Universitat Jaume I
Departament de Matemàtiques

Interpolation and equicontinuity sets in topological groups and spaces of continuous functions

A dissertation submitted in partial fulfillment of the requirements for the degree of Doctor of Philosophy

AUTHOR
Luis Tárrega Ruiz

ADVISORS
Dr. Salvador Hernández Muñoz
Dr. Mª Vicenta Ferrer González

Castellón de la Plana, December 2017

Universitat Jaume I
Departament de Matemàtiques

Conjuntos de interpolación y de equicontinuidad en grupos topológicos y en espacios de funciones continuas

Documento remitido en cumplimiento parcial de los requisitos para la obtención del grado de Doctor

AUTOR
Luis Tárrega Ruiz

DIRECTORES
Dr. Salvador Hernández Muñoz
Dr. Mª Vicenta Ferrer González

Castellón de la Plana, Diciembre 2017

To my family.

Acknowledgments

May the reader forgive me the discourtesy of following my wish to write the acknowledgements in my mother tongue.

En primer lugar, me gustaría dar las gracias a mis directores de tesis Salvador Hernández y Marita Ferrer por la confianza que depositaron en mi desde el primer momento, por el tiempo que me han dedicado, por los consejos que me han dado y por compartir sus inquietudes conmigo. También quisiera acordarme de todos los miembros del Departamento de Matemáticas con los que he podido interactuar durante estos años. Además, quiero agradecer a la *Universitat Jaume I* por haber financiado durante 30 meses la realización de esta tesis doctoral a través del *Pla de promoció de la investigació de la Universitat Jaume I 2014* (código de la ayuda PREDOC/2014/15).

Secondly, I would like to thank Prof. Helge Glöckner and the staff members of the Institut für Mathematik of Paderborn University, Germany, for their hospitality during a three-month stay in 2017.

En tercer lugar, quiero agradecer a mis amigos su gran apoyo. En especial a Dan por todos sus consejos y por estar siempre ahí. A Carmen y Txus, dos buenas amigas que me llevo de estos años en la UJI. No me quiero olvidar tampoco de todos los que nos reuniamos exactamente a las *13h30* todos los dias, de cuya compañia disfruté durante la hora de la comida en la cantina y, posteriomente, en el momento del café en *els Sentits*.

En cuarto lugar, quiero agradecer el apoyo de mis padres. Por todo el cariño que me han dado, por la educación y los valores que me han transmitido y por su apoyo incondicional. También quiero agradecer a mi hermano por los buenos ratos que hemos pasado. Asimismo, se la dedico a mi yaya, a mi abuela y a la memoria de mi yayo y de mi abuelo.

Finalmente, quiero dedicar estas últimas líneas a Ester. Si tuviera que destacar una única cosa de este periodo de mi vida, sería haberte conocido. Gracias por estar ahí en los buenos y malos momentos.

Contents

Introduction

Classical *Hadamard sets* (also called *lacunary sets*) have been studied since the beginning of the past century.

Definition. *A subset* $E = \{n_j\}_{j=1}^{\infty}$ *of* $\mathbb{N}$*, with* $n_1 < n_2 < \ldots$ *is said to be a* ***Hadamard set*** *if there exists some* $q > 1$ *(called Hadamard ratio) such that* $n_{j+1}/n_j \geq q$ *for all* $1 \leq j \leq \infty$*.*

From 1926 to 1941, Sidon showed that lacunary sets possessed various interesting properties. There is a great deal of results that illustrates an unexpected behaviour of functions associated to a Hadamard set. For instance:

Theorem. *(Classical Hadamard gap theorem [44, Th. 1.2.2.])*
Let $E = \{n_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$ *be a Hadamard set with ratio* $q > 1$*. Suppose the power series* $f(z) = \sum_{j=1}^{\infty} c_j z^{n_j}$ *has a radius of convergence equal to* 1*. Then* f *cannot be analytically continued across any portion of the arc* $|z| = 1$*.*

In the 1950's, some of these properties were more closely studied and finally became definitions. Since these properties were more functional analytic in nature, whereas the original lacunary property was arithmetic, it was no longer necessary to restrict attention to sets of integers.

Therefore, lacunary sets have also been studied in the dual set of compact topological groups and in more general settings. Kahane [62] first used the term *Sidon set* in 1957 and the modern point of view of the notion appeared in Rudin's Book [86] in 1962. Let $G = \mathbb{T}$, a Hadamard set $E \subseteq \widehat{G} = \mathbb{Z}$ is a special type of Sidon set.

Given a compact abelian group G, one of the several characterisations of a Sidon set (see [66], for instance), stated that a subset E of $\widehat{G}$ is a Sidon set if every bounded function on E is the restriction of the Fourier transform of a measure on G.

Since the dual of a compact group is a discrete group and the Sidon sets are situated in the dual set, we can study the Sidon sets as subsets of discrete groups. In this sense, Picardello [76] extended in 1973 the usual definition of Sidon set in a group G to the discrete non-abelian case.

A special kind of Sidon set is the I_0 sets. The concept of ***I_0 set*** in locally compact abelian groups was introduced by Hartman and Ryll-Nardzewski [48] in 1964, who considered the weak topology associated to a locally compact abelian (LCA, for short) group and introduced the notion of *interpolation set* or *I_0 set*. They defined that a subset A of a LCA group G is an I_0 set if every bounded function on A is the restriction of an *almost periodic function* on G (here, it is said that a complex-valued function f defined on G is *almost periodic* when it is the restriction of a continuous function defined on bG, the Bohr compactification of G).

Therefore, an I_0 set is a subset A of G such that any bounded map on A can be interpolated by a continuous function on bG. As a consecuence, if A is a countably infinite I_0 set, then $\overline{A}^{bG}$ is canonically homeomorphic to $\beta\omega$, the Stone-Čech compactification of ω.

The main result given by Hartman and Ryll-Nardzewski is the following:

Theorem. *([48]) Every LCA group G contains I_0 sets.*

For the particular case of discrete abelian groups, van Douwen achieved a remarkable progress by proving the existence of I_0 sets in very general situations. His main result can be formulated in the following way:

Theorem. *([94, Th. 1.1.3]) Let G be a discrete Abelian group and let A be an infinite subset of G. Then, there is a subset B of A with $|B| = |A|$ such that B is an I_0 set.*

In fact, van Douwen extended his result to the real line but left unresolved the question for LCA groups. In general, the weak topology of locally compact groups has been considered by many researchers so far; specially for abelian groups, where the amount of important results is vast.

Since the dual of a LCA group is also LCA, the definition of I_0 set in the dual group is analogous. In this respect, Kahane [61] proved in 1966 a characterisation of this notion without recurring to the Bohr compactification using the Fourier transform. He stated that a subset E of the dual group is an I_0 set if every bounded function on E is the restriction of the Fourier transform of a discrete measure on G.

Hadamard sets were the first examples of I_0 sets and many of the properties that Hadamard sets posses are held by general I_0 sets.

Hare and Ramsey [47] introduced in 2003 the notion of I_0 set in he dual set of a compact non-abelian group. I_0 sets are a special type of Sidon set in which the interpolating measure can be chosen to be discrete.

The search for interpolation sets is a main goal in harmonic analysis and the monograph by Graham and Hare [44] contains most of the recent results in this area. In this thesis, this question is approached from a topological viewpoint and the main goal is the understanding of the key (topological) facts that characterise the existence of interpolation sets.

Following this viewpoint, Galindo and Hernández [34] provided in 1994 sufficient conditions for the existence of I_0 sets for the duals of abelian, locally connected, Čech-complete groups and abelian, compact groups, respectively.

For those locally compact groups that can be injected in their Bohr compactification, the so-called *maximally almost periodic groups*, the existence of I_0 sets was clarified by Galindo and Hernández [35] in 2004. However, many (non-abelian) locally compact groups cannot be injected in their Bohr compactification (that can become trivial in some cases).

In Section 4.2, we define $\mathrm{w}G$, the weak compactification of a locally compact group. This compactification extends the Bohr compactification, since when G is an abelain group, we have that $\mathrm{w}G = bG$. Using this extension we are able to give an appropriate definition of an I_0 set for locally compact groups in general: we say that a subset A of G is an I_0 set if every complex valued function on A can be extended to a continuous function on $\mathrm{w}G$. In this way, we are able to analyse the conditions of the existence of I_0 sets in locally compact groups in general.

Bearing the original definition of an I_0 set in mind, we can extend it to the context of topological spaces introducing the notion of M-interpolation set: we say that a subset Y of X is a *M-interpolation set* (equivalently, an *interpolation set* for $C(X,M)$) when for every function $g \in M^Y$ with relatively compact range in M, there exists a map $f \in C(X,M)$ such that $f_{|Y} = g$.

The idea of this generalisation to the realm of continuous functions is to use it as a tool so as to obtain different results about interpolation sets in topological groups and in its duals.

In this thesis, I study the relation between the existence of a particular sort of subsets of metric-valued continuous functions on a topological space X and the properties of the topological space itself.

The dissertation relies on how the existence of subsets of continuous functions that possess one of these two antagonist properties, *almost equicontinuity* and being a *$\mathfrak{B}$-family*, affects the topological space.

The former property appears in the setting of dynamical systems in [2] and has the following definition: a subset E of $C(X,M)$ is *almost equicontinuous* if E is equicontinuous on a dense subset of X.

The latter is a property stronger that the concept of *non-equicontinuity* and it is motivated by a result of Bourgain in [12]. We say that $E \subseteq C(X,M)$ is a $\mathfrak{B}$-family if the following two conditions hold: (a) E is relatively compact in M^X, and (b) there exists a nonempty open set V of X and $\epsilon > 0$ such that for every finite collection $\{U_1, \ldots, U_n\}$ of nonempty relatively open sets of V there is a $f \in E$ satisfying $\operatorname{diam}(f(U_j)) \geq \epsilon$ for all $j \in \{1, \ldots, n\}$.

I do not know exactly the relation between the notion of being a $\mathfrak{B}$-family and the negation of being almost equicontinuous (or even hereditary almost equicontinuous). Nevertheless, when the topological space X is homogeneous (topological groups, for instance) we know that the concept of almost equicontinuity is equivalent to equicontinuity; and for every subset relatively compact in M^X the notion of being a $\mathfrak{B}$-family is equivalent to the property of being non-equicontinuous.

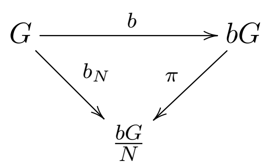
Throughout the thesis, I deal with the study of the existence and the properties of *interpolation sets* in different settings: (i) spaces of continuous functions, (ii) topological groups and (iii) the dual of a topological group. For (i) and (iii) the concept of $\mathfrak{B}$-family isolates a crucial fact for the existence of interpolation sets in fairly general circumstances, whereas for (ii), I confront this question by using an extension of Rosenthal's Theorem to general locally compact groups.

Theorem. *(H. P. Rosenthal [85]) Let X be a real Banach space and let $\{x_n\}_{n<\omega} \subseteq X$ be a bounded sequence. Then, either $\{x_n\}_{n<\omega}$ contains a weak-Cauchy subsequence or a subsequence which is homeomorphic to the ℓ^1 basis.*

However, the version of the Rosenthal's dichotomy result that I am going to use along the thesis is the following:

Theorem. *([91]) If X is a Polish space and $\{f_n\}_{n<\omega} \subseteq C(X)$ is a pointwise bounded sequence, then either $\{f_n\}_{n<\omega}$ contains a convergent subsequence or a subsequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\omega$.*

The question of the disposition or placement of a LCA group G within its Bohr compactification bG has been widely studied. Given a topological group G, let G^+ denote the algebraic group G equipped with the Bohr topology. Glicksberg [42] showed in 1962 that in a LCA group G, every compact subset in G^+ is compact in G. This result concerning LCA groups is one of the pivotal results of the subject, often referred to as *Glicksberg's theorem.* Glicksberg result was extended by Comfort, Trigos-Arrieta and Wu [21] in 1993 by the following remarkable result: let G be a LCA group and let N be a closed metrizable subgroup of its Bohr compactification bG. Denote by π the canonical projection from bG onto bG/N and set $b_N \stackrel{\text{def}}{=} \pi \circ b$ making the following diagram commutative:



Theorem. *(Comfort, Trigos-Arrieta and Wu) Let G be a LCA group and let N be a closed metrizable subgroup of its Bohr compactification bG. If A is a subset of G, then $A + (N \cap G)$ is compact in G if and only if the set $b_N(A)$ is compact in bG/N.*

In the same paper, the following classes of topological grous are introduced: a group G *respects compactness* (resp. *strongly respects compactness*) if satisfies Glicksberg's theorem (resp. satisfies the thesis of the previous theorem). The authors also propose the question of clarifying the relation between these two classes of groups and, furthermore, the characterization of the groups that strongly respect compactness. Observe that for the special case $N = \{0\}$, the condition is reduced to respecting compactness and, hence, every group which strongly respects compactness respects compactness.

The property of respecting compactness has also been generalised and studied in greater classes of abelian groups [8, 6, 82].

The existence of an interpolation set in a topological group is powerfully connected to the property of strongly respecting compactness. Therefore, using the results about interpolation sets in a topological group that we achieve, we are able to show that the family of locally quasiconvex, abelian, locally k_ω groups (that includes all the locally compact abelian groups, for instance) also respects compactness.

If the topological group is non-abelian, Hughes [56] proved in 1973 a generalization of Glicksberg's theorem to (not necessarily abelian) locally compact groups by considering the weak topology generated by the continuous irreducible unitary group representations. Taking this idea into account and considering the following extended definition: a locally compact group G *strongly respects compactness* if for any closed metrizable subgroup N of $inv(\mathrm{w}G)$, a subset A of G satisfies that $AN \cap G$ is compact in G if and only if AN is compact in $\mathrm{w}G$ ($inv(\mathrm{w}G)$ denotes the group of units of $\mathrm{w}G$), we also show that every locally compact group strongly respects compactness. This improves the aforementioned results by Comfort, Trigos-Arrieta and Wu [21] and Galindo and Hernández [35].

The results of this dissertation are mainly addressed in:

(1) M. Ferrer, S. Hernández, L. Tárrega (2017). 'Equicontinuity criteria for metric-valued sets of continuous functions.' *Topology and its Applications* 225, p. 220-236.

(2) M. Ferrer, S. Hernández, L. Tárrega (2017). 'A dichotomy property for locally compact groups'.
(To appear, https://arxiv.org/pdf/1704.03438.pdf).

(3) M. Ferrer, S. Hernández, L. Tárrega (2017). 'Interpolation sets in spaces of continuous metric-valued functions'.
(To appear, https://arxiv.org/pdf/1707.06550.pdf).

(4) M. Ferrer, S. Hernández, L. Tárrega (2017). 'Interpolation sets in the dual set of compact non-abelian groups'. *(Pending, the paper is still in progress)*.

• Contents and results:

The **first chapter** has a preliminary nature. It incorporates the notation, definitions and basic facts which are used along this dissertation.

In Section 1.1, we recall the definition of a topological space and we give some basic properties.

Section 1.2 is devoted to function spaces. It is divided into three parts. In Subsection 1.2.1, we define three useful topologies on a function space: the *topology of pointwise convergence*, the *compact-open topology* and the *topology of uniform convergence*. In Subsection 1.2.2, we recollect the definition of *Baire class* 1 *function* and we present some celebrated results on $C(X)$ and $B_1(X)$, which are useful for the subsequent work. We also add some direct consequences of these strong results. The aim of Subsection 1.2.3 is to obtain a way of extending results for $\mathbb{R}$-valued spaces of functions to metric-valued spaces of functions.

In Section 1.3, we recall the definition of topological group and we give some basic facts.

In Section 1.4, we present the definition of the dual set of a topological group and we introduce some terminology.

The **second chapter** deals with metric-valued sets of continuous functions.

Section 2.1 is devoted to the notion of almost equicontinuity. The main goal of this section is to extend this important notion to arbitrary topological spaces, which were introduced in the setting of topological dynamics studying the enveloping semigroup of a flow [2, 40, 41]. Combining ideas of Troallic [93] and Cascales, Namioka, and Vera [15], we prove several characterizations of *almost equicontinuity* and *hereditarily almost equicontinuity* for subsets of metric-valued continuous functions when they are defined on a Čech-complete space. We also obtain some applications of these results to topological groups and dynamical systems.

In Section 2.2, we study sets of continuous functions whose pointwise closure is compact and contained in the space of all Baire class 1 functions. We analyse the special case where the functions are defined from a Polish space X to a metric space M. Rosenthal [85], Bourgain [12], and Bourgain, Fremlin and Talagrand [13] and Todorčević [92] have extensively studied the compact subsets of $B_1(X)$. Our aim is to extend some of their fundamental results to the special case where the functions are metric-valued.

The **third chapter** is focused on the study of interpolation sets in the setting of metric-valued spaces of continuous functions.

In Section 3.1, we define the notion of M-interpolation set and we give some basic facts.

In Section 3.2, motivated by a result of Bourgain in [12], we introduce a property, called $\mathfrak{B}$-family, stronger than the mere *non-equicontinuity* of a family of continuous functions. It is important because it brings together the requisite for the existence of interpolation sets in metric-valued spaces of functions on Čech-complete spaces.

In Section 3.3, we focus on the applications to spaces of continuous homomorphisms on topological Čech-complete groups.

The **fourth chapter** is devoted to the analysis of the existence and properties of interpolation sets in topological groups. As we have previously said, the existence of this set helps us so as to determine which sort of topological groups possesses the property of strongly respecting compactness.

In Section 4.1, we deal with the family of locally k_ω-groups. It includes, for instance, all locally compact abelian groups, the free abelian groups on a compact space and all countable direct sum of compact groups. Glöckner, Gramlich and Hartnick [43] stated in 2010 that the dual group of an abelian locally k_ω group is an abelian Čech-complete group, and vice-versa. So, the proposed approach is a direct application of the results of Section 3.3.

Nevertheless, the approach in Section 4.2 for locally compact groups (not necessarily abelian) is completely different. Using the extension of the Rosenthal's theorem [85], which is presented in Section 2.2, we are able to extend Rosenthal's dichotomy theorem on Banach spaces to locally compact groups and their weak topologies. For this purpose, we use the notion of an I_0 set, which is analogous to the ℓ_1-basis in the realm of locally compact groups.

The weak topology of a topological group plays an analogous function to that of the weak topology in a Banach space. Therefore, it is often studied in connection to the original topology of the group. For instance, it can be said that the preservation of compact-like properties from G^w to G concerns "uniform boundedness" results and, in many cases, it can be applied to prove the continuity of certain related algebraic homomorphisms.

Our main result establishes that for every sequence $\{g_n\}_{n<\omega}$ in a locally compact group G, either $\{g_n\}_{n<\omega}$ has a weak Cauchy subsequence or contains a subsequence that is an I_0 set. This result is subsequently applied to obtain sufficient conditions for the existence of weak Sidon sets in locally compact groups.

Here, a subset E of G is called *weak Sidon set* when every bounded function can be interpolated by a continuous function defined on the Eberlein compactification eG. This is a weaker property than the classical notion of *Sidon set* in general (see [76]) but both notions coincide for abelian or amenable groups.

It is still an open question whether every infinite subset of a locally compact group G contains a weak Sidon subset (see [66, 31]).

The **fifth chapter** shows some reseach lines that we want to develop promptly. They deal with the analysis of the existence and properties of interpolation sets in the dual set of compact, not necessarily abelian, groups. In the realm of non-abelian groups, the theory of interpolation sets is different from the known facts in the commutative case. For instance, there exist non-abelian compact groups whose dual contains no infinite Sidon set.

This chapter presents some results that appear in article (4), which is still in internal revision. Our research is specially focussed on the case of non-tall compact groups.

In Section 5.1, we recall the definition of Sidon and I_0 sets.

In Section 5.2, we give a characterisation of the existence of I_0 sets in the dual of non-tall compact groups and some interesting corollaries.

In 1997, Hutchinson proved that every non-tall compact group contains an infinite Sidon set [57]. We do believe that this is also true for I_0 sets, and our characterisation can be useful to prove this fact. In this regard, taking into account [57, Corollary 2.5] and [47, Theorem 4.10], it is known that this fact is true if the group is also connected. This belief leads us to considering that we can ameliorate our present work.

In Section 5.3, we focus on the existence of central I_0 sets in the dual set of compact non-abelian groups. The study of central interpolation sets started in 1972, when Parker introduced the notion of central Sidon set [75]. Afterwards, Grow and Hare defined the concept of central (weighted) I_0 sets in 2004 [45]. Here, we present several characterisations of the existence of infinite central I_0 sets in the dual set of a compact group. The most noteworthy of them is connected with the property of containing a sequence equivalent to the unit basis ℓ_1. Moreover, we show that every infinite subset of $\widehat{G}$ contains an infinite central I_0 sets if G is a non-tall compact group. Finally, we present some results regarding the existence of that sort of *thin* sets when removing the condition of non-tallness.

We do not present the proofs in this chapter because article (4) has not yet been submitted. We plan to prove that the dual of every non-tall compact group contains an infinite I_0 set and to analyse the connection between I_0 sets and central I_0 sets as a subsequent research work.

Introducción

Los clásicos *conjuntos de Hadamard* (también llamados *conjuntos lacunarios*) se estudian desde principios del siglo pasado.

Definición. *Un subconjunto $E = \{n_j\}_{j=1}^{\infty}$ de $\mathbb{N}$, con $n_1 < n_2 < \ldots$ se dice que es un* ***conjunto de Hadamard*** *si existe algún $q > 1$ (llamado ratio de Hadamard) tal que $n_{j+1}/n_j \geq q$ para todo $1 \leq j \leq \infty$.*

Desde 1926 hasta 1941, Sidon mostró que los conjuntos lacunarios poseen propiedades interesantes. Existe una gran cantidad de resultados que ilustran un comportamiento inesperado en funciones asociadas a un conjunto de Hadamard. Por ejemplo:

Teorema. *(Classical Hadamard gap theorem [44, Th. 1.2.2.])*
Sea $E = \{n_j\}_{j=1}^{\infty} \subseteq \mathbb{N}$ un conjunto de Hadamard con ratio $q > 1$. Supongamos que la serie de potencias $f(z) = \sum\limits_{j=1}^{\infty} c_j z^{n_j}$ tiene un radio de convergencia igual a 1. Entonces f no puede extenderse analíticamente sobre ninguna porción del arco $|z| = 1$.

En la década de los 50, algunas de estas propiedades fueron analizadas más exhaustivamente y, finalmente, se convirtieron en definiciones. Como la propiedad original de lacunaridad es de naturaleza aritmética y estas propiedades se enmarcan mayoritariamente en el área del análisis funcional, se consideró que no era necesario centrarse únicamente al estudio del conjunto de los números enteros.

Todo ello motivó que los conjuntos lacunarios también se estudiaran dentro del conjunto dual de grupos topológicos compactos y en otros contextos. Fue Kahane [62] el primero que en usar el término *conjunto de Sidon* en el año 1957 y el punto de vista moderno de dicho concepto apareció en el libro de Rudin [86] en el año 1962. Sea $G = \mathbb{T}$, se tiene que todo conjunto de Hadamard $E \subseteq \widehat{G} = \mathbb{Z}$ es un tipo particular de conjunto de Sidon.

Dado un grupo compacto y abeliano G, una de las diferentes caracterizaciones de conjunto de Sidon (véase [66], por ejemplo) afirma que un subconjunto E de $\widehat{G}$ es un conjunto de Sidon si toda función acotada definida en E se puede obtener como restricción en E de la transformada de Fourier de una medida en G.

Como el dual de un grupo compacto es un grupo discreto y los conjuntos de Sidon se definieron, originalmente, en el conjunto dual, también se pueden estudiar los conjuntos de Sidon como subconjuntos de un grupo discreto. En este sentido, Picardello [76] extendió en 1973 la definición de conjunto de Sidon en un grupo G al caso discreto y no abeliano.

Un caso particular de conjunto de Sidon es el conjunto I_0. El concepto de ***conjunto*** $\boldsymbol{I_0}$ en grupos localmente compactos y abelianos fue introducido por Hartman y Ryll-Nardzewski [48] en 1964. En su artículo se trabaja con la topología débil asociada a un grupo localmente compacto y abeliano (escribiremos LCA, para abreviar) y se introduce la noción de *conjunto de interpolación* o *conjunto* I_0. Se dice que un subconjunto A de un grupo LCA G es un conjunto I_0 si toda función acotada en A se puede obtener como la restricción de una *función casi periódica* en G (se dice que una función complejo valuada f definida en G es *casi periódica* cuando se trata de la restricción de una función continua definida en bG, la compactación de Bohr de G).

Por lo tanto, un conjunto I_0 es un subconjunto A de G tal que cualquier función acotada definida en A puede ser interpolada por una función continua definida en bG. Por consiguiente, si A es un conjunto I_0 numerable e infinito, se tiene que $\overline{A}^{bG}$ es canónicamente homeomorfo a $\beta\omega$, la compactación de Stone-Čech de ω.

El resultado principal de Hartman y Ryll-Nardzewski es el siguiente:

Teorema. *([48]) Todo grupo LCA contiene un conjunto* I_0.

En el caso particular de los grupos discretos y abelianos, van Douwen realizó un progreso notable al demostrar la existencia de conjuntos I_0 en situaciones mucho más generales. Su resultado principal puede formularse de la siguiente manera:

Teorema. *([94, Teo. 1.1.3]) Sea G un grupo discreto y abeliano y sea A un subconjunto infinito de G. Entonces, existe un subconjunto B de A cumpliendo $|B| = |A|$ tal que B es un conjunto I_0.*

De hecho, van Douwen logró demostrar su resultado para el caso de la recta real, pero no llegó a probarlo para cualquier grupo LCA arbitrario. El estudio de la topología débil de un grupo localmente compacto ha sido considerado por gran cantidad de investigadores, especialmente para el caso de grupos abelianos, donde existe una gran cantidad de resultados relevantes.

Como el dual de un grupo LCA es también un grupo LCA, la definición de conjunto I_0 dentro del conjunto dual es análoga. En este sentido, Kahane [61] obtuvo en 1966 una caracterización del conjunto I_0 en el conjunto dual usando la transformada de Fourier y sin recurrir a la compactación de Bohr. Demostró que un subconjunto E del grupo dual es un conjunto I_0 si toda función definida en E se puede obtener como la restricción de la transformada de Fourier de una medida discreta en G.

Los conjuntos de Hadamard se pueden considerar como los primeros ejemplos de conjuntos I_0. Muchas de las propiedades que poseen los conjuntos de Hadamard se mantienen en los conjuntos I_0.

Hare y Ramsey [47] introdujeron en 2003 la noción de conjunto I_0 en el conjunto dual de un grupo compacto no abeliano. En este contexto, los conjuntos I_0 son conjuntos de Sidon en los que la medida interpoladora se puede tomar discreta.

La búsqueda de conjuntos de interpolación es un objetivo importante en el análisis armónico. El libro de Graham y Hare [44] contiene muchos de los resultados recientes en este área. En esta tesis, la búsqueda de conjuntos de interpolación se realiza desde un punto de vista topológico y el objetivo principal es poder comprender las claves (topológicas) que caracterizan la existencia de tales conjuntos.

Siguiendo este punto de vista, Galindo y Hernández [34] proporcionaron en 1994 condiciones suficientes para la existencia de conjuntos I_0 en el dual de un grupo abeliano, localmente conexo y Čech-completo y en un grupo abeliano y compacto.

En el año 2004, Galindo y Hernández [35] demostraron que los grupos localmente compactos *grupos maximalmente casi periódicos* (i.e. los grupos que pueden ser introducidos de manera inyectiva en su compactación de Bohr) contienen conjuntos I_0. Sin embargo, se sabe que muchos grupos localmente compactos (no abelianos) no son maximalmente casi periódicos.

En la Sección 4.2, se define wG, la compactación débil de un grupo localmente compacto. Esta compactación extiende la compactación de Bohr, ya que cuando G es abeliano se tiene que w$G = bG$. Usando esta extensión se consigue obtener una definición apropiada del conjunto I_0 para grupos localmente compactos en general: un subconjunto A de G es un conjunto I_0 si toda función complejo valuada en A se puede extender a una función continua en wG. De esta manera, se es capaz de analizar las condiciones necesarias para la existencia de conjuntos I_0 en grupos localmente compactos en general.

Teniendo en cuenta la definición original de conjunto I_0, se puede trasladar dicho concepto al contexto de espacios topológicos mediante la noción de conjunto de M-interpolación: un subconjunto Y de X es un *conjunto de M-interpolación* (equivalentemente, un *conjunto de interpolación* para $C(X, M)$) cuando para toda función $g \in M^Y$ con rango relativamente compacto en M, existe una función $f \in C(X, M)$ tal que $f_{|Y} = g$.

El análisis de esta propiedad sirve de ayuda para conseguir, además, resultados sobre conjuntos de interpolación en grupos topológicos y en sus conjuntos duales.

En esta tesis se estudia la relación que hay entre la existencia de ciertos tipos de subconjuntos de funciones valuadas en un espacio métrico y definidas en un espacio topológico X y las propiedades que posee el espacio topológico X en si.

La disertación se apoya en cómo la existencia de subconjuntos de funciones continuas que poseen una de estas dos propiedades antagonistas, *casi equicontinuidad* y ser una $\mathfrak{B}$*-familia*, afecta al espacio topológico.

La primera propiedad aparece en el contexto de los sistemas dinámicos en [2] y tiene la siguiente definición: un subconjunto E de $C(X, M)$ es *casi equicontinuo* si E is equicontinuo en un subconjunto denso de X.

La segunda propiedad es un concepto más fuerte que la *no equicontinuidad* y viene motivada por un resultado de Bourgain en [12]. Decimos que $E \subseteq C(X, M)$ es una $\mathfrak{B}$-familia si las siguientes dos condiciones se cumplen: (a) E es relativamente compacto in M^X, y (b) existe un conjunto no vacio y abierto V de X y un $\epsilon > 0$ tal que para toda colección finita $\{U_1, \ldots, U_n\}$ de conjuntos no vacios relativamente abiertos en V existe $f \in E$ tal que $\text{diam}(f(U_j)) \geq \epsilon$ para todo $j \in \{1, \ldots, n\}$.

No se sabe exactamente la relación existente entre la noción de ser una $\mathfrak{B}$-familia y la negación de casi equicontinuidad (o casi equicontuidad hereditaria). Sin embargo, cuando el espacio topológico X es homogéneo (un grupo topológico es homogéneo, por ejemplo) sabemos que el concepto de casi equicontinuidad es equivalente al de equicontinuidad; y para todo subconjunto relativamente compacto en M^X, la noción de ser una $\mathfrak{B}$-familia es equivalente a la propiedad de no ser equicontinuo.

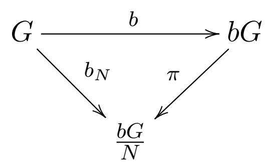
A lo largo de la tesis se lidia con el estudio de la existencia y propiedades de los *conjuntos de interpolación* en diferentes contextos: (i) espacios de funciones continuas, (ii) grupos topológicos y (iii) el dual de un grupo topológico. Para los casos (i) y (iii), el concepto de $\mathfrak{B}$-familia se trata de una propiedad crucial para la existencia de conjuntos de interpolación en circunstancias bastantes generales, mientras que para el caso (ii), se usa una extensión del Teorema de Rosenthal a grupos localmente compactos en general.

Teorema. *(H. P. Rosenthal [85]) Sea X un espacio de Banach real y sea $\{x_n\}_{n<\omega} \subseteq X$ una sucesión acotada. Entonces, o bien $\{x_n\}_{n<\omega}$ contiene una subsucesión débil-Cauchy, o bien contiene una subsucesión que es homeomorfa a la base del espacio ℓ^1.*

Sin embargo, la versión del resultado de dicotomía de Rosenthal que se va a usar a lo largo de la tesis es la siguiente:

Teorema. *([91]) Si X is un espacio polaco y $\{f_n\}_{n<\omega} \subseteq C(X)$ es una sucesión puntualmente acotada, entonces o bien $\{f_n\}_{n<\omega}$ contiene una subsucesión convergente, o bien continene una subsucesión cuya clausura en $\mathbb{R}^X$ es homeomorfa a $\beta\omega$.*

La cuestión acerca de la disposición o ubicación de un grupo LCA G dentro de su compactación de Bohr bG ha sido estudiada por muchos matemáticos. Dado un grupo topológico G, se denota por G^+ al grupo algebraico G equipado con la topología de Bohr. Glicksberg [42] demostró en 1962 que en un grupo LCA G todo subconjunto compacto en G^+ es compacto en G. Este resultado sobre grupos LCA es uno de los resultados clave en esta cuestión y se conoce como el *Teorema de Glicksberg*. Dicho resultado fue extendido por Comfort, Trigos-Arrieta y Wu [21] en 1993 de la siguiente manera: sea G un grupo LCA y sea N un subgrupo cerrado y metrizable de su compactación de Bohr bG. Sea π la proyección canónica desde bG hasta bG/N y sea $b_N \stackrel{\text{def}}{=} \pi \circ b$ la aplicación que hace el siguiente diagrama conmutativo:



Teorema. *(Comfort, Trigos-Arrieta and Wu) Sea G un grupo LCA y sea N un subgrupo cerrado y metrizable de su compactación de Bohr bG. Si A es un subconjunto de G, entonces $A + (N \cap G)$ es compacto en G si y solamente si el conjunto $b_N(A)$ es compacto en bG/N.*

En el mismo artículo, se introducen las siguientes clases de grupos topológicos: un grupo G *respeta la compacidad* (resp. *respeta fuertemente la compacidad*) si satisface el teorema de Glicksberg (resp. satisface la tesis del teorema previo). Los autores también plantean las siguientes cuestiones: clarificar la relación existente entre ambas clases de grupos y caracterizar los grupos que respetan fuertemente la compacidad. Nótese que en el caso particular en el que $N = \{0\}$ la condición se reduce a respetar la compacidad. Por lo tanto, todo grupo que respeta fuertemente la compacidad también respeta la compacidad.

La propiedad de respetar la compacidad ha sido también estudiada y generalizada para clases más grandes de grupos abelianos [8, 6, 82].

La existencia de un conjunto de interpolación en un grupo topológico está notablemente conectada con la propiedad de respetar fuertemente la compacidad. Por consiguiente, utilizando los resultados obtenidos sobre conjuntos de interpolación, es capaz de demostrar que la familia de grupos localmente cuasiconvexos, abelianos, localmente k_ω (la cual incluye todos los grupos LCA, por ejemplo) también respeta compacidad.

Hughes [56] demostró en 1973 una generalización del teorema de Glicksberg a grupos localmente compactos en general (no necesariamente abelianos) considerando la topología débil generada por las representaciones continuas irreducible y unitarias. Teniendo en cuenta este punto de vista y considerando la siguiente definición extendida: un grupo localmente compacto G *respeta fuertemente la compacidad* si para cualquier subgrupo cerrado y metrizable N de $inv(\mathrm{w}G)$, un subconjunto A de G satisface que $AN \cap G$ es compacto en G si y solamente si AN es compacto en $\mathrm{w}G$ ($inv(\mathrm{w}G)$ denota el grupo de unidades de $\mathrm{w}G$), se prueba que todo grupo localmente compacto respeta fuertemente la compacidad. Este resultado mejora los resultados previos de Comfort, Trigos-Arrieta y Wu [21] y Galindo y Hernández [35].

Los resultados de esta tesis se encuentran en los siguientes artículos:

(1) M. Ferrer, S. Hernández, L. Tárrega (2017). 'Equicontinuity criteria for metric-valued sets of continuous functions.' *Topology and its Applications* 225, p. 220-236.

(2) M. Ferrer, S. Hernández, L. Tárrega (2017). 'A dichotomy property for locally compact groups'.
(En fase de revisión, https://arxiv.org/pdf/1704.03438.pdf).

(3) M. Ferrer, S. Hernández, L. Tárrega (2017). 'Interpolation sets in spaces of continuous metric-valued functions'.
(En fase de revisión, https://arxiv.org/pdf/1707.06550.pdf).

(4) M. Ferrer, S. Hernández, L. Tárrega (2017). 'Interpolation sets in the dual set of compact non-abelian groups'. *(Pendiente, el artículo se encuentra todavía en fase de revisión interna)*.

• Contenidos y resultados:

El **primer capítulo** es de naturaleza preliminar. Presenta la notación, definiciones y resultados básicos que se usan a lo largo de la disertación.

En la Sección 1.1, se recuerda la definición de espacio topológico y se dan algunas propiedades básicas.

La Sección 1.2 se dedica a los espacios de funciones. Está dividida en tres partes. En la Subsección 1.2.1, se definen tres topologías conocidas para espacios de funciones: la *topología de la convergencia puntual*, la *topología compacto abierta* y la *topología de la convergencia uniforme*. En la Subsección 1.2.2, se presenta la definición de *función de la primera clase de*

Baire y se recogen algunos resultados conocidos en $C(X)$ y $B_1(X)$ que nos son de ayuda en nuestro trabajo subsecuente. También se añaden algunas consecuencias directas de estos resultados principales. El objetivo de la Subsección 1.2.3 es presentar una herramienta que sirva para extender resultados de espacios de funciones real valuadas a espacios de funciones valuadas en un espacio métrico arbitrario.

En la Sección 1.3, se recuerda la definición de grupo topológico y se dan algunos resultados básicos.

En la Sección 1.4, se presenta la definición de conjunto dual de un grupo topológico y se introduce terminología. Se distingue la naturaleza que posee el conjunto dual si el grupo es abeliano o no lo es.

El **segundo capítulo** trata sobre conjuntos de funciones continuas valuadas en un espacio métrico arbitrario.

La Sección 2.1 versa sobre la casi equicontinuidad. El objetivo principal de esta sección es extender esta noción a un espacio topológico arbitrario. La definición original fue introducida en el marco de los sistemas dinámicos [2, 40, 41]. Combinando ideas de Troallic [93] y Cascales, Namioka y Vera [15], se prueban varias caracterizaciones de *casi equicontinuidad* y *casi equicontinuidad hereditaria* para subconjuntos de funciones continuas valuadas en un espacio métrico y definidas en un espacio Čech-completo. También se obtienen algunas aplicaciones a grupos topológicos y a sistemas dinámicos.

En la Sección 2.2, se estudian los conjuntos de funciones continuas cuya clausura puntual es compacta y está contenida en el espacio de todas las funciones de la primera clase de Baire. Se analiza el caso especial en el que las funciones están definidas en un espacio polaco X y toman valores en un espacio métrico M. Rosenthal [85], Bourgain [12], y Bourgain, Fremlin y Talagrand [13] y, en una dirección diferente, Todorčević [92] han estudiado de manera exhaustiva los subconjuntos compactos de $B_1(X)$. El objetivo es extender algunos de estos resultados fundamentales al caso en el que la funciones toman valores en un espacio métrico.

El **tercer capítulo** está enfocado en el estudio de conjuntos de interpolación en el marco de las funciones continuas y valuadas en un espacio métrico.

En la Sección 3.1, se define la noción de conjunto de M-interpolación y se proporcionan algunos resultados básicos.

En la Sección 3.2, motivados por un resultado de Bourgain en [12], se introduce el concepto de $\mathfrak{B}$-familia. Se trata de una propiedad más fuerte que la no equicontinuidad. Además, se trata de un requisito indispensable para la existencia de conjuntos de interpolación en espacios de funciones definidos en un espacio Čech-completo y que toman valores en un espacio métrico.

En la Sección 3.3, nuestra atención se centra en las aplicaciones a espacios de homomorfismos continuos definidos en grupos Čech-completos.

El **cuarto capítulo** está dedicado al análisis de la existencia y las propiedades de los conjuntos de interpolación en grupos topológicos. Como se ha comentado anteriormente, la existencia de dicho tipo de conjuntos sirve para determinar qué clases de grupos topológicos poseen la propiedad de respetar fuertemente la compacidad.

En la Sección 4.1, se analiza la familia de los grupos localmente k_ω. Esta familia incluye, por ejemplo, todos los grupos localmente compactos abelianos, los grupos libres abelianos y toda suma directa numerable de grupos compactos. Glöckner, Gramlich y Hartnick [43] demostraron en 2010 que el dual de un grupo localmente k_ω y abeliano es un grupo Čech-completo y abeliano, y viceversa. Por consiguiente, la idea que se ha seguido en esta sección es la aplicación de los resultados que se obtienen en la Sección 3.3 a este contexto.

Sin embargo, la idea seguida en la Sección 4.2 para grupos localmente compactos (no necesariamente abelianos) es completamente diferente. Usando la extensión del teorema de Rosenthal [85], el cual se presenta en la Sección 2.2, se consigue demostrar una extensión del teorema de dicotomía de Rosenthal en espacios de Banach al caso de grupos localmente compactos y sus topologías débiles asociadas. En este contexto, el concepto de conjunto I_0 juega un rol análogo al de base del espacio ℓ_1.

Nótese que la topología débil de un grupo topológico juega un papel similar al de la topología débil en un espacio de Banach. Es por ello que a menudo es estudiada en conexión con la topología del grupo. Por ejemplo, se puede decir que las propiedades del tipo de conservación de la compacidad de G^w a G están relacionadas con los resultados de "acotación uniforme" y, en muchos casos, pueden ser aplicadas para probar la continuidad de ciertos homomorfismos algebraicos relacionados.

Nuestro resultado principal establece que toda sucesión $\{g_n\}_{n<\omega}$ en un grupo localmente compacto G, o bien $\{g_n\}_{n<\omega}$ posee una subsucesión débil Cauchy, o bien contiene una subsucesión que es un conjunto I_0. Este resultado se aplica, posteriormente, para obtener condiciones suficientes para la existencia de conjuntos débil Sidon en grupos localmente compactos.

Un subconjunto E de G se dice que es un conjunto *débil Sidon* si toda función acotada puede interpolarse por una función continua definida en la compactación de Eberlein eG. Se trata de una propiedad más débil que la noción clásica de *conjunto de Sidon* (véase [76]), aunque ambos conceptos coinciden para grupos abelianos o amenables.

Todavía es una cuestión abierta saber si todo subconjunto infinito de un grupo localmente compacto G contiene un conjunto débil Sidon (véase

[66, 31]).

El **quinto capítulo** muestra agunas de las lineas de invetigación fututa que queremos desarrollar a corto plazo. Estas tratan sobre el estudio de la existencia y las propiedades de los conjuntos de interpolación en el conjunto dual de un grupo compacto no necesariamente abeliano. En el contexto de los grupos no abelianos, la teoría sobre los conjuntos de interpolación es diferente a la que hay para grupos topológicos abelianos. Por ejemplo, se sabe que hay grupos compactos no abelianos cuyo dual no contiene ningún conjunto de Sidon.

Este capítulo presenta resultados que aparecen en el artículo (4), el cual se encuentra todavía en revisión interna. Nuestra investigación está particularmente enfocada en el estudio de los grupos compactos no *tall*.

En la Sección 5.1, se recuerda la definición de conjunto de Sidon y conjunto I_0.

En la Sección 5.2, se presenta una caracterización acerca de la existencia de conjuntos I_0 en el dual de un grupo compacto no tall, el cual nos proporciona algunos corolarios interesantes.

En el año 1997, Hutchinson demostró que todo grupo compacto no tall contiene un conjunto de Sidon [57]. Pensamos que dicha afirmación también es cierta para los conjuntos I_0, y que nuestra caracterización puede ser útil para probar esta conjetura. En este sentido, teniendo en cuenta [57, Corolario 2.5] y [47, Teorema 4.10], es conocido que este hecho se cumple si el grupo es, además, conexo. Esta creencia nos lleva a considerar que se puede mejorar el artículo en el que se está trabajando.

La Sección 5.3 se centra en el estudio de la existencia y de las propiedades de los conjuntos centrales I_0 en el conjunto dual de un grupo compacto no abeliano. El estudio de los conjuntos centrales de interpolación empezó en el año 1972, cuando Parker introdujo la noción de conjunto central Sidon [75]. Recientemente, Grow y Hare definieron los conjuntos centrales (ponderados) I_0 en 2014 [45]. En esta sección, se presentan algunas caracterizaciones para la existencia de conjuntos centrales I_0 infinitos en el conjunto dual de un grupo compacto. La más destacada de ellas está conectada con la propiedad de contener una sucesión equivalente a la base unidad del espacio ℓ_1. Además, se demuestra que todo subconjunto infinito de $\widehat{G}$ contiene un conjunto central I_0 infinito si G es compacto no tall. Finalmente, se presentan algunos resultados acerca de la existencia de este tipo de conjuntos de interpolación en los que no es necesaria la hipótesis de ser no tall.

No se presentan las pruebas de este capítulo porque el artículo (4) todavía no ha sido enviado para su publicación. Como subsecuentes lineas de investigación, cabe destacar la de conseguir demostrar que el dual de todo grupo compacto no tall contiene un conjunto infinito I_0 y la de analizar la conexión entre los conjuntos I_0 y los conjuntos centrales I_0.

Chapter 1

Preliminary results and terminology

In this chapter, we introduce some notation and results required along the dissertation. Our basic references are [71, 26, 98] for topological spaces and for topologies on function spaces, [85, 13, 89] for the study of real-valued functions, [53] for topological groups and [52, 32, 86, 44, 10] for the study of the dual set of a topological group.

1.1 Topological spaces

A ***topology*** on a set X is a collection τ of subsets of X satisfying the following conditions:

(i) $\emptyset \in \tau$ and $X \in \tau$.

(ii) If $U_1 \in \tau$ and $U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$.

(iii) If $\mathcal{A}_i \in \tau$, then $\bigcup_{i \in I} \mathcal{A}_i \in \tau$ for an arbitrary set of indices I.

A ***topological space*** is an ordered pair (X, τ) consisting of a set X and a topology τ on X. We often omit specific mention of τ if no confusion arises.

Let X be a topological space with topology τ and let U be a subset of X, we say that U is an *open set* of X if $U \in \tau$. We say that U is a *closed set* when its complement is open. The *closure* of a set U, denoted by $\overline{U}$, is the smallest closed set such that $U \subseteq \overline{U}$ and the *interior* of U, denoted by $Int(U)$, is the largest open set contained in U.

A topological space (X, τ) is called *Hausdorff* when distinct points can be separated by open sets. In this thesis, we always assume that all the topological spaces and groups are Hausdorff.

Let τ and τ' be two topologies on a set X. If $\tau \subseteq \tau'$ we write $\tau \leq \tau'$ and we say that τ is *weaker (coarser)* than τ'. We also say that τ' is *stronger (finer)* than τ.

A subset A of a topological space X is *dense* in X if $\overline{A} = X$. A subset B of X is said to be *nowhere dense* if the interior of the closure of B is the empty set. We say that a subset A of X is *a set of first category* if it can be expressed as the union of countably many nowhere dense subsets of X. The complement of a first category set is a *residual set*.

A family of nonempty sets $\mathcal{A} \subseteq \tau$ is called a *basis for a topological space* (X, τ) if every nonempty open subset of X can be represented as the union of a subfamily of $\mathcal{A}$. The family $\mathcal{A}$ is called *subbasis for a topological space* (X, τ) if it satisfies one of the two following equivalent conditions:

(i) The subcollection $\mathcal{A}$ generates the topology τ. This means that τ is the smallest topology containing $\mathcal{A}$.

(ii) The collection of open sets consisting of all finite intersections of elements of $\mathcal{A}$ forms a basis for τ.

Given a subset A of a topological space (X, τ), we denote by $\tau|_A$ the *topology induced by τ on A*; that is, the topology whose open sets are of the form $U \cap A$ with $U \in \tau$.

Let X and Y be two topological spaces. A function $f : X \to Y$ is said to be *continuous* if for each open subset V of Y, the set $f^{-1}(V)$ is an open subset of X. We denote by $C(X, Y)$ the set of all continuous functions from X to Y. When Y is the set of real numbers $\mathbb{R}$ we simply write $C(X)$.

A topological space (X, τ) is called *completely regular* if given any closed set F and any point x that does not belong to F, there is a continuous function f from X to $[0, 1]$ such that $f(x)$ is 0 and, for every y in F, $f(y)$ is 1. In other terms, this condition says that x and F can be separated by a continuous function. Moreover, X is a *Tychonoff space* if it is both completely regular and Hausdorff.

Let (X, τ) be a topological space, and let $\sim$ be an equivalence relation on X. We denote by $X/\!\sim$ the set of equivalence classes of $\sim$ and by q the mapping from X to $X/\!\sim$ assigning to the point $x \in X$ the equivalence class $[x] \in X/\!\sim$. We take the finest topology on $X/\!\sim$ that makes q continuous; that is, the family of all sets U such that $q^{-1}(U)$ is open in X. This topology is called the *quotient topology*, the set $X/\!\sim$ equipped with it is called the *quotient space*, and $q : X \to X/\!\sim$ is called the *quotient map*.

Let X be a set and let $E = \{f_\alpha : X \to Y_\alpha\}_{\alpha<\kappa}$ be a family of maps where Y_α is a topological space for every $\alpha < \kappa$. The *initial or weak topology* on X associated to E, $w(X, E)$, is the coarsest topology on X that makes continuous all the maps in E.

Let $X \stackrel{\text{def}}{=} \prod_{i \in I} X_i$ be a cartesian product of topological spaces X_i, $i \in I$. Consider the canonical projections $p_i : X \to X_i$. The *product topology* on X is defined to be the coarsest topology for which all the projections p_i are continuous. This topology is sometimes also called the Tychonoff topology.

We say that a topological space is *disconnected* if it is the union of two disjoint nonempty open sets. Otherwise, X is said to be *connected*. The maximal connected subsets (ordered by inclusion) of a nonempty topological space are called the *connected components* of the space.

Let X be a topological space, the *tightness of* X, denoted $tg(X)$, is the smallest infinite cardinal κ such that for any subset $A \subseteq X$ and any point $x \in \overline{A}$ there is a subset $B \subseteq A$ with $|B| \leq \kappa$ and $x \in \overline{B}$.

Let X be a Hausdorff topological space. Here we recall some topological properties that we use in the text:

- We say that X is *totally disconnected space* if the connected components in X are the one-point sets.

- A *metric space* is an ordered pair (M, d) where M is a set and d is a metric on M. Given $x \in M$, we define the open ball of radius $\epsilon > 0$ about x as the set $B(x, r) \stackrel{\text{def}}{=} \{y \in M : d(x, y) < \epsilon\}$. These open balls form a basis for a topology on M, making it a topological space. A metric space M is called *complete* if every Cauchy sequence in M converges in M. If it is also *separable* (i.e. it contains a countable dense subset), then it is said to be a *Polish space*.

- A topological space X is called *compact* (resp. *Lindelöf*) if every open cover has a finite subcover (resp. countable subcover). Every compact space is Lindelöf.

- A topological space X is said to be σ*-compact* if it is the union of countably many compact subspaces. Every compact space is σ-compact, and every σ-compact space is Lindelöf.

- A topological space X is called *locally compact* if every point of X has a compact neighbourhood. Clearly, every compact space is locally compact.

- A Tychonov space X is *Čech-complete* if it is a G_δ-subset (i.e. it is expressible as a countable intersection of open subsets) of its Stone-Čech compatification. The family of Čech-complete spaces includes all complete metric spaces and all locally compact spaces.

- A topological space is *hemicompact* if it has a sequence of compact subsets such that every compact subset of the space lies inside some compact set in the sequence. Every locally compact Lindelöf space is hemicompact. Every hemicompact space is σ-compact.

- We say that X is a k-*space* if the following condition holds: $A \subseteq X$ is open if and only if $A \cap K$ is open in K for each compact set K in X. Every Čech-complete space is a k-space [26, Th. 3.9.5.].

- A Hausdorff topological space X is a k_ω-*space* if there exists an ascending sequence of compact subsets $K_1 \subseteq K_2 \subseteq \ldots \subseteq X$ such that $X = \bigcup\limits_{n<\omega} K_n$ and $U \subseteq X$ is open if and only if $U \cap K_n$ is open in K_n for each $n < \omega$ (i.e. $X = \varinjlim K_n$) as a topological space. A Hausdorff topological space X is *locally* k_ω if each point has an open neighbourhood which is a k_ω-space in the induced topology. It is clear that every k_ω-space is a k-space (see [43]) and hemicompact.

1.2 Function spaces

Let X and Y be two topological spaces. We denote by Y^X the set of all functions from X to Y. The space Y^X can be viewed as a product of X copies of Y (i.e. $Y^X = \prod\limits_{x \in X} Y$). In particular, the set $C(X, Y)$ is a subspace of Y^X. In the following section we report on some topologies on the space Y^X and its subcollections, which are used along the thesis.

1.2.1 Topologies on function spaces

Definition 1.2.1. *We say that $E \subseteq Y^X$ has the* ***topology of pointwise convergence*** *if it is provided with the subspace topology induced by the Tychonoff product topology on Y^X.*

The reason for calling it the topology of pointwise convergence comes frome the following theorem:

Fact 1.2.2. *A net $\{f_\delta\}_{\delta\in\Delta}$ of functions converges to the function f in the topology of pointwise convergence if and only if for each $x \in X$, the sequence $\{f_\delta(x)\}_{\delta\in\Delta}$ of points of Y converges to the point $f(x)$.*

Given $F \subseteq X$, the symbol $t_p(F)$ denotes the topology on Y^X, of pointwise convergence on F. We denote by $C_p(X,Y)$ the topological space $(C(X,Y), t_p(X))$.

For a set E of functions from X to Y and $F \subseteq X$, the symbol $E|_F$ denotes the set $\{f|_F : f \in E\}$. We denote by $\overline{E}^{M^X}$ the closure of E in the Tychonoff product space M^X.

For each point $x \in X$, we consider the evaluation map $eval_x : Y^X \to Y$, defined by $eval_x(f) \stackrel{\text{def}}{=} f(x)$, for every $f \in Y^X$. Note that we can see each point $x \in X$ as a function from Y^X to Y if we associate the point x to the map $eval_x$. Therefore, the symbolism $(F, t_p(\overline{E}^{Y^X}))$ also denotes the set $F \subseteq X$ equipped with the weak topology generated by the functions in $\overline{E}^{M^X}|_F$; that is, the topological space $(F, w(F, \overline{E}^{M^X}|_F))$.

Definition 1.2.3. *The* ***compact-open topology*** *on $E \subseteq Y^X$ is the topology having for a subbase the sets*

$$[K, U] = \{f \in E : f(K) \subseteq U\}$$

for K compact in X, U open in Y. We denote this topology by τ_c.

We denote by $C_c(X,Y)$ the topological space $(C(X,Y), \tau_c)$.

Let (Y, d) be a metric space, let X be a topological space and let E be a subset of Y^X. It is known that the bounded metric $\overline{d}(x,y) = \min\{d(x,y), 1\}$ induces the same topology as d. For any set X and any two functions $f, g : X \to Y$, $\sup_{x \in X}\{d(f(x), g(x))\}$ is always a real number. Therefore, the function $d_\infty : E \times E \to [0, +\infty[$ defined by

$$d_\infty(f,g) = \sup_{x \in X}\{d(f(x), g(x))\}$$

is a metric on E.

Definition 1.2.4. *The metric d_∞ on E is called the uniform metric induced by d, and the topology it induces on E is called the* ***topology of uniform convergence****.*

Given $F \subseteq X$, the symbol $t_\infty(F)$ denotes the topology on Y^X, of uniform convergence on F. We denote by $C_\infty(X,Y)$ the topological space $(C(X,Y), t_\infty(X))$.

Let us see the following version of Arzelà-Ascoli's theorem that we can find in [98, Theorem 43.15] and is useful in Subsection 4.1.2. First, we introduce the definition of equicontinuity.

Definition 1.2.5. *Let X be a topological space and (M,d) be a metric space. A subset E is* ***equicontinuous at $x_0 \in X$*** *if for every $\epsilon > 0$, there is an open neighbourhood U of X such that $d(f(x_0), f(x)) < \epsilon$ for all $x \in U$ and for all $f \in E$. We say that E is* ***equicontinuous*** *provided it is equicontinuous at each point of X.*

Theorem 1.2.6. *(Arzelà-Ascoli Theorem [64, Th. 7.18.]) Let X be a Hausdorff, or regular, k-space, (M,d) a metric space, and E a subset of $C(X,Y)$. Then E is compact in the compact-open topology if an only if*

(a) E is pointwise closed,

(b) for each $x \in X$, $eval_x(E)$ has compact closure,

(c) E is equicontinuous on each compact subset of X.

Theorem 1.2.7. *(Namioka Theorem [72, Th. 2.3.]) Let X be a Čech-complete space, (M,d) a metric space, and E a subset of $C(X,Y)$. If E is compact relative to the pointwise topology, then E is equicontinuous at each point of a dense G_δ set in X.*

Corollary 1.2.8. *([91, Th. Section 4]) If X and Y are compact spaces and $f : X \times Y \to \mathbb{R}$ is separately continuous in each variable, then there is a dense G_δ set $U \subseteq X$ such that f is jointly continuous on $U \times Y$.*

1.2.2 Real-valued continuous functions and Baire class 1 functions

We begin by recalling the notion of a Baire class 1 function.

Definition 1.2.9. *A function $f : X \to M$ is said to be* ***Baire class 1*** *if there is a sequence of continuous functions that converges pointwise to f. We denote by $B_1(X,M)$ the set of all M-valued Baire 1 functions on X. If $M = \mathbb{R}$ we simply write $B_1(X)$.*

A compact space K is called *Rosenthal compactum* if K can be embedded in $B_1(X)$ for some Polish space X.

The first two basic results are Rosenthal's dichotomy theorem [85], which we present in the way they are formulated by Todorčević in [91], and a theorem by Bourgain, Fremlin and Talagrand about compact subsets of Baire class 1 functions [13].

Theorem 1.2.10. *(H. P. Rosenthal) If X is a Polish space and $\{f_n\}_{n<\omega} \subseteq C(X)$ is a pointwise bounded sequence, then either $\{f_n\}_{n<\omega}$ contains a convergent subsequence or a subsequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\omega$.*

Theorem 1.2.11. *(J. Bourgain, D.H. Fremlin, M. Talagrand) Let X be a Polish space and let $\{f_n\}_{n<\omega} \subseteq C(X)$ be a pointwise bounded sequence. The following assertions are equivalent (where the closure is taken in $\mathbb{R}^X$):*

(a) $\{f_n\}_{n<\omega}$ is sequentially dense in its closure.

(b) The closure of $\{f_n\}_{n<\omega}$ contains no copy of $\beta\omega$.

Our third starting fact is extracted from a result by Pol [78, p. 34], which again was formulated in different terms (cf. [15]). Here, we only use one of the implications established by Pol.

Theorem 1.2.12. *(R. Pol) Let X be a complete metric space and let E be an infinite subset of $C(X)$ which is uniformly bounded. If $\overline{E}^{\mathbb{R}^X} \nsubseteq B_1(X)$, then E contains a sequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\omega$.*

The following Remark show us how to translate some results in continuous functions defined on a Polish space into continuous functions on a compact space.

Remark 1.2.13. *Given a subset $E \subseteq C(X, M)$, it is possible to define an equivalence relation on X by $x \sim y$ if and only if $f(x) = f(y)$ for all $f \in E$. If $\widetilde{X} = X/\!\sim$ is the quotient space and $p : X \to \widetilde{X}$ denotes the canonical quotient map, each $f \in E$ has associated a map $\tilde{f} \in C(\widetilde{X}, M)$ defined as $\tilde{f}(\tilde{x}) \stackrel{\text{def}}{=} f(x)$ for any $x \in X$ with $p(x) = \tilde{x}$. Furthermore, if $\tilde{E} \stackrel{\text{def}}{=} \{\tilde{f} : f \in E\}$, we can extend this definition to the pointwise closure of $\tilde{E}$. Thus, each $f \in \overline{E}^{M^X}$ has associated a map $\tilde{f} \in \overline{\tilde{E}}^{M^{\tilde{X}}}$ such that $\tilde{f} \circ p = f$. We denote by X_E the topological space $(\widetilde{X}, t_p(\tilde{E}))$. Note that X_E is metrizable if E is countably infinite and it is Polish if X is compact and E is countably infinite.*

Proposition 1.2.14. *Let L be a countably infinite subset of $C(X, M)$ such that $\overline{L}^{M^X}$ is compact. We denote by X_L the topological space $(\widetilde{X}, t_p(\tilde{L}))$, which is metrizable because $\tilde{L}$ is countable. Consider the map*

$$p^* : (M^{\tilde{X}}, t_p(\tilde{X})) \to (M^X, t_p(X))$$

defined by $p^*(\tilde{f}) = \tilde{f} \circ p$, for each $\tilde{f} \in M^{\tilde{X}}$. Then p^* is a homeomorphism of $\overline{\tilde{L}}^{M^{\tilde{X}}}$ onto $\overline{L}^{M^X}$.

Proof. We observe that p^* is continuous, since a net $\{\tilde{f}_\alpha\}_{\alpha\in A}$ $t_p(\tilde{X})$-converges to $\tilde{f}$ in $\overline{\tilde{L}}^{M^{\tilde{X}}}$ if and only if $\{\tilde{f}_\alpha \circ p\}_{\alpha\in A}$ $t_p(X)$-converges to $\tilde{f}\circ p$ in $\overline{L}^{M^X}$.
Let us see that $p^*(\overline{\tilde{L}}^{M^{\tilde{X}}}) = \overline{L}^{M^X}$. Indeed, since p^* is continuous we have that $p^*(\overline{\tilde{L}}^{M^{\tilde{X}}}) \subseteq \overline{p^*(\tilde{L})}^{M^X} = \overline{L}^{M^X}$. We have the other inclusion because $\overline{L}^{M^X}$ is the smallest closed set that contains L and $L \subseteq p^*(\overline{\tilde{L}}^{M^{\tilde{X}}})$.
Let $\tilde{f}, \tilde{g} \in \overline{\tilde{L}}^{M^{\tilde{X}}}$ such that $\tilde{f} \neq \tilde{g}$. Then there exists $\tilde{x} \in \tilde{X}$ such that $\tilde{f}(\tilde{x}) \neq \tilde{g}(\tilde{x})$. Let $x \in X$ an element such that $\tilde{x} = p(x)$. Thus $(\tilde{f}\circ p)(x) \neq (\tilde{g}\circ p)(x)$. So, p^* is injective because $\tilde{f}\circ p \neq \tilde{g}\circ p$.
Finally, we arrive to the conclusion that $p^*|_{\overline{\tilde{L}}^{M^{\tilde{X}}}}$ is a homeomorphism because it is defined between compact spaces. □

A direct consequence of the strong results presented previously are the following corollaries. We frequently use $\mathfrak{c}$ to denote the cardinal of the continuum of real numbers, that is, $\mathfrak{c} = 2^{\aleph_0} = |\mathbb{R}|$.

Corollary 1.2.15. *If X is a Polish space and E is an infinite uniformly bounded subset of $C(X)$. Then $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$ if and only if $\overline{E}^{\mathbb{R}^X} \subseteq B_1(X)$.*

Proof. One implication is consequence of a well known result by Bourgain, Fremlin and Talagrand (see Theorem 1.2.11). So, assume that $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$. If $\overline{E}^{\mathbb{R}^X} \nsubseteq B_1(X)$, by Theorem 1.2.12, we can find a sequence $\{f_n\}_{n<\omega} \subseteq E$ whose closure in $\mathbb{R}^X$ is canonically homeomorphic to $\beta\omega$. This implies that $\overline{E}^{\mathbb{R}^X}$ contains a copy of $\beta\omega$, which is a contradiction. □

Corollary 1.2.16. *Let X be a Polish space and let E be an infinite uniformly bounded subset of $C(X)$. The following assertions are equivalent:*

(a) $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$.

(b) $\overline{E}^{\mathbb{R}^X} \subseteq B_1(X)$.

(c) E is sequentially dense in $\overline{E}^{\mathbb{R}^X}$.

(d) $|\overline{E}^{\mathbb{R}^X}| \leq \mathfrak{c}$.

(e) E does not contain any sequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\omega$.

Proof. $(a) \Leftrightarrow (b)$ is Corollary 1.2.15.

$(b) \Rightarrow (c)$ was proved by Bourgain, Fremlin and Talagrand (see [91]).

$(b) \Rightarrow (d)$, $(a) \Rightarrow (e)$ and $(c) \Rightarrow (a)$ are obvious.

$(d) \Rightarrow (b)$ and $(e) \Rightarrow (b)$ are a consequence of Theorem 1.2.12.

□

Furthermore, according to results by Rosenthal [85] and Talagrand [89] we can also add the property of containing a sequence equivalent to the unit basis ℓ_1.

Definition 1.2.17. *Let $\{g_n\}_{n<\omega}$ be a uniformly bounded real (or complex) sequence of continuous functions on a set X. We say that $\{g_n\}_{n<\omega}$ is* ***equivalent to the unit basis ℓ_1*** *if there exists a real constant $C > 0$ such that*

$$\sum_{i=1}^{N} |a_i| \leq C \cdot \| \sum_{i=1}^{N} a_i g_i \|_\infty$$

for all scalars $a_1, \ldots, a_N$ and $N \in \omega$.

Theorem 1.2.18. *(Talagrand [89]) Let X be a compact and metric space and let E be an infinite uniformly bounded subset of $C(X)$. The following assertions are equivalent:*

(a) $\overline{E}^{\mathbb{R}^X} \subseteq B_1(X)$.

(b) Every sequence in E has a weak-Cauchy subsequence.

(c) E does not contain any sequence equivalent to the ℓ_1 basis.

Remark 1.2.19. *It is pertinent to notice here that using Rosenthal-Dor Theorem [24], Talagrand's result formulated above also holds for complex valued continuous functions.*

A slight variation of Corollary 1.2.15 is also fulfilled if X is a compact space and E is countably infinite.

Corollary 1.2.20. *Let X be a compact space and let E be a countably infinite and uniformly bounded subset of $C(X)$. Then $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$ if and only if E does not contain any sequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\omega$.*

Proof. Let X_E be the quotient space associated to E equipped with the topology of pointwise convergence on X. According to Proposition 1.2.14, we may assume WLOG that $X = X_E$ and therefore that is a Polish space. It now suffices to apply Corollary 1.2.16. □

Corollary 1.2.21. *Let X be a compact space and let E be a countably infinite and uniformly bounded subset of $C(X)$. The following assertions are equivalent:*

(a) $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$.

(b) E does not contain any sequence whose closure in $\mathbb{R}^X$ is homeomorphic to $\beta\omega$.

(c) $\overline{E}^{\mathbb{R}^X}$ is a Rosenthal compactum.

(d) $|\overline{E}^{\mathbb{R}^X}| \leq \mathfrak{c}$.

(e) E does not contain any subsequence equivalent to the ℓ_1 basis.

Proof. If X_E denotes the quotient space associated to E, then Fact 1.2.14 implies that $\overline{E}^{\mathbb{R}^X}$ is canonically homeomorphic to $\overline{\tilde{E}}^{\mathbb{R}^{X_E}}$

$(a) \Leftrightarrow (b)$ is Corollary 1.2.20.

$(a) \Rightarrow (c)$ By Corollary 1.2.15, we have that $\overline{\tilde{E}}^{\mathbb{R}^{X_E}} \subseteq B_1(X_E)$. Thus $\overline{\tilde{E}}^{\mathbb{R}^{X_E}}$ and, consequently, also $\overline{E}^{\mathbb{R}^X}$ are Rosenthal compactum.

$(c) \Rightarrow (d)$ and $(d) \Rightarrow (b)$ are obvious.

$(a) \Leftrightarrow (e)$ It follows from Fact 1.2.14, Corollary 1.2.15 and Theorem 1.2.18. □

Corollary 1.2.22. *Let X be a compact space and let E be a countably infinite and uniformly bounded subset of $C(X)$. If $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$, then $\overline{E}^{\mathbb{R}^X} \subseteq B_1(X)$.*

Proof. Suppose that there is $f \in \overline{E}^{\mathbb{R}^X} \setminus B_1(X)$. Since $tg(\overline{E}^{\mathbb{R}^X}) \leq \omega$, there is $L \in [E]^{\leq\omega}$ such that $f \in \overline{L}^{\mathbb{R}^X}$. Therefore $f \in \overline{L}^{\mathbb{R}^X} \setminus B_1(X)$ and, by Fact 1.2.14, we deduce that $\tilde{f} \in \overline{\tilde{L}}^{\mathbb{R}^{X_L}} \setminus B_1(X_L)$. It now suffices to apply Corollaries 1.2.16 and 1.2.20. □

The following example shows that Corollary 1.2.21 may fail if one takes E of uncountable cardinality. Given a subset $A \subseteq X$, we denote by χ_A the characteristic function of A, that is, the map defined by setting $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ if $x \in X \setminus A$.

Example 1.2.23. *Let $X = [0, \omega_1]$ and set*

$$E = \{\chi_{[\alpha,\omega_1]} : \alpha < \omega_1, \quad \textit{and } \alpha \textit{ it is not a limit ordinal}\}.$$

Then $|\overline{E}^{\mathbb{R}^X}| = \mathfrak{c}$. However $\chi_{\{\omega_1\}}$ is in the closure of E but it does not belong to the closure of any countable subset of E.

1.2.3 Extensions to M-valued functions

The goal in this section is to provide two ways to extend some results obtained for real-valued continuous functions and Baire class 1 functions to functions that take values in a metric space. The former approach requires that M is a compact metric space, whereas the latter can be used for all metric spaces.

It is well known that for every compact metric space (M, d), there is a canonical continuous one-to-one mapping $\mathcal{E}_M : M \longrightarrow [0,1]^\omega$ that embeds M into $[0,1]^\omega$ as a closed subspace. Let $\rho_n : [-1,1] \longrightarrow [0,1]$ the map defined by $\rho_n(r) = \frac{|r|}{2^n}$ for every $n < \omega$. Along this subsection, we consider that $[0,1]^\omega$ is equipped with the metric ρ defined by

$$\rho((x_n),(y_n)) = \sum_{n<\omega} \rho_n(x_n - y_n)$$

The proof of the following lemma is obtained by a standard argument of compactness, using the continuity of $\mathcal{E}_M^{-1}$ and that every continuous map defined on a compact space is uniformly continuous.

Lemma 1.2.24. *Let (M, d) be a compact metric space. Let $\mathcal{E}_M : M \longrightarrow [0,1]^\omega$ denote its attached embedding into $[0,1]^\omega$, and let $\pi_n : [0,1]^\omega \to [0,1]$ denote the* nth *canonical projection. Then, for every $\epsilon > 0$, there is $\delta > 0$ and $n_0 < \omega$ such that if $(x,y) \in M \times M$ and $\rho_n(\pi_n(\mathcal{E}_M(x)) - \pi_n(\mathcal{E}_M(y))) < \delta/2n_0$ for $n \leq n_0$ then $d(x,y) < \epsilon$.*

Proof. As we said before, we know that $\mathcal{E}_M^{-1}$ is uniformly continuous on $\mathcal{E}_M(M)$. Let $\epsilon > 0$, then there is $\delta > 0$ such that for every $x, y \in M$ with $\rho(\mathcal{E}_M(x), \mathcal{E}_M(y)) < \delta$, we have that $d(x,y) < \epsilon$.

Let n_0 be the minimum natural number such that $1/2^{n_0} < \delta/2$. Let us take $(x,y) \in M \times M$ satisfying that $\rho_n(\pi_n(\mathcal{E}_M(x)) - \pi_n(\mathcal{E}_M(y))) < \delta/2n_0$ for $n \leq n_0$.

We have that:

$$\begin{aligned}
\rho(\mathcal{E}_M(x), \mathcal{E}_M(y)) &= \sum_{n<\omega} \rho_n(\pi_n(\mathcal{E}_M(x)) - \pi_n(\mathcal{E}_M(y))) \\
&\leq \sum_{n=1}^{n_0} \rho_n(\pi_n(\mathcal{E}_M(x)) - \pi_n(\mathcal{E}_M(y))) + \sum_{n=n_0+1}^{\infty} 1/2^n \\
&< \sum_{n=1}^{n_0} \delta/2n_0 + 1/2^{n_0} < \delta/2 + \delta/2 = \delta
\end{aligned}$$

Consequently, $d(x,y) < \epsilon$. □

We now recall some simple remarks that are used along the work.

Proposition 1.2.25. *Let X be a topological space and (M, d) a compact metric space. If π_n is the nth projection mapping defined above, then the following map is continuous if we consider that the two spaces have the topology of pointwise convergence.*

$$\pi_n^* : M^X \to [0,1]^X$$

defined by $\pi_n^*(f) \stackrel{\text{def}}{=} \pi_n \circ \mathcal{E}_M \circ f$, $f \in M^X$, *for each* $n < \omega$.

For each $S \subseteq M^X$ and each $n < \omega$ we define $S_n \stackrel{\text{def}}{=} \pi_n^*(S)$.

Proposition 1.2.26. *Let X be a Baire space, (M, d) be a compact metric space, $E \subseteq C(X, M)$ and $H \stackrel{\text{def}}{=} \overline{E}^{M^X}$. Then $\pi_n^*(H) = H_n = \overline{E_n}^{[0,1]^X}$.*

Proof. Indeed, since π_n^* is continuous we have that $H_n = \pi_n^*(H) = \pi_n^*(\overline{E}^{M^X}) \subseteq \overline{\pi_n^*(E)}^{[0,1]^X} = \overline{E_n}^{[0,1]^X}$. For the reverse inclusion, remark that $\overline{E_n}^{[0,1]^X}$ is the smallest closed subset that contains E_n and $E_n \subseteq H_n$. □

Proposition 1.2.27. *Let X be a Baire space, (M, d) be a compact metric space and $E \subseteq C(X, M)$. If E_n is almost equicontinuous (see Definition 2.1.1) for every $n < \omega$, then E is almost equicontinuous.*

Proof. For each $n \in \omega$ there exists a dense G_δ subset D_n of X such that E_n is equicontinuous on D_n. Since X is a Baire space, the $D = \bigcap_{n<\omega} D_n$ is dense in X. We claim that E is equicontinuous in D. Indeed, let $x_0 \in D$ and $\epsilon > 0$. By Lemma 1.2.24 we get $\delta > 0$ and $n_0 < \omega$. Take $\epsilon_0 = \frac{\delta}{2n_0}$. For each $n < n_0$, being E_n equicontinuous in x_0, there is an open neighbourhood U_n of x_0 such that $|g_n(x_0) - g_n(x)| < \epsilon_0$ for all $x \in U_n$ and $g_n \in E_n$. Consider the open neighbourhood $U = \bigcap_{n<n_0} U_n$ of x_0. So, let an arbitrary $g \in E$ and $x \in U$, then $\rho_n(\pi_n(\mathcal{E}_M(g(x_0))) - \pi_n(\mathcal{E}_M(g(x)))) = \rho_n(\pi_n^*(g)(x_0) - \pi_n^*(g)(x)) = \frac{|\pi_n^*(g)(x_0) - \pi_n^*(g)(x)|}{2^n} < \frac{\epsilon_0}{2^n} \leq \frac{\delta}{2n_0}$. Consequently, $d(g(x_0), g(x)) < \epsilon$ by Lemma 1.2.24. □

Proposition 1.2.28. *The diagonal map $\Delta : H \to \prod_{n<\omega} H_n$ defined by $\Delta(h) = (\pi_n \circ \mathcal{E}_M \circ h)_{n<\omega}$ for each $h \in H$, is a homeomorphism of H onto its image.*

All maps defined in the previous subsection can be conveyed canonically to a countably infinite subset L of E. Thus we have:

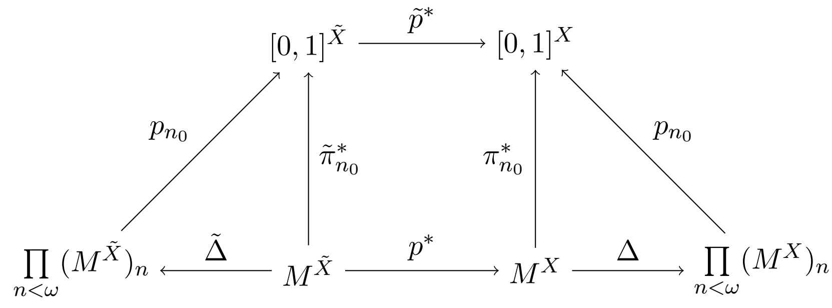
$$\tilde{\pi}_n^* : M^{\tilde{X}} \to [0,1]^{\tilde{X}}; \quad \tilde{f} \mapsto \pi_n \circ \mathcal{E}_M \circ \tilde{f}$$

$$\tilde{\Delta} : \overline{\tilde{L}}^{M^{\tilde{X}}} \to \prod_{n<\omega} \overline{\tilde{L}_n}^{[0,1]^{\tilde{X}}}; \quad \tilde{f} \mapsto \{\pi_n \circ \mathcal{E}_M \circ \tilde{f}\}_{n<\omega}$$

$$\overline{p}^* : [0,1]^{\tilde{X}} \to [0,1]^X; \quad \tilde{f} \mapsto \tilde{f} \circ p$$

Proposition 1.2.29. $\overline{p}^* : \overline{\tilde{L}}^{[0,1]^{\tilde{X}}} \longrightarrow \overline{L}^{[0,1]^X}$ *is an onto homeomorphism.*

So, for each $n_0 < \omega$ we have the following commutative diagram:



where p_{n_0} is the n_0th projection of the sequence.

In the second chapter of the thesis, we extend some results for real-valued functions spaces to metric-valued function spaces. This is accomplished using an idea of Christensen [19]. First, recall that given a metric space (M,d), it is well known that the metric $\bar{d} : M \times M \to \mathbb{R}$ defined by $\bar{d}(m_1,m_2) \stackrel{\text{def}}{=} \min\{d(m_1,m_2),1\}$ for all $m_1,m_2 \in M$ induces the same topology as d. So, without loss of generality, we work with this metric from here on.

Definition 1.2.30. *Let X be a topological space, (M,d) be a metric space that we always assume equipped with a bounded metric and E be a subset of $C(X,M)$ that we consider equipped with the pointwise convergence topology $t_p(X)$ in the sequel, unless otherwise stated.*
Set

$$\mathcal{K} \stackrel{\text{def}}{=} \{\alpha : M \to [-1,1] : |\alpha(m_1) - \alpha(m_2)| \leq d(m_1,m_2), \quad \forall m_1,m_2 \in M\}.$$

Being pointwise closed and equicontinuous by definition, it follows that $\mathcal{K}$ is a compact and metrizable subspace of $\mathbb{R}^M$. For $m_0 \in M$, define $\alpha_{m_0} \in \mathbb{R}^M$ by $\alpha_{m_0}(m) \stackrel{\text{def}}{=} d(m,m_0)$ for all $m \in M$. It is easy to check that $\alpha_{m_0} \in \mathcal{K}$.

Consider the evaluation map $\varphi : X \times E \to M$ defined by $\varphi(x,g) \stackrel{\text{def}}{=} g(x)$ for all $(x,g) \in X \times E$, which is clearly separately continuous. The map φ has *associated a separately continuous map* $\psi : X \times (E \times \mathcal{K}) \to [-1,1]$ *defined by* $\psi(x,(g,\alpha)) \stackrel{\text{def}}{=} \alpha(g(x))$ *for all* $(x,(g,\alpha)) \in X \times (E \times \mathcal{K})$.

Set

$$\nu : \overline{E}^{M^X} \times \mathcal{K} \to [-1,1]^X$$

defined by

$$\nu(h,\alpha) \stackrel{\text{def}}{=} \alpha \circ h \textit{ for all } h \in \overline{E}^{M^X} \textit{ and } \alpha \in \mathcal{K}.$$

Remark 1.2.31. ν *is continuous.*

Proof. Let $\{(h_\delta,\alpha_\delta)\}_{\delta\in\Delta} \subseteq \overline{E}^{M^X} \times \mathcal{K}$ be a net that converges to $(h,\alpha) \in \overline{E}^{M^X} \times \mathcal{K}$. Given $\epsilon > 0$ and $x \in X$, then there exists $\delta_0 \in \Delta$ such that $d(h_\delta(x), h_0(x)) < \epsilon/2$ and $|\alpha_\delta(h_0(x)) - \alpha_0(h_0(x))| < \epsilon/2$ for all $\delta > \delta_0$. Therefore, we have that $|\nu(h_0,\alpha_0)(x) - \nu(h_\delta,\alpha_\delta)(x)| = |\alpha_0(h_0(x)) - \alpha_\delta(h_\delta(x))| \leq |\alpha_0(h_0(x)) - \alpha_\delta(h_0(x))| + |\alpha_\delta(h_0(x)) - \alpha_\delta(h_\delta(x))| \leq |\alpha_0(h_0(x)) - \alpha_\delta(h_0(x))| + d(h_0(x), h_\delta(x)) < \epsilon$ for all $\delta > \delta_0$. □

Since $E \subseteq C(X,M)$*, we have that* $\nu(E \times \mathcal{K}) \subseteq C(X,[-1,1])$.

Given $f \in M^X$ *we can associate a map* $\check{f} \in \mathbb{R}^{X\times\mathcal{K}}$ *defined by*

$$\check{f}(x,\alpha) \stackrel{\text{def}}{=} \nu(f,\alpha)(x) = \alpha(f(x)), \textit{ for all } (x,\alpha) \in X \times \mathcal{K}.$$

In like manner, given any subset E *of* M^X*, we set* $\check{E} \stackrel{\text{def}}{=} \{\check{f} : f \in E\} \subseteq \mathbb{R}^{X\times\mathcal{K}}$.

Lemma 1.2.32. *Let* X *be a topological space,* (M,d) *a metric space and* E *a subset of* $C(X,M)$*. Let* $\mathcal{K}$ *and* ν *be the space and the map defined in Definition 1.2.30. Then, for every subset* F *of* X*, the identity map* $id : (F, t_p(\overline{E}^{M^X})) \to (F, t_p(\nu(\overline{E}^{M^X} \times \mathcal{K})))$ *is a homeomorphism.*

Proof. Let $\{x_\delta\}_{\delta\in\Delta} \subseteq F$ be a net that $t_p(\overline{E}^{M^X})$-converges to x. Since α is continuous, for any $(h,\alpha) \in \overline{E}^{M^X} \times \mathcal{K}$, we have $\lim_{\delta\in\Delta} \nu(h,\alpha)(x_\delta) = \lim_{\delta\in\Delta} \alpha(h(x_\delta)) = \alpha(h(x)) = \nu(h,\alpha)(x)$. So, id is continuous. Conversely, let $\{x_\delta\}_{\delta\in\Delta} \subseteq F$ be a net that $t_p(\nu(\overline{E}^{M^X} \times \mathcal{K}))$-converges to $x_0 \in F$. Given $h \in \overline{E}^{M^X}$ arbitrary, take $\alpha_{h(x_0)} \in \mathcal{K}$. So, fixed $\epsilon > 0$, there is $\delta_0 \in \Delta$ such that $\epsilon > |\nu(h,\alpha_{h(x_0)})(x_\delta) - \nu(h,\alpha_{h(x_0)})(x_0)| = |d(h(x_\delta), h(x_0)) - d(h(x_0), h(x_0))| = d(h(x_\delta), h(x_0))$ for every $\delta > \delta_0$. That is, the net $\{x_\delta\}_{\delta\in\Delta}$ converges to x_0 in $t_p(\overline{E}^{M^X})$, which completes the proof. □

Lemma 1.2.33. *Let X be a topological space, (M,d) a metric space and $E \subseteq C(X,M)$. Then:*

(a) $f \in C(X,M)$ if and only if $\check{f} \in C(X \times \mathcal{K})$.

(b) A net $\{g_\delta\}_{\delta\in \mathrm{w}} \subseteq C(X,M)$ converges to $f \in M^X$ if and only if the net $\{\check{g}_\delta\}_{\delta\in \mathrm{w}} \subseteq C(X\times\mathcal{K})$ converges to $\check{f} \in \mathbb{R}^{X\times\mathcal{K}}$.

(c) If $\overline{E}^{M^X}$ is compact, then $\overline{E}^{M^X}$ and $\overline{\check{E}}^{\mathbb{R}^{X\times\mathcal{K}}}$ are canonically homeomorphic.

Proof. (a) Suppose that $f \in C(X,M)$ and let $\{(x_\delta,\alpha_\delta)\}_{\delta\in\mathrm{w}} \subseteq X\times\mathcal{K}$ be a net that converges to $(x,\alpha) \in X\times\mathcal{K}$. For every $\delta\in\mathrm{w}$, we have

$$\begin{aligned}|\alpha_\delta(f(x_\delta)) - \alpha(f(x))| &\leq |\alpha_\delta(f(x_\delta)) - \alpha_\delta(f(x))| + |\alpha_\delta(f(x)) - \alpha(f(x))| \leq \\ &\leq d(f(x_\delta), f(x)) + |\alpha_\delta(f(x)) - \alpha(f(x))|\end{aligned}$$

Since $\{f(x_\delta)\}_{\delta\in\mathrm{w}}$ converges to $f(x)$ and $\{\alpha_\delta\}_{\delta\in\mathrm{w}}$ converges to α it follows that

$$\lim_{\delta\in\mathrm{w}} \check{f}(x_\delta,\alpha_\delta) = \lim_{\delta\in\mathrm{w}} \alpha_\delta(f(x_\delta)) = \alpha(f(x)) = \check{f}(x,\alpha).$$

Conversely, suppose that $\check{f} \in C(X\times\mathcal{K})$ and let $\{x_\delta\}_{\delta\in\mathrm{w}} \subseteq X$ a net that converges to $x \in X$. Consider the map $\alpha_{f(x)} \in \mathcal{K}$. We have

$$\lim_{\delta\in\mathrm{w}} d(f(x_\delta), f(x)) = \lim_{\delta\in\mathrm{w}} \alpha_{f(x)}(f(x_\delta)) = \lim_{\delta\in\mathrm{w}} \check{f}(x_\delta, \alpha_{f(x)}) = \check{f}(x,\alpha_{f(x)}) = 0,$$

That is, f is continuous.

(b) Suppose that $\{g_\delta\}_{\delta\in\mathrm{w}} \subseteq C(X,M)$ converges pointwise to $f \in M^X$ and take the associated sequence $\{\check{g}_\delta\}_{\delta\in\mathrm{w}} \subseteq C(X\times\mathcal{K})$. Then $\lim_{\delta\in\mathrm{w}} \check{g}_\delta(x,\alpha) = \lim_{\delta\in\mathrm{w}} \alpha(g_\delta(x)) = \alpha(\lim_{\delta\in\mathrm{w}} g_\delta(x)) = \alpha(f(x)) = \check{f}(x,\alpha)$ for all $(x,\alpha) \in X\times\mathcal{K}$.

Conversely, suppose that $\{\check{g}_\delta\}_{\delta\in\mathrm{w}} \subseteq C(X\times\mathcal{K})$ converges pointwise to $\check{f} \in \mathbb{R}^{X\times\mathcal{K}}$ and let us see that the sequence $\{g_\delta\}_{\delta\in\mathrm{w}} \subseteq C(X,M)$ converges pointwise to f. Indeed, it suffices to notice that for every $x\in X$ and its associated map $\alpha_{f(x)} \in \mathcal{K}$, we have

$$|\check{g}_\delta(x,\alpha_{f(x)}) - \check{f}(x,\alpha_{f(x)})| = |\alpha_{f(x)}(g_\delta(x)) - \alpha_{f(x)}(f(x))| = d(g_\delta(x), f(x)).$$

(c) Consider the map $\varphi : \overline{E}^{M^X} \to \overline{\check{E}}^{\mathbb{R}^{X\times\mathcal{K}}}$ defined by $\varphi(g) \stackrel{\text{def}}{=} \check{g}$ for all $g \in \overline{E}^{M^X}$. By compactness, it is enough to prove that φ is injective and continuous. The argument verifying the continuity of φ has been used in (b). Thus we only verify that φ is injective. Assume that $\varphi(f) = \varphi(g)$ with $f, g \in \overline{E}^{M^X}$, which means $\alpha(f(x)) = \alpha(g(x))$ for all $(x,\alpha) \in X\times\mathcal{K}$. Given $x \in X$, we have the map $\alpha_{g(x)} \in \mathcal{K}$ and, consequently, $\alpha_{g(x)}(f(x)) = \alpha_{g(x)}(g(x)) = 0$ for all $x\in X$. This yields $d(f(x),g(x)) = 0$ for all $x \in X$, which implies $f = g$. □

1.3 Topological groups

Definition 1.3.1. *Let $(G, \cdot)$ be a group and let τ be a topology on G. If the mappings*

$$m : G \times G \to G \qquad \text{and} \qquad i : G \to G$$
$$m(g, h) = g \cdot h \qquad\qquad i(g) = g^{-1}$$

are continuous, then τ is said to be a ***group topology*** *and (G, τ) is called* ***topological group****.*

The conditions on the continuity on m and on i are in general independent. In case where m is continuous but i need not be continuous, (G, τ) is called *paratopological group*. Compact paratopological groups are automatically topological groups because i is automatically continuous.

A group homomorphism is a map $g : G \to H$ between two groups such that the group operation is preserverd (i.e. $f(q_1 \cdot g_2) = f(g_1) \cdot f(g_2)$ for all $g_1, g_2 \in G$). The set of homomorphisms (resp. continuous homomorphisms) of G into H is denoted by $Hom(G, H)$ (resp. $CHom(G, H)$).

We say that a topological group G is *precompact* if it is isomorphic (as a topological group) to a subgroup of a compact group H (we may assume that G is dense in H).

Proposition 1.3.2. *([54, Prop. 1.8.])*

(i) The space underlying a topological group is homogeneous (i.e. for every $(g, h) \in G \times G$ there is a homeomorphism $f : G \to G$ such that $f(g) = h$).

(ii) Every quotient space $G/H \overset{\text{def}}{=} \{gH : g \in G\}$ with the quotient topology is a homogeneous space.

Proposition 1.3.3. *([54, Prop. 1.10.])*

(i) If H is a subgroup of a topological group G, then H is a topological group in the induced topology.

(ii) If $\{G_i\}_{i \in I}$ is a family of topological groups, then $\prod_{i \in I} G_i$ is a topological group.

(iii) If N is a normal subgroup of a topological group G, then the quotient group G/N is a topological group with respect to the quotient topology.

Example 1.3.4. *([54, Example 1.9.])*

(i) *Every group is a topological group when endowed with the discrete topology. A group G endowed with the discrete topology is denoted by G_d.*

(ii) *The group of real numbers $\mathbb{R}$ and the group of complex numbers $\mathbb{C}$ endowed with its usual topology are topological groups.*

(iii) *The general linear group $GL(n, \mathbb{C})$ of all invertible nxn matrices with complex entries can be viewed as a topological group with the topology defined by viewing $GL(n, \mathbb{C})$ as a subspace of Euclidean space $\mathbb{C}^{n\times n}$.*

(iv) *The unitary group of degree n, denoted $\mathbb{U}(n)$, is the group of $n \times n$ unitary matrices, with the group operation of matrix multiplication. The unitary group is a subgroup of the general linear group $GL(n, \mathbb{C})$.*

(v) *More generally, for a Hilbert space $\mathcal{H}$, $\mathbb{U}(\mathcal{H})$ is the group of unitary operators on that Hilbert space.*

Let us see the following structure theorem for compact connected groups.

Theorem 1.3.5. *([79, Th. 6.5.6.]) Let G be a compact connected group. Then, G is isomorphic to a quotient of $K \overset{\text{def}}{=} \prod_{i\in I} G_i \times A$ where each G_i is a compact simply-connected Lie group and A is a compact abelian group.*

The next definition is a purely algebraic concept.

Definition 1.3.6. *Let G be an arbitrary group, not necessarily topological. A complex-valued function f on G is said to be* ***positive-definite*** *if the inequality*

$$\sum_{j=1}^{m}\sum_{k=1}^{m} \overline{\alpha}_j \alpha_k f(x_j^{-1}x_k) \geq 0$$

holds for every choice of $x_1, \ldots, x_m$ in G and for every choice of complex numbers $\alpha_1, \ldots, \alpha_m$.

Example 1.3.7. *(a) Suppose $f \in L^2(G)$ (i.e. $\int_G |f|^2 dm_G < \infty$) and consider the map $\tilde{f}$ defined by $\tilde{f}(g) = \overline{f(x^{-1})}$ for every $x \in G$. Then the map $\varphi \overset{\text{def}}{=} f * \tilde{f}$ is positive-definite.*

(b) The map $h : \mathbb{C} \to \mathbb{C}$ defined by $h(x) = |x|$, for every $x \in \mathbb{C}$, is not positive-definite

Proof. (*a*)

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{\alpha}_j\alpha_k\varphi(x_j^{-1}x_k)=\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{\alpha}_j\alpha_k\int_G f(y)\overline{f(x_k^{-1}x_jy)}dm_G(y)=$$
$$=\sum_{j=1}^{m}\sum_{k=1}^{m}\overline{\alpha}_j\alpha_k\int_G f(x_j^{-1}y^{-1})\overline{f(x_k^{-1}y^{-1})}dm_G(y)=$$
$$=\int_G|\sum_{k=1}^{m}\alpha_k f(x_k^{-1}y^{-1})|^2dm_G(y)\geq 0$$

□

1.4 Dual set of a topological group

Definition 1.4.1. *Let (G,τ) be a locally compact group.*

- *An* ***unitary representation*** *of G is a homomorphism $\boldsymbol{\sigma}$ from G into the group $\mathbb{U}(\mathcal{H}_\sigma)$ that is continuous with respect the strong (equivalently, weak) operator topology (i.e. the topology of pointwise convergence on $\mathbb{U}(\mathcal{H}_\sigma)$).*

- *An unitary representation σ is* ***irreducible*** *if $\{0\}$ and $\mathcal{H}_\sigma$ are the only closed subspaces of $\mathcal{H}_\sigma$ invariant for all $\sigma(g)$, $g\in G$.*

- *If σ_1,σ_2 are two unitary representations of G, an* ***intertwining operator*** *for σ_1 and σ_2 is a bounded linear map $T:\mathcal{H}_{\sigma_1}\to\mathcal{H}_{\sigma_2}$ such that $T\sigma_1(g)=\sigma_2(g)T$ for all $g\in G$. Denote by $\mathcal{C}(\sigma_1,\sigma_2)$ the set of all such operators.*

- *Two unitary representations σ_1,σ_2 are* ***equivalent*** *if $\mathcal{C}(\sigma_1,\sigma_2)$ contains a unitary operator U (i.e. there is an unitary operator U such that $\sigma_2(g)=U\sigma_1(g)U^{-1}$, $g\in G$).*

Proposition 1.4.2. *([10, Prop C.4.3.]) Let σ be a unitary representation of G, and let u be a vector in $\mathcal{H}$. Then the diagonal matrix coefficient*

$$g\mapsto\langle\sigma(g)u,u\rangle$$

is a positive-definite function.

The functions of the type $\langle\sigma(\cdot)u,u\rangle$ are said to be *the positive-definite functions associated to σ*.

Definition 1.4.3. *Let σ_1 and σ_2 be two unitary representations of the topological group G. We say that σ_1 is **weakly contained** in σ_2 if every positive-definite function associated to σ_1 can be approximated, uniformly on compact subsets of G, by finite sums of positive-definite functions associated to σ_2. That is, for every u in $\mathcal{H}_{\sigma_1}$, every compact subset K of G and every $\epsilon > 0$, there exists $v_1, \ldots, v_n$ in $\mathcal{H}_{\sigma_2}$ such that, for all $x \in K$,*

$$|\langle \sigma_1(x)u, u\rangle - \sum_{i=1}^{n} \langle \sigma_2(x)v_i, v_i\rangle| < \epsilon.$$

We write for this $\sigma_1 \prec \sigma_2$.

Lemma 1.4.4. *(Schur's Lemma [32, Th. 3.5.])*

(i) A unitary representation σ of G is irreducible if and only if $\mathcal{C}(\sigma, \sigma)$ contains only scalar multiples of the identity.

(ii) Suppose σ_1 and σ_2 are irreducible unitary representations of G. If they are equivalent, then $\mathcal{C}(\sigma_1, \sigma_2)$ is one-dimensional; otherwise, $\mathcal{C}(\sigma_1, \sigma_2) = \{0\}$.

Given a locally compact group (G, τ), we denote by $Irr(G)$ the set of all continuous unitary irreducible representations σ defined on G. That is, continuous in the sense that each matrix coefficient function $g \mapsto \langle \sigma(g)u, v\rangle$ is a continuous map of G into the complex plane.

Thus, fixed $\sigma \in Irr(G)$, if $\mathcal{H}^\sigma$ denotes the Hilbert space associated to σ, we equip the unitary group $\mathbb{U}(\mathcal{H}^\sigma)$ with the strong operator topology. The dimension of $\mathcal{H}^\sigma$ is called *dimension* or *degree of* σ and it is denoted by d_σ.

A basic example is given when G is a locally compact group with a left Haar measure. Then left translations provide us a unitary representation λ_G of G on the Hilbert space $L^2(G)$ called the *left regular representation*, by taking $(\lambda_G(g)f)(h) \overset{\text{def}}{=} L_g f(h) = f(g^{-1}h)$, $f \in L^2(G)$, $g, h \in G$. Similarly, one can define the *right regular representation* ρ_G.

For two elements π and σ of $Irr(G)$, we write $\pi \sim \sigma$ to denote the relation of unitary equivalence and we denote by $\widehat{G}$ the ***dual set*** of G, which is defined as the set of equivalence classes in $(Irr(G)/\sim)$. We refer to [23, 10] for all undefined notions concerning the unitary representations of locally compact groups.

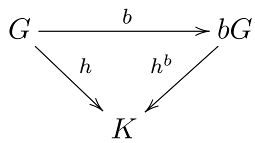
Adopting the terminology introduced by Ernest in [27], set $\mathcal{H}_n \overset{\text{def}}{=} \mathbb{C}^n$ for $n = 1, 2, \ldots$; and $\mathcal{H}_0 \overset{\text{def}}{=} l^2(\mathbb{Z})$. The symbol $Irr_n^C(G)$ denotes the set of irreducible unitary representations of G on $\mathcal{H}_n$. We assume that every set

$Irr^C_n(G)$ is equipped with the compact open topology. Finally, we define $Irr^C(G) = \bigsqcup_{n\geq 0} Irr^C_n(G)$ (the disjoint topological sum).

Theorem 1.4.5. *(Dixmier [23, Section 18.1.10]) If G is a Polish locally compact group, then $Irr^C_n(G)$ equipped with the compact-open topology is a Polish space for all $n \in \{0, 1, 2, \ldots\}$.*

We denote by $G^w = (G, w(G, Irr(G))$ (resp. $G^{w_C} = (G, w(G, Irr^C(G))))$ the group G equipped with the weak (group) topology generated by $Irr(G)$ (resp. $Irr^C(G)$). Since equivalent representations define the same topology, we have $G^w = (G, w(G, \widehat{G}))$. That is, the *weak topology of a group G* is the initial topology on G defined by the dual set. Moreover, in case G is a separable, metric, locally compact group, then every irreducible unitary representation acts on a separable Hilbert space and, as a consequence, it is unitary equivalent to a member of $Irr^C(G)$. Thus, $G^w = (G, w(G, Irr^C(G))) = G^{w_C}$ for separable, metric, locally compact groups. We make use of this fact in order to avoid the proliferation of isometries (see [23]). In case the group G is abelian, the dual object $\widehat{G}$ is a group, which is called *dual group*, and the weak topology of G reduces to the weak topology generated by all continuous homomorphisms of G into the unit circle $\mathbb{T}$. Therefore, the weak topology coincides with the so-called *Bohr topology* of G, which we recall in the next paragraph.

The *Bohr compactification* of a topological group G can be defined as a pair (bG, b), where bG is a compact Hausdorff group and b is a continuous homomorphism from G onto a dense subgroup of bG such that every continuous homomorphism $h\colon G \to K$ into a compact group K extends to a continuous homomorphism $h^b\colon bG \to K$, making the following diagram commutative:



The topology that b induces on G is referred to as the *Bohr topology*. In Anzai and Kakutani [4], bG is built when G is locally compact abelian (LCA). However, most authors agree that it was A. Weil [97] the first to build bG. Weil called bG "Groupe compact attaché à G ". The name of Bohr compactification was given by Alfsen and Holm [3] in the context of arbitrary topological groups. The Bohr topology of an abelian group G is commonly denoted by G^+.

A topological group G is said to be *maximally almost periodic* (MAP, for short) when the map b is one-to-one, which implies that the Bohr topology is Hausdorff.

Example 1.4.6. *(a)* $\mathbb{R}$ *is a MAP group.*

(b) $SL(2,\mathbb{C})$ *(i.e the special linear group of order 2) is not a MAP group (see [96]).*

1.4.1 Duality for abelian groups

We present some basic definitions and facts about the Pontryagin-van Kampen duality and the Bohr compactification of abelian groups.

As a corollary of the Schur's Lemma (see Lemma 1.4.4), we have that if G is abelian, then every irreducible representation of G is one-dimensional. Note that $\mathbb{U}(1) = \mathbb{T}$.

Let (G, τ) be an abelian topological group, with underlying group G and topology τ. In this case, the elements of the dual set $\widehat{G}$ are usually called *characters*. This set is a group when we consider the operation $(\chi_1 + \chi_2)(g) = \chi_1(g) + \chi_2(g)$ for all $g \in G$. Here, we use the additive notation and 0 for the trivial element. In particular, we identify $\mathbb{T}$ with the additive group $[-1/2, 1/2)$, having addition defined by identifying $\pm 1/2$.

For a topological abelian group G, let $\mathcal{K}(G)$ denote the family of all compact subsets of G. For a set $A \subseteq G$ and a positive real ϵ, define

$$[A, \epsilon] \stackrel{\text{def}}{=} \{\chi \in \widehat{G} : |\chi(a)| \leq \epsilon \text{ for all } a \in A\}.$$

The sets $[K, \epsilon] \subseteq \widehat{G}$, for $K \in \mathcal{K}(G)$ and $\epsilon > 0$, form a neighborhood base at the trivial character, defining the compact-open topology. We simply write $\widehat{G}$ for the topological abelian group obtained in this manner.

A topological abelian group G is *reflexive* if the evaluation map *eval*

$$eval \colon G \to \widehat{\widehat{G}},$$

defined by $eval(g)(\chi) = \chi(g)$ for all $g \in G$ and $\chi \in \widehat{G}$, is a topological isomorphism. By the Pontryagin–van Kampen theory, we know that every locally compact abelian group is reflexive. Furthermore, the dual of a compact group is discrete and the dual of a discrete group is compact. In general:

Theorem 1.4.7. *The dual of a locally compact abelian group is a locally compact abelian group.*

It follows that every compact abelian group is equipped with the topology of pointwise convergence on its dual group.

Definition 1.4.8. *Let G be a topological abelian group. For $A \subseteq G$, let $A^{\triangleright} \stackrel{\text{def}}{=} [A, 1/4]$. Similarly, for $X \subseteq \widehat{G}$, let*

$$X^{\triangleleft} \stackrel{\text{def}}{=} \left\{ g \in G : |\chi(g)| \leq \frac{1}{4} \textit{ for all } \chi \in X \right\}.$$

The following facts are well known (see [7]).

Lemma 1.4.9. *For each neighborhood U of 0 in G, we have that $U^{\triangleright} \in \mathcal{K}(\widehat{G})$ (i.e. $U^{\triangleright}$ is a compact subset of $\widehat{G}$).*

Definition 1.4.10. *Let G be a topological abelian group. A set $A \subseteq G$ is* **quasiconvex** *if $A^{\triangleright\triangleleft} = A$. The topological group G is* **locally quasiconvex** *if it has a neighborhood base at its identity, consisting of quasiconvex sets.*

For each set $A \subseteq G$, the set $A^{\triangleright}$ is a quasiconvex subset of $\widehat{G}$. Thus, the topological group $\widehat{G}$ is locally quasiconvex for all topological abelian groups G. Moreover, local quasiconvexity is hereditary for arbitrary subgroups.

The set $A^{\triangleright\triangleleft}$ is the smallest quasiconvex subset of G containing A. This set is closed.

In the case where G is a topological vector space, G is locally quasiconvex in the present sense if, and only if, G is a locally convex topological vector space in the ordinary sense.

If G is locally quasiconvex, its characters separate points of G, and thus the evaluation map $eval\colon G \to \widehat{\widehat{G}}$ is injective. For each quasiconvex neighborhood U of 0 in G, the set $U^{\triangleright}$ is a compact subset of $\widehat{G}$ (Lemma 1.4.9), and thus $U^{\triangleright\triangleright}$ is a neighborhood of 0 in $\widehat{\widehat{G}}$. As $eval[G] \cap U^{\triangleright\triangleright} = eval[U^{\triangleright\triangleleft}] = eval[U]$, we have that $eval$ is open [7, Lemma 14.3].

The duality theory can be used to represent the Bohr compactification of an abelian group as a group of homomorphisms. Indeed, if G is an abelian topological group and $\widehat{G}_d$ denotes its dual group equipped with the discrete topology then bG coincides with the dual group of $\widehat{G}_d$. This canonical representation of the Bohr compactification together with Pontryagin duality have made possible a much better understanding of the Bohr compactification and topology for locally compact abelian groups than in the general non-abelian case.

1.4.2 Dual set of a non-abelian group

When G is non-abelian, $\widehat{G}$ fails to be a group. Hence, it is called ***dual object*** in this context. Duality theory for non-abelian groups is much more involved because, in this case, the group $CHom(G, \mathbb{T})$ no longer determines the topological structure of G. In fact, given a finite simple group G, the only homomorphism that we can define between G and $\mathbb{T}$ is the trivial one. In the search of a duality in the case of non-abelian group, it is compulsory to remark the *Tannaka-Kreĭn* [33] and *Chu* duality [20].

In the remainder of the subsection we assume that the topological group G is compact, unless we indicate the contrary. In that case, the Peter-Weyl Theorem (see [53]) implies that all irreducible unitary representation of G are finite-dimensional and determine an embedding of G into the product of unitary groups $\mathbb{U}(n)$. Thus we may view $\widehat{G}$ as a set of matrix-valued functions $\sigma : G \to \mathbb{U}(d_\sigma)$.

Adopting the convention of McMullen and Price [69], we say that G is *tall* if for each positive integer n there are only finitely many elements of $\widehat{G}$ of degree n. Theorem 1.4.14 and Example 5.3.10 provides us with some examples of tall compact groups.

Given $\sigma \in \widehat{G}$, we denote by χ_σ the *character associated to* σ, which is defined by $\chi_\sigma(g) \stackrel{\text{def}}{=} tr(\sigma(f))$ for all $g \in G$, and by χ_σ^N the *normalised character associated to* σ; that is, the character associated to σ divided by the degree of the representation d_σ.

Theorem 1.4.11. *(Gallagher [36]) If G is a compact group, then there is $g \in G$ and $\sigma \in \widehat{G}$ such that $\chi_\sigma(g) = 0$.*

Every compact group admits a left Haar measure, m_G. Therefore, as in the abelian case, one can define the notion of *Fourier transform* for integrable functions and measures.

Given $f \in L^1(G)$, the *Fourier transform of f* is the function $\widehat{f}$ defined on $\widehat{G}$ by

$$\widehat{f}(\sigma) = \int_G f(x)\sigma(x) dm_G$$

Observe that $\widehat{f}(\sigma)$ is a matrix of size $d_\sigma \times d_\sigma$, for each $\sigma \in \widehat{G}$.

The Fourier transform can be extended to complex Radon measures on G by the following way: if $\mu \in M(G)$, its Fourier transform is the bounded continuous function $\widehat{\mu}$ on $\widehat{G}$ defined by:

$$\widehat{\mu}(\sigma) = \int_G \sigma(x) d\mu$$

It is easy to verify the following properties:

$$(\widehat{a\mu_1 + b\mu_2})(\sigma) = a\widehat{\mu_1}(\sigma) + b\widehat{\mu_2}(\sigma)$$

$$(\widehat{\mu_1 * \mu_2})(\sigma) = \widehat{\mu_1}(\sigma)\widehat{\mu_2}(\sigma)$$

where $\mu_1, \mu_2 \in M(G)$, $a, b \in \mathbb{C}$ and $\sigma \in \widehat{G}$.

The *Fourier series* of $f \in L^1(G)$ is given by

$$\sum_{\sigma \in \widehat{G}} d_\sigma tr(\widehat{f}(\sigma)\sigma(x))$$

where tr denotes the trace of a matrix.

A function $f \in L^1(G)$, $\quad f \sim \sum_{\sigma \in \widehat{G}} tr(\widehat{f}(\sigma)\sigma(x))$, is said to have an *absolutely convergent Fourier series* if

$$\sum_{\sigma \in \widehat{G}} d_\sigma \|\widehat{f}(\sigma)\|_{op} < \infty$$

here $\|A\|_{op}$ denotes the operator norm of the matrix A (i.e. the maximum eigenvalue of $|A|$, where $|A|$ is the positive square root of AA^*).

The set of functions with absolutely convergent Fourier series is denoted by $\mathfrak{R}(G)$. If G is compact, it is known that the space $\mathfrak{R}(G)$ coincides with the complex linear space of functions on G spanned by all continuous definite functions on G ([52, Th. 34.13]).

Theorem 1.4.12. *[52, Cor. 34.6] Let f be a function in $\mathfrak{R}(G)$, $\quad f \sim \sum_{\sigma \in \widehat{G}} tr(\widehat{f}(\sigma)\sigma(x))$. Then f is equal almost everywhere to the continuous function $\sum_{\sigma \in \widehat{G}} tr(\widehat{f}(\sigma)\sigma(x))$ and so can be regarded as an element of the set of complex valued continuous functions on G.*

Let $\ell^\infty(\widehat{G})$, denote the Banach space of all $\{A_\sigma\}_{\sigma \in \widehat{G}}$, where A_σ is a $d_\sigma \times d_\sigma$ matrix, with norm $\|\{A_\sigma\}_{\sigma \in \widehat{G}}\|_\infty = \sup_{\sigma \in \widehat{G}} \|A_\sigma\|_{op} < \infty$.

We define $\ell^\infty(E)$ similarly for $E \subseteq \widehat{G}$ by restricting the representations to E.

Sidon sets are an interesting subject of study in the Harmonic Analysis. They are defined as follows:

Definition 1.4.13. *Let E be a subset of $\widehat{G}$. We call E a* ***Sidon set*** *if every continuous function f such that $\widehat{f}(\sigma) = 0$ for $\sigma \notin E$ is necessarily in $\mathfrak{R}(G)$.*

It is trivial that every finite set E is a Sidon set. Unlike the abelian case, there are infinite non-abelian compact groups whose dual contains no infinite Sidon sets.

Theorem 1.4.14. *(Huchinson [57, Th. 3.2.]) Let G be a compact Lie group. The following assertions are equivalent:*

(i) G is semi-simple;

(ii) G is tall;

(iii) G admits no infinite Sidon sets.

In chapter 5, we use the following characterisation as the definition of Sidon set in the dual of a compact group:

Theorem 1.4.15. *[52, Th.37.2] Let E be a subset of $\widehat{G}$. Then the following assertions are equivalent:*

(i) E is a Sidon set;

(ii) given $\{A_\sigma\}_{\sigma\in E} \in \ell^\infty(E)$, there is a measure $\mu \in M(G)$ satisfying $\widehat{\mu}(\sigma) = A_\sigma$ for all $\sigma \in E$.

Let G be a locally compact group. We denote the space of all continuous functions vanishing at infinity on G by $C_0(G)$.

The spaces A(G) and B(G) are the *Fourier* and *Fourier-Stieltjes algebras*, as introduced in Eymard [28]. B(G) is the algebra defined as the matrix coefficients of the unitary representations of G. When G is abelian B(G) reduces to the set of Fourier transforms of measures of the dual group [25]. A(G) is the Banach subalgebra of B(G) spawned by the positive-definite functions with compact support. Let $\mathrm{B}_\lambda(\mathrm{G})$ be the closed subalgebra of B(G) defined as the matrix coefficients of the unitary representations of G that are weakly contained in λ_G, the left regular representation of G.

The set B(G) is a $*$-closed subalgebra of the C^*-algebra $\ell_\infty(G)$, where $\ell_\infty(G)$ is the space of all bounded functions $f : G \to \mathbb{C}$, with norm $\|f\|_\infty = \sup_{g \in G} f(g)$ and involution given by $f^*(g) = \overline{f(g)}$.

Let $\mathcal{E}(G) \stackrel{\text{def}}{=} \overline{\mathrm{B(G)}}^{\|\cdot\|_\infty}$ denote the commutative C^*-algebra consisting of the uniform closure of the Fourier-Stieltjes algebra of G. $\mathcal{E}(G)$ is commonly called the *Eberlein algebra of* G.

Let $\Psi(\mathcal{E}(G))$ denote the space of multiplicative linear functionals on $\mathcal{E}(G)$ (i.e. linear functionals $T : \mathcal{E}(G) \to \mathbb{C}$ with $T(f_1 f_2) = T(f_1)T(f_2)$, for all $f_1, f_2 \in \mathcal{E}(G)$). The set $\Psi(\mathcal{E}(G))$ with the topology of pointwise convergence on $\mathcal{E}(G)$ is a compact topological space called the spectrum of $\mathcal{E}(G)$. Every element $f \in \mathcal{E}(G)$ can be identified with a function $eval_f \in C(\Psi(\mathcal{E}(G)), \mathbb{C})$ via evaluations ($eval_f(T) = T(f)$, for every $T \in \Psi(\mathcal{E}(G))$).

The compact space $\Psi(\mathcal{E}(G))$ defines a compactification of G. Indeed, bearing in mind that the elements of $\mathcal{E}(G)$ are continuous functions on G, we have an evaluation mapping $j : G \to \Psi(\mathcal{E}(G))$ (given by $j(g)(T) = T(g)$) that defines a one-to-one continuous mapping with dense range.

Following [68] we call the spectrum of $\mathcal{E}(G)$ the *Eberlein compactification* of G, and we denote it with the symbol eG. Unlike the Bohr compactification, eG is no longer a topological group, only a semitopological semigroup.

The following lacunary sets in a discrete group have been deeply studied in the literature [76, 31].

Definition 1.4.16. *Let G be an infinite discrete group.*

(i) A subset S of G is called ***strong Sidon*** *if for every complex-valued function $f \in C_0(G)$ there is a function $g \in \mathrm{A(G)}$ such that $f(x) = g(x)$ for all $x \in S$.*

(ii) A subset S of G is called ***Sidon*** *if for every bounded function f on G there exists $g \in \mathrm{B}_\lambda(\mathrm{G})$ such that $f(x) = g(x)$ for all $x \in S$.*

(iii) A subset S of G is called ***weak Sidon*** *if for every bounded function f on G there exists $g \in \mathrm{B(G)}$ such that $f(x) = g(x)$ for all $x \in S$, or equivalently, if f can be interpolated by a continuous function defined on the Eberlein compactification eG.*

If G is abelian or amenable (see [10] for a proper definition of amenability) the three types of Sidon sets are equivalent [76]. The following result gives a characterisation of amenability of a locally compact group G in terms of the weak containment property. Recall that the symbol 1_G denotes the unit representation of the topological group G.

Theorem 1.4.17. *(Hulanicki-Reiter's Theorem [10, Th. G.3.2.]) Let G be a locally compact group. The following properties are equivalent:*

(*i*) *G is amenable;*

(*ii*) $1_G \prec \lambda_G$;

(*iii*) $\sigma \prec \lambda_G$ *for every unitary representation* σ *of* G.

Chapter 2

Metric-valued sets of continuous functions

This chapter is divided into two parts. In the first part, we focus on the notion of almost equicontinuity. In the second part, we extend some known results concerning $\mathbb{R}$-valued Baire class 1 functions to M-valued Baire class 1 functions. Throughout this chapter, we make use of the extension tools that are provided in Subsection 1.2.3.

2.1 Almost Equicontinuity

2.1.1 Main definitions and basic results

Within the setting of dynamical systems, the following definitions appear in [2].

Definition 2.1.1. *Let X and (M, d) be a topological space and a metric space respectively, and let $E \subseteq C(X, M)$. According to [2], we say that a point $x \in X$ is an* ***equicontinuity point*** *of E when for every $\epsilon > 0$ there is a neighborhood U of x such that $diam(f(U)) < \epsilon$ for all $f \in E$. We say that E is* ***almost equicontinuous*** *when the subset of equicontinuity points of E is dense in X. Furthermore, it is said that E is* ***hereditarily almost equicontinuous*** *if $E|_A$ is almost equicontinuous for every nonempty closed subset A of X.*

The proof of the following lemma is known (see [39, Prop 6.6.]). This characterisation is very useful in order to obtain subsets of continuous functions that are not almost equicontinuous.

Lemma 2.1.2. *Let X and (M, d) be a topological space and a metric space respectively, and let $E \subseteq C(X, M)$. Consider the following two properties:*

(a) E is almost equicontinuous.

(b) For every nonempty open subset U of X and $\epsilon > 0$, there exists a nonempty open subset $V \subseteq U$ such that $diam(f(V)) < \epsilon$ for all $f \in E$.

Then (a) implies (b). If X is a Baire space, then (a) and (b) are equivalent. Furthermore, in this case, the subset of equicontinuity points of E is a dense G_δ-set in X.

Proof. That (a) implies (b) is obvious. Assume that X is a Baire space and (b) holds. Given $\epsilon > 0$ arbitrary, we consider the open set

$$O_\epsilon \stackrel{\text{def}}{=} \bigcup \{U \subseteq X : \quad U \neq \emptyset \quad \text{and} \quad diam(f(U)) < \epsilon, \quad \forall f \in E\}.$$

By (b), we have that O_ϵ is nonempty and dense in X. Since X is Baire, taking $W \stackrel{\text{def}}{=} \bigcap_{n<\omega} O_{\frac{1}{n}}$, we obtain a dense G_δ subset which is the subset of equicontinuity points of E. □

Remark 2.1.3. *As a consequence of assertion (b) in Lemma 2.1.2, it follows that, when X is a Baire space, a subset of functions E is hereditarily almost equicontinuous if, and only if, $E|_A$ is almost equicontinuous for every nonempty (non necessarily closed) subset A of X. Since we mostly work with Baire spaces here, we make use of this fact in some parts along the section.*

Note that the set of equicontinuity points of a subset of functions E is a G_δ-set. The next corollary is a straightforward consequence of Lemma 2.1.2.

Corollary 2.1.4. *Let X and (M, d) be a topological space and a metric space respectively, and let $E \subseteq C(X, M)$. Suppose there is an open basis $\mathcal{V}$ in X and $\epsilon > 0$ such that for every $V \in \mathcal{V}$, there is $f_V \in E$ with $diam(f_V(V)) \geq \epsilon$. Then E is not almost equicontinuous.*

Let 2^ω be the Cantor space and let $2^{(\omega)}$ denote the set of finite sequences of 0's and 1's. For a $t \in 2^{(\omega)}$, we designate by $|t|$ the length of t. For $\sigma \in 2^\omega$ and $n > 0$ we write $\sigma|n$ to denote $(\sigma(0), \ldots, \sigma(n-1)) \in 2^{(\omega)}$. If $n = 0$ then $\sigma|0 \stackrel{\text{def}}{=} \emptyset$.

Applying Corollary 2.1.4, it is easy to obtain subsets of continuous functions that are not almost equicontinuous.

Example 2.1.5. *Let $X = 2^\omega$ be the Cantor space and let $E = \{\pi_n\}_{n<\omega}$ be the set of all projections of X onto $\{0, 1\}$. Then E is not almost equicontinuous.*

Proof. Let $U \neq \emptyset$ be an open subset in X. Then, for some index $n < \omega$ we have $\pi_n(U) = \{0, 1\}$, which implies $diam(\pi_n(U)) > 1/2$. Therefore, E is not almost equicontinuous by Corollary 2.1.4. □

The precedent result can be generalised in order to obtain a more general example of non-almost equicontinuous set of functions. It turns out that this example is universal in a sense that becomes clear along the section.

Example 2.1.6. *Let $X = 2^\omega$ be the Cantor space and let (M, d) be a metric space. Let $\{U_t : t \in 2^{(\omega)}\}$ be the canonical open basis of X. If $E = \{f_t\}_{t\in 2^{(\omega)}}$ is a set of continuous functions on X into M satisfying that $diam(f_t(U_t)) \geq \epsilon$ for some fixed $\epsilon > 0$ and all $t \in 2^{(\omega)}$, then E is not almost equicontinuous.*

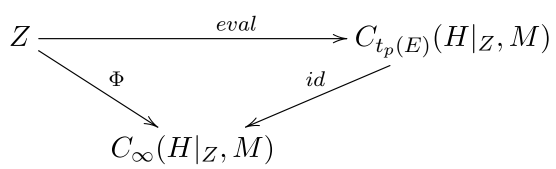
The next result gives a sufficient condition for the equicontinuity of a family of functions. It extends a well known result by Corson and Glicksberg [22]. However, we remark that the subset F found in the lemma below can become the empty set if Z is a first category subset of X.

Lemma 2.1.7. *Let X and (M, d) be a topological space and a separable metric space, respectively. If $E \subseteq C(X, M)$ and $(\overline{E}^{M^X})|_Z$ is metrizable and compact for some dense subset Z of X, then there is a residual subset F in Z such that E is equicontinuous at every point in F. In case Z is of second category in X, it follows that F is necessarily nonempty.*

Proof. Set $H \overset{\text{def}}{=} \overline{E}^{M^X}$ and consider the map $eval : X \to C(H, M)$, $x \mapsto eval_x$; defined by $eval_x(f) \overset{\text{def}}{=} f(x)$ for all $x \in X$ and $f \in H$.

For simplicity's sake, the symbols $C_{t_p(E)}(H|_Z, M)$ and $C_\infty(H|_Z, M)$ denotes the space $C(H|_Z, M)$ equipped with the pointwise convergence $t_p(E)$ and the uniform convergence topology, respectively.

Now set Φ such that the following diagram commutes



Remark that the evaluation map, $eval$, is continuous because $E \subseteq C(X, M)$. Since $H|_Z$ is $t_p(Z)$-compact and metrizable and Z is dense in X, it follows that $C_\infty(H|_Z, M)$ is separable and metrizable (see [26, Cor. 4.2.18]). Therefore, for every $n < \omega$, there is a sequence of closed balls $\{\overline{B}(u_i^{(n)}, 1/n) : i < \omega\}$ that covers $C_\infty(H|_Z, M)$. Furthermore, since E is dense in H, we have that each $\overline{B}(u_i^{(n)}, 1/n)$ is also closed in $C_{t_p(E)}(H|_Z, M)$. As a consequence

$K_{(i,n)} \overset{\text{def}}{=} \Phi^{-1}(\overline{B}(u_i^{(n)}, 1/n)) = eval^{-1}(\overline{B}(u_i^{(n)}, 1/n))$ is closed in Z for all $i, n < \omega$, because $eval$ is continuous.

We have that $Z \subseteq \bigcup_{i<\omega} K_{(i,n)}$ for every $n < \omega$, so $Z \subseteq \bigcap_{n<\omega} \bigcup_{i<\omega} K_{(i,n)}$. Observe that $\bigcup_{n<\omega} \bigcup_{i<\omega} (K_{(i,n)} \setminus int_Z(K_{(i,n)}))$ is a set of first category in Z. As a consequence

$$F \overset{\text{def}}{=} Z \setminus \bigcup_{n<\omega} \bigcup_{i<\omega} (K_{(i,n)} \setminus int_Z(K_{(i,n)}))$$

is a residual set in Z.

We now verify that E is equicontinuous at each point $z \in F$. Let $z \in F$ and $\epsilon > 0$ arbitrary. Take $n_0 < \omega$ such that $2/n_0 < \epsilon$. Since $z \in F \subseteq \bigcap_{n<\omega} \bigcup_{i<\omega} K_{(i,n)} \subseteq \bigcup_{i<\omega} K_{(i,n_0)}$ there is $i_0 < \omega$ such that $z \in K_{(i_0,n_0)}$. We claim that $z \in int_Z(K_{(i_0,n_0)})$. Indeed, if we assume that $z \notin int_Z(K_{(i_0,n_0)})$, then $z \in K_{(i_0,n_0)} \setminus int_Z(K_{(i_0,n_0)})$. Therefore, $z \in \bigcup_{n<\omega} \bigcup_{i<\omega} (K_{(i,n)} \setminus int_Z(K_{(i,n)}))$ and $z \notin F$, which is a contradiction.

Since $z \in int_Z(K_{(i_0,n_0)})$ there is a nonempty open set A in X such that $int_Z(K_{(i_0,n_0)}) = A \cap Z$. Note that $A \cap Z$ is dense on A because Z is dense in X. So, $z \in A = \overline{A \cap Z}^A \subseteq \overline{A \cap Z}^X$.

Let $a, b \in A \cap Z$. Then $\Phi(a) = eval_a, \Phi(b) = eval_b \in \overline{B}(u_{i_0}^{(n_0)}, 1/n_0)$. Consequently, $d(f(a), f(b)) \leq 2/n_0$ for every $f \in E$. So, given $x, y \in A \subseteq \overline{A \cap Z}^X$ we have that $d(f(x), f(y)) \leq 2/n_0$ for every $f \in E$. Then $diam(f(A)) \leq 2/n_0 < \epsilon$ for all $f \in E$. □

The following lemma reduces many questions related to a general metric space M to the interval $[-1, 1]$.

Lemma 2.1.8. *Let X and (M, d) be a topological and a metric space, respectively. If E is a subset of $C(X, M)$, then E is equicontinuous at a point $x_0 \in X$ if and only if $\nu(E \times \mathcal{K})$ is equicontinuous at it.*

Proof. Assume that E is equicontinuous at x_0. Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $d(f(x_0), f(x)) < \epsilon$ for all $x \in U$ and $f \in E$. Let $\alpha \in \mathcal{K}$, $x \in U$ and $f \in E$, then we have

$$|\nu(f,\alpha)(x_0) - \nu(f,\alpha)(x)| = |\alpha(f(x_0)) - \alpha(f(x))| \leq d(f(x_0), f(x)) < \epsilon.$$

Conversely, assume that $\nu(E \times \mathcal{K})$ is equicontinuous in x_0. Given $\epsilon > 0$, there is an open neighbouhood U of x_0 such that $|\nu(f,\alpha)(x_0) - \nu(f,\alpha)(x)| < \epsilon$ for all $x \in U$, $f \in E$ and $\alpha \in \mathcal{K}$.

For $f \in E$, consider the map $\alpha_{f(x_0)} \in \mathcal{K}$. In order to finish the proof, it suffices to observe that

$$|\alpha_{f(x_0)}(f(x_0)) - \alpha_{f(x_0)}(f(x))| = d(f(x), f(x_0))$$

for all $x \in U$ and $f \in E$. □

Corollary 2.1.9. *Let X and (M, d) be a topological and a metric space, respectively, and let E be an arbitrary subset of $C(X, M)$. Then E is (hereditarily) almost equicontinuous if and only if $\nu(E \times \mathcal{K})$ is (hereditarily) almost equicontinuous.*

The concept of fragmentability was introduced by Jayne and Rogers in 1985 [60]. This property is related with the notion of almost equicontinuity.

Definition 2.1.10. *A topological space X is said to be **fragmented by a pseudometric ρ** if for each nonempty subset A of X and for each $\epsilon > 0$ there exists a nonempty open subset U of X such that $U \cap A \neq \emptyset$ and ρ-diam$(U \cap A) \leq \epsilon$.*

There is a vast literature on this topic. It suffices to mention here the contribution by Namioka [73] and Ribarska [83].

Let X be a topological space, (M, d) a metric space and $E \subseteq M^X$ a family of functions. Whenever feasible, for example if $\overline{E}^{M^X}$ is compact, we consider the pseudometric $\rho_{E,d}$, defined as follows:

$$\rho_{E,d}(x, y) \stackrel{\text{def}}{=} \sup_{g \in E} d(g(x), g(y)), \quad \forall x, y \in X.$$

Therefore, taking into account Definition 2.1.1 and Lemma 2.1.2, we have the following proposition.

Proposition 2.1.11. *Let X and (M, d) be a topological space and a metric space, respectively, and let $E \subseteq C(X, M)$ such that $\overline{E}^{M^X}$ is compact. Consider the following two properties:*

(a) E is hereditarily almost equicontinuous.

(b) X is fragmented by $\rho_{E,d}$.

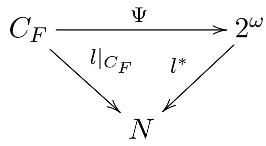
Then (a) implies (b). If X is a hereditarily Baire space, then (a) and (b) are equivalent.

2.1.2 Almost equicontinuity criteria

The following technical lemma is essential in most results along this section. The construction of the proof is based on an idea that appears in [88] and [15]. Throughout this section, the symbol $[A]^{\leq\omega}$ denotes the set of all countable subsets of A.

Lemma 2.1.12. *Let X and (M, d) be a Čech-complete space and a hemicompact metric space, respectively, and let E be an infinite subset of $C(X, M)$ such that $\overline{E}^{M^X}$ is compact. If E is not almost equicontinuous, then for every G_δ and dense subset F of X there exists a countably infinite subset L in E, a compact separable subset $C_F \subseteq F$, a compact subset $N \subseteq M$ and a continuous and surjective map Ψ of C_F onto the Cantor set 2^ω such that for every $l \in L$ there exists a continuous map $l^* : 2^\omega \to N$ satisfying that the following diagram is commutative*

Diagram 1:



Furthermore, the subset $L^* \stackrel{\text{def}}{=} \{l^* : l \in L\} \subseteq C(2^\omega, N)$ separates points in 2^ω and is not almost equicontinuous on 2^ω.

Proof. Let F be a G_δ and dense subset of X. Then there is a sequence $\{W_n\}_{n=1}^{\infty}$ of open dense subsets of X such that $W_s \subseteq W_r$ if $r < s$ and $F = \bigcap_{n=1}^{\infty} W_n$.

Since M is hemicompact, there is a sequence $\{M_n\}_{n<\omega}$ of compact subsets such that $M = \bigcup_{n<\omega} M_n$ and satisfying that for every compact subset $K \subseteq M$ there is $n < \omega$ such that $K \subseteq M_n$.

For each $n < \omega$ we consider the closed subset $X_n = \{x \in X : f(x) \in M_n \quad \forall f \in E\}$. We claim that $X = \bigcup_{n<\omega} X_n$. Indeed, let $x \in X$. Since $\overline{E}^{M^X} \subseteq M^X$ is compact and the xth projection π_x is continuous, then $\pi_x(\overline{E}^{M^X}) \subseteq M$ is compact. So, there is $n_x < \omega$ such that $\pi_x(\overline{E}^{M^X}) \subseteq M_{n_x}$ by hemicompactness. Therefore $x \in X_{n_x}$.

Since E is not almost equicontinuous there exists a nonempty open subset U of X and $\epsilon > 0$ such that for all nonempty open subset $V \subseteq U$ there exists a function $f_V \in E$ such that $diam(f_V(V)) \geq 2\epsilon > \epsilon$ by Lemma 2.1.2.

Note that U is Čech-complete. If we express $U = \bigcup_{n\in\omega} (U \cap X_n)$, by Baire's theorem, there is $n_0 < \omega$ such that $\tilde{U} \stackrel{\text{def}}{=} int_U(U \cap X_{n_0}) \neq \emptyset$ and open in X.

Set $C = \overline{\tilde{U}}^{X_{n_0}}$, which is closed in X, and $O_n = W_n \cap \tilde{U} = W_n \cap \tilde{U} \cap C$ that is open and dense in C for each $n < \omega$. Then $O_s \subseteq O_r$ if $r < s$ and $H = \bigcap_{n=1}^{\infty} O_n \subseteq F$ is a dense G_δ subset of C, which is a Baire space. Remark further that $f(x) \in M_{n_0}$ for all $x \in C$ and $f \in E$. Since M_{n_0} is compact, every function $g \in C(C, M_{n_0})$ can be extended to a continuous function $g^\beta \in C(\beta C, M_{n_0})$. Set $E^\beta = \{f^\beta : f \in E\} \subseteq C(\beta C, M_{n_0})$.

The space C, being Čech-complete, is a dense G_δ subset of its Stone-Čech compactification βC. Therefore, since H is a G_δ subset of C, it follows that H also is a dense G_δ subset of βC. Consider a sequence $\{B_n\}_{n=1}^\infty$ of open dense subsets of βC such that $B_s \subseteq B_r$ if $r < s$ and $H = \bigcap_{n=1}^\infty B_n$. We have that $H = \bigcap_{n=1}^\infty (B_n \cap O_n^\beta)$, where $O_n^\beta = \beta C \setminus \overline{(C \setminus O_n)}^{\beta C}$ is open in βC and $O_n^\beta \cap C = O_n$.

By induction on $n = |t|$ with $t \in 2^{(\omega)}$, we construct a family $\{U_t : t \in 2^{(\omega)}\}$ of nonempty open subsets of βC and a family of countable functions $L \stackrel{\text{def}}{=} \{f_t : t \in 2^{(\omega)}\} \subseteq E$, satisfying the following conditions for all $t \in 2^{(\omega)}$:

(*i*) $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta C} \subseteq O_0^\beta \stackrel{\text{def}}{=} \beta C \setminus \overline{(C \setminus \tilde{U})}^{\beta C}$ (remark that $O_0^\beta \cap C = \tilde{U}$);

(*ii*) $U_{ti} \subseteq \overline{U_{ti}}^{\beta C} \subseteq B_{|t|} \cap O_{|t|}^\beta \cap U_t$ for $i = 0, 1$ (where $B_0 \stackrel{\text{def}}{=} \beta C$);

(*iii*) $U_{t0} \cap U_{t1} = \emptyset$;

(*iv*) $d(f_t(x), f_t(y)) > \epsilon$, $\forall x \in U_{t0} \cap C$ and $\forall y \in U_{t1} \cap C$;

(*v*) whenever $s, t \in 2^{(\omega)}$ and $|s| < |t|$, $diam(f_s(U_{tj} \cap C)) < \frac{1}{|t|}$ for $j = 0, 1$.

Indeed, if $n = 0$, by regularity we can find $U_\emptyset$ an open set in βC such that $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta C} \subseteq B_0 \cap O_0^\beta$. For $n \geq 0$, suppose $\{U_t : |t| \leq n\}$ and $\{f_t : |t| < n\}$ have been constructed satisfying $(i) - (v)$. Fix a $t \in 2^{(\omega)}$ with $|t| = n$. Since U_t is open in βC, there is an open set A_t in X such that $U_t \cap C = A_t \cap C$. Therefore

$$U_t \cap C = (A_t \cap C) \cap O_0^\beta = A_t \cap (O_0^\beta \cap C) = A_t \cap \tilde{U}$$

is open in X and included in U.

By assumption there exist $f_t \in E$ such that $diam(f_t(U_t \cap C)) > \epsilon$. Consequently, we can find $x_t, y_t \in V_t \cap C$ such that $d(f_t(x_t), f_t(y_t)) > \epsilon$. By continuity, we can select two open disjoint neighbourhoods in βC, S_{t0} and S_{t1} of x_t and y_t, respectively, satisfying conditions (*iii*) and (*iv*).

If $i \in \{0, 1\}$, observe that $U_t \cap S_{ti} \cap O_0^\beta$ is open in βC and nonempty. Since $B_{|t|} \cap O_{|t|}^\beta$ is dense in βC then $U_t \cap S_{ti} \cap B_{|t|} \cap O_{|t|}^\beta$ is a nonempty open subset of βC. By regularity there exists a nonempty open subset U_{ti} of βC such that $U_{ti} \subseteq \overline{U_{ti}}^{\beta C} \subseteq U_t \bigcap S_{ti} \bigcap B_{|t|} \cap O_{|t|}^\beta$. Therefore, U_{t0} and U_{t1} satisfies conditions $(ii), (iii)$ and (iv) and, by continuity, we can adjust the open sets to satisfy (v).

Set $K \stackrel{\text{def}}{=} \bigcap_{n=0}^\infty \bigcup_{|t|=n} \overline{U_t}^{\beta C}$, which is closed in βC and, as a consequence, also compact. Remark that we can express $K = \bigcup_{\sigma \in 2^\omega} \bigcap_{n=0}^\infty \overline{U_{\sigma|n}}^{\beta C}$. Therefore, for each $\sigma \in 2^\omega$, we have $\bigcap_{n=0}^{\infty} \overline{U_{\sigma|n}}^{\beta C} \neq \emptyset$ by the compactness of βC, which implies $K \neq \emptyset$. Furthermore, since $K \subseteq \bigcap_{n=0}^{\infty} (B_n \cap O_n^\beta) = H \subseteq F$, it follows that K is contained in F.

Let $\Psi : K \to 2^\omega$ be the canonical map defined such that $\Psi^{-1}(\sigma) = \bigcap_{n=0}^{\infty} \overline{U_{\sigma|n}}^{\beta C}$ for all $\sigma \in 2^\omega$. Clearly Ψ is onto and continuous. Observe that for each $t \in 2^{(\omega)}$ and $\sigma \in 2^\omega$, $f_t(\Psi^{-1}(\sigma))$ is a singleton by (iv). Therefore, f_t lifts to a continuous function f_t^* on 2^ω such that $f_t(x) = f_t^*(\Psi(x))$ for all $x \in K$.

Take a countable subset D of K such that $\Psi(D) = 2^{(\omega)}$ and makes $\Psi|_D$ injective. Set $C_F \stackrel{\text{def}}{=} \overline{D}^K$. Note that $2^{(\omega)}$ is a countable dense subset of 2^ω.

We have that $\Psi_{|C_F} : C_F \to 2^\omega$ is an onto and continuous map. We consider the set $L^* \subseteq C(2^\omega, M_{n_0})$ defined by $L^* = \{l^* : l \in L|_{C_F}\}$ that makes the diagram 1 commutative. We claim that L^* separates points in 2^ω and, as a consequence, defines its topology. Indeed, let $\sigma, \sigma' \in 2^\omega$ be two arbitrary points such that $\sigma \neq \sigma'$. Since Ψ is an onto map there exist $x, y \in C_F$ such that $\sigma = \Psi(x)$ and $\sigma' = \Psi(y)$. Therefore, $x \in \bigcap_{n=0}^{\infty} \overline{U_{\sigma|n}}^{\beta C}$ and $y \in \bigcap_{n=0}^{\infty} \overline{U_{\sigma'|n}}^{\beta C}$. Since $\sigma \neq \sigma'$, there is $n_0 \in \omega$ such that $\sigma|n_0 = \sigma'|n_0$ and $\sigma(n_0) \neq \sigma'(n_0)$. Taking $t = \sigma|n_0$, then by (iv) we know that $d(f_t(x), f_t(y)) > \epsilon$. So, $f_t^*(\sigma) \neq f_t^*(\sigma')$.

On the other hand, by the commutativity of Diagram 1, and taking into account how L and L^* have been defined, it is easily seen that L^* is not almost equicontinuous on 2^ω using Example 2.1.6. □

Applying Corollary D of [15] by Cascales, Namioka and Vera, Lemma 1.2.24 and Propositions 1.2.25, 1.2.26 and 1.2.27, the next result follows easily.

Proposition 2.1.13. *Let X be a compact space, (M, d) be a compact metric space and let E be an infinite subset of $C(X, M)$. If $(X, t_p(\overline{E}^{M^X}))$ is Lindelöf, then E is hereditarily almost equicontinuous.*

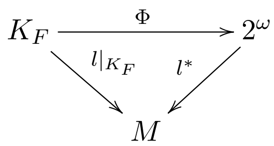
Using Lemma 2.1.12, the constraints in Proposition 2.1.13 can be relaxed as the following result shows.

Proposition 2.1.14. *Let X be a Čech-complete space, (M, d) be a compact metric space and let E be an infinite subset of $C(X, M)$. If there exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{E}^{M^X}))$ is Lindelöf, then E is almost equicontinuous.*

Proof. Reasoning by contradiction, suppose that E is not almost equicontinuous. By Lemma 2.1.12 there exists a compact separable subset C_F of F, a continuous onto map $\Psi : C_F \to 2^\omega$, and a countable subset L of E such that the subset $L^* \subseteq C(2^\omega, M)$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in C_F$ separates points in 2^ω and is not almost equicontinuous.

Let K_F be the closure of C_F in F with respect to the initial topology generated by the maps in L. Using a compactness argument, it follows that if $p \in K_F$ then there is $x_p \in C_F$ such that $l(p) = l(x_p)$ for all $l \in L$. Indeed, let $p \in K_F$. Then there is a net $\{x_\delta\}_{\delta\in\Delta} \subseteq C_F$ that $t_p(L)$-converges to p. Since C_F is compact there is a subnet $\{x_\gamma\}_{\gamma\in\Gamma}$ such that converges to $x_0 \in C_F$. Given $l \in L$, we know that $\lim\limits_{\gamma\in\Gamma} l(x_\gamma) = l(x_0)$ because l is continuous. Therefore, $l(x_0) = \lim\limits_{\gamma\in\Gamma} l(x_\gamma) = l(p)$. Consequently, we can extend Ψ to a map $\Phi : K_F \to 2^\omega$ by $\Phi(p) = \Psi(x_p)$ for all $p \in K_F$.

Let us see that Φ is well-defined. Let $p \in K_F$, suppose that there are $x_p, \tilde{x}_p \in C_F$ such that $x_p \neq \tilde{x}_p$ and $l(p) = l(x_p) = l(\tilde{x}_p)$ for all $l \in L$. Since the Diagram 1 commutes, we know that $l^*(\Psi(x_p)) = l^*(\Psi(\tilde{x}_p))$ for all $l^* \in L^*$. So, $\Psi(x_p) = \Psi(\tilde{x}_p)$ because L^* separates points in 2^ω.
Observe that the following diagram is commutative
Diagram 2:



Certainly, let $p \in K_F$, then there is $x_p \in C_F$ such that $\Phi(p) = \Psi(x_p)$. Given $l \in L$, we have that $l(p) = l(x_p) = l^*(\Psi(x_p)) = l^*(\Phi(p))$.
We claim that $\Phi : (K_F, t_p(L)) \to (2^\omega, t_p(L^*))$ is also continuous. Indeed, let $\{h_\delta\}_{\delta\in\Delta} \subseteq K_F$ a net that $t_p(L)$-converges to $h_0 \in K_F$. For each $\delta \in \Delta$ there is $x_\delta \in C_F$ such that $\Phi(h_\delta) = \Psi(x_\delta)$ and $l(h_\delta) = l(x_\delta)$ for all $l \in L$. Analogously, there is $x_0 \in C_F$ such that $\Phi(h_0) = \Psi(x_0)$ and $l(h_0) = l(x_0)$ for all $l \in L$.
Since C_F is compact there is a subnet $\{x_\gamma\}_{\gamma\in\Gamma}$ such that converges to $\tilde{x} \in C_F$. Given $l \in L$, we know that $\lim\limits_{\gamma\in\Gamma} l(x_\gamma) = l(\tilde{x})$ because l is continuous. On the other hand, we also have that $\lim\limits_{\gamma\in\Gamma} l(x_\gamma) = \lim\limits_{\gamma\in\Gamma} l(h_\gamma) = l(h_0) = l(x_0)$. Therefore, $l(\tilde{x}) = l(x_0)$ for all $l \in L$. So, $\Psi(\tilde{x}) = \Psi(x_0)$ because L^* separates points in 2^ω. The continuity follows because $\lim\limits_{\gamma\in\Gamma} \Phi(h_\gamma) = \lim\limits_{\gamma\in\Gamma} \Psi(x_\gamma) = \Psi(\tilde{x}) = \Psi(x_0) = \Phi(h_0)$.

Now, since K_F is $t_p(L)$-closed in F, it follows that it is also $t_p(\overline{E}^{M^X})$-closed in F.

By our initial assumption, we have that F is $t_p(\overline{E}^{M^X})$-Lindelöf, which implies that also K_F is $t_p(\overline{E}^{M^X})$-Lindelöf.

We claim that $(2^\omega, t_p(\overline{L^*}^{M^{2^\omega}}))$ is also Lindelöf. Indeed, it is enough to prove that Φ is continuous on K_F when it is equipped with the $t_p(\overline{E}^{M^X})$-topology and 2^ω is equipped with the $t_p(\overline{L^*}^{M^{2^\omega}})$-topology.

Take a map $k \in \overline{L^*}^{M^{2^\omega}}$ and let $\{l^*_\gamma\}_{\gamma\in\Gamma} \subseteq L^*$ be a net converging to k pointwise on 2^ω. Since $\overline{E}^{M^X}$ is compact, we may assume WLOG that $\{l_\gamma\}_{\gamma\in\Gamma} \subseteq L$ $t_p(X)$-converges to $h \in \overline{E}^{M^X}$. Therefore, for each $x \in K_F$ we have that $k(\Phi(x)) = \lim_{\gamma\in\Gamma} l^*_\gamma(\Phi(x)) = \lim_{\gamma\in\Gamma} l_\gamma(x) = h(x)$. That is $k \circ \Phi = h$. Since h is continuous on K_F, the continuity of Φ follows.

By Proposition 2.1.13, this implies that L^* is a hereditarily almost equicontinuous family on 2^ω, which is a contradiction. □

Proposition 2.1.15. *Let X be a Čech-complete space, (M, d) be a metric space and let E be an infinite subset of $C(X, M)$ such that $\overline{E}^{M^X}$ is compact. If there exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{E}^{M^X}))$ is Lindelöf, then E is almost equicontinuous.*

Proof. Let $\mathcal{K}$ and ν defined as in Definition 1.2.30. Since $\nu(\overline{E}^{M^X} \times \mathcal{K})$ is a compact subset of $[-1, 1]^X$, it follows that $\overline{\nu(E \times \mathcal{K})}^{[-1,1]^X} = \nu(\overline{E}^{M^X} \times \mathcal{K})$.

By Lemma 1.2.32 we know that $(F, t_p(\nu(\overline{E}^{M^X} \times \mathcal{K})))$ is Lindelöf . Now, applying Proposition 2.1.14 to the subset $\nu(E\times\mathcal{K}) \subseteq C(X, [-1, 1])$, it follows that $\nu(E \times \mathcal{K})$ is almost equicontinuous. Therefore, E is almost equicontinuous by Corollary 2.1.9. □

The following lemma is known. We refer to [30, Cor. 3.5] for its proof.

Lemma 2.1.16. *Let X be a Lindelöf space, (M, d) be a metric space. If E is an equicontinuous subset of $C(X, M)$, then $\overline{E}^{M^X}$ is metrizable.*

We are now in position of proving one of the two main theorems of this section.

Theorem 2.1.17. *Let X and (M, d) be a Čech-complete space and a separable metric space, respectively, and let E be an infinite subset of $C(X, M)$ such that $\overline{E}^{M^X}$ is compact. Consider the following three properties:*

(a) E is almost equicontinuous.

(b) There exists a dense Baire subset $F \subseteq X$ such that $(\overline{E}^{M^X})|_F$ is metrizable.

(c) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{E}^{M^X}))$ is Lindelöf.

Then (b) $\Rightarrow$ (c) $\Rightarrow$ (a). If X is also a hereditarily Lindelöf space, then all conditions are equivalent.

Proof. $(b) \Rightarrow (c)$ Since $(\overline{E}^{M^X})|_F$ is compact metric, it follows by Lemma 2.1.7 that there is a dense subset D such that E is equicontinuous at the points in D with respect to X. Since D is dense in F, which is dense in X, it follows that D is also be dense in X. Moreover, if Y denotes the G_δ subset of equicontinuity points of E in X, since $D \subseteq Y$, it follows that Y, the set of equicontinuity points of E is a dense G_δ-set in X. Set $K \stackrel{\text{def}}{=} (\overline{E}^{M^X})$. The equicontinuity of E at the points in Y combined with the density of $D \subseteq F$ in Y, implies that the map $\Theta : K|_F \longrightarrow K|_Y$ defined by $\Theta(f|_F) \stackrel{\text{def}}{=} f|_Y$ is a homeomorphism of $K|_F$ onto $K|_Y$.

By our initial assumption we have that $K|_F$ is compact and metrizable, which yields the metrizability of $K|_Y$. Thus, the evaluation map $Eval : Y \longrightarrow C_\infty(K|_Y, M)$ is a well defined and continuous map. We know that $C_\infty(K|_Y, M)$ is a separable space by [26, Cor. 4.2.18]. Therefore $(Eval(Y), t_\infty(K|_Y))$ and $(Y, t_\infty(K|_Y))$ are Lindelöf spaces. As a consequence $(Y, t_p(K|_Y))$ must be also Lindelöf and we are done.

$(c) \Rightarrow (a)$ This implication is Proposition 2.1.15

$(a) \Rightarrow (b)$ Suppose that X is Čech-complete and hereditarily Lindelöf. By Lemma 2.1.2, the subset, F, of equicontinuity points of E is a dense G_δ-set in X, which is a Lindelöf space by our initial assumption. Since E is equicontinuous on F, Lemma 2.1.16 implies that $(\overline{E}^{M^X})|_F$ must be metrizable. □

The following result can be found in [41, Prop. 2.5 and Section 5] in the setting of compact metric spaces. Notwithstanding this, the proof given there can be adapted easily for Čech-complete and hereditarily Lindelöf spaces, as it is formulated in the next proposition.

Proposition 2.1.18. *Let X be a hereditarily Lindelöf space, (M, d) is a metric space and $E \subseteq C(X, M)$. If $H \stackrel{\text{def}}{=} \overline{E}^{M^X}$ is compact and hereditarily almost equicontinuous, then H is metrizable.*

Proof. The symbol $C_\infty(H, M)$ denote the space $C(H, M)$ equipped with the uniform convergence topology. Consider the map $eval : X \to C_\infty(H, M)$ defined by $eval(x)[h] \stackrel{\text{def}}{=} h(x)$ for all $x \in X$ and $h \in H$.

By Proposition 2.1.11 X is fragmented by $\rho_{E,d}$. Thus, for each nonempty subset A of X and for each $\epsilon > 0$ there exists a nonempty open subset U

of X such that $U \cap A \neq \emptyset$ and $diam(h(U \cap A)) \leq \epsilon$ for all $h \in H$. Thus, d_∞-$diam(eval(U \cap A)) \leq \epsilon$.

We claim that $eval(X)$ is separable. Indeed, pick $\epsilon > 0$. Let $\mathcal{A}$ be the collection of all open subsets O of X such that $eval(O)$ can be covered by countably many sets of diameter less than ϵ. Since X is hereditarily Lindelöf there is a countable subfamily $\mathcal{B}$ of $\mathcal{A}$ such that $\bigcup_{A \in \mathcal{A}} A = \bigcup_{B \in \mathcal{B}} B$. Take $V \overset{\text{def}}{=} \bigcup_{A \in \mathcal{A}} A$. Observe that V is the largest element of $\mathcal{A}$. Let us see that $A \overset{\text{def}}{=} X \setminus V$ is empty. Assume that $A \neq \emptyset$. Then there is a nonempty set U of X such that $U \cap A \neq \emptyset$ and d_∞-$diam(eval(U \cap A)) \leq \epsilon$. Since $eval(U \cup V) = eval(U \cap A) \cup eval(V)$ we know that $eval(U \cup V)$ can be covered by countably many sets of diameter less than ϵ. So, $U \cup V \in \mathcal{A}$ and we arrive to a contradiction because $U \cap (X \setminus V) \neq \emptyset$. Since $X = V \in \mathcal{A}$ and ϵ was arbitrary $eval(X)$ is separable.

There is a dense and countable subset D of $eval(X)$. We know that D separates points of H because $eval(X)$ also separates points. Let $\Delta D : H \to M^D$ be the diagonal product. Since ΔD is an embedding and M^D is metrizable we conclude that H is metrizable. □

The next result is due basically to Namioka [73, Lemma 2.1]. It can also be found in [38, Lemma 6.4.], where the reference to Namioka is acknowledged.

Lemma 2.1.19. *Let X, Y and (M, d) be two arbitrary compact spaces and a metric space, respectively, and let E be a subset of $C(Y, M)$. Suppose that $p : X \longrightarrow Y$ is a continuous onto map. Then $E \circ p \overset{\text{def}}{=} \{g \circ p : g \in E\} \subseteq C(X, M)$ is hereditarily almost equicontinuous if and only if E is also hereditarily almost equicontinuous.*

Proof. In order to prove this result, we apply Lemma 2.1.2. Assume that $E \circ p$ is hereditarily almost equicontinuous. Let A be a closed (and compact) subset of Y, U be a nonempty relatively open set in A and $\epsilon > 0$. By Zorn's Lemma, there exists a minimal compact subset Z of X such that $p(Z) = A$. Since $\tilde{U} \overset{\text{def}}{=} p^{-1}(U) \cap Z$ is a nonempty relatively open set in Z and $(E \circ p)|_Z$ is almost equicontinuous there is a nonempty relatively open set $\tilde{V} \subseteq \tilde{U}$ in Z such that $diam((f \circ p)(\tilde{V})) < \epsilon$ for all $f \in E$. Let $V \overset{\text{def}}{=} A \setminus p(Z \setminus \tilde{V})$, which is relatively open set in A. We claim that $V \neq \emptyset$. Indeed, assume that $V = \emptyset$. Then $A = p(Z \setminus \tilde{V})$ and this contradicts the minimality of Z. Since $V \subseteq p(\tilde{V})$ we have that $diam(f(V)) < \epsilon$ for all $f \in E$.

Conversely, let Z be a closed subset of X, $\tilde{U}$ be a nonempty relatively open set in Z and $\epsilon > 0$. Consider the closed subset $W_0 \overset{\text{def}}{=} \overline{p(\tilde{U})}$ of Y. Since $E|_{W_0}$ is almost equicontinuous there is a nonempty relatively open set

V_0 in Y such that $V_0 \cap W_0 \neq \emptyset$ and $diam(f(V_0 \cap W_0)) < \epsilon$ for all $f \in E$. Take $\tilde{V} \overset{\text{def}}{=} p^{-1}(V_0) \cap \tilde{U}$. Since $\tilde{V}$ is a nonempty relatively open set in Z and $p(\tilde{V}) \subseteq V_0 \cap W_0$ we conclude that $diam(f(p(\tilde{V}))) < \epsilon$ for all $f \in E$. □

Remark 2.1.20. *If the map p of the previous lemma is open or quasi-open we obtain the same result for almost equicontinuity. Recall that a map $f : X \to Y$ between two topological spaces is quasi-open if for any nonempty open set $U \subseteq X$ the interior of $f(U)$ in Y is nonempty.*

Proof. Let U be a nonempty open set of Y and $\epsilon > 0$. Since $E \circ p$ is almost equicontinuous and $\tilde{U} = p^{-1}(U)$ is an open subset of X there is a nonempty open subset $\tilde{V} \subseteq \tilde{U}$ of X such that $diam((f \circ p)(\tilde{V})) < \epsilon$ for all $f \in E$. Since the nonempty open set $V \overset{\text{def}}{=} int(p(\tilde{V}))$ is included in $p(\tilde{V})$ we have that $diam(f(V)) < \epsilon$ for all $f \in E$.

Conversely, let $\tilde{U}$ be a nonempty open set of X and $\epsilon > 0$. Take $U \overset{\text{def}}{=} int(p(\tilde{U})) \neq \emptyset$. Since E is almost equicontinuous there is a nonempty open subset $V \subseteq U$ of Y such that $diam(f(V)) < \epsilon$ for all $f \in E$. So, taking the open subset $\tilde{V} \overset{\text{def}}{=} p^{-1}(V) \cap \tilde{U}$, we conclude that $diam((f \circ p)(\tilde{V})) < \epsilon$ for all $f \in E$. □

Proposition 2.1.21. *Let X be a Čech-complete space, (M, d) be a hemicompact metric space and E be an infinite subset of $C(X, M)$ such that $\overline{E}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) E is hereditarily almost equicontinuous.

(b) L is hereditarily almost equicontinuous on F, for all $L \in [E]^{\leq \omega}$ and F a separable and compact subset of X.

Proof. (a) implies (b) is trivial. To see the other implication, assume, reasoning by contradiction, that (a) does not hold. Then there must be some closed subset $A \subseteq X$ such that $E|_A$ is not almost equicontinuous. By Lemma 2.1.12 there exists a compact and separable subset F of X, an onto and continuous map $\Psi : F \to 2^\omega$, and a countable subset L of E such that the subset $L^* \subseteq C(2^\omega, M)$ defined by $l^*(\Psi(x)) = l(x)$ for all $x \in F$ is not almost equicontinuous. Therefore, L is not hereditarily almost equicontinuous on F by Lemma 2.1.19 and we arrive to a contradiction. □

The second main theorem of this section characterises the hereditarily almost equicontinuous families of functions defined on a Čech-complete space (this question has been studied in detail in [90] for compact spaces). It is a sort of generalisation of the result given by Troallic in [93, Corollary 3.2] due to the fact that we can reduce the verification of hereditarily almost equicontinuity to countable subsets. The equivalence $(a) \Leftrightarrow (b)$ bellow is a direct consequence of Troallic's result (op. cit.).

Theorem 2.1.22. *Let X and (M,d) be a Čech-complete space and a metric space, respectively, and let E be an infinite subset of $C(X,M)$ such that $\overline{E}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) E is hereditarily almost equicontinuous.

(b) L is hereditarily almost equicontinuous on F, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of X.

(c) $(\overline{L}^{M^X})|_F$ is metrizable, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of X.

(d) $(F, t_p(\overline{L}^{M^X}))$ is Lindelöf, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of X.

Proof. $(b) \Rightarrow (a)$ is a direct consequence of Proposition 2.1.21 and Corollary 2.1.9.

$(a) \Rightarrow (c)$ Let $L \in [E]^{\leq\omega}$ and let F be a separable and compact subset of X. L defines an equivalence relation on F by $x \sim y$ if and only if $l(x) = l(y)$ for all $l \in L$. If $\widetilde{F} = F/\sim$ is the compact quotient space and $p : F \to \widetilde{F}$ denotes the canonical quotient map, each $l \in L$ has associated a map $\tilde{l} \in C(\widetilde{F}, M)$ defined as $\tilde{l}(\tilde{x}) \stackrel{\text{def}}{=} l(x)$ for any $x \in F$ with $p(x) = \tilde{x}$. Furthermore, if $\tilde{L} \stackrel{\text{def}}{=} \{\tilde{l} : l \in L\}$, we can extend this definition to the closure of $\tilde{L}$ in $M^{\widetilde{F}}$. Thus, each $l \in \overline{L}^{M^F}$ has associated a map $\tilde{l} \in \overline{\tilde{L}}^{M^{\tilde{F}}}$ such that $\tilde{l} \circ p = l$. By construction, we have that $\tilde{L}$ separates the points in $\widetilde{F}$. Since $\tilde{L}$ is countable it follows that $(\widetilde{F}, t_p(\tilde{L}))$ is a compact metric space. On the other hand, E is hereditarily almost equicontinuous on X. Applying Lemma 2.1.19 to F and $\widetilde{F}$, it follows that $\widetilde{L}$ is hereditarily almost equicontinuous on $\widetilde{F}$. Therefore, the space $\overline{\tilde{L}}^{M^{\tilde{F}}}$ is metrizable by Proposition 2.1.18. In order to finish the proof, it suffices to remark that $\overline{L}^{M^F}$ is canonically homeomorphic to $\overline{\tilde{L}}^{M^{\tilde{F}}}$ (see Proposition 1.2.14).

$(c) \Rightarrow (d)$ Let $L \in [E]^{\leq\omega}$ and let F be a separable and compact subset of X. We know that $H \stackrel{\text{def}}{=} ((\overline{L}^{M^X})|_F, t_p(F))$ is compact metric. Since F is separable, we have that $l(F)$ is a separable for every $l \in L$. Hence $N \stackrel{\text{def}}{=} \overline{\bigcup_{l \in L} l(F)}^M$ is a separable subset of M. Now, remark that M can be replaced by N without loss of generality. On the other hand, since $F \subseteq C(H, M)$ and H is compact metric, it follows that $(F, t_\infty(H))$ is separable and metrizable by [26, Cor. 4.2.18], which implies that $(F, t_\infty(H))$ is Lindelöf. Since the the topology $t_p(H)$ is weaker than $t_\infty(H)$, we deduce that $(F, t_p(H))$ must be Lindelöf.

$(d) \Rightarrow (b)$ By Lemma 1.2.32, for all $L \in [E]^{\leq\omega}$ and F a separable compact subset of X, we have that $(F, t_p(\nu(\overline{L}^{M^X} \times \mathcal{K})))$ is Lindelöf . Applying [15, Corollary D], it follows that $\nu(\overline{L}^{M^X} \times \mathcal{K})$ is hereditarily almost equicontinuous for all $L \in [E]^{\leq\omega}$ and F a separable compact subset of X. Thus, Corollary 2.1.9 yields (b). □

Remark 2.1.23. *If E is an infinite subset of $C(X, M)$ such that $K \stackrel{\text{def}}{=} \overline{E}^{M^X}$ is contained in $C(X, M)$, then the implication $(c) \Rightarrow (a)$ in Theorem 2.1.22 provides a different proof of the celebrated Namioka Theorem (see Theorem 1.2.7). Indeed, given any $L \in [E]^{\leq\omega}$ and any separable compact subset F of X, since $K \subseteq C(X, M)$ and F is separable, it follows that $((\overline{L}^{M^X})|_F, t_p(F))$ is metrizable. Thus E (and therefore K) is hereditarily almost equicontinuous.*

Corollary 2.1.24. *With the same hypothesis of Theorem 2.1.22, consider the following three properties:*

(a) E is hereditarily almost equicontinuous.

(b) E is hereditarily almost equicontinuous on F, for all F a separable and compact subset of X.

(c) $(F, t_p(\overline{E}^{M^X}))$ is Lindelöf, for all F a separable and compact subset of X.

Then $(a) \Leftrightarrow (b) \Leftarrow (c)$.

2.1.3 Applications

The results formulated in the previous subsection have consequences in different settings. First, we consider an application of Theorem 2.1.22 to fragmentability.

Corollary 2.1.25. *Let X and (M, d) be a Čech-complete space and a metric space, respectively, and let E be an infinite subset of $C(X, M)$ such that $\overline{E}^{M^X}$ is compact. Then the following conditions are equivalent:*

(a) X is fragmented by $\rho_{E,d}$.

(b) F is fragmented by $\rho_{L,d}$, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of X.

(c) $((\overline{L}^{M^X})|_F, t_p(F))$ *is metrizable, for all* $L \in [E]^{\leq\omega}$ *and* F *a separable and compact subset of* X.

(d) $(F, t_p(\overline{L}^{M^X}))$ *is Lindelöf, for all* $L \in [E]^{\leq\omega}$ *and* F *a separable and compact subset of* X.

It is easy to check that, in the context of topological groups, the notion of almost equicontinuity is equivalent to equicontinuity. This fact allows us to characterise equicontinuous subsets of group homomorphisms using Theorem 2.1.17.

Recall that a topological group G is said to be *ω-narrow* if for every neighborhood V of the neutral element, there exists a countable subset H of E such that $G = HV$.

Corollary 2.1.26. *Let* G *and* (M, d) *be a Čech-complete topological group and a metric separable group, respectively, and let* E *be an infinite subset of* $CHom(G, M)$ *such that* $\overline{E}^{M^G}$ *is compact. Consider the following three properties:*

(a) E *is equicontinuous.*

(b) E *is relatively compact in* $CHom(G, M)$ *with respect to the compact open topology.*

(c) There exists a dense Baire subset $F \subseteq G$ *such that* $(\overline{E}^{M^G})|_F$ *is metrizable.*

(d) There exists a dense G_δ *subset* $F \subseteq G$ *such that* $(F, t_p(\overline{E}^{M^G}))$ *is Lindelöf.*

Then $(c) \Rightarrow (d) \Rightarrow (a) \Leftrightarrow (b)$. *If* G *is also ω-narrow, then all conditions are equivalent. Furthermore* (c) *and* (d) *are also true for* $F = G$.

Proof. The equivalence $(a) \Leftrightarrow (b)$ follows from Ascoli Theorem (see Theorem 1.2.6). So, after Theorem 2.1.17, it suffices to show the implication $(a) \Rightarrow (c)$ for an ω-narrow G. Now, assuming that E is equicontinuous, it follows that $K \stackrel{\text{def}}{=} \overline{E}^{M^G} \subseteq CHom(G, M)$. Thus K is an equicontinuous compact subset of continuous group homomorphisms. As a consequence, it is known that K is metrizable. (see [30, Cor. 3.5]). □

Applying Theorem 2.1.22 to the setting of topological groups, we obtain the next result.

Corollary 2.1.27. *Let* G *and* (M, d) *be a Čech-complete topological group and a metric group, respectively, and let* E *be an infinite subset of* $CHom(G, M)$ *such that* $\overline{E}^{M^G}$ *is compact. Then the following conditions are equivalent:*

(a) *E is equicontinuous.*

(b) *L is equicontinuous on F, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of G.*

(c) *$((\overline{L}^{M^G})|_F, t_p(F))$ is metrizable, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of G.*

(d) *$(F, t_p(\overline{L}^{M^G}))$ is Lindelöf, for all $L \in [E]^{\leq\omega}$ and F a separable and compact subset of G.*

For a function $f : X \times Y \to M$, let $f_x : Y \to M$ (resp. $f^y : X \to M$) be $f(x, \cdot)$ for a fixed $x \in X$ (resp. $f(\cdot, y)$ for a fixed $y \in Y$).

A variation of Corollary 1.2.8 is also obtained as a corollary of Theorems 2.1.17 and 2.1.22 (cf. [65, 88, 77, 9]).

Corollary 2.1.28. *Let X, H, and (M, d) be a Čech-complete space, a compact space, and a metric space, respectively, and let $f : X \times H \to M$ be a map satisfying that $f_x \in C(H, M)$ for every $x \in X$ and there is a dense subset E of H such that $f^g \in C(X, M)$ for every $g \in E$. Suppose that any of the two following equivalent conditions holds.*

(a) *There exists a dense Baire subset $F \subseteq X$ such that $(\overline{E}^{M^X})|_F$ is metrizable.*

(b) *There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{E}^{M^X}))$ is Lindelöf.*

Then there exists a G_δ and dense subset F in X such that f is jointly continuous at each point of $F \times H$.

Finally, we obtain some applications to dynamical systems [40, 39, 41]. Recall that a *dynamical system*, or a *G-space*, is a Hausdorff space X on which a topological group G acts continuously. We denote such a system by (G, X). For each $g \in G$ we have the self-homeomorphism $x \mapsto gx$ of X that we call g-translation.

Corollary 2.1.29. *Let X be a Polish G-space such that $\overline{G}^{X^X}$ is compact. The following properties are equivalent:*

(a) *G is almost equicontinuous.*

(b) *There exists a dense Baire subset $F \subseteq X$ such that $(\overline{G}^{X^X})|_F$ is metrizable.*

(c) There exists a dense G_δ subset $F \subseteq X$ such that $(F, t_p(\overline{G}^{X^X}))$ is Lindelöf.

Corollary 2.1.30. *Let X be a completely metrizable G-space such that $\overline{G}^{X^X}$ is compact. Then the following conditions are equivalent:*

(a) G is hereditarily almost equicontinuous.

(b) L is hereditarily almost equicontinuous on F, for all $L \in [G]^{\leq\omega}$ and F a compact subset of X.

(c) $((\overline{L}^{X^X})|_F, t_p(F))$ is metrizable, for all $L \in [G]^{\leq\omega}$ and F a compact subset of X.

(d) $(F, t_p(\overline{L}^{X^X}))$ is Lindelöf, for all $L \in [G]^{\leq\omega}$ and F a compact subset of X.

In [5, Problem 28], Arkhangel'skii raises the following question: Let X be a Lindelöf space and let K be a compact subset of $(C(X), t_p(X))$. Is it true that the tightness of K is countable? As far as we know, this question is still open in ZFC. Here we provide a partial answer to Arkhangel'skii's question.

Corollary 2.1.31. *Let X be a Lindelöf space and let K be a compact subspace of $(C(X), t_p(X))$. If there is a a dense subset $E \subseteq K$ such that $(X, t_p(E))$ is Čech-complete and hereditarily Lindelöf, then K is metrizable.*

Proof. The proof of this result is consequence of Theorem 2.1.22. Indeed, remark that, if F is a subset of X that is closed in the $t_p(E)$-topology, then F is Čech-complete and hereditarily Lindelöf as well. Moreover, since $E \subseteq K$, it follows that F is also closed in the $t_p(K)$-topology and, as a consequence, Lindelöf. Applying Corollary 2.1.24 to the (compact) space K, which is equipped with the $t_p(X)$-topology, it follows that E is hereditarily almost equicontinuous on X. Since $(X, t_p(E))$ is hereditarily Lindelöf, Proposition 2.1.18 yields the metrizability of $K = \overline{E}^{\mathbb{R}^X}$. □

2.2 Compact sets of metric-valued Baire class 1 functions

Using Lemma 1.2.33, we can generalise Theorem 1.2.10 to an arbitrary metric space. This result is very useful in the sequel.

Corollary 2.2.1. *Let X be a Polish space, (M,d) a metric space and a sequence $\{f_n\}_{n<\omega} \subseteq C(X,M)$ such that $\overline{\{f_n\}_{n<\omega}}^{M^X}$ is compact. Then, either $\{f_n\}_{n<\omega}$ contains a pointwise convergent subsequence or a subsequence whose closure in M^X is homeomorphic to $\beta\omega$.*

Now, we are in position of extending, to the setting of metric-valued functions, the results obtained in the previous section for real-valued functions.

Proposition 2.2.2. *Let X be a Polish space, (M,d) be a metric space and let E be an infinite subset of $C(X,M)$ such that $\overline{E}^{M^X}$ is compact. The following assertions are equivalent:*

(a) $tg(\overline{E}^{M^X}) \leq \omega$

(b) $\overline{E}^{M^X} \subseteq B_1(X,M)$.

(c) E is sequentially dense in $\overline{E}^{M^X}$.

(d) $|\overline{E}^{M^X}| \leq \mathfrak{c}$.

Proof. If follows from Lemma 1.2.33 and Corollary 1.2.16.

□

Proposition 2.2.3. *Let X be a Polish space, (M,d) be a metric space and a sequence $\{f_n\}_{n<\omega} \subseteq C(X,M)$ such that $\overline{\{f_n\}_{n<\omega}}^{M^X}$ is compact. The following assertions are equivalent:*

(a) $\{f_n\}_{n<\omega}$ is sequentially dense in its closure.

(b) The closure of $\{f_n\}_{n<\omega}$ contains no copy of $\beta\omega$.

Proof. $(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$ By Lemma 1.2.33 we know that $\overline{\{\check{f}_n\}_{n<\omega}}^{\mathbb{R}^{X\times\mathcal{K}}}$ contains no copy of $\beta\omega$. Thus by Theorem 1.2.11, it follows that $\{\check{f}_n\}_{n<\omega}$ is sequentially dense in its closure. By Lemma 1.2.33 we conclude that $\{f_n\}_{n<\omega}$ is sequentially dense in its closure. □

Corollary 2.2.4. *Let X be a compact space, (M,d) be a metric space and let E be a countably infinite subset of $C(X,M)$ such that $\overline{E}^{M^X}$ is compact. If $tg(\overline{E}^{M^X}) \leq \omega$, then $\overline{E}^{M^X} \subseteq B_1(X,M)$.*

Proof. It suffices to apply Corollary 1.2.22 and Lemma 1.2.33. □

Corollary 2.2.5. *Let X be a compact space, (M,d) be a metric space and let E be a countably infinite subset of $C(X,M)$ such that $\overline{E}^{M^X}$ is compact. The following assertions are equivalent:*

(a) $tg(\overline{E}^{M^X}) \leq \omega$.

(b) *E does not contain any sequence whose closure in M^X is homeomorphic to $\beta\omega$.*

(c) *$\overline{E}^{M^X}$ is a Rosenthal compactum.*

(d) $|\overline{E}^{M^X}| \leq \mathfrak{c}$.

Proof. It suffices to apply Corollary 1.2.21 and Lemma 1.2.33. □

Corollary 2.2.6. *Let X be a compact space, (M,d) be a metric space and let E be a countably infinite subset of $C(X,M)$ such that $\overline{E}^{M^X}$ is compact. If $|\overline{E}^{M^X}| \geq 2^{\mathfrak{c}}$, then there is a countable subset L of E such that its closure is canonically homeomorphic to $\beta\omega$.*

Proof. Use Corollary 2.2.5. □

In the case where the metric space M is $\mathbb{C}$ we have the following Corollary.

Corollary 2.2.7. *Let X be a compact space and E a uniformly and countably infinite subset of $C(X,\mathbb{C})$. The following assertions are equivalent:*

(a) $tg(\overline{E}^{\mathbb{C}^X}) \leq \omega$.

(b) *E does not contain any sequence whose closure in $\mathbb{C}^X$ is homeomorphic to $\beta\omega$.*

(c) *$\overline{E}^{\mathbb{C}^X}$ is a Rosenthal compactum.*

(d) $|\overline{E}^{\mathbb{C}^X}| \leq \mathfrak{c}$.

(e) *E does not contain a subsequence equivalent to the ℓ_1 basis.*

Proof. Apply the complex version of Theorem 1.2.18 and Corollary 2.2.5. □

Chapter 3

Interpolation sets in spaces of continuous functions

In this chapter, we introduce the notions of M-interpolation set and $\mathfrak{B}$-family and we analyse their fundamental properties. The study of interpolation sets in spaces of continuous functions is useful in the search of interpolation sets in a locally quasiconvex abelian locally k_ω-group and in the dual of a topological group.

3.1 Basic facts

Definition 3.1.1. *Let X and M be a topological space and metric space, respectively. If $C(X, M)$ denotes the set of all continuous functions from X to M, we say that a subset Y of X is a **M-interpolation set** (equivalently, an interpolation set for $C(X, M)$) when for every function $g \in M^Y$ with relatively compact range in M, there exists a map $f \in C(X, M)$ such that $f_{|Y} = g$.*

Definition 3.1.2. *Let X and M be a topological space and metric space, respectively, and let $C(X, M)$ denote the space of continuous functions of X into M. Given a subset $L \subseteq C(X, M)$, we say that $K \subseteq X$ **separates** L if for every subset $A \subseteq L$ there are two closed subsets in M, say D_1 and D_2, and $x_A \in K$ such that $dist(D_1, D_2) > 0$, $\chi(x_A) \subseteq D_1$ for all $\chi \in A$ and $\chi(x_A) \subseteq D_2$ for all $\chi \in L \setminus A$.*

The next lemma is folklore. We refer to [26, 37] for further information.

Lemma 3.1.3. *Let X be a topological space, M be a metric space and L a countably infinite subset of $C(X, M)$ such that $\overline{L}^{M^X}$ is compact. Consider the following four properties:*

(a) There is a nonempty subset Δ *of* X *such that* L *is separated by* Δ.

(b) Every two disjoint subsets of L *have disjoint closures in* M^X.

(c) $\overline{L}^{M^X}$ *is canonically homeomorphic to* $\beta\omega$.

(d) L *is a* M*-interpolation set for* $C(M^X, M)$.

Then $(a) \Rightarrow (b) \Leftrightarrow (c) \Leftarrow (d)$*. If* M *is a Banach space then the properties* (b)*,* (c) *and* (d) *are equivalent.*

Proof. $(b) \Leftrightarrow (c)$ is folklore. It is also clear that (d) implies (c). For (a) implies (b), let B_1 and B_2 two disjoint subsets of L, which is separated by Δ. Then, there are two closed sets D_1 and D_2 in M and $x_0 \in \Delta \subseteq X$ such that $d(D_1, D_2) \geq \epsilon_0$, for some $\epsilon_0 > 0$, $b_1(x_0) \in D_1$ for all $b_1 \in B_1$ and $\gamma(x_0) \in D_2$ for all $\gamma \in L \setminus B_1$ (in particular for all $\gamma \in B_2$). Thus, $\overline{B_1}^{M^X} \cap \overline{B_2}^{M^X} = \emptyset$.

Finally, let us see that (c) implies (d) if M is a Banach space. Let $f \in M^L$ with relatively compact range in M. By (c), there is a continuous function $\overline{f} : \overline{L}^{M^X} \longrightarrow M$ such that $\overline{f}_{|L} = f$. Now, since $\overline{f}(\overline{L}^{M^X})$ is compact metric, it must be separable. Therefore, there is a separable Banach subspace $F \subseteq M$ such that $\overline{f}(\overline{L}^{M^X}) \subseteq F$. Under such conditions, it is known that there is a continuous map $\widetilde{f} : M^X \longrightarrow F$ that extends $\overline{f}$ (see [51]). Thus $\widetilde{f}$ is the required extension of f to M^X. □

3.2 Continuous functions on a Čech-complete space

Given a Banach space M, our goal in this section is to obtain M-interpolation sets in subsets of M-valued continuous functions defined on a topological space X. Our procedure is the following: assume that E is an infinite subset of $C(X, M)$ and equip E with the pointwise convergence topology on X. If $\overline{E}^{M^X}$ is compact, in order to prove that E contains a (countably infinite) M-interpolation set, we find a (countably infinite) subset $L \subseteq E$ which is separated by X. Applying Lemma 3.1.3, it follows that L is a M-interpolation set. Previously, we need the following definitions and a technical lemma.

Definition 3.2.1. *Let* X *and* M *be a topological space and a metric space (respectively) and let* $f \in M^X$*. We say that* f *is* ***totally discontinuous*** *if there are two subsets* N_0 *and* N_1 *in* M *and two dense subsets* A_0 *and* A_1 *in* X *such that* $d(N_0, N_1) > 0$ *and* $f(A_j) \subseteq N_j$ *for* $j = 0, 1$.

We may assume that N_0 and N_1 are open sets, because otherwise we would work with $B(N_i, s/3) \overset{\text{def}}{=} \{m \in M : d(m, N_i) < s/3\}$, where $s = d(N_0, N_1)$ and $i = 0, 1$.

Lemma 3.2.2. *Let X and M be a Čech-complete space and a metric space, respectively. If E is an infinite subset of $C(X, M)$ where each element has relatively compact range in M such that $\overline{E}^{M^X}$ contains a totally discontinuous function f, then there is a nonempty compact subset Δ of X and a countably infinite subset L of E, which is separated by Δ. As a consequence, if M is a Banach space, it follows that L is a M-interpolation set*

Proof. Since X is Čech-complete, it is a G_δ-subset of its Stone-Cech compactification βX. Set $X = \bigcap_{n=0}^{\infty} W_n$, where W_n is a dense open subset of βX for each $n < \omega$ and $W_s \subseteq W_r$ if $r < s$. In the sequel, given a map $g \in C(X, M)$ with relatively compact range in M, we denote by g^β its continuous extension to βX.

Let N_0, N_1, A_0, A_1 as in Definition 3.2.1, recall that we can consider that N_0 and N_1 are open. By induction on $n = |t|$, $t \in 2^{(\omega)}$ (i.e. the set of finite sequences of 0's and 1's), we construct a family $\{U_t : t \in 2^{(\omega)}\}$ of nonempty open subsets in βX and a sequence of functions $\{h_n : n < \omega\} \subseteq E$, satisfying the following conditions for all $t \in 2^{(\omega)}$:

(i) $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta X} \subseteq W_0$;

(ii) $U_{ti} \subseteq \overline{U_{ti}}^{\beta X} \subseteq W_{|t|+1} \cap U_t$ for $i = 0, 1$;

(iii) $U_{t0} \cap U_{t1} = \emptyset$;

(iv) $h^\beta_{|t|}(U_{tj}) \subseteq N_j$ for $j = 0, 1$;

(v) whenever $s < |t|$, $diam(h^\beta_s(\overline{U_{tj}}^{\beta X})) < \frac{1}{|t|}$ for $j = 0, 1$.

Construction: If $n = 0$, by regularity we can find $U_\emptyset$ a nonempty open set in βX such that $U_\emptyset \subseteq \overline{U_\emptyset}^{\beta X} \subseteq W_0$. For $n \geq 0$, suppose $\{U_t : |t| \leq n\}$ and $\{h^\beta_{|t|} : |t| < n\}$ have been constructed satisfying $(i) - (v)$. Fix $t \in 2^{(\omega)}$ with $|t| = n$. Since U_t is open in βX, then $V_t \overset{\text{def}}{=} U_t \cap X \neq \emptyset$ is open in X. We can find $a_t, b_t \in V_t$ such that $f(a_t) \in N_0$ and $f(b_t) \in N_1$ because V_t is a relatively open subset of X and the sets A_0 and A_1 are dense in X. Since $f \in \overline{E}^{M^X}$, there is $h_n \in E$ such that $h_n(a_t) \in N_0$ and $h_n(b_t) \in N_1$.

Let h^β_n the continuous extension of h_n, then we can select two open disjoint neighbourhoods in βX, O_{t0} and O_{t1} of a_t and b_t, respectively, such that:

(1) $\overline{O_{t0}}^{\beta X} \cup \overline{O_{t1}}^{\beta X} \subseteq U_t$;

(2) $\overline{O_{t0}}^{\beta X} \cap \overline{O_{t1}}^{\beta X} = \emptyset$;

(3) $diam(h_n^{\beta}(O_{tj})) < \frac{1}{|t|}$;

(4) $h_n^{\beta}(O_{tj}) \subseteq N_j$, for $j = 0, 1$.

Since $W_{|t|+1}$ is dense in βX, then $W_{|t|+1} \cap O_{t0}$ and $W_{|t|+1} \cap O_{t1}$ are two nonempty open sets. By regularity, there exist two nonempty open sets U_{t0} and U_{t1} such that $U_{t0} \subseteq \overline{U_{t0}}^{\beta X} \subseteq W_{|t|+1} \cap O_{t0}$ and $U_{t1} \subseteq \overline{U_{t1}}^{\beta X} \subseteq W_{|t|+1} \cap O_{t1}$ respectively. Therefore, U_{t0} and U_{t1} satisfies the conditions $(ii), (iii)$ and (iv). Moreover, observe that by continuity we can adjust the open sets to satisfy (v).

Let $\Delta \stackrel{\text{def}}{=} \bigcap_{n<\omega} \bigcup_{|t|=n} \overline{U_t}^{\beta X}$, then Δ is a closed subset of βX. Consequently, Δ is compact. Note that we can express $\Delta = \bigcup_{\sigma \in 2^{\omega}} \bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X}$. For each $\sigma \in 2^{\omega}$, $\bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X} \neq \emptyset$ by compactness of βX. So $\Delta \neq \emptyset$. By construction we have that $\Delta \subseteq \bigcap_{n=0}^{\infty} W_n = X$. Consequently Δ is contained in X.

Define $\varphi : \Delta \to 2^{\omega}$ by $\varphi^{-1}(\sigma) = \bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X}$. Clearly φ is an onto and continuous map. For each $t \in 2^{(\omega)}$ and $\sigma \in 2^{\omega}$, $h_{|t|}(\varphi^{-1}(\sigma))$ is a singleton by (v). Therefore, $h_{|t|}$ lifts to a continuous function $h^*_{|t|}$ on 2^{ω} such that $h_{|t|}(x) = h^*_{|t|}(\varphi(x))$ for all $x \in \Delta$.

Let us see that $\{h_n\}_{n<\omega}$ is separated by Δ. Let $S \subseteq \omega$ an arbitrary subset, we can take $\sigma \in 2^{\omega}$ such that $\sigma(0) = 0$ and $\sigma(n+1) = 1$ if $n \in S$ or $\sigma(n+1) = 0$ if $n \notin S$. If we choose an element $z \in \bigcap_{n<\omega} \overline{U_{\sigma|n}}^{\beta X} \subseteq \Delta$, then we have that $h_n(z) \in N_1$ if $n \in S$ and $h_n(z) \in N_0$ if $n \notin S$. Finally, in case M is a Banach space, it suffices to apply Lemma 3.1.3, to obtain that $L = \{h_n\}_{n<\omega}$ is a M-interpolation set. □

The following definition is essential in the sequel. We refer to [12] for its motivation.

Definition 3.2.3. *Let X be a topological space and let M be a metric space. We say that $E \subseteq C(X, M)$ is a **$\mathfrak{B}$-family** if the following two conditions hold:*

(a) E is relatively compact in M^X.

(b) There exists a nonempty open set V of X and $\epsilon > 0$ such that for every finite collection $\{U_1, \ldots, U_n\}$ of nonempty relatively open sets of V there is a $f \in E$ satisfying diam$(f(U_j)) \geq \epsilon$ for all $j \in \{1, \ldots, n\}$.

It is pertinent to mention here a result by Pol [78], where the existence of interpolation subset in a set E of real-valued continuous functions defined on a metric complete space X can be obtained when $\overline{E}^X$ contains a function that is not Baire one. The main difference in our approach is that this propoperty is isolated within the set E.

In Section 2.1, we defined a subset E of $C(X, M)$ as *almost equicontinuous* (resp. *is hereditarily almost equicontinuous*) if E is equicontinuous on a dense subset of X (resp. if E is almost equicontinuous for every closed nonempty subset of X). We do not know which is the relation between the notions of being a $\mathfrak{B}$-family and the negation of being almost equicontinuous or hereditary almost equicontinuous when X is a Čech-complete space. However, in the cases in which this relation is known (topological groups, for instance), the existence of interpolation sets is assured as we show later.

We can now formulate the main result in this section. First, we recall that a map is said *quasi-open* when the closure of the image of an open subset has nonempty interior. In the proof of the next theorem we use the compact space $\mathcal{K}$ that was defined in Section 1.2.3.

Theorem 3.2.4. *Let X be a Čech-complete space, M a metric space, Y a metrizable separable space and $\Phi : X \to Y$ a continuous and quasi-open map. If an infinite subset E of $C(X, M)$ is a $\mathfrak{B}$-family where each $g \in E$ factors through Y; that is for every $g \in E$, there is a $\widetilde{g} \in C(Y, M)$ satisfying $g(x) = (\widetilde{g} \circ \Phi)(x)$ for all $x \in X$, then there is a nonempty compact subset Δ of X and a countably infinite subset L of E such that L is separated by Δ. As a consequence, if M is a Banach space, it follows that L is a M-interpolation set.*

Proof. We may assume without loss of generality that the map Φ is surjective because otherwise we would work with the separable and metrizable space $\Phi(X)$. Due to the fact that Čech-completeness is hereditary for closed subsets, we may assume, from here on, that $X = \overline{V}$ WLOG; where V is a nonempty open subset satisfying the following property: there is some fixed $\epsilon > 0$ such that for every finite collection $\{U_1, \ldots, U_n\}$ of nonempty open subsets contained in V, there is some element $g \in E$ with *diam*$(g(U_j)) \geq \epsilon$ for all $j \in \{1, \ldots, n\}$.

Let $\{\widetilde{V_k}\}_{k<\omega}$ be an arbitrary countable open basis in Y. We set $V_k \stackrel{\text{def}}{=} \Phi^{-1}(\widetilde{V_k})$ and pick and arbitrary point $x_k \in V_k$ for each $k < \omega$.

Since X is Čech-complete, there exists a sequence $\{\mathcal{A}_i\}_{i<\omega}$ of open coverings of X, such that, if a family $\mathcal{F}$ of closed subsets has the finite intersection property, and if for each $i < \omega$ there is an element of $\mathcal{F}$ such that is contained in a member of $\mathcal{A}_i$, then $\bigcap \mathcal{F} \neq \emptyset$ [26, Theorem 3.9.2]. In order to simplify the notation below, we say that a set of X is $\mathcal{A}_i$-small if it is contained in a member of $\mathcal{A}_i$.

Using an inductive argument, for every integer $n < \omega$, we find $f_n \in E$, $\alpha_n \in \mathcal{K}$ and a finite collection $\{U_{n,k}\}_{1\leq k\leq n}$ of nonempty open sets in X satisfying the following conditions (for each $n < \omega$ and each $k = 1, \ldots, n$):

(i) $U_{n,k} \subseteq V_k$;

(ii) $diam(f_n(U_{n,k})) \leq \frac{1}{n}$;

(iii) $\overline{U_{n+1,k}} \subseteq U_{n,k}$;

(iv) $d(f_n(x), f_n(x_k)) \geq \frac{\epsilon}{3}$, for all $x \in U_{n,k}$;

(v) $U_{n,k}$ is contained in a member of $\mathcal{A}_j$ for $1 \leq j \leq n$;

(vi) $|\alpha_n(f_n(x)) - \alpha_n(f_n(x_k)| \geq \frac{\epsilon}{3}$, for all $x \in U_{n,k}$.

Construction: If $n = 1$, since V_1 is an open subset in X there exists $f_1 \in E$ such that $diam(f_1(V_1)) \geq \epsilon$. By the continuity of f_1, it follows that there exists a nonempty open subset $W_{1,1}$ such that:

(a) $W_{1,1} \subseteq V_1$

(b) $d(f_1(x), f_1(x_1)) \geq \frac{\epsilon}{3}$, for all $x \in W_{1,1}$

Let $\alpha_1 \stackrel{\text{def}}{=} \alpha_{f_1(x_1)} \in \mathcal{K}$. Note that $|\alpha_1(f_1(x)) - \alpha_1(f_1(x_1))| \geq \frac{\epsilon}{3}$, for all $x \in W_{1,1}$.

Now, we take the open covering $\mathcal{A}_1$ of X. Then, there is $A \in \mathcal{A}_1$ such that $A \cap W_{1,1}$ is not empty. By regularity, we can find a nonempty open subset $U_{1,1}$ such that $U_{1,1} \subseteq \overline{U_{1,1}} \subseteq A \cap W_{1,1} \subseteq V_1$ and $diam(f_1(U_{1,1}) \leq 1$.

Assume now that f_n, α_n and $\{U_{n,k}\}_{1\leq k\leq n}$ have been obtained, with $n \geq 1$. By hypothesis, there exists $f_{n+1} \in E$ such that $diam(f_{n+1}(U_{n,k})) \geq \epsilon$ for all $k \in \{1, \ldots, n\}$ and $diam(f_{n+1}(V'_{n+1})) \geq \epsilon$, where $x_{n+1} \in V'_{n+1} \subseteq V_{n+1}$ and V'_{n+1} is $\mathcal{A}_j$-small for $1 \leq j \leq n$.

By the continuity of f_{n+1}, we can find a finite collection $\{W_{n+1,k}\}_{1\leq k\leq n+1}$ of nonempty open subsets such that:

(1) $W_{n+1,k} \subseteq U_{n,k}$, for all $1 \leq k \leq n$;

(2) $W_{n+1,n+1} \subseteq V_{n+1}$ and $W_{n+1,n+1}$ is $\mathcal{A}_j$-small for $1 \leq j \leq n$;

(3) $diam(f_{n+1}(W_{n+1,k})) \leq \frac{1}{n+1}$, for all $1 \leq k \leq n+1$;

(4) $d(f_{n+1}(x), f_{n+1}(x_k)) \geq \frac{\epsilon}{3}$, for all $x \in W_{n+1,k}$ and $1 \leq k \leq n+1$.

Let $\alpha_{n+1} \in [-1,1]^M$ defined by

$$\alpha_{n+1}(m) \stackrel{\text{def}}{=} \min_{1 \leq k \leq n+1} d(m, f_{n+1}(x_k)) \text{ for all } m \in M.$$

We claim that $\alpha_{n+1} \in \mathcal{K}$. Indeed, if $m_1, m_2 \in M$, then

$$|\alpha_{n+1}(m_1) - \alpha_{n+1}(m_2)| = |\min_{1 \leq k \leq n+1} d(m_1, f_{n+1}(x_k)) - \min_{1 \leq k \leq n+1} d(m_2, f_{n+1}(x_k))|.$$

Assume without loss of generality that

$$\min_{1 \leq k \leq n+1} d(m_1, f_{n+1}(x_k)) \geq \min_{1 \leq k \leq n+1} d(m_2, f_{n+1}(x_k))$$

and choose $k_0 \in \{1, \ldots, n+1\}$ such that

$$\min_{1 \leq k \leq n+1} d(m_2, f_{n+1}(x_k)) = d(m_2, f_{n+1}(x_{k_0})).$$

Then,

$$|\alpha_{n+1}(m_1) - \alpha_{n+1}(m_2)| = \min_{1 \leq k \leq n+1} d(m_1, f_{n+1}(x_k)) - d(m_2, f_{n+1}(x_{k_0})) \leq$$

$$d(m_1, f_{n+1}(x_{k_0})) - d(m_2, f_{n+1}(x_{k_0})) \leq d(m_1, m_2).$$

On the other hand, for all $x \in W_{n+1,k'}$ and $1 \leq k' \leq n+1$:

$$\begin{aligned} |\alpha_{n+1}(f_{n+1}(x)) - \alpha_{n+1}(f_{n+1}(x_{k'}))| &= |\alpha_{n+1}(f_{n+1}(x))| \\ &= \min_{1 \leq k \leq n+1} d(f_{n+1}(x), f_{n+1}(x_k)) \geq \frac{\epsilon}{3}, \end{aligned}$$

Take the open covering $\mathcal{A}_{n+1}$ of X. Then, for each $k \in \{1, \ldots, n+1\}$ there is $A_k \in \mathcal{A}_{n+1}$ such that $A_k \cap W_{n+1,k}$ is a nonempty open subset of X. By regularity we can find an open set $U_{n+1,k}$ such that:

- $U_{n+1,k} \subseteq \overline{U_{n+1,k}} \subseteq A_k \cap W_{n+1,k} \subseteq U_{n,k}$, if $1 \leq k \leq n$;
- $U_{n+1,n+1} \subseteq \overline{U_{n+1,n+1}} \subseteq A_{n+1} \cap W_{n+1,n+1} \subseteq V_{n+1}$, if $k = n+1$.

This completes the construction.

Now, for each $k < \omega$, the intersection $\bigcap_{n=k}^{\infty} U_{n,k}$ is nonempty by Čech-completeness. Therefore, we can fix a point $z_k \in \bigcap_{n=k}^{\infty} U_{n,k}$ for all $k < \omega$. Note that $\Phi(x_k) \in \widetilde{V_k}$ and $\Phi(z_k) \in \widetilde{V_k}$ for all $k \in \omega$.

Take an element $(f,\alpha) \in \overline{\{(f_n,\alpha_n)\}_{n<\omega}}^{(\overline{E}^{M^X}\times\mathcal{K})}$.
By (vi) we have:

$$|\alpha_n \circ \widetilde{f_n}(\Phi(z_k)) - \alpha_n \circ \widetilde{f_n}(\Phi(x_k))| = |\alpha_n \circ f_n(z_k) - \alpha_n \circ f_n(x_k)| \geq \frac{\epsilon}{3}, \quad \forall n \geq k.$$

Therefore, $osc(\alpha_n \circ \widetilde{f_n}, \widetilde{V_k}) \geq \frac{\epsilon}{3}$ for all $n \geq k$. As a consequence, we also have $osc(\alpha \circ \widetilde{f}, \widetilde{V_k}) \geq \frac{\epsilon}{3}$ for all $k < \omega$.

Let $\{r_m, \delta_m\}_{m<\omega}$ be an enumeration of all pairs of rational numbers (r,δ) with $\delta > 0$. For each $m < \omega$, define

$$\widetilde{F_m} = \{y \in Y : \inf(\alpha\circ\widetilde{f})(U) < r_m, \quad \sup(\alpha\circ\widetilde{f})(U) \geq r_m + \delta_m, \forall \text{nbd } U \text{ of } y\}.$$

It is easily seen that $\widetilde{F_m}$ is closed and, consequently, $F_m \stackrel{\text{def}}{=} \Phi^{-1}(\widetilde{F_m})$ is closed in X.
Observe that, since $\{\widetilde{V_k}\}_{k<\omega}$ is an open basis in Y, it follows that $Y = \bigcup_{m<\omega} \widetilde{F_m}$ and, hence $X = \bigcup_{m<\omega} F_m$. Being X Čech-complete, it is a Baire space. Therefore, there is some $m_0 < \omega$ such that F_{m_0} has nonempty interior U in X. Since Φ is a quasi-open map, we have that $\overline{\Phi(U)}$ has nonempty interior $\widetilde{U}$ included in $\widetilde{F}_{m_0}$. It follows that $\inf\left(\alpha \circ \widetilde{f}(\widetilde{U})\right) < r_{m_0}$ and $\sup\left(\alpha \circ \widetilde{f}(\widetilde{U})\right) \geq r_{m_0} + \delta_{m_0}$. Set $U_0 = \Phi^{-1}(\widetilde{U}) \subseteq U$ we have that $\inf(\alpha \circ f(U_0)) < r_{m_0}$ and $\sup(\alpha \circ f(U_0)) \geq r_{m_0} + \delta_{m_0}$.

Set $F = \overline{U_0}$, $r \stackrel{\text{def}}{=} r_{m_0}$ and $\delta \stackrel{\text{def}}{=} \delta_{m_0}$ and we consider the following sets:

$$A_0 = \{x \in F : \alpha \circ f(x) < r\} = \{x \in F : \alpha \circ f(x) \in I_0\}$$

$$A_1 = \{x \in F : \alpha \circ f(x) \geq r + \delta\} = \{x \in F : \alpha \circ f(x) \in I_1\}$$

where $I_0 = [-1, r)$ and $I_1 = (r+\delta, 1]$. Note that A_0 and A_1 are dense subsets in F. Define $N_0 \stackrel{\text{def}}{=} \alpha^{-1}(I_0)$ and $N_1 \stackrel{\text{def}}{=} \alpha^{-1}(I_1)$, which are disjoints. Moreover, since $\alpha \in \mathcal{K}$, it follows that $d(N_0, N_1) \geq \delta$ and $f(A_j) \subseteq N_j$ for $j = 0, 1$. Therefore f is totally discontinuous on F. It now suffices to apply Lemma 3.2.2 □

Remark 3.2.5. *Note that the result remains valid if we consider that for each residual subset R of X there is a separable metrizable space Y and a continuous and quasi-open map $\Phi : R \to Y$ such that for all $g \in E$ there is a $\widetilde{g} \in C(Y, M)$ satisfying $g(x) = (\widetilde{g} \circ \Phi)(x)$ for all $x \in R$.*

Corollary 3.2.6. *Let X be a Polish space and let (M, d) be a metric space. If an infinite subset E of $C(X, M)$ is a $\mathfrak{B}$-family, then there is a nonempty compact subset Δ of X and a countable subset L of E such that L is separated by Δ. As a consequence, if M is a Banach space, it follows that L is a M-interpolation set.*

3.3 Continuous functions on a Čech-complete group

In this section, we apply the results obtained previously to the context of topological groups. Our first result clarifies the relevance of the notion of $\mathfrak{B}$-family in the context of topological groups. From here on, we assume, without loss of generality, that every metrizable topological group M is equipped with a left-invariant metric. Furthermore, if M is in addition compact, then we assume that M is equipped with a bi-invariant metric.

Lemma 3.3.1. *Let G be a topological group, M a metric topological group and $E \subseteq CHom(G, M)$ such that $\overline{E}^{M^G}$ is compact. Then E is a $\mathfrak{B}$-family if and only if it is not equicontinuous.*

Proof. It is clear that, if E is a $\mathfrak{B}$-family, then it may not be equicontinuous. So, assume that E is not a $\mathfrak{B}$-family. Taking $V = G$ and $\epsilon > 0$ arbitrary, there exists a finite family $\{U_1, \dots, U_n\}$ open subsets in G (WLOG, we assume that $U_j = g_j V_j$, where V_j is a neighbourhood of the neutral element) such that for every $f \in E$ there is V_j, with $1 \leq j \leq n$, satisfying that $diam(f(g_j V_j)) < \epsilon$. Now, since f is a group homomorphism and d is left-invariant, it follows that $diam(f(V_j)) < \epsilon$ as well. Set $V_0 = V_1 \cap \dots \cap V_n$, then $diam(f(gV_0)) < \epsilon$ for all $f \in E$ and $g \in G$. Consequently E is equicontinuous. $\square$

The next result is a direct consequence of Lemma 3.3.1, Theorem 3.2.4 and Lemma 3.1.3. Previously, we need the following definition. Recall that $\mathbb{U}(n)$ denotes the unitary group of degree n.

Definition 3.3.2. *Let G be a topological group. If we equip $Hom(G, \mathbb{U}(n))$ with the pointwise convergence topology on G, it becomes a compact Hausdorff space (indeed, it is closed in the product $\mathbb{U}(n)^G$). We say that a subset E of $CHom(G, \mathbb{U}(n))$ is an **I_0^n set** when E is a $\mathbb{C}^{n^2}$-interpolation set in $Hom(G, \mathbb{U}(n))$; that is to say, when for every bounded function $f : E \longrightarrow \mathbb{C}^{n^2}$ there exists $\widetilde{f} \in C(Hom(G, \mathbb{U}(n)), \mathbb{C}^{n^2})$ such that $\widetilde{f}|_E = f$.*

One may look at the notion of I_0^n set as a generalization of I_0 set, given by Hartman and Ryll-Nardzewski for abelian groups [48].

Corollary 3.3.3. *Let G be a compact group, M a metric topological group and E an infinite subset of $CHom(G, M)$ such that $\overline{E}^{M^G}$ is compact. If E is not equicontinuous, then there is a nonempty compact subset Δ of G and a countably infinite subset L of E separated by Δ. As a consequence, if M is a Banach space, it follows that L is a M-interpolation set. In particular, if $M = \mathbb{U}(n)$ then E contains an I_0^n set.*

Proof. By Corollary 2.1.27, we may assume without loss of generality that E is countable and G is separable. By Lemma 3.3.1, E is a $\mathfrak{B}$-family. Define an equivalence relation on G by $g \sim h$ if and only if $f(g) = f(h)$ for all $f \in E$. Since E is countable and consists of group homomorphisms, it follows that the quotient space $\widetilde{G} = G/\sim$ is a compact metrizable group. Therefore, if $p : G \to \widetilde{G}$ denotes the canonical quotient map, each $f \in E$ factors through a map $\widetilde{f}$ defined on $CHom(\widetilde{G}, M)$; that is $\widetilde{f}(p(g)) \stackrel{\text{def}}{=} f(g)$ for any $g \in G$. Since every quotient group homomorphism is automatically open, Theorem 3.2.4 implies that there is a nonempty subset Δ of G and a subset L of E such that L is separated by Δ. In case $M = \mathbb{U}(n)$, applying Lemma 3.1.3, we obtain that L is a M-interpolation set. □

The next result is folklore but we include its proof for the sake of completeness.

Lemma 3.3.4. *Let G be a topological group, M a metric topological group, $E \subseteq C(X, M)$ and $h \in C(G, M)$. Set $Eh \stackrel{\text{def}}{=} \{fh : f \in E\}$. E is equicontinuous on G if and only if Eh is equicontinuous on G.*

Proof. It suffices to prove that Eh is equicontinuous if E is equicontinuous. Let g_0 be an arbitrary but fixed point in G. Since right translations are continuous mappings on a topological group, and E (resp. h) is equicontinuous (resp. continuous) on G, given $\epsilon > 0$, there is a neighbourhood U of g_0 such that $d(f(g_0)h(g_0), f(g)h(g_0)) < \epsilon/2$ and $d(h(g_0), h(g)) < \epsilon/2$ for all $g \in U$ and all $f \in E$. Thus, applying left invariance of the group metric, we obtain

$$\begin{aligned} d(f(g_0)h(g_0), f(g)h(g)) &\leq d(f(g_0)h(g_0), f(g)h(g_0)) \\ &+ d(f(g)h(g_0), f(g)h(g)) \\ &< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

for all $g \in U$, which completes the proof. □

With the hypothesis of the previous lemma, if $f \in CHom(G, M)$, the symbol f^{-1} denotes the map defined by $f^{-1}(g) = f(g)^{-1} = f(g^{-1})$ for all $g \in G$. Combining Lemmas 3.3.1 and 3.3.4, we obtain:

Corollary 3.3.5. *Let G be a topological group, M be a topological group with a bi-invariant metric, $E \subseteq CHom(X, M)$ such that $\overline{E}^{M^X}$ is compact and $f_0 \in E$. Then Ef_0^{-1} is a $\mathfrak{B}$-family if and only if it is not equicontinuous.*

Proof. It suffices to see that Ef_0^{-1} is equicontinuous if Ef_0^{-1} is not a $\mathfrak{B}$-family. Reasoning as in Lemma 3.3.4, let $V = G$ and $\epsilon > 0$, then there are

$\{U_1, \dots, U_n\}$ open subsets of G such that for all $f \in E$ there is $j \in \{1, \dots, n\}$ with $diam(ff_0^{-1}(U_j)) < \epsilon$. We can assume that $U_j = g_j V_j$, where V_j is a neighbourhood of the identity element of G, for all $1 \le j \le n$. Take $W \stackrel{\text{def}}{=} \bigcap_{1 \le j \le n} V_j$ and an arbitrary element $g_0 \in G$.

Given $f \in E$, there is $j \in \{1, \dots, n\}$ such that

$$\begin{aligned}
\epsilon &> diam(ff_0^{-1}(U_j)) = diam(ff_0^{-1}(g_j V_j)) \\
&= \sup_{g,h \in V_j} d(ff_0^{-1}(g_j g), ff_0^{-1}(g_j h)) \\
&= \sup_{g,h \in V_j} d(f(g_j) \left[ff_0^{-1}(g)\right] f_0^{-1}(g_j), f(g_j) \left[ff_0^{-1}(h)\right] f_0^{-1}(g_j)) \\
&= \sup_{g,h \in V_j} d(\left[ff_0^{-1}(g)\right], \left[ff_0^{-1}(h)\right]) = diam(ff_0^{-1}(V_j)) \\
&\ge diam(ff_0^{-1}(W)) = diam(ff_0^{-1}(g_0 W)).
\end{aligned}$$

□

We now formulate one of the main results in this section.

Theorem 3.3.6. *Let G be a Čech-complete group and K a compact group. If the infinite subset E of $CHom(G, K)$ is not equicontinuous, then E contains a countably infinite subset L such that $\overline{L}^{K^G}$ is canonically homeomorphic to $\beta\omega$. In case $K = \mathbb{U}(n)$, it follows that L is an I_0^n set.*

Proof. Since K is compact, there is a representation $\pi : K \to \mathbb{U}(n)$ such that $\{\pi \circ f : f \in E\}$ is not equicontinuous. Therefore, we assume that $K = \mathbb{U}(n)$ without loss of generality.

Applying Corollary 2.1.27, since $E \subseteq CHom(G, \mathbb{U}(n))$ is not equicontinuous, there exists $H \le G$ closed and separable and $L \subseteq E$ countable such that $L|_H$ is not equicontinuous . So we can assume WLOG that G is separable and E is countable. On the other hand, by Čech-completeness of G, there must be a compact subgroup C of G such that G/C is complete and metrizable [14], therefore, a Polish space.

Let $E|_C \stackrel{\text{def}}{=} \{f|_C : f \in E\} \subseteq CHom(C, \mathbb{U}(n))$. We have two possible cases:

(1) $E|_C$ contains infinitely many elements that are pairwise inequivalent (recall that $\gamma_1, \gamma_2 \in Hom(C, \mathbb{U}(n))$ are equivalent ($\gamma_1 \sim \gamma_2$) if there exists $U \in \mathbb{U}(n)$ such that $\gamma_1 = U^{-1}\gamma_2 U$).

(2) $E|_C$ only contains a finite subset of elements that are pairwise inequivalent.

- Case (1): We may suppose without loss of generality that all elements of $E|_C$ are pairwise inequivalent, which implies that $E|_C$ may not be equicontinuous on C. Applying Corollary 3.3.3, there is a nonempty subset Δ of C and a countable subset L of E such that L is separated by Δ. Thus, by Lemma 3.1.3, $\overline{L}^{\mathbb{U}(n)^G}$ is canonically homeomorphic to $\beta\omega$ and we are done.

- Case (2): Set $H \stackrel{\text{def}}{=} \{\varphi_1, \ldots, \varphi_m\} \subseteq E$ such that every $f \in E$ is equivalent to an element in H when they are restricted to C. If we define $E_i = \{f \in E : f|_C \sim \varphi_i|_C\}$, then $E = E_1 \bigcup \ldots \bigcup E_m$. Since E is not equicontinuous, there is $i \in \{1, \ldots m\}$ such that E_i is not equicontinuous. So, we assume without loss of generality that there is $f_0 \in E$ such that $f|_C \sim f_0|_C$ for all $f \in E$. Therefore, for each $f \in E$, there is $U_f \in \mathbb{U}(n)$ with $(U_f^{-1} f U_f)|_C = f_0|_C$. Denote by $\widetilde{f}$ the map $U_f^{-1} f U_f$ and set $\widetilde{E} \stackrel{\text{def}}{=} \{U_f^{-1} f U_f : f \in E\}$, which is a subset of $CHom(G, \mathbb{U}(n))$. It is easily seen that $\widetilde{E}$ is not equicontinuous on G. (Indeed, assume that $\widetilde{E}$ were equicontinuous and let W be an open neighbourhood of the identity matrix I_n in $\mathbb{U}(n)$. By [53, Corollary 1.12] there would exist an open neighbourhood V of e_G such that $\widetilde{f}(V) \subseteq \bigcap_{U \in \mathbb{U}(n)} U^{-1} W U$ for all $\widetilde{f} \in \widetilde{E}$. Therefore, we would have $f(v) = U_f \widetilde{f}(v) U_f^{-1} \in W$ for all $v \in V$. This implies that E is equicontinuous, which is a contradiction). Hence, $\widetilde{E} f_0^{-1}$ is a $\mathfrak{B}$-family on G by Lemmas 3.3.4 and 3.3.5.

  Let $\pi_C : G \to G/C$ the canonical quotient map, which is open an continuous. Since G/C is Polish and each $\widetilde{f} f_0^{-1}$ factors through G/C, we apply Theorem 3.2.4 and Lemma 3.1.3 in order to obtain $\Delta \subseteq G$ and $\widetilde{L} \subseteq \widetilde{E}$ such that

$$\overline{\widetilde{L}}^{\mathbb{U}(n)^\Delta} \simeq \overline{\widetilde{L}}^{\mathbb{U}(n)^\Delta} f_0^{-1} \simeq \beta\omega.$$

  Set $L \stackrel{\text{def}}{=} \{f : \widetilde{f} \in \widetilde{L}\} \subseteq E$ and consider the map

$$\begin{array}{rcl} \psi : (\widetilde{L}, t_p(\Delta)) & \longrightarrow & (L, t_p(\Delta)) \\ U_f^{-1} f U_f & \longmapsto & f \end{array}$$

  The map ψ is continuous because $\widetilde{L}$ is discrete. Moreover, using that $\overline{\widetilde{L}}^{\mathbb{U}(n)^G}$ is canonically homeomorphic to $\beta\omega$, there is a continuous extension map

$$\overline{\psi} : (\overline{\widetilde{L}}^{\mathbb{U}(n)^G}, t_p(\Delta)) \to (\overline{L}^{\mathbb{U}(n)^G}, t_p(\Delta)).$$

  A compactness argument on the group $\mathbb{U}(n)$, implies that if $p, q \in \overline{\widetilde{L}}^{\mathbb{U}(n)^G}$ and $\overline{\psi}(p) = \overline{\psi}(q)$ then p and q are equivalent. Since $Orbit(p) = \{U^{-1} p U : U \in \mathbb{U}(n)\}$ and $|\beta\omega| = |\overline{\widetilde{L}}^{\mathbb{U}(n)^G}| = 2^{\mathfrak{c}}$, we obtain:

$$2^{\mathfrak{c}} = |\overline{\overline{L}}^{\mathbb{U}(n)^G}| \leq |\overline{L}^{\mathbb{U}(n)^G}||\mathbb{U}(n)| = \max\{|\overline{L}^{\mathbb{U}(n)^G}|, \mathfrak{c}\}.$$

Therefore

$$|\overline{L}^{\mathbb{U}(n)^G}| \geq 2^{\mathfrak{c}}.$$

Applying Corollary 2.2.6, it follows that L contains a subset P such that $\overline{P}^{\mathbb{U}(n)^G}$ is canonically homeomorphic to $\beta\omega$. This completes the proof.

□

Corollary 3.3.7. *Let G be a Čech-complete abelian group. If the infinite subset E of $\widehat{G}$ is not equicontinuous, then E contains a countably infinite I_0 set.*

A consequence of this result is a variation of a well-know result of Corson and Glicksberg [22] asserting that if a subset E of continuous homomorphism defined on a hereditarily Baire group has a compact and metric closure, then it is equicontinuous. In case G is Čech-complete and K is a compact group, these constraints can be relaxed considerably.

Corollary 3.3.8. *Let G be a Čech-complete group, K be a compact group and E be an infinite subset of $CHom(G, K)$. If for every countable subset $L \subseteq E$ and compact separable subset $H \subseteq G$ we have that either $\overline{L}^{K^H}$ has countable tightness or $|\overline{L}^{K^H}| \leq \mathfrak{c}$, then E is equicontinuous.*

Proof. If for every countable subset $L \subseteq E$ and compact separable subset $H \subseteq X$ we have that either $\overline{L}^{K^H}$ has countable tightness or $|\overline{L}^{K^H}| \leq \mathfrak{c}$, then $\overline{L}^{K^H}$ may not contain a copy of $\beta\omega$. By Theorem 3.3.6, this implies that $L|_H$ is equicontinuous on H. Applying Theorem 2.1.22, it follows that E is hereditarily equicontinuous on G, which implies that E is equicontinuous because E consists of group homomorphisms.

□

Chapter 4

Interpolation sets in topological groups

4.1 Abelian locally k_ω groups

In this section, we study the existence of I_0 sets for abelian locally k_ω groups, which is a large family of topological groups that includes, for example, all locally compact abelian groups, the free abelian groups on a compact space and all countable direct sum of compact groups. The proof of our main results are obtained using methods of Pontryagin–van Kampen duality.

4.1.1 Basic facts

Definition 4.1.1. *A* ***k_ω-group*** *(resp.,* ***locally k_ω-group****) is a topological group where the underlying topological space is a k_ω-space (resp. locally k_ω).*

The following theorem of Glöckner, Gramlich and Hartnick [43] states that there exists a relation between the abelian locally k_ω groups and the abelian Čech-complete groups.

Theorem 4.1.2. *(Glöckner, Gramlich and Hartnick) If G is an abelian locally k_ω group, then $\widehat{G}$ is abelian Čech-complete. Conversely, $\widehat{G}$ is abelian locally k_ω, for each abelian Čech-complete topological group G.*

4.1.2 I_0 sets

Using the duality of Theorem 4.1.2 and Theorem 3.3.6, we prove the following result:

Theorem 4.1.3. *Let G be a locally quasiconvex, abelian, locally k_ω-group. If $\{g_n\}_{n<\omega}$ is a sequence in G that is not precompact in G, then $\{g_n\}_{n<\omega}$ contains an infinite I_0 set.*

Proof. Consider the abelian Čech-complete group $\widehat{G}$. By means of the evaluation map $eval\colon G \to \widehat{\widehat{G}} \subseteq C(\widehat{G}, \mathbb{T})$, we can look at the sequence $\{g_n\}_{n<\omega}$ as a subset of $C(\widehat{G}, \mathbb{T})$. Furthermore, since $\{g_n\}_{n<\omega}$ is not precompact in G, it follows that $\{g_n\}_{n<\omega}$ is not equicontinuous on $\widehat{G}$. Indeed, if it were equicontinuous on $\widehat{G}$, by Arzelà-Ascoli's theorem (see Theorem 1.2.6), then $\{g_n\}_{n<\omega}$ would be precompact in $C_c(\widehat{G}, \mathbb{T})$, the group $C(\widehat{G}, \mathbb{T})$ equipped with the compact open topology. Now, since G is a locally quasiconvex k-space, the evaluation map $eval\colon G \to \widehat{\widehat{G}}$ is a topological isomorphism in its image (see [49]). Thus $\{g_n\}_{n<\omega}$ would also be precompact in G, which is a contradiction.

Therefore, the sequence $\{g_n\}_{n<\omega}$ is not an equicontinuous set on $\widehat{G}$ and, by Corollary 3.3.6, contains an I_0 set. □

4.1.3 Property of strongly respecting compactness

The next result was proved in [34, Lemma 4.11].

Lemma 4.1.4. *Let G be a maximally almost periodic abelian group, A a subset of G and let N be a subset of bG containing the neutral element such that $A + N$ is compact in bG. If F is an arbitrary subset of A, there exists $A_0 \subseteq A$ with $|A_0| \leq |N|$ such that*

$$cl_{bG}F \subseteq A_0 + N + cl_{G^+}(F - F).$$

We are now in position of proving the main result in this section.

Corollary 4.1.5. *Every locally quasiconvex, abelian, locally k_ω, group strongly respects compactness.*

Proof. Let G be a locally quasiconvex, locally k_ω group and let bG denote its Bohr compactification. If N is a closed metrizable subgroup of bG and A is a subset G such that $A + (N \cap G)$ is compact in G, then $b_N(A)$ is trivially compact in bG/N. Note that $b_N(A + N \cap G) = b_N(A)$.

Reasoning by contradiction, assume that $b_N(A)$ is compact in bG/N but $A + (N \cap G)$ is not compact in G. This means, being closed in G, that $A + (N \cap G)$ is not precompact in the topology inherited from G. As in the proof of Theorem 4.1.3, if we take the abelian Čech-complete group $\widehat{G}$ and inject G in $C(\widehat{G}, \mathbb{T})$ by means of the evaluation map $eval\colon G \to \widehat{\widehat{G}} \subseteq C(\widehat{G}, \mathbb{T})$, it follows that $A + (N \cap G)$ is not equicontinuous on $\widehat{G}$. By Corollary 2.1.27, it follows that there exists a countable subset $F \subseteq A + (N \cap G)$ and a separable compact subset $X \subseteq \widehat{G}$ such that F is not equicontinuous on X. Taking the closure in $\widehat{G}$ of the subgroup generated by X, we may assume that X is a separable closed subset of $\widehat{G}$.

Set $X^{\perp} \stackrel{\text{def}}{=} \{g \in G : \chi(g) = 0 \text{ for all } \chi \in X\}$ and take the quotient $G/X^{\perp}$, which clearly is a maximally almost periodic group whose dual is X. Furthermore, the group $G/X^{\perp}$ is locally k_{ω} and X is Čech-complete. If $p\colon G \to G/X^{\perp}$ denotes the open quotient map, it follows that there is a canonical extension $p^b\colon bG \to b(G/X^{\perp})$. Therefore, we have that $p(A + (N \cap G))$ is contained in $p^b(A+N) = p(A) + p^b(N)$ that which is compact in $b(G/X^{\perp})$. Applying Lemma 4.1.4 to $p(F)$ and $p^b(N)$, we obtain that there exists $A_0 \subseteq p(A)$ with $|A_0| \leq |p^b(N)| \leq \mathfrak{c}$ such that

$$cl_{b(G/X^{\perp})}p(F) \subseteq A_0 + p^b(N) + cl_{(G/X^{\perp})^{+}}p(F - F).$$

Now, being the group X is separable, it follows that $G/X^{\perp}$ can be equipped with a metrizable precompact topology. As a consequence $|G/X^{\perp}| \leq \mathfrak{c}$. All in all, we obtain that $|cl_{bG/X^{\perp}}p(F)| \leq \mathfrak{c}$.

On the other hand $p(F)$ is not equicontinuous as a subset of $C(X, \mathbb{T})$ and, by Theorem 4.1.3, this means that it contains an I_0 set. This yields $|cl_{bG/X^{\perp}}p(F)| = |\beta\omega| = 2^{\mathfrak{c}} > \mathfrak{c}$. This is a contradiction that completes the proof. □

4.2 Locally compact groups

Let G be a locally compact group. Set $\mathcal{H}_n \stackrel{\text{def}}{=} \mathbb{C}^n$ for $n = 1, 2, \ldots$; $\mathcal{H}_0 \stackrel{\text{def}}{=} l^2(\mathbb{Z})$. Recall from the Section 1.4 that $Irr_n^C(G)$ denotes the set of irreducible unitary representations of G on $\mathcal{H}_n$ (where it is assumed that every set $Irr_n^C(G)$ is equipped with the compact open topology), and $Irr^C(G) = \bigsqcup_{n \geq 0} Irr_n^C(G)$ (the disjoint topological sum).

The symbol G^w (resp. G^{w_C}) designates the group G equipped with the weak (group) topology generated by $Irr(G)$ (resp. $Irr^C(G)$). As it was mentioned in the Introduction, if G is abelian, then the weak topology of G coincides with the so-called Bohr topology associated to G.

Definition 4.2.1. *We denote by* $\boldsymbol{P(G)}$ *the set of continuous positive-definite functions on* (G, τ). *If* $\sigma \in Irr(G)$ *and* $v \in \mathcal{H}^{\sigma}$, *then the positive-definite function*

$$\varphi : g \mapsto \langle \sigma(g)(v), v \rangle,\ g \in G$$

is called ***pure***, *and the family of all such functions is denoted by* $\boldsymbol{I(G)}$. *We also can define* $\boldsymbol{I^C(G)}$ *as the subset of* $I(G)$ *consisting of the elements whose irreducible representation is in* $Irr^C(G)$. *When* G *is abelian, the set* $I(G)$ *coincides with the dual group* $\widehat{G}$ *of the group* G.

The proof of the lemma below is straightforward.

Lemma 4.2.2. *Let G be a locally compact group. Then:*

(a) $G^{\mathrm{w}} = (G, \mathrm{w}(G, I(G)))$.

(b) $G^{\mathrm{w}_C} = (G, \mathrm{w}(G, I^C(G)))$.

Proof. Both proofs follow from the fact for any representation $\sigma \in Irr(G)$ and $v_1, v_2 \in \mathcal{H}^\sigma$ we have the following equality:

$$\begin{aligned}\langle \sigma(g)(u_1), u_2 \rangle = \; & \frac{1}{4}\langle \sigma(g)(u_1 + u_2), u_1 + u_2 \rangle + \frac{i}{4}\langle \sigma(g)(u_1 + iu_2), u_1 + iu_2 \rangle - \\ & \frac{1}{4}\langle \sigma(g)(u_1 - u_2), u_1 - u_2 \rangle - \frac{i}{4}\langle \sigma(g)(u_1 - iu_2), u_1 - iuu_2 \rangle\end{aligned}$$

and the fact that the strong operator topology and the weak operator topology coincides in $\mathbb{U}(\mathcal{H})$ for all Hilbert space $\mathcal{H}$. □

Remark 4.2.3. *We recall that $G^{\mathrm{w}} = G^{\mathrm{w}_C}$ if G is a separable, metrizable, locally compact group.*

Definition 4.2.4. *Let G be a locally compact group and consider the two following natural embeddings:*

$$\mathrm{w} : G \hookrightarrow \prod_{\varphi \in I(G)} \overline{\varphi(G)} \qquad \textit{and} \qquad \mathrm{w}_C : G \hookrightarrow \prod_{\varphi \in I^C(G)} \overline{\varphi(G)}$$

$$\mathrm{w}(g) = (\varphi(g))_{\varphi \in I(G)} \qquad\qquad \mathrm{w}_C(g) = (\varphi(g))_{\varphi \in I^C(G)}$$

We define the ***weak compactification*** $\mathrm{w}G$ *(resp.* ***C-weak compactification*** $\mathrm{w}_C G$*) of G as the pair* $(\mathrm{w}G, \mathrm{w})$ *(resp.* $(\mathrm{w}_C G, \mathrm{w}_C)$*), where* $\mathrm{w}G \overset{\mathrm{def}}{=} \overline{\mathrm{w}(G)}$ *(resp.* $\mathrm{w}_C G \overset{\mathrm{def}}{=} \overline{\mathrm{w}_C(G)}$*).*

This compactification has been previously considered in [16, 17] using different techniques. Also Akemann and Walter [1] extended Pontryagin duality to non-abelian locally compact groups using the family of pure positive-definite functions. Again, in case G is abelian, both compactifications, $(\mathrm{w}G, \mathrm{w})$ and $(\mathrm{w}_C G, \mathrm{w})$, coincide with the Bohr compactification of G.

The Eberlein compactification of a locally group G, eG, (see Subsection 1.4.2) is closely related to $\mathrm{w}G$. Since eG is defined using the family of all continuous positive-definite functions, it follows that $\mathrm{w}G$ is a factor of eG and, as a consequence, inherits most of its properties. In particular, $\mathrm{w}G$ is a compact involutive semitopological semigroup.

In the sequel, $inv(\mathrm{w}G) \overset{\mathrm{def}}{=} \{x \in \mathrm{w}G : xy = yx = 1 \text{ for some } y \in \mathrm{w}G\}$ designates the group of units of $\mathrm{w}G$.

The following definition was introduced by Hartman and Ryll-Nardzewski for abelian locally compact groups [48]. Here, we extend it to arbitrary not necessarily abelian locally compact groups.

Definition 4.2.5. *A subset A of a locally compact group G is an* ***I_0 set*** *if every bounded complex (or real) valued function on A can be extended to a continuous function on* wG*. This definition extends the classic one, since when G is an abelian group, we have that* w$G = bG$ *and $C(bG)|_G = AP(G)$ is the set of almost periodic functions on G.*

Remark 4.2.6. *Observe that if (G, τ) is a locally compact group and A be a countably infinite subset of G, then A is an I_0 set if and only if $\overline{A}^{wG}$ is canonically homeomorphic to $\beta\omega$ (see Lemma 3.1.3).*

The next Lemma can be found in [91, Section 14, Th.3].

Lemma 4.2.7. *Let X be a compact space and $f : X \to \beta\omega$ a continuous and onto map. If $f^{-1}(n)$ is a singleton for all $n < \omega$ and $f^{-1}(\omega)$ is dense in X. Then f is a homeomorphism.*

Lemma 4.2.8. *Let (G, τ) be a separable metric locally compact group and $\{g_n\}_{n<\omega}$ be a sequence on G such that $\overline{\{g_n\}_{n<\omega}}^{w_C G} \cong \beta\omega$, then $\overline{\{g_n\}_{n<\omega}}^{wG} \cong \beta\omega$.*

Proof. Let $\varphi : G^w \to G^{w_C}$ be the identity map, which is clearly a continuous group homomorphism and set $\overline{\varphi} : wG \to w_C G$ the continuous extension of φ. The result follows from Lemma 4.2.7. $\square$

We now recall some known results about unitary representations of locally compact groups that are needed in the proof of our main result in this section. One important point is the decomposition of unitary representations by direct integrals of irreducible unitary representations. This was established by Mautner [67] following the ideas introudced by von Neuman in [74].

Theorem 4.2.9 (*F. I. Mautner, [67]*)**.** *For any representation $(\sigma, \mathcal{H}_\sigma)$ of a separable locally compact group G, there is a measure space $(R, \mathcal{R}, r)$, a family $\{\sigma(p)\}$ of irreducible representations of G, which are associated to each $p \in R$, and an isometry U of $\mathcal{H}_\sigma$ such that*

$$U\sigma U^{-1} = \int_R \sigma(p) d_r(p).$$

Remark 4.2.10. *The proof of the above theorem given by Mautner assumes that the representation space $\mathcal{H}_\sigma$ is separable but, subsequently, Segal [87] removed this constraint. Furthermore, it is easily seen that we can assume that $\sigma(p)$ belongs to $Irr^C(G)$ locally almost everywhere in the theorem above (cf. [59]).*

A remarkable consequence of Theorem 4.2.9 is the following corollary about positive-definite functions.

Corollary 4.2.11. *Every Haar-measurable positive-definite function φ on a separable locally compact group G can be expressed for all $g \in G$ outside a certain set of Haar-measure zero in the form*

$$\varphi(g) = \int_R \varphi_p(g) d_r(p),$$

where φ_p is a pure positive-definite functions on G for all $p \in R$.

4.2.1 Dichotomy-type result

Definition 4.2.12. *Let U be an open neighbourhood of the identity of a topological group G. We say that a sequence $\{g_n\}_{n<\omega}$ is **U-discrete** if $g_n U \cap g_m U = \emptyset$ for all $n \neq m \in \omega$.*

Theorem 4.2.13. *Let (G, τ) be a metric locally compact group and let $\{g_n\}_{n<\omega}$ be a sequence in G. Then, either $\{g_n\}_{n<\omega}$ contains a weak Cauchy subsequence or an infinite I_0 set.*

Proof. Since G is metric, we may assume WLOG that the sequence is not $\{g_n\}_{n<\omega}$ is not τ-precompact. Otherwise, it would contain a τ-convergent subsequence that, as a consequence, would be weakly convergent and *a fortiori* weakly Cauchy.

Thus, $\{g_n\}_{n<\omega}$ must contain a subsequence that is U_0-discrete for some symmetric, relatively compact and open neighbourhood of the identity U_0 in G. For simplicity's sake, we assume WLOG that the whole sequence $\{g_n\}_{n<\omega}$ is U_0-discrete.

Take the σ-compact, open subgroup $H \stackrel{\text{def}}{=} \langle \overline{U}_0 \cup \{g_n\}_{n<\omega} \rangle$ of G. Since H is metric, σ-compact, it follows that H is a Polish locally compact group. Consequently, by [23, Section 18.1.10], we have that $Irr^C_m(H)$, equipped with the compact open topology, is a Polish space for all $m \in \{0, 1, 2, \ldots\}$.

Being $\mathcal{H}_m$ separable, for each $m \in \{0, 1, 2, \ldots\}$, there exists a countable subset $D_m \stackrel{\text{def}}{=} \{v^m_n\}_{n<\omega}$ that is dense in the unit ball of $\mathcal{H}_m$ (therefore, the linear subspace generated by D_m is dense in $\mathcal{H}_m$). Fix $m \in \{0,1,2,\ldots\}$ and let $\mathbb{D}$ denote the closed unit disk in $\mathbb{C}$. We have that $<\sigma(g)(v_n^m), v_n^m> \in \mathbb{D}$ for all $\sigma \in Irr_m^C(H)$, $g \in H$ and $n < \omega$.

For each $m \in \{0,1,2,\ldots\}$, let $\alpha_m : H \to C_p(Irr_m^C(H), \mathbb{D}^\omega)$ be the continuous and injective map defined by $\alpha_m(h)(\sigma) \stackrel{\text{def}}{=} (<\sigma(h)(v_n^m), v_n^m>)_{n\in\omega}$ for all $h \in H$ and $\sigma \in Irr_m^C(H)$. Since $\mathbb{D}^\omega$ is a compact metric space, it follows that

$$\overline{\{\alpha_m(g_n)\}_{n<\omega}}^{(\mathbb{D}^\omega)^{Irr_m^C(H)}}$$

is compact for all $m \in \{0,1,2,\ldots\}$.

Now, we successively apply Corollary 2.2.1 for each $m \in \{0,1,2,\ldots\}$ as follows.

For $m = 0$, $\{\alpha_0(g_n)\}_{n<\omega}$ contains either a pointwise convergent subsequence or a subsequence whose closure in $(\mathbb{D}^\omega)^{Irr_0^C(H)}$ is canonically homeomorphic to $\beta\omega$.

If there is a pointwise convergent subsequence $\{\alpha_0(g_{n_i^0})\}_{i<\omega}$, then we go on to the case $m = 1$. That is $\{\alpha_1(g_{n_i^0})\}_{i<\omega}$ contains either a pointwise convergent subsequence or a subsequence whose closure in $(\mathbb{D}^\omega)^{Irr_1^C(H)}$ is canonically homeomorphic to $\beta\omega$. If there is a pointwise convergent subsequence $\{\alpha_1(g_{n_i^1})\}_{i<\omega}$ we go on to the case $m = 2$, and so forth.

Assume that we can find a pointwise convergent subsequence in each step and take the diagonal subsequence $\{g_{n_i^i}\}_{i<\omega}$. We have that $\{\alpha_m(g_{n_i^i})\}_{i<\omega}$ is pointwise convergent for each $m \in \{0,1,2,\ldots\}$. We claim that the the subsequence $\{g_{n_i^i}\}_{i<\omega}$ is Cauchy in the weak topology of G.

Indeed, take an arbitrary element $\varphi \in I^C(H)$, then there is $t \in \{0,1,2,\ldots\}$, $\sigma \in Irr_t^C(H)$ and $v \in \mathcal{H}_t$ such that $\varphi(h) =<\sigma(h)(v), v>$ for all $h \in H$, where we may assume that $\|v\| \leq 1$ WLOG.

Let $\epsilon > 0$ be an arbitrary positive real number. By the density of D_t, there is $u \in D_t$ such that $\|u - v\| < \epsilon/6$.

For every $h \in H$, we have

$$\begin{aligned}
|<\sigma(h)(v), v> - <\sigma(h)(u), u>| &= |<\sigma(h)(v), v> - <\sigma(h)(u), v> + \\
&\quad <\sigma(h)(u), v> - <\sigma(h)(u), u>| \\
&\leq |<\sigma(h)(v-u), v>| + \\
&\quad |<\sigma(h)(u), v-u>| \\
&\leq 2\|v-u\| < \epsilon/3.
\end{aligned}$$

On the other hand, we know that $\{\alpha_t(g_{n_i^i})\}_{i<\omega}$ is a pointwise Cauchy sequence in $(\mathbb{D}^\omega)^{Irr_t^C(H)}$. Thus, from the definition of α_t and, since $u \in D_t$, it follows that

$$\{<\sigma(g_{n_i^i})(u), u>\}_{i<\omega}$$

is a Cauchy sequence in $\mathbb{D}$. Hence, there is $i_0 < \omega$ such that

$$| < \sigma(g_{n_i^i})(u), u > - < \sigma(g_{n_j^j})(u), u > | < \epsilon/3 \text{ for all } i, j \geq i_0.$$

This yields

$$\begin{aligned} | < \sigma(g_{n_i^i})(v), v > &- < \sigma(g_{n_j^j})(v), v > | \\ &\leq | < \sigma(g_{n_i^i})(v), v > - < \sigma(g_{n_i^i})(u), u > | \\ &+ | < \sigma(g_{n_i^i})(u), u > - < \sigma(g_{n_j^j})(u), u > | \\ &+ | < \sigma(g_{n_j^j})(u), u > - < \sigma(g_{n_j^j})(v), v > | \\ &< \epsilon/3 = \epsilon. \end{aligned}$$

We conclude that $\{< \sigma(g_{n_i^i})(v), v >\}_{i<\omega} = \{\varphi(g_{n_i^i})\}_{i<\omega}$ is a Cauchy sequence in $\mathbb{D}$ for all $\varphi \in I^C(H)$. Since H is a locally compact Polish group, we have that $G^{\mathrm{w}} = G^{\mathrm{w_C}}$ by Remark 4.2.3. As a consequence, it follows that which proves that $\{g_{n_i^i}\}_{i<\omega}$ is weakly Cauchy in H. We must now verify that $\{g_{n_i^i}\}_{i<\omega}$ is weakly Cauchy in G.

In order to do so, take a map $\psi \in I(G)$. Since H is separable, by Corollary 4.2.11, there is a measure space $(R, \mathcal{R}, r)$, a family $\{\psi_p\}$ of pure positive-definite functions on H, which are associated to each $p \in R$, such that

$$\psi(h) = \int_R \psi_p(h) d_r(p) \text{ for all } h \in H.$$

Therefore

$$\psi(g_{n_i^i}) = \int_R \psi_p(g_{n_i^i}) d_r(p)) \text{ for all } i < \omega.$$

Now, for each $i < \omega$, consider the map f_i on R by $f_i(p) \stackrel{\text{def}}{=} \psi_p(g_{n_i^i})$. Then f_i is integrable on R and, since $\{g_{n_i^i}\}_{i<\omega}$ is weakly Cauchy in H, it follows that $\{f_i\}$ is a pointwise Cauchy sequence on R. Furthermore, if $\psi_p(h) =< \sigma_p(h)[v_p], v_p >$ for some $\sigma_p \in Irr(H)$ and $v_p \in \mathcal{H}_{\sigma_p}$, it follows that

$$|f_i(p)| = |\psi_p(g_{n_i^i})| = | < \sigma_p(g_{n_i^i})[v_p], v_p > | \leq \|v_p\|^2.$$

Thus defining f on R as the pointwise limit of $\{f_i\}$, we are in position to apply Lebesgue's dominated convergence theorem in order to obtain that

$$\int_R f(p) d_r(p) = \lim_{i\to\infty} \int_R \psi_p(g_{n_i^i}) d_r(p) = \lim_{i\to\infty} \psi(g_{n_i^i}).$$

In other words, the sequence $\{\psi(g_{n_i^i})\}$ converges and, therefore, is Cauchy for all $\psi \in I(G)$. Hence $\{g_{n_i^i}\}$ is weakly Cauchy in G and we are done.

Suppose now that there exists an index $m_0 \in \{0, 1, 2, \ldots, \infty\}$ such that $\{\alpha_{m_0}(g_{n_i^{m_0}})\}_{i<\omega}$ contains a subsequence $\{\alpha_{m_0}(g_{n(j)})\}_{j<\omega}$ whose closure in $(\mathbb{D}^\omega)^{Irr^C_{m_0}(H)}$ is homeomorphic to $\beta\omega$. Applying Lemma 4.2.7, we know that $\overline{\{g_{n(j)}\}_{j<\omega}}^{w_C H} \cong \beta\omega$. Consequently, by Lemma 4.2.8, we obtain that $\overline{\{g_{n(j)}\}_{j<\omega}}^{wH} \cong \beta\omega$.

On the other hand, by Bichteler's [11, Lemma 3.2], we have that the irreducible representations of H are the restrictions of irreducible representations of G, which implies that the identity map $id : (H, w(G, I(G))_{|H}) \to (H, w(H, I(H)))$ is a continuous group isomorphism that can be extended canonically to a homeomorphism between their associated compactifications $\overline{id} : \overline{H}^{wG} \to wH$. By Lemma 4.2.7 again, we obtain that $\overline{\{g_{n(j)}\}_{j<\omega}}^{wG} \cong \beta\omega$. Thus $\{g_{n(j)}\}_{j<\omega}$ is an I_0 set, which completes the proof. □

Remark 4.2.14. *Theorem 4.2.13 fails if we try to extend it to every locally compact group (by removing the metrizability condition) or even to every compact group. Indeed, Fedorčuk [29] has proved that the existence of a compact space K of cardinality $\mathfrak{c}$ without convergent sequences is compatible with ZFC. If we take the Bohr compactification of the free abelian group generated by K, then every sequence contained in K does not fulfil any of the two choices established in Rosenthal's dichotomy.*

4.2.2 I_0 and Sidon sets

Hartman and Ryll-Nardzewski [48] proved that every abelian locally compact group contains an I_0 set. This result was improved in [34], where it was proved that every non-precompact subset of an abelian locally compact group contains an I_0 set. These sort of results do not hold for general locally compact groups unfortunately. Indeed, the Eberlein compactification of the group $SL_2(R)$ coincides with its one-point compactification, which means that each continuous positive-definite function on $SL_2(R)$ converges at infinity (see [18]). Therefore, for this group, only the first case of the dichotomy result in Theorem 4.2.13 holds. If we search for interpolation sets, some extra conditions have to be assumed.

In this section we explore the application of the results in the previous sections in the study of interpolation sets in locally compact groups. First, we need the following result, which was established by Ernest [27] (cf. [59]) for separable metric locally compact groups and convergent sequences and subsequently extended to locally compact groups and compact subsets by Hughes [55].

Proposition 4.2.15. *(J. Ernest, J.R. Hughes) Let* (G, τ) *be a locally compact group. Then* (G, τ) *and* G^w *contain the same compact subsets.*

In some special cases, Hughes' result implies the convergence of weakly Cauchy sequences.

Proposition 4.2.16. *Let* (G, τ) *be a locally compact group and suppose that* $\{g_n\}_{n<\omega}$ *is a Cauchy sequence in* G^w*. If* $\overline{\{g_n\}_{n<\omega}}^{\mathrm{w}G} \subseteq inv(\mathrm{w}G)$*, then* $\{g_n\}_{n<\omega}$ *is* τ*-convergent in* G*.*

Proof. Assume that $\{g_n\}_{n<\omega}$ is a Cauchy sequence in G^w. First, we verify that the sequence is a precompact subset of (G, τ).

Indeed, we have that $\{g_n\}_{n<\omega}$ converges to some element $p \in inv(\mathrm{w}G)$. If $\{g_n\}_{n<\omega}$ were not precompact in (G, τ), there would be a neighbourhood of the neutral element U and a subsequence $\{g_{n(m)}\}_{m<\omega}$ such that $g_{n(m)}^{-1} \cdot g_{n(l)} \notin U$ for each $m, l < \omega$ with $m \neq l$. On the other hand, the sequence $\{g_{n(m)}^{-1} \cdot g_{n(m+1)}\}_{m<\omega}$ converges to $p^{-1}p$, the neutral element in G^w. This takes us to a contradiction because, by Proposition 4.2.15, it follows that $\{g_{n(m)}^{-1} \cdot g_{n(m+1)}\}_{m<\omega}$ must also converge to the neutral element in (G, τ).

Therefore, the sequence $\{g_n\}_{n<\omega}$ is a precompact subset of (G, τ). This implies that $p \in G$ and we are done. □

Lemma 4.2.17. *Let* (G, τ) *be a locally compact group and let* B *be a non-precompact subset of* G *such that* $\overline{B}^{\mathrm{w}G} \subseteq inv(\mathrm{w}G)$*. Then there exist a open subgroup* H *of* G*, a compact and normal subgroup* K *of* H*, a quotient map* $p : H \to H/K$ *and a sequence* $\{g_n\}_{n<\omega} \subseteq B \cap H$ *such that* H/K *is a Polish group and* $\overline{p(\{g_n\}_{n<\omega})}^{\mathrm{w}H/K} \cong \beta\omega$*.*

Proof. Since B is non-precompact there exists an open, symmetric and relatively compact neighbourhood of the identity U in G such that B contains a U-discrete sequence $\{g_n\}_{n<\omega}$.

Consider the subgroup $H \overset{\mathrm{def}}{=} < \overline{U} \cup \{g_n\}_{n<\omega} >$, which σ-compact and open in G. By Kakutani-Kodaira's theorem, there exists a normal, compact K of H such that $K \subseteq U$ and H/K is metrizable, and consequently Polish. Let $p : H \to H/K$ be the quotient map and let $\overline{p} : \mathrm{w}H \to \mathrm{w}H/K$ denote the canonical extension to the weak compactifications. Therefore, we have that $\overline{p}(inv(\mathrm{w}H) \subseteq inv(\mathrm{w}H/K)$. Furthermore, since $\overline{H}^{\mathrm{w}G}$ is canonically homeomorphic to $\mathrm{w}H$, it follows that $\overline{\{g_n\}_{n<\omega}}^{\mathrm{w}H} \subseteq inv(\mathrm{w}H)$. Hence $\overline{\{p(g_n)\}_{n<\omega}}^{\mathrm{w}H/K} \subseteq inv(\mathrm{w}H/K)$. Thus, we are in position of applying Proposition 4.2.16.

Assume that there is a weakly Cauchy subsequence $\{p(g_s)\}_{s<\omega}$ in $(H/K)^{\mathrm{w}}$, which would be τ/K-convergent by Proposition 4.2.16. Then by a theorem of Varopoulos [95], the sequence $\{p(g_s)\}_{s<\omega}$ could be lifted to a sequence $\{x_s\}_{s<\omega} \subseteq H$ converging to some point $x_0 \in H$. This would entail that $x_s^{-1} g_s \in K$ for all $s \in \omega$. Thus the sequence $\{g_s\}_{s<\omega}$ would be contained in the compact subset $(\{x_s\}_{s<\omega} \cup \{x_0\})K$, which is a contradiction since $\{g_n\}_{n<\omega}$ was supposed to be U-discrete. This contradiction completes the proof. □

Theorem 4.2.18. *Every non-precompact subset of a locally compact group whose closure is placed in inv*$(\mathrm{w}G)$ *contains an infinite* I_0 *set.*

Proof. Apply Lemmata 4.2.7 and 4.2.17. □

Corollary 4.2.19. *Let* G *be a discrete group and let* $\{g_n\}_{n<\omega}$ *be an infinite sequence in* G*. If* $\overline{\{g_n\}_{n<\omega}}^{\mathrm{w}G} \subseteq$ *inv*$(\mathrm{w}G)$*, then* $\{g_n\}_{n<\omega}$ *contains an infinite* I_0 *set.*

Remark 4.2.20. *In case the group* G *is abelian, Corollary 4.2.19 is a variant of van Douwen's Theorem (see page 2).*

We now look at *Sidon sets*, a well known family of interpolation sets in harmonic analysis. Recall that a subset E of G is called *weak Sidon set* when every bounded function can be interpolated by a continuous function defined on the Eberlein compactification eG and it is equivalent to a *Sidon set* if G is abelian or amenable [76]. We notice that the following question still remains open (see [66] and [31, p. 57]).

Question 4.2.21. *(Figà-Talamanca, 1977) Do infinite weak Sidon exist in every infinite discrete non-abelian group?*

Since every I_0 set is automatically weak Sidon (because $\mathrm{w}G$ is a factor of eG), Theorem 4.2.18 and Corollary 4.2.19 give sufficient conditions for the existence of weak Sidon sets.

Theorem 4.2.22. *Every non-precompact subset of a locally compact group whose closure is placed in inv*$(\mathrm{w}G)$ *contains an infinite weak Sidon set.*

Let us see some results for discrete groups. First, we need the following definitions.

Definition 4.2.23. *Let G be a group. A sequence $\{x_n\}_{n<\omega} \subseteq G$ is called* ***independent*** *if for every $n_0 < \omega$ the element $x_{n_0} \notin \langle \{x_n\}_{n\in\omega\setminus\{n_0\}} \rangle$. A group G is called* ***locally finite*** *if every finite subset of the group generates a finite subgroup. The group G is* ***residually finite*** *if for every non-identity element g of G there exists a normal subgroup N of finite index in G such that $g \notin N$. Finally, the group G is called an* ***FC-group*** *if every conjugacy class of G is finite (i.e. for all $g \in G$, we have that $\mathcal{O}_g \stackrel{\text{def}}{=} \{hfh^{-1} : h \in G\}$ is finite.)*

Proposition 4.2.24. *Every independent sequence in a discrete group G is a weak Sidon set.*

Proof. Let $E = \{x_n\}_{n<\omega}$ be an independent sequence in G. It suffices to show that $\overline{E}^{eG}$ is homeomorphic to $\beta\omega$ or, equivalently, that every pair of disjoint subsets in E have disjoint closures in eG.

Indeed, for any pair A, B of arbitrary disjoint subsets of ω, set $X_A \stackrel{\text{def}}{=} \{x_n\}_{n\in A}$, and $X_B \stackrel{\text{def}}{=} \{x_n\}_{n\in B} \subseteq G \setminus \langle X_A \rangle$. Since $\langle X_A \rangle$ is an open subgroup of G, the positive-definite function h defined by $h(x) = 1$ if $x \in \langle X_A \rangle$ and $h(x) = 0$ if $x \in G \setminus \langle X_A \rangle$ is continuous (see [52, 32.43(a)]). Thus $\overline{X_A}^{eG} \cap \overline{X_B}^{eG} = \emptyset$, which completes the proof. □

Corollary 4.2.25. *Let G be a discrete group and let $\{x_n\}_{n<\omega}$ be a sequence in G. If the sequence contains either an independent subsequence or its* w*G-closure is contained in $inv(\mathrm{w}G)$, then the sequence contains an infinite weak Sidon set.*

Remark 4.2.26. *Note that if G is a discrete FC-group we can find an independent sequence within every infinite subset of G. Therefore, every infinite subset of a discrete FC-group contains an infinite weak Sidon set [70].*

We finish this subsection with an example of a sequence that is a weak Sidon set and converges to the neutral element in the Bohr topology. Recall that, given two elements g and h of a group G, the *commutator of g and h* is $[g, h] \stackrel{\text{def}}{=} g^{-1}h^{-1}gh$. We denote by G' the *commutator subgroup of G* generated by all the commutators of the group.

Proposition 4.2.27. *Let G be a discrete, residually finite, locally finite group that is not abelian by finite. Then G contains a sequence that is weak Sidon and converges in the Bohr topology.*

Proof. Since G is not abelian by finite there are $x_1, x_2 \in G$ such that $a_1 \stackrel{\text{def}}{=} [x_2, x_2] \neq e_G$. Take the finite subgroup $B_1 \stackrel{\text{def}}{=} \langle x_1, x_2 \rangle$ of G. Note that $a_1 \in B_1'$. Suppose we have defined $L = \sum_{i=1}^{n} B_i$. Since L is finite and G is residually finite, we can get $m \in \mathbb{N}$ and a representation σ of G with dimension m such that $\sigma_{|L}$ is faithful and $\sigma(G)$ is finite. Indeed, consider the finite subgroup $L \stackrel{\text{def}}{=} \{l_1, \ldots, l_r\}$, where l_1 is the identity element. Since G is residually finite, we know that for each $i \in \{2, \ldots, r\}$ there exists a normal subgroup M_i of finite index in G such that $l_i \notin M_i$. Take the normal subgroup $M \stackrel{\text{def}}{=} \bigcap_{i=2}^{r} M_i$ of finite index in G. Observe that $l_i \notin M$ for all $i \in \{2, \ldots, r\}$. Now, since G/M is a a finite group, there is a faithful representation ψ of G with dimension $m \in \mathbb{N}$ such that $\psi(G/M)$ is finite. Take the quotient map $\pi : G \to G/M$ and define the representation $\sigma \stackrel{\text{def}}{=} \psi \circ \pi$ of G. Note that σ has dimension m and $\sigma(G)$ is finite. Finally, we know that $\sigma_{|L}$ is faithful because $ker(\sigma_{|L}) = e_L$.

Set $N = ker(\sigma)$. If N were Abelian, it would follow that G is Abelian by finite, which is impossible. Thus, we may assume WLOG that $N' \neq \{1\}$ and we can replace G by N in order to obtain a finite non-abelian subgroup B_{n+1} of N and a non-trivial element $a_{n+1} \in B_{n+1}'$. Using an inductive argument, we obtain $B = \sum_{i=1}^{\infty} B_i$, a subgroup of G, such that $B_i' \neq \{1\}$, for all $i \in \mathbb{N}$. Under such circumstances, it is known that the sequence $\{a_n\}_{n=1}^{\infty}$ is convergent in the Bohr topology (cf. [50, Cor. 3.10]). On the other hand, the sequence $\{a_n\}_{n=1}^{\infty}$ is an independent set by definition and, therefore, a weak Sidon subset in G. This completes the proof. □

Remark that a positive answer to the following question also solves Question 4.2.21.

Problem 4.2.28. *Does every discrete group contain an infinite I_0 set?*

4.2.3 Property of strongly respecting compactness

We introduce the following definition, which extends the concept of strongly respecting compactness, given by Comfort, Trigos-Arrieta and Wu [21] in the setting of locally compact abelian groups, to groups that are not necessarily abelian equipped with the weak topology.

Definition 4.2.29. *We say that a locally compact group G **strongly respects compactness** if for any closed metrizable subgroup N of $inv(\mathrm{w}G)$, a subset A of G satisfies that $AN \cap G$ is compact in G if and only if AN is compact in $\mathrm{w}G$.*

Let G be a locally compact group and let H be an open subgroup of G. We discuss in the proof of Theorem 4.2.13 that the identity map $id : (H, \mathrm{w}(G, I(G))_{|H}) \to H^{\mathrm{w}}$ is a continuous isomorphism that can be extended to a continuous map $\overline{id} : \overline{H}^{\mathrm{w}G} \to \mathrm{w}H$ defined on their respective compactifications. Moreover, using that wG is factor of the Eberlein compactification eG, it follows that wG is a compact involutive semitopological semigroup for every locally compact group G. Taking this fact into account, the following lemma is easily verified.

Lemma 4.2.30. *Let G and H be locally compact groups and let $h\colon G \to H$ be a continuous homomorphism. Then there is a canonical continuous extension $\overline{h} : \mathrm{w}G \to \mathrm{w}H$ such that for every p, q in wG, we have $\overline{h}(pq) = \overline{h}(p)\overline{h}(q)$.*

The next result is a variation of [35, Lemma 3.6].

Lemma 4.2.31. *Let G be a locally compact group, H an open subgroup of G, A a subset of G, and let N be a subgroup of $inv(\mathrm{w}G)$, containing the identity, such that AN is compact in wG. If F is an arbitrary subset of $AN \cap H$, then there exists $A_0 \subseteq A$ with $|A_0| \leq |N|$ such that*

$$\overline{F}^{\mathrm{w}H} \subseteq \overline{(FF^{-1})}^{H^{\mathrm{w}}} \cdot \overline{id}(A_0 N).$$

Proof. We first verify that $\overline{F}^{\mathrm{w}H} \subseteq \overline{id}(AN \cap \overline{H}^{\mathrm{w}G})$.

Indeed, since $AN \cap H \subseteq AN \cap \overline{H}^{\mathrm{w}G}$ and $AN \cap \overline{H}^{\mathrm{w}G}$ is compact, we have $AN \cap H \subseteq \overline{id}(AN \cap \overline{H}^{\mathrm{w}G})$ and, as a consequence, it follows that $\overline{AN \cap H}^{\mathrm{w}H} \subseteq \overline{id}(AN \cap \overline{H}^{\mathrm{w}G})$. Hence $\overline{F}^{\mathrm{w}H} \subseteq \overline{id}(AN \cap \overline{H}^{\mathrm{w}G})$ and $\overline{F}^{\mathrm{w}H}$ is compact.

For any $x \in N$ such that $\overline{F}^{\mathrm{w}H} \cap \overline{id}(Ax) \neq \emptyset$, pick $a_x \in A$ with $\overline{id}(a_x x) \in \overline{F}^{\mathrm{w}H}$. We define $A_0 \stackrel{\text{def}}{=} \{a_x \in A : x \in N \text{ and } \overline{id}(a_x x) \in \overline{F}^{\mathrm{w}H}\}$. We have $A_0 \subseteq A$ and $|A_0| \leq |N|$.

Pick an arbitrary point $b \in \overline{F}^{\mathrm{w}H}$. Since $\overline{F}^{\mathrm{w}H} \subseteq \overline{id}(AN \cap \overline{H}^{\mathrm{w}G})$ we can find $a \in A$ and $y \in N$ such that $b = \overline{id}(ay)$. Set $b' = \overline{id}(a_y y) \in \overline{F}^{\mathrm{w}H}$. Then $bb'^{-1} = \overline{id}(ay)\overline{id}(a_y y)^{-1} \in \overline{F}^{\mathrm{w}H}\overline{F^{-1}}^{\mathrm{w}H} = \overline{FF^{-1}}^{\mathrm{w}H}$. Observe also that, by Lemma 4.2.30, we have $bb'^{-1} = \overline{id}(ayy^{-1}a_y^{-1}) = \overline{id}(aa_y^{-1}) = aa_y^{-1} \in \mathrm{w}H \cap G = H$. Therefore $bb'^{-1} \in \overline{FF^{-1}}^{\mathrm{w}H} \cap H$. Since H is an open subgroup, by [55, Cor. 14.2], we deduce that $bb'^{-1} \in \overline{FF^{-1}}^{\mathrm{w}H} \cap H = \overline{FF^{-1}}^{H^{\mathrm{w}}}$. Thus $b = bb'^{-1}b' \in \overline{FF^{-1}}^{H^{\mathrm{w}}} \cdot \overline{id}(A_0 N)$ and we are done. □

Theorem 4.2.32. *Every locally compact group G strongly respects compactness.*

Proof. Let N be a metrizable subgroup of $inv(\mathrm{w}G)$ and let A a subset of G such that AN is compact in $\mathrm{w}G$. Since $AN \cap G$ is closed in G it suffices to see that it is precompact. By reduction to absurd, assume that $AN \cap G$ is non-precompact. By Theorem 4.2.17 there exist an open subgroup H of G, a compact and normal subgroup K of H, a quotient map $p : H \to H/K$ and a sequence $F \subseteq AN \cap H$ such that H/K is a Polish group and $\overline{p(F)}^{\mathrm{w}H/K} \cong \beta\omega$. Thus, $|\overline{p(F)}^{\mathrm{w}H/K}| \geq 2^{\mathfrak{c}}$.

By Lemma 4.2.31 there is $A_0 \subseteq A$ with $|A_0| \leq |N|$ such that $\overline{F}^{\mathrm{w}H} \subseteq \overline{(FF^{-1})}^{H^{\mathrm{w}}} \overline{id}(A_0N)$. Since $|\overline{id}(A_0N)| \leq \mathfrak{c}$ we can enumerate it as $\{a_\alpha\}_{\alpha<\mathfrak{c}}$. Therefore, we can write $\overline{F}^{\mathrm{w}H} \subseteq \bigcup_{\alpha<\mathfrak{c}} \overline{FF^{-1}}^{H^{\mathrm{w}}} a_\alpha$.

Let $\overline{p} : \mathrm{w}H \to \mathrm{w}H/K$ be the canonical extension of p to the respective compactifications of H and H/K. Using Lemma 4.2.30, for each $z \in \mathrm{w}H$ consider the map T_z defined on $\mathrm{w}H/K$ by $T_z(\overline{p}(x)) = \overline{p}(xz) = \overline{p}(x)\overline{p}(z)$ for all $x \in \mathrm{w}H$. Hence, from the previous inclusion we obtain that $\overline{p(F)}^{\mathrm{w}H/K} = \overline{p}(\overline{F}^{\mathrm{w}H}) \subseteq \bigcup_{\alpha<\mathfrak{c}} T_{a_\alpha}(\overline{p(FF^{-1})}^{H^{\mathrm{w}}/K})$. Since the topology of H^{w}/K is finer than that of $(H/K)^{\mathrm{w}}$ we have that $\overline{p(FF^{-1})}^{H^{\mathrm{w}}/K} \subseteq \overline{p(FF^{-1})}^{(H/K)^{\mathrm{w}}}$. Furthermore $|\overline{p(FF^{-1})}^{(H/K)^{\mathrm{w}}}| \leq \mathfrak{c}$ because H/K is a Polish space. Therefore, $|\overline{p(F)}^{\mathrm{w}H/K}| \leq \mathfrak{c}$. This is a contradiction that completes the proof. □

Chapter 5

Future research lines: Interpolation sets in the dual set of a topological group

5.1 Main definitions and basic results

In this setting, we use the characterisation of Sidon set given in Theorem 1.4.15 as its definition. Moreover, we present the notion of I_0 set in the dual set as a particular case of Sidon set, as it was introduced by Hare and Ramsey [47] in 2003.

Definition 5.1.1. *A subset $E \subseteq \widehat{G}$ is called* ***Sidon set*** *if whenever element $\{A_\sigma\}_{\sigma\in E} \in \ell^\infty(E)$, there is a measure μ on G satisfying $\widehat{\mu}(\sigma) = A_\sigma$ for all $\sigma \in E$. If, in addition, μ can be chosen to be discrete (or discontinuous), then E is said to be an* ***I_0 set****.*

Proposition 5.1.2. *([47, Prop. 2.2.]) Any finite set in $\widehat{G}$ is an I_0 set.*

I_0 sets have been investigated by several authors. First, let us see a characterization of I_0 set.

Theorem 5.1.3 *([47, Prop. 2.1.]).* *A subset E of $\widehat{G}$ is an I_0 set if and only if there exists $0 < \epsilon < 1$ (equivalently, for every $0 < \epsilon < 1$) such that for each $\{A_\sigma\}_{\sigma\in E} \in \ell^\infty(E)$ with $\|\{A_\sigma\}_{\sigma\in E}\|_\infty \leq 1$, there is a discrete measure μ such that $\|\widehat{\mu}(\sigma) - A_\sigma\| < \epsilon$ for all $\sigma \in E$.*

If we look at the elements in $\ell^\infty(E)$ as bounded functions defined on E, then we have that E is a Sidon set (resp. I_0 set) when for every bounded function $f : E \longrightarrow \bigcup\limits_{\sigma\in E} \mathbb{C}^{d_\sigma^2}$ there is a measure (resp. discrete measure) μ

such that $\widehat{\mu}(\sigma) = f(\sigma)$ for all $\sigma \in E$.

Fixed $n < \omega$, we denote by $[\widehat{G}]_n$ the set of continuous irreducible unitary representations of G of dimension n. Selecting, as usual, a convenient representative from each class, we may assume that $[\widehat{G}]_n \subseteq CHom(G, \mathbb{U}(n))$.

Bearing in mind Proposition 3.3.3, the following proposition is immediate.

Proposition 5.1.4. *Let G be a compact group and let E be an infinite subset of $[\widehat{G}]_n$, for some $n < \omega$. Then, there exists a countably infinite subset L of E that is an I_0^n set. Consequently, if G is a non-tall compact group, then $\widehat{G}$ contains a countably infinite I_0^n set, for some $n < \omega$.*

Proof. It is well known that the dual space of a compact group is discrete, which means that no infinite subset of inequivalent representations of the same dimension can be equicontinuous. □

The next section explores how to relate the notion of I_0^n set with the classical notion of I_0 set mentioned above.

5.2 I_0 sets in the dual of compact non-abelian groups

As we have said in the introduction, we think that every non-tall compact group contains an I_0 set. Since we consider that we can ameliorate our current work and demonstrate the conjecture, we present the results that we have obtained here but without the proof.

The main result of this section is a characterisation of the existence of I_0 sets in the dual of non-tall compact groups.

Theorem 5.2.1. *Let G be a compact group and let E be an infinite subset of $[\widehat{G}]_n$. The following conditions are equivalent:*

(a) E is an I_0 set.

(b) For every (constant) $n \times n$ matrix A, there is a discrete measure μ such that $\widehat{\mu}(\sigma) = A$ for all $\sigma \in E$.

Let us see some applications of this theorem. Now, the following known result is immediate.

Corollary 5.2.2. *Let G be a compact abelian group and let E be an infinite subset of $\widehat{G}$. Then E contains an infinite I_0 set.*

Corollary 5.2.3. *Let $G = \prod_{\alpha\in I} G_\alpha$ be a non-tall compact group formed by an infinite product of compact groups. Let $n < \omega$ be the minimum non-negative integer such that $|[\widehat{G}]_n| \geq \aleph_0$. Suppose that each G_α satisfies that $|[\widehat{G_\alpha}]_m| < \aleph_0$, for every $m \leq n$. Then, every infinite subset E of $[\widehat{G}]_n$ contains an infinite I_0 set.*

Proposition 5.2.4. *Let $G = A_1 \times A_2$ be a non-tall compact group and let $n < \omega$ be the minimum non-negative integer such that $|[\widehat{G}]_n| \geq \aleph_0$. Suppose that each A_i $(i = 1, 2)$ is a compact group such that if $|[\widehat{A_i}]_{m_i}| \geq \aleph_0$, for some $m_i \leq n$, it follows that every infinite subset $E_i \subseteq [\widehat{A_i}]_{m_i}$ contains an I_0 set. Then, every infinite subset E of $[\widehat{G}]_n$ contains an infinite I_0 set.*

Lemma 5.2.5. *Let G be a simple, simply connected compact Lie group. Then G is a tall group.*

Tacking into account the previous characterisation of I_0 set, we obtain a different approach to prove the result of Hare and Ramsey [47, Theorem 4.10].

Corollary 5.2.6. *Let G be a non-tall compact connected group. Then, every infinite subset E of $\widehat{G}$ contains an infinite I_0 set.*

Proof. Since G is non-tall there is an infinite subset E' of E such that $E' \subseteq [\widehat{G}]_n$, for some non-negative integer n. By the structure theorem for compact connected groups (Theorem 1.3.5), we know that G is isomorphic to a quotient of $K \stackrel{\text{def}}{=} \prod_{i\in I} G_i \times A$, where each G_i is a compact simply-connected Lie group and A is a compact abelian group. Let $\varphi : K \to G$ be the quotient map. We claim that if $\tilde{E} \stackrel{\text{def}}{=} \{\sigma \circ \varphi : \sigma \in E'\}$ contains an I_0 set, then E' contains an I_0 set. Indeed, given a constant $n \times n$ matrix A, we know that there is a discrete measure $\mu_1 \stackrel{\text{def}}{=} \sum_{i<\omega} a_i \delta_{x_i} \in M_d(K)$ such that $\widehat{\mu_1}(\tau) = A$ for every $\tau \in \tilde{E}$. Take $\mu_2 \stackrel{\text{def}}{=} \sum_{i<\omega} a_i \delta_{y_i} \in M_d(G)$, where $y_i \stackrel{\text{def}}{=} \varphi(x_i)$ for each $i < \omega$. Then, it follows that $\widehat{\mu_2}(\sigma) = A$ for every $\sigma \in E'$.

By Lemma 5.2.5, we know that each G_i is tall. Taking $A_1 \stackrel{\text{def}}{=} \prod_{i\in I} G_i$ and $A_2 \stackrel{\text{def}}{=} A$, the hypothesis of Corollary 5.2.4 are satisfied thanks to Theorem 5.2.3 and Corollary 5.2.2. Therefore, there exists an I_0 set in $\tilde{E}$, and hence E contains an infinite I_0 set. □

Corollary 5.2.7. *Let* $G \overset{\text{def}}{=} \prod_{n<\omega} F$ *a countably infinite cartesian product of infinitely many copies of a finite group* F*. Then* $\widehat{G}$ *contains an infinite* I_0 *set.*

5.3 Central I_0 sets in the dual of compact non-abelian groups

Central interpolation sets are related with the concept of central measures, which are characterised by the property that their Fourier transforms are scalar multiples of identity matrices. Recall that a measure is *central* if it commutes with all other measures on the group under convolution. We do not present all the proofs in this section.

Continuing with the idea of the definition of Sidon sets, Parker [75] introduces the notion of central Sidon set in 1972: a subset E of $\widehat{G}$ is said ***central Sidon set*** if whenever $\{a_\sigma I_{d_\sigma}\}_{\sigma\in E} \in \ell^\infty(E)$ there is a central measure μ on G satisfying $\widehat{\mu}(\sigma) = a_\sigma I_{d_\sigma}$ for all $\sigma \in E$. Hence, every Sidon set is a central Sidon set.

Now, one could analogously define central I_0 sets, but in that case, since central discrete measures must be supported on the centre of G [80], there would be groups for which not even all finite sets would be central I_0 sets (see [44, 45]). As a consequence, there would be I_0 sets that are not central I_0 sets. In order to avoid this issue, in 2004 Grow and Hare [45] change the definition replacing central discrete measure by a linear combination of orbital measures.

Definition 5.3.1. *A subset* $E \subseteq \widehat{G}$ *is said* ***central*** $\boldsymbol{I_0}$ ***set*** *if any* $\{a_\sigma I_{d_\sigma}\}_{\sigma\in E} \in \ell^\infty(E)$ *can be interpolated by the Fourier transform of a linear combination of orbital measures.*

The orbital measures are the central, probability measures, μ_x, supported in the conjugacy class containing $x \in G$ and defined by

$$\int_G f d\mu_x = \int_G f(gxg^{-1})dm_G(g)$$

for all continuous functions on G. It is know that any I_0 set is a central I_0 set and all finite subsets of $\widehat{G}$ are central I_0 sets[45]. Moreover, any central I_0 set is a central Sidon set.

There is an equivalent characterisation of central I_0 set in the same spirit as the Kalton's characterisation for I_0 sets (see [63], [81] and Theorem 5.1.3) that we can find in [46].

Proposition 5.3.2. *Let G be compact group. A subset E of $\widehat{G}$ is a central I_0 set if and only if for some $0 < \epsilon < 1$ (equivalently, for every $0 < \epsilon < 1$) there is a constant $C > 0$ so that for all choices of $\{a_\sigma\}_{\sigma \in E}$, $a_\sigma = \pm 1$, there is a finite measure $\mu = \sum_{i=1}^{m} \lambda_i \mu_{g_i}$ such that $\|\mu\| \leq C$ and $\|\widehat{\mu}(\sigma) - a_\sigma I_{d_\sigma}\|_{op} \leq \epsilon$ for all $\sigma \in E$.*

Recall that, given $\sigma \in \widehat{G}$, we denote by χ_σ the character associated to σ and by χ_σ^N the normalised character associated to σ; that is, the character divided by the degree of the representation d_σ.

Remark 5.3.3. *Since the Fourier transform of any orbital measure satisfies:*

$$\widehat{\mu_g}(\sigma) = \chi_\sigma^N(g) I_{d_\sigma}$$

for every $\sigma \in \widehat{G}$, we can say that $E \subseteq \widehat{G}$ is a central I_0 set *if for every $\{a_\sigma I_{d_\sigma}\}_{\sigma \in E} \in \ell^\infty(E)$ there exists $\{\lambda_i\}_{i<\omega} \subseteq \mathbb{C}$ and $\{g_i\}_{i<\omega} \subseteq G$ such that $\sum_{i<\omega} \lambda_i \chi_\sigma^N(g_i) = a_\sigma$ for every $\sigma \in E$.*

Using some results from Section 2.2, it is possible to relate the notion of infinite central I_0 set with the property of containing a sequence equivalent to the unit basis ℓ_1. Again, we don't include any proof in this section.

Theorem 5.3.4. *Let G be a compact group and let E be a countably infinite subset of $\widehat{G}$. The following conditions are equivalent:*

(a) There is a countably infinite subset L of E that is a central I_0 set.

(b) There is a countably infinite subset L of E such that $\{\chi_\sigma^N\}_{\sigma \in L}$ contains a subsequence equivalent to the ℓ_1 basis.

Corollary 5.3.5. *Let G be a compact group and let E be a countably infinite subset of $\widehat{G}$. The following conditions are equivalent:*

(a) There is a countably infinite subset L of E that is a central I_0 set.

(b) There is a countably infinite subset L of E such that $\{\chi_\sigma^N\}_{\sigma \in L}$ contains a subsequence equivalent to the ℓ_1 basis.

(c) There is a countably infinite subset L of E such that $tg(\overline{\{\chi_\sigma^N\}_{\sigma \in L}}^{\mathbb{C}^G}) > \omega$.

(d) There is a countably infinite subset L of E such that $\overline{\{\chi_\sigma^N\}_{\sigma \in L}}^{\mathbb{C}^G}$ is not a Rosenthal compactum.

(e) There is a countably infinite subset L of E such that $|\overline{\{\chi_\sigma^N\}_{\sigma \in L}}^{\mathbb{C}^G}| > \mathfrak{c}$.

(f) There is a countably infinite subset L of E such that every two disjoint subsets of $\{\chi^N_\sigma\}_{\sigma\in L}$ have disjoint closures in $\mathbb{C}^G$.

(g) There is a countably infinite subset L of E such that $\overline{\{\chi^N_\sigma\}_{\sigma\in L}}^{\mathbb{C}^G}$ is canonically homeomorphic to $\beta\omega$.

Corollary 5.3.6. *Let G be a compact group and let E be an infinite subset of $[\widehat{G}]_n$. Then, E contains an infinite central I_0 set. Consequently, if G is a non-tall compact group, then $\widehat{G}$ contains an infinite central I_0 set.*

Corollary 5.3.7. *Let G be a compact group and let E be a countably infinite subset of $\widehat{G}$. Then, either E contains a countably infinite subset L such that $\{\chi^N_\sigma\}_{\sigma\in L}$ is a pointwise Cauchy sequence, or E contains a countably infinite central I_0 set.*

Remark 5.3.8. *In [80] Ragozin proves that, given a compact simple Lie group G, μ is a continuous central measure measure in G if and only if $\{a_\sigma\}_{\sigma\in\widehat{G}} \in c_0(\widehat{G})$, where the element a_σ is the scalar associated to the Fourier transform $\widehat{\mu}(\sigma)(= a_\sigma I_\sigma)$, for each $\sigma \in \widehat{G}$. If we consider an arbitrary sequence $\{\chi^N_{\sigma_n}\}_{n\in\omega}$ of normalised characters, we claim that it is a pointwise convergent sequence. Indeed, take a point $g \in G$. Since the orbital measure μ_g is continuous and central, we know that $\widehat{\mu_g}(\sigma_n) = \chi^N_{\sigma_n}(g)I_{d_\sigma}$ converges as $n \to \infty$. Thus, $\chi^N_{\sigma_n}(g)$ converges as $n \to \infty$. By Corollary 5.3.7 we can conclude that there are not infinite central I_0 sets in $\widehat{G}$ if G is a compact simple Lie group.*

Corollary 5.3.9. *Let $G = \prod_{\alpha\in I} G_\alpha$, where each G_α is a compact group, and I is an infinite index set. Then, $\widehat{G}$ contains an infinite central I_0 set.*

By Corollary 5.3.6 we know that every non-tall compact group contains an infinite central I_0 set. However, by Ragozin's result mentioned above, some tall compact groups neither contain an infinite Sidon nor an infinite I_0 set. Using Corollary 5.3.9, it is possible to establish the existence of central I_0 sets for some classes of tall groups.

Example 5.3.10. *Consider the following tall groups:*

1) $G_1 = \prod_{n=6}^{\infty} A_n$, where A_n is the alternating group on n letters for each n.

2) $G_2 = \prod_{n=1}^{\infty} SL(2, p^n)$, *where* $SL(2, p^n)$ *denotes the special linear group of order* 2 *with coefficients in the Galois field* $GF[p^n]$.

3) $G_3 = \prod_{n=2}^{\infty} SU(n)$, *where* $SU(n)$ *denotes the special unitary group.*

4) $G_4 = \prod_{n=2}^{\infty} SO(n)$, *where* $SO(n)$ *denotes the special orthogonal group.*

Observe that G_1 *and* G_2 *are profinite and* G_3 *and* G_4 *are connected. In [57, 58] Hutchinson has demonstrated that* $\widehat{G_3}$ *and* $\widehat{G_4}$ *contain an infinite Sidon set and that* $\widehat{G_1}$ *and* $\widehat{G_2}$ *doesn't. He also proves that all of them contain an infinite central Sidon set. In view of Corollary 5.3.9, we demonstrate that the dual of all these groups also admit an infinite central* I_0 *set.*

Note that the group G_2 from the previous example has an infinite center. The following Theorem assures us that this property implies the existence of an infinite central I_0 set.

Theorem 5.3.11. *Let* G *be a compact group with an infinite center. Then* $\widehat{G}$ *contains an infinite central* I_0 *set.*

As a consequence, if G is a connected compact group, we get a generalization of a result from Rider [84, Th. 9.], which states that $\widehat{G}$ contains an infinite central Sidon set if and only if G is not a semi-simple Lie group.

Corollary 5.3.12. *Let* G *be a connected compact group. Then* $\widehat{G}$ *has an infinite central* I_0 *set if and only if* G *is not a semi-simple Lie group.*

Remark 5.3.13. (i) *The converse implication of Theorem* 5.3.11 *is not true in general because* G_1 *is a compact group with a finite center that contains an infinite central* I_0 *set.* (ii) *By the previous Corollary we obtain the same conclusion that which we have reached in Remark 5.3.8.*

Index

Bibliography

[1] C. A. Akemann and M. E. Walter. Non abelian Pontriagin duality. *Duke Mathematical Journal*, 39(3):451–463, Sept. 1972.

[2] E. Akin, J. Auslander, and K. Berg. Almost equicontinuity and the enveloping semigroup. *Contemp. Math.*, 215:75–81, 1998.

[3] E. Alfsen and P. Holm. A note on compact representations and almost periodicity in topological groups. *Mathematica Scandinavica*, 1962.

[4] H. Anzai and S. Kakutani. Bohr Compactifications of a Locally Compact Abelian Group I. *Proceedings of the Imperial Academy*, 1943.

[5] A. Arkhangel'skii. Problems in Cp-theory. *Open Problems in Topology, North-Holland*, 1990.

[6] L. Auß enhofer. On the Glicksberg theorem for locally quasi-convex Schwartz groups. *Fundamenta Mathematicae*, 201(2):163–177, 2008.

[7] W. Banaszczyk. *Additive subgroups of topological vector spaces.* Springer-Verlag, Berlin, 1991.

[8] W. Banaszczyk and E. Martín-Peinador. The Glicksberg Theorem on Weakly Compact Sets for Nuclear Groups. *Annals of the New York Academy of Sciences*, 788(1):34–39, May 1996.

[9] A. Bareche and A. Bouziad. Some results on separate and joint continuity. *Topology and its Applications*, 157(2):327–335, Feb. 2010.

[10] B. Bekka, P. de la Harpe, and A. Valette. *Kazhdan's Property (T).* 2008.

[11] K. Bichteler. Locally compact topologies on a group and the corresponding continuous irreducible representations. *Pacific Journal of Mathematics*, 31:583–593, 1969.

[12] J. Bourgain. Compact sets of first baire class. *Bull. Soc. Math. Belg.*, 29(2):135–143, 1977.

[13] J. Bourgain, D. Fremlin, and M. Talagrand. Pointwise compact sets of Baire-measurable functions. *American Journal of Mathematics*, 100(4):845–886, 1978.

[14] L. Brown. Topologically complete groups. *Proceedings of the American Mathematical Society*, 1972.

[15] B. Cascales, I. Namioka, and G. Vera. The Lindelöf property and fragmentability. *Proc. Amer. Math. Soc.*, 128(11):3301–3309, 2000.

[16] Y. H. Cheng. Subalgebras generated by extreme points in Fourier Stieltjes algebras of locally compact groups. *Studia Mathematica*, 202(3):289–302, 2011.

[17] Y. H. Cheng, B. E. Forrest, and N. Spronk. On the subalgebra of a Fourier Stieltjes algebra generated by pure positive definite functions. *Monatshefte für Mathematik*, 171(3-4):305–314, Sept. 2013.

[18] C. Chou. Minimally weakly almost periodic groups. *Journal of Functional Analysis*, 36:1–17, 1980.

[19] J. Christensen. Joint Continuity of separately continuous functions. *Proceedings of the American Mathematical Society*, 82(3):455–461, 1981.

[20] H. Chu. Compactification and Duality of Topological Groups. *Transactions of the American Mathematical Society*, 123(2):310, June 1966.

[21] W. Comfort, F. Trigos-Arrieta, and T.-S. Wu. The Bohr compactification , modulo a metrizable subgroup. *Fund. Math.*, 143:119–136, 1993.

[22] H. Corson and I. Glicksberg. Compactness in Hom (G, H). *Canad. J. Math.*, 22:164–170, 1970.

[23] J. Dixmier. *C* Algèbres et leurs representations.* 1969.

[24] L. E. Dor and H. P. Rosenthal. On sequences spanning a complex l_1 space. *Proceedings of the American Mathematical Society*, 47(2):515–515, Feb. 1975.

[25] C. F. Dunkl and D. E. Ramirez. *Topics in harmonic analysis.* Appleton-Century- Crofts [Meredith Corporation], New York, 1971.

[26] R. Engelking. *General topology.* Heldermann Verlag, 1989.

[27] J. Ernest. A strong duality theorem for separable locally compact groups. *Transactions of the American Mathematical Society*, 156:287–307, 1971.

[28] P. Eymard. L'algèbre de Fourier d'un groupe localement compact. *Bulletin de la Société Mathématique de France*, 92:181–236, 1964.

[29] V. Fedorčuk. A compact space having the cardinality of the continuum with no convergent sequences. *Math. Proc. Cambridge Philos. Soc.*, 81(2):177–181, 1977.

[30] M. Ferrer and S. Hernández. Dual topologies on non-abelian groups. *Topology and its Applications*, 159(9):2367–2377, 2012.

[31] A. Figà-Talamanca. Insiemi lacunari nei gruppi non commutativi. *Milan Journal of Mathematics*, 47:45–59, 1977.

[32] G. Folland. *A course in abstract harmonic analysis.* 1995.

[33] W. French, J. Luukkainen, and J. Price. The Tannaka-KreÄŋn duality principle. *Advances in Mathematics*, 43:230–249, 1982.

[34] J. Galindo and S. Hernández. The concept of boundedness and the Bohr compactification of a MAP abelian group. *Fundamenta Mathematicae*, 159(3):195–218, 1999.

[35] J. Galindo and S. Hernández. Interpolation sets and the Bohr topology of locally compact groups. *Advances in Mathematics*, 188:51–68, 2004.

[36] P. Gallagher. Zeros of group characters. *Mathematische Zeitschrift*, 87:363–364, 1965.

[37] L. Gillman and M. Jerison. *Rings of continuous functions.* Literary Licensing, 1960.

[38] E. Glasner and M. Megrelishvili. Linear representations of hereditarily non-sensitive dynamical systems. *arXiv preprint math.DS/0406192*, 2004.

[39] E. Glasner and M. Megrelishvili. Hereditarily non-sensitive dynamical systems and linear representations. *Colloq. Math.*, 104(2):223–283, 2006.

[40] E. Glasner and B. Weiss. Locally equicontinuous dynamical systems. *Colloq. Math*, 84/85(2):345–361, 2000.

[41] E. L. I. Glasner, M. Megrelishvili, and V. V. Uspenskij. On metrizable enveloping semigroups. *Israel Journal of Mathematics*, 164(1):317–332, 2006.

[42] I. Glicksberg. Uniform boundedness for groups. *Canad. J. Math.*, 14:269–276, 1962.

[43] H. Glöckner, R. Gramlich, and T. Hartnick. Final group topologies, Kac-Moody groups and Pontryagin duality. *Israel Journal of Mathematics*, 177(1):49–101, 2010.

[44] C. Graham and K. Hare. *Interpolation and Sidon sets for compact groups*. Springer US, Boston, MA, 2013.

[45] D. Grow and K. Hare. The independence of characters on non-abelian groups. *Proceedings of the American Mathematical Society*, 132(12):3641–3652, 2004.

[46] D. Grow and K. Hare. Central interpolation sets for compact groups and hypergroups. *Glasgow Mathematical Journal*, 51:593–603, 2009.

[47] K. Hare and L. T. Ramsey. I_0 sets in non-abelian groups. *Mathematical Proceedings of the Cambridge Philosophical Society*, 135(1):81–98, July 2003.

[48] S. Hartman and C. Ryll-Nardzewski. Almost periodic extensions of functions. *Colloquium Mathematicae*, 12(1):29–39, 1964.

[49] S. Hernández. Pontryagin duality for topological abelian groups. *Mathematische Zeitschrift*, 238(3):493–503, 2001.

[50] S. Hernández and T.-S. Wu. Some New Results on the Chu Duality of Discrete Groups. *Monatshefte für Mathematik*, 149(3):215–232, Nov. 2006.

[51] S. Hernández-Muñoz. Approximation and extension of continuous functions. *Journal of the Australian Mathematical Society*, 57(02):149, Oct. 1994.

[52] E. Hewitt and K. A. Ross. *Abstract harmonic analysis. Volume II, Structure and Analysis for Compact Groups Analysis on Locally Compact Abelian Groups*. Springer, 1970.

[53] K. Hofmann and S. Morris. The Structure of Compact Groups: A Primer for Students-A Handbook for the Expert. 2006.

[54] K. H. Hofmann. *Introduction to Topological Groups*. 2005.

[55] J. Hughes. Weak compactness and topological groups. *Thesis (Ph.D.)âĂŞThe University of North Carolina at Chapel Hill*, 1972.

[56] R. Hughes. Compactness in locally compact groups. *Bulletin of the American Mathematical Society*, 79:122–123, 1973.

[57] M. Hutchinson. Non-tall compact groups admit infinite Sidon sets. *Journal of the Australian Mathematical*, 23(4):467–475, 1977.

[58] M. Hutchinson. Local Λ sets for profinite groups. *Pacific Journal of Mathematics*, 89(1):81–88, 1980.

[59] K. Ikeshoji and Y. Nakagami. On a strong duality for separable locally compact groups. *Mem. Fac. Sci. Kyushu Univ. Ser. A*, 33(2):377–389, 1979.

[60] J. Jayne and C. Rogers. Borel selectors for upper semi-continuous set-valued maps. *Acta Math.*, 155(1-2):41–79, 1985.

[61] J. Kahane. Ensembles de Ryll-Nardzewski et ensembles de Helson. *Colloquium Mathematicae*, 15:87–92, 1966.

[62] J.-P. Kahane. Sur les fonctions moyenne-périodiques bornées. *Annales de lâĂŹinstitut Fourier*, 7:293–314, 1957.

[63] N. Kalton. On vector-valued inequalities for Sidon sets and sets of interpolation. *Colloq. Math.*, 64(2):233–244, 1993.

[64] J. Kelley. *General topology.* 1991.

[65] P. Kenderov and W. Moors. Separate continuity, joint continuity and the Lindelöf property. *Proceedings of the American Mathematical Society*, 2006.

[66] J. M. López and K. A. Ross. *Sidon sets.* M. Dekker, 1975.

[67] F. Mautner. Unitary representations of locally compact groups II. *Annals of Mathematics (2)*, 52:528–556, 1950.

[68] M. Mayer. Strongly mixing groups. *Semigroup Forum*, 54(3):303–316, 1997.

[69] J. McMullen and J. Price. Rudin-Shapiro sequences for arbitrary compact groups. *Journal of the Australian*, 22(4):421–430, 1976.

[70] L. D. Michele and P. M. Soardi. Existence of Sidon Sets in Discrete FC-Groups. *Proceedings of the American Mathematical Society*, 55(2):457, Mar. 1976.

[71] J. R. Munkres. *Topology.* Prentice Hall, Incorporated, 2000.

[72] I. Namioka. Separate continuity and joint continuity. *Pacific J. Math.*, 51(2):515–531, 1974.

[73] I. Namioka. Radon Nikodym compact spaces and fragmentability. *Mathematika*, 34(2):258–281, 1987.

[74] J. V. Neumann. On rings of operators. Reduction theory. *Annals of Mathematics (2)*, 50:401–485, 1949.

[75] W. A. Parker. Central Sidon and Central Ap sets. *J. Austral. Math. Soc.*, 14:62–74, 1972.

[76] M. A. Picardello. Lacunary sets in discrete noncommutative groups. *Boll. Un. Mat. Ital.*, 8(4):494–508, 1973.

[77] Z. Piotrowski. Separate and joint continuity. *Real Anal. Exchange*, 1985.

[78] R. Pol. On pointwise and weak topology in function spaces. 1984.

[79] J. Price. *Lie groups and compact groups.* 1977.

[80] D. Ragozin. Central measures on compact simple Lie groups. *Journal of Functional Analysis*, 10:212–229, 1972.

[81] L. Ramsey. Comparisons of Sidon and I0 sets. *Colloq. Math*, 70(1):103–132, 1996.

[82] D. Remus and F. Trigos-Arrieta. Abelian groups which satisfy Pontryagin duality need not respect compactness. *Proceedings of the American Mathematical*, 117(4):1195–1200, 1993.

[83] N. Ribarska. Internal characterization of fragmentable spaces. *Mathematika*, 34(2):243–257, 1987.

[84] D. Rider. Central lacunary sets. *Monatshefte für Mathematik*, 76(4):328–338, Aug. 1972.

[85] H. Rosenthal. A characterization of Banach spaces containing l1. *Proc. Nat. Acad. Sci. USA*, 71(6):2411–2413, 1974.

[86] W. Rudin. *Fourier analysis on groups.* 1962.

[87] I. Segal. Decompositions of operator algebras, I and II. 1951.

[88] M. Talagrand. Deux généralisations d'un théoreme de I. Namioka. *Pacific J. Math*, 81(1):239–251, 1979.

[89] M. Talagrand. Pettis integral and measure theory. 1984.

[90] L. Tárrega. El teorema de Namioka y algunas generalizaciones. *Master Thesis*, June, 2014.

[91] S. Todorcevic. Topics in topology. 1997.

[92] S. Todorcevic. Compact subsets of the first Baire class. *Journal of the American Mathematical Society*, 12(4):1179–1212, 1999.

[93] J. Troallic. Sequential criteria for equicontinuity and uniformities on topological groups. *Topology and its Applications*, 68(1):83–95, 1996.

[94] E. van Douwen. The maximal totally bounded group topology on G and the biggest minimal G-space, for abelian groups G. *Topology and its Applications*, 34(1):64–91, 1990.

[95] N. Varopoulos. A theorem on the continuity of homomorphisms of locally compact groups. *Proc. Cambridge Philos. Soc.*, 60:449–463, 1964.

[96] J. von Neumann and E. P. Wigner. Minimally Almost Periodic Groups. *Annals of Math. (Series 2)*, 41:746–750, 1940.

[97] A. Weil. L'intégration dans les groupes topologiques et ses applications. 1965.

[98] S. Willard. General topology, 1970. *Addison-Wesley, Reading, MA*.