APPENDIX A

DELAY OSCILLATOR

Dynamics of a single delay oscillator have six different zones as depicted in Figure A.1.

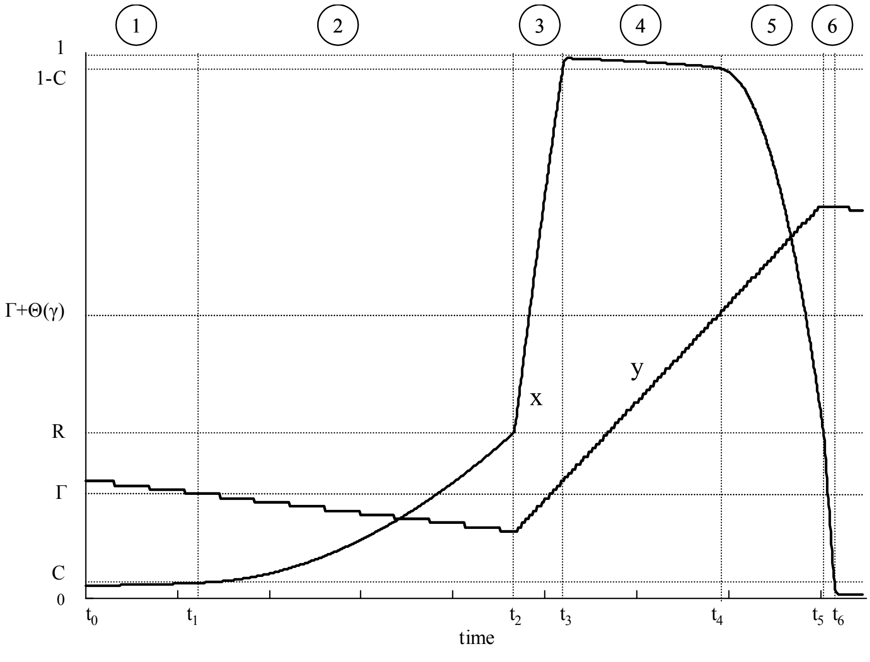

Figure A.1: Temporal evolution of the delay oscillator during one cycle with its different six zones.

Differential equations that describe oscillator behavior are:

$$\begin{cases} \frac{dy}{dt} = v \\ \frac{dx}{dt} = \frac{w-z}{\varepsilon} \end{cases} \qquad \text{Eq.A.1}$$

where w is the hysteresis comparator threshold; z the nonlinear function and v the integrator slope:

$$w = \begin{cases} \gamma; & x < R \\ \gamma + \theta; & R \le x < 1 - C \\ (\gamma + \theta)(1 - x)/C; & 1 - C \le x \end{cases}$$
$$z = \begin{cases} ky^2 \frac{x}{C}; & x < C \\ ky^2; & C \le x \end{cases} \qquad \text{Eq.A.2}$$
$$v = \begin{cases} -q; & x < R \\ p; & R \le x \end{cases}$$

Zone 1. Silent state, *z* state variable limited.

In this zone, state variables are:

$w=\gamma$; $v=-q$; $x<C$.

Equations and initial conditions (chosen for simplicity) for this system are:

$$\begin{cases} \frac{dy}{dt} = -q \\ \frac{dx}{dt} = \frac{\gamma - ky^2 \frac{x}{C}}{\varepsilon} \\ y(t_o = 0) = y_0 \;\; where \;\; \gamma < y_0 < \gamma + \theta \\ x(t_o = 0) = x_0 \;\; where \;\; 0 < x_0 < C \end{cases} \qquad \text{Eq.A.3}$$

y is a linear function of time and, as v and *(w-z)* are on the same order of magnitude and ε is much smaller than 1, dynamics of y state variable are much slower than variations of x; thus, y can be considered constant when solving the second equation.

The result of Eq.A.3:

$$\begin{cases} y = -qt + y_0 \\ x = \frac{\gamma C}{ky^2} + \left(x_0 - \frac{\gamma C}{ky^2} \right) \exp\left(-\frac{ky^2}{\varepsilon C} t \right) \xrightarrow[t \to \infty]{} \frac{\gamma C}{ky^2} \end{cases} \qquad \text{Eq.A.4}$$

For the time scale of oscillations, x exponential tends to 1, thus x is proportional to the inverse of the square of y, which is a first order polynomial. y decreases linearly and x increases monotonically until $t=t_1$ when it reaches C and w and z are approximately equal as demonstrated in Eq.A.5:

$$\left.\begin{array}{ll} Eq.A.4 \Rightarrow & C = x(t_1) \cong \dfrac{\gamma C}{ky^2} \Rightarrow y^2 \cong \dfrac{\gamma}{k} \\ Eq.A.2 \Rightarrow & y^2 = \dfrac{z}{k} \end{array}\right\} \gamma \cong z \qquad \text{Eq.A.5}$$

Zone 2. Silent state, integrator state variables not limited.

In this zone, *z* and *w* are very similar, thus their difference is very small and x has to shift from *C* to *1-C*. It causes that dynamics of *x* must be taken into account and it cannot be done a similar approximation as in the previous zone.

State variables for this zone are:

$w=\gamma$; $v=-q$; $C<x<R$

Thus equations and initial conditions are:

$$\begin{cases} \dfrac{dy}{dt} = -q \\ \dfrac{dx}{dt} = \dfrac{\gamma - ky^2}{\varepsilon} \\ y(t_1 = 0) = \Gamma = \sqrt{\dfrac{\gamma}{k}} \\ x(t_1 = 0) = C \end{cases} \qquad \text{Eq.A.6}$$

The first equation is independent of the second one and is easily solved taking t_1 as the time reference:

$$y = -qt + \Gamma \qquad \text{Eq.A.7}$$

After applying this value and initial conditions to the second equation:

$$x = -\frac{kq^2}{3\varepsilon}t^3 + \frac{kq\Gamma}{\varepsilon}t^2 + C \qquad \text{Eq.A.8}$$

Which is still a monotonically growing function between $C<x<R$ but faster than in the previous zone.

This equation allows calculating how much time (t_2-t_1) it takes to shift *w* since *y* has reached threshold Γ (at $t=t_1$), thus, delay of oscillators to shift from the silent state to the active one. We only have to fix $x(t_2)=R$. In addition to this, using Eq.A.7 it is easy to calculate state variable *z* at $t=t_2$ and, thus, *y*-threshold with delay Γ_{-D}.

$$\Gamma_{-D} = y(t_2) = -q(t_2 - t_1) + \Gamma \qquad \text{Eq.A.9}$$

Zone 3. Active state, integrator state variables not limited.

x has already reached *R*, thus, *y* changes its slope to *p* and *x*-threshold increases to $\Gamma+\Theta(\gamma)$.

Equations and initial conditions in this zone are:

$$\begin{cases} \dfrac{dy}{dt} = p \\ \dfrac{dx}{dt} = \dfrac{\gamma + \theta - ky^2}{\varepsilon} \\ y(t_2 = 0) = y_2 = \Gamma_{-D} \\ x(t_2 = 0) = R \end{cases} \qquad \text{Eq.A.10}$$

Again, the first equation is easily solved and the second one, as in the previous zone, is solved by substituting the value of y and initial conditions:

$$y = pt + \Gamma_{-D}$$
$$x = -\frac{kp^2}{3\varepsilon}t^3 - \frac{kp\Gamma_{-D}}{\varepsilon}t^2 + \left[\frac{\gamma+\theta}{\varepsilon} - \frac{k\Gamma_{-D}^2}{\varepsilon}\right]t + R \qquad \text{Eq.A.11}$$

Zone 4. Active state, *w* state variable limited.

Equations and initial conditions for this zone are:

$$\begin{cases} \dfrac{dy}{dt} = p \\ \dfrac{dx}{dt} = \dfrac{(\gamma+\theta)\dfrac{1-x}{C} - ky^2}{\varepsilon} \\ y(t_3 = 0) = y_3 \\ x(t_3^+ = 0) = x_3 \end{cases} \qquad \text{Eq.A.12}$$

As in zone 1, dynamics of x are faster than dynamics on the slow integrator, thus, to solve the second equations we can consider that y is constant. This assumption produces a small discontinuity at $t=t_3$, $1-C=x(t_3^-)\neq x(t_3^+)$ that it is not important for analysis purposes due to the fast dynamics of the output node at this stage.

Solution for this system is:

$$\begin{cases} y = pt + y_3 \\ x = \dfrac{a}{b} + \left(x_3 - \dfrac{a}{b}\right)\exp(-bt) \xrightarrow[t\to\infty]{} \dfrac{a}{b} = \ldots \\ \ldots = 1 - \dfrac{ky^2}{\gamma+\theta}C \end{cases} \qquad \text{Eq.A.13}$$

$$where\ \ a = \frac{\gamma+\theta}{C\varepsilon} - \frac{ky^2}{\varepsilon};\ \ b = \frac{\gamma+\theta}{C\varepsilon}$$

As in Eq.A.3, the exponential of x tends to 1 due to its fast dynamics; it makes x decrease proportionally to the square of y. When y, which increases in this zone, reaches the threshold that makes $z=\gamma+\theta$, the positive current source leaves saturation and starts the next zone.

Zone 5. Active state, integrator state variables not limited.

Equations for this zone are equal to zone 3 but initial conditions differ. In addition, dynamics of nodes *x* and *y* must be considered. Differential equations are shown in Eq.A.10 and initial conditions are:

$$\begin{cases} y(t_4 = 0) = \Gamma + \Theta(\gamma) \\ x(t_4 = 0) = 1 - C \end{cases} \qquad \text{Eq.A.14}$$

The upper threshold of *y* has been reached but as dynamics of the output node are not instantaneous, appears a delay until *x* reaches the threshold voltage that changes *v* and *w* and making the oscillator to enter back to the silent state.

Solution for the equation system is:

$$\begin{aligned} y &= pt + \Gamma + \Theta(\gamma) \\ x &= -\frac{kp^2}{3\varepsilon}t^3 - \frac{kp(\Gamma + \Theta(\gamma))}{\varepsilon}t^2 + 1 - C \end{aligned} \qquad \text{Eq.A.15}$$

These equations allow us to calculate the time it takes the oscillator to shift to the active state and thus threshold $\Gamma+\Theta(\gamma)_{+D}$.

$$\Gamma + \Theta(\gamma)_{+D} = y(t_5) = pt_5 + \Gamma + \Theta(\gamma) \qquad \text{Eq.A.16}$$

Zone 6. Silent state, integrator state variables not limited.

After switching from the active state to the silent state, the oscillator enters in zone 6 where equations are equal to zone 2 but initial conditions change to:

$$\begin{cases} y(t_5 = 0) = \Gamma + \Theta(\gamma)_{+D} \\ x(t_5 = 0) = R \end{cases} \qquad \text{Eq.A.17}$$

Solution to the system is:

$$\begin{cases} y = -qt + \Gamma + \Theta(\gamma)_{+D} \\ x = -\dfrac{kq^2}{3\varepsilon}t^3 + \dfrac{kq\left(\Gamma + \Theta(\gamma)_{+D}\right)}{\varepsilon}t^2 + \left[\dfrac{\gamma}{\varepsilon} - \dfrac{k\left(\Gamma + \Theta(\gamma)_{+D}\right)^2}{\gamma}\right]t + R \end{cases} \qquad \text{Eq.A.18}$$

Then, oscillator is governed by these equations until *x* reaches *C* where the negative current source saturates and it enters to zone 1 of the next cycle.

Oscillator Characteristics

Once calculated the analitical approximation of the oscillator, its fundamental characteristics as frequency and duty cycle can be easily calculated by solving equations for each zone, which are simple first and third order polynomials. Thus frequency (f_0) and duty cycle (Δ) are:

$$f_0 = (t_6 - t_0)^{-1}$$
$$\Delta = (t_5 - t_2)/(t_6 - t_0) \qquad \text{Eq.A.19}$$

To check the validity of the results and approximations made in zones 1 and 4, next, we present some examples of oscillators whose characteristics have been estimated by solving analytical equations presented in this chapter and also simulating differential equations that describe oscillators. From typical parameter values, a variation of each one has been performed to verify the very similar behavior of the model compared to the numerical simulation.

Characteristics estimated are T_0:period; f_0:frequency; T_A:Time active during one period; Δ_0=Duty cycle; Γ_{-D}:Equivalent Γ threshold and $\Gamma+\Theta(\gamma)_{+D}$:Equivalent $\Gamma+\Theta(\gamma)$ threshold.

Example #1

γ	θ	p	q	R	ε
0.3	1	0.9	0.1	0.3	0.165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	6.64	0.151	0.669	10.08	0.2298	0.8262
Analytical	6.69	0.150	0.669	10.01	0.2304	0.8318

Example #2

γ	θ	p	q	R	ε
0.1	1	0.9	0.1	0.3	0.165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	7.90	0.127	0.794	10.05	0.0743	0.7847
Analytical	7.93	0.126	0.794	10.01	0.0754	0.7889

Example #3

γ	θ	p	q	R	ε
0.3	2	0.9	0.1	0.3	0.165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	8.66	0.115	0.870	10.04	0.2299	1.0088
Analytical	8.71	0.115	0.871	10.00	0.2304	1.0138

Example #4

γ	θ	P	q	R	ε
0.3	1	1.9	0.1	0.3	0.165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	7.19	0.139	0.365	5.08	0.2298	0.9118
Analytical	7.25	0.138	0.364	5.02	0.2304	0.9193

Example #5

γ	θ	p	q	R	ε
0.3	1	0.5	0.5	0.3	0.165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	2.58	0.388	1.294	50.23	0.1314	0.7771
Analytical	2.58	0.387	1.292	49.98	0.1349	0.7808

Example #6

γ	θ	p	q	R	ε
0.3	1	0.9	0.1	0.6	0.165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	6.47	0.156	0.649	10.03	0.1963	0.7782
Analytical	6.48	0.154	0.649	10.02	0.1967	0.7796

Example #7

γ	θ	p	q	R	ε
0.3	1	0.9	0.1	0.3	0.0165

	T_0	f_0	T_A	Δ_0 [%]	Γ_{-D}	$\Gamma+\Theta(\gamma)_{+D}$
Num. sim.	4.59	0.218	0.463	10.08	0.2787	0.6925
Analytical	4.60	0.218	0.460	10.01	0.2788	0.6923