Understanding, Evaluating and Selecting Voting Rules Through Games and Axioms

Danilo Santa Cruz Coelho

Advisor: Dr. Salvador Barberà

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Para *Ana e meus pais.*

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Chapter 1

Introduction

This dissertation provides a contribution to social choice theory. Broadly speaking, the main subject of this branch of economic theory is the process of collective decision making.

This dissertation is composed of four chapters. Each one of them can be read independently of the others. This introduction is the first one and describes briefly the motivation and the contents of the following chapters.

The theoretical analysis of Chapter 2 is motivated by the debate on the principles that guided the voting reform in the European Union Council of Ministers following the arrival of new members. A principle that is written in many official documents states that the voting reform should preserve the dual nature of the Union which is both a Union of States and a Union of People. Another declared propose of the reform is to extend the use of majority voting to areas currently subject to unanimity (see Baldwin, Berglof and Giavazzi, 2001).

In this debate, inter-governmentalism and federalism are two different point of views about the future of the European Union. Romanno Prodi, a declared federalist and former President of the European Commission, compared the European Union to a train in motion and the unanimity rule as an impediment to its speed. In contrast, the supporters of inter-governmentalism, such as the UK's Conservative Party, say that because governments can veto decisions in any area where they are specially sensitive, no EU country is forced to accept a change from the status quo that it simply finds terrible.[1]

I analyze the case of a society or a committee facing the choice of a voting rule, which will then be used in a sequence of decisions involving the rejection or the adoption of proposed changes to the status quo.

In the analysis, I use a standard probabilistic voting model which was proposed by Douglas Rae in 1969. This model can be described as follows: (1) A voting rule is characterized by the minimum number of votes needed to approve a proposal of change from the status quo; (2) a voter is characterized by a probability that he will support a proposal, and this probability is common knowledge; (3) each voter's probability distribution of supporting a proposal is independent of any other voter's probability distribution; (4) each voter casts only one vote, and he gets utility of value equal to 1 if his preferred alternative is chosen in the vote, and utility of value 0 otherwise.

Given these assumptions, the voters are able to assign an expected utility to each different voting rule. Earlier studies on this model are, among others, Rae (1969), Curtis (1972), Badger (1972) and Barberà and Jackson (2004). Rae (1969) and Curtis (1972) point out the supremacy of simple majority over alternative voting rules by showing that simple majority always maximizes the sum of the voters' expected utilities. Badger (1972) proves the existence of a Condorcet winner voting rule, i.e. a voting rule that is never defeated by any other alternative rule on the basis of simple majority rule. Barberà and Jackson (2004) investigate self stable voting rules, i.e. rules that cannot be defeated by any other alternative rule when the choice between the rules is based on these same rules.

In contrast to Barberà and Jackson (2004) and Badger (1972), who concentrate on the voting rules that might be chosen if voters themselves vote on rules, I take a

[1] See "Don't call me greedy" in The Economist, 12/08/2001 Issue 8251, page 50.

normative point of view, and characterize the voting rules that satisfy Rawls' maximin criterion. That is, those rules that maximizes the expected utility of the voter who is worst off in the committee. This characterization provides a benchmark against which the utilitarian choice of voting rules, which Rae (1969) and Curtis (1972) showed leads to the choice of simple majority rule, can be compared.

In Chapter 2, I restrict my attention to voting rules that are specialized in selecting one alternative from a set of two alternatives (adopting or not adopting a proposal of change from the status quo). This is not the case of the voting rules under analysis in Chapters 3 and 4. The system that I study in Chapter 3 selects one alternative from a set of several alternatives, while those studied in Chapter 4 select a fixed number of alternatives from a set of several alternatives.

Chapters 3 and 4 are motivated by the ongoing debate in Brazil about whether or not the system used to elect rectors of public universities should be changed. The system under scrutiny is the rule of three names, known in Brazil as "sistema por lista tríplice". In Brazil as well as in many Latin American countries, this nomination/decision system is also used to designate the members of several decision bodies of the Judiciary system. In Spain, it is called "La terna" and was largely used during Franco's regime. In fact, this system is still used in Spain in the Judiciary system.

Eventually, despite of its widespread use by many institutions around world, we realized that the theoretical literature about this system was scarce. Moreover, most of the examples that we have found come from countries with a Christian tradition.

Later, we found, in the Catholic encyclopedia, a explanation for this concentration in these countries: This system has been used for more than fifteen centuries by the Church to elect bishops. Nowadays, the Pope chooses one of the candidates in the list with three names proposed by the members of the ecclesiastical province. In Ireland, each canon of the cathedral and parish priest can cast a vote for three candidates to bishop. In England, the canons vote three times, and select each time the most voted candidate. In both cases, the list is made with the three most voted candidates (see

Catholic Encyclopedia). We have found a diversity of voting procedures been used to choose the three names to make up the list. These types of voting procedures are interesting per se, Chapter 4 is devoted specifically to study them.

We have found examples where the procedure requires a list with more than three names. Hence, we shall from now on refer to the "rule of k names". The rule of k names can be formally described as follows: given a set of candidates for office, a committee chooses k members from this set by voting, and makes a list with their names. Then a single individual from outside the committee selects one of the listed names for the office.

In Chapter 3, after providing examples of the widespread use of the rule of k names, a game theoretical analysis of this rule is provided. We concentrate on the plausible outcomes induced by the rule of k names when the agents involved act strategically and cooperatively.

In this game theoretical analysis, we adopt the cooperative approach introduced by Aumann (1959). In this approach, the cooperative strategic behaviour problem is modelled using the strategic form framework of non-cooperative game theory. The strategic form of a game is the specification of the set of players, players' strategy sets and payoffs. The game solution concept that we adopt is the Strong Nash Equilibrium also proposed by Aumann (1959). As can be inferred by its name, it is an adaptation of a non-cooperative equilibrium proposed by Nash (1951).

Two strategic form games that can be induced by the rule of k names under somewhat different scenarios are proposed. These scenarios differ in the ability of the chooser, who appoints the winning candidate, to make credible threats in order to influence the choice of the k names by the committee members. In the most favorable scenario for the chooser, he can credibly declare in advance who is the winning candidate for every list that can be proposed by the committee.

We characterize the sets of all strong equilibrium outcomes of each of the games. This task was facilitated by Sertel and Sanver's (2004) characterization of the set of strong Nash equilibrium outcomes of voting games. They consider a standard voting game where a committee elects a candidate for office without any external interference.

We use extensively the concept of effectivity functions associated with a game form. This type of functions expresses the strategic possibilities for every admissible coalition of players. In other words, it expresses which outcomes can be achieved by a coalition given the rules of the game[2].

These characterizations enable us to examine the consequences of changing the parameter k, of adding undesirable candidates and of replacing a majoritarian screening rule by non-majoritarian screening rule. By knowing these consequences, we are able to infer what might be the preferences of the chooser over different variants of the rule of k names.

An important part of a rule of k names is given by the screening rule used by the committee in order to select, or screen out, those k candidates to be presented to the chooser. In Chapter 4, we investigate whether or not the six screening rules documented in the third chapter satisfy stability. The definition of stability is based on a specific definition of weak Condorcet sets proposed by Gehrlein (1985). A set with k candidates is a weak Condorcet set if no candidate in the set can be defeated by any other candidate outside the set on the basis of simple majority rule. A screening rule for k names is said to be stable if it always selects a weak Condorcet set, whenever such a set exists.

After discussing the scenarios where stability can be considered to be a desirable property, we show that all these six screening rules violate it. We show, however, that it is not difficult to create stable screening rules. Here we propose two stable screening rules. We also proves two results which hint at possible reasons why there is a widespread use of unstable screening rules. The first result states that the stability property is incompatible with other equally desirable properties. The second one

[2] Two excellent books on effectivity functions are Abdou and Keiding (1991) and Peleg (1984).

states that many unstable screening rules tend to become stable if the voters act strategically and cooperatively.

1.1 References

Abdou, J. and Keiding, H. (1991) Effectivity Functions in Social Choice. Kluwer Academic Publishers, The Netherlands.

Aumann, R. J. (1959) Accetable points in general cooperative N-person games. In: R.D. Luce and A.W. Turcker (eds.) Contributions to the Theory of Games IV, Annals of Mathematical Studies 40, Princeton University Press, 287-324.

Badger, W.W. (1972) Political individualism, positional preferences and optimal decision-rules. In: R.G. Niemi, H.F. Weisberg (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.

Baldwin, R., Berglof, E. and Giavazzi, F. (2001) Nice Try: Should the Treaty of Nice be Ratified. Monitoring European Integration 11, Centre for Economic Policy Research, London.

Barberà, S. and Jackson, M. (2004) Choosing how to choose: Self-stable majority rules and constitutions. Quarterly Journal of Economics, forthcoming in 119(3).

Catholic Encyclopedia, ed New Advent.
[Internet]. In: http://www.newadvent.org/cathen/02581b.htm.

Curtis, R. (1972) Decision rules collective values in constitutional choice. In: R.G. Niemi, H.F. Weisberg (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.

Gehrlein, W. (1985) The Condorcet criterion and committee selection. Mathematical Social Sciences 10:199-209.

Nash, J. (1951) Non-cooperative equilibria. Econometrica, 21:129-140.

Peleg, B. (1984) Game Theoretic Analysis of Voting in Committees. Econometric Society Monographs in Pure Theory, Cambridge University Press, Cambridge.

Rae, D. (1969) Decision rules and individual values in constitutional choice. American Political Science Review 63: 40-56.

Sertel, M. R., and Sanver M. R. (2004) Strong equilibrium outcomes of voting games are the generalized Condorcet winners. Social Choice and Welfare 22: 331-347.

Chapter 2

Maximin Choice of Voting Rules for Committees

2.1 Introduction

Members of a committee, with conflicting interests, need to choose a threshold voting rule to be used in a series of dichotomous choices involving rejection or adoption of proposed changes from the status quo. A threshold voting rule is the minimum number of votes needed to approve a proposal of change from the status quo. There is uncertainty about the future proposals that will be voted on. However, voters are able to form expectations about the proposals and about the behavior of other members. Based on these expectations, they form preferences over (threshold) voting rules that best satisfy their own interests. The debate about which voting system should be adopted by the European Union Council of Ministers, following the arrival of new members, is an example of where this theoretical framework could be applied.

In this context voters might have different preferences over voting rules. We explore the idea that voters tend to agree on rules that generate patterns of outcomes that are considered reasonable by all of them. In other words, voting rules are chosen according to principles such as fairness or Pareto efficiency.

We analyze the endogenous choice of voting rules in a probabilistic voting model first proposed by Rae (1969). We consider the choice of voting rules according to the Rawls' maximin criterion, with the suggestion that fairness considerations may recommend the choice of a rule that maximizes the expected utility of the worst off voter.

In this model, each voter is characterized by a probability of being in favor of the status quo and the action each voter takes on any given proposal is completely independent of the action taken by others. Each voter casts only one vote and he gets utility 1 if his preferred alternative is chosen in the vote, and utility 0 otherwise. Given this setup, the voter's expected utility over voting rules is the frequency, generated by the rule, with which his opinion about proposed changes of the status quo coincides with the decisions taken by the committee.

As a part of our analysis of voting rules that satisfies the maximin criterion, we investigate the characteristics of the worst-off voters in a committee. We show that there is an endogenous threshold such that for any voting rule lower than it (i.e. any voting rule that requires less votes to adopt a proposal than it), the worst-off voter is the most conservative among the members of the committee. For voting rules higher than this threshold the reverse holds, i.e. the worst off voter is the most radical voter.

After investigating the relationship between the distribution of well being across voters and voting rules, we are able to prove that there are at most two maximin voting rules, at least one is Pareto efficient and is often different to the simple majority rule. If a committee is formed only by "conservative voters" (i.e. voters who are more likely to prefer the status quo to a change) then the maximin criterion recommends voting rules that require not more votes than the simple majority rule. If there are only "radical voters", then this criterion recommends voting rules that are not lower than half of the total number of voters.

Early proponents of this model concentrated on the utilitarian perspective, which always recommends the choice of the simple majority rule (Rae, 1969 and Curtis, 1972). In a similar model, Guttman (1998) uses Harsanyi's construction of veil of ignorance to justify the use of this criterion in the choice of voting rules. He assumes that each voter would participate in the choice of the voting rule with no knowledge whatsoever of whether his expected utility over voting rules will be that of voter 1, voter 2, etc. There is equal probability of being any particular voter. Under this assumption, Guttman proves that the optimal voting rule for any voter is the one that maximizes the sum of voters' expected utilities and this is the simple majority rule in our context. See Buchanan, 1998; Tullock, 1998 and Arrow, 1998 for further discussion on this issue.

By contrast, Badger (1972) and Barberà and Jackson (2004) have discussed the choice of voting rules by means of a vote. Badger (1972) shows that the Condorcet winner always exists proving that voters' preferences over voting rules are single peaked. Barberà and Jackson (2004) claim that a voting rule is likely to persist in a group if it cannot be defeated by any other alternative rule when the choice between the rules is based on this same rule. They call this property self stability. They argue that the possible lack of self-stable rules among voting rules could be an explanation why most states' constitutions require super majorities in order to change the voting system used for day-to-day decisions.[1]

Other authors use different lines of argument to defend the choice of some voting rule over another.[2] The rules used by society to make constitutional choices are called meta rules by Brennan and Buchanan (1985). They argue that any rule is " just " if it is chosen by a society using an agreed meta rule. Moreover, they claim that the use of unanimity criterion to choose between rules is feasible because consensus about the choice of rules can be easily reached: "... the uncertainty introduced in any choice among rules or institutions serves the salutary function of making potential agreement more rather than less likely...."(Brennan and Buchanan, 1985, page 29).

[1] Wakayama (2003) extended Barberà and Jackson's (2004) analysis adding the possibility of voters' abstention.

[2] See Grofman (1979) and Esteban and Ray (2001).

The outline of this chapter is as follows: In Section 2.2, we describe the model. In Section 2.3, we present some known properties of individual preferences over voting rules, proved in Badger (1972), Barberà and Jackson (2004) and Rae (1969), this serves as an introduction to our results. Our characterization of voting rules that satisfy the maximin criteria are presented in Section 2.4. Finally, in Section 2.5, we close with some final remarks.

2.2 The model

Let us represent the set of voters by $N = \{1, ..., n\}$. We shall assume that N is finite and $n \geq 3$. The voters, with conflicting interests, need to choose the voting rule to be used in a series of dichotomous choices involving the rejection or the adoption of proposed changes from the status quo. Each voter casts a vote in {yes, no}. Voting for "yes" is interpreted as being in favor of the proposed change. Voting for "no" is interpreted as being against the change. A voting rule is a number $s \in \{1, ..., n\}$. Given a voting rule s, the proposed change is adopted if there are at least s voters in favor of it.

The voters have expectations over future issues that will be voted on, but do not know their exact realization. The voters are simply characterized by a parameter $p_i \in (0, 1)$. This represents the probability that they will support change at the time of the vote. The realizations of voters' support for the alternatives are independent. Badger (1972) offers a convincing justification for this assumption:

> "We shall also make the admittedly highly unrealistic assumption that the action each legislator takes on any given proposal is completely independent of the action taken by others. This eliminates the consideration of factional disputes, logrolling, and the entire gamut of political and historical dynamics which are basic to the evolution of any real legislative

> structure. But then we shall not attempt to analyze such structures. By eliminating "interactive" political dynamics entirely, we hope to get a much narrower yet somewhat clearer view of the relationship between an individual legislative will and optimal collective policy."(Badger, 1972, page 35).

A voter gets utility 1 if his preferred alternative is chosen in the vote, and utility 0 otherwise. Henceforth, a committee is a set of voters $N = \{1, ..., n\}$ associated with a vector $p = (p_1, ..., p_n)$.

For any $m \in \{1, ..., n-1\}$, let $P_i(m)$ denote the probability that exactly m individuals in $N \setminus \{i\}$ support change:

$$P_i(m) = \sum_{\mathrm{B} \subset \mathrm{N} \setminus \{i\}:|B|=m} \times_{j \in B} p_j \times_{l \notin B} (1 - p_l). \tag{2.1}$$

Let $U_i(s)$ be the expected utility of voter i when voting rule s is used. This is expressed as follows:

$$U_i(s) = p_i \sum_{m=s-1}^{n-1} P_i(m) + (1 - p_i) \sum_{m=0}^{s-1} P_i(m). \tag{2.2}$$

In the right hand side of expression (2.2) above, the first term is the probability, under voting rule s, of a proposal of change being accepted when i supports it. The second term is the probability, under voting rule s, of a proposal being rejected when i opposes it. Thus $U_i(s)$ can be interpreted as the frequency, generated by the rule s, with which voter i expects to support a proposal and have it adopted and to oppose a proposal and have it defeated.

Notice that if we move from s to s' $(s' > s)$ then the first term of the right hand side decreases while the second term increases. Whether or not i's expected utility will increases with the movement from s to s', will depend on the intensity of these two effects.

2.3 Choosing how to choose: Utilitarianism, Condorcet winner and self-stability

Rae (1969), the proposer of this model, considers only homogeneous committees, i.e. $p_i = p_j$ for every $i, j \in N$. It is easy to see from expression (2.2) that for any $i, j \in N$, if $p_i = p_j$ then $U_i(s) = U_j(s)$ for every $s \in \{1, ..n\}$. So, in any homogeneous committees, all the voters have the same expected utilities over voting rules. Let the simple majority rule be referred as s^{maj} and defined as $s^{maj} \equiv \frac{(n+1)}{2}$ if n is odd and $\frac{n}{2} + 1$ if n is even. Rae (1969) shows, for any homogeneous committee, that for any value of the parameters p, the voters' preferred voting rule is s^{maj} if n is odd and s^{maj} and $s = \frac{n}{2}$, if n is even. This result is due two facts: The first one is that only under voting rules s^{maj} and $s = \frac{n}{2}$, any collective decision is never taken in disagreement with the majority of voters. The second one is that, in a homogeneous committee, for any voter $i \in N$ the probability that voter *i's* opinion about a proposal coincides with the majoritarian one is higher than fifty percent.

Notice that this last fact would not hold if voter $i \in N$ had a very small parameter p compared with the other voters. For example, consider a committee represented $N = \{1, 2, 3\}$ with $p_1 = 0.1$ and $p_2 = p_3 = 0.90$. It follows that the probability of voter *1's* opinion about a proposal coincides with the majoritarian one is equal to 0.27.

Curtis (1972) considers heterogeneous committees. He generalizes Rae's (1969) result by showing that even in heterogeneous committee, the only rules that maximizes the sum of voters' expected utilities (i.e., that satisfies the utilitarian criterion) is s^{maj} if n is odd and s^{maj} and $s = \frac{n}{2}$ if n is even (see for example Muller, 1989, page 100). The intuition of this result is that only simple majority rule (and $s = \frac{n}{2}$ if n is even) maximizes the probability that a decision taken by the committee is supported by the majority of the voters[3].

[3] Schofield (1972) points out that the marginal advantage of simple majority over any other voting

If the committee is heterogeneous, i.e. not homogeneous, the distribution of well being across voters in terms of expected utilities may depend drastically on the voting rule. So, consensus over the choice of a voting rule to be adopted by the committee may be difficult if the voters are not utilitarians or if utilities are not transferable. Example 1 below illustrates this point.

Example 1. *Let* $N = \{1, 2, 3, 4, 5, 6, 7\}$, $p_1 = 0.9$, $p_2 = 0.8, p_3 = 0.7$, $p_4 = 0.6$, $p_5 = 0.5$, $p_6 = 0.4$ *and* $p_7 = 0.1$ *be a representation of a committee. Knowing the parameters p's for each voter, expression (2.2) can be applied to compute the voters expected utility generated by each voting rule. The voters' expected utilities over voting rules are illustrated in Figure 1 below.*[4]

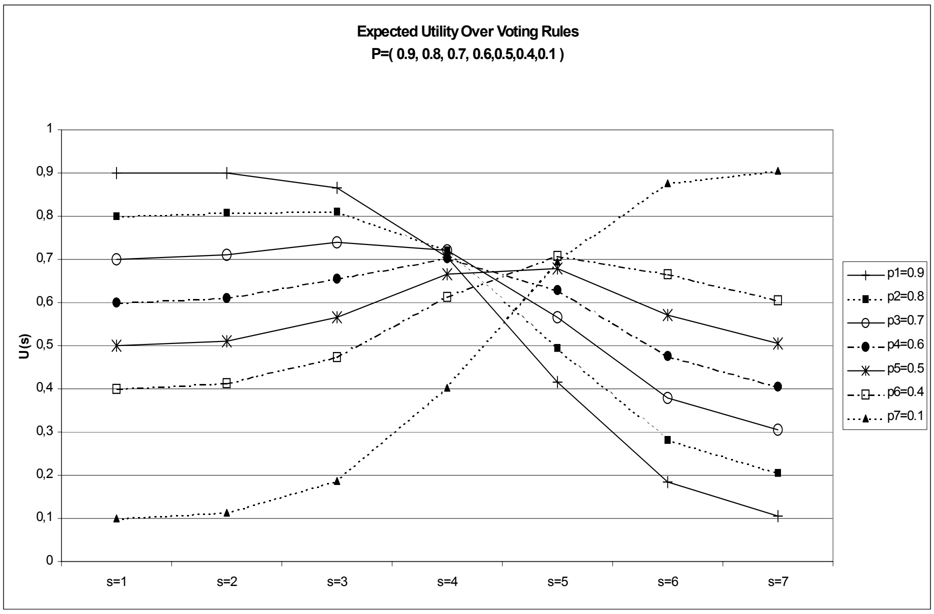

Figure 1

Badger (1972) shows that there exist at most two voting rules that maximize voter i's expected utility. In the case where s' and s" both maximize $U_i(\cdot)$ and $s' < s$", he

rule does become vanishingly small as the size of the committee increases.

[4]The author has written a program in Matlab that computes the voters' expected utilities over voting rules. This program is available upon request.

proves that s' and s" are adjacent, i.e. $s' = s" - 1$.

Let $\widehat{s}_i$ denote the peak for voter i, i.e. the voting rule that maximizes the voter i's expected utility. If s^* and $s^* - 1$ are both peaks of voter i then $\widehat{s}_i = s^*$ and we say that s^* is a twin-peak. Examining Figure 1, we can see that $s^{maj} = 4$, $\widehat{s}_1 = 2$, $\widehat{s}_2 = 3$, $\widehat{s}_3 = 3$, $\widehat{s}_4 = 4$, $\widehat{s}_5 = 5$, $\widehat{s}_6 = 5$ and $\widehat{s}_7 = 7$. For any $m \in \{1, ..., n\}$, let us denote by $q_i(m)$ the conditional probability that voter i supports a proposal given that the number of voters that support it is exactly equal to m. And let $Z(m)$ be the probability that exactly m voters in N support a proposal. Notice that $q_i(m) = \frac{p_i P_i(m-1)}{Z(m)}$ and $1 - q_i(m) = \frac{(1-p_i)P_i(m-1)}{Z(m)}$. After some algebraic manipulations of expression (2.2), it can be shown that the difference between $U_i(s+1)$ and $U_i(s)$ can be expressed as follows:

$$U_i(s+1) - U_i(s) = (1 - 2q_i(s))Z(s) \text{ for every } s \in \{1, ..., n-1\} \tag{2.3}$$

The assumption of independence of voters support implies that $Z(m) > 0$ for any $m \in \{1, ..., n\}$ and $0 < q_i(1) < q_i(m-1) < q_i(m) < q_i(n) = 1$ for any $m \in \{3, ..., n-1\}$. Hence with the help of expression (2.3) it can be easily proved that the peak for voter i, $\widehat{s}_i$, can be characterized as follows: $\widehat{s}_i$ is the largest $s' \in \{1, ..., n\}$ such that $q_i(s') \geq 1/2$ and $q_i(s) \leq 1/2$ for any $s < s'$.[5] Moreover if $q_i(\widehat{s}_i) = 1/2$ then $\widehat{s}_i$ is a twin-peak.

As can be verified in Figure 1, the expected utility of any voter $i \in N$ is strictly increasing in $\{1, ..., \widehat{s}_i - 1\}$ and strictly decreasing in $\{\widehat{s}_i, ..., 7\}$. Badger (1972) proves that this is a regularity of this model. Thus, following the literature, Badger (1972) proves that, for any committee, the voters' preferences over voting rules belong to the domain of single-plateaued preferences. However, Badger (1972) and Barberà and Jackson (2004) adopted the term single-peaked preferences since, in this model, indifferences can occur only between two adjacent rules on top and happens non-generically (in p).

[5] This characterization was first proposed by Barberà and Jackson (2004).

Again in Figure 1, notice that $p_1 \geq ... \geq p_7$ and $\widehat{s}_1 \leq \widehat{s}_2 \leq ... \leq \widehat{s}_7$. That is, if voter i expects to support proposals more often than voter j, then voter i's peak $\widehat{s}_i$ cannot be larger than $\widehat{s}_j$. This is a very intuitive property of this model pointed out by Barberà and Jackson (2004). Writing it formally, for any committee (N, p) and any $i, j \in N$ we have that $\widehat{s}_j \geq \widehat{s}_i$ whenever $p_i \geq p_j$. They also show that for any committee there is a pair of voters where one of them has a peak smaller than or equal to the simple majority rule and the other has a peak higher than or equal to it. That is, for any committee (N, p), there exist $i, j \in N$ such that $\widehat{s}_j \geq s^{maj} \geq \widehat{s}_i$. In Example 1, $s^{maj} = 4$ and $2 = \widehat{s}_1 \geq s^{maj} \geq \widehat{s}_7 = 7$.

Take the voters characterized by the highest p (smallest p) in the committee and select only one of them to be referred as *voter R* (*voter C*). Thus R (C) is the voter that has the highest probability of supporting (reject) a proposal of change at the time of the vote. Let $\widehat{s}_R$ ($\widehat{s}_C$) denote the peak for *voter R* (*voter C*). Thus, in Example 1, voter 1 with $p_1 = 0.9$ is *voter R* since he has the highest p, while voter 7 with $p_7 = 0.1$ is *voter C*. A direct corollary of the properties proved by Barberà and Jackson (2004), is that for any committee $\widehat{s}_C \geq s^{maj} \geq \widehat{s}_R$.

Badger (1972) studies the set of weak Condorcet winner voting rules. A voting rule $s \in \{1, ..., n\}$ is a weak Condorcet winner if $|\{i \in N | U_i(s') > U_i(s)\}| < s^{maj}$ for every $s' \in \{1, ..., n\} \backslash \{s\}$. Such voting rules are particular interesting in situations where voting rules are chosen on the basis of simple majority rule. The single peakedness property guarantees that the set of weak Condorcet winner rules is never empty. In Example 1, there is only one weak Condorcet winner rule and it is the simple majority rule.[6]

Barberà and Jackson (2004) also analyse the case where voting rules are chosen by voting. They say that a voting rule s is self-stable if $|\{i \in N | U_i(s') > U_i(s)\}| < s$ for every $s' \in \{1, ..., n\} \backslash \{s\}$. That is, a rule $s \in \{1, ..., n\}$ is self stable if it cannot be defeated by any other rule when s is used to choose between rules. They argue

[6]This is only a coincidence.

that a self stable rule tends to prevail in a committee. They provide examples of committees where self-stable rules do not exist.[7] In Example 1, $s = 4$ is a self stable voting rule since there is no rule $s' \in \{1, 2, 3, 5, 6, 7\}$ such that the number of voters that prefer s' to $s = 4$ is lower than four. Notice also that $s = 6$ is not self stable since there are six voters that prefer $s' = 5$ to $s = 6$. For that committee the set of self stable rules is $\{4, 5, 7\}$.

2.4 Choosing voting rules according to the maximin criterion

We follow the approach adopted by Rae (1969) and Curtis (1972), where voting rules are chosen according to a criterion.

2.4.1 The maximin criterion

We consider the possibility of choosing among voting rules according to the maximin criterion which requires the choice of a rule that maximizes the expected utility of the worst off voter on the basis of fairness.

Definition 1. *A voting rule $s \in \{1, ..., n\}$ satisfies the maximin criterion if $Min\{U_1(s), ..., U_n(s)\} \geq Min\{U_1(s'), ..., U_n(s')\}$ for every $s' \in \{1, ..., n\}$. We denote by S_{Rawls} the set of voting rules that satisfy the maximin criterion in a committee.*

It follows that, in Example 1, $S_{Rawls} = \{5\}$ so for that committee, the recommendation of the maximin and the utilitarian principles do not coincide.

Our aim is to provide a characterization of the voting rules that satisfy the maximin criterion as a function of the distribution of voters' probabilities to favor change

[7] They also show that in any dichotomous committee there exists at least one self-stable rule. A committee (N, p) is dichotomous if for every $i \in N_1 \neq \{\emptyset\}$ and $j \in N_2 \neq \{\emptyset\}$ we have that $p_i = p_1$ and $p_j = p_2$ such that $N = N_1 \cup N_2$, $N_1 \cap N_2 = \{\emptyset\}$ and $p_1 \neq p_2$.

from the status quo. It is easy to see that if a committee is homogeneous then maximin and utilitarian criteria give the same recommendation. The reason is that expression (2.2) implies that: if $p_i = p_j$ then $U_i(s) = U_j(s)$ for any $s \in \{1, ..., n\}$.

Proposition 1. *For any homogeneous committee, if n is odd then* $S_{Rawls} = \left\{s^{maj}\right\}$ *otherwise* $S_{Rawls} = \left\{\frac{n}{2}, s^{maj}\right\}$.

The proof is trivial since it is a direct consequence of the utilitarian characterization of voting rules proved by Rae (1969).

Before presenting our characterization result for a heterogeneous committee, we first need to study how the distribution of expected utilities across voters change with voting rules.

2.4.2 Distribution of voters' expected utility over voting rules

It is straightforward to see in Example 1, that $Min\{U_1(s), U_2(s), ..., U_7(s)\} = U_7(s)$ for any $s < 5$ and $Min\{U_1(s), U_2(s), ..., U_7(s)\} = U_1(s)$ for any $s \geq 5$. In this subsection we will show that it is a regularity of this model. More specifically, we will prove that in our model voters' preferences over voting rules satisfy the *strict single-crossing* and *p-monotonic strict single-crossing* properties. These two properties together imply the existence of a voting rule, denoted by $s_{R,C}$, that will play a important role in our analysis. As will be shown, for any rule lower than $s_{R,C}$, the worst off agent is the *voter* C, and for any voting rule larger or equal than it, *voter* R is the worst off agent. Thus in Example 1, we have that $s_{R,C} = 5$ since $Min\{U_1(s), U_2(s), ..., U_7(s)\} = U_C(s)$ for any $s < 5$ and $Min\{U_1(s), U_2(s), ..., U_7(s)\} = U_R(s)$ for any $s \geq 5$.

Definition 2. *We say that a committee has preferences over voting rules that satisfy the strict single-crossing property, if for any pair of voters* $i, j \in N$, *with* $p_i > p_j$, *there is a threshold* $s_{i,j} \in \{2, ..., n\}$ *such that: (1)* $U_i(s) > U_j(s)$ *for any* $s < s_{i,j}$, *(2)* $U_i(s) \leq U_j(s)$ *for* $s = s_{i,j}$ *and (3)* $U_i(s) < U_j(s)$ *for any* $s > s_{i,j}$.

Putting it differently, a committee has preferences over voting rules that satisfy the *strict single-crossing property,* if for any pair of voters $i, j \in N$, with $p_i > p_j$, there is a threshold $s_{i,j} \in \{2, ..., n\}$ such that, for any voting rule larger or equal to it, the one that rejects proposals more often has higher utility than the other, and for any voting rule lower than it, the reverse holds.

Proposition 2. *Every committee has preferences over voting rules that satisfy the strict single-crossing property.*

The proof of Proposition 2 is in the Appendix. In Example 1, $s_{1,7} = 5$. Notice that $U_1(s) > U_7(s)$ for any $s \in \{1, 2, 3, 4\}$ and $U_1(s) < U_7(s)$ for any $s \in \{5, 6, 7\}$.

In order to give an intuition about why Proposition 2 holds we need the following definition: For any $m \in \{2, ..., n\}$ and $i, j \in N$, let denote by $G_{i,j}(m)$ be the probability that there are no more than $m - 2$ voters, other than i and j, that support change. Let $G_{i,j}(1) \equiv 0$.

After some algebraic manipulations of expression (2.2), it can be shown that the difference between $U_i(s)$ and $U_j(s)$ can be expressed as follows:[8]

$$U_i(s) - U_j(s) = (p_i - p_j)(1 - 2G_{i,j}(s)) \ for \ any \ s \in \{2, ..., n\} \tag{2.4}$$

Recall that $s_{i,j}$ is only defined for heterogeneous committees. Its existence is guaranteed because $G_{i,j}(n) = 1, G_{i,j}(1) = 0$ and $G_{i,j}(\cdot)$ is a strictly increasing.

The intuition behind Proposition 2 is that $G_{i,j}(s) > \frac{1}{2}$ means that under the voting rule s if voter i or voter j do not support a proposal the probability of the proposal being rejected is higher than fifty percent. Thus voters i and j are decisive under voting rule s. Moreover, since $p_i > p_j$, the probability that voter i supports change and voter j does not is higher than the reverse since $p_i(1 - p_j) > (1 - p_i)p_j$. This means that voter j is more decisive that voter i under voting rule s. Therefore, under this rule, voter j will have higher expected utility than voter i. Not only for s,

[8]Lemma 1 in the Appendix shows how to reach expression (2.4) from (2.2).

but for any $s' \geq s$ since $G_{i,j}(s)$ is strictly increasing. When $G_{i,j}(s) < \frac{1}{2}$, the situation is reversed and then voter i has higher expected utility than voter j. Therefore $s_{i,j}$ can be characterized as the largest $s' \in \{1, ..., n\}$ such that $G_{i,j}(s') \geq 1/2$ and $G_{i,j}(s) \leq 1/2$ for any $s < s'$. Note that for any $p_i > p_j$, $\{s_{i,j}\} \neq \{1\}$ since $G_{i,j}(1) \equiv 0$.

Now, let us introduce the *p-monotonic strict single-crossing* property.

Definition 3. *We say that a committee has preferences over voting rules that satisfy p-monotonic strict single-crossing property if for any $i, j, k \in N$ we have that $s_{i,j} \leq s_{i,k} \leq s_{j,k}$ whenever $p_i > p_j > p_k$.*

Proposition 3. *Every committee has preferences over voting rules that satisfy the p-monotonic strict single crossing property.*

In Example 1, we have that $s_{1,3} \leq s_{1,7} \leq s_{3,7}$ since $s_{1,3} = 4$ and $s_{1,7} = s_{3,7} = 5$.

The proof of Proposition 3 is in the Appendix. We prove Proposition 3 by showing that for any $p_i > p_j > p_k$, we have that: (1) $G_{i,j}(s) \geq G_{i,k}(s)$, (2) $G_{i,j}(s) \geq G_{j,k}(s)$ and (3) $G_{i,k}(s) \geq G_{j,k}(s)$. Notice that it implies that $s_{i,j} \leq s_{i,k} \leq s_{j,k}$.

Notice that Propositions 2 and 3 imply that $s_{R,j} \leq s_{R,C} \leq s_{j,C}$ for any $j \in N$ with $p_j \neq p_C$ and $p_j \neq p_R$. Thus, we have the following corollary.

Corollary 1. *For any heterogeneous committee there is a $s_{R,C} \in \{2, .., n\}$, such that for all $j \in N$, we have that: (1) $U_j(s) \geq U_C(s)$ whenever $s < s_{R,C}$ and (2) $U_j(s) \geq U_R(s)$ whenever $s \geq s_{R,C}$.*

Corollary 1 tells us that for any heterogeneous committee there is a rule $s_{R,C} \in \{2, .., n\}$ such that for any rule lower than it, the most conservative member of the committee (every $i \in N$ such that $p_i = p_C$) is the worst-off voter. And for rules higher than $s_{R,C}$ the reverse holds (i.e., the worst-off voter is the most radical voter (every $i \in N$ such that $p_i = p_R$)).

Notice that in a heterogeneous committee only $s_{R,C}$ and $s_{R,C} - 1$ can minimize the difference between $U_R(s)$ and $U_C(s)$. At first glance one could imagine that only

$s_{R,C}$ and $s_{R,C}-1$ can satisfy the maximin criterion. However, this is not always true. To clarify this point let us examine Figure 2 below.

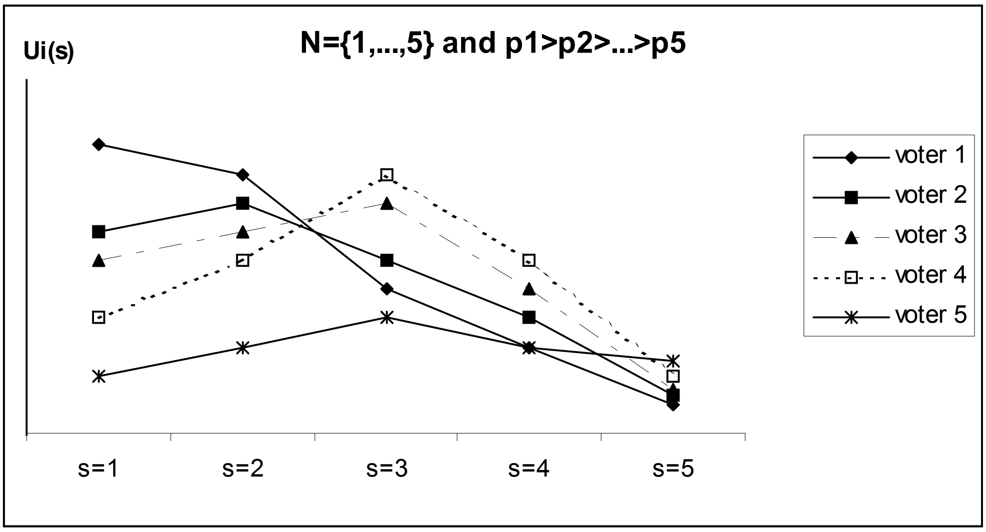

Figure 2

Notice that in Figure 2 above, $s_{R,C}$ is equal to five. If we move from $s_{R,C}$ or $s_{R,C}-1$ to $s=3$ then all the voters will be better off. Thus, $s_{R,C}$ and $s_{R,C}-1$ do not satisfy the maximin criterion. As it can be verified in Figure 2, $S_{Rawls}=\{3\}$.

Definition 4. *A voting rule $s\in\{1,...,n\}$ is Pareto efficient if there is no other voting rule $s'\in\{1,...,n\}$ such that $U_i(s')\geq U_i(s)$ for all $i\in N$ and $U_j(s')>U_j(s)$ for some $j\in N$.*

In Figure 2, only $s_{R,C}$ and $s_{R,C}-1$ are not Pareto efficient. The following proposition identifies the voting rules that are Pareto efficient.

Proposition 4. *If $s\in\{1,...,n\}$ is Pareto efficient, then $\hat{s}_R-1\leq s\leq\hat{s}_C$;*
If $s\in\{1,...,n\}$ is not Pareto efficient, then $s\geq\hat{s}_C$ or $s<\hat{s}_R$.

Proof. It follows by single peakedness and by the fact that for any committee, if $p_i\geq p_j$ then $\hat{s}_i\leq\hat{s}_j$. □

Remark 1. *Notice that* $\widehat{s}_C$ *is not Pareto efficient if and only if* $\widehat{s}_C$ *is a twin-peak and* $\widehat{s}_R \neq \widehat{s}_C$. *Moreover* $\widehat{s}_R - 1$ *is Pareto efficient if and only if* $\widehat{s}_R$ *is a twin-peak and the committee is homogeneous.*

2.4.3 Maximin characterization

The next result characterizes the voting rules that satisfy the maximin criterion in a heterogeneous committees.

Theorem 1. *For any heterogeneous committee,*
$S_{Rawls} \subseteq \{s_{R,C} - 1, s_{R,C}\}$ *whenever* $s_{R,C}$ *is Pareto efficient;*
$S_{Rawls} \subseteq \{\widehat{s}_C - 1, \widehat{s}_C\}$ *whenever* $s_{R,C}$ *is Pareto inefficient and larger than* s^{maj};
$S_{Rawls} \subseteq \{\widehat{s}_R - 1, \widehat{s}_R\}$ *whenever* $s_{R,C}$ *is Pareto inefficient and smaller than* s^{maj}.

Remark 2. *Let* $s_{R,C}$ *be Pareto inefficient and larger than* s^{maj}, *if* $\widehat{s}_C$ *is a twin-peak then* $S_{Rawls} = \{\widehat{s}_C - 1, \widehat{s}_C\}$ *otherwise* $S_{Rawls} = \{\widehat{s}_C\}$. *Let* $s_{R,C}$ *be Pareto inefficient and smaller than* s^{maj}, *if* $\widehat{s}_R$ *is a twin-peak then* $S_{Rawls} = \{\widehat{s}_R - 1, \widehat{s}_R\}$ *otherwise* $S_{Rawls} = \{\widehat{s}_R\}$.

Notice also that the theorem above implies that the set of voting rules that satisfy the maximin criterion, S_{Rawls}, has at most two voting rules and at least one is Pareto efficient. The intuition behind it is that there are only two configurations in which a Pareto inefficient voting rule satisfies the maximin criterion. The first one is when $s_{R,C} = \widehat{s}_C$ and $\widehat{s}_C$ is a twin peak and the second is when $s_{R,C} = \widehat{s}_R - 1$ and $\widehat{s}_R$ is a twin peak. If the first configuration occurs then $S_{Rawls} = \{\widehat{s}_C - 1, \widehat{s}_C\}$ and $\widehat{s}_C$ is Pareto inefficient. If the second one occurs then $S_{Rawls} = \{\widehat{s}_R - 1, \widehat{s}_R\}$ and $\widehat{s}_R - 1$ is Pareto inefficient.

Theorem 1 follows basically from the single-peakedness property, Proposition 4 and by the fact that $\widehat{s}_C \geq s^{maj} \geq \widehat{s}_R$. Its proof is in the Appendix. Next we present a direct corollary of Theorem 1.

Corollary 2. *For any committee, there are at most two voting rules that satisfy the maximin criterion and at least one is Pareto efficient.*

In what follows, we study the maximin criterion recommendations in two special cases. The first case is when the committee is formed only by "conservative voters", i.e. $p_i \leq 0.5$ for any $i \in N$ and the second one is when a committee is formed only by "radical voters", , i.e. $p_i \geq 0.5$ for any $i \in N$.

Theorem 2. *For any committee and any $s \in S_{Rawls}$ we have that*
$s^{maj} - 1 \leq s \leq \widehat{s}_C$ *whenever* $p_i \geq \frac{1}{2}$ *for every* $i \in N$;
$\widehat{s}_R - 1 \leq s \leq s^{maj}$ *whenever* $p_i \leq \frac{1}{2}$ *for every* $i \in N$.

The theorem states that if a committee is formed by "conservative voters" then the maximin criterion recommends voting rules that are not higher than the simple majority rule and not lower than the optimal voting rule of the least conservative among these voters. If a committee is formed only by "radical voters" then the maximin criterion recommends voting rules that are not lower than simple majority rule minus one vote[9] and not higher than the peak of the least radical among these voters. The intuition behind this result is that in a committee formed only by radical voters, i.e. $p_i \geq 1/2$ for every $i \in N$, is most unlikely that a proposal of change be rejected under any rule lower than $s^{maj} - 1$. It means that voter C, the one with lowest p, would be at a severe disadvantage compared with other voters if the voting rule is lower than $s^{maj} - 1$. This is the intuition why maximin criterion recommends voting rules that are higher than $s^{maj} - 1$ in committees formed by only radical voters. A similar argument explains the maximin recommendation for conservative committees.

The proof of Theorem 2 is in the appendix.

[9]Notice that $s^{maj} - 1$ is equal to fifty percent majority if n is even.

2.5 Concluding remarks

In contrast to Barberà and Jackson (2004) and Badger (1972), who concentrate on the voting rules that might be chosen if voters vote on rules, we take a normative point of view and investigate the choice of voting rules according to Rawls's maximin criterion. Specifically, we complement the utilitarian view (which leads to the choice of simple majority as proved in Rae, 1969 and Curtis, 1972) with the suggestion that fairness considerations may recommend the choice of a rule that maximizes the expected utility of the individual that is worst off. By doing this we hope to expand awareness of choice subject to criteria.

As part of our comparison of different rules, we are led to study their implications on the distribution of well being across voters. We have pointed out new properties of the model in terms of voters' preferences over voting rules. The main property which we have proved is that, for any pair of voters, with different probabilities of being in favor of the status quo, there is a threshold such that, for any rule higher than it, the one with the highest probability to reject changes of the status quo has a higher utility. And for any rule lower than this threshold, the reverse holds. Moreover, this threshold changes depending on the pair of voters under analysis, but in a particular way such that there is a threshold rule such that, for any rule lower than it, the most conservative voter among the members of the committee (every $i \in N$ such that $p_i = Min\{p_1, ..., p_n\}$) is the worst off. For rules higher than this threshold the reverse holds, i.e. the worst off voter is the most radical voter (every $i \in N$ such that $p_i = Max\{p_1, ..., p_n\}$).

This last result is important because it restricts the set of rules which are candidates to satisfy the maximin criterion. We proved that there are at most two voting rules that maximizes this function, at least one is Pareto efficient and it is often different from the simple majority. Indeed, if a committee is formed only by "conservative voters" (i.e. $p_i \leq 0.5$ for every $i \in N$) then the maximin criterion recommends voting rules that are between the optimal voting rule of the least conservative among the voters and that of the simple majority rule (i.e. $\widehat{s}_R - 1 \leq s \leq s^{maj}$). If a committee is formed only by "radical voters" ($p_i \geq 0.5$ for every $i \in N$) then the maximin criterion recommends voting rules that are between fifty percent majority and the optimal voting rule of the less radical among these voters (i.e. $s^{maj} - 1 \leq s \leq \widehat{s}_C$).

2.6 Appendix

The following lemma will be needed to prove Proposition 2.

Lemma 1. *For any $i, j \in N$ we have that: $U_i(s) - U_j(s) = (p_i - p_j)(1 - 2G_{i,j}(s))$ for any $s \in \{1, ..., n\}$.*

Proof. Take any $i, j \in N$. For any $m \in \{0, ..., n-2\}$, denote by $P_{i,\,j}(m)$ the probability that exactly m of the voters other than i and j support a proposal. First note that for any $m \in \{1, ..., n-2\}$ we have that $P_i(m) = p_j P_{i,j}(m-1) + (1-p_j)P_{i,j}(m)$ and $P_j(m) = p_i P_{i,j}(m-1) + (1-p_i)P_{i,j}(m)$ for any $m \in \{1, ..., n-2\}$. After some algebraic manipulation with these expressions we have that:

$$p_i P_i(m) - p_j P_j(m) = (p_i - p_j)P_{i,j}(m) \; for \; any \; m \in \{0, ..., n-2\} \tag{2.5}$$

$$(1-p_i)P_i(m) - (1-p_j)P_j(m) = (p_j - p_i)P_{i,j}(m-1) \; for \; any \; m \in \{1, ..., n-1\} \tag{2.6}$$

Notice also that expression (2.2) implies that:

$$U_i(s) - U_j(s) = \sum_{m=s-1}^{n-1} (p_i P_i(m) - p_j P_j(m)) + \sum_{k=0}^{s-1}((1-p_i)P_i(m) - (1-p_j)P_j(m)). \tag{2.7}$$

After some algebraic manipulation in (2.7) using expressions (2.5) and (2.6) and knowing that $p_i P_i(n-1) - p_j P_j(n-1) = 0$ and $(1-p_i)P_i(0) = (1-p_j)P_j(0)$,we have that:

$$U_i(s) - U_j(s) = (p_i - p_j)(1 - 2\sum_{m=o}^{s-2} P_{i,j}(m)) \; for \; any \; s \in \{2, ..., n\}, \tag{2.8}$$

$$U_i(1) - U_j(1) = p_i - p_j \tag{2.9}$$

Recall that for any $m \in \{2, ..., n\}$ and $i, j \in N$, $G_{i,j}(m)$ is the probability that there are no more than $m-2$ voters, other than i and j that support change and $G_{i,j}(1) \equiv 0$. Hence expressions (2.8) and (2.9) imply that: $U_i(s) - U_j(s) = (p_i - p_j)(1 - 2G_{i,j}(s))$ for any $s \in \{1, ..., n\}$. Therefore Lemma 1 is established. □

Proof of Proposition 2. Notice that $G_{i,j}(\cdot)$ is a strictly increasing function and $0 = G_{i,j}(1) < G_{i,j}(s) < G_{i,j}(n) = 1$ for any $s \in \{2, ..., n-1\}$. These two properties hold because for any $m \in \{1, ..., n\}$, $P_{i,j}(m) > 0$ (This later argument follows by the assumption of independence of the realization of voters' support). Given these two properties and Lemma 1 we have that: (1) $U_i(s) > U_j(s)$ whenever $G_{i,j}(s) < \frac{1}{2}$, (2) $U_i(s) = U_j(s)$ whenever $G_{i,j}(s) = \frac{1}{2}$ and (3) $U_i(s) < U_j(s)$ whenever $G_{i,j}(s) > \frac{1}{2}$. Therefore Proposition 2 is established. □

The following lemma will be needed to prove Proposition 3.

Lemma 2. *For any $p_i > p_j > p_k$, we have that:* (1) $G_{i,j}(s) \geq G_{i,k}(s)$, (2) $G_{i,j}(s) \geq G_{j,k}(s)$ *and* (3) $G_{i,k}(s) \geq G_{j,k}(s)$.

Proof. For any $m \in \{0, ..., n-3\}$ and $i, j, k \in N$, $P_{i,\,j,k}(m)$ is the probability that exactly m of the voters other than i, j and k support the change. Notice also that for any $m \in \{2, ..., n-3\}$ we have that $P_{i,j}(m) = [p_k P_{i,j,k}(m-1) + (1-p_k)P_{i,j,k}(m)]$ and $P_{i,k}(m) = [p_j P_{i,j,k}(m-1) + (1-p_j)P_{i,j,k}(m)]$. It follows that:

$$P_{i,j}(m) - P_{i,k}(m) = (p_j - p_k)(P_{i,j,k}(m) - P_{i,j,k}(m-1)) \ for \ any \ m \in \{2, ..., n-3\} \tag{2.10}$$

Taking (2.10) and summing up over s we have that:

$$\sum_{m=1}^{s-2} (P_{i,j}(m) - P_{i,k}(m)) = (p_j - p_k)(P_{i,j,k}(s-2) - P_{i,j,k}(0)) \ for \ any \ s \in \{3, ..., n-1\} \tag{2.11}$$

Notice that $P_{i,j}(0) = (1-p_k)P_{i,j,k}(0)$, $P_{i,k}(0) = (1-p_j)P_{i,j,k}(0), P_{i,j}(n-2) = p_k P_{i,j,k}(n-3)$ and $P_{i,k}(n-2) = p_j P_{i,j,k}(n-3)$. Hence,

$$P_{i,j}(0) - P_{i,k}(0) = (p_j - p_k)P_{i,j,k}(0). \tag{2.12}$$

$$P_{i,j}(n-2) - P_{i,k}(n-2) = (p_k - p_j)P_{i,j,k}(n-3) \tag{2.13}$$

By definition we have that: $G_{i,j}(s) - G_{i,k}(s) = \sum_{m=0}^{s-2}(P_{i,j}(m) - P_{i,k}(m))$ for any $s \in \{2, ..., n\}$ and $G_{i,j}(0) - G_{i,j}(0) = 0$. After some algebraic manipulation using (2.11),(2.12) and (2.13) imply that: $G_{i,j}(s) - G_{i,k}(s) = (p_j - p_k)P_{i,j,k}(s-2)$ for any $s \in \{2, ..., n-1\}$, $G_{i,j}(1) - G_{i,k}(1) = 0$ and $G_{i,j}(n) - G_{i,k}(n) = 0$. Therefore we have established Lemma 2. □

Proof of Proposition 3. Take any $i, j, k \in N$ such that $p_i > p_j > p_k$. It follows by Lemma 2 that $p_i > p_j > p_k$ implies that: (a) $G_{i,j}(s) \geq G_{i,k}(s)$, (b) $G_{i,j}(s) \geq G_{j,k}(s)$ and (c) $G_{i,k}(s) \geq G_{j,k}(s)$. Notice also that by Lemma 1 we have that for any $i, j \in N$:

$$\{s_{ij}\} = \{s' \in \{2, ..., n\} | G_{i,j}(s') \geq 1/2 \textit{ and } G_{i,j}(s) < 1/2 \textit{ for any } s < s'\}. \tag{2.14}$$

It follows that: $G_{i,j}(s) \geq G_{i,k}(s)$ and (2.14) imply that $s_{i,j} \leq s_{i,k}$; $G_{i,j}(s) \geq G_{j,k}(s)$ and (2.14) imply that $s_{i,j} \leq s_{j,k}$; $G_{i,k}(s) \geq G_{j,k}(s)$ and (2.14) imply that $s_{i,k} \leq s_{j,k}$. Therefore we have that $s_{i,j} \leq s_{i,k} \leq s_{j,k}$ and the proof of Proposition 3 is established. □

The following lemmas will be needed to prove Theorem 1:

Lemma 3. *For any committee,* $\widehat{s}_C \geq s^{maj} \geq \widehat{s}_R$.

Proof. This statement is a direct consequence of Barberà and Jackson's (2004) propositions that tell us that for any $i, j \in N$ we have that $\widehat{s}_j \geq \widehat{s}_i$ whenever $p_i \geq p_j$ and for any committee (N, p), there exist $i, j \in N$ such that $\widehat{s}_j \geq s^{maj} \geq \widehat{s}_i$. □

Lemma 4. *a) If $s_{R,C}$ is Pareto inefficient and higher than s^{maj} then $s_{R,C} \geq \widehat{s}_C$.*
b) If $s_{R,C}$ is Pareto inefficient and smaller than s^{maj} then $s_{R,C} < \widehat{s}_R$.

Proof. It follows by Proposition 4 and Lemma 3. Recall that $\widehat{s}_C$ is Pareto inefficient if and only if $\widehat{s}_C$ is a twin-peak and $\widehat{s}_R \neq \widehat{s}_C$. □

Lemma 5. *If $\widehat{s}_R \leq s_{R,C} \leq \widehat{s}_C$ then $S_{Rawls} \subseteq \{s_{R,C} - 1, s_{R,C}\}$*

Proof. Let $\widehat{s}_R \leq s_{R,C} \leq \widehat{s}_C$. Take any $s' \in S_{Rawls}$. Suppose by contradiction that $s' > s_{R,C}$. Note that by Corollary 1 and single peakedness, we have that

$$Min\{U_1(s'), ..., U_n(s')\} = U_R(s') < U_R(s_{R,C}) = Min\{U_1(s_{R,C}), ..., U_n(s_{R,C})\}$$

The inequality above contradicts the maximin criterion so $s' \leq s_{R,C}$. Take any $s' \in S_{Rawls}$. Suppose by contradiction that $s' < s_{R,C} - 1$. Note that by Corollary 1 and single peakedness, we have that

$$Min\{U_1(s'), ..., U_n(s')\} = U_C(s') < U_C(s_{R,C} - 1) = Min\{U_1(s_{R,C} - 1), ..., U_n(s_{R,C} - 1)\}$$

The inequality above contradicts the maximin criterion so $s' \geq s_{R,C} - 1$. Therefore we can conclude that $s' \in \{s_{R,C} - 1, s_{R,C}\}$ but then $S_{Rawls} \subseteq \{s_{R,C} - 1, s_{R,C}\}$. □

Lemma 6. *a) If $\widehat{s}_C < s_{R,C}$ then $S_{Rawls} \subseteq \{\widehat{s}_C - 1, \widehat{s}_C\}$*
b) If $s_{R,C} < \widehat{s}_R$ then $S_{Rawls} \subseteq \{\widehat{s}_R - 1, \widehat{s}_R\}$

Proof. Let $\widehat{s}_C < s_{R,C}$. Take any $s' \in S_{Rawls}$. First suppose by contradiction that $s' < \widehat{s}_C - 1$. But then it implies that $s' < s_{R,C}$. Note that by Corollary 1 and single peakedness, $s' < s_{R,C}$ implies that:

$$Min\{U_1(s'), ..., U_n(s')\} = U_C(s') < U_C(\widehat{s}_C - 1) = Min\{U_1(\widehat{s}_C - 1), ..., U_n(\widehat{s}_C - 1)\}.$$

The inequality above contradicts the maximin criterion so $s' \geq \widehat{s}_C - 1$. Now suppose that $s' > \widehat{s}_C$. First note that it implies that $s' > \widehat{s}_C \geq \widehat{s}_R$. Thus by single peakedness: $U_R(s') < U_R(\widehat{s}_C)$ and $U_C(s') < U_C(\widehat{s}_C)$. This leads to a contradiction since by

Corollary 1 and single peakedness, we have that

$$Min\{U_1(s'),...,U_n(s')\} = Min\{U_R(s'),U_C(s')\} < Min\{U_R(\widehat{s}_C),U_C(\widehat{s}_C)\}$$

The inequality above contradicts the maximin criterion. Thus $s' \leq \widehat{s}_C$. Therefore $S_{Rawls} \subseteq \{\widehat{s}_C - 1, \widehat{s}_C\}$. Notice that $\widehat{s}_C - 1 \in S_{Rawls}$ only if $\widehat{s}_C$ is a twin peak. The proof of the part (b) of the lemma is very similar from the part (a) so it is omitted. □

Proof of Theorem 1. Proposition 4 and Lemma 5 imply that $S_{\max\min} \subseteq \{s_{R,C} - 1, s_{R,C}\}$ whenever $s_{R,C}$ is Pareto efficient. Lemma 4a and Lemma 6a imply that $S_{Rawls} \subseteq \{\widehat{s}_C - 1, \widehat{s}_C\}$ whenever $s_{R,C}$ is Pareto inefficient and larger than s^{maj}. Lemma 4b and Lemma 6b imply that $S_{Rawls} \subseteq \{\widehat{s}_R - 1, \widehat{s}_R\}$ whenever $s_{R,C}$ is Pareto inefficient and smaller than s^{maj}. Therefore Theorem 1 is proved. □

The following lemma will be needed to prove Theorem 2.

Lemma 7. *a)* $s_{R,C} \geq s^{maj}$ *whenever* $p_i \geq 0.5$ *for every* $i \in N$

b) $s_{R,C} \leq s^{maj}$ *whenever* $p_i \leq 0.5$ *for every* $i \in N$.

Proof. First notice that:

1) For any $m \in \{0, ..., n-2\}, P_{R,C}(m) \leq P_{R,C}(n-2-m)$ whenever $p_i \geq 0.5$ for every $i \in N$.

2) For any $m \in \{0, ..., n-2\}, P_{R,C}(m) \geq P_{R,C}(n-2-m)$ whenever $p_i \leq 0.5$ for every $i \in N$.

But then since $\sum_{m=0}^{n-2} P_{R,C}(m) = 1$, the informations in (1) and (2) above imply that:

3) $G_{R,C}(s^{maj}) \equiv \sum_{m=0}^{s^{maj}-2} P_{R,C}(m) \geq \frac{1}{2}$ whenever $p_i \leq 0.5$ for every $i \in N$

4) $G_{R,C}(s^{maj} - 1) \equiv \sum_{m=0}^{s^{maj}-3} P_{R,C}(m) < \frac{1}{2}$ whenever $p_i \geq 0.5$ for every $i \in N$ and $\exists$ $j \in N$ such that $p_j > 0.5$.

Recall that Lemma 1 implies that $\{s_{R,C}\} = \{s' \in \{1, ..., n\} | G_{R,C}(s') \geq 1/2 \textit{ and } G_{R,C}(s) < 1/2 \textit{ for any } s < s'\}$. Thus Lemma 1 with the informations in (3) and

(4) imply that: $s_{R,C} \leq s^{maj}$ whenever $p_i \leq 0.5$ for every $i \in N$ and $s_{R,C} \geq s^{maj}$ whenever $p_i \geq 0.5$ for every $i \in N$. Therefore the proof of Lemma 7 is established. □

Proof of Theorem 2. Theorem 1, Lemma 5, Lemma 6a and Lemma 7a imply that $S_{Rawls} \subseteq \{s^{maj} - 1, ..., \widehat{s}_C\}$ whenever $p_i \geq 0.5$ for every $i \in N$. Theorem 1, Lemma 5, Lemma 6b and Lemma 7b imply that $S_{Rawls} \subseteq \{\widehat{s}_R - 1, ..., s^{maj}\}$ whenever $p_i \leq 0.5$ for every $i \in N$. Therefore the proof of Theorem 2 is established. □

2.7 References

Arrow, K.J. (1998) The external costs of voting rules: A note on Guttman, Buchanan and Tullock. European Journal of Political Economy 14: 219-222.

Badger, W.W. (1972) Political individualism, positional preferences and optimal decision-rules. In: R.G. Niemi, H.F. Weisberg (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.

Barberà, S. and Jackson, M. (2004) Choosing how to choose: Self-stable majority rules and constitutions. Quarterly Journal of Economics, forthcoming in 119(3).

Brennan, G. and Buchanan, M. (1985) The Reason of Rules: Constitutional Political Economy. Cambridge University Press, Cambridge, MA.

Buchanan, J. (1998) Agreement and efficiency: Response to Guttman. European Journal of Political Economy 14: 209-213.

Curtis, R. (1972) Decision rules collective values in constitutional choice. In: R.G. Niemi, H.F. Weisberg (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.

Esteban, J. and Ray, D. (2001) Social decision rules are not immune to conflicts. Economics of Governance 2: 59-67.

Grofman, B. (1979) A preliminary model of jury decision making as a function of jury size, effective jury decision rule and mean juror judgmental competence. In: Tullock, G. (ed.) Frontiers of Economics. Blacksburg, VA: Center for Study of Public Choice.

Guttman, J.M. (1998) Unanimity and majority rules: The calculus of consent reconsidered. European Journal of Political Economy 14: 189-207.

Muller, D.C. (1989) Public Choice II. Cambridge University Press, Cambridge, MA.

Rae, D. (1969) Decision rules and individual values in constitutional choice. American Political Science Review 63: 40-56.

Schofield, N. J. (1972) Is majority rule special? In: R.G. Niemi, H.F. Weisberg (eds.) Probability Methods for Collective Decision Making. Merril Publishing, Columbus, Ohio.

Tullock, G. (1998) Reply to Guttman. European Journal of Political Economy 14: 215-218.

Wakayama, T. (2003) Endogenous choice of voting rules with abstention. mimeo.

Chapter 3

On the Rule of k Names

3.1 Introduction

In the beginning of the sixth century, the clergy and the chief of the citizens of some Eastern European countries chose three names from whom the archbishop selected the bishop. Nowadays, several institutions around the world use variants of this system to fill public offices. For example, this system is known as the "rule of three names" in the United States, "regla de la terna" in Spain and "lista tríplice" in Brazil. Sometimes the list consists of more than three names. Since this does not complicate our analysis, we shall from now on refer to the "rule of k names".

The rule of k names can be formally described as follows: given a set of candidates for office, a committee chooses k members from this set by voting, and makes a list with their names. Then a single individual from outside the committee selects one of the listed names for the office.

We emphasize that an important part of a rule of k names is the procedure used by the committee in order to screen out those k candidates that will be presented to the final chooser. A diversity of screening rules are actually used to select the k names. Because of this diversity, the "rule of k names" is in fact a family of different rules. Let us review some variants of the rule that have been used in the past and are used in the present.

In the Catholic Church, the bishops are appointed under the rule of three names. According the Code Canon Law #377, the Pope may accept one of the candidates in the list proposed by the Apostolic Nuncio (papal ambassador), or consult further. The list is made after consultation with the members of the ecclesiastical province. In many countries the list is decided by means of voting. In Ireland, each canon of the cathedral and parish priest can cast a vote for three candidates to bishop[1]. In England, the canons vote three times, and select each time the most voted candidate. In both cases, the list is made with the three most voted candidates (see Catholic Encyclopedia and Code Canon Law #375, #376 and #377).

In the nomination of the rectors in Brazilian federal public universities, the university councils are permitted, since 1996, to consult their university communities. The law requires that during this consultation, each voter shall cast a vote for only one candidate, and that the three most voted candidates will form the list.[2] The President of the Republic shall select one of the listed names (see Decreto n°1916, May 23th, 1996, Brazil).

The committee that takes the final decision is in most of the cases a single individual and this is what we shall study here. But we could easily embed our definition of the rule of k names, with one committee and one chooser, into a larger class of procedures where both the screening and the choice are made by more than one agent. Here are two examples in which the first committee is smaller than the second

[1] Most of Brazilian states adopts this procedure to make the list in the choice of Prosecutor-General. The article 128 paragraph 3 of the Brazilian Constitution states that the governors of the states shall choose one name of those three names proposed by the members of the State Public Prosecute.

[2] In additional, the sum of the weighted votes of teaching staff need to be a minimum of 70% of the total. In Brazil, before 1996, the list was made with six names proposed by the university council without consulting the university community (see Decreto-Lei n°5540, November 28th, 1968, Brazil).

committee. One is in the Article 76, par.5° of the Mexican Constitution that states that the President of the Republic shall propose three names to the Senate, which shall appoint one of them to become member of the Supreme Court of Justice. Another, is the Brazilian law of corporate finance, approved in 2001, that states that the preferred stockholders who hold at least 10% of the company capital stock shall choose one name among the three names listed by the controller of the company to become their representative on the company's board of administration (see Lei n° 10303, October 31th, 2001, Brasil).[3]

There are many variations of the rule of k names, involving more than two committees. In Chile, according to Article 75 of the Chilean Constitution, the members of the Superior Court of Justice are designated by the President of the Republic among those in the list with five names proposed by the Superior Court of Justice[4], and must get the approval of two thirds of the Senate. If the Senate does not approve the proposal of the President, then the Superior Court must substitute the rejected name in the list, and the procedure is repeated until a presidential nominee is finally approved by the Senate. Another example also comes from Brazil. According to the Brazilian Constitution, one-third of the members of the Superior Court of Justice shall be chosen in equal parts among lawyers and members of the Public Prosecution nominated in a list of six names by the entities representing their respective classes. Upon receiving the nominations, the court shall organize a list of three names and send it to the President of the Republic, who selects one of the listed names for appointment.

Since two or more candidates may have the same number of votes during the preparation of the list, a tie-breaking criterion is often used. In some institutions ties are broken randomly and in others by some deterministic rule. For example, in the nomination of the minister of the Superior Court of Justice in Chile, ties are broken

[3] After 2006, the law states that there will be not such restriction.

[4] Each member of the Court cast a vote for three candidates, the list is made with the five most voted candidates.

randomly, while in Brazil, the age and tenure in the public service of the candidates are used to break ties.

Let us mention that there are versions of the rule of k names in which one of the parties is supposed to be impartial regarding the alternatives. We have two examples and both come from the US. In the first example, the neutral party is the one who makes the list; in the second example, it is the one that makes the final choice.

For the first example, consider the following variant of the rule of three names, which the US Federal Government uses to recruit and select new employees. The US civil service law requires federal examining officers to assign each job applicant a numerical score, based on assessment tools, performance tests or by evaluations of her training and experience. Then a manager hiring people into the civil service must select one from among the top three candidates available (US Merit Systems Protection Board, 1995). Notice that this may be viewed as a variant of the rule of three names, but one where the list is not obtained by voting. Rather, the list must appear as being obtained by applying a neutral scoring test.

Let us turn to the second example. Under the 1993 Labor Reform, California's Labor Code 4065 states that the workers' compensation judge (WCJ) is constrained, in determining a permanent disability rating, to choose among the offers of one of the two parties. This procedure to settle disputes is referred to in the literature as the final-offer arbitration (FOA) (Neuhauser and Swezey, 1999). It is also known as "baseball arbitration" since US major baseball leagues used it to determine wages in disputed contracts. This system is often presented as an improvement over the conventional arbitration procedure, where the arbitrator is not constrained to choose only among the parties' offers. The final-offer arbitrage model was first proposed by Stevens (1966), who argued that it would induce convergence among the offers of the two parties,and presented this conjectured property as an advantage over the conventional arbitration scheme. However, the theoretical literature does not support this conjecture. Faber (1980), Chatterjee (1981), Crawford (1982), Whitman (1986) and Brams and Merrill (1983) show that the offers still diverge under FOA (Brams and Merrill, 1986). Whatever its formal properties, FOA can be interpreted as a rule of 2 names, with the added qualification that the chooser is not guided by self interest, and is assumed to choose in the name of fairness.

Surprisingly, despite of the extensive economics and political science literature on voting, we do not know any article that is specifically devoted to study the rule of k names when both parties are not neutral regarding the alternatives. This chapter is an attempt to fill this gap.

Many questions come to mind. Why are these rules used? What type of decisions are they well suited for? What could be the intentions and expectations of those who decided to set them up? Is there reason to believe that such expectations could be fulfilled? What is the type of strategic behavior that these rules induce on the different agents involved? Do these rules satisfy good properties that make them defensible in a public debate? Why choose three names in some cases, six in other occasions? We cannot give an answer to all of these questions, but we advance some hypothesis and then provide an analysis of different aspects relating to the rule. Hopefully, this analysis provides a first step toward a good understanding of the rule of k names and of its implications.

One of the most reasonable assumptions about the circumstances that recommend the use of the rule point to the existence of some balance between the ability to make decisions on the part of the committee and on the part of the final chooser. Indeed, if k was equal to one, this would amount to give all decisive power to the committee. At the other extreme, when k equals the number of alternatives, then no alternative is eliminated from the list, in which case the chooser decides everything. However, in order to be precise about the type of balances involved, we need a full game theoretic analysis of the rule. This is the main objective of this chapter. We study what outcomes one may expect from applying the rule of k names, when agents act strategically and cooperatively. Our analysis shows how the parameter k, the screening rule and nature of candidacies act as a means to balance the power of the committee with that of the chooser.

3.2 An introductory example: The case of one proposer and one chooser

The rule of k names is definitely a procedure to balance the power of the committee and that of the chooser, though making this statement more precise will take some effort. Before engaging in any complicated analysis, let us consider a simple and suggestive case involving only two agents: one of them, the proposer, selects a subset of k alternatives, from which the other agent has to choose one. This situation is reminiscent of the classical problem of how to cut a cake, though here we are dealing with a finite set of possibilities and we are not introducing any prior normative notion regarding the outcome.

So, let us consider two agents, 1 and 2, facing four alternatives a, b, c and d. Assume that their preferences over alternatives are as follows: $a \succ_1 b \succ_1 c \succ_1 d$ and $c \succ_2 b \succ_2 a \succ_2 d$ where $x \succ_i y$ means that agent i prefers x to y. Assume that agent 1 can propose k alternatives, from which agent 2 makes a final choice. Clearly, $k = 1$ is the case where 1's choice is final, and 2 has no influence, whereas, $k = 4$ gives all decision power to 2. What about the intermediate cases where $k = 2$ or $k = 3$? Let us informally discuss what outcomes we might expect under different strategic assumptions.

First, assume that the choice of strategies is sequential. So, using the language of game theory, 1's strategy is a list with k alternatives selected from the set $\{a, b, c, d\}$ and 2's strategy is a plan in advance regarding the alternative he will choose from every list which can be proposed by 1.

In this case, the only reasonable behavior for 2 is to choose the best alternative out of those proposed by 1. In practice, then, the only strategic player is 1. This is exactly the notion of backward induction equilibrium of this game. Since in any backward equilibrium strategy profile, 2's strategy prescribes the choice of 2's best alternative from every list which can be proposed by 1. Thus, when $k = 2$, the best strategy for 1 is to propose the set $\{a, d\}$, to let a be chosen by 2. When $k = 3$, a cannot be elected since either b or c will be in the list, so proposing a, b, d is 1's best strategy to let b be chosen by 2.

The set of backward induction equilibrium outcomes for different values of k is displayed in the table below:

Set of backward induction equilibrium outcomes

$k = 1$	$\{a\}$
$k = 2$	$\{a\}$
$k = 3$	$\{b\}$
$k = 4$	$\{c\}$

Let us now consider the case where both players choose strategies simultaneously. Now, 2's strategy is a choice rule that dictates the winning alternative from every list which can be proposed by 1. The reader could think that, given our description of the rule of k names as the result of a well defined sequence, where the proposer goes first and the chooser goes last, there is no point in considering this case. However, we think that it is worth studying, for the following reason. Our model of the interaction between the committee and the chooser is a very simple one. We do not model some important facts that will arise in real contexts, like the fact that the relationship among these main actors is a repeated one, and that the choice of alternatives is only a part of it. Since introducing these unmodelled aspects would complicate our analysis very much, we simply admit that threats from the chooser may sometimes be credible. Turning attention to the simultaneous game is the simplest device to study the consequences of such threats.

In this simultaneous game, a strictly Pareto dominated alternative can be the outcome of a Nash equilibrium. An alternative is strictly Pareto dominated if some other alternatives is considered better for both agents 1 and 2. To understand this point, let $k = 3$ and suppose that agents 1 and 2 have the same preferences over alternatives, $a \succ_i b \succ_i c \succ_i d$ for $i = 1, 2$. Given this preference profile a is the only alternative that is not Pareto dominated. It turns out that there exists a strategy profile that can sustain b as a Nash equilibrium outcome. Agent 1 proposes a list with b, c and d and agent 2 declares a choice rule $C(\cdot)$ such that $C(a, b, c) = C(a, b, d) = C(b, d, c) = b$ and $C(a, c, d) = c$. Notice that under this strategy profile, b is the winning alternative and it is a Nash equilibrium since no agent can profitably deviate, given that the other keeps its strategy unchanged. However both 1 and 2 would be better off if agent 1 substituted d by a in the proposed list, and 2 changed the choice rule from C to C', so that $C'(B) = C(B)$ unless $B = \{a, b, d\}$, and $C'(B) = a$. In other words, the previous Nash equilibrium strategy is not a strong Nash equilibrium. A strategy profile is said to be a pure strategy strong Nash equilibrium of a game, if no coalition of players (maybe singletons) can profitably deviate from this strategy profile, given that the strategies of other players remain unchanged. Notice that, under this equilibrium concept, any Pareto dominated alternative is ruled out as an equilibrium outcome. This is basically the reason why we will study the strong Nash equilibria of the simultaneous game.

Let us go back to the previous preference profile where $a \succ_1 b \succ_1 c \succ_1 d$ and $c \succ_2 b \succ_2 a \succ_2 d$. Now, if $k = 2$, since the game is simultaneous, agent 2 can threaten 1 by pledging to choose d if $\{a, d\}$ is proposed. Under this threat, 1's best response would be to propose the set $\{a, b\}$ to let b be chosen by 2. The outcome b is now the result of a strong Nash equilibrium play. However, a is still the outcome of another strong equilibrium where 2 does not threaten and 1 proposes a and d. The table below presents the set of pure strong Nash equilibrium outcomes of the simultaneous game for different values of k.

Set of pure strong Nash equilibrium outcomes

$k = 1$	$\{a\}$
$k = 2$	$\{a, b\}$
$k = 3$	$\{c, b\}$
$k = 4$	$\{c\}$

Notice that the passage from $k = 2$ to $k = 3$ or $k = 4$ makes a difference. We can see that the case $k = 3$ still leaves room for 2 to get the preferred outcome c, while this is out of the question with $k = 1$ and $k = 2$.

This simple example suggests that one important concern of a careful study of rules of k names is precisely to asses the impact of the choice of k, from the point of view of the different parties involved. A second hint is that, in order to evaluate the likely consequences of establishing a rule of k names, we'll have to analyze the game that naturally arises, and that this analysis will be quite different depending on whether or not we think that the chooser can make credible threats. Because of that, in the sequel we analyze several games, and let the reader decide which one will suit each practical situation better. Notice that, in any case, the backward induction equilibrium outcome of the sequential game will be unique, and that it will coincide with agent 2's worst equilibrium outcome for the simultaneous game.

The consequences of the choice of k cannot be analyzed independently of the cardinality of the set of alternatives. Even the alternatives that no one likes play a role in the functioning of the rule: since numbers count, having an undesirable alternative is not the same as not having that alternative at all. To illustrate this simple point, let us go back to the previous example. Clearly, alternative d could never be a strong Nash equilibrium outcome since it is the last option for both agents. However its presence as an alternative makes a difference. Had it not been there, the set of equilibrium outcomes of the simultaneous game for different values of k would have been as shown in the table below.

Set of pure strong Nash equilibrium outcomes

$k = 1$	$\{a\}$
$k = 2$	$\{b\}$
$k = 3$	$\{c\}$

The reader can verify that the presence of d helps 1 and harms 2 for some k's, but never the reverse. For $k = 2$, the presence of d is crucial for agent 1 because, without d, agent 1 will not able to propose a list with two alternatives in which agent 2's best listed name is a. For $k = 3$, agent 2's favorite alternative c is elected in equilibrium if d is out. Yet, in the presence of d, b is chosen.

Since, for a given k, the number of alternatives affects the outcome, there is a lot of room to study how and why alternatives emerge. Adding undesirable alternatives to the contest, or introducing very similar alternatives (clones) to run are obvious forms to manipulate a rule of k names. We shall not pursue formally the related issue of strategic candidacies in the present chapter.

Here is a last point regarding equilibria under the rule of k names, for the two games we have considered. We can express the equilibria as the result of a procedure where agent 2 takes the initiative of vetoing $k - 1$ alternatives and agent 1 chooses one out of those not vetoed for appointment. It turns out that, in terms of strong equilibrium outcomes, the rule of k names is equivalent to the rule of $k - 1$ vetoes where agent 2 is the one who has the right to veto.

The aim of the next section will be to extend our preceding remarks to the case where the proposer is not a single person, but a committee.

3.3 General results: Several proposers and one chooser

The case of one proposer and one chooser is interesting per se. Moreover it allows us to understand some interesting features of the rule of k names. However, our rules will involve, in general, not one but many proposers, whose interests may be at least partially divergent. The choice of the best set to propose will then no longer be a matter of one agent, but the result of a collective decision.

Let us describe formally the rule of k names: given a finite set of candidates for office denoted by $\mathbf{A}$, a committee of proposers $\mathbf{N} = \{1, ..., n\}$ chooses k members from $\mathbf{A}$ by voting, and makes a list with their names. Then a single individual from outside the committee, called the chooser, selects one of the listed names for appointment.

3.3.1 Screening rules for k names

A screening rule for k names is a voting procedure that selects k alternatives from a given set, based on the actions of the proposers. These actions may consist of single votes, sequential votes, the submission of preference of rankings, the filling of ballots, etc...[5] We have found six different screening rules that are actually used by institutions around the world. We divide them into two different groups depending on their properties. The first group we call majoritarian screening rules. The second consists of rules that are not majoritarian, but they still satisfy a weaker condition.

We say that a screening rule is majoritarian if and only if for any set with k candidates there exists an action such that every strict majority coalition of proposers can impose the choice of this set provided that all of its members choose this action. We say that a screening rule is weakly majoritarian if and only if for any candidate there exists an action such that every strict majority coalition of proposers can impose the inclusion of this candidate among the k chosen candidates provided that all of its members choose this action.[6] Notice that by definition any majoritarian screening rule is weakly majoritarian.

[5] In other studies, procedures that select sets have been analized, but then they focus on the problem of selecting a committee of representatives of a fixed size. Fishburn (1981), Gehrlein (1985), Kaymak and Sanver (2003) discuss the Condorcet winner criterion for this type of rules. In Barberà, Sonnenschein and Zhou (1991), the sets that can be selected by the rule may be of variable size.

[6] In Appendix 1, we give exact definitions for these two properties.

Below we present two majoritarian screening rules.

- Each proposer votes for three candidates and the list has the names of the three most voted candidates, with a tie break when needed. It is used in the election of Irish Bishops and that of Prosecutor-General in Brazilian states.
- The list is made with the names of the winning candidates in three successive rounds of plurality voting, with a tie break when needed. It is used in the election of English Bishops.

Notice that these two rules above guarantee the election of any set of k names, provided that a strict majority votes for them (in the same order).

The following rules are only weakly majoritarian but not majoritarian.

- Each proposer votes for 3 candidates and the list has the names of the five most voted candidates, with a tie break when needed. It is used in the election of the members of the Superior Court of Justice in Chile.
- Each proposer votes for 2 candidates and the list has the names of the three most voted candidates, with a tie break when needed. It is used in the election of the members of the Court of Justice in Chile.
- Compute the plurality score of the candidates and include in the list the names of the three most voted candidates, with a tie break when needed. It is used in the election of rectors of public universities in Brazil.
- This is a sequential rule adopted by the Brazilian Superior Court of Justice to choose three names from a set with six names. At each stage there are twice as many candidate as there are positions to be filled in the list. Hence, if the list needs to have three names, we start by six candidates. Each proposer votes for one, and if there is an absolute majority winner, it has his name included in the list. Then, since there are two positions left, the candidate with less votes

is eliminated, so as to leave four candidates to the next round. If the procedure keeps producing absolute majority winners, then the process is continued until three names are chosen. It may be that, at some stage (including the first one), no absolute majority winner arises. Then the voters are asked to reconsider their vote and vote again. Notice that, if they persist in their initial vote, the rule leads to stalemate. Equivalently, we could say that the rule is not completely defined. However, in practice, agents tend to reassess their votes on the basis of strategic cooperative actions.

Of course, many others screening rules are conceivable, with or without the properties described above. In what follows, our analysis is general, since we are not specific about the exact form of the screening rule. The results, however, will depends on some of the properties of the screening rule that is used.

We now propose and analyze two games with complete information induced by the rule of k names.

3.3.2 A game theoretical analysis

In the first game, called the Sincere Chooser Game, it is assumed that the chooser is not a player. The strategy space of the players, i.e. the proposers, is the space of admissible messages associated with the screening rule used to select the k names. Based on these messages a list with k names is made and the winning candidate is the chooser's preferred listed name. It is assumed that the players only care about the identity of the winning candidate. This game is intended to reflect a two-stage process, where the chooser acts after the proposers have already decided whom to propose. It is a way to refine the strong Nash equilibria in the spirit of backward induction equilibria, by not allowing the chooser to send threats, which would be non-credible given the sequential nature of the play.

In the second game, called the Strategic Chooser Game, we assume that the chooser and the proposers play simultaneously. 2's strategy is a choice rule that dictates the winning alternative from every list which can be proposed by the committee. As a consequence, this game has more equilibria than the Sincere Chooser Game. In fact, the additional equilibria would not pass the test of backward induction if the chooser was playing last in the two-stage game. Such additional equilibria are based on the strategies by the chooser that reflect non-credible threats, within the context of the game. Why, then, do we study this second game? We do it because our model is bit narrow, and we feel that knowing about these additional equilibria is interesting, because in real life, choosers are in a position to threaten. True, their possibility to threaten depends on aspects of the problem that are not modelled here: For example, the fact that these elections are embedded within a lasting set of relationships that allow the chooser to retaliate. Since modelling these further opportunities will complicate our model very much, we allow for simultaneous moves as the simplest way to include threats into our analysis.[7]

3.3.3 Strong Nash equilibrium outcomes

We investigate what are the equilibrium outcomes for these two games, when agents act strategically and cooperatively. More specifically, we study their pure strong Nash equilibrium outcomes when the screening rule used to select the k names is majoritarian.[8] For simplicity, we assume that the number of proposers is odd and that all agents have strict preference over the set of candidates. Individual indifferences are ruled out. These two assumptions are convenient because they eliminate the necessity of specifying a tie-breaking criterion, if the screening rule is majoritarian.[9]

[7] See Appendix 1 for more formal descriptions of each game.

[8] See Appendix 1 for a formal definition of a pure strong Nash equilibrium for each one of the games.

[9] In Appendix 3, we present sufficient conditions for a candidate to be a strong Nash equilibrium outcome for each game provided that the screening rule is "unanimous". We also present the necessary conditions for a candidate to be a strong Nash equilibrium outcome for each game provided

We provide a full characterization of the set of strong Nash equilibrium outcomes of the games when the screening rule is majoritarian. Our main result holds for each one of the games: All majoritarian screening rules generate the same set of strong Nash equilibrium outcomes.

Before introducing the characterization results, we need to provide three definitions. The first one is the standard definition of a Condorcet winner. A candidate $x \in \mathbf{B} \subseteq \mathbf{A}$ is the Condorcet winner over $\mathbf{B}$ if $\#\{i \in \mathbf{N} | x \succ_i y\} > \#\{i \in \mathbf{N} | y \succ_i x\}$ for any $y \in \mathbf{B} \backslash \{x\}$[10]. In words, a candidate is the Condorcet Winner over a subset of $\mathbf{A}$ if and only if it belongs to this subset and it defeats any other candidate in this subset in pairwise majority contests among proposers. It is important to note that the chooser's preferences over candidates are not taken into account in this definition. Notice also that there is at most one Condorcet winner over any set, and that such an alternative may not exist. In particular, if there is only one proposer, then the Condorcet winner over a set coincides with the proposer's preferred candidate in this set.

In the two following definitions, the preferences of the chooser will matter. A candidate is dominated if and only if there exists another candidate that is considered better than him by the chooser and by a strict majority of the proposers. A candidate is a chooser's ℓ-top candidate if and only if he is among the ℓ best ranked candidates according to the chooser's preference. These two definitions are important because only those candidates that are undominated and $(\#\mathbf{A} - k + 1)$-top candidates for the chooser can be strong equilibrium outcomes.

As we shall see later, even when the chooser is a player, the equilibrium conditions still require him to choose his truly preferred candidate among the k listed names. This remark is important because if a candidate is not a chooser's $(\#\mathbf{A} - k + 1)$-

that the screening rule is "anonymous". In both cases we do not make any restriction whether the number of proposers is odd or even.

[10] Where $\#\{i \in \mathbf{N} | x \succ_i y\}$ stands for the cardinality of $\{i \in \mathbf{N} | x \succ_i y\}$ and $\mathbf{B} \subseteq \mathbf{A}$ means that B is contained in $\mathbf{A}$.

top candidate then it cannot be the best listed name for the chooser in any list with k names. This is why only those candidates that are $(\#\mathbf{A} - k + 1)$-top for the chooser can be strong equilibrium outcomes. Such outcomes need to be undominated, because any coalition formed by the chooser and by the strict majority of proposers is a winning coalition, i.e. are able to induce the election of any candidate.

Propositions 1 and 2 below characterize the pure strong Nash equilibrium outcomes of the Sincere Chooser Game and the Strategic Chooser Game, respectively.[11]

Proposition 1. *A candidate is a strong Nash equilibrium outcome of the Sincere Chooser Game if and only if it is the Condorcet winner over the chooser's* $(\#\mathbf{A} - k + 1)$*-top candidates.*

Notice that the Condorcet winner over the chooser's $(\#\mathbf{A} - k + 1)$-top candidates is an undominated candidate.

The proofs of all propositions appear in Appendix 2.

Proposition 2. *A candidate is a strong Nash equilibrium outcome of the Strategic Chooser Game if and only if*

1. *it is an undominated and chooser's* $(\#\mathbf{A} - k + 1)$*-top candidate, and*
2. *it is the Condorcet winner over some set of candidates with cardinality larger or equal than* $\#\mathbf{A} - k + 1$.

Hence, given a set of candidates, a preference profile over this set and a value for the parameter k, we can easily identify the set of strong Nash equilibrium outcomes of the games with the help of Propositions 1 and 2. The following example illustrate it.

[11] We follow closely the approach of Sertel and Sanver (2004). They consider a standard voting game where a committee elects a candidate for office, without any external interference. They show that the set of strong equilibrium outcomes of their voting game is the set of generalized Condorcet winners.

Example 1. *Let* $\mathbf{A} = \{a, b, c, d\}$*,* $N = \{1, 2, 3\}$ *and a majoritarian screening rule. The preferences of the chooser and the committee members are as follows:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Chooser*
a	a	b	d
b	b	c	b
c	c	a	a
d	d	d	c

Following Propositions 1 and 2, the first step in describing the equilibrium outcomes for each $k \in \{1, 2, 3, 4\}$*, is to identify the set of undominated candidates. The second step is to find, for each undominated candidate, the largest set in which it is the Condorcet winner. The third and final step is to know the set of the chooser's* $(\#A - k + 1)$*-top candidates.*
Inspecting the preference profile above and recalling that $\#A = 4$*, we have that:*
1. The set of undominated candidates is $\{a, b, d\}$*.*
2. Candidate a *is the Condorcet winner over* $\{a, b, c, d\}$*, candidate* b *is the Condorcet winner over* $\{b, c, d\}$ *and candidate* d *is the Condorcet winner over* $\{d\}$*.*
3. The chooser's $(\#A - k - 1)$*-top candidates are: for* $k = 1$*,* $\{a, b, c, d\}$*, for* $k = 2$*,* $\{a, b, d\}$*, for* $k = 3$*,* $\{b, d\}$ *and for* $k = 4$*,* $\{d\}$*.*
Combining the informations in steps 1-3 above and Propositions 1 and 2 we have the following:
For the Sincere Chooser Game, when $k = 1$ *or* $k = 2$*, candidate* a *is the strong equilibrium outcome. The outcome is* b *when* $k = 3$*, and it is* d *when* $k = 4$*.*
For the Strategic Chooser Game, the only change is that when $k = 2$*,* $\{a, b\}$ *is the set of strong equilibrium outcomes.*

Propositions 1 and 2 imply three corollaries. The first two refer to the existence and the number of strong equilibrium outcomes.

Corollary 1. *The set of strong Nash equilibrium outcomes of the Sincere Chooser Game is either a singleton or empty.*

Corollary 1 follows from Proposition 1 and from the fact that a Condorcet winner, if it exists, is unique. Since Condorcet winner may not exist, a strong equilibrium of the Sincere Chooser Game may not exist either.

Corollary 2. *The set of strong Nash equilibrium outcomes of the Strategic Chooser Game may be empty, and its cardinality cannot be higher than the minimum between* k *and* $\#\mathbf{A} - k + 1$.

Corollary 2 follows from Proposition 2, the uniqueness of a Condorcet winner and because there are at most k candidates that can be a Condorcet winner over some set with cardinality $\#\mathbf{A} - k + 1$ and, by definition, there are exactly $\#\mathbf{A} - k + 1$ candidates that are chooser's $(\#\mathbf{A} - k + 1)$-top candidates.

The third corollary states the connection between the equilibrium outcomes of the two games we have studied.

Corollary 3. *If the strong Nash equilibrium outcome of the Sincere Chooser Game exists then it is the chooser's worst strong Nash equilibrium outcome of the Strategic Chooser Game.*

The example below shows that our characterization results are not valid when the screening rule is only weakly majoritarian.

Example 2. *Let* $\mathbf{A} = \{a, b, c, d, e, f\}$ *and let* $N = \{1, 2, 3\}$. *Assume that each proposer casts a vote for one candidate and the list is formed with the names of the three most voted candidates (a tie-breaking criterion is used when needed). So, this screening rule is only weakly majoritarian. The preferences of the chooser and the committee members are as follows:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Chooser*
a	a	f	f
b	b	e	e
c	c	d	a
d	d	c	b
e	e	b	c
f	f	a	d

Notice that, in both games, if the screening rule was majoritarian then candidate a would be the unique strong equilibrium outcome. However, the screening rule considered here is weakly majoritarian but not majoritarian. As can be verified, proposer 3 is able to force the inclusion of candidate f, his preferred candidate, in the chosen list independently of what the other proposers do. Notice that f is also the chooser's favorite candidate. Therefore candidate f is the unique strong Nash equilibrium outcome of both games.

We finish the presentation of the characterization results by admitting that, like in many others cases, our analysis of the strategic behavior of agents under the rule of k names is marred by the fact that strong equilibria may fail to exist. Since this is pervasive, we do not need to be apologetic about it. But we offer a simple example about how easy it is for existence to fail.

Example 3. *Let $\mathbf{A} = \{a, b, c, d, e\}$, $N = \{1, 2, 3\}$ and a majoritarian screening rule. The preferences of the chooser and the committee members are as follows:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Chooser*
e	e	e	a
b	d	c	c
d	a	a	b
c	b	b	d
a	c	d	e

After a quick inspection of the preference profile above we have that: The set of undominated candidates is $\{a, c, e\}$, candidates a, c and e are the Condorcet winners over $\{a, b\}$, $\{a, c\}$ and $\{a, b, c, d, e\}$ respectively.
Applying Propositions 1 and 2, we have that the two games, for each k, share the same set of strong Nash equilibrium outcomes.

Set of strong Nash equilibrium outcomes

k=1	$\{e\}$
k=2	$\{\emptyset\}$
k=3	$\{\emptyset\}$
k=4	$\{c\}$
k=5	$\{a\}$

The table above is interesting because we can examine the effects of changing the parameter k on the set of strong equilibrium outcomes. For a moment, consider only those rules for which an equilibrium exists. Those are the rules with $k = 1$, $k = 4$ and $k = 5$. According to the agents' preferences over candidates, the chooser prefers $k = 5$ to $k = 4$ and $k = 4$ to $k = 1$. All the proposers agree that the best scenario is when $k = 1$. However, proposers 1 and 2 prefer $k = 4$ to $k = 5$ while 3 prefers $k = 5$ to $k = 4$. So proposer 3 does not have monotonic preferences over k's. It is easy to find a preference profile where one of the proposers always prefers a higher k. For instance, this may happen when one of the proposers shares with the chooser the same preferences over the candidates.

3.3.4 Comparative statics

Our purpose now is to examine the consequences of changing the parameter k, of adding undesirable candidates and of replacing a majoritarian screening rule by non-majoritarian screening rule. By knowing these consequences, we can infer the agent's preferences over different variants of the rule of k names. This can provide some insights into the questions raised in the introduction. Let us recall some of these questions: Why are these rules used? What could be the intentions and the expectations of those who decided to set them up? What is the type of strategic behavior that these rules induce on the different agents involved? Why choose three names in some cases, six in other occasions?

We have already discussed partially some of these issues for the one proposer case. Allowing several proposers complicates our analysis because the strong equilibrium may fail to exist. In the latter example, does the chooser prefer $k = 1$ to $k = 3$ or the reverse? This is a difficult question since for $k = 3$ there is no equilibrium. We could partially avoid this problem by assuming that the preference profile satisfies single-peakedness[12], since under this assumption there is always a Condorcet winner and thus an equilibrium. Unfortunately, this assumption would not avoid the non existence of equilibrium when screening rules are not majoritarian. Thus it cannot help us to compare the performance of majoritarian screening rules with others that are only weakly majoritarian. Another drawback is the possibility of multiple equilibrium outcomes in the Strategic Chooser Game.

In what follows, and with this warning, we'll try to make our best in tackling with those added difficulties. We assume that the agents have preferences over sets of strong Nash equilibrium outcomes that satisfy two mild requirements: Let $\mathbf{P}$ denote a generic individual strict preference relation on $2^A \equiv \{B \subseteq A | B \neq \{\varnothing\}\}$. Consider

[12] We say that a preference profile satisfies single peakedness if and only if the elements of $\mathbf{A}$ can be linearly ordered as $x_1 > x_2 > ... > x_{\#A}$ such that for every $i \in N$ and $a, b \in A$ we have that if $b > a > \alpha(\mathbf{A}, \succ_i)$ or $\alpha(\mathbf{A}, \succ_i) > a > b$ then $a \succ_i b$, where $\alpha(A, \succ_i)$ is i's preferred candidate in $\mathbf{A}$.

any $X, Y \in 2^A$ and $X \neq Y$. (1) If $X \subset Y$ then $X\mathbf{P}_iY$ if $x \succ_i y$ for all $x \in X$ and for all $y \in Y \backslash X$. (2) If $X \nsubseteq Y$ then $X\mathbf{P}_iY$ if $x \succ_i y$ for all $x \in X\backslash Y$ and for all $y \in Y$. Notice that if an agent i prefers x to y then condition (1) implies that $\{x\}$ is preferred to $\{x, y\}$ and condition (2) implies that $\{x\}$ and $\{x, y\}$ are preferred to $\{y\}$. These are very natural conditions since the elements of a set of strong Nash equilibrium outcomes are mutually exclusive alternatives (see Barbera, Bossert and Pattanaik, 2004).

Consider any of the two games. Let denote by $\mathbf{SET}(\mathbf{S}'; \mathbf{A}'; k')$ the set of strong Nash equilibrium outcomes of this game when $k = k'$, the set of candidates is $\mathbf{A}'$ and the screening rule is $\mathbf{S}'$. We say that the agent i prefers the triple $(\mathbf{S}''; \mathbf{A}''; k'')$ to $(\mathbf{S}'; \mathbf{A}'; k')$ if and only if $\mathbf{SET}(\mathbf{S}''; \mathbf{A}''; k'')\mathbf{P}_i\mathbf{SET}(\mathbf{S}'; \mathbf{A}'; k')$**.**

In the context of the Sincere Chooser Game, we know by Corollary 1 that the set of equilibrium outcomes is singleton or empty. So, the condition above says that the agent i prefers the triple $(\mathbf{S}''; \mathbf{A}''; k'')$ to $(\mathbf{S}'; \mathbf{A}'; k')$ if and only if the strong equilibrium outcome under $(\mathbf{S}''; \mathbf{A}''; k'')$ is preferred to the respective outcome under $(\mathbf{S}'; \mathbf{A}'; k')$ according to agent i preferences over candidates.

The next proposition states that, in the context of the Sincere Chooser Game, if the chooser is asked to choose between a rule of k' names and of k'' names, and both rules use majoritarian screening rules then the chooser prefers the rule with the highest k.

Proposition 3. *For the Sincere Chooser Game the following statement holds: If* $\{\emptyset\} \neq \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}, k'') \neq \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}, k') \neq \{\emptyset\}$, $k'' > k'$ *and both* $\mathbf{S}'$ *and* $\mathbf{S}''$ *are majoritarian then the chooser prefers the triple* $(\mathbf{S}''; \bar{\mathbf{A}}, k'')$ *to* $(\mathbf{S}'; \bar{\mathbf{A}}, k')$.

Surprisingly, this proposition is not valid in the context of the Strategic Chooser Game. It can be seen in the following example.

Example 4. *Let* $\mathbf{A} = \{a, b, c, d\}$, *and let* $N = \{1, 2, 3\}$. *The preferences of the chooser and the committee members are as follows:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Chooser*
d	c	b	b
a	a	c	a
b	b	d	c
c	d	a	d

Given the preference profile above, the set of undominated candidates is $\{a, b\}$*, candidates* a *and* b *are the Condorcet winners over* $\{a, b\}, \{b, c, d\}$ *respectively.*
Applying Proposition 2, we have the following equilibrium outcomes:

Set of strong Nash equilibrium outcomes

	Strategic Chooser Game
k=1	$\{\emptyset\}$
k=2	$\{b\}$
k=3	$\{a, b\}$
k=4	$\{b\}$

Examining the table above, we can see that the chooser prefers $k = 2$ *to* $k = 3$*, while the majority of the proposers, 1 and 3, prefer* $k = 3$ *to* $k = 2$*.*

Now, let us analyze the role of a candidacy that, at first glance, one could imagine that has no influence in the game. We say that a candidate is an undesirable candidate in $\mathbf{A}$ if the chooser and all proposers dislike him more than any other candidate in $\mathbf{A}$.

The next results show that the withdrawal of an undesirable candidate has an effect similar to that of passing from k to $k + 1$.

Proposition 4. *For both games the following statement holds:*
If candidate u *is an undesirable candidate of* $\bar{\mathbf{A}}$ *and both* $\mathbf{S}'$ *and* $\mathbf{S}''$ *are majoritarian then* $\mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}} \backslash \{u\}; k) \subseteq \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k + 1)$.

Corollary 4. *For the Sincere Chooser Game:*
If candidate u is an undesirable candidate of $\bar{\mathbf{A}}$*,* $\mathbf{S}'$ *is majoritarian and the proposers have single peaked preferences then* $\mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}\backslash\{u\}; k) = \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}; k+1) \neq \{\emptyset\}$.

The next proposition states that the chooser cannot be worst off if an undesirable candidate decides to withdraw from the contest.

Proposition 5. *For both games the following statement holds:*
If $\{\emptyset\} \neq \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k) \neq \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}\backslash\{u\}; k) \neq \{\emptyset\}$*, candidate u is an undesirable candidate of* $\bar{\mathbf{A}}$ *and both* $\mathbf{S}'$ *and* $\mathbf{S}''$ *are majoritarian, then the chooser prefers the triple* $(\mathbf{S}'; \bar{\mathbf{A}}\backslash\{u\}; k)$ *to* $(\mathbf{S}''; \bar{\mathbf{A}}; k)$*.*

How about the chooser's preferences over screening rules? This is a natural question since half of the screening rules that are actually used are not majoritarian. The next proposition tells us that, in the Sincere Chooser Game, the chooser cannot be worst off if a majoritarian screening rule is substituted by another that is only weakly majoritarian.

Proposition 6. *For the Sincere Chooser Game the following statement holds:*
If $\{\emptyset\} \neq \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k) \neq \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}; k) \neq \{\emptyset\}$*, both* $\mathbf{S}'$ *and* $\mathbf{S}''$ *are weakly majoritarian but only* $\mathbf{S}''$ *is majoritarian then the chooser prefers the triple* $(\mathbf{S}'; \bar{\mathbf{A}}; k)$ *to* $(\mathbf{S}''; \bar{\mathbf{A}}; k)$*.*

The proposition above is not valid for the Strategic Chooser Game as proven by the following example.

Example 5. *Let* $\mathbf{A} = \{a, b, c, d, e\}$*, and let* $N = \{1, 2, 3\}$*. Consider the following screening rule: Each proposer casts a vote for list A or list B. List A is formed by candidates* a, b *and* e *and list B is formed by candidates* b, c *and* d*. The screened list is the most voted list. Notice that this screening rule is weakly majoritarian but not majoritarian.*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Chooser*
b	b	e	e
c	c	a	a
a	a	b	b
d	d	c	c
e	e	d	d

For $k = 3$*, under this screening rule, candidate* b *is the unique strong Nash equilibrium outcome of the Strategic Chooser Game. However under any majoritarian screening,* $\{a, b\}$ *is the set of strong Nash equilibrium outcomes of the Strategic Choose Game. Therefore, the chooser is better off under a majoritarian screening rules. Notice that the reverse can be said to the majority of the proposers.*

3.3.5 Some voting paradoxes

In this subsection we formulate two axioms that express consistency properties of the election of outcomes from different bodies of proposers. These axioms are: (1) If there are two committee members who rank the candidates exactly as the chooser does then the chooser cannot be better off if these two members decide not to participate in the decision about the list. (2) If a committee member is substituted by an agent who ranks the candidates exactly as the chooser does, then the chooser cannot be worst off.

Notice that axioms 1 and 2 are closely related with two standard axioms of voting literature: Participation and reinforcement axioms (see Moulin, 1988, page 237).

The Sincere Chooser Game satisfies axioms 1 and 2. It turns out that the Strategic Chooser Game violates them as proven by the following example.

Example 6. *Let* $k = 3$*,* $\mathbf{A} = \{a, b, c, d, e\}$*,* $N = \{1, 2, 3, 4, 5\}$*, a majoritarian screening rule and the following preference profile:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Proposer 4*
c	c	d	d
b	b	e	e
a	a	b	b
d	d	a	a
e	e	c	c

Proposer 5	*Proposer 6*	*Proposer 7*	*Chooser*
d	a	a	a
e	b	b	b
b	c	c	c
a	d	d	d
c	e	e	e

Notice that proposers 6 and 7 and the chooser have the same preferences over the candidates. For k=3, the set of strong Nash equilibrium outcomes of the Strategic Chooser Game is $\{a, b\}$.

Now let us examine what would happen if proposers 6 and 7 decided not to participate. The preference profile without proposers 6 and 7' s preferences is displayed below.

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Proposer 4*	*Proposer 5*	*Chooser*
c	c	d	d	a	a
b	b	e	e	d	b
a	a	b	b	e	c
d	d	a	a	c	d
e	e	c	c	b	e

Now, candidate a is the unique strong Nash equilibrium outcome of the Strategic Chooser Game. Thus both proposers 6 and 7 and the chooser are better off with this *new situation.*

Suppose that proposer 5 is substituted by proposer 5', who ranks the candidates as the chooser does. The preference profile of this new committee is displayed below.

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Proposer 4*	*Proposer 5'*	*Chooser*
c	c	d	d	a	a
b	b	e	e	b	b
a	a	b	b	c	c
d	d	a	a	d	d
e	e	c	c	e	e

In this new situation, $\{a, b\}$ is the set of strong Nash equilibrium outcomes of the Strategic Chooser Game. Hence the chooser is worse off with the substitution of proposer 5 by someone who ranks the candidate as the chooser do.

3.3.6 The rule of q vetoes

As we said in the introduction, in Mexico, the President of the Republic shall propose three names to the Senate, which shall appoint one of them to become member of the Supreme Court of Justice. Since vetoing q names is equivalent to selecting $\#\mathbf{A} - q$ names, we can say that the Mexican president vetoes $\#\mathbf{A} - 3$ candidates, and then the Senate chooses one of the remaining candidates for appointment. This system is thus a member of the family of the "rules of q vetoes" that can be described as follows: given a set of candidates for office, a single individual vetoes q members from this set. Then a committee selects one candidate by plurality voting, among those not vetoed, for appointment.

Proposition 7. *In terms of strong Nash equilibrium outcomes of the Sincere Chooser Game, the rule of k names is equivalent to the rule of $k-1$ vetoes whenever the sincere chooser is the one who vetoes.*

Proposition 8. *In terms of strong Nash equilibrium outcomes of the Strategic Chooser Game, the rule of k names is equivalent to the rule of $k-1$ vetoes whenever the strategic chooser is the one who vetoes.*

A interesting implication of the propositions above is that the balance of power between the Mexican President and the Senate would not changed, if this nomination system was substituted by the rule of $\#\mathbf{A}-2$ names where the Senate is the committee of proposers and the screening rule is majoritarian.

3.3.7 Remarks about the case where the number of proposers is even

If the number of proposers is even the set of strong Nash equilibrium outcomes may depend on how ties are broken. It can be seen in the following example.

Example 7. *Let $\mathbf{A} = \{a, b, c, d\}$, and let $N = \{1, 2, 3, 4\}$. Assume that each proposer casts votes for two candidates, and that the list is formed with the names of the two most voted candidates with a tie breaking rule in case of need. Notice that this screening rule is majoritarian. The preferences of the chooser and the committee members are as follows:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Proposer 4*	*Chooser*
a	a	b	b	b
d	d	d	d	a
c	c	c	c	d
b	b	a	a	c

The set of strong equilibrium outcomes of this game will depend on how ties are broken under this screening rule:
1) Suppose that the tie breaking criterion is as follows: $\{a, b\} \succ \{a, c\} \succ \{a, d\} \succ$

$\{b, c\} \succ \{b, d\} \succ \{c, d\}$.

Notice that proposers 3 and proposer 4 are able to include in the chosen list their favorite candidate, which is b. Since candidate b is also the chooser's favorite candidate, there is no doubt that candidate b is the unique strong equilibrium outcome of the Sincere Chooser Game.

2) Suppose that the tie breaking criterion is as follows: $\{a, c\} \succ \{a, d\} \succ \{a, b\} \succ \{c, d\} \succ \{c, b\} \succ \{d, b\}$.

Now proposers 1 and 2 are able to guarantee the choice of the list formed by candidates a and c. If this list is chosen candidate a, which is proposers 1 and 2's preferred candidate will be elected, since he is the best listed name according to the chooser's preferences. Thus it is clear that candidate a is the unique strong equilibrium outcome of the Sincere Chooser Game.

If the number of proposers is even, the set of strong Nash equilibrium outcomes of the Strategic Chooser Game may not contain the set of strong Nash equilibrium outcomes of the Sincere Chooser Game. See the example below.

Example 8. *Let* $\mathbf{A} = \{a, b, c, d\}$, *and let* $N = \{1, 2, 3, 4\}$. *Consider the following majoritarian screening rule for two names: The chosen list of two names is* $\{b, d\}$ *unless the strict majority of the proposers agrees with another list. Notice that this screening rule is majoritarian. The preferences of the chooser and the committee members are as follows:*

Preference Profile

Proposer 1	*Proposer 2*	*Proposer 3*	*Proposer 4*	*Chooser*
a	a	b	b	d
b	b	a	a	b
c	c	c	c	a
d	d	d	d	c

The set of strong Nash equilibrium outcomes of the Sincere Chooser Game is $\{a, b\}$.

Notice that all the proposers announcing that they support $\{a, c\}$ is a strong Nash equilibrium outcome of the Sincere Chooser Game and it leads the victory of candidate a. All the proposers supporting $\{b, c\}$ is a strong equilibrium as well. In this equilibrium, candidate b is the winner.
However in the context of the Strategic Chooser Game, candidate b is the unique strong Nash equilibrium outcome. There exists no strategy profile that can sustain candidate a as a strong equilibrium outcome under this game because the coalition formed by the chooser and by proposers 3 and 4 can always find a profitable deviation.

3.4 Concluding remarks

As shown by our analysis the rule of k names is a method to balance the power of the two parties involved in decisions: the committee and the final chooser. We have provided examples of several institutions around the world that use the rule of k names to take decisions. We described six different screening rules that are actually used. Two of them are majoritarian and the others are only weakly majoritarian.

As part of our attempt to understand the widespread use of these rules, we have engaged in a game theoretical analysis of two games induced by them. We have shown that the choice of the screening procedure to select the k names is not too crucial when agents act strategically and cooperatively. This is because rules of k names based on different majoritarian screening rules lead to the same sets of strong equilibrium outcomes. We characterized the set of strong equilibrium outcomes of these games under any majoritarian screening rules.

For both games, we determined the effects on the equilibria of increasing k, adding undesirable candidates and substituting a majoritarian screening rule for another not majoritarian. Knowing these effects, we were able to derive endogenously the agents' preferences over different variants of the rule of k names.

For both games, adding to the contest an undesirable candidate, i.e. a candidate that nobody likes, goes against the chooser's interests. For the Sincere Chooser Game, the chooser weakly prefers high k's as well as any weakly majoritarian rule to any majoritarian screening rule. We showed with examples that the same cannot be said in the context of the Strategic Chooser Game. We provided an example where the chooser strictly prefers $k = 2$ to $k = 3$ while a majority of proposers strictly prefers $k = 3$ to $k = 2$. In another example, the chooser strictly prefers a weakly majoritarian screening to any majoritarian rule.

We have also shown the equivalence of rule of $k - 1$ vetoes with the rule of k names in terms of strong Nash equilibrium outcomes. In other words, we proved that the set of strong equilibrium outcomes would not change if instead of taking the final decision, the chooser vetoes $k - 1$ candidates and then let the committee select by plurality one of the remaining candidates for appointment.

We interpret our present work as a first step for understanding the implications of using such methods, and hope to generate interest in its further study.

3.5 Appendix

3.5.1 Appendix 1

Denote by W the set of all strict orders (transitive[13], asymmetric[14] and irreflexive[15]) on $\mathbf{A}$. Each member $i \in N \cup \{chooser\}$ has a strict preference $\succ_i \in W$. For any nonempty subset B of $\mathbf{A}, B \subseteq \mathbf{A}\backslash\{\varnothing\}$, we denote by $\alpha(B, \succ_i) \equiv \{x \in B | x \succ_i y$ for all $y \in B\backslash\{x\}\}$ the preferred candidate in B according to preference profile $\succ_i$. Denote by $\mathbf{A}_k \equiv \{B \subseteq A | \#B = k\}$ the set of all possible subsets of A with cardinality k where $\#B$ stands for the cardinality of B and $\mathbf{B} \subseteq \mathbf{A}$ means that B is contained

[13]Transitive: For all $x, y, z \in A : (x \succ y$ and $y \succ z)$ implies that $x \succ z$.
[14]Asymmetric: For all $x, y \in A : x \succ y$ implies that $\neg(x \succ x)$.
[15]Irreflexive: For all $x \in A, \neg(x \succ x)$.

in **A**.

Definition 1. *Let* $M^N \equiv M_1 \times ... \times M_n$ *with* $M_i = M_j = M$ *for all* $i, j \in N$ *where* M *is the space of actions of a proposer in* N*. For example, if the actions in* M^N *are casting single votes then* $M \equiv \mathbf{A}$*. If the actions in* M^N *are submissions of strict preference relation then* $M \equiv W$*. Given* $k \in \{1, 2, ..., \#\mathbf{A}\}$*, a* ***screening rule for k names*** *is a function* $S_k : M^N \longrightarrow \mathbf{A}_k$ *associating to each action profile* $m_N \equiv \{m_i\}_{i \in N} \in M^N$ *the k-element set* $S_k(m_N)$*.*

Definition 2. *We say that a screening rule* $S_k : M^N \longrightarrow \mathbf{A}_k$ *is* ***majoritarian*** *if and only if for every set* $B \in A_k$ *there exists* $m \in M$ *such that for every strict majority coalition* $C \subseteq N$ *and every profile of the complementary coalition* $m_{N \backslash C} \in M^{N \backslash C}$ *we have that* $S_k(m_{N \backslash C}, m_C) = B$ *provided that* $m_i = m$ *for every* $i \in C$*.*

Definition 3. *We say that a screening rule* $S_k : M^N \longrightarrow \mathbf{A}_k$ *is* ***weakly majoritarian*** *if and only if for every candidate* $x \in A$ *there exists* $m \in M$ *such that for every strict majority coalition* $C \subseteq N$ *and every profile of the complementary coalition* $m_{N \backslash C} \in M^{N \backslash C}$ *we have that* $x \in S_k(m_{N \backslash C}, m_C)$ *provided that* $m_i = m$ *for every* $i \in C$*.*

Definition 4. *Given* $k \in \{1, 2, ..., \#\mathbf{A}\}$*, a screening rule for k names* $S_k : M^N \longrightarrow \mathbf{A}_k$ *and a preference profile* $\succ \equiv \{\succ_i\}_{i \in N \cup \{chooser\}} \in W^{N+1}$*, the* ***Sincere Chooser Game*** *can be described as follows: It is a simultaneous game with complete information where each player* $i \in N$ *chooses a strategy* $m_i \in M_i$*. Given* $m_N \equiv \{m_i\}_{i \in N} \in M^N$*,* $S_k(m_N)$ *is the chosen list with k names and the winning candidate is* $\alpha(S_k(m), \succ_{chooser})$*.*

Definition 5. *Given* $k \in \{1, 2, ..., \#\mathbf{A}\}$*, a screening rule for k names* $S_k : M^N \longrightarrow \mathbf{A}_k$ *and a preference profile* $\succ \equiv \{\succ_i\}_{i \in N \cup \{chooser\}} \in W^{N+1}$*, a joint strategy* $m_N \equiv \{m_i\}_{i \in N} \in M^N$ *is a* ***pure strong Nash equilibrium of the Sincere Chooser Game*** *if and only if, given any coalition* $C \subseteq N$*, there is no* $m'_N \equiv \{m'_i\}_{i \in N} \in M^N$

with $m'_j = m_j$ *for every* $j \in N \backslash C$ *such that* $\alpha(m'_N, \succ_{chooser}) \succ_i \alpha(m_N, \succ_{chooser})$ *for each* $i \in C$.

Definition 6. *Given* $k \in \{1, 2, ..., \#\mathbf{A}\}$*, a screening rule for k names* $S_k : M^N \longrightarrow \mathbf{A}_k$ *and a preference profile* $\succ \equiv \{\succ_i\}_{i \in N \cup \{chooser\}} \in W^{N+1}$*, the* ***Strategic Chooser Game*** *can be described as follows: It is a simultaneous game with complete information where each player* i*'s* $\in N$ *strategy space is* M*, while the chooser's strategy space is* $M_{chooser} \equiv \{f : \mathbf{A}_k \longrightarrow \mathbf{A} | f(B) \in B \text{ for every } B \in \mathbf{A}_k\}$*. Given* $m_{N \cup \{Chooser\}} = (m_N \equiv \{m_i\}_{i \in N}, m_{chooser}) \in M^N \times M_{chooser}$*,* $S_k(m_N)$ *is the chosen list with k names and the winning candidate is* $m_{chooser}(S_k(m_N))$*.*

Definition 7. *Given* $k \in \{1, 2, ..., \#\mathbf{A}\}$*, a screening rule for k names* $S_k : M^N \longrightarrow \mathbf{A}_k$ *and a preference profile* $\succ \equiv \{\succ_i\}_{i \in N \cup \{chooser\}} \in W^{N+1}$*, a joint strategy* $m_{N \cup \{Chooser\}} \equiv (m_N \equiv \{m_i\}_{i \in N}, m_{chooser}) \in M^N \times M_{chooser}$ *is a* ***pure strong Nash equilibrium of the Strategic Chooser Game*** *if and only if, given any coalition* $C \subseteq N \cup \{chooser\}$*, there is no* $m'_{N \cup \{Chooser\}} = (m'_N, m'_{chooser}) \in M^N \times M_{chooser}$ *with* $m'_j = m_j$ *for every* $j \in N \cup \{chooser\} \backslash C$ *such that* $m'_{N \cup \{Chooser\}}(S_k(m'_N)) \succ_i m_{N \cup \{Chooser\}}(S_k(m_N))$ *for each* $i \in C$*.*

3.5.2 Appendix 2

Proof of Proposition 1. Suppose that candidate x is the outcome of a strong equilibrium of the Sincere Chooser Game. In any strong Nash equilibrium where x is the outcome, the screened set is such that x is the best candidate in this set according to the chooser's preferences. So x is a chooser's $(\#\mathbf{A} - k + 1)$-top candidate. Since the screening rule is majoritarian there exists no other chooser's $(\#\mathbf{A} - k + 1)$-top candidate that is considered better that x by a strict majority of proposers. Otherwise this coalition could impose the choice of a set where this candidate would be the preferred candidate according to the chooser's preference. Therefore candidate x is the Condorcet winner over the set of chooser's $(\#\mathbf{A} - k + 1)$-top candidates, and the first part of the proposition is proved.

To complete the proof we need to show that if a candidate is the Condorcet winner over the set of chooser's $(\#\mathbf{A} - k + 1)$-top candidates then there exists a strategy profile that sustains him as a strong Nash equilibrium outcome. Let x be such a candidate. Take any set with k candidates contained in $\mathbf{A}$ such that x is the chooser's best candidate in this set. Notice that this set exists, since candidate x is a chooser's $(\#\mathbf{A} - k + 1)$-top candidate. Let B be such a set. Since the screening rule is majoritarian there exists an action such that every majority coalition of proposers can impose the choice of B provided that all of its members choose this action. Let m be such an action. Consider the strategy profile, where all proposers choose action m. Then, candidate x will be elected since the screening rule is majoritarian. By this same reason, any coalition with less than half of the proposers cannot change the outcome. Notice also that any majoritarian coalition does not have any incentive to deviate, since there is no candidate among the chooser's $(\#\mathbf{A} - k + 1)$-top candidate that is considered better than x by all proposers in the coalition (recall that only the chooser's $(\#\mathbf{A} - k + 1)-$top candidates can be the chooser' best name among the candidates of a set with cardinality k). Otherwise x would not be a Condorcet winner over the set of the chooser's $(\#\mathbf{A} - k + 1)$-top candidates. Therefore this strategy profile is a strong Nash equilibrium of the Sincere Chooser Game. □

Proof of Proposition 2. Suppose that candidate x is the outcome of a strong equilibrium of the Strategic Chooser Game. In any strong Nash equilibrium where x is the outcome, the screened set is such that x is the best candidate in this set according to the chooser's preferences. Otherwise the chooser would have incentives to choose another name in this set. So x is a chooser's $(\#\mathbf{A} - k + 1)$-top candidate. Since the screening rule is majoritarian there exists no other set with k names where all the candidates in this set are considered better than x by a strict majority of proposers. Otherwise this coalition would have incentives to impose the choice of this set. This is only true when candidate x is the Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A} - k + 1$. For this same reason there exists no candidate that is considered better than x by a strict majority of proposers and the chooser. This implies that candidate x is a undominated candidate. Therefore the first part of the proposition is proved.

To complete the proof we need to show that if a candidate is (1) undominated and chooser's $(\#\mathbf{A} - k + 1)$-top candidate, and (2) Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A} - k + 1$ then there exists a strategy profile that sustains him as a strong Nash equilibrium outcome. Let us call this candidate x. Take any set with k candidates contained in $\mathbf{A}$ such that x is the chooser's best candidate in this set. Notice that this set exists, since candidate x is a chooser's $(\#\mathbf{A} - k + 1)$-top candidate. Let B be such a set. Since the screening rule is majoritarian there exists an action such that every majority coalition of proposers can impose the choice of B provided that all of its members choose this action. Let m be such an action. Suppose the strategy profile where all proposers choose the action m. Let the chooser declare a choice rule such that if candidate x is in the screened set, x is the winning candidate. Otherwise, the winner is a candidate in the screened set that is considered worse than x by a strict majority of proposers (notice that this choice rule exists since x is the Condorcet winner over a set with cardinality $\#\mathbf{A} - k + 1$). Under this strategy profile, candidate x will be elected since the screening rule is majoritarian. Notice that the chooser's strategy eliminates any incentive of any majority coalition of proposers to deviate. Notice also that candidate x is the chooser's best candidate in the screened set, so that the chooser has no incentive to unilaterally deviate either. No coalition formed by a majority of proposers and the chooser has incentives to deviate either, since x is a undominated candidate. Therefore this strategy profile is a strong Nash equilibrium of the Strategic Chooser Game. Therefore the proof of the proposition is established. □

Proof of Corollary 3. Let x be the strong equilibrium outcome of the Sincere Chooser Game and z be a strong equilibrium outcome of the Strategic Chooser Game. By

Proposition 1, x is the Condorcet winner over the set of chooser's $(\#\mathbf{A}-k+1)$-top candidates. This information and Proposition 2 imply that x is also a strong equilibrium outcome of the Strategic Chooser Game. Again by Proposition 2, z is (1) undominated and chooser's $(\#\mathbf{A}-k+1)$-top candidate, and (2) the Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A}-k+1$. Let us prove that the chooser does not prefer x to z. Suppose by contradiction that the chooser prefers x to z. This implies that the strict majority of proposers prefers z to x. Otherwise z would not be undominated. Since z is chooser's $(\#\mathbf{A}-k+1)$-top candidate, it implies that x is not the Condorcet winner over the set of chooser's ($\#\mathbf{A}$-k+1)-top candidates. This is a contradiction. □

Proof of Proposition 3. Suppose $k" > k'$, both **S**' and **S**" are majoritarian screening rules and $\{\emptyset\} \neq \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k'') \neq \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}; k') \neq \{\emptyset\}$.
Let $x \in \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k'')$ and $y \in \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}; k')$. It will suffice to show that $x \succ_{chooser} y$. Suppose by contradiction that the chooser prefers y to x. By Proposition 1, we have that x is the Condorcet winner over the set of the chooser's $(\#\bar{\mathbf{A}}-k''+1)$-top candidates. And y is the Condorcet winner over the of chooser $(\#\bar{\mathbf{A}}-k'+1)$-top candidates.
Since x is one of the chooser's $(\#\bar{\mathbf{A}}-k''+1)$-top candidate, the chooser prefers y to x and $k'' > k'$, we also have that y is a chooser's $(\#\bar{\mathbf{A}}-k''+1)$-top candidate. But then this contradicts the fact that x is the Condorcet winner over the set of a chooser's $(\#\bar{\mathbf{A}}-k''+1)$-top candidates. □

Proof of Proposition 4. Consider first the Strategic Chooser Game. Let u be an undesirable candidate of $\bar{\mathbf{A}}$. It is easy to see that the set of the chooser's $(\#\bar{\mathbf{A}}\backslash\{u\}-k+1)$-top candidates is equal to the set of the chooser's $(\#\bar{\mathbf{A}}-(k+1)+1)$-top candidates. Moreover, the set of undominated candidates does not change when the set of the candidates is $\bar{\mathbf{A}}\backslash\{u\}$ or $\bar{\mathbf{A}}$. We also have that the set of candidates that are Condorcet winners over some set with cardinality $\#\bar{\mathbf{A}}\backslash\{u\}-k+1$ is contained in the set of candidates that are Condorcet winners over some set with cardinality $\#\bar{\mathbf{A}}-(k+1)+1$. By Proposition 2, these informations imply that $\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k) \subseteq \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k+1)$. For the Sincere Chooser Game, the proof requires a similar argument. For this reason it is omitted. □

Proof of Proposition 5. Let u be the undesirable candidate of the set $\bar{\mathbf{A}}$ and $\{\emptyset\} \neq \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k) \neq \mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k) \neq \{\emptyset\}$.

First let us prove that the statement holds for the Strategic Chooser Game. Notice that the set of undominated candidates does not change whenever the set of candidates is $\bar{\mathbf{A}}\backslash\{u\}$ or $\bar{\mathbf{A}}$. We also have that the set of candidates that are Condorcet winners over some set with cardinality $\#\bar{\mathbf{A}}\backslash\{u\}-k+1$ is equal to the set of candidates that are Condorcet winners over some set with cardinality $\#\bar{\mathbf{A}}-k+1$. Moreover, any chooser's $(\#\bar{\mathbf{A}}\backslash\{u\}-k+1)$-top candidate is also chooser's $(\#\bar{\mathbf{A}}-k+1)$-top candidate. Therefore by Proposition 2, we have that $\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k) \subset \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k)$. Thus it suffices to show that we have $x \succ_i y$ for all $x \in \mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$ and $y \in \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k)\backslash\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$.

Suppose by contradiction that there is $y \in \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k)\backslash\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$ and $x \in \mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$ such that the chooser prefers y to x. Notice that this information implies that y is a chooser's $(\#\bar{\mathbf{A}}\backslash\{u\}-k+1)$-top candidate. But then, we have y is undominated, chooser's $(\#\bar{\mathbf{A}}\backslash\{u\}-k+1)$-top candidate and the Condorcet winner of a set with cardinality $\#\bar{\mathbf{A}}\backslash\{u\}-k+1$. So, by Proposition 2, we have that $y \in \mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$. This is a contradiction, since we had assumed that $y \in \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k)\backslash\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$.

Now let us prove that the statement for the Sincere Chooser Game. By proposition 4, we have that $\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k) = \mathbf{SET}(\mathbf{S}'';\bar{\mathbf{A}};k+1)$ since $\mathbf{SET}(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k) \neq \{\emptyset\}$. Hence, by Proposition 3, we have that the chooser prefers the triple $(\mathbf{S}';\bar{\mathbf{A}}\backslash\{u\};k)$ **to** $(\mathbf{S}'';\bar{\mathbf{A}};k)$. □

Proof of Proposition 6. First let us prove that if a screening rule is weakly majoritarian and x is an outcome of a strong equilibrium of the Sincere Chooser Game then x is undominated and chooser's $(\#\mathbf{A}-k+1)$-top candidate. Notice that in any strong Nash equilibrium in which x is the outcome, the screened set is such that x is the best candidate in this set according to the chooser's preferences. So x is a chooser's $(\#\mathbf{A}-k+1)$-top candidate. Since the screening rule is weakly majoritarian there exists no other candidate that is considered better that x by a strict majority of proposers and the chooser. Otherwise these proposers could impose the inclusion of this candidate in the screened set and this candidate would win. Therefore candidate x is undominated and chooser's $(\#\mathbf{A}-k+1)$-top candidate.

Let $\mathbf{S}'$ and $\mathbf{S}''$ be weakly majoritarian screening rules but only $\mathbf{S}''$ is majoritarian such that $\{\emptyset\} \neq \mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k) \neq \mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}; k) \neq \{\emptyset\}$. Suppose by contradiction that $\mathbf{SET}(\mathbf{S}''; \bar{\mathbf{A}}; k) = \{y\}$ and $\mathbf{SET}(\mathbf{S}'; \bar{\mathbf{A}}; k) = \{x\}$ such that chooser prefers y to x. By the previous paragraph y and x are undominated and chooser's $(\#\bar{\mathbf{A}}-k+1)$-top candidates. We also know, by Proposition 1, that y is also the Condorcet winner over the set of chooser's $(\#\bar{\mathbf{A}}-k+1)$-top candidates.

Since the chooser prefers y to x, a strict majority of proposers prefers x to y. Otherwise x would not be undominated. Thus y is not the Condorcet winner over the set of chooser's $(\#\bar{\mathbf{A}}-k+1)$-top candidates. Therefore we have reached a contradiction. □

Proof of Proposition 7. Since the chooser is always sincere (i.e. he is not a player), the vetoed candidates under rule of $k-1$ are those that are not chooser's $(\#\mathbf{A}-k+1)$-top candidates. We know that under plurality voting the unique strong equilibrium outcome is the Condorcet winner (see Sertel and Sanver, 2004). Therefore, by Proposition 1, the strong equilibrium outcomes under rule of k names and rule $k-1$ vetoes coincides and it is the Condorcet winner over the set of chooser's $(\#\mathbf{A}-k+1)$-top candidates. □

Proof of Proposition 8. Here we assumed that the chooser is a player. So, by Proposition 2, it suffices to show that a candidate is a strong equilibrium outcome under the rule of $k-1$ vetoes if and only if it is (1) undominated and chooser's ($\#\mathbf{A}-k+1$)-top candidate, and (2) Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A}-k+1$.

Suppose that candidate x is the outcome of a strong equilibrium outcome under rule of $k-1$ vetoes. In any strong Nash equilibrium where x is the outcome, x is the Condorcet winner over the set of not vetoed candidates. Otherwise there would be a coalition formed by a majority of proposers that would have incentives in voting for another not vetoed candidate and this candidate would be elected. So x is the Condorcet winner over a set with cardinality $\#\mathbf{A}-k+1$, since this is the cardinality of the set of available candidates after the veto made by the chooser. Notice that there exists no subset with cardinality $\#\mathbf{A}-k+1$ where all candidates are considered better than x by the chooser. Otherwise the chooser would have an incentive to veto all but those in this subset. This is only true when x is a chooser's ($\#\mathbf{A}-k+1$)-top candidate. Since the proposers use plurality, there exists no other candidate that is considered better than x by a majority of proposers and by the chooser. Otherwise this coalition would be able to elect this candidate. This is only true when x is an undominated candidate. Thus we have proved that if a candidate is a strong equilibrium outcome under rule of $k-1$ vetoes then it is (1) undominated and chooser's ($\#\mathbf{A}-k+1$)-top candidate, and (2) Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A}-k+1$. So we have completed the first part of the proof.

To finish the proof we need to show that if a candidate is (1) undominated and chooser's ($\#\mathbf{A}-k+1$)-top candidate, and (2) Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A}-k+1$ then there exists a strategy profile that sustains him as a strong Nash equilibrium outcome. Let x be such a candidate. Consider the following strategy profile: The chooser vetoes all candidates but the set with cardinality $\#\mathbf{A}-k+1$ in which candidate x is the Condorcet winner (if there are more than one set with this characteristic choose one of them). All proposers unanimously cast a vote for candidate x if candidate x is a not vetoed; otherwise they vote for candidate considered worse than x according to the chooser's preference (this strategy is feasible since x is a chooser's $(\#\mathbf{A}-k+1)$-top candidate). Notice that under this strategy profile, candidate x will be elected. Notice that the proposers' threat eliminates any chooser's incentive in unilaterally deviating. Notice that candidate x is the Condorcet winner over the set of not vetoed candidates. So the majority of proposers do not have incentive to deviate. Notice also that a coalition formed by a majority of the proposers and the chooser does not have any incentive to deviate either, since x is an undominated candidate. Thus this strategy profile is a strong equilibrium under rule of $k-1$ vetoes. Therefore the proof of the proposition is established. □

3.5.3 Appendix 3

In this appendix, we present sufficient conditions for a candidate to be a strong Nash equilibrium outcome for each of our games provided that the screening rule is "unanimous". We also present the necessary conditions for a candidate to be a strong Nash equilibrium outcome for each of our games provided that the screening rule is "anonymous".

Here, the number of proposers can be even and individual indifferences over the alternatives are not ruled out. We follow closely the approach of Sertel and Sanver (2004). They consider a standard voting game where a committee elects a candidate for office, without any external interference. In their voting game, the strategies of the voters are expressions of the agents' preferences regarding candidates. They provide a quasi-characterization of the set of strong equilibrium outcomes of their voting game under any anonymous and top-unanimous voting rule.

Denote by Θ the set of all reflexive[16], transitive[17] and complete[18] orders on A. Each member $i \in N \cup \{chooser\}$ has a preference $\succsim_i \in \Theta$. Given any $i \in N \cup \{chooser\}$ and any $\succsim_i \in \Theta$, $\succ_i$ stands the strict counterpart of $\succsim_i$. Denote by $\mathbf{A}_k \equiv \{B \subseteq \mathbf{A} | \#B = k\}$ the set of all possible subsets of $\mathbf{A}$ with cardinality k.

Definition 8. *Let $M^N \equiv M_1 \times ... \times M_n$ with $M_i = M_j = M$ for all $i, j \in N$ where M is the space of actions of a proposer in N. For example, if the actions in M^N are casting single votes then $M \equiv A$. If the actions in M^N are submissions of strict preference relation then $M \equiv \Theta$. Given $k \in \{1, 2, ..., \#A\}$, a* ***screening rule for k names*** *is a function $S_k : M^N \longrightarrow A_k$ associating to each action profile $m_N \equiv \{m_i\}_{i \in N} \in M^N$ the k-element set $S_k(m_N)$.*

Definition 9. *We say that a screening rule $S_k : M^N \longrightarrow \mathbf{A}_k$ is* ***anonymous*** *if and only if given any permutation $\rho : N \rightarrow N$ of voters and any $(m_i)_{i \in N} \in M^N$, we have $S_k(\{m_i\}_{i \in N}) = S_k((m_{\rho(i)})_{i \in N})$.*

All the six screening rules described in Subsection 3.3.1 are anonymous.

Definition 10. *We say that a screening rule $S_k : M^N \longrightarrow A_k$ is* ***unanimous*** *if and only if for every set $B \in \mathbf{A}_k$ there exists $m \in M$ such that $S_k(m_N) = B$ if $m_i = m$ for every $i \in N$.*

Among all six screening rules described in Subsection 3.3.1 only the two majoritarian screening rules are unanimous.

Remark 1. *A scoring screening rule for k names is characterized by a nondecreasing sequence of real numbers $s_0 \leq s_1 \leq ... \leq s_{\#\mathbf{A}-1}$. Voters are required to rank the candidates, thus giving $s_{\#\mathbf{A}-1}$ points to the one ranked first, $s_{\#\mathbf{A}-2}$ to one ranked second, and so on. The selected list is formed by the candidates with the k highest total*

[16] Reflexive: For all $x \in A : x \succsim x$.
[17] Transitive: For all $x, y, z \in A : (x \succsim y$ and $y \succsim z)$ implies that $x \succsim z$.
[18] Complete: For all $x, y \in A : (x \neq y)$ implies $x \succsim y$ or $y \succsim x$.

point score. Any scoring screening rule for k names characterized by a nondecreasing sequence of real numbers $s_0 \leq ... \leq s_{\#\mathbf{A}-k-1} < s_{\#\mathbf{A}-k} \leq ... \leq s_{\#\mathbf{A}-1}$ *is unanimous and anonymous.*

Before presenting the results, we need first to introduce several concepts of effectivity of a coalition. They are direct extensions of the concepts of effectivity functions studied by, among others, Peleg (1984), Abdou and Keiding (1991) and Sertel and Sanver (2004). These concepts of effectivity refer to the ability of agents to ensure an outcome, under the given rule.

Definition 11. *Given a screening rule for k names* $S_k : M^N \longrightarrow \mathbf{A}_k$ *we say that a coalition* $C \subseteq N$ *of voters is* β^+-**effective** *for* $B \in \mathbf{A}_k$ *if and only if there exists* $m \in M$ *such that for every profile of the complementary coalition* $m_{N\backslash C} \in M^{N\backslash C}$ *we have that* $S_k(m_{N\backslash C}, m_C) = B$ *provided that* $m_i = m$ *for every* $i \in C$.

Notice that if C is $\beta^+ - effective$ for B then all supersets of C are $\beta^+ - effective$ for B. Denote the set of $\beta^+ - effective$ coalitions for the set $B \in \mathbf{A}_k$ by $\beta^+ s(B)$. Let $bs^+(B)$ stand for the cardinality of minimal coalition in $\beta^+ s(B)$. By convention, we $bs^+(B) = n + 1$ whenever $\beta^+ s(B)$ is empty.

Definition 12. *Given* $S_k : M^N \longrightarrow \mathbf{A}_k$, $bs_k^+ \equiv \max_{B \in \mathbf{A}_k}\{bs^+(B)\}$.

Definition 13. *Given a screening rule for k names* $S_k : M^N \longrightarrow \mathbf{A}_k$ *we say that a coalition* $C \subseteq N$ *of voters is* $\boldsymbol{\beta}^0-$**effective** *for* $x \in \mathbf{A}$ *if and only if there exists* $m \in M$ *such that for every profile of the complementary coalition* $m_{N\backslash C} \in M^{N\backslash C}$ *we have that* $x \in S_k(m_{N\backslash C}, m_C)$ *provided that* $m_i = m$ *for every* $i \in C$.

Notice that if C is $\beta^0 - effective$ for B then all supersets of C are $\beta^0 - effective$ for B. Denote the set of $\beta^0 - effective$ coalitions for $x \in \mathbf{A}$ by $\beta^0 s(x)$. Let $bs^0(x)$ stand for the cardinality of minimal coalition in $\beta^0 s(x)$. By convention, we $bs_k^0(x) = n + 1$ whenever $\beta^0 s(x)$ is empty.

Definition 14. *Given* $S_k : M^N \longrightarrow \mathbf{A}_k$, $bs_k^0 \equiv \max_{x \in \mathbf{A}}\{bs^0(x)\}$.

Definition 15. *Given a screening rule for k names $S_k : M^N \longrightarrow \mathbf{A}_k$, a coalition $C \subseteq \mathbf{N}$ of voters is* $\boldsymbol{\beta}^-$**−effective** *for $x \in \mathbf{A}$ if and only if for some set $D \in \{H \in \mathbf{A}_k | x \notin H\}$ and a $m' \in \{m \in M \mid S_k(\{m_i\}_{i\in N}) = D$ provided that $m_i = m$ for every $i \in N\}$, there exists a profile $m_C \in M^C$ such that $x \in S_k(m_C, m_{N\backslash C})$ given that $m_i = m'$ for every $i \in N\backslash C$.*

Notice that if C is $\beta^- - effective$ for B then all supersets of C are $\beta^- - effective$ for B. Denote the set of $\beta^- - effective$ coalitions for $x \in \mathbf{A}$ by $\beta^- s(x)$. Let $bs^-(x)$ stand for the cardinality of the minimal coalition belonging to $\beta^- s(x)$. By convention, set $bs^-(x) = n + 1$ whenever $\beta^- s(x)$ is empty.

Definition 16. *Given $S_k : M^N \longrightarrow \mathbf{A}_k$, $bs_k^- \equiv \min_{x\in\mathbf{A}}\{bs^-(x)\}$.*

Notice that for any screening rule for k names we have that $bs_k^+ \geq bs_k^0 \geq bs_k^-$.

Remark 2. *Let $S_k : M^N \longrightarrow A_k$ be an anonymous screening rule, if n is odd then $bs_k^+ \geq \frac{n+1}{2}$ otherwise $bs_k^+ \geq \frac{n}{2} + 1$.*

Remark 3. *Let S_k be a weakly majoritarian screening rule, if n is odd then $bs_k^+ \geq bs_k^0 \geq \frac{n+1}{2}$ otherwise $bs_k^+ \geq bs_k^0 \geq \frac{n}{2} + 1$.*

Remark 4. *Let $S_k : M^N \longrightarrow A_k$ be a majoritarian screening rule, if n is odd then $bs_k^+ = bs_k^0 = bs_k^- = \frac{n+1}{2}$ otherwise $bs_k^+ = bs_k^0 = \frac{n}{2} + 1$ and $bs_k^- = \frac{n}{2}$.*

Definition 17. *Given $q \in \{1, ..., n + 1\}$, we say that $x \in B \subseteq \mathbf{A}$ is a q-generalized Condorcet winner over B if and only if $\#\{i \in N | y \succ_i x\} < q$ for all $y \in B\backslash\{x\}$.*

Definition 18. *Given $q \in \{1, ..., n + 1\}$, we say that $x \in \mathbf{A}$ is a q-undominated candidate if and only if there exists no $y \in \mathbf{A}\backslash\{x\}$ such that $\#\{i \in N | y \succ_i x\} \geq q$ and $y \succ_{chooser} x$.*

Definition 19. *Given $l \in \{1, ..., \#A\}$, we say that $x \in \mathbf{A}$ is a chooser's ℓ-top candidate if and only if $\#\{y \in \mathbf{A}\backslash\{x\} | x \succsim_i y\} \geq \mathbf{A} - q$.*

Proposition 9. *Let $S_k : M^N \longrightarrow A_k$ be an anonymous screening rule for k names:*

1. *If a candidate x is a strong Nash equilibrium outcome of the Sincere Chooser Game then x is b_k^0-undominated and b_k^+-generalized Condorcet winner over the chooser's ($\#A - k + 1$)-top candidates.*

2. *If a candidate x is a strong Nash equilibrium outcome of the Strategic Chooser Game then x is (1) bs_k^0-undominated and chooser's ($\#A - k + 1$)-top candidate, and (2) b_k^+-generalized Condorcet winner over some set of candidates with cardinality larger or equal than $\#A - k + 1$.*

Proof. Suppose that candidate x is the outcome of a strong equilibrium of the Sincere Chooser Game. In any strong Nash equilibrium where x is the outcome, the screened set is such that x is the best candidate in this set according to the chooser's preferences. So x is a chooser's ($\#\mathbf{A} - k + 1$)-top candidate. Since the screening rule is anonymous, there exists no other chooser's ($\#\mathbf{A} - k + 1$)-top candidate that is considered better that x by b_k^+ proposers or more. Otherwise these proposers could form a coalition to impose the choice of another set where this candidate would be the preferred candidate according to the chooser's preference. Hence candidate x is a b_k^+-generalized Condorcet winner over the set of chooser's ($\#\mathbf{A} - k + 1$)-top candidates. For this same reason there exists no candidate that is considered better than x by at least b_k^0 proposers and the chooser. Hence x is b_k^0-undominated,.and the first part of the proposition is proved.

Suppose that candidate x is the outcome of a strong equilibrium of the Strategic Chooser Game. In any strong Nash equilibrium where x is the outcome, the screened set is such that x is the best candidate in this set according to the chooser's preferences. Otherwise the chooser would have incentives to choose another candidate in this set. So x is a chooser's ($\#\mathbf{A} - k + 1$)-top candidate. Since the screening rule is anonymous, there exists no set with k candidates where all the candidates in this set are considered better than x by any coalition of proposers with cardinality higher or equal than b_k^+. Otherwise this coalition would have incentives to impose this list. This is only true when candidate x is a b_k^+-generalized Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A} - k + 1$. For this same reason there exists no candidate that is considered better than x by at least bs_k^0 proposers and the chooser. This implies that candidate x is a bs_k^0-undominated candidate. Therefore the proof of the proposition is established. □

Proposition 10. *Let $S_k : M^N \longrightarrow \mathbf{A}_k$ be an unanimous screening rule for k names:*

1. *If a candidate x is a b_k^--generalized Condorcet winner over the chooser's $(\#\mathbf{A} - k+1)$-top candidates then x is a strong Nash equilibrium outcome of the Sincere Chooser Game.*

2. *If a candidate x is (1) b_k^--undominated and chooser's $(\#\mathbf{A} - k + 1)$-top candidate, and (2) b_k^--generalized Condorcet winner over some set of candidates with cardinality larger or equal than $\#\mathbf{A} - k + 1$ then x is a strong Nash equilibrium outcome of the Strategic Chooser Game.*

Proof. First let us show that if a candidate is bs_k^-−undominated and b_k^--generalized Condorcet winner over the chooser's $(\#\mathbf{A} - k + 1)$-top candidates then x is a strong Nash equilibrium outcome of the Sincere Chooser Game. Let x be such a candidate. Take any set with k candidates contained in $\mathbf{A}$ such that x is the chooser's best candidate in this set. Notice that this set exists, since candidate x is a chooser's $(\#\mathbf{A} - k + 1)$-top candidate. Let B be such a set. Since the screening rule is unanimous there exists an action such that all proposers can impose the choice of B provided that all them choose this action. Let m be such an action. Consider the strategy profile where all proposers choose action m.

Then, candidate x will be elected since the screening rule is unanimous. Notice that there exists no coalition of players that can make a profitable deviation. Because any coalition of proposers with size higher or equal than b_k^- does not have incentive in deviating since there is no candidate among the chooser's ($\#\mathbf{A}-k+1$)-top candidate that is considered better than x by all proposers in the coalition (recall that only the chooser's ($\#\mathbf{A}-k+1$)$-$top candidates can be the chooser' best name among the candidates of a set with cardinality k). Otherwise x would not be a b_k^--generalized Condorcet winner over the set of the chooser's ($\#\mathbf{A}-k+1$)-top candidates. Therefore this strategy profile is a strong Nash equilibrium of the Sincere Chooser Game.

To finish the proof we need to show that if a candidate is (1) a bs_k^--undominated and chooser's ($\#\mathbf{A}-k+1$)-top candidate, and (2) b_k^--generalized Condorcet winner over some set of candidates with cardinality higher or equal than $\#\mathbf{A}-k+1$ then there exists a strategy profile that sustains him as a strong Nash equilibrium outcome of the Strategic Chooser Game. Let us call this candidate x. Take any set with k candidates contained in $\mathbf{A}$ such that x is the chooser's best candidate in this set. Notice that this set exists, since candidate x is a chooser's ($\#\mathbf{A}-k+1$)-top candidate. Let B be such a set. Since the screening rule is unanimous there exists an action such that all proposers can impose the choice of B provided that all them choose this action. Let m be such an action. Consider the strategy profile where all proposers choose action m. Let the chooser declare a choice rule such that if candidate x is in the screened set, x is the winning candidate. Otherwise, the winner is a candidate in the screened set that is considered not better than x by more than $n-b_k^-$ proposers (notice that this choice rule exists since x is a b_k^--generalized Condorcet winner over a set with cardinality $\#\mathbf{A}-k+1$). Under this strategy profile, candidate x will be elected since the screening rule is unanimous. Notice that the chooser's strategy eliminates any incentive of any coalition of proposers with size higher or equal to bs_k^- to deviate. Notice also that candidate x is the chooser's best candidate in the screened set, so the chooser has no incentive to unilaterally deviate either. No coalition formed by at least bs_k^- proposers and the chooser has no incentive to deviate either since x is a $bs_k^-$$-$undominated candidate. Therefore this strategy profile is a strong Nash equilibrium of the Strategic Chooser Game. Therefore the proof of the proposition is established. □

3.6 References

Abdou, J. and Keiding, H. (1991) Effectivity Functions in Social Choice. Kluwer Academic Publishers, The Netherlands.

Barberà, S., Sonnenschein, H. and Zhou, L. (1991) Voting by committees. Econometrica 59:595–609.

Barberà, S., Bossert, W., Pattanaik, P. (2004) Ranking sets of objects. In: Barberà, S., Hammond, P. and Seidl, C.(Eds.) Handbook of Utility Theory, Volume II Extensions. Kluwer Academic Publishers. 893–977

Base de Datos Políticos de las Américas. Designacion de los magistrados de la corte suprema. Análisis comparativo de constituciones de los regímenes presidenciales. Georgetown University y Organización de Estados Americanos.
[Internet]. In: http://www.georgetown.edu/pdba/Comp/Judicial/Suprema /designacion.html.

Base de Datos Políticos de las Américas. Constituição da República Federativa do Brasil 1988. Georgetown University y Organización de Estados Americanos.
[Internet]. In: http://www.georgetown.edu/pdba/Constitutions/Brazil/brtitle4.html.

Base de Datos Políticos de las Américas. Constitución Política de los Estados Unidos Mexicanos, actualizada hasta Reforma de 14.08.2001. Georgetown University y Organización de Estados Americanos.
[Internet]. In: http://www.georgetown.edu/pdba/Constitutions/Mexico/ mexico2001.html

Brams, S. J., and Merrill, S. (1983) Equilibrium strategies for final-offer arbitration: there is no median convergence. Management Science 29:927-941.

Brams, S. J., and Merrill, S. (1986) Binding versus final-offer arbitration: A combination is best. Management Science 32:1346-1355.

Brasil, Governo Federal, Decreto-Lei n°5540, de 28 de novembro de 1968.

Brasil, Governo Federal, Decreto n°1916, de 23 de maio de 1996.
[Internet]. In: http://www.planalto.gov.br/ccivil_03/decreto/Antigos/D1916.htm.

Brasil, Governo Federal, Decreto n°2014 de 26 de setembro de 1996.
[Internet]. In: http://www.planalto.gov.br/ccivil_03/decreto/1996/D2014.htm.

Brasil, Governo Federal, Lei n° 10303, de 31 de Outubro de 2001.
[Internet]. In: http://www.planalto.gov.br/ccivil_03/Leis/LEIS_2001/L10303.htm.

Brasil, Superior Tribunal de Justiça, Regimento Interno do Superior Tribunal de Justiça.
[Internet]. In: http://www.dji.com.br/normas_inferiores/regimento_interno_e_sumula_stj/regimento_interno_do_superior_tribunal_de_justica.htm

Catholic Encyclopedia, ed New Advent.
[Internet]. In: http://www.newadvent.org/cathen/02581b.htm.

Código de Derecho Canónico, Libro II El Pueblo de Dios, Parte II La constitución jerárquica de la Iglesia, Sección II De las iglesias particulares y de sus agrupaciones, Título I - De las Iglesias particulares y de la autoridad constituida en ellas, Capítulo II De los Obispos, Art. 1 - De los Obispos en general, Can 375, 376, 377.
[Internet]. In: http://www.servidoras.org.ar/cgi-bin/folioisa.dll/codigola.nfo?

Crawford, V.P. (1982) Arbitration and conflict resolution in labor-management bargaining. AEA Papers and Proc. 71:205-202.

Chatterjee, K. (1981) Comparison of arbitration procedures: models with complete and incomplete information. IEEE Trans. System, Man, Cybernet, SMC-11, 101-109.

Farber, H. S. (1980) An analysis of final offer arbritation. Journal of Conflict Resolution 24:683-705.

Gehrlein, W. (1985) The Condorcet criterion and committee selection. Mathematical Social Sciences 10:199-209.

Kaymak, B. and Sanver, M. R. (2003) Sets of alternatives as Condorcet winners. Social Choice and Welfare 20:477-494.

Moulin, H. (1988) Axioms of Cooperative Decision Making. Cambridge University Press, Cambridge.

Neuhauser, F. Swezey, C. L. (1999) Preliminary evidence on the implementation of "baseball arbritation" in Workers' Compensation. Commission on Health and Safety and workers' Compesation.

Peleg, B. (1984) Game Theoretic Analysis of Voting in Committees. Econometric Society Monographs in Pure Theory, Cambridge University Press, Cambridge.

Sertel, M. R., and Sanver M. R. (2004) Strong equilibrium outcomes of voting games are the generalized Condorcet winners. Social Choice and Welfare 22: 331-347.

Stevens, C.M. (1966) Is compulsory arbitration compatible with bargaining. Industrial Relations 5:38-52.

U.S. Merit Systems Protection Board (1995) The rule of three in federal hiring: Boon or bane? Report to the President and the Congress of the United States, Washington, DC.
[Internet]. In: http://www.mspb.gov/studies/rule.pdf.

Whitman, D. (1986) Final-offer arbitration. Management Science 32:1551-1561.

Chapter 4

How to Choose a Non-controversial List with k Names

4.1 Introduction

The study of set-valued functions has a long tradition in economics, in general, and in social choice theory, in particular. The Walrasian correspondence is a salient example. More specific to social choice theory is the study of social choice correspondence and of set valued social choice functions.

The specific meaning attached to these rules can be very diverse. But there are two types of competing interpretations, depending on the nature of the objects to be chosen.

In a first interpretation, the chosen sets are sets of alternatives - that is, of mutually exclusive objects. They may be sets of candidates for office, sets of alternative policies to solve the same social problem, etc... In this case, the set cannot be seen as a full solution of the social choice problem, a further resolution is necessary, and the underlying procedure to solve the remainder of the problem is a necessary reference to complete the interpretation. To give some examples:

- It is sometimes assumed that the final choice will be made through some random procedure. (See Barberà, Dutta and Sen, 2001 and references therein).
- It is sometimes assumed that a new decision process will take place to choose from the pre-selected alternatives. This covers a wide range of possibilities, and it includes one of the leading interpretations to be referred to later, namely the use of a screening rule as part of what we will call a "rule of k names".

Turning now to a second interpretation, the elements chosen by a social choice rule need not be mutually exclusive. Here are some examples:

- The choice of new members for a club, or of several compatible projects, as in Barberà, Sonnenschein and Zhou (1991). In that case, the sets in the range can be of different cardinalities.
- The choice of locations for a fixed number of public facilities, as in Barberà and Beviá (2002).
- The choice of candidates to form a delegation, or to represent a district in a legislative body, as in Dodgson (1884, 1885a, 1885b).

In the last two cases, the cardinality of the sets to be chosen is exogenously given.

Under each of these interpretations, and many others, set valued social choices become objects of theoretical and practical interest. What questions to ask, and to eventually solve about them do depend very much on our specific interpretations and of the kind of phenomena we want to focus on.

In what follows, we shall present an analysis of set-valued functions in terms of their ability to satisfy a property that we call stability.

Our main source of motivation comes from the fact that set valued (screening) rules are part of the description of the rule of k names, an interesting and widely used set of procedures for choosing one candidate to an office. However, our analysis does only contemplate some cases where the rule of k names might be used, and it is not useful in other cases.

On the other hand, the reader may find that some of our results can be used under other circumstances, to discuss issues relating to other types of choice procedures. For example, stability may be attractive in some cases for rules that choose sets of representatives, and not in other cases. Our discussion of Dodgson's classical pamphlet is a proof of this.

4.1.1 Stability and related literature

An important part of the rule of k names is given by the procedure used by the committee in order to select, or screen out, those k candidates to be presented to the chooser. One property that we could ask from these rules is that, once the set with k candidates is chosen under some of these rules, any proposal to change this set is never supported by a strict majority of the voters.

The following definition of weak Condorcet consistency formalizes this idea: A set with k candidates is a weak Condorcet set *a la'* Fishburn (1981) if it cannot be defeated by any other set with the same cardinality on the basis of simple majority rule. A screening rule for selecting a set with k candidates is Condorcet consistency if it always selects a weak Condorcet set whenever such a set exists.

A well known weak Condorcet consistent rule is the Simpson rule: Voters are required to rank all possible sets of candidates with cardinality k. Then each set is compared with every other set. Let N(X,Y) be the number of voters ranking subset X above the subset Y. The Simpson score of X is the minimum of N(X,Y) over all Y that belongs to the set of all sets of candidates with cardinality k. The winner of the election, called a Simpson winner, is the set with the highest total point score.

In practice, however, the screening rules that are used to select sets of candidates with fixed size do not require that the voters declare their preferences over sets of candidates. In Chapter 3 we documented six different screening rules that are used by different decision bodies to make a list with k names in the first stage of the rule of k names. None of these rules require information on voters's preferences over sets of candidates. Kaymak and Sanver (2003) make an observation in the same line:

"Even when we have to choose more than one alternative from an existing set of alternatives, we use social choice rules defined on the domain of preference profiles where individual preferences are over alternatives. Hence, the final outcome, which is a set of alternatives, is determined without referring to individual preferences over sets of alternatives." (Kaymak and Sanver, 2003, page 478).

Knowing the voters's preferences over candidates is not enough to know the voters' preferences over sets of candidates. Therefore, there is no hope to find a weak Condorcet consistent rule for selecting more than one alternative and that only requires voters' information about their preferences over candidates.

Kaymak and Sanver (2003) say that a set with cardinality k strongly respects the Condorcet principle if and only if this set is a weak Condorcet set *a la'* Fishburn (1981) for any preference profile over sets of alternatives. They provide a characterization of the sets that satisfy this property in terms of individual preferences over alternatives when the preference profiles over sets of alternatives satisfy a monotonicity axiom. Among other things, they show that there is at most one set with cardinality k that satisfies this property, and that it may not exist. Moreover, a necessary but not sufficient condition for a set to satisfy this property is to be a weak Condorcet set *a la'* Gehrlein. A set with k candidates is a weak Condorcet set *a la'* Gehrlein (1985) if no candidate in this set can be defeated by any other candidate outside the set on the basis of simple majority rule.

The monotonicity axiom used by Kaymak and Sanver (2003) can be described as follows: If a voter prefers candidate a to b and b is substituted by a in the elected set then this voter cannot be worst off. Under the rule of k names, monotonicity would be a natural assumption in a scenario where no proposer has any knowledge whatsoever of the chooser's preferences over the candidates. Thus, for them, each listed name would have the same probability of being the chooser's selected candidate for the office. On the other hand, we can also easily imagine other scenarios where monotonicity axiom would not be natural. For instance, suppose an election for an office under the rule of two names, let a proposer rank candidate a first, b second and c third, and let the chooser rank b first, a second and c third. So, under the complete information assumption, the proposer's preferred list would be $\{a, c\}$ instead of $\{a, b\}$, thus violating monotonicity.

Ratliff (2003) proposes two procedures that always select Condorcet sets *a la'* Gehrlein (1985), when such a set exists: These are the Dodgson Method and the Kemeny Method. The Dodgson Method selects the set with k candidates that requires the fewest adjacent switches in the voters' preferences to become the Condorcet set. The Kemeny Method selects the set with k candidates with the smallest total margin loss against the remaining $m - k$ candidates (where m stands for the number of candidates).

By contrast, we investigate procedures that always select weak Condorcet sets. We say that screening rule for selecting k names is stable if it always selects a weak Condorcet set *a la'* Gehrlein, whenever one exists.

We show that all of the six screening rules documented in Chapter 3 which are used in reality do violate stability if the voters do not act strategically. In fact, these rules perform quite badly. They even violate other properties that are less demanding than stability. For instance, we provide an example where the lists formed under five of these rules would have the Condorcet and Borda loser candidate but not the Condorcet and Borda winner candidate!

In our search of stable procedures, we prove that most of screening rule based on standard voting rules violate stability. However, we show that it is not difficult to create stable rules. Here we make two proposals. So, why are unstable rules so popular? One possible justification, and we are aware, is that stability is not a guarantee that the elected set is a weak Condorcet set *a la'* Fishburn (1981) even under the assumption of monotonicity[1]. Our goal is to provide two other justifications. The first result states that the stability is incompatible with a another desirable property. The second one states that several unstable screening rules tend to be stable whenever the voters act strategically and cooperatively.

This chapter proceeds as follows: In the next section, weak Condorcet set *a la'* Gehrlein (1985) is defined and discussed. In Section 4.3, we give a formal definition of stability and show that most of the standard voting rules violate it. Three screening rules that satisfy stability are presented in Section 4.4. Finally, in Section 4.5, we provide two results that can explain the widespread use of unstable screening rules.

4.2 Weak Condorcet sets

For $n \geq 2$, consider a polity $N = \{1, ..., n\}$, whose members confront a nonempty finite set of candidates $\mathbf{A}$. Writing W for the set of all strict orders (transitive[2], asymmetric[3] and irreflexive[4]) on $\mathbf{A}$. Each member $i \in \mathbf{N}$ has a strict preference $\succ_i \in W$, and we let $\succ_{\mathbf{N}} \equiv \{\succ_i\}_{i \in \mathbf{N}} \in W^{\mathbf{N}}$. Given $k \in \{1, 2, ..., \#\mathbf{A}\}$, let $2^{\mathbf{A}}$ be the set of all non empty subsets of $\mathbf{A}$ and $\mathbf{A}_k \equiv \{B \in 2^{\mathbf{A}} | \#B = k\}$ be the set of all possible subsets of $\mathbf{A}$ with cardinality equal to k.

Definition 1 (Gehrlein, 1985). *Given $k \in \{1, 2, ..., \#\mathbf{A}\}$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, a set $B \in \mathbf{A}_k$ is a weak Condorcet set if for any $a \in B$ and $b \in \mathbf{A} \backslash B$ we have that $\#\{i \in \mathbf{N} | a \succ_i b\} \geq \#\{i \in \mathbf{N} | b \succ_i a\}$. Let denote by $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ the set of all weak Condorcet sets that belong to $\mathbf{A}_k$.*[5]

[1] We can learn it from Kaymak and Sanver (2003).

[2] Transitive: For all $x, y, z \in A : (x \succ y$ and $y \succ z)$ implies that $x \succ z$.

[3] Asymmetric: For all $x, y \in A : x \succ y$ implies that $\neg(x \succ x)$.

[4] Irreflexive: For all $x \in A, \neg(x \succ x)$.

[5] In fact, Gehrlein (1985) only defines Condorcet sets. A set is a Condocert set if each candidate

In other words, a set $B \in \mathbf{A}_k$ is a weak Condorcet set if no candidate that belongs to B can be defeated by any other candidate that belongs to $\mathbf{A} \backslash B$ on the basis of simple majority rule.

Example 1. *Consider the following preference profile:*

No. of voters:	1	1	1
	a	c	a
	b	b	b
	c	a	c

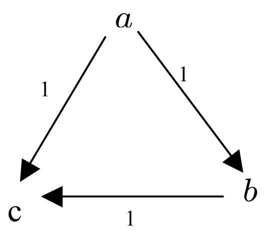

Figure 1

Figure 1 above displays the binary relations induced by the preference profile where $\#\{i \in \mathbf{N} | x \succ_i y\} > \#\{i \in \mathbf{N} | y \succ_i x\}$ *if and only if there is a line from* x *to* y *induced by the preference profile below. For any* $x, y \in \mathbf{A}$, *the number above the line from* x *to* y *indicates the difference between the number of voters that rank* x *higher than* y *and the number of voters that rank* y *higher than* x.
Examining Figure 1, we can see that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \{\{a\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, b\}\}$ *and* $E(\mathbf{A}, \succ_{\mathbf{N}}, 3) = \{\{a, b, c\}\}$.

The following proposition shows that a weak Condorcet set may not exist as pointed out first by Dodgson (1885). In the appendix, we discuss Dodgson's classical pamphlet related with this issue.

Proposition 1 (Dodgson, 1885). *For any* $k \in \{1, 2, ..., \#\mathbf{A} - 1\}$, *there exists some profile of preferences* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$ *where* $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *is empty.*

Proof. For $k = 1$, we need only to show a preference profile where there is no weak Condorcet winner candidate[6]. A typical example is the following profile:

in this set defeats any other candidate from outside the set on the basis of simple majority rule.

[6] A candidate is a weak Condorcet candidate if it cannot be defeated by any other candidate on the basis of simple majority rule.

No. of voters:	1	1	1
	a	b	c
	b	c	a
	c	a	b

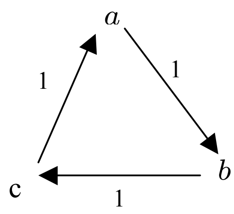

Figure 2

Given the preference profile above we have that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \emptyset$. For $k \geq 2$, we will prove the statement by construction. First, choose a preference profile with 3 candidates without a Condorcet winner. The preference profile above can be used. Then add an extra $k-1$ candidates and put them on the top of all voters preferences. The set $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ will be empty. For example, for $k = 2$ consider the following preference profile:

No. of voters:	1	1	1
	d	d	d
	a	b	c
	b	c	a
	c	a	b

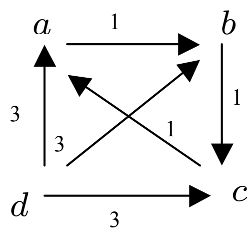

Figure 3

It can be verified in Figure 3 that $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \emptyset$. □

Remark 1. *Given $k \in \{1, 2, ..., \#\mathbf{A} - 1\}$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, if n is odd then $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ is either empty or singleton.*

In view of these negative results, it is interesting to identify some circumstances under which the existence of a weak Condorcet set is guaranteed. One case is provided below.

Definition 2 (Black, 1958). *We say that a preference profile satisfies single peakedness if and only if the elements of A can be linearly ordered as $x_1 > x_2 > ... > x_{\#\mathbf{A}}$ such that for every $i \in \mathbf{N}$ and for every $a, b \in \mathbf{A}$ we have that if $b > a > \alpha(A, \succ_i)$ or $\alpha(A, \succ_i) > a > b$ then $a \succ_i b$, where $\alpha(\mathbf{A}, \succ_i)$ is i's preferred candidate in A.*

The next result shows a sufficient condition that guarantees the existence of weak Condorcet sets.

Proposition 2. *$E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ is never empty whenever the preference profile satisfies single peakedness.*

Proof. Take any k and any profile of single peaked preferences. Let $X(1)$ be the set of all weak Condorcet winner candidates over $\mathbf{A}$. A well known result in social choice theory, proved by Black (1958), states that whenever the preference profile is single peaked there is at least one weak Condorcet winner candidate. Thus $X(1)$ is not empty.

Remove the elements of $X(1)$ from $\mathbf{A}$, and find all the weak Condorcet winner candidates over this restricted set. Denote this new set by $X(2)$. Keep defining $X(3)$, etc...until you exhaust the alternatives.

These sets cannot not be empty by the following reason: Take any strict preference profile $\succ_{\mathbf{N}} \in W^N$ that is single peaked over $\mathbf{A}$ and any subset $B \subseteq \mathbf{A}$. Restrict the preference profile $\succ_{\mathbf{N}}$ over B in the usual manner. It turns out that the restrict preference profile $\succ_{\mathbf{N}}$ over B will be single peaked. Therefore there will be a candidate $x \in B$ such that it is a weak Condorcet winner candidate over B.

For $k \leq \#X(1)$, any set formed by any k elements of $X(1)$ is a weak Condorcet set. Since the elements of this set are weak Condorcet winner candidates, they cannot be defeated by any candidate outside this set.

For $k > \#X(1)$, let i be an integer number such that $\# \bigcup_{j=1}^{i} X(j) \leq k \leq \# \bigcup_{j=1}^{i+1} X(j)$. Let $r \equiv k - \# \bigcup_{j=1}^{i} X(j)$.

Construct a set B with all elements of $\bigcup_{j=1}^{i} X(j)$ and r elements of $X(i+1)$. Notice that B will have size k (see the definition of r) and will be a weak Condorcet set. Stability comes from the fact that there will be no candidate outside this set that can defeat by strictly majority any candidate that belongs to this set. Therefore the proof is completed. □

4.3 Almost all screening rules are unstable

A screening rule for k names is a voting procedure that selects k alternatives from a given set, on the basis of actions of the voters. These actions may consist of single votes, sequential votes, the submission of preference of rankings, the filling of ballots, etc...

Definition 3. *Let* $M^N \equiv M \times ... \times M$ *where* M *is the space of actions of a voter in* N*. For example, if the actions in* M^N *are casting single votes then* $M \equiv \mathbf{A}$*. If the actions in* M^N *are submissions of strict preference relation then* $M \equiv W$*. Given* $k \in \{1, 2, ..., \#\mathbf{A}\}$*, a screening rule for* k *names is a function* $S_k : M^N \longrightarrow \mathbf{A}_k$ *associating to each action profile* $m_N \equiv \{m_i\}_{i \in N} \in M^N$ *the* k*-element set* $S_k(m_N)$*.*

Definition 4. *Given* $k \in \{1, 2, ..., \#\mathbf{A} - 1\}$ *and a preference profile* $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$*, we say that a screening rule* $S_k : M^{\mathbf{N}} \rightarrow \mathbf{A}_k$ *is stable if* $S_k(m_{\mathbf{N}}) \in E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *whenever* $E(\mathbf{A}, \succ_{\mathbf{N}}, k)$ *is not empty and* m_N *is a profile of sincere actions. For example, if* S_k *is plurality rule then a voter's sincere action is casting a vote for its preferred candidate. If* S_k *is Borda rule*[7] *then a voter's sincere action is declaring its true preferences over candidates.*

We will show in this section that almost all standard voting rules do not satisfy stability.

Example 2. *In this example, we provide a preference profile in which all screening rules documented in Chapter 3 fail simultaneously to select a weak Condorcet set.*

[7] The Borda rule is defined as follows: Voters are required to rank the candidates, thus giving $\#A - 1$ points to the one ranked first,$\#A - 2$ to one ranked second, and so on. The Borda winner is the candidate with the highest total point score. The Borda loser is the candidate with the lowest total point score.

These screening rules which are used in reality by different decision bodies around the world can be described as follows:

1) Screening 3 names by 3-votes plurality (P_3S_3): Each proposer votes for three candidates and the list has the names of the three most voted candidates, with a tie-break when needed. It is used in the election of Irish Bishops and that of Prosecutor-General in most of Brazilian states.

2) Screening 3 names by 1-vote sequential plurality (SP_1S_3): The list is made with the names of the winning candidates in three successive rounds of plurality voting. It is used in the election of English Bishops.

3) Screening 5 names by 3-votes plurality (P_5S_3): Each proposer votes for three candidates and the list has the names of the five most voted candidates, with a tie break when needed. It is used in the election of the members of Superior Court of Justice in Chile.

4) Screening 3 names by 2-votes plurality (P_3S_2): Each proposer votes for two candidates and the list has the names of the three most voted candidates, with a tie break when needed. It is used in the election of the members of Court of Justice in Chile.

5) Screening 3 names by 1-vote plurality (P_1S_3): Compute the plurality score of the candidates and include in the list the names of the three most voted candidates, with a tie break when needed. It is used in the election of rectors of public universities in Brazil.

6) Screening 3 names by 1-vote sequential strict plurality (SSP_1S_3): This is a sequential rule adopted by the Brazilian Superior Court of Justice to choose three names from a set with six names. At each stage there are twice as many candidate as there are positions to be filled in the list. Hence, if the list needs to have three names, we start by six candidates. Each proposer votes for one, and if there is an absolute majority winner, it has his name included in the list. Then, since there are two positions left, the candidate with less votes is eliminated, so as to leave four candidates to the next round. If the procedure keeps producing absolute majority winners, then

the process is continued until three names are chosen. It may be that, at some stage (including the first one), no absolute majority winner arises. Then the voters are asked to reconsider their vote and vote again. Notice that, if they persist in their initial vote, the rule leads to stalemate. Equivalently, we could say that the rule is not completely defined. However, in practice, agents tend to reassess their votes on the basis of strategic cooperative actions.
Having defined these six rules, let us now propose a case where they all fail to work properly.
Consider the preference profile below with 11 voters and 9 candidates:

Name of the voter:	1	2	3	4	5	6	7	8	9	10	11
	i	d	g	f	a	b	g	g	h	b	i
	e	b	f	a	h	d	c	b	f	c	h
	c	e	i	i	g	g	i	d	e	e	b
	d	h	h	d	e	a	a	a	a	a	a
	a	c	a	c	d	f	f	h	g	f	f
	f	a	c	g	f	e	e	f	c	h	e
	g	f	d	e	c	h	h	e	d	g	d
	h	i	e	h	i	c	d	c	i	d	c
	b	g	b	b	b	i	b	i	b	i	g

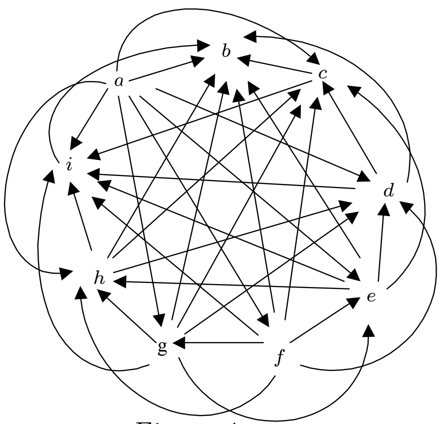

Figure 4

As it can be verified with the help of Figure 4 that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \{\{a\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, f\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 3) = \{\{a, f, g\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 4) = \{\{a, f, g, e\}\}$, $E(\mathbf{A}, \succ_{\mathbf{N}}, 5) = \{\{a, f, g, e, h\}\}$.

Now let us check whether or not the screening rules listed above select weak Condorcet sets. Assume that the ties are broken according to the following order: $a \succ b \succ c \succ d \succ e \succ f \succ g \succ h \succ i$. *It follows that:*

$$
\begin{aligned}
PS_3 &= \{b, g, i\} \\
P_3S_3 &= \{b, g, i\} \\
SPS_3 &= \{b, d, g\} \\
P_5S_3 &= \{b, c, e, g, i\} \\
P_3S_2 &= \{b, f, g\} \\
SSP_1S_3 &= \textit{Not defined}
\end{aligned}
$$

Therefore all these six rules fail to satisfy the stability.

The table below shows the Borda score of each candidate.

Candidates	*Borda score*
a	56
f	49
g	47
h	45
e	44
d	43
c	39
i	37
b	36

Notice that Candidate a *is the Condorcet and the Borda winner candidate, and yet does not belong to the outcomes of those screening rules. Moreover, five of the rules do select candidate* b, *who is the Condorcet and the Borda loser candidate.*

Our next proposition states that any screening method based on scoring voting rules fails to satisfy stability.[8]

Definition 5. *A scoring voting rule is characterized by a nondecreasing sequence of real numbers* $s_0 \leq s_1 \leq ... \leq s_{\#\mathbf{A}-1}$ *with* $s_0 < s_{\#\mathbf{A}-1}$. *Voters are required to rank the candidates, thus giving* $s_{\#\mathbf{A}-1}$ *points to the one ranked first,* $s_{\#\mathbf{A}-2}$ *to one ranked second, and so on. The winner of the election is the candidate with the highest total point score (see Moulin, 1988).*

Proposition 3. *When a scoring rule or any sequential application of a scoring rule are used to make a list of k names, the resulting screening rule does not satisfy stability provided that ties are broken according to a fixed ordering over* $\boldsymbol{A}$.

Proof. For $k = 1$. Consider the following profile with 17 voters and 3 candidates.[9]

No. of voters:	3	6	4	4
	c	a	b	b
	a	b	a	c
	b	c	c	a

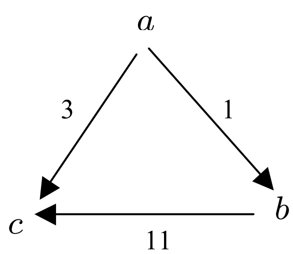

Figure 5

Here candidate a is the Condorcet winner. However for any scoring method candidate b will be elected. So the elected outcome will not be a weak Condorcet. Let us show why a cannot be elected.

score of $a = 6s_2 + 7s_1 + 4s_0$

[8] Gehrlein (1985) provides estimations of the conditional probability of one-stage constant scoring rules selecting the Condorcet set given that such a set exists, in a context with m candidates and an infinitely large number of voters. One-stage constant scoring rules can be described as follows: Each voter is instructed to vote for q candidates and the k most voted candidates are selected.

[9] This preference profile was used in Fishburn (1984) to prove that the scoring voting rules do not satisfy Condorcet consistency (see Moulin, 1988, page 232).

score of $b = 8s_2 + 6s_1 + 3s_0$

(score of b) $-$ (score of a) $= (s_2 - s_1) + (s_2 - s_0) > 0$

The inequality above is strict because $(s_2 - s_1)$ is nonnegative and $(s_2 - s_0)$ is strict positive.

For $k \geq 2$, we only need to add $k - 1$ candidates at the top of this preference profile. See the example for $k = 2$. We add a candidate d, putting him at the top of $\succ_{\mathbf{N}}$.

No. of voters:	3	6	4	4
	d	d	d	d
	c	a	b	b
	a	b	a	c
	b	c	c	a

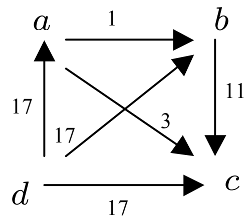

Figure 6

Notice that the only weak Condorcet set is $\{d, a\}$. However for any sequential application of a scoring method the elected set is $\{d, b\}$. This set is not weak Condorcet set, since the majority of the voters prefers a to b. For the case of a simultaneous application of a scoring method the proof need to be a little bit more elaborated. Notice that

score of $a = 6s_2 + 7s_1 + 4s_0$.

score of $b = 8s_2 + 6s_1 + 3s_0$.

score of $c = 3s_2 + 4s_1 + 10s_0$.

score of $d = 17s_3$.

(score d+score b) $\geq$ (score x+score y) for every $x, y \in \{a, b, c, d\}$.

Take any scoring rule such that the inequalities above are strict so we have that $\{d, b\}$ is elected and then this screening rule violates stability.

Now take any scoring rule such that there is $\{x, y\} \subset \{a, b, c, d\}$ with $\{x, y\} \neq \{d, b\}$ such that (score d+score b) $=$ (score x+score y). Since the ties are broken according to a fixed ordering over $\mathbf{A}$, there exists a permutation of the names of the candidates that breaks the ties in favor of the set $\{d, b\}$. Therefore we have completed the proof. $\square$

We now explore the consequence of trying to use a different rationale, that of Copeland's rule, in order to select our sets of candidates.

Definition 6. *Compare candidate a with every other candidate x. Give a score +1 if a majority prefers a to x, -1 if a majority prefers x to a, and 0 if it is a tie. Adding up those scores over all* $x \in \mathbf{A}\backslash\{a\}$ *yields the Copeland score of a. The winner of the election, called a Copeland winner, is the candidate with the highest total point score (see Moulin 1988).*

The Copeland method always selects a Condorcet winner candidate whenever one exists.

Proposition 4. *For any* $k \in \{1, ..., \#A-1\}$*, screening a list of k names by applying the Copeland rule, either sequentially or one shot, does not guarantee stability.*

Proof. In the preference profile below we have that $E(\mathbf{A}, \succ_{\mathbf{N}}, 1) = \{\{a\}, \{c\}\}$ and $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, c\}\}$. However, applying sequentially Copeland rule or taking those with highest scores leads to $\{d\}$ when $k = 1$ and $\{d, c\}$ when $k = 2$.

No. of voters:	1	1	1	1
	a	e	d	b
	c	f	h	d
	b	g	g	a
	d	c	f	e
	h	h	c	c
	g	a	e	f
	f	b	a	g
	e	d	b	h

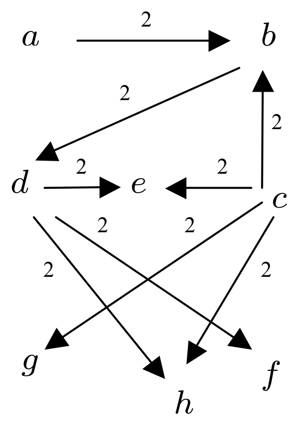

Figure 7

Candidates	1^{st} stage Copeland score	2^{st} stage Copeland score
a	1	1
b	0	-1
c	2	2
d	3	-
e	-1	0
f	-1	0
g	-1	0
h	-2	-1

Therefore the proof is established for $k \in \{1, 2\}$. To prove the result for $k > 2$, we need just to add $k - 2$ candidates at the top of this preference profile. □

We now turn to a similar analysis of screening rules based on the Simpson score.

Definition 7. *Compare candidate a with every other candidate x. Let $N(a, x)$ be the number of voters preferring a to x. The Simpson score of a is the minimum of $N(a, x)$ over all $x \in \mathbf{A} \backslash \{a\}$. The winner of the election, called a Simpson winner, is the candidate with the highest total point score (see Moulin, 1988).*

The Simpson method always selects a weak Condorcet candidate whenever one exists.

Proposition 5. *For $k = 2$, making a list of two names by selecting the two candidates with highest Simpson score (one shot method) does not guarantee stability. However, applying the Simpson rule sequentially does.*

Proof. First let us prove that making a list of two names by selecting the two candidates with highest Simpson score (one shot method) does not satisfy stability. We will prove it through an example with 3 voters and 3 candidates.

No. of voters:	2	1
	a	c
	b	a
	c	b

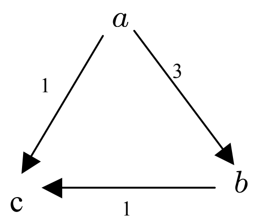

Figure 8

Candidates	Simpson score
a	2
b	0
c	1

Thus the elected outcome is $\{a, c\}$. However $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, b\}\}$.

Now let us prove that for $k = 2$, applying Simpson rule sequentially satisfies stability. The proof is trivial since it can be easily proved that $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{x, y\} \subseteq \mathbf{A} | x$ is a weak Condorcet winner over $\mathbf{A}$ and y is a weak Condorcet winner over $\mathbf{A} \setminus \{x\}\}$. In addition, the set of winning candidates under the Simpson rule is the set of all weak Condorcet winners. □

Proposition 6. *For $k \geq 3$, screening k names applying Simpson rule, either sequentially or in one shot, does not satisfy stability.*

Proof. This proposition will be proved with an example with 6 voters and 5 candidates.

No. of voters:	1	1	1	1	1	1
	d	a	d	c	d	b
	a	b	c	a	b	c
	b	c	a	b	c	a
	c	d	b	d	a	d

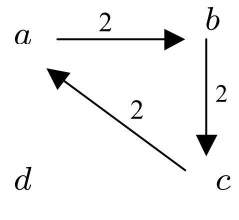

Figure 9

Candidates	Simpson score
a	2
b	2
c	2
d	3

Notice that $E(\mathbf{A}, \succ_{\mathbf{N}}, 3) = \{\{a, b, c\}\}$. However it is easy to see that if we apply the Simpson rule, either sequentially or one shot, the elected set must contains d. To prove this for $k > 3$, we need just to substitute, in the preference profile above, the cycle of size 3 for another cycle with size k such that candidate d still is the unique weak Condorcet winner. Therefore this completes the proof. □

Now let us turn our attention to a method that was proposed specifically to select a Condorcet set provided that one exists.

Definition 8. *The Dodgson method for selecting a set with cardinality k: Compute for each set $B \in \mathbf{A}_k$ the minimum number of adjacency switches on the voters' preferences required for B to become the Condorcet set. The winner is the set with k candidates that requires the fewest adjacency switches (see Ratliff, 2003).*

The proposition below shows for some preference profiles the Dodgson method fails to select a weak Condorcet set.

Proposition 7. *For $k \geq 2$, the Dodgson method for selecting a set with cardinality k does not satisfy stability.*

Proof. In the preference profile below we have that $E(\mathbf{A}, \succ_{\mathbf{N}}, 2) = \{\{a, c\}\}$. However, applying Dodgson method leads to $\{d, c\}$.

No. of voters:	1	1	1	1
	a	c	d	b
	c	g	g	d
	b	f	f	a
	d	e	e	c
	e	a	c	e
	f	b	a	g
	g	d	b	f

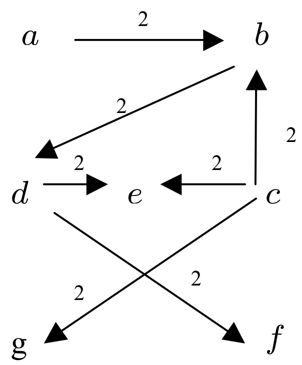

Figure 10

Therefore the proof is established. To prove for $k > 2$, we need just to add $k - 2$ candidates at the top of this preference profile. □

The proposition above shows that stability is stronger than the requirement of choosing the Condorcet set when such a set exists.

4.4 Some stable screening rules

In this section, we present three stable screening rules for selecting a set with cardinality k.

Definition 9. *The total margin of loss of a set $S \in 2^{\mathbf{A}}$ to the candidates in $\mathbf{A} \backslash S$ induced by a profile of preferences $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$ is denoted by $KE(\mathbf{A}, \succ_{\mathbf{N}}, S)$ and defined over* ***A*** *as follows:*

$$KE(\mathbf{A}, \succ_{\mathbf{N}}, S) = \sum_{y \in \mathbf{A} \backslash S \text{ and } x \in S} Max\{0, \#\{i \in \mathbf{N} | y \succ_i x\} - \#\{i \in \mathbf{N} | x \succ_i y\}\}$$

The following method was proposed by Ratliff (2003). It is a generalization of the procedure proposed by John Kemeny in 1959.

Definition 10. *The Kemeny Method (KE_k): Compute the KE score for all subsets of candidates with cardinality k. The elected set is the set with the lowest KE score.*

We also propose the following two stable procedures to select sets with k candidates.

We first introduce the Minimal Number of External Defeats Rule.

Definition 11. *The number of external defeats of a set $S \in 2^{\mathbf{A}}$ induced by a profile of preferences $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$ is denoted by $NED(\mathbf{A}, \succ_{\mathbf{N}}, S)$ and defined over $\mathbf{A}$ as follows:*

$$NED(\mathbf{A}, \succ_{\mathbf{N}}, S) = \sum_{x \in S} \#\{y \in \mathbf{A} \backslash S | \#\{i \in \mathbf{N} | y \succ_i x\} > \#\{i \in \mathbf{N} | x \succ_i y\}\}$$

If the monotonicity axiom holds, then the NED score of a set can be interpreted as the number of proposals of substitution of a candidate in the set for another from outside this set that would receive a majority support.[10]

The Minimal Number of External Defeats Rule for selecting a set with cardinality k chooses the set that minimizes the NED score.

Definition 12. *The Minimal Number of External Defeats Rule (NED_k): Compute the NED score for all subsets of candidates with cardinality k. The elected set is the set with the lowest NED score.*

Now let us present the Minimal Size of External Opposition Rule.

Definition 13. *The size of external opposition of a set $S \in 2^{\mathbf{A}}$ induced by a profile of preferences $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$ is denoted by $SEO(\mathbf{A}, \succ_{\mathbf{N}}, S)$ and defined over $\mathbf{A}$ as follows:*

$$SEO(\mathbf{A}, \succ_{\mathbf{N}}, S) = \underset{y \in \mathbf{A} \backslash S \text{ and } x \in S}{Max} \#\{i \in \mathbf{N} | y \succ_i x\}$$

If the monotonicity axiom holds, then the SEO score of a set can be interpreted as the maximum number of voters that would support a proposal of substitution of a candidate in the set for another from outside this set.

[10] Monotonicity axiom: If a voter prefers candidate a to b and b is substituted by a in the elected set then this voter cannot be worst off.

The Minimal Size of External Opposition Rule for selecting a set with cardinality k chooses the set that minimizes the NED score.

Definition 14. *The Minimal Size of External Opposition Rule (SEO_k): Compute the SEO score for all subsets of candidates with cardinality k. The elected set is the set with the lowest SEO score.*

Note that SEO_k rule can be viewed as an adaptation of the Simpson rule for selecting sets with fixed size.

Example 3. *Let us give an example in order to clarify the definitions above.*

No. of voters:	3	1	1	1
	d	d	d	d
	a	b	c	b
	b	c	b	c
	c	a	a	a

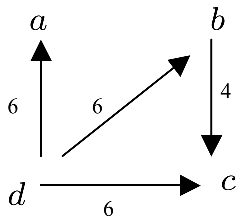

Figure 11

Sets with cardinality three	*SEO*	*NED*	*KE*
$\{a, b, c\}$	*6*	*3*	*18*
$\{a, b, d\}$	*3*	*0*	*0*
$\{a, c, d\}$	*5*	*1*	*4*
$\{b, c, d\}$	*3*	*0*	*0*

The sets $\{a, b, d\}$ and $\{b, c, d\}$ are the weak Condorcet sets with cardinality three. Notice that both sets are the winning sets under the KE_3, NED_3 and SEO_3 methods.

Remark 2. *Given a $k \in \{1, ..., \#\mathbf{A}\}$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, the following statements are equivalent:*

1. $B \in E(\mathbf{A}, \succ_{\mathbf{N}}, k)$.
2. $NED(\mathbf{A}, \succ_{\mathbf{N}}, B) = 0$.

3. $SEO(\mathbf{A}, \succ_{\mathbf{N}}, B) \leq n/2$.

4. $KE(\mathbf{A}, \succ_{\mathbf{N}}, B) = 0$.

Remark 3. *Given a $k \in \{1, ..., \#\mathbf{A}\}$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, the following statements are equivalent:*

1. $B, C \in E(\mathbf{A}, \succ_{\mathbf{N}}, k)$.

2. $NED(\mathbf{A}, \succ_{\mathbf{N}}, B) = NED(\mathbf{A}, \succ_{\mathbf{N}}, C) = 0$.

3. $SEO(\mathbf{A}, \succ_{\mathbf{N}}, B) = SEO(\mathbf{A}, \succ_{\mathbf{N}}, C) = n/2$.

4. $KE(\mathbf{A}, \succ_{\mathbf{N}}, C) = KE(\mathbf{A}, \succ_{\mathbf{N}}, C) = 0$.

Remark 4. *These three stable methods KE_k, NED_k and SEO_k may select different sets if there exists no weak Condorcet sets.*

No. of voters:	2	1	1	2
	a	d	c	b
	b	a	d	c
	c	b	a	d
	d	c	b	a

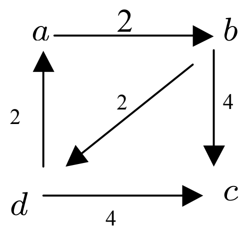

Figure 12

Candidates	*SEO*	*NED*	*KE*
$\{a, b, c\}$	*4*	*1*	*4*
$\{a, b, d\}$	5	*1*	*4*
$\{a, c, d\}$	5	*2*	*6*
$\{b, c, d\}$	*4*	*1*	*2*

Notice that there is no weak Condorcet set with cardinality three. The SEO_3 method selects $\{a, b, c\}$ and $\{b, c, d\}$, NED_3 method selects $\{a, b, c\}, \{a, b, d\}$ and $\{b, c, d\}$ and the KE_3 methods selects $\{b, c, d\}$.

4.5 Why are unstable screening rules so popular?

We have shown in the previous sections that unstable screening rules are often used. In fact, we do not have any example of stable screening rules that are actually used by some decision body. We also have shown that is not difficult to create stable screening rules. In this section we provide two results that can be viewed as a solution for this puzzle.

4.5.1 Impossibility result

The following proposition shows that stability is incompatible with another desirable property that one might expect from a screening rule.

Proposition 8. *There exists no screening rule that for every k satisfies the following natural axioms:*

Axiom I: Any listed name should not be excluded if the list is enlarged. In other words, if a candidate is included in the chosen list with k names then he should be also in the chosen list with $k+1$ names.

Axiom II (stability): A chosen list with k names should be formed from a weak Condorcet set with cardinality k, whenever such one exists.

Proof. The proof of this proposition is very simple. Let us prove by contradiction. Suppose that there exist screening rules for k names that satisfy both axioms I and II. Consider the following preference profile:

No. of voters:	1	1	1	1	1	1
	a	a	a	c	e	d
	b	b	b	d	c	e
	c	e	d	e	d	c
	d	c	e	a	a	a
	e	d	c	b	b	b

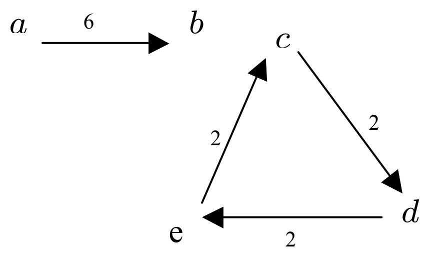

Figure 13

Notice by Axiom II we have that for $k = 2$, the selected set have to be $\{a, b\}$ and for $k = 3$, the selected set have to be $\{c, e, d\}$. Thus Axiom I is violated since $\{a, b\}$ is not contained in $\{c, e, d\}$. Therefore the proof of the proposition is completed. □

Remark 5. *It turns out that all the screening rules documented in Chapter 3 and any screening rule based on the sequential application of any voting rules satisfy Axiom I.*

4.5.2 A strategic analysis: The Random Chooser Game

We now study the case where the voters act strategically and cooperatively. More specifically, we propose a voting game where the players choose by voting a subset of candidates with a fixed size from a given set of candidates. We study the properties of the strong Nash equilibrium outcomes of this game.

Definition 15. *Given $k \in \{1, 2, ..., \#A\}$, a screening rule for k names $S_k : M^{\mathbf{N}} \longrightarrow A_k$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, the Random Chooser Game can be described as follows: It is a simultaneous game with complete information where each voter $i \in \mathbf{N}$ chooses a message $m_i \in M$. Given $m_N \equiv \{m_i\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$, $S_k(m) \in \mathbf{A}_k$ is the screened set. Each voter $i \in N$ has a payoff function $u_i : M^{\mathbf{N}} \rightarrow \mathbf{R}$ that satisfies the following axioms: (Axiom 1) For any $m_{\mathbf{N}}, m'_{\mathbf{N}} \in M^{\mathbf{N}}$ we have that $u_i(m_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ only if $S_k(m_{\mathbf{N}}) \neq S_k(m'_{\mathbf{N}})$, and (Axiom 2) for any $m_{\mathbf{N}}, m'_{\mathbf{N}} \in M^{\mathbf{N}}$ and any $y, x \in A$ we have that $u_i(m_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ if $x \succ_i y$, $y \in S_k(m'_{\mathbf{N}})$ and*

$S_k(m_{\mathbf{N}}) = \{x\} \cup (S_k(m'_{\mathbf{N}}) \backslash \{y\})$.[11]

Axiom 2 is a modified version of the monotonicity axiom of Kannai and Peleg (1984), used among others by Roth and Sotomayor(1990) and Kaymak and Sanver (2003). Under the rule of k names, this assumption would be a very natural one only if the committee members, who are supposed to choose by voting a list with k names, do not have any knowledge whatsoever of the chooser's preferences over the candidates. Thus for them, each listed name would have the same probability of being the chooser's selected candidate for the office.

Let us introduce the solution concept that we will use to analyze this game.

Definition 16. *Given $k \in \{1, 2, ..., \#A\}$, a screening rule for k names $S_k : M^{\mathbf{N}} \longrightarrow A_k$ and a preference profile $\succ_{\mathbf{N}} \in W^{\mathbf{N}}$, a joint strategy $m_{\mathbf{N}} = \{m_i\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$ is a pure strong Nash equilibrium of the Random Chooser Game if and only if, given any coalition $C \subset N$, there exists no $m'_{\mathbf{N}} \equiv \{m'_i\}_{i \in \mathbf{N}} \in M^{\mathbf{N}}$ with $m'_j = m_j$ for every $j \in N \backslash C$ such that $u_i(m'_{\mathbf{N}}) > u_i(m_{\mathbf{N}})$ for each $i \in C$.*

Two out of the six screening rules documented in the third chapter are majoritarian.

Definition 17. *We say that a screening rule $S_k : M^{\mathbf{N}} \longrightarrow \mathbf{A}_k$ is majoritarian if and only if for every set $B \in A_k$ there exists $m \in M$ such that for every strict majority coalition $C \subseteq \mathbf{N}$ and every profile of the complementary coalition $m_{\mathbf{N} \backslash C} \in M^{\mathbf{N} \backslash C}$ we have that $S_k(m_{\mathbf{N} \backslash C}, m_C) = B$ provided that $m_i = m$ for every $i \in C$.*

Proposition 9. *Let $S_k : M^{\mathbf{N}} \longrightarrow A_k$ be a majoritarian screening rule. If a set is a pure strong Nash equilibrium outcome of the Random Chooser Game then it is a weak Condorcet set.*

[11] This is a modified version of the monotonicity axiom of Kannai and Peleg (1984), used by Kaymak and Sanver (2003) (see Kaymak and Sanver, 2003).

Proof. Suppose that a subset $B \subset \mathbf{A}$ with cardinality k is an outcome of a strong equilibrium of the Random Chooser Game. Thus there exists a strong Nash equilibrium strategy profile $m'_{\mathbf{N}} \equiv \{m'_i\}_{i\in\mathbf{N}} \in M^{\mathbf{N}}$ such that $S_k(m'_{\mathbf{N}}) = B$. Suppose by contradiction that B is not a weak Condorcet set. Then there exists $x \in B$ and $y \in A\backslash B$ such that a strict majority of the voters prefers y to x. Let $D \equiv \{y\} \cup B\backslash\{x\}$ and $C \equiv \{i \in \mathbf{N}| y \succ_i x\}$. Since the screening rule is majoritarian and $\#C > \frac{n}{2}$, there exists $m''_{\mathbf{N}} \equiv \{m''_i\}_{i\in\mathbf{N}} \in M^{\mathbf{N}}$ with $m''_j = m'_j$ for every $j \in N\backslash C$ such that $S_k(m''_{\mathbf{N}}) = D$. By Axiom 2, we have that $u_i(m''_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ for every $i \in C$. This is a contradiction since $m'_{\mathbf{N}}$ is a strong Nash equilibrium. Therefore any Strong Nash equilibrium outcome need to be a weak Condorcet set. $\square$

This result implies that any majoritarian screening rule tends to be stable if the voters act strategically and cooperatively provided that the monotonicity axiom holds.

In the example below, we provide a preference profile over candidates in which there is an unique Condorcet set with cardinality two. However, for a given players' payoff functions that satisfy both axioms 1 and 2, the set of Strong Nash equilibrium outcome of the Random Chooser Game is empty provided that the screening rule is majoritarian.

Example 4. *Consider the following preference profile:*

No. of the voter:	1	1	1
	a	c	d
	b	b	a
	c	a	b
	d	d	c

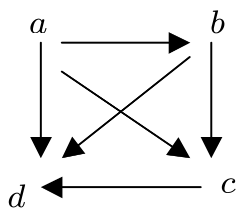

Figure 14

Notice that the any candidate of the set $\{a, b\}$ defeats any other candidate of $\mathbf{A}\backslash\{a, b\}$ on the basis of simple majority rule. Hence $\{a, b\}$ is a Condorcet set a la' Gerhlein (1985).

Consider the following preference profile over sets of candidates:

No. of the voter:	1	1	1
	$\{a,b\}$	$\{b,c\}$	$\{a,d\}$
	$\{a,c\}$	$\{a,c\}$	$\{b,d\}$
	$\{a,d\}$	$\{c,d\}$	$\{c,d\}$
	$\{b,c\}$	$\{a,b\}$	$\{a,b\}$
	$\{b,d\}$	$\{b,d\}$	$\{a,c\}$
	$\{c,d\}$	$\{a,d\}$	$\{b,c\}$

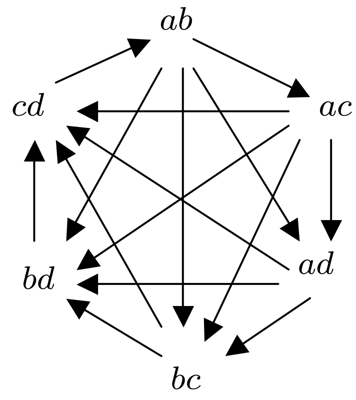

Figure 15

First notice that preference profile above is a lexicographic extension of the preference profile over candidates.

Let $S_2 : M^{\mathbf{N}} \longrightarrow A_2$ *be a majoritarian screening rule. Let the voters' payoff function be represented by any function* $u_i : M^{\mathbf{N}} \to \mathbf{R}$ *that satifies the following condition : For any* $m_{\mathbf{N}}, m'_{\mathbf{N}} \in M^{\mathbf{N}}$ *we have that* $u_i(m_{\mathbf{N}}) > u_i(m'_{\mathbf{N}})$ *if and only if voter* i *prefers* $S_2(m_{\mathbf{N}})$ *to* $S_2(m'_{\mathbf{N}})$*. Notice that the players' payoff functions satisfy axioms 1 and 2. As we can see in Figure 15, there exists a strict majority of voters that prefers* $\{c,d\}$ *to* $\{a,b\}$*. Hence* $\{a,b\}$ *cannot be a strong Nash equilibrium outcome of the Ramdon Chooser Game since the screening rule is majoritarian. Therefore, by Proposition 9, the set of strong Nash equilibrium outcome of the Random Chooser Game is empty.*

4.6 Concluding remarks

We have shown that all of the six screening rules documented in Chapter 3 violate stability if the voters do not act strategically. In our search for stable procedures, we have proved that any procedure based on scoring rules or resulting from a sequential use of standard Condorcet consistent methods such as those of Simpson and

Copeland, will also violate this property.

We have shown that it is not difficult to create stable rules. Here we made two proposals. They are the Minimal Number of External Defeats (NED) and the Minimal Size of External Opposition (SEO) procedures. The selected sets under NED and SEO procedures are those that have the lowest, respectively, NED and SEO scores. The NED score of a set is the number of proposals of substitution of a candidate in the set for another from outside this set that would receive a majority support. The SEO score of a set is the maximum number of voters that would support a proposal of substitution of a candidate in the set for another from outside this set.

We provide two results that can explain the widespread use of unstable screening rules. The first one states that there exists no stable screening rule that satisfies the following natural axiom: Any listed name should not be excluded if the list is enlarged. In other words, if a candidate is included in the chosen list with k names then he should be also in a larger list. It turns out that all the screening rules documented in Chapter 3 satisfy the axiom above. Therefore, leaving aside stability can be seen as a price to pay for a rule to keep an even more important or desirable property.

The second justification comes from the remark that any majoritarian procedure tends to select weak Condorcet sets if the agents act strategically and cooperatively. More specifically, we propose a voting game where under any majoritarian procedure, a set is a strong Nash equilibrium outcome only if it is a weak Condorcet winner set *a la'* Gehrlein (1985). Two out of the six screening rules documented in Chapter 3 satisfy majoritarian property.

4.7 Appendix

Considered one of the founders of the Social Choice theory, Rev Charles Lutwidge Dodgson (Lewis Carroll) wrote in 1884 a pamphlet entitled "The principle of parliamentary representation"[12]. In this pamphlet, Dodgson argued in favor of the contention of the Society for Proportional Representation that each district should return several representatives instead of only one. He also proposed a modification of the Society's method for transferring the spare votes of a candidate that has more votes than they need to be returned. The Society's method specified that each voter should mark on his paper his second-best man, his third-best, and so on: and, if his first man already returned, his vote would be used for his second, and so on. Dodgson argued that this method would not secure a just result. His proposal was that each elector cast a vote for one candidate only, and the candidates, after the announcement of how many votes each of them received, can freely distributed their votes to others, so as to bring in others beside those already announced as returned.

In the supplement of this pamphlet dated February 1885, Dodgson provided the following example to show the superiority of his method:

Let a district with 11999 electors have to return 3 representatives. Let 5 candidates stand, 3 liberals (Chamberlain, Gladstone and Goschen), 1 Independent Liberal (Hartington), and 1 Conservative (Northcote). Suppose the method used to return the representatives is the one proposed by the Society of Proportional Representation. Let 11999 voting-papers be filled up as follows:

CHAMBERLAIN	4	4	2	1	4	-
GLADSTONE	1	2	1	2	2	-
GOSCHEN	3	3	4	4	1	-
HARTINGTON	2	1	3	3	3	-
NORTHCOTE	-	-	-	-	-	1
Nos. of papers	3030	2980	2020	1100	790	2079

[12] William Gehrlein informed us about the existense of this pamphlet and gave us a copy of it. He got it from a Duncan Black's former student after he has published a paper about Condorcet winner sets. Very few people know about this pamphlet since it is not one of the famous three pamphlets reproduced by Duncan Black in his classical book entitled "The theory of committees and elections" (Black, 1958).

It can easily verified that under the Society's method Gladstone, Hartington, and Goschen would be returned. Under Dodgson's method Gladstone, Hartington, and Chamberlain would club their votes, making 9130 votes, which would suffice to return all three.

Dodgson wrote:

> " Also it is clear that, as a matter of justice, Gladstone, Hartington, and Chamberlain ought to be returned, since there are 6010 electors who put Gladstone and Hartington as their first two favorites, and, over and above these, 3120 who put Gladstone and Chamberlain as their first two...
>
> May I, in conclusion, point out that the method advocated in my pamphlet (where each elector names one candidate only, and the candidates themselves can after the number are announced club their votes, so as to bring in others beside those already announced as returned) would be at once perfectly simple and perfectly equitable in its result?" (Dodgson, 1985a, pages 5 and 7).

Dodgson's remark stated above did not convince the members of the Society for Proportional Representation. They did not agree that Chamberlain was the right man to be returned. Since the majority of the voters would prefer Goschen to Chamberlain. So, Dodgson was forced to write a postcript to supplement of his pamphlet dated February 1885. In this postcript to supplement, Dodgson wrote:

> "Objection has been taken to my statement on page 5 ("it is clear that, as a matter of justice, Gladstone, Hartington, and Chamberlain ought to be returned") on the ground that, of the 9920 Liberal electors there are 6800 who prefer Goschen to Chamberlain, while there are only 3120 who prefer Chamberlain to Goschen. And it has been pressed upon me that,

after all, Goschen is the right man to return, so that the Society's method does not break down in this instance." (Dodgson, 1985b, page 1).

Dodgson rejected the Society's criticism using two different arguments:

"Now, first, we might almost on a priory considerations reject such a test as manifestly unfair. For does it not involve the using an elector's voting-power more than once? We first let an elector exhaust his full voting-power in helping to return Gladstone; and after that, we allow his opinion to have weight in deciding between two other candidates. Is not this to abandon the principle, adopted by the society, that each elector shall have one vote only? But, secondly, this test may be easily proved to be valueless, by a simple reduction ad absurdum." (Dodgson, 1985b, pages 1 and 2).

He showed that in some circumstances there exists no set with three candidates such that any candidate in the set cannot be defeated by any one outside it on the basis of simple majority. Dodgson changed the voting configuration of the table above to show this point, as follows:

CHAMBERLAIN	4	4	2	1	3	-
GLADSTONE	1	2	1	2	2	-
GOSCHEN	3	3	4	4	1	-
HARTINGTON	2	1	3	3	4	-
NORTHCOTE	-	-	-	-	-	1
Nos. of papers	1826	1712	1826	1712	1910	3013

Note that the majority of voters prefers Chamberlain to Hartington, Hartington to Goschen, Goschen to Chamberlain. So for any set of candidates with cardinality three, there will be always one candidate in the set that is considered worst than some candidate outside the set by the majority of the voters. After showing this example Dodgson concluded:

> "This lands us in a hopeless circle: and the logical conclusion I believe to be that the proposed test is absolutely valueless." (Dodgson,1985b, page 3).

We have three comments about this debate between Dodgson and the Society of Proportional Representation. First, the Society's method does not pass the "test", either. We propose the following example:

Let 17000 voting papers be filled up as follows:

CHAMBERLAIN	1	1	1	1
GLADSTONE	2	2	2	2
GOSCHEN	4	3	4	5
HARTINGTON	3	5	5	4
NORTHCOTE	5	4	3	3
Nos. of papers	3000	6000	4000	4000

Here the necessary quota is 4251.

Following the Society's method, in the first count would give:

Chamberlain 17000

Hartington 0

Gladstone 0

Northcote 0

Goschen 0

Thus Chamberlain is returned, with 12749 votes to spare, the whole of wish must go to Hartington.

In the second count would give:

Chamberlain 4251

Gladstone 12749

Hartington 0

Northcote 0

Goschen 0

Thus Gladstone is returned, with 8498 votes to spare, which must divided between Hartington, Goschen and Northcote in the proposition of 3:6:8.

In the third count would give:

Chamberlain 4251

Gladstone 4251

Hartington 1500

Goschen 2999

Northcote 3999

Thus Northcote is returned.

Therefore Chamberlain, Gladstone and Northcote would be returned. However, the majority of the voters prefers Goschen to Northcote. The set {Chamberlain, Gladstone and Goschen} is Condorcet set since all candidates in this set defeats any other candidate from outside it. Note that Dodgson's method does not break down in this instance.

Our second comment is that the fact that sometimes there exists no weak Condorcet set cannot be the basis for a criticism to any particular rule, since this non-existence is prior to it. The most we can ask for a rule is to select such an alternative when it exists.

Finally, our last comment is that probably Dodgson rejected this "test" because he believed that it would give full power to the majority. In some circumstances, 49% of the electors would not return any candidates, and using his own words, 49% of the votes would be wasted. This is true. However, this criticism would not apply for the case of screening rules, since only one candidate from its elected outcome will be chosen for office. This proves that although formally we move in the same framework, the interpretation given to set-valued rules is crucial in order to appreciate the validity of certain axioms or the criticisms to any given rule.

4.8 References

Barberà, S., Sonnenschein, H. and Zhou, L. (1991) Voting by committees. Econometrica, 59:595–609.

Barberà, S., Dutta, B. and Sen, A. (2001) Strategy-proof social choice correspondence. Journal of Economic Theory, 101:374-394,

Barberà, S. and Beviá, C. (2002) Self-selection consistent functions. Journal of Economic Theory, 105: 263-277.

Black, D. (1958). The Theory of Committees and Elections. Cambridge University Press, Cambridge.

Dodgson, C.L. (1884) The Principle of Parliamentary Representation. mimeo.

Dodgson, C.L. (1885a) The Principle of Parliamentary Representation: Supplement. mimeo.

Dodgson, C.L. (1885b) The Principle of Parliamentary Representation: Postscript to Supplement. mimeo.

Fishburn, P.C. (1981) An analysis of simple voting systems for electing committees. SIAM Journal on Applied Mathematics, 33:469-489.

Fishburn, P.C.(1984) Discrete mathematics in voting and group choice. SIAM Journal of Algebraic and Discrete Methods, 5:263-75.

Gehrlein, W. (1985) The Condorcet criterion and committee selection. Mathematical Social Sciences, 10:199-209.

Kannai, Y. and Peleg, B. (1984) A note on the extension of an order on a set to the power set. Journal of Economic Theory, 32:172-175.

Kaymak, B. and Sanver M. R. (2003) Sets of alternatives as Condorcet winners. Social Choice and Welfare, 20:477-494.

Kemeny, J. (1959) Mathematics without numbers. Daedalus, 88:577-591.

Moulin, H. (1988) Axioms of Cooperative Decision Making. Cambridge University Press, Cambridge.

Ratiff, T. (2003) Some starlling inconsistencies when electing committees. Social Choice and Welfare 21:433-454.

Roth, A. and Sotomayor, M. (1990) Two-sided matching: A study in game theoretic modeling and analysis. Cambridge university Press, Cambridge.

Errata

1. In Acknowledgements, line 8: "Shmuel Nitzan" *instead of* "Shamuel Nitzan".

2. In Acknowledgements, line 8: "Ugur Akgün" *instead of* "Akgün Ugur".

3. Page 18, line 17: "$q_i(\hat{s}_i - 1) = 1/2$" *instead of* "$q_i(\hat{s}_i) = 1/2$".

4. Page 20, line 4 : "higher or equal than four" *instead of* "lower than four".

5. Page 48, line 1. "chooser's strategy" *instead of* "2'strategy".

6. Page 65, line 18 and page 90, line 15: "(transitive[13], asymmetric[14], irreflexive[15] and complete)" *instead of* "(transitive[13], asymmetric[14], irreflexive[15])".

7. Page 65, footnote 13 and page 90, footnote 3: "Asymmetric: For all $x, y \in A : x \succ y$ implies that $\neg(y \succ x)$" *instead of* "Asymmetric: For all $x, y \in A : x \succ y$ implies that $\neg(x \succ x)$".

8. Page 91, in Proposition 1: "For any $k \geq 1$, there exists $\mathbf{A}$ and some..." *instead of* "For any $k \in \{1, 2, ..., \#\mathbf{A} - 1\}$, there exists some..".

9. Page 95, line 10: "3) Screening 5 names by 3-votes plurality (P_3S_5)" *instead of* " 3) Screening 5 names by 3-votes plurality (P_5S_3)".

10. Page 95, line 14: "4) Screening 3 names by 2-votes plurality (P_2S_3)" *instead of* " 4) Screening 3 names by 2-votes plurality (P_3S_2)".

11. Page 97 line 10: "P_3S_5" *instead of* "P_5S_3".

12. Page 97 line 11: "P_2S_3" *instead of* "P_3S_2".

13. Page 98, in Proposition 3. "For any $k \geq 1$, when a scoring..." *instead of* "When a scoring..."

14. Page 100, in Proposition 4: "For any $k \geq 1$" *instead of* "For any $k \in \{1, 2, ..., \#\mathbf{A} - 1\}$".

15. Page 120 in line 10: "Ratliff, T. (2003)" *instead of* "Ratiff, T. (2003)".