Introduction

In this memoir, the main topic is the study of the dynamics close to resonant (in the sense we are going to precise) periodic orbits. We focus on real analytic three-degree of freedom Hamiltonian systems and our objective is to investigate the quasi-periodic bifurcation phenomena linked to one parameter families of periodic orbits undergoing $1 : -1$ resonance. To be more concrete, we assume that the orbits of the family are first linearly stable; for a *critical* value of the parameter, the nontrivial (i. e., those different from one) characteristic multipliers of the corresponding periodic orbit collide (the so called Krein collision) on the unit circle: this corresponds to the *critical* $1 : -1$ *resonant* or simply *resonant* periodic orbit. Then, if certain generic conditions are met, the characteristic multipliers leave out the unit circle to the complex plane, hence the family looses its (linear) stability and the periodic orbits become *complex unstable*.

This transition stable-complex unstable is not a strange or uncommon phenomenon, so examples can be found in several fields of science, from astronomy –galactic dynamics (see Martinet, 1984; Pfenniger, 1985b, 1990; Ollé and Pfenniger, 1998), planetary theory (e. g. in Hadjidemetriou, 1985)–, to particle accelerators (Howard et al., 1986). Moreover, not only in three degrees of freedom Hamiltonian systems, but also in higher dimensional problems. For example, in Ollé et al. (1999) were found families of periodic orbits with transitions stable-complex unstable for the spatial elliptic three-body problem (three and a half degrees of freedom): two pair of characteristic multipliers collide, while the third stays on the unit circle.

On the other hand, three-degree of freedom Hamiltonian systems can be investigated through Poincaré (or first return) four dimensional maps (see appendix B, section B.2 for a short description of Poincaré maps paraphernalia). This reduces the study of a Hamiltonian system in $\mathbb{R}^6$ to that of symplectic maps defined on a certain four dimensional *surface of section*. It turns out that each element in the dynamical backbone of the flow has its map counterpart. So for instance (and particularly interesting for us): periodic orbits and two dimensional invariant tori on the system are pictured as fixed points and invariant curves on the map respectively. Even more, the eigenvalues of a fixed point (we mean, those of the linearization of the map around that fixed point) are given by the nontrivial characteristic multipliers of the periodic orbit of the flow it comes from.

Hence (despite the interest they could have on their own), the study of symplectic maps are often used to envisage some qualitative properties of Hamiltonian systems. With this aim, in order to explore the motion close to complex instability, several researchers have investigated one-parameter families of symplectic diffeomorphisms with a fixed point undergoing Krein collisions between its characteristic multipliers. Particularly, in Pfenniger (1985a), and Ollé and Pfenniger (1999) two symplectic generalizations of the Froeschlé's map, $T_s, T_t : \mathbb{R}^4 \to \mathbb{R}^4$, were explored (see section 1.3.2 of chapter 1 for a more detailed description). Both T_s and T_t have a fixed point at 0 and depend on a parameter L in such a way that, for values L less than some critic value, L_{crit}, the fixed point is linearly stable (its characteristic multipliers lie on the unit circle), when $L = L_{crit}$ a pairwise eigenvalue collision there takes place, and for $L > L_{crit}$ they leave the unit circle to the complex plane. In addition, we shall assume that the argument of the two complex conjugate multipliers of the fixed point for L_{crit} (i. e., when eigenvalue collision takes place) are not commensurable with 2π –one speaks then about *irrational* collision–. Therefore, for T_s, stable invariant curves unfold for each $L > L_{crit}$ (so, on the "unstable side", since for these values of the parameter, the fixed point is complex-unstable). This is known as the *direct* bifurcation. For the map T_t, though, unstable invariant curves rise, for $L < L_{crit}$ "on the stable side": they appear thus in a similar way as limit cycles do in the classical Andronov-Hopf bifurcation (see Andronov et al., 1959, chap. VI, §4). We shall also mention that rational collisions (where the argument of the characteristic multipliers is 2π-commensurable at collision) can occur. Then, generically, multiple periodic points unfold. This situation is likewise investigated at the first of the papers quoted at the beginning of the paragraph (and analytically, in a more general context, in Bridges, 1990, 1991). However, as we are interested in quasi-periodic motions, irrationality is assumed throughout.

With T_s and T_t as paradigmatic examples, we wonder if such behavior (well understood from *numerical* research) could be dealt analytically in a general symplectic context, as Bridges, Cushman and Mackay (1995) did for symplectic maps and Van der Meer (1985) for the Hamiltonian Hopf bifurcation at equilibrium points in two-degrees of freedom Hamiltonian systems (also, see Meyer and Schmidt, 1971; Schmidt, 1994; Meyer, 1998). The improvements of this work can be summarized as: (i) We rely on normal forms as *one* of the key tools of our approach, deriving in a *constructive* way and up to *any* (arbitrary) order, a versal normal form of the Hamiltonian around the resonant periodic orbit. Analyzing the (truncated) normal form, we describe the mechanism of the 2D-invariant tori unfolding, according to the Hamiltonian Hopf pattern, identifying those parameters which govern not only the bifurcation, but also its character. We remark that this is not a merely qualitative process for, in addition, accurate parametrizations of the families of invariant tori are derived in this way. (ii) We compute "optimal" bounds for the remainder of the normal form, so one expects to prove the preservation of a higher number (in measure sense) of invariant tori –than, indeed, with a less sharp estimates–. (iii) And, finally, we apply KAM methods to establish the persistence of most (in the measure sense) of the bifurcated invariant tori. A more detailed explanation of these points is given below.

In chapter 1 we state the problem. So let us consider a real analytic three-degree of freedom Hamiltonian, $H(\boldsymbol{\zeta})$, $\boldsymbol{\zeta}^* = (\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3) \in \mathbb{R}^6$ with the corresponding system

$$\dot{\boldsymbol{\zeta}} = J_3 \mathrm{grad}\, H(\boldsymbol{\zeta}), \tag{1}$$

being J_3 the matrix of the standard canonical 2-form in $\mathbb{R}^6$ (see appendix B, section B.1). Suppose that this system has a non-degenerate family of periodic orbits, $\{\mathcal{M}_\sigma\}_{\sigma\in\mathbb{R}}$, such that for some value of the parameter, say $\sigma = 0$, the corresponding (critical, resonant) periodic orbit, $\mathcal{M}_0$, has a collision of its nontrivial characteristic multipliers. To be more precise (see figure 1.2 in chapter 1), suppose that, for $\sigma < 0$, these characteristic multipliers of $\mathcal{M}_\sigma$ lie on the unit circle, they approach pairwise as σ goes to 0, for this value they collide and separate towards the complex plane when $\sigma > 0$. Moreover:

(i) As it has been already mentioned above, we assume the collision is irrational: more precisely, if $2\pi\nu_0$ is the characteristic exponent corresponding to the characteristic multiplier λ_0 of the resonant periodic orbit $\mathcal{M}_0$ (so $\lambda_0 = \mathrm{e}^{2\pi\, i\nu_0}$), then $\nu_0 \notin \mathbb{Q}$.

(ii) We suppose both, non-degeneracy of the family of periodic orbits, –that is, we ask the twist condition[(1)]–, and non-degeneracy of the collision, since one requires the eigenvalues to leave the unit circle for $\sigma > 0$. These assumptions imply that the monodromy matrix of $\mathcal{M}_0$ should not have trivial (diagonal) Jordan blocks, neither those corresponding to the (double) eigenvalues λ, $\overline{\lambda}$ nor the (non-diagonal) block associated to the (double) eigenvalue 1.

Prior to the normal form computations, we need to perform some previous reductions. The objective is to transform the initial raw Hamiltonian into a new one that eases the application of the normal form algorithm. This process involves: (i) the introduction of local coordinates around the orbit $\mathcal{M}_0$, (ii) Floquet reduction of the quadratic part of the Hamiltonian and (iii) complexification to make this quadratic part as simple as possible.

Next, we use Giorgilli-Galgani algorithm (see definition 1.7 in chapter 1 and the references given there) to carry out the nonlinear reduction. The reasons for the choice of the Giorgilli-Galgani machinery to compute the nonlinear normalization are basically two: one of them is motivated by the practical implementation of this methodology. So, if one plans to apply normal form computations to our problem (we mean, a three-degree of freedom Hamiltonian system with a family of periodic orbits undergoing a transition stable to complex-unstable), then the Giorgilli-Galgani algorithm is a very efficient way to carry this process out (see Giorgilli et al., 1989; Simó, 1989; Delshams and Gutiérrez, 1996). In this sense, we want to stress that the solvability of the homological equations (with the intricacies due to its Jordan block structure) has been discussed in chapter 1 in a constructive way, we mean: not only the resonant terms are identified, but also we issue an algorithm to compute them explicitly, as well as the coefficients of the generating function (compare with Schmidt, 1994; Bridges, Cushman and Mackay, 1995).

The second reason is of a deeper technical nature, so it is a more involved task to discuss it here. However, we can try to give a systematic outline: first of all, it is well known that in presence of resonances, normal forms do not converge in general. Hence, a natural question (that in the present context is discussed in chapter 2) is to ask what is the optimal order up to which the normal form should be computed. As this order is not known *a priori*, a good idea to obtain it *a fortiori* is to perform nonlinear normalization as a composition of canonical transformations such that, at any step, the first order correction of the corresponding transformation kills all the terms not left in normal form after the previous step (see, for instance Nekhoroshev, 1977, 1979; Perry and Wiggins, 1994). In the process, one sets up and solves homological equations holding monomials of arbitrary high order, and finally is precisely the number of steps what determines the order of the normal form. However, though this process works well in the semi-simple[(2)] case, in our context, if we try to solve the homological equations at any order, the solution is no longer convergent (even when we have good arithmetic properties for the frequencies). Thus it will be clear that, to derive the optimal normalization order, one has to proceed order by order and then, closed algorithms become much more efficient and, among them, that of Giorgilli-Galgani, fits specially well to our purposes.

Consequently, it is worth remarking here that the *homological equations* we have to solve in order to determine the generating function cannot be transformed into diagonal form –as happens when one normalizes around semi-simple periodic orbits–. This makes the derivation of the structure of the normal form a more involved task. At the end, we are able to give, in

[(1)]At least locally at $\sigma = 0$, so if $\omega(0)$ is the frequency of the critical periodic orbit, $\mathcal{M}_0$, then it must be $\omega'(0) \neq 0$

[(2)]We mean, when the monodromy matrix of the periodic orbit has pairwise different normal eigenvalues.

theorem 1.24 on page 37, a versal normal form for three-degree of freedom Hamiltonian around the $1:-1$ resonant periodic orbit $\mathcal{M}_0$. Below, a (short) version of this result follows.

Theorem 1. *Consider the above Hamiltonian system (1). Under the forementioned conditions, H can be reduced, by means of a symplectic change defined in a neighborhood of the periodic orbit $\mathcal{M}_0$, to a real Hamiltonian which (keeping the same name for the old and the transformed one), is given by*

$$H(\theta_1,\boldsymbol{x},I_1,\boldsymbol{y}) = \omega_1 I_1 + \omega_2 \boldsymbol{y}\times\boldsymbol{x} + \tfrac{1}{2}|\boldsymbol{y}|_2^2 + \mathcal{Z}_r\left(\tfrac{1}{2}|\boldsymbol{x}|_2^2, I_1, \boldsymbol{y}\times\boldsymbol{x}\right) + \mathfrak{R}^{(r)}(\theta_1,\boldsymbol{x},I_1,\boldsymbol{y}),$$

with the notation,

$$|\boldsymbol{x}|_2 = (x_1^2+x_2^2)^{1/2}, \quad |\boldsymbol{y}|_2 = (y_1^2+y_2^2)^{1/2}, \quad \boldsymbol{x}\times\boldsymbol{y} = x_1y_2 - x_2y_1,$$

ω_1 being the frequency of the resonant periodic orbit and ω_2/ω_1 (assumed not commensurable with 2π), the characteristic exponent of the critical periodic orbit, $\mathcal{M}_0$, respectively; $\mathcal{Z}_r(u,v,w)$ a polynomial of degree[(3)] $\lfloor r/2\rfloor$ beginning with quadratic terms and $\mathfrak{R}^{(r)}(\theta_1,\boldsymbol{x},I_1,\boldsymbol{y})$ 2π-periodic in θ_1 holding terms of "degree" higher than r (see chapter 1, section 1.7, for a precise definition of what means degree in our context).

The quadratic part of the Hamiltonian plus $\mathcal{Z}_r(\cdots)$ in the statement of this last theorem will be denoted by $Z^{(r)}$, i. e.,

$$Z^{(r)}(\boldsymbol{x},I_1,\boldsymbol{y}) = \omega_1 I_1 + \omega_2\boldsymbol{y}\times\boldsymbol{x} + \tfrac{1}{2}|\boldsymbol{y}|_2^2 + \mathcal{Z}_r\left(\tfrac{1}{2}|\boldsymbol{x}|_2^2, I_1, \boldsymbol{y}\times\boldsymbol{x}\right).$$

Moreover, to formulate some results, it will be convenient to express the polynomial $\mathcal{Z}_2$ as,

$$\mathcal{Z}_2(u,v,w) = \frac{1}{2}(au^2+bv^2+cw^2) + duv + euw + fvw, \tag{2}$$

with a, b, c, d, e f real coefficients. $Z^{(r)}$ is the (real) normal form up to order (degree) r and is proven to be integrable, whereas $\mathfrak{R}^{(r)}$ stands for the (non-integrable) remainder. Then, from section 1.8, and up to the end of chapter 1, $\mathfrak{R}^{(r)}$ is dropped so only the dynamics of the normal form is accounted for.

Parametrization of the family of periodic orbits: it is immediately seen that

$$\left.\begin{aligned} &\theta_1 = (\omega_1 + \partial_2\mathcal{Z}_r(0,I_1,0))t + \theta_1^0,\\ &I_1 = \text{const.},\\ &x_1 = x_2 = y_1 = y_2 = 0, \end{aligned}\right\} \tag{3}$$

is a family (with I_1 as parameter) of periodic orbits of the Hamiltonian system $Z^{(r)}$, which will match the family $\mathcal{M}_\sigma$ of the complete Hamiltonian. Whence, the characteristic exponents associated to the normal directions can be computed in terms of the action I_1 to give

$$[rl]\alpha_{I_1}^{\pm} = \mathrm{i}(\omega_2 + fI_1) \pm \sqrt{-dI_1 + \mathrm{O}(I_1^2)} + \mathrm{O}(I_1^2), \tag{4a}$$
$$\beta_{I_1}^{\pm} = -\mathrm{i}(\omega_2 + fI_1) \pm \sqrt{-dI_1 + \mathrm{O}(I_1^2)} + \mathrm{O}(I_1^2), \tag{4b}$$

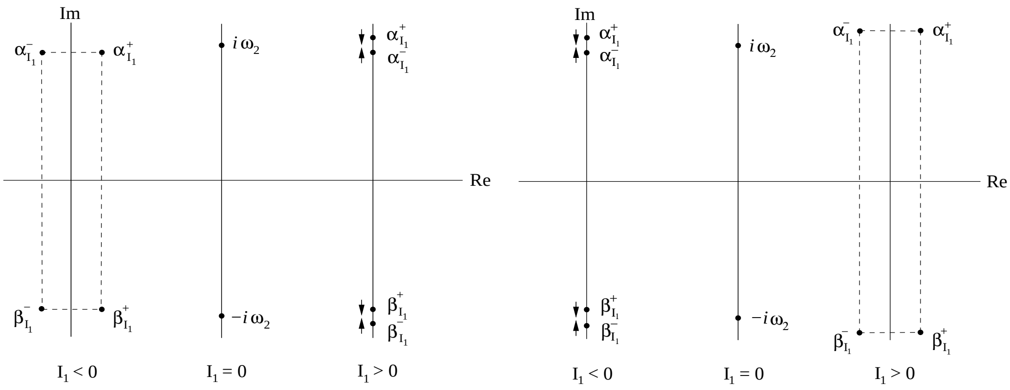
from these expressions we see that if $|I_1|$ is small enough, the sign of the content inside the square roots, depends mainly on the sign of $-dI_1$. This opens two possibilities:

(3) Throughout the text, the symbol $\lfloor\cdot\rfloor$ will mean the integral part, i. e., for $x\in\mathbb{R}$,

$$\lfloor x\rfloor = \max\{z\in\mathbb{Z} : z\le x\}.$$

Case 1. $d > 0$, The family of periodic orbits (3) is complex unstable for $I_1 < 0$, and (linearly) stable for $I_1 > 0$. See figure below.

Case 2. $d < 0$. The family turns out to be (linearly) stable for $I_1 < 0$ and complex unstable for $I_1 > 0$, as it can be appreciated in the figure.



(a) $d > 0$. (b) $d < 0$.

Figure 1: We note that when $I_1 = 0$, then $\alpha_0^- = \alpha_0^+ = i\omega_2$ and $\beta_0^- = \beta_0^+ = -i\omega_2$ (collision of characteristics exponents). Therefore, the family changes its linear character from complex-unstable to stable (when $d > 0$, fig. 1(a)), or vice-versa (when $d < 0$, fig.1(b)).

On the quasi-periodic solutions: further, the quasi-periodic solutions (note, solutions of $Z^{(r)}$) we seek are more easily derived if first the symplectic change,

$$\begin{aligned} x_1 &= \sqrt{2q}\cos\theta_2, & y_1 &= -\frac{I_2}{\sqrt{2q}}\sin\theta_2 + p\sqrt{2q}\cos\theta_2, \\ x_2 &= -\sqrt{2q}\sin\theta_2, & y_2 &= -\frac{I_2}{\sqrt{2q}}\cos\theta_2 - p\sqrt{2q}\sin\theta_2, \end{aligned} \tag{5}$$

is applied to $Z^{(r)}$ introducing thus an extra angle, θ_2, and its conjugate action, I_2. Taking these new coordinates: $(\theta_1, \theta_2, q, I_1, I_2, p)$, the corresponding Hamiltonian equations are bound to be,

$$\begin{aligned} \dot\theta_1 &= \omega_1 + \partial_2 \mathcal{Z}_r(q, I_1, I_2), \\ \dot\theta_2 &= \omega_2 + \frac{I_2}{2q} + \partial_3 \mathcal{Z}_r(q, I_1, I_2), \\ \dot q &= 2qp, \\ \dot I_1 &= 0, \\ \dot I_2 &= 0, \\ \dot p &= -p^2 + \frac{I_2^2}{4q^2} - \partial_1 \mathcal{Z}_r(q, I_1, I_2). \end{aligned} \tag{6}$$

So one might compute 2D-invariant tori as equilibrium points of $\dot q = 0$, $\dot p = 0$. With this idea, the next theorem follows naturally (see section 1.9, page 40).

Theorem 2. *Let the coefficients a, b and d be those in $\mathcal{Z}_2$. Then:*

(i) If $a \neq 0$, there exists a real analytic function $\varUpsilon : \mathcal{U} \subset \mathbb{R}^2 \to \mathbb{R}$, defined in some neighborhood of the origin $\mathcal{U} \subset \mathbb{R}^2$, such that the functions $\{\mathcal{T}(t; \boldsymbol{J}, \boldsymbol{\theta}^0), t \in \mathbb{R}\}_{\boldsymbol{J} \in \mathcal{U}, \boldsymbol{\theta}^0 \in \mathbb{T}^2}$, with $\boldsymbol{J}^* = (J_1, J_2)$ and,

$$
\begin{aligned}
\mathcal{T}(t; \boldsymbol{J}, \boldsymbol{\theta}^0) &= \begin{pmatrix} \boldsymbol{\varOmega}(\boldsymbol{J})t + \boldsymbol{\theta}^0 \\ \varUpsilon(\boldsymbol{J}) \\ J_1 \\ 2J_2\varUpsilon(\boldsymbol{J}) \\ 0 \end{pmatrix} \in \mathbb{T}^2 \times \mathbb{R}^4 \qquad (\mathbb{T}^2 = R^2/2\pi\mathbb{Z}^2), && (7) \\
\varOmega_1(\boldsymbol{J}) &= \omega_1 + \partial_2 \mathcal{Z}_r \left(\varUpsilon(\boldsymbol{J}), J_1, 2J_2\varUpsilon(\boldsymbol{J})\right), && (8) \\
\varOmega_2(\boldsymbol{J}) &= \omega_2 + J_2 + \partial_3 \mathcal{Z}_r \left(\varUpsilon(\boldsymbol{J}), J_1, 2J_2\varUpsilon(\boldsymbol{J})\right), && (9)
\end{aligned}
$$

are solutions winding a two-parameter family of 2D-tori of the system (6) with intrinsic (or basic) frequencies given by (8) and (9).

(ii) If, in addition, $b - \frac{d^2}{a} \neq 0$, the invariant tori of the first item are non-degenerate, in the sense that the matrix of the derivatives of $\boldsymbol{\varOmega}^* = (\varOmega_1, \varOmega_2)$ with respect the parameters $\boldsymbol{J}^* = (J_1, J_2)$ is not singular at the origin, i. e., $\det\left(\partial_{\boldsymbol{J}} \boldsymbol{\varOmega}(\mathbf{0})\right) \neq 0$.

In particular, the second item means that the frequencies $\varOmega_1$, $\varOmega_2$ in (8), (9) map diffeomorphically (locally at the origin) the space of the parameters into the space of frequencies, so the tori in the family could be described equally well using the frequencies $\boldsymbol{\varOmega}$ instead of the parameters $\boldsymbol{J}$.

Remark 3. From the expression for the polar coordinates in (5), and the parametrization of the family of invariant tori follows that only those values of the parameters J_1, J_2 making $\varUpsilon(J_1, J_2) > 0$ will determine real quasi-periodic solutions. As $\varUpsilon(J_1, J_2)$ is obtained applying the implicit function theorem (section 1.9, proof of theorem 1.27) at $(0, 0)$, in particular, one can compute its expansion around the origin. Up to second order in J_1, J_2, results:

$$
\varUpsilon(J_1, J_2) = -\frac{d}{a} J_1 + \frac{1}{a} J_2^2 + \dots,
$$

so real invariant tori are assured for $\boldsymbol{J}$ in a neighborhood of the origin if, for instance, the two additional conditions

$$
-\frac{d}{a} J_1 > 0 \quad \text{and} \quad |J_2| \leq |J_1|^\alpha,
$$

with $\alpha > 1/2$ are simultaneously fulfilled. ♠

Before studying the linear character of the invariant tori, we note that, if through the change (5) we transform back to the normal form coordinates, the family of invariant tori given above is expressed as $\theta_1(t; \boldsymbol{J}, \boldsymbol{\theta}^0) = \varOmega_1(\boldsymbol{J})t + \theta_1^{(0)}$, $I_1 = J_1$ and,

$$
\begin{aligned}
\boldsymbol{x}(t; \boldsymbol{J}, \boldsymbol{\theta}^0) &= \begin{pmatrix} -\sqrt{\varUpsilon(\boldsymbol{J})} \sin(\varOmega_2(\boldsymbol{J})t + \theta_2^0) \\ \sqrt{\varUpsilon(\boldsymbol{J})} \cos(\varOmega_2(\boldsymbol{J})t + \theta_2^0) \end{pmatrix}, \\
\boldsymbol{y}(t; \boldsymbol{J}, \boldsymbol{\theta}^0) &= \begin{pmatrix} -\sqrt{2J_2\varUpsilon(\boldsymbol{J})} \sin(\varOmega_2(\boldsymbol{J})t + \theta_2^0) \\ -\sqrt{2J_2\varUpsilon(\boldsymbol{J})} \cos(\varOmega_2(\boldsymbol{J})t + \theta_2^0) \end{pmatrix}.
\end{aligned}
$$

Therefore, in the phase plane (x_1, x_2), the family of tori plots as a family of invariant circles of radii $(\Upsilon(\boldsymbol{J}))^{1/2}$ and centered at the origin where the periodic orbits are placed (analogously in (y_1, y_2)). In this sense, the (bifurcated) 2D-invariant tori appear "around" the periodic orbits.

Normal behavior of the invariant tori. Next, to study the linear stability of the family of invariant tori (7) one first sets up the variational equations of system (6) around one of the real torus of the family in the normal directions, say $\mathcal{T}(t; \boldsymbol{J}, \boldsymbol{\theta}^0)$, for some $J \in \mathcal{U}$:

$$\begin{aligned} \dot{X} &= 2\Upsilon(\boldsymbol{J})\, Y, \\ \dot{Y} &= -\left(\frac{2J_2^2}{\Upsilon(\boldsymbol{J})} + \partial_{1,1}^2 \mathcal{Z}_r(\Upsilon(\boldsymbol{J}), J_1, 2J_2\Upsilon(\boldsymbol{J})) \right) X, \end{aligned} \tag{10}$$

and sees that,

$$\varrho_{J_1,J_2}^{\pm} = \pm\sqrt{-4J_2^2 - 2\Upsilon(\boldsymbol{J})\partial_{1,1}^2 \mathcal{Z}_r(\Upsilon(\boldsymbol{J}), J_1, 2J_2\Upsilon(\boldsymbol{J}))}$$

are its characteristic exponents, but expansion of the stuff inside the square root with respect to $\boldsymbol{J}$ yields:

$$\varrho_{J_1,J_2}^{\pm} = \pm\sqrt{2dJ_1 - 6J_2^2 + \dots} \tag{11}$$

so, under the reality conditions in remark 3 the normal behavior of the invariant tori is determined –for $|J_1|$, $|J_2|$ sufficiently small–, just by the first term $2dJ_1$ inside the square root. Hence, suppose first that the coefficient a is positive. Then, under the reality condition dJ_1 must be negative, so by (11) the invariant tori will be elliptic and the periodic orbit with $I_1 = J_1$ is –linear–, unstable (see figure 1). Otherwise, let a be negative. Now the reality condition forces $dJ_1 > 0$, which implies hyperbolic tori and the periodic orbit surrounded (in the sense specified above) by the invariant tori is stable for $I_1 = J_1$. These considerations motivate the proposition 1.29 at the end of chapter 1, which we also write here.

Proposition 4. *Under the assumptions of theorem 2 –including the reality conditions of remark 3–, the type of the bifurcation is determined by the sign of the coefficient a. More precisely:*

Case 1. $a > 0$ *then, we say that the bifurcation is "direct": there appear elliptic tori around complex-unstable periodic orbits; and if*

Case 2. $a < 0$ *the bifurcation is "inverse": hyperbolic invariant tori unfold around stable periodic orbits. In this case, the family contains also parabolic and elliptic tori.*

In this way, we have determined the parameter, a, that governs both, the existence and the type of quasi-periodic bifurcation.

Chapter 2. Thus far the *formal* part of the memoir. Indeed, it is worth studying the whole Hamiltonian (normal form the plus remainder) and to inquire whether quasi-periodic solutions still persist (or not) after the remainder is added. So, one has to mess around with some type of KAM perturbation methods, but this implies some knowledge –or, at least, some hypothesis–, about how large the perturbation could be (because is this size which determines the order of the measure of the gaps). For us, two approaches are possible at this point: one can deal with the normal form as an integrable Hamiltonian and then add a generic perturbation –considered as small as one might need– or work in a more quantitative direction, asking, if R is the distance to the resonant periodic orbit, what could be the order, $r_{opt} = r_{opt}(R)$, of the normal form leading to the smallest remainder[4] in this R-neighborhood. Thus, one surely will be able to derive R-dependent "optimal" effective bounds, say $M(R)$, such that $\|\mathfrak{R}^{(r_{opt})}\| \le M(R)$. In the analytical context, and assuming Diophantine frequencies, one typically expects to obtain exponentially small bounds for the remainder as a function of R. Something like:

$$M(R) \sim \mathrm{O}\left(\exp\left(-\frac{c_1}{R^{c_2}}\right)\right),$$

for certain $c_1, c_2 > 0$ (c_2 typically depends on the exponent of the Diophantine conditions). However, this works for semi-simple homological equations (even when resonant periodic orbits are dealt). In our case, though, the Jordan structure of the homological equations causes that when one solves for the coefficients associated to a monomial of certain degree, n, the associated small divisors appear not just raised up to the first power (as happens in the semi-simple case), but up to a power which depends on the degree n. Heuristically speaking, the amplification factor n^τ, ($\tau > 1$ independent of n) of the terms of degree n of the solutions of the homological equations, is replaced by a factor that can be guessed to be close to $n^{\tau n}$. The bounds obtained in the present resonant non semi-simple case are given in proposition 2.19 on page 78. We summarize below –omitting technicalities–, the main results described there:

Theorem 5. *If the frequencies $\boldsymbol{\omega}^* = (\omega_1, \omega_2)$ in the Hamiltonian H of theorem 1 satisfy the Diophantine conditions,*

$$|\langle \boldsymbol{k}, \boldsymbol{\omega}\rangle| \ge \frac{\gamma}{|\boldsymbol{k}|^\tau}, \qquad \boldsymbol{k} \in \mathbb{Z}^2 \setminus \{\boldsymbol{0}\},$$

for $\tau > 1$ and for a certain $\gamma > 0$. Then:

(i) There is $R_0 > 0$ such that, for a given distance R to the critical periodic orbit with $0 < R < R_0$, the optimal normalizing order –according to our bounds–, is given by $r_{opt} = \lfloor \check{r} \rfloor$, with $\check{r}$ depending on R through,

$$\check{r} = \mathrm{e}^{W_0\left(\ln\left(c_3 \frac{R}{R_0}\right)^{-1/c_4}\right)},$$

where $\mathrm{e} = \exp(1)$; c_3, c_4 are two constants independent of R and W_0 denotes the principal branch of the Lambert-W function –see section 2.6.1 and references therein[5]–.

(ii) The remainder of the normal form (theorem 1) is bounded by,

$$|||\mathfrak{R}^{(r_{opt})}||| \le c_5 \left(1 - \frac{R}{R_0}\right)^{-1} \left(c_3 \frac{R}{R_0}\right)^{\frac{r_{opt}(R)}{2} - 1}, \tag{12}$$

in this R-neighborhood of the critical periodic orbit. Here, c_5 is a constant independent of R.

(iii) Moreover, $\mathfrak{R}^{(r_{opt})}$ goes to zero faster than any analytic order, i. e.,

$$\mathfrak{R}^{(r_{opt})} = \mathrm{o}\left(\left(\frac{R}{R_0}\right)^n\right), \qquad (R/R_0 \to 0^+),$$

for any positive integer n.

[4] or, at least, that optimizes some suitable bound of the remainder.

[5] Here, it is enough to known that for any $z \in \mathbb{R}$, $z > -1/\mathrm{e}$, is $w = W_0(z) \Leftrightarrow w\mathrm{e}^w = z$ with $w > -1$.

Therefore, in a small enough neighborhood of the critical periodic orbit, the remainder, $\mathfrak{R}^{(r_{opt})}$, can be thought of as a perturbation of the normal form, and justifies the application of KAM methods in the next chapter. Of course, the setting of a very explicit constructive scheme to compute the normal form plays an essential rôle in the derivation of the bounds given above.

Chapter 3 is devoted to the discussion of the persistence of the bifurcated invariant tori derived in chapter 1. First, we note that we have different possible situations where persistence can be investigated: we can consider the case of direct bifurcation ($a > 0$) or the inverse ($a < 0$). Furthermore (and depending on the case) we can study the persistence of (real) *elliptic*, *parabolic* or *hyperbolic* tori. What we have done here is to study in detail the case of elliptic tori in the direct bifurcation. We have chosen the *elliptic* case because the context of elliptic tori is always the most difficult to deal with, and contains almost all the difficulties inherent to this (degenerate) problem. Likewise, we think that is important to stress here the main difficulties (and differences) of this problem with respect to others results of persistence of invariant tori (see Sevryuk, 1997; Pöschel, 1989) which make interesting by itself the methodology that we have followed in chapter 3. To discuss this, let us start giving a new parametrization of the unperturbed (i. e., those coming from the normal form) bifurcated families of 2D-tori. This parametrization will be more suitable if –as in our case– one wants to control the real character of the tori (see theorem 3.1 o page 96):

Theorem 6. *If the coefficient d in (2) is $d \neq 0$, then the function $\mathcal{T} : \mathbb{R} \times \Gamma \times \mathbb{T}^2 \to \mathbb{R}^6$, defined by*

$$\mathcal{T}(t;\xi,\eta;\boldsymbol{\theta}^{(0)}) = \left(\boldsymbol{\Omega}(\xi,\eta)t + \boldsymbol{\theta}^{(0)}, \xi, \mathcal{I}(\xi,\eta), 2\xi\eta, 0\right) \tag{13}$$

with: $\Gamma \subset \mathbb{R}^2$ a neighborhood of $(0,0)$, $\mathcal{I}$ an analytic function on Γ, defined implicitly by the equation

$$\eta^2 = \partial_1 \mathcal{Z}_r \left(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta\right), \tag{14}$$

and the vector of frequencies, $\boldsymbol{\Omega}^* = (\Omega_1, \Omega_2)$ given by the components,

$$\begin{aligned} \Omega_1(\xi,\eta) &= \omega_1 + \partial_2 \mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta), \\ \Omega_2(\xi,\eta) &= \omega_2 + \eta + \partial_3 \mathcal{Z}_r(\xi, \mathcal{I}(\xi,\eta), 2\xi\eta); \end{aligned} \tag{15}$$

constitutes a two-parameter family of solutions of (6) winding the corresponding family of two-dimensional invariant tori.

In terms of these new parameters, the characteristic exponents of the unperturbed family are given by,

$$\lambda_\pm(\xi,\eta) = \pm\sqrt{-4\eta^2 - 2a\xi - 2\xi\partial^2_{1,1}\mathcal{Z}_3(\xi,\mathcal{I}(\xi,\eta),2\xi\eta)}, \quad (\xi,\eta) \in \Gamma,\ \xi > 0.$$

Then, the first difficulty arises when we have to choose the suitable set of parameters to characterize the tori of the family along the iterative KAM process. Let us mention that we have three frequencies to control: the two intrinsic ones, Ω_1, Ω_2 of the quasi-periodic motion and the normal one (the real part of λ_+), but just two parameters to control them ξ, η. So, we have to face the so called "lack of parameters" problem (see Moser, 1967; Sevryuk, 1999, or chapter 3 of this memoir for a more detailed explanation).

Typically, on applying KAM techniques for low-dimensional tori, one sets a diffeomorphism between some neighborhood of the origin in the parameter space (ξ,η) and a vicinity of (ω_1, ω_2) in the space of intrinsic frequencies (Ω_1, Ω_2) (similarly as stated in item (ii), theorem 2). Hence, the characteristic exponents $\lambda_\pm$ may be put also as a function of the intrinsic frequencies. For elliptic tori, besides the non-degeneracy of the these frequencies, one needs to ask the normal frequencies to "move" as a function of $\boldsymbol{\Omega}$, this forces to impose suitable "transversal" conditions in the denominators of the KAM process (see Sevryuk, 1999; Jorba and Villanueva, 1997a). In our case, for $\boldsymbol{\zeta} = (\xi, \eta)$ in a small neighborhood of the origin the invariant tori will be elliptic when $a > 0$ (and $\xi > 0$), as follows easily form the expression for $\lambda_\pm$. However, the typical transversal conditions,

$$\mathrm{Im}\ (\mathrm{grad}_{\boldsymbol{\Omega}} \langle \boldsymbol{\ell}, \boldsymbol{\lambda}(\boldsymbol{\Omega}) \rangle |_{\boldsymbol{\Omega}=\boldsymbol{\omega}}) \notin \mathbb{Z}^2, \qquad \text{for any } \boldsymbol{\ell} \in \mathbb{Z}^2 \text{ with } 0 \leq |\ell_1| + |\ell_2| \leq 2,\ \ell_1 \neq \ell_2,$$

(where $\boldsymbol{\lambda}^*(\boldsymbol{\Omega}) = (\lambda_+(\boldsymbol{\Omega}), \lambda_-(\boldsymbol{\Omega}))$), does not work, because the derivatives of $\boldsymbol{\lambda}(\boldsymbol{\Omega})$ are not defined for $\boldsymbol{\Omega} = \boldsymbol{\omega}$ (the elliptic invariant tori are too close to parabolic). We have overcome this situation taking as *basic frequencies* for the unperturbed tori not $\boldsymbol{\Omega}^* = (\Omega_1, \Omega_2)$, the intrinsic frequencies, but $\boldsymbol{\Lambda}^* = (\mu, \Omega_2)$ with $\mu = |\lambda_+|$ and then, the first component of the intrinsic frequencies, Ω_1, as a function of $\boldsymbol{\Lambda}$, i. e.: $\Omega_1 = \Omega_1(\boldsymbol{\Lambda})$. In other words: we "label" the (elliptic) invariant tori with their normal frequency and second intrinsic frequency. It is checked that, with this parametrization, the small divisors do change in the normal directions, so one can proceed with to the KAM iterative scheme, which –due to the forementioned proximity of parabolic tori–, involves a more tricky control on the different terms of the Hamiltonian appearing at each successive step.

As we have implicitly mentioned, we will look for reducible elliptic tori (see J. Puig, 2002, for a survey on quasi-periodic reducibility), and hence the normal frequency will be well defined. So, in our approach we have not taken into account the machinery allowing to work with non-reducible elliptic tori (see Bourgain, 1997). A non-reducible approach complicates strongly the formulation, with no significative gain in the measure estimates. However, reducibility is a very important trick in order to simplify the resolution of the homological equations, which, in our case (and in spite of these simplifications) have some additional difficulties. To be more precise: the way we choose the basic frequencies, forced by the nondegeneracy conditions yields a coupling between some of the equations. This bind us to a very careful determination of the compatibility term which allows to keep the basic frequencies fixed at any step of the iterative process. On page 104 we state theorem 3.9, gathering the results related with the preservation of the (direct) bifurcated elliptic invariant tori. Here we present a shortened version:

Theorem 7. *Consider the Hamiltonian in theorem 1, assuming that the coefficient a of the quadratic part of the normal form (see (2)) is positive. Moreover, we define $\mu = |\lambda_+|$, $\boldsymbol{\Lambda}^* = (\mu, \Omega_2)$. Then, there exists a symplectic change:*

$$\Psi : \mathcal{D} \times \mathfrak{A} \subseteq \mathbb{T}^2 \times \mathbb{R} \times \mathbb{R}^2 \times \mathbb{R} \times \mathcal{A} \longrightarrow \mathbb{T}^1 \times \mathbb{R}^2 \times \mathbb{R} \times \mathbb{R}^2,$$

where $\mathfrak{A}$ is a Cantorian of the initial set of basic frequencies $\mathcal{A}$, such that:

(1) The measure of $\mathfrak{A}$ is plenty in the following sense: let $M(R)$ denote the bound for $\mathfrak{R}^{(r_{opt})}$ given in theorem 5 and $\mathcal{A}_R$ denote the subset of basic frequencies of $\mathcal{A}$ in a R-neighborhood of $\boldsymbol{\omega}^* = (0, \omega_2)$, then the Lebesgue measure:

$$\textit{meas}\,(\mathcal{A}_R \setminus \mathfrak{A}) \sim (M(R))^{\frac{\alpha}{2}},$$

being $0 < \alpha < 1$, a fixed constant.

(2) The transformed Hamiltonian $\mathcal{H} = H \circ \Psi_{\boldsymbol{\Lambda}}$ *can be cast into:*

$$\mathcal{H}(\boldsymbol{\theta}, q, \boldsymbol{I}, p; \boldsymbol{\Lambda}) = \phi(\boldsymbol{\Lambda}) + \langle \boldsymbol{\Omega}(\boldsymbol{\Lambda}), \boldsymbol{I} \rangle + \frac{1}{2}\langle \boldsymbol{z}, \mathcal{B}(\boldsymbol{\Lambda})\boldsymbol{z} \rangle + \\ + \frac{1}{2}\langle \boldsymbol{I}, \mathcal{C}(\boldsymbol{\theta}; \boldsymbol{\Lambda})\boldsymbol{I} \rangle + \langle \boldsymbol{z}, \mathcal{E}(\boldsymbol{\Lambda})\boldsymbol{I} \rangle + \mathcal{H}_*(\boldsymbol{\theta}, q, \boldsymbol{I}, p; \boldsymbol{\Lambda}), \quad \boldsymbol{\Lambda} \in \mathfrak{A}, \tag{16}$$

where $\boldsymbol{\Omega}^*(\boldsymbol{\Lambda}) = (\Omega_1(\boldsymbol{\Lambda}), \Omega_2)$; $\mathcal{B}$, $\mathcal{E}$, $\mathcal{C}$ *are* 2×2 *matrices (*$\mathcal{B}$, $\mathcal{E}$ *depending only on* $\boldsymbol{\Lambda}$ *whereas the matrix* $\mathcal{C}$ *depends also on* $\boldsymbol{\theta}$*), and* $\mathcal{H}_*$ *holds the terms of order greater than two in* $\boldsymbol{z}$, $\boldsymbol{I}$.

(3) For every $\boldsymbol{\Lambda}$*, the corresponding Hamiltonian* (16) *has a (reducible) invariant torus at* $\boldsymbol{z} = 0$, $\boldsymbol{I} = 0$*, with vector of intrinsic frequencies* $\boldsymbol{\Omega}(\boldsymbol{\Lambda})^* = (\Omega_1(\boldsymbol{\Lambda}), \Omega_2)$*, and normal frequency given by* μ.

(4) $(\boldsymbol{\theta}, \boldsymbol{\Lambda}) \in \mathbb{T}^2 \times \mathfrak{A} \mapsto \Psi(\boldsymbol{\theta}, 0, \mathbf{0}, 0, \boldsymbol{\Lambda})$ *is a parametrization of a Whitney regular Cantorian manifold holding the family of invariant tori, which can be embedded in a* C^∞ *regular manifold, in such a way that the measure of the extension of the Cantorian manifold to this regular manifold, is of the same order than the measure of the gaps coming from the elimination of frequencies in the KAM process.*

In appendix A, we gather some auxiliary lemmas used to prove the different results spread along the text and finally, appendix B includes some basic background on Hamiltonian systems, stability of periodic orbits and transformation theory (Lie method series). Longer than our initial purpose was, it is included only to make this work as self contained as possible.