On the diagonals of a Rees algebra

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Chapter 1

Multigraded rings

In this chapter we collect some basic definitions and facts of the theory of multigraded rings which we will need in the next chapters. We also state the multigraded versions of some well-known results in the category of graded rings. Rings are always assumed to be noetherian.

1.1 Multigraded rings and modules

The general theory of multigraded rings and modules is analogous to that of graded rings and modules. We first recall some basic definitions. The main sources are [BH1], [HHR] and [GW1].

We use the following multi-index notation. For $\mathbf{n} = (n^1, \ldots, n^r) \in \mathbb{Z}^r$, we set $|\mathbf{n}| = n^1 + \ldots + n^r$, and for $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$, we define their sum $\mathbf{n} + \mathbf{m} = (n^1 + m^1, \ldots, n^r + m^r)$, and we set $\mathbf{n} < \mathbf{m}$ ($\mathbf{n} \leq \mathbf{m}$) if $n^i < m^i$ ($n^i \leq m^i$) for every i.

A $\mathbb{Z}^r$-graded ring (or r-graded ring) is a ring S endowed with a direct sum decomposition $S = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} S_{\mathbf{n}}$, such that $S_{\mathbf{n}} S_{\mathbf{m}} \subset S_{\mathbf{n}+\mathbf{m}}$ for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$. An r-graded S-module is an S-module M endowed with a decomposition $M = \bigoplus_{\mathbf{n} \in \mathbb{Z}^r} M_{\mathbf{n}}$, such that $S_{\mathbf{n}} M_{\mathbf{m}} \subset M_{\mathbf{n}+\mathbf{m}}$ for all $\mathbf{n}, \mathbf{m} \in \mathbb{Z}^r$. We shall call $M_{\mathbf{n}}$ the homogeneous component of M of degree $\mathbf{n}$. An element $x \in M$ is homogeneous of degree $\mathbf{n}$ if $x \in M_{\mathbf{n}}$. The degree of x is then denoted by $\deg x$. For any r-graded S-module M, we define the support of M to be the set $\operatorname{supp} M = \{\mathbf{n} \in \mathbb{Z}^r \mid M_{\mathbf{n}} \neq 0\}$.

For a given r-graded ring S, we may consider the category of r-graded S-modules $M^r(S)$. Its objects are the r-graded S-modules, and a morphism $f : M \to N$ in $M^r(S)$ is an S-module morphism such that $f(M_{\mathbf{n}}) \subset N_{\mathbf{n}}$ for all $\mathbf{n} \in \mathbb{Z}^r$.

Given an r-graded S-module M, an r-graded submodule is a submodule $N \subset M$ such that $N = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} N \cap M_{\mathbf{n}}$, equivalently, N is generated by homogeneous elements. The r-graded submodules of S are called homogeneous ideals. For an arbitrary ideal I of S, the homogeneous ideal I^* is defined to be the ideal generated by all the homogeneous elements of I.

As a first example of r-graded ring we have the polynomial ring $S = A[X_1, \dots, X_n]$ defined over an arbitrary ring A. For every choice of elements $\mathbf{d}_1, \dots, \mathbf{d}_n \in \mathbb{Z}^r$, we have a unique r-grading on S such that $\deg X_i = \mathbf{d}_i$ and $\deg a = 0$ for all $a \in A$.

For an r-graded S-module M and $\mathbf{k} \in \mathbb{Z}^r$, then $M(\mathbf{k})$ denotes the S-module M with the grading given by $M(\mathbf{k})_{\mathbf{n}} = M_{\mathbf{k}+\mathbf{n}}$.

If M, N are r-graded S-modules, we denote by $\underline{\mathrm{Hom}}_S(M,N)_0$ the abelian group of all the homomorphisms of r-graded S-modules from M into N. We set $\underline{\mathrm{Hom}}_S(M,N) = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} \underline{\mathrm{Hom}}_S(M, N(\mathbf{n}))_0$. Note that $\underline{\mathrm{Hom}}_S(M,N)_{\mathbf{k}}$ is nothing but the abelian group of S-module homomorphisms $f : M \to N$ such that $f(M_{\mathbf{n}}) \subset N_{\mathbf{n}+\mathbf{k}}$ for all $\mathbf{n} \in \mathbb{Z}^r$. The derived functors of $\underline{\mathrm{Hom}}_S(\ , \ )$ are $\underline{\mathrm{Ext}}^i_S(\ , \ )$, with $i \in \mathbb{N}$.

1.2 Multigraded cohomology

Next we are going to introduce the local cohomology functor in the category of multigraded modules, mainly following [HHR]. The basic results are the multigraded version of the Local Duality Theorem and the good behaviour of the local cohomology modules under a change of grading.

From now on in this chapter, we assume that $S = \bigoplus_{\mathbf{n}\in\mathbb{N}^r} S_{\mathbf{n}}$ is an r-graded ring defined over a local ring $S_0 = A$. Then S has a unique homogeneous maximal ideal $\mathcal{M} = \mathfrak{m} \oplus (\bigoplus_{\mathbf{n}\neq 0} S_{\mathbf{n}})$, where $\mathfrak{m}$ is the maximal ideal of A. Set $d = \dim S$.

If $I \subset S$ is a homogeneous ideal and M is an r-graded S-module, we denote by $\underline{\mathrm{H}}^0_I(M) = \Gamma_I(M) = \{x \in M : I^k x = 0 \text{ for some } k \geq 0\}$. Note that $\underline{\mathrm{H}}^0_I(M)$

is an r-graded submodule of M. The local cohomology functors $\underline{\mathrm{H}}_I^i(\ )$ are the right derived functors of $\Gamma_I(\ )$ in the category of r-graded S-modules. If no confusion, we will usually denote them by $H_I^i(\ )$.

An r-graded S-module K_S is called a canonical module of S if

$$K_S \otimes_A \widehat{A} \cong \underline{\mathrm{Hom}}_S(\underline{\mathrm{H}}_{\mathcal{M}}^d(S), \underline{\mathrm{E}}_S(k)),$$

where k is the residue field of A and $\underline{\mathrm{E}}_S(k)$ is the injective envelope of k in the category of r-graded S-modules. The injective envelope $\underline{\mathrm{E}}_S(k)$ of k is $\underline{\mathrm{Hom}}_A(S, \mathrm{E}_A(k))$, where A is thought as an r-graded ring concentrated in degree 0, and both S and $\mathrm{E}_A(k)$ are considered as r-graded A-modules. Therefore, we have

$$K_S \otimes_A \widehat{A} \cong \underline{\mathrm{Hom}}_A(\underline{\mathrm{H}}_{\mathcal{M}}^d(S), \mathrm{E}_A(k)) = \bigoplus_{\mathbf{n}\in\mathbb{Z}^r} \mathrm{Hom}_A([\underline{\mathrm{H}}_{\mathcal{M}}^d(S)]_{-\mathbf{n}}, E_A(k)).$$

If a canonical module exists, it is finitely generated and unique up to an isomorphism. In the particular case where $A = k$ is a field, the canonical module of S exists and

$$K_S \cong \underline{\mathrm{Hom}}_k(\underline{\mathrm{H}}_{\mathcal{M}}^d(S), k).$$

The next results are the extension to the r-graded case of two of the main properties of the canonical module, well-known for the graded case (see [GW2, Theorem 2.2.2]).

Theorem 1.2.1 *(Local Duality) Let S be an r-graded ring defined over a complete local ring A. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Then S is Cohen-Macaulay if and only if every finitely generated r-graded S-module M satisfies*

$$\underline{\mathrm{Hom}}_S(\underline{\mathrm{H}}_{\mathcal{M}}^i(M), \underline{\mathrm{E}}_S(k)) \cong \underline{\mathrm{Ext}}_S^{d-i}(M, K_S)\ ,\ i = 0, \ldots, d.$$

Corollary 1.2.2 *Let S be a Cohen-Macaulay r-graded ring with canonical module K_S. Let T be an r-graded ring defined over a local ring such that there exists a finite r-graded ring morphism $S \to T$. Then T has canonical module*

$$K_T = \underline{\mathrm{Ext}}_S^e(T, K_S),$$

where $e = \dim S - \dim T$.

Often we are going to consider the ring S endowed with a different grading obtained in the following way: given a group morphism $\varphi : \mathbb{Z}^r \to \mathbb{Z}^q$ such that $\varphi(\text{supp } S) \subset \mathbb{N}^q$, we can define the $\mathbb{N}^q$-graded ring

$$S^\varphi := \bigoplus_{\mathbf{m}\in\mathbb{N}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} S_{\mathbf{n}} \Big).$$

Similarly, given an r-graded S-module M, we may define the q-graded S^φ-module M^φ as

$$M^\varphi := \bigoplus_{\mathbf{m}\in\mathbb{Z}^q} \Big(\bigoplus_{\varphi(\mathbf{n})=\mathbf{m}} M_{\mathbf{n}} \Big).$$

Then $(\ )^\varphi : \mathcal{M}^r(S) \to \mathcal{M}^q(S^\varphi)$ is an exact functor. By considering $\varphi_j : \mathbb{Z}^r \to \mathbb{Z}$ the projection on the j-component, that is, $\varphi_j(\mathbf{n}) = n^j$, we denote by $S_j = S^{\varphi_j}$ and by $M_j = M^{\varphi_j}$. Note that S_j is just the ring S graded by the j-th partial degree.

The next lemma shows that the local cohomology modules behave well under a change of grading.

Lemma 1.2.3 *[HHR, Lemma 1.1] Let S be an r-graded ring defined over a local ring. Let $\mathcal{M}$ be the homogeneous maximal ideal of S. Let $\varphi : \mathbb{Z}^r \to \mathbb{Z}^q$ be a morphism such that $\varphi(\text{supp}\, S) \subset \mathbb{N}^q$. For every r-graded S-module L, we have*

$$(\underline{H}^i_{\mathcal{M}}(L))^\varphi = \underline{H}^i_{\mathcal{M}^\varphi}(L^\varphi),\ \forall i.$$

1.3 Multigraded a-invariants

We begin this section by extending the definition of the a-invariants of a graded module to the multigraded case. After that, under some mild assumptions, we relate the multigraded a-invariants of a multigraded module to the shifts which appear in its multigraded minimal free resolution. This result will be essential in the next chapters. In the graded case, a similar result can be found in [BH1, Example 3.6.15] for Cohen-Macaulay modules.

Let S be a d-dimensional $\mathbb{N}^r$-graded ring defined over a local ring. For each $i = 0, ..., d$, the multigraded a_i-invariant of S is $\mathbf{a}_i(S) = (a_i^1(S), ..., a_i^r(S))$, where

$$a_i^j(S) = \max\{m \in \mathbb{Z} \mid \exists \mathbf{n} \in \mathbb{Z}^r : \varphi_j(\mathbf{n}) = m, [\underline{\mathrm{H}}^i_{\mathcal{M}}(S)]_{\mathbf{n}} \neq 0\}$$

if $\underline{\mathrm{H}}^i_{\mathcal{M}}(S) \neq 0$ and $a^j_i(S) = -\infty$ otherwise. We will denote by $\mathbf{a}(S) = \mathbf{a}_d(S)$. Note that by Lemma 1.2.3

$$a^j_i(S) = \max\{m \in \mathbb{Z} \mid [\underline{\mathrm{H}}^i_{\mathcal{M}_j}(S_j)]_m \neq 0\} = a_i(S_j).$$

Following N.V. Trung [Tr2], the multigraded a_*-invariant of S is defined as $\mathbf{a}_*(S) = (a^1_*(S), ..., a^r_*(S))$, where $a^j_*(S) = \max\{a^j_0(S), \ldots, a^j_d(S)\}$. Similarly, for any finitely generated r-graded S-module M we may define the a-invariants $\mathbf{a}_i(M)$ of M and the a_*-invariant $\mathbf{a}_*(M)$ of M.

Observe that if there exists K_S the canonical module of S , then

$$a^j(S) = a^j_d(S) = -\min\{m \in \mathbb{Z} \mid \exists \mathbf{n} \in \mathbb{Z}^r : \varphi_j(\mathbf{n}) = m, [K_S]_{\mathbf{n}} \neq 0\}.$$

If S has a canonical module K_S, S is said to be quasi-Gorenstein if there exists an r-graded isomorphism $K_S \cong S(\mathbf{a}(S))$, and Gorenstein if in addition S is Cohen-Macaulay.

From now on in this section we assume that S is a noetherian $\mathbb{N}^r$-graded algebra defined over a field k, and let $\mathcal{M}$ be its homogeneous maximal ideal. Our main purpose is then to compute the multigraded a-invariants of a finitely generated r-graded S-module M from an r-graded minimal finite free resolution of M over S, whenever it exists and S is Cohen-Macaulay. To begin with, let us consider

$$\ldots \to D_t \to \ldots \to D_1 \to D_0 \to 0$$

an exact sequence of finitely generated r-graded S-modules such that $\mathrm{Im}(D_{p+1}) \subset \mathcal{M}D_p$, for all $p \geq 0$. Let us denote by $\{\mathbf{v}_{pq}\}$ the set of degree vectors of a minimal homogeneous system of generators of D_p. Note that this set is uniquely determined because it can be obtained as the homogeneous components of the vector space $D_p \otimes_S k$ which are not zero. We set $\mathbf{m}_p = \min_{\leq_{lex}}\{\mathbf{v}_{pq}\}$ and $\mathbf{M}_p = \max_{\leq_{lex}}\{\mathbf{v}_{pq}\}$, where $\leq_{lex}$ is the lexicographic order. Let us denote by $n^j_p = \min_q\{v^j_{pq}\}$, $t^j_p = \max_q\{v^j_{pq}\}$, where $\mathbf{v}_{pq} = (v^1_{pq}, \ldots, v^r_{pq})$, and $\mathbf{n}_p = (n^1_p, ..., n^r_p)$, $\mathbf{t}_p = (t^1_p, ..., t^r_p)$. Let us also consider $\leq$ the partial order in $\mathbb{Z}^r$ defined coefficientwise. Then we have

Lemma 1.3.1 *(i)* $\mathbf{n}_p \leq \mathbf{n}_{p+1}$.

(ii) $\mathbf{m}_p <_{lex} \mathbf{m}_{p+1}$

Proof. Let $C_p = \text{Coker}(D_{p+1} \to D_p)$, $\forall p \geq 1$. Then there are short exact sequences

$$0 \to C_{p+2} \to D_{p+1} \to C_{p+1} \to 0, \quad \forall p \geq 0.$$

Applying the functor $- \bigotimes_S k$, we get exact sequences

$$C_{p+2}/\mathcal{M}C_{p+2} \to D_{p+1}/\mathcal{M}D_{p+1} \to C_{p+1}/\mathcal{M}C_{p+1} \to 0, \quad \forall p \geq 0.$$

Since $C_{p+2} \subset \mathcal{M}D_{p+1}$, then the first map is the zero morphism. Therefore we get isomorphisms

$$D_{p+1}/\mathcal{M}D_{p+1} \xrightarrow{\cong} C_{p+1}/\mathcal{M}C_{p+1}.$$

Let us denote by $\{e_{pq}\}$ a minimal homogeneous system of generators of D_p with $\deg(e_{pq}) = \mathbf{v}_{pq}$, and let f be the map from D_{p+1} to D_p. From the isomorphism it follows that $f(e_{p+1,q}) \neq 0$, for all q. Now let us fix q. We can write $f(e_{p+1,q}) = \sum_l \lambda_l e_{pl}$, where λ_l are homogeneous elements of $\mathcal{M}$. Set $\deg(\lambda_l) = (\lambda_l^1, ..., \lambda_l^r) \in \mathbb{N}^r$ and note that $\deg(\lambda_l) \neq 0$ if $\lambda_l \neq 0$. Looking at the j-th component of the degree, we get $v^j_{p+1,q} \geq \min_l\{ v^j_{pl}\} = n^j_p$, and so $n^j_{p+1} \geq n^j_p$ for all j.

To obtain (ii), it is enough to prove that $\mathbf{v}_{p+1,q} >_{lex} \mathbf{m}_p$ for all q. We have already shown that $v^1_{p+1,q} \geq \min_l\{ v^1_{pl}\} = m^1_p$. If $v^1_{p+1,q} > m^1_p$, we are done. Otherwise, $v^1_{p+1,q} = m^1_p$ and so $\lambda^1_l = 0$ for each l such that $\lambda_l \neq 0$. Then we have $v^2_{p+1,q} \geq \min_l\{ v^2_{pl} \mid v^1_{pl} = m^1_p\} = m^2_p$. By repeating this argument, we get the result since there exist l, j such that $\lambda^j_l > 0$. $\square$

Let S be a d-dimensional r-graded Cohen-Macaulay k-algebra. Assume that M is a finitely generated r-graded S-module with a finite minimal r-graded free resolution over S

$$0 \to D_l \to \ldots \to D_1 \to D_0 \to M \to 0,$$

with $D_p = \bigoplus_q S(a^1_{pq}, ..., a^r_{pq})$. Set $m = \dim M$, $\rho = \operatorname{depth} M$. Note that $l = d - \rho$ by the graded Auslander-Buchsbaum formula. Next we are going to study the shifts which appear in this resolution.

Note that, with the notation introduced before,

$$n^j_p = \min_q\{-a^j_{pq}\},$$
$$t^j_p = \max_q\{-a^j_{pq}\},$$
$$\mathbf{m}_p = \min_{\leq_{lex}}\{ (-a^1_{pq}, ..., -a^r_{pq})\},$$

$$\mathbf{M}_p = \max_{\leq_{lex}}\{(-a^1_{pq}, ..., -a^r_{pq})\}.$$

We will also denote by $t^j_p(M) = t^j_p$, $t^j_*(M) = \max\{t^j_0, \ldots, t^j_l\}$, $\mathbf{t}_*(M) = (t^1_*(M), \ldots, t^r_*(M))$. From Lemma 1.3.1, we have $\mathbf{n}_p \leq \mathbf{n}_{p+1}$, $\mathbf{m}_p <_{lex} \mathbf{m}_{p+1}$. Furthermore,

Lemma 1.3.2 *(i)* $\mathbf{M}_0 <_{lex} \mathbf{M}_1 <_{lex} \cdots <_{lex} \mathbf{M}_{d-m-1} <_{lex} \mathbf{M}_{d-m}$.

(ii) $\mathbf{t}_0 \leq \mathbf{t}_1 \leq \cdots \leq \mathbf{t}_{d-m-1} \leq \mathbf{t}_{d-m}$.

Proof. Let K_S be the canonical module of S. Note that it exists because S is a finitely generated k-algebra. By setting $C_p = \mathrm{Coker}(D_{p+1} \to D_p)$ for $p \geq 0$, we get short exact sequences

$$0 \to C_{p+1} \to D_p \to C_p \to 0,$$

for $0 \leq p \leq l-1$, where $C_0 = M$, $C_l = D_l$. For any $p < d-m-1$, we have

$$\underline{\mathrm{Ext}}^1_S(C_p, K_S) \cong \underline{\mathrm{Ext}}^2_S(C_{p-1}, K_S) \cong \cdots \cong \underline{\mathrm{Ext}}^{p+1}_S(M, K_S) = 0$$

by Theorem 1.2.1. Therefore, by applying the functor $(\ )^* = \underline{\mathrm{Hom}}_S(\ , K_S)$ to the sequences above for $p \leq d-m-1$, we get exact sequences

$$0 \to C^*_p \to D^*_p \to C^*_{p+1} \to 0, \quad \text{for } p \leq d-m-2,$$

$$0 \to C^*_{d-m-1} \to D^*_{d-m-1} \to C^*_{d-m} \to H^m_{\mathcal{M}}(M)^\vee \to 0,$$

where $(\ )^\vee = \underline{\mathrm{Hom}}_k(\ , k)$. By gluing these exact sequences, we get the r-graded exact sequence

$$0 \to D^*_0 \to \ldots \to D^*_{d-m-1} \to C^*_{d-m} \to H^m_{\mathcal{M}}(M)^\vee \to 0.$$

Observe that $D^*_p = \bigoplus_q K_S(-a^1_{pq}, ..., -a^r_{pq})$. One can also check that $\mathrm{Im}(D^*_p) \subset \mathcal{M}D^*_{p+1}$ for all $p \leq d-m-2$.

Let $\{\mathbf{b}_1, ..., \mathbf{b}_k\}$ be the set of degree vectors of a minimal homogeneous system of generators of K_S. If we denote by $\mathbf{a}_{pq} = (a^1_{pq}, ..., a^r_{pq})$, then the vectors $\mathbf{a}_{pq}+\mathbf{b}_i$ are the degrees of a minimal homogeneous system of generators of D^*_p. For $p \leq d-m-1$, let us consider

$$\widetilde{\mathbf{m}}_p = \min_{\leq_{lex}}\{\mathbf{a}_{pq} + \mathbf{b}_i\} = -\mathbf{M}_p + \min_{\leq_{lex}}\{\mathbf{b}_i\},$$

$$\widetilde{n}^j_p = \min_{q,i}\{a^j_{pq} + b^j_i\} = -t^j_p + \min_i\{b^j_i\}.$$

According to Lemma 1.3.1, we have $\widetilde{n}^j_{p+1} \leq \widetilde{n}^j_p$ and $\widetilde{\mathbf{m}}_{p+1} <_{lex} \widetilde{\mathbf{m}}_p$, so

$$\mathbf{t}_0 \le \mathbf{t}_1 \le \cdots \le \mathbf{t}_{d-m-2} \le \mathbf{t}_{d-m-1}$$

$$\mathbf{M}_0 <_{lex} \mathbf{M}_1 <_{lex} \cdots <_{lex} \mathbf{M}_{d-m-2} <_{lex} \mathbf{M}_{d-m-1}.$$

Next we want to show that $\mathbf{M}_{d-m} >_{lex} \mathbf{M}_{d-m-1}$. To this end, let us study the morphism $D_{d-m} \to D_{d-m-1}$ and for that denote by $\nu : C_{d-m} \to D_{d-m-1}$. Assume that there is an element u in the basis of D_{d-m-1} of degree $\mathbf{M}_{d-m-1} \ge_{lex} \mathbf{M}_{d-m}$. If g is a homogeneous minimal generator of C_{d-m}, then g has trivial terms in u: Otherwise, we would have that $\mathbf{M}_{d-m-1} <_{lex} \deg g \le_{lex} \mathbf{M}_{d-m}$ because $C_{d-m} \subset \mathcal{M}D_{d-m-1}$. Let $\mathbf{b} = \min_{<_{lex}}\{\mathbf{b}_1, ..., \mathbf{b}_k\}$, and let us take $c \in [K_S]_{\mathbf{b}}$, $c \neq 0$. Let $w : D_{d-m-1} \to K_S$ defined by $w(u) = c$, $w(v) = 0$ for any $v \neq u$ homogeneous element in the basis of D_{d-m-1}. Then $\nu^* : D^*_{d-m-1} \to C^*_{d-m}$ satisfies $\nu^*(w) = 0$, hence ν^* is not a monomorphism in degree $\deg w = \deg w(u) - \deg(u) = \mathbf{b} - \mathbf{M}_{d-m-1}$. Therefore $[C^*_{d-m-1}]_{\mathbf{b}-\mathbf{M}_{d-m-1}} \neq 0$, and then $[D^*_{d-m-2}]_{\mathbf{b}-\mathbf{M}_{d-m-1}} \neq 0$, so there exists a shift $\mathbf{a} = (a^1, \ldots, a^r)$ in D_{d-m-2} such that $-\mathbf{a} \ge_{lex} \mathbf{M}_{d-m-1}$. So we obtain $\mathbf{M}_{d-m-2} \ge_{lex} \mathbf{M}_{d-m-1}$ which is a contradiction.

Furthermore, note that the first component of $\mathbf{M}_p$ is t^1_p. Therefore, we have $t^1_{d-m-1} \le t^1_{d-m}$ since $\mathbf{M}_{d-m-1} <_{lex} \mathbf{M}_{d-m}$. The inequalities $t^j_{d-m-1} \le t^j_{d-m}$ for $j = 2, \ldots, r$ follow directly from the next remark. □

Remark 1.3.3 Given a permutation σ of $\{1, \ldots, r\}$, we may define $\le_\sigma$ to be the order in $\mathbb{Z}^r$ defined by

$$(u_1, ..., u_r) \le_\sigma (v_1, ..., v_r) \iff (u_{\sigma(1)}, ..., u_{\sigma(r)}) \le_{lex} (v_{\sigma(1)}, ..., v_{\sigma(r)}).$$

Then Lemmas 1.3.1 and 1.3.2 also hold if we define

$$\mathbf{m}^\sigma_p = \min_{\le_\sigma}\{(-a^1_{pq}, ..., -a^r_{pq})\},$$

$$\mathbf{M}^\sigma_p = \max_{\le_\sigma}\{(-a^1_{pq}, ..., -a^r_{pq})\}.$$

The following result gives a formula for the multigraded a_*-invariant of M by means of the shifts which arise in its resolution over S (see [BH1, Example 3.6.15] for the case of a $\mathbb{Z}$-graded Cohen-Macaulay module).

Theorem 1.3.4 *For each $j = 1, \ldots, r$, we have*

(i) $a^j_{d-p}(M) \le t^j_p(M) + a^j(S)$, for $d - m \le p \le d - \rho$.

(ii) *Assume that for some p there exists σ s.t. $\sigma(1) = j$ and $\mathbf{M}_p^\sigma >_\sigma \mathbf{M}_{p+1}^\sigma$. Then $a_{d-p}^j(M) = t_p^j(M) + a^j(S)$.*

(iii) *$a_*^j(M) = t_*^j(M) + a^j(S)$. That is, $\mathbf{a}_*(M) = \mathbf{t}_*(M) + \mathbf{a}(S)$.*

Proof. From the minimal r-graded free resolution of M over S

$$0 \to D_l \to \dots \to D_0 \to M \to 0,$$

by setting $C_p = \text{Coker}(D_{p+1} \to D_p)$, we have short exact sequences

$$0 \to C_{p+1} \to D_p \to C_p \to 0,$$

for $0 \leq p \leq l-1$. By Theorem 1.2.1, if we apply the functor $(\ )^* = \underline{\text{Hom}}_S(\ , K_S)$ to the sequences above we get exact sequences

$$(1)\ \ 0 \to D_0^* \to \dots \to D_{d-m-1}^* \to C_{d-m}^* \to H_{\mathcal{M}}^m(M)^\vee \to 0,$$

and

$$(2)\ \ 0 \to C_p^* \to D_p^* \to C_{p+1}^* \to 0, \quad \text{for } p \leq d-m-2,$$

$$(3)\ \ 0 \to C_{p-1}^* \to D_{p-1}^* \to C_p^* \to H_{\mathcal{M}}^{d-p}(M)^\vee \to 0, \quad \text{for } p \geq d-m\,,$$

where $(\ )^\vee = \underline{\text{Hom}}_k(\ , k)$. Note that for $d - m \leq p \leq d - \rho$ we have monomorphisms

$$0 \to C_p^* \to D_p^* = \bigoplus_q K_S(-a_{pq}^1, \dots, -a_{pq}^r),$$

and so $[C_p^*]_{-\mathbf{i}} = 0$ for any $\mathbf{i}$ such that $i^1 > t_p^1 + a^1(S)$. Now from the epimorphisms

$$C_p^* \to H_{\mathcal{M}}^{d-p}(M)^\vee \to 0,$$

we get $H_{\mathcal{M}}^{d-p}(M)_{\mathbf{i}} = 0$ if $i^1 > t_p^1 + a^1(S)$, and therefore $a_{d-p}^1(M) \leq t_p^1(M) + a^1(S)$. This proves (*i*) for the case $j = 1$.

Assume now that there exists p with $\mathbf{M}_p >_{lex} \mathbf{M}_{p+1}$ (then $p \geq d - m$ according to Lemma 1.3.2). Let $\mathbf{b} = (b^1, \dots, b^r)$ be the minimum with respect to the lexicographic order such that $[K_S]_{\mathbf{b}} \neq 0$. Note that $b^1 = -a^1(S)$. Let $\mathbf{i} = \mathbf{M}_p - \mathbf{b}$. Since $[D_{p+1}^*]_{-\mathbf{i}} = 0$ because $\mathbf{M}_p >_{lex} \mathbf{M}_{p+1}$, we have $[C_{p+1}^*]_{-\mathbf{i}} = 0$ by (3), and so $[C_p^*]_{-\mathbf{i}} = [D_p^*]_{-\mathbf{i}}$ also by (3). Then, denoting by $f : D_p \to D_{p-1}$, we get an exact sequence

$$[D_{p-1}^*]_{-\mathbf{i}} \xrightarrow{f^*} [D_p^*]_{-\mathbf{i}} \to [H_{\mathcal{M}}^{d-p}(M)]_{\mathbf{i}} \to 0.$$

Let $e_1, \ldots, e_s$ be the elements of the canonical basis of D_p with degree $\mathbf{M}_p$, and let $v_1, \ldots, v_m$ be the canonical basis of D_{p-1}. Since $f^*(D^*_{p-1}) \subset \mathcal{M}D^*_p$ and $[D^*_p]_{-\mathbf{i}} = [K_S]_{\mathbf{b}}e^*_1 \oplus \ldots \oplus [K_S]_{\mathbf{b}}e^*_s$, we have that $[K_S]_{\mathbf{b}}e^*_1 \not\subset \mathrm{Im} f^*$. In particular, f^* is not an epimorphism, and so $[H^{d-p}_{\mathcal{M}}(M)]_{\mathbf{i}} \neq 0$. Therefore, $a^1_{d-p}(M) \geq i^1 = M^1_p - b^1 = t^1_p(M) + a^1(S)$. This proves (ii) for the case $j = 1$, $\sigma = Id$.

Let p be the greatest integer such that $\mathbf{M}_p = \max_{\leq_{lex}}\{\mathbf{M}_0, \ldots, \mathbf{M}_l\}$. Then, $\mathbf{M}_{p+1} <_{lex} \mathbf{M}_p$, so $a^1_{d-p}(M) = t^1_p(M) + a^1(S)$ by (ii). Therefore, $a^1_*(M) = t^1_*(M) + a^1(S)$ and we have (iii) for $j = 1$. The proof of the statement for $j = 2, ..., r$ follows from Remark 1.3.3. □

1.4 Scheme associated to a multigraded ring

Let S be a noetherian $\mathbb{N}^r$-graded ring. We call S standard if S may be generated over S_0 by elements in degrees $(1, 0, \ldots, 0), \ldots, (0, \ldots, 0, 1)$. Similarly to the graded case, we may associate to such a ring a multigraded scheme in a natural way (see [Hy]). Our purpose is to extend this construction to a more general class of multigraded rings, which will recover the standard case as well as the Rees algebra of any homogeneous ideal in a graded k-algebra.

Let S be a noetherian $\mathbb{N}^r$-graded ring finitely generated over S_0 by homogeneous elements $x_{11}, \ldots, x_{1k_1}, \ldots, x_{r1}, \ldots, x_{rk_r}$ of degrees $\deg(x_{ij}) = (d^1_{ij}, \ldots, d^{i-1}_{ij}, 1, 0, \ldots, 0)$, where d^l_{ij} are non-negative integers, and set $d^l_i = \max_j\{d^l_{ij}\}$. This class of rings includes for instance any standard $\mathbb{N}^r$-graded ring by taking $d^l_{ij} = 0$. For every $i = 1, \ldots, r$, let I_i be the ideal of S generated by the homogeneous components of S of degree $\mathbf{n} = (n_1, \ldots, n_r)$ such that $n_i > 0, n_{i+1} = \ldots = n_r = 0$. Then we define the irrelevant ideal of S as $S_+ = I_1 \cdots I_r$. We are going to associate a scheme to S in the following way. A homogeneous prime ideal P of S is said to be relevant if P does not contain S_+. Then we define the set $\mathrm{Proj}^{\,r}(S)$ to be the set of all relevant homogeneous prime ideals P. It is easy to check that $\dim S/P \geq r$ for any relevant prime ideal (see the proof of Lemma 1.4.1). Following [STV] (where the standard bigraded case was studied), we define the relevant dimension of S as

$$\mathrm{rel.dim}\, S = \begin{cases} r-1 & \text{if } \mathrm{Proj}^{\,r}(S) = \emptyset \\ \max\{\dim S/P \mid P \in \mathrm{Proj}^{\,r}(S)\} & \text{if } \mathrm{Proj}^{\,r}(S) \neq \emptyset \end{cases}.$$

If I is a homogeneous ideal of S, we define the subset $V_+(I) := \{P \in \text{Proj}\,^r(S) \mid I \subset P\}$. We can define a topology on Proj $^r(S)$ by taking as closed subsets the subsets of the form $V_+(I)$. Next, to define a sheaf of rings $\mathcal{O}$ in Proj $^r(S)$, we first consider for each $P \in \text{Proj}\,^r(S)$ the homogeneous localization by P

$$S_{(P)} = \{ \frac{a}{s} \mid s \notin P,\ a, s \in S_{\mathbf{n}},\ \mathbf{n} \in \mathbb{Z}^r \}.$$

For any open subset $U \subset \text{Proj}\,^r(S)$, we define $\mathcal{O}(U)$ to be the set of functions $t : U \to \bigsqcup_{P \in U} S_{(P)}$ such that for each $P \in U$, $t(P) \in S_{(P)}$ and t is locally a quotient of elements of S. Then, $\mathcal{O}$ is a sheaf of rings. We call Proj $^r(S)$ the r-projective scheme associated to S. Defining for any homogeneous $f \in S_+$ the set $D_+(f) = \{P \in \text{Proj}\,^r(S) \mid f \notin P\}$ we have an open cover of Proj $^r(S)$, and for each such open set we have an isomorphism of locally ringed spaces

$$(D_+(f), \mathcal{O}|D_+(f)) \cong \text{Spec}(S_{(f)}).$$

Moreover, $\mathcal{O}_P \cong S_{(P)}$ for any relevant prime ideal P, hence Proj $^r(S)$ is a scheme in a natural way. This construction extends the usual one given in the standard case (see [Hy]).

The next lemma computes the dimension of Proj $^r(S)$. Its proof follows the same arguments as in [Hy, Lemma 1.2], but we include it for completeness.

Lemma 1.4.1 $\dim \text{Proj}\,^r(S) = \text{rel.dim}\, S - r$.

Proof. We may assume that Proj $^r(S) \neq \emptyset$ (otherwise the result is trivial). Let $P \in \text{Proj}\,^r(S)$ be a closed point. Since the projection Proj $^r(S) \to \text{Spec}(S_0)$ is proper, we have that $P_0 = P \cap S_0$ is a closed point of $\text{Spec}(S_0)$, so $(S/P)_0 = S_0/P_0$ is a field. Let us denote by $T = S/P$, and note that $\dim \text{Proj}\,^r(T) = 0$. For $j = 1, \dots, r$, let J_j be the ideal of T generated by the homogeneous components of T of degree $\mathbf{n}$ such that $n_j > 0, n_{j+1} = \dots = n_r = 0$. We have a maximal chain of homogeneous prime ideals

$$0 \subset J_r \subset J_{r-1} + J_r \subset \dots \subset J_1 + \dots + J_r,$$

so $\dim T = r$ because T is a catenary ring. On the other hand, for a given minimal prime $Q_0 \in \text{Proj}\,^r(S)$, we have a chain of homogeneous prime ideals of type $Q_0 \subset \dots \subset Q_s \subset \dots \subset Q_{s+r}$, with Q_s a closed point of Proj $^r(S)$. Therefore,

$$\begin{aligned}\dim \operatorname{Proj}^{r}(S) &= \sup \{\operatorname{ht} Q : Q \in \operatorname{Proj}^{r}(S)\} \\ &= \sup \{\dim S/Q : Q \in \operatorname{Proj}^{r}(S)\} - r \\ &= \operatorname{rel.dim} S - r. \square\end{aligned}$$

Next we are going to define the diagonal functor. Given $e_1, \ldots, e_r$ positive integers, the set

$$\Delta := \{(e_1 s, \ldots, e_r s) \mid s \in \mathbb{Z}\}$$

is called the $(e_1, \ldots, e_r)$-diagonal of $\mathbb{Z}^r$. We may then define the diagonal of S along Δ as the graded ring $S_\Delta := \bigoplus_{s \in \mathbb{Z}} S_{(e_1 s, \ldots, e_r s)}$. Similarly, given an r-graded S-module M we define the diagonal of M along Δ as the graded S_Δ-module $M_\Delta := \bigoplus_{s \in \mathbb{Z}} M_{(e_1 s, \ldots, e_r s)}$. Then we have an exact functor

$$(\ )_\Delta : M^r(S) \to M^1(S_\Delta),$$

called diagonal functor.

Let us denote by $X = \operatorname{Proj}^{r}(S)$, and for each Δ, let $X_\Delta = \operatorname{Proj}(S_\Delta)$. By considering diagonals $\Delta = (e_1, \ldots, e_r)$ such that $e_r > 0$, $e_{r-1} > d_r^{r-1} e_r, \ldots, e_1 > d_2^1 e_2 + \ldots + d_r^1 e_r$, then the sheaf of ideals $\mathcal{L} = (S_{(e_1, \ldots, e_r)}) \mathcal{O}_X$ defines an isomorphism $X \xrightarrow{\cong} X_\Delta$. In particular, this isomorphism allows us to compute the dimension of S_Δ, extending [STV, Proposition 2.3] where this dimension was computed for bigraded standard k-algebras.

Lemma 1.4.2 *Assume that S_0 is an artinian local ring. Then* $\dim S_\Delta = \operatorname{rel.dim} S - r + 1$, *for any* $\Delta = (e_1, \ldots, e_r)$ *with* $e_r > 0$, $e_{r-1} > d_r^{r-1} e_r, \ldots, e_1 > d_2^1 e_2 + \ldots + d_r^1 e_r$.

Proof. From the isomorphism $X \cong X_\Delta$, we have that $\operatorname{rel.dim} S_\Delta = \operatorname{rel.dim} S - r + 1$ by Lemma 1.4.1. Moreover, since S_0 is artinian, any minimal prime ideal of S_Δ is relevant, and so $\operatorname{rel.dim} S_\Delta = \dim S_\Delta$. $\square$

Classically, S is the multihomogeneous coordinate ring of a multiprojective variety V contained in some multiprojective space $\mathbb{P}_k^{n_1} \times \ldots \times \mathbb{P}_k^{n_r}$. By taking the $(1, \ldots, 1)$-diagonal, S_Δ is then the homogeneous coordinate ring of the image of V via the Segre embedding $\mathbb{P}_k^{n_1} \times \ldots \times \mathbb{P}_k^{n_r} \to \mathbb{P}_k^N$, where $N = (n_1 + 1) \ldots (n_r + 1) - 1$.

1.5 Hilbert polynomial of multigraded modules

Let $S = \bigoplus_{\mathbf{n}\in\mathbb{N}^r} S_{\mathbf{n}}$ be an r-graded ring defined over an artinian local ring $S_0 = A$. If S is standard, then we have that the Hilbert function of any finitely generated r-graded S-module L, $H(L,\mathbf{n}) = \mathrm{length_A(L_n)}$, is a polynomial function; that is, there exists a polynomial $P_L(t_1,\ldots,t_r) \in \mathbb{Q}[t_1,\ldots,t_r]$, called Hilbert polynomial of L, such that for any $\mathbf{n} \gg 0$, $P_L(n_1,\ldots,n_r) = \mathrm{length}_A(L_{\mathbf{n}})$ (see [HHRT], [KMV]). In this section we are going to extend the existence of such a polynomial for the larger class of finitely generated r-graded modules defined over the multigraded rings introduced in Section 1.4. Furthermore, we will state a formula for the difference between the Hilbert polynomial and the Hilbert function of any finitely generated r-graded module analogous to the one known in the graded case.

Let S be a noetherian $\mathbb{N}^r$-graded ring generated over $S_0 = A$ by homogeneous elements $x_{11},\ldots,x_{1k_1},\ldots,x_{r1},\ldots,x_{rk_r}$ in degrees $\deg(x_{ij}) = (d_{ij}^1,\ldots,d_{ij}^{i-1},1,0,\ldots,0)$, where $d_{ij}^l \geq 0$. Set $d_i^l = \max_j\{d_{ij}^l\}$.

Given a finitely generated r-graded S-module L, let us define its homogeneous support as $\mathrm{Supp}_+(L) = \{P \in \mathrm{Proj}^{\,r}(S) \mid L_P \neq 0\}$. Note that $\mathrm{Supp}_+(L) = V_+(\mathrm{Ann}\, L)$ is a closed subset of $\mathrm{Proj}^{\,r}(S)$. We define the relevant dimension of L as

$$\mathrm{rel.dim}\, L = \begin{cases} r-1 & \text{if } \mathrm{Supp}_+(L) = \emptyset \\ \max\{\dim S/P \mid P \in \mathrm{Supp}_+(L)\} & \text{if } \mathrm{Supp}_+(L) \neq \emptyset \end{cases}.$$

One can check that $\mathrm{rel.dim}\, L = \dim \mathrm{Supp}_+ L + r$.

From now on in this section we will assume that A is an artinian local ring. Given a finitely generated r-graded S-module L, its homogeneous components $L_{\mathbf{n}}$ are finitely generated A-modules, and hence have finite length. The numerical function $H(L,\,\cdot\,) : \mathbb{Z}^r \to \mathbb{Z}$ with $H(L,\mathbf{n}) = \mathrm{length_A(L_n)}$ is the Hilbert function of L. Next result shows the existence of the Hilbert polynomial for any finitely generated r-graded S-module.

Proposition 1.5.1 *Let L be a finitely generated r-graded S-module of relevant dimension δ. Then there exists a polynomial $P_L(t_1,\ldots,t_r) \in \mathbb{Q}[t_1,\ldots,t_r]$ of total degree $\delta - r$ such that $H(L,i_1,\ldots,i_r) = P_L(i_1,\ldots,i_r)$ for $i_1 \gg d_2^1 i_2 + \ldots + d_r^1 i_r$, $\ldots$, $i_{r-1} \gg d_r^{r-1} i_r$, $i_r \gg 0$. Moreover,*

$$P_L(t_1,\ldots,t_r) = \sum_{|\mathbf{n}|\leq\delta-r} a_{\mathbf{n}} \binom{t_1 - d_2^1 t_2 - \ldots - d_r^1 t_r}{n_1} \cdots \binom{t_{r-1} - d_r^{r-1} t_r}{n_{r-1}} \binom{t_r}{n_r},$$

where $a_{\mathbf{n}} \in \mathbb{Z}$, $a_{\mathbf{n}} \geq 0$ *if* $|\mathbf{n}| = \delta - r$.

Proof. Given a finitely generated r-graded S-module L, first note that there is a chain

$$0 = L_0 \subset L_1 \subset \ldots \subset L_s = L$$

of r-graded submodules of L such that for each $i \geq 1$, $L_i/L_{i-1} \cong (S/P_i)(\mathbf{m}_i)$, where $P_i \in \operatorname{Supp} L$ is a homogeneous prime ideal and $\mathbf{m}_i \in \mathbb{Z}^r$. Indeed, we may assume $L \neq 0$. Choose $P_1 \in \operatorname{Ass} L$. Then P_1 is a homogeneous prime ideal, and there exists an r-graded submodule $L_1 \subset L$ such that $L_1 \cong (S/P_1)(\mathbf{m}_1)$. If $L_1 \neq L$, by repeating the procedure with L/L_1 we get an r-graded submodule $L_2 \subsetneq L$ such that $L_2/L_1 \cong (S/P_2)(\mathbf{m}_2)$. Since L is noetherian, this process finishes after a finite number of steps. From this chain, we obtain

$$H(L, \mathbf{n}) = \sum_{i=1}^{s} H(S/P_i, \mathbf{n} + \mathbf{m}_i).$$

So it is enough to prove the result for the rings $T = S/P$, with P a homogeneous prime ideal. To this end, we will reduce the problem to the standard case where the result is already known.

Set $B = T_0$. Let us consider $\overline{T} \subset T$ the B-algebra generated by the homogeneous elements of T of degree $(e_1, \ldots, e_r)$ such that

$$\begin{cases} e_{r-1} \geq d_r^{r-1} e_r \\ e_{r-2} \geq d_{r-1}^{r-2} e_{r-1} + d_r^{r-2} e_r \\ \cdots\cdots \\ e_1 \geq d_2^1 e_2 + \ldots + d_r^1 e_r. \end{cases}$$

Then one has $\overline{T}_{\mathbf{n}} = T_{\mathbf{n}}$ for each $\mathbf{n} \in \mathbb{N}^r$ satisfying the inequalities before. Let us consider the morphism

$$\begin{array}{cccc} \psi: & \mathbb{Z}^r & \longrightarrow & \mathbb{Z}^r \\ & (x_1, \ldots, x_r) & \mapsto & (x_1 - d_2^1 x_2 - \cdots - d_r^1 x_r, \ldots, x_{r-1} - d_r^{r-1} x_r, x_r) \end{array}$$

Note that $\psi(\operatorname{supp} \overline{T}) \subset \mathbb{N}^r$, so $\overline{T}^{\psi}$ is again a $\mathbb{N}^r$-graded ring. Furthermore, we have $\operatorname{rel.dim} \overline{T}^{\psi} = \operatorname{rel.dim} \overline{T} = \operatorname{rel.dim} T = \delta$. If $\overline{T}^{\psi}$ is standard, by [HHRT, Theorem 4.1] there exists a polynomial $Q(t_1, \ldots, t_r) \in \mathbb{Q}[t_1, \ldots, t_r]$ of total degree $\delta - r$

$$Q(t_1, \ldots, t_r) = \sum_{|\mathbf{n}| \leq \delta - r} a_{\mathbf{n}} \binom{t_1}{n_1} \cdots \binom{t_r}{n_r},$$

with $a_{\mathbf{n}} \in \mathbb{Z}$, $a_{\mathbf{n}} \geq 0$ if $|\mathbf{n}| = \delta - r$ such that for $\mathbf{i} \gg 0$

$$Q(i_1, \dots, i_r) = \mathrm{length_B}[\overline{\mathrm{T}}^{\psi}]_{(\mathrm{i}_1,\dots,\mathrm{i}_r)}.$$

Then, by defining $P(t_1, \dots, t_r) = Q(t_1 - d_2^1 t_2 - \dots d_r^1 t_r, \dots, t_{r-1} - d_r^{r-1} t_r, t_r)$, let us observe that for $i_1 \gg d_2^1 i_2 + \dots + d_r^1 i_r, \dots, i_{r-1} \gg d_r^{r-1} i_r, i_r \gg 0$, we have

$$\begin{aligned} P(i_1, \dots, i_r) &= \mathrm{length_B}\ [\overline{\mathrm{T}}^{\psi}]_{(\mathrm{i}_1 - \mathrm{d}_2^1 \mathrm{i}_2 - \dots - \mathrm{d}_r^1 \mathrm{i}_r, \dots, \mathrm{i}_{r-1} - \mathrm{d}_r^{r-1} \mathrm{i}_r, \mathrm{i}_r)} \\ &= \mathrm{length_B}\ [\overline{\mathrm{T}}]_{(\mathrm{i}_1,\dots,\mathrm{i}_r)} \\ &= \mathrm{length_A}\ [\mathrm{T}]_{(\mathrm{i}_1,\dots,\mathrm{i}_r)}, \end{aligned}$$

so we get the statement.

Therefore we only have to prove that $\overline{T}^{\psi}$ is standard or, equivalently, that $\overline{T}$ can be generated over B by homogeneous elements in degrees $(e_1, \dots, e_r)$ such that $e_{i+1} = \dots = e_r = 0$, $e_i = 1$, $e_{i-1} = d_i^{i-1} e_i$, $e_{i-2} = d_{i-1}^{i-2} e_{i-1} + d_i^{i-2} e_i$, $\dots$, $e_1 = d_2^1 e_2 + \dots + d_i^1 e_i$. Assume that T is generated over B by homogeneous elements $z_{11}, \dots, z_{1k_1}, \dots, z_{r1}, \dots, z_{rk_r}$ in degrees $\deg(z_{ij}) = (d_{ij}^1, \dots, d_{ij}^{i-1}, 1, 0, \dots, 0)$. Let us take a homogeneous element z in $\overline{T}$, with $\deg z = (\alpha_1, \dots, \alpha_r)$. Let j be such that $\alpha_j \neq 0$, $\alpha_{j+1} = \dots = \alpha_r = 0$ (j is 0 if $z \in B$). We are going to prove by induction on j that z can be generated over B by the homogeneous elements whose degrees satisfy the equalities before. If $j = 0$, there is nothing to prove. If $j = 1$, then deg $z = (\alpha_1, 0, \dots, 0)$ and we can write z as a linear combination with coefficients in B of products of α_1 elements among $z_{11}, \dots, z_{1k_1}$, so the result is trivial. Assume now that $j > 1$. By forgetting the first component of the degree, we have by induction hypothesis that z can be written as a sum of terms of the type $\lambda w_1 \dots w_l$ with $\lambda \in B[z_{11}, \dots, z_{1k_1}]$, and the degree of the elements w_i satisfying the $r-1$ first equalities. Set $\deg w_j = (s_j^1, \dots, s_j^r)$, $\deg \lambda = (s, 0, \dots, 0)$. We will finish if we prove that

$$\alpha_1 \geq \sum_{j=1}^{l} (d_2^1 s_j^2 + \dots + d_r^1 s_j^r).$$

But note that $\alpha_1 \geq d_2^1 \alpha_2 + \dots + d_r^1 \alpha_r = \sum_{j=1}^{l} d_2^1 s_j^2 + \dots + d_r^1 s_j^r$. $\square$

Our next aim will be to study for a given finitely generated r-graded S-module L, the A-modules $H_{S_+}^i(L)_{\mathbf{n}}$ for $i \geq 0$, $\mathbf{n} \in \mathbb{Z}^r$. We need two previous lemmas.

Lemma 1.5.2 *Let L be a finitely generated r-graded S-module such that $(S_+)^u L = 0$ for an integer u. Then there exits $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ such that*

$$L_{\mathbf{n}} = 0,$$

for $\mathbf{n} = (n_1, \dots, n_r)$ such that $n_1 > d_2^1 n_2 + \dots + d_r^1 n_r + m_1, \dots, n_{r-1} > d_r^{r-1} n_r + m_{r-1}, n_r > m_r$.

Proof. Since there exists $u \in \mathbb{Z}$ such that $(S_+)^u L = 0$, we have $\operatorname{Supp}_+(L) = V_+(\operatorname{Ann} L) \subset V_+(S_+) = \emptyset$, so $\operatorname{rel.dim} L = r - 1$. Then the result follows from Proposition 1.5.1. □

Lemma 1.5.3 *(Homogeneous Prime Avoidance) Let $P_1, \dots, P_m \in \operatorname{Proj}^r(S)$. If I is any homogeneous ideal of S such that $I \not\subset P_i$ for $i = 1, \dots, m$, then there is a homogeneous element a such that $a \in I$, $a \notin P_1 \cup \dots \cup P_m$.*

Proof. We may assume that $P_j \not\subset P_i$ for $i \neq j$, so for a given i, we have that for any $j \neq i$ there exists a homogeneous element $p_{ij} \in P_j$, $p_{ij} \notin P_i$. Then $p_i = \prod_{j \neq i} p_{ij}$ satisfies that $p_i \notin P_i$, but $p_i \in P_j$ for all $j \neq i$. Next we may take homogeneous elements $a_i \in I$, $a_i \notin P_i$ for $i = 1, \dots, m$. Set $\deg a_i p_i = (\alpha_{i1}, \dots, \alpha_{ir})$. Since $S_+ \not\subset P_i$, there exists an element of the type $x_{1j_1} \dots x_{rj_r} \notin P_i$. So multiplying each $a_i p_i$ by a power of the corresponding x_{rj_r} we can assume that $\alpha_{1r} = \dots = \alpha_{mr} = \alpha_r$. Then, multiplying by suitable powers of each $x_{r-1,j_{r-1}}$ we may also assume that $\alpha_{1,r-1} = \dots = \alpha_{m,r-1} = \alpha_{r-1}$. By repeating this procedure as many times as necessary, we can assume at the end that $\deg(a_1 p_1) = \dots = \deg(a_m p_m) = (\alpha_1, \dots, \alpha_r)$. Now $a = a_1 p_1 + \dots + a_m p_m$ is homogeneous and $a \in I$, $a \notin P_1 \cup \dots \cup P_m$. □

Now we are ready to prove that if L is a finitely generated r-graded S-module, then the A-modules $H^i_{S_+}(L)_{\mathbf{n}}$ are finitely generated for all $\mathbf{n} \in \mathbb{Z}^r$, $i \geq 0$, and vanish for all sufficiently large $\mathbf{n}$. Here, the artinian assumption on A is not necessary. In the graded case, this is a classical result due to J.P. Serre.

Proposition 1.5.4 *Let L be a finitely generated r-graded S-module. Then*

(i) For all $i \geq 0$, $\mathbf{n} \in \mathbb{Z}^r$, the A-module $H^i_{S_+}(L)_{\mathbf{n}}$ is finitely generated.

(ii) There exits $\mathbf{m} = (m_1, \dots, m_r) \in \mathbb{Z}^r$ such that $H^i_{S_+}(L)_{\mathbf{n}} = 0$ for all $i \geq 0$, $\mathbf{n} = (n_1, \dots, n_r)$ such that $n_1 > d_2^1 n_2 + \dots + d_r^1 n_r + m_1, \dots, n_{r-1} > d_r^{r-1} n_r + m_{r-1}, n_r > m_r$.

Proof. We will follow the same lines as the proof of the graded version in [BroSha, Proposition 15.1.5]. We will prove by induction on i that $H^i_{S_+}(L)_\mathbf{n}$ is a finitely generated A-module for all $\mathbf{n} \in \mathbb{Z}^r$, and that it is zero for all sufficiently large values of $\mathbf{n}$. This proves the statement because $H^i_{S_+}(L) = 0$ for all i greater than the minimal number of generators of S_+.

Assume $i = 0$. Then $H^0_{S_+}(L)$ is a finitely generated r-graded S-module since it is a submodule of L, and so $H^0_{S_+}(L)_\mathbf{n}$ is a finitely generated A-module and there exists $u \in \mathbb{N}$ such that $(S_+)^u H^0_{S_+}(L) = 0$. Then, according to Lemma 1.5.2 there exists $\mathbf{m} \in \mathbb{Z}^r$ such that $H^0_{S_+}(L)_\mathbf{n} = 0$ for $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + m_1, \ldots, n_{r-1} > d^{r-1}_r n_r + m_{r-1}, n_r > m_r$.

Now let us assume $i > 0$. From the r-graded isomorphism $H^i_{S_+}(L) \cong H^i_{S_+}(L/H^0_{S_+}(L))$, we may replace L by $L/H^0_{S_+}(L)$ and then assume that $H^0_{S_+}(L) = 0$. Then $S_+ \not\subset P$ for all $P \in \mathrm{Ass}(L)$, and so by the Prime Avoidance Lemma there exists a homogeneous element $x \in S_+$ of degree $\mathbf{k} = (k_1, \ldots, k_r)$ such that $x \notin P$ for all $P \in \mathrm{Ass}(L)$. Looking at the proof of the Prime Avoidance Lemma, notice that we can choose x such that $\mathbf{k}$ satisfies $k_1 > d^1_2 k_2 + \cdots + d^1_r k_r, \ldots, k_{r-1} > d^{r-1}_r k_r$. Then we get an r-graded exact sequence

$$0 \to L(-\mathbf{k}) \xrightarrow{\cdot x} L \to L/xL \to 0,$$

which induces for all $\mathbf{n} \in \mathbb{Z}^r$ the exact sequence of A-modules

$$H^{i-1}_{S_+}(L/xL)_\mathbf{n} \to H^i_{S_+}(L)_{\mathbf{n}-\mathbf{k}} \xrightarrow{\cdot x} H^i_{S_+}(L)_\mathbf{n} \ .$$

By the induction hypothesis, there exists $\overline{\mathbf{m}} \in \mathbb{Z}^r$ such that $H^{i-1}_{S_+}(L/xL)_\mathbf{n} = 0$ for all $\mathbf{n} = (n_1, \ldots, n_r)$ such that $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + \overline{m}_1, \ldots, n_{r-1} > d^{r-1}_r n_r + \overline{m}_{r-1}, n_r > \overline{m}_r$. Now let $\mathbf{n}$ verifying these inequalities. Then note that for any $s \geq 1$, $\mathbf{n} + s\mathbf{k}$ also satifies them, and so we have exact sequences

$$0 \to H^i_{S_+}(L)_{\mathbf{n}-\mathbf{k}} \xrightarrow{\cdot x^s} H^i_{S_+}(L)_{\mathbf{n}+(s-1)\mathbf{k}} \ .$$

Since $H^i_{S_+}(L)$ is S_+-torsion and $x \in S_+$, we have $H^i_{S_+}(L)_{\mathbf{n}-\mathbf{k}} = 0$. Therefore, by taking $\mathbf{m} = \overline{\mathbf{m}} - \mathbf{k}$, we obtain $H^i_{S_+}(L)_\mathbf{n} = 0$ for all $\mathbf{n}$ such that $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + m_1, \ldots, n_{r-1} > d^{r-1}_r n_r + m_{r-1}, n_r > m_r$.

Now let us fix $\mathbf{n} \in \mathbb{Z}^r$. If $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + m_1, \ldots, n_{r-1} > d^{r-1}_r n_r + m_{r-1}, n_r > m_r$, we have that $H^i_{S_+}(L)_\mathbf{n} = 0$, and so it is a finitely generated A-module. Otherwise, let us take $y \in S_+$ such that $y \notin \bigcup_{P \in \mathrm{Ass}(L)} P$ with degree $\mathbf{l} = (l_1, \ldots, l_r)$ such that $\mathbf{n} + \mathbf{l}$ satisfies the previous inequalities (we can find such a y by the Prime Avoidance Lemma). Then we have the graded exact sequence

$$H^{i-1}_{S_+}(L/yL)_{\mathbf{n}+1} \to H^i_{S_+}(L)_{\mathbf{n}} \xrightarrow{\cdot y} H^i_{S_+}(L)_{\mathbf{n}+1} = 0,$$

and from the induction hypothesis we also have that $H^{i-1}_{S_+}(L/yL)_{\mathbf{n}+1}$ is a finitely generated A-module, and so $H^i_{S_+}(L)_{\mathbf{n}}$. □

We have already shown that the Hilbert function of any finitely generated r-graded S-module is a polynomial function for large $\mathbf{n}$. Our next result precises the difference between the Hilbert function and the Hilbert polynomial for any $\mathbf{n}$.

Proposition 1.5.5 *Let L be a finitely generated r-graded S-module. Then for all $\mathbf{n} \in \mathbb{Z}^r$*

$$H(L,\mathbf{n}) - P_L(\mathbf{n}) = \sum_q (-1)^q \operatorname{length}_A(H^q_{S_+}(L)_{\mathbf{n}}).$$

Proof. We will follow the proof of the graded version from [BH1, Theorem 4.3.5]. For an arbitrary finitely generated r-graded S-module L, let us define the series

$$H'_L(u_1,\ldots,u_r) = \textstyle\sum_{\mathbf{n}\in\mathbb{Z}^r}(H(L,\mathbf{n}) - P_L(\mathbf{n}))u^{\mathbf{n}}$$

$$H''_L(u_1,\ldots,u_r) = \textstyle\sum_{\mathbf{n}\in\mathbb{Z}^r}(\sum_q(-1)^q \operatorname{length}_A(H^q_{S_+}(L)_{\mathbf{n}}))u^{\mathbf{n}}.$$

We will prove the statement by induction on $\delta = \operatorname{rel.dim} L$. If $\delta = r-1$, then $\operatorname{Supp}_+ L = \emptyset$, and so there exists m such that $S^m_+ \subset \operatorname{Ann}(L)$. Therefore $H^0_{S_+}(L) = L$, and hence the result is trivial. Assume now $\delta \geq r$, and let us consider $\overline{L} = L/H^0_{S_+}(L)$. Since $H^0_{S_+}(L)$ is a finitely generated r-graded S-module which is vanished by some power of S_+, there are integers $i_1,\ldots,i_r$ such that $H^0_{S_+}(L)_{\mathbf{n}} = 0$ for $n_1 > d^1_2 n_2 + \ldots + d^1_r n_r + i_1, \ldots, n_{r-1} > d^{r-1}_r n_r + i_{r-1}$, $n_r > i_r$. So we have $P_L(\mathbf{t}) = P_{\overline{L}}(\mathbf{t})$, and it is enough to prove the result for $\overline{L}$ because then, for all $\mathbf{n} = (n_1,\ldots,n_r)$

$$\begin{aligned} H(L,\mathbf{n}) - P_L(\mathbf{n}) &= H(\overline{L},\mathbf{n}) + \operatorname{length}_A(H^0_{S_+}(L)_{\mathbf{n}}) - P_{\overline{L}}(\mathbf{n}) \\ &= \textstyle\sum_q(-1)^q \operatorname{length}_A(H^q_{S_+}(\overline{L})_{\mathbf{n}}) + \operatorname{length}_A(H^0_{S_+}(L)_{\mathbf{n}}) \\ &= \textstyle\sum_q(-1)^q \operatorname{length}_A(H^q_{S_+}(L)_{\mathbf{n}}). \end{aligned}$$

So let us assume $H^0_{S_+}(L) = 0$. Then $S_+ \not\subset P$ for all $P \in \operatorname{Ass}(L)$, and so by the Prime Avoidance Lemma there exists a homogeneous element $x \in S_+$ of degree $\mathbf{k} = (k_1, \ldots, k_r)$ such that $x \notin P$ for all $P \in \operatorname{Ass}(L)$. Then we have the r-graded exact sequence

$$0 \to L(-\mathbf{k}) \to L \to L/xL \to 0,$$

with $\operatorname{rel.dim} L/xL < \operatorname{rel.dim} L$. Note that $H(L/xL, \mathbf{n}) = H(L, \mathbf{n}) - H(L, \mathbf{n}-\mathbf{k})$ for all $\mathbf{n}$, and so $P_{L/xL}(\mathbf{t}) = P_L(\mathbf{t}) - P_L(\mathbf{t} - \mathbf{k})$. We conclude $H'_{L/xL}(\mathbf{u}) = (1 - \mathbf{u}^{\mathbf{k}})H'_L(\mathbf{u})$. From the long exact sequence of local cohomology, we also get $H''_{L/xL}(\mathbf{u}) = (1 - \mathbf{u}^{\mathbf{k}})H''_L(\mathbf{u})$. By the induction hypothesis, we have $H'_{L/xL}(\mathbf{u}) = H''_{L/xL}(\mathbf{u})$ and so $H'_L(\mathbf{u}) = H''_L(\mathbf{u})$. □