On the diagonals of a Rees algebra

Olga Lavila Vidal

UNIVERSITAT DE BARCELONA

Departament d'Àlgebra i Geometria

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Chapter 2

The diagonals of a bigraded module

Throughout this chapter we will study in more detail the diagonal functor in the category of bigraded S-modules, where $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ is the polynomial ring in $n+r$ variables with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$, and $d_1, \ldots, d_r \geq 0$. This category includes any standard bigraded k-algebra, by taking $d_1 = \ldots = d_r = 0$, as well as the Rees ring and the form ring of a homogeneous ideal in a graded k-algebra, when those rings are endowed with an appropiate bigrading (see Section 2.3).

For a given c, e positive integers, let Δ be the (c, e)-diagonal of $\mathbb{Z}^2$. Our purpose is to study the exact functor $(\ )_\Delta : M^2(S) \to M^1(S_\Delta)$ (see Chapter 1, Section 4). We are mainly interested in studying how the arithmetic properties of a bigraded S-module L and its diagonals L_Δ are related. Most of these properties, like the Cohen-Macaulayness or the Gorenstein property, can be characterized by means of the local cohomology modules. So it would be very useful to relate the local cohomology modules of L with the local cohomology modules of its diagonals. This has been done by A. Conca et al. in [CHTV] from the study of the bigraded minimal free resolution of L over S, after developing a theory of generalized Segre products of bigraded algebras. In Section 2.1 we are going to present their results by a different and somewhat easier approach. In addition, this approach will provide more detailed information about several problems concerning to the behaviour of the local cohomology when taking diagonals.

In Section 2.2 we focus our study on standard bigraded k-algebras. For a such k-algebra R, let $\mathcal{R}_1 = \bigoplus_{i\in\mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = \bigoplus_{j\in\mathbb{N}} R_{(0,j)}$. In this case, we give a characterization for R to have a good resolution in terms of the a_*-invariants of $\mathcal{R}_1$ and $\mathcal{R}_2$ which, in particular, provides a criterion for the Cohen-Macaulayness of its diagonals. We also find necessary and sufficient conditions on the local cohomology of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the existence of Cohen-Macaulay diagonals of R, whenever R is Cohen-Macaulay.

Given a homogeneous ideal I in a graded k-algebra A, the Rees algebra $R_A(I) = \bigoplus_{n\geq 0} I^n$ of I can be endowed with the bigrading $R_A(I)_{(i,j)} = (I^j)_i$. The last section of the chapter is devoted to study the diagonals of the Rees algebra. In the case where A is the polynomial ring, we will show that if the Rees algebra is Cohen-Macaulay then there exists some diagonal with this property, thus proving a conjecture stated in [CHTV]. Furthermore, we will give necessary and sufficient conditions on the ring A for the existence of a Cohen-Macaulay diagonal of a Cohen-Macaulay Rees algebra.

2.1 The diagonal functor on the category of bigraded modules

Let $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ be the polynomial ring in $n + r$ variables over a field k with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d_j, 1)$, where $d_1, \dots, d_r \geq 0$. Set $d = \max\{d_1, \dots, d_r\}$, $u = \sum_{j=1}^{r} d_j$. Let us denote by $\mathcal{M}$ the homogeneous maximal ideal of S. Note that the irrelevant ideal S_+ of S is the ideal generated by the products X_iY_j, for $i = 1, \dots, n$, $j = 1, \dots, r$.

Given c, e positive integers, let Δ be the (c, e)-diagonal of $\mathbb{Z}^2$. For any bigraded S-module L, let us recall that the diagonal of L along Δ is defined as $L_\Delta := \bigoplus_{s\in\mathbb{Z}} L_{(cs,es)}$, which is a graded module over the graded ring $S_\Delta := \bigoplus_{s\geq 0} S_{(cs,es)}$. Our first lemma computes the dimension of the diagonals of a finitely generated bigraded S-module.

Lemma 2.1.1 *Let L be a finitely generated bigraded S-module. For $\Delta = (c, e)$ with $c \geq de + 1$, $\dim L_\Delta = \text{rel.dim } L - 1$.*

Proof. The proof follows the same lines as the one given for the bigraded standard case by A. Simis et al. in [STV, Proposition 2.3]. Set $\delta = \text{rel.dim} L$.

According to Proposition 1.5.1, there is a polynomial $P(s,t) \in \mathbb{Q}[s,t]$ of total degree $\delta - 2$ of the type

$$P(s,t) = \sum_{k+l \leq \delta-2} a_{kl} \binom{s-dt}{k} \binom{t}{l},$$

with $a_{kl} \geq 0$ for any k,l verifying $k+l = \delta-2$ such that for $i \gg dj$, $j \gg 0$, $P(i,j) = \dim_k L_{(i,j)}$. For any $c \geq de+1$, let us consider the polynomial $Q(u) = P(cu, eu) \in \mathbb{Q}[u]$. Then $Q(u) = \dim_k L_{(cu,eu)} = \dim_k (L_\Delta)_u$ for u large enough and deg $Q(u) = \delta - 2$. Therefore $\dim L_\Delta = \delta - 1$. □

From now on in the chapter we will always consider diagonals $\Delta = (c,e)$ with $c \geq de+1$. The next two propositions are inspired in some results and techniques used by E. Hyry in [Hy]. The first one shows how the local cohomology modules of L with respect to S_+ are related to the local cohomology modules of L_Δ with respect to $\mathcal{M}_\Delta$.

Proposition 2.1.2 *Let L be a finitely generated bigraded S-module. Then there are graded isomorphisms*

$$H^q_{S_+}(L)_\Delta \cong H^q_{\mathcal{M}_\Delta}(L_\Delta) \quad ,\forall q.$$

Proof. Let $\mathcal{N}$ be the ideal of S generated by $\mathcal{M}_\Delta$. Observe that $\sqrt{S_+} = \sqrt{\mathcal{N}}$, so we immediately get a bigraded isomorphism $H^q_{S_+}(L) \cong H^q_{\mathcal{N}}(L)$, $\forall q \geq 0$. Denoting by $g_1, \ldots, g_s$ a k-basis of $S_{(c,e)}$, we have that $\mathcal{N}$ can be generated by $g_1, \ldots, g_s$. So we may compute the local cohomology modules of L with respect to $\mathcal{N}$ from the Cěch complex built up from these elements

$$\mathbf{C}^\cdot : 0 \to C^0 \to C^1 \to \ldots \to C^s \to 0,$$

$$C^t = \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq s} L_{g_{i_1} g_{i_2} \cdots g_{i_t}},$$

with the differentation $d^t : C^t \to C^{t+1}$ defined on the component

$$L_{g_{i_1} g_{i_2} \cdots g_{i_t}} \longrightarrow L_{g_{j_1} g_{j_2} \cdots g_{j_t} g_{j_{t+1}}}$$

to be the homomorphism $(-1)^{m-1}\text{nat} : L_{g_{i_1} g_{i_2} \cdots g_{i_t}} \to (L_{g_{i_1} g_{i_2} \cdots g_{i_t}})_{g_{j_m}}$ if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \widehat{j_m}, \ldots, j_{t+1}\}$, and 0 otherwise. Then $H^q_{\mathcal{N}}(L) \cong H^q(\mathbf{C}^\cdot)$. We can also consider the Cěch complex associated to L_Δ built up from $g_1, \ldots, g_s$

$$\mathbf{D}^\cdot : 0 \to D^0 \to D^1 \to \ldots \to D^s \to 0,$$

$$D^t = \bigoplus_{1 \leq i_1 < i_2 < \ldots < i_t \leq s} (L_\Delta)_{g_{i_1} g_{i_2} \cdots g_{i_t}},$$

with the differentation $\delta^t : D^t \to D^{t+1}$ defined on the component

$$(L_\Delta)_{g_{i_1} g_{i_2} \cdots g_{i_t}} \longrightarrow (L_\Delta)_{g_{j_1} g_{j_2} \cdots g_{j_t} g_{j_{t+1}}}$$

to be the homomorphism $(-1)^{m-1}\text{nat} : (L_\Delta)_{g_{i_1} g_{i_2} \cdots g_{i_t}} \to ((L_\Delta)_{g_{i_1} g_{i_2} \cdots g_{i_t}})_{g_{j_m}}$ if $\{i_1, \ldots, i_t\} = \{j_1, \ldots, \widehat{j_m}, \ldots, j_{t+1}\}$, and 0 otherwise. Then $H^q_{\mathcal{M}_\Delta}(L_\Delta) \cong H^q(\mathbf{D}^\cdot)$. Note that $d^t|D^t = \delta^t$, so we have $(\operatorname{Ker} d^t)_\Delta = \operatorname{Ker} \delta^t$, $(\operatorname{Im} d^t)_\Delta = \operatorname{Im} \delta^t$. Therefore we may conclude $H^q_{\mathcal{N}}(L)_\Delta \cong H^q_{\mathcal{M}_\Delta}(L_\Delta)$. □

Now, let S_1, S_2 be the bigraded subalgebras of S defined by $S_1 = k[X_1, \ldots, X_n]$, $S_2 = k[Y_1, \ldots, Y_r]$, and note that the ideals $\mathfrak{m}_1 = (X_1, ..., X_n)$ and $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$ are the homogeneous maximal ideals of S_1 and S_2 respectively. Then let us define $\mathcal{M}_1$ to be the ideal of S generated by $\mathfrak{m}_1$ and $\mathcal{M}_2$ to be the ideal of S generated by $\mathfrak{m}_2$. Note that $\mathcal{M}_1 + \mathcal{M}_2 = \mathcal{M}$ and $\mathcal{M}_1 \cap \mathcal{M}_2 = S_+$. Therefore we have

Proposition 2.1.3 *Let L be a finitely generated bigraded S-module. There is a natural graded exact sequence*

$$\ldots \to H^q_{\mathcal{M}}(L)_\Delta \to H^q_{\mathcal{M}_1}(L)_\Delta \oplus H^q_{\mathcal{M}_2}(L)_\Delta \to H^q_{\mathcal{M}_\Delta}(L_\Delta) \xrightarrow{\varphi^q_L} H^{q+1}_{\mathcal{M}}(L)_\Delta \to \ldots$$

Proof. We get the result by applying the diagonal functor to the Mayer-Vietoris sequence associated to $\mathcal{M}_1$, $\mathcal{M}_2$ and by then using Proposition 2.1.2. □

As a first consequence we may recover the following result by A. Conca et al. in [CHTV].

Corollary 2.1.4 *[CHTV, Theorem 3.6] Let L be a finitely generated bigraded S-module. For all $q \geq 0$, there exists a canonical graded homomorphism*

$$\varphi^q_L : H^q_{\mathcal{M}_\Delta}(L_\Delta) \to H^{q+1}_{\mathcal{M}}(L)_\Delta,$$

which is an isomorphism for $q > \max\{n, r\}$.

Proof. Since $\mathcal{M}_1$ is generated by n elements, we have that $H^q_{\mathcal{M}_1}(L) = 0$ for any $q > n$. Similarly, $H^q_{\mathcal{M}_2}(L) = 0$ for any $q > r$. Now, the corollary follows from Proposition 2.1.3. □

Moreover, let us also notice that Proposition 2.1.3 precises the obstructions for φ^q_L to be isomorphism. Denote by $[\varphi^q_L]_s : H^q_{\mathcal{M}_\Delta}(L_\Delta)_s \to H^{q+1}_{\mathcal{M}}(L)_{(cs,es)}$ the component of degree s of the map φ^q_L. Then we have

Corollary 2.1.5 *Let L be a finitely generated bigraded S-module. For a given $s \in \mathbb{Z}$, the following are equivalent*

(i) $[\varphi_L^q]_s$ *is an isomorphism, for all* $q \geq 0$.

(ii) $H^q_{\mathcal{M}_1}(L)_{(cs,es)} = H^q_{\mathcal{M}_2}(L)_{(cs,es)} = 0$, *for all* $q \geq 0$.

In particular, φ_L^q *is an isomorphism for all* $q \geq 0$ *if and only if* $H^q_{\mathcal{M}_1}(L)_\Delta = H^q_{\mathcal{M}_2}(L)_\Delta = 0$ *for all* $q \geq 0$.

Therefore, the obstructions for the maps φ_L^q to be isomorphisms are located in the vanishing of the local cohomology modules with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$. So our next goal will be to study these local cohomology modules. For that, let us consider

$$0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$$

the $\mathbb{Z}^2$-graded minimal free resolution of L over S. For every p, D_p is a finite direct sum of S-modules of the type $S(a,b)$. If we apply the diagonal functor to this resolution, we get a resolution of L_Δ by means of the modules $S(a,b)_\Delta$. Let us begin by studying the local cohomology modules of the bigraded S-modules obtained by shifting S with degree (a,b).

First, let us fix some notations. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$, and $\beta = (\beta_1, \ldots, \beta_r) \in \mathbb{Z}^r$, we write X^α for the monomial $X_1^{\alpha_1} \cdots X_n^{\alpha_n}$ and Y^β for the monomial $Y_1^{\beta_1} \cdots Y_r^{\beta_r}$. Note that $\deg(X^\alpha) = (\sum_{i=1}^n \alpha_i, 0)$, $\deg(Y^\beta) = (\sum_{i=1}^r d_i\beta_i, \sum_{i=1}^r \beta_i)$. We will write $\alpha < 0$ (or $\alpha \geq 0$) if all the components of α satisfy this condition, and the same for β. Then we have:

Proposition 2.1.6 *Let* $a, b \in \mathbb{Z}$.

(i)

$$H^q_{\mathcal{M}_1}(S(a,b)) = \begin{cases} 0 & \text{if } q \neq n \\ (\bigoplus_{\alpha<0,\beta\geq 0} kX^\alpha Y^\beta)(a,b) & \text{if } q = n \end{cases}$$

(ii)

$$H^q_{\mathcal{M}_2}(S(a,b)) = \begin{cases} 0 & \text{if } q \neq r \\ (\bigoplus_{\alpha\geq 0,\beta< 0} kX^\alpha Y^\beta)(a,b) & \text{if } q = r \end{cases}$$

Proof. Since $S(a,b)$ is a free S_1-module with basis the monomials in the variables $Y_1, \ldots, Y_r$, we have that $H^q_{\mathcal{M}_1}(S(a,b)) = 0$ for all $q \neq n$, and $H^n_{\mathcal{M}_1}(S(a,b)) = (\bigoplus_{\beta\geq 0} H^n_{\mathfrak{m}_1}(S_1)Y^\beta)(a,b) = (\bigoplus_{\alpha<0,\beta\geq 0} kX^\alpha Y^\beta)(a,b)$. By taking into account that $S(a,b)$ is a free S_2-module with basis the monomials in the variables $X_1,\dots,X_n$, we also get $H^q_{\mathcal{M}_2}(S(a,b)) = 0$ for all $q \neq r$, and $H^r_{\mathcal{M}_2}(S(a,b)) = (\bigoplus_{\alpha\geq 0} H^r_{\mathfrak{m}_2}(S_2)X^\alpha)(a,b) = (\bigoplus_{\alpha\geq 0,\beta<0} kX^\alpha Y^\beta)(a,b)$. □

Corollary 2.1.7 *Let $a,b\in\mathbb{Z}$.*

(i)

$$\operatorname{supp}(H^q_{\mathcal{M}_1}(S(a,b))_\Delta) = \begin{cases} \emptyset & \text{if } q\neq n \\ \{ s\in\mathbb{Z} \mid \frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed} \} & \text{if } q = n \end{cases}$$

(ii)

$$\operatorname{supp}(H^q_{\mathcal{M}_2}(S(a,b))_\Delta) = \begin{cases} \emptyset & \text{if } q\neq r \\ \{ s\in\mathbb{Z} \mid \frac{(b+r)d-u-a}{c-ed} \leq s \leq \frac{-b-r}{e} \} & \text{if } q = r \end{cases}$$

Proof. From Proposition 2.1.6, a straightforward computation gives the support by taking into account that a monomial $X^\alpha Y^\beta$ in $H^n_{\mathcal{M}_1}(S(a,b))$ has degree (p,q) with $p = \sum_{i=1}^n \alpha_i + \sum_{j=1}^r d_j\beta_j - a$ and $q = \sum_{j=1}^r \beta_j - b$. Similarly one gets *(ii)*. □

For a real number x, let us denote by $[x] = \max\{n\in\mathbb{Z} \mid n\leq x\}$ the integral part of x. The following corollary gives necessary and sufficient numerical conditions for $S(a,b)_\Delta$ to be Cohen-Macaulay in terms of the diagonal Δ and the shift (a,b). In particular, notice that S_Δ is Cohen-Macaulay for any Δ.

Corollary 2.1.8 *[CHTV, Proposition 3.4] Assume $n,r\geq 2$. For any $a,b\in\mathbb{Z}$, $S(a,b)_\Delta$ is a Cohen-Macaulay S_Δ-module if and only if $[\frac{bd-a-n}{c-ed}] < \frac{-b}{e}$ and $[\frac{-b-r}{e}] < \frac{(b+r)d-u-a}{c-ed}$.*

Proof. Since S is a domain, we have that $\operatorname{rel.dim} S(a,b) = \operatorname{rel.dim} S = \dim S = n+r$, and so $\dim S(a,b)_\Delta = n+r-1$ by Lemma 2.1.1. Therefore, $S(a,b)_\Delta$ is Cohen-Macaulay if and only if $H^q_{\mathcal{M}_\Delta}(S(a,b)_\Delta) = 0$ for any $q < n+r-1$. By Proposition 2.1.3, note that for $q < n+r-1$ we have that

$$H^q_{\mathcal{M}_\Delta}(S(a,b)_\Delta) \cong H^q_{\mathcal{M}_1}(S(a,b))_\Delta \oplus H^q_{\mathcal{M}_2}(S(a,b))_\Delta.$$

Since $n,r\geq 2$, we get $n+r-2\geq n,r$, and then the result follows from Corollary 2.1.7. □

Remark 2.1.9 Note that if $n=r=1$, the proof above shows that $S(a,b)_\Delta$ is always Cohen-Macaulay. In the case where $n\geq 2$, $r=1$, we get that $S(a,b)_\Delta$ is Cohen-Macaulay if and only if $[\frac{-b-r}{e}] < \frac{(b+r)d-u-a}{c-ed}$, while if $n=1$, $r\geq 2$, $S(a,b)_\Delta$ is Cohen-Macaulay if and only if $[\frac{bd-a-n}{c-ed}] < \frac{-b}{e}$.

For simplicity, from now on we will assume $n, r \geq 2$. Now, let L be a finitely generated bigraded S-module. For any $p \geq 0$, let us denote by $\Omega_{p,L}$ the set of shifts (a,b) which appear in the place p of its bigraded minimal free resolution, and Ω_L the union of all these sets. Often we will write Ω_p, Ω if there is not danger of confusion with respect to the module L. The next result relates the local cohomology of the diagonals L_Δ of L to the local cohomology of the diagonals $S(a,b)_\Delta$ of the modules $S(a,b)$ which arise in its minimal free resolution.

Proposition 2.1.10 *Let L be a finitely generated bigraded S-module. Then*

(i) If $H^q_{\mathcal{M}_1}(L)_{(cs,es)} \neq 0$, then there exists a shift $(a,b) \in \Omega_{n-q,L}$ such that $H^n_{\mathcal{M}_1}(S(a,b))_{(cs,es)} \neq 0$, and so $\frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}$.

(ii) If $H^q_{\mathcal{M}_2}(L)_{(cs,es)} \neq 0$, then there exists a shift $(a,b) \in \Omega_{r-q,L}$ such that $H^r_{\mathcal{M}_2}(S(a,b))_{(cs,es)} \neq 0$, and so $\frac{(b+r)d-u-a}{c-ed} \leq s \leq \frac{-b-r}{e}$.

Proof. To prove (i), let $0 \to D_t \to \ldots \to D_0 \to L \to 0$ be the bigraded minimal free resolution of L over S. By considering $C_p = \operatorname{Coker}(D_{p+1} \to D_p)$ for $p \geq 0$, this yields the short exact sequences

$$0 \to C_{p+1} \to D_p \to C_p \to 0, \quad \forall p \geq 0.$$

If $H^q_{\mathcal{M}_1}(L) \neq 0$, then $q \leq n$ because $\mathcal{M}_1$ is generated by n elements. In the case $q = n$, from the short exact sequence $0 \to C_1 \to D_0 \to L \to 0$, we obtain a bigraded epimorphism $H^n_{\mathcal{M}_1}(D_0) \to H^n_{\mathcal{M}_1}(L)$. Therefore, if $H^n_{\mathcal{M}_1}(L)_{(cs,es)} \neq 0$ then $H^n_{\mathcal{M}_1}(D_0)_{(cs,es)} \neq 0$, so by Corollary 2.1.7 there exists a shift $(a,b) \in \Omega_0$ such that $\frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}$. If $q < n$, since $H^v_{\mathcal{M}_1}(D_p) = 0$ for any $v \neq n$, we have bigraded isomorphisms

$$H^q_{\mathcal{M}_1}(L) \cong H^{q+1}_{\mathcal{M}_1}(C_1) \cong H^{q+2}_{\mathcal{M}_1}(C_2) \cong \ldots \cong H^{n-1}_{\mathcal{M}_1}(C_{n-q-1}),$$

a bigraded monomorphism

$$0 \to H^{n-1}_{\mathcal{M}_1}(C_{n-q-1}) \to H^n_{\mathcal{M}_1}(C_{n-q}),$$

and a bigraded epimorphism

$$H^n_{\mathcal{M}_1}(D_{n-q}) \to H^n_{\mathcal{M}_1}(C_{n-q}) \to 0.$$

Therefore, $H^q_{\mathcal{M}_1}(L)_{(cs,es)} \neq 0$ implies $H^n_{\mathcal{M}_1}(D_{n-q})_{(cs,es)} \neq 0$, and we are done by Corollary 2.1.7. Similarly one can prove (ii). □

Remark 2.1.11 Given a finitely generated bigraded S-module L, for each diagonal $\Delta = (c, e)$ let us consider the sets of integers

$$X_p^\Delta = \bigcup_{(a,b)\in\Omega_{p,L}} \operatorname{supp}(H^n_{\mathcal{M}_1}(S(a,b))_\Delta),$$

$$Y_p^\Delta = \bigcup_{(a,b)\in\Omega_{p,L}} \operatorname{supp}(H^r_{\mathcal{M}_2}(S(a,b))_\Delta),$$

where $X_p^\Delta = Y_p^\Delta = \emptyset$ if $p < 0$. Let $X^\Delta = \bigcup_p X_p^\Delta$, $Y^\Delta = \bigcup_p Y_p^\Delta$. Then, Proposition 2.1.10 jointly with Proposition 2.1.3 says that if $s \notin X^\Delta_{n-q} \cup Y^\Delta_{r-q}$, then $[\varphi^q_L]_s$ is a monomorphism and $[\varphi^{q-1}_L]_s$ is an epimorphism. In particular, for an integer $s \notin X^\Delta \cup Y^\Delta$ then $[\varphi^q_L]_s$ is an isomorphism for any q. (In fact, it is enough to define $X^\Delta = \bigcup_{p\leq n} X^\Delta_p$ and $Y^\Delta = \bigcup_{p\leq r} Y^\Delta_p$).

Note that the set of integers s satisfying that $\frac{-b}{e} \leq s \leq \frac{bd-a-n}{c-ed}$ and $\frac{(b+r)d-u-a}{c-ed} \leq s \leq \frac{-b-r}{e}$ is empty or it only contains the integer 0 for suitable c, e, that is, the sets X^Δ and Y^Δ are contained in $\{0\}$ for those $\Delta = (c, e)$. So we immediately get:

Corollary 2.1.12 *Let L be a finitely generated bigraded S-module. There exist positive integers e_0, α such that for any $e > e_0$, $c > de + \alpha$, we have isomorphisms $[\varphi^q_L]_s : H^q_{\mathcal{M}_\Delta}(L_\Delta)_s \to H^{q+1}_{\mathcal{M}}(L)_{(cs,es)}$ for all $q \geq 0$ and $s \neq 0$.*

Proof. It is enough to take $e_0 \geq \max\{b, -b-r : (a,b) \in \Omega_L\}$ and $\alpha \geq \max\{bd - a - n, u + a - (b+r)d : (a,b) \in \Omega_L\}$. □

A similar result has been obtained in [CHTV, Lemma 3.8]. Therefore, we have that for diagonals large enough the only obstruction for the map φ^q_L to be an isomorphism is located in the component of degree 0.

Definition 2.1.13 Let L be a finitely generated bigraded S-module and let

$$0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$$

be the bigraded minimal free resolution of L over S. Let Δ be a diagonal. We say that the resolution is good for Δ if all the modules $(D_p)_\Delta$ are Cohen-Macaulay, that is, $X^\Delta = Y^\Delta = \emptyset$. We say that the resolution is good if there exists Δ such that the resolution is good for Δ.

From Remark 2.1.11, we immediately get that if L has a good resolution for Δ then the corresponding maps φ^q_L are isomorphisms.

Corollary 2.1.14 *Let L be a finitely generated bigraded S-module whose resolution is good for Δ. Then we have graded isomorphisms*

$$\varphi_L^q : H^q_{\mathcal{M}_\Delta}(L_\Delta) \to H^{q+1}_{\mathcal{M}}(L)_\Delta, \ \forall q \geq 0.$$

Our next goal is to study the existence of diagonals Δ for which the bigraded minimal free resolution of L is good for Δ. To this end, following [CHTV] we define:

Definition 2.1.15 We say that a property holds for $c \gg 0$ relatively to $e \gg 0$ if there exists e_0 such that for all $e > e_0$ there exists a positive integer $c(e)$ (depending on e) such that this property holds for all (c, e) with $c > c(e)$. We will often write $c \gg e \gg 0$. In fact, in the statements we will prove we could replace the condition $c \gg e \gg 0$ by the stronger one that there exist positive integers e_0, α such that the property holds for $e > e_0$, $c > de + \alpha$. For simplicity, we will keep the notation and definition of $c \gg e \gg 0$ from [CHTV].

Next result provides necessary and sufficient numerical conditions for the Cohen-Macaulayness of $S(a,b)_\Delta$ for $c \gg e \gg 0$. Namely,

Proposition 2.1.16 *[CHTV, Corollary 3.5] Let $a, b \in \mathbb{Z}$. Then $S(a,b)_\Delta$ is a Cohen-Macaulay module for $c \gg e \gg 0$ if and only if a, b satisfy one of the following conditions:*

(i) $b \leq -r$ *and* $(b+r)d - u - a > 0$,

(ii) $-r < b < 0$,

(iii) $b \geq 0$ *and* $bd - a - n < 0$.

Proof. From Proposition 2.1.3, we have that $S(a,b)_\Delta$ is Cohen-Macaulay for $c \gg e \gg 0$ if and only if $0 \notin \operatorname{supp}(H^n_{\mathcal{M}_1}(S(a,b))_\Delta) \cup \operatorname{supp}(H^r_{\mathcal{M}_2}(S(a,b))_\Delta)$ for $c \gg e \gg 0$. Then the result follows from Corollary 2.1.7. □

Notice that, for a given diagonal Δ, we have that $S(a,b)_\Delta$ is Cohen-Macaulay if and only if the corresponding maps $\varphi^q_{S(a,b)}$ are isomorphisms for all $q \geq 0$. On the other hand, from the proof of Proposition 2.1.16, observe that if (a,b) does not satisfy any of the conditions above then $S(a,b)_\Delta$ is never Cohen-Macaulay. Therefore, we can not hope to extend Corollary 2.1.12 to the component of degree 0 of the maps φ_L^q. In fact, the proof of Proposition 2.1.3 shows that $[\varphi_L^q]_0$ does not depend on the diagonal Δ.

Furthermore, note that the proof of Proposition 2.1.16 also shows that if there exists Δ such that $S(a,b)_\Delta$ is Cohen-Macaulay then $S(a,b)_\Delta$ is Cohen-Macaulay for $c \gg e \gg 0$. Therefore, if a finitely generated bigraded S-module L has a good resolution, then the resolution of L is good for diagonals $\Delta = (c,e)$ with $c \gg e \gg 0$.

Up to now we have related the vanishing of the local cohomology with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ of a bigraded S-module L with the vanishing of the local cohomology with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$ of the modules $S(a,b)$ which arise in the bigraded minimal free resolution of L over S. This study has led us to get sufficient conditions on the shifts (a,b) in order to φ_L^q to be isomorphisms. In the rest of the section we shall deal with the computation of the local cohomology modules of a bigraded S-module L with respect to the ideals $\mathcal{M}_1$ and $\mathcal{M}_2$ by themselves.

In Corollary 2.1.14 we have given sufficient conditions on the shifts in Ω_L to get that the maps φ_L^q are isomorphisms for large diagonals. Next we give necessary and sufficient conditions for the maps φ_L^q to be isomorphisms in terms of the local cohomology modules of L with respect to $\mathcal{M}_1$ and $\mathcal{M}_2$. Namely,

Proposition 2.1.17 *Let L be a finitely generated bigraded S-module. Then the following are equivalent:*

(i) There exists Δ such that φ_L^q is an isomorphism for all $q \geq 0$.

(ii) For large diagonals Δ, φ_L^q is an isomorphism for all $q \geq 0$.

(iii) $H^q_{\mathcal{M}_1}(L)_{(0,0)} = H^q_{\mathcal{M}_2}(L)_{(0,0)} = 0$ for all $q \geq 0$.

For an integer e and a bigraded S-module L, let us define the graded S_1-module $L^e = \oplus_{i\in\mathbb{Z}} L_{(i,e)}$. Then we have an exact functor $(\ )^e : M^2(S) \to M^1(S_1)$. The bigraded initial degree of a bigraded S-module L is defined by $\mathbf{indeg}(L) = (\text{indeg}_1(L), \text{indeg}_2(L))$, where

$$\text{indeg}_1(L) = \min\{i \mid \exists j \ s.t.\ L_{(i,j)} \neq 0\},$$

$$\text{indeg}_2(L) = \min\{j \mid \exists i \ s.t.\ L_{(i,j)} \neq 0\}.$$

Proposition 2.1.18 *Let L be a finitely generated bigraded S-module. Then:*

(i) $H^q_{\mathcal{M}_1}(L)_{(i,j)} = H^q_{\mathfrak{m}_1}(L^j)_i$. *In particular,* $H^q_{\mathcal{M}_1}(L)_{(i,j)} = 0$ *for* $i > a_q(L^j)$ *or* $j < \operatorname{indeg}_2(L)$.

(ii) $H^q_{\mathcal{M}_2}(L)_{(i,j)} = 0$ *for* $j > a^2_*(L)$.

Proof. As S_1-module, L is the direct sum of the modules $L^e = \oplus_i L_{(i,e)}$. Since $\mathcal{M}_1$ is the ideal of S generated by $\mathfrak{m}_1 = (X_1, \ldots, X_n)$, we have that $H^q_{\mathcal{M}_1}(L) = \oplus_j H^q_{\mathfrak{m}_1}(L^j)$, and so we get (i).

Now let $0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$ be the bigraded minimal free resolution of L over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. By taking short exact sequences as in Proposition 2.1.10, it is just enough to prove that if $j > a^2_*(L)$ then $H^q_{\mathcal{M}_2}(S(a,b))_{(i,j)} = 0$ for any $(a,b) \in \Omega_L$ and $q \geq 0$. The case $q \neq r$ is trivial. From Proposition 2.1.6, we may deduce that $H^r_{\mathcal{M}_2}(S(a,b))_{(i,j)} = 0$ for $j > -b-r$. This finishes the proof because, according to Theorem 1.3.4, $a^2_*(L) \geq -b-r$ for any $(a,b) \in \Omega_L$. □

In the particular case $d_1 = \ldots = d_r = d$, S can be thought as a standard bigraded k-algebra by a change of grading. If we consider the morphism $\varphi(p,q) = (p-dq, q)$, observe that $\varphi(\operatorname{supp} S) \subset \mathbb{N}^2$, so S^φ is a $\mathbb{N}^2$-graded ring with $[S^\varphi]_{(p,q)} = S_{(p+dq,q)}$. Noting that $\deg(X_i) = (1,0)$ for $i = 1, \ldots, n$, and $\deg(Y_j) = (0,1)$ for $j = 1, \ldots, r$ as elements of S^φ, we have that S^φ is standard. For a bigraded S-module L, let us recall that the S^φ-module L^φ is the S-module L with the grading defined by $[L^\varphi]_{(p,q)} = L_{(p+dq,q)}$.

Furthermore, in this case, given an integer e we can define an exact functor $(\ )_e : M^2(S) \to M^1(S_2)$ in the following way: For any bigraded S-module L, we define L_e to be the graded S_2-module $L_e = \bigoplus_{j\in\mathbb{Z}} L_{(e+dj,j)}$. Then we have

Proposition 2.1.19 *Assume that $d_1 = \ldots = d_r = d$. For any finitely generated bigraded S-module L, we have*

(i) $H^q_{\mathcal{M}_1}(L)_{(i,j)} = 0$ *for* $i > dj + a^1_*(L^\varphi)$.

(ii) $H^q_{\mathcal{M}_2}(L)_{(i,j)} = H^q_{\mathfrak{m}_2}(L_{i-dj})_j$. *In particular,* $H^q_{\mathcal{M}_2}(L)_{(i,j)} = 0$ *for* $j > a_q(L_{i-dj})$.

Proof. Let $0 \to D_t \to \cdots \to D_1 \to D_0 \to L \to 0$ be the bigraded minimal free resolution of L over S. Observe that

$$S(a,b)^\varphi = \bigoplus_{(i,j)} S(a,b)_{(i+dj,j)}$$

$$= \bigoplus_{(i,j)} S_{(a+i+dj,b+j)}$$
$$= \bigoplus_{(i,j)} S_{(a-db+i+d(b+j),b+j)}$$
$$= S^{\varphi}(a-db,b),$$

so in particular $S(a,b)^{\varphi}$ is a free S^{φ}-module. Therefore, by applying the exact functor $(\ )^{\varphi}$ to the resolution of L we get that

$$0 \to D_t^{\varphi} \to \ldots \to D_0^{\varphi} \to L^{\varphi} \to 0$$

is a bigraded minimal free resolution of L^{φ} over S^{φ}. Since $a^1(S^{\varphi}) = -n$, from Theorem 1.3.4 it follows that

$$a_*^1(L^{\varphi}) = \max\{\, db - a \mid (a,b) \in \Omega_L\} - n.$$

If i, j are such that $i > dj + a_*^1(L^{\varphi})$, then we have that $i > dj + db - a - n$ for any shift $(a,b) \in \Omega_L$, and so from Proposition 2.1.6 we have that $H^q_{\mathcal{M}_1}(S(a,b))_{(i,j)} = 0$ for any $q \geq 0$. By taking short exact sequences as in Proposition 2.1.18, we then obtain $H^q_{\mathcal{M}_1}(L)_{(i,j)} = 0$ for $q \geq 0$, $i > dj + a_*^1(L^{\varphi})$.

To prove (ii), note that since $d_1 = \ldots = d_r = d$ we may decompose L as the direct sum of the S_2-modules L_i. Then, by using that $\mathcal{M}_2$ is the ideal of S generated by $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$, we obtain $H^q_{\mathcal{M}_2}(L) = \bigoplus_i H^q_{\mathfrak{m}_2}(L_i)$. Noting that $\deg(Y_1) = \ldots = \deg(Y_r) = (d,1)$, we finally get $H^q_{\mathcal{M}_2}(L)_{(i,j)} = H^q_{\mathfrak{m}_2}(L_{i-dj})_j$. □

2.2 Case study: Standard bigraded k-algebras

Our aim in this section is to particularize and improve for standard bigraded k-algebras several results proved in Section 2.1. So let R be a standard bigraded k-algebra generated by homogeneous elements $x_1, \ldots, x_n, y_1, \ldots, y_r$ in degrees $\deg(x_i) = (1,0)$, $i = 1, \ldots, n$, $\deg(y_j) = (0,1)$, $j = 1, \ldots, r$. By taking the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ with the bigrading given by $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$, we have that R is a finitely generated bigraded S-module in a natural way.

In this case, denote by $\mathcal{R}_1 = R^0 = \bigoplus_{i \in \mathbb{N}} R_{(i,0)}$, $\mathcal{R}_2 = R_0 = \bigoplus_{j \in \mathbb{N}} R_{(0,j)}$. Observe that $\mathcal{R}_1$ and $\mathcal{R}_2$ are graded k-algebras, and denote by $\mathfrak{m}_1$ and $\mathfrak{m}_2$ their homogeneous maximal ideals. Given $e \in \mathbb{Z}$, we may define the graded $\mathcal{R}_1$-module $R^e = \oplus_{i \in \mathbb{Z}} R_{(i,e)}$ and the graded $\mathcal{R}_2$-module $R_e = \oplus_{j \in \mathbb{Z}} R_{(e,j)}$. By a straightforward application of Proposition 2.1.3, we get

Proposition 2.2.1 *There is a natural graded exact sequence*

$$\dots \to H^q_{\mathcal{M}}(R)_\Delta \to H^q_{\mathcal{M}_1}(R)_\Delta \oplus H^q_{\mathcal{M}_2}(R)_\Delta \to H^q_{\mathcal{M}_\Delta}(R_\Delta) \xrightarrow{\varphi^q_R} H^{q+1}_{\mathcal{M}}(R)_\Delta \to \dots$$

In particular, given $s \in \mathbb{Z}$, the following are equivalent:

(i) $[\varphi^q_R]_s$ *is isomorphism,* $\forall q \geq 0$.

(ii) $H^q_{\mathfrak{m}_1}(R^{es})_{cs} = 0$ *and* $H^q_{\mathfrak{m}_2}(R_{cs})_{es} = 0$, $\forall q \geq 0$.

Proof. It follows from Proposition 2.1.3, Proposition 2.1.18 and Proposition 2.1.19. □

As a direct consequence of Proposition 2.2.1 we have:

Corollary 2.2.2 φ^q_R *is an isomorphism for* $q > \max\{\dim \mathcal{R}_1, \dim \mathcal{R}_2\}$.

Proof. Set $d_1 = \dim \mathcal{R}_1$, $d_2 = \dim \mathcal{R}_2$. It is enough to prove that $H^q_{\mathfrak{m}_1}(R^e) = H^q_{\mathfrak{m}_2}(R_e) = 0$ for any $e \in \mathbb{Z}$, $q > \max\{d_1, d_2\}$. But note that R^e is a graded $\mathcal{R}_1$-module, so $H^q_{\mathfrak{m}_1}(R^e) = 0$ for $q > d_1$. Similarly, $H^q_{\mathfrak{m}_2}(R_e) = 0$ for $q > d_2$ and we are done. □

From Proposition 2.2.1 we can also determine a set of integers s, depending on the diagonal Δ, for which $[\varphi^q_R]_s$ is an isomorphism for all $q \geq 0$. More explicitly,

Corollary 2.2.3 *(i)* $[\varphi^q_R]_s$ *is isomorphism for* $s < 0$.

(ii) $[\varphi^q_R]_s$ *is isomorphism for* $s > \max\{a^1_*(R)/c, a^2_*(R)/e\}$. *In particular,* $a_*(R_\Delta) \leq \max\{a^1_*(R)/c, a^2_*(R)/e\}$.

Proof. It is a direct consequence of Proposition 2.2.1, Proposition 2.1.18 and Proposition 2.1.19. □

We have shown that $[\varphi^q_R]_s$ is an isomorphism for any $s < 0$. Moreover, note that if $c > a^1_*(R)$ and $e > a^2_*(R)$, then $[\varphi^q_R]_s$ is an isomorphism for any $s > 0$. We may ensure that $[\varphi^q_R]_0$ is an isomorphism for any q if R has a good resolution. Next we study the existence of a such resolution. The following result provides a useful characterization for a standard bigraded k-algebra R to have a good resolution by means of the a_*-invariants of $\mathcal{R}_1$ and $\mathcal{R}_2$. Namely,

Proposition 2.2.4 *The following are equivalent:*

(i) R has a good resolution.

(ii) $a_*(\mathcal{R}_1) < 0$, $a_*(\mathcal{R}_2) < 0$.

Proof. Let us consider

$$\mathbf{D}_. : 0 \to D_t \to \cdots \to D_1 \to D_0 = S \to R \to 0$$

the bigraded minimal free resolution of R over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. Note that a, b are non-positive integers. So, by Proposition 2.1.16 the resolution is good if and only if all the shifts (a,b) satisfy one of the three following conditions

(i) $-r < b < 0$.

(ii) $b = 0$ and $-n < a$.

(iii) $b \leq -r$ and $a < 0$.

It is not hard to check that these conditions are equivalent to that for any shift $(a,0) \in \Omega_R$ we have $a > -n$, and for any shift $(0,b) \in \Omega_R$ we have $b > -r$. Observe that

$$S(a,b)^0 = \bigoplus_j S_{(a+j,b)} = \begin{cases} 0 & \text{if } b < 0 \\ S_1(a) & \text{if } b = 0 \end{cases}$$

So by applying the functor $(\ )^0$ to the resolution of R we obtain a graded free resolution of $\mathcal{R}_1$ over S_1:

$$\mathbf{F}_. : 0 \to F_t \to \cdots \to F_1 \to F_0 = S_1 \to \mathcal{R}_1 \to 0\ ,$$

where $F_p = (D_p)^0 = \bigoplus_{a\in\gamma_p} S_1(a)$, and $\gamma_p = \{a \in \mathbb{Z} : (a,0) \in \Omega_p\}$ ($F_p = 0$ if $\gamma_p = \emptyset$). Furthermore, we have that $\mathrm{Im}(F_p) \subset \mathfrak{m}_1 F_{p-1}$ for all $p = 1, ..., t$. Hence this resolution is in fact the graded minimal free resolution of $\mathcal{R}_1$ over S_1. Then we can use Theorem 1.3.4 to compute $a_*(\mathcal{R}_1)$:

$$a_*(\mathcal{R}_1) = \max\{-a \mid a \in \cup_p \gamma_p\} + a(S_1) =$$

$$= \max\{-a \mid (a,0) \in \Omega_R\} - n.$$

Therefore, any shift $(a,0) \in \Omega_R$ satisfies $a > -n$ if and only if $a_*(\mathcal{R}_1) < 0$. Similarly, any shift $(0,b) \in \Omega_R$ satisfies $b > -r$ if and only if $a_*(\mathcal{R}_2) < 0$. □

As an immediate consequence we get a criterion for the existence of Cohen-Macaulay diagonals of a standard bigraded k-algebra which extends [CHTV, Corollary 3.12]. More explicitly,

Corollary 2.2.5 *Let R be a standard bigraded k-algebra with $a_*(\mathcal{R}_1) < 0$, $a_*(\mathcal{R}_2) < 0$. Then* $\operatorname{depth} R_\Delta \geq \operatorname{depth} R - 1$ *for large Δ. In particular, if R is Cohen-Macaulay, then so R_Δ for large Δ.*

For a standard bigraded ring R defined over a local ring with $a^1(R), a^2(R) < 0$, it has been shown in [Hy, Theorem 2.5] that if R is Cohen-Macaulay, then its $(1,1)$-diagonal inherits this property. This result can be extended to any diagonal of a standard bigraded k-algebra.

Proposition 2.2.6 *Let R be a standard bigraded Cohen-Macaulay k-algebra with $a^1(R), a^2(R) < 0$. Then R_Δ is Cohen-Macaulay for any diagonal Δ.*

Proof. The bigraded standard k-algebra R has a presentation as a quotient of the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ bigraded by $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$. According to Theorem 1.3.4, for any shift $(a,b) \in \Omega_R$ we have that

$$0 \leq -a \leq a^1(R) - a^1(S) < n$$

$$0 \leq -b \leq a^2(R) - a^2(S) < r.$$

Then note that for any diagonal $\Delta = (c,e)$ with $c, e > 0$,

$$X^\Delta = \bigcup_{(a,b)\in\Omega_R} \{\, s \in \mathbb{Z} \mid \frac{-b}{e} \leq s \leq \frac{-a-n}{c} \,\} = \emptyset$$

$$Y^\Delta = \bigcup_{(a,b)\in\Omega_R} \{\, s \in \mathbb{Z} \mid \frac{-a}{c} \leq s \leq \frac{-b-r}{e} \,\} = \emptyset,$$

so the resolution is good for any Δ. Now, by Corollary 2.1.14 we have $H^q_{\mathcal{M}_\Delta}(R_\Delta) \cong H^{q+1}_{\mathcal{M}}(R)_\Delta = 0$ for $q < \dim R - 1$, so we are done. □

We finish this section by giving necessary and sufficient conditions on the local cohomology of $\mathcal{R}_1$ and $\mathcal{R}_2$ for the existence of a Cohen-Macaulay diagonal of a standard bigraded Cohen-Macaulay k-algebra R. Namely,

Proposition 2.2.7 *Let R be a standard bigraded Cohen-Macaulay k-algebra of relevant dimension δ. Then there exists Δ such that R_Δ is Cohen-Macaulay if and only if $H^q_{\mathfrak{m}_1}(\mathcal{R}_1)_0 = H^q_{\mathfrak{m}_2}(\mathcal{R}_2)_0 = 0$ for any $q < \delta - 1$.*

Proof. According to Lemma 2.1.1, we have that $\dim R_\Delta = \delta - 1$ for any Δ. By taking into account Corollary 2.1.12, there exists Δ such that R_Δ is

Cohen-Macaulay if and only if there exists Δ such that $H^q_{\mathcal{M}_\Delta}(R_\Delta)_0 = 0$ for any $q < \delta - 1$. But from Proposition 2.2.1, for any $q < \delta - 1$ we have

$$H^q_{\mathcal{M}_\Delta}(R_\Delta)_0 \cong H^q_{\mathfrak{m}_1}(\mathcal{R}_1)_0 \oplus H^q_{\mathfrak{m}_2}(\mathcal{R}_2)_0.$$

This finishes the proof. □

2.3 Case study: Rees algebras

Let A be a noetherian graded algebra generated in degree 1 over a field k. Then A has a presentation $A = k[X_1, \dots, X_n]/K = k[x_1, \dots, x_n]$, where K is a homogeneous ideal of the polynomial ring $k[X_1, \dots, X_n]$ with the usual grading. Let $\mathfrak{m}$ be the graded maximal ideal of A. For a homogeneous ideal I of A, let us consider the Rees algebra

$$R = R_A(I) = \bigoplus_{n \geq 0} I^n t^n \subset A[t]$$

of I endowed with the natural bigrading given by

$$R_{(i,j)} = (I^j)_i,$$

introduced by A. Simis et al. in [STV]. If I is generated by forms $f_1, \dots, f_r$ in degrees $d_1, \dots, d_r$ respectively, note that R is a k-algebra finitely generated by $x_1, \dots, x_n, f_1 t, ..., f_r t$ with $\deg(x_i) = (1,0)$, $\deg(f_j t) = (d_j, 1)$, and that it has a unique homogeneous maximal ideal $\mathcal{M} = (x_1, \dots, x_n, f_1 t, ..., f_r t)$. By considering the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ with the grading determined by setting $\deg(X_i) = (1,0)$ and $\deg(Y_j) = (d_j, 1)$, we have a bigraded epimorphism:

$$\begin{array}{ccc} S & \longrightarrow & R \\ X_i & \longmapsto & x_i \\ Y_j & \longmapsto & f_j t \end{array}$$

so that R has a natural structure as finitely generated bigraded S-module. Set $d = \max\{d_1, \dots, d_r\}$. For any $c \geq de + 1$, the $\Delta = (c, e)$-diagonal of the Rees algebra is

$$R_A(I)_\Delta = \bigoplus_{s \geq 0} (I^{es})_{cs} = k[(I^e)_c].$$

Note that $k[(I^e)_c]$ is a graded k-algebra with a unique homogeneous maximal ideal $m = \mathcal{M}_\Delta$. The interest of these algebras $k[(I^e)_c]$ is, as we will show in the next chapter, that for any $c \geq de+1$ there is a projective embedding of the blow-up X of Proj (A) along $\widetilde{I}$ so that $X \cong \text{Proj}\,(k[(I^e)_c])$.

Set $\overline{n} = \dim A$. The next lemma computes the dimension of the rings $k[(I^e)_c]$ extending [CHTV, Lemma 1.3] where the case $A = k[X_1, \ldots, X_n]$ was studied.

Lemma 2.3.1 *Assume $I \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in \text{Ass}\,(A)$. Then $\dim k[(I^e)_c] = \overline{n}$ for all $c \geq de+1$.*

Proof. Since I is not contained in any associated prime of A, we have that any associated prime ideal of the Rees algebra R is relevant. So rel.dim $R = \dim R$. Furthermore, $\dim R = \overline{n}+1$ by [BH1, Exercise 4.4.12]. So we may conclude $\dim k[(I^e)_c] = \overline{n}$ by Lemma 2.1.1. □

From now on we will always assume that $I \not\subset \mathfrak{p}$, for all $\mathfrak{p} \in \text{Ass}\,(A)$. The following result relates the local cohomology of the graded k-algebras $k[(I^e)_c]$ to the local cohomology of the Rees algebra. By setting $\mathfrak{n} = (It) = (f_1t, \ldots, f_rt) \subset k[It] = k[f_1t, \ldots, f_rt]$, we have

Proposition 2.3.2 *Let I be an ideal of A generated by forms of degree $\leq d$. For any diagonal $\Delta = (c,e)$ with $c \geq de+1$, there is a natural graded exact sequence*

$$\ldots \to H^q_{\mathcal{M}}(R)_\Delta \to H^q_{\mathfrak{m}R}(R)_\Delta \oplus H^q_{\mathfrak{n}R}(R)_\Delta \to H^q_{m}(k[(I^e)_c]) \xrightarrow{\varphi^q_R} H^{q+1}_{\mathcal{M}}(R)_\Delta \to \ldots$$

Proof. It is clear that $H^q_{\mathcal{M}_1}(R) = H^q_{\mathfrak{m}R}(R)$ and $H^q_{\mathcal{M}_2}(R) = H^q_{\mathfrak{n}R}(R)$. Then the result follows immediately by applying Proposition 2.1.3 to the Rees algebra R of I.□

In Corollary 2.1.4 we proved that the maps φ^q_L become isomorphisms for $q > \max\{n, r\}$. This bound was refined for standard bigraded k-algebras in Corollary 2.2.2. Next we want to consider the case of the Rees algebras. To this end, we are going to study the vanishing of the local cohomology modules of R with respect to $\mathfrak{n}R$. For any ideal I of A, the fiber cone of I is defined as the graded k-algebra $F = F_{\mathfrak{m}}(I) = \bigoplus_{n \geq 0} I^n/\mathfrak{m}I^n$. The analytic spread $l(I)$ of I is then the dimension of the fiber cone, that is, $l(I) = \dim F$. Note that if I is generated by forms of the same degree d, the fiber cone is nothing but $F_{\mathfrak{m}}(I) = k[I_d]$. The following lemma shows the known result that the local cohomology modules of R with respect to $\mathfrak{n}R$ vanish in order $q > l(I)$, but not in order $l(I)$. We include the proof for the sake of completeness.

Lemma 2.3.3 *Let I be a homogeneous ideal of A. Set $l = l(I)$. Then* $H^q_{\mathfrak{n}R}(R) = 0$, $\forall q > l$ *and* $H^l_{\mathfrak{n}R}(R) \neq 0$.

Proof. We may assume that the field k is infinite. Then there exists an ideal $J \subset I$ generated by $l = l(I)$ elements of A such that $I^m = JI^{m-1}$ for $m \gg 0$, that is, there exists a reduction J of I generated by l elements (see [BH1, Proposition 4.6.8]). Note that IR and JR are ideals with the same radical, so $H^q_{\mathfrak{n}R}(R) = H^q_{IR}(R) = H^q_{JR}(R) = 0, \forall q > l$. Moreover, from the presentation $R \to R/\mathfrak{m}R = F_{\mathfrak{m}}(I)$ we get the epimorphism

$$H^l_{\mathfrak{n}R}(R) \to H^l_{\mathfrak{n}R}(F) \neq 0,$$

so $H^l_{\mathfrak{n}R}(R) \neq 0$. □

As a consequence, we get:

Corollary 2.3.4 *Let I be an ideal of A generated by forms of degree $\leq d$. For any Δ, we have a graded epimorphism*

$$H^{\overline{n}}_{m}(k[(I^e)_c]) \xrightarrow{\varphi^{\overline{n}}_R} H^{\overline{n}+1}_{\mathcal{M}}(R)_\Delta.$$

From Proposition 2.3.2 we may also deduce that for diagonals large enough the positive components of the local cohomology of the diagonals of the Rees algebra coincide with the positive components of the local cohomology of the powers of the ideal. Namely,

Corollary 2.3.5 *Let I be an ideal generated by forms of degree $\leq d$. For any $c \geq de + 1$, $e > a^2_*(R)$, $s > 0$, there are isomorphisms*

$$H^q_m(k[(I^e)_c])_s \cong H^q_{\mathfrak{m}}(I^{es})_{cs} \ , \forall q \geq 0.$$

Proof. Let c, e be integers such that $c \geq de + 1$, $e > a^2_*(R)$. For any $s > 0$, we have that $H^q_{\mathcal{M}_1}(R)_{(cs,es)} = H^q_{\mathfrak{m}}(I^{es})_{cs}$ and $H^q_{\mathcal{M}_2}(R)_{(cs,es)} = 0$ by Proposition 2.1.18. Thus from Proposition 2.3.2 we get the isomorphisms $H^q_m(k[(I^e)_c])_s \cong H^q_{\mathfrak{m}}(I^{es})_{cs}$.□

In the case where I is generated by forms in the same degree d (that is, I is equigenerated) the Rees algebra is a standard bigraded k-algebra by setting

$$R_A(I)_{(i,j)} = (I^j)_{i+dj}.$$

Then we may apply the results in Section 2.2 to these Rees algebras. From Lemma 2.2.4, we get a useful characterization for the Rees algebra to have a good resolution by means of the a_*-invariants of the ring A and the fiber cone of I. Namely,

Proposition 2.3.6 *Let I be an ideal of A generated by forms in degree d. The following are equivalent:*

(i) The Rees algebra $R_A(I)$ has a good resolution.

(ii) $a_*(A) < 0$, $a_*(F_{\mathfrak{m}}(I)) < 0$.

Proof. We have already noted that the Rees algebra is a standard bigraded ring by means of $R_{(i,j)} = (I^j)_{i+dj}$. With this degree, notice that $\mathcal{R}_1 = A$ and $\mathcal{R}_2 = k[I_d] = F_{\mathfrak{m}}(I)$. Then the result follows from Lemma 2.2.4. □

To apply Proposition 2.3.6, we need to know the a_*-invariant of the fiber cone. The next two lemmas bound it. The first one gives a lower bound by means of the reduction number of I (compare with [Tr1], [Sch2]), while the second one gives an upper bound by means of the a_*-invariant of the Rees algebra.

Lemma 2.3.7 *Let $(A, \mathfrak{m})$ be a local noetherian ring with an infinite residue field. Let $I \subset \mathfrak{m}$ be an arbitrary ideal of A, J a minimal reduction of I and l the analytic spread of I. Then*

$$a_l(F_{\mathfrak{m}}(I)) + l \leq r_J(I) \leq \max\{a_i(F_{\mathfrak{m}}(I)) + i\} = \operatorname{reg}(F_{\mathfrak{m}}(I)).$$

Proof. Let $a_1, \ldots, a_l$ be a minimal system of generators of J. For $a \in I$, denote by a^0 the class of a in I/mI. Then $a_1^0, \ldots, a_l^0$ are a (homogeneous) system of parameters of $F_{\mathfrak{m}}(I)$ (see [HIO, Proposition 10.17]). According to [HIO, Lemma 45.1], we have

$$a_l(F_{\mathfrak{m}}(I)) + l \leq \max\{n \mid \left[\frac{F_{\mathfrak{m}}(I)}{(a_1^0, \ldots, a_l^0)}\right]_n \neq 0\} \leq \max\{a_i(F_{\mathfrak{m}}(I)) + i\}$$

On the other side,

$$\begin{aligned} r_J(I) &= \min\{n \mid I^{n+1} = JI^n\} \\ &= \min\{\, n \mid \tfrac{I^{n+1}}{mI^{n+1}} = \tfrac{JI^n + mI^{n+1}}{mI^{n+1}} \,\} \\ &= \min\{\, n \mid \left[\tfrac{F_{\mathfrak{m}}(I)}{(a_1^0, \ldots, a_l^0)}\right]_{n+1} = 0 \,\} \end{aligned}$$

$$= \max\{ n \mid \left[\frac{F_{\mathfrak{m}}(I)}{(a_1^0,\ldots,a_l^0)}\right]_n \neq 0 \}.$$

This concludes the lemma. □

The next lemma bounds the a_*-invariant of the fiber cone by means of the a_*-invariant of the Rees algebra. Namely,

Lemma 2.3.8 *Let I be an equigenerated homogeneous ideal of A. Then $a_*(F_{\mathfrak{m}}(I)) \leq a_*^2(R_A(I))$.*

Proof. Let

$$\mathbf{D}_. : \ 0 \to D_t \to \cdots \to D_1 \to D_0 = S \to R = R_A(I) \to 0$$

be the bigraded minimal free resolution of the Rees algebra R over S, where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. Note that a, b are non-positive integers with $a \leq db$. Therefore, we have that

$$S(a,b)_0 = \bigoplus_j S_{(a+dj,b+j)} = \begin{cases} 0 & \text{if } a < db \\ S_2(b) & \text{if } a = db \end{cases}$$

Then by applying the functor $(\ )_0$ to the resolution $\mathbf{D}_.$ we get a graded free resolution of $R_0 = F_{\mathfrak{m}}(I)$ over S_2:

$$\mathbf{F}_. : \ 0 \to F_t \to \cdots \to F_1 \to F_0 = S_2 \to F \to 0\ ,$$

where $F_p = (D_p)_0 = \bigoplus_{b\in\gamma_p} S_2(b)$, and $\gamma_p = \{b \in \mathbb{Z} : (db, b) \in \Omega_p\}$. Moreover, for any $p = 1, \ldots, t$ we have that $\mathrm{Im}(F_p) \subset \mathfrak{m}_2 F_{p-1}$, so $\mathbf{F}_.$ is in fact the graded minimal free resolution of $F_{\mathfrak{m}}(I)$ over S_2. Then by Theorem 1.3.4 we have:

$$\begin{aligned} a_*(F) &= \max\{-b \mid b \in \cup_p \gamma_p\} + a(S_2) \\ &\leq \max\{-b \mid (a,b) \in \Omega_R\} + a(S_2) \\ &= a_*^2(R). \quad \square \end{aligned}$$

Now we are ready to exhibit some families of ideals such that the diagonal functor and the local cohomology functor commute whenever we take diagonals large enough.

Example 2.3.9 Let I be an equigenerated ideal in a ring A with $a_*(A) < 0$ (for instance, we may take $A = k[X_1, \ldots, X_n]$). Set $r(I)$ the reduction number of I and assume that $F_{\mathfrak{m}}(I)$ is Cohen-Macaulay with negative a-invariant. Note that $a(F) < 0$ is equivalent to $r(I) < l(I)$ by Lemma 2.3.7. This class of ideals includes:

(i) ideals I with reduction number $r(I) = 0$ (for instance, complete intersection ideals and ideals of linear type).

(ii) $\mathfrak{m}$-primary ideals with $r(I) \leq 1 < l(I)$ in a Cohen-Macaulay ring A (see [HS]) .

(iii) equimultiple ideals with $r(I) \leq 1 < l(I)$ in a Cohen-Macaulay ring A (see [Sha]).

(iv) generically complete intersection ideals with $\mathrm{ad}(I) = 1$, $r(I) \leq 1 < l(I)$ in a Cohen-Macaulay ring A (see [CZ]).

For all these families of ideals, we have that the Rees algebra has a good resolution according to Lemma 2.3.6. Then we have graded isomorphisms

$$H^q_m(k[(I^e)_c]) \cong H^{q+1}_{\mathcal{M}}(R)_\Delta,$$

for $c \gg e \gg 0$ by Corollary 2.1.14. Therefore, we have that for large diagonals of the Rees algebra $\mathrm{depth}(k[(I^e)_c]) \geq \mathrm{depth}(R) - 1$. In particular, if the Rees algebra is Cohen-Macaulay then its large diagonals will be also Cohen-Macaulay.

Remark 2.3.10 Recall that the form ring $G_A(I)$ of an ideal I in A is

$$G = G_A(I) = \bigoplus_{n \geq 0} I^n/I^{n+1} = R_A(I)/IR_A(I).$$

If I is a homogeneous ideal, the form ring has a natural bigrading by means of $G_{(i,j)} = (I^j/I^{j+1})_i$. We can get for the form ring similar results to the ones obtained for the Rees algebra. For instance, for an equigenerated ideal I we have that $G_A(I)$ has a good resolution if and only if $a_*(A/I) < 0$, $a_*(F_{\mathfrak{m}}(I)) < 0$ and it holds $a_*(F_{\mathfrak{m}}(I)) \leq a^2_*(G_A(I))$.

Remark 2.3.11 For an equigenerated ideal I of A, note that we can recover several relationships between $r(I)$, $l(I)$ and $a^2(G)$ proved with more generality in [AHT]. By applying the diagonal functor to the minimal bigraded free resolution of $R_A(I)$ or $G_A(I)$, we obtain the minimal graded free resolution of $F_{\mathfrak{m}}(I) = k[I_d]$, and so $a_*(F_{\mathfrak{m}}(I)) \leq a^2_*(R_A(I))$ and $a_*(F_{\mathfrak{m}}(I)) \leq a^2_*(G_A(I))$. Now, according to Lemma 2.3.7, given J an arbitrary minimal reduction of I we have $r_J(I) - l(I) \leq a^2_*(R_A(I))$ and the same formula for $G_A(I)$. In particular, if $R_A(I)$ is CM we get $r_J(I) \leq l(I) + a^2(R_A(I)) \leq l(I) - 1$. We can also obtain that if $R_A(I)$ is CM then $\mathrm{reltype}(I) \leq \mu(I) - 1$.