On the diagonals of a Rees algebra

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Chapter 3

Cohen-Macaulay coordinate rings of blow-up schemes

After introducing in the previous chapters the basic tools we will need along this work, we are now ready to study in detail the Cohen-Macaulayness of the coordinate rings of blow-ups of projective varieties. Let k be a field, and let Y be a closed subscheme of $\mathbb{P}_k^{n-1}$ with coordinate ring $A = k[X_1, \dots, X_n]/K$, where $K \subset k[X_1, \dots, X_n]$ is a homogeneous ideal. Given $I \subset A$ a homogeneous ideal, let denote by $\mathcal{I} = \widetilde{I}$ the sheaf associated to I in $Y = \text{Proj }(A)$. Let X be the projective scheme obtained by blowing up Y along $\mathcal{I}$, that is, $X = \mathcal{P}roj(\bigoplus_{n \geq 0} \mathcal{I}^n)$. If I is generated by forms of degree $\leq d$, then $(I^e)_c$ corresponds to a complete linear system on X very ample for $c \geq de+1$ which gives a projective embedding of X so that $X \cong \text{Proj }(k[(I^e)_c]) \subset \mathbb{P}_k^{N-1}$, where $N = \dim_k (I^e)_c$ (see [CH, Lemma 1.1]).

For a given homogeneous ideal $I \subset A$, we can consider the Rees algebra $R_A(I) = \oplus_{j \geq 0} I^j$ of I endowed with the natural bigrading $R_A(I)_{(i,j)} = (I^j)_i$. By taking diagonals $\Delta = (c, e)$ with $c \geq de+1$, we have that $R_A(I)_\Delta = k[(I^e)_c]$. In Chapter 2 we used this fact to study the existence of algebras $k[(I^e)_c]$ which are Cohen-Macaulay in the case where the Rees algebra also has this property (see Theorems 2.3.12 and 2.3.13). Our aim in this chapter is to get some general criteria for the existence of (at least) one coordinate ring $k[(I^e)_c]$ with the Cohen-Macaulay property. In Section 3.2 we will give sufficient and necessary conditions to ensure this existence by means of the local cohomology of $R_A(I)$ and the sheaf cohomology $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X})$. This result will be applied in

Section 3.3 to exhibit several situations in which we can ensure the existence of Cohen-Macaulay coordinate rings for X. We also give a criterion for the existence of Buchsbaum coordinate rings, proving in particular a conjecture stated by A. Conca et al. [CHTV].

Once we have studied the existence of Cohen-Macaulay diagonals of a Rees algebra, in Section 3.4 our aim will be to precise these diagonals. This is a difficult problem which has only been completely solved for complete intersection ideals in the polynomial ring [CHTV, Theorem 4.6]. We will give several criteria to decide if a given diagonal is Cohen-Macaulay, which will allow us to recover and extend the result on complete intersection ideals as well as to determine the Cohen-Macaulay diagonals for new families of ideals. If the Rees algebra is Cohen-Macaulay, we can also determine a family of Cohen-Macaulay diagonals. The section finishes by studying the coordinate rings of the embeddings of the blow-up of a projective space along an ideal of fat points.

The last section is devoted to study sufficient conditions for the existence of a constant f ensuring that $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq ef$ and $e > 0$, a question that has been treated by S.D. Cutkosky and J. Herzog in [CH]. The main result shows that this holds for homogeneous ideals in a Cohen-Macaulay ring A whose Rees algebra is Cohen-Macaulay at any $\mathfrak{p} \in$ Proj (A).

3.1 The blow-up of a projective variety

From now on in this chapter we will have the following assumptions. Let k be a field and A a noetherian graded k-algebra generated in degree 1. Then A has a presentation $A = k[X_1, \ldots, X_n]/K = k[x_1, \ldots, x_n]$, where K is a homogeneous ideal in the polynomial ring $k[X_1, \ldots, X_n]$ with the usual grading. We will denote by $\mathfrak{m}$ the graded maximal ideal of A. Let Y be the projective scheme $\text{Proj}(A) \subset \mathbb{P}_k^{n-1}$. Let I be a homogeneous ideal not contained in any associated prime ideal of A, and let $\mathcal{I}$ be the sheaf associated to I in Y. Then $\mathcal{I}$ can be blown up to produce the projective scheme $X = \mathcal{P}roj(\bigoplus_{n\geq 0} \mathcal{I}^n)$ together with a natural morphism $\pi : X \to Y$. Let us recall the construction of the $\mathcal{P}roj$ of a sheaf of graded algebras $\mathcal{R}$ over a scheme Y (see [Har, Chapter II, Section 7]). For each open affine subset $U = \text{Spec}(B)$ of Y, let $\mathcal{R}(U)$ be the graded B-algebra $\Gamma(U,\mathcal{R}|U)$. Then we can consider $\text{Proj}(\mathcal{R}(U))$ and its natural morphism $\pi|U : \text{Proj}(\mathcal{R}(U)) \to U$. These schemes can be glued to obtain the scheme $\mathcal{P}roj(\mathcal{R})$ with the morphism $\pi : \mathcal{P}roj(\mathcal{R}) \to Y$ such that for each open affine $U \subset Y$, $\pi^{-1}(U) \cong \text{Proj}(\mathcal{R}(U))$.

Assume that I is generated by forms $f_1,\dots,f_r$ in degrees $d_1,\dots,d_r$ respectively. Let $d = \max\{d_1,\dots,d_r\}$. For any $c \geq d+1$, let us consider the invertible sheaf of ideals $\mathcal{L} = \mathcal{I}(c)\mathcal{O}_X$. We are going to show that $\mathcal{L}$ defines a morphism of X in a projective space $\varphi : X \to \mathbb{P}_k^{N-1}$ which is a closed immersion so that $X \cong \text{Proj }(k[I_c])$. Since the blow-up of Y along $\mathcal{I}^e$ is isomorphic to X, we will also have $X \cong \text{Proj }(k[(I^e)_c])$ for any $c \geq de+1$. For that, we are going to follow the proof of [CH, Lemma 1.1]. First of all, notice that we have an affine cover of X by considering the set $\{U_{ij} \mid 1 \leq i \leq n, 1 \leq j \leq r\}$, where $U_{ij} = \text{Spec}(R_{ij})$, and

$$R_{ij} = \left(k\left[\frac{X_1}{X_i},\dots,\frac{X_n}{X_i}\right]/K_i\right)\left[\frac{f_1X_i^{d_j-d_1}}{f_j},\dots,\frac{f_rX_i^{d_j-d_r}}{f_j}\right].$$

Furthermore, $\Gamma(U_{ij},\mathcal{I}(c)\,\mathcal{O}_X) = f_jx_i^{c-d_j}R_{ij}$. Since $f_jx_i^{c-d_j} \in I_c$ and $I_c \subset \Gamma(X,\mathcal{I}(c)\,\mathcal{O}_X)$, we have that $\mathcal{L} = (I_c)\,\mathcal{O}_X$.

I_c is a k-vector space generated by the elements s of the type $s = f_jx_1^{l_1}\dots x_n^{l_n}$ with degree c, that is, such that $d_j+l_1+\dots+l_n = c$. By considering $X_s = \{P \in X \mid s_P \notin \mathfrak{m}_P\mathcal{L}_P\}$ with $s \in I_c$, we have an open covering of X. Since $c > d$, there exists some i with $l_i > 0$, so denoting by $u = (\frac{x_1}{x_i})^{l_1}\dots(\frac{x_n}{x_i})^{l_n}$ we have that

$$X_s = \text{Spec}((R_{ij})_u) = \text{Spec}\left(R_{ij}\left[\left(\frac{x_i}{x_1}\right)^{l_1}\cdots\left(\frac{x_i}{x_n}\right)^{l_n}\right]\right)$$

is an open affine.

Set $N = \dim_k I_c$. Let $\mathbb{P}_k^{N-1} = \text{Proj }(k[\{Z_s\}_{s\in\Lambda}])$, where Λ is a k-basis of I_c, and $V_s = D_+(Z_s) \subset \mathbb{P}_k^{N-1}$. The k-linear maps defined by

$$\begin{array}{rcl}\Gamma(V_s,\mathcal{O}_{V_s}) = k[T_{\overline{s}} : \overline{s} \neq s] & \longrightarrow & \Gamma(X_s,\mathcal{O}_{X_s}) \\ T_{\overline{s}} & \mapsto & \frac{\overline{s}}{s}\end{array}$$

are epimorphisms which define morphisms of schemes $X_s \to V_s$. By gluing them, we get a closed immersion $\varphi : X \to \mathbb{P}_k^{N-1}$ so that $X \cong \text{Proj }(k[I_c])$ by [Har, Proposition II.7.2].

Let $\mathcal{L} = \widetilde{I}\mathcal{O}_X$, $\mathcal{M} = \pi^*\mathcal{O}_Y(1)$, so that $(I^e)_c\mathcal{O}_X = \mathcal{L}^e \otimes \mathcal{M}^c$. Then, the classical Serre's exact sequence allows to relate the local cohomology of the rings $k[(I^e)_c]$ and the global cohomology.

Remark 3.1.1 *[CH, Lemma 1.2]* There is an exact graded sequence

$$0 \to H^0_m(k[(I^e)_c]) \to k[(I^e)_c] \to \bigoplus_{s\in\mathbb{Z}} \Gamma(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) \to H^1_m(k[(I^e)_c]) \to 0$$

and isomorphisms

$$H^i_m(k[(I^e)_c]) \cong \bigoplus_{s\in\mathbb{Z}} H^{i-1}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs})$$

for $i > 1$. In particular, looking at the homogeneous component of degree 0 we get the exact sequence

$$0 \to H^0_m(k[(I^e)_c])_0 \to k \to \Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) \to H^1_m(k[(I^e)_c])_0 \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_0 \cong H^{i-1}(\mathrm{X}, \mathcal{O}_\mathrm{X})$ for $i > 1$.

For a homogenous ideal I of A, let us consider the Rees algebra $R = R_A(I) = \bigoplus_{n\geq 0} I^n t^n \subset A[t]$ of I endowed with the natural bigrading given by $R_{(i,j)} = (I^j)_i$. Then, by taking a diagonal $\Delta = (c, e)$ with $c \geq de + 1$, we have that $R_A(I)_\Delta = \bigoplus_{s\geq 0}(I^{es})_{cs} = k[(I^e)_c]$. The natural inclusion $k[(I^e)_c] = R_\Delta \hookrightarrow R$, gives the isomorphism of schemes $\mathrm{Proj}\,^2(R_A(I)) \cong \mathrm{Proj}\,(k[(I^e)_c])$. Summarizing, we have:

Proposition 3.1.2 *Let X be the blow-up of $Y = \mathrm{Proj}\,(A)$ along $\mathcal{I} = \widetilde{I}$, where I is a homogeneous ideal of A generated by forms of degree $\leq d$. For any $c \geq de + 1$, we have isomorphisms of schemes*

$$X \cong \mathrm{Proj}\,^2(R_A(I)) \cong \mathrm{Proj}\,(k[(I^e)_c]).$$

In Chapter 2, Section 3, we have followed an algebraic approach to study the local cohomology modules of the rings $k[(I^e)_c]$ in terms of the local cohomology of the Rees algebra and the diagonal functor. Next we give a new approach to these modules by using sheaf cohomology.

Notice that from Remark 3.1.1 we may determine the local cohomology modules of the k-algebras $k[(I^e)_c]$ by means of the cohomology modules

$H^i(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs})$. On the other hand, we can get some information about these modules from the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathrm{Y}, R^j\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs})) \Rightarrow H^{i+j}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}),$$

and the vanishing of the higher direct-image sheaves $R^j\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs})$. First of all, let us study the vanishing of $R^j\pi_*(\mathcal{L}^{es})$.

Theorem 3.1.3 *Set* $e_0 = \max\{ a_*(R_{A_\mathfrak{p}}(I_\mathfrak{p})) : \mathfrak{p} \in \mathrm{Proj}\ (A) \}$. *For any* $e > e_0$, $j > 0$, $R^j\pi_*\mathcal{L}^e = 0$ *and* $\pi_*\mathcal{L}^e = \widetilde{I^e}$.

Proof. Let us denote by $A_i = A_{(x_i)}$, $I_i = I_{(x_i)}$ and $R_i = R_{A_i}(I_i) = A_i[I_i t]$. Note that by defining $Y_i = Y - V_+(x_i) = D_+(x_i) \cong \mathrm{Spec}(A_i)$, we have that $\{Y_i : 1 \le i \le n\}$ is an open affine cover of Y. Then, given $j > 0, e > 0$, $R^j\pi_*\mathcal{L}^e = 0$ if and only if $(R^j\pi_*\mathcal{L}^e) \mid Y_i = 0$ for all i. Denoting by $X_i = \pi^{-1}Y_i = \mathrm{Proj}\ (R_i)$, by [Har, Corollary III.8.2 and Proposition III.8.5] we have that for $j > 0$

$$R^j\pi_*(\mathcal{L}^e) \mid Y_i = R^j\pi_*(\mathcal{L}^e \mid X_i) = H^j(X_i, \mathcal{L}^e \mid \mathrm{X}_i)^\sim = H^{j+1}_{(R_i)_+}((I_i)^e R_i)_0{}^\sim .$$

From the graded exact sequence

$$0 \to (I_i)^e R_i(-e) \to R_i \to \bigoplus_{q<e} (I_i)^q \to 0 ,$$

it follows that $H^{j+1}_{(R_i)_+}((I_i)^e R_i)_0 = H^{j+1}_{(R_i)_+}((I_i)^e R_i(-e))_e = H^{j+1}_{(R_i)_+}(R_i)_e$. Similarly, $\pi_*\mathcal{L}^e = \widetilde{I^e}$ if $H^0_{(R_i)_+}(R_i)_e = H^1_{(R_i)_+}(R_i)_e = 0$ for all i. Therefore, we have reduced the problem to prove that $H^j_{(R_i)_+}(R_i)_e = 0$ for all i, j if $e > e_0$.

Set $\overline{R}_i = R_{A_{x_i}}(I_{x_i})$. We can think $\overline{R}_i$ as a $\mathbb{Z}$-graded ring with $\deg(\frac{x_j}{x_i^m}) = 1 - m$, $\deg(\frac{f_j t}{x_i^m}) = d_j - m$. Note that with this grading we have $[\overline{R}_i]_0 = R_i$ and $\frac{x_i}{1}$ is an invertible element in $\overline{R}_i$ of degree 1. Then we may define the graded isomorphism

$$\begin{array}{ccc} R_i[T, T^{-1}] & \xrightarrow{\psi} & \overline{R}_i \\ T & \mapsto & \frac{x_i}{1} , \end{array}$$

where $\psi | R_i = id$ and $\deg(T) = 1$. Since $R_i \hookrightarrow \overline{R}_i$ is a flat morphism, we have that

$$H^j_{(\overline{R}_i)_+}(\overline{R}_i) = H^j_{(R_i)_+}(R_i) \otimes_{R_i} \overline{R}_i = H^j_{(R_i)_+}(R_i)[T, T^{-1}],$$

so that $H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e = H^j_{(R_i)_+}(R_i)_e\,[T, T^{-1}]$. Therefore, it suffices to prove that $H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e = 0$ for all i, j if $e > e_0$.

Given a homogeneous prime $\mathfrak{q} \in \text{Spec}(A_{x_i})$, we have that $\mathfrak{q} = \mathfrak{p}A_{x_i}$ with $\mathfrak{p} \in \text{Proj}\,(A)$. Localizing $\overline{R}_i$ at $\mathfrak{q}$, we have that $\overline{R}_i \otimes (A_{x_i})_{\mathfrak{q}} = R_{A_\mathfrak{p}}(I_\mathfrak{p})$. Denoting by $B = R_{A_\mathfrak{p}}(I_\mathfrak{p})$, note that B is a standard graded ring whose homogeneous component of degree 0 is the local ring $A_\mathfrak{p}$. So B has a unique homogeneous maximal ideal $\mathfrak{n}$, with $\mathfrak{n} = \mathfrak{p}A_\mathfrak{p} \oplus B_+$. Since $H^j_\mathfrak{n}(B)_e = 0$ for all $j \geq 0$ and $e > e_0$, according to [Hy, Lemma 2.3] we also have $H^j_{B_+}(B)_e = 0$ for all $j \geq 0$ and $e > e_0$. Therefore,

$$[H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e]_\mathfrak{q} = [H^j_{(\overline{R}_i)_+}(\overline{R}_i)_\mathfrak{q}]_e = H^j_{B_+}(B)_e = 0.$$

Hence $(H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e)_\mathfrak{q} = 0$ for any homogeneous ideal $\mathfrak{q} \in \text{Spec}(A_{x_i})$, and we conclude $H^j_{(\overline{R}_i)_+}(\overline{R}_i)_e = 0$ for $j \geq 0$ and $e > e_0$. □

Corollary 3.1.4 *Assume that $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{Proj}\,(A)$. Then, for any $e \geq 0$, $j > 0$, $R^j\pi_*\mathcal{L}^e = 0$ and $\pi_*\mathcal{L}^e = \widetilde{I^e}$.*

Given a homogeneous ideal I of A, let us denote by $I^* = \{f \in A \mid \mathfrak{m}^k f \subset I \text{ for some } k\}$ the saturation of I. Note that $H^0_\mathfrak{m}(A/I) = I^*/I$. Next we use Proposition 3.1.3 to relate the local cohomology modules of the k-algebras $k[(I^e)_c]$ and the local cohomology of the powers of the ideal (compare with Corollary 2.3.5).

Corollary 3.1.5 *Let I be a homogeneous ideal of A, and $e_0 = \max\{\, a_*(R_{A_\mathfrak{p}}(I_\mathfrak{p})) : \mathfrak{p} \in \text{Proj}\,(A)\}$. For any $c \geq de+1$, $e > e_0$, $s > 0$, there is an exact sequence*

$$0 \to H^0_m(k[(I^e)_c])_s \to (I^{es})_{cs} \to (I^{es})^*_{cs} \to H^1_m(k[(I^e)_c])_s \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_s \cong H^i_\mathfrak{m}(I^{es})_{cs}$ for $i > 1$.

Proof. By the Leray spectral sequence we have

$$H^i(\mathrm{Y}, R^j\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs})) \Rightarrow H^{i+j}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}).$$

On the other hand, by the Projection formula [Har, Exercise III.8.3] and Proposition 3.1.3, we get that for $e > e_0$, $s > 0$,

$$\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = \pi_*(\mathcal{L}^{es}) \otimes \mathcal{O}_\mathrm{Y}(cs) = \widetilde{I^{es}}(cs)$$

$$R^j\pi_*(\mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = R^j\pi_*(\mathcal{L}^{es}) \otimes \mathcal{O}_\mathrm{Y}(cs) = 0, \text{ for all } j > 0.$$

Therefore we may conclude that for any $i \geq 1$, $H^i(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = H^i(Y, \widetilde{I^{es}}(cs)) = H^{i+1}_{\mathfrak{m}}(I^{es})_{cs}$, and $\Gamma(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = \Gamma(Y, \widetilde{I^{es}}(cs)) = (I^{es})^*_{cs}$. Now the result follows from Remark 3.1.1. □

We have shown that the positive components of the local cohomology modules of the rings $k[(I^e)_c]$ are closely related to the local cohomology modules of the powers of the ideal I^e. Next we want to study the negative components. In the case where X is Cohen-Macaulay, we will express them by means of the local cohomology of the canonical module of the Rees algebra. Recall that the canonical module of the Rees algebra is defined in the category of bigraded S-modules, so let us write $K_R = \bigoplus_{(i,j)} K_{(i,j)}$. For any integer e, we denote by $K^e = (K_R)^e = \bigoplus_i K_{(i,e)}$. Then we have

Proposition 3.1.6 *Assume that X is Cohen-Macaulay. For any $c \geq de+1$, $e > a^2_*(K_R)$, $s > 0$, $1 \leq i < \overline{n}$,*

$$H^i_m(k[(I^e)_c])_{-s} \cong H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs}.$$

Proof. First we will show that X is equidimensional. Let $\mathfrak{p} \in X_\Delta$ be a closed point. Then $\dim \mathcal{O}_{X_\Delta,\mathfrak{p}} = \dim(R_\Delta)_{(\mathfrak{p})} = \dim(R_\Delta)_{\mathfrak{p}}$. As $X_\Delta \cong X$ is Cohen-Macaulay, we have that R_Δ is generalized Cohen-Macaulay and so $\dim(R_\Delta)_{\mathfrak{p}} = \dim R_\Delta - \dim R_\Delta/\mathfrak{p}$ by [HIO, Corollary 37.6]. Recall from Lemma 1.4.1 that $\dim R_\Delta/\mathfrak{p} = 1$, and so $\dim \mathcal{O}_{X_\Delta,\mathfrak{p}} = \overline{n} - 1 = \dim X_\Delta$. Therefore, $X \cong X_\Delta$ is equidimensional.

Now by Serre's duality we have that for any $s > 0$, $i \geq 1$,

$$H^i_m(k[(I^e)_c])_{-s} \cong H^{i-1}(X, \mathcal{L}^{-es} \otimes \mathcal{M}^{-cs}) \cong H^{\overline{n}-i}(X, w_X \otimes \mathcal{L}^{es} \otimes \mathcal{M}^{cs}).$$

Then, by taking $c \geq de+1$, $e > a^2_*(K_R)$, $i < \overline{n}$ we get

$$\begin{aligned} H^i_m(k[(I^e)_c])_{-s} &= H^{\overline{n}+1-i}_{R_+}(K_R)_{(cs,es)} \\ &= H^{\overline{n}+1-i}_{\mathfrak{m}}(K^{es})_{cs}, \text{ by Proposition 2.1.18. } \square \end{aligned}$$

To finish the section, note that according to [HHK, Theorem 2.1] we can also express the negative components of the local cohomology of the rings $k[(I^e)_c]$ by means of the local cohomology of their canonical modules whenever X is Cohen-Macaulay. Namely,

Proposition 3.1.7 *Assume that X is Cohen-Macaulay. Then for all $s > 0$, $1 \leq i < \overline{n}$, we have isomorphisms*

$$H^i_m(R_\Delta)_{-s} = H^{\overline{n}+1-i}_m(K_{R_\Delta})_s.$$

3.2 Existence of Cohen-Macaulay coordinate rings

Our aim in this section is to find necessary and sufficient conditions for the existence of integers c, e, with $c \geq de+1$, such that the ring $k[(I^e)_c]$ is Cohen-Macaulay. Before proving our main result, we need two previous lemmas. The first one may be seen as a Nakayama's Lemma adapted to our situation, and in fact it is just Lemma 1.5.2 for the case $r=2$.

Lemma 3.2.1 *Let L be a finitely generated bigraded R-module and m an integer such that $R_+^m L = 0$. Then, there exist integers q_0, t such that $L_{(p,q)} = 0$ for all $p > dq+t$, $q > q_0$.*

The second lemma provides restrictions on the local cohomology modules of the Rees algebra whenever X is Cohen-Macaulay.

Lemma 3.2.2 *If X is Cohen-Macaulay, then there are integers q_0, t such that $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ for all $i < \overline{n}+1$, $q < q_0$ and $p < dq+t$.*

Proof. Let $\mathrm{P} \in X$. Then $R_+ \not\subseteq \mathrm{P}$ and so there exist i, j such that $x_i \notin \mathrm{P}$, $f_j t \notin \mathrm{P}$. Denote by $R_{<\mathrm{P}>} = T^{-1}R$, where T is the multiplicative system consisting of all homogeneous elements of R which are not in P. Note that $R_{(\mathrm{P})} = [R_{<\mathrm{P}>}]_{(0,0)}$. Furthermore, $\frac{x_i}{1}$ and $\frac{f_j t}{x_i^{d_j}}$ are invertible elements in $R_{<\mathrm{P}>}$ with

$$\deg \frac{x_i}{1} = (1,0), \quad \deg \frac{f_j t}{x_i^{d_j}} = (0,1).$$

Then we may define a bigraded isomorphism ψ:

$$\begin{array}{ccc} R_{(\mathrm{P})}[U, U^{-1}, V, V^{-1}] & \stackrel{\psi}{\longrightarrow} & R_{<\mathrm{P}>} \\ U & \longmapsto & \frac{x_i}{1} \\ V & \longmapsto & \frac{f_j t}{x_i^{d_j}} \end{array}$$

where $\psi | R_{(\mathrm{P})} = id$, and $\deg(U) = (1,0)$, $\deg(V) = (0,1)$. Since X is CM, $\mathcal{O}_{X,\mathrm{P}} = R_{(\mathrm{P})}$ is CM and so $R_{<\mathrm{P}>}$ too. Then, localizing at $\mathrm{P}R_{<\mathrm{P}>}$, we have that R_P is CM.

Now let $\mathrm{P} \in \mathrm{Spec}(R)$ and denote by P^* the ideal generated by the homogeneous elements of P. By [GW2, Corollary 1.2.4], R_P is CM if and only if R_{P^*} is CM, so we have that R_P is CM for any prime ideal P such that $R_+ \not\subseteq \mathrm{P}$. Localizing the Rees algebra R at the homogeneous maximal ideal $\mathcal{M}$ we then have that $R_{\mathcal{M}}$ is a generalized Cohen-Macaulay module with respect to $R_+R_{\mathcal{M}}$ [HIO, Lemma 43.2]. Therefore there exists $m \geq 0$ such that $R_+^m H_{\mathcal{M}}^i(R) = 0$ for all $i < \overline{n}+1$. From the presentation of R as a quotient of the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$, by Theorem 1.2.1 we get

$$H_{\mathcal{M}}^i(R) = \underline{\mathrm{Ext}}_S^{n+r-i}(R, K_S)^{\vee},$$

and so $R_+^m \underline{\mathrm{Ext}}_S^{n+r-i}(R, K_S) = 0$ for $i < \overline{n}+1$. By Lemma 3.2.1 we then obtain that there exist integers q_1, t_1 such that $\underline{\mathrm{Ext}}_S^{n+r-i}(R, K_S)_{(p,q)} = 0$ for all $q > q_1$, $p > dq + t_1$ and $i < \overline{n}+1$. The proof finishes by dualizing again. □

Now we may formulate the main result of this section.

Theorem 3.2.3 *The following are equivalent:*

(i) There exist c, e such that $k[(I^e)_c]$ is Cohen-Macaulay.

(ii) (1) There exist integers q_0, t such that $H_{\mathcal{M}}^i(R_A(I))_{(p,q)} = 0$ for all $i < \overline{n}+1$, $q < q_0$ and $p < dq + t$.

(2) $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for $0 < i < \overline{n}-1$.

In this case, $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg 0$ relatively to $e \gg 0$.

Proof. If (i) is satisfied, then the scheme $X = \mathrm{Proj}\ ^2(R) \cong \mathrm{Proj}\ (k[(I^e)_c])$ is CM and by Lemma 3.2.2 we have (1) of (ii). Furthermore, $H_m^i(k[(I^e)_c])_0 = 0$ for any $i < \overline{n}$ and then by using Remark 3.1.1 we get (2) of (ii).

Assume now that (ii) is satisfied. We want to find a diagonal Δ such that $R_\Delta = k[(I^e)_c]$ is CM. By Remark 3.1.1 and (2) of (ii), we have that $H_m^i(R_\Delta)_0 = 0$ for any diagonal Δ and $i < \overline{n}$. On the other hand, since $H_{\mathcal{M}}^i(R)$ are artinian modules there exists p_1 such that $H_{\mathcal{M}}^i(R)_{(p,q)} = 0$ for all i and $p > p_1$. Furthermore, by Corollary 2.1.12 there are positive integers e_0, α such that for $e > e_0$, $c > de + \alpha$ we have

$$H_m^i(R_\Delta)_j \cong H_{\mathcal{M}}^{i+1}(R)_{(cj,ej)},\ \forall i, \forall j \neq 0.$$

Now, let us consider q_0, t given by (1) of (ii). Note that we can assume that q_0, t are negative. Then, by taking diagonals $\Delta = (c, e)$ with $e > \max\{e_0, -q_0\}$, $c > \max\{de + \alpha, p_1, de - t\}$, we have that $H_m^i(R_\Delta)_j = 0$ for all j and $i < \overline{n}$, and therefore $k[(I^e)_c]$ are CM for all these c, e. □

Remark 3.2.4 Assume that $(A,\mathfrak{m})$ is a noetherian local ring and let $I \subset \mathfrak{m}$, $I \neq 0$ be an ideal. Denote by $X = \text{Proj}\,(R_A(I))$ the blow-up of Spec(A) along I. Then, it was proved by J. Lipman [Li, Theorem 4.1] that there exists a positive integer e such that $R_A(I^e)$ is Cohen-Macaulay if and only if X is Cohen-Macaulay, $\Gamma(\mathrm{X},\mathcal{O}_{\mathrm{X}}) = A$ and $H^i(\mathrm{X},\mathcal{O}_{\mathrm{X}}) = 0$ for all $i > 0$. The following corollary may be seen as a projective version of this result.

Corollary 3.2.5 *The following are equivalent:*

(i) There exist c, e *such that* $k[(I^e)_c]$ *is Cohen-Macaulay.*

(ii) X *is Cohen-Macaulay,* $\Gamma(\mathrm{X},\mathcal{O}_{\mathrm{X}}) = k$ *and* $H^i(\mathrm{X},\mathcal{O}_{\mathrm{X}}) = 0$ *for all* $0 < i < \overline{n} - 1$.

(iii) X *is Cohen-Macaulay and* $H^i_{R_+}(R)_{(0,0)} = 0$ *for all* $i < \overline{n}$.

Proof. It is enough to note that we have an exact bigraded sequence

$$0 \to H^0_{R_+}(R) \to R \to \bigoplus_{(p,q)} \Gamma(\mathrm{X},\mathcal{O}_{\mathrm{X}}(p,q)) \to H^1_{R_+}(R) \to 0,$$

and isomorphisms $H^{i+1}_{R_+}(R) \cong \bigoplus_{(p,q)} H^i(\mathrm{X},\mathcal{O}_{\mathrm{X}}(p,q))$ for $i > 0$. □

We can also give sufficient and necessary conditions for the existence of generalized Cohen-Macaulay or Buchsbaum diagonals of the Rees algebra, in particular proving a conjecture of A. Conca et al. in [CHTV].

Proposition 3.2.6 *The following are equivalent:*

(i) $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ *for* $i < \overline{n}+1$ *and* $p \ll 0$ *relatively to* $q \ll 0$.

(ii) $k[(I^e)_c]$ *is a generalized Cohen-Macaulay module for* $c \gg e \gg 0$.

(iii) There exist c, e *such that* $k[(I^e)_c]$ *is generalized Cohen-Macaulay.*

(iv) $k[(I^e)_c]$ *is a Buchsbaum ring for* $c \gg e \gg 0$.

(v) There exist c, e *such that* $k[(I^e)_c]$ *is a Buchsbaum ring.*

(vi) There exist integers q_0, t *such that* $H^i_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$ *for* $i < \overline{n}+1$, $q < q_0$ *and* $p < dq + t$.

Proof. $(i) \Rightarrow (ii)$ Assume that (i) is satisfied. By Corollary 2.1.12, we get $H^i_m(R_\Delta)_s = 0$ for $c \gg 0$ relatively to $e \gg 0$, $s \neq 0$ and $i < \overline{n}$. So $k[(I^e)_c]$ is generalized CM for $c \gg e \gg 0$.

$(ii) \Rightarrow (iii)$ Obvious.

$(iii) \Rightarrow (vi)$ Let Δ be a diagonal such that R_Δ is generalized CM. Then $(R_\Delta)_\mathfrak{p}$ is CM for any $\mathfrak{p} \in \text{Proj}\ (R_\Delta)$ by [HIO, Lemma 43.3], and so $X \cong \text{Proj}\ (R_\Delta)$ is CM. By using Lemma 3.2.2 one obtains (vi).

$(vi) \Rightarrow (i)$ Obvious.

The implications $(i) \Rightarrow (iv) \Rightarrow (v) \Rightarrow (vi)$ may be proved similarly. □

3.3 Applications

In this section we show several situations in which we can ensure the existence of Cohen-Macaulay coordinate rings for the blow-up scheme by using Theorem 3.2.3. First lemma provides sufficient conditions to have $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) = 0$ for all $0 < i < \overline{n} - 1$.

Lemma 3.3.1 *Assume* $a^2_*(R) < 0$, $a_*(A) < 0$. *Then* $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) = k$ *and* $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) = 0$ *for* $0 < i < \overline{n} - 1$.

Proof. Note that for any i, we have $H^i_{\mathcal{M}}(R)_{(0,0)} = H^i_{\mathcal{M}_2}(R)_{(0,0)} = 0$ and $H^i_{\mathcal{M}_1}(R)_{(0,0)} = H^i_{\mathfrak{m}}(A)_0 = 0$ by Proposition 2.1.18. Then, from the Mayer-Vietoris exact sequence associated to $\mathcal{M}_1$ and $\mathcal{M}_2$, we get $H^i_{R_+}(R)_{(0,0)} = 0$ for any i, so we are done. □

As an immediate consequence we get:

Corollary 3.3.2 *Suppose that* $a^2_*(R) < 0$, $a_*(A) < 0$. *If* X *is Cohen-Macaulay, then* $k[(I^e)_c]$ *is Cohen-Macaulay for* $c \gg e \gg 0$.

It is known that there are smooth projective varieties with no arithmetically Cohen-Macaulay embeddings (see for instance [Mat, Theorem 3.4]). Next we exhibit a situation where this implication is true.

Proposition 3.3.3 *Let* X *be the blow-up of* $\mathbb{P}^{n-1}_k$ *along a closed subscheme, where* k *has* $\text{char} k = 0$. *Assume that* X *is smooth or with rational singularities. Then* X *is arithmetically Cohen-Macaulay.*

Proof. Let $\pi : X \to \mathbb{P}_k^{n-1}$ be the blow-up morphism. From [KKMS], we have that $\pi_* \mathcal{O}_X = \mathcal{O}_{\mathbb{P}_k^{n-1}}$ and $R^j \pi_* \mathcal{O}_X = 0$ for all $j > 0$. This implies that the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathbb{P}_k^{n-1}, R^j \pi_* \mathcal{O}_X) \Longrightarrow H^{i+j}(X, \mathcal{O}_X)$$

degenerates. Therefore we have $\Gamma(X, \mathcal{O}_X) = \Gamma(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}) = k$ and $H^i(X, \mathcal{O}_X) = H^i(\mathbb{P}_k^{n-1}, \mathcal{O}_{\mathbb{P}_k^{n-1}}) = 0$ for all $i > 0$. Then the result follows from Corollary 3.2.5. □

Assume that A is Cohen-Macaulay. S.D. Cutkosky and J. Herzog proved in [CH] that the Rees algebra has Cohen-Macaulay diagonals for locally complete intersection ideals and for ideals whose homogeneous localizations are strongly Cohen-Macaulay satisfying condition $(\mathcal{F}_1)$. In the first case, observe that $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{Proj}\ (A)$, while in the second one $R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})})$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{Proj}\ (A)$. Next we want to study those examples.

Proposition 3.3.4 *The following are equivalent:*

(i) $R_{A_{x_i}}(I_{x_i})$ *is Cohen-Macaulay for all* $1 \leq i \leq n$.

(ii) $R_{A_{(x_i)}}(I_{(x_i)})$ *is Cohen-Macaulay for all* $1 \leq i \leq n$.

(iii) $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \text{Proj}\ (A)$.

(iv) $R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})})$ *is Cohen-Macaulay for all* $\mathfrak{p} \in \text{Proj}\ (A)$.

Proof. Set $\overline{R}_i = R_{A_{x_i}}(I_{x_i})$, $R_i = R_{A_{(x_i)}}(I_{(x_i)})$. We have already shown in the proof of Proposition 3.1.3 that there exists an isomorphism $\overline{R}_i \cong R_i[T, T^{-1}]$. Therefore, R_i is CM if and only if $\overline{R}_i$ is CM, and so the two first conditions are equivalent.

Now let us prove *(i)* $\Longleftrightarrow$ *(iii)*. First assume *(i)*, and for any prime ideal $\mathfrak{p} \in \text{Proj}\ (A)$ let us take $x_i \notin \mathfrak{p}$. Note that $R_{A_\mathfrak{p}}(I_\mathfrak{p}) = \overline{R}_i \otimes_{A_{x_i}} (A_{x_i})_\mathfrak{p}$, and so $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay. Now assume *(iii)*, and let us think $\overline{R}_i$ as a bigraded ring. Then, to prove *(i)*, it is enough to show that for any homogeneous prime ideal $Q \in \text{Spec}(\overline{R}_i)$, we have that $(\overline{R}_i)_Q$ is CM. Given such a Q, denote by $\mathfrak{q}A_{x_i} = Q \cap A_{x_i}$, where $\mathfrak{q} \in \text{Spec}(A)$ is a homogeneous prime which does not contain x_i, that is, $\mathfrak{q} \in \text{Spec}(A_{(x_i)}) \subset \text{Proj}\ (A)$. Then we have $(\overline{R}_i)_Q = (R_{A_\mathfrak{q}}(I_\mathfrak{q}))_Q$, and so $(\overline{R}_i)_Q$ is CM.

Finally, let us prove $(ii) \iff (iv)$. Given $\mathfrak{p} \in \text{Proj}\,(A)$ and $x_i \notin \mathfrak{p}$, let $\mathfrak{q} = \mathfrak{p}A_{x_i} \cap A_{(x_i)} \in \text{Spec}(A_{(x_i)})$. Then, $(A_{(x_i)})_{\mathfrak{q}} = A_{(\mathfrak{p})}$ and so $R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})}) = R_i \otimes_{A_{(x_i)}} (A_{(x_i)})_{\mathfrak{q}}$. Therefore, (ii) implies (iv). Now let us assume (iv). Given any homogeneous prime ideal $Q \in \text{Spec}(R_i)$, let $\mathfrak{q} = Q \cap A_{(x_i)} \in \text{Spec}(A_{(x_i)}) \subset \text{Proj}\,(A)$, and let $\mathfrak{p} \in \text{Proj}\,(A)$ such that $\mathfrak{p}A_{x_i} = \mathfrak{q}[x_i, x_i^{-1}]$. Since $(R_i)_Q = (R_{A_{(\mathfrak{p})}}(I_{(\mathfrak{p})}))_Q$, we have that $(R_i)_Q$ is Cohen-Macaulay, and so R_i is CM. □

Now we can prove that the Rees algebra of a homogeneous ideal I in a Cohen-Macaulay ring A satisfying any of the equivalent conditions above has Cohen-Macaulay diagonals. More generally, we have:

Theorem 3.3.5 *Assume that $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in$ Proj (A). Then $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg 0$ relatively to $e \gg 0$ if and only if $H^i_\mathfrak{m}(A)_0 = 0$ for all $i < \overline{n}$.*

Proof. Given $P \in X$, let us denote by $\mathfrak{p} = P \cap A \in \text{Proj}\,(A)$. Then $R_A(I)_P = (R_{A_\mathfrak{p}}(I_\mathfrak{p}))_P$ is CM and so X is CM. Then, by Corollary 3.1.4 and the Leray spectral sequence

$$E_2^{i,j} = H^i(Y, R^j\pi_*\mathcal{O}_X) \Longrightarrow H^{i+j}(\mathrm{X}, \mathcal{O}_\mathrm{X}),$$

we get $H^j(\mathrm{X}, \mathcal{O}_\mathrm{X}) = H^j(\mathrm{Y}, \mathcal{O}_\mathrm{Y}) = H^{j+1}_\mathfrak{m}(A)_0$ for $0 < j < \overline{n} - 1$, and the exact sequence $0 \to k \to \Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) = \Gamma(\mathrm{Y}, \mathcal{O}_\mathrm{Y}) \to H^1_\mathfrak{m}(A)_0 \to 0$, so we get the statement. □

Denote by E the exceptional divisor of the blow-up and by w_E its dualizing sheaf. The last result of the section shows that weaker assumptions on [CH, Lemma 2.1] are enough to ensure that the rings $k[(I^e)_c]$ are Cohen-Macaulay for $c \gg e \gg 0$.

Proposition 3.3.6 *Suppose that A is Cohen-Macaulay, X is a Cohen-Macaulay scheme, $\pi_*\mathcal{O}_E(m) = \widetilde{I}^m/\widetilde{I}^{m+1}$ for $m \geq 0$ and $R^i\pi_*\mathcal{O}_E(m) = 0$ for $i > 0$ and $m \geq 0$. Then $k[(I^e)_c]$ is Cohen-Macaulay for $c >> 0$ relatively to $e >> 0$.*

Proof. $R^i\pi_*\mathcal{O}_\mathrm{X} = 0$ for $i > 0$ and $\pi_*\mathcal{O}_\mathrm{X} = \mathcal{O}_\mathrm{Y}$ by [CH, Lemma 2.1]. Then, from the Leray spectral sequence, we obtain $H^i(\mathrm{X}, \mathcal{O}_\mathrm{X}) = H^i(\mathrm{Y}, \mathcal{O}_\mathrm{Y}) = H^{i+1}_\mathfrak{m}(A)_0 = 0$ for $0 < i < \overline{n} - 1$ and $\Gamma(\mathrm{X}, \mathcal{O}_\mathrm{X}) = \Gamma(\mathrm{Y}, \mathcal{O}_\mathrm{Y}) = k$. Now, the proposition follows from Corollary 3.2.5. □

3.4 Cohen-Macaulay diagonals

Once we have studied the problem of the existence of Cohen-Macaulay diagonals of a Rees algebra, now we would like to study in more detail which diagonals are Cohen-Macaulay. This question has been totally answered only for complete intersection ideals in the polynomial ring [CHTV, Theorem 4.6]. Our approach to this problem will give us criteria to decide if a diagonal is Cohen-Macaulay, which will allow us to recover and extend the result in [CHTV] to any Cohen-Macaulay ring as well as to precise the Cohen-Macaulay diagonals for new families of ideals.

The first criterion gives necessary and sufficient conditions for a diagonal of a Cohen-Macaulay Rees algebra to have this property in the case where I is equigenerated. Namely,

Proposition 3.4.1 *Let $I \subset A$ be a homogeneous ideal generated by forms of degree d whose Rees algebra is Cohen-Macaulay. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$, for $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$, for $i < \overline{n}$, $s > 0$.

Proof. First, recall that the assumptions on the local cohomology of A are necessary and sufficient conditions for the existence of Cohen-Macaulay diagonals (Theorem 2.3.13). Then, for any $c \geq de+1$ and $i < \overline{n}$, we have $H^i_m(k[(I^e)_c])_0 = 0$ by Theorem 3.2.3 and Remark 3.1.1.

On the other hand, by applying Proposition 2.1.18 and Proposition 2.1.19, for any $s < 0$ we have:

$$H^q_{\mathcal{M}_1}(R)_{(cs,es)} = H^q_{\mathfrak{m}_1}(R^{es})_{cs} = 0$$

$$H^q_{\mathcal{M}_2}(R)_{(cs,es)} = H^q_{\mathfrak{m}_2}(R_{cs-des})_{es} = 0$$

because $R^{es} = 0$ and $R_{cs-des} = 0$. Therefore, for any diagonal and any $i < \overline{n}$, $s < 0$, we get $H^i_m(k[(I^e)_c])_s = 0$ according to Proposition 2.1.3. The statement, then, follows from Corollary 2.3.5. □

We may apply Proposition 3.4.1 to study in detail the following example considered by L. Robbiano and G. Valla in [RV].

Corollary 3.4.2 *Let $\{L_{ij}\}$ be a set of $d \times (d+1)$ homogeneous linear forms of a polynomial ring $A = k[X_1, \dots, X_n]$, $i = 1, \dots, d$; $j = 1, \dots, d+1$, and let M be the matrix (L_{ij}). Let $I_t(M)$ be the ideal generated by the $t \times t$ minors of M and assume that $\operatorname{ht}(I_t(M)) \geq d - t + 2$ for $1 \leq t \leq d$. Denoting by $I = I_d(M)$, then $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + 1$.*

Proof. The ideal I is generated by $d+1$ forms of degree d, and the Rees algebra has a presentation of the form

$$R_A(I) = k[X_1, \dots, X_n, Y_1, \dots, Y_{d+1}]/(\phi_1, \dots, \phi_d),$$

where the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_{d+1}]$ is bigraded by $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d,1)$, and $\phi_1, \dots, \phi_d$ is a regular sequence in S with $\deg(\phi_l) = (d+1, 1)$ (see the proof of [RV, Theorem 5.11]). Then we have a bigraded minimal free resolution of the Rees algebra $R_A(I)$ as S-module given by the Koszul complex associated to $\phi_1, \dots, \phi_d$:

$$0 \to F_d \to \cdots \to F_1 \to F_0 = S \to R_A(I) \to 0,$$

with $F_p = S(-(d+1)p, -p)^{\binom{d}{p}}$. By applying the functor $(\ )^e$ to this resolution, we have a graded free resolution of I^e over A:

$$0 \to F_p^e \to \cdots \to F_1^e \to F_0^e = S^e \to I^e \to 0,$$

with $p = \min\{e, d\}$, $F_p^e = A(-p - de)^{\rho_p^e}$ for certain $\rho_p^e \in \mathbb{Z}$. The minimal graded free resolution of I^e is then obtained by picking out some terms, but in any case it must have length p because the Hilbert series of A/I^e is given by ([RV, Example 6.1])

$$H_{A/I^e}(z) = \frac{1 - \sum_{j=0}^{d} (-1)^j \binom{d}{j} \binom{d+e-j}{e-j} z^{de+j}}{(1-z)^n}$$

(note that z^{p+de} appears in the numerator). So by Theorem 1.3.4 we can compute the a_*-invariant of I^e and we get

$$a_*(I^e) = \begin{cases} de + e - n & \text{if } e < d \\ de + d - n & \text{if } e \geq d. \end{cases}$$

On the other hand, since $n \geq \operatorname{ht}(I_1(M)) \geq d + 1$, we have that $d \leq n - 1$, and so $a_*(I^e) < de$. Therefore, for any $c \geq de + 1$, $s \geq 1$, we have that $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ for all i. So $k[(I^e)_c]$ is Cohen-Macaulay by Proposition 3.4.1. Furthermore, note that $a(k[(I^e)_c]) < 0$. □

For arbitrary homogeneous ideals, we can also get a criterion for the Cohen-Macaulayness of the diagonals by means of the local cohomology of the powers of the ideal and the local cohomology of the graded pieces of the canonical module of the Rees algebra. More explicitly,

Theorem 3.4.3 *Let I be a homogeneous ideal in A generated by forms of degree $\leq d$ whose Rees algebra is Cohen-Macaulay. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$ *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ *for* $i < \overline{n}$, $s > 0$.

(iii) $H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs} = 0$ *for* $1 \leq i < \overline{n}$, $s > 0$.

Proof. As in the proof of Proposition 3.4.1, the assumptions on the local cohomology of A are necessary and sufficient conditions for the existence of Cohen-Macaulay diagonals. Then we have that $H^i_m(k[(I^e)_c])_0 = 0$ for $i < \overline{n}$.

Since R is Cohen-Macaulay, we have that K_R is Cohen-Macaulay with $a^2_*(K_R) = 0$. Therefore, for any $s > 0$, $1 \leq i < \overline{n}$, $H^i_m(k[(I^e)_c])_{-s} = H^{\overline{n}-i+1}_{\mathfrak{m}}(K^{es})_{cs}$ by Proposition 3.1.6. Moreover, note that $H^0_m(k[(I^e)_c])_{-s} = 0$ for any $s > 0$. Then the statement follows from Corollary 2.3.5. □

Let us denote by $G = G_A(I) = \bigoplus_{n \geq 0} I^n/I^{n+1}$ the form ring of I with the natural bigrading as a quotient of the Rees algebra. For ideals whose form ring is quasi-Gorenstein, we may get necessary and sufficient conditions for a diagonal to be Cohen-Macaulay only in terms of the powers of the ideal.

Corollary 3.4.4 *Let I be a homogeneous ideal in A generated by forms of degree $\leq d$. Assume that the Rees algebra is Cohen-Macaulay and the form ring is quasi-Gorenstein. Let $a = -a^2(G_A(I))$, $b = -a(A)$. For any $c \geq de+1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if*

(i) $H^i_{\mathfrak{m}}(A)_0 = 0$, *for* $i < \overline{n}$.

(ii) $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$, *for* $i < \overline{n}$, $s > 0$.

(iii) $H^i_{\mathfrak{m}}(I^{es-a+1})_{cs-b} = 0$, *for* $1 < i \leq \overline{n}$, $s > 0$.

Proof. Under these assumptions K_R has the expected form, that is, there is a bigraded isomorphism

$$K_{R_A(I)} \cong \bigoplus_{(l,m),\, m \geq 1} [I^{m-a+1}]_{l-b}$$

(see Corollary 4.1.7 for more details about the isomorphism). Then, for any $s > 0$ we have $K^{es} \cong I^{es-a+1}(-b)$, and now the result follows from Theorem 3.4.3. □

We can use Corollary 3.4.4 to precise the Cohen-Macaulay diagonals for a complete intersection ideal of a Cohen-Macaulay ring. In particular, this gives a new proof of [CHTV, Theorem 4.6] where the case $A = k[X_1, \dots, X_n]$ was studied.

Proposition 3.4.5 *Let I be a complete intersection ideal of a Cohen-Macaulay ring A minimally generated by r forms of degrees $d_1, \dots, d_r$. Set $u = \sum_{i=1}^{r} d_i$. For any $c \geq de + 1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if $c > (e-1)d + u + a(A)$.*

Proof. From the bigraded isomorphism $G_A(I) \cong A/I[Y_1, \dots, Y_r]$, with $\deg(Y_j) = (d_j, 1)$, it is easy to prove by induction on e that the a_*-invariant of A/I^e is:

$$a_*(A/I^e) = a(A/I^e) = (e-1)d + u + a(A).$$

On the other hand, we also have that $H_{\mathfrak{m}}^{\bar{n}-r}(A/I^e)_s \neq 0$, for all $s \leq a(A/I^e) = (e-1)d + u + a(A)$ (see Lemma 5.2.19).

Let $\Delta = (c, e)$ be a diagonal with $c \geq de + 1$. Since $a^2(G) = -\mathrm{ht}\,(I) = -r$, by Corollary 3.4.4 we have that $k[(I^e)_c]$ is Cohen-Macaulay if and only if $cs > a(A/I^{es})$ and $cs + a(A) > a(A/I^{es-r+1})$ for all $s > 0$. The first condition is equivalent to $(c-de)s > u-d+a(A)$ for all $s > 0$, that is, $c-de > u-d+a(A)$. The other one is equivalent to $(c-de)s > u - dr$ for all $s > 0$, and this always holds because $u - dr \leq 0$. □

Until now we have given criteria to decide if a diagonal $k[(I^e)_c]$ is Cohen-Macaulay once we know the local cohomology of the powers of I, and the local cohomology of the graded pieces of the canonical module of the Rees algebra. We will apply them in Chapter 5, Section 2, after computing the local cohomology of the powers of certain families of ideals.

The following result shows the behaviour of the a_*-invariant for the graded pieces of any finitely generated bigraded S-module, so in particular for the powers of an ideal and the pieces of the canonical module by applying it to the Rees algebra and its canonical module respectively. This fact has been also obtained independently by S.D. Cutkosky, J. Herzog and N. V. Trung [CHT] and V. Kodiyalam [Ko2] by different methods (see Chapter 5 for more details).

Theorem 3.4.6 *Let L be a finitely generated bigraded S-module. Then there exists α such that for any e*

$$a_*(L^e) \leq de + \alpha.$$

Proof. Let $e_0 = a_*^2(L)$. By Proposition 2.1.18, $H^i_{\mathcal{M}_2}(L)_{(c,e)} = 0$ for $i \geq 0$, $e > e_0$. Then, by Proposition 2.1.3 and Proposition 2.1.18, we have that for any $c \geq de + 1$, $e > e_0$, $i \geq 0$, there are isomorphisms

$$H^i_m(L_\Delta)_1 \cong H^i_{\mathcal{M}_1}(L)_{(c,e)} \cong H^i_{\mathfrak{m}}(L^e)_c.$$

On the other hand, from Corollary 2.1.12 there exist positive integers e_1, α_1 such that $H^i_m(L_\Delta)_s \cong H^{i+1}_{\mathcal{M}}(L)_{(cs,es)}$ for $s \neq 0$, $e > e_1$, $c > de + \alpha_1$. Therefore, we have $H^i_{\mathfrak{m}}(L^e)_c = 0$ for $e > \max\{e_0, e_1\}$, $c > de + \alpha_1$, $i \geq 0$. This proves the statement. □

Next we will show how to obtain a family of Cohen-Macaulay diagonals from the bound on the shifts in the bigraded minimal free resolution of the Rees algebra given by Theorem 1.3.4. To begin with, let us study the bigraded a-invariant of the Rees algebra.

Lemma 3.4.7 *(i) $a^1(R) \leq a(A)$.*

(ii) If R is Cohen-Macaulay and $a^2(G) < -1$, then $a^1(R) = a(A)$.

Proof. By setting $R_{++} = \bigoplus_{j>0} R_{(i,j)}$, we have the following bigraded exact sequences:

$$0 \to R_{++} \to R \to A \to 0$$

$$0 \to R_{++}(0,1) \to R \to G \to 0.$$

For each (i,j), we get exact sequences:

$$\cdots \to H^{\overline{n}}_{\mathcal{M}}(A)_{(i,j)} \to H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j)} \to H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \quad (1)$$

$$\cdots \to H^{\overline{n}}_{\mathcal{M}}(G)_{(i,j)} \to H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \to H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \quad (2)$$

Note that $A_{(i,j)} = 0$ if $j \neq 0$ and so $H^{\overline{n}}_{\mathcal{M}}(A)_{(i,j)} = 0$ if $j \neq 0$.

We want to determine $a^1(R) = \max\{i \mid \exists j : H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \neq 0\}$. Suppose $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)} \neq 0$. Since $a^2(R) = -1$, we have $j \leq -1$. Then, from (2), we get $H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \neq 0$. If $j+1 < 0$, from (1) we obtain $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j+1)} \cong H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \neq 0$. By repeating this argument, we obtain $H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(i,0)} \neq 0$ and, since $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,0)} = 0$, from (1) we get $H^{\overline{n}}_{\mathfrak{m}}(A)_i = H^{\overline{n}}_{\mathcal{M}}(A)_{(i,0)} \neq 0$. Thus $i \leq a(A)$, and then it follows that $a^1(R) \leq a(A)$.

Assume now that R is Cohen-Macaulay and $a^2(G) < -1$. From (2), we have that if $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(a(A),-1)} = 0$ then $H^{\overline{n}+1}_{\mathcal{M}}(R_{++})_{(a(A),0)} = 0$. Since R is Cohen-Macaulay, from (1) we get $H^{\overline{n}}_{\mathfrak{m}}(A)_{a(A)} = H^{\overline{n}}_{\mathcal{M}}(A)_{(a(A),0)} = 0$, which is a contradiction. □

Remark 3.4.8 Note that in the proof of the Lemma 3.4.7 (*ii*) it is enough to assume $H^{\overline{n}}_{\mathcal{M}}(G)_{(a(A),-1)} = 0$ and $H^{\overline{n}}_{\mathcal{M}}(R)_{(a(A),0)} = 0$.

Remark 3.4.9 Let us consider the group morphism $\psi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\psi(i,j) = i+j$. By Lemma 1.2.3, $H^{\overline{n}+1}_{\mathcal{M}}(R^{\psi})_l = \bigoplus_{i+j=l} H^{\overline{n}+1}_{\mathcal{M}}(R)_{(i,j)}$. Then, by applying Lemma 3.4.7 we get $a(R^{\psi}) \leq a(A) - 1$. If R is Cohen-Macaulay and $a^2(G) < -1$, we have proved $H^{\overline{n}+1}_{\mathcal{M}}(R)_{(a(A),-1)} \neq 0$ and so $a(R^{\psi}) = a(A) - 1$.

We can use the upper bound for the bigraded a-invariant of the Rees algebra found in Lemma 3.4.7 to get bounds for the shifts (a,b) in its resolution. Namely,

Lemma 3.4.10 *Let I be an ideal of A generated by r forms in degrees $d_1 \leq \ldots \leq d_r$ whose Rees algebra is Cohen-Macaulay. Set $u = \sum_{j=1}^{r} d_j$. Let*

$$0 \to D_m \to \ldots \to D_1 \to D_0 = S \to R_A(I) \to 0$$

be the minimal bigraded free resolution of $R_A(I)$ over S. Given $p \geq 1$ and $(a,b) \in \Omega_p$, we have

(*i*) $a \leq 0$, $b \leq 0$, $a \leq d_1 b$.

(*ii*) $-a - b \leq u + \overline{n} + a(A) + p$.

(*iii*) $-a \leq u + \overline{n} + a(A) + p - (r-1)$. *In particular,* $-a \leq u + n + a(A)$.

(*iv*) $-b < r$.

Proof. It is clear that $a \leq 0$, $b \leq 0$, $a \leq d_1 b$. Also note that $m = \text{proj.dim}_S R = n + r - \overline{n} - 1$. To prove (*ii*), let us consider the morphism $\psi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\psi(i,j) = i + j$, and note that $S(a,b)^{\psi} = S^{\psi}(a+b)$. Applying the functor $(\ )^{\psi}$ to the resolution, we get a $\mathbb{Z}$-graded minimal free resolution of R^ψ over S^ψ. Moreover $a(S^\psi) = -n-u-r$ and $a(R^\psi) \le a(A)-1$ (see Remark 3.4.9). Given $(a,b) \in \Omega_p$, from Theorem 1.3.4 we get:

$$-a-b \le \max\{\, -\alpha-\beta \mid (\alpha,\beta) \in \Omega_p\} \le$$
$$\le \max\{\, -\alpha-\beta \mid (\alpha,\beta) \in \Omega_m\} + p - m =$$
$$= a(R^\psi) - a(S^\psi) + p - m \le u + a(A) + \overline{n} + p.$$

To prove (iii), observe that by Theorem 1.3.4 we have

$$-a \le \max\{-\alpha \mid (\alpha,\beta) \in \Omega_p\} \le$$
$$\le \max\{\, -\alpha \mid (\alpha,\beta) \in \Omega_m\} + p - m =$$
$$= a^1(R) - a^1(S) + p - m \le u + a(A) + \overline{n} - r + 1 + p.$$

Finally, by using Theorem 1.3.4 we also obtain:

$$-b \le \max\{\, -\beta \mid (\alpha,\beta) \in \Omega_p\} \le$$
$$\le \max\{\, -\beta \mid (\alpha,\beta) \in \Omega_R\} =$$
$$= a^2(R) - a^2(S) = -1 + r\ ,$$

so (iv) is proved. □

Remark 3.4.11 When I is a complete intersection ideal of the polynomial ring $A = k[X_1, \ldots, X_n]$, all the shifts in the resolution may be explicitly computed. In fact, by the Eagon-Northcott complex the shifts $(a,b) \in \Omega_p$ are of the type:

$$a = -d_{j_1} - \ldots - d_{j_{p+1}},\quad b = -m$$

where $1 \le j_1 \le \ldots \le j_{p+1} \le r$, $1 \le m \le p$ (see [CHTV, Lemma 4.1]). Note that b takes all the values between $-r$ and 0 and the bounds of Lemma 3.4.10 (ii), (iii) are sharp for $p = r-1$.

Now we are ready to determine a family of diagonals of the Rees algebra with the Cohen-Macaulay property when the Rees algebra is Cohen-Macaulay. Namely,

Theorem 3.4.12 *Let $I \subset A$ be a homogeneous ideal generated by r forms of degrees $d_1 \le \ldots \le d_r = d$. Assume that $H^i_{\mathfrak{m}}(A)_0 = 0$ for all $i < \overline{n}$. Set $u = \sum_{j=1}^r d_j$. If the Rees algebra is Cohen-Macaulay, then*

(i) $k[(I^e)_c]$ *is Cohen-Macaulay for* $c > \max\{d(e-1)+u+a(A), d(e-1)+u-d_1(r-1)\}$.

(ii) *If* I *is equigenerated by forms of degree* d, $k[(I^e)_c]$ *is Cohen-Macaulay for* $c > d(e-1+l)+a(A)$.

Proof. We have already shown that the assumptions on the local cohomology of A imply that $H^i_m(k[(I^e)_c])_0 = 0$ for $i < \overline{n}$. Now, let us consider the bigraded minimal free resolution of R over S:

$$0 \to D_m \to \ldots \to D_0 = S \to R \to 0 \,,$$

where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. From Remark 2.1.11, recall that if we define

$$X^\Delta = \bigcup_{(a,b)\in\Omega_R}\{\, s \in \mathbb{Z} \mid \tfrac{-b}{e} \leq s \leq \tfrac{bd-a-n}{c-ed}\,\},$$

$$Y^\Delta = \bigcup_{(a,b)\in\Omega_R}\{\, s \in \mathbb{Z} \mid \tfrac{(b+r)d-u-a}{c-ed} \leq s \leq \tfrac{-b-r}{e}\,\},$$

then we have $H^i_m(k[(I^e)_c])_s = H^{i+1}_{\mathcal{M}}(R)_{(cs,es)} = 0$ for $i < \overline{n}$, $s \notin X^\Delta \cup Y^\Delta$. Therefore, $k[(I^e)_c]$ is Cohen-Macaulay for any diagonal $\Delta = (c,e)$ such that $X^\Delta \cup Y^\Delta \subset \{0\}$. Since $b \leq 0$, any $s \in X^\Delta$ satisfies $s \geq 0$. If $b \leq -1$, then $bd-a-n \leq -d+u+a(A)$ by Lemma 3.4.10. If $b=0$, then note that $[\mathbf{D}.]_0$ is a graded minimal free resolution of A over S_1 with $[D_p]_0 = \bigoplus_{(a,0)\in\Omega_p} S_1(a)$, so $bd-a-n = -a-n \leq a(A)$ by Theorem 1.3.4. Therefore, by taking $c > (e-1)d+u+a(A)$, we have $\frac{bd-a-n}{c-ed} < 1$ and so $X^\Delta \subset \{0\}$. On the other hand, any shift $(a,b) \in \Omega_R$ satisfies $b > -r$ by Lemma 3.4.10, so if $s \in Y^\Delta$ then $s \leq -1$. By taking $c > d(e-1)+u-d_1(r-1)$, one can check $\frac{(b+r)d-u-a}{c-ed} > -1$, so $Y^\Delta = \emptyset$. This proves (*i*).

Now, let us assume that I is generated in degree d. From the proof of Proposition 3.4.1, we have that $H^i_m(k[(I^e)_c])_s = 0$ for $i < \overline{n}$, $s < 0$. So it is just enough to study the positive components of these local cohomology modules. Tensoriazing by $k(T)$ we may assume that the field k is infinite. Then, since the fiber cone $F_{\mathfrak{m}}(I)$ of I is a k-algebra generated by homogeneous elements in degree $(d,1)$, there exists a minimal reduction J of I generated by l forms of degree d. Now, by considering the polynomial ring $S = k[X_1,\ldots,X_n,Y_1,\ldots,Y_l]$, we have a natural epimorhism $S \to R_A(J)$. Then $R_A(J)$ is a finitely generated bigraded S-module, and so $R_A(I)$ because it is a finitely generated $R_A(J)$-module. Then we may consider the bigraded minimal free resolution of $R_A(I)$ over S, and it suffices to check that the sets

X^{Δ} and Y^{Δ} associated to this resolution do not have positive integers for $c > d(e-1+l) + a(A)$. □

In the case where $A = k[X_1, \ldots, X_n]$ we can improve the bounds slightly. More explicitly,

Theorem 3.4.13 *Let $I \subset A = k[X_1, \ldots, X_n]$ be a homogeneous ideal generated by r forms of degrees $d_1 \leq \ldots \leq d_r$. Assume that the Rees algebra $R_A(I)$ is Cohen-Macaulay. Then, by defining*

$$\alpha = \min\{d(e-1) + u - n, e(u-n)\},$$

$$\beta = \min\{d(e-1) + u - d_1(r-1), e(u-d_1)\},$$

we have that $k[(I^e)_c]$ is Cohen-Macaulay for all $c > \max\{de, \alpha, \beta\}$.

Proof. Note that the first homomorphism in the resolution of the Rees algebra is:

$$\begin{array}{ccc} D_0 = S & \longrightarrow & R_A(I) \\ X_i & \mapsto & X_i \\ Y_j & \mapsto & f_j t \end{array},$$

so any shift $(a,b) \in \Omega_p$, with $p \geq 1$, satisfies $b < 0$. Note that if $\frac{bd-a-n}{c-ed} < \frac{-b}{e}$ for all $(a,b) \in \Omega_p$, $p \geq 1$, then X^{Δ} is empty. This condition is equivalent to $e(-a-n) < -bc$. Since $e(-a-n) \leq e(-n-u)$ by Lemma 3.4.10 and $-bc \geq c$, it suffices to take $c > e(u-n)$ to get this condition. Similarly, if $c > e(u-d_1)$ then $Y^{\Delta} = \emptyset$ and we are done. □

Remark 3.4.14 With the notation above, note that $\alpha = e(u-n)$ if and only if $u - d - n < 0$, and $\beta = e(u-d_1)$ if and only if $u - d - d_1 < 0$ and $e > \frac{u-d-d_1(r-1)}{u-d_1-d}$. For instance, if $u - d < n$ then $k[(I^e)_c]$ is Cohen-Macaulay for all $c > \max\{de, \beta\}$.

We finish this section with an application of Corollary 3.1.4 to the study of the $(n-1)$-folds obtained from $\mathbb{P}_k^{n-1}$ by blowing-up a finite set of distinct points. Let $P_1, \ldots, P_s \in \mathbb{P}_k^{n-1}$ be distinct points, and for each $i = 1, \ldots, s$, denote by $\mathcal{P}_i \subset A = k[X_1, \ldots, X_n]$ the homogeneous prime ideal which corresponds to P_i. Let us consider the ideal of fat points $I = \mathcal{P}_1^{m_1} \cap \ldots \cap \mathcal{P}_s^{m_s}$, with $m_1, \ldots, m_s \in \mathbb{Z}_{\geq 1}$. Next we study the embeddings of the blow-up of $\mathbb{P}_k^{n-1}$ along $\mathcal{I}$ via the linear systems $(I^e)_c$, whenever these linear systems are very ample, slightly extending [GGP, Theorem 2.4] where only the divisors (I_c) were considered.

Theorem 3.4.15 *Let $I \subset A = k[X_1, \dots, X_n]$ be an ideal of fat points, where k is a field with characteristic 0. Then:*

(i) $k[(I^e)_c]$ is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ for any $s > 0$, $i < n$.

(ii) For $c > \operatorname{reg}(I)\, e$, $k[(I^e)_c]$ is Cohen-Macaulay with $a(k[(I^e)_c]) < 0$. In particular, $\operatorname{reg}(k[(I^e)_c]) < n - 1$.

Proof. Let X be the blow-up of the projective space $\mathbb{P}^{n-1}_k$ along $\mathcal{I}$. Assume that I is generated by forms in degree $\leq d$. Then we have shown that $\mathcal{L}^e \otimes \mathcal{M}^c$ is very ample if $c > de$. Therefore, for any $s < 0$, $i < n - 1$, $c > de$, we have that $H^i(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = 0$ by the Kodaira vanishing theorem (see for instance [Har, Remark III.7.5]). Then, $H^i_m(k[(I^e)_c])_s = 0$ for $i < n$, $s < 0$ by Remark 3.1.1.

On the other hand, from Proposition 3.3.3 we get $\Gamma(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = k$ and $H^i(\mathrm{X}, \mathcal{O}_{\mathrm{X}}) = 0$ for all $i > 0$. Then, according to Remark 3.1.1, we have $H^i_m(k[(I^e)_c])_0 = 0$ for any i.

Finally, note that for a given $\mathfrak{p} \in \operatorname{Proj}(A)$ we have:

$$I_{\mathfrak{p}} = \begin{cases} A_{\mathfrak{p}} & \text{if } \mathfrak{p} \notin \{\mathcal{P}_1, \dots, \mathcal{P}_s\} \\ \mathcal{P}_i^{m_i} A_{\mathcal{P}_i} & \text{if } \mathfrak{p} = \mathcal{P}_i \ . \end{cases}$$

In both cases, $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay. So, according to Corollary 3.1.4, $\pi_* \mathcal{L}^e = \widetilde{I^e}$ and $R^j \pi_* \mathcal{L}^e = 0$ for $e > 0$, $j > 0$. Then, by the Leray spectral sequence

$$E_2^{i,j} = H^i(\mathrm{Y}, R^j \pi_* (\mathcal{L}^{es} \otimes \mathcal{M}^{cs})) \Rightarrow H^{i+j}(\mathrm{X}, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}),$$

we have that for $s > 0$

$$\Gamma(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = \Gamma(\mathbb{P}^{n-1}_k, \mathcal{I}^{es}(cs)) = (I^{es})^*_{cs},$$

$$H^i(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = H^i(\mathbb{P}^{n-1}_k, \mathcal{I}^{es}(cs)) = H^{i+1}_{\mathfrak{m}}(I^{es})_{cs} \ , \forall i \geq 1.$$

Therefore, we immediately get (i) by Remark 3.1.1. From [GGP, Theorem 1.1] or [Cha, Theorem 6], we have $a_*(I^e) \leq \operatorname{reg}(I^e) \leq e \operatorname{reg}(I)$. Furthermore, $(I^e)^*_c = (I^e)_c$ for $c \geq \operatorname{reg}(I) e$ by [GGP, Corollary 1.4]. Then, by taking $c > \operatorname{reg}(I) e$, we have $\Gamma(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = (I^{es})_{cs}$ and $H^i(X, \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) = 0$ for any $i \geq 1$. Now, by Remark 3.1.1, we obtain $H^i_m(k[(I^e)_c])_s = 0$ for any $s > 0$. So $k[(I^e)_c]$ is Cohen-Macaulay with $a(k[(I^e)_c]) < 0$. □

3.5 Linear bounds

S.D. Cutkosky and J. Herzog [CH] studied sufficient conditions for the existence of a constant f satisfying that the rings $k[(I^e)_c]$ are Cohen-Macaulay for all $c \geq ef$ and $e > 0$, that is, for the existence of a linear bound on c and e ensuring that $k[(I^e)_c]$ is Cohen-Macaulay. Note that, according to Theorem 3.4.12, this holds for any homogeneous ideal in a Cohen-Macaulay ring whose Rees algebra is Cohen-Macaulay. Our first purpose is to show that this also holds under the weaker assumption that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for any $\mathfrak{p} \in \text{Proj }(A)$. This would recover for instance locally complete intersection ideals.

Let $K = K_R = \bigoplus_{(i,j)} K_{(i,j)}$ be the canonical module of $R = R_A(I)$, and let K^e be the graded A-module $K^e = \bigoplus_i K_{(i,e)}$. Then we have

Theorem 3.5.1 *Assume that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for all $\mathfrak{p} \in$ Proj (A). Then $\pi_*(w_X \otimes \mathcal{L}^e) = \widetilde{K^e}$ and $R^j\pi_*(w_X \otimes \mathcal{L}^e) = 0$ for $e > 0$, $j > 0$.*

Proof. Let $A_i = A_{(x_i)}$, $I_i = I_{(x_i)}$, $R_i = A_i[I_it]$ and $K_i = K_R \otimes R_i$. Let us consider the affine cover $\{Y_i : 1 \leq i \leq n\}$ of Y, where $Y_i = Y - V_+(x_i) \cong \text{Spec}(A_i)$. Denote by $X_i = \pi^{-1}Y_i = \text{Proj }(R_i)$. Then, for a given j and $e > 0$ we have that $R^j\pi_*(w_X \otimes \mathcal{L}^e) = 0$ if and only if $R^j\pi_*(w_X \otimes \mathcal{L}^e) \mid Y_i = 0$ for all $1 \leq i \leq n$. Furthermore, we have a diagram

$$\begin{array}{ccc} X_i = \text{Proj}(R_i) & \hookrightarrow & X = \text{Proj }^2(R) \\ \pi' \downarrow & & \pi \downarrow \\ Y_i = \text{Spec}(A_i) & \hookrightarrow & Y = \text{Proj}(A) \end{array}$$

Now, by Corollary III.8.2 and Proposition III.8.5 of [Har], for any $e > 0$ and $j > 0$ we have

$$R^j\pi_*(w_X \otimes \mathcal{L}^e) \mid Y_i = R^j\pi'_*((w_X \otimes \mathcal{L}^e) \mid X_i) = H^j(X_i, (w_X \otimes \mathcal{L}^e) \mid X_i)^{\sim}.$$

Since $(w_X \otimes \mathcal{L}^e) \mid X_i = \widetilde{K_i(e)}$, we have reduced the problem to show that $H^{j+1}_{(R_i)_+}(K_i)_e = 0$. Similarly, $\pi_*(w_X \otimes \mathcal{L}^e) = \widetilde{K^e}$ if $H^0_{(R_i)_+}(K_i)_e = H^1_{(R_i)_+}(K_i)_e = 0$.

Denote by $\overline{R}_i = R_{A_{x_i}}(I_{x_i})$. Tensoriazing by $\overline{R}_i$, we have

$$H^j_{(\overline{R}_i)_+}(K \otimes \overline{R}_i)_e = H^j_{(R_i)_+}(K_i)_e[T, T^{-1}],$$

so it is enough to show that $H^j_{(\overline{R_i})_+}(K \otimes \overline{R_i})_e = 0$ for any i, j and $e > 0$. Let $\mathfrak{q} \in \operatorname{Spec}(A_{x_i})$ be a homogeneous prime, and let $\mathfrak{p} \in \operatorname{Proj}(A)$ be such that $\mathfrak{q} = \mathfrak{p}A_{x_i}$. Denote by $B = R_{A_\mathfrak{p}}(I_\mathfrak{p})$. Then

$$[H^j_{(\overline{R_i})_+}(K \otimes \overline{R_i})_e]_\mathfrak{q} = [H^j_{(\overline{R_i})_+}(K \otimes \overline{R_i})_\mathfrak{q}]_e = [H^j_{B_+}(K \otimes_A A_\mathfrak{p})]_e.$$

By taking into account that B is Cohen-Macaulay, standard arguments allow to check that $K \otimes_A A_\mathfrak{p} = K_B$ or $K \otimes_A A_\mathfrak{p} = 0$. In any case, we have that $[H^j_{B_+}(K \otimes_A A_\mathfrak{p})]_e = 0$ for any j and $e > 0$, so we are done. □

From this result we can obtain a simple criterion for having a linear bound for the Cohen-Macaulay property. First, let us notice the following interesting fact.

Proposition 3.5.2 *Assume $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Proj}(A)$. Let $c \geq de + 1$, $e > 0$. Then:*

(i) For $s > 0$, there is an exact sequence

$$0 \to H^0_m(k[(I^e)_c])_s \to (I^{es})_{cs} \to (I^{es})^*_{cs} \to H^1_m(k[(I^e)_c])_s \to 0$$

and isomorphisms $H^i_m(k[(I^e)_c])_s \cong H^i_\mathfrak{m}(I^{es})_{cs}$ for $i > 1$.

(ii) For $s > 0$, $1 \leq i \leq \overline{n} - 1$, $H^i_m(k[(I^e)_c])_{-s} \cong H^{\overline{n}-i+1}_\mathfrak{m}(K^{es})_{cs}$.

Proof. The first part of the statement follows directly from Corollary 3.1.5. To prove (ii), let $s > 0$, $i \geq 1$. Then

$$\begin{aligned} H^i_\mathfrak{m}(k[(I^e)_c])_{-s} &\cong H^{i-1}(X, \mathcal{L}^{-es} \otimes \mathcal{M}^{-cs}) \quad \text{by Remark 3.1.1} \\ &= H^{\overline{n}-i}(X, \omega_X \otimes \mathcal{L}^{es} \otimes \mathcal{M}^{cs}) \quad \text{by Serre's duality} \\ &= H^{\overline{n}-i}(Y, \pi_*(\omega_X \otimes \mathcal{L}^{es}) \otimes \mathcal{M}^{cs}) \quad \text{by Theorem 3.5.1} \\ &= H^{\overline{n}-i}(Y, \widetilde{K^{es}}(cs)) \quad \text{by Theorem 3.5.1} \\ &= H^{\overline{n}-i+1}_\mathfrak{m}(K^{es})_{cs}. \ \square \end{aligned}$$

Theorem 3.5.3 *Assume that A is a ring with $H^i_\mathfrak{m}(A)_0 = 0$ for $i < \overline{n}$. If I is a homogeneous ideal of A such that $R_{A_\mathfrak{p}}(I_\mathfrak{p})$ is Cohen-Macaulay for all $\mathfrak{p} \in \operatorname{Proj}(A)$, then there exists α such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq de + \alpha$, $e > 0$.*

Proof. From Proposition 3.3.5 we have that $k[(I^e)_c]$ is Cohen-Macaulay for $c \gg e \gg 0$. So, in particular, by Theorem 3.2.3 and Remark 3.1.1 we have that $H^i_m(k[(I^e)_c])_0 = 0$ for any $c \geq de + 1$, $i < \overline{n}$. On the other hand, according to Theorem 3.4.6, there exists $\alpha > 0$ such that $a_*(I^e) < de + \alpha$, $a_*(K^e) < de + \alpha$, for all e. Then, $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + \alpha$ by Proposition 3.5.2. □

In particular, we can recover Corollary 4.2 and Corollary 4.4 in [CH]. Furthermore, note that the bound has been improved slightly.

Corollary 3.5.4 *Let I be a locally complete intersection ideal in a Cohen-Macaulay ring A. Then there exists α such that such that $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + \alpha$ and $e > 0$.*

Corollary 3.5.5 *Let I be a strongly Cohen-Macaulay ideal such that for any prime ideal $\mathfrak{p} \supseteq I$, $\mu(I_\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p})$ in a Cohen-Macaulay ring A. Then there exists α such that such that $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + \alpha$ and $e > 0$.*

We can also characterize the existence of linear bounds for the Cohen-Macaulay property of the rings $k[(I^e)_c]$ by means of the local cohomology modules of the Rees algebra and its canonical module. Namely,

Proposition 3.5.6 *Assume that there exist c, e such that $k[(I^e)_c]$ is Cohen-Macaulay. Then the following are equivalent*

(i) *There exists f such that $k[(I^e)_c]$ is Cohen-Macaulay for $c \geq ef$, $e > 0$.*

(ii) *There exists f such that $H^i_{R_+}(R)_{(c,e)} = 0$, $H^{\overline{n}-i+1}_{R_+}(K_R)_{(c,e)} = 0$, for $i < \overline{n}$, $c \geq ef$ and $e > 0$.*

(iii) *There exists f such that $H^i_{\mathcal{M}_2}(R)_{(c,e)} = 0$, $H^{\overline{n}-i+1}_{\mathcal{M}_2}(K_R)_{(c,e)} = 0$, for $i < \overline{n}$, $c \geq ef$ and $e > 0$.*

Proof. From Lemma 2.1.2, we have $H^i_m(k[(I^e)_c])_s = H^i_{R_+}(R)_{(cs,es)}$ for any $s > 0$. Moreover, since X is Cohen-Macaulay we also have that $H^i_m(k[(I^e)_c])_{-s} = H^{\overline{n}-i+1}_{R_+}(K_R)_{(cs,es)}$ for $1 \leq i < \overline{n}$, $s > 0$ (see the proof of Proposition 3.1.6). Therefore two first conditions are equivalent.

For *(ii)* $\Longleftrightarrow$ *(iii)*, first we will show that there exists $\overline{f}$ such that for all i, $e > 0$, $c \geq e\overline{f}$ it holds

$$H^i_{\mathcal{M}}(R)_{(c,e)} = H^i_{\mathcal{M}_1}(R)_{(c,e)} = 0,$$

$$H^i_{\mathcal{M}}(K_R)_{(c,e)} = H^i_{\mathcal{M}_1}(K_R)_{(c,e)} = 0.$$

Then, from the Mayer-Vietoris exact sequence, we get that for any i, $e > 0$, $c \geq e\overline{f}$,

$$H^i_{R_+}(R)_{(c,e)} = H^i_{\mathcal{M}_2}(R)_{(c,e)},$$

$$H^i_{R_+}(K_R)_{(c,e)} = H^i_{\mathcal{M}_2}(K_R)_{(c,e)},$$

and so (ii) and (iii) are equivalent. To get the vanishing of the local cohomology modules with respect to the maximal ideal $\mathcal{M}$, it is just enough to take $c > \max\{a^1_*(R), a^1_*(K_R)\}$. By Theorem 3.4.6 there exists $\alpha > 0$ such that $a_*(I^e) < de + \alpha$, $a_*(K^e) < de + \alpha$, so by taking $c \geq de + \alpha$ we have $H^i_{\mathcal{M}_1}(R)_{(c,e)} = H^i_{\mathfrak{m}}(I^e)_c = 0$ and $H^i_{\mathcal{M}_1}(K_R)_{(c,e)} = H^i_{\mathfrak{m}}(K^e)_c = 0$. □

Also note that the last proposition holds if we replace the condition for all $c \geq ef$ and $e > 0$ by the following one: for all $c \geq de + \alpha$ and $e > 0$. As a direct consequence, we obtain:

Corollary 3.5.7 *Assume that R has some Cohen-Macaulay diagonal. If $a^2_*(R) \leq 0$ and $a^2_*(K_R) \leq 0$, there exists α such that $k[(I^e)_c]$ is Cohen-Macaulay for all $c \geq de + \alpha$ and $e > 0$.*

Proof. It is a direct consequence of Proposition 3.5.6 by noting that for any i and $e > 0$ we have $H^i_{\mathcal{M}_2}(R)_{(c,e)} = 0$, $H^i_{\mathcal{M}_2}(K_R)_{(c,e)} = 0$ by Proposition 2.1.18. □

Remark 3.5.8 The converse of the last corollary is not true. Let us take the homogeneous ideal $I = (x^7, y^7, x^6y + x^2y^5)$ in the polynomial ring $A = k[x, y]$. We have $\mathfrak{m}^{14} \subset I$, so $(I^e)_c = A_c$ for any $c \geq 14e$, and then $k[(I^e)_c] = k[x^a y^b \mid a + b = c]$ is Cohen-Macaulay for all these c, e. But $a^2_*(R) = 4 > 0$ by [HM, Example 3.13].