On the diagonals of a Rees algebra

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Chapter 6

Asymptotic behaviour of the powers of an ideal

Let $A = k[X_1, \ldots, X_n]$ be a polynomial ring over a field k, and let I be a homogeneous ideal in A. In this chapter we are concerned with the asymptotic behaviour of the powers of I. We will use the bigraded structure of the Rees algebra to get information about the Hilbert polynomials, the Hilbert series and the graded minimal free resolutions of the powers of I.

In Section 6.1 we show that the Hilbert polynomials of the powers of the ideal I have a uniform behaviour. In particular, the Hilbert polynomials of a finite set of these powers allow to compute the Hilbert polynomials of its Rees algebra and its form ring, without needing an explicit presentation of these algebras. In Section 6.2, similar results are stated for the Hilbert series of the powers of I.

The last section begins by studying the projective dimension of the powers of I. The approach to this question by means of the bigrading of the Rees algebra allows to recover some classical results as the constant asymptotic value for the projective dimension, as well as to determine the powers of the ideal which take the asymptotic value whenever the form ring is Gorenstein. After that, we study the graded minimal free resolutions of the powers of an ideal. In the equigenerated case, it is proved that the shifts are given by linear functions asymptotically and the graded Betti numbers of these resolutions are given by polynomials asymptotically. This result is then applied to guess the resolutions of the powers of some families of ideals from a finite set of these resolutions.

6.1 Hilbert polynomial of the powers of an ideal

First of all, let us recall some standard definitions and notations referred to the Hilbert polynomial (see for instance [BH1]). Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring over a field k, and let M be a finitely generated graded A-module. The numerical function

$$\begin{aligned} H(M,\ ) : \mathbb{Z} &\longrightarrow \mathbb{Z} \\ j &\mapsto \dim_k M_j \end{aligned}$$

is the Hilbert function of M. Denoting by $d = \dim M$, there exists an unique polynomial $P_M(s) \in \mathbb{Q}[s]$, of degree $d-1$, for which $H(M,j) = P_M(j)$ for all $j \gg 0$. We can write

$$P_M(s) = \sum_{k=0}^{d-1} (-1)^{d-1-k} e_{d-1-k} \binom{s+k}{k},$$

with $e_0, \ldots, e_{d-1} \in \mathbb{Z}$. $P_M(s)$ is called the Hilbert polynomial of M.

Our first result shows the uniform behaviour of the Hilbert polynomial of the powers of any homogeneous ideal in a polynomial ring.

Theorem 6.1.1 *Let I be a homogeneous ideal in A. Set $c = a_*^2(R_A(I))$, $h = \mathrm{ht}\,(I)$. Then there are polynomials $e_0(j), \ldots, e_{n-h-1}(j)$ with integer values such that for all $j \geq c+1$*

$$P_{A/I^j}(s) = \sum_{k=0}^{n-h-1} (-1)^{n-h-1-k} e_{n-h-1-k}(j) \binom{s+k}{k}.$$

Furthermore, $\deg e_{n-h-1-k}(j) \leq n-k-1$ *for all k.*

Proof. Assume that I is generated by forms $f_1, \ldots, f_r$ in degrees $d_1 \leq \ldots \leq d_r = d$ respectively. Then the Rees algebra $R = R_A(I)$ of I can be endowed with the bigrading given by $R_{(i,j)} = (I^j)_i$, so that R is a bigraded S-module, for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$. Since R is a domain, it has relevant dimension $n+1$. Then by Proposition 1.5.1 and Proposition 1.5.5 there exists a polynomial $P_R(s,t)$ of degree $n-1$ such that for all (i,j)

$$\dim_k R_{(i,j)} - P_R(i,j) = \sum_q (-1)^q \dim_k H^q_{R_+}(R)_{(i,j)},$$

where R_+ is the ideal generated by the products $X_i f_j t$ for $1 \leq i \leq n$, $1 \leq j \leq r$. By taking in S the homogeneous ideals $\mathcal{M}_1 = (X_1, \ldots, X_n)S$ and $\mathcal{M}_2 = (Y_1, \ldots, Y_r)S$, the Mayer-Vietoris long exact sequence gives then

$$\cdots \to H^q_{\mathcal{M}_1}(R) \oplus H^q_{\mathcal{M}_2}(R) \to H^q_{S_+}(R) \to H^{q+1}_{\mathcal{M}}(R) \to \cdots$$

Notice that for $j > c$ we have $H^q_{\mathcal{M}}(R)_{(i,j)} = 0$ for all i, q. Then, by Proposition 2.1.18 we also get $H^q_{\mathcal{M}_2}(R)_{(i,j)} = 0$ for $j > c$, for all i, q. Furthermore, $H^q_{\mathcal{M}_1}(R)_{(i,j)} = H^q_{\mathfrak{m}}(I^j)_i$ for any $j \geq 0$ by Proposition 2.1.18. Therefore, for any $j > c$ there exists an integer $i_0 = a_*(I^j)$ (depending on j) such that $H^q_{R_+}(R)_{(i,j)} = H^q_{S_+}(R)_{(i,j)} = H^q_{\mathfrak{m}}(I^j)_i = 0$ for all q and $i > i_0$. Hence $P(i,j) = \dim_k R_{(i,j)} = \dim_k (I^j)_i$ for any $j > c$ and $i > i_0$.

Now, by defining $P_j(s) = \binom{n+s-1}{n-1} - P(s,j)$, for $j > c$, $s \gg 0$ we have that $P_j(s) = \dim_k (A/I^j)_s$. Hence $P_j(s)$ is the Hilbert polynomial of A/I^j. Furthermore, we can write

$$\begin{aligned} P_j(s) &= \tbinom{n+s-1}{n-1} - P(s,j) \\ &= \tbinom{n+s-1}{n-1} - \textstyle\sum_{l+m \leq n-1} a_{lm} \binom{s-dj}{l}\binom{j}{m} \\ &= \textstyle\sum_{k=0}^{n-1} b_k(j) \binom{s+k}{k}, \end{aligned}$$

with $b_k(j)$ polynomials in j. Since $\deg P_j(s) = n-h-1$ for any $j > c$, we have $b_k(j) = 0$ for $k \geq n-h$ and $j > c$, so $b_k(j) \equiv 0$ for $k \geq n-h$. Then we may write

$$P_j(s) = \sum_{k=0}^{n-h-1} (-1)^{n-h-1-k} e_{n-h-1-k}(j) \binom{s+k}{k},$$

for $j > c$. Moreover, since $P_R(s,t)$ has total degree $n-1$ and $P_R(s,t) = \binom{n+s-1}{n-1} - P_t(s)$, we easily obtain that $\deg e_{n-h-1-k}(j) \leq n-k-1$. □

Remark 6.1.2 We have seen that $\deg e_{n-h-1-k}(j) \leq n-k-1$ for all k, so in particular the polynomial $e_0(j)$ which gives the multiplicity of A/I^j has degree $\leq h$. By Nagata's formula,

$$e_0(j) = e(A/I^j) = \sum_{\mathfrak{p} \in \operatorname{Assh}(A/I)} \operatorname{length}(A_{\mathfrak{p}}/I^j_{\mathfrak{p}}) e(A/\mathfrak{p}),$$

with $\operatorname{Assh}(A/I) = \{\mathfrak{p} \in \operatorname{Ass}(A/I) \mid \dim A/\mathfrak{p} = \dim A/I\}$. Note that for all those $\mathfrak{p}$, we have that $\dim A_{\mathfrak{p}} = h$ and then

$$\operatorname{length}(A_{\mathfrak{p}}/I^j_{\mathfrak{p}}) = e(IA_{\mathfrak{p}}, A_{\mathfrak{p}}) \binom{h+j}{j} + \text{polynomial in } j \text{ of degree lower than } h.$$

Therefore $e_0(j)$ has degree h, so let us write

$$e_0(j) = \lambda_h \binom{j}{h} + \text{polynomial in } j \text{ of degree lower than } h.$$

We can give an upper bound for the leading coefficient λ_h. According to [HS, Corollary 3.8], we have

$$e_0(j) \leq \binom{\operatorname{reg}(I^j) + h - 1}{h}.$$

Assume that I is generated by forms in degree $\leq d$. Then there exists a positive integer α such that $\operatorname{reg}(I^j) \leq dj + \alpha$ by Theorem 3.4.6, and so $\lambda_h \leq d^h$. In Proposition 6.1.4 we will show that λ_h and, more generally, the leading coefficients of the polynomials $e_{n-h-1-k}(j)$ play an important role in the mixed multiplicities of the Rees algebra and the form ring.

Now let us consider a homogeneous ideal I generated by forms in degree d. Let us take the Hilbert polynomial $P_R(s,t)$ of its Rees algebra with the usual bigrading, and let us write

$$P_R(s + dt, t) = \sum_{k+m \leq n-1} a_{km} \binom{s}{k} \binom{t}{m}.$$

Following [HHRT], we call $e_i(R) = a_{i,n-1-i}$ the mixed multiplicity of R of type i for $i = 0, \ldots, n-1$. According to Proposition 1.5.1 we have $e_i(R) \geq 0$, and then $e(R) = \sum_{i=0}^{n-1} e_i(R)$ by [HHRT, Theorem 4.3]. Next we are going to study the multiplicity of the Rees ring and to relate it to the multiplicity of the form ring. First, we need to compute the relevant dimension of the form ring.

Lemma 6.1.3 *Let I be a homogeneous ideal in A generated by forms in degree d. Then the relevant dimension of G is n if and only if I is not $\mathfrak{m}$-primary.*

Proof. If I is $\mathfrak{m}$-primary, then $G_+ \subset P$ for all $P \in \operatorname{Proj}^2(G)$ because $G_{(1,0)}$ is nilpotent, and so $\operatorname{rel.dim} G = 1 < \dim G = n$. If I is not $\mathfrak{m}$-primary, for $k = 0, \ldots, n-h-1$ let us write

$$e_{n-h-1-k}(j) = \lambda_{n-k-1} \binom{j}{n-k-1} + \text{polynomial in } j \text{ of lower degree.}$$

Then for $s \gg 0$, $t \gg 0$, we have

$$P_G(s+dt,t) = P_{A/I^{t+1}}(s+dt) - P_{A/I^t}(s+dt)$$

$$= \sum_{k=0}^{n-h-1}(-1)^{n-h-1-k}(e_{n-h-1-k}(t+1) - e_{n-h-1-k}(t))\binom{s+dt+k}{k}$$

$$= \sum_{k=0}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}(\binom{t+1}{n-k-1} - \binom{t}{n-k-1})\binom{s+dt+k}{k} +$$

$$+ \text{ polynomial in } s,t \text{ of lower total degree}$$

$$= \sum_{k=0}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}\binom{t}{n-k-2}\frac{(s+dt)^k}{k!} +$$

$$+ \text{ polynomial in } s,t \text{ of lower total degree}$$

$$= \sum_{i=0}^{n-h-1}\left[\sum_{k=i}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}d^{k-i}\binom{n-2-i}{k-i}\right]\binom{s}{i}\binom{t}{n-2-i} +$$

$$+ \text{ polynomial in } s,t \text{ of lower total degree}$$

In particular, λ_h is the coefficient of $\binom{s}{n-h-1}\binom{t}{h-1}$ which is not zero by Remark 6.1.2. So the total degree of the Hilbert polynomial of the form ring is $n-2$, and then the relevant dimension of G is n by Proposition 1.5.1. □

If I is a homogeneous ideal generated by forms in degree d which is not $\mathfrak{m}$-primary, let us consider the Hilbert polynomial of its form ring

$$P_G(s+dt,t) = \sum_{k+m\leq n-2} b_{km}\binom{s}{k}\binom{t}{m}.$$

We call $e_i(G) = b_{i,n-2-i} \geq 0$ the mixed multiplicity of G of type i for $i = 0,\ldots,n-2$. Then $e(G) = \sum_{i=0}^{n-2} e_i(G)$ again by [HHRT, Theorem 4.3].

Now we can give the mixed multiplicities of the Rees algebra and the form ring of an equigenerated ideal by means of the leading coefficients of the polynomials $e_{n-h-1-k}(j)$ given by Theorem 6.1.1, and to relate the mixed multiplicities of both rings.

Proposition 6.1.4 *Let I be a homogeneous ideal generated in degree d which is not $\mathfrak{m}$-primary. Set $h = \operatorname{ht}(I)$, $l = l(I)$. For each k, let us write*

$$e_{n-h-1-k}(j) = \lambda_{n-k-1}\binom{j}{n-k-1} + \text{polynomial in } j \text{ of lower degree.}$$

Then

(i) $e_i(G) = 0$ if $i \geq n-h$ or $i \leq n-l-2$. For each $n-l-2 < i < n-h$, we have

$$e_i(G) = \sum_{k=i}^{n-h-1}(-1)^{n-h-1-k}\lambda_{n-k-1}d^{k-i}\binom{n-2-i}{k-i}.$$

(ii) $e_i(R) = 0$ *if* $i \leq n-l-1$. *For each* $i > n-l-1$, *we have*

$$e_i(R) = \begin{cases} d^{n-1-i} & \text{if } i \geq n-h \\ d^{n-1-i} - \sum_{k=i}^{n-h-1} (-1)^{n-h-1-k} \lambda_{n-k-1} d^{k-i} \binom{n-1-i}{k-i} & \text{otherwise} \end{cases}$$

(iii) $e_i(G) = de_{i+1}(R) - e_i(R)$, *for* $i = 0, \ldots, n-2$. *In particular, we have* $e_i(R) \leq de_{i+1}(R)$, *for* $i = 0, \ldots, n-2$. *Furthermore,*

$$e(G) = \begin{cases} (d-1)e(R) + 1 & \text{if } l \leq n-1 \\ (d-1)e(R) + 1 - de_0(R) & \text{if } l = n \end{cases}$$

Proof. Let us fix $j > a_*^2(G)$. Then we have that for $s \gg 0$,

$$\dim_k \left(\frac{I^j}{I^{j+1}}\right)_{s+dj} = P_G(s+dj, j) = \sum_{k+m \leq n-2} b_{km} \binom{s}{k} \binom{j}{m},$$

so $P_G(s+dj, j)$ is the Hilbert polynomial of the A/I-module I^j/I^{j+1} for large j. Hence $b_{km} = 0$ for any $k \geq n-h$, so in particular $e_i(G) = 0$ for $i \geq n-h$.

Let us fix now $i > a_*^1(G^\varphi)$. Then we have that $H^q_{\mathcal{M}}(G)_{(i+dj,j)} = H^q_{\mathcal{M}_1}(G)_{(i+dj,j)} = 0$ for all q and j by Remark 5.2.2. From the Mayer-Vietoris long exact sequence, we have that for $t \gg 0$,

$$\dim_k \left(\frac{I^t}{I^{t+1}}\right)_{i+dt} = P_G(i+dt, t) = \sum_{k+m \leq n-2} b_{km} \binom{i}{k} \binom{t}{m}.$$

Therefore, we have that $P_G(i+dt, t)$ is the Hilbert polynomial of the $F_{\mathfrak{m}}(I) = k[I_d]$-module $E_i = \bigoplus_{j \geq 0} (I^j/I^{j+1})_{i+dj}$. Hence $b_{km} = 0$ for $m \geq l$. Then the first part of (i) is proved, and for the rest it suffices to notice that for $s, t \gg 0$, $P_G(s+dt, t) = P_{A/I^{t+1}}(s+dt) - P_{A/I^t}(s+dt)$.

To get (ii) and (iii), it is just enough to take into account that for $s, t \gg 0$ we have

$$P_R(s+dt, t) = P_A(s+dt) - P_{A/I^t}(s+dt),$$

$$P_G(s+dt, t) = P_R(s+dt, t) - P_R(s+dt, t+1). \quad \square$$

Remark 6.1.5 Let $(A, \mathfrak{m}, k)$ be a local ring, and $I \neq A$ an ideal. Set $n = \dim A$, $l = l(I)$, $h = \mathrm{ht}(I)$. Let us denote by $\overline{G} = G_{\mathfrak{m}}(G_I(A))$ bigraded by means of

$$\overline{G}_{(i,j)} = \frac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}}.$$

Since $\overline{G}$ is a standard bigraded k-algebra, we may consider its Hilbert polynomial

$$P_{\overline{G}}(s,t) = \sum_{k+m \leq n-2} c_{km} \binom{s}{k} \binom{t}{m},$$

and let us denote by $c_i(\overline{G}) = c_{i-1,n-1-i}$ for $1 \leq i \leq n-1$. R. Achilles and M. Manaresi [AM] show that $c_i(\overline{G}) = 0$ if $i > \dim A/I$ or $i < n-l$.

This definition and the results proved in [AM] can be extended to the graded case, and then we get that $c_i(\overline{G}) = 0$ if $i > n-h$ or $i < n-l$. For a homogenous ideal I in $A = k[X_1, \ldots, X_n]$ generated by forms of degree d, note that

$$\overline{G}_{(i,j)} = \left(\frac{I^j}{I^{j+1}}\right)_{i+dj} = G_{(i+dj,j)},$$

so $e_i(G) = c_{i+1}(\overline{G})$, and we get part of (i) of the previous proposition. In fact, the idea for proving this part of (i) is similar to [AM].

Remark 6.1.6 If I is a complete intersection ideal then $l = h$, and from Lemma 6.1.4 (ii) we have $e_i(R) = 0$ if $i \leq n-l-1$, and $e_i(R) = d^{n-1-i}$ if $i \geq n-l$. This result was proved in [STV, Theorem 3.6].

Corollary 6.1.7 *For a homogeneous ideal I generated by forms in degree d of height h, we have*

(i) $e(R) \geq 1 + d + \ldots + d^{h-1}$.

(ii) If I is equimultiple, $e(R) = 1 + d + \ldots + d^{h-1}$. *Assume further that I is not a $\mathfrak{m}$-primary ideal. Then* $e(G) = e_{n-h-1}(G) = \lambda_h = d^h$.

(iii) Assume that I is not a $\mathfrak{m}$-primary ideal. If A/I^j is Cohen-Macaulay for $j \gg 0$ (or Buchsbaum), $e_{n-h-1}(G) = d^h$. *Therefore,* $e(G) \geq d^h$.

Proof. (i) and (ii) are trivial. If we assume (iii), according to [HRTZ, Proposition 2.3] for $j \gg 0$ we have that $e(A/I^j) \geq \binom{dj+h-2}{h}$, and so $\lambda_h \geq d^h$. Furthermore, $\lambda_h \leq d^h$ because $e_{n-h-1}(R) = d^h - \lambda_h \geq 0$ by Lemma 6.1.4. We conclude $e_{n-h-1}(G) = \lambda_h = d^h$, and so $e(G) \geq e_{n-h-1}(G) = d^h$. □

Notice that as a consequence of Theorem 6.1.1 we have that with the Hilbert polynomials of a finite set of the powers of an ideal we can compute the Hilbert polynomials of its Rees algebra and its form ring, without needing an explicit presentation of these bigraded algebras. For equigenerated ideals, we may also compute the multiplicities of the Rees algebras and the form ring. Namely,

Corollary 6.1.8 *Let I be a homogeneous ideal in A. Set $c = a_*^2(R_A(I))$, $h = \mathrm{ht}\,(I)$. Then the Hilbert polynomials of I^j for $c+1 \le j \le c+n$ determine*

(i) The polynomials $e_{n-h-1-k}(j)$ for $k = 0, \dots, n-h-1$.

(ii) The Hilbert polynomials of A/I^j for $j > c+n$.

(iii) The Hilbert polynomial of the Rees algebra of I and the Hilbert polynomial of the form ring of I.

(iv) If I is equigenerated and not $\mathfrak{m}$-primary, the multiplicity of the Rees algebra of I and the multiplicity of the form ring of I.

We describe all these computations by means of an explicit example.

Example 6.1.9 Let us consider $I \subset A = k[X_1, X_2, X_3, X_4]$ the defining ideal of the twisted cubic in $\mathbb{P}_k^3$. Recall from Example 5.2.3 that the Rees algebra of I is Cohen-Macaulay, and so $a_*^2(R_A(I)) = -1$. Moreover, I is an ideal of height 2 generated by forms in degree 2. Then, according to Corollary 6.1.8 we can get the Hilbert polynomials of A/I^j for $j > 3$ from the Hilbert polynomials of I, I^2 and I^3. By using CoCoa, we have

$$P_{A/I}(s) = 3s + 1$$
$$P_{A/I^2}(s) = 9s - 7$$
$$P_{A/I^3}(s) = 18s - 34$$

By imposing $e_0(0) = 0$, $e_0(1) = 3$ and $e_0(2) = 9$, we get the multiplicity function $e_0(t) = \frac{3}{2}t(t+1)$. Similarly, one gets $e_1(t) = \frac{5}{3}t(t+1)(t-\frac{2}{5})$. Then the Hilbert polynomial of A/I^j is

$$P_{A/I^j}(s) = e_0(j)\binom{s+1}{1} - e_1(j)$$

and the Hilbert polynomial of the Rees algebra R of I is

$$P_R(s,t) = \binom{s+3}{3} - e_0(t)\binom{s+1}{1} + e_1(t).$$

In this case $\lambda_2 = 3$, $\lambda_3 = 10$, so by Lemma 6.1.4 we have $e_3(R) = 1$, $e_2(R) = 2$, $e_1(R) = 1$, $e_0(R) = 0$, and $e_2(G) = 0$, $e_1(G) = 3$, $e_0(G) = 2$. Therefore, the multiplicity of the Rees algebra is $e(R) = \sum_{i=0}^{3} e_i(R) = 4$ and the multiplicity of the form ring is $e(G) = \sum_{i=0}^{2} e_i(G) = 5$.

6.2 Hilbert series of the powers of an ideal

Let $A = k[X_1, \ldots, X_n]$ be the polynomial ring over a field k in n variables. For any finitely generated graded A-module M, recall that the Hilbert series of M is defined as

$$H_M(s) = \sum_{j \in \mathbb{Z}} H(M,j)s^j = \sum_{j \in \mathbb{Z}} \dim_k M_j s^j \in \mathbb{Z}[[s]].$$

Following A. Conca and G. Valla [CV], for a given class $\mathcal{C}$ of homogeneous ideals in A, we say that $\mathcal{C}$ has rigid powers if for any ideals I, J in $\mathcal{C}$ such that $H_{A/I}(s) = H_{A/J}(s)$ then $H_{A/I^j}(s) = H_{A/J^j}(s)$ for all j. For example, the class of complete intersection ideals has rigid powers. The class of the homogeneous ideals in A which are Cohen-Macaulay of codimension 2 and the class of the homogeneous ideals in A which are Gorenstein of codimension 3 do not have rigid powers, but their subclasses consisting of the ideals of linear type have this property as it has been proved in [CV].

Our first aim in this section is to show that for an equigenerated ideal I we can compute the Hilbert series of A/I^j for $j \geq 1$ from a finite set among these Hilbert series, and so we can also compute the bigraded Hilbert series of its Rees algebra. This fact will be a direct consequence of the noetherian property of the Rees algebra, and the finite set of Hilbert series will be found thanks to the bounds for the shifts of the bigraded minimal free resolution of the Rees algebra given by Theorem 1.3.4. In particular, we will have that if the Hilbert series of the powers of two ideals I, J coincide for certain exponents then all the Hilbert series of the powers of I and J must coincide.

Theorem 6.2.1 *Let I be an equigenerated homogeneous ideal. Set $l = l(I)$, $c = a_*^2(R_A(I))$. The Hilbert series of I^j for $c+1 \leq j \leq c+l$ determine the Hilbert series of I^j for $j > c+l$.*

Proof. Let us assume that I is generated by forms in degree d. Then we have that $R = R_A(I)$ is a finitely generated bigraded S-module in a natural way, for $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$ the polynomial ring with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d,1)$. Let $H_R(s,t)$ be the bigraded Hilbert series of R, that is

$$H_R(s,t) = \sum_{i,j} \dim_k R_{(i,j)} s^i t^j = \sum_{i,j} \dim_k (I^j)_i \, s^i t^j = \sum_j H_{I^j}(s) t^j.$$

By considering the bigraded minimal free resolution of R as S-module

$$0 \to D_t \to \ldots \to D_0 \to R \to 0,$$

with $D_p = \bigoplus_{(a,b)\in\Omega_p} S(a,b)$, we can write

$$H_R(s,t) = \frac{Q(s,t)}{(1-s)^n(1-s^d t)^l} \quad ,$$

with $Q(s,t) = \sum_{p=0}^{t}(-1)^p \sum_{(a,b)\in\Omega_p} s^{-a}t^{-b} \in \mathbb{Z}[s,t]$. Now let us fix $\alpha \in \mathbb{Z}_{\geq 0}$. Denoting by $\beta_p^j = \dim_k \operatorname{Tor}_p^A(k, I^j)_{\alpha+dj}$, then

$$\begin{aligned}
\textstyle\sum_p(-1)^p\beta_p^j &= [(1-s)^n H_{I^j}(s)\ ]_{\deg s=\alpha+dj} \\
&= [(1-s)^n H_R(s,t)\ ]_{\substack{\deg s = \alpha + dj \\ \deg t = j}} \\
&= \left[\tfrac{Q(s,t)}{(1-s^d t)^l}\right]_{\substack{\deg s = \alpha + dj \\ \deg t = j}}
\end{aligned}$$

Let us write $Q(s,t) = \sum_k m_k s^{\alpha+dk}t^k + \overline{Q}(s,t)$, with $\overline{Q}(s,t)$ containing all the monomials of the type $s^{\beta+dk}t^k$ for any $\beta \neq \alpha$ and any k. The pairs $(-\alpha - dk, -k)$ are shifts in the bigraded minimal free resolution of R as S-module, so $k \leq t_*^2(R) = a_*^2(R) + l = k_0$ by Theorem 1.3.4. Then we have that for any $j \geq k_0$,

$$\begin{aligned}
\textstyle\sum_p(-1)^p\beta_p^j &= \left[\left(\sum_{v=0}^{j}\binom{v+l-1}{l-1}s^{dv}t^v\right)\left(\sum_{k=0}^{k_0} m_k s^{\alpha+dk}t^k\right)\right]_{\substack{\deg s = \alpha + dj \\ \deg t = j}} \\
&= m_0\tbinom{j+l-1}{l-1} + \ldots + m_{k_0}\tbinom{j+l-k_0-1}{l-1} \\
&= P_\alpha(j).
\end{aligned}$$

It is easy to prove that this equality holds for $j \geq k_0 - l + 1 = a_*^2(R) + 1$. So we have found a polynomial $P_\alpha(j)$ of degree $\leq l-1$ such that $P_\alpha(j) = \sum_p(-1)^p\beta_p^j$ for any $j \geq a_*^2(R) + 1$. Hence the Hilbert series of the powers I^j for $a_*^2(R) + 1 \leq j \leq a_*^2(R) + l$ will determine the Hilbert series of I^j for any $j > a_*^2(R) + l$. □

Corollary 6.2.2 *Let I be an equigenerated homogeneous ideal whose Rees algebra is Cohen-Macaulay, and $l = l(I)$. Then the Hilbert series of I^j for $j \leq l-1$ determine the bigraded Hilbert series of the Rees algebra of I.*

Recent papers by A. M. Bigatti, A. Capani, G. Niesi and L. Robbiano [BCNR] and L. Robbiano and G. Valla [RV] treat the problem of computing the Hilbert series of the powers of a homogeneous ideal I in the polynomial ring $A = k[X_1, \ldots, X_n]$. The strategy followed there to solve this problem is to compute the Rees algebra $R_A(I)$ of I and then a Gröbner basis of it, from which one can get easily the bigraded Hilbert series of the Rees algebra, and so the Hilbert series of all the powers of I. Notice that we can use Theorem 6.2.1 to give another approach to this problem: To get the Hilbert series of the powers of an equigenerated ideal I it suffices to compute the Hilbert series of $l(I)$ of its powers. Next we apply this procedure to the following example studied by A. Bigatti et al. [BCNR, Example 5.4].

Example 6.2.3 Let us consider the ideal I generated by the 2 by 2 minors of the generic symmetric 3 by 3 matrix

$$M = \begin{pmatrix} X_1 & X_2 & X_3 \\ X_2 & X_4 & X_5 \\ X_3 & X_5 & X_6 \end{pmatrix}.$$

The Rees algebra of I is Cohen-Macaulay, so $a_*^2(R) = -1$. Therefore, the Hilbert series of I^j for $j \leq 5$ will determine the rest of the Hilbert series. By using CoCoa, we obtain:

$$H_I(s) = \frac{6s^2-8s^3+3s^4}{(1-s)^6},$$

$$H_{I^2}(s) = \frac{21s^4-45s^5+38s^6-18s^7+6s^8-s^9}{(1-s)^6},$$

$$H_{I^3}(s) = \frac{56s^6-150s^7+165s^8-100s^9+36s^{10}-6s^{11}}{(1-s)^6},$$

$$H_{I^4}(s) = \frac{126s^8-385s^9+486s^{10}-330s^{11}+125s^{12}-21s^{13}}{(1-s)^6},$$

$$H_{I^5}(s) = \frac{252s^{10}-840s^{11}+1155s^{12}-840s^{13}+330s^{14}+56s^{15}}{(1-s)^6}.$$

Then the polynomials $P_\alpha(j)$ defined in the proof of Theorem 6.2.1 are

$$P_\alpha(j) = 0 \text{ for } \alpha \neq 0, \ldots, 5\ ,$$

$$P_0(j) = 1 + \tfrac{137}{60}j + \tfrac{15}{8}j^2 + \tfrac{17}{24}j^3 + \tfrac{1}{8}j^4 + \tfrac{1}{20}j^5,$$

$$P_1(j) = -\tfrac{7}{4}j - \tfrac{83}{24}j^2 - \tfrac{53}{24}j^3 - \tfrac{13}{24}j^4 - \tfrac{1}{24}j^5,$$

$$P_2(j) = -\tfrac{3}{2}j + \tfrac{13}{12}j^2 + \tfrac{29}{12}j^3 + \tfrac{11}{12}j^4 + \tfrac{1}{12}j^5,$$

$$P_3(j) = \tfrac{7}{6}j + \tfrac{3}{4}j^2 - \tfrac{13}{12}j^3 - \tfrac{3}{4}j^4 - \tfrac{1}{12}j^5,$$

$$P_4(j) = -\tfrac{1}{4}j - \tfrac{7}{24}j^2 + \tfrac{5}{24}j^3 + \tfrac{7}{24}j^4 + \tfrac{1}{24}j^5,$$

$$P_5(j) = \tfrac{1}{20}j + \tfrac{1}{24}j^2 - \tfrac{1}{24}j^3 - \tfrac{1}{24}j^4 - \tfrac{1}{120}j^5,$$

and the Hilbert series of I^j is

$$\frac{P_0(j)s^{2j} + P_1(j)s^{2j+1} + P_2(j)s^{2j+2} + P_3(j)s^{2j+3} + P_4(j)s^{2j+4} + P_5(j)s^{2j+5}}{(1-s)^6}.$$

Next we also compute the Hilbert series of the powers of the ideal of the twisted cubic in $\mathbb{P}_k^3$ studied in the previous section.

Example 6.2.4 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}_k^3$. From Example 5.2.3, let us recall that I is generated by quadrics with $l(I) = \mu(I) = 3$ and $R_A(I)$ is Cohen-Macaulay. Therefore, according to Theorem 6.2.1, we can get the Hilbert series of I^j for $j > 2$ from the Hilbert series of I and I^2. By using CoCoa, we have

$$H_I(s) = \tfrac{3s^2-2s^3}{(1-s)^4},$$

$$H_{I^2}(s) = \tfrac{6s^4-6s^5+s^6}{(1-s)^4}.$$

Then the polynomials $P_\alpha(j)$ defined in the proof of Theorem 6.2.1 are

$$P_\alpha(j) = 0, \text{ for } \alpha \neq 0, 1, 2,$$

$$P_0(j) = \tfrac{1}{2}(j+1)(j+2),$$

$$P_1(j) = -j(j+1),$$

$$P_2(j) = \tfrac{1}{2}j(j-1),$$

and the Hilbert series of I^j is then

$$H_{I^j}(s) = \frac{P_0(j)s^{2j} + P_1(j)s^{2j+1} + P_2(j)s^{2j+2}}{(1-s)^4}.$$

Now, the bigraded Hilbert series of its Rees algebra is

$$H_R(s,t) = \sum_j H_{I^j}(s)t^j = \frac{1 - 2s^3t + s^6t^2}{(1-s)^4(1-s^2t)^3}.$$

Remark 6.2.5 Similarly we can prove the following statement for the Hilbert series of the form ring of an equigenerated ideal I: If $l = l(I)$ and $e = a_*^2(G_A(I))$, then the Hilbert series of I^j/I^{j+1} for $e+1 \leq j \leq e+l$ determine the Hilbert series of I^j/I^{j+1} for $j > e+l$. In fact, for any $a \geq e$ the Hilbert series of I^j/I^{j+1} for $a+1 \leq j \leq a+l$ determine the rest.

For any $m \geq 0$, let us define C_m to be the class of equigenerated homogeneous ideals I in A such that $a_*^2(G_A(I)) + l(I) \leq m$. Note that C_0 contains the class of complete intersection ideals, and we have the chain

$$C_0 \subset C_1 \subset \cdots \subset C_m \subset C_{m+1} \subset \cdots$$

As a corollary, we get

Corollary 6.2.6 *Let $I, J \in C_m$ be such that*

$$H_{I^j/I^{j+1}}(s) = H_{J^j/J^{j+1}}(s), \text{ for } m-l+1 \leq j \leq m.$$

Then $H_{I^j/I^{j+1}}(s) = H_{J^j/J^{j+1}}(s)$, $\forall j$. Therefore $H_{I^j}(s) = H_{J^j}(s)$, for all j, and in particular C_0 has rigid powers.

For an arbitrary homogeneous ideal I in A, we can also show that a finite set of Hilbert series of the powers of I determine the rest. But in this case, the bound we get is worse.

Proposition 6.2.7 *Let I be a homogeneous ideal in A. Set $r = \mu(I)$, $c = a_*^2(R_A(I))$. The Hilbert series of I^j for $j \leq c+r$ determine the Hilbert series of I^j for $j > c+r$.*

Proof. Assume that I is minimally generated by forms $f_1,\ldots,f_r$ of degrees $d_1, \ldots, d_r$ respectively. Then let us consider the presentation of the Rees algebra R of I as a quotient of the polynomial ring $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$, with $\deg(X_i) = (1,0)$, $\deg(Y_j) = (d_j, 1)$. From the bigraded minimal free resolution of R as S-module, we have that there is a polynomial $Q(s,t) \in \mathbb{Z}[s,t]$ such that

$$H_R(s,t) = \frac{Q(s,t)}{(1-s)^n(1-s^{d_1}t)\ldots(1-s^{d_r}t)} .$$

According to Theorem 1.3.4 we can write $Q(s,t) = \sum_{i=0}^{m} Q_i(s)t^i$, with $m = a_*^2(R) + r$. Since $H_R(s,t) = \sum_{j\geq 0} H_{I^j}(s)t^j$, we have

$$Q(s,t) = (1-s)^n(1-s^{d_1}t)\dots(1-s^{d_r}t)(\sum_{j\geq 0} H_{I^j}(s)t^j),$$

and then the result follows immediately. □

6.3 Minimal graded free resolutions of the powers of an ideal

The results about the Hilbert series of the powers of a homogeneous ideal imply in some particular cases the estability (in a meaning which we will precise immediately) of the minimal graded free resolutions of the powers of the ideal. For instance,

Proposition 6.3.1 *Let I be an ideal generated in degree d with $l(I) = 2$ whose Rees algebra is Cohen-Macaulay. Then the minimal graded free resolution of I determines the minimal graded free resolutions of all its powers. Namely, if the minimal graded free resolution of I is*

$$0 \to A(-\alpha_1 - d)^{\beta_1} \oplus \dots \oplus A(-\alpha_m - d)^{\beta_m} \to A(-d)^{\beta} \to I \to 0,$$

then for any $j \geq 1$ the minimal graded free resolution of I^j is

$$0 \to A(-\alpha_1 - dj)^{\beta_1 j} \oplus \dots \oplus A(-\alpha_m - dj)^{\beta_m j} \to A(-dj)^{(\beta-1)j+1} \to I^j \to 0.$$

Proof. First, note that for any $j \geq 1$ we have that proj.dim$_A I^j \leq l(I) - 2 = 1$ because R is Cohen-Macaulay (see Proposition 6.3.2). Therefore, we can conclude that proj.dim$_A I^j = 1$ for any j. On the other hand, since R is Cohen-Macaulay, we have $a_*^2(R) = -1$, so the Hilbert series of I^j for $j \leq l - 1 = 1$ determine the Hilbert series of I^j for $j > 1$ according to Theorem 6.2.1. The polynomials $P_\alpha(j)$ defined there are

$$P_\alpha(j) = 0, \text{ for } \alpha \notin \{0, \alpha_1, \dots, \alpha_m\},$$

$$P_0(j) = (\beta - 1)j + 1,$$

$$P_{\alpha_i}(j) = -\beta_i j, \text{ for } i \in \{0, \dots, m\}.$$

Then the Hilbert series of I^j is

$$H_{I^j}(s) = \frac{\sum_\alpha P_\alpha(j) s^{\alpha+dj}}{(1-s)^n},$$

so the minimal graded free resolution of I^j must be

$$0 \to A(-\alpha_1 - dj)^{-P_{\alpha_1}(j)} \oplus \ldots \oplus A(-\alpha_m - dj)^{-P_{\alpha_m}(j)} \to A(-dj)^{P_0(j)} \to I^j \to 0.$$

□

This result leads to the question of when a finite number of minimal graded free resolutions of the powers of I determine the rest (and, in this case, which set of resolutions determine the others).

Let us begin by studying the behaviour of the projective dimension of the powers of I. It is well-known that these projective dimensions are asymptotically constant (see [Bro, Theorem 2]), but not for which powers of the ideal the projective dimension takes the asymptotic value. We will precise these powers for ideals whose form ring is Gorenstein by considering the Koszul homology of the Rees algebra R of I with respect to $X_1, \ldots, X_n$. This also provides new proofs of well-known results as the Burch's inequality or the constant asymptotic value for the depth.

Proposition 6.3.2 *Let I be a homogeneous ideal in A, and set $l = l(I)$. Then:*

(i) $\text{proj.dim}_A(I^j) \leq n - \text{depth}_{(\mathfrak{m}R)}(R)$ *for all j, and the equality holds for $j \gg 0$. So,* $\inf_{j\geq 0}\{\text{depth}(A/I^j)\} = n - l - (\text{ht}\,(\mathfrak{m}R) - \text{depth}_{(\mathfrak{m}R)}(R))$.

(ii) If R is Cohen-Macaulay, $\text{proj.dim}_A(I^j) \leq l - 1$ *for any j and* $\text{proj.dim}_A(I^j) = l - 1$ *for $j \gg 0$. Furthermore,* $\text{proj.dim}_A(I^j) = l - 1$ *implies* $\text{proj.dim}_A(I^{j+1}) = l - 1$.

(iii) If G is Gorenstein, $\text{proj.dim}_A I^j = l - 1$ *if and only if $j > a^2(G) - a(F)$, and* $\text{proj.dim}_A I^j/I^{j+1} = l - 1$ *if and only if $j \geq a^2(G) - a(F)$.*

Proof. Let us consider the Koszul complex $K.(\underline{X}, R) = K.(X_1, \ldots, X_n, R)$ of the Rees algebra R with respect to $\underline{X} = X_1, \ldots, X_n$. We have the natural bigrading in the Rees algebra R by means of $R_{(i,j)} = (I^j)_i$, and then the modules $K_p(\underline{X}, R)$ of the Koszul complex are also bigraded in a natural way. Denoting by $F = F_\mathfrak{m}(I)$, we have that for any p the Koszul homology module $H_p = H_p^S(\underline{X}; R)$ is a finitely generated bigraded F-module. Moreover, since $K_p(\underline{X}, R)_{(i,j)} = K_p(\underline{X}, I^j)_i$ we have

$$H_p^S(\underline{X}, R)_{(i,j)} = H_p^A(\underline{X}, I^j)_i = \text{Tor}_p^A(k, I^j)_i \ ,$$

so the Koszul homology modules H_p contain all the information about the graded minimal free resolutions of the powers of I.

Now set $s = n - \mathrm{depth}_{(\mathfrak{m}R)}(R)$. Recall that H_p is zero for any $p > s$, and so $\mathrm{proj.dim}_A(I^j) \leq s$ for any j. Moreover, since H_s is a F-module of dimension l [Hu2, Remark 1.5] we can find for any $j \gg 0$ an integer i (depending on j) such that $[H_s]_{(i,j)} \neq 0$. Therefore we obtain that $\mathrm{proj.dim}_A(I^j) = s$ for $j \gg 0$. By the graded Auslander-Buchsbaum formula, $\mathrm{proj.dim}_A(I^j) = n - \mathrm{depth}\, A/I^j - 1$, and noting that $\mathrm{depth}_{(\mathfrak{m}R)}(R) \leq \mathrm{ht}(\mathfrak{m}R) = n + 1 - l$ we get (i).

To prove (ii), let us denote by $t = \mathrm{depth}_{(\mathfrak{m}R)}(R) = n + 1 - l$. We may assume that k is infinite, and then there exists a homogeneous regular sequence $b_1, \ldots, b_t \in \mathfrak{m}R$ of degree $(1,0)$. Then

$$H_s = \frac{(b_1, \ldots, b_t) : (\mathrm{X}_1, \ldots, \mathrm{X}_n)}{(b_1, \ldots, b_t)}(t - n, 0).$$

Note that $s = l - 1$ because R is CM. Now, observe that $\mathrm{proj.dim}_A(I^j) = l - 1$ implies that there exists i such that $[H_s]_{(i,j)} \neq 0$; so let us take $f \in [H_s]_{(i,j)}, f \neq 0$. For a positively graded ring $A = \bigoplus_{j \geq 0} A_j$, let us denote by $A_+ = \bigoplus_{j>0} A_j$, and in the following let us consider the fiber cone F and the Rees algebra R graded by means of $F_j = I^j/\mathfrak{m}I^j$, $R_j = I^j$. If $F_+ f = 0$, then $I^m f \subset (b_1, \ldots, b_t)$ for any $m \geq 1$. So, denoting by $\overline{R} = R/(b_1, \ldots, b_t)$, we have that $\overline{R}_+ \subset \mathrm{Ann}(f) \subset \mathfrak{p} \in \mathrm{Ass}\,(\overline{R})$ and so $\mathrm{ht}(\overline{R}_+) = 0$. But $\mathrm{ht}(\overline{R}_+) = \dim \overline{R} - \dim \overline{R}/\overline{R}_+ = 1$. As a consequence, $(F_+)_1 f \neq 0$ and so there exists d such that $[H_s]_{(i+d,j+1)} \neq 0$ and we have (ii).

Finally, we are going to determine the powers of I whose projective dimension is $l-1$ if G is Gorenstein. To this end, let us consider the Koszul homology modules of the form ring G with respect to $\underline{X}$, which will be also denoted by H_p. Set $s = n - \mathrm{depth}_{(\mathfrak{m}G)}(G) = l$, $t = \mathrm{depth}_{(\mathfrak{m}G)}(G)$. As before, H_p is zero for $p > s$ and there exists a homogeneous regular sequence $b_1, \ldots, b_t \in \mathfrak{m}G$ of degree $(1,0)$, such that

$$H_l = \frac{(b_1, \ldots, b_t) : (\mathrm{X}_1, \ldots, \mathrm{X}_n)}{(b_1, \ldots, b_t)}(t - n, 0).$$

On the other hand, from the natural bigraded epimorphism $G \to G/\mathfrak{m}G = F$, we can compute the canonical module of the fiber cone F by using Corollary 1.2.2 :

$$K_F = \underline{\mathrm{Ext}}_G^{n-l}(F, K_G).$$

Since G is Gorenstein, we have a bigraded isomorphism $K_G \cong G(-n, a)$ with $a = a^2(G)$ by Corollary 4.1.7. Therefore,

$$\begin{aligned} K_F &= \underline{\mathrm{Ext}}_G^{n-l}(F, G)(-n, a) \\ &= \underline{\mathrm{Hom}}_G(F, G/(b_1, \ldots, b_t))(-n+t, a) \\ &= \tfrac{(b_1,\ldots,b_t):(X_1,\ldots,X_n)}{(b_1,\ldots,b_t)}(-n+t, a) \\ &= H_l(0, a). \end{aligned}$$

Now, observe that $\mathrm{proj.dim}_A(I^j/I^{j+1}) = l$ if and only if there exists i such that $[H_l]_{(i,j)} \neq 0$ if and only if there exists i such that $[K_F]_{(i,j-a)} \neq 0$, that is, $j \geq a - a(F)$. From the exact sequences

$$0 \to I^{j+1} \to I^j \to I^j/I^{j+1} \to 0 ,$$

it is then easy to check that $\mathrm{proj.dim}_A(I^j) = l - 1$ if and only if $j > a - a(F)$, and so we are done.□

Example 6.3.3 Let I be a strongly Cohen-Macaulay ideal such that $\mu(I_{\mathfrak{p}}) \leq \mathrm{ht}\,(\mathfrak{p})$ for any prime ideal $\mathfrak{p} \supseteq I$. Set $l = l(I)$, $h = \mathrm{ht}(I)$. Recall from Corollary 5.2.11 that $G_A(I)$ is Gorenstein, $a^2(G_A(I)) = -h$ and $a(F_{\mathfrak{m}}(I)) = -l$. So, by Proposition 6.3.2 we have $\mathrm{depth}(A/I^j) = n - l$ if and only if $j > l - h$.

Example 6.3.4 Let $\mathbf{X} = (X_{ij})$ be a generic matrix, with $1 \leq i \leq d$, $1 \leq j \leq n$ and $d \leq n$. Let $I \subset A = k[\mathbf{X}]$ be the ideal generated by the maximal minors of $\mathbf{X}$. Recall from Example 5.2.10 that the Rees algebra R is Cohen-Macaulay and the form ring G is Gorenstein with $a^2(G) = -\mathrm{ht}\,(I) = -(n - d + 1)$. Furthermore, $l(I) = d(n - d) + 1$ and $a(F) = -n$. Now by Proposition 6.3.2, we get that $\mathrm{depth}(A/I^j) = d^2 - 1$ if and only if $j > d - 1$. In the case $n = d + 1$, this was proved in [BV, Example 9.27].

Example 6.3.5 Let $\mathbf{X} = (X_{ij})$ be a generic skew-symmetric matrix, with $1 \leq i < j \leq n$, and n odd. Let $I \subset A = k[\mathbf{X}]$ be the ideal generated by the $(n-1)$-pfaffians of $\mathbf{X}$, where k is a field. In this case, the form ring G is Gorenstein [CD] and $l(I) = n$, $a(F_{\mathfrak{m}}(I)) = -n$ [Hu3]. So $\mathrm{depth}(A/I^j)$ takes the asymptotic value $\frac{n(n-1)}{2} - n$ for some $j \leq n$, and by Proposition 6.3.2 for all $j \geq n$.

G. Boffi and R. Sánchez [BoSa] have constructed a family of complexes which give a resolution for all the powers I^j, for $j \geq 1$, in particular proving that $\text{proj.dim}_A(A/I^j) = n$ if and only if $j \geq n-2$. Then Proposition 6.3.2 shows that $a^2(G_A(I)) = -3$.

Our next aim is to study the graded minimal free resolutions of the powers of an equigenerated ideal by doing a deeper study of the Koszul homology of the Rees algebra with respect to $X_1, \ldots, X_n$. The general case will be studied later by different methods.

6.3.1 Case study : Equigenerated ideals

First of all, we show that the shifts in the graded minimal free resolutions of the powers of an equigenerated ideal are given by linear functions asymptotically and the graded Betti numbers of these resolutions are given by polynomials asymptotically.

Proposition 6.3.6 *Let I be an ideal generated by forms in degree d. Set $l = l(I)$, $s = n - \text{depth}_{(\mathfrak{m}R)}(R)$. Then there is a finite set of integers*

$$\{\alpha_{pi} \mid 0 \leq p \leq s, 1 \leq i \leq k_p\}$$

and polynomials of degree $\leq l-1$

$$\{Q_{\alpha_{pi}}(j) \mid 0 \leq p \leq s, 1 \leq i \leq k_p\}$$

such that the graded minimal free resolution of I^j for j large enough is

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = \bigoplus_i A(-\alpha_{pi} - dj)^{\beta_{pi}^j}$ and $\beta_{pi}^j = Q_{\alpha_{pi}}(j)$.

Proof. Let us consider again the Koszul homology of the Rees algebra R of I with respect to $\underline{X} = X_1, \ldots, X_n$, and let us denote by $F = \bigoplus_{j \geq 0} I^j/\mathfrak{m}I^j$ the fiber cone of I and by $F_+ = \bigoplus_{j>0} I^j/\mathfrak{m}I^j$. For every $p \leq s$, $H_p = H_p^S(\underline{X}; R)$ is a finitely generated bigraded F-module. Let g be a homogeneous generator of H_p with $\deg(g) = (a, b)$, and set $\alpha = a - db$. If $F_+ \subset \text{rad}\,(\text{Ann}\,(g))$, there exists j such that $F_+^j g = 0$, and so $F_j g = 0$ for all $j \gg 0$. Otherwise, there exists a homogeneous element $f \in F$ of degree d such that $f \notin \text{rad}\,(\text{Ann}\,(g))$. Then $f^j g \neq 0$ for all j, and so we have $[H_p]_{(\alpha+dj,j)} \neq 0$ for all $j \gg 0$. Let $g_1, \ldots, g_m$ be the homogeneous generators of H_p with this property, and set

$\deg(g_i) = (a_i, b_i)$, $\alpha_i = a_i - db_i$. Then, for j large enough we have that $[H_p]_{(a,j)} \neq 0$ if, and only if, there exists $i \in \{1, \dots, m\}$ such that $a = \alpha_i + dj$. Since $[H_p]_{(\alpha_i+dj,j)} = \operatorname{Tor}_p^A(k, I^j)_{\alpha_i+dj}$, we obtain that $\alpha_i + dj$, for $1 \leq i \leq m$, are the only shifts in the place p of the graded minimal free resolution of I^j for $j \gg 0$.

For $\alpha \in \{\alpha_1, \dots, \alpha_m\}$, let us define $H_p^\alpha = \bigoplus_j [H_p]_{(\alpha+dj,j)}$. Notice that $\dim H_p^\alpha \leq \dim H_p = l$ by [Hu2, Remark 1.5]. Since H_p^α is a finitely generated graded F-module, there exists a polynomial $Q_\alpha(j)$ of degree $\dim H_p^\alpha - 1 \leq l-1$ such that for j large enough

$$Q_\alpha(j) = \dim_k [H_p^\alpha]_j = \dim_k \operatorname{Tor}_p^A(k, I^j)_{\alpha+dj},$$

so $Q_\alpha(j)$ is the Betti number of I^j corresponding to $\alpha + dj$ in the place p. □

Example 6.3.7 Let I be a Cohen-Macaulay homogeneous ideal of codimension two in the polynomial ring $A = k[X_1, \dots, X_n]$ such that:

(i) The entries of the Hilbert-Burch matrix of I are linear forms.

(ii) I verifies G_n.

(iii) $\mu(I) \leq n$.

This example has been studied by A. Conca and G. Valla in [CV]. Set $r = \mu(I)$, $d = r-1$, $S = A[Y_1, \dots, Y_r]$. Then

$$R_A(I) \cong \operatorname{Sym}_A(I) \cong S/(F_1, \dots, F_{r-1}),$$

with $F_1, \dots, F_{r-1}$ a regular sequence of degree $(d, 1)$. So the Koszul complex of S with respect to $F_1, \dots, F_{r-1}$ gives the bigraded minimal free resolution of $R_A(I)$. From this resolution one can get the minimal graded free resolutions of I^j, for all $j \geq 0$. Namely, $\text{proj.dim}_A(I^j) = \min\{j, r-1\}$ and the minimal free resolution of I^j is

$$0 \to D_{r-1}^j \to \dots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = A(-p-dj)^{\beta_p^j}$, $\beta_p^j = \binom{r-1}{p}\binom{r+j-p-1}{r-1}$.

Remark 6.3.8 Let $(A, \mathfrak{m}, k)$ be a noetherian local ring, and let $I \subset A$ be an ideal. Set $l = l(I)$, $r = \mu(I)$. Denote by R the Rees algebra of I graded by $R_j = I^j$, and let $S = A[Y_1, \dots, Y_r]$ be a polynomial ring over A with $\deg Y_j = 1$ so that R is a finitely generated graded S-module. As above, we can prove that there are polynomials $Q_p(j)$ of degree $\leq l-1$, the Hilbert polynomials of $\operatorname{Tor}_p^S(S/\mathfrak{m}S, R)$, such that the minimal free resolutions of I^j for $j \gg 0$ are

$$\dots \to A^{Q_p(j)} \to \dots \to A^{Q_0(j)} \to I^j \to 0.$$

This result was proved by V. Kodiyalam in [Ko1]. If A is regular, let $\underline{x}$ be a regular sequence generating $\mathfrak{m}$. Since $\operatorname{Tor}_p^S(S/\mathfrak{m}S, R) \cong H_p(\underline{x}, R)$, the module $\operatorname{Tor}_p^A(k, R)$ has dimension l if it is not zero. Therefore, the polynomial $Q_p(j)$ has degree $l-1$ if $Q_p(j) \neq 0$. This answers affirmatively [Ko1, Question 13] for any regular ring A.

Observe that Proposition 6.3.6 says that we can compute the graded minimal free resolution of any power of an equigenerated ideal from a finite set among these resolutions. Now we consider the problem of determining this finite set of resolutions. To begin with, let us study the asymptotic shifts of Proposition 6.3.6.

Lemma 6.3.9 *Let I be a homogeneous ideal generated in degree d and let $R = R_A(I)$. Then*

(i) For all p and i, there exists $(a, b) \in \Omega_{p,R}$ such that $\alpha_{pi} = db - a$.

(ii) For each α, let

$$p = \min\{\, q \mid \exists\, b \; s.t.\, (\alpha + db, b) \in \Omega_{q,R}\},$$

and let

$$b_0 = \max\{\, b \mid (\alpha + db, b) \in \Omega_{p,R}\}.$$

Then $\alpha + db_0 \in \Omega_{p,I^{-b_0}}$, that is, $\alpha + db_0$ is a shift that appears in the graded minimal free resolution of I^{-b_0} at the place p.

Proof. Let $0 \to D_m \to \ldots \to D_0 \to R \to 0$ be the bigraded minimal free resolution of R over $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_l]$. By applying the functor $(\ )^j$ to this resolution, we get a graded free resolution of I^j over A

$$0 \to D_m^j \to \ldots \to D_1^j \to D_0^j \to I^j \to 0\,,$$

with $D_p^j = \bigoplus_{(a,b)\in\Omega_{p,R}} A(a - db - dj)^{\rho_{ab}^j}$, for some $\rho_{ab}^j \in \mathbb{Z}$. This resolution is the direct sum of the minimal graded free resolution of I^j and the trivial complex [Eis, Exercise 20.1], so we obtain that for $j \gg 0$

$$\{\alpha_{pi} + dj\}_{p,i} \subset \{db - a + dj \mid (a, b) \in \Omega_{p,R}\},$$

and so (i) is already shown.

Now let α be such that there exists b with $(\alpha + db, b) \in \Omega_R$. Let p be the first integer such that $(\alpha + db, b) \in \Omega_{p,R}$, and let b_0 be the maximum of these b's. We must show that

$$\operatorname{Tor}_p^S(S/\mathfrak{m}S, R)_{(-\alpha-db_0,-b_0)} = \operatorname{Tor}_p^A(k, I^{-b_0})_{-\alpha-db_0} \neq 0.$$

We will proceed as in Theorem 5.1.1: Let

$$D_{p+1} \xrightarrow{\psi_{p+1}} D_p \xrightarrow{\psi_p} D_{p-1}$$

be the differential maps appearing in the resolution of R. Tensorazing by $\otimes_S S/\mathfrak{m}S$, we have the sequence

$$D_{p+1}/\mathfrak{m}D_{p+1} \xrightarrow{\overline{\psi}_{p+1}} D_p/\mathfrak{m}D_p \xrightarrow{\overline{\psi}_p} D_{p-1}/\mathfrak{m}D_{p-1}.$$

Now let $v \in D_p$ be an element of the homogeneous basis of D_p as free S-module with $\deg(v) = (-\alpha - db_0, -b_0)$. If $w_1, \dots, w_s$ is the basis of D_{p-1}, we can write

$$\psi_p(v) = \sum_{j=1}^{s} \lambda_j w_j,$$

with $\lambda_j \in \mathcal{M}$ homogeneous. Set $\deg(w_j) = (-\alpha_j - db_j, -b_j)$. By looking at the degree of the elements, we have that λ_j must be zero for all j such that $-b_j > -b_0$. For the integers j such that $-b_j = -b_0$, we have that $\lambda_j \in \mathfrak{m}S$ necessarily. Finally, for j such that $-b_j < -b_0$ we also have $\lambda_j \in \mathfrak{m}S$ because $\alpha_j \neq \alpha$. We may conclude $\overline{\psi}_p(v) = 0$, that is, $v \in \operatorname{Ker}\overline{\psi}_p$. It is clear that $v \notin \operatorname{Im}\overline{\psi}_{p+1}$ because $\operatorname{Im}\overline{\psi}_{p+1} \subset \mathcal{M}D_p$. So $v \in \operatorname{Tor}_p^S(S/\mathfrak{m}S, L)_{(-\alpha-db_0,-b_0)}$, $v \neq 0$ and we are done. □

As a consequence of this lemma we have that all the differences $a - db$ for $(a, b) \in \Omega_R$ appear in the minimal graded free resolution of some power I^j of I for $j \leq a_*^2(R) + l(I)$. The problem is to distinguish which of these shifts will appear asymptotically, and the place from where on the resolutions are stable. We can solve this problem for ideals with a very particular nice behaviour. For instance, we get

Proposition 6.3.10 *Let I be an equigenerated homogeneous ideal, and set $b = a_*^2(R_A(I)) + l(I)$. If the graded minimal free resolutions of $I, I^2, \dots, I^b$ are linear, then the graded minimal free resolutions of I^j are also linear for any j. Furthermore, the minimal free resolutions of $I, I^2, \dots, I^b$ determine the minimal graded free resolutions of I^j for any j.*

Proof. Assume that I is generated by forms in degree d and set $s = \sup_{j=1,\dots,b}\{\text{proj.dim}_A I^j\}$. According to Lemma 6.3.9, we have that the shifts in Ω_R are of the type $(p+db, b)$ with $0 \leq p \leq s$. Furthermore, there exists b_0 such that $(p+db_0, b_0) \in \Omega_{p,R}$, but for any b, $q < p$, $(p+db, b) \notin \Omega_{q,R}$. Therefore, $\Omega_{p,R}$ has only shifts of the form $(a+db, b)$ for $0 \leq a \leq p$. Again by Lemma 6.3.9, we get

$$\{\alpha_{pi}\}_i \subset \{0, \dots, p\}.$$

Finally, since $\min\{-\beta : \beta \in \Omega_{p+1,I^j}\} > \min\{-\beta : \beta \in \Omega_{p,I^j}\}$, we have that $\min\{-\beta : \beta \in \Omega_{p,I^j}\} \geq p + dj$. Therefore I^j must have a linear minimal free resolution.

Moreover, by Theorem 6.2.1 we also have that for $p = 0, \dots, s$ there exists a polynomial $Q_p(j)$ of degree $\leq l-1$ such that

$$Q_p(j) = \dim_k \operatorname{Tor}_p^A(k, I^j)_{p+dj},$$

for $j \geq a_*^2(R) + 1$. So, if we know the minimal graded free resolutions of $I^{b-l+1}, \dots, I^b$, we may determine the polynomials $Q_p(j)$, and then the minimal graded free resolution of I^j for $j > b$.□

Remark 6.3.11 The first part of Proposition 6.3.10 can be also obtained from Theorem 5.2.1 (ii).

Remark 6.3.12 Given an equigenerated homogeneous ideal I with a linear minimal free resolution, it can happen that I^2 has a non linear minimal free resolution (see [Con, Remark 3]). We have shown in Proposition 6.3.10 that if certain powers of I have linear resolution, then the rest of the powers have this property too.

We may apply this result to guess the minimal graded free resolutions of the powers of the ideal defining the twisted cubic in $\mathbb{P}_k^3$.

Example 6.3.13 Let $I \subset A = k[X_1, X_2, X_3, X_4]$ be the defining ideal of the twisted cubic in $\mathbb{P}_k^3$, and let us study the graded minimal free resolutions of its powers. I is generated by forms in degree 2 with $l(I) = 3$ and $b = a_*^2(R) + l(I) = 2$. The minimal resolutions of I and I^2 (computed with CoCoa) are:

$$0 \to A(-3)^2 \to A(-2)^3 \to I \to 0\ ,$$

$$0 \to A(-6) \to A(-5)^6 \to A(-4)^6 \to I^2 \to 0\ .$$

Since these resolutions are linear, we have that the minimal graded free resolutions of I^j for $j > 2$ are also linear by Proposition 6.3.10, and we may compute them:

$$0 \to A(-2-2j)^{Q_2(j)} \to A(-1-2j)^{Q_1(j)} \to A(-2j)^{Q_0(j)} \to I^j \to 0\ ,$$

with $Q_0(j) = \frac{1}{2}(j+1)(j+2)$, $Q_1(j) = j(j+1)$ and $Q_2(j) = \frac{1}{2}j(j-1)$.

Similarly, one can prove the following statement.

Proposition 6.3.14 *Let I be an ideal generated in degree d, and set $b = a_*^2(R_A(I)) + l(I)$. Assume that there are integers $\alpha_1, \dots, \alpha_s$ such that the graded minimal free resolutions of $I, I^2, \dots, I^b$ take the form*

$$0 \to D_s^j \to \dots \to D_1^j \to D_0^j \to I^j \to 0,$$

with $D_p^j = A(-\alpha_p - dj)^{\beta_p^j}$, for $\beta_p^j \geq 0$. Then the graded minimal free resolutions of I^j are of this type too. Furthermore, the minimal graded free resolutions of $I, I^2, \dots, I^b$ determine the minimal graded free resolutions of I^j for any j.

The following example does not belong to the family of ideals considered in the previous propositions, but we can also guess the asymptotic resolution of its powers.

Example 6.3.15 Let $I = (X^7, Y^7, X^6Y + X^2Y^5) \subset A = k[X,Y]$. Note that I is a $\mathfrak{m}$-primary ideal generated by forms of degree 7 with $l(I) = 2$. Since $\text{proj.dim}_A I^j = 1$ for any $j \geq 1$, we have that the shifts in the place 0 and 1 of the resolution of I^j can not coincide. Then, according to Theorem 6.2.1 we have that for any $\alpha \neq 0$ there is a polynomial $P_\alpha(j)$ of degree ≤ 1 such that

$$P_\alpha(j) = \dim_k \operatorname{Tor}_1^A(k, I^j)_{\alpha+dj},$$

for all $j \geq a_*^2(R) + 1$.

This example was studied by S. Huckaba and T. Marley [HM, Example 3.13]. Denoting by $G_{++} = \bigoplus_{j>0} I^j/I^{j+1}$ and by $\underline{a}_i(G) = \max\{j \mid H^i_{G_{++}}(G)_j \neq 0\}$, it was proved that the form ring G has depth 0, and $\underline{a}_0(G) < \underline{a}_1(G) < \underline{a}_2(G) = 4$. Now $a_*^2(G) = \max\{\underline{a}_i(G) : i = 0,1,2\} = 4$ according to [Hy, Lemma 2.3], and then the short exact sequences

$$0 \to R_{++} \to R \to A \to 0,$$

$$0 \to R_{++}(1) \to R \to G \to 0,$$

where $R_{++} = \bigoplus_{j>0} I^j$, show $a_*^2(R) = 4$. The graded minimal free resolutions of I^5 and I^6 (computed with CoCoa) are :

$$0 \to A(-37)^{15} \oplus A(-36)^5 \to A(-35)^{21} \to I^5 \to 0\ ,$$

$$0 \to A(-44)^{15} \oplus A(-43)^{12} \to A(-42)^{28} \to I^6 \to 0\ ,$$

Then we may compute the polynomials $P_\alpha(j)$, so the graded minimal free resolutions of I^j for $j \geq 5$ are:

$$0 \to A(-2-7j)^{15} \oplus A(-1-7j)^{7j-30} \to A(-7j)^{7j-14} \to I^j \to 0\ .$$

Furthermore, in this case we check that the bound can not be improved because the resolution of I^4 is

$$0 \to A(-30)^{14} \to A(-28)^{15} \to I^4 \to 0\ .$$

Open Question Let I be a homogeneous ideal generated by forms in degree d. Denote by $l = l(I)$, $s = n - \text{depth}_{(\mathfrak{m}R)}(R)$, $c = a_*^2(R)$. By Proposition 6.3.6, there are integers $\{\alpha_{pi}\}$ and polynomials $\{Q_{\alpha_{pi}}(j)\}$ of degree $\leq l-1$ such that for $j \gg 0$ the graded minimal free resolution of I^j is

$$0 \to D_s^j \to \ldots \to D_0^j \to I^j \to 0\ ,$$

with $D_p^j = \bigoplus_i A(-\alpha_{pi} - dj)^{\beta_{pi}^j}$ and $\beta_{pi}^j = Q_{\alpha_{pi}}(j)$. In some particular cases, we have shown that this holds for $j \geq c+1$. The question is if this bound holds for any equigenerated ideal I.

6.3.2 General case

We may also study the minimal graded free resolutions of the powers of any arbitrary homogeneous ideal in the polynomial ring although in this case the asymptotic result is not so nice. Our approach will be based on a detailed study of the proof of [CHT, Theorem 3.4]. We need to introduce some notation.

Let I be a homogeneous ideal in A generated by r forms of degrees $d_1, \ldots, d_r$. Let us consider $S = k[X_1, \ldots, X_n, Y_1, \ldots, Y_r]$ the polynomial ring with $\deg X_i = (1,0)$, $\deg Y_j = (d_j, 1)$, and let $S_2 = k[Y_1, \ldots, Y_r]$. For any finitely generated bigraded module L over S_2, let us define the set

$$\delta_L(j) = \{i : L_{(i,j)} \neq 0\}.$$

Given $\underline{c} = (c_1, \ldots, c_r) \in \mathbb{N}^r$, let us denote by $v(\underline{c}) = d_1 c_1 + \ldots + d_r c_r$ and by $| \underline{c} |= c_1 + \ldots + c_r$. Given a set $C \subset \mathbb{N}^r$, $C + \mathbb{N}^r$ denotes the set of points of $\mathbb{N}^r$ of the type $\underline{c} + \underline{c}'$ with $\underline{c} \in C$, $\underline{c}' \in \mathbb{N}^r$. Then we have:

Lemma 6.3.16 *Let L be a finitely generated bigraded S_2-module. Then there are pairs $(\alpha_i, \beta_i) \in \mathbb{Z}^2$ and finite subsets C_i of $\mathbb{N}^r$, $1 \leq i \leq m$, such that for any j*

$$\delta_L(j) = \bigcup_i \{v(\underline{c}) + \alpha_i : \underline{c} \notin C_i + \mathbb{N}^r, | \underline{c} |= j - \beta_i\}.$$

Furthermore,

$$\dim_k L_{(l,j)} = \sum_{i=1}^{m} \#\{\underline{c} \in \mathbb{N}^r : \underline{c} \notin C_i + \mathbb{N}^r, | \underline{c} |= j - \beta_i, v(\underline{c}) = l - \alpha_i\}.$$

Proof. As said, the proof is based on [CHT, Theorem 3.4]. Given any finitely generated bigraded S_2-module L, there exists a sequence of bigraded submodules

$$0 = L_0 \subset L_1 \subset \ldots \subset L_{m-1} \subset L_m = L$$

of L such that $M_i = L_i / L_{i-1} \cong S_2/\mathfrak{p}_i(-\alpha_i, -\beta_i)$, $1 \leq i \leq m$, with $\mathfrak{p}_i$ homogeneous prime ideals in S_2. Note that $\delta_L(j) = \bigcup_i \delta_{M_i}(j) = \bigcup_i \delta_{S_2/\mathfrak{p}_i}(j - \beta_i) + \alpha_i$, and so we can assume that L is cyclic.

Now let $L = S_2/J$, with $J \subset S_2$ a homogeneous ideal. By fixing a term order $<$ in S_2, then L has a k-basis consisting of the classes of the monomials which do not belong to the initial ideal $\text{in}(J)$ of J. So we get $\delta_{S_2/J}(j) = \delta_{S_2/\text{in}(J)}(j)$, and we may assume J is a monomial ideal.

Let us write $J = (Y_1^{c_{11}} \cdots Y_r^{c_{1r}}, \ldots, Y_1^{c_{p1}} \cdots Y_r^{c_{pr}})$, and $\underline{c_i} = (c_{i1}, \ldots, c_{ir})$ for $1 \leq i \leq p$. For any $\underline{c} \in \mathbb{N}^r$, note that $Y_1^{c_1} \cdots Y_r^{c_r} \in J$ if and only if there exists i such that $\underline{c} = \underline{c_i} + \underline{c}'$, for some $\underline{c}' \in \mathbb{N}^r$, i.e. $\underline{c} \in C + \mathbb{N}^r$, where $C = \{\underline{c}_1, \ldots, \underline{c}_p\}$. Therefore

$$\delta_L(j) = \{v(\underline{c}) : \underline{c} \notin C + \mathbb{N}^r, | \underline{c} |= j\},$$

and we are done.□

Now we can show the asymptotic minimal graded free resolution of the powers of an arbitrary homogeneous ideal I.

Proposition 6.3.17 *Let I be a homogeneous ideal in the polynomial ring $A = k[X_1, \dots, X_n]$ minimally generated by forms $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$. Then there are pairs $(\alpha_{pi}, \beta_{pi}) \in \mathbb{Z}^2$ and sets $C_{pi} \subset \mathbb{N}^r$, for $0 \leq p \leq s, 0 \leq i \leq k_p$, such that for j large enough the graded minimal free resolution of I^j is*

$$0 \to D_s^j \to \dots \to D_0^j \to I^j \to 0 \ ,$$

with $D_p^j = \bigoplus_{i,\underline{c}} A(-\alpha_{pi} - v(\underline{c}))$, for α_{pi} and $\underline{c}$ such that $\underline{c} \notin C_{pi} + \mathbb{N}^r$ and $| \underline{c} | = j - \beta_{pi}$.

Proof. Let us consider the Koszul homology modules $H_p = H_p(\underline{X}, R)$ of the Rees algebra R with respect to $X_1, \dots, X_n$. For any p, H_p is a finitely generated bigraded S_2-module with $[H_p]_{(i,j)} = \mathrm{Tor}_p^A(k, I^j)_i$. By Lemma 6.3.16, there exist $(\alpha_{pi}, \beta_{pi}) \in \mathbb{Z}^2$ and sets $C_{pi} \subset \mathbb{N}^r$ such that

$$\delta_{H_p}(j) = \bigcup_i \{v(\underline{c}) + \alpha_{pi} : \underline{c} \notin C_{pi} + \mathbb{N}^r, | \underline{c} | = j - \beta_{pi}\},$$

so we get the statement. □

Similarly to the equigenerated case, we can also prove that the rank of the modules of the graded minimal free resolution of the powers of an ideal behaves as a polynomial.

Proposition 6.3.18 *Let I be a homogeneous ideal of the polynomial ring $A = k[X_1, \dots, X_n]$ minimally generated by forms $f_1, \dots, f_r$ of degrees $d_1, \dots, d_r$, and set $l = l(I)$. For any pair (α, β) and set C in the previous proposition, there exist a polynomial $Q(j)$ of degree $\leq l - 1$ such that for any $j \gg 0$*

$$Q(j) = \#\{\underline{c} : \underline{c} \notin C + \mathbb{N}^r, | \underline{c} | = j - \beta\} \ .$$

Proof. Given (α, β), let us define $H_p^{(\alpha,\beta)} = \bigoplus_j (\bigoplus_{|\underline{c}|=j-\beta} [H_p]_{(\alpha+v(\underline{c}),j)})$. Since $H_p^{(\alpha,\beta)}$ is a finitely generated graded F- module of dimension $\leq l$, there exists a polynomial $Q(j)$ of degree $\leq l - 1$ such that for $j \gg 0$

$$\begin{aligned} Q(j) &= \dim_k [H_p^{(\alpha,\beta)}]_j \\ &= \#\{\underline{c} : \underline{c} \notin C + \mathbb{N}^r, | \underline{c} | = j - \beta\}. \ \square \end{aligned}$$

Remark 6.3.19 Given a homogeneous ideal I in the polynomial ring A generated by r forms of degree d, we considered its Rees algebra R with a natural bigrading. By defining $S = k[X_1, \dots, X_n, Y_1, \dots, Y_r]$ the polynomial ring with the bigrading deg$X_i = (1,0)$, deg$Y_j = (d,1)$, R is a bigraded finite S-module in a natural way. Now let E be any bigraded finitely generated S-module, and let consider the graded A-modules $E^j = \oplus_i E_{(i,j)}$. By taking E instead of R, we can get analogous results for the asymptotic behaviour of the A-modules E^j. In particular, by considering E to be the form ring G of I, the integral clousure of the Rees algebra $\overline{R} = \bigoplus_j \overline{I^j}$ or the symmetric algebra Sym$_A(I)$ of I we have the asymptotic behaviour of I^j/I^{j+1}, $\overline{I^j}$ and Sym$_j(I)$.