On the diagonals of a Rees algebra

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Chapter 5

The a-invariants of the powers of an ideal

Our aim in this chapter is to study in more detail the bigraded a-invariant and the bigraded regularity of any finitely generated bigraded S-module L, for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring with $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$.

In Section 5.1 we will give a new description of the a_*-invariant $\mathbf{a}_*(L)$ of L and the regularity $\mathbf{reg}(L)$ of L by means of the a_*-invariants and the regularities of the graded S_1-modules L^e and the graded S_2-modules L_e.

This result is used in Section 5.2 to study the behaviour of the a_*-invariant of the powers of a homogeneous ideal in the polynomial ring. In particular, we will bound it for several families of ideals such as equimultiple ideals and strongly Cohen-Macaulay ideals. Those results will be then applied to determine Cohen-Macaulay diagonals of their Rees algebras.

The last section is devoted to study the regularity of homogeneous ideals I in the polynomial ring S. First, we will provide a bigraded version of the well-known Bayer-Stillman's Theorem characterizing the regularity of I in terms of generic forms. After that, similarly to the graded case, we define the bigraded generic initial ideal $\mathbf{gin}\, I$ of I and we establish its basic properties. In the graded case, a classical result due to D. Bayer and M. Stillman states the existence of an order such that for any homogeneous ideal I it holds $\operatorname{reg} I = \operatorname{reg}(\operatorname{gin} I)$. We will show that the analogous bigraded statement is not true.

We finish the chapter by explaining how these results can be used to study the Koszulness of the diagonals $k[(I^e)_c]$.

5.1 The a-invariant of a standard bigraded algebra

Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring over a field k in $n+r$ variables with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (0,1)$, and let us distinguish two bigraded subalgebras: $S_1 = k[X_1, \ldots, X_n]$, $S_2 = k[Y_1, \ldots, Y_r]$, with homogeneous maximal ideals $\mathfrak{m}_1 = (X_1, ..., X_n)$, $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$ respectively. Given $e \in \mathbb{Z}$ and a bigraded S-module L, recall that we may define the graded S_1-module $L^e = \bigoplus_{i \in \mathbb{Z}} L_{(i,e)}$ and the graded S_2-module $L_e = \bigoplus_{j \in \mathbb{Z}} L_{(e,j)}$.

The first result shows how to compute the bigraded a_*-invariant of any finitely generated bigraded S-module L by means of the a_*-invariants of the graded S_1-modules L^e and the graded S_2-modules L_e. Namely,

Theorem 5.1.1 *Let L be a finitely generated bigraded S-module. Then :*

(i) $a_*^1(L) = \max_e\{a_*(L^e)\} = \max_e\{a_*(L^e) \mid e \leq a_*^2(L) + r\}$.

(ii) $a_*^2(L) = \max_e\{a_*(L_e)\} = \max_e\{a_*(L_e) \mid e \leq a_*^1(L) + n\}$.

Proof. Let us consider

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0$$

the minimal bigraded free resolution of L over S, where $D_p = \bigoplus_{(a,b) \in \Omega_p} S(a,b)$. We have $a_*^1(L) = \max\{-a \mid (a,b) \in \Omega_L\} - n$ by Theorem 1.3.4.

Let us denote by $\underline{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ and $|\underline{\beta}| = \beta_1 + \cdots + \beta_r$. By applying the functor $(\ )^e$ to the resolution note that

$$\begin{aligned} S(a,b)^e &= \textstyle\bigoplus_{i \in \mathbb{Z}} S(a,b)_{(i,e)} = \bigoplus_{i \in \mathbb{Z}} S_{(a+i, b+e)} \\ &= \textstyle\bigoplus_{i \in \mathbb{Z}} \bigoplus_{|\underline{\beta}| = b+e} [S_1]_{a+i} Y_1^{\beta_1} \ldots Y_r^{\beta_r} \\ &= S_1(a)^{\rho_{ab}^e} \end{aligned}$$

for certain $\rho_{ab}^e \in \mathbb{Z}$ ($\rho_{ab}^e = 0$ if $b + e < 0$). In this way, we have obtained a graded free resolution of L^e over S_1

$$0 \to D_t^e \to \ldots \to D_1^e \to D_0^e \to L^e \to 0\,,$$

with $D_p^e = \bigoplus_{(a,b)\in\Omega_p} S_1(a)^{\rho_{ab}^e}$. The minimal graded free resolution of L^e may be obtained by picking out some terms [Eis, Exercise 20.1]. Therefore,

$$a_*(L^e) \le \max\{-a \mid (a,b) \in \Omega_L\} - n = a_*^1(L).$$

Now let $\alpha = \max\{-a \mid (a,b) \in \Omega_L\}$. Let p be the first place in the resolution of L with a shift of the form $(-\alpha, b)$, and let β be one of these $-b$'s. We are done if we prove that $-\alpha$ is a shift which appears in the place p of the minimal graded free resolution of L^β. Note that it is enough to show that

$$\operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)} = \operatorname{Tor}_p^{S_1}(k, L^\beta)_\alpha \neq 0.$$

Let us consider

$$D_{p+1} \xrightarrow{\psi_{p+1}} D_p \xrightarrow{\psi_p} D_{p-1}$$

the differential maps appearing in the resolution of L. Tensorazing by $S/\mathfrak{m}_1 S$, we have the sequence

$$D_{p+1}/\mathfrak{m}_1 D_{p+1} \xrightarrow{\overline{\psi}_{p+1}} D_p/\mathfrak{m}_1 D_p \xrightarrow{\overline{\psi}_p} D_{p-1}/\mathfrak{m}_1 D_{p-1}\,.$$

Now let us take $v \in D_p$ one of the elements of the homogeneous basis of D_p as free S-module with $\deg(v) = (\alpha, \beta)$. If $w_1, \dots, w_s$ is the homogeneous basis of D_{p-1}, we can write

$$\psi_p(v) = \sum_{j=1}^{s} \lambda_j w_j,$$

with $\lambda_j \in \mathcal{M}$ homogeneous. Set $\deg(w_j) = (\alpha_j, \beta_j)$. By taking into account the way we have choosen α and p, we have $\alpha > \alpha_j$ for any j. Therefore the first component of the degree of λ_j is positive, so $\lambda_j \in \mathfrak{m}_1 S$. We conclude $\overline{\psi}_p(v) = 0$, that is, $v \in \operatorname{Ker} \overline{\psi}_p$. Furthermore, notice that $v \notin \operatorname{Im} \overline{\psi}_{p+1}$ because $\operatorname{Im} \overline{\psi}_{p+1} \subset \mathcal{M}(D_p/\mathfrak{m}_1 D_p)$. So $v \in \operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)}$, $v \neq 0$. By symmetry, we get (ii). □

Next we are going to consider the bigraded regularity of a finitely generated bigraded S-module L. Assume that

$$0 \to D_t \to \dots \to D_1 \to D_0 \to L \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$, is the minimal bigraded free resolution of L over S. The bigraded regularity of L is defined by $\mathbf{reg}(L) = (\operatorname{reg}_1 L, \operatorname{reg}_2 L)$, where

$$\operatorname{reg}_1 L = \max_p \{-a - p : (a,b) \in \Omega_p\}$$

$$\operatorname{reg}_2 L = \max_p \{-b - p : (a,b) \in \Omega_p\}.$$

Let $A = k[X_1, \dots, X_n]$ be the polynomial ring with the usual grading. For any finitely generated graded A-module L, it is well known that

$$\operatorname{reg}(L) = \max_{p \geq 0} \{ t_p(L) - p\} = \max_{p \geq 0} \{ a_p(L) + p\}.$$

This equality does not hold in the bigraded case. For instance, let us consider $f_1, \dots, f_r \in A$ a regular sequence of forms in degree d, and $I = (f_1, \dots, f_r)$. Let $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ be the polynomial bigraded by setting $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d, 1)$, and let R be the Rees algebra of I. Since R is Cohen-Macaulay, we immediately get $a^2_{n+1}(R) = -1$, $a^2_i(R) = 0$ for $i \neq n+1$. Furthermore, the Eagon-Northcott complex gives the bigraded minimal free resolution of R over S:

$$0 \to D_{r-1} \to \dots \to D_0 = S \to R \to 0,$$

with $D_p = \bigoplus_{m=1}^{p} S(-(p+1)d, -m)^{\binom{r}{p+1}}$ for $p \geq 1$. Therefore,

$$\max_{p \geq 0} \{ t^2_p(R) - p\} = 0$$

$$\max_{p \geq 0} \{ a^2_p(L) + p\} = n,$$

which are different.

The following result shows that the regularity of L can also be described by means of the regularity of the graded S_1-modules L^e and the graded S_2-modules L_e. Namely,

Theorem 5.1.2 *Let L be a finitely generated bigraded S-module. Then :*

(i) $\operatorname{reg}_1(L) = \max_e\{\operatorname{reg}(L^e)\} = \max_e\{\operatorname{reg}(L^e) \mid e \leq a^2_*(L) + r\}$.

(ii) $\operatorname{reg}_2(L) = \max_e\{\operatorname{reg}(L_e)\} = \max_e\{\operatorname{reg}(L_e) \mid e \leq a^1_*(L) + n\}$.

Proof. The proof follows the same lines as Theorem 5.1.1. By applying the functor $(\ )^e$ to the minimal bigraded free resolution of L over S, we obtain a graded free resolution of L^e over S_1

$$0 \to D^e_t \to \dots \to D^e_1 \to D^e_0 \to L^e \to 0,$$

with $D_p^e = \bigoplus_{(a,b)\in\Omega_p} S_1(a)^{\rho^e_{ab}}$. Since the minimal graded free resolution of L^e is then obtained by picking out some terms, we have

$$\operatorname{reg}(L^e) \le \max_p\{-a-p \mid (a,b)\in\Omega_p\} = \operatorname{reg}_1 L.$$

Hence $\max_e\{\operatorname{reg}(L^e)\} \le \operatorname{reg}_1 L$. To prove the equality, let us take $(a,b)\in\Omega_p$ such that $\operatorname{reg}_1 L = -a-p$, and set $\alpha=-a$, $\beta=-b$. We are done if we prove $a\in\Omega_{p,L^\beta}$, that is, a is a shift which appears in the place p of the minimal graded free resolution of L^β. So we want to show that

$$\operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)} = \operatorname{Tor}_p^{S_1}(k, L^\beta)_\alpha \neq 0.$$

Let us consider

$$D_{p+1} \xrightarrow{\psi_{p+1}} D_p \xrightarrow{\psi_p} D_{p-1}$$

the differential maps appearing in the resolution of L. Tensorazing by $S/\mathfrak{m}_1 S$, we have the sequence

$$D_{p+1}/\mathfrak{m}_1 D_{p+1} \xrightarrow{\overline{\psi}_{p+1}} D_p/\mathfrak{m}_1 D_p \xrightarrow{\overline{\psi}_p} D_{p-1}/\mathfrak{m}_1 D_{p-1}.$$

Now let $v\in D_p$ be an element of the homogeneous basis of D_p as free S-module with $\deg(v)=(\alpha,\beta)$. If $w_1,\dots,w_s$ is the homogeneous basis of D_{p-1}, we can write

$$\psi_p(v) = \sum_{j=1}^s \lambda_j w_j,$$

with $\lambda_j\in\mathcal{M}$ homogeneous. Set $\deg(w_j)=(\alpha_j,\beta_j)$. Since $\alpha-p\ge\alpha_j-(p-1)$ for any j, we have that $\alpha>\alpha_j$, and so the first component of the degree of λ_j is positive. Therefore $\lambda_j\in\mathfrak{m}_1 S$, and we can conclude $\overline{\psi}_p(v)=0$, that is, $v\in\operatorname{Ker}\overline{\psi}_p$. It is clear that $v\notin\operatorname{Im}\overline{\psi}_{p+1}$ because $\operatorname{Im}\overline{\psi}_{p+1}\subset\mathcal{M}(D_p/\mathfrak{m}_1 D_p)$. So $v\in\operatorname{Tor}_p^S(S/\mathfrak{m}_1 S, L)_{(\alpha,\beta)}$, $v\neq 0$. We get (*ii*) by symmetry. □

5.2 The a-invariants of the powers of an ideal

Let $A=k[X_1,\dots,X_n]$ be the usual polynomial ring over a field k, and let I be a homogeneous ideal in A. Recently, the question of how the regularity changes with the powers of I has been studied by many authors. I. Swanson in [Swa] proved that there exists an integer B such that $\operatorname{reg}(I^e)\le Be$ for all e. The problem is then to make B explicit.

For ideals such that $\dim(A/I) = 1$, A. Geramita, A. Gimigliano and Y. Pitteloud [GGP] and K. Chandler [Cha] had shown that $\operatorname{reg}(I^e) \leq \operatorname{reg}(I)\, e$; and this bound also holds for Borel-fixed monomial ideals by using the Eliahou-Kervaire resolution [EK].

Another kind of bound is given by R. Sjögren [Sjo]: If I is an ideal generated by forms in degree $\leq d$ with $\dim(A/I) \leq 1$, then $\operatorname{reg}(I^e) < (n-1)de$. Also A. Bertram, L. Ein and R. Lazarsfeld [BEL] gave a bound for the regularity of the powers of an ideal in terms of the degrees of its generators. More explicitly, if I is the ideal of a smooth complex subvariety X in $\mathbb{P}^{n-1}_{\mathbb{C}}$ of codimension c and I is generated by forms in degrees $d_1 \geq d_2 \geq \ldots \geq d_r$, then

$$H^i(\mathbb{P}^{n-1}_{\mathbb{C}}, \mathcal{I}^e(k)) = 0, \ \ \forall i \geq 1, \forall k \geq ed_1 + d_2 + \ldots + d_c - (n-1).$$

This result has been improved by A. Bertram [Ber] for some determinantal varieties.

Recently, work by S.D. Cutkosky, J. Herzog and N.V. Trung [CHT], V. Kodiyalam [Ko2] and O. Lavila–Vidal (see Theorem 3.4.6) provides by different methods bounds for arbitrary graded ideals by means of the degrees of the generators similar to the ones given in [Sjo] and [BEL]. Namely, if I is a graded ideal generated by forms in degree $\leq d$, then there exists β such that

$$\operatorname{reg}(I^e) \leq de + \beta, \ \forall e.$$

We are also interested in the behaviour of the a_*-invariant of the powers of I, which can be used to apply the criteria seen in Chapter 3 for the Cohen-Macaulayness of the diagonals. We have already proved in Theorem 3.4.6 the existence of an integer α such that $a_*(I^e) \leq de + \alpha$ for all e. Our first purpose will be to find for any graded ideal an explicit α. Furthermore, for equigenerated ideals we will compute the best α we can take in terms of an appropiate a-invariant of the Rees algebra. After that, these results will be applied to give bounds for the a_*-invariant of the powers of several families of ideals such as equimultiple ideals and strongly Cohen-Macaulay ideals. Finally, we will use those bounds to study the Cohen-Macaulay property of the diagonals of the Rees algebra.

Let k be a field, A a standard noetherian graded k-algebra, $\overline{n} = \dim A$. Then A has a presentation $A = k[X_1, \ldots, X_n]/K = k[x_1, \ldots, x_n]$, where K is a homogeneous ideal and each X_i has degree 1. Let I be a homogeneous ideal in A generated by forms of degree $\leq d$. From Theorem 3.4.6, there exists α such that

$$a_*(I^e) \leq de + \alpha, \ \forall e.$$

Now let us assume that I is generated by forms in degree d. By defining $\varphi(p,q) = (p - dq, q)$, we have that R^φ is a standard bigraded k-algebra with $[R^\varphi]_{(p,q)} = R_{(p+dq,q)}$. The next result precises the best α we can take.

Theorem 5.2.1 *Let I be a homogeneous ideal of A generated by forms in degree d. Set $l = l(I)$. Then*

(i) $a_*^1(R^\varphi) = \max_e\{\, a_*(I^e) - de\} = \max\{\, a_*(I^e) - de \mid e \leq a_*^2(R) + l\}$.

(ii) $\operatorname{reg}_1(R^\varphi) = \max_e\{\operatorname{reg}(I^e) - de\} = \max\{\operatorname{reg}(I^e) - de \mid e \leq a_*^2(R) + l\}$.

Proof. We may assume that k is infinite (tensorazing by $k(T)$). Then there exists a minimal reduction J of I generated by l forms in degree d. By considering the polynomial ring $S = k[X_1, \dots, X_n, Y_1, \dots, Y_l]$, we have a natural epimorhism $S \to R_A(J)$. Then $R_A(J)$ is a finitely generated bigraded S-module, and so $R = R_A(I)$ because it is a finitely generated $R_A(J)$-module. Note that S^φ is standard and R^φ is a finitely generated bigraded S^φ-module, so according to Theorem 5.1.1

$$a_*^1(R^\varphi) = \max_e\{a_*([R^\varphi]^e)\} = \max_e\{a_*([R^\varphi]^e) \mid e \leq a_*^2(R^\varphi) + l\}.$$

First, observe that $a_*^2(R^\varphi) = a_*^2(R)$ by Lemma 1.2.3. Moreover, for each $e \geq 0$, we have $[R^\varphi]^e = \bigoplus_i (I^e)_{i+de} = (I^e)^\psi$, where $\psi : \mathbb{Z} \to \mathbb{Z}$ is defined by $\psi(j) = j - de$. From Lemma 1.2.3 we have $a_*((I^e)^\psi) = a_*(I^e) - de$, and then we obtain (i). The proof of (ii) follows the same lines. □

Remark 5.2.2 Let I be a homogeneous ideal in A generated by forms in degree d. By repeating the previous arguments for the form ring, we also get

(i) $a_*^1(G^\varphi) = \max_e\{\, a_*(I^e/I^{e+1}) - de\}$
$\qquad = \max_e\{\, a_*(I^e/I^{e+1}) - de \mid e \leq a_*^2(G) + l\}$.

(ii) $\operatorname{reg}_1(G^\varphi) = \max_e\{\operatorname{reg}(I^e/I^{e+1}) - de\}$
$\qquad = \max_e\{\operatorname{reg}(I^e/I^{e+1}) - de \mid e \leq a_*^2(G) + l\}$.

Example 5.2.3 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}^3_k$, that is,

$$I = (X_1X_4 - X_2X_3, X_2^2 - X_1X_3, X_3^2 - X_2X_4).$$

It is well known that I is the ideal of the Veronese embedding of $\mathbb{P}^1_k$ in $\mathbb{P}^3_k$:

$$\begin{array}{ccc} \mathbb{P}^1_k & \xrightarrow{\mu} & \mathbb{P}^3_k \\ (u:v) & \longmapsto & (u^3 : u^2v : uv^2 : v^3) \,. \end{array}$$

I is licci because it is linked to $J = (X_1, X_2)$ by the regular sequence $\underline{\alpha} = X_2^2 - X_1X_3, X_3^2 - X_2X_4$ [Ul, Example 2.3], so I is a strongly Cohen-Macaulay ideal [Hu1, Theorem 1.14]. Since I is a prime ideal, we easily get $\mu(I_\mathfrak{p}) \leq \text{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Therefore $R_A(I)$ is Cohen-Macaulay by [HSV1, Theorem 2.6], so in particular $a_*^2(R_A(I)) = -1$. On the other hand, I is an ideal generated by forms of degree 2 with $l(I) = \mu(I) = 3$. By using CoCoa [CNR], we have that the minimal graded free resolutions of I and I^2 are:

$$0 \to A(-3)^2 \to A(-2)^3 \to I \to 0\,,$$

$$0 \to A(-6) \to A(-5)^6 \to A(-4)^6 \to I^2 \to 0\,,$$

so according to Theorem 1.3.4 we have $a_*(I) = -1$, $a_*(I^2) = 2$. By Theorem 5.2.1 we get

$$a_*(I^e) \leq 2(e-1), \ \forall e.$$

Furthermore, notice that since $\text{reg}\,(I) = 2$, $\text{reg}\,(I^2) = 4$ we also get

$$\text{reg}\,(I^e) \leq 2e, \ \forall e.$$

Therefore, we have that I^e has a linear resolution for any $e \geq 1$. This has already been proved by A. Conca [Con] by different methods.

Remark 5.2.4 Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring bigraded by setting $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$, with $d_1, \ldots, d_r \in \mathbb{Z}_{\geq 0}$. For a finitely generated bigraded S-module L, let us consider the minimal bigraded free resolution of L over S

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to L \to 0,$$

where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. By applying the functor $(\ )^e$, note that

$$S(a,b)^e = \bigoplus_{|\underline{\beta}|=b+e} S_1(a - d_1\beta_1 - \ldots - d_r\beta_r),$$

where $\underline{\beta} = (\beta_1, \ldots, \beta_r) \in \mathbb{N}^r$ and $|\underline{\beta}| = \beta_1 + \cdots + \beta_r$. So we get a graded free resolution of L^e over S_1

$$0 \to D_t^e \to \ldots \to D_1^e \to D_0^e \to L^e \to 0\,,$$

with $D_p^e = \bigoplus_{(a,b)\in\Omega_p} \bigoplus_{|\underline{\beta}|=b+e} S_1(a - d_1\beta_1 - \ldots - d_r\beta_r)$. The minimal graded free resolution is then obtained by picking out some terms. Therefore, for any $i \leq n$ we have that

$$a_i(L^e) \leq \max\{d_1\beta_1 + \ldots + d_r\beta_r - a \mid (a,b) \in \Omega_{n-i}, |\underline{\beta}| = b + e\} - n$$

$$\leq de - n + \max\{db - a \mid (a,b) \in \Omega_{n-i}\}.$$

Therefore, $a_*(L^e) \leq de - n + \max\{db - a \mid (a,b) \in \Omega_L\} \leq d(e - \mathrm{indeg}_2 L) + a_*^1(L)$. In particular, for any homogeneous ideal I of A we have

$$a_*(I^e) \leq de + a_*^1(R).$$

5.2.1 Explicit bounds for some families of ideals

The next purpose is to get explicit bounds for the a_*-invariant of the powers of an ideal, and we will focus our attention to the case of ideals in the polynomial ring. Throughout the rest of this section, $A = k[X_1, \ldots, X_n]$ will denote the usual polynomial ring in n variables over a field k and I will be a homogeneous ideal in A. First of all, for equigenerated ideals whose Rees algebra is Cohen-Macaulay we have

Proposition 5.2.5 *Let I be a homogeneous ideal generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. Then*

$$-n + d(-a^2(G) - 1) \leq \max_{e\geq 0}\{a_*(I^e) - de\} \leq -n + d(l - 1).$$

Proof. As in the proof of Theorem 5.2.1, we may assume that k is infinite. By considering then the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$, we have that $R_A(I)$ is a finitely generated bigraded S-module in a natural way. Then let

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to R \to 0$$

be the minimal bigraded free resolution of the Rees algebra R over S, with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. The shifts $(a,b) \in \Omega_R$, $(a,b) \neq (0,0)$, satisfy $b \leq -1$

and $-a \leq dl + n + a^1(R) \leq dl$ by Lemma 3.4.7. Therefore, we have $a_*(I^e) \leq de + d(l-1) - n$ by Remark 5.2.4.

Let $R_{++} = \bigoplus_{(i,j),j>0} R_{(i,j)}$. From the bigraded exact sequences

$$0 \to R_{++} \to R \to A \to 0,$$

$$0 \to R_{++}(0,1) \to R \to G \to 0,$$

we get the following exact sequences of local cohomology

$$0 \to H^n_{\mathfrak{m}}(A)_{(i,j)} \to H^{n+1}_{\mathcal{M}}(R_{++})_{(i,j)} \to H^{n+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \quad (\star),$$

$$0 \to H^n_{\mathcal{M}}(G)_{(i,j)} \to H^{n+1}_{\mathcal{M}}(R_{++})_{(i,j+1)} \to H^{n+1}_{\mathcal{M}}(R)_{(i,j)} \to 0 \quad (\star\star).$$

Since $a^2(R) = -1$, from the above exact sequences we have $a^2(G) \leq -1$. If $a^2(G) = -1$, the lower bound is obvious by considering $e = 0$. So we may assume $a^2(G) < -1$, and by Theorem 5.2.1 we must prove $a^1(R^\varphi) \geq -n - d(a^2(G)+1)$. The local cohomology modules behave well under a change of grading by Lemma 1.2.3, hence we have

$$H^{n+1}_{\mathcal{M}}(R^\varphi)_{(p,q)} = H^{n+1}_{\mathcal{M}}(R)^\varphi_{(p,q)} = H^{n+1}_{\mathcal{M}}(R)_{(p+dq,q)},$$

so $a^1(R^\varphi) = \max\{p \mid \exists\ q \ s.t.\ H^{n+1}_{\mathcal{M}}(R)_{(p+dq,q)} \neq 0\}$. Since $H^n_{\mathfrak{m}}(A)_{(-n,0)} \neq 0$ we have $H^{n+1}_{\mathcal{M}}(R_{++})_{(-n,0)} \neq 0$ from the exact sequence $(\star)$. As $a^2(G) < -1$, from the second exact sequence $(\star\star)$ we get $H^{n+1}_{\mathcal{M}}(R)_{(-n,-1)} \neq 0$, and by using once more $(\star)$ we have $H^{n+1}_{\mathcal{M}}(R_{++})_{(-n,-1)} \neq 0$. Note that we can repeat this procedure while the second component of the degree be greater than $a^2(G)$, and finally we get $H^{n+1}_{\mathcal{M}}(R)_{(-n,a^2(G)+1)} \neq 0$. In particular, $a^1(R^\varphi) \geq -n - d(a^2(G)+1)$. □

Remark 5.2.6 Let I be an ideal generated by forms of degree d in a general standard graded noetherian k-algebra A. By setting $l = l(I)$, one can similarly prove that if the Rees algebra is Cohen-Macaulay then

$$a(A) + d(-a^2(G) - 1) \leq \max\{a_*(I^e) - de\} \leq a(A) + dl.$$

For non-equigenerated ideals, we can also give an upper bound. A similar result for the regularity was already proved in [CHT, Corollary 2.6].

Remark 5.2.7 Let I be a homogeneous ideal generated by forms $f_1, \dots, f_r$ in degrees $d_1 \leq \dots \leq d_r = d$ whose Rees algebra is Cohen-Macaulay. Set $u = \sum_{i=1}^r d_i$. Then $a_*(I^e) \leq d(e-1) + u - n$.

Proof. By considering the bigraded minimal free resolution of R over $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$, we have that any shift $(a, b) \in \Omega_p$ with $p \geq 1$ satisfies $b \leq -1$ and $-a \leq u$ by Lemma 3.4.10 and Ω_0 only contains the shift $(0, 0)$. Therefore, $a_*(I^e) \leq d(e-1) + u - n$ by Remark 5.2.4. □

The a_*-invariant of the powers of an ideal can be computed for complete intersection ideals (see the proof of Proposition 3.4.5), and then we have that the inequalities in Proposition 5.2.5 and Remark 5.2.7 are sharp. Next we are going to compute explicitly $\max_{e \geq 0}\{a_*(I^e) - de\} = a^1(R^\varphi)$ for several families of ideals. First we consider the case of equimultiple ideals.

Proposition 5.2.8 *Let I be an equimultiple ideal generated in degree d. Set $h = \mathrm{ht}\,(I) \geq 1$. If the Rees algebra is Cohen-Macaulay,*

(i) $a(I^e/I^{e+1}) = de + a(A/I)$. In particular, $a^1(G^\varphi) = a(A/I)$.

(ii) $a_{n-h+1}(I^e) = d(e-1) + a(A/I)$. In particular, $a^1(R^\varphi) = a(A/I) - d$.

Proof. We may assume that k is infinite. Then there exist $f_1, \dots, f_{n-h} \in A$ of degree 1 such that $\overline{f}_1, \dots, \overline{f}_{n-h} \in A/I$ is a homogeneous system of parameters. Denoting by f^* the initial form of $f \in A$ in G, let us consider $\overline{G} = G/(f_1^*, \dots, f_{n-h}^*)$. Since $\mathrm{rad}\,((f_1^*, \dots, f_{n-h}^*)G) = \mathrm{rad}\,(\mathfrak{m}G)$, we have that a system of parameters of $F_\mathfrak{m}(I)$ is also a system of parameters of $\overline{G}$. As $F_\mathfrak{m}(I)$ is a bigraded k-algebra generated by forms in degree $(d, 1)$, there exist $F_1, \dots, F_h \in I$ of degree d such that $\overline{F}_1, \dots, \overline{F}_h$ is a system of parameters of $F_\mathfrak{m}(I)$. Then $f_1^*, \dots, f_{n-h}^*, F_1^*, \dots, F_h^*$ is a homogeneous system of parameters of G, and so algebraically independent over k. Therefore, there is a finite extension

$$T = k[U_1, \dots, U_h, V_1, \dots, V_{n-h}] \to G,$$

where T is a polynomial ring with $\deg(U_i) = (d, 1)$, $\deg(V_j) = (1, 0)$ for $i = 1, \dots, h$, $j = 1, \dots, n-h$. Since G is Cohen-Macaulay and T is regular, we have that G is a free T-module, that is, $G = \bigoplus_{(a,b)\in\Lambda} T(a, b)$, where $\Lambda \subset \mathbb{Z}^2$ is a finite set. Let us denote by $T_1 = k[V_1, \dots, V_{n-h}]$, $m = (V_1, \dots, V_{n-h})$, and for a given $\underline{\beta} = (\beta_1, \dots, \beta_h) \in \mathbb{N}^h$, let $|\,\underline{\beta}\,| = \beta_1 + \cdots + \beta_h$. Note that

$$\begin{aligned} T(a,b)^e &= \textstyle\bigoplus_i T(a,b)_{(i,e)} = \bigoplus_i T_{(a+i,b+e)} \\ &= \textstyle\bigoplus_i \bigoplus_{|\underline{\beta}|=b+e} [T_1]_{a+i-d(b+e)} U_1^{\beta_1} \dots U_h^{\beta_h} \\ &= T_1(a - db - de)^{\rho_b^e}, \end{aligned}$$

with $\rho_b^e \in \mathbb{N}$ and $\rho_b^e = 0$ if $b + e < 0$. Therefore,

$$I^e/I^{e+1} = G^e = \bigoplus_{(a,b)\in\Lambda} T_1(a - db - de)^{\rho_b^e},$$

and by taking local cohomology

$$H_{\mathfrak{m}}^{n-h}(I^e/I^{e+1}) = \bigoplus_{(a,b)\in\Lambda} H_m^{n-h}(T_1(a - db - de))^{\rho_b^e}.$$

Hence $a_*(I^e/I^{e+1}) = a(I^e/I^{e+1}) = \max\{-(n-h) - a + db + de : -b \le e\}$. In particular, $a(A/I) = \max\{-(n-h) - a + db : b = 0\}$, and so we get $a(I^e/I^{e+1}) \ge de + a(A/I)$ for all e. On the other hand, since the modules I^e/I^{e+1} are A/I-modules of maximal dimension, we have an epimorhism $\bigoplus A/I(-de) \to I^e/I^{e+1}$ and we may deduce that $a(I^e/I^{e+1}) \le de + a(A/I)$ for all e. To get (ii), it is just enough to consider the short exact sequences

$$0 \to I^{e+1} \to I^e \to I^e/I^{e+1} \to 0\ ,$$

and then the result follows from (i) by induction on e. □

Next we study equigenerated ideals whose form ring is Gorenstein. In this case, we prove that the lower bound given in Proposition 5.2.5 is sharp.

Proposition 5.2.9 *Let I be a homogeneous ideal equigenerated in degree d whose form ring is Gorenstein. Set $l = l(I)$. Then*

(i) $\max_{e\ge 0}\{a_*(I^e) - de\} = d(-a^2(G) - 1) - n$.

(ii) For $e > a^2(G) - a(F)$, $\operatorname{depth}(A/I^e) = n - l$ *and* $a_*(I^e) = a_{n-l}(A/I^e) = d(e - a^2(G) - 1) - n$.

Proof. We may assume that the field k is infinite. Since I is generated by forms in degree d, there exists a minimal reduction J of I generated by forms $g_1, \dots, g_l$ of degree d. By considering $S = k[X_1, \dots, X_n, Y_1, \dots, Y_l]$ bigraded by setting $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (d, 1)$, we have a bigraded epimorphism $S \to R_A(J)$. Suppose that $I^{m+1} = JI^m$. Then $R_A(I)$ is finitely generated over $R_A(J)$ by the generators of A, I, ..., I^m; so in particular by homogeneous elements in degree (di, i) for $i = 0, \dots, m$. Then we have an epimorphism $F \to R_A(I)$, where F is a finite free S-module with a basis of elements in degrees (di, i) for $i = 0, \dots, m$.

Let us consider the minimal bigraded free resolution of G over S

$$0 \to D_l \to \ldots \to D_1 \to D_0 \to G \to 0,$$

where $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$. From Remark 5.2.4, for all $e \geq 0$ we have

$$a_*(I^e/I^{e+1}) \leq de - n + \max\{db - a \mid (a,b) \in \Omega_G\}.$$

Assume we prove that the maximum is accomplished for a shift $(a,b) \in \Omega_l$. Denoting by $(\ )^* = \underline{\mathrm{Hom}}_S(\ ,K_S)$, then

$$0 \to D_0^* \to \ldots \to D_l^* \to \underline{\mathrm{Ext}}_S^l(G,K_S) = K_G \to 0$$

is the minimal bigraded free resolution of the canonical module K_G of G over S. Since G is Gorenstein, according to Corollary 4.1.7 there is a bigraded isomorphism

$$K_G \cong G(-n, a^2(G)).$$

Now the shifts $(a,b) \in \Omega_l$ are of the type $(di, i - a^2(G))$ for certain integers i, so we get $a_*(I^e/I^{e+1}) \leq de - n + \max_i\{d(i - a^2(G)) - di\} = d(e - a^2(G)) - n$. From Remark 5.2.5 we have $a^1(G^\varphi) = \max\{a_*(I^e/I^{e+1}) - de\} \leq -n - da^2(G)$, and $a^1(R^\varphi) \leq -n - da^2(G) - d$. Observe that Proposition 5.2.5 gives the other inequality, so we obtain $a^1(R^\varphi) = d(-a^2(G) - 1) - n$.

Now let us prove that the maximum for the differences $db - a$ is taken for $(a,b) \in \Omega_l$. Let us consider $\phi : \mathbb{Z}^2 \to \mathbb{Z}$ defined by $\phi(i,j) = i - dj$. Note that $X_i \in S^\phi$ has degree 1 and $Y_j \in S^\phi$ has degree 0, and

$$0 \to D_l^\phi \to \ldots \to D_0^\phi \to G^\phi \to 0$$

is a graded free resolution of G^ϕ over S^ϕ. Since $S(a,b)^\phi = S^\phi(a - db)$, it is clear that

$$\min\{db - a \mid (a,b) \in \Omega_{p+1}\} \geq \min\{db - a \mid (a,b) \in \Omega_p\}.$$

Applying the same argument for the resolution of K_G, one gets that

$$\max\{db - a \mid (a,b) \in \Omega_{p+1}\} \geq \max\{db - a \mid (a,b) \in \Omega_p\},$$

so (i) is already proved.

To prove the rest of the statement, we will use that $\mathrm{proj.dim}_A(I^e/I^{e+1}) = l$ if and only if $e \geq a^2(G) - a(F)$ (see Proposition 6.3.2). By applying the functor $(\ )^e$ to the resolution of G over S we obtain a free resolution of I^e/I^{e+1} as

A-module, and the shifts appearing in the place p of these resolutions of the type $a - db - de$, with $(a, b) \in \Omega_p$. Therefore, $t_p(I^e/I^{e+1}) \leq de - da^2(G)$ for any p, and for $e \geq a^2(G) - a(F)$ we have $t_l(I^e/I^{e+1}) = de - da^2(G)$. Now, since $\text{proj.dim}_A I^e \leq l - 1$ by Proposition 6.3.2, from the short exact sequences

$$0 \to I^{e+1} \to I^e \to I^e/I^{e+1} \to 0,$$

we get $t_{l-1}(I^e) \geq d(e - a^2(G) - 1)$ for $e > a^2(G) - a(F)$. On the other hand, we have $t_{l-1}(I^e) \leq t_*(I^e) = a_*(I^e) + n \leq d(e - a^2(G) - 1)$, so we get the equality. We finally obtain $a_{n-l}(A/I^e) = a_{n-l+1}(I^e) = d(e - a^2(G) - 1) - n$ for $e > a^2(G) - a(F)$ by Theorem 1.3.4. □

Example 5.2.10 Let $\mathbf{X} = (X_{ij})$ be a $d \times n$ generic matrix, with $1 \leq i \leq d$, $1 \leq j \leq n$ and $d \leq n$, and let $A = k[\mathbf{X}]$ be the polynomial ring in the entries of $\mathbf{X}$. Let $I = I_d(\mathbf{X})$ be the ideal generated by the maximal minors of $\mathbf{X}$. We are going to apply to this example the different bounds we have found.

- The Rees algebra of I is Cohen-Macaulay, so by applying Remark 5.2.7 we get

  $$a_*(I^e) \leq d(e-1) + \binom{n}{d} d - nd.$$

  (a similar bound for the regularity of the powers has been given in [CHT, Example 2.7]).

- Note that $F_{\mathfrak{m}}(I) = k[I_d]$ is the coordinate ring of the Grassmannian $G(d, n)$, so we have that the analytic spread of I is $l(I) = d(n - d) + 1$. Therefore by Proposition 5.2.5 we get the bound

  $$a_*(I^e) \leq de + d^2(n - d) - nd.$$

- Since $G_A(I)$ is Gorenstein with $a^2(G_A(I)) = -\text{ht}\,(I) = -(n - d + 1)$, by using Proposition 5.2.9 we get the better bound

  $$a_*(I^e) \leq d(e + n - d) - nd = de - d^2.$$

  Furthermore, the a-invariant of F is $-n$ by [BH2, Corollary 1.4]. So we also obtain $a_{d^2}(I^e) = de - d^2$ for any $e > d - 1$.

K. Akin et al. [ABW] have constructed resolutions for the powers of I, in particular showing that all the powers of I have linear resolutions. Note that this fact also allows to prove the last bound: According to Proposition 6.3.2 and Theorem 1.3.4, for any $e > a^2(G) - a(F)$ we have $a_*(I^e) = a_{nd-l+1}(I^e) = t_{l-1}(I^e) - nd = (de + l - 1) - nd = de - d^2$.

We may also use Proposition 5.2.9 to study the a_*-invariant of the powers of a strongly Cohen-Macaulay ideal. Let I be an ideal of A, and let $\underline{f} = f_1, \ldots, f_r$ be a system of generators of I. Recall that I is a strongly Cohen-Macaulay ideal if for any $p \geq 0$ the Koszul homology $H_p(K(\underline{f}))$ is a Cohen-Macaulay A/I-module.

Corollary 5.2.11 *Let I be a strongly Cohen-Macaulay ideal generated in degree d such that $\mu(I_\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Let $h = \operatorname{ht}(I)$, $l = l(I)$. Then*

(i) $a_*(I^e) \leq d(e + h - 1) - n$, $\forall e$.

(ii) For $e > l - h$, $\operatorname{depth}(A/I^e) = n - l$ *and* $a_*(I^e) = a_{n-l}(A/I^e) = d(e + h - 1) - n$.

Proof. In this situation, $G_A(I)$ is Gorenstein and I is an ideal of linear type by [HSV1, Theorem 2.6], so $a(F_\mathfrak{m}(I)) = -l$. Furthermore, according to [HRZ, Proposition 2.5] we have $a^2(G_A(I)) = -h$. Then the result follows from Proposition 5.2.9.□

Example 5.2.12 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}^3_k$. From Example 5.2.3, recall that I is a strongly Cohen-Macaulay ideal generated in degree 2 with $\operatorname{ht}(I) = 2$, $l(I) = \mu(I) = 3$. Now, by Corollary 5.2.11, for any $e > 1$ we have that $\operatorname{depth}(A/I^e) = 1$, $a_1(A/I^e) = 2e - 2$ and $a_2(A/I^e) \leq 2e - 2$. In the case $e = 1$, since I is linked to $J = (X_1, X_2)$ by the regular sequence $\underline{\alpha} = X_2^2 - X_1X_3, X_3^2 - X_2X_4$, we have that A/I is Cohen-Macaulay and there is a graded isomorphism $K_{A/I} \cong J/(\underline{\alpha})$. In particular, $a(A/I) = -1$.

In trying to extend the bounds in Proposition 5.2.9 to the non-equigenerated case many difficulties appear. Next we will use approximation complexes to do this for strongly Cohen-Macaulay ideals.

Let I be a homogeneous ideal in the polynomial ring $A = k[X_1, \ldots, X_n]$ and $\underline{f} = f_1, \ldots, f_r$ a homogeneous system of generators of I, with d_i =deg(f_i), and let us consider the graded Koszul complex $K(\underline{f})$ of A with respect to $\underline{f}$. Denote by $S = A[Y_1, \dots, Y_r]$ with the bigrading $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$. Then the approximation complex of I is

$$\mathcal{M}(\underline{f}) : 0 \to \mathcal{M}_r \to \dots \to \mathcal{M}_1 \to \mathcal{M}_0 \to 0,$$

with $\mathcal{M}_p = H_p(K(\underline{f})) \otimes_A S(0,-p)$, and the differential maps are homogeneous. Assume that I is a strongly Cohen-Macaulay ideal with $\operatorname{ht}(I) \geq 1$ such that for any prime ideal $\mathfrak{p} \supseteq I$, $\mu(I_\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p})$. Then $\mathcal{M}(\underline{f})$ is exact and provides a resolution of $\operatorname{Sym}_A(I/I^2) \cong G_A(I)$ [HSV1, Theorem 2.6]. We will use it to get a bound for the a-invariants of the powers of these ideals.

Proposition 5.2.13 *Let I be a strongly Cohen-Macaulay ideal such that for any prime ideal $\mathfrak{p} \supseteq I$, $\mu(I_\mathfrak{p}) \leq \operatorname{ht}(\mathfrak{p})$. Assume that I is minimally generated by forms $f_1, \dots, f_r$ of degree $d = d_1 \geq \dots \geq d_r$, and set $h = \operatorname{ht}(I)$, $t = r - h$. Then:*

(i) $a(H_m(\underline{f})) \leq -n + d_1 + \dots + d_{h+m}$, *for all* $m \leq t$.

(ii) If $1 \leq e \leq t$, $\operatorname{depth}(A/I^e) \geq n - h - e + 1$ *and for any* $0 \leq m \leq e-1$,

$$a_{n-h+1-m}(I^e) \leq -n + d_1 + \dots + d_{h+m} + d(e-m-1).$$

(iii) If $e > t$, $\operatorname{depth}(A/I^e) = n - r$ *and for any* $0 \leq m \leq t$,

$$a_{n-h+1-m}(I^e) \leq -n + d_1 + \dots + d_{h+m} + d(e-m-1).$$

Proof. Recall that $H_p = H_p(K(\underline{f})) = 0$ for all $p > t$. So the resolution of G given by the approximation complex is

$$0 \to \mathcal{M}_t \to \dots \to \mathcal{M}_0 \to G \to 0,$$

with $\mathcal{M}_p = H_p \otimes_A S(0,-p)$. Let us denote by $\underline{\beta} = (\beta_1, \dots, \beta_r) \in \mathbb{N}^r$, and $|\underline{\beta}| = \beta_1 + \dots + \beta_r$. Applying the functor $(\,)^e$ to the modules of this resolution:

$$G^e = \textstyle\bigoplus_i G_{(i,e)} = I^e/I^{e+1},$$

$$\mathcal{M}_p^e = \textstyle\bigoplus_i H_p[Y_1, \dots, Y_r]_{(i,e-p)} = \begin{cases} 0 & \text{if } e < p \\ \bigoplus_{|\underline{\beta}|=e-p} H_p(-d_1\beta_1 - \dots - d_r\beta_r) & \text{if } e \geq p. \end{cases}$$

So we get graded exact sequences

$$0 \to \mathcal{M}_q^e \to \dots \to \mathcal{M}_0^e \to I^e/I^{e+1} \to 0,$$

with $q = \min\{e, t\}$. Since I is a strongly Cohen-Macaulay ideal, we have that $\mathcal{M}_p^e$ is a maximal CM A/I-module for any $p \leq q$. By taking short exact sequences, we obtain that if $e < t$, $\operatorname{depth}(I^e/I^{e+1}) \geq n - h - e$ and if $e \geq t$, $\operatorname{depth}(I^e/I^{e+1}) \geq n - h - t = n - r$. On the other hand, by Proposition 6.3.2 we also have that $\operatorname{depth}(I^e/I^{e+1}) = n - r$ if and only if $e \geq t$. Furthermore, we get $a_{n-h-m}(I^e/I^{e+1}) \leq a(\mathcal{M}_m^e) = a(H_m) + d(e-m)$ for all $0 \leq m \leq \min\{e, t\}$. From the exact sequences

$$0 \to I^e/I^{e+1} \to A/I^{e+1} \to A/I^e \to 0,$$

we have now

(a) $\operatorname{depth}(A/I^e) \geq n - h - e + 1$ if $1 \leq e \leq t$

(b) $\operatorname{depth}(A/I^e) = n - r$ if $e > t$

(c) $a_{n-h-m}(A/I^e) \leq \max_{0 \leq j \leq e-1}\{a_{n-h-m}(I^j/I^{j+1})\} \leq a(H_m) + d(e-m-1)$,

for $0 \leq m \leq \min\{e-1, t\}$.

So, if we prove the bound for the a-invariant of the Koszul homology we have finished. Let us assume that among the forms $f_1, \ldots, f_r$ we can choose a regular sequence of length h. Let $g_1 = f_{j_1}, \ldots, g_h = f_{j_h}$ be this sequence, and $g_1, \ldots g_r$ the minimal system of generators of I.

Let us consider the morphism from A to $A/(g_1)$. By [Hu2, Lemma 1.1], there is a graded exact sequence

$$0 \to H_m(I; A) \to H_m(I/(g_1); A/(g_1)) \to H_{m-1}(I; A)(-\deg g_1) \to 0$$

for all $m \geq 1$; where $H_m(I/(g_1); A/(g_1))$ denotes the Koszul homology of the elements $0, \overline{g}_2, \ldots, \overline{g}_r \in A/(g_1)$. From this exact sequence, we have in particular $a(H_m(I; A)) \leq a(H_m(I/(g_1); A/(g_1)))$. Denote by "$-$" the morphism from A to $\overline{A} = A/(g_1, \ldots, g_{h-1})$. Repeating $h-1$ times the previous procedure, we get $a(H_m(I; A)) \leq a(H_m(\overline{I}; \overline{A}))$ for all $m \geq 1$. But now $\overline{I}$ is a height one ideal in the CM ring $\overline{A}$. Let us denote the Koszul complex of $\overline{I}$ by $\overline{K} = K(\overline{I}; \overline{A})$, and the differential from $\overline{K}_{m+1}$ to $\overline{K}_m$ by d_{m+1}. Set $\overline{Z}_m = \operatorname{Ker}(d_m)$, $\overline{B}_m = \operatorname{Im}(d_{m+1})$, $\overline{H}_m = H_m(\overline{I}; \overline{A})$. Then there are exact sequences

$$0 \to \overline{B}_m \to \overline{Z}_m \to \overline{H}_m \to 0,$$

$$0 \to \overline{Z}_{m+1} \to \overline{K}_{m+1} \to \overline{B}_m \to 0.$$

By [Hu1, Lemma 1.6] $\overline{H}_m$ are CM modules for all m, and then by [Hu1, Lemma 1.8], $\overline{Z}_m$ and $\overline{B}_m$ are maximal CM modules for $\overline{A}$. The exact sequences imply now $a(\overline{H}_m) \leq a(\overline{B}_m) \leq a(\overline{K}_{m+1})$. Denoting by δ_i =deg(g_i),

$$a(\overline{K}_{m+1}) = a(\overline{A}) + \max\{\delta_{i_1} + \ldots + \delta_{i_{m+1}} \mid h \leq i_1 < \ldots < i_{m+1} \leq r\}$$

$$= a(A) + \delta_1 + \ldots + \delta_{h-1} + \max\{\delta_{i_1} + \ldots + \delta_{i_{m+1}} \mid h \leq i_1 < \ldots < i_{m+1} \leq r\}$$

$$\leq -n + d_1 + \ldots + d_{h+m}.$$

So we are done if we prove the following lemma. □

Lemma 5.2.14 *Let $A = k[t_1, \ldots, t_s]$ be a CM graded algebra over an infinite field k, with $\deg(t_i) = 1$. For any homogeneous ideal I, there exists a minimal homogeneous system of generators $g_1, \ldots g_r$ of I such that $g_1, \ldots g_h$ is a maximal regular sequence in I.*

Proof. Set $r = \mu(I)$, $h = \text{ht}\,(I)$. Let $f_1, \ldots, f_r$ be a minimal homogeneous system of generators of I, with $d_i = \deg(f_i)$, $d_1 \leq \ldots \leq d_r = d$. We are going to prove the statement by induction on h. If $h = 0$ there is nothing to prove. Assume $h \geq 1$. Then $I \not\subset z(A) = \bigcup_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p}$, and so $I_d \not\subset \mathfrak{p}$, $\forall \mathfrak{p} \in \text{Ass}\,(A)$ (otherwise, we would have $f_i^d \in \mathfrak{p}$ for all i, and then $I \subset \mathfrak{p}$). Since k is infinite, we get $I_d \not\subset \bigcup_{\mathfrak{p} \in \text{Ass}(A)} \mathfrak{p} \cap A_d$, and so there exists $g \in I_d$ such that $g \notin \mathfrak{p}$ for all $\mathfrak{p} \in \text{Ass}\,(A)$. Note that I_d is a k-vector space generated by the forms $f_{j_1}, \ldots, f_{j_i}$ in degree d and the forms Mf_j, with $d_j < d$ and M a monomial in $t_1, \ldots, t_s$ of degree $d - d_j$. Now we can write

$$g = \lambda_1 f_{j_1} + \ldots + \lambda_i f_{j_i} + \sum \mu_{jM} M f_j,$$

with $\lambda_1, \ldots, \lambda_i, \mu_{jM} \in k$. If there exists p such that $\lambda_p \neq 0$, we can replace f_{j_p} by g in the minimal system of generators of I. Otherwise, we have an element g in the ideal generated by the forms in I of degree $d' = d_{r-1}$ with the property that $g \notin \mathfrak{p}$, $\forall \mathfrak{p} \in \text{Ass}\,(A)$. We can repeat the arguments for $I_{d'}$, and finally we will replace one of the forms f_j by g. By considering $\overline{A} = A/(g)$, the ideal $\overline{I} = I/(g)$ has $\mu(\overline{I}) = r - 1$, ht$\,(\overline{I}) = h - 1$. Then we get the result by induction. □

5.2.2 Applications to the study of the diagonals of the Rees algebra

Next we are going to apply the results about the a-invariant of the powers of an ideal to study the Cohen-Macaulay property of the diagonals of the Rees algebra. If the Rees algebra is Cohen-Macaulay, according to Theorem 3.4.13 there exists $\alpha \in \mathbb{Z}$ such that $k[(I^e)_c]$ is Cohen-Macaulay for any $c > de + \alpha$ and $e > 0$. For equigenerated ideals, we obtained $\alpha = d(l-1)$ as upper bound. The following result precises the best α.

Proposition 5.2.15 *Let I be an ideal in $A = k[X_1, \ldots, X_n]$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l = l(I)$. For $\alpha \geq 0$, the following are equivalent*

(i) For all $c > de + \alpha$, $k[(I^e)_c]$ is CM.

(ii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e$.

(iii) $a_i(I^e) \leq de + \alpha$, $\forall i$, $\forall e \leq l - 1$.

(iv) $H^{n+1}_{\mathcal{M}}(R_A(I))_{(p,q)} = 0$, $\forall p > dq + \alpha$, that is, $\alpha \geq a^1(R^\varphi)$.

(v) The minimal bigraded free resolution of $R_A(I)$ is good for any diagonal $\Delta = (c, e)$ such that $c > de + \alpha$.

Proof. If $k[(I^e)_c]$ is CM for $c > de + \alpha$ then we have $H^i_{\mathfrak{m}}(I^e)_c = 0$ for any $i < n$ and $c > de + \alpha$ by Proposition 3.4.1, so $a_*(I^e) \leq de + \alpha$, $\forall e$. The converse follows similarly, and we get the equivalence between (i) and (ii).

Since the Rees algebra R of I is Cohen-Macaulay, we have $a^2_*(R) = a^2(R) = -1$. Then, conditions (ii) and (iii) are equivalent to $a^1(R^\varphi) = a^1_*(R^\varphi) \leq \alpha$ by Theorem 5.2.1.

Finally, we want to prove the equivalence to (v). If the resolution of R is good for diagonals $\Delta = (c, e)$ such that $c > de+\alpha$, then we have $H^i_m(k[(I^e)_c]) = H^{i+1}_{\mathcal{M}}(R)_\Delta = 0$ for $i < n$ by Corollary 2.1.14, so $k[(I^e)_c]$ is Cohen-Macaulay for any $c > de + \alpha$ and we obtain (i). Now assume that $a^1(R^\varphi) \leq \alpha$, and let us consider the minimal bigraded free resolution of R over S

$$0 \to D_t \to \ldots \to D_1 \to D_0 \to R \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a, b)$. By applying the functor $(\ )^\varphi$, we have that

$$0 \to D^\varphi_t \to \ldots \to D^\varphi_0 \to R^\varphi \to 0$$

is the bigraded minimal free resolution of R^φ over S^φ, with $D_p^\varphi = \bigoplus_{(a,b)\in\Omega_p} S^\varphi(a-db,b)$. Therefore, according to Theorem 1.3.4, for any $(a,b)\in\Omega_R$ we have $db-a-n\leq\alpha$. Then the sets X^Δ, Y^Δ introduced in Remark 2.1.11 are empty for diagonals $\Delta=(c,e)$ with $c>de+\alpha$, so the resolution is good for these Δ. □

If the form ring is Gorenstein, we can express this criterion by means of the second a-invariant of the form ring. Namely,

Corollary 5.2.16 *Let I be an ideal in $A=k[X_1,\dots,X_n]$ generated by forms of degree d whose form ring is Gorenstein. For $\alpha\geq 0$, the following are equivalent*

(i) For all $c>de+\alpha$, $k[(I^e)_c]$ is CM.

(ii) $\alpha\geq d(-a^2(G)-1)-n$.

Proof. By Proposition 5.2.9, $a^1(R^\varphi)=d(-a^2(G)-1)-n$. Then the result follows from Proposition 5.2.15. □

Let I be an equigenerated ideal in A. If the Rees algebra is Cohen-Macaulay, it can happen that some of its diagonals are not Cohen-Macaulay. Now, by taking $\alpha=0$ in Proposition 5.2.15 we have a criterion to decide when all the diagonals of a Cohen-Macaulay Rees algebra are Cohen-Macaulay.

Corollary 5.2.17 *Let I be an ideal in $A=k[X_1,\dots,X_n]$ generated by forms in degree d whose Rees algebra is Cohen-Macaulay. Set $l=l(I)$. Then the following are equivalent*

(i) For all $c\geq de+1$, $k[(I^e)_c]$ is CM.

(ii) $a_i(I^e)\leq de$, $\forall i$, $\forall e\leq l-1$.

(iii) $H^{n+1}_{\mathcal{M}}(R_A(I))_{(p,q)}=0$, $\forall p>dq$.

(iv) The minimal bigraded free resolution of $R_A(I)$ is good for any Δ.

Assuming that $G_A(I)$ is Gorenstein, these conditions are also equivalent to

(v) $-a^2(G)\leq \frac{n}{d}+1$.

Example 5.2.18 We may recover Corollary 3.4.2 as an easy application of Corollary 5.2.17. Let $\{L_{ij}\}$ be a set of $d \times (d+1)$ homogeneous linear forms in a polynomial ring $A = k[X_1, \dots, X_n]$, and let M be the matrix (L_{ij}). Let $I_t(M)$ be the ideal generated by the $t \times t$ minors of M and assume that $\mathrm{ht}(I_t(M)) \geq d - t + 2$ for $1 \leq t \leq d$. Set $I = I_d(M)$. The ideal I is generated by $d+1$ forms of degree d, and we have a presentation of the Rees algebra of the form

$$R_A(I) = k[X_1, \dots, X_n, Y_1, \dots, Y_{d+1}]/(\phi_1, \dots, \phi_d),$$

with $\deg(Y_j) = (d, 1)$, $\deg(\phi_i) = (d+1, 1)$, such that $\phi_1, \dots, \phi_d$ is a regular sequence. Therefore $R_A(I)$ is Gorenstein, and so $a^2(G_A(I)) = -2$. Since $d \leq n-1$, we have that $-a^2(G) \leq \frac{n}{d}+1$. Therefore $k[(I^e)_c]$ is Cohen-Macaulay for any $c \geq de + 1$.

Next, we are going to use the bounds of the a-invariants of the families of ideals considered in Subsection 5.3.1 to study the Cohen-Macaulayness of the diagonals of their Rees algebras. First we recall a well-known result about the vanishing of the graded pieces of the local cohomology modules.

Lemma 5.2.19 *Let A be a standard noetherian graded k-algebra with graded maximal ideal $\mathfrak{m}$. Let L be a finitely generated graded A-module with $d = \dim L > 0$. Then*

$$H^d_{\mathfrak{m}}(L)_j \neq 0, \ \forall j \leq a(L).$$

Proof. Since $d > 0$, we can assume $H^0_{\mathfrak{m}}(L) = 0$ because otherwise by considering $\overline{L} = L/H^0_{\mathfrak{m}}(L)$ we have $H^0_{\mathfrak{m}}(\overline{L}) = 0$ and $H^d_{\mathfrak{m}}(\overline{L}) = H^d_{\mathfrak{m}}(L)$. We may also assume that the field k is infinite. Then there exists $x \in A_1$ such that $x \notin z_A(L)$, and the exact sequence

$$0 \to L(-1) \xrightarrow{\cdot x} L \to L/xL \to 0$$

induces the graded exact sequence of local cohomology modules

$$H^{d-1}_{\mathfrak{m}}(L/xL) \to H^d_{\mathfrak{m}}(L)(-1) \to H^d_{\mathfrak{m}}(L) \to 0.$$

From this exact sequence, we have that $H^d_{\mathfrak{m}}(L)_s = 0$ implies $H^d_{\mathfrak{m}}(L)_j = 0$ for $j \geq s$, so we are done. □

Proposition 5.2.20 *Let I be an equimultiple ideal in A generated in degree d whose Rees algebra is Cohen-Macaulay. For any $c \geq de + 1$, $k[(I^e)_c]$ is Cohen-Macaulay if and only if $c > d(e-1) + a(A/I)$.*

Proof. We have proved in Proposition 5.2.8 that $a^1(R^\varphi) = a(A/I) - d$. Therefore, $k[(I^e)_c]$ is Cohen-Macaulay for any $c > d(e-1) + a(A/I)$ by Proposition 5.2.15.

On the other hand, since $a_{n-h}(I^e/I^{e+1}) = de + a(A/I)$ by Proposition 5.2.8, we have $H^{n-h}_{\mathfrak{m}}(I^e/I^{e+1})_s \neq 0$ for all $s \leq de + a(A/I)$ according to Lemma 5.2.19. By considering the short exact sequences

$$0 \to I^{e+1} \to I^e \to I^e/I^{e+1} \to 0,$$

and by induction on e, we get $H^{n-h+1}_{\mathfrak{m}}(I^e)_s \neq 0$ for all $s \leq d(e-1) + a(A/I)$. Now, if $k[(I^e)_c]$ is Cohen-Macaulay then $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$ for $i < n$ and $s > 0$ by Proposition 3.4.1. In particular, it holds $H^{n-h+1}_{\mathfrak{m}}(I^e)_c = 0$, and so $c > d(e-1) + a(A/I)$. This proves the converse. □

Proposition 5.2.21 *Let I be a strongly Cohen-Macaulay ideal such that $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Assume that I is minimally generated by r forms of degree $d = d_1 \geq \ldots \geq d_r$, and let $h = \mathrm{ht}(I)$. For $c > d(e-1) + d_1 + \ldots + d_h - n$, $k[(I^e)_c]$ is Cohen-Macaulay.*

Proof. According to Corollary 3.4.4, for a given $c \geq de + 1$ we have that $k[(I^e)_c]$ is Cohen-Macaulay if and only if $H^i_{\mathfrak{m}}(I^{es})_{cs} = 0$, for $i < n$, $s > 0$, and $H^i_{\mathfrak{m}}(I^{es-h+1})_{cs-n} = 0$, for $1 < i \leq n$, $s > 0$.

From Proposition 5.2.13, note that $a_*(I^e) \leq (e-1)d + d_1 + \ldots + d_h - n$. Therefore, to get the vanishing of the cohomology modules it suffices to see that $cs > (es-1)d + d_1 + \ldots + d_h - n$ and $cs - n > (es-h)d + d_1 + \ldots + d_h - n$ for any $s \geq 1$. The first condition is equivalent to $(c - de)s > d_1 + \ldots + d_h - d - n$ for $s \geq 1$, that is, $c - de > d_1 + \ldots + d_h - d - n$. The second one is equivalent to $(c - de)s > d_1 + \ldots + d_h - dh$ for $s \geq 1$, that is, $c - de > d_1 + \ldots + d_h - dh$; and this always holds because $d_1 + \ldots + d_h - dh \leq 0$.□

To finish, let us consider the case that the Rees algebra has rational singularities. Then all the diagonals $k[(I^e)_c]$ have rational singularities by [Bou], so in particular the Rees algebra and its diagonals are Cohen-Macaulay. By Proposition 3.4.1, we get immediately

Proposition 5.2.22 *Let I be a homogeneous ideal in $A = k[X_1, \ldots, X_n]$ generated by forms of degree $\leq d$, where k is a field with $\mathrm{char} k = 0$. If $R_A(I)$ has rational singularities, then $a_*(I^e) \leq de$ for all e.*

Example 5.2.23 Let $\mathbf{X} = (X_{ij})$ be an $m \times n$ matrix of indeterminates, with $1 \leq i \leq m$, $1 \leq j \leq n$ and $m \leq n$. Let $A = k[\mathbf{X}]$ be the polynomial ring with variables the entries in $\mathbf{X}$, where k is a field with char $k = 0$ and $k = \overline{k}$. Let $I = I_d(\mathbf{X})$ be the ideal generated by the d-minors of $\mathbf{X}$, $1 < d < m$.

By [Bru, Theorem 3.2], $R_A(I)$ has rational singularities. So we have $a_*(I^e) \leq de$, for all e. This also holds for ideals generated by minors of symmetric generic matrices and ideals generated by pfaffians of alternating generic matrices by [Bru, Remark 3.4]. The defining ideals of the varieties considered by A. Bertram in [Ber] are:

(a) $I_2(\mathbf{X})$, with $\mathbf{X}$ a generic matrix, for the defining ideals of the products $\mathbb{P}^r_k \times \mathbb{P}^s_k$.

(b) $I_2(\mathbf{X})$, with $\mathbf{X}$ a generic symmetric matrix, for the defining ideals of quadratic Veronese embeddings of $\mathbb{P}^r_k$.

(c) $Pf_2(\mathbf{X})$, the ideal generated by the pfaffians of a generic alternating matrix, for the defining ideal of the Plücker embedding of $G(2, r)$.

In these cases we get $a_*(I^e) \leq 2e$, for all e. A. Bertram gets the following bounds for $M = \max_{i \geq 2}\{a_i(I^e)\}$:

(a) $M \leq 2e - 4$.

(b) $M \leq 2e - 3$.

(c) $M \leq 2e - 6$.

5.3 Bayer–Stillman Theorem

Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring in $n + r$ variables with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$, so that S is a standard bigraded k-algebra. For a homogeneous ideal I in S, we have already defined in Section 5.1 the bigraded regularity $\mathbf{reg}(I)$ of I. The aim of this section is to give a new description of the regularity of I analogous to the one given by D. Bayer and M. Stillman [BaSt] in the graded case. To this end, we are going to prove several technical lemmas which are the bigraded version of the ones in [BaSt]. To state them, we need to introduce the *saturations* of I

with respect to the variables $\underline{X}$ and $\underline{Y}$ (I^{*1} and I^{*2}), and the generic forms for I with respect to $\underline{X}$ and $\underline{Y}$. Furthermore, Theorem 5.1.1 and Theorem 5.1.2 will be needed to prove some of these lemmas. We will include all the proofs for the completeness.

For a given finitely generated bigraded S-module L, we say that L is $(m,\cdot)$-regular if $\operatorname{reg}_1 L \le m$. Similarly, L is $(\cdot,m)$-regular if $\operatorname{reg}_2 L \le m$. Denote by $\mathcal{M}_1$ and $\mathcal{M}_2$ the ideals of S generated by $\mathfrak{m}_1 = (X_1, ..., X_n)$ and $\mathfrak{m}_2 = (Y_1, \ldots, Y_r)$ respectively. Then we have

Proposition 5.3.1 *Let L be a finitely generated bigraded S-module. Then the following are equivalent:*

(i) L is $(m,\cdot)$-regular.

(ii) $H^i_{\mathcal{M}_1}(L)_{(p,q)} = 0$ for all i, q, $p \ge m - i + 1$.

Proof. By Theorem 5.1.2, L is $(m,\cdot)$-regular if and only if $\operatorname{reg}(L^q) \le m$ for any q, that is, $H^i_{\mathfrak{m}_1}(L^q)_p = 0$ for all i, q and $p \ge m - i + 1$. Now the result follows from Proposition 2.1.18. □

Given a homogeneous ideal I, let us define I^{*1} and I^{*2} to be the homogeneous ideals

$$I^{*1} = \{f \in S : \exists\, k \ s.t. \ {\mathcal{M}_1}^k f \subset I\},$$
$$I^{*2} = \{f \in S : \exists\, k \ s.t. \ {\mathcal{M}_2}^k f \subset I\}.$$

Lemma 5.3.2 *Assume k infinite, and let $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\}$. Then,*

(i) If $s = 0$, $I^{*1} = S$.

(ii) If $s > 0$, there exists $h \in S_{(1,0)}$ such that $(I^{*1} : h) = I^{*1}$.

Proof. First note that

$$H^0_{\mathcal{M}_1}(S/I) = \{\overline{f} \in S/I \mid \exists\, k \ s.t. \ {\mathcal{M}_1}^k\, \overline{f} = 0\} =$$
$$= \{f \in S \mid \exists\, k \ s.t. \ {\mathcal{M}_1}^k\, f \subset I\}/I = I^{*1}/I.$$

If $s = 0$, we have $H^i_{\mathfrak{m}_1}((S/I)^q)_p = H^i_{\mathcal{M}_1}(S/I)_{(p,q)} = 0$ for any p, q, $i > 0$, so $(S/I)^q$ has dimension 0 as graded S_1-module, and then $H^0_{\mathfrak{m}_1}((S/I)^q) = (S/I)^q$. Therefore, $H^0_{\mathcal{M}_1}(S/I) = S/I$, and we get $I^{*1} = S$.

If $s > 0$, note that $I^{*1} \neq S$ because otherwise $H^0_{\mathcal{M}_1}(S/I) = S/I$ and then we would have $H^i_{\mathcal{M}_1}(S/I) = 0$ for all $i > 0$. Now consider

$\overline{S} = S/I^{*1}$, and denote by $T = \overline{S}^0 = S_1/(I^{*1})^0$, $\overline{\mathfrak{m}_1} = \mathfrak{m}_1 T$. We have that $H^0_{\mathcal{M}_1}(\overline{S}) = (I^{*1})^{*1}/I^{*1} = 0$, and so $H^0_{\mathfrak{m}_1}(\overline{S}^q) = 0$ for all q. Therefore, $\overline{\mathfrak{m}_1} \notin \bigcup_q \operatorname{Ass}_T(\overline{S}^q)$. On the other hand, according to [HIO, Proposition 23.6] we have that $\bigcup_q \operatorname{Ass}_T(\overline{S}^q)$ is a finite set. Since k is infinite, we can find $h \in S_1$ of degree 1 such that $\overline{h} \notin z_T(\overline{S}^q)$ for all q. Then $h \in S_{(1,0)}$ satisfies that $h \notin z_{S_1}(S/I^{*1})$. Therefore, $(I^{*1} : h) = I^{*1}$. □

From now on in this section we will assume that the field k is infinite. Let $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\}$. If $s > 0$, $h \in S_{(1,0)}$ is generic for I if $h \notin z_{S_1}(S/I^{*1})$, that is, $(I^{*1} : h) = I^{*1}$. If $s = 0$, we say that any $h \in S_{(1,0)}$ is generic for I. Given $j > 0$, we define $U^1_j(I)$ to be the set

$$\{(h_1, \dots, h_j) \in S^j_{(1,0)} \mid h_i \text{ is generic for } (I, h_1, \dots, h_{i-1}), 1 \le i \le j\}.$$

Lemma 5.3.3 *Let $h \in S_{(1,0)}$. The following are equivalent:*

(i) $(I : h)_{(p,q)} = I_{(p,q)}$ for $p \geq m$.

(ii) h is generic for I and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for $p \geq m$.

Proof. First, let us notice that for $p > a^1_*(S/I)$ we have

$$(I^{*1}/I)_{(p,q)} = H^0_{\mathcal{M}_1}(S/I)_{(p,q)} = H^0_{\mathfrak{m}_1}((S/I)^q)_p = 0$$

by Theorem 5.1.1. Therefore, for p large enough it holds $(I^{*1})_{(p,q)} = I_{(p,q)}$, $\forall q$.

Now let us assume that (i) holds, and let $f \in I^{*1}$ be a homogeneous element not in I such that $\deg_1 f$ is maximum. Then $hf \in I^{*1}$ has $\deg_1(hf) > \deg_1 f$, so $hf \in I$. Hence $\deg_1 f < m$, and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for any $p \geq m$. To show that h is generic for I, we may assume that $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\} > 0$ (if not, any element in $S_{(1,0)}$ is generic for I). Then we want to prove $h \notin z_{S_1}(S/I^{*1})$. Otherwise, there exists a homogeneous element $f \notin I^{*1}$ such that $hf \in I^{*1}$. By Lemma 5.3.2, there exists $g \in S_{(1,0)}$ such that $g \notin z_{S_1}(S/I^{*1})$. Then, for any $s \geq 0$ we have $g^s f \notin I^{*1}$ and $hg^s f \in I^{*1}$, so $(I^{*1} : h)_{(p,q)} \neq (I^{*1})_{(p,q)}$ for all $p \gg 0$. But note that for any $p \geq m$,

$$(I^{*1} : h)_{(p,q)} = (I : h)_{(p,q)} = I_{(p,q)} = (I^{*1})_{(p,q)},$$

and we get a contradiction.

Now assuming (ii), we have that for $p \geq m$

$$(I : h)_{(p,q)} = (I^{*1} : h)_{(p,q)} = (I^{*1})_{(p,q)} = I_{(p,q)}. \ \square$$

Lemma 5.3.4 *Let $h \in S_{(1,0)}$ be generic for I. The following are equivalent:*

(i) I is $(m,\cdot)$-regular.

(ii) (I,h) is $(m,\cdot)$-regular and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \geq m$.

Proof. If I is $(m,\cdot)$-regular, then S/I is $(m-1,\cdot)$-regular. Then for any i, q and $p \geq m-i$, we have $H^i_{\mathcal{M}_1}(S/I)_{(p,q)} = 0$ by Proposition 5.3.1. In particular, for $p \geq m$

$$0 = H^0_{\mathcal{M}_1}(S/I)_{(p,q)} = (I^{*1}/I)_{(p,q)},$$

and so $(I^{*1})_{(p,q)} = I_{(p,q)}$ for $p \geq m$.

Let us consider $Q := (I:h)/I$. In the assumptions of (i) or (ii), observe that for $p \geq m$, $(I:h)_{(p,q)} = (I^{*1}:h)_{(p,q)} = (I^{*1})_{(p,q)} = I_{(p,q)}$, so $Q_{(p,q)} = 0$ for $p \geq m$. Therefore $H^i_{\mathcal{M}_1}(Q) = 0$ for all $i > 0$ and $H^0_{\mathcal{M}_1}(Q) = Q$. From the bigraded exact sequence

$$0 \to I \to (I:h) \to Q \to 0,$$

the long exact sequence of local cohomology gives

$$H^i_{\mathcal{M}_1}(I)_{(p,q)} \cong H^i_{\mathcal{M}_1}((I:h))_{(p,q)}\ , \forall i, p \geq m-i+1.$$

Assume first (i). We have already shown that $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \geq m$. Since I is $(m,\cdot)$-regular, we have $H^i_{\mathcal{M}_1}((I:h))_{(p,q)} = 0$ for all i, $p \geq m-i+1$. By considering the exact sequence

$$0 \to I \cap (h) = (I:h)h = (I:h)(-1,0) \to I \oplus (h) \to (I,h) \to 0,$$

we get $H^i_{\mathcal{M}_1}((I,h))_{(p,q)} = 0$ for all i, $p \geq m-i+1$, so (I,h) is $(m,\cdot)$-regular.

Now by assuming (ii), from the previous exact sequence we obtain that $H^i_{\mathcal{M}_1}((I:h))_{(p-1,q)} \cong H^i_{\mathcal{M}_1}(I)_{(p,q)}$ for $p \geq m-i+2$. For $p \geq m-i+1$, we then have that $H^i_{\mathcal{M}_1}(I)_{(p,q)} \cong H^i_{\mathcal{M}_1}((I:h))_{(p,q)} \cong H^i_{\mathcal{M}_1}(I)_{(p+1,q)}$. Therefore $H^i_{\mathcal{M}_1}(I)_{(p,q)} = 0$ for $p \geq m-i+1$, so I is $(m,\cdot)$-regular. □

Lemma 5.3.5 *Let I be an ideal generated by forms in* $\deg_1 \leq m$ *and $h \in S_{(1,0)}$. If (I,h) is $(m,\cdot)$-regular, then $(I:h)$ is generated by forms in* $\deg_1 \leq m$.

Proof. Let $f_1, \ldots, f_u, hf_{u+1}, \ldots, hf_v$ be a minimal system of homogeneous generators for I, where $f_1, \ldots, f_u, h$ is a minimal system of generators for (I,h). If $f \in (I:h)$, then

$$hf = g_1 f_1 + \ldots + g_u f_u + h(g_{u+1} f_{u+1} + \ldots + g_v f_v),$$

for $g_1, \dots, g_v \in S$. Thus

$$(f - g_{u+1}f_{u+1} - \dots - g_v f_v)h - g_1 f_1 - \dots - g_u f_u = 0.$$

The first map in the bigraded minimal free resolution of (I, h) is

$$\begin{array}{ccc} Se \oplus Se_1 \oplus \dots \oplus Se_u & \longrightarrow & (I, h) \\ e & \longmapsto & h \\ e_j & \longmapsto & f_j \end{array}$$

and we have that

$$(f - g_{u+1}f_{u+1} - \dots - g_v f_v)e - g_1 e_1 - \dots - g_u e_u$$

is a first syzygy of (I, h). Conversely, if $le + l_1 e_1 + \dots + l_u e_u$ is a first syzygy of (I, h) then $lh + l_1 f_1 + \dots + l_u f_u = 0$, so $lh \in (f_1, \dots, f_u) \subset I$, and $l \in (I : h)$. Because (I, h) is $(m, \cdot)$-regular, each first syzygy of (I, h) can be expressed in terms of syzygies of (I, h) in $\deg_1 \leq m + 1$. Then

$$(f - g_{u+1}f_{u+1} - \dots - g_v f_v)e - g_1 e_1 - \dots - g_u e_u = \sum_i \lambda_i (\gamma_i e + \gamma_{i1} e_1 + \dots + \gamma_{iu} e_u),$$

with $\deg_1(\gamma_i e + \gamma_{i1} e_1 + \dots + \gamma_{iu} e_u) \leq m + 1$. So

$$f = g_{u+1}f_{u+1} + \dots + g_v f_v + \sum_i \lambda_i \gamma_i,$$

with $\gamma_i \in (I : h)$, $\deg_1 \gamma_i \leq m$. Since $f_{u+1}, \dots, f_v$ also belong to $(I : h)$ and have $\deg_1 \leq m$, we finally obtain that $(I : h)$ can be generated by elements in $\deg_1 \leq m$. □

We are now ready to prove a bigraded version of the Bayer-Stillman's Theorem characterizing the regularity of a homogeneous ideal in terms of generic forms.

Theorem 5.3.6 *Let I be a homogeneous ideal in S generated by forms in $\deg_1 \leq m$. Then the following are equivalent:*

(i) I is $(m, \cdot)$-regular.

(ii) There exist $h_1, \dots, h_j \in S_{(1,0)}$ for some $j \geq 0$ such that

$$((I, h_1, \dots, h_{i-1}) : h_i)_{(m,q)} = (I, h_1, \dots, h_{i-1})_{(m,q)}, \ \forall q, \ 1 \leq i \leq j.$$

$$(I, h_1, \dots, h_j)_{(m,q)} = S_{(m,q)}, \ \forall q.$$

(iii) Let $s = \max\{i \mid H^i_{\mathcal{M}_1}(S/I) \neq 0\}$*. For all* $(h_1, \ldots, h_s) \in U^1_s(I)$*,* $p \geq m$*,*

$$((I, h_1, \ldots, h_{i-1}) : h_i)_{(p,q)} = (I, h_1, \ldots, h_{i-1})_{(p,q)}, \ \forall q, \ 1 \leq i \leq s.$$

$$(I, h_1, \ldots, h_s)_{(p,q)} = S_{(p,q)}, \ \forall q.$$

Proof. Note that *(iii)* $\Rightarrow$ *(ii)* is obvious. Now we are going to show that *(ii)* $\Rightarrow$ *(i)* by induction on j. If $j = 0$, we have that $I_{(m,q)} = S_{(m,q)}$ for all q, so $I_{(p,q)} = S_{(p,q)}$ for all q and $p \geq m$. Therefore,

$$H^i_{\mathcal{M}_1}(S/I) = \begin{cases} 0 & \text{if } i > 0 \\ S/I & \text{if } i = 0 \end{cases}.$$

In particular, we have that $H^i_{\mathcal{M}_1}(S/I)_{(p,q)} = 0$ for all i, q and $p \geq m - i$, so I is $(m, \cdot)$-regular. If $j > 0$, we have that (I, h_1) is $(m, \cdot)$-regular by the induction hypothesis. Since I is generated by forms in $\deg_1 \leq m$, we have that $(I : h_1)$ is generated by forms in $\deg_1 \leq m$ by Lemma 5.3.5. As $(I : h_1)_{(m,q)} = I_{(m,q)}$, we then conclude $(I : h_1)_{(p,q)} = I_{(p,q)}$ for all $p \geq m$. According to Lemma 5.3.3, we have that h_1 is generic for I and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \geq m$. Then I is $(m, \cdot)$-regular by Lemma 5.3.4.

Now let us prove *(i)* $\Rightarrow$ *(iii)* by induction on s. If $s = 0$, since I is $(m, \cdot)$-regular we have $H^0_{\mathcal{M}_1}(S/I)_{(p,q)} = (I^{*1}/I)_{(p,q)} = 0$ for $p \geq m$, and $I^{*1} = S$ by Lemma 5.3.2. Therefore, $I_{(p,q)} = (I^{*1})_{(p,q)} = S_{(p,q)}$ for $p \geq m$. Assume now $s > 0$. Since I is $(m, \cdot)$-regular and h_1 is generic for I, we get that (I, h_1) is $(m, \cdot)$-regular and $(I^{*1})_{(p,q)} = I_{(p,q)}$ for all $p \geq m$ by Lemma 5.3.4. As $(h_2, \ldots, h_s) \in U^1_{s-1}((I, h_1))$, by the induction assumption it is just enough to show $(I : h_1)_{(p,q)} = I_{(p,q)}$ for $p \geq m$, which is satisfied by Lemma 5.3.3. □

We are going to use this criterion to compute the bigraded regularity of the generic initial ideal of a homogeneous ideal I in S. Let $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ be the polynomial ring over an infinite field k with the bigrading given by $\deg(X_i) = (1, 0)$, $\deg(Y_j) = (0, 1)$. Let $<$ be an order on the monomials of S. Let us denote by $\mathcal{G} = \mathcal{G}_1 \times \mathcal{G}_2$, with $\mathcal{G}_1 = GL(n, k)$, $\mathcal{G}_2 = GL(r, k)$. Given an element $g = (f, h) \in \mathcal{G}$, where $f = (f_{ij})_{1 \leq i,j \leq n}$ and $h = (h_{ij})_{1 \leq i,j \leq r}$, g acts on S by acting on the variables in the following way

$$X_j \longmapsto \sum_{i=1}^{n} f_{ij} X_i \quad , \quad Y_j \longmapsto \sum_{i=1}^{r} h_{ij} Y_i.$$

We will denote by $\mathcal{B} = \mathcal{B}_1 \times \mathcal{B}_2$, where $\mathcal{B}_1, \mathcal{B}_2$ are the Borel subgroups of $\mathcal{G}_1, \mathcal{G}_2$ consisting of upper triangular matrices, and by $\mathcal{B}' = \mathcal{B}'_1 \times \mathcal{B}'_2$, where $\mathcal{B}'_1, \mathcal{B}'_2$ are the Borel subgroups of $\mathcal{G}_1, \mathcal{G}_2$ consisting of lower triangular matrices. We will denote by $\mathcal{U} = \mathcal{U}_1 \times \mathcal{U}_2$, where $\mathcal{U}_1, \mathcal{U}_2$ are the unipotent matrices. By bigrading the proof of [Eis, Theorem 15.18], we get

Theorem 5.3.7 *(Galligo, Bayer–Stillman) Let $I \subset S$ be a homogeneous ideal. There exists a non-empty Zariski open $U = \mathcal{B}'U \subset \mathcal{G}$, $U \cap \mathcal{U} \neq Id$, and a monomial ideal J such that*

$$\mathrm{in}(gI) = J, \; \forall g \in U.$$

We call J the (bi)graded generic initial ideal of I, written $J = \mathbf{gin}(I)$. Given a homogeneous ideal $I \subset S$, we say that I is Borel-fix if $gI = I$ for any $g \in \mathcal{B}$. It was proved that the generic initial ideal of a graded ideal is Borel fix. By bigrading the proof of [Eis, Theorem 15.20], we easily obtain that the generic initial ideal is Borel-fix.

Theorem 5.3.8 *Let $I \subset S$ be a homogeneous ideal. For any $g \in \mathcal{B}$,*

$$g(\mathbf{gin}(I)) = \mathbf{gin}(I).$$

Let $p \geq 0$. Given $s, t \in \mathbb{N}$, we define $s <_p t \iff \binom{t}{s} \not\equiv 0 \; (mod\, p)$. We also can give an equivalent characterization of the Borel-fix bihomogeneous ideals analogous to the one in the graded case [Eis, Theorem 15.23]. Namely,

Theorem 5.3.9 *Let I be a homogeneous ideal of S. Let $p = char k \geq 0$. Then*

(i) I is diagonal-fix iff I is monomial.

(ii) I is Borel-fix iff I is generated by monomials m such that satisfy the following conditions

- *If m is divisible by X_j^t but by no higher power of X_j, then $(X_i/X_j)^s m \in I$, $\forall i < j$, $s <_p t$.*
- *If m is divisible by Y_j^t but by no higher power of Y_j, then $(Y_i/Y_j)^s m \in I$, $\forall i < j$, $s <_p t$.*

For a homogeneous ideal I, let us denote by $\delta_1(I)$ the maximum first component of the degree in a minimal system of generators of I. In a similar way, we may define $\delta_2(I)$. Then we have

Proposition 5.3.10 *Let $I \subset S$ be a Borel-fix ideal. If* $\operatorname{char} k = 0$, *then*

$$\operatorname{reg}_1(I) = \delta_1(I),$$

$$\operatorname{reg}_2(I) = \delta_2(I).$$

Proof. Set $m = \delta_1(I)$. From the definition of the regularity it is clear that $\operatorname{reg}_1(I) \geq m$. According to Theorem 5.3.6, to prove the equality it is enough to show that for $p \geq m$, $i \leq n$, we have

$$((I, X_n, \dots, X_{i+1}) : X_i)_{(p,q)} = (I, X_n, \dots, X_{i+1})_{(p,q)}.$$

Let $f \in ((I, X_n, \dots, X_{i+1}) : X_i)$ be a monomial with $\deg_1 f \geq m$. If there exists $k \geq i+1$ such that $X_k | f$, we immediately have $f \in (I, X_n, \dots, X_{i+1})$. Otherwise,

$$X_i f = X^\alpha Y^\beta (X^A Y^B),$$

where $X^A Y^B \in I$, $\deg_1 X^A \leq m$, $\deg_1 X^\alpha \geq 1$. If $X_i | X^\alpha$, we then easily get $f \in I$. If not, by taking $k \leq i$ such that $X_k | X^\alpha$, we can write

$$f = \frac{X^\alpha}{X_k} Y^\beta \left(\frac{X_k}{X_i} X^A Y^B\right).$$

Since I is Borel-fix, we have that $\frac{X_k}{X_i} X^A Y^B \in I$ by Theorem 5.3.9, and so $f \in I \subset (I, X_n, \dots, X_{i+1})$. □

This result has been proved independently by A. Aramova et al. [ACD] by other methods.

In the graded case, it was proved by D. Bayer and M. Stillman [BaSt] that there exists an order in $A = k[X_1, \dots, X_n]$ (the reverse lexicographic order) with the property that $\operatorname{reg}(I) = \operatorname{reg}(\operatorname{gin} I)$ for any homogeneous ideal I in A. We may wonder if the analogous bigraded result also holds, that is, if there exists an order in the polynomial ring S such that $\mathbf{reg}(I) = \mathbf{reg}(\mathbf{gin}\, I)$ for any homogeneous ideal I. We show that this is not true by giving a homogeneous ideal in S such that $\mathbf{reg}(I) \neq \mathbf{reg}(\mathbf{gin}\, I)$ for any order on S.

Example 5.3.11 Let us consider the polynomial ring $S = k[X_1, X_2, Y_1, Y_2]$, with $\deg(X_1) = \deg(X_2) = (1, 0)$, $\deg(Y_1) = \deg(Y_2) = (0, 1)$. Let $>$ be a term order in S, that is, an order satisfying

(i) $X_1 > X_2$, $Y_1 > Y_2$.

(ii) For monomials m, m_1, m_2 in S, if $m_1 > m_2$ then $mm_1 > mm_2$.

Let I be the homogeneous ideal in S generated by the forms $f_1 = X_1Y_1$ and $f_2 = X_1Y_2+X_2Y_1$ in degree $(1,1)$. Note that f_1, f_2 is a regular sequence, so the Koszul complex of these forms provides the minimal bigraded free resolution of I:

$$0 \to S(-2,-2) \to S(-1,-1)^2 \to I \to 0.$$

Then the regularity of I is $\mathbf{reg}(I) = (1,1)$. Note that $X_1Y_1 > X_1Y_2$, $X_2Y_1 > X_2Y_2$. Therefore, if we want to define an order on the monomials of S we only must decide if $X_1Y_2 > X_2Y_1$ or $X_1Y_2 < X_2Y_1$ in degree $(1,1)$. Assume first that $X_1Y_2 > X_2Y_1$. Recall that $g \in GL(2,k) \times GL(2,k)$, with

$$g = (A,B) = \left(\begin{pmatrix} a & c \\ b & d \end{pmatrix}, \begin{pmatrix} \alpha & \gamma \\ \beta & \rho \end{pmatrix} \right),$$

operates in S by means of

$$\begin{array}{rcl} X_1 & \longmapsto & aX_1 + bX_2 \\ X_2 & \longmapsto & cX_1 + dX_2 \\ Y_1 & \longmapsto & \alpha Y_1 + \beta Y_2 \\ Y_2 & \longmapsto & \gamma Y_1 + \rho Y_2 \end{array}$$

Since $\dim_k \mathbf{gin}(I)_{(i,j)} = \dim_k I_{(i,j)}$ for any (i,j), we have that $\mathbf{gin}(I)_{(i,j)} = 0$ for $(i,j) \in \{(0,0),(1,0),(0,1)\}$. In degree $(1,1)$, the forms f_1, f_2 are a k-basis of $I_{(1,1)}$. By computing $g(f_1 \wedge f_2)$, we get

$$g(f_1 \wedge f_2) = a^2(\alpha\rho - \beta\gamma)X_1Y_1 \wedge X_1Y_2 + \dots$$

so $\mathbf{gin}(I)_{(1,1)}$ is the k-vector space generated by X_1Y_1, X_1Y_2. If $\mathbf{gin}(I) = (X_1Y_1, X_1Y_2)$, then $\dim_k \mathbf{gin}(I)_{(1,2)} = 3$ because $X_1Y_1^2$, $X_1Y_1Y_2$, $X_1Y_2^2$ is a k-basis, which is a contradiction because $\dim_k I_{(1,2)} \geq 4$. Therefore, $\mathbf{gin}(I)$ has minimal generators with $\deg_1 \geq 2$ or $\deg_2 \geq 2$, so $\operatorname{reg}_1(\mathbf{gin}\, I) \geq 2$ or $\operatorname{reg}_2(\mathbf{gin}\, I) \geq 2$. In the case $X_1Y_2 < X_2Y_1$, it can be proved that $\mathbf{reg}(\mathbf{gin}\, I) \neq (1,1)$ by similar arguments. Therefore, we get $\mathbf{reg}(I) \neq \mathbf{reg}(\mathbf{gin}\, I)$ for any order in S.

Finally, note that these results can be applied to study the Koszul property of the diagonals of a bigraded standard k-algebra. By using [ERT, Theorem 18], and following the same lines as [ERT, Theorem 2] in the graded case, it can be proved that that for a homogeneous ideal I of S, $(S/I)_\Delta$ has a Gröbner basis of quadrics for $c \gg 0$, $e \gg 0$ (see also [ACD]).