Multigraded Structures and the Depth of Blow-up Algebras

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A first goal of this thesis is to contribute to the knowledge of cohomological properties of non-standard multigraded modules. In particular we study the Hilbert function of a non-standard multigraded module, the asymptotic depth of the homogeneous components of a multigraded module and the asymptotic depth of the Veronese modules. To reach our purposes, we generalize some cohomological invariants to the non-standard multigraded case and we study properties on the vanishing of local cohomology modules. In particular we study the generalized depth of a multigraded module.

In chapters 2, 3 and 4, we consider multigraded rings S, finitely generated over the local ring $S_{\underline{0}}$ by elements of degrees $\gamma_1, \ldots, \gamma_r$ with $\gamma_i = (\gamma_1^i, \ldots, \gamma_i^i, \ldots, 0) \in \mathbb{N}^r$ and $\gamma_i^i \neq 0$ for $i = 1, \ldots, r$. In Chapter 2, we prove that the Hilbert function of a multigraded S-module is quasi-polynomial in a cone of $\mathbb{N}^r$. Moreover the Grothendieck-Serre formula is satisfied in our situation as well.

In Chapter 3, using the quasi-polynomial behavior of the Hilbert function of the Koszul homology modules of a multigraded S-module M with respect to a system of generators of the maximal ideal of $S_{\underline{0}}$, we can prove that the depth of the homogeneous components of M is constant for degrees in a subnet of a cone of $\mathbb{N}^r$ defined by $\gamma_1, \ldots, \gamma_r$. In some cases we can assure constant depth in all the cone. By considering the multigraded blow-up algebras associated to ideals $I_1, \ldots, I_r$ in a Noetherian local ring $(R, \mathfrak{m})$, we can prove that the depth of $R/I_1^{n_1} \cdots I_r^{n_r}$ is constant for $n_1, \ldots, n_r$ large enough.

In Chapter 4, we study the depth of Veronese modules $M^{(\underline{a},\underline{b})}$ for $\underline{a}, \underline{b}$ large enough. In particular we prove that in almost-standard case (i.e. with $\gamma_i = (0, \ldots, 0, \gamma_i^i, 0, \ldots, 0)$, $\gamma_i^i > 0$, for $i = 1, \ldots, r$) with $S_{\underline{0}}$ a quotient of a regular local ring, this depth is constant for $\underline{a}, \underline{b}$ in some regions of $\mathbb{N}^r$. To reach this result we need a previous study about Veronese modules and about the vanishing of local cohomology modules. In particular we prove that, in the more general case, if $S_{\underline{0}}$ is a quotient of a regular local ring, the generalized depth is invariant by taking Veronese transforms. Moreover in the almost-standard case the generalized depth coincides with the index of finite graduation of the local cohomology modules with respect to the homogeneous maximal ideal.

A second goal of the thesis is the study of the depth of blow-up algebras associated to an ideal. In Chapter 5 we obtain refined versions of some conjectures on the depth of the associated graded ring of an ideal. By using certain non-standard bigraded structures, the integers that appear in Guerrieri's Conjecture and in Wang's Conjecture can be interpreted as a multiplicities of some bigraded modules. In particular we have given an answer to the question formulated by A. Guerrieri and C. Huneke in 1993. We have proved that given an $\mathfrak{m}$-primary ideal I in a Cohen-Macaulay local ring $(R, \mathfrak{m})$ of dimension $d > 0$ with minimal reduction J, assuming that the lengths of the homogeneous components of the Valabrega-Valla module of I and J are less than or equal to 1, then the depth of the associated graded ring of I is greater than or equal to $d - 2$.

Finally, in Chapter 6, the study of the Hilbert function of certain submodules of the bigraded modules studied before, allows us to prove some cases in which the Hilbert function of an $\mathfrak{m}$-primary ideal in a one-dimensional Cohen-Macaulay local ring is non-decreasing.