On the diagonals of a Rees algebra

Olga Lavila Vidal

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UNIVERSITAT DE BARCELONA

Departament d'Àlgebra i Geometria

ON THE DIAGONALS OF A REES ALGEBRA

Olga Lavila Vidal

MATEMÀTIQUES

Chapter 4

Gorenstein coordinate rings of blow-up schemes

Let $Y = \mathrm{Proj}(A)$ be a closed subscheme of $\mathbb{P}_k^{n-1}$, and let X be the blow-up of Y along $\mathcal{I} = \widetilde{I}$, where I is a homogeneous ideal in A. If I is generated by forms of degree $\leq d$, we have already shown that for any $c \geq de + 1$ the ring $k[(I^e)_c]$ is the homogeneous coordinate ring of a projective embedding of X in $\mathbb{P}_k^{N-1}$, where $N = \dim_k(I^e)_c$. In this chapter we are interested in the (quasi-) Gorenstein property of the rings $k[(I^e)_c]$. The results work in the case of the blow-up of the projective space $\mathbb{P}_k^{n-1}$.

If the Rees algebra is Cohen-Macaulay and the associated graded ring is Gorenstein we will determine exactly for which pairs (c, e) the ring $k[(I^e)_c]$ is quasi-Gorenstein and, in particular, we will obtain that there is just a finite set of diagonals with this property. This result can be applied to several families of ideals. In particular, to any complete intersection ideal, extending in this way [CHTV, Corollary 4.7], and to the ideal generated by the maximal minors of a generic matrix.

After that, we show that there are always at most a finite number of rings $k[(I^e)_c]$ which are quasi-Gorenstein and we give upper bounds for such diagonals whenever $R_A(I)$ is Cohen-Macaulay.

Finally, we prove that under some restrictions the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the associated graded ring to be Gorenstein.

At the end of the chapter we apply our results to Room surfaces. These surfaces are obtained by blowing-up $\mathbb{P}_k^2$ along $\binom{d+1}{2}$ points, $d \geq 2$, which do not lie in any curve of degree $d-1$ and then embedding in $\mathbb{P}_k^{2d+2}$. We will show that the only Room surface which is Gorenstein is the del Pezzo sestic surface in $\mathbb{P}^6$, so recovering that well known result (see [GG, Example 1]).

4.1 The case of ideals whose form ring is Gorenstein

Throughout all the chapter we shall use the following notations. $A = k[X_1, ..., X_n]$ will denote the usual polynomial ring with coefficients in a field k, and $I \subset A$ a homogeneous ideal minimally generated by forms $f_1, ..., f_r$ of degrees $d_1 \leq ... \leq d_r = d$. We put $u = \sum_{j=1}^r d_j$. Let $S = k[X_1, ..., X_n, Y_1, ..., Y_r]$ be the polynomial ring with the grading obtained by setting $\deg X_i = (1,0)$ for $i = 1, ..., n$, $\deg Y_j = (d_j, 1)$ for $j = 1, ..., r$; so that $R = R_A(I)$ and $G = G_A(I)$ can be seen in a natural way as bigraded S-modules. We will assume $n \geq r \geq 2$.

Notice that any diagonal S_Δ of the polynomial ring S is Cohen-Macaulay by Corollary 2.1.8. We begin this section by showing that, on the contrary, S_Δ is Gorenstein only for a finite number of diagonals. Furthermore, we may determine them.

Proposition 4.1.1 *S_Δ is Gorenstein if and only if $\frac{r}{e} = \frac{n+u}{c} = l \in \mathbb{Z}$. Then $a(S_\Delta) = -l$.*

Proof. Let $T = S_\Delta = \bigoplus_{s \geq 0} U_s$, where U_s is the k-vector space generated by the monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_r^{\beta_r}$ with $\alpha_i, \beta_j \geq 0$ satisfying the equations

$$(\star) \quad \begin{cases} \sum_{i=1}^n \alpha_i + \sum_{j=1}^r d_j \beta_j = cs \\ \\ \sum_{j=1}^r \beta_j = es \end{cases}$$

By Corollary 2.1.4, we have

$$H_m^{n+r-1}(T) \cong H_{\mathcal{M}}^{n+r}(S)_\Delta \cong (\bigoplus_{\alpha<0, \beta<0} kX^\alpha Y^\beta)_\Delta.$$

Therefore, $K_T = \underline{\mathrm{Hom}}_k(H_m^{n+r-1}(T), k) = \bigoplus_{s \geq 1} V_s$ with V_s the k-vector space generated by the monomials $X_1^{\alpha_1} \cdots X_n^{\alpha_n} Y_1^{\beta_1} \cdots Y_r^{\beta_r}$, and $\alpha_i > 0, \beta_j > 0$ which satisfy $(\star)$. Since T is Cohen-Macaulay, T is Gorenstein if and only if $K_T \cong T(a(T))$.

Assume first that $\frac{r}{e} = \frac{n+u}{c} = l \in \mathbb{Z}$. Then, the multiplication by $X_1 \cdots X_n Y_1 \cdots Y_r \in T_l$ induces an isomorphism $T \cong K_T(l)$ and so T is Gorenstein with $a(T) = -l$.

To prove the converse set $(\alpha, \beta) = (\alpha_1, ..., \alpha_n, \beta_1, ..., \beta_r)$ with $\alpha_i, \beta_j > 0$ and assume the contrary. This means that $(\mathbf{1}, \mathbf{1})$ is not a solution of $(\star)$ for any s. On the other hand, the set of solutions of $(\star)$ for some s is partially ordered by means of $(\alpha, \beta) \leq (\gamma, \rho) \iff \alpha_i \leq \gamma_i,\ \beta_j \leq \rho_j,\ \forall i, j$. Then, one can easily check that for any $i = 1, \ldots, n$, $j = 1, \ldots, r$ there exists a solution of $(\star)$ for some s such that $\alpha_i = \beta_j = 1$. This implies the existence of at least two minimal solutions, and so T is not Gorenstein. □

Remark 4.1.2 Note that the number of minimal elements in the set of solutions of the system $(\star)$ coincides with the type of S_Δ. It is not difficult to see that if S_Δ is not Gorenstein, then its type is $\geq r$.

Remark 4.1.3 Throughout the chapter we assume $r \geq 2$. In the case that $r = 1$ we have that I is a principal ideal, and $R_A(I) \cong S = k[X_1, \ldots, X_n, Y]$. Then, it is easy to check that S_Δ is always Cohen-Macaulay, and S_Δ is Gorenstein if and only if $\Delta = (n + d, 1)$.

The last proposition leads to the question of whether there exist diagonals (c, e) such that $k[(I^e)_c]$ be quasi-Gorenstein, and how we can determine them. Our answer will be partially based on the following proposition which links the diagonal of the canonical module of $R_A(I)$ to the canonical module of the diagonal of $R_A(I)$. It is stated and proved for complete intersection ideals in [CHTV, Proposition 4.5], but in fact the same statement and proof are valid in general. We include the proof for completeness.

Proposition 4.1.4 $K_{R_\Delta} = (K_R)_\Delta$.

Proof. Let us consider a presentation of R as S-module

$$0 \to C \to S \to R \to 0,$$

which leads to the bigraded exact sequence of local cohomology modules

$$0 \to H^{n+1}_{\mathcal{M}}(R) \to H^{n+2}_{\mathcal{M}}(C) \to H^{n+2}_{\mathcal{M}}(S) \to 0.$$

Similarly, we get the graded exact sequence

$$0 \to H^n_m(R_\Delta) \to H^{n+1}_m(C_\Delta) \to H^{n+1}_m(S_\Delta) \to 0.$$

On the other hand, by Corollary 2.1.4 we have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & H^{n+1}_{\mathcal{M}}(R)_\Delta & \to & H^{n+2}_{\mathcal{M}}(C)_\Delta & \to & H^{n+2}_{\mathcal{M}}(S)_\Delta & \to & 0 \\ & & \varphi^n_R \uparrow & & \varphi^{n+1}_C \uparrow & & \varphi^{n+1}_S \uparrow & & \\ 0 & \to & H^n_m(R_\Delta) & \to & H^{n+1}_m(C_\Delta) & \to & H^{n+1}_m(S_\Delta) & \to & 0 \end{array}$$

where φ^{n+1}_C, φ^{n+1}_S are isomorphisms, and so φ^n_R is also an isomorphism. Therefore $H^n_m(R_\Delta) \cong H^{n+1}_{\mathcal{M}}(R)_\Delta$ and we get

$$K_{R_\Delta} = \underline{\mathrm{Hom}}_k(H^n_m(R_\Delta), k) = \underline{\mathrm{Hom}}_k(H^{n+1}_{\mathcal{M}}(R)_\Delta, k) =$$

$$= \underline{\mathrm{Hom}}_k(H^{n+1}_{\mathcal{M}}(R), k)_\Delta = (K_R)_\Delta. \ \square$$

Remark 4.1.5 The hypothesis $n \geq r \geq 2$ fixed before is only used in this chapter to prove Proposition 4.1.4, and of course its applications. Nevertheless, the isomorphism $K_{R_\Delta} = (K_R)_\Delta$ is also valid if $n, r \geq 2$, I is equigenerated and R is Cohen-Macaulay. To prove this, assume $r > n$ (otherwise we may apply Proposition 4.1.4). Let us consider the bigraded minimal free resolution of R over S

$$0 \to D_{r-1} \to \ldots \to D_1 \to D_0 = S \to R \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. Since R is Cohen-Macaulay, we have $-b \leq r-1$ and $-a \geq -bd$ for all $(a,b) \in \Omega_R$ by Lemma 3.4.10. On the other hand, recall from Corollary 2.1.7 that $H^r_{\mathcal{M}_2}(S(a,b))_{(cs,es)} \neq 0$ if and only if $\frac{bd-a}{c-ed} \leq s \leq \frac{-b-r}{e}$, so we get $H^r_{\mathcal{M}_2}(D_p)_\Delta = 0$ for all p. Since $r > n$, we also have that $H^i_{\mathcal{M}_1}(D_p)_\Delta = H^i_{\mathcal{M}_2}(D_p)_\Delta = 0$ for all $i > n$. Then, by Proposition 2.1.10 and Proposition 2.1.3 we have that φ^{n+1}_C is an isomorphism, and the same proof as in Proposition 4.1.4 shows that $K_{R_\Delta} = (K_R)_\Delta$.

This means that all the results we are going to prove in this chapter are also valid if $n, r \geq 2$, I is equigenerated and $R_A(I)$ is Cohen-Macaulay.

In view of Proposition 4.1.4 any information on the bigraded structure of K_R will be of interest. Let B be a d-dimensional local ring, $d \geq 1$, which has a canonical module K_B and $I \subset B$ an ideal of positive height such that $R_B(I)$ is Cohen-Macaulay. In [TVZ, Theorem 2.2] it is given a description of $K_{R_B(I)}$

in terms of a filtration of submodules of K_B. Assume now that $B = \bigoplus_{n\geq 0} B_n$ is a positively graded ring of positive dimension over a local ring B_0, which has a canonical module K_B. Let $I \subset B$ be a homogeneous ideal of positive height. Then, the Rees algebra $R_B(I)$ has a bigraded structure by means of $[R_B(I)]_{(i,j)} = (I^j)_i t^j$ for all $i, j \geq 0$. We also have a bigraded structure on the form ring by means of $[G_B(I)]_{(i,j)} = (I^j)_i/(I^{j+1})_i$ for all $i, j \geq 0$.

Then, the proof of [TVZ, Theorem 2.2] may be "bigraded" and we thus obtain a description of the bigraded structure of $K_{R_B(I)}$. Namely, we get:

Theorem 4.1.6 *With the notation above assume that $R_B(I)$ is Cohen-Macaulay. Then, there exists a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_B and isomorphisms of bigraded modules such that*

$$K_{R_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [K_m]_l\,,$$

$$K_{G_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [K_{m-1}]_l/[K_m]_l.$$

Several other results of [TVZ] may also be "bigraded". In particular [TVZ, Lemma 4.1] which makes precise when the canonical module of the Rees algebra has the expected form. Recall that $K_{R_B(I)}$ has the expected form if

$$K_{R_B(I)} \cong Bt \oplus Bt^2 \oplus \cdots \oplus Bt^l \oplus It^{l+1} \oplus I^2t^{l+2} \oplus \cdots\,,$$

for some $l \geq 0$. This definition was introduced by J. Herzog, A. Simis and W. Vasconcelos in [HSV2]. We still use the same notation and again omit the proof.

Corollary 4.1.7 *Assume $R_B(I)$ is Cohen-Macaulay and $G_B(I)$ is quasi-Gorenstein. Set $\mathbf{a}(G_B(I)) = (-b, -a)$. Then $K_B \cong B(-b)$ and*

$$K_{R_B(I)} \cong \bigoplus_{(l,m),\, m\geq 1} [I^{m-a+1}]_{l-b},$$

where $I^n = B$ if $n \leq 0$.

Note that $-a$ coincides with the usual a-invariant of $G_B(I)$. By Ikeda-Trung's criterion [IT] it is always negative if $R_B(I)$ is Cohen-Macaulay, and it has been calculated in many cases (see for instance [HRZ], [GH]). As for

b, it is clear that under the hypothesis of Corollary 4.1.7 we get $-b = a(B)$. It is then also easy to compute the bigraded a-invariant of $R_B(I)$. Namely, we get that if $a = 1$ then $\mathbf{a}(R_B(I)) = (-d_1 + a(B), -1)$, and if $a > 1$ then $\mathbf{a}(R_B(I)) = (a(B), -1)$.

Remark 4.1.8 Assume that $B = A = k[X_1, \ldots, X_n]$ and I is a complete intersection ideal. Then, the Eagon-Northcott complex provides a $\mathbb{Z}^2$-graded minimal free resolution of $R_A(I)$. Following the proof of Yoshino [Yos] it is possible to see that

$$K_{R_A(I)} = J((r-2)d_1 - n, -1)$$

with $J = (f_1^{r-2}, f_1^{r-2}\, t, ..., f_1^{r-2}\, t^{r-2})R_A(I)$.

Observe that in this case $a(G_A(I)) = (-n, -r)$ and by Corollary 4.1.7

$$K_{R_A(I)} = \bigoplus_{(l,m),\, m \geq 1} [I^{m-r+1}]_{l-n}.$$

A straightforward computation shows that, in fact, multiplication by f_1^{r-2} provides an explicit isomorphism

$$\bigoplus_{(l,m),\, m \geq 1} [I^{m-r+1}]_{l-n} \cong J((r-2)d_1 - n, -1).$$

Let us now assume that $I \subset A = k[X_1, \ldots, X_n]$ is a homogeneous ideal whose form ring is Gorenstein. We are now ready to prove the main result of this section determining the quasi-Gorenstein diagonals of $R_A(I)$. Namely,

Theorem 4.1.9 *Assume* $\mathrm{ht}\,(I) \geq 2$, $\dim(A/I) > 0$, *and* $G_A(I)$ *is Gorenstein. Set* $a = -a^2(G_A(I))$. *Then* $k[(I^e)_c]$ *is quasi-Gorenstein if and only if* $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$. *In this case,* $a(k[(I^e)_c]) = -l_0$.

Proof. Note that the Rees algebra R is Cohen-Macaulay by using a result of Lipman [Li, Theorem 5]. Now, by applying Corollary 4.1.7 we have that $b = -a(A) = n$ and $K_R = \bigoplus_{(l,m),m\geq 1}[I^{m-a+1}]_{l-n}$, so that by Proposition 4.1.4 we get $K_{R_\Delta} = (K_R)_\Delta = \bigoplus_{l \geq 1}[I^{el-a+1}]_{cl-n}$. Let $l_0 = \min\{l \in \mathbb{Z} \mid l \geq \frac{n}{c}\}$, $s = a - 1 - el_0$. We shall now distinguish three cases.

If $s = 0$, then the first non-zero component of K_{R_Δ} is $[K_{R_\Delta}]_{l_0} = [I^{el_0-a+1}]_{cl_0-n} = A_{cl_0-n}$, so that if R_Δ is quasi-Gorenstein $cl_0 - n = 0$ and we get that $l_0 = \frac{n}{c} = \frac{a-1}{e}$ and $a(R_\Delta) = -l_0$. Conversely, if $l_0 = \frac{n}{c} = \frac{a-1}{e}$ then

$[K_{R_\Delta}]_{l_0+m} = [I^{el_0-a+1+em}]_{cl_0+cm-n} = [I^{em}]_{cm} = [R_\Delta]_m$ for all m and so R_Δ is quasi-Gorenstein.

If $s < 0$, let $l_1 = \min\{l \mid el-a+1 > 0, cl-n \geq d_1(el-a+1)\}$. Then $l_1 \geq l_0$ and the first non-zero component of K_{R_Δ} is $[K_{R_\Delta}]_{l_1} = [I^{el_1-a+1}]_{cl_1-n}$. In particular, $a(R_\Delta) = -l_1$. Assume R_Δ is quasi-Gorenstein. Then $K_{R_\Delta} \cong R_\Delta(-l_1)$ and so $[K_{R_\Delta}]_{l_1} \cong k$. This implies that $cl_1 - n = d_1(el_1 - a + 1)$: If $cl_1 - n - d_1(el_1 - a + 1) = r > 0$ we may choose two lineary independent forms $g, h \in A_r$ such that $gf_1^{el_1-a+1}, hf_1^{el_1-a+1} \in [I^{el_1-a+1}]_{cl_1-n} \cong k$, which is a contradiction. From the isomorphism one gets that K_{R_Δ} is generated by $f_1^{el_1-a+1}$ as R_Δ-module. Now let $f_j \notin \mathrm{rad}\,(f_1)$ (it exists because $\mathrm{ht}\,(I) \geq 2$), and choose m such that $m(c - d_je) > d_j - d_1$ and there exists $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, f_1) = 1$. Then $f_1^{el_1-a} f_j^{em+1} f \in [I^{el_1-a+1+em}]_{d_1(el_1-a+1)+cm} = f_1^{el_1-a+1}[I^{em}]_{cm}$, and we get $f_j^{em+1} f \in (f_1)$ which is a contradiction.

If $s > 0$, the first non-zero component of K_{R_Δ} is $[I^{el_0-a+1}]_{cl_0-n} = A_{cl_0-n}$, so if R_Δ is quasi-Gorenstein we get $cl_0 - n = 0$. Furthermore, for all $m \geq 1$ we have $[K_{R_\Delta}]_{l_0+m} = [I^{-s+em}]_{cl_0-n+cm} = [I^{-s+em}]_{cm} \cong [I^{em}]_{cm}$. Since $s > 0$ and $[I^{em}]_{cm} \subset [I^{-s+em}]_{cm}$ this isomorphism is possible if and only if $[I^{em}]_{cm} = [I^{-s+em}]_{cm}$. Now choose X_i such that $X_i \notin \mathrm{rad}\,(I)$ (it always exists because $\dim(A/I) > 0$) and m with $em - s \geq 1$. For any j consider $F_j = X_i^{\alpha_j} f_j^{em-s}$ where $\alpha_j = cm - d_j(em-s) \geq 1$, and assume $[I^{em}]_{cm} = [I^{-s+em}]_{cm}$. Then $F_j \in [I^{em-s}]_{cm}$ and so $X_i^{\alpha_j} f_j^{em-s} \in I^{em}$. Now let $f_1^{c_1} \dots f_r^{c_r}$ such that $c_1 + \ldots + c_r \geq r(em - s)$. This implies that there exists l with $c_l \geq em - s$ and so $X_i^{\alpha_1} f_1^{c_1} \dots f_r^{c_r} = X_i^{\alpha_1} f_l^{em-s} f_1^{c_1} \dots f_l^{c_l-em+s} \dots f_r^{c_r} \in I^{c_1+\ldots+c_r+s}$, since $\alpha_1 \geq \alpha_i$ for all i. Thus we get $X_i^\alpha I^h \subset I^{h+s}$ for $h >> 0$, which implies that $X_i^\alpha \in I^s \subset I$ since $R_A(I)$ is Cohen-Macaulay (see [TVZ, Lemma 4.3]). But this contradicts $X_i \notin \mathrm{rad}\,(I)$ and so R_Δ is not quasi-Gorenstein. □

The remaining cases $\mathrm{ht}\,(I) = 1, n$ in the above theorem are studied separately in the following remarks.

Remark 4.1.10 If $\mathrm{ht}\,(I) = 1$ then $k[(I^e)_c]$ is never quasi-Gorenstein. In fact, by [TVZ, Proposition 4.6], $a^2(G_A(I)) = -1$ and so $a = 1$. Following the same proof as in Theorem 4.1.9 we have that $s = -el_0 < 0$. On the other hand, since $\mathrm{ht}\,(I) = 1$ we may write $I = gJ$, with $\mathrm{ht}\,(J) \geq 2$, $J = (\overline{f}_1, \ldots, \overline{f}_r)$ and $f_j = \overline{f}_j g$ for all j. The same argument as in Theorem 4.1.9 for the case $s < 0$ but taking $\overline{f}_j \notin \mathrm{rad}\,(\overline{f}_1)$ and $f \in A_{d_1+cm-d_j(em+1)}$ such that $(f, \overline{f}_1) = 1$ leads to $\overline{f}_j^{em+1} f \in (\overline{f}_1)$, which is a contradiction.

Remark 4.1.11 If $\dim(A/I) = 0$, then the condition $\frac{n}{c} = \frac{a-1}{e} = l_0 \in \mathbb{Z}$ is sufficient but not necessary for $k[(I^e)_c]$ to be quasi-Gorenstein. For instance, let $A = k[X_1, X_2, X_3]$ and $I = (X_1, X_2, X_3)$. Note that $n = 3 \geq r = 3 \geq 2$, G is Gorenstein and $a = 3$. Then, by Corollary 4.1.7 we have that $K_R = \bigoplus_{(l,m),m\geq 1}[I^{m-2}]_{l-3}$. According to Proposition 4.1.4, by taking the $(3,1)$-diagonal we have

$$K_{R_\Delta} = \bigoplus_{l\geq 1}[I^{l-2}]_{3(l-1)} = \bigoplus_{l\geq 1} A_{3(l-1)} = (\bigoplus_{l\geq 0} A_{3l})(-1) = R_\Delta(-1),$$

and so $R_\Delta = k[I_3]$ is quasi-Gorenstein. In this case, $\frac{n}{c} = 1 \neq 2 = \frac{a-1}{e}$.

As a consequence of Theorem 4.1.9 we obtain the following result for the case of complete intersection ideals. It generalizes [CHTV, Corollary 4.7] where the case of ideals generated by two elements was considered.

Corollary 4.1.12 *Let $I \subset k[X_1, \ldots, X_n]$ be a complete intersection ideal minimally generated by r forms of degrees $d_1 \leq \ldots \leq d_r = d$, with $r < n$. Then for $c \geq de + 1$, $k[(I^e)_c]$ is Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. In this case, $a(k[(I^e)_c]) = -l_0$.*

Proof. Since $a^2(G_A(I)) = -r$, by Theorem 4.1.9 we have that $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{n}{c} = \frac{r-1}{e} = l_0 \in \mathbb{Z}$. But then $u + (e-1)d - n \leq rd + ed - d - n = (r-1)d + de - n = e\frac{n}{c}d + de - n = n(\frac{ed-c}{c}) + de \leq de < c$, and according to Proposition 3.4.5, $k[(I^e)_c]$ is also Cohen-Macaulay and so Gorenstein. □

We may also study the ideals generated by the maximal minors of a generic matrix. We thank A. Conca for suggesting to consider this case.

Example 4.1.13 Let $\mathbf{X} = (X_{ij})$ be a generic matrix, with $1 \leq i \leq n$, $1 \leq j \leq m$ and $m \leq n$. Let us consider $I \subset A = k[\mathbf{X}]$ the ideal generated by the maximal minors of $\mathbf{X}$, where k is a field. In this case, the Rees algebra $R_A(I)$ is Cohen-Macaulay and the form ring $G_A(I)$ is Gorenstein [EH, Theorem 3.5]. Moreover, it has been proved by A. Conca (personal communication) that all the diagonals of $R_A(I)$ are Cohen-Macaulay (see also Example 5.2.23).

Now we want to study the Gorenstein property of these diagonals. Note that I is an equigenerated ideal whose Rees algebra is Cohen-Macaulay, so we can apply Theorem 4.1.9 thanks to Remark 4.1.5. Since I is generically a complete intersection, we have that $a^2(G_A(I)) = -\mathrm{ht}(I) = -(n-m+1)$. We shall distinguish two cases.

If $m < n$, then $\mathrm{ht}(I) \geq 2$, and we get that $k[(I^e)_c]$ is Gorenstein if and only if $\frac{nm}{c} = \frac{n-m}{e} \in \mathbb{Z}$. So there exists always at least one diagonal which is Gorenstein by taking $c = nm, e = n-m$.

If $m = n$, note that I is a principal ideal and so the Rees algebra is isomorphic to a polynomial ring. Then the only diagonal which is Gorenstein occurs when $c = n(n+1), e = 1$ by Remark 4.1.3.

4.2 Restrictions to the existence of Gorenstein diagonals. Applications.

In Section 4.1 we have proved that under the assumptions of Theorem 4.1.9 there is just a finite set of diagonals (c,e) such that $k[(I^e)_c]$ is quasi-Gorenstein. Our next result shows that this holds in general.

Proposition 4.2.1 *There exist at most a finite number of diagonals (c,e) such that $k[(I^e)_c]$ is quasi-Gorenstein.*

Proof. Let $w_1, \ldots, w_m \in K_R$ be a homogeneous system of generators of K_R as R-module with $\deg w_i = (\alpha_i, \beta_i)$ for all i, and so $K_R = \sum_{i=1}^m Rw_i$. Note that since R is a domain K_R is torsion free. For any diagonal $\Delta = (c,e)$ we then have by Proposition 4.1.4 that for all $l \geq 1$

$$[K_{R_\Delta}]_l = \sum_{i=1,\ldots,m, el-\beta_i \geq 0} [I^{el-\beta_i}]_{cl-\alpha_i} w_i.$$

If R_Δ is quasi-Gorenstein there exists an integer l such that $[K_{R_\Delta}]_l \cong k$ and so $[I^{el-\beta_i}]_{cl-\alpha_i} \neq 0$ for some i $(\star)$. We shall distinguish two cases.

Assume first that I is an equigenerated ideal in degree d. Then condition $(\star)$ implies that $el-\beta_i = 0$ and $cl-\alpha_i \geq 0$ or $el-\beta_i > 0$ and $cl-\alpha_i \geq d(el-\beta_i)$. If $el-\beta_i = 0$, then $k \cong [K_{R_\Delta}]_l \supset A_{cl-\alpha_i} w_i$ and since K_R is torsion-free we get $cl - \alpha_i = 0$. Hence (c,e) satisfies $\frac{\beta_i}{e} = \frac{\alpha_i}{c} = l \in \mathbb{Z}$ and the statement holds. If $el - \beta_i > 0$ then $k \cong [K_{R_\Delta}]_l \supset [I^{el-\beta_i}]_{cl-\alpha_i} w_i$ which is impossible since K_R is torsion free and $cl - \alpha_i \geq d(el - \beta_i)$.

Assume now that I is not equigenerated. Condition $(\star)$ implies that $el - \beta_i = 0$ and $cl - \alpha_i \geq 0$ or $el - \beta_i > 0$ and $cl - \alpha_i \geq d_1(el - \beta_i)$. In the first case we may proceed as before to get the statement. In the second case we have that $k \cong [K_{R_\Delta}]_l \supset [I^{el-\beta_i}]_{cl-\alpha_i} w_i$ and so $cl - \alpha_i = d_1(el - \beta_i)$ and $d_1 < d_2$. Then $\alpha_i - d_1\beta_i = cl - d_1 el \geq c - d_1 e \geq (d - d_1)e$ since $l \geq 1$ and $c \geq de + 1 > de$. Thus we obtain the inequality $e \leq \frac{\alpha_i - d_1\beta_i}{d - d_1}$ and for each e, we have $c \leq d_1 e + \alpha_i - d_1\beta_i$. In any case, these inequalties hold for at most a finite number of diagonals and so we get the result. □

For a real number x, let us denote by $\lceil x \rceil = \min\{m \in \mathbb{Z} \mid m \geq x\}$. If the Rees algebra $R_A(I)$ is Cohen-Macaulay we can also give bounds for the diagonals (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein.

Proposition 4.2.2 *Assume that* $\mathrm{ht}\,(I) \geq 2$ *and* $R_A(I)$ *is Cohen-Macaulay. Let* $a = -a^2(G_A(I))$. *If* $k[(I^e)_c]$ *is quasi-Gorenstein, then* $e \leq a - 1$ *and* $c \leq n$. *Moreover, if* $\dim(A/I) > 0$ *then* $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} \in \mathbb{Z}$. *In particular, if* $a = 1$ *there are no diagonals* (c, e) *such that* $k[(I^e)_c]$ *is quasi-Gorenstein.*

Proof. By Theorem 4.1.6, there exists a homogeneous filtration $\{K_m\}_{m\geq 0}$ of K_A such that $K_R \cong \bigoplus_{m\geq 1} K_m$ and $K_G \cong \bigoplus_{m\geq 1} K_{m-1}/K_m$. Bigrading the proof of [TVZ, Corollary 2.5], we have that $K_m = \mathrm{Hom}_A(I, K_{m+1})$ for every $m \geq 0$. Note that K_A may be viewed as an ideal of A. Assume that R_Δ is quasi-Gorenstein. Then there is an integer l_0 such that $[K_{R_\Delta}]_{l_0} \cong k$. By Proposition 4.1.4 we may find an element $f \in [K_{el_0}]_{cl_0} = [K_R]_{(cl_0, el_0)}$, $f \neq 0$, $K_{R_\Delta} = R_\Delta f$.

CLAIM: $K_{el_0} = Af$.

To prove the claim we first show that for any $g \in K_{el_0}$, $g \neq 0$, then g has degree $\geq cl_0$. Assume the contrary: $\deg g = k < cl_0$. Then $[Ag]_{cl_0} = A_{cl_0 - k} g \subset [K_{el_0}]_{cl_0} \cong k$. But since $cl_0 - k > 0$, $\dim_k A_{cl_0-k} > 1$, so we get a contradiction.

Now let $g \in K_{el_0}$. If $\deg g = cl_0$, then $g \in Af$ because $[K_{el_0}]_{cl_0} = kf$. Let us assume that $\deg g = k > cl_0$. Then, for each $l > 0$, $[I^{el}]_{cl} f + [I^{el}]_{c(l_0+l)-k} g \subset [K_{e(l_0+l)}]_{c(l_0+l)} \cong [I^{el}]_{cl}$ as k-vector spaces, and so $[I^{el}]_{c(l_0+l)-k} g \subset [I^{el}]_{cl} f$. Now let $I^{el} = (F_1, \ldots, F_t)$ where F_i is a homogeneous polynomial of degree $\leq del$ for all i, and set $\alpha = c(l_0 + l) - k - \deg F_i$. Note that for $l >> 0$, $\alpha \geq c(l_0 + l) - k - del = (c - de)l + cl_0 - k > 0$ and we can find $h \in A_\alpha$ such that $(h, f) = 1$. Then $hgF_i \in [I^{el}]_{c(l_0+l)-k} g \subset [I^{el}]_{cl} f \subset Af$ and we have that $gF_i \in Af$ for all i. Thus $I^{el} g \subset (f)$ and writing $g = d\overline{g}$, $f = d\overline{f}$ with $(\overline{f}, \overline{g}) = 1$ we get $I^{el}\overline{g} \subset A\overline{f}$. If $g \notin Af$, then $\overline{f} \notin k$ and so $I^{el} \subset (\overline{f})$ which is absurd because $\mathrm{ht}\,(I) \geq 2$.

Now, as $\text{grade}(I) \geq 2$ we have $K_m = K_{el_0}$ for all $m \leq el_0$, which implies that $K_A = K_{el_0}$ and so $c \leq cl_0 = n$. Furthermore, $e \leq el_0 \leq \min\{m \mid K_m \subsetneq K_{m-1}\} - 1 = a - 1$.

Finally assume that $\dim(A/I) > 0$. We shall distinguish two cases. If $e = 1$ we have that $K_{l_0+1} \subsetneq K_{l_0}$: If not, then $I_c \cong [K_{l_0+1}]_{c(l_0+1)} = [Af]_{c(l_0+1)} \cong A_c$ which is absurd if $\dim(A/I) > 0$. Therefore $a = l_0 + 1 = \frac{n}{c} + 1$. If $e > 1$, let $\widetilde{\Delta} = (c, 1)$ and $\widetilde{R} = R_A(I^e)$. Note that $\widetilde{R}_{\widetilde{\Delta}} = R_\Delta$ is quasi-Gorenstein. Applying the case before we obtain that $-a(G_A(I^e)) = \frac{n}{c} + 1$. By [HRZ, Proposition 2.6], $a(G_A(I^e)) = [\frac{-a}{e}] = -\lceil \frac{a}{e} \rceil$ and so $\lceil \frac{a}{e} \rceil - 1 = \frac{n}{c} = l_0 \in \mathbb{Z}$. □

Our next result shows that in some cases the existence of a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein forces the form ring to be Gorenstein. It may be seen as a converse of Theorem 4.1.9 for those cases.

Theorem 4.2.3 *Assume that $R_A(I)$ is Cohen-Macaulay,* $\text{ht}(I) \geq 2$, $l(I) < n$ *and I is equigenerated. If there exists a diagonal (c, e) such that $k[(I^e)_c]$ is quasi-Gorenstein then $G_A(I)$ is Gorenstein.*

Proof. Let $\Delta = (c, e)$. Assume first that $e = 1$. We have seen in the proof of Proposition 4.2.2 that there exists a homogeneous filtration $\{K_m\}_{m \geq 0}$ of K_A such that $K_R \cong \bigoplus_{m \geq 1} K_m$ and $K_G \cong \bigoplus_{m \geq 1} K_{m-1}/K_m$, and an integer $l_0 = -a(R_\Delta)$ such that $K_0 = \ldots = K_{l_0} = Af$, with $f \in K_R$ and $\deg f = cl_0$. It is then clear that for all $m \geq 0$, $I^m f \subset K_{l_0+m}$ and so $[I^m]_{cm} f \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}$ since R_Δ is quasi-Gorenstein. This implies that $[K_{l_0+m}]_{c(l_0+m)} = [I^m]_{cm} f$.

We want to show that $K_{l_0+m} = I^m f$ for all $m \geq 0$. Suppose that there exists m_0 such that $I^{m_0} f \subsetneq K_{l_0+m_0}$. Then let $g \in K_{l_0+m_0}$, $g \notin I^{m_0} f$ be a homogeneous element of degree k. Note that from the inclusion $K_{l_0+m_0} \subset K_{l_0} = Af$ we also have $g = f\overline{g}$ with $\overline{g} \notin I^{m_0}$.

If $k \geq c(l_0 + m_0)$ then for any $m > m_0$ we have $I^m f + I^{m-m_0} g \subset K_{l_0+m}$ and so $[I^m]_{cm} f + [I^{m-m_0}]_{c(l_0+m)-k}\, g \subset [K_{l_0+m}]_{c(l_0+m)} \cong [I^m]_{cm}$. Hence $[I^{m-m_0}]_{c(l_0+m)-k}\, g \subset [I^m]_{cm} f$ and we get that $[I^{m-m_0}]_{c(l_0+m)-k}\, \overline{g} \subset [I^m]_{cm}$. Let $\lambda = c(l_0 + m) - k - d(m - m_0) = (c - d)m + cl_0 + dm_0 - k$. For $m >> 0$ we have that $\lambda > 0$. Then, if $A_\lambda \overline{g} \subset I^{m_0}$ we would have that $\overline{g} \in (I^{m_0})^*$, the saturation of I^{m_0}. Since $G_A(I)$ is Cohen-Macaulay, we have that the inequality of Burch becomes an equality by [EH, Proposition 3.3], that is,

$$\inf_{j \geq 0}\{\text{depth}(A/I^j)\} = \dim A - l(I).$$

Since $l(I) < n$, we then get depth $A/I^{m_0} > 0$, and so $(I^{m_0})^* = I^{m_0}$. Hence $\overline{g} \in I^{m_0}$, which is a contradiction. We may conclude that there exist $\lambda > 0$, $h \in A_\lambda$ such that $\overline{g}h \notin I^{m_0}$. On the other hand, $\overline{g}h[I^{m-m_0}]_{d(m-m_0)} \subset \overline{g}[I^{m-m_0}]_{c(l_0+m)-k} \subset [I^m]_{cm}$. Since I is equigenerated we get $\overline{g}hI^{m-m_0} \subset I^m$. Therefore, $\overline{g}h \in (I^m : I^{m-m_0}) = I^{m_0}$ because R is Cohen-Macaulay. This is a contradiction.

If $k < c(l_0 + m_0)$, let us write $k = c(l_0 + m_0) - s$ with $s > 0$. Then $A_s g \subset [K_{l_0+m_0}]_{c(l_0+m_0)} = [I^{m_0}]_{cm_0} f$, and $\overline{g} \in (I^{m_0})^* = I^{m_0}$ which, as before, is a contradiction.

Hence we have proved that $K_{l_0+m} = I^m f$ for all $m \geq 0$, so

$$K_R = f(At \oplus \cdots \oplus At^{l_0} \oplus It^{l_0+1} \oplus \cdots),$$

i.e. K_R has the expected form. By [TVZ, Theorem 4.2] this implies that both $R_A(I^{l_0})$ and $G_A(I)$ are Gorenstein.

Finally assume $e > 1$, and denote by $\widetilde{\Delta} = (c, 1)$ and $\widetilde{R} = R_A(I^e)$. Then $\widetilde{R}_{\widetilde{\Delta}} = R_\Delta$ is quasi-Gorenstein and so there exists l_0 such that $R_A(I^{el_0})$ is Gorenstein. By [TVZ, Theorem 4.2] this implies again that $G_A(I)$ is Gorenstein. □

Example 4.2.4 *(Room surfaces)* Let k be an algebraically closed field. Set $t = \binom{d+1}{2}$, with $d \geq 2$. Let $P_1, \ldots, P_t$ be a set of t distinct points in $\mathbb{P}^2_k$ which do not lie on a curve of degree $d-1$. We assume further that there is not a subset of d points on a line if $d \geq 3$. We are going to study the rational projective surfaces which arise as embeddings of blowing-ups of $\mathbb{P}^2_k$ at this set of points via the linear system I_{d+1}.

Let I be the ideal defining the set of points $\{P_1, \ldots, P_t\}$. Since the points are not on a curve of degree $d-1$, we have that I is generated by forms in degree d [GG], and so for any $c \geq d+1$ the linear system I_c gives a projective embedding of the blow-up. For $c = d+1$ the surface obtained is called Room surface.

Assume $d \geq 3$. Since there are not d points on a line, we also have that the rational map defined by the linear system I_d give an embedding of the blow-up in the projective space $\mathbb{P}^d_k$ (see [GG]), and the resulting surface is called White surface. A. Gimigliano proved that White surfaces have the defining ideal given by the 3×3 minors of a $3 \times d$ matrix of linear forms, and it has a linear minimal graded free resolution which comes from the Eagon-Northcott complex [Gi, Proposition 1.1]. By applying Theorem 1.3.4, we obtain $a(k[I_d]) = -1$ and so by Lemma 2.3.7 the reduction number of I is $r(I) = a(k[I_d]) + l(I) = -1+3 = 2$. Moreover, the analytic deviation of I is $\mathrm{ad}(I) = l(I) - \mathrm{ht}\,(I) = 1$ and I is generically a complete intersection ideal. So according to [GN] we may conclude that $G_A(I)$ is Cohen-Macaulay and hence $a^2(G_A(I)) = r(I) - \mathrm{ht}\,(I) - 1 = -1$ by [GH, Proposition 2.4]. Therefore, $R_A(I)$ is also Cohen-Macaulay by using Ikeda-Trung's criterion. From Proposition 4.2.2 we get that there are not diagonals (c, e) such that $k[(I^e)_c]$ is Gorenstein, that is, there are not Gorenstein embeddings for the blow-up. In particular, $k[I_{d+1}]$ is not Gorenstein for $d \geq 3$.

If $d = 2$, by choosing the points to be [1:0:0], [0:1:0] and [0:0:1], we have $I = (X_1X_2, X_1X_3, X_2X_3)$. Notice that I has $\mu(I) = 3 = \mathrm{ht}\, I + 1$ and A/I is Cohen-Macaulay. Moreover, $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supset I$. Then $G_A(I)$ is Gorenstein with and $a^2(G_A(I)) = -\mathrm{ht}\,(I) = -2$ by [HRZ]. Now, according to Theorem 4.1.9, $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{3}{c} = \frac{1}{e} \in \mathbb{Z}$. Hence $(3,1)$ is the only diagonal with the quasi-Gorenstein property. This embedding corresponds to the del Pezzo sestic surface in $\mathbb{P}^6$.

With more generality, we may consider the blow-up of $\mathbb{P}^2_k$ at a set of t arbitrary distinct points.

Example 4.2.5 Let k be an algebraically closed field. Let $P_1, \ldots, P_t$ be a set of t distinct points in $\mathbb{P}^2_k$, and let $I = \mathcal{P}_1 \cap \ldots \cap \mathcal{P}_t$, where $\mathcal{P}_i \subset A = k[X_1, X_2, X_3]$ is the defining ideal of P_i. Now we consider the surfaces which arise as embeddings of the blow-up of $\mathbb{P}^2_k$ at these points via the linear systems $(I^e)_c$. We want to study the Gorenstein property of the rings $k[(I^e)_c]$.

Set $d = \mathrm{reg}(I)$. We will assume that $P_1, \ldots, P_t$ do not lie on a curve of degree $d-1$ and that there is not a subset of d points on a line. Then, I is generated by forms in degree d and I_d defines a projective embedding of the blow-up [GG, Theorem A].

On the other hand, observe that $R_{A_{\mathfrak{p}}}(I_{\mathfrak{p}})$ is Cohen-Macaulay for all $\mathfrak{p} \in \mathrm{Proj}\ (A)$. Let $\mathcal{L} = \widetilde{I}\mathcal{O}_X$, $\mathcal{M} = \pi^*\mathcal{O}_{\mathbb{P}^2}(1)$, where $\pi : X \longrightarrow \mathbb{P}^2_k$ is the blow-up of $\mathbb{P}^2_k$ along $\mathcal{I}$. Then we have that $R^j\pi_*\mathcal{L}^e = 0$ for all $e \geq 0$, $j > 0$ and $\pi_*\mathcal{L}^e = \widetilde{I}^e$ for all $e \geq 0$ by Corollary 3.1.4. Therefore, for any $s \geq 0$, $\Gamma(X, \mathcal{L}^s \otimes \mathcal{M}^{sd}) = \Gamma(\mathbb{P}^2, \widetilde{I}^s(sd)) = (I^s)^*_{sd}$ and $H^i(X, \mathcal{L}^s \otimes \mathcal{M}^{sd}) = H^i(\mathbb{P}^2, \widetilde{I}^s(sd)) = H^{i+1}_{\mathfrak{m}}(I^s)_{sd}$ for $i \geq 1$. By [GGP, Theorem 1.1 and Corollary 1.4], we have $a_*(I^s) < \mathrm{reg}\,(I^s) \leq sd$ and $(I^s)^*_{sd} = (I^s)_{sd}$. Then, by Remark 3.1.1 we get $H^i_{\mathfrak{m}}(k[I_d])_s = 0$ for all $s \geq 0$. Furthermore, recall that the fiber cone F of I coincides with

$k[I_d]$ because I is generated in degree d, so we have $a_*(F) \leq -1$, and then $r(I) \leq \max_{i \leq 3}\{a_i(F)+i\} \leq 2$ by Lemma 2.3.7 . The analytic deviation of I is $\operatorname{ad}(I) = l(I) - \operatorname{ht}(I) = 1$ and I is generically a complete intersection ideal, so we may conclude by [GN] that $G_A(I)$ is Cohen-Macaulay and hence by [GH, Proposition 2.4]

$$a^2(G_A(I)) \leq r(I) - \operatorname{ht}(I) - 1 \leq -1.$$

So $R_A(I)$ is also Cohen-Macaulay by Ikeda-Trung's criterion.

If $r(I) = 2$ then $a^2(G_A(I)) = -1$ by [GH, Proposition 2.4]. Then, according to Proposition 4.2.2, there are not quasi-Gorenstein diagonals $k[(I^e)_c]$.

Otherwise, $r(I) \leq 1$. By [GN, Theorem 1.3], the case $r(I) = 1$ is not possible, so $r(I) = 0$ and then $G_A(I)$ is Gorenstein. Furthermore, $a^2(G_A(I)) = -\operatorname{ht}(I) = -2$ by [GH, Proposition 2.4]. Therefore, by Theorem 4.1.9, $k[(I^e)_c]$ is quasi-Gorenstein if and only if $\frac{3}{c} = \frac{1}{e} \in \mathbb{Z}$. So $k[I_3]$ is the only quasi-Gorenstein diagonal.

Remark 4.2.6 All throughout this chapter we have treated the case where $A = k[X_1, \ldots, X_n]$ is the polynomial ring and I is a homogeneous ideal in A satisfying $r \leq n$ or the assumptions in Remark 4.1.5. This set up was used to study the relationship between the canonical module of the Rees algebra R and the canonical modules of its diagonals R_Δ. Now let A be an arbitrary standard k-algebra and let I be a homogeneous ideal in A generated by r forms in degree $\leq d$. Set $\overline{n} = \dim A$. For any $c \geq de+1$, from the Mayer-Vietoris sequence (see Proposition 2.1.3) we have a graded monomorphism

$$\psi : (K_R)_\Delta \to K_{R_\Delta}$$

such that

(i) If $l(I) < \overline{n}$ or $r < \overline{n}$ or I equigenerated, then ψ_s is an isomorphism for any $s > 0$.

(ii) Assume $l(I) < \overline{n}$ or $a_2^*(R) < 0$. If $\operatorname{ht}(I) \geq 2$, $H^{\overline{n}}_{\mathfrak{m}}(A)_0 = 0$, $a(A) < c$, then ψ_s is an isomorphism for any $s \leq 0$.

(iii) Assume that R is Cohen-Macaulay. Then

$$\psi \text{ isomorphism} \iff \begin{cases} H^{\overline{n}}_{\mathfrak{m}}(A)_0 = 0 & \\ H^{\overline{n}}_{\mathfrak{m}}(I^{es})_{cs} = 0 & \text{for } s > 0 \\ H^{\overline{n}}_{\mathcal{M}_2}(R)_{(cs,es)} = 0 & \text{for } s \in \mathbb{Z} \end{cases}$$

In the cases where ψ is an isomorphism, some of the results of the chapter can be extended. For instance, if R is Cohen-Macaulay and G is quasi-Gorenstein, for any c, e such that ψ is an isomorphism and $\frac{n}{c} = \frac{a-1}{e} \in \mathbb{Z}$ the ring $k[(I^e)_c]$ is quasi-Gorenstein.