end
```

Algorithm A.21. Computing if a polytope is spiked and/or boxed. The routine `quasi_box_spi` used for this computation has several parts:

- First, for each vertex v we compute the width of P^v: if it equals zero (P^v is lower dimensional) or one, then v is essential. If so, we store the functionals that give the corresponding width to P^v. The polytope is quasi-minimal if, and only if, all but perhaps one of the vertices are essential.

The next two steps aim to decide whether the polytope is boxed and/or spiked. For this:

- We want to have the affine functionals that give width one or zero to P^v (for v essential vertex) chosen canonically in the following way: f is the unique primitive affine functional such that $f(P^v) = 0$ or $f(P^v) = [0, 1]$, and such that $f(v) > 1$.
- Finally, for each choice $(f_v)_{v \text{ essential vertex}}$ of one such functional for each essential vertex v, we check whether $\{\hat{f}_v\}_v$ spans the 3-dimensional dual space ($\hat{f}_v$ is the vector of coefficients of the linear part):
  - If they span the 3-dimensional space, we select all 3-tuples $\{\hat{f}_u, \hat{f}_v, \hat{f}_w\}$ that are linearly independent functionals. Then P is be boxed with respect to the parallelepiped $f_u^{-1}([0,1]) \cap f_v^{-1}([0,1]) \cap f_w^{-1}([0,1])$. We store the tuple plus the determinant of the vectors, which is the inverse of the Euclidean volume of the parallelepiped.

- If not, we compute the unique primitive direction that is orthogonal to all of them. If the polytope is also quasi-minimal, then it is spiked projecting in this direction.

```
function [q,e,b,s,b_p,s_d]=quasi_box_spi(LP,SS,V_full,VI,F2dim)

%N is the number of lattice points in the polytope
N=size(LP,2);

%q=1 if the polytope is quasi-minimal, 0 otherwise
%e is the set of essential indices
e=zeros(1,N);
%b=1 if the polytope is boxed (not necessarily quasi-minimal), 0
%otherwise s=1 if the polytope is spiked, 0 otherwise
%p_b stores the (maybe several) parallelepiped Q which P is boxed with
%respect to.
b_p=cell(1,2);
%p_b{j,1} stores the three affine functionals of the parallelepiped Qj.
%p_b{j,2} stores the (inverse of its) Euclidean volume (of Qj)

%d_s will be empty if the polytope is not spiked, or otherwise it will
%be a matrix with 3 columns and each row is a direction with respect to
%which P is spiked
s_d=[];

%SS is fulldim_subpolytopes
%Vfull is fulldim_indices
%VI is vertices_indices
%F2dim is a 1 x N array with the hyperplanes that contain P^v lower-
%dimensional

%In FF we will store the functionals that give width one or zero to each
%of P^v
FF=cell(1,N);

for i=1:N
    %If i is a vertex, and subpolytope P^i is lower-dimensional, then
    %vertex i is essential and the functional that gives width zero to
    %it is stored in F2dim{i}
    if VI(i)==1 && V_full(i)==0
        e(i)=1;
        FF{i}=F2dim{i};

    %If i is a vertex and P^i is fulldimensional, we want to see if it
    %has width one or not
    elseif VI(i)==1 && V_full(i)==1
        LP0=SS{i};
        HP0=convex_hull(LP0);
        VP0=vertex_int_boun_pt(HP0,LP0);
        [bool,func]=width_one(VP0,HP0);
        if bool==1
            e(i)=1;
            FF{i}=func;
        end
    end
```

```
end

%The polytope is quasi-minimal if there is at most one vertex that is
%non-essential

K=sum(e); %number of essential vertices

if K<sum(VI)-1
    q=0;
else
    q=1;
end

%If there is less than 3 essential vertices, the polytope will not be
%quasi-minimal, nor boxed or spiked, and we have finished
if K<3
    b=0;
    s=0;
    return
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

%If the program continues, it means that there are at least 3 essential
%vertices.
%Now we want to find out whether the polytope is boxed and, in the case
%of it being quasi-minimal, whether it is spiked.

%So far we have all the affine functionals that give width 0 or 1.
%Each row [a,b,c,d] is a functional of the form ax+by+cz+d
%Now we want the affine functionals so that the values of the
%functionals in the points are 0,1 (0 in the case of lower dimensional)
%and >1 for the removed point.
for i=1:N
    if e(i)==1
        %fu is the list of functionals
        fu=FF{i};

        %for each of those functionals
        for b=1:size(fu,1)

            %vals is the vector 1 x N of values that functional f(b,:)
            %takes in the points LP
            vals=fu(b,:)*[LP;ones(1,N)];

            %The value of the functional in point LP(:,i) is either the
            %maximum or the minimum. We want it to be the maximum:
            if vals(i)==min(vals)
                fu(b,:)=-fu(b,:);
                vals=-vals;
            end

            %Now we want the minimum value of the functional in the list
            %of lattice points to be 0:
            fu(b,4)=fu(b,4)-min(vals);
        end
        FF{i}=fu;
```

```
    end
end

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

l=find(e); %l has the positions of the essential vertices
M=FF{l(1)}';
for i=l(1)+1:N
    if e(i)==1
        M=combvec(M,FF{i}');
    end
end
M=M';
% M is now a matrix eval x 4K, where each row is
% [a1 b1 c1 d1 a2 b2 c2 d2 ..... aK bK cK dK]
% with all the possible choices of [ai bi ci di] the coefficients of an
% affine functional giving with 0 or 1 to P^i, for i an essential vertex

%The number of all possible combinations of functionals.
eval=size(M,1);

%c_box and c_spi will keep count of how many parallelepipeds and
%directions the polytope is boxed or spiked with respect to,
%respectively
c_box=0;
c_spi=0;
for r=1:eval
    Maux=M(r,1:3);
    Maux0=M(r,4);
    for m=1:K-1
        Maux=[Maux;M(r,(4*m+1):(4*m+3))];
        Maux0=[Maux0;M(r,4*m+4)];
    end

    %Maux is now a K x 3 matrix, and Maux0 is a K x 1 matrix.
    %Rows of respective matrices are [ai bi ci] and [di] as stored in M.

    ra=rank(Maux);

    %If the rank of Maux is 3, the polytope is boxed with respect to a
    %parallelepiped defined by the minors with non-zero determinant.
    if ra==3
        %Let us check all the minors
        for i=1:K-2
            for j=i+1:K-1
                for k=j+1:K
                    TT=Maux([i j k],:);
                    TT0=Maux0([i j k]);
                    x=round(abs(det(TT)));
                    if x~=0
                        %Check that that 3-tuple of functionals is not
                        %stored already
                        bool=0;
                        for t=1:c_box
                            if compare_cols([TT,TT0]', b_p{t,1}')==1
                                bool=1;
                            end
```

```
                        end
                        if bool==0
                            c_box=c_box+1;
                            b_p{c_box,1}=[TT,TT0];
                            b_p{c_box,2}=x;
                        end
                    end
                end
            end
        end
    %If the rank is 2, the polytope is spiked in the direction that is
    %orthogonal to all the rows of Maux. To compute this direction it
    %suffices to compute the vector product of two linearly independent
    %rows
    elseif ra==2 && q==1
        dir=0;
        for i=1:K-1
            for j=i+1:K
                TT=Maux([i j],:);
                if dir==0 && rank(TT)==2
                    %If the rank of those two vectors is 2, it means
                    %they are linearly independent, and we want to
                    %compute the unique primitive vector that
                    %is orthogonal to them
                    v=round(cross(TT(1,:),TT(2,:)));
                    d=gcd(v(1),gcd(v(2),v(3)));
                    v=v./d;

                    %For uniqueness we will choose the first non-zero
                    %coordinate of v to be positive
                    vf=find(v);
                    if v(vf(1))<0
                        v=-v;
                    end

                    %We have found the direction so we put dir=1 and the
                    %loop will finish
                    dir=1;

                    %We want to add this direction only if it was not
                    %considered before
                    bool=0;
                    if c_spi>0
                        if ismember(v,s_d,'rows')==1
                            bool=1;
                        end
                    end
                    if bool==0
                        c_spi=c_spi+1;
                        s_d=[s_d;v];
                    end
                end
            end
        end
    end
end
```

```
%The polytope is boxed if c_box>0
if c_box>0
    b=1;
else
    b=0;
end

%The polytope is spiked if it is quasi-minimal (q=1) and c_spi>0
if c_spi>0 && q==1
    s=1;
else
    s=0;
end
```

Algorithm A.22. Computing if a polytope is normal and normal minimal. Deciding whether a polytope is normal or not is, as explained in Section 3.4.5, simply checking that

$$\#\Big(2P \cap \mathbb{Z}^3\Big) = \#\Big(P \cap \mathbb{Z}^3 + P \cap \mathbb{Z}^3\Big).$$

A normal polytope is normal minimal if all the fulldimensional subpolytopes P^v are also normal.

```
function bool=is_normal(LP)

%LP is the list of lattice points of polytope P
N=size(LP,2);

%LPLP contains the lattice points in (P \cap \Z^d) + (P \cap \Z^d)
LPLP=minkowski_sum(LP,LP);

%In LPP we will store all the lattice points in (P + P) \cap \Z^d
HPP=convex_hull(LPLP);
LPP=lattice_points(LPLP,HPP);

%The next condition would imply that the number of points in
%   (P + P) \cap \Z^d
% is the same as in (P \cap \Z^d) + (P \cap \Z^d)

if size(LPP,2)==size(LPLP,2)
    bool=1;
else
    bool=0;
end
```

```
function [nor_ind]=normal_subpolytopes(SS,V_full)

%V_full is a 1 x N vector where V_full(i)=1 if i is a vertex and P^i is
% full-dimensional, or 0 otherwise.
%SS is an 3 X N-1 X N matrix where SS{i} is the set of lattice points
% in P^i if V_full(i)=1, or [] otherwise

N=size(Vfull,2);
```

```
%In nor_ind we keep the indices of the vertices such that P^v is normal
nor_ind=zeros(1,N);

for i=1:N
    if V_full(i)==1
        LP=SS{i};
        if is_normal(LP)==1
            nor_ind(i)=1;
        end
    end
end
```

Algorithm A.23. Properties of a polytope. Routine `propertiesof_polytope` (page 170) computes all the properties of a polytope $P = \mathrm{conv}(A)$ given $A \in \mathbb{Z}^d$ as input. We also input the name of the file and folder where we want to store this information.

Routine `propertiesof_3polytopesofsizeN` (page 173) computes the properties of all lattice 3-polytopes of size N and width larger than one, from the previously computed file `PP_3polytopesofsizeN.mat`, and stores them in a folder `SizeN_configurations` under the names `N_c.mat`, where N is the size and c is a counter.

```
function []=propertiesof_polytope(A,folder,name)

%A is a list of lattice points. The polytope is P=conv(A)
%folder is the string name of the folder we want to save the properties
if exist(folder,'dir')~=7
    mkdir(folder)
end
%name is the string name we will give to this specific file

fileName=fullfile(folder,name);


t=cputime;


%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%hyperplanes,hyperplanes_values,lattice_pts,n_lattice_pts
hyperplanes=convex_hull(A);
lattice_pts=lattice_points(A,hyperplanes);
n_lattice_pts=size(lattice_pts,2);
hyperplanes_values=hyperplanes*[lattice_pts;ones(1,n_lattice_pts)];
hyperplanes_signofvalues=sign(hyperplanes_values);
save(fileName,'hyperplanes','hyperplanes_values')
save(fileName,'n_lattice_pts','lattice_pts','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%volume_vector,volume_vector_abs
[volume_vector,volume_vector_abs]=volumevectors(lattice_pts);
save(fileName,'volume_vector','volume_vector_abs','-append')
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
% TF_primitive polytope
```

```
% sublattice_index
g=0;
for i=1:nchoosek(n_lattice_pts,4)
    g=gcd(g,volume_vector_abs(i));
end
if g==1
    TF_primitive=1;
    sublattice_index=1;
else
    TF_primitive=0;
    sublattice_index=g;
end
save(fileName,'TF_primitive','sublattice_index','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%TF_dps
TF_dps=is_dps(lattice_pts);
save(fileName,'TF_dps','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%facets_indices
%n_facets
%edges_indices
%n_edges
n_facets=size(hyperplanes,1);
facets_indices=+~hyperplanes_signofvalues;
edges_indices=edges(facets_indices);
n_edges=size(edges_indices,1);
save(fileName,'facets_indices','n_facets','-append')
save(fileName,'edges_indices','n_edges','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%n_vertices
%n_interior_pts
%n_boundary_pts
%vertices
%vertices_indices
%interior_pts_indices
%boundary_pts_indices
[V,V_ind,I_ind,B_ind]=vertex_int_boun_pt(hyperplanes,lattice_pts);
vertices=V;
vertices_indices=V_ind;
interior_pts_indices=I_ind;
boundary_pts_indices=B_ind;
n_vertices=sum(vertices_indices);
n_interior_pts=sum(interior_pts_indices);
n_boundary_pts=sum(boundary_pts_indices);
save(fileName,'n_vertices','vertices','vertices_indices','-append')
save(fileName,'n_interior_pts','interior_pts_indices','-append')
save(fileName,'n_boundary_pts','boundary_pts_indices','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%width
```

```
%func_min_width
[width,func_min_width]=width_p(vertices);
save(fileName,'width','func_min_width','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%TF_hollow  %TF_canonical   %TF_clean   %TF_terminal   %TF_empty
TF_hollow=0; TF_canonical=0; TF_clean=0; TF_terminal=0; TF_empty=0;
if n_interior_pts==0
    TF_hollow=1;
end
if n_interior_pts==1
    TF_canonical=1;
end
if n_boundary_pts==n_vertices
    TF_clean=1;
end
if TF_canonical==1 && TF_clean==1
    TF_terminal=1;
end
if TF_hollow==1 && TF_clean==1
    TF_empty=1;
end
save(fileName,'TF_hollow','TF_canonical','TF_terminal','-append')
save(fileName,'TF_clean','TF_empty','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%ehrhart_coeff,hstar_vector
%volume
[E,H]=ehrhart(n_lattice_pts,vertices,n_interior_pts,hyperplanes);
ehrhart_coeff=E;
hstar_vector=H;
volume=ehrhart_coeff(4);
save(fileName,'ehrhart_coeff','volume','hstar_vector','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%fvector
fvector=[n_vertices,n_edges,n_facets];
save(fileName,'fvector','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%fulldim_subpolytopes,fulldim_indices
%TF_quasiminimal,essential_indices
[SS,V_full,F2dim]=subpolytopes(lattice_pts,V_ind,hyperplanes);
fulldim_subpolytopes=SS;
fulldim_indices=V_full;
[q,e,b,s,b_p,s_d]=quasi_box_spi(lattice_pts,SS,V_full,V_ind,F2dim);
TF_quasiminimal=q;
essential_indices=e;
TF_boxed=b;
TF_spiked=s;
boxed_parallelepiped=b_p;
spiked_direction=s_d;
save(fileName,'fulldim_subpolytopes','fulldim_indices','-append')
```

```
save(fileName,'TF_quasiminimal','essential_indices','-append')
save(fileName,'TF_boxed','boxed_parallelepiped','-append')
save(fileName,'TF_spiked','spiked_direction','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%TF_normal, normal_indices, TF_normal_minimal
TF_normal=is_normal(lattice_pts);
normal_indices=normal_subpolytopes(fulldim_subpolytopes,fulldim_indices);
TF_normal_minimal=0;
if TF_normal==1 && sum(normal_indices)==0
    TF_normal_minimal=1;
end
save(fileName,'TF_normal','TF_normal_minimal','normal_indices','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%
%automorfisms
[v,v0]=volumevectors(vertices);
[~,Mx,bx,sx]=maps(vertices,v,v0,vertices,v,v0);
automorphisms=zeros(3,4,sx);
for i=1:sx
    automorphisms(:,:,i)=[Mx(:,:,i),bx(:,:,i)];
end
save(fileName,'automorphisms','-append')
%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

cputime-t
```

```
function []=propertiesof_3polytopesofsizeN(N)

t=cputime;

%Load the vertices of lattice 3-polytopes of size N (variable V)
file=sprintf('PP_3polytopesofsize%d.mat',N);
load(file,'V','C')

%We will want to store the properties in the following folder
foldername=sprintf('Size%d_configurations',N);

c=0; % configuration number
for v=4:N-1 %v is the number of vertices
    for k=1:C(v) %k is the configuration number of lattice 3-polytopes
                 %of size N and with v vertices in V{v}
        c=c+1;

        %We want to save the file with the following name
        fname=sprintf('%d_%d.mat',[N,c]);
        propertiesof_polytope(V{v}(:,:,k),foldername,fname);
    end
end

cputime-t
```

A.3.2 Sample output files

Polytope `5_9.mat`

```
Number of lattice points: 5
Lattice points:
   -3    0    0    1    2
   -5    0    0    0    5
   -2    0    1    0    1
---------------------------------------------------------------------
Number of vertices: 4
Vertices:
   -3     0     1     2
   -5     0     0     5
   -2     1     0     1
Indices of vertices:
    1    0    1    1    1
---------------------------------------------------------------------
Number of interior points: 1
Indices of interior points:
    0    1    0    0    0
---------------------------------------------------------------------
There are no other boundary points.
---------------------------------------------------------------------
Number of facets: 4
Facet defining hyperplanes:
   - 5x + 6y - 5z + 5
   15x - 6y - 5z + 5
   - 5x - 2y + 15z + 5
   - 5x + 2y - 5z + 5
Facet-point distances:
    0    5    0    0   20
    0    5    0   20    0
    0    5   20    0    0
   20    5    0    0    0
Facet-point incidences:
    1    0    1    1    0
    1    0    1    0    1
    1    0    0    1    1
    0    0    1    1    1
---------------------------------------------------------------------
Number of edges: 6
Edge-point incidences:
    1    0    1    0    0
    1    0    0    1    0
    0    0    1    1    0
    1    0    0    0    1
    0    0    1    0    1
    0    0    0    1    1
---------------------------------------------------------------------
f-vector: (4,6,4)
---------------------------------------------------------------------
Width: 2
Achieved with respect to the following functionals:
   x - z
Values of the functionals in the points:
   -1    0   -1    1    1
```

```
----------------------------------------------------------------------
Volumes of 4-tuples of points:
   (5,20)

Total volume: 20
----------------------------------------------------------------------
The polytope IS:
  - Boxed
  - Canonical
  - Clean
  - Dps
  - Quasi-minimal
  - Terminal
----------------------------------------------------------------------
The polytope IS NOT:
  - Empty
  - Hollow
  - Normal
  - Normal_minimal
  - Primitive: its sublattice index is 5
  - Spiked
----------------------------------------------------------------------
Ehrhart polynomial: 1 - (1/3)*n + n^2 + (10/3)*n^3
h*-vector: (1,1,17,1)
----------------------------------------------------------------------
Indices of full-dimensional subpolytopes of size 4:
    1    0    1    1    1
----------------------------------------------------------------------
There are no normal subpolytopes of size 4.
----------------------------------------------------------------------
Indices of essential vertices:
    1    0    1    1    1
----------------------------------------------------------------------
Non-trivial automorphisms:
   ( 2x - 3z, 5x - y - 5z, x - 2z)
   ( - y + 2z, - 2y + 5z, x - y + z)
   ( - 3x + y + z, - 5x + 2y, - 2x + y)
----------------------------------------------------------------------
The polytope is NOT spiked.
----------------------------------------------------------------------
The polytope is boxed with respect to the following parallelepipeds:
                           - z + 1
Euclidean volume 1         - x + 2z + 1
                           2x - y + 1

                           - z + 1
Euclidean volume 1         - x + 2z + 1
                           - x + y - z + 1

                           - z + 1
Euclidean volume 1         2x - y + 1
                           - x + y - z + 1

                           - x + 2z + 1
Euclidean volume 1         2x - y + 1
                           - x + y - z + 1
```

Polytope 6_6.mat

```
Number of lattice points: 6
Lattice points:
   -1    0    0    0    0    1
   -1    0    0    1    2    0
    0    0    1    0   -1    0
------------------------------------------------------------------------
Number of vertices: 4
Vertices:
   -1     0     0     1
   -1     0     2     0
    0     1    -1     0
Indices of vertices:
    1    0    1    0    1    1
------------------------------------------------------------------------
Number of interior points: 1
Indices of interior points:
    0    1    0    0    0    0
------------------------------------------------------------------------
Indices of other boundary points:
    0    0    0    1    0    0
------------------------------------------------------------------------
Number of facets: 4
Facet defining hyperplanes:
   2x - y - z + 1
   - x + 2y - z + 1
   - x + 2y + 5z + 1
   - x - y - z + 1
Facet-point distances:
    0    1    0    0    0    3
    0    1    0    3    6    0
    0    1    6    3    0    0
    3    1    0    0    0    0
Facet-point incidences:
    1    0    1    1    1    0
    1    0    1    0    0    1
    1    0    0    0    1    1
    0    0    1    1    1    1
------------------------------------------------------------------------
Number of edges: 6
Edge-point incidences:
    1    0    1    0    0    0
    1    0    0    0    1    0
    0    0    1    1    1    0
    1    0    0    0    0    1
    0    0    1    0    0    1
    0    0    0    0    1    1
------------------------------------------------------------------------
f-vector: (4,6,4)
------------------------------------------------------------------------
Width: 2
Achieved with respect to the following functionals:
   x
```

```
   x + z
   x
   x - z
   x - y - z
   y + z
   z
Values of the functionals in the points:
   -1    0    0    0    0    1
   -1    0    1    0   -1    1
   -1    0    0    0    0    1
   -1    0   -1    0    1    1
    0    0   -1   -1   -1    1
   -1    0    1    1    1    0
    0    0    1    0   -1    0
--------------------------------------------------------------------
Volumes of 4-tuples of points:
   (0,1,2,3,6)

Total volume: 6
--------------------------------------------------------------------
The polytope IS:
  - Boxed
  - Canonical
  - Normal
  - Primitive
  - Quasi-minimal
  - Spiked
--------------------------------------------------------------------
The polytope IS NOT:
  - Clean
  - Dps
  - Empty
  - Hollow
  - Normal_minimal
  - Terminal
--------------------------------------------------------------------
Ehrhart polynomial: 1 + (5/2)*n + (3/2)*n^2 + n^3
h*-vector: (1,2,2,1)
--------------------------------------------------------------------
Indices of full-dimensional subpolytopes of size 5:
    1    0    1    0    1    1
--------------------------------------------------------------------
Indices of normal subpolytopes of size 5:
    1    0    1    0    1    1
--------------------------------------------------------------------
Indices of essential vertices:
    1    0    1    0    1    1
--------------------------------------------------------------------
Non-trivial automorphisms:
   ( x, y + 2z, - z)
   ( - x, - x + y, z)
   ( - x, - x + y + 2z, - z)
--------------------------------------------------------------------
The polytope is spiked in the following directions:
    0    1    0
--------------------------------------------------------------------
The polytope is boxed with respect to the following parallelepipeds:
```

```
                          - x + 1
Euclidean volume 1        z + 1
                          2x - y - z + 1

                          - x + 1
Euclidean volume 1        - z + 1
                          2x - y - z + 1

                          - x - y - z + 1
Euclidean volume 1/3      z + 1
                          2x - y - z + 1

                          - x - y - z + 1
Euclidean volume 1/3      - z + 1
                          2x - y - z + 1

                          - y - z + 1
Euclidean volume 1/2      z + 1
                          2x - y - z + 1

                          - y - z + 1
Euclidean volume 1/2      - z + 1
                          2x - y - z + 1

                          - x - y - z + 1
Euclidean volume 1        z + 1
                          x + 1

                          - x - y - z + 1
Euclidean volume 1        - z + 1
                          x + 1

                          - y - z + 1
Euclidean volume 1        z + 1
                          x + 1

                          - y - z + 1
Euclidean volume 1        - z + 1
                          x + 1

                          - x + 1
Euclidean volume 1        z + 1
                          x - y - z + 1

                          - x + 1
Euclidean volume 1        - z + 1
                          x - y - z + 1

                          - x - y - z + 1
Euclidean volume 1/2      z + 1
                          x - y - z + 1

                          - x - y - z + 1
Euclidean volume 1/2      - z + 1
                          x - y - z + 1

                          - y - z + 1
```

```
Euclidean volume 1         z + 1
                           x - y - z + 1

                           - y - z + 1
Euclidean volume 1         - z + 1
                           x - y - z + 1
```

Polytope 8_3.mat

```
Number of lattice points: 8
Lattice points:
   -6   -3   -1    0    0    0    1    1
   -3   -1    0    0    1    2    0    1
   -4   -2    0    0    0    0    0    2
---------------------------------------------------------------------
Number of vertices: 4
Vertices:
   -6     0     1     1
   -3     2     0     1
   -4     0     0     2
Indices of vertices:
    1    0    0    0    0    1    1    1
---------------------------------------------------------------------
Number of interior points: 4
Indices of interior points:
    0    1    1    1    1    0    0    0
---------------------------------------------------------------------
There are no other boundary points.
---------------------------------------------------------------------
Number of facets: 4
Facet defining hyperplanes:
   - 8x - 4y + 17z + 8
   14x - 8y - 11z + 16
   - 2x + 14y - 7z + 2
   - 4x - 2y + z + 4
Facet-point distances:
    0    2   16    8    4    0    0   30
    0    4    2   16    8    0   30    0
    0    8    4    2   16   30    0    0
   30   16    8    4    2    0    0    0
Facet-point incidences:
    1    0    0    0    0    1    1    0
    1    0    0    0    0    1    0    1
    1    0    0    0    0    0    1    1
    0    0    0    0    0    1    1    1
---------------------------------------------------------------------
Number of edges: 6
Edge-point incidences:
    1    0    0    0    0    1    0    0
    1    0    0    0    0    0    1    0
    0    0    0    0    0    1    1    0
    1    0    0    0    0    0    0    1
    0    0    0    0    0    1    0    1
    0    0    0    0    0    0    1    1
```

```
------------------------------------------------------------------
f-vector: (4,6,4)
------------------------------------------------------------------
Width: 3
Achieved with respect to the following functionals:
   x - z
   x - y - z
   y - z
Values of the functionals in the points:
   -2   -1   -1    0    0    0    1   -1
    1    0   -1    0   -1   -2    1   -2
    1    1    0    0    1    2    0   -1
------------------------------------------------------------------
Volumes of 4-tuples of points:
   (0,2,4,8,16,30)

Total volume: 30
------------------------------------------------------------------
The polytope IS:
  - Clean
------------------------------------------------------------------
The polytope IS NOT:
  - Boxed
  - Canonical
  - Dps
  - Empty
  - Hollow
  - Normal
  - Normal_minimal
  - Primitive: its sublattice index is 2
  - Quasi-minimal
  - Spiked
  - Terminal
------------------------------------------------------------------
Ehrhart polynomial: 1 + n + n^2 + 5*n^3
h*-vector: (1,4,21,4)
------------------------------------------------------------------
Indices of full-dimensional subpolytopes of size 7:
    1    0    0    0    0    1    1    1
------------------------------------------------------------------
There are no normal subpolytopes of size 7.
------------------------------------------------------------------
There are no essential vertices.
------------------------------------------------------------------
Non-trivial automorphisms:
   ( 2x + y - 4z - 1, x - 2z, 2x - 3z)
   ( - 3x + 2y + 2z - 3, - 2x + y + 2z - 1, - 2x + 2y + z - 2)
   ( - 3y + 2z, x - 2y + 1, - 2y + z)
------------------------------------------------------------------
The polytope is NOT spiked.
------------------------------------------------------------------
The polytope is NOT boxed.
```

Polytope `9_269.mat`

```
Number of lattice points: 9
Lattice points:
   -1   -1    0    0    0    0    0    0    2
   -1    1    0    0    0    0    0    0    1
    0   -5    0    1    2    3    4    5    0
------------------------------------------------------------------
Number of vertices: 4
Vertices:
   -1    -1     0     2
   -1     1     0     1
    0    -5     5     0
Indices of vertices:
    1    1    0    0    0    0    0    1    1
------------------------------------------------------------------
Number of interior points: 5
Indices of interior points:
    0    0    1    1    1    1    1    0    0
------------------------------------------------------------------
There are no other boundary points.
------------------------------------------------------------------
Number of facets: 4
Facet defining hyperplanes:
   15x - 5y - 2z + 10
   - 10x + 15y + 6z + 5
   - 10x + 15y - z + 5
   5x - 25y - 3z + 15
Facet-point distances:
    0    0   10    8    6    4    2    0   35
    0    0    5   11   17   23   29   35    0
    0   35    5    4    3    2    1    0    0
   35    0   15   12    9    6    3    0    0
Facet-point incidences:
    1    1    0    0    0    0    0    1    0
    1    1    0    0    0    0    0    0    1
    1    0    0    0    0    0    0    1    1
    0    1    0    0    0    0    0    1    1
------------------------------------------------------------------
Number of edges: 6
Edge-point incidences:
    1    1    0    0    0    0    0    0    0
    1    0    0    0    0    0    0    1    0
    0    1    0    0    0    0    0    1    0
    1    0    0    0    0    0    0    0    1
    0    1    0    0    0    0    0    0    1
    0    0    0    0    0    0    0    1    1
------------------------------------------------------------------
f-vector: (4,6,4)
------------------------------------------------------------------
Width: 2
Achieved with respect to the following functionals:
   y
Values of the functionals in the points:
   -1    1    0    0    0    0    0    0    1
------------------------------------------------------------------
```

```
Volumes of 4-tuples of points:
   (0,1,2,3,4,5,6,8,9,10,11,12,15,17,23,29,35)

Total volume: 35
--------------------------------------------------------------------
The polytope IS:
  - Clean
  - Primitive
  - Quasi-minimal
  - Spiked
--------------------------------------------------------------------
The polytope IS NOT:
  - Boxed
  - Canonical
  - Dps
  - Empty
  - Hollow
  - Normal
  - Normal_minimal
  - Terminal
--------------------------------------------------------------------
Ehrhart polynomial: 1 + (7/6)*n + n^2 + (35/6)*n^3
h*-vector: (1,5,24,5)
--------------------------------------------------------------------
Indices of full-dimensional subpolytopes of size 8:
    1    1    0    0    0    0    0    1    1
--------------------------------------------------------------------
Indices of normal subpolytopes of size 8:
    0    1    0    0    0    0    0    0    0
--------------------------------------------------------------------
Indices of essential vertices:
    1    1    0    0    0    0    0    0    1
--------------------------------------------------------------------
There are no non-trivial automorphisms.
--------------------------------------------------------------------
The polytope is spiked in the following directions:
    0    0    1
--------------------------------------------------------------------
The polytope is NOT boxed.
```

Polytope 10_454.mat

```
Number of lattice points: 10
Lattice points:
   -1    0    0    0    1    1    1    1    1    1
    0    0    0    1    0    0    0    1    1    2
    0    0    1    0    0    1    2    0    1    0
--------------------------------------------------------------------
Number of vertices: 4
Vertices:
   -1     1     1     1
    0     0     0     2
    0     0     2     0
Indices of vertices:
```

```
    1    0    0    0    1    0    1    0    0    1
------------------------------------------------------------------
There are no interior points.
------------------------------------------------------------------
Indices of other boundary points:
    0    1    1    1    0    1    0    1    1    0
------------------------------------------------------------------
Number of facets: 4
Facet defining hyperplanes:
   y
   z
   x - y - z + 1
   - x + 1
Facet-point distances:
    0    0    0    1    0    0    0    1    1    2
    0    0    1    0    0    1    2    0    1    0
    0    1    0    0    2    1    0    1    0    0
    2    1    1    1    0    0    0    0    0    0
Facet-point incidences:
    1    1    1    0    1    1    1    0    0    0
    1    1    0    1    1    0    0    1    0    1
    1    0    1    1    0    0    1    0    1    1
    0    0    0    0    1    1    1    1    1    1
------------------------------------------------------------------
Number of edges: 6
Edge-point incidences:
    1    1    0    0    1    0    0    0    0    0
    1    0    1    0    0    0    1    0    0    0
    0    0    0    0    1    1    1    0    0    0
    1    0    0    1    0    0    0    0    0    1
    0    0    0    0    1    0    0    1    0    1
    0    0    0    0    0    0    1    0    1    1
------------------------------------------------------------------
f-vector: (4,6,4)
------------------------------------------------------------------
Width: 2
Achieved with respect to the following functionals:
   x
   x
   x - z
   x - y
   x - y - z
   y + z
   y
   z
Values of the functionals in the points:
   -1    0    0    0    1    1    1    1    1    1
   -1    0    0    0    1    1    1    1    1    1
   -1    0   -1    0    1    0   -1    1    0    1
   -1    0    0   -1    1    1    1    0    0   -1
   -1    0   -1   -1    1    0   -1    0   -1   -1
    0    0    1    1    0    1    2    1    2    2
    0    0    0    1    0    0    0    1    1    2
    0    0    1    0    0    1    2    0    1    0
------------------------------------------------------------------
Volumes of 4-tuples of points:
   (0,1,2,4,8)
```

```
Total volume: 8
---------------------------------------------------------------------
The polytope IS:
  - Boxed
  - Hollow
  - Normal
  - Primitive
  - Quasi-minimal
---------------------------------------------------------------------
The polytope IS NOT:
  - Canonical
  - Clean
  - Dps
  - Empty
  - Normal_minimal
  - Spiked
  - Terminal
---------------------------------------------------------------------
Ehrhart polynomial: 1 + (11/3)*n + 4*n^2 + (4/3)*n^3
h*-vector: (1,6,1,0)
---------------------------------------------------------------------
Indices of full-dimensional subpolytopes of size 9:
    1    0    0    0    1    0    1    0    0    1
---------------------------------------------------------------------
Indices of normal subpolytopes of size 9:
    1    0    0    0    1    0    1    0    0    1
---------------------------------------------------------------------
Indices of essential vertices:
    1    0    0    0    1    0    1    0    0    1
---------------------------------------------------------------------
Non-trivial automorphisms:
   ( x, z, y)
   ( x, y, x - y - z + 1)
   ( x, z, x - y - z + 1)
   ( x, x - y - z + 1, z)
   ( x, x - y - z + 1, y)
   ( - x + y + z, y, z)
   ( - x + y + z, z, y)
   ( - z + 1, y, x - y - z + 1)
   ( - y + 1, z, x - y - z + 1)
   ( - y + 1, x - y - z + 1, z)
   ( - z + 1, x - y - z + 1, y)
   ( - z + 1, y, - x + 1)
   ( - y + 1, z, - x + 1)
   ( - x + y + z, y, - x + 1)
   ( - x + y + z, z, - x + 1)
   ( - z + 1, x - y - z + 1, - x + 1)
   ( - y + 1, x - y - z + 1, - x + 1)
   ( - y + 1, - x + 1, z)
   ( - z + 1, - x + 1, y)
   ( - y + 1, - x + 1, x - y - z + 1)
   ( - z + 1, - x + 1, x - y - z + 1)
   ( - x + y + z, - x + 1, z)
   ( - x + y + z, - x + 1, y)
---------------------------------------------------------------------
The polytope is NOT spiked.
```

```
--------------------------------------------------------------------
The polytope is boxed with respect to the following parallelepipeds:
                           - x + 1
Euclidean volume 1         x - y - z + 1
                           z

                           - x + 1
Euclidean volume 1         x - y - z + 1
                           y

                           - x + 1
Euclidean volume 1         z
                           y

                           x - y - z + 1
Euclidean volume 1         z
                           y
```

List of algorithms

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