UNIVERSITAT DE BARCELONA

EL GRUP FONAMENTAL DE LES VARIETATS KÄHLER

per

Jaume Amorós Torrent

Facultat de Matemàtiques

1996

EL GRUP FONAMENTAL

DE LES VARIETATS KÄHLER

Memòria presentada per Jaume Amorós
per aspirar al grau de Doctor en Matemàtiques.

Departament d'Àlgebra i Geometria, Universitat de Barcelona.
Programa de Doctorat d'Àlgebra i Geometria 1991-93.
Doctorand: Jaume Amorós Torrent.
Tutor: Vicenç Navarro Aznar.
Director de Tesi: Vicenç Navarro Aznar.

Vicenç Navarro Aznar
catedràtic del Departament d'Àlgebra i Geometria
de la Universitat de Barcelona

CERTIFICA:

Que la present memòria ha estat realitzada sota la seva direcció per Jaume Amorós; i que constitueix la tesi d'aquest per a aspirar al grau de Doctor en Matemàtiques.

Barcelona, Novembre de 1996.

Agraïments.

En primer lloc, desitjo agrair a Vicenç Navarro la seva labor de direcció, que ha estat essencial per a la completació d'aquesta memòria, i també les nombroses i molt formatives discussions matemàtiques que hem mantingut en aquests anys.

També vull agrair a Agustí Roig el haver–me ensenyat pacientment la major part de la Homotopia Racional que sé.

En l'agraïment per informacions valuoses, observacions oportunes i converses interessants per a aquest treball, he de destacar principalment a Norbert A'Campo, Sebastià del Baño, Miguel Angel Barja, Marc Burger, José Ignacio Burgos, Carles Casacuberta, Paco Guillén, Manfred Hartl, János Kollár, Dieter Kotschick, Pere Pascual, Francesc Planas, Fernando Serrano i Domingo Toledo. També estic agraït per aquestes raons col.lectivament als participants en els seminaris de Geometria en la Universitat de Barcelona i Borel 1995 en la Universitat de Berna.

He passat el temps de l'elaboració d'aquesta tesi en el Departament d'Àlgebra i Geometria de la U.B., el Mathematisches Institut de la Universitat de Basilea, el Departament de Matemàtica Aplicada I de la U.P.C. en la ETSEIB, i l'Institut de Ciències de Materials de Barcelona (CSIC). En tots aquests llocs de treball he rebut un generós recolzament material i d'altres formes a la meva recerca, que ara els agraeixo. També agraeixo els diferents ajuts econòmics que durant la realització d'aquest projecte he rebut de la Generalitat de Catalunya, la Dirección General de Ciencia y Tecnología, i el Fons Nacional Suís de Recerca.

Finalment, vull agrair a la Cristina i als meus pares pel suport que m'han donat des del pricipi d'aquesta tasca.

Contents

Introducció

1. Els grups Kähler

1.1. Els grups fonamentals en la classificació de varietats. Quan hom estudia la classificació homotòpica, topològica o diferenciable de varietats PL, diferencials, complexes, ... , apareix una dicotomia molt marcada en els resultats assolits fins el moment present entre el cas simplement connex i el no simplement connex.

En el cas simplement connex s'han fet avanços notables fins el moment present. Per exemple, podem destacar entre aquests la teoria de torres de Postnikov per CW–complexes, i els treballs recents sobre varietats $\mathcal{C}^\infty$ compactes de dimensió 4, culminant en els teoremes de Freedman i Donaldson que caracteritzen el tipus topològic d'aquestes varietats. Aquests teoremes de classificació tenen en comú que codifiquen la classe d'equivalència mòdul isomorfisme d'un espai mitjançant uns invariants lineals, grups abelians, elements seus, morfismes entre ells, en nombre finit o numerable, i que poden ser relativament coneguts en molts casos: la torre de Postnikov mòdul torsió es pot calcular via models minimals de Sullivan, el tipus topològic es pot deduir de l'àlgebra de cohomologia entera i certes classes característiques de la varietat.

En contrast, en el cas no simplement connex el nostre coneixement es troba molt més endarrerit. Max Dehn va provar el 1912 que tot grup finit presentat és grup fonamental d'una varietat compacta llisa orientable de dimensió 4. Això fa que el problema de classificació fins i tot homotòpica d'aquestes varietats contingui al problema de classificació de grups finit presentats mòdul isomorfisme. El problema essencial d'aquesta classificació ha estat resumit per Mikhail Gromov com

"Qualsevol afirmació sobre tots els grups és trivial o falsa"

Fins i tot si ens restringim a la categoria de grups finit presentats, questions tan bàsiques com ara decidir si dues presentacions defineixen grups isomorfs, si dues paraules defineixen elements conjugats o fins i tot el mateix en una presentació d'un grup són en general indecidibles. Qualsevol sistema d'invariants lineals ofereix només una informació grollera, i és trivial per àmplies famílies de grups.

Així la impossibilitat de classificació de grups és heretada per les varietats llises. Davant d'aquesta dificultat, la línea de recerca bàsica ha estat estudiar les representacions lineals i en general les accions dels grups fonamentals en espais distingits, amb les propietats d'origen geomètric que aquestes accions satisfan. En aquesta direcció s'emmarca la present memòria.

1.2. Uns grups molt especials. Quan hom restringeix el seu estudi de grups fonamentals a categories més i més restringides de varietats, s'observa a partir de la dimensió real 4 una divisió nítida: qualsevol grup finit presentat és grup fonamental d'una varietat $\mathcal{C}^\infty$ compacta (Dehn, 1912, [**30**]), d'una varietat quasi–complexa de dimensió 4 (Kotschick, 1992 [**62**]), simplèctica de dimensió 4 (Gompf, 1995 [**39**]), fins i tot complexa i simplèctica de dimensió 6 (Taubes [**92**],Gompf). Quan hom augmenta la dimensió no es perden en cap cas grups fonamentals.

En canvi, hi ha una classe de varietats molt propera a les anteriors citades de la que es coneix de fa anys que imposa restriccions als seus grups fonamentals: es tracta de les varietats Kähler compactes. La descomposició de Hodge de la cohomologia complexa de les varietats compactes Kähler té com a consequència elemental que els nombres de Betti senars de tals varietats són parells. En particular, $b_1(X) = \operatorname{rang} \pi_1(X,*)^{ab}$ és parell, i els grups amb abelianitzat de rang senar, com ara $\mathbb{Z}$, no poden ser grups fonamentals de varietats compactes Kähler.

Si hom restringeix encara més la classe de varietats estudiades, no s'apercebeix l'aparició gradual de més restriccions topològiques. Tot al contrari, hom es troba amb que:
- no es coneix cap exemple de varietat Kähler compacta que no sigui difeomorfa a una varietat projectiva sobre $\mathbb{C}$,
- per la classificació de Kodaira, es sap que totes les superfícies Kähler compactes són difeomorfes a superfícies projectives llises sobre $\mathbb{C}$,
- totes les varietats projectives llises sobre $\mathbb{C}$ són difeomorfes a varietats definides sobre $\bar{\mathbb{Q}}$.

És a dir, dins de les principals classes de varietats compactes estudiades actualment en Geometria, les varietats Kähler són la classe més gran en la que hom observa restriccions en el grup fonamental, i aquestes restriccions semblen ser les mateixes que en el cas de varietats projectives sobre un cos de nombres! Aquest fet fa molt especials i interessants els grups fonamentals de les varietats Kähler compactes, que reben el nom de *grups Kähler* (vegi's [**3**]).

En el context de la Geometria Algebraica, és també interessant conéixer els grups fonamentals de varietats algebraiques obertes llises o singulars. En el cas obert, es sap que el grup fonamental d'una varietat quasi–projectiva llisa satisfà també restriccions, com ara les que provenen de l'estructura de Hodge mixta posada per Morgan i Hain a la seva completació unipotent ([**70**], [**47**]). En el cas singular, hom pot realitzar qualsevol grup fonamental finit presentat mitjançant un poliedre afí complex, però en canvi es sap molt poc sobre els grups fonamentals de varietats singulars irreductibles.

1.3. L'estudi dels grups Kähler. L'estudi dels grups Kähler és un tema recent, que es troba en la confluència de la Teoria de Grups, la Geometria Algebraica i la Topologia Diferencial.

A continuació introduïm breument les principals línees d'investigació sobre el tema amb les que aquesta memòria entronca:

1: La completació unipotent dels grups Kähler. Aquesta és la nostra línea de treball fonamental. Sigui k un cos de característica zero. La completació k–unipotent d'un grup Γ, denotada $\Gamma \otimes k$, és el límit projectiu del sistema invers de morfismes de grups

$$\Gamma \longrightarrow U\,,$$

on U és un grup k–algebraic unipotent. Per un grup Γ finit presentat, la seva completació k–unipotent és un pro–grup k–algebraic unipotent, equivalent pels treballs de Malcev a la seva àlgebra de Lie pro–nilpotent, l'*àlgebra de Malcev* $\mathcal{L}(\Gamma, k)$. Aquesta completació classifica les representacions unipotents del grup, o equivalentment els sistemes locals/fibrats integrables unipotents sobre un espai X tal que $\pi_1(X) \cong \Gamma$. Debut a l'existència de reticles provinents del grup Γ en tota representació unipotent, el completat k–unipotent $\Gamma \otimes k$ i la seva àlgebra de Malcev $\mathcal{L}(\Gamma, k)$ per k cos de característica zero s'obtenen per extensió d'escalars del cas racional $\Gamma \otimes \mathbb{Q}, \mathcal{L}(\Gamma, \mathbb{Q})$.

Un altre tret especial de la completació unipotent que simplifica el seu estudi és el fet de que si $\Gamma \cong \pi_1(X)$, amb X varietat diferenciable, la completació $\Gamma \otimes \mathbb{R}$ pot ser calculada a partir del complex de formes diferencials de de Rham de X. Aquest càlcul es pot fer via la teoria de models 1–minimals de Sullivan, o mitjançant les integrals iterades de K.T. Chen. A més, en el cas de les varietats Kähler compactes el teorema de formalitat de Deligne–Griffiths–Morgan–Sullivan permet calcular el model 1–minimal directament a partir de la cohomologia de la varietat, i traduir propietats de la cohomologia com l'estructura i l'aparellament de Hodge, a propietats de la completació unipotent.

Entre els resultats principals obtinguts per aquesta via destaquem la propietat de que els productes triples de Massey de grups Kähler són zero, cas particular de 1–formalitat conegut ja per J.-P. Serre, i l'isomorfisme de l'àlgebra de Malcev d'una varietat Kähler compacta amb la d'un model llis de la seva imatge per l'aplicació d'Albanese, resultat debut a Campana ([**20**]).

2: La cohomologia L^2 dels grups Kähler. El càlcul de la cohomologia L^2 d'una varietat Kähler iniciat per M. Gromov i continuat per Arapura, Bressler i Ramachandran dóna condicions suficients per a fibrar varietats Kähler simplement connexes sobre el disc de Poincaré. La consequència al aplicar aquests resultats als recobridors universals de varietats compactes és que l'extensió d'un grup amb infinits finals per un grup finit generat (i en particular un producte lliure de grups) no pot ser Kähler ([**46**],[**5**]).

3: Aplicacions harmòniques sobre varietats Kähler. El punt de partida d'aquesta via d'estudi és el teorema de Siu i Sampson que diu que tota aplicació harmònica d'una varietat compacta Kähler a una varietat riemanniana amb curvatura seccional Hermítica negativa és pluriharmònica. Siu, Sampson, Carlson i Toledo parteixen d'aquesta propietat per a estudiar les aplicacions harmòniques de varietats compactes Kähler a espais hermítics localment simètrics, obtenint teoremes de factorització d'aquestes aplicacions a través de superfícies que mostren que els reticles co–compactes en $SO(1,n)$ no són Kähler per $n \geq 2$. Hi ha una versió de Gromov i Schoen d'aquesta teoria per a aplicacions harmòniques cap a arbres.

4: Teoria de Hodge no abeliana. Aquest camp constitueix una versió equivariant de l'anterior. L'objectiu d'aquesta teoria és l'estudi d'espais de moduli de representacions de grups fonamentals de varietats projectives i quasi–projectives, i dels seus resultats se n'extreuen algunes restriccions que aquests grups han de satisfer, com ara el fet de que les singularitats en l'espai tangent de Zariski de l'espai de moduli de representacions de dimensió n d'un grup Kähler són quadràtiques (Goldman–Millson, [**38**]), o que la clausura Zariski real de la monodromia d'una $\mathbb{R}$–variació d'estructures de Hodge sobre una varietat Kähler compacta ha de tenir un subgrup de Cartan compacte (Simpson, [**85**]).

1.4. Varietats quasi–projectives i grups fonamentals relatius. A més dels grups Kähler, en Geometria Algebraica apareixen de manera natural el grups fonamentals de varietats quasi–projectives llises. Aquests grups també satisfan algunes de les propietats dels grups Kähler abans citades, com ara l'existència d'una estructura de Hodge mixta en la seva completació unipotent. En el cas de varietats algebraiques llises sobre un cos de nombres, aquests grups admeten diferents realitzacions: Betti, Hodge, de Rham, étale, cristal.lina, construides per Deligne en [**34**] en el cas de varietats X tals que una completació verifica $H^1(\bar{X}, \mathcal{O}) = 0$. En aquesta memòria estudiem les realitzacions de Betti, Hodge i de Rham del grup fonamental sense aquesta condició cohomològica, manifestament molt forta. En particular, en el Capítol 5 compararem en el cas de corbes els resultats que s'obtenen per varietats que compleixen $H^1(\bar{X}, \mathcal{O}) = 0$ amb el d'algunes que no la compleixen.

El fet de que el grup fonamental es un invariant dels espais puntejats obliga també a tractar el cas relatiu, ja que el que podem associar naturalment a una varietat algebraica X no és un grup fonamental, sinó la família donada per la projecció en el primer factor $X \times X \to X$ amb punt base diagonal, que dóna la variació del grup fonamental de X al variar el punt base. Hom considera aleshores el cas relatiu, de famílies de varietats algebraiques $f : X \to S$ on f és llís i topològicament localment trivial, amb una secció punt base $\sigma : S \to X$. En aquest cas els grups fonamentals de les fibres formen un fibrat principal de grups discrets $\{\pi_1(X_s, \sigma(s))\}_{s \in S}$, i les àlgebres de Malcev de les fibres formen sistemes locals d'àlgebres de Lie.

Aquests sistemes locals d'àlgebres de Malcev que denotem $\mathcal{L}(X|S) = \{\mathcal{L}(\pi_1(X_s, \sigma(s)))\}_{s \in S}$ són equivalents a fibrats holomorfs sobre la base S amb una connexió integrable $(\mathcal{L}(X|S) \otimes \mathcal{O}_S, d_S)$. Aquesta connexió és la *connexió de Gauss–Manin en l'àlgebra de Malcev.* La connexió de Gauss–Manin en l'àlgebra de Malcev és d'origen algebraic, i singular-regular (Navarro Aznar, [**73**]).

2. Continguts d'aquesta memòria

El propòsit d'aquesta memòria ha estat l'estudi del grup fonamental de les varietats algebraiques complexes, en les seves realitzacions Betti, Hodge i de Rham. L'estudi s'ha fet tant en el cas absolut, és a dir grups fonamentals de les varietats esmentades, com en el cas relatiu, en el que s'estudia la monodromia en el grup fonamental i la connexió de Gauss–Manin associada. Les tres pricipals direccions de treball han estat:

(i) La completació unipotent dels grups Kähler, mitjançant els models 1–minimals de Sullivan i el teorema de formalitat de Deligne–Griffiths–Morgan–Sullivan.
(ii) La monodromia en el grup fonamental en pinzells de Lefschetz de corbes, és a dir, famílies de corbes amb singularitats quadràtiques ordinàries.

(iii) El model 1–minimal de la connexió de Gauss–Manin en la cohomologia de varietats algebraiques llises.

Fem a continuació una descripció més detallada dels continguts d'aquesta memòria:

El **Capítol 1** és una introducció al grup fonamental de de Rham i algunes de les seves propietats. Les completacions nilpotent, nilpotent sense torsió, i k–unipotent d'un grup Γ són definides categòricament i caracteritzades a continuació en termes de la sèrie central descendent del grup. En el cas d'un cos k de característica zero, la completació unipotent $\Gamma \otimes k$ és un grup pro–algebraic k–unipotent, equivalent per tant a la seva algebra de Lie pro–nilpotent. Aquesta àlgebra rep el nom d'àlgebra de Malcev de Γ sobre k, i es denota $\mathcal{L}(\Gamma, k)$. Es descriu a continuació com aquesta àlgebra de Malcev i la completació unipotent per un cos k de característica zero s'obtenen a partir dels homòlegs racionals per extensió d'escalars, així en particular $\mathcal{L}(\Gamma, k) \cong \mathcal{L}(\Gamma, \mathbb{Q}) \otimes_{\mathbb{Q}} k$. Aquesta propietat ens ha sigut molt convenient, ja que permet utilitzar un cos k particular, usualment $\mathbb{Q}$ o $\mathbb{R}$, i les propietats observades s'estenen a tots els cossos k de característica zero. La relació d'aquesta completació k–unipotent amb la completació nilpotent sense torsió utilitzada per Campana és descrita en el Lema 1.18.

Una altra àlgebra de Lie pro–nilpotent naturalment associada a un grup Γ és l'àlgebra graduada $\operatorname{gr}\Gamma = \oplus_{n\geq 1}\Gamma_n/\Gamma_{n+1}$. Aquesta àlgebra, definida ja sobre $\mathbb{Z}$, és una forma entera de la graduada de l'àlgebra de Malcev, ço és

$$\operatorname{Gr}\mathcal{L}(\Gamma, k) \cong (\operatorname{gr}\Gamma) \otimes_{\mathbb{Z}} k\,.$$

Descrivim a continuació algoritmes per a calcular els primers quocients d'aquesta àlgebra graduada $\Gamma/\Gamma_2 \otimes k, \Gamma_2/\Gamma_3 \otimes k$ a partir d'una presentació del grup Γ, derivats de [**88**], [**27**]. L'ús de coeficients en un cos de característica zero permet fer aquest càlcul mitjançant l'àlgebra del grup $k\,\Gamma$ ([**79**]). Com a pas previ a la presentació de l'algoritme, estudiem aquestes àlgebres de Malcev i de grup en el cas d'un grup lliure finit generat F_r; en aquest cas apareixen les corresponents algebres de Lie nilpotents lliures.

L'avantatge principal de la completació unipotent i l'àlgebra de Malcev respecte de completacions comparables és que pot ser calculada explícitament, i relacionada directament amb l'àlgebra de cohomologia, mitjançant els models 1–minimals de Sullivan o alternativament via les integrals iterades de Chen. Després d'una breu introducció als models 1–minimals i la seva dualitat amb l'àlgebra de Malcev, entrem en la discussió de la formalitat. La formalitat significa bàsicament que el model minimal pot ser calculat a partir de la cohomologia. El tipus de formalitat que requerim és la 1–formalitat; aquesta propietat dels espais és present en la literatura sota tres formes diferents: la 1–formalitat, la presentació quadràtica de l'àlgebra de Malcev, i l'annulació dels productes de Massey. Mostrem com aquestes tres presentacions d'aquesta propietat són equivalents, per mitjà d'accions del grup $\mathbb{G}_m(k)$ i de la filtració pel pes en l'àlgebra de Malcev. Una altra consequència d'aquesta caracterització és que en el cas de grups 1–formals, l'àlgebra de Malcev és naturalment isomorfa a la seva graduada $\mathrm{Gr}\,\mathcal{L}(\Gamma, k) \cong \mathrm{gr}\,\Gamma \otimes_{\mathbb{Z}} k$, i per tant l'àlgebra graduada del grup és una forma entera per l'àlgebra de Malcev dels grups Kähler.

Finalment, hom estudia la relació entre el grup d'automorfismes d'un grup Γ, dels seus quocients nilpotents Γ/Γ_n i de l'àlgebra de Malcev. Es demostra que el grup d'automorfismes de l'àlgebra de Malcev $\mathcal{L}\,\Gamma$ és una extensió pro–unipotent del grup d'automorfismes de l'abelianitzat $\mathrm{GL}\,(\Gamma/\Gamma_2 \otimes k)$, i en particular la restricció d'un automorfisme de $\mathcal{L}\,\Gamma$ als quocients de la sèrie central descendent $\Gamma_n/\Gamma_{n+1} \otimes k$ és la imatge per un automorfisme algebraic de l'automorfisme induït en $\Gamma/\Gamma_2 \otimes k$. Aquesta propietat servirà per a caracteritzar els grups de Galois diferencials de l'àlgebra de Malcev en famílies de varietats.

El **Capítol 2** estudia la completació unipotent dels grups Kähler, i en ell s'inclouen els resultats de l'autor ja publicats a [**2**] i [**3**].

El coneixement dels grups Kähler ha progressat a grans passos en els darrers anys (veure [**3**]), però resten obertes qüestions fonamentals, com ara

- És tot grup Kähler el grup fonamental d'una varietat projectiva llisa?

i, completant a l'anterior,

- És tot grup fonamental d'una varietat projectiva llisa una extensió quasi–abeliana del d'una varietat de tipus general? (Kollár, [**61**])

Donem una resposta afirmativa a totes dues qüestions per a les àlgebres de Malcev, és a dir, per a les representacions unipotents dels grups en lloc dels grups mateixos. L'eina usada per a obtenir aquests resultats és l'aplicació d'Albanese $\alpha_X : X \to Alb(X)$. Demostrem que les àlgebres de Malcev de X i d'un model llis de la imatge d'Albanese de X són isomorfes. Per tant, n'hi ha prou amb examinar les subvarietats dels tors complexes i els seus models llisos, un tipus de varietats bastant estudiat (cf. [**97**]). Es presenten a continuació algunes altres aplicacions d'aquestes idees, calculant l'àlgebra de Malcev de les varietats compactes Kähler amb dimensió de Kodaira 0 o 1, i establint a nivell de representacions unipotents la predicció de Kollár de que el problema de caracterització dels grups Kähler es troba fonamentalment en les superfícies de tipus general. També es mostra que els grups Kähler definits per una sola relació tenen àlgebra de Malcev zero o isomorfa a la d'una corba.

Previament en aquest capítol mostrem que la 1–formalitat de les varietats Kähler compactes implica que les seves àlgebres de Malcev estan determinades pels seus quocients $\mathcal{L}_2\Gamma \cong \Gamma/\Gamma_2 \otimes k \oplus \Gamma_2/\Gamma_3 \otimes k$ que hem calculat en el capítol anterior, i que com a conseqüència de l'aparellament Q en cohomologia els grups tals que el quocient $\mathcal{L}_2\Gamma$ és lliure no poden ser Kähler. Entre els grups amb quocient de Malcev $\mathcal{L}_2\Gamma$ lliure destaquem els grups para–lliures de G. Baumslag ([**11**]).

Finalment, recordem la dicotomia entre els grups Kähler establerta per Beauville i Siu: un grup Kähler $\Gamma = \pi_1(X)$ és fibrat si admet un morfisme exhaustiu $\Gamma \to \Gamma_g \to 1$ amb Γ_g el grup fonamental d'una corba de gènere $g \geq 2$, i Γ és no fibrat si no existeix tal morfisme. Pels resultats de Beauville i Siu, que es remunten al teorema de Castelnuovo–de Franchis, aquesta condició és equivalent a la de l'existència de pinzells de gènere ≥ 2 per la varietat X. Els únics exemples de grups Kähler no fibrats coneguts són racionalment nilpotents, i s'ignora encara si aquests són els únics grups possibles. Donem una cota superior pel rang del quocient Γ_2/Γ_3, o equivalentment una cota inferior pel rang del segon nombre de Betti $b_2(\Gamma) \leq b_2(X)$ en el cas de grups Kähler no fibrats $\Gamma = \pi_1(X)$. Els càlculs de l'àlgebra de Malcev de classe 2 $\mathcal{L}_2\Gamma$ del capítol anterior permeten obtenir a partir d'aquestes cotes una cota inferior per la deficiència de Γ (diferència mínima entre el nombre de relacions i generadors entre totes les presentacions finites de Γ), que depén linealment de la irregularitat $q = \frac{1}{2}b_1(X)$. Aquestes cotes ens permeten donar uns quants exemples de grups que no poden ser Kähler no fibrats, ni tan sols Kähler en general.

El **Capítol 3** estudia la monodromia geomètrica i en el grup fonamental de pinzells de Lefschetz de corbes, és a dir famílies de corbes sobre $\mathbb{P}^1_{\mathbb{C}}$ amb singularitats quadràtiques ordinàries, tant projectives com quasi–projectives. Aquest estudi es basa en el lema de Morse complex i els grups de trenes sobre les corbes, i és comparable al de [**8**], on s'estudien famílies versals de corbes projectives i quasi–projectives amb seccions holomorfes com a complement. Tota superfície projectiva llisa admet un pinzell de Lefschetz després d'un nombre finit d'explosions en punts, i com aquest procés no varia el grup fonamental, tot grup fonamental de varietat projectiva llisa és grup fonamental d'un pinzell de Lefschetz de corbes projectives. Mostrem a continuació com obtenir una presentació del grup fonamental de l'espai total d'un pinzell de Lefschetz de corbes projectives a partir de la monodromia del pinzell en el grup fonamental. Aprofitem l'extensió de la descripció a pinzells quasi–projectius per a calcular la monodromia en el grup fonamental de la família de Legendre de cúbiques afins, així com de famílies de corbes racionals punxades, de cara a la seva comparació amb els resultats dels capítols posteriors.

La caracterització dels difeomorfismes de monodromia geomètrica en termes de trenes i twists de Dehn mostra que ténen entropia zero. Aquest fet, junt amb la quasi–isometria entre el recobriment universal d'una corba i el seu grup fonamental amb la mètrica de la longitud de les paraules (veure [**37**]) impliquen la quasi–unipotència de la monodromia en la cohomologia de les fibres per famílies de corbes ([**67**]). En aquesta memòria mostrem que la monodromia en el grup fonamental de famílies de varietats projectives satisfà una propietat de creixement lineal que implica la quasi–unipotència de la monodromia en el H^1 de les fibres, i la entropia zero en el cas de corbes. Hom ilustra amb un exemple el fet de que les propietats de creixement lineal i entropia zero són més restrictives que la de quasi–unipotència en cohomologia per un difeomorfisme de monodromia.

Per a concloure el capítol, s'estudien les propietats de formalitat de la monodromia en famílies de corbes. La monodromia d'una tal família en cohomologia no determina la monodromia en el grup fonamental, ni tan sols la monodromia en l'àlgebra de Malcev (veure [**73**]). En contrast, demostrem que la monodromia en el quocient nilpotent d'ordre 3 del grup fonamental Γ/Γ_4 sí determina la monodromia en el grup fonamental i geomètrica d'una família de corbes. Aquesta conclusió estén un resultat comparable de [**8**] per a famílies de corbes estables, on la filtració per la sèrie central descendent és reemplaçada per una filtració pel pes similar. A més, és una versió topològica del teorema de Torelli puntejat de Pulte ([**77**], veure també [**49**]).

Els capítols 4 i 5 estan dedicats a la realització de Hodge i de de Rham de l'àlgebra de Malcev i de la connexió de Gauss–Manin sobre ella en una família algebraica. El cas absolut i el relatiu estan lligats per la dependència respecte del punt base, que associa a cada varietat la família sobre ella obtinguda pel punt base diagonal. Aquesta connexió de Gauss–Manin en el grup fonamental ha estat construida per Deligne com a sistema de realitzacions Betti, Hodge, de Rham, cristal.lí en el cas de varietats X tals que una completació seva verifica $H^1(\bar{X}, \mathcal{O}) = 0$. Navarro Aznar construeix la versió de de Rham en [**73**], per famílies arbitràries de varietats algebraiques.

El **Capítol 4** està dedicat al càlcul de la variació de cohomologia i estructures de Hodge. Aquest estudi es fa mitjançant la introducció de complexes de Dolbeault analítics reals de diverses menes: logarítmics, relatius, i logarítmics relatius horizontals. Es mostra com aquests complexes calculen la variació de la cohomologia de les fibres per morfismes analítics reals localment trivials, i com dónen una resolució acíclica dels complexes anàlegs holomorfs. Aquest complexes de Dolbeault calculen els fibrats analítics reals induïts pels sistemes locals de cohomologia de les fibres, i com els complexes holomorfs relatius calculen el fibrat holomorf pla associat a aquest sistema local, ens ha calgut introduir el concepte de variació d'estructura de Hodge analítica real i explicar la seva relació amb la variació complexa. Sembla prou clar que la presentació de la connexió de Gauss–Manin en un complexe d'àlgebres diferencials graduades commutatives acícliques ha de permetre el càlcul de models 1–minimals en un futur immediat, i obtenir així la realització de Hodge de la connexió en les àlgebres de Malcev de les fibres, tal com s'obté la realització de de Rham en el següent capítol.

El **Capítol 5** està dedicat a la realització de de Rham de l'àlgebra de Malcev i la connexió de Gauss–Manin en ella. En primer lloc, hom descriu el model 1–minimal de la connexió de Gauss–Manin de [**73**] per a varietats quasi–projectives. Comparem aquest model 1–minimal en el cas de famílies de corbes racionals punxades, cobert per Deligne en [**34**], amb el de la família de Legendre de corbes el.líptiques afins

$$E_t = \{(x, y, t) \mid y^2 = x(x-1)(x-t)\}\,.$$

Els resultats observats han estat molt diferents en un cas i en l'altre. Mentre en el cas de corbes racionals punxades el càlcul esdevé formal després d'un parell de passos i està definit sobre $\mathbb{Z}$ invertint només un nombre finit de primers, el càlcul del model 1–minimal de la connexió de Gauss–Manin en la família de Legendre requereix en tots els seus infinits passos més informació provinent del complexe de formes relatives de la família, i comprovem que demana la inversió de tots els nombres primers, pel que la connexió de Gauss–Manin en la família de Legendre, a diferència de les famílies de corbes racionals de Deligne, sembla tenir sentit només amb coeficients en $\mathbb{Q}$.

Finalment, per a concloure el capítol s'estudia el grup de Galois diferencial de la connexió de Gauss–Manin en l'àlgebra de Malcev. Aquest és el grup de Galois diferencial de l'equació satisfeta per les seccions horizontals del sistema local d'àlgebres de Malcev, els *periodes no abelians*. Hom prova mitjançant la comparació de grups d'automorfismes del capítol 1 entre grup, àlgebra de Malcev i abelianitzat, i el teorema de Schlesinger sobre grup de monodromia i grup de Galois diferencial, que aquests darrers en les àlgebres de Malcev són extensions unipotents dels grups de Galois diferencials de la connexió de Gauss–Manin en el primer grup de cohomologia. Això significa que els periodes no abelians de l'àlgebra de Malcev poden ser obtinguts a partir dels abelians per un procés succesiu de calcular primitives. Il.lustrem aquests resultats generals amb càlculs sobre els grups de Galois diferencials de l'àlgebra de Malcev de la família afí de Legendre.

3. Resultats assolits i conclusions

En compliment de la normativa de la Universitat de Barcelona, expliquem a continuació els principals resultats originals i conclusions d'aquesta memòria.

Els resultats originals assolits per aquest treball són:

- Per un grup finit presentat Γ i qualsevol cos de característica zero, el grup d'automorfismes de l'àlgebra de Malcev $\mathcal{L}(\Gamma, k)$ és una extensió pro–unipotent del grup d'automorfismes de l'abelianitzat $\mathrm{GL}\,(\Gamma/\Gamma_2 \otimes k)$.
- Els grups finit presentats Γ tals que la seva àlgebra de Malcev 2–nilpotent $\mathcal{L}_2\Gamma \cong \Gamma/\Gamma_2 \otimes \mathbb{Q} \oplus \Gamma_2/\Gamma_3 \otimes \mathbb{Q}$ és lliure no poden ser grups fonamentals de varietats compactes Kähler. Entre els exemples donats de grups amb $\mathcal{L}_2\Gamma$ lliure, destaquem els grups para–lliures de Baumslag.
- Si Γ és un grup Kähler que admet una presentació amb una sola relació, aleshores $\Gamma \cong \mathbb{Z}/n\mathbb{Z}$ o l'àlgebra de Malcev $\mathcal{L}\Gamma$ és isomorfa a l'àlgebra de Malcev del grup fonamental d'una superfície de gènere g.
- L'àlgebra de Malcev d'una varietat compacta Kähler X amb dimensió de Kodaira $\kappa(X) = 1$ és suma directa $\mathcal{L}\Gamma \cong \mathcal{L}\Gamma_g \oplus \mathbb{Q}^{2m}$, amb Γ_g el grup fonamental d'una superfície compacta llisa de gènere g, i $\mathbb{Q}^{2m}$ l'àlgebra de Lie abeliana de rang $2m$.
- Si Γ és un grup Kähler no fibrat amb irregularitat $q = \frac{1}{2}b_1(\Gamma)$, el seu segon nobre de Betti ha de verificar

$$b_2(\Gamma) \geq 6q - 7\,,$$

i també $b_2(X) \geq 6q - 7$ per tot espai topològic X amb $\pi_1(X) \cong \Gamma$. A més, si Γ admet una presentació amb n generadors i s relacions, es verifica la següent desigualtat

$$s - n \geq 4q - 7\,,$$

si $q \geq 2$, $s - n \geq -1$ si $q = 1$, i finalment $s - n \geq 0$ si $q = 0$. Aquesta cota millora la cota previament coneguda de **[41]**, que era $s - n \geq -3$ en tots els casos.
- Es calcula la monodromia en el grup fonamental per la familia de cúbiques afins de Legendre i per altres famílies de corbes racionals punxades. Aquests són els primers exemples coneguts per l'autor de càlculs de monodromia en grups fonamentals no abelians.

- La monodromia en el grup fonamental per famílies de varietats projectives té creixement lineal. Aquesta propietat implica en particular les ja conegudes de que la monodromia geomètrica té entropia zero i la quasi–unipotència de la monodromia en el primer grup de cohomologia.
- Tot grup finit presentat és grup fonamental d'un pinzell de Lefschetz $\mathcal{C}^\infty$ de corbes completes sobre $\mathbb{C}$. Aquesta propietat contrasta amb les fortes restriccions induïdes en Γ per l'existència d'un pinzell holomorf de corbes amb $\pi_1(X) \cong \Gamma$.
- La monodromia en el quocient nilpotent d'ordre tres del grup fonamental d'una família de corbes projectives determina la monodromia en el grup fonamental i la monodromia geomètrica de la família.
- Per famílies llises de varietats compactes Kähler o de varietats quasi–projectives, es construeixen complexes de Dolbeault relatius $\mathcal{A}^{*,*}_{X|S}$, logarítmics relatius $\mathcal{A}^{*,*}_{X|S}(\log H)$ respectivament, es demostra que les imatges directes $f_*\mathcal{A}^{*,*}_{X|S}(\log H)$ d'aquests complexes resolen els feixos derivats $\mathbb{R}^p f_*\mathbb{C}_X, \mathbb{R}^p f_*\mathbb{C}_{X\setminus H}$, i que la connexió de Gauss–Manin de la família ja està definida sobre ells. Aquesta construcció dóna a la connexió de Gauss–Manin una estructura real natural que no té en el cas holomorf clàssic. Es demostra que les propietats de transversalitat de Griffiths i de preservació del pes de la connexió de Gauss–Manin provenen de la connexió definida en les formes, i s'introdueix el concepte de variació d'estructura de Hodge analítica real per a comparar la connexió de Gauss–Manin en aquests feixos amb la versió holomorfa clàssica.
- Calculem la connexió de Gauss–Manin en l'àlgebra de Malcev per a la família de Legendre de corbes el.líptiques afins i per a algunes famílies de corbes racionals punxades, via l'algoritme de [**73**]. El cas de la connexió de Gauss–Manin en la família de Legendre té unes propietats molt diferents de les dels exemples de famílies de varietats amb $H^1(\bar{X}, \mathcal{O}) = 0$ presentades aquí o calculades per Deligne en [**34**].
- El grup de Galois diferencial de l'equació integrable dels periodes no abelians, associada al sistema local d'àlgebres de Malcev en una família de varietats llises, fins i tot analítiques reals, és una extensió unipotent del grup de Galois diferencial dels periodes abelians de la cohomologia.
- Es mostren algunes propietats de 1–connexió i sobre la classe de nilpotència dels grups de Galois diferencials de les àlgebres de Malcev en el cas de la família de Legendre.

Finalment, tot i no ser un resultat totalment original, mereix ser destacat l'estudi de la relació entre l'aplicació d'Albanese i la completació unipotent del grup fonamental en varietats compactes Kähler. Aquesta relació va ser trobada abans per Campana ([**22**]), usant una altra completació comparable del grup fonamental. Establim la relació entre les dues completacions, i el canvi d'aquesta per la completació unipotent i l'àlgebra de Malcev $\mathcal{L}\Gamma$, més l'ús de models 1–minimals de Sullivan ha permés una notable simplificació de les proves originals de Campana, alhora que la demostració de resultats que estenen els d'ell.

També volem consignar com a resultat assolit, encara que no completament original, l'estudi de la monodromia de famílies de corbes projectives que es fa en el Capítol 3: ens basem en una prova clàssica de la fòrmula de monodromia de Picard–Lefschetz en cohomologia, i a partir d'ella donem una demostració completa de la fòrmula de Picard–Lefschetz en monodromia geomètrica i en el grup fonamental. Fòrmules comparables ja apareixen a la literatura (en [**8**], per exemple), però l'autor no coneix cap referència que contingui una demostració completa.

CHAPTER 1

The de Rham fundamental group

1. Nilpotent groups and completions

Let Γ be a group. For any two elements $a, b \in \Gamma$, their commutator is defined as $[a, b] = a^{-1}b^{-1}ab$ (we follow [**65**] in this definition; the choice $[a, b] = aba^{-1}b^{-1}$ is very usual in the literature). The commutator of two subgroups $G, H \subset \Gamma$ is defined to be the subgroup $[G, H]$ of Γ generated by the commutators $[a, b]$ with $a \in G, b \in H$, and it is a normal subgroup if so are G, H. The *lower central series* of a group Γ is defined recursively by

$$\Gamma_1 = \Gamma\,, \qquad \Gamma_{n+1} = [\Gamma_n, \Gamma]\,.$$

A group Γ is nilpotent when $\Gamma_n = \{1\}$ for some finite n, and in such case we define the nilpotency class of Γ as the last n such that $\Gamma_n \neq \{1\}$, and call Γ a step n nilpotent group. Nilpotent groups may be obtained from abelian groups by iterated central extensions, this feature allows its study by starting with commutative groups and continuing by induction up the nilpotency class studying central extensions (see [**51**]).

The filtration of a group Γ given by the lower central series satisfies the condition that $[\Gamma_i, \Gamma_j] \subset \Gamma_{i+j}$ (see [**65**], Thm. 5.3). This fact together with the Witt–Hall identities (see Thm. 5.1 in [**65**]) allow the definition of a graded Lie algebra naturally associated to the group:

DEFINITION 1.1. The graded Lie algebra of a group Γ is the $\mathbb{Z}$–module

$$\operatorname{gr}\Gamma = \oplus_{n\geq 1}\Gamma_n/\Gamma_{n+1}$$

with the bracket induced by the group bracket.

Let R be a ring. The graded R–Lie algebra of a group Γ is the Lie algebra $\operatorname{gr}\Gamma \otimes R$.

The torsion elements in the graded Lie algebra of a group form an ideal $Tor \subset \operatorname{gr}\Gamma$. This allows the definition of the *torsion free graded Lie algebra* of a group Γ as

$$\operatorname{gr}_0\Gamma = \operatorname{gr}\Gamma/Tor\,.$$

For $R = k$ a field of characteristic zero one has $\operatorname{gr}\Gamma \otimes k \cong \operatorname{gr}_0\Gamma \otimes k$ naturally.

Denote by $n-\mathcal{G}r$ the category of nilpotent groups, which is a full subcategory of that of groups, $\mathcal{G}r$. One may consider the projective closure of $n-\mathcal{G}r$ in $\mathcal{G}r$, to be denoted $pro-n-\mathcal{G}r$. Its objects are pronilpotent groups, i.e. projective limits of nilpotent groups, and as $pro-n-\mathcal{G}r$ is projectively closed, the inclusion functor $pro-n-\mathcal{G}r \hookrightarrow \mathcal{G}r$ has a left adjoint

$$\begin{aligned} .^{nilp} : \mathcal{G}r &\longrightarrow n-\mathcal{G}r \\ \Gamma &\longmapsto \Gamma^{nilp} \end{aligned}$$

characterized by the natural bijections

$$\mathrm{Hom}_{\mathcal{G}r}(\Gamma, N) = \mathrm{Hom}_{pro-n-\mathcal{G}r}(\Gamma^{nilp}, N) \tag{1}$$

DEFINITION 1.2. The nilpotent completion of a group Γ is the group morphism

$$j : \Gamma \longrightarrow \Gamma^{nilp},$$

where j is the morphism corresponding by the adjointness natural bijection to $\mathrm{Id} \in \mathrm{Hom}_{pro-n-\mathcal{G}r}(\Gamma^{nilp}, \Gamma^{nilp})$.

The nilpotent completion may also be characterized by the universality property that it satisfies: every group morphism from Γ to a nilpotent group N factors uniquely through $j : \Gamma \to \Gamma^{nilp}$. In fact, the nilpotent completion of a group Γ is the limit of the projective system formed by morphisms from Γ to nilpotent groups. It may be checked from its definition that the lower central series quotient $\Gamma \to \Gamma/\Gamma_{n+1}$ has this universality property for morphisms into step n nilpotent groups, and that the nilpotent completion of Γ is its natural projection to the limit of the tower of quotients

$$\cdots \longrightarrow \Gamma/\Gamma_3 \longrightarrow \Gamma/\Gamma_2$$

The following property of nilpotent groups may be seen to extend from the abelian case:

LEMMA 1.3. *Let N be a nilpotent group. The set of its torsion elements $\mathrm{Tor}\, N$ forms a normal subgroup. Moreover, if N is finitely generated, $\mathrm{Tor}\, N$ is finite.*

A nilpotent group is torsion–free when $\mathrm{Tor}\, N = \{1\}$. We will write as $n-\mathbb{Z}-\mathcal{G}r$ the category of torsion–free nilpotent groups. The torsion subgroups $\mathrm{Tor}\, N$ are natural, so there is a modulo torsion functor

$$\begin{aligned} ._0 : n-\mathcal{G}r &\longrightarrow n-\mathbb{Z}-\mathcal{G}r \\ N &\longmapsto N_0 = N/\mathrm{Tor}\, N \end{aligned}$$

One may proceed now as in the nilpotent case, considering the category $pro-n-\mathbb{Z}-\mathcal{G}r$ of pro–torsion–free nilpotent groups, which is the full subcategory of $\mathcal{G}r$ obtained as the projective closure of $n-\mathbb{Z}-\mathcal{G}r$. The inclusion functor $pro-n-\mathbb{Z}-\mathcal{G}r \hookrightarrow \mathcal{G}r$ has a left adjoint, which gives rise to the torsion–free nilpotent completion $j_0 : \Gamma \to \Gamma_0^{nilp}$. As in the nilpotent case, the torsion–free nilpotent completion of a group Γ is the tower of projections from Γ to its torsion–free nilpotent quotients

$$\cdots \longrightarrow (\Gamma/\Gamma_3)_0 \longrightarrow (\Gamma/\Gamma_2)_0$$

Rational homotopy theory studies the unipotent representations of the fundamental group. These may be summed up, as in the nilpotent and torsion–free nilpotent cases, in the unipotent completion of a group Γ, which may be defined analogously:

Let k be a field, and let $U(k)$ be the category of k–unipotent algebraic groups, with algebraic morphisms. Let $pro-U(k)$ be the category of pro–k–unipotent groups, which is the projective closure of $U(k)$ in $\mathcal{G}r$. The inclusion functor $pro-U(k) \hookrightarrow \mathcal{G}r$ has a left adjoint functor $.\otimes k : \mathcal{G} \to pro-U(k)$, satisfying natural bijections

$$\mathrm{Hom}_{pro-U(k)}(\Gamma \otimes k, U) = \mathrm{Hom}_{\mathcal{G}r}(\Gamma, U)\,. \qquad (2)$$

DEFINITION 1.4. The k–unipotent completion of a group Γ is the morphism

$$j \otimes k : \Gamma \longrightarrow \Gamma \otimes k\,,$$

where $j \otimes k$ is the morphism corresponding to $\mathrm{Id} \in \mathrm{Hom}_{\mathcal{G}r}(\Gamma,\Gamma)$ in the natural bijection (2).

This is the abstract definition of the k–unipotent completion of a group Γ. It is equivalent to the universality property with respect to morphisms from Γ to a k–unipotent group, and not very helpful in computational terms. In the case of a field k of characteristic zero and a finitely presented group Γ there are alternative ways to construct and compute unipotent completions, which will be the subject of the next sections.

Before that, we will give an equivalent presentation of unipotent completions. Malcev showed that over a field of characteristic zero, the addition laws defined by the Baker–Campbell–Hausdorff formula induce an equivalence between nilpotent Lie algebras and unipotent algebraic groups ([**66**]). The projectively completed version of the correspondence is:

THEOREM 1.5 (Malcev). *Let k be a field of characteristic zero. The correspondence between Lie groups and Lie algebras gives a categorical equivalence between the category of pro–k–unipotent Lie groups and that of pro–nilpotent k–Lie algebras.*

DEFINITION 1.6. Let Γ be a group and k a field of characteristic zero. The Malcev algebra of Γ over k, denoted $\mathcal{L}(\Gamma, k)$, is the Lie algebra of the pro–k–unipotent completion $\Gamma \otimes k$.

Finally, a first example is presented in order to justify the notation for unipotent completions.

EXAMPLE 1.7. Let M be a finitely generated abelian group. Its nilpotent completion is $\mathrm{Id} : M \to M$, as M is nilpotent and it fulfills the unique factorization condition. Moreover, every morphism from M to a torsion–free nilpotent group N sends torsion elements in M to the identity, thus factors uniquely through the quotient $M \to M/\mathrm{Tor}\, M$. This quotient is torsion–free nilpotent, so $M_0^{nilp} = M/\mathrm{Tor}\, M$. The same holds for any nilpotent group M.

Fix now a field k of characteristic zero, and consider the morphisms $M \to U$ to k–unipotent groups. Unipotent groups over a zero characteristic field are torsion–free nilpotent, so the completion morphism $M \to M \otimes k$ must factor through the torsion–free nilpotent completion $M \to M/\mathrm{Tor}\, M \cong \mathbb{Z}^{\mathrm{rank}\, M}$. Given a morphism $\varphi : M/\mathrm{Tor}\, M \to U$ to a unipotent group, a basis $x_1, \dots, x_m$ of M must be mapped to commuting elements $a_1, \dots, a_m \in U$. The logarithm is well defined in U, so we may take the ordinary tensor product and define a map

$$\begin{aligned} \varphi \otimes k : (M/\mathrm{Tor}\, M) \otimes_{\mathbb{Z}} k &\longrightarrow U \\ \lambda_1 x_1 + \cdots + \lambda_m x_m &\longmapsto \exp(\lambda_1 \log(a_1)) \cdot \cdots \cdot \exp(\lambda_m \log(a_m)) \end{aligned}$$

As $a_1, \dots, a_m$ commute, so do the terms $\exp(\lambda_i \log(a_i))$, thus $\varphi \otimes k$ is a group morphism. This shows that every morphism $M \to U$ factors uniquely through the natural morphism $M \to M \otimes_{\mathbb{Z}} k$, which is therefore the k–unipotent completion of M.

2. The Malcev algebra and the de Rham fundamental group

In this section the unipotent completion of a group, which is called the de Rham fundamental group in the case of Γ a fundamental group, is constructed for fields of characteristic zero. This is done by means of the group algebra. The equivalence of the pro–k–unipotent groups with pro–k–nilpotent Lie algebras, the Malcev algebras, is also described. Finally, the torsion–free nilpotent and k–unipotent completions are compared.

We will assume throughout this section that k is a field of characteristic zero and that Γ is a finitely generated group.

Let $k\Gamma$ be the group k–algebra of Γ. This is an augmented algebra, with augmentation

$$\begin{aligned} \varepsilon : k\Gamma &\longrightarrow k \\ \sum \lambda_i g_i &\longmapsto \sum \lambda_i \end{aligned}$$

Its kernel $J = \ker \varepsilon$ admits a linear basis $\{g - 1 \mid g \in \Gamma\}$. This ideal is closely related to the lower central series of Γ, as we proceed to explain.

Let $\widehat{k\Gamma}$ be the J–adic completion of $k\Gamma$, and $\hat{J}$ its augmentation ideal, which is the completion of J. This algebra is the complete augmented k–algebra generated by the group Γ. It contains a multiplicative group $1 + \hat{J}$, together with a morphism

$$\begin{aligned} j : \Gamma &\longrightarrow 1 + \hat{J} \\ g &\longmapsto 1 + (g - 1) \end{aligned}$$

and a linear space $\hat{J}$, with a k–Lie algebra structure given by the bracket $[x, y] = xy - yx$. There is a *set* bijection

$$\hat{J} \xrightarrow{\phi} 1 + \hat{J}\,,$$

where ϕ is any map such that $\phi(x) = 1 + x + o(x^2)$. This bijection respects the filtration induced by the powers of $\hat{J}$. Moreover, the group bracket in the group $1+\hat{J}$ verifies that $[1+\hat{J}^m, 1+\hat{J}^n] \subset 1+\hat{J}^{m+n}$. This endows the graded k–linear space $\mathrm{Gr}\,(1+\hat{J}) = \oplus_{n\geq 1}(1+J^n)/(1+J^{n+1})$ with a Lie algebra structure, and $\mathrm{Gr}\,\phi$ induces a graded Lie algebra isomorphism between $\mathrm{Gr}\,(1+\hat{J})$ and $\mathrm{Gr}\,\hat{J} = \oplus_{n\geq 1} J^n/J^{n+1}$.

An elementary recursive computation shows that if $g \in \Gamma_n$, then $g - 1 \in J^n$. This allows the definition of a k–Lie algebra morphism

$$\mathrm{gr}\, j : \mathrm{gr}\,\Gamma \otimes k \longrightarrow \oplus_{n\geq 1} J^n/J^{n+1}$$

by sending the homogeneous elements $\bar{g} \in \Gamma_n/\Gamma_{n+1}$ to $\overline{g-1} \in J^n/J^{n+1}$, which induces an associative algebra morphism between the universal enveloping algebra $U(\mathrm{gr}\,\Gamma \otimes k)$ and $\oplus J^n/J^{n+1}$.

THEOREM 1.8. (Quillen, [**78**]) *Let k be a field of characteristic zero, and Γ a group. The morphism $gr\,j$ induces an isomorphism of complete associative algebras between $U(gr\,\Gamma \otimes k)$ and $\oplus_{n\geq 1} J^n/J^{n+1}$.*

Theorem 1.8 may be reformulated as follows:

COROLLARY 1.9. *The group morphism*

$$j : \Gamma \longrightarrow 1 + \hat{J}$$
$$g \longmapsto 1 + (g - 1)$$

verifies that $j(g) \in 1 + \hat{J}^n$ *if and only if* $g^m \in \Gamma_n$ *for some nonzero integer* m.

In the case of free groups the identity $j^{-1}(1 + \hat{J}^n) = \Gamma_n$ is satisfied (see [**83**]), but this is not the case in general. The equality for all finitely presented Γ was known as the *dimension subgroup conjecture* until it was disproved.

Moreover, the algebras $k\Gamma$ have a coalgebra structure, with coproduct given by $\Delta g = g \otimes g$ for $g \in \Gamma$. This coproduct extends to the J-adic completion, making $\widehat{k\Gamma}$ a complete Hopf algebra. The coproduct gives rise to two sets of distinguished elements in $\widehat{k\Gamma}$:

DEFINITION 1.10. **(i)** The group-like elements are the elements of the set

$$\mathcal{G}(\widehat{k\Gamma}) = \{x \in 1 + \hat{J} \mid \Delta x = x \otimes x\}.$$

(ii) The primitive elements are the elements of the set

$$\mathcal{P}(\widehat{k\Gamma}) = \{y \in \hat{J} \mid \Delta y = 1 \otimes y + y \otimes 1\}.$$

The image of Γ in $\widehat{k\Gamma}$ obviously lies in $\mathcal{G}(\widehat{k\Gamma})$. The theory of Hopf algebras shows that these sets have additional structures and relations (see [**69**]):

- Group-like elements form a subgroup of $1 + \hat{J}$.
- Primitive elements form a sub-Lie algebra of $\hat{J}$.
- Consider the formal power series $\exp(y) = \sum \frac{y^n}{n!}$ and $\log(1+x) = \sum (-1)^{n+1} \frac{x^n}{n}$, inverse to each other. These series induce filtered set bijections

  $$\mathcal{P}(\widehat{k\Gamma}) \underset{\log}{\overset{\exp}{\rightleftarrows}} \mathcal{G}(\widehat{k\Gamma})$$

  and produce k-Lie algebra isomorphisms between their graduates.

The unipotent completion of finitely generated groups may be obtained by means of the above constructions ([**79**], Appendix A).

THEOREM 1.11 (Malcev). *Let* Γ *be a finitely generated group and* k *a field of characteristic zero.*
(i) *The* k*-unipotent completion of* Γ *is the morphism* $j : \Gamma \to \mathcal{G}(\widehat{k\Gamma})$.
(ii) *The Malcev algebra of* Γ *over* k *is* $\mathcal{P}(\widehat{k\Gamma})$.

Some properties of the unipotent completion and its Malcev algebra relevant for our purposes (see [**66**], [**79**] Append. A, [**80**] Thm. 2.18) are:

PROPOSITION 1.12. *Let N be a finitely generated nilpotent group, and $j : N \to N \otimes k$ its k–unipotent completion.*
(i) *The kernel of j is the torsion subgroup $Tor\,N$.*
(ii) *For every element $h \in N \otimes \mathbb{Q}$ there is an integer m and an element $g \in Im\,j$ such that $h = g^m$.*

PROPOSITION 1.13. *Let Γ be a finitely generated group and k a field of characteristic zero.*
(i) *The k–unipotent completion group $\Gamma \otimes k$ is the projective limit of the tower of unipotent groups*

$$\cdots \longrightarrow (\Gamma/\Gamma_3) \otimes k \longrightarrow (\Gamma/\Gamma_2) \otimes k\,,$$

and the completion morphism is induced by the tower of natural morphisms $\Gamma \to \Gamma/\Gamma_n \to (\Gamma/\Gamma_n) \otimes k$.
(ii) *The Malcev algebra of Γ over k is the projective limit of the tower of nilpotent k–Lie algebras*

$$\cdots \longrightarrow \mathcal{L}(\Gamma/\Gamma_3, k) \longrightarrow \mathcal{L}(\Gamma/\Gamma_2, k)$$

(iii) *The filtrations by the corresponding lower central series induce isomorphisms of graded k–Lie algebras*

$$Gr\,\mathcal{L}(\Gamma, k) \overset{\exp\cong}{\longrightarrow} Gr\,(\Gamma \otimes k) \overset{j\cong}{\longleftarrow} (gr\Gamma) \otimes_{\mathbb{Z}} k\,.$$

(iv) *The k–Malcev algebra of Γ is*

$$\mathcal{L}(\Gamma, k) \cong \mathcal{L}(\Gamma, \mathbb{Q}) \otimes_{\mathbb{Q}} k$$

REMARK 1.14. It follows from Proposition 1.13 that for any field k of characteristic zero, the k–unipotent completion and its Malcev algebra are obtained from the corresponding rational completions $\Gamma \otimes \mathbb{Q}, \mathcal{L}(\Gamma, \mathbb{Q})$ by extension of scalars. This is a consequence of the fact that every k–unipotent representation of a discrete group Γ factors through a rational unipotent representation, and contradicts the ordering of k–unipotent completions that the lattice of subcategories $U(k)$ suggests.

Due to this fact, the change from the rational to a k–unipotent completion of Γ may coarsen the isomorphism type of the completion. An example of this phenomenon is given in Remark II.2.15 of [**80**].

Propositions 1.12 and 1.13 allow also the comparison between the torsion–free nilpotent and $\mathbb{Q}$–unipotent completions of a group. First let us recall a fact about lattices in $\mathbb{Q}$–unipotent groups ([**80**], Thm. 2.12 and Rmk. 2.16):

LEMMA 1.15. *A $\mathbb{Q}$–unipotent group U contains discrete lattices. Any two such lattices are commensurable.*

Given a unipotent group U defined over $\mathbb{Q}$, a lattice $N \subset U$ is called *an integral form* of U. Different commensurable lattices in a nilpotent group need not be isomorphic:

EXAMPLE 1.16. The real Heisenberg group is the unipotent matrix group $\mathcal{H}_3(\mathbb{R}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}\,(3, \mathbb{R}) \right\}$. Two examples of non–isomorphic lattices in $\mathcal{H}_3(\mathbb{R})$ are:

(i) The integer Heisenberg group $\mathcal{H}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in \mathrm{GL}\,(3, \mathbb{Z}) \right\}$.

(ii) The congruence lattice $N = \left\{ \begin{pmatrix} 1 & 2x & 2z \\ 0 & 1 & 2y \\ 0 & 0 & 1 \end{pmatrix} \mid x, y, z \in \mathbb{Z} \right\}$.

There is an important difference between the abelian quotients of the two lattices: while $\mathcal{H}_3(\mathbb{Z})/\mathcal{H}_3(\mathbb{Z})_2 \cong \mathbb{Z}^2$ is torsion–free, the abelian quotient N/N_2 is generated by the matrices

$$\begin{pmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and is isomorphic to $\mathbb{Z}^2 \oplus \mathbb{Z}/2\mathbb{Z}$, with the torsion given by the last generator. Using Prop. 2.17 of [**80**], one may check that any lattice in $\mathcal{H}_3(\mathbb{R})$ with a torsion–free abelianization is isomorphic to $\mathcal{H}_3(\mathbb{Z})$.

This behaviour of lattices in the Heisenberg group seems to be generic among (pro–)unipotent groups known to the author, motivating the following conjecture:

CONJECTURE 1.17. *Let $U = \varprojlim U^n(\mathbb{R})$ be a pro–$\mathbb{R}$–unipotent group, where the groups $U^n(\mathbb{R})$ have nilpotency class n. Let $N = \varprojlim N^n, G = \varprojlim G^n$ be two projective systems of lattices in the tower of groups*

$$\cdots \longrightarrow U^{n+1} \longrightarrow U^n \longrightarrow U^{n-1} \longrightarrow \cdots$$

such that $N^n \cong N^{n+1}/(N^{n+1})_{n+1}, G^n \cong G^{n+1}/(G^{n+1})_{n+1}$ and these quotients are lattices in the unipotent groups U^n for every n. Then the two towers of lattices $\cdots \to N^n \to \dots$ and $\cdots \to G^n \to \dots$ are isomorphic.

If Conjecture 1.17 is true, the isomorphism type of the towers of lattices would be the integral form of the tower of unipotent groups

$\varprojlim U^n$. In the case of the unipotent completion $U = \Gamma \otimes \mathbb{R}$ of a finitely presented group Γ, we know after Malcev that such a pro–lattice always exists: it is the torsion–free nilpotent completion Γ_0^{nilp} of Γ, and it would follow from Conjecture 1.17 that $\Gamma_0^{nilp} \cong \Delta_0^{nilp}$ if and only if $\Gamma \otimes \mathbb{Q} \cong \Delta \otimes \mathbb{Q}$, i.e. the $\mathbb{Q}$–unipotent and the torsion–free nilpotent completions would determine each other. As this is only conjectural, we will prove a partial result that allows the extension of our results of Chapter II from unipotent completions/Malcev algebras to the torsion–free nilpotent completions used by Campana in [**20**], [**22**].

LEMMA 1.18. *Let $f\colon \Gamma \to \Delta$ be a homomorphism between finitely presentable groups.*

(i) *If $Im(f) \subset \Delta$ has finite index, then $f \otimes k$ is an isomorphism if and only if f_0^{nilp} is injective and has finite index image.*

(ii) *If f is surjective, then $f \otimes k$ is an isomorphism if and only if f_0^{nilp} is.*

PROOF. By the extension of scalars property of Proposition 1.13 (iv) it suffices to prove the case of $k = \mathbb{Q}$.

(i) Let $f \otimes \mathbb{Q}$ be an isomorphism. As $\Gamma_0^{nilp} \to \Gamma \otimes \mathbb{Q}$ is injective, the map f_0^{nilp} must also be injective. Moreover, the fact that $\mathrm{Im}(f) \subset \Delta$ has finite index implies that the induced homomorphisms

$$(\Gamma/\Gamma_n)/_{Tor} \longrightarrow (\Delta/\Delta_n)/_{Tor}$$

have images with finite index. Consequently, there is a tower of surjective homomorphisms from the finite quotient $\Delta/\mathrm{Im}(f)$ to the projective system of quotients

$$\cdots \longrightarrow ((\Delta/\Delta_n)/_{Tor})\,/\mathrm{Im}(f) \longrightarrow \ldots$$

These homomorphisms extend to a surjection from $\Delta/\mathrm{Im}(f)$ to its projective limit $\Delta_0^{nilp}/\mathrm{Im}(f_0^{nilp})$, which must therefore be finite.

Conversely, assume that f_0^{nilp} is injective and almost surjective. This implies that all the finite steps of the projective system

$$(f_0^{nilp})_n : (\Gamma/\Gamma_n)/_{Tor} \longrightarrow (\Delta/\Delta_n)/_{Tor}$$

must also be injective and almost surjective homomorphisms. Consider now the projective system of maps

$$(f \otimes \mathbb{Q})_n : (\Gamma/\Gamma_n) \otimes \mathbb{Q} \longrightarrow (\Delta/\Delta_n) \otimes \mathbb{Q}\ .$$

By Proposition 1.12 (ii) every element of $\ker(f \otimes \mathbb{Q})_n$ has a power in $\ker(f_0^{nilp})_n = \{1\}$. The groups $(\Gamma/\Gamma_n) \otimes \mathbb{Q}$ are torsion free, so the homomorphisms $(f \otimes \mathbb{Q})_n$ must be injective. Moreover, every element of $(\Delta/\Delta_n) \otimes \mathbb{Q}$ has a power in $(\Delta/\Delta_n)/_{Tor}$, thus also a possibly higher power in $\mathrm{Im}(f_0^{nilp})_n \subset \mathrm{Im}(f \otimes \mathbb{Q})_n$. As $(f \otimes \mathbb{Q})_n$ is a homomorphism of $\mathbb{Q}$-unipotent groups, this means that it is onto.

The completion $f \otimes \mathbb{Q}$ is the projective limit of the isomorphisms $(F \otimes \mathbb{Q})_n$, so it must also be an isomorphism.

(ii) can be proved analogously. □

The tower in Proposition 1.13 (ii) motivates the following

DEFINITION 1.19. The step n Malcev algebra of a group Γ over a field k is the nilpotent Lie algebra

$$\mathcal{L}_n(\Gamma, k) = \mathcal{L}(\Gamma/\Gamma_{n+1}, k)$$

By Proposition 1.13 and Theorem 1.5, the step n Malcev algebra is equivalent to the group $(\Gamma/\Gamma_n)/_{Torsion}$ Another immediate consequence of Proposition 1.13 and the properties of the lower central series in groups is that the tower of step n Malcev algebras either stations or is strictly growing:

COROLLARY 1.20. *A finitely generated group Γ and its Malcev algebra $\mathcal{L}(\Gamma, k)$ satisfy either of the mutually excluding properties:*
(i) *The abelian lower central series quotient Γ_n/Γ_{n+1} is formed by torsion elements for some n, and $\mathcal{L}_n(\Gamma, k) = \mathcal{L}_{n+1}(\Gamma, k) = \cdots = \mathcal{L}(\Gamma, k)$ for any field k of characteristic zero.*
(ii) *The lower central series quotients Γ_n/Γ_{n+1} have nontorsion elements for every n, and the step n Malcev algebras $\mathcal{L}_n(\Gamma, k)$ have nilpotency class n.*

This motivates the following

DEFINITION 1.21. A group Γ is rationally nilpotent if $\mathcal{L}\Gamma = \mathcal{L}_n\Gamma$ for some integer n. The first such integer is the rational nilpotency class of n.

The concept of a rationally nilpotent group parallels that of almost nilpotent group. Nevertheless there is no inclusion between the two classes of groups. The Higman 4–group (see [**84**]) is rationally nilpotent, indeed it has Malcev algebra 0, yet it is not almost nilpotent. Likewise there exist almost nilpotent groups which are not rationally nilpotent. All such examples in both cases known to the author are not Kähler groups.

3. Malcev algebras of free groups

In this section we compute the $\mathbb{Q}$-unipotent completion of free groups, and then apply it to give an algorithm for the computation of $\mathcal{L}_2\Gamma$ from a finite presentation of the group.

When the group Γ is F_r, the free group generated by $\{a_1, a_2, \dots, a_r\}$, its associated towers of groups $F_r/(F_r)_n \otimes \mathbb{Q}$ and corresponding Lie algebras can be further identified.

Let $A_0(\mathbb{Q}, r)$ be the free associative $\mathbb{Q}$-algebra generated by r elements $x_1, \dots, x_r$. It has an augmentation ε defined by $\varepsilon(1) = 1$, $\varepsilon(x_i) = 0$ for $i = 1, \dots, r$. Let $A(\mathbb{Q}, r)$ be the completion of $A_0(\mathbb{Q}, r)$ with respect to $\ker \varepsilon$: it is the associative free algebra of formal power series in the non-commuting variables $x_1, \dots, x_r$. It is filtered by the powers of its maximal ideal, which we shall denote $J_\mathbb{Q}$. As in the algebra $\widehat{\mathbb{Q}\Gamma}$ described in the previous section, the set $1 + J_\mathbb{Q}$ forms a group with the algebra product and $J_\mathbb{Q}$ has a $\mathbb{Q}$-Lie algebra structure with linear addition and bracket $[x, y] = xy - yx$. There is good reason for this similarity:

PROPOSITION 1.22. (i) *The group morphism defined by*

$$\begin{aligned} \nu : F_r &\longrightarrow 1 + J_\mathbb{Q} \\ a_i &\longrightarrow 1 + x_i \end{aligned}$$

is injective, and it induces a complete filtered augmented $\mathbb{Q}$-algebra isomorphism $\check{\nu}$ between $\widehat{\mathbb{Q}F_r}$ and $A(\mathbb{Q}, r)$.

(ii) $\nu^{-1}(1 + J_\mathbb{Q}^n) = (F_r)_n \quad \forall\, n \in \mathbb{N}^*$

PROOF. (i) The injectivity of ν is well known, see for instance [**65**], §5.5, Thm. 5.6. This inclusion gives $A(\mathbb{Q}, r)$ the same universal property for complete associative algebras as $\widehat{\mathbb{Q}F_r}$, thus they are isomorphic.

(ii) is Thm. 6.3 in [**83**], I, 4.

□

Due to the isomorphism $\widehat{\mathbb{Q}F_r} \cong A(\mathbb{Q}, r)$, the group $1 + J_\mathbb{Q}$, the Lie algebra $J_\mathbb{Q}$ and their graduated rings are isomorphic to the corresponding structures in $\widehat{\mathbb{Q}F_r}$, and there are also set maps

$$J_\mathbb{Q} \underset{\log}{\overset{\exp}{\rightleftarrows}} 1 + J_\mathbb{Q}$$

mutually inverse and commuting with $\check{\nu}$. Again, these maps restrict to bijections between $1 + J_\mathbb{Q}^n$, $J_\mathbb{Q}^n$ and they induce $\mathbb{Q}$-Lie algebra isomorphisms of $\operatorname{gr}(1 + J_\mathbb{Q})$ with $\operatorname{gr} J_\mathbb{Q}$. Another useful consequence of Prop. 1.22 is the following:

PROPOSITION 1.23. (i) *The map ν above defined induces morphisms*

$$\bar{\nu}_n : F_r/(F_r)_n \longrightarrow (1 + J_\mathbb{Q})/(1 + J_\mathbb{Q}^n)$$

which are injective, and induce complete augmented filtered $\mathbb{Q}$-algebra isomorphisms $\check{\nu}_n$ between $\mathbb{Q}\widehat{F_r/(F_r)_n}$ and $A(\mathbb{Q},r)/J^n_{\mathbb{Q}}$ for all $n \geq 2$.

(ii) $\check{\nu}_n^{-1}\left((1+J^k_{\mathbb{Q}})/(1+J^n_{\mathbb{Q}})\right) = (F_r)_k/(F_r)_n$ *for* $k \leq n$.

PROOF. (i) The group inclusions come straight from Prop. 1.22 (ii). The algebra isomorphisms are a consequence of the same universality property invoked in the proof of Prop. 1.22 (i), and the isomorphisms $(1+J_{\mathbb{Q}})/(1+J^n_{\mathbb{Q}}) \cong 1+J_{\mathbb{Q}}/J^n_{\mathbb{Q}} \subset A(\mathbb{Q},r)/J^n_{\mathbb{Q}}$, which are easy to compute.

(ii) comes also straight from Prop. 1.22 (ii). □

We will now study the graded Lie algebra $J_{\mathbb{Q}}$.

Let R be a ring which is an integral domain. We define $L_0(R,r)$ to be the free R-Lie algebra generated by elements $\xi_1,\dots,\xi_r$, with augmentation ε sending scalars to themselves and the ξ_i to zero. Let $L(R,r)$ the completion of $L_0(R,r)$ with respect to ε; $L(R,r)$ is a profinite Lie algebra and may be graded by the weight of the brackets; we denote as $\mathcal{M}$ its maximal ideal.

We will use the Lie algebras $L(\mathbb{Z},r), L(\mathbb{Q},r)$, which are directly related to our group algebra constructions.

PROPOSITION 1.24. (i) *There is an isomorphism of graded Lie algebras*

$$\psi : L(\mathbb{Z},r) \longrightarrow gr(F_r)$$

determined by

$$\xi_i \longrightarrow x_i$$
$$(\alpha,\beta) \longrightarrow (\psi(\alpha),\psi(\beta))$$

(ii) *The map ψ also induces for all $n \geq 1$ isomorphisms of graded $\mathbb{Q}$-Lie algebras*

$$\bar{\psi}_n : L(\mathbb{Q},r)/\mathcal{M}^n \longrightarrow gr(F_r/(F_r)_n \otimes \mathbb{Q}) \cong \mathcal{L}_n(F_r,\mathbb{Q})$$

PROOF. (i) This is [**83**], I, 4 §6, Thm. 6.1, or [**65**], 5.7, Thm. 5.12.

(ii) The $\mathbb{Z}$-Lie algebra $\mathrm{gr}\,(F_r/(F_r)_n)$ is $\bigoplus_{i=1}^{n-1}(F_r)_i/(F_r)_{i+1}$. As it preserves degrees, applying the isomorphism of (i) tensored by $\mathbb{Q}$ yields another isomorphism

$$L(\mathbb{Q},r)/\mathcal{M}^n \longleftarrow \mathrm{gr}\,(F_r/(F_r)_n)\otimes\mathbb{Q}$$

The isomorphism of Proposition 1.23 (ii) completes the proof. □

Proposition 1.24 implies that a $\mathbb{Q}$-linear basis of the Lie brackets of weight m is mapped by ψ to a $\mathbb{Q}$-linear basis of $\mathrm{gr}\,(F_r/(F_r)_n \otimes \mathbb{Q})^m$ for $m < n$. This fact allows us to compute the dimension of each piece of the graduate of $F_r/(F_r)_n \otimes \mathbb{Q}$ with one of Witt's formulae ([**65**],[**17**]):

$$\dim\left(\mathrm{gr}\,(F_r/(F_r)_n \otimes \mathbb{Q})\right) = \dim(\mathcal{L}_m(\mathbb{Q}, r)) = \frac{1}{n}\sum_{d|n} \mu(d) r^{\frac{n}{d}}$$

where μ is the Möbius function. Furthermore there are algorithms that produce an ordered homogeneous basis, called the *Hall basis*, of $\mathcal{L}(\mathbb{Q}, r)$, thus of $J_{\mathbb{Q}}$. Its elements of degree $< n$ are mapped by ψ on a basis of $\mathrm{gr}\,(F_r/(F_r)_n \otimes \mathbb{Q})$.

To close this paragraph, we give the brackets in F_2 with generators a, b that form a Hall basis up to weight 5. It is printed in [**17**], II §2.10 and [**83**],I, 4 §5, with every word in the opposite order due to differing bracket conventions between them and [**65**], which we follow. Bracket writing conventions also vary in literature, and after [**65**] we will henceforth write bracket arrangements of the form $[[\ldots[a_1, a_2], a_3], \ldots], a_n]$ formed by nested brackets on the left as $[a_1, \ldots, a_n]$, thus we will denote $[[[b, a], a], b]$ as $[b, a, a, b]$, $[[[b, a], a], [b, a]]$ as $[[b, a, a], [b, a]]$ and so on, except in some explicit computations where the full bracket arrangements will be written to avoid causing doubts.

Weight 1	a	b	
Weight 2	$[b, a]$		
Weight 3	$[b, a, a]$	$[b, a, b]$	
Weight 4	$[b, a, a, a]$	$[b, a, a, b]$	$[b, a, b, b]$
Weight 5	$[b, a, a, a, a]$	$[b, a, a, a, b]$	$[b, a, a, b, b]$
	$[b, a, b, b, b]$	$[[b, a, a], [b, a]]$	$[[b, a, b], [b, a]]$

4. Malcev algebras of finitely presented groups

The Malcev algebras of free groups have been studied in the previous section. We will continue this study with finitely presented groups. When the group Γ is the fundamental group of a topological space X, the abelian algebra $\mathcal{L}_1\Gamma$ is just $H_1(X; \mathbb{Q})$. We will consider in this section the following simplest algebra, $\mathcal{L}_2\Gamma \cong (\Gamma/\Gamma_2) \otimes \mathbb{Q} \oplus (\Gamma_2/\Gamma_3) \otimes \mathbb{Q}$. The algebra $\mathcal{L}_2\Gamma$ is the quotient of the Malcev algebra $\mathcal{L}\Gamma$ by its third commutator ideal $\mathcal{L}\Gamma^{(3)}$, and is also the quotient of the holonomy algebra of Γ $\mathfrak{g}_\Gamma$ (cf. [**25**],[**60**]) by its third commutator ideal. As will be seen in Proposition 1.49 and Corollary 2.6, in the case of Kähler groups and 1–formal groups in general, the finite–dimensional algebra $\mathcal{L}_2\Gamma$ determines the full Malcev algebra. Rational coefficients will be used thoroughout the section, but the fact that the corresponding k–Malcev algebras are obtained by extension of scalars from the $\mathbb{Q}$–Malcev algebra makes all the results in this section hold verbatim for any field k of characteristic zero.

The algorithmic constructions appearing in this section were described to the author by Manfred Hartl.

The groups Γ we will study will be given by finite presentations $\Gamma = \langle x_1, \ldots, x_n\ ;\ r_1, \ldots, r_s\rangle$. This means that Γ is defined by

$$1 \longrightarrow N \longrightarrow F \longrightarrow \Gamma \longrightarrow 1 \qquad (1.3)$$

where F is the free group generated by the generator set $\{x_1, \ldots, x_n\}$, and N is the normal subgroup of F spanned by the relation set $\{r_1, \ldots, r_s\} \subset F$.

The above constructions in the case of free groups have been described in section 3:

EXAMPLE 1.25. Free groups.
Let $\Gamma = F_n = F_{\{x_1,\ldots,x_n\}}$. Its Malcev completion and Lie algebras $\mathcal{L}_m F_n$ has been computed by means of its group algebra in Proposition 1.24 (cf. also [**65**],[**79**],[**83**]). The conclusion is that, denoting by $\mathcal{L}(S)$ the free $\mathbb{Q}$-Lie algebra spanned by a set S, there are isomorphisms

$$\mathcal{L}_m F_n \cong L(\{X_1, \ldots, X_n\})/L(\{X_1, \ldots, X_n\})^{(m+1)}$$

In particular, $\Gamma/\Gamma_2 F_n \otimes \mathbb{Q} \cong \mathbb{Q}x_1 \oplus \cdots \oplus \mathbb{Q}x_n$, $\Gamma_2/\Gamma_3 F_n \otimes \mathbb{Q} \cong \mathbb{Q}(x_1, x_2) \oplus \cdots \oplus \mathbb{Q}(x_{n-1}, x_n)$, and the brackets in $\mathcal{L}_2 F_n$ are the group ones in $\Gamma/\Gamma_2 F_n$ and zero all others.

The Lie algebra $\mathcal{L}_2\Gamma$ for a finitely presented Γ may be obtained from its presentation and $\mathcal{L}_2 F$. We will use an algorithm for computing them derived from [**88**], where a spectral sequence that computes all $J_\Gamma^m / J_\Gamma^{m+1}$ is described, and communicated to the author by Manfred Hartl.

Consider a group presentation $\Gamma = \langle x_1, \ldots, x_n\ ;\ r_1, \ldots, r_s\rangle$, which induces the exact sequence given in (1.3). Let $\mathbb{Q}F, \mathbb{Q}\Gamma$ be the $\mathbb{Q}$-group algebras of F, Γ, and denote by J_F, J_Γ their respective augmentation ideals. The sequence (1.3) induces an exact sequence of $\mathbb{Q}$-algebras

$$0 \longrightarrow K \longrightarrow \mathbb{Q}F \longrightarrow \mathbb{Q}\Gamma \longrightarrow 0 \qquad (1.5)$$

where K is the two-sided ideal generated by the $\mathbb{Q}$-vector space $D = \langle r_1 - 1, \ldots, r_s - 1\rangle \subset J_F$. This sequence restricts to exact sequences

$$0 \longrightarrow K \longrightarrow J_F^m + K \longrightarrow J_\Gamma^m \longrightarrow 0 \qquad (1.6)$$

for all $m \geq 1$. We will compute J_Γ/J_Γ^2, J_Γ^2/J_Γ^3 from those sequences:

PROPOSITION 1.26. *Consider the linear map $f : \bigoplus_{i=1}^s \mathbb{Q}r_i \longrightarrow J_F$ determined by $r_i \mapsto r_i - 1$.*

(i) *Let $d_0 : \oplus \mathbb{Q}r_i \to J_F/J_F^2$ be the projection of f. Then $\operatorname{coker} d_0 \cong J_\Gamma/J_\Gamma^2$.*

(ii) *The map f induces a linear map*

$$\begin{aligned} d_1 : \ker d_0 &\longrightarrow J_F^2/(J_F^3 + J_F \cdot D + D \cdot J_F) \\ \sum \lambda_i r_i &\longmapsto \sum \lambda_i (r_i - 1) \end{aligned}$$

and $\operatorname{coker} d_1 \cong J_\Gamma^2/J_\Gamma^3$.

PROOF. (i) The exact sequences of (1.6) induce an isomorphism $J_\Gamma/J_\Gamma^2 \cong J_F/J_F^2 + K$. As K is the two-sided ideal spanned by D and $\mathbb{Q}F \cong \mathbb{Q} \oplus J_F$, actually $J_F^2 + K = J_F^2 + D$, and thus $J_\Gamma/J_\Gamma^2 \cong J_F/J_F^2 + D$. By its construction, $\operatorname{Im} d_0 = D$, and this proves (i).

(ii) Again by (1.6) we have

$$J_\Gamma^2/J_\Gamma^3 \cong \left(J_F^2/(J_F^2 \cap K)\right) / \left(J_F^3/(J_F^3 \cap K)\right) \cong J_F^2/(J_F^3 + J_F^2 \cap K)$$

The last denominator is $J_F^3 + J_F^2 \cap K = J_F^3 + J_F \cdot D + D \cdot J_F + D \cap J_F^2$. Obviously $f(\ker d_0) \subset J_F^2$ and thus d_1 is well defined. Moreover, its image is precisely $D \cap J_F^2$, and (ii) follows from this. □

We now relate the computed modules J_Γ/J_Γ^2, J_Γ^2/J_Γ^3 with the sought ones Γ/Γ_2, $\Gamma_2/\Gamma_3\Gamma \otimes \mathbb{Q}$ applying Quillen's Theorem 1.8 ([**78**]):

Following the notation of [**88**], we will use the *wedge product*, or alternating product, of the associative algebra $\mathbb{Q}\Gamma$, which is

$$x \wedge y := xy - yx$$

The wedge product of two linear subspaces $A, B \subset \mathbb{Q}\Gamma$ is the linear subspace

$$A \wedge B = \{w = \sum \lambda_i a_i \wedge b_i \in \mathbb{Q}\Gamma \mid a_i \in A\,,\ b_i \in B\,,\ \lambda_i \in \mathbb{Q}\}$$

Quillen's theorem implies that the Lie algebra $\oplus \Gamma_n/\Gamma_{n+1}\Gamma \otimes \mathbb{Q}$ is contained in the J_Γ-adic graduate of the group algebra, $\oplus J_\Gamma^n/J_\Gamma^{n+1}$. This inclusion sends the brackets of the Lie algebra to wedge products in $\oplus J_\Gamma^n/J_\Gamma^{n+1}$. In the cases $n = 1, 2$ this means:

COROLLARY 1.27. (i) $\Gamma/\Gamma_2 \otimes \mathbb{Q} \cong J_\Gamma/J_\Gamma^2$.

(ii) *Consider the inclusion* $J_\Gamma \wedge J_\Gamma \hookrightarrow J_\Gamma^2$. *Then*

$$\Gamma_2/\Gamma_3 \otimes \mathbb{Q} \cong (J_\Gamma \wedge J_\Gamma + J_\Gamma^3)/J_\Gamma^3 \subset J_\Gamma^2/J_\Gamma^3$$

Corollary 1.27 allows us to adapt the algorithm of Prop. 1.26 to compute Γ/Γ_2, $\Gamma_2/\Gamma_3\Gamma \otimes \mathbb{Q}$:

LEMMA 1.28. *The image of the restriction* $f : \ker d_0 \to J_F^2$ *lies in* $J_F \wedge J_F + J_F^3 \subset J_F^2$.

PROOF. Denote F_s the free group generated by $\{y_1, \dots, y_s\}$, and the map $r : F_s \to F$ sending y_i to r_i. The map $d_0 : \oplus \mathbb{Q} r_i \to J_F/J_F^2 \cong F/F_2 \otimes \mathbb{Q}$ is the map induced by r, $r \otimes \mathbb{Q} : F_s/(F_s)_2 \otimes \mathbb{Q} \to F/F_2 \otimes \mathbb{Q}$. Furthermore $\ker(r \otimes \mathbb{Q}) \cong \ker(r) \otimes \mathbb{Q}$, as $F_s/(F_s)_2$ is a free abelian group. Thus $\ker d_0$ admits a basis $\bar{w}_1, \dots, \bar{w}_k$, with the w_i words in F_s mapping to F_2 by r.

Now, the map $F_2 \to J_F^2$ sends a bracket (a, b) to $(a-1)(b-1) - (b-1)(a-1) +$ terms in J_F^3, and a product $\prod(a_i, b_i)$ to $\sum(a_i - 1)(b_i - 1) - (b_i - 1)(a_i - 1) +$ terms in J_F^3. Therefore, all the $w_i = \prod(a_{j_i}, b_{j_i})$ map to $J_F \wedge J_F + J_F^3$. □

Lemma 1.28 allows us to define a map $d_1 : \ker d_0 \to \bigwedge^2(\Gamma/\Gamma_2 \otimes \mathbb{Q})$ by composing

$$\ker d_0 \longrightarrow (J_F \wedge J_F + J_F^3)/J_F^3 \cong \bigwedge^2(F/F_2 \otimes \mathbb{Q}) \longrightarrow \bigwedge^2(\Gamma/\Gamma_2 \otimes \mathbb{Q})$$

PROPOSITION 1.29. $coker d_1 \cong \Gamma_2/\Gamma_3 \otimes \mathbb{Q}$.

PROOF. As we have previously explained, $\Gamma/\Gamma_2 \otimes \mathbb{Q} \cong J_\Gamma/J_G^2 \cong J_F/(J_F^2 + K) \cong J_F/(J_F^2 + D)$. Thus

$$\bigwedge^2(\Gamma/\Gamma_2 \otimes \mathbb{Q}) \cong \left(J_F \wedge J_F + (J_F^3 + J_F \cdot D + D \cdot J_F)\right)/(J_F^3 + J_F \cdot D + D \cdot J_F).$$

Also $f(\ker d_0) = D \cap J_F^2 \subset J_F \wedge J_F + J_F^3$ by Lemma 1.28, so

$$\begin{aligned} \operatorname{coker} d_1 &\cong \left(J_F \wedge J_F + J_F^3 + J_F \cdot D + D \cdot J_F + D \cap J_F^2\right) / \\ &\quad \left(J_F^3 + J_F \cdot D + D \cdot J_F + D \cap J_F^2\right) \\ &\cong \left(J_F \wedge J_F + J_F^3 + K \cap J_F^2\right) / \left(J_F^3 + K \cap J_F^2\right) \\ &\cong (J_\Gamma \wedge J_\Gamma + J_\Gamma^3)/J_\Gamma^3 \cong \Gamma_2/\Gamma_3 \otimes \mathbb{Q} \end{aligned}$$

the last isomorphism being given by Cor. 1.27. □

COROLLARY 1.30. $\dim \Gamma_2/\Gamma_3 \otimes \mathbb{Q} = \binom{\dim \Gamma/\Gamma_2 \otimes \mathbb{Q}}{2} - \dim \ker d_0 + \dim \ker d_1$

We are now able to determine the structure of the 2-step nilpotent Lie algebra $\mathcal{L}_2\Gamma$ of a finitely presented group $\Gamma = \langle x_1, \dots, x_n \; ; \; r_1, \dots, r_s \rangle$:

PROPOSITION 1.31. *Let $\bigwedge^{\leq 2}(\Gamma/\Gamma_2 \otimes \mathbb{Q})$ be the free exterior algebra generated by $\Gamma/\Gamma_2 \otimes \mathbb{Q}$ modulo the ideal $\bigwedge^{\geq 3}(\Gamma/\Gamma_2 \otimes \mathbb{Q})$ generated by wedges of length 3 or more. There is an isomorphism*

$$\mathcal{L}_2\Gamma \cong \left(\bigwedge^{\leq 2}(\Gamma/\Gamma_2 \otimes \mathbb{Q})\right) / (\ker d_0 / \ker d_1)$$

PROOF. There is an obvious map of exterior algebras, which is a linear isomorphism in every degree by the above results. □

Thus $\mathcal{L}_2\Gamma$ is the quotient of a free 2-step nilpotent $\mathbb{Q}$-Lie algebra $\bigwedge^{\leq 2}(H_1(\Gamma;\mathbb{Q}))$ by a subspace of 2-brackets $\ker d_0/\ker d_1$, which corresponds to the relations of the holonomy algebra. We have stated in Ex. 1.25 the case of free groups. Let us examine this structure in some other simple cases:

COROLLARY 1.32. *Let $\Gamma = \langle x_1, \dots, x_n \ ; \ r\rangle$ be a group admitting a presentation with a single relation. Then:*

(i) *If $r \notin F_2$, there is an isomorphism $\mathcal{L}_2\Gamma \cong \mathcal{L}_2 F_{n-1}$ with F_{n-1} a free group of rank $n-1$.*
(ii) *If $r \in F_2 \setminus F_3$, there is an isomorphism $\mathcal{L}_2\Gamma \cong \mathcal{L}_2 F/d_1(r)$.*
(iii) *If $r \in F_3$, there is an isomorphism $\mathcal{L}_2\Gamma \cong \mathcal{L}_2 F$.*

PROOF. All cases are found by applying Prop. 1.31.

(i) In this case $\Gamma/\Gamma_2 \otimes \mathbb{Q} \cong F_{n-1}/(F_{n-1})_2 \otimes \mathbb{Q}$, and as $r \notin F_2$, $\ker d_0 = \{0\}$.
(ii) In this case the map $F \to \Gamma$ induces an isomorphism $F/F_2 \otimes \mathbb{Q} \cong \Gamma/\Gamma_2 \otimes \mathbb{Q}$, $\ker d_0 = \mathbb{Q}r$, and as $r \notin F_3$, the coincidence of the lower central series and augmentation ideal power filtrations in free groups ([**65**] 5.12,[**83**]) shows that $r - 1 \notin J_F^3$, hence $d_1(r) \neq 0$.
(iii) In this case, $\ker d_0 = \mathbb{Q}r$ and again by the above coincidence of filtrations, $d_1(r) = 0$.

□

COROLLARY 1.33. *Let $\Gamma = \langle x_1, \dots, x_n \ ; \ r_1, \dots, r_s\rangle$ be a finitely presented group such that its defining relations may be divided in two sets: $\{r_1, \dots, r_k\}$ such that $\bar{r}_1, \dots, \bar{r}_k$ are linearly independent in $F/F_2 \otimes \mathbb{Q}$ and $\{r_{k+1}, \dots, r_s\}$ which belong to F_3. Then there is an isomorphism $\mathcal{L}_2\Gamma \cong \mathcal{L}_2 F_{n-k}$, where F_{n-k} is a free group of rank $n-k$.*

PROOF. In this case $\Gamma/\Gamma_2 \otimes \mathbb{Q}$ has rank $n-k$, $\ker d_0 = \mathbb{Q}r_{k+1} \oplus \cdots \oplus \mathbb{Q}r_n$ because those r_j are commutators and the other relations form a basis of $\operatorname{Im} f$, and $\ker d_1 = \ker d_0$ because $r_{k+1}, \dots, r_n \in \Gamma_3 F$. □

REMARK 1.34. We will be interested in this note in which groups Γ have a free 2-step nilpotent Lie algebra $\mathcal{L}_2\Gamma$, which by Prop. 1.31 is equivalent to $\ker d_0 = \ker d_1$.

Generic presentations with less relations than generators produce a free $\mathcal{L}_2\Gamma$. The reason is that given a group presentation $\Gamma = \langle x_1, \dots, x_n \mid r_1, \dots, r_s \rangle$ with a number of relations $s \leq n$, $\ker d_0 = 0$ and therefore $\mathcal{L}_2\Gamma$ is free, unless the classes $\bar{r}_1, \dots, \bar{r}_s \in F_n/(F_n)_2 \otimes \mathbb{Q}$ are linearly dependent. But the sets of linearly dependent $\bar{r}_1, \dots, \bar{r}_s$ form a codimension $n - s + 1$ closed subset of $(F_n/(F_n)_2 \otimes \mathbb{Q})^s$.

The hypotheses of Corollary 1.33 may be weakened by requiring only that $\{r_1, \dots, r_k\}$ map on a basis of $\operatorname{Im} d_0$, and the remaining relations $\{r_{k+1}, \dots, r_s\}$ belong to $F_3 \cdot N_k$, where N_k is the normal closure in F of $\{r_1, \dots, r_k\}$.

5. 1–minimal models and unipotent completions

As follows from Theorem 1.11, the de Rham fundamental group contains the torsion–free nilpotent completion of the fundamental group as a pro–cocompact lattice, and so both completions are very closely related. The de Rham fundamental group owes its name to the fact that when X is a smooth manifold, this group can be computed from the de Rham complex. Sullivan's construction of 1–minimal models gives us an algorithm for doing this. In order to describe it, we need first to define a few concepts. We cite as generic references for the contents of this section [**45**],[**91**],[**18**].

DEFINITION 1.35. (i) A *commutative differential graded algebra* (CDGA) is a graded algebra A which is graded–commutative, i.e.

$$y \wedge x = (-1)^{|x||y|} x \wedge y$$

for any two homogeneous elements $x, y \in A$ of degree $|x|, |y|$ respectively; and has a differential operator, which is a map of degree one $d\colon A \to A$ such that $d^2 = 0$ and

$$d(x \wedge y) = dx \wedge y + (-1)^{|x|} x \wedge dy \ .$$

Morphisms of CDGAs must respect the degree and the boundary operator.

(ii) A *quasi–isomorphism* of CDGAs is a CDGA morphism inducing an isomorphism in cohomology.

Quasi–isomorphisms do not necessarily have an inverse in the category of CDGAs, the following definition is a remedy for this problem:

DEFINITION 1.36. Two CDGAs A, B are *weakly equivalent* if there is a finite diagramme of CDGAs

$$A \to C_1 \leftarrow C_2 \to \cdots \leftarrow B$$

such that all the morphisms are quasi–isomorphisms.

The basic example of CDGA is the de Rham algebra $\mathcal{E}^*(X)$ of smooth forms on a smooth manifold X. We will deal with $\mathbb{R}$–CDGAs, unless something else is specified. Two other important notions in the CDGA category are:

DEFINITION 1.37. (i) A *base point* of a CDGA A is a CDGA morphism

$$\varepsilon \colon A \longrightarrow \mathbb{R} ,$$

where $\mathbb{R}$ is a CDGA with differential $d = 0$.
(ii) A *basepointed homotopy* between two CDGA morphisms $\varphi_0, \varphi_1 : A \to B$ is a CDGA morphism

$$\Phi : A \longrightarrow \mathbb{R}(t, dt) \otimes B ,$$

where $\mathbb{R}(t, dt)$ is the CDGA determined by assigning degree 0 to t and setting the obvious differential, such that given the two base points of $\mathbb{R}(t, dt)$, $\varepsilon_0, \varepsilon_1$, obtained by sending t to 0 and 1 respectively and dt to 0, one has the identities

$$\varphi_0 = (\varepsilon_0 \otimes \mathrm{Id}) \circ \Phi , \qquad \varphi_1 = (\varepsilon_1 \otimes \mathrm{Id}) \circ \Phi .$$

Again, the motivating examples for these definitions are the notions of base point of a smooth manifold $x \in X$, with the evaluation map of forms at x, and of the codifferential of a homotopy of smooth maps $H \colon [0,1] \times X \to Y$.

Let X be a smooth manifold, and $\mathcal{E}^*_X$ its de Rham complex. The theory of minimal models developed by Sullivan shows that the CDGA of global forms $\mathcal{E}^*(X)$ has a 1–minimal model. This is a certain free commutative differential graded algebra $M_X(2,0)$, or simply M_X, defined as the limit of an inductive system of CDGAs

$$M_X(1,1) \hookrightarrow M_X(1,2) \hookrightarrow M_X(1,3) \hookrightarrow \dots ,$$

together with a morphism $\rho \colon M_X \to \mathcal{E}^*(X)$ such that in cohomology ρ^* induces isomorphisms in H^0 and H^1 and a monomorphism $H^2(M_X) \xrightarrow{\rho^*} H^2(\mathcal{E}^*(X))$.

We review the construction of the 1–minimal model up to the second step $M(1,2)$, which will be used to relate $\pi_1(X)$ to the cup products of 1–forms. For a more detailed discussion of 1–minimal models we refer the reader to [**45**].

Define $M(1,1)$ as the free CDGA $\wedge(V_1^1)$, where V_1^1 is the $\mathbb{R}$-vector space $H^1(X,\mathbb{R})$. Every element of V_1^1 is defined to have degree one and boundary zero, and the map $\rho\colon M(1,1) \to \mathcal{E}^*(X)$ sends every $x \in V_1^1 = H^1(X,\mathbb{R})$ to its image in a fixed arbitrary linear section of the modulo boundary map $H^1(X,\mathbb{R}) \to Z^1(\mathcal{E}^*(X))$.

The $(1,2)$–minimal model is defined as an extension of $M(1,1)$: $M(1,2) = \wedge(V_1^1 \oplus V_2^1)$, where V_2^1 is the $\mathbb{R}$-vector space $\ker(H^2M(1,1) \xrightarrow{\rho^*} H^2(X,\mathbb{R}))$. For any $v \in V_2^1$ we define its boundary dv as the element of $\ker H^2\rho \subset V_1^1 \wedge V_1^1$ defining its cohomology class, and if $dv = \sum x_i y_i \in M(1,1)$, $\rho(v)$ is a linearly varying primitive of $\sum \rho(x_i)\rho(y_i)$ in $\mathcal{E}^*(X)$.

REMARK 1.38. By definition, $H^2M(1,1) \cong H^1(X,\mathbb{R}) \wedge H^1(X,\mathbb{R})$, hence there is an isomorphism $V_2^1 \cong \ker(\cup : H^1(X,\mathbb{R}) \wedge H^1(X,\mathbb{R}) \to H^2(X,\mathbb{R}))$.

The subsequent steps $M(1,n)$ are constructed similarly, defining V_n^1 as $\ker(H^2M(1,n-1) \to H^2\mathcal{E}^*(X))$ and d, ρ as for $n = 2$. The inductive limit, $M(2,0)$ or M_X, is the 1–minimal model of $\mathcal{E}^*(X)$. To achieve functoriality, one must fix a base point for the de Rham algebra and select ρ at every step so that the morphism $\rho\colon M(1,n) \to \mathcal{E}^*(X)$ preserves the base point. Our subsequent use of the 1–minimal model allows us to ignore this issue.

The 1–minimal model is the first step in an inductive system which forms Sullivan's (full) minimal model of a CDGA A. This concept was developed as the simplest CDGA which is weakly equivalent to the original algebra A. Some relevant properties of the 1–minimal model are:

PROPOSITION 1.39. (Sullivan) (i) *All connected CDGAs have a* 1–*minimal model.*
(ii) *After fixing a basepoint, the* 1–*minimal model of a CDGA is well-defined up to isomorphism.*
(iii) *The 1–minimal model is functorial i.e. any basepoint–preserving CDGA morphism* $\mathcal{E}^*(Y) \xrightarrow{f} \mathcal{E}^*(X)$ *may be lifted to a morphism* $M_Y \xrightarrow{M(f)} M_X$.
(iv) *Weakly equivalent CDGAs have isomorphic* 1–*minimal models.*

By (iv) we may compute the 1–minimal model of a manifold X by computing it for any CDGA linked by a chain of quasi–isomorphisms to $\mathcal{E}^*(X)$. Another interesting consequence of this proposition is:

COROLLARY 1.40. *Let* $f\colon X \to Y$ *be a map between smooth manifolds such that the induced map* f^* *on real cohomology is an isomorphism on* H^0 *and* H^1 *and a monomorphism on* H^2. *Then* $f^* \circ \rho_Y\colon M_Y \to \mathcal{E}^*(X)$ *is a* 1–*minimal model for* $\mathcal{E}^*(X)$.

REMARK 1.41. For every CW–complex X there exists an aspherical space $Y = K(\pi_1(X), 1)$ and a map $X \to Y$ which satisfy the hypotheses of the Corollary. The space Y may be obtained by attaching to X cells of dimension 3 and more to kill the higher homotopy classes.

We now recall the dualizing process between Lie algebras and free commutative differential graded algebras generated by elements of degree one.

Let L be a finite–dimensional $\mathbb{R}$–Lie algebra. Its bracket is a bilinear alternating map

$$[.,.] : L \wedge L \longrightarrow L \ .$$

Dualizing on both sides, the bracket $[.,.]$ has an adjoint map

$$d\colon L^\vee \longrightarrow L^\vee \wedge L^\vee \ .$$

The map d may be extended as a graded derivation to the free graded algebra $\bigwedge L^\vee$, defining the degree of elements in $V = L^\vee$ to be one. Then the Jacobi identity satisfied by $[.,.]$ dualizes as $d^2 = 0$.

Conversely, if $M = \bigwedge W$ is a free CDGA and $\deg W = 1$, the differential restricts to a map $d\colon W = M^1 \to M^2 = W \wedge W$, which dualizes to a map $[.,.]\colon W^\vee \wedge W^\vee \to W^\vee$, and the fact $d^2 = 0$ in M translates as the Jacobi identity in $W^\vee$.

DEFINITION 1.42. A Lie algebra L and a free CDGA generated by elements of degree one are *dual* when each one yields the other by the above processes.

THEOREM 1.43. (Sullivan (1977)) *Let X be an arc–connected smooth manifold with a finitely presentable fundamental group $\pi_1(X, *)$. The inductive system*

$$M(1,1) \hookrightarrow M(1,2) \hookrightarrow \dots$$

formed by the $(1,n)$–minimal models of X and the projective system of real Malcev algebras of the fundamental group

$$\dots \to \mathcal{L}_2(\pi_1(X), \mathbb{R}) \to \mathcal{L}_1(\pi_1(X), \mathbb{R})$$

are dual to each other.

This theorem has important consequences for our purposes. Foremost is the following duality between the linear spaces V_n^1 and the quotients of the lower central series.

COROLLARY 1.44. (i) $V_n^1 \cong (\pi_1(X)_n/\pi_1(X)_{n+1} \otimes \mathbb{R})^\vee$.

(ii) $\ker\Big(H^1(X;\mathbb{R}) \wedge H^1(X;\mathbb{R}) \overset{\cup}{\to} H^2(X;\mathbb{R})\Big) \cong V_2^1 \cong (\pi_1(X)_2/\pi_1(X)_3 \otimes \mathbb{R})^\vee$

Sullivan's theory of minimal models has two main geometric applications. It allows the computation of the Malcev algebra, and thus of the de Rham fundamental group, of many smooth manifolds, and in the case of simply connected manifolds there is a theorem of Sullivan's, analogous to Theorem 1.43, stating that the (full) minimal model of the manifold is equivalent to its real Postnikov tower, therefore yielding its real homotopy type. For an introduction to minimal models of simply connected spaces, we refer the reader to [**45**], and for a unified approach complete with proofs, to [**18**].

6. 1–formality and quadratic presentations

We shall describe the prime consequence of 1–formality for the de Rham fundamental group. This is the existence of a quadratic presentation of the Malcev algebra $\mathcal{L}\pi_1(X)$, which is actually equivalent to 1–formality. It is easy to write down examples of groups that cannot be Kähler because their Malcev algebras do not fulfill this property.

This section has been clarified through suggestions of D. Toledo. The reader is referred to [**23**] for a broader discussion of the topic.

Recall the following concepts of Lie theory. Given a finite–dimensional $\mathbb{R}$–vector space H, the *free Lie algebra* spanned by H, which we will denote by $L(H)$, is the sub–Lie algebra of the tensor algebra $T(H) = \bigoplus_{n\geq 0} T^n(H) = \bigoplus_{n\geq 0} H^{\otimes n}$ generated by H, with the bracket given by

$$[u,v] = u \otimes v - v \otimes u\,.$$

The free Lie algebra $L(H)$ may be alternatively characterised by a universal property, as the functor $H \mapsto L(H)$ is the left adjoint of the inclusion of $\mathbb{R}$–Lie algebras into $\mathbb{R}$–vector spaces. Another alternative presentation in terms of Malcev algebras is the isomorphism

$$L(H) \cong \mathcal{L}(F_{\dim H})\,,$$

where $F_{\dim H}$ is the free group of rank $\dim H$. Let us fix some notation:

- The *lower central series* of a Lie algebra will be denoted by $\mathcal{C}^1 L = L, \mathcal{C}^2 L = [L, L], \ldots, \mathcal{C}^n L = [\mathcal{C}^{n-1}L, L], \ldots$.
- The *quadratic elements* of $L(H)$ are the elements of the linear subspace $(L(H) \cap T^2(H)) \cong \wedge^2 H$.
- An ideal $J \subset L(H)$ is *quadratically generated* if it is generated by quadratic elements.
- A *quadratically presented* Lie algebra is the quotient $L(H)/J$ of a free Lie algebra $L(H)$ by a quadratically generated ideal J.

It is clear that the class of quadratically presented Lie algebras is very narrow. However, it has been shown by Carlson–Toledo and by

S. Chen in [**28**] that this class contains nilpotent algebras of arbitrarily large nilpotency class.

All Lie algebras are quotients of free Lie algebras, but in the case of Malcev algebras such a quotient presentation can be given naturally:

LEMMA 1.45. *Let* Γ *be a finitely presentable group and* $\mathcal{L}\Gamma$ *its real Malcev algebra. There is an isomorphism of Lie algebras*

$$\mathcal{L}\Gamma \cong L(H_1(\Gamma, \mathbb{R}))/J\,,$$

where $J \subset \mathcal{C}^2 L(H)$, *and is a finitely generated ideal.*

For the proof, the reader may consider a morphism from a free group $F_{\dim H_1\Gamma} \to \Gamma$ inducing an isomorphism on $H_1/_{torsion}$.

Next, following Morgan, we shall give a cohomological characterisation of quadratically presented Malcev algebras. First, let us recall that the tower of n–step Malcev algebras

$$\cdots \to \mathcal{L}_n\Gamma \to \cdots \to \mathcal{L}_1\Gamma$$

or, equivalently, the dual inductive system of minimal CDGAs

$$M_\Gamma(1,1) \hookrightarrow \cdots \hookrightarrow M_\Gamma(1,n) \hookrightarrow \ldots$$

define an inductive system of cohomology maps

$$H^2(\mathcal{L}_n\Gamma) \cong H^2 M_\Gamma(1,n) \xrightarrow{\nu_n^*} H^2 M_\Gamma(2,0) \cong H^2(\mathcal{L}\Gamma)\ .$$

The following reformulation of the quadratic presentation property based on [**91**] was suggested to the author by D. Toledo.

LEMMA 1.46. *Let* Γ *be a finitely presentable group. Then the following are equivalent:*
(i) *its Malcev algebra* $\mathcal{L}\Gamma$ *admits a quadratic presentation,*
(ii) *the map*

$$H^2(\mathcal{L}_1\Gamma) \xrightarrow{\nu_1^*} H^2(\mathcal{L}\Gamma)$$

is surjective,
(iii) *there is an action of the multiplicative group* $\mathbb{R}^*$ *by automorphisms on* $\mathcal{L}\Gamma$ *so that* $\lambda \in \mathbb{R}^*$ *acts as multiplication by* λ *on* $H^1(\mathcal{L}\Gamma)$ *and as multiplication by* λ^2 *on* $H^2(\mathcal{L}\Gamma)$.

REMARK 1.47. The action in statement (iii) is necessarily by semi–simple automorphisms.

PROOF. Consider the presentation $\mathcal{L}\Gamma \cong L(H)/J$, where $H = H_1(\Gamma, \mathbb{R})$ and J is a finitely generated ideal in $\mathcal{C}^2 L(H)$. This presentation arises from an exact sequence of $L(H)$–modules

$$0 \longrightarrow J \longrightarrow L(H) \longrightarrow \mathcal{L}\Gamma \longrightarrow 0\,. \qquad (3)$$

Taking cohomology with coefficients in the trivial module $\mathbb{R}$, and the action of $L(H)$ on the cohomology groups, there is an exact sequence of cohomology groups (see [**99**])

$$0 \longrightarrow H^1(\mathcal{L}\Gamma) \longrightarrow H^1(L(H)) \longrightarrow H^1(J)^{L(H)} \longrightarrow H^2(\mathcal{L}\Gamma) \longrightarrow H^2(L(H))\,.$$

As $L(H)$ is a free algebra, $H^2(L(H)) = 0$. Moreover, the fact that $J \subset \mathcal{C}^2 L(H)$ also means that the map $H^1(\mathcal{L}\Gamma) \to H^1(L(H))$ is an isomorphism. The action of $L(H)$ on $H^1(J)$ yields an isomorphism

$$H^2(\mathcal{L}\Gamma) \cong H^1(J)^{L(H)} \cong (J/[J, L(H)])^\vee\,. \tag{4}$$

For every n–step Malcev algebra one may repeat this reasoning with the presentation

$$0 \longrightarrow J + \mathcal{C}^{n+1}L(H) \longrightarrow L(H) \longrightarrow \mathcal{L}_n\Gamma \longrightarrow 0\,,$$

and thus obtain an isomorphism

$$\begin{aligned} H^2(\mathcal{L}_n\Gamma) \cong H^2 M_\Gamma(1,n) \cong (J/([J, L(H)] + \mathcal{C}^{n+2}L(H) \cap J) \\ + \mathcal{C}^{n+1}L(H)/([J, L(H)] \cap \mathcal{C}^{n+1}L(H) + \mathcal{C}^{n+2}L(H)))^\vee\,. \end{aligned}$$

The second term $(\mathcal{C}^{n+1}L(H)/([J, L(H)] \cap \mathcal{C}^{n+1}L(H) + \mathcal{C}^{n+2}L(H))^\vee$ lies in the kernel of the cohomology map $H^2(M(1,n)) \to H^2(M(1,n+1))$, hence it has trivial image in $H^2(\mathcal{L}\Gamma)$ and the latter is the inductive limit

$$(J/[J, L(H)])^\vee \cong \varinjlim \left(J/([J, L(H)] + \mathcal{C}^{n+2}L(H) \cap J)\right)^\vee\,.$$

Thus the morphism $H^2(\mathcal{L}_1\Gamma) \xrightarrow{\nu_1^*} H^2(\mathcal{L}\Gamma)$ is onto if and only if

$$\mathcal{C}^3 L(H) \cap J \subset [J, L(H)]\,,$$

and this inclusion is equivalent to J being generated by the finite–dimensional linear space of quadratic elements $J \cap T^2 H$. This proves the equivalence between conditions (1) and (2).

To prove the equivalence between conditions (1) and (3), first observe that $\mathbb{R}^*$ acts on $\mathcal{L}\Gamma$, with the action on H^1 being multiplication by λ, if and only if J is a homogeneous ideal. Namely the $\mathbb{R}^*$–action on H extends uniquely to the $\mathbb{R}^*$–action on $L(H)$, where the action on homogeneous elements of degree k in $L(H)$ is multiplication by λ^k. This action is the only possible lifting of each automorphism in $\mathbb{R}^*$ from $L(H)/J$ to $L(H)$. Moreover, this action on $L(H)$ descends to an action on $L(H)/J$ if and only if J is invariant under the action, which is equivalent to the definition of a homogeneous ideal.

Note that this argument shows that any action on $L(H)/J$ which is multiplication by λ on H must be by semi–simple automorphisms. In particular, this justifies Remark 1.47.

Now equation (4) shows that the induced action on $H^2(\mathcal{L}\Gamma)$ is multiplication by λ^2 if and only if

$$[J, L(H)] = \mathcal{C}^3 L(H) ,$$

i.e., if and only if $\mathcal{L}\Gamma$ is quadratically presented. □

REMARK 1.48. One may check in the same way the following generalisation of the first equivalence in Lemma 1.46: For any $m \geq 2$, $\mathcal{L}\Gamma$ admits a presentation $L(H)/J$ where J has generators in $T^2H \oplus \cdots \oplus T^mH$ if and only if the morphism $H^2(\mathcal{L}_{m-1}\Gamma) \to H^2(\mathcal{L}\Gamma)$ is onto.

We are now ready to establish the equivalence of the two versions of 1–formality present in the literature, which are quadratic presentation of the Malcev algebra and the rational homotopy definition given in our Definition 2.5.

PROPOSITION 1.49 (Morgan). *Let X be a topological space with a finitely presentable fundamental group. Then X is 1–formal if and only if its Malcev algebra $\mathcal{L}\pi_1(X)$ is quadratically presented.*

PROOF. Due to Lemma 1.46 and the property of the Malcev algebra that the morphism

$$H^2(\mathcal{L}\pi_1(X)) \cong H^2(M_X) \xrightarrow{H^2\rho} H^2(X)$$

is a monomorphism, it suffices to show that X is 1–formal if and only if $\operatorname{Im} H^2\rho = \operatorname{Im} H^2\rho_{(1,1)}$, i.e., that the images in $H^2(X)$ of

$$\rho_{(1,n)} \colon M(1,n) \longrightarrow \mathcal{E}^*(X) ,$$

which form an increasing chain of subspaces by the definition of the minimal CDGAs $M(1,n)$, stabilise at the step $n = 1$.

If X is 1–formal, then its 1–minimal model is isomorphic to that of the algebra $H^*(X)$. In this case we can build a 1–minimal model for $H^*(X)$ such that, if $M(1,n) = \Lambda(V_1^1 \oplus \cdots \oplus V_n^1)$, then the morphism $\rho : M(2,0) \to H^*(X)$ verifies that $\rho_{|V_m^1} = 0$ for $m \geq 2$. This may be done as follows:

Let $M(1,1) = \Lambda(V_1^1)$ with $\rho_{(1,1)} : V_1^1 \xrightarrow{\cong} H^1(X)$ be the first step of the minimal model. The following steps are defined by adjoining spaces

$$V_n^1 = \ker\left(H^2(M(1,n-1)) \xrightarrow{\rho_{(1,n-1)}} H^2(X)\right) ,$$

and we can define $\rho_{(1,n)}$ over V_n^1 as any linear map such that $d\rho_{(1,n)}v = \rho_{(1,n-1)}dv = 0 \in H^2(X)$. Therefore we may set $\rho_{|V_n^1} = 0$, for all $n > 1$.

The 1–minimal model of $H^*(X)$ that we have just described obviously verifies that $\operatorname{Im} H^2(M(1,1)) = \operatorname{Im} H^2(M(2,0)) \subset H^2(X)$, therefore the Malcev algebra $\mathcal{L}\pi_1(X)$ admits a quadratic presentation.

Conversely, assume that $\mathcal{L}\pi_1(X)$ admits a quadratic presentation. Lemma 1.46 implies that there is an action by Lie algebra automorphisms of $\mathbb{R}^*$ on $\mathcal{L}\Gamma$, and consequently an action by CDGA automorphisms on $M(2,0)$. By Remark 1.47 these automorphisms are semi–simple. This action is equivalent to a grading of $M(2,0)$, namely $M(2,0) = \bigoplus_{k\geq 0} M(2,0)(k)$, where $M(2,0)(k)$ is the subspace of $M(2,0)$ where all $\lambda \in \mathbb{R}^*$ act as λ^k. This gives a *weight filtration* $W_\bullet$ on $M(2,0)$, defined by

$$W_n = \bigoplus_{k\leq n} M(2,0)(k) .$$

This filtration is determined by the facts that it is multiplicative and that the homogeneous elements $v \in V_n^1$ have weight n. It is *strongly graded* as in §4,5 of [**32**] and has the property that $H^2(M(2,0))$ is of pure weight 2. This allows the definition of a CDGA morphism

$$\varphi\colon M_X(2,0) \longrightarrow H^*(X)$$

defined by setting

$$\varphi_{|V_1^1}\colon V_1^1 \xrightarrow{\cong} H^1(X)$$
$$\varphi_{|V_n^1}\colon V_n^1 \xrightarrow{0} 0 \in H^1(X) \qquad \text{for } n \geq 2 .$$

This is well–defined because for $v \in V_m^1$, with $m \geq 3$, $d\varphi(v) = 0$ and $\varphi(dv) = 0$ as, due to the weight filtration, the monomials in dv contain a factor in V_k^1 with $k \geq 2$, while for $v \in V_1^1, V_2^1$ the identity $d\varphi(v) = \varphi(dv) = 0$ is a consequence of the defining properties of V_1^1, V_2^1.

By construction, the morphism $\varphi\colon M_X(2,0) \to H^*(X)$ induces an isomorphism on H^0 and H^1 and an injection of the subspace $\operatorname{Im} H^2(M(1,1)) \subset H^2(M(2,0))$. As this subspace is the full group $H^2(M(2,0))$, we reach the conclusion that $H^2\varphi$ is a monomorphism, thus X must be 1–formal. □

REMARK 1.50. In the case of compact Kähler manifolds the filtration $W_\bullet$ is indeed the weight filtration of a mixed Hodge structure in the 1–minimal model ([**47**], [**70**]).

REMARK 1.51. The property of a minimal model over a field of characteristic zero to have an action of the multiplicative group inducing given weights on cohomology is independent of the field. This is how Sullivan deduces, in §12 of [**91**], that formality is independent of the field.

Proposition 1.49 implies a restrictive necessary condition for a group to be Kähler (cf. the next Chapter).

EXAMPLE 1.52. The group $\Gamma = \langle x, y, z, t \,|\, [x, y][z, t], [[[[y, x], x], x], y] \rangle$, all of whose Massey triple products are zero (cf. subsection 2.2 in the next chapter), cannot be Kähler, because its Malcev algebra does not admit a quadratic presentation.

Another consequence of the quadratic presentation of the Malcev algebra, originally observed by Morgan in the case of Kähler groups, is:

COROLLARY 1.53. *Let Γ be a finitely presentable group such that its Malcev algebra is quadratically presented. Its n–step Malcev algebras $\mathcal{L}_n(\Gamma, \mathbb{R})$ are isomorphic to the graded Lie algebras induced by the group bracket $gr_n\Gamma \otimes \mathbb{R} = \oplus_{k=1}^{n} \Gamma_k/\Gamma_{k+1} \otimes \mathbb{R}$.*

PROOF. The weight filtration $W_\bullet$ and its associated $\mathbb{R}^*$–action induces a canonical splitting of the algebras $\mathcal{L}_n(\Gamma, \mathbb{R})$, which is respected by the Lie bracket. The filtration induced by weight coincides with that induced by the lower central series on $\mathcal{L}_n\Gamma$, and the graded Lie algebra induced by the lower central series in $\mathcal{L}_n\Gamma$ is isomorphic to $gr_n\Gamma \otimes \mathbb{R}$ (see the Appendix to [**79**]). □

7. Automorphisms of group origin

In this section we depart from the line of the previous ones, and study the relation between the automorphisms of free groups, of their Malcev algebras and nilpotent quotients. The properties of the action of Aut F_r on the Malcev algebra of F_r will be used in the final chapter to characterize the differential Galois groups of the Malcev algebra periods of algebraic families.

Because of the functorial character of the $\Gamma_i/\Gamma_{i+1} \otimes \mathbb{Q}$, $\mathcal{L}_n\Gamma$, the group Aut Γ acts on all those constructions. The automorphisms it induces may be called of group origin, and by a cumbersome use of standard group action notation they will be designed as $\mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, \Gamma_i/\Gamma_{i+1} \otimes \mathbb{Q}$, $\mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, \mathcal{L}_n\Gamma \otimes \mathbb{Q}$, etcetera. We will also denote as

$$p_n : \mathrm{Aut}\,\Gamma \longrightarrow \mathrm{Aut}\ (\Gamma_n/\Gamma_{n+1})$$
$$p_{\mathrm{l}n} : \mathrm{Aut}\,\Gamma \longrightarrow \mathrm{Aut}\, \mathcal{L}_n\Gamma$$

the respective actions of Aut Γ, and their extensions to the same constructions $\otimes k$, with k a field of characteristic zero.

We will establish now relations between automorphisms of geometric origin in the lower central series of Γ and in the Lie algebras $\mathcal{L}_n$.

The graded $\mathbb{Z}$-Lie algebra gr Γ is generated by its component of degree one $\mathrm{Gr}^1\,\Gamma \cong \Gamma/\Gamma_2$. The maps p_n factor through the functor gr : $\mathcal{G}r \to \mathcal{L}ie\ Alg$; this means that all the p_n are determined by p_1, as we directly prove in the following lemma:

LEMMA 1.54. *Let Γ be a group. For all $n \in \mathbb{N}^*$ there are group morphisms $\phi_n : Im\, p_1 \to Aut\,(\Gamma_n/\Gamma_{n+1})$ such that $p_n = \phi_n \circ p_1$.*

PROOF. It suffices to check that $\ker p_1 \subset \ker p_n$. Let φ be an element of $\ker p_1$. This means that for all $g \in \Gamma$ $\varphi(g) = gw_g$, with $w_g \in \Gamma_2$. The subgroup Γ_n is generated by brackets of length n $(g_1, g_2, \dots, g_n)$, with $g_1, \dots, g_n \in \Gamma$. The map φ sends such elements to the bracket $(g_1w_1, g_2w_2, \dots, g_nw_n)$, with $w_i \in \Gamma_2$. This bracket equals $(g_1, g_2, \dots, g_n)$ as a consequence of the Witt-Hall identities; more precisely as an iterated application of Theorem 5.3 of [**65**], Chap.5. □

REMARK 1.55. It is easily checked that Lemma 1.54 holds as well with coefficients in k, either directly or using the observation preceding the Lemma.

The study of the action of $\mathrm{Aut}\,\Gamma$ on the Lie algebras $\mathcal{L}_n(\Gamma) \otimes \mathbb{Q}$ yields the dual of a result by Sullivan (cf. [**91**], 6.1): There is a group morphism given by restriction of this action to the degree one component:

$$\psi_n : \mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, \mathcal{L}_n\Gamma \otimes \mathbb{Q} \longrightarrow \mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, (\Gamma/\Gamma_2 \otimes \mathbb{Q})$$

PROPOSITION 1.56. *There is an exact sequence*

$$1 \to \ker \psi_n \to Aut_{Aut\Gamma}\, \mathcal{L}_n\Gamma \otimes \mathbb{Q} \xrightarrow{\psi_n} Aut_{Aut\Gamma}\, (\Gamma/\Gamma_2 \otimes \mathbb{Q}) \to 1$$

with $\ker \psi_n$ *a unipotent group.*

PROOF. If $\eta \in \mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, (\Gamma/\Gamma_2 \otimes \mathbb{Q}) = \mathrm{Im}\, p_1$, then $\eta = p_1(\varphi)$ and $\psi_n(p_{1n}(\varphi)) = p_1(\varphi) = \eta$. Therefore ψ_n is exhaustive.

Let us fix a homogeneous $\mathbb{Q}$-linear basis for $\mathcal{L}_n\Gamma$. It will induce a block decomposition in the matrices $M \in \mathrm{GL}\,(\mathcal{L}_n\Gamma \otimes \mathbb{Q})$: The block (i,j) of M, denoted $M^{i,j}$ will be the component in $\Gamma_i/\Gamma_{i+1} \otimes \mathbb{Q}$ of $M|_{\Gamma_j/\Gamma_{j+1}\otimes\mathbb{Q}}$. Since $\mathrm{Aut}\,\Gamma$ respects the filtration by the lower central series of $\mathcal{L}_n\Gamma\otimes\mathbb{Q}$, the matrices of $\mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, (\mathcal{L}_n\Gamma \otimes \mathbb{Q})$ will be block lower triangular. Moreover, by the graded isomorphism of Proposition 1.13 the diagonal blocks $p_{1n}(\varphi)^{i,i}$ express the action of $\mathrm{Aut}\,\Gamma$ on $\Gamma_i/\Gamma_{i+1}\otimes\mathbb{Q}$, that is

$$p_{1n}(\varphi)^{i,i} = p_i(\varphi) = \phi_i(p_1(\varphi)) ,$$

the latter equality given by Lemma 1.54.

It is now obvious that if $\eta \in \ker \psi_n$, $\eta^{1,1} = \mathrm{Id}$, and $\eta^{i,i} = \phi_i(\eta^{1,1}) = \mathrm{Id}$, both for all i. Thus η is the sum of the identity and a block strictly lower triangular matrix, and this completes our proof. □

REMARK 1.57. As the filtration of $\mathcal{L}_n\Gamma \otimes \mathbb{Q}$ has length $n-1$, the nilpotence class of the kernel of ψ_n satisfies the inequality nil $(\ker \psi_n) \leq n-2$.

REMARK 1.58. There is a linear group tower formed by the kernels of the maps ψ_n

$$\cdots \to \ker \psi_3 \to \ker \psi_2$$

Its projective limit is the kernel of the map

$$\mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, \mathcal{L}\Gamma \longrightarrow \mathrm{Aut}_{\mathrm{Aut}\,\Gamma}\, \mathrm{gr}\,\Gamma\,.$$

It is a pro-unipotent group.

When Γ is a free group F_r, our knowledge of its lower central series enables us to further characterise these maps:

LEMMA 1.59. *The map* $p_1 : Aut\, F_r \to Aut\,(F_r/(F_r)_2)$ *is exhaustive.*

PROOF. If F_r is the free group spanned by $a_1, \ldots, a_r$ then $F_r/(F_r)_2 \cong \mathbb{Z}^r$ is the free abelian group generated by $a_1, \ldots, a_r$ and therefore $\mathrm{Aut}\,(F_r/(F_r)_2) \cong \mathrm{GL}\,(r, \mathbb{Z})$.

It is well known (see for instance [**53**], sect. 14.3, Thm. 3.2) that $\mathrm{GL}(r, \mathbb{Z})$ is generated by the three matrices

$$U_1 = \begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{r-1} \\ 1 & 0 & \cdots & 0 & 0 \\ & \ddots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1 & 0 \end{pmatrix}, \qquad U_2 = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix},$$

$$U_3 = \begin{pmatrix} -1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

These matrices, in the base $a_1, \ldots, a_r$ correspond to the automorphisms of F_r determined by

$$\varphi_1 : \begin{cases} a_1 & \to & a_2 \\ a_2 & \to & a_3 \\ \cdots & \cdots & \cdots \\ a_r & \to & a_1^{(-1)^{r-1}} \end{cases}, \qquad \varphi_2 : \begin{cases} a_1 & \to & a_1 \\ a_2 & \to & a_1 a_2 \\ a_3 & \to & a_3 \\ \cdots & \cdots & \cdots \\ a_r & \to & a_r \end{cases}, \qquad \varphi_3 : \begin{cases} a_1 & \to & a_1^{-1} \\ a_2 & \to & a_2 \\ \cdots & \cdots & \cdots \\ a_r & \to & a_r \end{cases}$$

therefore p_1 is onto. □

A consequence of Lemma 1.59 is that the maps ϕ_n are defined over all automorphisms of $F_r/(F_r)_2$:

$$\phi_n : \mathrm{Aut}\ (F_r/(F_r)_2) \cong \mathrm{GL}\,(r,\mathbb{Z}) \longrightarrow \mathrm{Aut}\ ((F_r)_n/(F_r)_{n+1}) \cong \mathrm{GL}\,(N_n,\mathbb{Z})$$

with N_n given by one of Witts formulæ([**65**], §5.6). To describe more explicitly ϕ_n we will consider its extension $\mathrm{Aut}\,(F_r/(F_r)_2 \otimes \mathbb{C}) \to$- $\mathrm{Aut}\,((F_r)_n/(F_r)_{n+1} \otimes \mathbb{C})$ through its image in $1 + J_{\mathbb{C}} \subset A(\mathbb{C}, r)$. We may take the Hall basis of $(F_r)_n/(F_r)_{n+1}$ formed by brackets on the generators $a_1, \ldots, a_r$; if $\varphi \in \mathrm{Aut}\ (F_r/(F_r)_2 \otimes \mathbb{C})$, $\varphi(a_i) = \sum_j \lambda_{ji} a_j$, then $\phi_n(\varphi)$ sends every bracket on the a_i to the same bracket on the $\varphi(a_i)$ and since we are computing modulo Γ_{n+1} and there is a Lie algebra isomorphism between $\mathrm{gr}\,(1 + J_{\mathbb{C}})$ and $\mathrm{gr}\,(J_{\mathbb{C}})$, the latter with bracket $[x, y] = xy - yx$ we may thus expand the brackets on the $\varphi(a_i)$ and find their coefficients in the Hall basis.

EXAMPLE 1.60. We compute as an example ϕ_2, ϕ_3 for F_2. The computation of ϕ_2 will be explicitly used in Proposition 1.61 (i) to determine its kernel, that of ϕ_3 is identical to the computation of ϕ_n for $n > 3$ and sheds light on the method of proof followed in Prop. 1.61 (i),(ii) for $n \geq 3$. The computation holds verbatim with coefficients in $\mathbb{Z}$, $\mathbb{Q}$ or $\mathbb{C}$. Denoting the two generators of F_2 as a_1, a_2, we have seen at the end of 2.3 that

$$\begin{array}{lr} F_2/(F_2)_2 \cong \mathbb{Z}^2\,, & \text{with basis} \quad a_1, a_2 \\ (F_2)_2/(F_2)_3 \cong \mathbb{Z}\,, & \text{with basis} \quad [a_2, a_1] \\ (F_2)_3/(F_2)_4 \cong \mathbb{Z}^2\,, & \text{with basis} \quad [[a_2, a_1], a_1]\,,\ [[a_2, a_1], a_2]\,. \end{array}$$

If $\varphi(a_1) = \lambda_{11} a_1 + \lambda_{21} a_2$, $\varphi(a_2) = \lambda_{12} a_1 + \lambda_{22} a_2$, then

$$\begin{aligned} \phi_2(\varphi)([a_2, a_1]) &= [\lambda_{12} a_1 + \lambda_{22} a_2, \lambda_{11} a_1 + \lambda_{21} a_2] \\ &= [\lambda_{12}\lambda_{11}[a_1, a_1] + \lambda_{12}\lambda_{21}[a_1, a_2] + \lambda_{22}\lambda_{11}[a_2, a_1] + \lambda_{22}\lambda_{21}[a_2, a_2] \\ &= (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})[a_2, a_1] \\ &= (\det \varphi)[a_2, a_1] \end{aligned}$$

$$\begin{aligned} \phi_3(\varphi)([[a_2, a_1], a_1]) &= [[\lambda_{12} a_1 + \lambda_{22} a_2, \lambda_{11} a_1 + \lambda_{21} a_2], \lambda_{11} a_1 + \lambda_{21} a_2] \\ &= [(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})[a_2, a_1], \lambda_{11} a_1 + \lambda_{21} a_2] \\ &= \lambda_{11}(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})[[a_2, a_1], a_1] + \lambda_{21}(\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})[[a_2, a_1], a_2] \\ &= (\det \varphi)(\lambda_{11}[[a_2, a_1], a_1] + \lambda_{21}[[a_2, a_1], a_2] \end{aligned}$$

and in the same manner we obtain

$$\phi_3(\varphi)[[a_2, a_1], a_2] = \lambda_{12}(\det \varphi)[[a_2, a_1], a_1] + \lambda_{22}(\det \varphi)[[a_2, a_1], a_2]$$

In this way the kernel of the ϕ_n may be determined. We will find it in the most general case, i.e. with coefficients in $\mathbb{C}$. The kernel for any ring $R \subset \mathbb{C}$ is then found by restriction.

PROPOSITION 1.61. *Let* $\phi_n : Aut\ (F_r/(F_r)_2 \otimes \mathbb{C}) \to Aut\ ((F_r)_n/(F_r)_{n+1} \otimes \mathbb{C})$ *be the morphism defined in Lemma 1.54, and denote* $\mu_n Id = \left\{ e^{\frac{2\pi i k}{n}} Id \mid 0 \leq k < n \right\}$.

(i) *For* F_2, $\ker \phi_2 = SL(2, \mathbb{Z})$, *and* $\ker \phi_n$ *is* $\mu_n Id$.
(ii) *For* F_r *with* $r > 2$, $\ker \phi_n$ *is* $\mu_n Id$.

PROOF. (i) As we have already seen, $(F_2)_2/(F_2)_3 \otimes \mathbb{C} \cong \mathbb{C}$, with basis $[a_2, a_1]$. The computation of Example 1.60 yields

$$\phi_2(\varphi)[a_2, a_1] = (\det \varphi)[a_2, a_1]$$

from which the kernel of ϕ_2 is deduced.

For $n > 2$, an element of the Hall basis for $(F_2)_n/(F_2)_{n+1} \otimes \mathbb{C}$ is

$$w_{12} = [a_2, a_1, a_1, \dots, a_1]$$

If $\varphi(a_i) = \sum_j \lambda_{ji} a_j$ then $\phi_n(\varphi) w_{12}$ is

$$[\lambda_{12}a_1 + \lambda_{22}a_2, \lambda_{11}a_1 + \lambda_{21}a_2, \dots, \lambda_{11}a_1 + \lambda_{21}a_2]$$

We can expand it by linearity into a sum of 2^n elementary bracket terms on a_1, a_2

$$\lambda_{12}\lambda_{11}^{n-1}[a_1, a_1, a_1, \dots, a_1] + \lambda_{22}\lambda_{11}^{n-1}[a_2, a_1, a_1, \dots, a_1] + \dots + \lambda_{22}\lambda_{21}^{n-1}[a_2, a_2, a_2, \dots, a_2]$$

and then compute all these brackets to find the expression of $\phi_n(\varphi) w_{12}$ in $J^n_{\mathbb{Q}}/J^{n+1}_{\mathbb{Q}}$. It turns out that the monomials of $\phi_n(\varphi) w_{12}$ with $n-1$ factors a_2 and one factor a_1 are exactly those coming from the elementary bracket terms

$$\begin{aligned} &\lambda_{12}\lambda_{21}^{n-1}[a_1, a_2, a_2, \dots, a_2] + \lambda_{22}\lambda_{11}\lambda_{21}^{n-2}[a_2, a_1, a_2, \dots, a_2] \\ &= (\lambda_{11}\lambda_{22} - \lambda_{12}\lambda_{21})\lambda_{21}^{n-2}[a_2, a_1, a_2, \dots, a_2] \end{aligned}$$

because such monomials must come from elementary bracket terms with $n-1$ a_2 components and one a_1 component, and the a_1 component must be in the innermost bracket or else the whole elementary bracket term is zero. Since $\lambda_{22}\lambda_{11} - \lambda_{12}\lambda_{21} = \det \varphi \neq 0$, then $\phi_n(\varphi) = \mathrm{Id}$ implies $\lambda_{21} = 0$. An identical computation with $w_{21} = [a_2, a_1, a_2, \dots, a_2]$ yields $\lambda_{12} = 0$. Therefore if $\varphi \in \ker \phi_n$ then φ must be diagonal. In such case

$$\begin{aligned} \varphi[a_2, a_1, \dots, a_1] &= \lambda_{11}^{n-1}\lambda_{22}[a_2, a_1, \dots, a_1] \\ \varphi[a_2, a_1, \dots, a_1, a_2] &= \lambda_{11}^{n-2}\lambda_{22}^2[a_2, a_1, \dots, a_1, a_2] \end{aligned}$$

and since $n > 2$ $[a_2, a_1, \dots, a_1]$, $[a_2, a_1, \dots, a_1, a_2]$ are two distinct elements of the Hall basis of length n. Now $\phi_n(\varphi) = \mathrm{Id}$ implies $\lambda_{11}^{n-1}\lambda_{22} =$

1, $\lambda_{11}^{n-2}\lambda_{22}^2 = 1$, thus $\lambda_{11} = \lambda_{22}$ and $\lambda_{11}^n = 1$. By linearity and length n of the brackets, all such candidates belong to the kernel.
(ii) The Hall basis of $(F_r)_2/(F_r)_3 \otimes \mathbb{C}$ is the set $\{[a_k, a_i]\}_{k>i}$. Its images by $\phi_2(\varphi)$ are:

$$\begin{aligned}\phi_2(\varphi)[a_k, a_i] &= [\sum \lambda_{hk}a_h, \sum \lambda_{ji}a_j] \\ &= \sum_{h>j} (\lambda_{hk}\lambda_{ji} - \lambda_{hi}\lambda_{jk})\,[a_h, a_j]\end{aligned}$$

Therefore $\phi_2(\varphi) = \mathrm{Id}$ means that all 2×2 minors of the matrix (λ_{ji}) of φ in the basis $a_1, \ldots, a_r$ which are centered on the diagonal have determinant one, and the others zero. This implies that $\varphi = \pm\mathrm{Id}$. Both choices clearly belong to the kernel.

In the case $n > 2$ we study as in the case of F_2 Hall basis elements of the form

$$w_{ik} = [a_k, a_i, a_i, \ldots, a_i], \qquad k > i$$

$$\phi_n(\varphi)w_{ik} = [\sum \lambda_{hk}a_h, \sum \lambda_{ji}a_j, \ldots, \sum \lambda_{ji}a_j]$$

Again, the monomials in $\phi_n(\varphi)w_{ik}$ with exactly $n-1$ factors a_i and one factor a_k come from

$$(\lambda_{kk}\lambda_{ii} - \lambda_{ki}\lambda_{ik})\,\lambda_{ii}^{n-2}[a_k, a_i, a_i, \ldots, a_i]$$

Therefore $\lambda_{kk}\lambda_{ii} - \lambda_{ki}\lambda_{ik}, \lambda_{ii} \neq 0$.

The terms with $n-1$ factors a_k and one factor a_i come from

$$(\lambda_{kk}\lambda_{ii} - \lambda_{ki}\lambda_{ik})\,\lambda_{ki}^{n-2}[a_k, a_i, a_k, \ldots, a_k]$$

Since there are no such terms in w_{ik} and $\lambda_{kk}\lambda_{ii} - \lambda_{ki}\lambda_{ik} \neq 0$ then it must be that $\lambda_{ki} = 0$. This holds for all $k > i$, but if $k < i$ the same conclusion is reached evaluating $\phi_n(\varphi)[a_i, a_k, a_i, \ldots, a_i]$. Hence φ has a diagonal matrix. As in the case of F_2, we examine the image by φ of two distinct Hall basis elements for every pair $k > i$

$$\varphi[a_k, a_i, \ldots, a_i] = \lambda_{ii}^{n-1}[a_k, a_i, \ldots, a_i]$$

$$\varphi[a_k, a_i, \ldots, a_i, a_k] = \lambda_{ii}^{n-2}\lambda_{kk}^2[a_k, a_i, \ldots, a_i, a_k]$$

So as in item (i) $\phi_n(\varphi) = \mathrm{Id}$ implies $\lambda_{ii} = \lambda_{kk}$ and $\lambda_{ii}^n = 1$. The only possibilities for φ are again the matrices of $\mu_n\,\mathrm{Id}$, which are in the kernel by linearity and length of the brackets. □

CHAPTER 2

Kähler groups

This chapter recalls the notion of formality in rational homotopy theory ([**91**], [**35**], [**70**]) and applies it to the study of the fundamental group of compact Kähler manifolds, i.e. Kähler groups. We retrieve in this way some known properties and restrictions verified by Kähler groups and extend them. Basic results in Hodge theory, such as the Q pairing, the dd^c lemma and the Hard Lefschetz theorem are used thoroughout this chapter, for a generic reference to them see [**101**].

1. Formality of compact Kähler manifolds

In the previous chapter we described the construction of Sullivan's 1–minimal model and its relation to the de Rham fundamental group in the case of arbitrary smooth manifolds. We shall now study a special property of compact Kähler manifolds with respect to their real homotopy.

DEFINITION 2.1. A smooth manifold X is *formal* if the CDGAs $\mathcal{E}^*(X)$ and $H^*(X,\mathbb{R})$ are weakly equivalent.

This is equivalent to $\mathcal{E}^*(X)$ and $H^*(X,\mathbb{R})$ having isomorphic minimal models. Thus by Sullivan's theory, the real homotopy type of X is determined by its real cohomology algebra.

EXAMPLE 2.2. Formality is a common property among manifolds with a simple cohomology algebra. Some particular examples are:

(i) spheres and wedges of spheres,
(ii) compact connected Lie groups,
(iii) Eilenberg–Mac Lane spaces $K(\pi,n)$ for $n>1$,
(iv) Riemannian symmetric spaces,
(v) complements of hyperplane arrangements in $\mathbb{C}^n$.

Another important class of formal spaces is that of compact Kähler manifolds. These are formal as an immediate consequence of Hodge theory, notably of the dd^c–Lemma, and this is the basis for the results of this Chapter.

THEOREM 2.3 (Deligne–Griffiths–Morgan–Sullivan [**35**]). *Compact Kähler manifolds are formal.*

We shall concentrate on the implications of Theorem 2.3 for the 1–minimal model and the de Rham fundamental group.

COROLLARY 2.4. *The de Rham fundamental group of a compact Kähler manifold is determined by the cup product* $\cup$: $H^1(X,\mathbb{R})\otimes H^1(X,\mathbb{R}) \to H^2(X,\mathbb{R})$.

PROOF. By the categorical equivalence of Malcev's Theorem 1.5, the de Rham fundamental group $\pi_1(X)\otimes\mathbb{R}$ is determined by the real Malcev algebra $\mathcal{L}(\pi_1(X),\mathbb{R})$, which is dual to the 1–minimal model M_X of X.

In the case when X is a compact Kähler manifold, by Theorem 2.3 the 1–minimal model M_X is also the 1–minimal model of the cohomology algebra $H^*(X,\mathbb{R})$. As may be seen from its construction in the previous Chapter, this 1–minimal model is determined by the cohomology group $H^1(X,\mathbb{R})$ and the cup product $H^1(X,\mathbb{R})\otimes H^1(X,\mathbb{R}) \to H^2(X,\mathbb{R})$. □

Topological spaces X for which Corollary 2.4 holds are called 1–formal spaces. An equivalent and more precise definition is:

DEFINITION 2.5. A topological space X is 1*–formal* if there exists a CDGA morphism

$$\rho\colon M_X(2,0) \longrightarrow H^*(X)$$

such that $H^0(\rho)$ and $H^1(\rho)$ are isomorphisms and $H^2(\rho)$ is a monomorphism.

It follows from the definitions that formal spaces are 1–formal. Therefore:

COROLLARY 2.6. *Compact Kähler manifolds are 1–formal.*

According to Proposition 1.49 and Corollary 1.53 in the first chapter, 1–formality of compact Kähler manifolds has the following consequences for its Malcev algebras:

COROLLARY 2.7. *Let* Γ *be a Kähler group and* k *a field of characteristic zero.*
(i) *The Malcev algebra* $\mathcal{L}(\Gamma,k)$ *is quadratically presented.*
(ii) *There is an isomorphism of pro–nilpotent Lie algebras*

$$\mathcal{L}(\Gamma,k)\cong(gr\Gamma)\otimes k\,.$$

Corollary 2.7 implies that the Malcev algebra, thus also the unipotent completion, of a Kähler group are originally defined over the integers, and the completion over a field k of characteristic zero is just an extension of scalars on this integral form $\operatorname{gr}\Gamma=\oplus_{n\geq 1}\Gamma_n/\Gamma_{n+1}$.

2. Examples and applications

2.1. Groups with free Malcev completions. As they always contain an odd rank free group with finite index, free groups may be seen not to be Kähler by an immediate covering argument. More generally, we can now show that groups with a free Malcev algebra, or even those "free up to order two brackets", cannot be Kähler.

We will use here the 2–*step nilpotent de Rham fundamental group* $(\pi_1(X)/\pi_1(X)_3)\otimes\mathbb{R}$, or equivalently the 2–*step nilpotent Malcev algebra*
$\mathcal{L}_2(\pi_1(X),\mathbb{R})$, which is isomorphic, though not canonically so, to $\operatorname{Gr}\mathcal{L}_2(\pi_1(X),\mathbb{R}) = (\pi_1(X)/\pi_1(X)_2\otimes\mathbb{R})\oplus(\pi_1(X)_2/\pi_1(X)_3\otimes\mathbb{R})$. This is the simplest non–Abelian quotient after $H^1(X,\mathbb{R})$.

PROPOSITION 2.8. *If $\mathcal{L}_2\Gamma\cong\mathcal{L}_2F_n$ for some free group F_n, then Γ cannot be a Kähler group.*

PROOF. If $\mathcal{L}_2\Gamma\cong\mathcal{L}_2F_n$, then $\dim\Gamma/\Gamma_2\otimes\mathbb{R}=n$, and $\dim\Gamma_2/\Gamma_3\otimes\mathbb{R}=\dim(F_n)_2/(F_n)_3\otimes\mathbb{R}=\binom{n}{2}$. Thus if $\Gamma=\pi_1(X)$ with X compact Kähler, by the formality of X and the duality of the de Rham fundamental group with the 1–minimal model, we would have that

$$\dim H^1(X,\mathbb{R})=\dim V_1^1=\dim\Gamma/\Gamma_2\otimes\mathbb{R}=n$$

and

$$\dim\ker(H^1(X,\mathbb{R})\wedge H^1(X,\mathbb{R})\stackrel{\cup}{\to}H^2(X,\mathbb{R}))=\dim V_2^1=\dim\Gamma_2/\Gamma_3\otimes\mathbb{R}=\binom{n}{2}.$$

But the first equality implies that $\dim H^1(X,\mathbb{R})\wedge H^1(X,\mathbb{R})=\binom{n}{2}$, so in fact, all exterior products of 1–forms on X would be exact. This is not possible for X compact Kähler, because of the Hard Lefschetz Theorem. Hence the statement. □

EXAMPLE 2.9 (parafree groups (see [**11**])). A group Γ is parafree of rank r if for every $n\in\mathbb{N}$ there are isomorphisms $\Gamma/\Gamma_n\cong F_r/(F_r)_n$. Parafree groups were introduced by G. Baumslag, who showed that there exist many nonfree examples. The isomorphism $\Gamma/\Gamma_3\cong F_r/(F_r)_3$ induces an isomorphism $\mathcal{L}_2\Gamma\cong\mathcal{L}_2F_r$, thus Γ may not be Kähler.

EXAMPLE 2.10 (1–relator groups). If Γ is a Kähler group admitting a presentation with only one relation, $\Gamma=\langle x_1,\ldots,x_n\mid r\rangle$, then either $n=1$ and $\Gamma\cong\mathbb{Z}/m\mathbb{Z}$, or r must lie in $F\{x_1,\ldots,x_n\}_2$, otherwise by Corollary 1.32 $\mathcal{L}_2\Gamma\cong\mathcal{L}_2F_{n-1}$ or $\mathcal{L}_2F_n$.

For instance, the groups $\langle x,y,z|xyxzxzxy\rangle$, or $\langle x,y|[[x,y],y]\rangle$ cannot be Kähler.

EXAMPLE 2.11. (Generic groups with few defining relations) Let Γ be a group admitting a presentation with n generators $x_1, \dots, x_n$ and $s < n$ defining relations $r_1, \dots, r_s$, such that their images in the Abelianised group $\mathbb{Z}\bar{x}_1 \oplus \cdots \cong \mathbb{Z}^n$ are linearly independent. Then Γ cannot be Kähler, as $\mathcal{L}_2\Gamma \cong \mathcal{L}_2F_{n-s}$. This is the generic case among presentations with fewer relations than generators.

For instance, the group $\Gamma = \langle x_1, \dots, x_5 | x_1^2x_2^2x_3^2, x_2^2x_3^2x_4^2, x_3^2x_4^2x_5^2\rangle$ cannot be Kähler.

2.2. Massey products and Heisenberg groups. The consequences of formality for the topology of compact Kähler manifolds were first realized in terms of vanishing of Massey products. We will present some well-known results in this subsection, in order to show how these vanishings follow from formality, and give instances of its effect on Kähler groups.

We now define Massey triple products. These are cohomological operations which, in the case of 1-forms or of spherical cohomology classes in general, are dual to the group bracket, respectively the Whitehead bracket, of representing homotopy classes of loops or spheres.

Let $\alpha, \beta, \gamma \in H^*(X, \mathbb{R})$, of degrees p, q, r respectively, such that $\alpha \cup \beta = 0$, $\beta \cup \gamma = 0$. Choose corresponding cocycles a, b, c, and select primitive cochains f, g such that $df = a \cup b$, $dg = b \cup c$. We define the *Massey triple product* $\langle \alpha, \beta, \gamma \rangle$ as the class of $f \cup c + (-1)^{p-1} a \cup g$ in $H^{p+q+r-1}(X)/(\alpha \cup H^{q+r-1}(X) + \gamma \cup H^{p+q-1}(X))$. One can check that this is well-defined in the quotient, although it would not be well-defined as a cohomology class. This is the definition of the Massey triple product in the singular cochain algebra of a topologic space. The definition actually extends to any CDGA as above, and its functoriality follows from the definition. A consequence of its naturality is the following

LEMMA 2.12. *Let A and B be weakly equivalent algebras. The isomorphism $H^*A \cong H^*B$ preserves Massey triple products.*

In the case of compact Kähler manifolds, formality together with the above Lemma allow us to compute Massey triple products rather easily.

PROPOSITION 2.13. *All Massey triple products on a compact Kähler manifold are zero.*

PROOF. Let X be compact Kähler. By the formality of X, the algebras $\mathcal{E}^*(X)$ and $H^*(X, \mathbb{R})$ are weakly equivalent, so we can compute Massey triple products in $H^*(X, \mathbb{R})$. The differential is zero by definition, so all Massey products will be zero. □

From Proposition 2.13 we derive a restriction on Kähler groups, which we will state in terms of the algebra of singular cochains $C^*(\Gamma, \mathbb{R})$. It can be seen directly as a consequence of the quadratic presentation of its Malcev algebra, which implies that all types of Massey products of 1–forms in a Kähler group are zero:

COROLLARY 2.14. *Let Γ be a Kähler group. Then all Massey triple products of classes of $H^1(\Gamma, \mathbb{R})$ must be zero.*

PROOF. This follows from Proposition 2.13 and from the fact that if $\Gamma = \pi_1(X)$, there is a map $c\colon X \to K(\Gamma, 1)$ inducing an isomorphism of fundamental groups, an isomorphism on H^0 and H^1, and a monomorphism $H^2(\Gamma) \to H^2(X, \mathbb{R})$, cf. Remark 1.41. Therefore, for Massey products of 1–classes to be zero in a quotient of $H^2(X, \mathbb{R})$, they must be zero in the corresponding quotient of $H^2(\Gamma)$. □

EXAMPLE 2.15 (Serre). The Heisenberg group $\mathcal{H}_3(\mathbb{Z})$ is the group of matrices

$$\mathcal{H}_3(\mathbb{Z}) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{Z}) \right\} .$$

This group is not Kähler, because its cohomology contains nontrivial Massey products.

To check this, let us first observe that $\mathcal{H}_3$ has a dimension 3 Malcev algebra $\langle X, Y, Z \mid [X, Y] = Z, [X, Z] = 0, [Y, Z] = 0 \rangle$. Dualisation of this nilpotent Lie algebra yields the 1–minimal model of any topological space having $\pi_1(X) \cong \mathcal{H}_3$, which is

$$M = \wedge(x, y, z) \qquad \deg x, y, z = 1 \qquad dx = 0,\ dy = 0,\ dz = xy .$$

The Massey triple product $\langle x, x, y \rangle$ is well–defined in $H^2(\mathcal{H}_3)$, and it is xz, which is a non–zero cohomology class.

EXAMPLE 2.16. Consider the Heisenberg group of Gaussian integers

$$\mathcal{H}_3(\mathbb{Z}[i]) = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} \in GL(3, \mathbb{Z} \oplus \mathbb{Z}i) \right\} .$$

As in the case of the integral Heisenberg group $\mathcal{H}_3(\mathbb{Z})$, we can check that there are nontrivial Massey triple products of 1–forms, and therefore $\mathcal{H}_3(\mathbb{Z}[i])$ is not a Kähler group.

3. The Albanese map and the de Rham fundamental group

In this section we describe how the de Rham fundamental group of a compact Kähler manifold is determined by that of its Albanese image.

We then discuss some consequences that follow from this combined with knowledge of the structure of the Albanese map.

Let X be a compact Kähler manifold, and $\alpha_X \colon X \longrightarrow Alb(X)$ its Albanese map. Denote as $Y = \alpha_X(X)$ its image, which may be singular. We consider a desingularisation $\varepsilon \colon \tilde{Y} \to Y$, and a desingularisation $\tilde{X}$ of the pullback of α_X:

$$\begin{array}{ccc} \tilde{X} & \xrightarrow{\tilde{\alpha}_X} & \tilde{Y} \\ {\scriptstyle \varepsilon_X}\downarrow & & \downarrow{\scriptstyle \varepsilon} \\ X & \xrightarrow{\alpha_X} & Y \end{array}$$

It is clear that the manifold $\tilde{X}$ is also compact Kähler and that the map ε_X is a birational morphism and thus induces an isomorphism of fundamental groups $\varepsilon_{X*} \colon \pi_1(\tilde{X}) \longrightarrow \pi_1(X)$.

We will call the map $\tilde{\alpha}_X \colon \tilde{X} \longrightarrow \tilde{Y}$ a *smoothing of the Albanese map* of X. The properties of the original Albanese map α_X relate X, $\tilde{X}$ and $\tilde{Y}$ as follows:

PROPOSITION 2.17 (Campana). *Let X be a compact Kähler manifold and $\tilde{\alpha}_X \colon \tilde{X} \longrightarrow \tilde{Y}$ a smoothing of its Albanese map. Then ε_X and $\tilde{\alpha}_X$ induce an isomorphism $\pi_1(X) \otimes \mathbb{R} \xrightarrow{\cong} \pi_1(\tilde{Y}) \otimes \mathbb{R}$.*

PROOF. As ε_X induces an isomorphism of fundamental groups, $\varepsilon_X^* \colon H^1(X) \longrightarrow H^1(\tilde{X})$ is also an isomorphism. This implies that $Alb(X)$ is the Albanese torus of $\tilde{X}$ and $\alpha_X \circ \varepsilon_X = \varepsilon \circ \tilde{\alpha}_X$ its Albanese map. As a consequence, $\tilde{\alpha}_X^* \colon H^1(\tilde{Y}) \longrightarrow H^1(\tilde{X})$ is surjective. As α_X itself is also surjective, $\tilde{\alpha}_X^*$ is also injective for H^*. Therefore $\tilde{\alpha}_X$ induces an isomorphism $H^1(\tilde{Y}) \cong H^1(\tilde{X})$ and an injection $H^2(\tilde{Y}) \hookrightarrow H^2(\tilde{X})$. Thus, by the universality of the 1–minimal model, $\tilde{\alpha}_X$ induces an isomorphism $M_{\tilde{Y}}(2,0) \cong M_{\tilde{X}}(2,0)$. Dualizing, we obtain our statement for the Malcev algebras $\mathcal{L}(\pi_1\tilde{X})$ and $\mathcal{L}(\pi_1\tilde{Y})$. The categorical equivalence between real Malcev algebras and de Rham fundamental groups, and the fact that ε_X induces an isomorphism of fundamental groups $\pi_1(\tilde{X}) \cong \pi_1(X)$, complete the proof. □

Thus the study of de Rham fundamental groups, or equivalently Malcev completions of fundamental groups, of compact Kähler manifolds may be reduced to the study of smoothings of its Albanese images. This is particularly convenient in the following cases:

COROLLARY 2.18 (Campana). *Let X be a compact Kähler manifold with surjective Albanese map $\alpha_X : X \longrightarrow Alb(X)$. Then the Albanese map induces an isomorphism of de Rham fundamental groups*

$$\alpha_{X*} : \pi_1(X) \otimes \mathbb{R} \xrightarrow{\cong} \pi_1(Alb(X)) \otimes \mathbb{R} \cong \mathbb{R}^{b_1(X)} .$$

PROOF. As α_X is surjective, its image is smooth, and so $\tilde{Y} = Alb(X)$. □

EXAMPLE 2.19. Some examples of Kähler manifolds with surjective Albanese map are: manifolds with Kodaira dimension $\kappa(X) = 0$, manifolds with algebraic dimension $a(X) = 0$ and manifolds with first Betti number $b_1(X) = 0$ or 2.

The image of the Albanese maps of a compact Kähler manifold X is a subvariety of the Albanese torus $Alb(X)$. The following theorem on the structure of subvarieties of complex tori is very useful in the study of their de Rham fundamental groups.

THEOREM 2.20 ([**97**], Theorem 10.9). *Let Y be a subvariety of a complex torus T. Then there exists a complex subtorus $A_1 \subset T$ such that $A_2 = T/A_1$ is an Abelian variety, and a projective subvariety $W \subset A_2$ such that:*

(i) *the natural projection $\pi : T \to A_2$ satisfies $Y = \pi^{-1}(W)$, and*

(ii) *there is an equality of Kodaira dimensions*

$$\kappa(W) = \dim(W) = \kappa(Y) .$$

Kollár pointed out to the author that this can be reformulated in the following useful form:

COROLLARY 2.21. *Let Y be a subvariety of a complex torus T. Then*

(i) *there are a projective variety W with dimension $\dim W = \kappa(Y)$, and a subtorus $A_1 \subset T$ such that there is a holomorphic map $Y \to W$ and a diffeomorphism over W*

$$Y \cong A_1 \times W ;$$

(ii) *the subvariety Y admits a desingularisation $\tilde{Y}$, together with a holomorphic map $\tilde{Y} \to \tilde{W}$ to a projective desingularisation of W, such that $\tilde{Y}$ is diffeomorphic to $A_1 \times \tilde{W}$.*

PROOF. Real tori are semi–simple. Thus, considering the real torus underlying T and its subtorus A_1 in Theorem 2.20, there is a diffeomorphism

$$f : T \xrightarrow{\cong} A_1 \times A_2$$

with the inclusion $A_1 \hookrightarrow T$ as a factor. Thus by Theorem 2.20 the map f restricts to the desired diffeomorphism $Y \cong A_1 \times W$.

Consider now a projective desingularisation $\tilde{W} \to W$ and the pull-back $\tilde{Y}$ in the diagram of holomorphic maps

$$\begin{array}{ccc} \tilde{Y} & \longrightarrow & T \\ \downarrow & & \downarrow \\ \tilde{W} & \longrightarrow & A_2 \end{array}$$

The holomorphic map $\tilde{Y} \to \tilde{W}$ is smooth by base change, and $\tilde{W}$ is smooth. Thus $\tilde{Y}$ is a desingularisation of Y. Moreover, T is diffeomorphic over A_2 to the trivial family $A_1 \times A_2$, and so $\tilde{Y}$ is diffeomorphic over $\tilde{W}$ to the trivial family $A_1 \times \tilde{W}$. □

REMARK 2.22. The varieties $W, \tilde{W}$ of Theorem 2.20 and Corollary 2.21 are of general type.

Here is another application of Theorem 2.20 to de Rham groups of Kähler manifolds, which had already been proved by Campana when the image of the Albanese mapping is a divisor in $Alb(X)$.

COROLLARY 2.23. *Let X be a compact Kähler manifold with Kodaira dimension $\kappa(X) = 1$. Then there is a noncanonically split exact sequence*

$$1 \longrightarrow \mathbb{R}^{b_1(X)-2g} \longrightarrow \pi_1(X) \otimes \mathbb{R} \longrightarrow \pi_1(C_g) \otimes \mathbb{R} \longrightarrow 1 ,$$

where C_g is a compact curve of genus $g \geq 0$.

PROOF. The Kodaira dimension of the Albanese image Y of X satisfies the inequality

$$\kappa(Y) \leq \kappa(X) .$$

The fact that Y is contained in a complex torus rules out the possibility that $\kappa(Y) = -\infty$. Moreover, if $\kappa(Y) = 0$, the submanifold Y must be a translation of a complex subtorus of $Alb(X)$ because of Theorem 2.20 and must generate $Alb(X)$ because it is the Albanese image of X. Therefore $Y = Alb(X)$ and $\pi_1(X) \otimes \mathbb{R} \cong \mathbb{R}^{b_1(X)}$.

If $\kappa(Y) = 1$, by Corollary 2.21 there is a diffeomorphism $\tilde{Y} \cong A_1 \times C_g$ for some smooth compact curve C_g, and thus the split exact sequence follows from Proposition 2.17. □

REMARK 2.24. Let us note in addition to the previous proof that when the Albanese image has Kodaira dimension $\kappa(Y) = 1$, then the curve C_g has genus $g \geq 2$, and X is a fibered Kähler manifold.

Moreover, if the Albanese image Y has dimension one, then it must be a smooth curve, the fibers of the Albanese map will be connected

([**97**], Proposition 9.19), and the isomorphism of de Rham fundamental groups comes from an isomorphism of torsion–free nilpotent completions $\pi_1(X)_0^{nilp} \cong \pi_1(C_g)_0^{nilp}$. It is shown in [**41**] that this is the case for Kähler groups admitting a presentation with n generators and s relations where $s \leq n-2$.

An important open question is whether all Kähler groups are fundamental groups of complex projective manifolds. The following Corollary, based on Theorem 2.20, clarifies the question at the de Rham level:

COROLLARY 2.25 (Campana). *Every de Rham fundamental group of a compact Kähler manifold is the de Rham fundamental group of a complex projective manifold, and the direct product of the de Rham fundamental group of a general type projective manifold and an abelian group* $\mathbb{R}^{2k}$.

PROOF. By Proposition 2.17, for every Kähler group $\Gamma = \pi_1(X)$, one has an isomorphism of de Rham fundamental groups $\pi_1(X) \otimes \mathbb{R} \cong \pi_1(\tilde{Y})$, where $\tilde{Y}$ is a smoothing of the Albanese image. Moreover, Corollary 2.21 shows that one may choose $\tilde{Y}$ diffeomorphic to $A_1 \times \tilde{W}$, where A_1 is a torus and $\tilde{W}$ a projective manifold of general type. Therefore

$$\pi_1(X) \otimes \mathbb{R} \cong \mathbb{R}^{2\dim A_1} \times \pi_1(\tilde{W}) \otimes \mathbb{R} .$$

The latter group is the de Rham fundamental group of $A \times \tilde{W}$, with A any Abelian variety of the same rank as A_1. □

Corollary 2.25 solves the rational homotopy analogue of the following conjecture by J. Kollár:

CONJECTURE 2.26 ([**61**]). *The fundamental group of a compact Kähler manifold is commensurable to the direct product of* $\mathbb{Z}^{2k}$ *and the fundamental group of a general type projective manifold.*

REMARK 2.27. As we remarked at the beginning of this Chapter, all the results in this section are valid for rational coefficients in cohomology algebras, de Rham groups and Malcev algebras. Moreover, by taking the Stein factorisation of the Albanese map, one can extend Corollary 2.25 and show that all torsion–free nilpotent completions of Kähler groups are torsion–free nilpotent completions of fundamental groups of projective manifolds.

3.1. One- and two–relator Kähler groups. We will use Proposition 2.17 now to relate the dimension of the Albanese image of a compact Kähler manifold X and the rank of the second order brackets quotient $\pi_1(X)_2/\pi_1(X)_3$. As an application, we will determine the

Malcev algebras of Kähler groups with one or two defining relations, and the one relator- groups themselves.

LEMMA 2.28. *Let X be compact Kähler, Y the Albanese image of X and $m = \dim_{\mathbb{C}} \tilde{Y}$. Then the graded algebra $H^*(X;\mathbb{C})$ contains a free graded exterior algebra $\bigwedge(V)$, where V is a complex vector space of dimension m and degree 1 spanned by holomorphic forms.*

PROOF. (cf. [**12**] V.18) Let $y \in Alb(X)$ be a regular point of the Albanese image $Y = \alpha_X(X)$. As $\dim Y = m$, there are local coordinates $u_1, \ldots u_n$ of $Alb(X)$ in a neighbourhood U of y such that $Y \cap U$ is defined as $u_{m+1} = 0, \ldots, u_n = 0$. The holomorphic forms $du_1, \ldots, du_m$ are defined on U and, as $Alb(X)$ is parallelizable, the forms in $\bigwedge(du_1, \ldots, du_m)$ extend to global holomorphic forms on $Alb(X)$. Its pull–back $\bigwedge(\alpha_X^* du_1, \ldots, \alpha_X^* du_m)$ defines a subalgebra of holomorphic cohomology classes in $H^*(X)$ which is free on a neighbourhood of y, hence is free. □

The above Lemma together with the correspondence of Corollaries 1.30, 1.44 may be used to bound from below the number of defining relations for Kähler groups, and to study those admitting a one- or two-relation presentation.

PROPOSITION 2.29. *Let Γ be a Kähler group, X a compact Kähler manifold such that $\pi_1 X \cong \Gamma$ and Y its Albanese image. Then:*

(i) *If $\dim Y = 1$, there is an isomorphism $\mathcal{L}\Gamma \cong \mathcal{L}\pi_1 C_g$ with C_g a compact Riemann surface of genus g, induced by a group map $\Gamma \to \pi_1 C_g$.*
(ii) *If $\dim Y = m > 1$, $\dim\ker(d_0 : \mathbb{R}r_1 \oplus \cdots \oplus \mathbb{R}r_s \to F/F_2 \otimes \mathbb{R}) \geq 2\binom{m}{2}+1$. In particular, any presentation $\Gamma = \langle x_1, \ldots, x_n \,;\, r_1, \ldots, r_s\rangle$ must have defining relations $r_1, \ldots, r_k$ such that they form a basis of $Im f$ and at least another $2\binom{m}{2}+1$ defining relations.*

PROOF. (i) is just Prop. 2.17 with $\tilde{Y}$ as C_g.

(ii) By Lemma 2.28, the algebra $H^*(X;\mathbb{C})$ contains a free algebra $\bigwedge(V)$ generated by m linearly independent holomorphic 1–forms. By the Hodge structure of $H^*(X)$ it contains an isomorphic algebra $\bigwedge(\bar{V})$ spanned by m independent antiholomorphic 1–forms. Both algebras being free, one obtains the lower bound $\dim\left[\operatorname{Im}\cup : \bigwedge^2 H^1(X) \to H^2(X)\right] \geq 2\binom{m}{2}$ considering either holomorphic or antiholomorphic products alone. Finally, due to the properties of the Q pairing in $H^1(X)$ ([**101**] 5.6), the product of a holomorphic 1–form with its conjugate cannot be zero, so $\dim(\mathrm{Im}\cup)\cap H^{1,1}(X)\geq 1$. By the correspondence of Corollary 1.44 this produces the sought bound.

□

REMARK 2.30. If X is compact Kähler and satisfies (i) in the above Proposition 2.29, it is not hard to check that the map $\pi_1 X=\Gamma\to\pi_1 C_g$ is onto. If the genus g is ≥ 2, such groups Γ are examples of what we will call fibered Kähler groups, to be defined in 2.34.

We are now able to characterize de Rham fundamental groups of compact Kähler manifolds with one or two defining relations.

THEOREM 2.31. *Let Γ be a Kähler group admitting a presentation with only one or two defining relations. Then either $\Gamma/\Gamma_2\otimes\mathbb{R}=0$ or $\mathcal{L}\Gamma\cong\mathcal{L}\pi_1 C_g$ with C_g a compact Riemann surface.*

PROOF. If $\Gamma/\Gamma_2\otimes\mathbb{R}\neq 0$, then $\mathcal{L}_2\Gamma\neq 0$, and by Prop. 2.29 any presentation of Γ must have at least $2\binom{\dim Y}{2}+1$ defining relations, with $Y=Alb(X)$. Thus the only possible case is $\dim Y=1$, and Proposition 2.29 (i) completes the proof. □

REMARK 2.32. (i) The 1–relator groups Γ with $\Gamma/\Gamma_2\otimes\mathbb{R}=0$ are exactly the $\Gamma\cong\mathbb{Z}/n\mathbb{Z}$.
(ii) The 2-relator groups Γ with $\Gamma/\Gamma_2\otimes\mathbb{R}=0$ are those with a presentation $\langle x_1,x_2\ ;\ r_1,r_2\rangle$ with $\bar{r}_1,\bar{r}_2$ linearly independent in $F\{x_1,x_2\}/(F\{x_1,x_2\})_2$. This is immediately derived from the exact sequence (1.5).

EXAMPLE 2.33. Denote C_g a compact Riemann surface of genus g.

(i) The group Γ defined in Example 2.11 can also be seen not to be Kähler by Theorem 2.31, as $\Gamma/\Gamma_2\otimes\mathbb{R}\cong\mathbb{R}^2$ but $\mathcal{L}_2\Gamma\not\cong\mathcal{L}_2\pi_1(C_1)$.
(ii) The group $\Gamma=\langle x_1,x_2,x_3,x_4\ ;\ (x_1x_2,x_3^2),(x_1x_3x_1,x_4^3)\rangle$ has a Malcev algebra which fulfills the quadratic presentation condition imposed by Morgan (Prop. 1.49). Yet Γ cannot be Kähler because $\Gamma/\Gamma_2\otimes\mathbb{R}\cong\mathbb{R}^4$ but $\dim\Gamma_2/\Gamma_3\otimes\mathbb{R}=4\neq 5=\dim\pi_1(C_2)_2/\pi_1(C_2)_3$, contradicting Theorem 2.31.

4. Non–fibered Kähler groups

Here we establish a dicothomy between fibered and nonfibered Kähler groups, arising from a result by A. Beauville and Y.T. Siu on the existence of irregular pencils on compact Kähler manifolds. We skip the fibered case, and we give in Proposition 2.42 an upper bound for $\dim\Gamma_2/\Gamma_3\otimes\mathbb{R}$ and a lower bound for the second Betti number $b_2(\Gamma)$

in the case of nonfibered groups. This translates as a lower bound for the number of relations that their presentations must have.

Let $\Gamma = \pi_1(X,*)$ be a fundamental group. By Corollary 1.44 $\dim \Gamma_2/\Gamma_3 \otimes \mathbb{R} = \dim \bigwedge^2 H^1(X) - \dim \operatorname{Im} \left(\cup : \bigwedge^2 H^1(X) \to H^2(X)\right)$. As we have used in Proposition 2.8, if X is compact Kähler, by the properties of the Q pairing $\operatorname{Im} \cup$ must be nonzero. Now we will establish a lower bound on its dimension in the case of nonfibered manifolds, by recalling a result of Castelnuovo-De Franchis and its extension to arbitrary dimension.

DEFINITION 2.34. Let Γ be a Kähler group.

(i) We call Γ a *fibered Kähler group* when $\Gamma = \pi_1(X,*)$ with X compact Kähler admitting a nonconstant holomorphic map $f : X \to C_g$, with C_g a compact Riemann surface of genus $g \geq 2$.
(ii) We call Γ a *nonfibered Kähler group* when $\Gamma = \pi_1(X,*)$ with X compact Kähler not admitting any nonconstant holomorphic map to a compact Riemann surface of genus $g \geq 2$.

A. Beauville and Y.T. Siu independently proved that the above definitions make sense:

PROPOSITION 2.35 ([**13**],[**86**]). *Let X be a compact Kähler manifold, write $\Gamma = \pi_1(X,*)$, and let $g \geq 2$ be an integer. Then X admits a nonconstant holomorphic map to a compact Riemann surface of a genus $h \geq g$ if and only if there is an epimorphic group morphism $\Gamma \to \pi_1(C_g,*)$, with $\pi_1(C_g,*)$ the fundamental group of a compact Riemann surface of genus g.*

Proposition 2.35 means that a Kähler group Γ is either fibered or nonfibered, and that the former are characterised by admitting a $\pi_1(C_g)$ as a quotient.

If we have an onto map $\Gamma \to H \to 1$, it induces onto maps $\Gamma_n/\Gamma_{n+1} \otimes \mathbb{R} \to H_n/H_{n+1} \otimes \mathbb{R} \to 0$ for all n. This together with the fact that the lower central series quotients of the $\pi_1 C_g$ have all nonzero rank shows that nilpotent or rationally nilpotent Kähler groups must be nonfibered. Campana gave recently examples of Kähler groups with a Malcev algebra of nilpotency class 2, and Shirping Chen has found quadratically presented Malcev algebras of arbitrary nilpotency class. A question asked by D. Toledo to the author is:

QUESTION 2.36. *Are there non-fibered Kähler groups which are not rationally nilpotent?*

We now study the cup products of 1–forms in the case of nonfibered compact Kähler manifolds. We begin with an extension of a classical result (see [**24**]):

PROPOSITION 2.37 (Castelnuovo-De Franchis). *Let X be a compact Kähler manifold. If there exist ω_1, ω_2 linearly independent holomorphic 1-forms such that $\omega_1 \wedge \omega_2 = 0$ then there is a holomorphic map $f : X \to C$ with C a curve of genus $g(C) \geq 2$, such that ω_1, ω_2 belong to $Im\, f^*$.*

REMARK 2.38. The form equality $\omega_1 \wedge \omega_2 = 0$ is equivalent to $\omega_1 \wedge \omega_2$ being exact. This is a result of Hodge theory, showing that a nonzero holomorphic form over a compact Kähler manifold cannot be exact.

The Castelnuovo-De Franchis theorem together with the conic structure of the set of products in $H^{2,0}(X)$ yield the following corollary (see [**10**] IV, Prop. 4.2):

COROLLARY 2.39. *If X is a nonfibered compact Kähler manifold, then* $\dim Im(\cup : \bigwedge^2 H^{1,0}(X) \to H^{2,0}(X)) \geq 2\dim H^{1,0}(X) - 3$.

Corollary 2.39 gives a bound for the products of holomorphic 1–forms, and by conjugation, of antiholomorphic 1–forms. The dimension of products of holomorphic-antiholomorphic 1–forms has been bounded for compact complex surfaces in [**10**], IV, Prop. 4.3. We slightly alter their proof to extend it to compact Kähler manifolds of arbitrary dimension:

PROPOSITION 2.40. *Let X be a nonfibered compact Kähler manifold. Then* $\dim Im\,(\cup : H^{1,0}(X) \otimes H^{0,1}(X) \to H^{1,1}(X)) \geq 2\dim H^{1,0}(X) - 1$.

PROOF. Denote $n = \dim X \geq 2$, $V = \operatorname{Im} \cup : H^{1,0}(X) \otimes H^{0,1}(X) \to H^{1,1}(X)$ and fix ω a fundamental Kähler form on X. We begin by showing that the pairing $\cup : H^{1,0}(X) \otimes H^{0,1}(X) \to V$ becomes injective when we fix a nonzero $\xi \in H^{1,0}(X)$ or $\bar{\eta} \in H^{0,1}(X)$.

Suppose there are holomorphic 1–forms ξ, η such that $\xi \wedge \bar{\eta} = d\alpha$. Then obviously $\xi \wedge \eta \wedge \bar{\xi} \wedge \bar{\eta} = d\alpha'$, and

$$\int_X \xi \wedge \eta \wedge \bar{\xi} \wedge \bar{\eta} \wedge \omega^{n-2} = 0$$

By the properties of the pairing Q of compact Kähler manifolds (see [**101**] 5.6), this implies that $\xi \wedge \eta = 0$, thus by the Castelnuovo-De Franchis theorem ξ and η are linearly dependent. Take $\xi = a\eta$, with $a \in \mathbb{C}^*$. Then $0 = \xi \wedge \bar{\eta} = a\eta \wedge \bar{\eta}$. Again by the properties of the pairing Q, this means that $\xi, \eta = 0$.

Thus a map may be defined

$$\mathbb{P}(H^{1,0}(X)) \times \mathbb{P}(H^{0,1}(X)) \longrightarrow \mathbb{P}(V)$$

with injective restrictions fixing a point in either factor of the source. We apply now the following result from [**81**]:

PROPOSITION 2.41. *Let $\varphi : \mathbb{P}^m(\mathbb{C}) \times \mathbb{P}^k(\mathbb{C}) \to \mathbb{P}^l(\mathbb{C})$ be a holomorphic mapping, with $l < m+k$. Then φ factors through one of the projections $\mathbb{P}^m \times \mathbb{P}^k \to \mathbb{P}^m$, $\mathbb{P}^m \times \mathbb{P}^k \to \mathbb{P}^k$.*

In our case, $\cup$ cannot factor through any of the projections because it is fiberwise injective in both cases, so it holds that $\dim V \geq 2\dim H^{1,0}(X) - 1$ as was wanted. □

We have now all the required pieces to study $\Gamma_2/\Gamma_3 \otimes \mathbb{R}$ of nonfibered groups. We return to the notations defined in the first chapter.

PROPOSITION 2.42. *Let X be a nonfibered compact Kähler manifold with $q = \frac{1}{2}\dim H^1(X) = \dim \frac{1}{2}\Gamma/\Gamma_2\pi_1(X,*)) \otimes \mathbb{R}$. Then:*

(i) *If $q = 0, 1$, $\dim \pi_1(X,*)_2/\pi_1(X,*)_3 \otimes \mathbb{R} = 0$ and $b_2(\pi_1(X)) \geq 1$.*
(ii) *If $q \geq 2$, $\dim \pi_1(X,*)_2/\pi_1(X,*)_3 \otimes \mathbb{R} \leq 2q^2 - 7q + 7$ and $b_2(\pi_1(X)) \geq 6q - 7$.*

PROOF. We have seen in Corollary 1.44 that $\dim \pi_1(X,*)_2/\pi_1(X,*)_3 \otimes \mathbb{R} = \dim \bigwedge^2 H^1(X) - \dim \operatorname{Im}\left(\cup : \bigwedge^2 H^1(X) \to H^2(X)\right) = \frac{2q(2q-1)}{2} - \dim \operatorname{Im} \cup$.

Thus if $q = 0$, $\dim \pi_1(X,*)_2/\pi_1(X,*)_3 \otimes \mathbb{R} = 0$.

If $q = 1$, $\dim \operatorname{Im} \cup \leq 1$, so $\dim \pi_1(X,*)_2/\pi_1(X,*)_3 \otimes \mathbb{R} \leq 1$. Let a, b be a basis of $\pi_1(X,*)/\pi_1(X,*)_2 \otimes \mathbb{R}$. The equality $\dim \pi_1(X,*)_2/\pi_1(X,*)_3 \otimes \mathbb{R} = 1$ would imply that $(a,b) \neq 0$ in $\mathcal{L}_2\pi_1(X,*)$ by Proposition 1.31. Therefore there would be an isomorphism $\mathcal{L}_2F_2 \xrightarrow{\cong} \mathcal{L}_2\pi_1(X,*)$ sending the generators X_1, X_2 of $\mathcal{L}_2F_2$ to a, b respectively. By Proposition 2.8 this would mean that $\pi_1(X,*)$ is not Kähler, leading to a contradiction. Hence our statement follows.

For $q \geq 2$, we break $H^1(X)$ into its Hodge components. By Cor. 2.39 $\dim\left(\operatorname{Im} \bigwedge^2 H^{1,0}(X) \to H^{2,0}(X)\right) \geq 2q - 3$. The same holds by conjugation for $\bigwedge^2 H^{0,1}(X) \to H^{0,2}(X)$. Prop. 2.40 gives the inequality $\dim\left(H^{1,0}(X) \otimes H^{0,1}(X) \to H^{1,1}(X)\right) \geq 2q - 1$ and our statement follows from the addition of bounds. □

Proposition 2.42 roughly means that nonfibered Kähler groups need many defining relations. M. Green and R. Lazarsfeld give a bound ([**41**],Thm. 5.4), establishing that given X nonfibered compact Kähler in the sense of Def. 2.34, that is admitting no pencil of genus $g \geq 2$, and a presentation of its fundamental group $\pi_1(X) = \langle x_1, \ldots, x_n \ ; \ r_1, \ldots, r_s \rangle$, then

$$s \geq n - 3$$

Proposition 2.42 above allows us to establish a more accurate bound:

COROLLARY 2.43. *Let* $\Gamma = \langle x_1, \ldots, x_n \ ; \ r_1, \ldots, r_s \rangle$ *be a finite group presentation. If* $\Gamma = \pi_1(X)$, *with* X *nonfibered compact Kähler, and writing* $q = \frac{1}{2} b_1(X)$, *the total number of relations must satisfy*

(i) *If* $q = 0$, $s \geq n$.
(ii) *If* $q = 1$, $s \geq n - 1$.
(iii) *If* $q \geq 2$, $s \geq n + 4q - 7$.

PROOF. The group presentation $\Gamma = \langle x_1, \ldots, x_n \ ; \ r_1, \ldots, r_s \rangle$ induces an exact sequence $1 \to N \to F \to \Gamma \to 1$ described in (1.3). Let $d_0 : \oplus \mathbb{R} r_i \to J_F / J_F^2 \cong F/F_2 \otimes \mathbb{R}$ be the map defined in Proposition 1.26. We may suppose the relations $r_1, \ldots, r_s$ ordered so that the images of $r_1, \ldots, r_k$, with $k \leq s$ form a basis of $\operatorname{Im} d_0 \cong N/N \cap F_2 \otimes \mathbb{R} \hookrightarrow F/F_2 \otimes \mathbb{R}$.

By Proposition 1.26 and Corollary 1.27 (i), $\dim \Gamma/\Gamma_2 \otimes \mathbb{R} = \dim F/F_2 \otimes \mathbb{R} - \dim N/N \cap F_2 \otimes \mathbb{R}$, so there is an equality

$$n = k + 2q$$

Let us remark also that $\dim \ker d_0 = s - k$.

Thus if $q = 0$ we have $n = k \leq s$ as was wanted.

If $q = 1$, by Corollary 1.30 and Proposition 2.42 (i) we have that

$$\begin{aligned} 0 = \dim \Gamma_2/\Gamma_3 \otimes \mathbb{R} &= \binom{2}{2} - (s - k) + \dim \ker d_1 \\ &= 1 - s + k + \dim \ker d_1 \geq 1 - s + k \end{aligned}$$

As $n = k + 2$ in this case, this yields the sought bound.

If $q \geq 2$, again by Corollary 1.30 and Proposition 2.42,

$$\dim \Gamma_2/\Gamma_3 \otimes \mathbb{R} = \binom{2q}{2} - (s - k) + \ker d_1 \leq 2q^2 - 7q + 7$$

which implies

$$k + 2q + \dim \ker d_1 \leq s - 4q + 7$$

and as $n = k + 2q$,

$$s \geq n + 4q - 7$$

□

EXAMPLE 2.44. (i) A group $\Gamma = \langle x_1, \ldots, x_{2q} \ ; \ w_1, \ldots, w_s \rangle$ with $w_1, \ldots, w_s \in F_2$ can be nonfibered Kähler only if $s \geq 6q - 7$ for $q \geq 2$, and $s \geq 1$ for $q = 1$.

(ii) *Chain link groups* (see [**82**], 3.G) The group $G_{2q} = \langle x_1, \dots, x_{2q} \mid (x_1, x_2), \dots, (x_{2q-1}, x_{2q}), (x_{2q}, x_1)\rangle$ is the fundamental group of a link of $2q$ circumferences forming a circular chain, for $q \geq 2$. This group verifies $k = \dim F/F_2 \otimes \mathbb{R} - \dim G_{2q}/(G_{2q})_2 \otimes \mathbb{R} = 0$, and $s = 2q < 6q - 7$, and therefore G_{2q} cannot be nonfibered Kähler. Broadly speaking, if a link is not very intertwined, its group is not going to be nonfibered Kähler. The group G_4 verifies that $\dim (G_4)_2/(G_4)_3 \otimes \mathbb{R} = 2$, and therefore it cannot be fibered Kähler either, as it cannot map onto $\pi_1(C_g, *)$ for any $g \geq 2$. The groups G_{2q} with $q \geq 3$ do admit onto mappings to $\pi_1(C_2, *)$, and the author does not know if they are fibered Kähler.

(iii) Let $\Gamma = \langle x_1, \dots, x_5 \,;\, x_1^2 x_2^{-2} x_4^2, (x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5)\rangle$. In this case $n = 5$, $k = 1$, $q = 2$ as $\operatorname{Im} d_0 = \langle 2\bar{x}_1 - 2\bar{x}_2 + 2\bar{x}_4\rangle$, and $s = 5 < n + 4q - 7 = 6$. Therefore Γ cannot be nonfibered Kähler. The group Γ cannot either map onto $\pi_1(C_g)$, with C_g a smooth projective curve of genus $g \geq 2$ because $\dim \Gamma_2/\Gamma_3 \otimes \mathbb{R} = 2$, $\dim \pi_1(C_g)_2/\pi_1(C_g)_3 \otimes \mathbb{R} = \frac{2g(2g-1)}{2} - 1 \geq 5$, so we reach the conclusion that Γ cannot be Kähler.

EXAMPLE 2.45. (Groups of planar hyperplane arrangements are not non–fibered Kähler (cf.[**75**])) Let $\mathcal{A} = \{H_1, \dots, H_n\}$ be a planar hyperplane arrangement, i.e., a finite set of hyperplanes in $\mathbb{C}^2$, and let $\alpha_j = 0$ be a defining linear equation for every line H_j. The complement of the lines is a smooth complex manifold $M(\mathcal{A}) = \mathbb{C}^2 \backslash (H_1 \cup \dots \cup H_n)$, and its integral cohomology algebra was shown by Brieskorn to be

$$H^*(M; \mathbb{Z}) \cong \Big\langle \frac{1}{2\pi i}\frac{d\alpha_1}{\alpha_1}, \dots, \frac{1}{2\pi i}\frac{d\alpha_n}{\alpha_n}\Big\rangle \subset \mathcal{E}^*_{\mathbb{C}}(M) \ ,$$

that is, the subalgebra of the complex–valued de Rham complex of M generated by the forms $\omega_j = \frac{1}{2\pi i}\frac{d\alpha_j}{\alpha_j}$.

The above inclusion induces a weak equivalence between the cohomology algebra $H^*(M, \mathbb{R})$ and the de Rham complex of M. Therefore the space M is formal, and all the Massey triple products of 1–forms in its cohomology are zero.

Brieskorn's explicit computation of the cohomology of M allows us to present bases for $H^1(M, \mathbb{R})$, $H^2(M.\mathbb{R})$ (see [**75**], Example 7.4):

$$\begin{aligned} H^1(M, \mathbb{R}) &= \langle \omega_1, \dots, \omega_n \rangle \cong \mathbb{R}^n \\ H^2(M, \mathbb{R}) &= \operatorname{Im}\Big(\cup : H^1(M) \wedge H^1(M) \to H^2(M)\Big) \\ &= \langle \omega_1 \wedge \omega_n, \omega_2 \wedge \omega_n, \dots, \omega_{n-1} \wedge \omega_n \rangle \cong \mathbb{R}^{n-1} \ . \end{aligned}$$

Therefore, if $n \geq 3$ then $n - 1 \leq 3n - 7$, and by Proposition 2.42 the fundamental group $\Gamma = \pi_1(M)$ cannot be non–fibered Kähler.

On the other hand, in the case $n = 2$, the line arrangement $\{x = 0, y = 0\} \subset \mathbb{C}^2$ yields $\pi_1(M) \cong \mathbb{Z}^2$, which is non-fibered Kähler.

EXAMPLE 2.46. Fundamental groups of compact oriented 3-folds with first Betti number $b_1(X) \geq 4$ are not non-fibered Kähler. This is due to the fact that every such 3-fold admits a Heegaard splitting, and a presentation with n generators and only n defining relations (see [**90**]).

CHAPTER 3

Geometric monodromy

We will study the geometric monodromy of Lefschetz pencils of curves and some related families, and obtain formulae for the monodromy automorphisms of the fundamental group in the case of proper families. Our formulae parallel those of [8], where a combinatoric and group–theoretic approach is followed. Properties of quasi–unipotence and formality of the family will arise from these formulae.

1. Geometric monodromy of pencils of curves

1.1. The projective case. We begin by fixing our notations.

DEFINITION 3.1. A *Lefschetz pencil* of curves over a simply connected open domain $B \subset \mathbb{P}^1_{\mathbb{C}}$ is a proper holomorphic map $f : X \to B$, where X is a smooth complex surface, such that it only has a finite number of critical points $p_1, \dots, p_n$, and all of them are nondegenerate, that is, the matrices $D^2 f(p_i)$ are invertible at every critical point. We do not suppose the critical values $z_i = f(p_i)$ nonequal.

We will denote the set of singular fibres as $\Sigma = f^{-1}(z_1) \cup \cdots \cup f^{-1}(z_n)$, and the set of regular values as $S = B \setminus \{z_1, \dots, z_n\}$.

The following is a classical result, deduced from the theorem of Ehresmann and the Implicit Function Theorem.

PROPOSITION 3.2. *The restricted map $f : X \setminus \Sigma \to S$ is a $\mathcal{C}^\infty$ locally trivial fibration, and its fibres are compact Riemann surfaces.*

Thus we may define a $\mathcal{C}^\infty$ parallel transport: every path $\gamma : I \to S$ may be lifted to $X \setminus \Sigma$ and produces a diffeomorphism $X_{\gamma(0)} \xrightarrow{\cong} X_{\gamma(1)}$ which is well defined up to diffeotopy and orientation preserving as the family $X \setminus \Sigma \to S$ is oriented. The liftings are compatible with path composition, and therefore, fixing a basepoint $s_0 \in S$, we obtain the *geometric monodromy* map of this family

$$\rho : \pi_1(S, s_0) \longrightarrow \mathrm{Aut}^+(X_{s_0})/\text{diffeotopies} = M(g, 0)$$

where $M(g, 0)$ is the mapping class group of the topological surface X_{s_0}.

The action of the fundamental group $\pi_1(S, s_0)$ on the cohomology groups of the fibre is described by the Picard-Lefschetz formula. A standard proof of it is based on the computation of the geometric monodromy of the pencil (see [**7**],vol. 2, Ch. 1,2 or [**63**]). As this is not very explicitly displayed in our sources, we provide a separate proof.

To study the local monodromy around critical values, we separately study neighbourhoods of the critical points and the rest of the fibre. The situation around the critical point has been thoroughly studied (see [**68**]):

LEMMA 3.3 (complex Morse lemma). *Every nondegenerate critical point p_i of f admits a coordinate neighbourhood V_i such that p_i has coordinates $(0,0)$ and the function f is $f(x,y) = z_i + x^2 + y^2$.*

LEMMA 3.4 (conic structure, Milnor). *For every critical point p_i there is a small enough ball $B_{\varepsilon_i} \subset V_i$ centered at p_i such that:*

(i) *The point p_i is the only singular point of $f_{|V_i}$.*
(ii) *Its boundary $\partial B_{\varepsilon_i} = S_{\varepsilon_i}$ cuts transversally every fibre $X_z \cap V_i$.*
(iii) *The singular fibre $X_{z_i} \cap B_{\varepsilon_i}$ is homeomorphic to the cone of $X_{z_i} \cap S_{\varepsilon_i}$.*

The same holds for any radius $r < \varepsilon_i$.

NOTATION 3.5. Fix now a $\delta > 0$ such that $B(z_i, \delta) \subset f(B_{\varepsilon_{i_j}})$ for every critical value z_i and critical point above p_{i_j}, define loops $\beta_i(t) = z_i + \frac{3}{4}\delta e^{2\pi i t}$, and open sets $U_{i_j} = B_{\varepsilon_{i_j}} \cap f^{-1}(B(z_i, \delta))$.

We proceed to compute the monodromy in $f^{-1}(B(z_i, \delta))$. Let $p_{i_1}, \ldots, p_{i_k}$ be the singular points of f above the critical value z_i. We will first compute monodromies in the manifolds ∂U_{i_j} and

$$Y_i = f^{-1}(B(z_i, \delta)) \setminus (U_{i_1} \cup \cdots \cup U_{i_k}),$$

and afterwards in the Morse balls $\bar{U}_{i_j}$, to glue all of them together in Theorem 3.18.

PROPOSITION 3.6. *The restricted maps*

$$f : \partial U_{i_j} \longrightarrow B(z_i, \delta)$$

are C^∞ globally trivial fibrations over $B(z_i, \delta)$.

PROOF. We have defined U_{i_j} as $B_{\varepsilon_{i_j}} \cap f^{-1}(B(z_i, \delta)) \subset X$, with $B_{\varepsilon_{i_j}}$ a 4-ball such that its boundary $S_{\varepsilon_{i_j}}$ cuts transversally every fibre of f. Thus $\partial U_{i_j} = S_{\varepsilon_{i_j}} \cap f^{-1}(B(z_i, \delta))$ is a 3-manifold that also cuts transversally every fibre. As the fibres X_z have codimension 2 in X, $X_z \cap \partial U_{i_j}$ also has codimension 2 in ∂U_{i_j} for every $z \in B(z_i, \delta)$, and thus

$df(T_p\partial U_{i_j})$ has rank 2 for every $p \in \partial U_i$, and again by Ehresmann's theorem $f : \partial U_{i_j} \to B(z_i, \delta)$ is a $\mathcal{C}^\infty$ locally trivial fibration. As the base space is contractible, it is also globally trivial. □

Next we consider the manifolds with boundary Y_i.

PROPOSITION 3.7. *The restricted map*

$$f : Y_i \longrightarrow B(z_i, \delta)$$

is a $\mathcal{C}^\infty$ globally trivial fibration of manifolds with boundary over $B(z_i, \delta)$.

PROOF. The map $f : Y_i \longrightarrow B(z_i, \delta)$ is proper, and its tangent bundle maps $df_p : T_p\overset{\circ}{Y_i} \to T_{f(p)}B(z_i, \delta)$ and $df_q : T_q\partial Y_i \to T_{f(q)}$ are onto for every $p \in \overset{\circ}{Y_i}$, $q \in \partial Y_i = \coprod \partial U_{i_j}$, by Prop. 3.2 and 3.6 respectively. Therefore, by the Ehresmann theorem for manifolds with boundary f is a $\mathcal{C}^\infty$ locally trivial fibration with fibre a manifold with boundary, and as the base space is contractible it must be globally trivial. □

Finally, we must compute the geometric monodromy in the closed balls $\bar{U}_{i_j}$ around the critical points of the mapping. Each of these local families is biholomorphic to a family

$$\begin{aligned} f : \bar{B}((0,0), \varepsilon) \cap f^{-1}(B(0, \delta)) &\longrightarrow B(0, \delta) \\ (x, y) &\longmapsto x^2 + y^2 \end{aligned}$$

with $f(x, y) = x^2 + y^2$, ε small enough so that the conic structure Lemma 3.4 holds, and δ such that $B(0, \delta) \subset f(B(0, 0), \varepsilon)$. The fibration defined by f is trivial outside these balls and on its boundary, so we may assume by increasing its radius that every family $f : \bar{U}_{i_j} \to B(z_i, \delta)$ is topologically trivial in a neighbourhood of its boundary.

We will briefly recall the classical Picard–Lefschetz theory of the pencil of curves (5) after [**7**], and show how the homological concepts are equivalent to their mapping class group counterparts.

The fibers of the pencil (5) are topological cylinders with boundary, i.e., denoting by F the $\mathcal{C}^\infty$ regular fiber there is a diffeomorphism $(F, \partial F) \cong (S^1 \times [-1, 1], S^1 \times \{-1\} \cup S^1 \times \{1\})$. The integral homology groups $H_1(F, \partial F) \cong \mathbb{Z}$, $H_1(F) \cong \mathbb{Z}$ admit as generators the classes of the paths $\nabla(t) = (1, 2t - 1)$, $\Delta(t) = (e^{2\pi i t}, 0)$ respectively, and the complex orientation of the fiber defines an *intersection pairing*

$$\begin{aligned} i : H_1(F, \partial F) \times H_1(F) &\longrightarrow H_0(F) \cong \mathbb{Z} \\ (c, d) &\longmapsto c \circ d \end{aligned}$$

which is nondegenerate, as $\nabla \circ \Delta = -1$.

The geometric monodromy φ of the pencil along a loop around its critical value is defined up to isotopy, and may be chosen to be the identity in a neighbourhood of the boundary ∂F. This property allows the definition, already at the cycle level, of a *variation morphism*

$$\begin{aligned} \mathrm{Var}_\varphi : H_1(F, \partial F) &\longrightarrow H_1(F) \\ c &\longmapsto \mathrm{Var}_\varphi(c) = c - \varphi_*(c) \end{aligned}$$

of the pencil. The variation morphism may be defined for Lefschetz pencils of any dimension.

In the case of the pencil (5), the intersection pairing and variation morphism allow the definition of another homological invariant of the monodromy diffeomorphism: Let $\varphi : (F, \partial F) \to (F, \partial F)$ be a relative oriented diffeomorphism, such that it is the identity in a neighbourhood of ∂F. We may define a quadratic form

$$\begin{aligned} q_\varphi : H_1(F, \partial F) &\longrightarrow H_0(F) \cong \mathbb{Z} \\ c &\longmapsto c \circ \mathrm{Var}_\varphi(c) \end{aligned}$$

As $H_1(F, \partial F) \cong \mathbb{Z}\nabla$, the quadratic form is determined by $q_\varphi(\nabla)$. This motivates our next definition:

DEFINITION 3.8. Let $(F, \partial F)$ be a cylinder with boundary as above, and $\varphi : (F, \partial F) \to (F, \partial F)$ be a relative oriented diffeomorphism such that its restriction to the boundary ∂F is the identity. The *index* $k(\varphi)$ of φ is the integer $q_\varphi(\nabla)$.

The index of a diffeomorphism of the cylinder does not depend on the generator that has been selected for $H_1(F, \partial F)$. Furthermore, it is not hard to show that the index is invariant modulo relative diffeotopy. Therefore the index induces a mapping

$$k : \mathrm{Diff}^+(F, \partial F)/\mathrm{Diff}_0(F, \partial F) \longrightarrow \mathbb{Z},$$

where $\mathrm{Diff}^+(F, \partial F)$ is the group of orientation–preserving relative $\mathcal{C}^\infty$ diffeomorphisms of the fiber $(F, \partial F)$, such that they are the identity in the boundary ∂F, and $\mathrm{Diff}_0(F, \partial F)$ is the group of relative $\mathcal{C}^\infty$ diffeomorphisms isotopic to the identity.

PROPOSITION 3.9. *The map k is a group isomorphism.*

PROOF. We must first check that k is indeed a group morphism. This follows from the decomposition

$$\nabla - \phi_* \circ \varphi_* \nabla = \nabla - \phi_* \nabla + \phi_*(\nabla - \varphi_* \nabla),$$

and the fact that $\nabla - \varphi_* \nabla$ is a cycle with support away from the boundary, so it is homologous to its image by ϕ_*. Moreover, if a diffeomorphism belongs to $\mathrm{Diff}_0(F, \partial F)$ then its index is zero.

It is harder to show that $k(\varphi) = 0$ only if φ is diffeotopic to the identity. Since this is a classical result, we will only sketch here a method of proof. Let $\varphi \in \mathrm{Diff}^+(F, \partial F)$ be a diffeomorphism such that it is the identity in a neighbourhood of ∂F. We claim that the diffeotopy class of φ is determined by the isotopy class of the image $\varphi(\nabla)$ of the generating vertical path $\nabla(t) = (1, 2t-1) \in S^1 \times [-1, 1]$. This is the case because:

- First, the boundary of the cylinder and the image path $\varphi(\nabla)$ form the boundary of a 2–disk. If $\phi(\nabla)$ has the same image as $\varphi(\nabla)$, by our orientedness assumption the two parametrizations of the path are isotopic. We may extend the isotopy to a tubular neighbourhood of the path $\varphi(\nabla)$ and find after it that φ, ϕ are induced by identification on the boundary by two diffeomorphisms of the 2–disk φ', ϕ' such that they are the identity in a neighbourhood of the disk. These diffeomorphisms are diffeotopic, thus so are φ, ϕ.

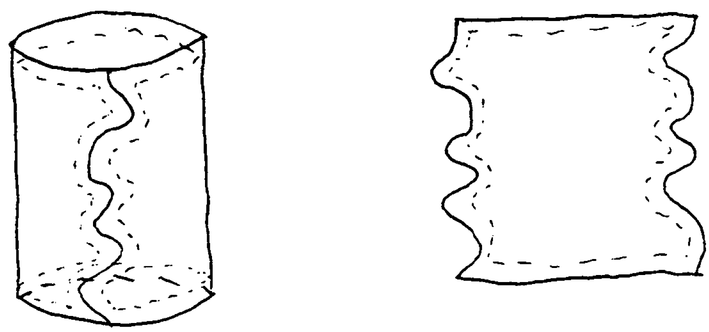

Fig. F3.1 Disk diffeomorphisms.

- Second, denote $\gamma := \varphi(\nabla)$ the image of the vertical path ∇ by a diffeomorphism $\varphi \in \mathrm{Diff}_0(F, \partial F)$. If we take any oriented vertical path $\nabla_z = \{z\} \times [-1, 1]$ and deform it in a very small neighbourhood so as to make it transverse to γ, the intersection of γ and ∇_z is a set of $2m$ points with total sum of intersection indices $k(\varphi) = 0$. As γ is a simple path, there must be two intersection points consecutive both in γ and in ∇_z and with opposite intersection numbers. The pieces of γ, ∇_z between these two points form a simple closed loop enclosing a disk. We may use the tubular neighbourhood on the other side of γ, and apply the following relative diffeotopy theorem (cf. Proposition 4.15 of [**103**] for a proof in the PL category):

THEOREM 3.10. *Let $\mathbb{D}^n$ be the closed unit disk in $\mathbb{R}^n$ and $\mathbb{D}' \hookrightarrow \mathbb{D}^n$ an embedded C^∞ closed disk, such that the intersection of boundaries $\partial\mathbb{D}^n \cap \partial\mathbb{D}'$ is a $(n-1)$-disk. Then there exists a diffeotopy F_t of $\mathbb{R}^n$ such that $F_1(\mathbb{D}^n) = \mathbb{D}'$, and the diffeomorphisms F_t are the identity on the common boundary $\partial\mathbb{D}^n \cap \partial\mathbb{D}'$ and outside a compact neighbourhood of the disk $\mathbb{D}^n$.*

By this theorem on the uniqueness of disk embeddings, we may send the path γ by an isotopy to a path γ' that has lost the two intersection points and runs parallel to ∇_z (see the figure).

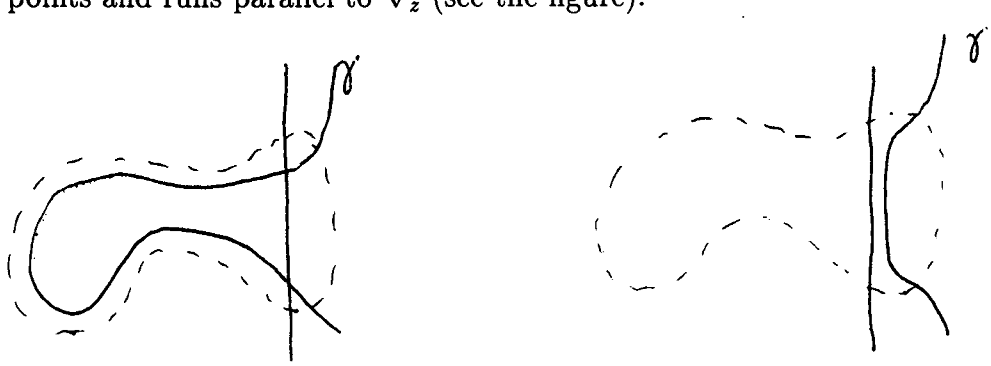

Fig. F3.2 Path straightening.

This disk retraction process may be iterated until the resulting path, which we may denote again by γ, and the fixed vertical path ∇_z are disjoint.

– Third, by repeating this procedure for suitable vertical paths ∇_z we may obtain a new path $\bar{\gamma}$ which is isotopic to γ, is contained in a vertical strip as narrow as it may be wished. One may then apply the above uniqueness of disk embeddings theorem to the path γ' and two embedded strips, as shown in the figure, and conclude that γ' is isotopic to the vertical path ∇.

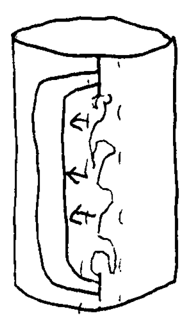

Fig. F3.3 Final isotopy.

□

Proposition 3.9 shows that the relative isotopy class of a diffeomorphism φ of $(F, \partial F)$ is determined by its index $k(\varphi)$, which may be interpreted as the number of twists that φ induces on the cylinder. The mapping class group isomorphism $\mathrm{Diff}^+(F, \partial F)/\mathrm{Diff}_0(F, \partial F) \cong \mathbb{Z}$ has been obtained by means of an orientation of the cyclinder, induced by the complex orientation of the fiber in our case. The opposite orientation would change the sign of the intersection product, thus of all indices. On the other hand, due to the quadratic nature of the index,

$$(-\nabla) \circ (-\nabla - \varphi_*(-\nabla)) = \nabla \circ (\nabla - \varphi_* \nabla) .$$

Therefore, once an orientation of F has been fixed we may compute the index of a diffeomorphism using either of the generators $\nabla, -\nabla$ of the relative homology group $H_1(F, \partial F)$.

We will give now a primary example of diffeomorphisms:

DEFINITION 3.11. Let C be an open set in an orientable topological surface, and $c \subset C$ a simple closed curve in C, such that it has a bicollar open neighbourhood $c \subset N \subset C$. Take a cylindrical chart in $N \cong S^1 \times (-1,1)$, with c corresponding to $S^1 \times \{0\}$. Then a *Dehn twist* about c is the map $g_c : C \to C$ defined as

$$g_c(\theta, y) = (\theta + \pi(y+1), y)$$

in N, and extended by the identity map outside N.

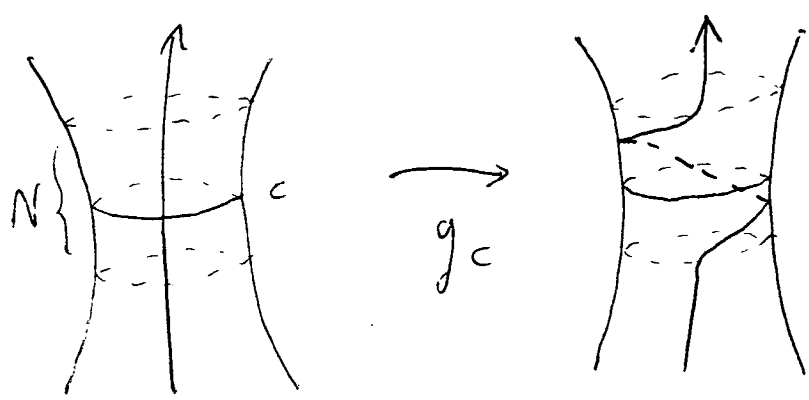

Fig. F3.4 Dehn twist.

The choice of a chart $S^1 \times (-1,1)$ in the bicollar neighbourhood N fixes an orientation for N. A parametrization of N with opposite orientation would yield the inverse of this Dehn twist, i.e. a twist in the other direction.

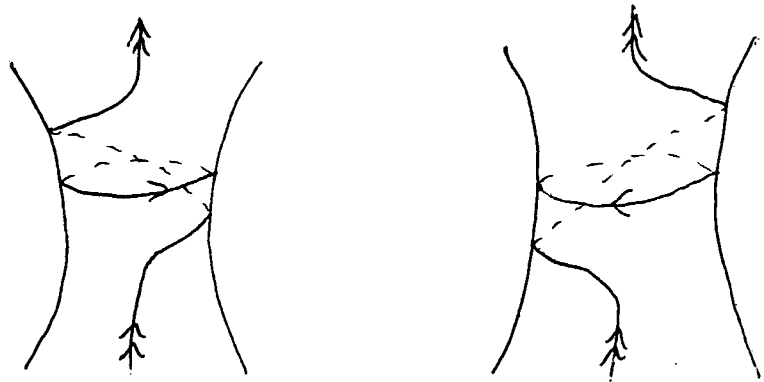

Fig. F3.5 The two inverse Dehn twists.

Other standard properties of Dehn twists that will be required are:

(i) Dehn twists about isotopic paths are isotopic.
(ii) Selection of different bicollar neighbourhoods produces different Dehn twists about the same path c. Nevertheless, all Dehn twists produced by bicollar parametrizations with the same orientation are isotopic. Thus, by requiring that a Dehn twist be determined up to isotopy, we may suppose it to be the identity outside an arbitrarily small neighbourhood of the path c.
(iii) The mapping class groups $M(g,0)$ are generated by Dehn twists.

For proofs and more information about Dehn twists and mapping class groups, see [**15**]. We will limit ourselves here to computing the index of a Dehn twist, after precising our orientation conventions:

CONVENTION 3.12. We will refer to the orientation of $S^1 \times [-1,1]$ such that the paths $\Delta(t) = (e^{2\pi i t}, 0)$ and $\nabla(t) = (1, 2t-1)$ have intersection number $\Delta \circ \nabla = -1$ as the orientation induced by the chart $S^1 \times [-1,1]$.

LEMMA 3.13. *Let $(F, \partial F)$ be a topological cylinder with a chart $S^1 \times [-1,1]$. A Dehn twist along the simple closed loop $\Delta = S^1 \times \{0\}$ with the bicollar orientation induced by that of the chart $F \cong S^1 \times [-1,1]$ has index 1.*

PROOF. We may take as a parametrized bicollar neighbourhood of Δ the open set $N = S^1 \times (-1,1)$ of the given chart of F. The image of the vertical path $\nabla(t) = (1, 2t-1)$, which generates $H_1(F, \partial F)$, by the Dehn twist g_Δ is homologous to the cycle $\nabla + \Delta$ in N.

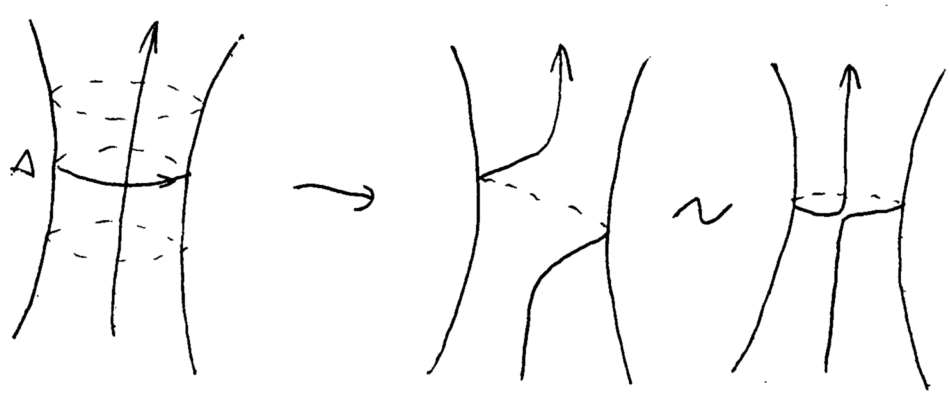

Fig. F3.6 Index of Dehn twist.
Thus the computation of the index yields

$$\nabla \circ (\nabla - g_{\Delta *}\nabla) = \nabla \circ (-\Delta) = 1 .$$

□

We are able now to determine without ambiguity the local geometric monodromy around a critical point in a Lefschetz pencil of curves. We will associate first a path to every critical point:

DEFINITION 3.14. Let p be a nondegenerate critical point of a Lefshetz pencil of curves $f : X \to S$, U a coordinate neighbourhood of p satisfying the complex Morse lemma 3.3, and β a loop around p in $f(U)$ with origin s. A *vanishing path* associated to p is a C^∞ simple closed path $d \subset X_s \cap U$, such that its homology class is the vanishing cycle $\Delta \in H_1(X_s \cap U, \mathbb{Z})$.

The conic structure lemma 3.4 shows that vanishing paths always exist. The vanishing cycle in homology is defined up to sign, i.e. orientation, and likewise a vanishing path may be defined with any of the two possible orientations. Classical results in surface topology (see [**90**]), or a direct proof along the lines of Proposition 3.9 show that

LEMMA 3.15. *Two vanishing paths d, d' associated to the critical point p in the same conic structure neighbourhood U and with the same homology class are isotopic.*

Given our complex Morse lemma coordinate neighbourhoods U_{i_j}, we may select as vanishing paths in $X_{\beta_i(0)} \cap U_{i_j}$ those with image $d_{i_j} = \{(x, y) \in X_{\beta(0)} \mid \operatorname{Im} x = \operatorname{Im} y = 0\}$.

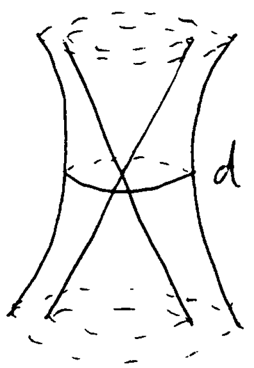

Fig. F3.7 Regular fiber near singular fiber.

PROPOSITION 3.16. *Let $f : \bar{U} \to B(0,\delta)$ be a holomorphic pencil of curves as defined in equation* (5).

(i) *The pencil f restricts to a C^∞ locally trivial fibration of manifolds with boundary over $B(0,\delta) \setminus \{0\}$.*
(ii) *The geometric monodromy along the loop $\beta(t) = \frac{3}{4}\delta e^{2\pi i t}$ is a Dehn twist along a bicollar neighbourhood with complex orientation of the vanishing path of the basepoint fiber and has index 1, also with the complex orientation of the fiber.*

PROOF. The proof of (i) is wholly identical to that of Proposition 3.7. To prove (ii), let us observe that the map $f(x,y) = x^2 + y^2$ factors as

$$\begin{array}{ccc} X = \mathbb{C}^2 & \xrightarrow{p} & Y = \mathbb{C}^2 \\ f & \searrow & \downarrow g \\ & & \mathbb{C} \end{array}$$

where $p(x,y) = (y, x^2 + y^2)$ and $g(x,s) = s$. The restriction of p to a regular fiber X_s of f is a 2 to 1 covering map of the complex plane Y_s, with two branchpoints $(\pm\sqrt{s}, s)$, and the map p is a holomorphic family of such double covers outside the critical fiber X_0.

The map $g : Y \to \mathbb{C}$ is a trivial fibration on $\mathbb{C}$. We will use for this family a geometric monodromy over the base loop $\alpha(t) = e^{2\pi i t}$. This monodromy will be trivial only after isotopy, and will preserve the branchpoint locus of the map p. The branchpoints for $X_{\alpha(t)} \to Y_{\alpha(t)}$ are $e^{\pi i t}, e^{(\pi t + \pi)i}$, and we define after [7] a parallel transport over α by a family of diffeomorphisms

$$\begin{aligned} h_t : Y_{\alpha(0)} &\longrightarrow Y_{\alpha(t)} \\ (y, \alpha(0)) &\longmapsto (e^{\pi i t \Phi(|y|)} y, \alpha(t)) \end{aligned}$$

where the function $\Phi : [0, \infty) \to [0, 1]$ is a $\mathcal{C}^\infty$ plateau function with value 1 on $[0, \varepsilon]$ and 0 on $[2\varepsilon, \infty)$ for ε suitably small.

Fig. F3.8 Plane monodromy.

Our choice of h_t rather than the classical one $(y, 1) \mapsto (e^{\pi i t} y, e^{\pi i t})$ makes the parallel transport and geometric monodromy the identity outside the balls $B(0, 2\varepsilon) \times \{s\} \subset Y_s$.

The diffeomorphisms $h_t : Y_1 \to Y_{\alpha(t)}$ preserve the branchpoint locus of the covers $X_{\alpha(t)} \to Y_{\alpha(t)}$. Therefore they can be lifted to diffeomorphisms $X_1 \to X_{\alpha(t)}$. There are two possible lifts, but only one of them is the identity outside $V_t = p^{-1}(B(0, 2\varepsilon) \times \{\alpha(t)\})$, as the other possible lift interchanges the leaves of the cover. Making ε small enough, and choosing the lifts $\tilde{h}_t$ that are the identity outside the bounded regions V_t we obtain a parallel transport and geometric monodromy for the family $f : X = \mathbb{C}^2 \to \mathbb{C}$ over the loop α, such that they restrict to the family $f : \bar{U} \to B(0, \delta)$ and are the identity in a neighbourhood of its boundary.

After computing a geometric monodromy for the family $f : \bar{U} \to B(0, \delta)$, in order to complete the proof we must show that this monodromy is a Dehn twist of index 1 with the complex orientation. As the geometric monodromies over isotopic paths are isotopic, it suffices to check the index for the global family of conics

$$\begin{aligned} f : X = \mathbb{C}^2 &\longrightarrow \mathbb{C} \\ (x, y) &\longmapsto x^2 + y^2 \end{aligned}$$

over the simple loop $\alpha(t) = e^{2\pi i t}$. This will be done by using the above described monodromy diffeomorphisms and an explicit diffeomorphism between the topological cylinder $S^1 \times \mathbb{R}$ and the fiber X_1.

For the rest of this proof, we will parametrize the unit circumference as $S^1 = \{(u_1, u_2) \in \mathbb{R}^2 \mid u_1^2 + u_2^2 = 1\}$. We select the following diffeomorphism between the basepoint fiber X_1 and the cylinder $S^1 \times \mathbb{R}$:

$$\begin{aligned} \Psi : S^1 \times \mathbb{R} &\longrightarrow X_1 = \{(x,y) \in \mathbb{C}^2 \mid x^2 + y^2 = 1\} \\ (u_1, u_2, t) &\longmapsto (\sqrt{1+t^2}\, u_1 - i\, t u_2, \sqrt{1+t^2}\, u_2 + i\, t u_1) \end{aligned}$$

If we fix the complex orientation on the fiber X_1 and the orientation defined in Convention 3.12 for the cylinder $S^1 \times \mathbb{R}$, the diffeomorphism Ψ preserves the orientation.

The closed curve $\Delta = \{(u_1, u_2, 0) \in S^1 \times \mathbb{R}\}$ is mapped by Ψ onto the vanishing path $\{(u_1, u_2) \in \mathbb{C}^2 \mid u_1, u_2 \in \mathbb{R}\}$ associated to the critical point $(0,0)$. The vertical path $\nabla = \{(1, 0, t) \in S^1 \times \mathbb{R}\}$ is mapped to the path $\{(\sqrt{1+t^2}\, u_1, i\, t) \in X_1\}$.

The composition of the diffeomorphism Ψ with the covering map $p : X_1 \to \mathbb{C}$, $p(x,y) = y$, is a topological double cover of the complex plane with branchpoints ± 1, and has equation

$$\begin{aligned} p \circ \Psi : S^1 \times \mathbb{R} &\longrightarrow \mathbb{C} \\ (u_1, u_2, t) &\longmapsto \sqrt{1+t^2}\, u_2 + i\, t u_1 \end{aligned}$$

The loop Δ is folded by $p \circ \Psi$ onto the interval $[-1, 1]$, and the vertical path ∇ is sent diffeomorphically to the path $\{it \mid t \in \mathbb{R}\}$. We will describe more closely this double cover in order to characterize the lifts of paths in the complex plane.

The positive semi–cylinder $\{(u_1, u_2, t) \in S^1 \times [0, \infty)\}$ is mapped by $p \circ \Psi$ onto the complex plane. This mapping is one–to–one outside Δ, and sends the loops $\Delta_t(\theta) = (\cos(2\pi\theta), \sin(2\pi\theta), t), t_0)$ to parametrized ellipses $(\sqrt{1+t^2}\, \sin(2\pi\theta), t\cos(2\pi\theta))$, the clockwise parametrization of the loops Δ_t being sent to a counter–clockwise parametrization in the ellipses. The points (u_1, u_2, t) with $u_1 > 0$ cover the upper half–plane of $\mathbb{C}$, while the points with $u_1 < 0$ are sent to the lower half–plane.

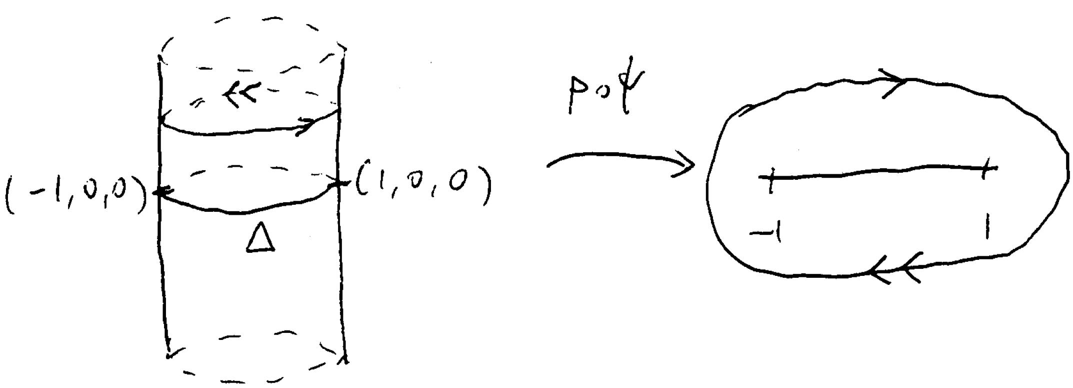

Fig. F3.9 Upper cylinder.
In the same manner, the negative semi–cylinder $\{(u_1, u_2, t) \in S^1 \times (-\infty, 0]\}$ is mapped by $p \circ \Psi$ onto the complex plane, one–to–one outside the boundary Δ. The level loops $\Delta_{-t}(\theta) = (\cos(2\pi\theta), \sin(2\pi\theta), -t)$ are mapped to ellipses $(\sqrt{1+t^2}\, u_2, -tu_1)$, and this time the clockwise parametrization is preserved. The points $(u_1, u_2, -t)$ with $u_1 > 0$ are sent to the lower half–plane, while the points with $u_1 < 0$ cover the upper half–plane.

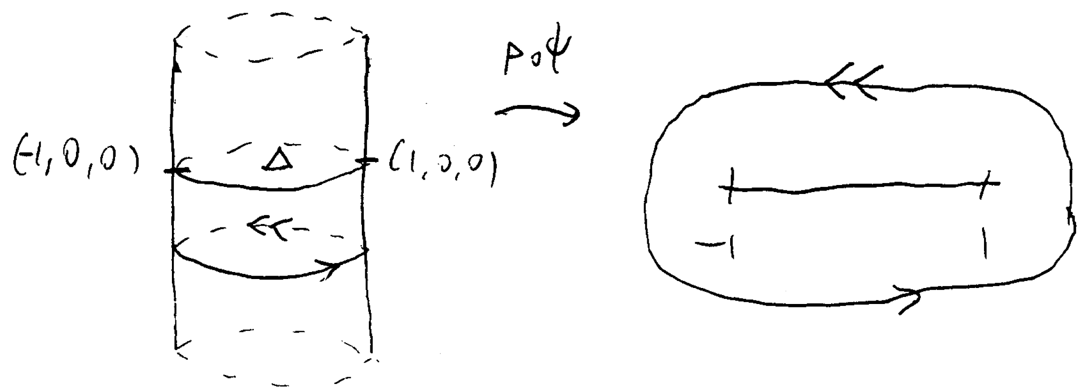

Fig. F3.10 Lower cylinder.
Our study of the covering map $p \circ \Psi$ enables us to present explicitly the cylinder $S^1 \times \mathbb{R}$ as a double cover of the complex plane with branchpoints ± 1. This double cover has been classically described as cutting the complex plane along the real interval $[-1, 1]$ and glueing two copies, corresponding to the positive and negative semi–cylinders.

Fig. F3.11 Glueing of two planes.

Consider now the vertical path $\nabla(t) = (1, 0, t) \subset S^1 \times \mathbb{R}$, mapped by $p \circ \Psi$ onto the path it. The points $\nabla(t)$ lie in the positive or negative semi–cylinder leaf of the double cover according to the sign of t. We will henceforth denote by ∇_+, ∇_- these two halves of the path.

Let us look now at the projection of ∇ on the plane, and apply the parallel transport diffeomorphisms h_t above defined to the paths ∇_+, ∇_-. The monodromy diffeomorphism h_1 is a rotation of angle π in a disk $B(0, r)$ centered in the origin and containing the interval $[-1, 1]$, glued by rotations of decreasing positive argument to the identity outside a larger disk $B(0, R)$. Thus the path $p \circ \Psi(\nabla_-)$ is mapped to itself for $t \in (-\infty, -R)$. The final piece $\{p \circ \Psi(\nabla(t)) \mid -r < t \leq 0\}$ is rotated an angle π, i.e. multiplied by -1, and the piece $\{p \circ \Psi(\nabla(t)) \mid -R \leq t \leq r\}$ is mapped to a simple path joining the two points $p \circ \Psi(\nabla(-R)) = -iR, p \circ \Psi(\nabla(-r)) = ir$ and contained in the half-plane $\{\operatorname{Re} z \geq 0\}$. These facts determine the homotopy class, in fact even the isotopy class, of the image $h_1(p \circ \Psi(\nabla_-))$.

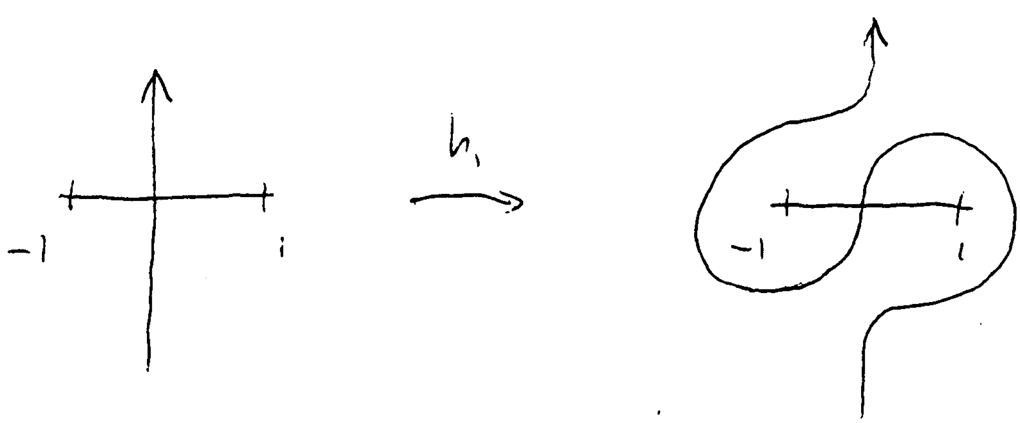

Fig. F3.12 The plane geometric monodromy.

The monodromy image of the half–path $p \circ \Psi(\nabla_+)$ may be likewise computed. The result is a rotation of angle π of $h_1(p \circ \Psi(\nabla_-))$. As the half–paths ∇_+, ∇_- are contained in single leaves of the covering p, their monodromy images are determined by their images by p just computed and the fact that the monodromy is the identity outside a compact neighbourhood. Using the diffeomorphism Ψ and the fact that the paths $h_1(p \circ \Psi(\nabla_-)_{[-R,-r]}$, resp. $(\nabla_+)_{[r,R]}$ may be isotoped to a straight line plus an arc of the above described ellipses $p \circ \Psi(\Delta_t)$, we reach the conclusion that the monodromy image of ∇ is homologous to $\nabla + \Delta$.

Fig. F3.13 The curve monodromy.

This homology implies that the monodromy mapping on X_1 has index 1 with the selected orientation, and thus it is by Proposition 3.13 the Dehn twist defined by the complex orientation on the bicollar neighbourhood of the vanishing path of the singularity. □

We are now ready to compute the geometric monodromy of the Lefschetz pencil $f : X \to \mathbb{C}$. First we will fix a presentation for $\pi_1(S, *)$:

NOTATION 3.17. Select a basepoint $s_0 \in S$ and smooth paths α_j from s_0 to $\beta_j(0)$ for every distinct critical value $z_1, \ldots, z_m$. The fundamental group $\pi_1(S, s_0)$ is then the free group spanned by the homotopy classes of the paths $\gamma_j = \alpha_j \beta_j \alpha_j^{-1}$ for $1 \leq j \leq m$. We will fix as well a parallel transport along the α_j, such that all the neighbourhoods U_{i_j} of the singular points introduced in Notation 3.5 are sent to disjoint open sets of X_{s_0}, and denote also as d_{i_j} the transported vanishing paths on X_{s_0}.

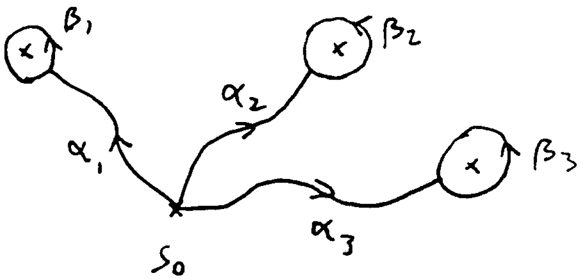

Fig. F3.14 Base space loops.

THEOREM 3.18. *Let $f : X \to \mathbb{C}$ be a Lefschetz pencil of curves. The geometric monodromy map induced by $f : X \setminus \Sigma \to S$ is*

$$\rho : \pi_1(S, s_0) \longrightarrow Aut^+(X_{s_0})/\textit{diffeotopies} = M(g,0)$$
$$\gamma_i \longmapsto \tau_{d_{i_1}} \circ \cdots \circ \tau_{d_{i_k}}$$

where $d_{i_1}, \ldots, d_{i_k}$ are the vanishing paths associated to the singular points $p_{i_1}, \ldots, p_{i_k}$ lying over the critical value z_i.

PROOF. We will compute the monodromy around a loop β_i.

Let $p_{i_1}, \ldots, p_{i_k}$ be the singular points of f above the critical value z_i, $U_{i_1}, \ldots, U_{i_k}$ the corresponding neighbourhoods defined after Lemma 3.4, and $Y_i = f^{-1}(B(z_i, \delta)) \setminus (U_{i_1} \cup \cdots \cup U_{i_k})$. By Prop. 3.7, the family $f : Y_i \to B(z_i, \delta)$ is globally trivial, so it admits the identity as the monodromy along β_i. The monodromy along β_i for the families $f : \bar{U}_{i_j} \to B(z_i, \delta)$ has been computed in Prop. 3.16, and has been shown to be the identity in a neighbourhood of ∂U_{i_j}. It is therefore possible to glue the monodromy maps on Y_i, $\bar{U}_{i_j}$ and obtain a map that will be the global monodromy map of $f : f^{-1}(B(z_i, \delta) \setminus \{z_i\}) \to B(z_i, \delta) \setminus \{z_i\}$ by the uniqueness up to diffeotopy of the monodromy.

This resulting map consists of Dehn twists $\tau_{d_{i_1}}, \ldots, \tau_{d_{i_k}}$ which are the identity outside the disjoint open sets $U_{i_j} \cap X_{\beta_i(0)}$, and hence commute. Transport along α_i completes our proof. □

EXAMPLE 3.19. The Legendre family:

This is the elliptic surface $E = \{(x, y, t) \in \mathbb{C}^3 \mid y^2 = x(x-1)(x-t)\}$, with parametrizing map $f(x, y, t) = t$. Neither E is smooth nor f is proper. Therefore, we begin by completing the fibers by considering the projective family $\bar{E} = \{([X : Y : Z], t) \in \mathbb{P}^2(\mathbb{C}) \times \mathbb{C} \mid Y^2 Z = X(X - Z)(X - tZ)\}$. The surface $\bar{E}$ has two singular points $P_0 = ([0 : 0 : 1], 0)$

and $P_1 = ([1:0:1], 1)$; its singular fibers $\bar{E}_0$, $\bar{E}_1$ are nodal cubics. We blow up $\bar{E}$ at P_0, P_1 and in this way obtain a smooth elliptic surface $\tilde{E}$, and a smooth proper map $f : \tilde{E} \to \mathbb{C}$. The fibers $\tilde{E}_0, \tilde{E}_1$ consist each of two rational curves, one the strict transform of $\bar{E}_0$ resp. $\bar{E}_1$, and the other an exceptional smooth conic, both components meeting transversely at two points.

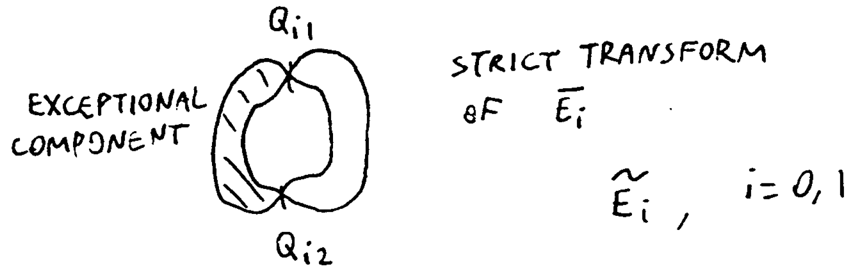

Fig. F3.15 Components of singular fiber $\tilde{E}_i$.
Let Q_{1i}, Q_{2i} be the antiimages of every P_i, and the function f has a critical point set $\{Q_{10}, Q_{20}, Q_{11}, Q_{21}\}$ and critical values $\{0, 1\}$. Thus $S = \mathbb{C} \setminus \{0, 1\}$, and fixing $s_0 = \frac{1}{2}$, there is a geometric monodromy map

$$\rho : \pi_1(S, s_0) \longrightarrow M(1, 0)$$

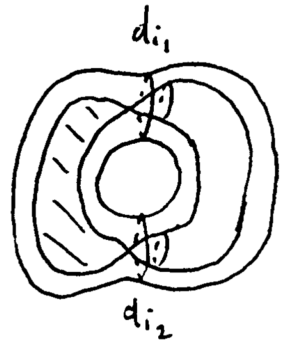

Fig. F3.16 Regular fiber near a singular fiber.
The vanishing paths associated to every critical point are two loops over every critical value i, both isotopic to the path d_i of $\tilde{E}_{s_0} = \bar{E}_{s_0}$ that collapses to the original node P_i. The collapsing paths in E_{s_0} are d_0, d_1 as shown in Fig. F3.17.

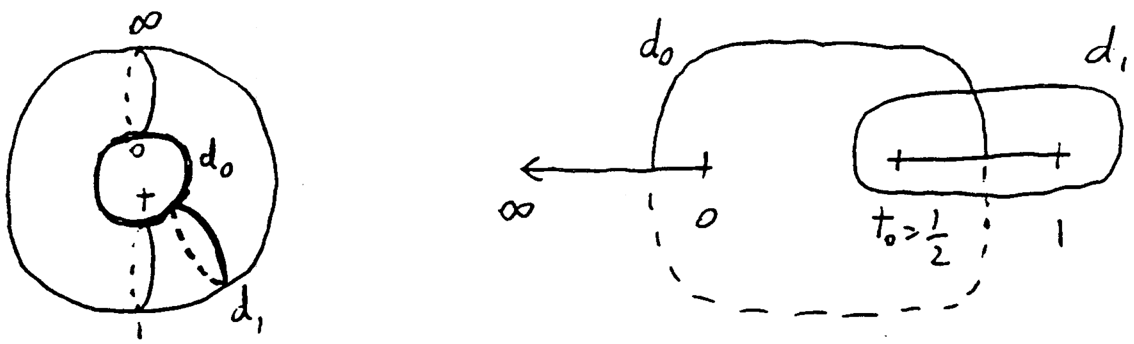

Fig. F3.17 Vanishing paths in torus and their projection to complex plane.
This is easily seen regarding the curves $\bar{E}_t$ as double covers of $\mathbb{P}^1(\mathbb{C})$ ramified over the four points $0, 1, t, \infty$. The singular fibers are obtained as $t \to 0, 1$ and observing the effect on the covering.

Denoting as γ_0, γ_1 the simple loops around $0, 1$ with positive orientation and also their homotopy classes, the geometric monodromy is

$$\begin{aligned} \rho : \pi_1(S, \frac{1}{2}) &\longrightarrow M(1,0) \\ \gamma_0 &\longrightarrow \tau_{d_0}^2 \\ \gamma_1 &\longrightarrow \tau_{d_1}^2 \end{aligned}$$

as every map $\rho(\gamma_i)$ consists of two Dehn twists along paths isotopic to d_i.

For a direct computation of the geometric monodromy in this case, carried out with great detail, see [**19**], 9.3 Example 2.

1.2. Pencils of quasi–projective curves. We will study now the case of a pair $f : (X, D) \to \mathbb{P}^1_{\mathbb{C}}$. This situation arises when considering base points, and when studying Lefschetz pencils of noncomplete curves.

DEFINITION 3.20. Let $B \subset \mathbb{P}^1_{\mathbb{C}}$ be a simply connected open domain. A relative Lefschetz pencil of curves over B is a map $f : (X, D) \to B$ such that:

(i) $f : X \to B$ is a Lefschetz pencil of curves.
(ii) D is a subvariety of X such that no singular point of $f : X \to B$ lies in D.

PROPOSITION 3.21. *There is a Zariski open subset $S \subset B$ such that the restriction of $f : (X, D) \to \mathbb{C}$ over S is a C^∞ locally trivial fibration, with fibre a pair $(X_s, D \cap X_s)$, where the intersection is a set of l distinct points $D \cap X_s = \{q_1, \ldots, q_l\}$.*

PROOF. Let us decompose the curve $D \subset X$ in its irreducible components $D = D_1 \cup \cdots \cup D_m$. Some of them may be fibres of f, $D_1 = X_{r_1}, \ldots, D_h = X_{r_h}$, and the others $D_{h+1}, \ldots, D_m$ are horizontal irreducible subvarieties of X. Each horizontal D_i has a generic intersection with fibres X_z consisting of a finite number of points $\{q_{i_1}, \ldots, q_{i_k}\}$ with multiplicity 1 for $z \in W_i$, a Zariski open subset. Also, as the curves D_i are irreducible and different, given $D_i \neq D_j$, $X_z \cap D_i$ and $X_z \cap D_j$ may only have common points for z in a proper closed set of $\mathbb{C}$. Thus there is a Zariski open set S such that for $s \in S$ the fibres of f are pairs $(X_s, \{q_1, \ldots, q_l\}$ with l fixed and the points q_i all different.

As $q_1, \ldots, q_l$ are points of $X_s \cap D$ with multiplicity 1, D is transverse to X_s in them, and so over S $f_{|D}$ is proper and smooth. Therefore, the relative Ehresmann theorem yields our statement. □

We will denote again $\Sigma = f^{-1}(\mathbb{C} \setminus S)$, and write D instead of $D \cap (X \setminus \Sigma)$. The C^∞ relative fibration

$$f : (X \setminus \Sigma, D) \longrightarrow S$$

has an associated parallel transport, which assigns to every path $\gamma : I \to S$ a relative diffeomorphism

$$f^*\gamma : (X_{\gamma(0)}, \{q_1, \ldots, q_l\}_{\gamma(0)}) \longrightarrow (X_{\gamma(1)}, \{q_1, \ldots, q_l\}_{\gamma(1)})$$

well defined up to relative diffeotopy, and compatible with path product. Fixing a base point $s_0 \in S$, we thus obtain the geometric monodromy map of this relative fibration

$$\rho : \pi_1(S, s_0) \longrightarrow \mathrm{Aut}^+(X_{s_0}, D \cap X_{s_0})/\text{rel. diffeotopies} = M(g, l)$$

where $M(g, l)$ is the mapping class group of the fibre X_{s_0} and the distinguished subset $X_{s_0} \cap D = \{q_1, \ldots, q_l\}$.

Before computing the monodromy of $f : (X, D) \to B$, let us recall an elementary description of the mapping class group $M(g, l)$ after [**15**], chapters 1 and 4. Denote C_g the compact topological orientable surface of genus g.

LEMMA 3.22. *There is an onto group morphism*

$$M(g, l) \xrightarrow{j_*} M(g, 0) \longrightarrow 0$$

given by forgetting the distinguised set $\{q_1, \ldots, q_l\} \subset C_g$.

We will see in Theorem 3.25 that, given the monodromy $\rho : \pi_1(S, s_0) \to M(g,l)$ of the relative Lefschetz pencil $f : (X, D) \to \mathbb{C}$, its projection to $M(g,0)$ is the monodromy of the absolute pencil $f : X \to \mathbb{C}$.

To learn about the kernel of $j_* : M(g,l) \to M(g,0)$, we introduce the braid groups of C_g:

Consider C_g^l, and its generalized diagonal $\Delta = \{(x_1, \ldots, x_l) \in C_g^l \mid \exists i \neq j \text{ s.t. } x_i = x_j\}$. The symmetric group S_l acts on C_g^l by permuting coordinates, and this action restricts to the open set $C_g^l \setminus \Delta$, where S_l acts freely.

DEFINITION 3.23. Let $l \in \mathbb{N}$. The l-*braid group* of C_g is the group

$$B_l(C_g) = \pi_1\left((C_g^l \setminus \Delta)/S_l, *\right)$$

We condense the relations between braid groups and mapping class groups that we will use in the following theorem (see [**15**] Ch. 4):

THEOREM 3.24. *Fix a base point* $(q_1, \ldots, q_l) \in (C_g^l \setminus \Delta)/S_l$. *Consider the set* $Aut^+(C_g)$ *of oriented self-diffeomorphisms of* C_g *with the compact-open topology. The map*

$$\begin{aligned} Aut^+(C_g) &\longrightarrow (C_g^l \setminus \Delta)/S_l \\ h &\longmapsto (h(q_1), \ldots, h(q_l)) \end{aligned}$$

is a topological fibration. Its fibre is $Aut^+(C_g, \{q_1, \ldots, q_l\})$, *and the long exact sequence associated to this fibration*

$$\begin{aligned} \ldots \longrightarrow \pi_1((C_g^l \setminus \Delta)/S_l, (q_1, \ldots, q_l)) &\longrightarrow \pi_0 Aut^+(C_g, \{q_1, \ldots, q_l\}) \\ \longrightarrow \pi_0 Aut^+(C_g) &\to \pi_0(C_g^l \setminus \Delta)/S_l = * \end{aligned}$$

yields a group exact sequence

$$B_l(C_g) \xrightarrow{\varepsilon} M(g,l) \xrightarrow{j_*} M(g,0) \longrightarrow 1 \tag{5}$$

The exact sequence (5) implies that to every l-braid b in C_g we can associate a relative diffeomorphism $\varepsilon(b) : (C_g, \{q_1, \ldots, q_l\}) \to (C_g, \{q_1, \ldots, q_l\})$ such that $\varepsilon(b) : C_g \to C_g$ is diffeotopic to the identity map, and $\varepsilon(b)$ is well determined up to relative diffeotopy. It may be obtained by integrating a vector field v in $C_g \times I$ such that v is transverse to the level surfaces $C_g \times \{t\}$ and tangent to the braid b.

Every braid induces a permutation of $\{q_1, \ldots, q_l\}$ that may be obtained by integrating the above cited vector field v. This induces an onto group morphism $B_l(C_g) \to S_l$ on the symmetric group. We will denote also as b_i the permutation thus induced by every braid b_i.

We are now able to describe the geometric monodromy of

$$f : (X \setminus \Sigma, D) \longrightarrow S = B \setminus \{z_1, \ldots, z_m\}$$

First of all, let us fix neighbourhoods U_{i_j}, $\pi_1(S, s_0)$ and relative parallel transport over the α_i, as in Not. 3.5 and 3.17.

THEOREM 3.25. *Let $f : (X, D) \to B$ be a relative Lefschetz pencil of curves. The geometric monodromy map induced by $f : (X \setminus \Sigma, D) \to S$ is*

$$\rho : \pi_1(S, s_0) \longrightarrow Aut^+(X_{s_0}, D \cap X_{s_0}) = M(g, l)$$
$$\gamma_i \longmapsto \tau_{d_{i_1}} \circ \cdots \circ \tau_{d_{i_k}} \circ \varepsilon(b_i)$$

where the d_{i_j} are the vanishing paths associated to the singular points $p_{i_1}, \ldots, p_{i_k}$ lying over the critical value z_i, and b_i is the braid described by $\{q_1, \ldots, q_l\}$ in its parallel transport along β_i.

PROOF. The proof is analogous to that of the absolute case (Thm. 3.18).

Let $\{p_{i_1}, \ldots, p_{i_k}\}$ be the set of singular points above z_i, possibly empty. We may take their neighbourhoods U_{i_j} small enough so that $\bar{U}_{i_j} \cap D = \emptyset$. Thus Prop. 3.16 tells us that the monodromy in $\bar{U}_{i_j}$ along β_i is the Dehn twist $\tau_{d_{i_j}}$, and is the identity on $\partial U_{i_j} \cap X_{\beta_i(0)}$, by Prop. 3.6 and also 3.16.

Denote again $Y_i = f^{-1}(B(z_i, \delta)) \setminus (U_{i_1} \cup \cdots \cup U_{i_k})$. By Prop. 3.7, the family $f : Y_i \to B(z_i, \delta)$ is trivial. Take a trivialization

$$\begin{array}{ccc} Y_i & \xrightarrow{\varphi_i} & Y_i \cap X_{\beta_i(0)} \times B(z_i, \delta) \\ f & \searrow & \downarrow \\ & & B(z_i, \delta) \end{array} \qquad (6)$$

As the set $D \cap X_{\beta_i(t)} = \{q_1, \ldots, q_l\}$ is preserved by relative parallel transport, the points $\varphi_i(D \cap X_{\beta_i(t)})_{t \in I}$ describe a braid in $X_{\beta_i(0)} \cap Y_i$, thus a braid b_i in $X_{\beta_i(0)}$. The relative monodromy on Y_i is thus the diffeomorphism associated to the braid b_i, which gives $\varepsilon(b_i)$ when extended as the identity outside $Y_i \cap X_{\beta_i(0)}$.

As in the absolute case, we may glue together the monodromy automorphisms on Y_i and the $\bar{U}_{i_j}$ by extending them as the identity outside their domain in $X_{\beta_i(0)}$ and so obtain the monodromy along β_i. Parallel transport along α_i completes our proof. □

EXAMPLE 3.26. Let us consider the pair (X, D) where $X = \mathbb{C}^2$ and D is the line arrangement $D = \{x = 0, x = s, s = 0, s = 1\}$ and $f(x, s) = s$. The fibers are $X_s = \mathbb{C} \setminus 0, s$. The function f is smooth over X, and therefore the geometric monodromy will be produced by the braids in X_s described by $D_s = \{0, s\}$ around the critical values of $f_{|D}$ 0, 1. Choosing $s_0 = \frac{1}{2}$, and generators γ_0, γ_1 of $\pi_1(S, s_0)$ positively circling $0, 1$, the monodromy maps $\rho(\gamma_0), \rho(\gamma_1)$ turn out to be the self-diffeomorphisms of $\mathbb{C} \setminus \{0, \frac{1}{2}\}$ associated to the braids

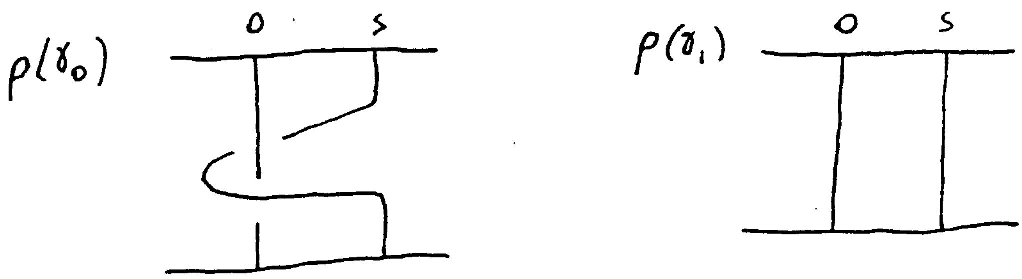

Fig. F3.18 Braids in the punctured complex plane.

EXAMPLE 3.27. In Example 3.19 of the previous section we have computed the monodromy of the Legendre family of complete cubics $\bar{E} = \{([X : Y : Z], t) \in \mathbb{P}^2_{\mathbb{C}} \times \mathbb{C} \,|\, Y^2 Z = X(X - Z)(X - tZ)\}$. The added points at infinity in the fibers $\bar{E}_s$ do not lie on the vanishing paths. Therefore the family is globally trivial in the neighbourhood of the infinity. The isotopies between vanishing paths we have used are still valid in the punctured curves E_t and therefore the geometric monodromy we have computed restricts to the monodromy in $M(1, 1)$ of the Legendre affine family of cubics

$$E = \{(x, y, z) \in \mathbb{C}^3 \mid y^2 = x(x - 1)(x - t)\}$$

2. Monodromy in the fundamental group

2.1. The projective case. We will study now the monodromy action in the fundamental group of the fibre of Lefschetz pencils of curves. This monodromy is trivial when the fibres are simply connected or its dimension is greater than one. The remaining cases are aspherical spaces, so the geometric monodromy is equivalent to the monodromy action on the π_1. Therefore, all that is required is to translate the formulas given in Thm. 3.18 and 3.25 taking account of base points.

Let $f : X \to B$ be a Lefschetz pencil of curves, $S = B \setminus \{z_1, \dots, z_m\}$ the set of regular values of f, and let $\sigma : S \to X$ be a section of f over S such that the singular points of f, $p_1, \dots, p_n$, do not lie in $\overline{\operatorname{Im} \sigma}$.

PROPOSITION 3.28. *The fundamental groups $\{\pi_1(X_s, \sigma(s)) \mid s \in S\}$ form a flat $\pi_1(X_{s_0}, \sigma(s_0))$-principal bundle.*

PROOF. By Prop. 3.21, $f : (X \setminus \Sigma, \sigma(S)) \to S$ is a $\mathcal{C}^\infty$ locally trivial fibration, with fibre the pair $(X_{s_0}, \sigma(s_0))$, and there is a base point-preserving parallel transport over S. As f is a topological fibration, the parallel transport along two homotopic paths γ_0, γ_1 produces homotopic diffeomorphisms between $X_{\gamma_i(0)}$ and $X_{\gamma_i(1)}$. Therefore $\{\pi_1(X_s, \sigma(s)) \mid s \in S\}$ has a homotopy-invariant parallel transport defined, or equivalently a flat bundle structure. □

We will compute the monodromy action

$$\rho : \pi_1(S, s_0) \longrightarrow \text{Aut}\,(\pi_1(X_{s_0}, \sigma(s_0)))$$

Let us make first some conventions:

NOTATION 3.29. (i) Choose the neighbourhoods U_{i_j} defined in Not. 3.5 such that $\bar{U}_{i_j} \cap \overline{\sigma(S)} = \emptyset$.

(ii) Fix a generating system for $\pi_1(S, s_0) = \langle \gamma_1, \dots, \gamma_m \rangle$ as in Not. 3.17 and basepoint preserving parallel transport along the α_i such that $\sigma(s_0)$ does not belong to the image of any $\bar{U}_{i_j} \cap X_{\beta_i(0)}$.

(iii) Select vanishing paths $d_1, \dots, d_n$ in X_{s_0} corresponding to the singular points of f, and collar neighbourhoods of the d_j $N_j \subset f^*\alpha_j(\bar{U}_j \cap X_{\beta_j(0)})$.

(iv) Pick a set of generators $c_1, \dots, c_{2g}$ for $\pi_1(X_{s_0}, \sigma(s_0))$ such that all the c_j are $\mathcal{C}^\infty$ paths cutting transversally and only a finite number of times every vanishing path d_j, and such that for every intersection point R_{ij} between a c_i and a d_j, the connected component of $c_i \cap N_j$ containing R_{ij} is exactly the fibre over R_{ij} of N_j as a normal bundle over d_j.

(v) Single out over $Y_i = f^{-1}(B(z_i, \delta)) \setminus (U_{i_1} \cup \dots \cup U_{i_k})$ a global trivialization φ_i as in (6).

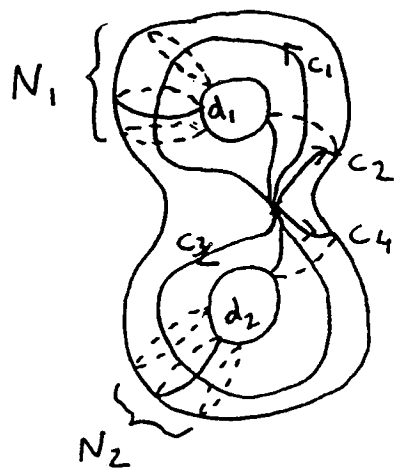

Fig. F3.19 Generating loops in a quasi–projective fiber.

DEFINITION 3.30. Given a global trivialization φ_i as in Not. 3.29 (v), we define the basepoint loop around β_i, σ_i in $Y_i \cap X_{\beta_i(0)}$ as

$$\sigma_i(t) = p_1\varphi_i(\sigma(\beta_i(t)))$$

DEFINITION 3.31. Let γ and d be C^∞ transverse loops, and $R = \gamma(t_R) \in \gamma \cap d$. We define the Dehn twist loop on d corresponding to R, γ_R, as the product of the paths

$$\gamma_R = \gamma_{[0,t_R]} \cdot d \cdot \gamma^{-1}_{|[0,t_R]}$$

where the path d is given the orientation of its monodromy Dehn twist.

We will also need a classical result on the monodromy of the fundamental grupoid (for a proof see [**74**]):

PROPOSITION 3.32. *Let X be an arc-connected, locally arc-connected topological space. Denote as Θ_X its continuous path space $\{\gamma : I \to X\}$, endowed with the compact open topology, and let $\Pi(X)$ be the fundamental grupoid of X with the induced topology. Then:*

(i) *The evaluation map*

$$e_0 : \Pi(X) \longrightarrow X$$
$$[\gamma] \longmapsto \gamma(0)$$

gives $\Pi(X)$ a local system structure over X, with structural group $\pi_1(X,*)$. The parallel transport over a path σ of $\Pi(X) \xrightarrow{e_0} X$ sends γ to $\sigma^{-1}\gamma$.

(ii) *The evaluation map*

$$\begin{aligned} e_1 : \Pi(X) &\longrightarrow X \\ [\gamma] &\longmapsto \gamma(1) \end{aligned}$$

gives $\Pi(X)$ *a local system structure over* X, *with structural group* $\pi_1(X,*)$. *The parallel transport over a path* σ *of* $\Pi(X) \xrightarrow{e_1} X$ *sends* γ *to* $\gamma\sigma$.

We are now ready to compute the monodromy action in the fundamental group of fibres for Lefschetz pencils of curves.

PROPOSITION 3.33. *Let* γ *be a* C^∞ *path on a surface* C, *such that it cuts transversally a* C^∞ *simple closed loop d in points* $R_1 = \gamma(t_1), \dots, R_m = \gamma(t_m)$, *with* $0 < t_1 < \cdots < t_m < 1$. *Let* $\tau_{d*} : \Pi(C) \to \Pi(C)$ *be the fundamental groupoid morphism induced by a Dehn twist above d.*

Then, the class of the path γ *is sent by* τ_{d*} *to the class of* $\gamma_{R_1} \cdot \dots \cdot \gamma_{R_m} \cdot \gamma$, *where the* γ_{R_j} *are the Dehn twist loops defined in Def. 3.31.*

PROOF. By our transversality assumption, there exists a collar neighbourhood N of the loop d such that $\gamma \cap N = N_{R_1} \sqcup \cdots \sqcup N_{R_m}$, the fibres over the points R_j of N as a normal bundle. We may assume the Dehn twist τ_d to be the identity outside N.

Parametrize by $\theta \in [0, 2\pi]$ the loop d, and write $R_j = d(\theta_j)$. Then the normal bundle N admits coordinates $[-1, 1] \times [0, 2\pi]$. The fibre paths $N_R(t) = (t, \theta_R)$ are sent by the Dehn twist τ_d to $\tau_d(N_R)(t) = (t, \theta_R + \pi(t+1))$, as is seen in Figure F3.20.

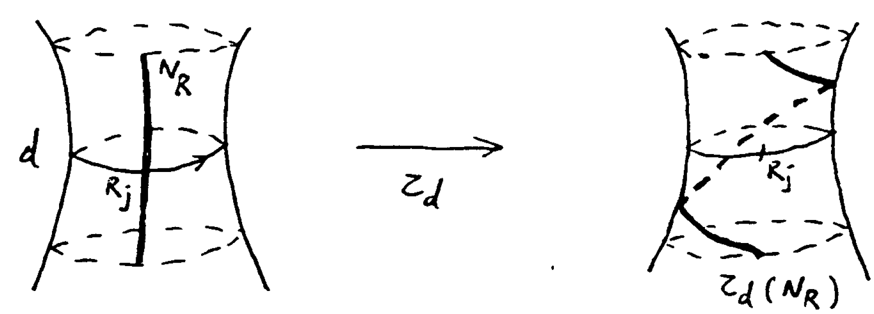

Fig. F3.20 Dehn twist on N_R.

The path $\tau_d(N_R)$ is homotopic to the path $N_R([-1,0]) \cdot d \cdot N_R([0,1])$ relatively leaving its extremes fixed. We may glue those homotopies for every R_j and thus obtain the homotopy between $\tau_{d*}(\gamma)$ and $\gamma_{R_1}\dots\gamma_{R_m}\gamma$. □

THEOREM 3.34. *Let $f : X \to B$ be a Lefschetz pencil of curves, and $\sigma : S \to X$ a section over the set of regular values. With the notation of 3.29, the monodromy action $\rho : \pi_1(S, s_0) \to Aut\ (\pi_1(X_{s_0}, \sigma(s_0)))$ is determined by*

$$\rho(\gamma_k) : \pi_1(X_{s_0}, \sigma(s_0)) \longrightarrow \pi_1(X_{s_0}, \sigma(s_0))$$
$$c_i \longmapsto \sigma_k^{-1} c_{R_1} \dots c_{R_w} c_i \sigma_k$$

where $(R_1, \dots, R_w)$ is the ordered set of intersection points of c_i with the union of the vanishing paths $d_{k_1} \cup \dots \cup d_{k_r}$ of all singular points over z_k, and σ_k is the basepoint loop around β_i described in Def. 3.30.

PROOF. We will compute the monodromy action along β_i on $\pi_1(X_{\beta_i(0)}, \sigma(\beta_i(0)))$. As in the case of the geometric monodromy, decompose

$$f^{-1}(B(z_i, \delta)) = Y_i \cup (\bar{U}_{i_1} \cup \dots \cup \bar{U}_{i_k})$$

where the U_{i_j} have been selected small enough so the collar neighbourhoods $N_{i_j} = U_{i_j} \cap X_{\beta_i(0)}$ satisfy 3.29 (iv).

We may decompose every path c_h as $c_h = c_{h_1} \dots c_{h_m}$, where every $c_{h_r} = c_{h|[t_{r-1}, t_r]}$ is a path contained either in Y_i or in a $\bar{U}_{i_j}$. We apply Prop. 3.33 to the paths $c_{h_r} \subset \bar{U}_{i_j}$. By our transversality assumption 3.29 (iv), $c_{h_r} \cap d_{i_j} = \{R\}$, and the path c_{h_r} is the fibre $N_R = \{(t, \theta_R) \mid t \in [-1, 1]\}$ with a suitably oriented parametrization, and thus by Prop. 3.33 c_{h_r} is sent to $(c_{h_r})_R \cdot c_{h_r}$.

The geometric monodromy in Y_i is trivial, but this is not the case for the relative family $(Y_i, \operatorname{Im}\sigma)$ over the punctured ball $B(z_i, \delta) \setminus \{z_i\}$. The parallel transport of the initial path c_{h_1} must have starting point $\sigma(\beta_i(t))$, and the final path c_{h_m} must also have a parallel transport with endpoint $\sigma(\beta_i(t))$. As the family is geometrically trivial and only the basepoint changes, we may apply Prop. 3.32 and conclude that c_{h_1} is sent by monodromy to $\sigma_k^{-1} c_{h_1}$, and c_{h_m} to $c_{h_m} \sigma_k$.

The paths c_{h_r} in Y_i with $r \neq 1, m$ have initial and end points in ∂Y_i. As the geometric monodromy of the family $(Y_i, \operatorname{Im}\sigma)$ may be chosen to be the identity on a neighbourhood of ∂Y_i, and the absolute monodromy on Y_i is trivial, those c_{h_r}, $r \neq 1, m$ are sent to paths homotopic to them.

The monodromy given in our statement is now obtained by glueing the separate monodromy transformations on the c_{h_i} we have determined and transporting it to s_0 over α_i. □

2.2. The quasiprojective case and the weight filtration. We will study as a final case families of affine curves.

DEFINITION 3.35. Let $f : (X, D) \to B$ be a relative Lefschetz pencil of curves. An affine Lefschetz pencil of curves is the map

$$f : X \setminus D \longrightarrow B$$

A relative affine Lefschetz pencil of curves is a map $f : (X \setminus D, C) \to B$ such that:

(i) $f : X \setminus D \to B$ is an affine Lefschetz pencil.
(ii) C is an algebraic subvariety of X such that it does not contain any singular point of f and does not intersect D.

From a geometric monodromy viewpoint, affine and relative Lefschetz pencils are the same, as the following Proposition shows:

PROPOSITION 3.36. *Let C_g be a topological compact orientable surface of genus g, and $\{q_1, \dots, q_l\}$ a distinguished subset of l distinct points. There is a group isomorphism in the topological category*

$$Aut^+(C_g \setminus \{q_1, \dots, q_l\}) \cong Aut^+(C_g, \{q_1, \dots, q_l\})$$

which conserves isotopies, and induces an isomorphism

$$Aut^+(C_g \setminus \{q_1, \dots, q_l\})/isotopies \cong Aut^+(C_g, \{q_1, \dots, q_l\})/relative\ isotopies = M(g, l)$$

PROOF. Let $\{K_i\}_{i \in I}$ be a final directed system of compacts of $C_g \setminus \{q_1, \dots, q_l\}$, with the ordering given by inclusion. The connected components of the complement of K_i in $C_g \setminus \{q_1, \dots, q_l\}$ for i big enough are disjoint neighbourhoods of the points $q_1, \dots, q_l$. Any homeomorphism h of $C_g \setminus \{q_1, \dots, q_l\}$ sends $\{K_i\}$ to $\{h(K_i)\}$, which is another final directed system of compact sets, and thus h sends small enough neighbourhoods of $q_1, \dots, q_l$ to neighbourhoods of $q_{\sigma(1)}, \dots, q_{\sigma(l)}$ for a fixed permutation $\sigma \in S_l$. We may extend h by defining $h(q_i) = q_{\sigma(i)}$, and the map thus defined is a relative self-homeomorphism of $(C_g, \{q_1, \dots, q_l\})$.

As $\{q_1, \dots, q_l\}$ is a subset of isolated points in C_g, the above construction may be extended to isotopies in $C_g \setminus \{q_1, \dots, q_l\}$ and yields relative isotopies in $(C_g, \{q_1, \dots, q_l\})$.

Restriction of homeomorphisms and relative isotopies gives an inverse arrow, and completes our proof. □

Prop. 3.36 tells us that the geometric monodromy of an affine Lefschetz pencil of curves is that of the relative Lefschetz pencil of its completion, which has been computed in Thm. 3.25.

We have introduced relative affine Lefschetz pencils because we will need basepoint sections for affine Lefschetz pencils. One may check by repeating the proof of Prop. 3.21 that a relative affine Lefschetz pencil has a relative parallel transport well defined up to relative diffeotopy.

In the case of our concern, if $f : X \setminus D \to B$ is an affine Lefschetz pencil of curves which is $\mathcal{C}^\infty$ locally trivial over $S \subset B$ and $\sigma : S \to X \setminus D$ is a section such that $\overline{\operatorname{Im}\sigma}$ does not meet D, there are basepoint preserving parallel transport and monodromy in the family $f : (X \setminus D) \setminus \Sigma \to S$. The missing points $X_s \cap D_s = \{q_1, \dots, q_l\}$ and the base point $\sigma(s)$ describe an $(l+1)$-braid around every point $z_i \in B \setminus S$. This braid satisfies the additional condition that the basepoint strand must always return to itself, while the puncture strands may permute.

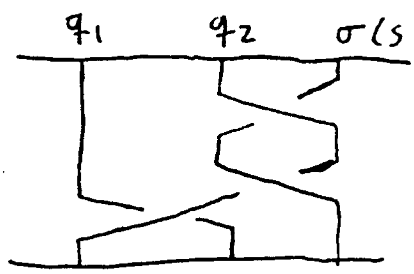

Fig. F3.21 Admisible braid

The fundamental group of a topological surface of genus g with $l \geq 1$ punctures, $C_g \setminus \{q_1, \dots, q_l\}$, is a free group of rank $2g + l - 1$. One may choose as a generating set for $\pi_1(C_g \setminus \{q_1, \dots, q_l\}, *)$ a set of $2g$ generators of $\pi_1(C_g, *)$ and $l - 1$ loops, each going around one and only one distinct q_i. We will fix such a generating set in the case of $(X_{s_0} \setminus D_{s_0}, \sigma(s_0))$.

NOTATION 3.37. (i) Fix neighbourhoods Y_i over the singular points of f, a generating system for $\pi_1(S, s_0) = \langle \gamma_1, \dots, \gamma_m \rangle$, vanishing paths $d_1, \dots, d_n$ and collar neighbourhoods $N_1, \dots, N_n$ in X_{s_0} and a set of generators $c_1, \dots, c_{2g}$ for $\pi_1(X_{s_0}, \sigma(s_0))$ as in Not. 3.29.

(ii) Select simple closed $\mathcal{C}^\infty$ loops $u_1, \dots, u_l \subset X_{s_0} \setminus D_{s_0}$, each of them contained in a small enough neighbourhood W_j of the corresponding q_j and with the direct orientation. Select also simple $\mathcal{C}^\infty$ paths $a_1, \dots, a_l$ from $\sigma(s_0)$ to $u_1(0), \dots, u_l(0)$ respectively. Define loops $g_i = a_i u_i a_i^{-1}$.

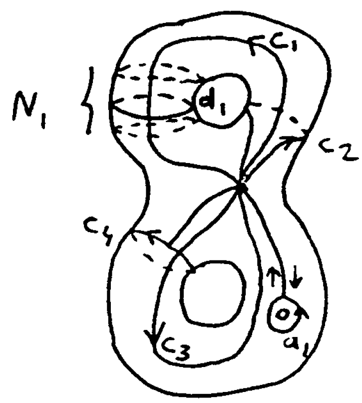

Fig. F3.22 Generating loops on the fiber.

We must also fix a relative trivialization $\varphi : (Y_i, D \cap Y_i, \sigma(B(z_i, \delta)) \to (Y_i \cap X_{\beta_i(0)}, D \cap X_{\beta_i(0)}, \sigma(\beta_i(0))) \times B(z_i, \delta)$ as in 3.29 (v). Such a trivialization is equivalent to a relative parallel transport $T_i = \varphi_i^{-1}$, and we will select one that satisfies further conditions of transversality and of moving the strands of the braid b_i described by $\{q_1, \dots, q_l\}$ over β_i and the base point one at a time. That such a parallel transport exists may be seen by a simple local definition piecewise and a tedious glueing process, we will only list its relevant properties:

LEMMA 3.38. *We may choose a parallel transport T_i such that its restriction over β_i verifies:*

(i) $\frac{\partial T_i}{\partial t}(., t) = 0$ *outside neighbourhoods of the paths $T_i(q_j, \beta_i(t))$, σ_i.*
(ii) *The paths $T_i(q_j, \beta_i(.))$ cut each other transversally. The paths c_h, a_m are also transversally cut, and may only be cut outside the neighbourhoods $W_1, \dots, W_l$ of 3.37 (ii).*
(iii) *The intersection of a path $T_i(q_j, .)$ with the neighbourhoods $W_1, \dots, W_l$ has exactly two connected components $T_i(q_j, [0, t_0)) \subset W_j$ and $T_i(q_j, (t_1, 1]) \subset W_{b_i(j)}$ corresponding to the two edges of the path.*
(iv) *For $j = 1, \dots, l$ and $t \in [\frac{0.9(j-1)}{l}, \frac{0.9j}{l}]$, $\frac{\partial T_i}{\partial t} = 0$ outside a neighbourhood of q_j, and $T_i(q_j, .)$ moves from q_j to $W_{b_i(j)}$.*
(v) *For $t \in [0.9, 0.95]$, $\frac{\partial T_i}{\partial t} = 0$ outside $W_1 \cup \dots \cup W_l$.*
(vi) *For $t \in [0.95, 1]$, $\frac{\partial T_i}{\partial t} = 0$ outside a neighbourhood of σ_i, and $T_i(\sigma_i(0), .) = \sigma_i$.*

The fundamental group $\pi_1(X_{s_0} \backslash D_{s_0}, \sigma(s_0))$ is generated by $c_1, \ldots, c_{2g}, g_1, \ldots, g_l$, and the only defining relation among the generators is $g_1 \ldots g_l = w$, with w a word in $c_1, \ldots, c_{2g}$. We will name the different paths according to the properties of their images in $H_1(X_{s_0} \setminus D_{s_0})$.

REMARK 3.39. The loops $c_1, \ldots, c_{2g}$ are the *weight minus one* loops, and $g_1, \ldots, g_l$ are the *weight minus two* loops, according to the weight filtration in the fundamental group $\pi_1(X_{s_0} \setminus D_{s_0}, *)$ of [**8**].

DEFINITION 3.40. (Driving a strand across a path)
Let γ be a $\mathcal{C}^\infty$ path in a smooth surface C, s a transverse $\mathcal{C}^\infty$ path such that $\gamma \cap s$ is the ordered set $\{R_1 = \gamma(t_1) = s(y_1), \ldots, R_w = \gamma(t_w) = s(y_w)\}$ and let $u(s(1))$ be a $\mathcal{C}^\infty$ simple loop around $s(1)$ with direct orientation and contained in a small enough neighbourhood. We define the transported path of γ by the strand s as:

$$T_s\gamma = \gamma_{R_1} \cdot \cdots \cdot \gamma_{R_w} \cdot \gamma$$

where

$$\gamma_R = \gamma_{|[0,t_R]} \cdot s_{|[y_R,1]} \cdot u_{s(1)}^{\varepsilon(R)} \cdot s^{-1}_{|[y_R,1]} \cdot \gamma^{-1}_{|[0,t_R]}$$

and $\varepsilon(R)$ is the sign of the inner product $\langle \frac{\partial s}{\partial y}(y_R), \frac{\partial \gamma}{\partial t}(t_R) \rangle$.

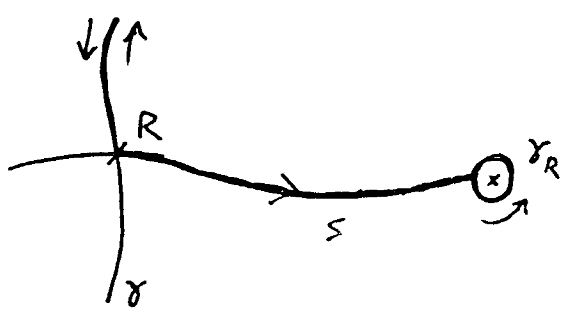

Fig. F3.23 The braid loops

DEFINITION 3.41. We define the conjugation path of q_j over the critical value $z_i \in B \setminus S$ as the product

$$\psi_{ij} = a_j \cdot T_{i\,u(j)}(0) \cdot a^{-1}_{b_i(j)}$$

LEMMA 3.42. *Let γ be a $\mathcal{C}^\infty$ path on a smooth surface C. Let s be another $\mathcal{C}^\infty$ path in C transversally cutting γ along a finite number of points. Then the parallel transport of γ in $C \times I \setminus \{(s(t), t)\}$ is the transported path of γ by the strand s as defined in Def. 3.40.*

Everything has been prepared for our final result in this section:

THEOREM 3.43. *Let $f : X \setminus D \to B$ be an affine Lefschetz pencil of curves, and $\sigma : S \to X \setminus D$ a section over the regular set of values. With the conventions of 3.29, 3.37, 3.39, and denoting $s_j := T_i(q_j, .)$, the monodromy action $\rho : \pi_1(S, s_0) \to Aut\ (\pi_1(X_{s_0} \setminus D_{s_0}, \sigma(s_0)))$ is determined by:*

(i) Weight minus one paths:

$$\rho(\gamma_k)(c_i) = \sigma_k^{-1} T_{s_l} \dots T_{s_1}(c_{R_1} \dots c_{R_w} c_i)\sigma_k$$

where $(R_1, \dots, R_w)$ is the ordered set of intersection points of c_i with the vanishing paths over z_k $d_{k_1} \cup \dots \cup d_{k_r}$, c_{R_j} denotes the Dehn twist loop (Def. 3.31) corresponding to R_j if $R_j \in c_i \cap d_j$, and $T_{s_j}(.)$ is the transport over the strand $T_i(q_j, .)$, and σ_k is the base point path over β_k.

(ii) Weight minus two paths:

$$\rho(\gamma_k)(g_i) = \sigma_k^{-1} \psi g_{b_i(j)} \psi^{-1} \sigma_k$$

where the path ψ is $\psi = T_{s_l} \dots T_{s_1}(a_{j\ S_1} \dots a_{j\ S_m} a_j) T_{s_l} \dots T_{s_{j+1}}(T(u_j(0), .)) a_{b_i(j)}^{-1}$, with $S_1, \dots, S_m$ the ordered set of intersection points of a_j with the vanishing paths over z_k and $a_{j\ S_n}$ is the corresponding Dehn twist loop (Def. 3.31).

EXAMPLE 3.44. On polylogarithms:
Let us consider the family given by $S = B \setminus \{0, 1\}$,

$$f : X = S \times S \longrightarrow S$$
$$(x, s) \longrightarrow s$$

and diagonal basepoint $\sigma : S \to S \times S$, $\sigma(s) = (s, s)$. From the geometric viewpoint this is a trivial fibration with fiber $X_s = \mathbb{C} \setminus \{0, 1\}$ and its monodromy is the identity. However, the monodromy in the principal F_2-bundle $\{\pi_1(X_s, \sigma(s)) \mid s \in S\}$ is not trivial, as the selected basepoint section is not constant. Fixing $s_0 = \frac{1}{2}$ and g_0, g_1 simple paths positively around $0, 1$, the monodromy automorphisms ρ_0, ρ_1 are

$$\rho_i(\alpha) = g_i^{-1} \alpha g_i$$

For any topological space S, the monodromy group of this family is always the group of inner automorphisms of $\pi_1(S, *)$. In our case $S = \mathbb{C} \setminus \{0, 1\}$, by the theory of iterated integrals of Chen ([**25**]),nontriviality of the monodromy is equivalent to the fact that the polilogarithms on $\mathbb{C} \setminus \{0, 1\}$ are multivalued.

EXAMPLE 3.45. The Legendre family revisited:

The example to be studied with more detail in this work is that of the affine Legendre family of Ex. 3.19. There we have computed its geometric monodromy over $S = \mathbb{C} \setminus \{0, 1\}$, and here we extend the computation to the fundamental group.

Fix a constant basepoint section P in a neighbourhood of infinity, and a basis a, b of $\pi_1(E_{s_0}, P)$ as indicated in Figure F3.24.

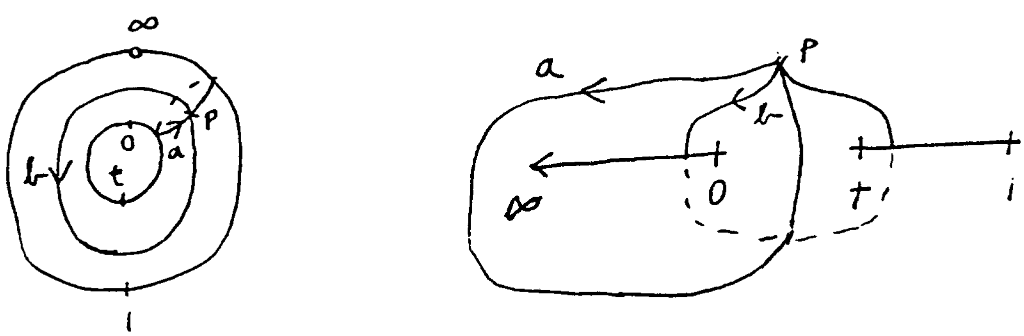

Fig. F3.24 Basis of fundamental group in punctured torus and its projection to the complex plane.

Figure F3.25 shows the effect of τ_{d_0}, τ_{d_1} on the loops a, b: The paths c_{ij} of Def. 3.31 here are $a_0 = ab^{-1}a^{-1}$, $a_1 = \underline{P}$, $b_0 = \underline{P}$, $b_1 = bab^{-1}$. The monodromy representation is now immediate.

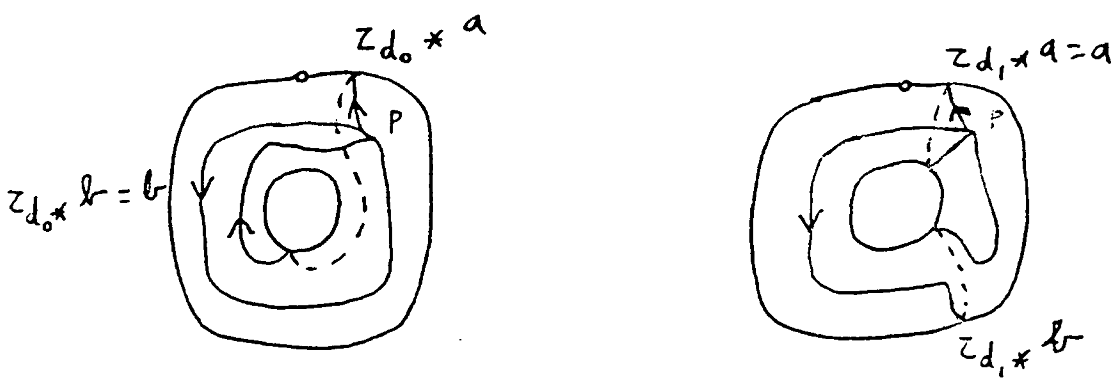

Fig. F3.25 Vanishing paths in the Legendre family.

PROPOSITION 3.46. *The monodromy representation* $\rho : \pi_1(\mathbb{C}\setminus\{0,1\}, \frac{1}{2}) \to$ $Aut(\pi_1(E_{\frac{1}{2}}, P))$ *induced by the Legendre family of affine elliptic curves* *is determined by:*

$$\begin{aligned}\rho([\gamma_0]) : \pi_1(E_{\frac{1}{2}}, P) &\longrightarrow \pi_1(E_{\frac{1}{2}}, P)\\ a &\longrightarrow ab^{-2}\\ b &\longrightarrow b\end{aligned}$$

$$\begin{aligned}\rho([\gamma_1]) : \pi_1(E_{\frac{1}{2}}, P) &\longrightarrow \pi_1(E_{\frac{1}{2}}, P)\\ a &\longrightarrow a\\ b &\longrightarrow ba^2\end{aligned}$$

3. Monodromy properties of pencils of projective curves

3.1. Quasi–unipotence and zero entropy. We will study now the monodromy of a pencil of projective curves around a critical value. By Grothendieck's theorem on the quasi–unipotence of the monodromy in algebraic families, the monodromy in the first cohomology group of a pencil of curves around a critical value is quasi–unipotent. A'Campo showed in [**1**] that the geometric monodromy of an affine family given by a holomorphic map $F : \mathbb{C}^{n+1} \to \mathbb{D} \subset \mathbb{C}$, which is well–defined up to diffeotopy, is realized by a *distal* map. This means that the dynamical system formed by the Milnor fiber F_z and the distal geometric monodromy diffeomorphism has zero entropy, and a theorem by A. Manning shows that a diffeomorphism with zero entropy induces a quasi–unipotent monodromy isomorphism in $H^1(F_z, \mathbb{R})$ ([**67**]). Subsequent independent work by Bowen, Gromov, Shub and others extended Manning's theorem to the fundamental group, showing how the topological entropy of a map $f : X \to X$ bounds the growth rate of the endomorphism $f_* : \pi_1(X) \to \pi_1(X)$ (see [**36**] for a very complete discussion of the topic).

The author is grateful to N. A'Campo for originally bringing his attention to this field. This section contains an adaptation of these results to the case of Lefschetz pencils of curves. We work with the property of linear growth of group endomorphisms (see Definition 3.52 below). This property implies rate of growth zero, and is actually slightly stronger than zero entropy in the case of curves. It is shown in Proposition 3.59 that the monodromy of a projective family around a critical value with reduced fiber has linear growth in the fundamental group. Finally, Proposition 3.60 shows how linear growth in the fundamental group implies quasi–unipotence in the first cohomology group.

We start with some generalities on metric spaces and dynamical systems (see [**37**], [**36**]).

DEFINITION 3.47. Let X, Y be metric spaces.

A mapping $f : X \to Y$ is a quasi–isometry if there exist fixed positive constants λ, C verifying

$$\frac{1}{\lambda}d(x,y) - C \leq d(f(x), f(y)) \leq \lambda d(x,y) + C$$

for all $x, y \in X$.

The metric spaces X, Y are quasi–isometric if there exist a pair of mappings $f : X \to Y, g : Y \to X$ and fixed positive constants λ, C verifying

$$\begin{aligned} d(f(x), f(y)) &\leq \lambda d(x,y) + C \\ d(g(x'), g(y')) &\leq \lambda d(x', y') + C \\ d(g(f(x)), x) &\leq C \\ d(f(g(x')), x') &\leq C \end{aligned}$$

for all $x, y \in X,\ x', y' \in Y$.

The reader is remainded that a quasi–isometry needs not be continuous according to our definition. Quasi–isometry defines an equivalence relation among metric spaces.

EXAMPLE 3.48. A metric space X is quasi–isometric to a point if and only if X has a bounded diameter. In particular, compact metric spaces are quasi–isometric to points.

There is an equivariant version of Example 3.48; its description requires some combinatorial group theory.

DEFINITION 3.49. Let Γ be a group.

(i) A finite symmetric generating set $S \subset \Gamma$ is a finite generating set for Γ such that $1 \notin \Gamma$, and if $x \in S$, also $x^{-1} \in S$.
(ii) The length of an element $g \in \Gamma$ with respect to S is the least length of a word in the elements of S representing g. The length of 1 is defined to be zero.
(iii) The word length distance defined by S in Γ is defined by setting as $d_S(g_1, g_2)$ the least length of a word on the generators of S representing $g_1^{-1}g_2$.

This definition makes Γ a discrete metric space, whith a free isometric Γ–action given by the group product. The word length distance extends to a metric in the Cayley graph of Γ, S by isometrically identifying the edges to the unit interval. This distance d_S depends on the choice of a finite,symmetric generating set S, but in a controlled way:

LEMMA 3.50. *Let Γ be a group, and S_1, S_2 two finite symmetric generating sets. The metric spaces $(\Gamma, d_{S_1}), (\Gamma, d_{S_2})$ are quasi–isometric.*

PROOF. The generators of S_2 have a bounded length in terms of S_1, and vice versa. The identity mapping already induces the quasi–isometry. □

Thus the word length metric associates an equivalence class of quasi–isometric spaces to every finitely generated group Γ. This class gives rise to the equivariant version of Example 3.48, for its proof the reader is referred to [**37**], Ch. 3, Prop. 19.

PROPOSITION 3.51 (Milnor). *Let X be a compact Riemannian manifold, $\tilde{X} \to X$ a universal covering space, and $\pi_1(X)\tilde{x} \subset \tilde{X}$ the $\pi_1(X)$–orbit of a point, with the metric and distance induced by those of X. Then the metric spaces $\tilde{X}$, $\pi_1(X)\tilde{x}$ and $\pi_1(X)$ are quasi–isometric. In fact, the inclusion $\pi_1(X)\tilde{x} \hookrightarrow \tilde{X}$ and the orbit identification $\pi_1(X) \xrightarrow{\cong} \pi_1(X)\tilde{x}$ induce the quasi–isometries.*

The case that will be discussed here is that of $X = C_g$ a compact Riemann surface of positive genus. In this context the fundamental group and the topology of C_g are specially close: C_g is an aspherical space and, moreover, it was shown by Dehn, Baer and others that homotopic homeomorphisms of C_g are isotopic, and thus the groups of homotopy and isotopy equivalence of C_g are isomorphic:

$$\operatorname{Out} \pi_1(C_g) \cong M(g, 0) \, .$$

Let $f : X \to \mathbb{D}$ be a Lefschetz pencil of curves, as defined in Def. 3.1, over a disk, and such that 0 is the only critical value. The geometric and fundamental group monodromy around the critical value have been discussed in Theorems 3.18, 3.34. Selection of a geometric monodromy homeomorphism $h : X_s \to X_s$ and of a basepoint section and presentation of the fundamental group of the fiber $\pi_1(X_s)$ gives rise to two topological dynamical systems (X_s, h) and $(\pi_1(X_s), \varphi)$, where $\pi_1(X_s)$ has a word length metric and φ is the monodromy automorphism. What follows may be seen as a discussion of the entropy of those dynamical systems, although we will stick to more elementary concepts.

DEFINITION 3.52. Let (Γ, S) be a group with a finite symmetric set of generators and its word length metric. A morphism $\varphi : \Gamma \to \Gamma$

has linear growth if there exists a fixed constant $\Lambda \in \mathbb{R}$ such that for any element $g \in \Gamma$ and $n \in \mathbb{N}$, the length of the elements $\varphi^n(g)$ satisfies a bound

$$\text{length}_S(\varphi^n(g)) \leq \Lambda \, \text{length}_S(g) n \, .$$

It is not hard to check that if a morphism φ has linear growth for a finite symmetric set of generators S then the property holds for any such set of generators, therefore we will often omit it. As first examples of automorpisms with linear growth we may cite:

EXAMPLE 3.53. (i) The inner automorphisms of a group.
(ii) The composition of a linear growth automorphism with an inner automorphism.

Example 3.53 shows that the linear growth property descends to outer automorphisms of the group. We will see in Example 3.57 that the composition of two linear growth automorphisms need not have linear growth.

Another source of linear growth automorphisms that we will require later is:

LEMMA 3.54. *Let $\varphi : \Gamma \to \Gamma$ be a group morphism such that its power φ^m has linear growth for some m. Then φ has linear growth.*

PROOF. Fix a finite symmetric generating system S to carry the computations. By the definition of linear growth applied to φ^m, there exists a fixed $\Lambda \in \mathbb{R}$ such that for all $g \in \Gamma$ one has

$$\text{length}\,(\varphi^m)^n(g) \leq \Lambda \text{length}\,(g) n \, .$$

Moreover, denoting $\varphi^0 = \text{Id}$, there exists a fixed $\Lambda_1 \in \mathbb{R}$ such that for every $g \in \Gamma$, and every $k \in \{0, \dots, m-1\}$ one has

$$\text{length}\,\varphi^k(g) \leq \Lambda_1 \text{length}(g) \, ,$$

one such constant is $\Lambda_1 = \max_{\substack{h \in S \\ k \in \{0,\dots,m-1\}}} \text{length}\,\varphi^k(h)$.

Now, for every $N \in \mathbb{N}$ and $g \in \Gamma$, let $N = mq + k$ be the euclidean division of N by m. By the above bounds we have

$$\text{length}\,\varphi^N(g) = \text{length}\,(\varphi^m)^q(\varphi^k(g)) \leq \Lambda\Lambda_1 \text{length}\,(g) q \leq (\Lambda\Lambda_1) \text{length}\,(g) N \, .$$

Hence our thesis. □

Linear growth of an automorphism φ implies that the rate of growth of φ, defined as

$$\sup_{g \in \Gamma} \limsup_n \frac{1}{n} \log \text{length}_S(\varphi^n g)$$

is zero. Given a self homeomorphism of a topological space, the rate of growth of the induced morphism in the fundamental group provides a bound for the entropy of the topological system. In the case of a closed orientable surface C_g linear growth also implies that the dynamical system $(\pi_1(C_g), \varphi)$ has zero entropy ([**36**]). The geometric monodromy around the critical value 0 has been seen to be the composition of Dehn twists along vanishing paths associated to the critical points in the singular fiber. A Dehn twist is a fibered isometry and has zero entropy ([**1**]). The zero entropy properties for both dynamical systems indeed correspond.

PROPOSITION 3.55. *Let $f : X \to \mathbb{D}$ be a family of compact Riemann surfaces with the single critical value 0. The associated monodromy automorphism $\varphi : \pi_1(X_s) \to \pi_1(X_s)$ has linear growth.*

PROOF. Assume first that the family $f : X \to \mathbb{D}$ is a Lefschetz pencil, i.e. that the singular fiber is reduced and has only ordinary uadratic singularities.

Let $p_1, \ldots, p_k$ be the critical points in the singular fiber X_0. By Theorem 3.18 the geometric monodromy around 0 is the composition of Dehn twists along the vanishing paths $d_1, \ldots, d_k$ corresponding to the critical points. The vanishing paths are disjoint, and the Dehn twists may be assumed to be the identity outside arbitrarily small bicollar neighbourhoods $U_1, \ldots, U_k$ of the vanishing paths, also disjoint. We may choose a monodromy homeomorphism h and an adapted set of generators $a_1, \ldots, a_{2g}$ for $\pi_1(X_s)$ as in Not. 3.29, cutting transversally the vanishing paths. In particular, every generator a_i intersects a finite number of times every vanishing path d_j, and thus taking as λ the maximal such number we have:

LEMMA 3.56. *There exists a constant λ such that every element $g \in \pi_1(X_s)$ admits a representing path γ that intersects at most λlength(g) times the vanishing paths, always transversally.*

Let now L be an upper bound for the length in X of the vanishing paths. As has been seen in Theorems 3.18,3.34, the effect of the monodromy map on a path γ cutting transversally all vanishing paths is homotopic to inserting a copy of the corresponding vanishing path in every intersection point. The resulting path $h_*(\gamma)$ is isotopic to a path γ_1, which is a copy of the path γ with the vanishing path inserted just outside the bicollar neighbourhood, as may be seen in Figure F3.26.

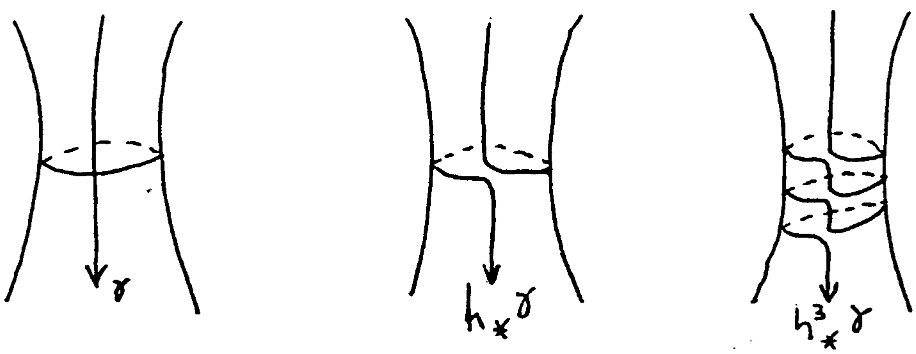

Fig. F3.26 Iteration of a Dehn twist.

If one iterates the monodromy homeomorphism h, one may recursively choose as a representing path for $h_*^n(g)$ the path γ_n, obtained from γ_{n-1} by isotopically moving outside the collar neighbourhoods U_j the inserted copies of the vanishing path. The result is that $h_*^n(g)$ admits a representing path γ_n which is the original path γ with n copies of the vanishing path d_j inserted after every intersection point $\gamma \cap d_j$. Therefore, the length of γ_n in X is bounded above by $l(\gamma) + nL\lambda\text{length}\,(g)$, where $l(\gamma)$ stands for the length of γ in X. If C is an upper bound for the length in X of the selected generating loops a_i of $\pi_1(X_s)$, then $l(\gamma) \leq C\text{length}(g)$, and thus

$$l(\gamma_n) \leq (C + nL)\text{length}(g)$$

Fix now a lift $\tilde{x}_0$ of the basepoint of the fiber $x \in X_s$ to its universal cover $\tilde{X}_s$. The loops γ_n may be lifted to paths $\tilde{\gamma}_n$ starting in $\tilde{x}_0$. As the metric in $\tilde{X}_s$ is the lift of that of X_s, we have $l(\tilde{\gamma}_n) \leq (C+nL)\text{length}(g)$. Due to the quasi–isometry of Proposition 3.51, this implies that

$$\text{length}(\varphi^n(g)) \leq \lambda(C + nL)\text{length}(g) + C'$$

As the least length is one, the choice $\Lambda = \lambda(C + C' + L)$ completes our proof in the case of a Lefschetz pencil of curves.

Let $f : X \to \mathbb{D}$ be now an arbitrary family of compact Riemann surfaces. By the Semistable Reduction Theorem (see Thm. 1.1 in [**9**]), there is a Lefschetz pencil of curves $\tilde{X} \to \mathbb{D}$ obtained by pulling back the family $f : X \to \mathbb{D}$ along the map $z \mapsto z^n$ for an adequate n, and blowing up the singular fiber a finite number of times. As the map $z \mapsto z^n$ induces multiplication by n in $\pi_1(\mathbb{D}^*, *) \cong \mathbb{Z}$, and blow ups on the singular fiber do not change the regular fibers, the relation between the monodromy automorphisms in the fundamental group φ of $X \to \mathbb{D}$

and Φ of $\tilde{X} \to \mathbb{D}$ is

$$\Phi = \varphi^n .$$

Our previous discussion has shown that Φ has linear growth, so by Lemma 3.54 φ has also linear growth, as was sought. □

EXAMPLE 3.57. Take C_2 a closed genus two surface, and a, b simple loops generating the homology of one of the handles, as in Figure F3.27.

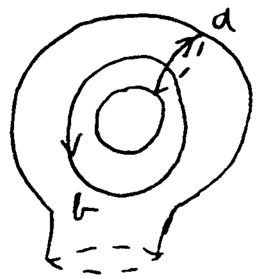

Fig. F3.27 Generators of the homology of a handle

Consider now the homeomorphism $h = \tau_b \circ \tau_a$, composition of Dehn twists around the loops a first and b afterwards. By selecting a nearby basepoint and a, b as two of the generators of $\pi_1(C_g)$ we get an induced monodromy automorphism of $\pi_1(C_2, *)$ such that

$$\varphi(a) = ab$$
$$\varphi(b) = ab^2$$

It may be checked by induction that the length of $\varphi^n(a)$ is the term $2n + 3$ of the Fibonacci sequence, thus its growth is exponential.

By Proposition 3.55 both τ_a, τ_b induce linear growth automorphisms of the fundamental group, so this example shows that the composition of linear growth automorphisms need not have also linear growth.

The following result will be valuable for our applications:

LEMMA 3.58. *Let Γ be a finitely generated group and $\Delta \subset \Gamma$ a normal subgroup. If a morphism $\varphi : \Gamma \to \Gamma$ has linear growth and $\varphi(\Delta) \subset \Delta$, the induced morphism $\bar{\varphi} : \Gamma/\Delta \to \Gamma/\Delta$ also has linear growth.*

PROOF. Our statement is a consequence of the fact that if $S = \{x_1, \ldots, x_n\}$ is a finite symmetric generating set for Γ, its image in Γ/Δ becomes a finite symmetric generating set $\bar{S}$ after removing the $x_i \in \Delta$.

Take $\bar{g} \in \Gamma/\Delta$, there exists an antiimage $g \in \Gamma$ such that $\text{length}_{\bar{S}}(\bar{g}) \geq \alpha \text{length}_S(g)$, as every element in $\bar{S}$ has a lift in S. As φ has linear growth, there exists a fixed $\Lambda \in \mathbb{R}$ such that $\text{length}_S(\varphi^n(g)) \leq \Lambda n \text{length}_S(g)$. As the generators of S descend to $\bar{S}$ it holds that $\text{length}_{\bar{S}}(\bar{\varphi}^n(\bar{g})) \leq \text{length}_S(\varphi^n(g))$. Our statement follows from the concatenation of all these inequalities. □

The first application of Lemma 3.58 is the study of higher dimensional families when hyperplane sections are available.

PROPOSITION 3.59. *Let X be a complex manifold, $f : X \to \mathbb{D}$ a projective holomorphic mapping, and $0 \in \mathbb{D}$ be a critical value. The monodromy automorphism around 0 $\varphi \in Out(\pi_1(X_s, *))$ has linear growth.*

PROOF. Let $X \subset \mathbb{P}^N_{\mathbb{C}} \times \mathbb{D}$ be a projective embedding. By Bertini's lemma we may choose a $N-2$–dimensional linear subvariety $H \subset \mathbb{P}^N_{\mathbb{C}}$ such that $Y = X \cap (H \times \mathbb{D})$ is a smooth surface and $Y_s = X_s \cap (H \times \{s\}$ are smooth curves for regular values s near zero. The fiber $Y_0 = X_0 \cap H$ is a possibly singular curve, and we have a commutative diagram

$$\begin{array}{ccc} Y & \hookrightarrow & X \\ & \searrow \quad \swarrow & \\ & \mathbb{D} & \end{array}$$

By the Lefschetz hyperplane section theorem the regular fiber inclusions $Y_s \hookrightarrow X_s$ induce epimorphisms $\pi_1(Y_s, *) \to \pi_1(X_s, *)$.

Consider now the family of curves $f : Y \to \mathbb{P}^1_{\mathbb{C}}$. By Proposition 3.55, fixing any topological basepoint section σ as we have done in the previous sections the monodromy automorphism φ of the fundamental groups $\pi_1(Y_s, *)$ has linear growth.

The same topological section σ serves as basepoint section for X. By the relative Ehresmann theorem the triple $(X, Y, \text{Im}\,\sigma)$ is a $\mathcal{C}^\infty$–locally trivial fibration over $\mathbb{D}$, and we may select a parallel transport and geometric monodromy diffeomorphism preserving Y_s and $\sigma(s)$. Consequently, the monodromy automorphism φ of $\pi_1(Y_s, \sigma(s))$ preserves the kernel of the epimorphism $\pi_1(Y_s, \sigma(s)) \to \pi_1(X_s, \sigma(s))$, the monodromy in $\pi_1(X_s, \sigma(s))$ is the quotient of φ, and by Lemma 3.58 the latter also has linear growth. □

The second application of Lemma 3.58 is a linear growth version of Manning's theorem on entropy and eigenvalues of the homology mapping:

PROPOSITION 3.60. *Let Γ be a finitely generated group, and $\varphi : \Gamma \to \Gamma$ a linear growth automorphism. The induced automorphism $\varphi^* : H^1(\Gamma, \mathbb{Q}) \to H^1(\Gamma, \mathbb{Q})$ is quasi-unipotent.*

PROOF. It suffices to show quasi-unipotence in $H^1(\Gamma, \mathbb{R})$. If φ has linear growth, by Lemma 3.58 so has the induced automorphism φ_* of $(\Gamma/\Gamma_2)/_{\text{Torsion}}$. This quotient is a lattice in $H_1(\Gamma, \mathbb{R})$, and φ_* is actually the monodromy automorphism of the first homology group.

Let S be a symmetric generating set for Γ. It defines a word length metric on $\Gamma, \Gamma/\Gamma_2$. The image of the elements of S under the chain of morphisms $\Gamma \to \Gamma/\Gamma_2 \to H_1(\Gamma, \mathbb{R})$ contains a basis for the latter linear space. We may endow $H_1(\Gamma, \mathbb{R})$ with an euclidean metric by setting one such basis as an orthonormal basis, and it may be immediately checked that the lattice inclusion $(\Gamma/\Gamma_2)/_{\text{Torsion}} \hookrightarrow H_1(\Gamma, \mathbb{R})$ induces a quasi-isometry between both spaces. As φ_* has linear growth in $(\Gamma/\Gamma_2)/_{\text{Torsion}}$, the norm of the images $\varphi_*^n(e_i)$ grows linearly on n for the vectors e_i belonging to the basis of $H_1(\Gamma, \mathbb{R})$ induced by S, thus there is a bound

$$||\varphi_*^n(v)|| \leq \Lambda ||v|| n$$

with Λ fixed and determined by S in Γ, and valid for any vector $v \in H_1(\Gamma, \mathbb{R})$.

If the automorphism φ_* of $H_1(\Gamma, \mathbb{R})$ has a complex eigenvalue $z = a + bi = re^{i\theta}$, with $r > 1$, there exists either a real eigenvector v if $z \in \mathbb{R}$, such that $||\varphi_*^n(v)|| = r^n ||v||$, or an invariant bidimensional subspace in $H_1(\Gamma, \mathbb{R})$ with a basis v_1, v_2 such that the restriction to it of φ has matrix $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$. In the latter case it turns out that $||\varphi_*^n(v_1)|| = 2r^n \cos(n\theta) ||v_1||, ||\varphi_*^n(v_2)|| = 2r^n \sin(n\theta)$. As the eigenvalues of φ_* are algebraic the argument is $\theta = \frac{p}{q}\pi$, and the growth of these norms cannot be linear either.

We conclude from the previous discussion that linear growth automorphisms cannot have eigenvalues with a norm greater than one. As their inverse automorphisms also have linear growth, we conclude that all eigenvalues must have norm one, and our statement follows now from the algebraicity of those eigenvalues. □

Thus linear growth of the monodromy in the fundamental group provides an alternative proof of some quasi-unipotence results:

COROLLARY 3.61. (i) *Let $f : X \to \mathbb{D}$ be a holomorphic family of projective curves with a single critical value $0 \in \mathbb{D}$. The monodromy automorphisms of $H^*(Y_s, \mathbb{Q})$ are quasi-unipotent.*

(ii) *Let* $f : X \to \mathbb{D}$ *a projective family of complex manifolds such that* 0 *is an isolated critical value. The monodromy automorphism of* $H^1(X_s, \mathbb{Q})$ *around 0 is quasi-unipotent.*

Finally, let us remark that linear growth of a group automorphism is a more restrictive property than quasi-unipotence of the induced morphism in the first cohomology group. We provide an example to show this.

EXAMPLE 3.62. Consider the fundamental group of a genus 2 surface $\Gamma = \langle a_1, b_1, a_2, b_2 \mid (a_1, b_1)(a_2, b_2)\rangle$, and the automorphism

$$\begin{aligned} \varphi : \Gamma &\longrightarrow \Gamma \\ a_1 &\longmapsto a_1(b_1, a_1) = a_1 b_1^{-1} a_1^{-1} b_1 a_1 \\ b_1 &\longmapsto b_1 \\ a_2 &\longmapsto a_2 \\ b_2 &\longmapsto b_2 \end{aligned}$$

The automorphism φ is the identity modulo Γ_2, thus it induces the identity morphism in the first cohomology group. Yet a computation by induction shows that, with the set of generators S formed by the a_i, b_j and their inverses, $\text{length}_S(\varphi^n(a)) \geq 3^n$, thus φ does not have linear growth.

3.2. Topological formality. Compact Kähler manifolds and smooth algebraic varieties satisfy rigidity properties over its cohomology algebra and the first nilpotent quotients of the fundamental group, as has been explained in Chapter 2 ([**35**],[**70**]). Those rigidity properties, such as formality, do not carry verbatim from the absolute to the relative case; nevertheless, weaker versions still hold. The description of the monodromy in the fundamental group of Lefschetz pencils of curves in the previous sections may be applied to show that monodromy in the fundamental group of families of curves satisfies one such rigidity property: it was first established by M. Asada, M. Matsumoto and T. Oda in [**8**] in the case of stable families, and it roughly means that its order 3 nilpotent quotient determines monodromy in the fundamental group. This is a topological analogue of the pointed Torelli theorem by Hain and Pulte ([**77**]).

Let $f : X \to \mathbb{D}$ be a Lefschetz pencil of projective curves over a disk, such that 0 is the only critical value. As has been shown in Theorem 3.18, the geometric monodromy around 0 is the isotopy class of the homeomorphism $\tau_{d_1} \circ \cdots \circ \tau_{d_k}$, where $d_1, \ldots, d_k$ are the vanishing paths associated to the critical points of the singular fiber X_0, and

τ_{d_i} are the corresponding Dehn twists. The vanishing paths d_i are determined up to isotopy, and by the complex Morse lemma they may be assumed to be $\mathcal{C}^\infty$ simple loops in $X_s \cong C_g$. Those loops correspond to conjugation classes in $\pi_1(C_g, *)$, and their characterization was a fundamental achievement in the classification theory of surfaces (cf. [**90**]). Indeed, we are able to present a short, complete proof by using the classification theorem and Alexander duality:

PROPOSITION 3.63. *Let C_g be a $\mathcal{C}^\infty$ closed surface of genus g, and $d \subset C_g$ a $\mathcal{C}^\infty$ simple loop. Then either of the following statements holds:*

(i) *The loop d is contractible in C_g.*
(ii) *There exists a basepoint $p \in d \subset C_g$ and a presentation $\pi_1(C_g, p) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid (a_1, b_1) \dots (a_g, b_g) \rangle$ such that the homotopy class of d is the generator a_1.*
(iii) *There exists a basepoint $p \in d \subset C_g$ and a presentation $\pi_1(C_g, p) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid (a_1, b_1) \dots (a_g, b_g) \rangle$ such that the homotopy class of d is the bracket product $(a_1, b_1) \dots (a_k, b_k)$ for some $k < g$.*

PROOF. The vanishing path d is actually a smooth proper embedding $d : S^1 \hookrightarrow C_g$, although we will also denote by d its image. As d is smoothly embedded, the pair (C_g, d) is taut, and there is an Alexander duality isomorphism

$$H_0(C_g \setminus d; \mathbb{Z}) \cong H^2(C_g, d; \mathbb{Z})$$

The pair (C_g, d) has a cohomology exact sequence with coefficients in $\mathbb{Z}$

$$0 \to H^1(C_g, d) \to H^1(C_g) \xrightarrow{d^*} H^1(S^1) \to H^2(C_g, d) \to H^2(C_g) \to 0$$

The morphism $d^* : H^1(C_g) \cong \mathbb{Z}^{2g} \to H^1(S^1) \cong \mathbb{Z}$ is given by cup product with the Poincaré dual of d. As $H^2(C_g, d) \cong H_0(C_g \setminus d)$ is a free abelian group of rank at least one, there are only two possibilities for d^*:

(i) The morphism d^* is onto, and $C_g \setminus d$ is connected.
(ii) The morphism d^* is the zero morphism, and $C_g \setminus d$ has two connected components.

In the first case, cutting C_g along d we obtain a compact surface S with a boundary consisting of two connected components. By the classification theory of surfaces with boundary, its fundamental group with basepoint $p = d(0)$ on the boundary admits a presentation $\pi_1(S, p) \cong \langle d, a_2, \dots, a_g, b_2, \dots, b_g, d' \mid (a_2, b_2) \dots (a_g, b_g) = dd' \rangle$, where d is the edge of the cut where we have placed our basepoint and d' is a loop formed by a path c joining the two copies of $d(0)$ on the boundary, followed by the loop around the other boundary component and finally c^{-1}.

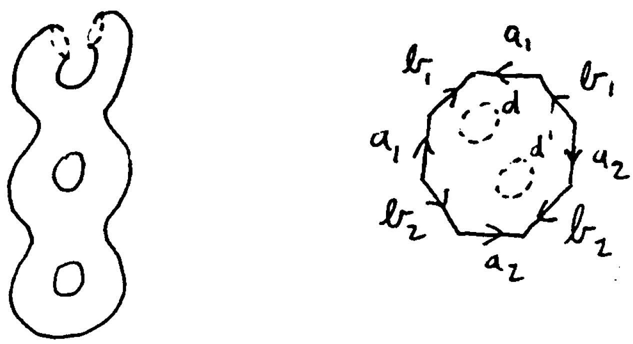

Fig. F3.28 Connected surface with 2 boundary components.

Identification of the two connected components of the boundary of S along d yields now the presentation of $\pi_1(C_g, *)$ stated in the second case of the proposition.

If d^* is the zero morphism, then $C_g \setminus d$ has two connected components. These are surfaces S_1, S_2 such that their boundary has one connected component, and the loop d is a parametrization of it for either surface. Again the classification theory of surfaces shows that selecting as a basepoint $d(0)$ in both cases there exist presentations $\pi_1(S_1, *) = \langle a_1, \ldots, a_k, b_1, \ldots, b_k, d \mid (a_1, b_1) \ldots (a_k, b_k) = d \rangle$ and $\pi_1(S_2, *) = \langle a_{k+1}, \ldots, a_g, b_{k+1}, \ldots, b_g, d \mid (a_{k+1}, b_{k+1}) \ldots (a_g, b_g) = d^{-1} \rangle$, with $g - k \leq k \leq g$.

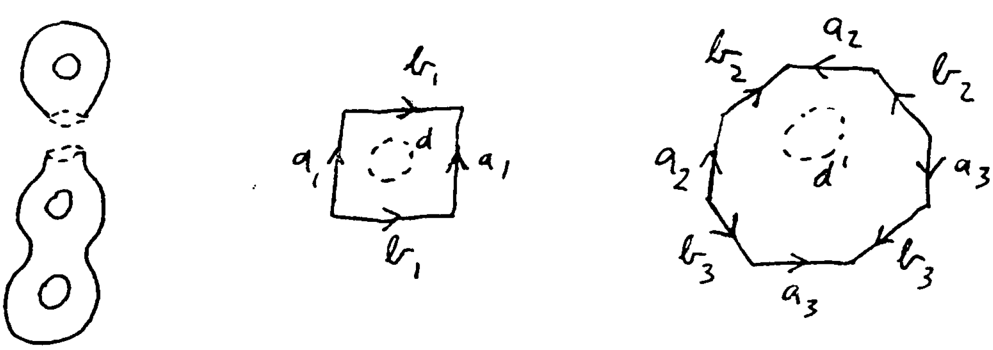

Fig. F3.29 Two surfaces with connected boundary.

An immediate application of the Seifert-Van Kampen theorem shows that either the first or the third option in our proposition holds, depending on whether $k = g$ or $k < g$. □

Proposition 3.63 characterizes the homotopy class of every vanishing path *in a presentation of the fundamental group depending on the path itself.* If one fixes a presentation $\pi_1(C_g,*) \cong \langle a_1,\dots,b_g \mid (a_1,b_1)\dots(a_g,b_g)\rangle$ there does not exist such a simple classification of the conjugation classes of simple loops on C_g. Nevertheless, the classification of Proposition 3.63 may be extended by the same methods to finite sets of disjoint simple loops. Such sets appear as sets of vanishing paths for the monodromy of a pencil of curves around a critical value with several critical points over it.

PROPOSITION 3.64. *Let C_g be a C^∞ closed orientable surface of genus g, and $d_1,\dots,d_n \subset C_g$ be pairwise disjoint simple loops. There exists then a base point and presentation $\pi_1(C_g,*) \cong \langle a_1,\dots,a_g,b_1,\dots,b_g \mid (a_1,b_1)\dots(a_g,b_g)\rangle$ such that the conjugation class of every loop d_i is either of the following:*

(i) *Trivial.*
(ii) *The class of $\prod_{j\in J}(a_j,b_j)^{\varepsilon_j}$ for some $J\subset\{1,\dots,g\}$ and $\varepsilon_j=\pm1$.*
(iii) *The class of $\prod_{i\in I}a_i^{\rho_i}\prod_{j\in J}(a_j,b_j)^{\varepsilon_j}$ for some disjoint subsets $I,J\subset\{1,\dots,g\}$ and $\rho_i,\varepsilon_j=\pm1$.*

PROOF. Let us examine first the homology classes of the loops. As $d_1,\dots,d_n$ are pairwise disjoint, the subspace $V\subset H_1(C_g,\mathbb{Z})\cong\mathbb{Z}^{2g}$ spanned by their homology classes is isotropic. Therefore it has rank $0\le k\le g$.

Pick a set of k loops, which we may assume to be $d_1,\dots,d_k$, such that their homology classes form a basis for $V\otimes\mathbb{Q}$. The cohomology exact sequence for the pair $(C_g,d_1\cup\dots\cup d_k)$ with rational coefficients is

$$0\longrightarrow H^0(C_g)\longrightarrow H^0(d_1\cup\dots\cup d_k)\longrightarrow H^1(C_g,d_1\cup\dots\cup d_k)\longrightarrow H^1(C_g)$$
$$\longrightarrow H^1(d_1\cup\dots\cup d_k)\longrightarrow H^2(C_g,d_1\cup\dots d_k)\longrightarrow H^2(C_g)\longrightarrow 0\,.$$

The morphism $H^1(C_g)\to H^1(d_1\cup\dots\cup d_k)$ is onto because its dual is one–to–one. Therefore $H^2(C_g,d_1\cup\dots\cup d_k)\cong\mathbb{Z}$. By Alexander duality $H^2(C_g,d_1\cup\dots\cup d_k)\cong H^0(C_g\setminus(d_1\cup\dots\cup d_k))$. Therefore the complementary surface $C_g\setminus(d_1\cup\dots\cup d_k)$ is connected.

Consider now the surface obtained by removing all the loops. The cohomology exact sequence of the pair with rational coefficients is

$$0\longrightarrow H^0(C_g)\longrightarrow H^0(d_1\cup\dots\cup d_n)\longrightarrow H^1(C_g,d_1\cup\dots\cup d_n)\longrightarrow H^1(C_g)$$
$$\longrightarrow H^1(d_1\cup\dots\cup d_n)\longrightarrow H^2(C_g,d_1\cup\dots d_n)\longrightarrow H^2(C_g)\longrightarrow 0\,.$$

As the homology classes of $d_1,\dots,d_k$ span all the image of the morphism $H_1(d_1\cup\dots\cup d_k,\mathbb{Q})\to H_1(C_g,\mathbb{Q})$, its dual in the exact sequence has an image of dimension k and a kernel of dimension $2g-k$. The morphism $H^0(C_g) \to H^0(d_1 \cup \cdots \cup d_n) \cong \mathbb{Q}^n$ is one–to–one, thus computing the ranks and applying Alexander duality we have that

$$H^0(C_g \setminus (d_1 \cup \cdots \cup d_n)) \cong H^2(C_g, d_1 \cup \cdots \cup d_n) \cong \mathbb{Q}^{n-k+1}$$
$$H^1(C_g \setminus (d_1 \cup \cdots \cup d_n)) \cong H^1(C_g, d_1 \cup \cdots \cup d_n) \cong \mathbb{Q}^{2g-k+n-1}$$

Thus cutting the surface C_g along all the loops $d_1, \ldots, d_n$ produces $n-k+1$ connected surfaces $S_1, \ldots, S_{n-k+1}$. If the surface S_i has genus g_i and a boundary with n_i connected components, its first homology group is $H_1(S_i) \cong \mathbb{Q}^{2g_i+n_i-1}$. The surfaces $S_1, \ldots S_{n-k+1}$ have in total $2n$ circumferences as boundaries, thus the sum of their genuses is

$$g_1 + \cdots + g_{n-k+1} = g - k\,,$$

i.e., every cut that is nonhomologically trivial lowers the genus by one unit, while new cuts that are homologous to a combination of previus ones split the surface but preserve the total sum of genuses.

By the classification theory of surfaces, there exist simple closed loops $a_{k+1}, \ldots, a_g, b_{k+1}, \ldots, b_g$ such that every pair a_j, b_j is contained in $S_i \setminus \partial S_i$ for some i, and for every connected surface S_i its fundamental group is freely generated by the loops $a_j, b_j \subset S_i \setminus \partial S_i$ and $d_{i_1}, \ldots, d_{i_l} \subset \partial S_i$, with the single defining relation $\prod_{i_h} d_{i_h} \cdot \prod_{a_j \in S_i}(a_j, b_j) = 1$.

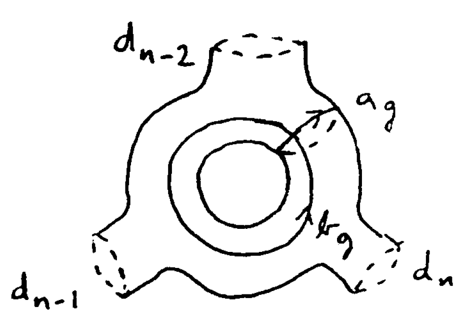

Fig. F3.30 A connected component of of the complement of a cut system.

The cohomology exact sequence and Alexander duality show as before that the g loops $d_1, \ldots, d_k, a_{k+1}, \ldots, a_g$ form a complete cut system for C_g. By surface classification theory we may choose a basepoint and loops $b_1, \ldots, b_k$ such that $a_1 = d_1, \ldots, a_k = d_k, a_{k+1}, \ldots a_g, b_1, \ldots, b_g$ generate $\pi_1(C_g, *)$ with the single defining relation $(a_1, b_1) \ldots (a_g, b_g) =$

1. The loops $a_{k+1},\dots,a_g,b_{k+1},\dots,b_g$ do not intersect any of the original loops $d_1,\dots,d_n$.

With the above presentation of the fundamental group of C_g, which still allows any basepoint in $C_g\setminus(d_1\cup\cdots\cup d_n)$, the conjugation classes of $d_1,\dots,d_k$ are $a_1,\dots,a_k$ respectively. We shall investigate the rest of loops. For every d_j with $k+1\le j\le n$, our standard argument of the cohomology exact sequence plus Alexander duality shows that if we cut C_g along $d_1,\dots,d_k,d_j$ we obtain two connected surfaces S,S'. As the cut along $d_1,\dots,d_k$ produces only one connected component, each of S,S' has a copy of the extra loop d_j in its boundary. Fix a base point $p\in S$, and a presentation of $\pi_1(C_g,p)$ by linking in $C_g\setminus(d_1\cup\dots d_n)$ the base point to all the loops $a_1,\dots,b_g$ previously determined. The fundamental group of S is generated by the loops forming its boundary and the paths $a_j,b_j\subset S\setminus\partial S$. The paths $b_1,\dots,b_k$ have been cut open to produce S, so if $a_j,b_j\subset S\setminus\partial S$ then $j>k$. The boundary loops are d_j and up to two copies of some of the loops $d_1,\dots,d_k$. If a loop d_i with $i<k$ appears twice, it does so because the two sides of the cut lie in the same connected component, and an orientation of d_i in C_g produces opposite orientations for the two bounding loops in ∂S. The single relation in this presentation of $\pi_1(S,p)$ is

$$d_j\prod_{d_i\subset\partial S} d_i^{\varepsilon_i}\cdot\prod_{a_j,b_j\in S\setminus\partial S}(a_j,b_j)^{\rho_j}$$

with $\varepsilon_i,\rho_j=\pm1$. The ordering of the factors $d_i\in\partial S$ may be selected so that if the two copies of the same d_i with opposite orientations appear, they appear together and cancel out. As $d_1,\dots,d_k$ are $a_1,\dots,a_k$ in this presentation our statement is proved. □

By Proposition 3.63 the nontrivial vanishing paths do not correspond to arbitrary conjugation classes in the fundamental group, but to generators of the abelian quotient $H_1(C_g,\mathbb{Z})$ or of the second quotient of the lower central series $\pi_1(C_g,*)_2/\pi_1(C_g,*)_3$. This fact, arising from the classification of surfaces, makes the geometric monodromy of families of curves rigid over the monodromy in the lower central series quotients. The first such rigidity result is:

PROPOSITION 3.65. *Let $f:X\to\mathbb{D}$ be a Lefschetz pencil of projective curves over the disk $\mathbb{D}$, such that $0\in\mathbb{D}$ is the only critical value. Let $\rho\in Out(\pi_1(X_z,*)),\tilde\rho\in Out(\pi_1(X_z,*)/\pi_1(X_z,*)_4)$ be the corresponding monodromy automorphisms modulo conjugation. Then $\tilde\rho=Id$ if and only if $\rho=Id$.*

PROOF. We will assume that the fibers have genus $g>0$, otherwise our statement is empty. Let X_z be a regular fiber of the pencil, and

$d_1, \dots, d_n \subset X_z$ a set of vanishing paths corresponding to the critical points of f. Fix a pesentation $\pi_1(X_z, *) \cong \langle a_1, \dots, a_g, b_1, \dots, b_g \mid (a_1, b_1) \dots (a_g, b_g) = 1 \rangle$ adapted to the set of vanishing paths, as in Proposition 3.64. The coefficients of the homology classes of the vanishing paths $[d_i] = \alpha_{i1}[a_1] + \dots \alpha_{ig}[a_g] + \beta_{i1}[b_1] + \dots + \beta_{ig}[b_g]$ in the basis of $H_1(X_z, \mathbb{Z})$ induced by the selected presentation of $\pi_1(X_z)$ form a matrix

$$(A \mid B) = \begin{pmatrix} \alpha_{11} & \dots & \alpha_{1g} & \beta_{11} & \dots & \beta_{1g} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \alpha_{n1} & \dots & \alpha_{ng} & \beta_{n1} & \dots & \beta_{ng} \end{pmatrix}$$

According to the Picard–Lefschetz formula (see [**7**] vol. 2), the monodromy around 0 in the first homology group is

$$\rho_*(g) = g + (g \cdot d_1)d_1 + \dots (g \cdot d_n)d_n \,,$$

where the products $g \cdot d_i$ are the intersection products in homology. Therefore, if we decompose the monodromy in $H_1(X_z, \mathbb{Z})$ as $\rho_* = \mathrm{Id} + \mathrm{Var}$, the variation morphism Var has matrix

$$\begin{pmatrix} B^t \\ A^t \end{pmatrix} \begin{pmatrix} A & B \end{pmatrix} = \begin{pmatrix} B^t A & B^t B \\ A^t A & A^t B \end{pmatrix}$$

in the basis $[a_i], [b_j]$. Thus $\rho_* = \mathrm{Id}$ if and only if all the vanishing paths are homologically trivial. Hence if there exists any homologically nontrivial vanishing path, the monodromy in $\mathrm{GL}\,(H_1(X_z, \mathbb{Z})) \cong \mathrm{Out}\,(\pi_1(X_z)/\pi_1(X_z)_2)$, thus also in $\mathrm{Out}\,(\pi_1(X_z)/\pi_1(X_z)_4)$ is not the identity.

Assume now that all vanishing paths are homologically trivial, and some of them are homotopically nontrivial. The cohomology exact sequence of the pair $(X_z, d_1 \cup \dots \cup d_n)$ plus Alexander duality show, as in Propositions 3.63, 3.64, that the complement $X_z \setminus (d_1 \cup \dots \cup d_n)$ has $n+1$ connected components. At least two of these components must have positive genus, or else all the vanishing paths would be nullhomotopous, and the total sum of genuses is g. Since a change of basepoint does not vary the monodromy outer automorphism, we may assume that the base point p lies in a component S_1 of $X_z \setminus (d_1 \cup \dots \cup d_n)$ with positive genus. The monodromy is the identity on the loops in this component. Choose now a path γ in X_z from the base point p to another conneced component of $X_z \setminus (d_1 \cup \dots \cup d_n)$ of positive genus, such that γ is transversal to all vanishing paths and intersects each of them at most once. Let γ' be a path along γ that stops at the first connected component S' of $X_z \setminus (d_1 \cup \dots \cup d_n)$ encountered. Denote $d_1, \dots, d_l$ the vanishing paths intersected by γ'. These vanishing paths are homotopic, or else a shorter path γ' would reach a positive genus component. By Proposition 3.64 their conjugacy class is $\prod_{j\in J}(a_j,b_j)^{\epsilon_j}$ for some $J \subset \{1,\ldots,g\}$. Given our choice of basepoint p, the action of the monodromy automorphism on the loops in S' joined to the basepoint p through γ', γ'^{-1} is conjugation by $\left(\prod_{j\in J}(a_j,b_j)^{\epsilon_j}\right)^l$.

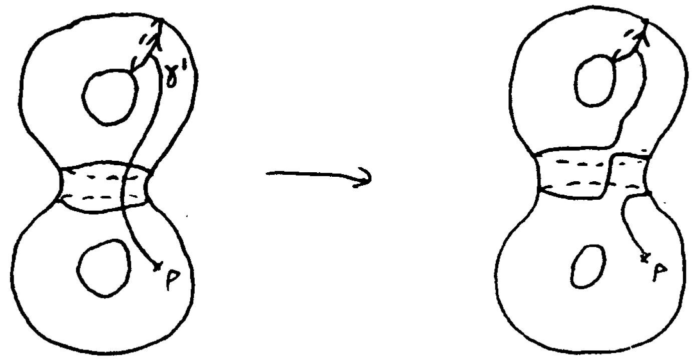

Fig. F3.31 Monodromy around a second type vanishing path.

Thus the monodromy automorphism with the given base point p is the identity on the loops of S_1 and conjugation by an element of $\pi_1(X_z,p)_2 \setminus \pi_1(X_z,p)_3$. Such an automorphism is trivial modulo $\pi_1(X_z,p)_3$, but it is neither trivial nor inner modulo $\pi_1(X_z,p)_4$. □

Proposition 3.65 may be extended to a more general setting with the help of the semistable reduction theorem and Nielsen realization:

THEOREM 3.66. *Let $f : X \to \mathbb{D}$ be a holomorphic family of projective curves over the disk $\mathbb{D}$, such that $X_0 = f^{-1}(0)$ is the only singular fiber. Let $\rho \in Out(\pi_1(X_z,*)), \rho_4 \in Out(\pi_1(X_z,*)/\pi_1(X_z,*)_4)$ be the corresponding monodromy automorphisms modulo conjugation. Then $\rho_4 = Id$ if and only if $\rho = Id$.*

PROOF. The difference between the case of a Lefschetz pencil, settled in Proposition 3.65, and an arbitrary family $f : X \to \mathbb{D}$ is that in the latter case the singular fiber may have non–quadratic singularities and a multiplicity greater than one.

By the semistable reduction theorem (Thm. 1.1 in [**9**]), there exists a Lefschetz pencil of curves $\tilde{X} \to \mathbb{D}$ obtained by pulling back the family $X \to \mathbb{D}$ along the map $z \mapsto z^n$ of $\mathbb{D}$ for an adequate integer n, and afterwards blowing up the singular fiber a finite number of times. The morphism $z \mapsto z^n$ induces multiplication by n in the fundamental group $\pi_1(\mathbb{D}^*,*) \cong \mathbb{Z}$, and the blow ups on the singular fiber do not alter the family over $\mathbb{D}^*$. Therefore, the monodromy automorphism in the fundamental group of the fibers of $\tilde{X} \to \mathbb{D}$ is the power ρ^n of the monodromy automorphism of the original family $f : X \to \mathbb{D}$. Likewise, the induced automorphism in $\pi_1(\tilde{X}_z, *)/\pi_1(\tilde{X}_z, *)_4$ is ρ_4^n.

Let us show now the nontrivial implication in our statement. If $\rho_4 = \mathrm{Id}$, then $\rho_4^n = \mathrm{Id}$. As the family $\tilde{X} \to \mathbb{D}$ is a Lefschetz pencil of projective curves, by Proposition 3.65 $\rho^n = \mathrm{Id}$. This means that the monodromy automorphism ρ of the family $f : X \to \mathbb{D}$ generates a finite cyclic subgroup $\{\mathrm{Id}, \rho, \ldots, \rho^{n-1}\}$ of the mapping class group $M(g, 0)$ of the smooth fiber $X_s \cong C_g$.

Nielsen showed that any finite cyclic subgroup of the mapping class group $M(g, 0)$ is induced by a finite cyclic subgroup of the group of homeomorphisms of the topological surface C_g. This result started a deep study of the problem of realizing finite subgroups of $M(g, 0)$ by finite groups of homeomorphisms, which culminated in the following theorem by S. Kerckhoff:

THEOREM 3.67 ([**59**]). *Every finite subgroup G of $M(g, 0)$ can be realized as a group of isometries of a hyperbolic surface.*

By Kerckhoff's theorem there exists a hyperbolic structure, i.e. a metric with constant Gaussian curvature -1, on the regular fiber $X_z \cong C_g$, and an isometry h of this metric surface, such that h induces the automorphism ρ of the fundamental group. The hyperbolic structure is equivalent to a holomorphic structure on C_g, such that h is a holomorphic automorphism with this structure. But any such holomorphic automorphism inducing the identity morphism in homology must be the identity itself (see for instance Thm. 2.2.1 in [**96**]). Therefore $h = \mathrm{Id}$, and $\rho = \pi_1(h) = \mathrm{Id}$, which completes our proof. □

Proposition 3.65 and Theorem 3.66 parallel the results of [**8**], and extend the rigidity property from stable to arbitrary holomorphic families of curves. In that paper Asada, Matsumoto and Oda study the versal deformation of a n-pointed stable curve, which is a Lefschetz pencil of curves over a polydisk $\mathbb{D}^n$. This study is performed by combinatoric and group-theoretic means, but its essentials translate to our more geometric setting: the bridges in the curve graph correspond to vanishing paths of the form $\prod(a_j, b_j)^{\epsilon_j}$, and maximal cut systems to sets of vanishing paths $\{d_m, \ldots, d_n\}$ such that they are homologically nontrivial but yield the same homology class. The weight filtration of [**8**] is the lower central series filtration in the case of projective curves, and our methods allow us to retrieve the formulae in their Theorem 1.1 on the induced filtration in the monodromy group.

4. Fundamental groups of Lefschetz pencils

The study in the previous sections of the monodromy in the fundamental group of families of projective curves may be applied to compute the fundamental group of the source manifold. Here a major contrast appears: while semistable families of projective manifolds of dimension $d \geq 2$ have trivial monodromy in the π_1, for families of curves one has the following situation (see [**4**]):

PROPOSITION 3.68. *Every smooth projective surface X admits a blow-up in a finite number of points $\varepsilon : \tilde{X} \to X$ such that there exists a Lefschetz pencil of curves $f : \tilde{X} \to \mathbb{P}^1_{\mathbb{C}}$.*

In terms of fundamental groups, this means:

COROLLARY 3.69. *Let Γ be the fundamental group of a smooth projective manifold. Then $\Gamma \cong \pi_1(X, *)$ for some Lefschetz pencil of projective curves $f : X \to \mathbb{P}^1_{\mathbb{C}}$.*

PROOF. By the Lefschetz hyperplane section theorem, every projective manifold has the same fundamental group as some smooth projective surface. The blow-up of points in a smooth surface does not change the fundamental group. Therefore our statement is an immediate consequence of Proposition 3.68. □

Corollary 3.69 provides us with a motivation to study the fundamental group of a Lefschetz pencil of projective curves. With a view towards Donaldson theory and its search of elementary building blocks for smooth 4–folds, we will carry out this study in a slightly more general context.

DEFINITION 3.70. A proper smooth map $f : X \to \mathbb{P}^1_{\mathbb{C}}$ with X a closed oriented 4–manifold is a *smooth Lefschetz pencil of curves* if it has only a finite number of critical points $p_1, \ldots, p_n$, all of them are nondegenerate, and for every critical point p_i there are $\mathcal{C}^\infty$ coordinate charts of p_i in $\mathbb{C}^2$ and of $f(p_i)$ in $\mathbb{C}$ such that in the new coordinates f has the form $f(z_1, z_2) = z_1^2 + z_2^2$.

Smooth Lefschetz pencils of curves are thus an analogue of holomorphic Lefschetz pencils, and they still have the monodromic properties of the latter. Namely, Theorems 3.18, 3.34, Proposition 3.55 and Proposition 3.65 hold for smooth Lefschetz pencils of curves, because their proof only uses the $\mathcal{C}^\infty$ structure of the fibration and singularities, given by Lemma 3.70.

Our following goal will be to describe the fundamental group of the total space X in a smooth Lefschetz pencil of curves $f : X \to \mathbb{P}^1_{\mathbb{C}}$ in terms of the monodromy of the family. This may be done piecewise, by examining the smooth part of the fibration first, neighbourhoods of the singular fibers then, and glueing all the pieces applying the Seifert–Van Kampen theorem.

LEMMA 3.71 (Le Dung Trang). *Let $f : X \to \mathbb{P}^1_{\mathbb{C}}$ be a smooth Lefschetz pencil of curves of genus g, $S = \mathbb{P}^1_{\mathbb{C}} \setminus \{z_1, \dots, z_n\}$ its set of regular values and $U = f^{-1}(S)$ the open subset formed by the regular fibers. Assume that there exists a C^∞ basepoint section $\sigma : S \to U$, and select a basepoint $s_0 \in S$. Then the fundamental group $\pi_1(U, \sigma(s_0))$ admits the following presentation:*

$$\begin{aligned}\langle a_1, \dots, a_g, b_1, \dots, b_g, T_1, \dots, T_n \mid (a_1, b_1) \cdots (a_g, b_g) = 1, T_1 \cdots T_n = 1,\\ T_i a_j T_i^{-1} = \varphi_i(a_j), T_i b_j T_i^{-1} = \varphi_i(b_j) \quad 1 \le i \le n,\ 1 \le j \le g\rangle,\end{aligned}$$

where $\langle a_1, \dots, b_g \mid (a_1, b_1) \cdots (a_g, b_g)\rangle$ is a presentation of $\pi_1(X_{s_0}, \sigma(s_0))$ and $\varphi_1, \dots, \varphi_n$ are the monodromy automorphisms of this presentation around the critical values $z_1, \dots, z_n$.

PROOF. As the restriction $f : U \to S$ is a locally trivial fibration, and $\sigma : S \to U$ a section, there exists a semidirect product presentation

$$\pi_1(U, \sigma(s_0)) = \pi_1(X_{s_0}, \sigma(s_0)) \rtimes \pi_1(S, s_0)$$

with the action of $\pi_1(S, *)$ on $\pi_1(X_{s_0}, *)$ given by the basepoint–preserving parallel transport. The fundamental group $\pi_1(S, s_0)$ admits a presentation $\langle T_1, \dots, T_n \mid T_1 \cdots T_n = 1\rangle$ with the generators T_i corresponding to loops around the critical values z_i. The parallel transport around z_i induces the corresponding monodromy automorphism, as described in Theorem 3.34, and sends a_j, b_j to $\varphi_i(a_j), \varphi_i(b_j)$ respectively. □

As has been outlined in Theorems 3.18, 3.34, the vanishing paths in a singular fiber determine the monodromy around it. The same holds, in a direct way, for the fundamental group of the singular fiber.

LEMMA 3.72. *Let $z_i \in \mathbb{P}^1_{\mathbb{C}}$ be a critical value of a smooth Lefschetz pencil of curves, $\mathbb{D}_i$ a disk centered on z_i containing no other critical value, $U_i = f^{-1}(\mathbb{D}_i)$, Fix a basepoint $z \in \mathbb{D}_i^*$, and let $d_1, \dots, d_k$ be the vanishing paths in X_z associated to the critical points over z_i. There is an isomorphism*

$$\pi_1(U_i, *) \cong \pi_1(X_{z_i}, *)/\langle d_1, \dots, d_k\rangle$$

where we also denote by d_j the conjugation classes of the vanishing paths in X_z.

PROOF. The contraction of the disk $\mathbb{D}_i$ to its center may be lifted to a deformation retraction of the open set $U_i \subset X$ onto the singular fiber X_{z_i}, therefore $\pi_1(U_i, *) \cong \pi_1(X_{z_i}, *)$. The singular fiber X_{z_i} is homotopy equivalent to the smooth fiber X_z with disks attached along every vanishing path, thus our statement. □

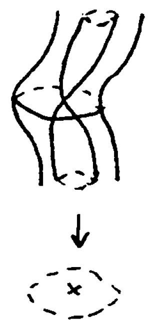

Fig. F3.32 Singular fiber vs regular fiber.

All that remains now is to assemble the different pieces of the Lefschetz pencil.

THEOREM 3.73. *Let $f : X \to S$ be a smooth Lefschetz pencil of curves of genus g over $S = \mathbb{C}$ or $S = \mathbb{P}^1_{\mathbb{C}}$, $z \in \mathbb{P}^1_{\mathbb{C}}$ a regular value, and $d_1, \ldots, d_n$ the vanishing paths of all critical points of f transported to X_z. The fundamental group of X admits a presentation*

$$\langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid (a_1, b_1) \cdots (a_g, b_g), d_1, \ldots, d_n \rangle$$

where $d_1, \ldots, d_n$ denote the conjugation classes of the vanishing paths in $\pi_1(X_z, *) \cong \langle a_1, \ldots, a_g, b_1, \ldots, b_g \mid (a_1, b_1) \cdots (a_g, b_g) \rangle$.

PROOF. As has been discussed previously to Theorem 3.34, Lefschetz pencils of curves always admit C^∞ basepoint sections, hence we may choose one such section $\sigma : S \to X$, a set of loops $T_1, \ldots, T_m$ around the critical values, and denote by φ_i the monodromy in the fundamental group along every loop T_i.

The departing point of the proof is Lemma 3.71, which computes the fundamental group of the open set $U \subset X$ formed by the regular fibers of f. Then we must adjoin neighbourhoods U_i of the singular fibers one by one, using the Seifert–Van Kampen theorem. Let us complete the first step in the process:

By Lemma 3.71,

$$\pi_1(U,*) \cong \langle a_1,\dots,b_g,T_1,\dots,T_m \mid (a_1,b_1)\cdots(a_g,b_g),T_1\cdots T_m,\\ T_i a_j T_i^{-1} = \varphi_i(a_j), T_i b_j T_i^{-1} = \varphi_i(b_j)\rangle\,.$$

By Lemma 3.72, $\pi_1(U_i,*) \cong \langle a_1,\dots,b_g \mid (a_1,b_1)\cdots(a_g,b_g),d_1,\dots,d_k\rangle$, where $d_1,\dots,d_k$ correspond to the vanishing paths of the critical points over the critical value z_i.

Lemma 3.71 actually holds for Lefschetz pencils of curves with basis any domain in $\mathbb{P}^1_{\mathbb{C}}$. Therefore, using the same basepoint section σ, the fundamental group of the intersection $U \cap U_i$ admits a presentation

$$\pi_1(U\cap U_i,*) \cong \langle a_1,\dots,b_g,T \mid (a_1,b_1)\cdots(a_g,b_g), T a_j T^{-1} = \varphi_i(a_j), T b_j T^{-1} = \varphi_i(b_j)\rangle\,.$$

We must examine now the fibered product of $\pi_1(U_i,*)$ and $\pi_1(U,*)$ over $\pi_1(U\cap U_i,*)$. The generators $a_1,\dots,b_g$ correspond to generators of the fundamental group of a smooth fiber X_z containing the base point, and may therefore be identified in the three groups. The generator $T \in \pi_1(U\cap U_i,*)$ is a lift of the loop around the critical value z_i and maps to T_i in $\pi_1(U,*)$. On the other hand it maps to the trivial loop in $\pi_1(U_i,*)$. We can thus conclude that the group $\pi_1(U\cup U_i,*)$ admits a presentation

$$\langle a_1,\dots,a_g,b_1,\dots,b_g \mid (a_1,b_1)\cdots(a_g,b_g), a_j = \varphi_i(a_j), b_j = \varphi_i(b_j), d_1,\dots,d_k\rangle$$

As has been seen in Theorem 3.34, the relations $\varphi_i(a_j)a_j^{-1}, \varphi_i(b_j)b_j^{-1}$ are actually products of elements in the conjugation classes of $d_1,\dots,d_k$. Hence they are superfluous for the presentation of the group.

The proof of our statement may be completed in this manner by induction on the number of critical values. For the final glueing, and in order to avoid basepoint inconveniences, one may take $U_i = f^{-1}(D_i)$, where the D_i are domains containing each exactly one critical value z_i, and intersecting all in a common disk, in such a way that $\mathbb{C}$ retracts over $\cup U_i$.

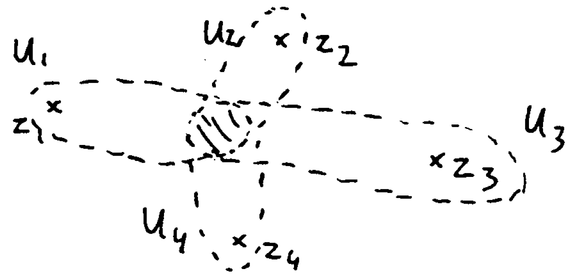

Fig F3.33 Neighbourhoods of critical values.

To complete the proof in the case of $S = \mathbb{P}^1_{\mathbb{C}}$, choose as a point at infinity a regular value of the pencil. The pencil will then be trivial around ∞, so glueing the pencil over a disk $\mathbb{D}_\infty$ around the infinity point to the pencil over $\mathbb{C}$, we have a trivializing diffeomorphism $f^{-1}(\mathbb{D}_\infty) \cong C_g \times \mathbb{D}_{infty}$ defined already over $\mathbb{P}^1_{\mathbb{C}}$. The intersection $f^{-1}(\mathbb{D}_\infty) \cap f^{-1}(\mathbb{C})$ admits as a deformation retract a trivial family diffeomorphic to $C_g \times S^1$. Therefore, by Seifert–Van Kampen's theorem, glueing $f^{-1}(\mathbb{D}_\infty)$ does not add any new relation, and our statement holds. □

Theorem 3.73 gives a monodromic presentation for fundamental groups of smooth Lefschetz pencils of curves, in particular for fundamental groups of projective manifolds. They turn out to be quotients of the fundamental group of the fibers, and the new defining relations given by the vanishing paths are not arbitrary, but only those listed in Proposition 3.63: for every vanishing path d, there exists a presentation $\pi_1(X_z, *) = \langle a_1, \dots, b_g \mid (a_1, b_1) \cdots (a_g, b_g) \rangle$ such that $d = a_1$ or $d = (a_1, b_1) \cdots (a_k, b_k)$. This might seem at first sight related to the quadratic presentation results of [**70**],[**47**] for the Malcev algebra of Kähler groups, but the fact that the required presentation of $\pi_1(X_z, *)$ is specific to every critical value of the pencil allows the class of Lefschetz pencil groups to be larger. A first result in this direction is

PROPOSITION 3.74. *Every finitely presented group is the fundamental group of a smooth Lefschetz pencil of curves over* $\mathbb{C}$.

PROOF. Let $\Gamma = \langle x_1, \dots, x_n \mid r_1, \dots, r_s \rangle$ be a finite presentation of a group. We seek another presentation of Γ that complies with Theorem 3.73 and Proposition 3.63.

The new presentation will have the following generators: First, $x_1, \dots, x_n$ as the given presentation of Γ. Second, for every relation

r_j written in reduced form $r_j = x_{i_1}^{e_1} \dots x_{i_k}^{e_k}$ we will add as generators $y_{ji_1}, \dots, y_{ji_k}$. Lastly, the set of generators is completed by appending a generator c_i for every x_i and a c_{ji_l} for every y_{ji_l}, thus doubling the number of generators. The set of relations will be the following:

(i) For every generator y_{ji_l}, a relation $y_{ji_l} x_{i_l}^{-e_l}$.
(ii) For every relation $r_j = x_{i_1}^{e_1} \dots x_{i_k}^{e_k}$ in the original presentation, a relation $y_{ji_1} \cdots y_{ji_k}$.
(iii) Every generator c_i, c_{ji_l} appears as a relation.
(iv) The symplectic relation $(x_1, c_1) \cdots (x_n, c_n)(y_{1i_1}, c_{1i_1}) \cdots (y_{si_k}, c_{si_k})$.

The last relation is the symplectic relation of a curve group, and the other relations belong to the second type in Proposition 3.63. It is not hard to check that the morphism from Γ to this new presentation identifying the x_i generators in both presentations is well defined and an isomorphism.

Let now C_g be a curve of genus half the number of generators in the second presentation of Γ. The conjugacy class of any defining relation $y_{ji_l} x_{i_l}^{-el}$, $y_{ji_1} \dots y_{ji_k}$, c_i, c_{ji_l} is realized in $\pi_1(C_g, *)$ by a simple closed loop d in C_g, of second type in Proposition 3.63. We may assume this loop to be $\mathcal{C}^\infty$.

Let $U \subset C_g$ be a bicollar neighbourhood of the loop d, and $Y = C_g \backslash U$. We will piecewise define a Lefschetz pencil over the disk $\mathbb{D}$ with a single critical point and d as its vanishing path. The trivial component is $Y \times \mathbb{D}$. This may be $\mathcal{C}^\infty$ glued over $\mathbb{D}$ along its boundary with a small neighbourhood of $(0, 0)$, fibered over $\mathbb{D}$ by the map $(x, y) \mapsto x^2 + y^2$.

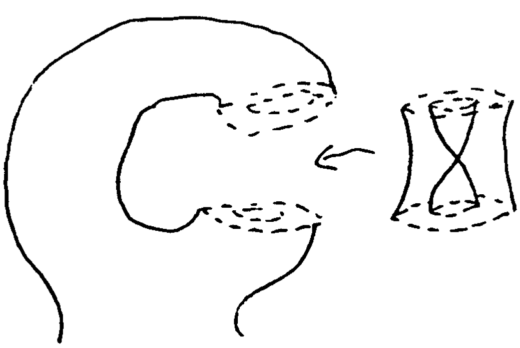

Fig.F3.34 Glueing process.

The result is a smooth Lefschetz pencil with a single critical value and vanishing path d. In this way we may obtain a Lefschetz pencil over a disk $\mathbb{D}_j \subset \mathbb{C}$ for every nonsymplectic relation in the second presentation of Γ. The disks $\mathbb{D}_j$ may be chosen to be pairwise disjoint.

Then we may join all those disks to a basepoint $p \in \mathbb{C}$ by simple nonintersecting paths γ_j. A trivial family of curves on the paths γ_j may be glued to the families on the disks, and as the set formed by the disks and the path system is a deformation retract of $\mathbb{C}$, taking pullbacks we obtain a family over $\mathbb{C}$, which by Theorem 3.73 has fundamental group Γ. □

We conclude from Proposition 3.74 that smooth Lefschetz pencils of curves over $\mathbb{C}$ are relatively flexible. The question becomes harder when we ask the same question for Lefschetz pencils over $\mathbb{P}^1_{\mathbb{C}}$: choosing a critical value to be he point at infinity, the monodromy around ∞ must also be a product of Dehn twists along disjoint simple loops. Yet this monodromy is the composition of all the monodromies around the other critical values. The algorithm that we have used in the proof of Proposition 3.74 might not meet that requirement, as the vanishing paths corresponding to the relations $y_j x_{i_l}^{-e_l}$ and c_{i_l} have nontrivial homologic intersection.

It is also a harder problem to characterize which smooth Lefschetz pencils admit a holomorphic structure, such that the pencil becomes holomorphic; indeed solving such a problem would provide a list of all fundamental groups of projective manifolds. As first steps in this direction, one may study the following questions:

QUESTION 3.75. (i) *Is it possible to realize every finitely presented group Γ as the fundamental group of a Lefschetz pencil of curves over $\mathbb{P}^1_{\mathbb{C}}$?*

(ii) *Does the existence of a smooth Lefschetz pencil of curves with fundamental group Γ and a quasi-complex structure on the fibers preserved by parallel transport place any restriction on Γ?*

(iii) *Does the existence of a smooth Lefschetz pencil of curves with fundamental group Γ satisfying Deligne's semisimplicity theorem on the first cohomology group of the fibers place any restriction on Γ?*

It seems to the author that the answers to the questions may vary: even in a pencil over $\mathbb{C}$, semisimplicity of the monodromy action in the first cohomology group of the fibers places strong restrictions on Γ, that will be studied in the continuation of this work. On the other hand, extension to $\mathbb{P}^1_{\mathbb{C}}$ or existence of a quasi-complex pencil do not seem to pose any restriction on Γ.

5. Monodromy in the Malcev algebras of the Legendre family

In a previous section the monodromy of the Legendre affine family in the fundamental groups of the fibres $\pi_1(E_t, P_t) \cong F_2$ has been determined by geometric methods. In this paragraph we derive from it the monodromy representation in the $(F_2)_{i-1}/(F_2)_i$ and $\mathcal{L}(F_2/(F_2)_i \otimes \mathbb{Q})$ up to $i = 4$ by algebraic computation.

As it has been explained in Section 3 of the first chapter, the elements of the Hall basis for the free Lie algebra $L(\mathbb{Q}, 2) \hookrightarrow \operatorname{gr} A(\mathbb{Q}, 2)$ up to weight 3 are:

Weight 1	a	b
Weight 2	$[b,a]$	
Weight 3	$[[b,a],a]$	$[[b,a],b]$

The monodromy representation $\rho : \pi_1(\mathbb{P}^1(\mathbb{C})\backslash\{0,1,\infty\}, \frac{1}{2}) \to \operatorname{Aut}(\pi_1(E_t, P_t)) \cong \operatorname{Aut} F_2$ established in Proposition 3.46 is determined by:

$$\rho([\gamma_0]) : \begin{cases} a & \to & ab^{-2} \\ b & \to & b \end{cases}$$

$$\rho([\gamma_1]) : \begin{cases} a & \to & a \\ b & \to & ba^2 \end{cases}$$

In the sequel, endangering clarity for the sake of convenience, we will denote as ρ_0, ρ_1 the automorphisms induced by $\rho([\gamma_0])$, $\rho([\gamma_1])$ respectively in the succesive algebras, Lie algebras and group quotients to be obtained from $\pi_1(E_t, P_t)$.

Our first goal is to determine ρ_0, ρ_1 in $\mathcal{L}(F_2/(F_2)_4) \cong \mathcal{P}(A(\mathbb{Q}, 2)/J^4_{\mathbb{Q}})$. The graded Lie algebra isomorphisms

$$\begin{aligned} L(\mathbb{Q},2)/\bigoplus_{i\geq 4} L(\mathbb{Q},2)^i &\longrightarrow \operatorname{Gr}(F_2/(F_2)_4 \otimes \mathbb{Q}) \cong \operatorname{Gr}(\mathcal{G}(A(\mathbb{Q},2)/J^4_{\mathbb{Q}})) \\ &\longrightarrow \operatorname{Gr}(\mathcal{P}(A(\mathbb{Q},2)/J^4_{\mathbb{Q}})) \cong \operatorname{Gr}(\mathcal{L}(F_2/(F_2)_4)) \end{aligned}$$

seen in Proposition 1.24, send the elements of the Hall basis of $L(\mathbb{Q}, 2)$ up to weight 3 to the classes a, b, $[b,a]$, $[[b,a],a]$, $[[b,a],b] \in \operatorname{Gr}(\mathcal{P}(A(\mathbb{Q},2)/J^4_{\mathbb{Q}}))$. The elements of $\mathcal{P}(A(\mathbb{Q},2)/J^4_{\mathbb{Q}})$ $X = \log(1+a)$, $Y = \log(1+b)$, $[Y,X]$, $[[Y,X],X]$, $[[Y,X],Y]$ are sent to those classes by the natural $\mathbb{Q}$-Lie algebra isomorphism $\mathcal{P}(A(\mathbb{Q},2)/J^4_{\mathbb{Q}}) \to \operatorname{Gr}(\mathcal{P}(A(\mathbb{Q},2)/J^4_{\mathbb{Q}}))$. Therefore they form an homogeneous basis of $\mathcal{L}(F_2/(F_2)_4)$. Their expressions in

$A(\mathbb{Q},2)/J_{\mathbb{Q}}^4$ are:

$$X = a - \frac{a^2}{2} + \frac{a^3}{3}$$

$$Y = b - \frac{b^2}{2} + \frac{b^3}{3}$$

$$[Y,X] = -ab + ba + \frac{a^2b}{2} + \frac{ab^2}{2} - \frac{ba^2}{2} - \frac{b^2a}{2}$$

$$[[Y,X],X] = a^2b - 2aba + ba^2$$

$$[[Y,X],Y] = -ab^2 + 2bab - b^2a$$

We may now determine ρ_0, ρ_1 in $\mathcal{L}(F_2/(F_2)_4) \cong \mathcal{P}(A(\mathbb{Q},2)/J_{\mathbb{Q}}^4)$ because we know their values for $1+a, 1+b \in \bar{\nu}_4(F_2)$ and they commute with the formal series log. We begin with the automorphism ρ_0:

$$\begin{aligned}\rho_0(1+a) &= \rho_0(\bar{\nu}_4(a)) \\ &= \bar{\nu}_4(ab^{-2}) \\ &= (1+a)(1+b)^{-2} \\ &= 1 + a - 2b - 2ab + 3b^2 + 3ab^2 - 4b^3\end{aligned}$$

$$\begin{aligned}\rho_0(1+b) &= \rho_0(\bar{\nu}_4(b)) \\ &= 1 + b\end{aligned}$$

Thus

$$\begin{aligned}\rho_0(X) &= \rho_0(\log(1+a)) \\ &= \log(\rho_0(1+a)) \\ &= a - 2b - \frac{1}{2}a^2 - ab + ba + b^2 + \frac{1}{3}a^3 + \frac{1}{3}a^2b + \frac{1}{3}aba + \frac{5}{6}ab^2 \\ &\quad - \frac{2}{3}ba^2 - \frac{2}{3}bab - \frac{1}{6}b^2a - \frac{2}{3}b^3 \\ &= X - 2Y + [Y,X] - \frac{1}{6}[[Y,X],X] - \frac{1}{3}[[Y,X],Y]\end{aligned}$$

and

$$\begin{aligned}\rho_0(Y) &= \rho_0(\log(1+b))\\ &= \log(1+b)\\ &= Y\end{aligned}$$

$$\begin{aligned}\rho_0([Y,X]) &= [\rho_0(Y), \rho_0(X)]\\ &= [Y, X - 2Y + [Y,X] - \frac{1}{6}[[Y,X],X] - \frac{1}{3}[[Y,X],Y]]\\ &= [Y,X] - [[Y,X],Y]\end{aligned}$$

$$\begin{aligned}\rho_0([[Y,X],X]) &= [[\rho_0 Y, \rho_0 X], \rho_0 X]\\ &= [[Y,X],X] - 2[[Y,X],Y]\end{aligned}$$

$$\rho_0([[Y,X],Y]) = [[Y,X],Y]$$

The procedure for ρ_1 is wholly identical. The result is:

$$\rho_1 X = X$$

$$\begin{aligned}\rho_1 Y &= \log\left((1+b)(1+a)^2\right)\\ &= 2X + Y + [Y,X] + \frac{1}{3}[[Y,X],X] - \frac{1}{6}[[Y,X],Y]\end{aligned}$$

$$\rho_1[Y,X] = [Y,X] + [[Y,X],X]$$

$$\rho_1[[Y,X],X] = [[Y,X],X]$$

$$\rho_1[[Y,X],Y] = 2[[Y,X],X] + [[Y,X],Y]$$

These results may be summed up in the following

PROPOSITION 3.76. *The monodromy representation*

$$\rho : \pi_1(\mathbb{P}^1(\mathbb{C}) \setminus \{0,1,\infty\}, \frac{1}{2}) \longrightarrow Aut(\mathcal{L}_4\pi_1(E_{\frac{1}{2}}, P) \otimes \mathbb{Q}) \cong GL(\mathbb{Q}, 5)\,,$$

with the latter isomorphism given by the above used basis, is determined by

$$\rho([\gamma_0]) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ -\frac{1}{6} & 0 & 0 & 1 & 0 \\ -\frac{1}{3} & 0 & -1 & -2 & 1 \end{pmatrix}$$

$$\rho([\gamma_1]) = \begin{pmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & \frac{1}{3} & 1 & 1 & 2 \\ 0 & -\frac{1}{6} & 0 & 0 & 1 \end{pmatrix}$$

Two properties of the monodromy just computed in Prop. 3.76 should be observed:

First, both $\rho([\gamma_0])$,$\rho([\gamma_1])$ are unipotent matrices.

Second, if we divide them in blocks according to the decomposition of $\mathcal{L}_4(\pi_1(E_{\frac{1}{2}}, P)) \otimes \mathbb{Q})$ by the length of the brackets of the Hall basis we have used, the monodromy matrices $\rho([\gamma_0])$ and $\rho([\gamma_1])$ are simultaneously block lower triangular, with the diagonal blocks containing representations of the $H_1(E_t; \mathbb{Q})$-monodromy.

We will show in Chapter 5 how these properties extend to all $\mathcal{L}_n\pi_1(E_{\frac{1}{2}}, P) \otimes \mathbb{Q})$.

REMARK 3.77. The monodromy representations of $\pi_1(\mathbb{P}^1(\mathbb{C})\backslash\{0, 1, \infty\}, \frac{1}{2})$ in $\pi_1(E_{\frac{1}{2}}, P)_i/(\pi_1(E_{\frac{1}{2}}, P))_{i+1} \otimes \mathbb{Q}$, $\mathcal{L}_i\pi_1(E_{\frac{1}{2}}, P) \otimes \mathbb{Q})$ for $i < 4$ are easily deduced from that in $\mathcal{L}_4\pi_1(E_t, *) \otimes \mathbb{Q}$ computed in Prop. 3.76 using the graded Lie algebra isomorphisms of Proposition 1.13. An immediate consequence of them is that the monodromy in $\pi_1(E_t, *)_i/(\pi_1(E_t, *))_{i+1} \otimes \mathbb{Q}$ is the projection to the component of weight i of the representation ρ from Proposition 3.76, and the monodromy in the $\mathcal{L}_i(\pi_1(E_t, *))$ is the projection of ρ in the homogeneous subspace of weight lower than i. Thus the ρ_0, ρ_1 for these representations are the minors of the $\rho([\gamma_0])$, $\rho([\gamma_1])$ of Proposition 3.76 indicated in Figure F3.35.

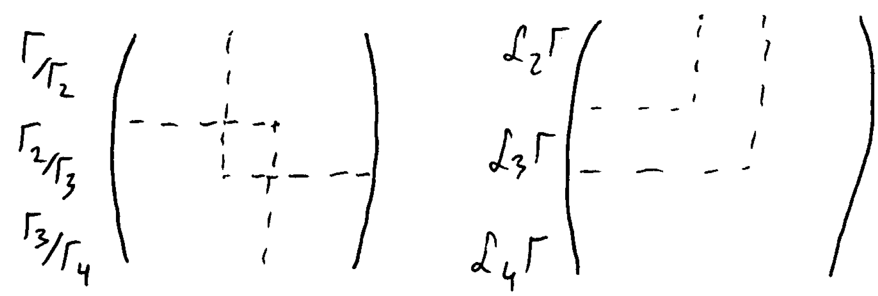

Fig. F3.35 Graduate quotient minors of the monodromy matrix.

In the case of the abelianised $\pi_1(E_t, P_t)/(\pi_1(E_t, P_t))_2 \otimes \mathbb{Q} \cong H_1(E_t; \mathbb{Q})$, the monodromy just computed is the classical monodromy in $H_1(\bar{E}_t)$ of the projective Legendre family.

It may also be noted that the coefficients of the monodromy in the $\pi_1(E_t, *)_i/(\pi_1(E_t, *))_{i+1}$ are integers. This is due to the fact that the monodromy in those $\mathbb{Q}$-vector spaces comes from the free abelian groups $(F_2)_i/(F_2)_{i+1}$, and the Hall basis of $(F_2)_i/(F_2)_{i+1} \otimes \mathbb{Q}$ we have used also come from $(F_2)_i/(F_2)_{i+1}$.

CHAPTER 4

Dolbeault realization

Let X, S be complex analytic manifolds, $f : X \to S$ a smooth, proper map and $H \subset X$ a relative divisor with normal crossings. We will describe a resolution of the local systems $\mathbb{R}^p f_* \mathbb{C}_{(X \setminus H)}$ by a complex of real analytic forms $f_* \mathcal{A}^{*,*}_{X|S}(\log H)$ with a natural real structure. This complex yields the variation of Hodge structures associated to the cohomology of the fibers; more precisely, it yields *real analytic variations of Hodge structure*, to be defined in Section 3, which carry also a natural real structure and are seen to be naturally equivalent to complex variations.

The Gauss–Manin connection of these real analytic Dolbeault complexes has a 1–minimal model, which may be computed using the techniques of [**73**] that we apply in the next chapter to the relative holomorphic de Rham complexes. Such a construction should lead to the same results of [**48**] on the variation of Malcev algebras in complex algebraic families.

We describe first the absolute case $S = \{*\}$ and that of $H = \{\emptyset\}$, as they will be of use in the case of our main concern, and we presume that they are not devoid of interest.

REMARK 4.1. - We will only use manifolds with finite Betti numbers, and we will omit to mention this condition for the sake of agility.
- The structural sheaves we will consider for real analytic manifolds are the sheaves of *complex-valued* real analytic functions on them. These sheaves arise from the sheaves of real–valued analytic functions and forms by tensoring with $\mathbb{C}$. Because of this, the complex–valued real analytic sheaves are endowed with a natural real structure, which is obtained by complex conjugation of the coefficients in the power series defining every function.

1. Acyclicity in the real analytic category

We will work in this chapter with real analytic sheaves. These are not fine as their $\mathcal{C}^\infty$ analogues. Nevertheless, they are acyclic, and this may be seen using results of Whitney, Bruhat and Grauert which we sum up in the following

THEOREM 4.2 ([**102**],[**40**]). *Let $X_{\mathbb{R}}$ be a real analytic manifold of real dimension n. There exists a complex analytic manifold $\tilde{X}$ and a real analytic embedding $j : X_{\mathbb{R}} \hookrightarrow \tilde{X}$, such that*

(i) *There is a covering of $\tilde{X}$ with coordinate neighbourhoods $\mathcal{U} = \{U_i, z_1, \ldots, z_n\}$ such that $X_{\mathbb{R}} \cap U_i$ has equations $Im\, z_1 = 0, \ldots, Im\, z_n = 0$.*
(ii) *The closed submanifold $j(X_{\mathbb{R}}) \subset \tilde{X}$ admits a neighbourhood base formed by Stein open sets.*

COROLLARY 4.3. *Let X be a real analytic manifold.*

(i) *The structural sheaf $\mathcal{A}_X$ is a Oka sheaf of rings.*
(ii) *Any coherent sheaf over X is acyclic.*

PROOF. Let $j : X \hookrightarrow \tilde{X}$ be a closed imbedding of X in a complex analytic manifold satisfying Theorem 4.2. The real analytic structural sheaf $\mathcal{A}_X$ is the restriction to the closed submanifold X of the holomorphic structural sheaf $\mathcal{O}_{\tilde{X}}$. Therefore, any $\mathcal{A}_X$-linear map $\varphi : \mathcal{A}_X^q \to \mathcal{A}_X^p$ extends in some open neighbourhood of X, which we may assume by restriction to be $\tilde{X}$ itself, to a $\mathcal{O}_{\tilde{X}}$-linear map $\tilde{\varphi}$. Writing this down with kernels and cokernels shows that the exact sequence of $\mathcal{A}_X$-sheaves

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{A}_X^q \xrightarrow{\varphi} \mathcal{A}_X^p \longrightarrow \mathcal{F} \longrightarrow 0 \tag{7}$$

is the restriction to X of the exact sequence of $\mathcal{O}_{\tilde{X}}$-sheaves

$$0 \longrightarrow \tilde{\mathcal{K}} \longrightarrow \mathcal{O}_{\tilde{X}}^q \xrightarrow{\tilde{\varphi}} \mathcal{O}_{\tilde{X}}^p \longrightarrow \tilde{\mathcal{F}} \longrightarrow 0 \tag{8}$$

The structural sheaf $\mathcal{O}_{\tilde{X}}$ is a Oka sheaf of rings. Hence the kernel $\tilde{\mathcal{K}}$ is finitely generated over it, and by restriction $\mathcal{K}$ is a finitely generated $\mathcal{A}_X$-module. This proves our assertion (i).

Any coherent sheaf $\mathcal{F}$ admits a cokernel presentation as (7). By restricting our domain $\tilde{X}$, we may find an extension of this presentation to $\tilde{X}$ as in (8), and thus an isomorphism $\tilde{\mathcal{F}}_{|X} \cong \mathcal{F}$. Consequently, for any $i > 0$,

$$H^i(X, \mathcal{F}) = H^i(X, \tilde{\mathcal{F}}_{|X}) = \lim_{\substack{X \subset V \\ V \text{ open}}} H^i(V, \tilde{\mathcal{F}}_{|V}) = \lim_{\substack{X \subset V \\ V \text{ Stein open}}} H^i(V, \tilde{\mathcal{F}}_{|V}) = 0\,,$$

which proves statement (ii). □

The category of coherent sheaves will not suffice us in the sequel, so we extend in part the previous results to quasi-coherent sheaves. Its definition is the same in the real analytic case as in the holomorphic setting.

DEFINITION 4.4. A sheaf $\mathcal{F}$ of $\mathcal{A}_X$-modules on a real analytic manifold X is *quasi-coherent* if it is a strict inductive limit of coherent sheaves.

PROPOSITION 4.5. *Let X be a real analytic manifold. Any quasi-coherent sheaf $\mathcal{F}$ on X is acyclic over the compact subspaces of X.*

PROOF. Sheaf cohomology commutes with strict inductive limits on compact spaces, so our statement is a consequence of Corollary 4.3 (ii). □

We finish this section by stating the relative versions of Corollary 4.3, Proposition 4.5, which follow immediately from them.

COROLLARY 4.6. *Let $f : X \to S$ be a smooth locally trivial map between real analytic manifolds.*

(i) *Any coherent sheaf on X is f_*-acyclic.*
(ii) *If the map f is proper, any quasi-coherent sheaf on X is f_*-acyclic.*

2. Dolbeault lemmas and variations of Hodge structure

2.1. The Dolbeault lemma for complex manifolds. We restate the classical Dolbeault Lemma substituting real analytic for C^∞ functions. Its proof is analogous to that of the C^∞ case, and it is included here for the reader's convenience.

Let X be throughout this section a complex analytic manifold, with $\dim_{\mathbb{C}} X = n$. The sheaves of real analytic forms $\mathcal{A}_X^{*,*}$ are acyclic by Corollary 4.3.

PROPOSITION 4.7 (Real analytic Dolbeault lemma). *The sequences*

$$0 \longrightarrow \Omega_X^p \longrightarrow \mathcal{A}_X^{p,0} \xrightarrow{\bar\partial} \mathcal{A}_X^{p,1} \longrightarrow \dots$$

are exact, and they define an acyclic resolution $\Omega_X^* \xrightarrow{\sim} \mathcal{A}_X^{*,*}$.

PROOF. The proof is analogous to that of the C^∞ case. As the question is local, by restriction to coordinate open sets it suffices to prove it for germs of forms on $0 \in \mathbb{C}^n$.

Let ω be a differential form of type p, q in a neighbourhood $0 \in U \subset \mathbb{C}^n$ such that $\bar\partial\omega = 0$. We may assume U is small enough so that all coefficients admit a global power series development on it.

Let m be the highest subindex such that $d\bar z_m$ appears in ω. We will prove our assertion by induction on m.

We can write our form in a unique way as

$$\omega = d\bar z_m \wedge \alpha + \beta$$

with $\alpha = 0$ when $m = 0$. The forms α, β involve only the conjugate differentials $d\bar{z}_1, \dots, d\bar{z}_{m-1}$. As

$$0 = \bar{\partial}\omega = d\bar{z}_m \wedge \bar{\partial}\alpha + \bar{\partial}\beta$$

it is easily checked that the coefficients of α, β are holomorphic on the variables $z_{m+1}, \dots, z_n$. If $m = 0$ this establishes our initial step.

If $m \neq 0$, denote $\alpha = \sum a_{I,J} dz_I d\bar{z}_J$. As the coefficients $a_{I,J}$ are power series on U, there are real analytic functions $g_{I,J} \in \mathcal{A}_U$ such that $\frac{\partial g_{I,J}}{\partial \bar{z}_m} = a_{I,J}$. Define a form $\gamma = \sum g_{I,J} dz_I d\bar{z}_J$. We have now that

$$\bar{\partial}\gamma = d\bar{z}_m \wedge \alpha + \delta$$

where the form δ does not involve any of the differentials $d\bar{z}_m, \dots, d\bar{z}_n$. The differential form

$$\varphi = \omega - \bar{\partial}\gamma$$

is also closed, and involves only $d\bar{z}_1, \dots, d\bar{z}_{m-1}$. By our induction hypothesis $\varphi = \bar{\partial}\psi$ for some $\psi \in \mathcal{A}_U^{p,q-1}$, and thus $\omega = \bar{\partial}(\gamma + \psi)$. □

We may define the Hodge filtration in $\mathcal{A}_X^{*,*}$ as $F^p\mathcal{A}_X^{*,*} = \oplus_{s \geq p} \mathcal{A}_X^{s,*}$. The Laplacian and Green operators may be defined on real analytic forms as in the well-known $\mathcal{C}^\infty$ case, and this fact allows a real analytic construction of the Hodge theory of compact Kähler manifolds. The basic step would be:

LEMMA 4.8 (Real analytic $\partial\bar{\partial}$ lemma). *Let X be a compact Kähler manifold, and $u \in \mathcal{A}^{p,q}(X)$ be a form such that $du = 0$. Then the following are equivalent:*

(i) *u is d–exact.*
(ii) *u is ∂–exact.*
(iii) *u is $\bar{\partial}$–exact.*
(iv) *u is $\partial\bar{\partial}$–exact.*

PROOF. The proof of this statement is exactly the proof of the $\mathcal{C}^\infty$ $\partial\bar{\partial}$ lemma, regarding all the intervening forms and operators as real analytic. □

Proceeding further as in the $\mathcal{C}^\infty$ case, we retrieve the sought Hodge structure:

PROPOSITION 4.9. *Let X be a compact Kähler manifold. The Hodge filtration $F^\bullet$ on the real analytic Dolbeault complex $\mathcal{A}_X^{*,*}$ induces a pure Hodge structure of weight n on the cohomology groups $H^n(X, \mathbb{C})$ for every n. The inclusion in the $\mathcal{C}^\infty$ Dolbeault complex $\mathcal{A}_X^{*,*} \hookrightarrow \mathcal{E}_X^{*,*}$ preserves the Hodge filtration and induces an isomorphism of Hodge structures on $H^*(X, \mathbb{C})$.*

2.2. The Dolbeault lemma in the logarithmic case. Let X be a complex analytic manifold of dimension n and $Y \subset X$ a normal crossing divisor with smooth irreducible components. The real analytic logarithmic Dolbeault complex of the pair (X, Y) has been studied by Navarro Aznar in [**72**]. We sum up some of his results on it.

Denote $U = X \setminus Y$, and the inclusion as $j : U \hookrightarrow X$. Consider a coordinate cover $\{V\}$ of X such that for every $x \in Y$ there is a V such that $x = 0$ and Y has equation $z_1 \dots z_r = 0$. The real analytic logarithmic Dolbeault complex $\mathcal{A}_X^{*,*}(\log Y)$ is defined on every V as the sub-$\mathcal{A}_V$ algebra of $j_* \mathcal{A}_{V \setminus Y}^{*,*}$ spanned by

$$\frac{dz_i}{z_i}, \quad \frac{d\bar{z}_i}{\bar{z}_i}, \quad \log |z_i|, \qquad 1 \leq i \leq r \tag{9}$$

and

$$dz_i \, , d\bar{z}_i \, , \qquad r+1 \leq i \leq n \tag{10}$$

The weight filtration W on $\mathcal{A}_X^{*,*}(\log Y)$ is defined by assigning weight 1 to the generators of (9), weight 0 to those of (10) and applying multiplicativity of weight. This filtration is actually an extension of a weight filtration defined on the sheaf $\mathcal{A}_X^*(\log Y)$ of real valued logarithmic forms.

The Hodge filtration is defined as

$$F^p \mathcal{A}_X^{*,*}(\log Y) = \oplus_{s \geq p} \mathcal{A}_X^{s,*}(\log Y)$$

With all these definitions, we have

THEOREM 4.10 ([**72**], 8.8). *The natural map*

$$(\Omega_X^*(\log Y), W, F) \longrightarrow (\mathcal{A}_X^{*,*}(\log Y), W, F)$$

is a bifiltered quasi-isomorphism.

As in the complete case of the previous section, when X is compact Kähler we retrieve the mixed Hodge structure defined by Deligne on $H^*(X \setminus Y)$, induced now by weight and Hodge filtrations on the differential form algebras.

2.3. Relative Dolbeault complexes. We will study now the relative version of the real analytic Poincaré and Dolbeault lemmas. The comparison in this case between the holomorphic and real analytic constructions is less direct than in the absolute case of 2.1, because we must compare holomorphic vs. real analytic bundles. We will start with the Poincaré lemma and Gauss–Manin connection in the real analytic category, and deal later with the specific case of complex analytic manifolds and the relative Dolbeault lemma.

Let $f : X \to S$ be a smooth locally trivial map between real analytic manifolds, such that its fibers have finite Betti numbers.

The relative real analytic de Rham complex of forms over X, denoted $\mathcal{A}^{*}_{X|S}$, is defined as its $\mathcal{C}^{\infty}$ analogue: one starts by defining the degree one piece with the exact sequence

$$0 \longrightarrow f^{*}\mathcal{A}^{1}_{S} \longrightarrow \mathcal{A}^{1}_{X} \longrightarrow \mathcal{A}^{1}_{X|S} \longrightarrow 0\,,$$

and then sets $\mathcal{A}^{n}_{X|S} = \bigwedge^{n} \mathcal{A}^{1}_{X|S}$.

The relative sheaves $\mathcal{A}^{n}_{X|S}$ are coherent $\mathcal{A}_X$–modules, therefore f_*–acyclic by Corollary 4.6. We proceed to check that they provide a resolution of the constant sheaf $\mathbb{C}_X$.

LEMMA 4.11. *Let Y be a paracompact, locally contractible space, with finite Betti numbers, and S a real analytic manifold. Denote as $\pi : Y \times S \to S$ the projection. The natural morphisms of sheaves of $\mathcal{A}_S$-modules*

$$H^p(Y, \mathbb{C}) \underset{\mathbb{C}}{\otimes} \mathcal{A}_S \longrightarrow \mathbb{R}^p\pi_*\pi^{-1}\mathcal{A}_S$$

are isomorphisms for all $p \geq 0$.

PROOF. Take $V \subset S$ a small disk. As $H^*(V, \mathcal{A}_S)$ is free, the Künneth exact sequence yields an isomorphism

$$\left(H^*(Y, \mathbb{C}) \underset{\mathbb{C}}{\otimes} H^*(V, \mathcal{A}_S)\right)^p \xrightarrow{\cong} H^p(Y \times V, \mathbb{C} \underset{\mathbb{C}}{\otimes} \pi^{-1}\mathcal{A}_S)$$

The sheaf $\mathcal{A}_S$ is acyclic by Cor. 4.3, so this is actually an isomorphism

$$H^p(Y, \mathbb{C}) \otimes \Gamma(V, \mathcal{A}_S) \xrightarrow{\cong} H^p(Y \times V, \pi^{-1}\mathcal{A}_S)$$

□

LEMMA 4.12. *Let X, S be real analytic manifolds, and $f : X \to S$ a real analytic map such that it is smooth, locally trivial and its fibers have finite Betti numbers. The natural morphisms of sheaves of $\mathcal{A}_S$-modules*

$$(\mathbb{R}^p f_*\mathbb{C}_X) \underset{\mathbb{C}}{\otimes} \mathcal{A}_S \longrightarrow \mathbb{R}^p f_* \left(f^{-1}\mathcal{A}_S\right)$$

are isomorphisms for all $p \geq 0$.

PROOF. The isomorphism has to be verified locally over S. Take $V \subset S$ a trivializing open set for f. There is then an isomorphism of fibrations

$$\begin{array}{ccc} f^{-1}(V) & \overset{\varphi}{\underset{\cong}{\longrightarrow}} & X_s \times V \\ f \searrow & & \swarrow \pi \\ & V & \end{array}$$

Therefore $\varphi_*\mathbb{C}_{f^{-1}(V)} \cong \mathbb{C}_{X_s \times V}$, and $\varphi_*\varphi^{-1} = \mathrm{Id}$. It holds over V as a consequence that

$$(\mathbb{R}^p f_*\mathbb{C}_X) \underset{\mathbb{C}}{\otimes} \mathcal{A}_S \cong (\mathbb{R}^p \pi_*\varphi_*\mathbb{C}_X) \otimes \mathcal{A}_S \cong (\mathbb{R}^p \pi_*\mathbb{C}_{X_s \times V}) \underset{\mathbb{C}}{\otimes} \mathcal{A}_S \tag{11}$$

and

$$\mathbb{R}^p f_*(f^{-1}\mathcal{A}_S) \cong \mathbb{R}^p \pi_*\varphi_*\varphi^{-1}\pi^{-1}\mathcal{A}_S \cong \mathbb{R}^p \pi_*\pi^{-1}\mathcal{A}_S \tag{12}$$

The final expressions of 11 and 12 are isomorphic by Lemma 4.11. □

PROPOSITION 4.13 (relative Poincaré lemma). *Let $f : X \to S$ be a smooth locally trivial map between real analytic manifolds. The natural map*

$$f^{-1}\mathcal{A}_S \longrightarrow \mathcal{A}^*_{X|S}$$

induces a quasi-isomorphism of sheaves.

PROOF. This is a local question on X, so we may assume f to be a trivial fibration, $f : X = Y \times S \to S$ with Y, S disks in $\mathbb{R}^m, \mathbb{R}^d$ respectively. It is then immediate that $f^{-1}\mathcal{A}_S$ is the kernel of the map $d : \mathcal{A}_{X|S} \to \mathcal{A}^1_{X|S}$.

To complete our proof we produce a homotopy operator, which is a real analytic version of those used in the $\mathcal{C}^\infty$ or holomorphic context. For every multiindex $I = \{i_1, \dots, i_p\}$, denote $dy_I = dy_{i_1} \wedge \cdots \wedge dy_{i_p}$, and define a form

$$\omega_I = \sum_{k=0}^{p} (-1)^{k-1} y_{i_k} dy_{i_1} \wedge \dots \widehat{dy_{i_k}} \dots dy_{i_p}$$

As partial integration of a real analytic form over a compact domain produces another real analytic form, we may define a relative version of the classical homotopy operator

$$H : \mathcal{A}^p_{X|S} \longrightarrow \mathcal{A}^{p-1}_{X|S}$$

$$\alpha = \sum_I f_I(y,s) dy_I \longmapsto \sum_I \left(\int_0^1 t^{p-1} f_I(ty,s) dt \right) \omega_I$$

All that remains to be verified is that $dH + Hd = \mathrm{Id}$. This is a straightforward computation. □

COROLLARY 4.14. *Let $f : X \to S$ be a smooth locally trivial map between real analytic manifolds. The natural map of sheaves $\mathbb{C}_X \to \mathcal{A}^*_{X|S}$ induces isomorphisms of $\mathcal{A}_S$-modules*

$$\mathbb{R}^p f_*\mathbb{C}_X \underset{\mathbb{C}}{\otimes} \mathcal{A}_S \longrightarrow H^p(f_*\mathcal{A}^*_{X|S}, f_* d_{X|S})$$

for all $p \geq 0$.

PROOF. Let us observe first that $H^p(f_*\mathcal{A}^*_{X|S}, f_*d) \cong \mathbb{R}^p f_*\mathcal{A}^*_{X|S}$, as the sheaves $\mathcal{A}^{p,q}_{X|S}$ are f_*-acyclic. The local isomorphisms $H^p(f^{-1}(V), \mathbb{C}_X) \otimes \mathcal{A}_S \cong H^p(f^{-1}(V), \mathcal{A}^*_{X|S})$ are a consequence of the application of Lemma 4.12 and Prop. 4.13 to the restrictions of f. □

Thus we have obtained a f_*-acyclic resolution of $\mathbb{C}_X$ with a natural real structure. This resolution induces the Gauss–Manin connection in the derived sheaves $\mathbb{R}^p f_*\mathcal{A}^*_{X|S}$:

DEFINITION 4.15. We define the Gauss–Manin connection ∇ on $\mathbb{R}^p f_*\mathcal{A}^*_{X|S}$ as the real analytic connection

$$\nabla : \mathbb{R}^p f_*\mathcal{A}^*_{X|S} \longrightarrow \mathcal{A}^1_S \otimes \mathbb{R}^p f_*\mathcal{A}^*_{X|S}$$

which has the local system $\mathbb{R}^p f_*\mathbb{C}_X$ as horizontal sections.

The Gauss–Manin connection is basically parallel transport along suitable vector fields, and due to the acyclicity of the structural sheaf its computation in the real analytic case is somewhat simpler than that of its holomorphic counterpart.

PROPOSITION 4.16. *Let v_0 be a real analytic tangent field defined on an open set $V \subset S$. Then:*

(i) *The field v_0 admits a real analytic lifting v defined over $f^{-1}(V) \subset X$.*
(ii) *The Lie derivative $L_v : \mathcal{A}^*_X \to \mathcal{A}^*_X$ of a lift v of the vector field v_0 induces a derivation $L_v : \mathcal{A}^*_{X|S} \to \mathcal{A}^*_{X|S}$, and if v, v' are two lifts of v_0, the induced derivations in $\mathcal{A}^*_{X|S}$ are homotopic as morphisms of sheaves.*
(iii) *The induced derivation $\nabla_{v_0} := \mathbb{R}^p f_* L_v$ on $\mathbb{R}^p f_*\mathcal{A}^*_{X|S}$ depends only on v_0, and it is the Gauss-Manin connection along v_0.*
(iv) *If u_0, v_0 are vector fields on $V \subset S$ such that $[u_0, v_0] = 0$ and u, v are lifts to $f^{-1}(V) \subset X$, then the bracket $[L_u, L_v]$ of induced derivations in $\mathcal{A}^*_{X|S}$ is null-homotopic.*

PROOF. (i) is due to the exact sequence of acyclic sheaves over X

$$0 \longrightarrow T_{X|S} \longrightarrow T_X \xrightarrow{df} f^*T_S \longrightarrow 0$$

To show (ii) we begin by checking that the kernel $0 \longrightarrow K^* \longrightarrow \mathcal{A}^*_X \longrightarrow \mathcal{A}^*_{X|S} \longrightarrow 0$ is L_v-stable. This is a local question on X, so we may restrict ourselves to the projection $Y \times V \to V$, where $\{Y, (y_1, \ldots, y_m)\}$ and $\{V, (s_1, \ldots, s_d)\}$ are balls. The kernel K is in this case that of forms with a ds_j factor on every sumand.

The Cartan homotopy formula $L_v\omega = di_v\omega + i_v d\omega$ shows that the question is trivial for forms with a $ds_i \wedge ds_j$ factor, so we just have to study the case

$$\varphi = \alpha_1 \wedge ds_1 + \cdots + \alpha_d \wedge ds_d$$

with $\alpha_i \in \mathcal{A}^p_X$ containing no factors ds_j. By linearity on v of the Lie derivation, it is sufficient to compute the case $v = \frac{\partial}{\partial s_1}$. With this assumption and the notational convention $\frac{\partial(\sum f_I dx_I)}{\partial x_k} := \sum \frac{\partial f_I}{\partial x_k} dx_k \wedge dx_I$ we have

$$\begin{aligned}
L_v\varphi &= di_v\varphi + i_v d\varphi \\
&= d\left((-1)^p \alpha_1\right) + i_v\Big(\sum_{i=1}^{m}(-1)^p \frac{\partial \alpha_1}{\partial y_i} \wedge dy_i \wedge ds_1 + \sum_{j=2}^{d}(-1)^p \frac{\partial \alpha_1}{\partial s_j} \wedge ds_j \wedge ds_1 \\
&+ (-1)^{p+1} \sum_{k=2}^{d} \frac{\partial \alpha_k}{\partial s_1} \wedge ds_k \wedge ds_1 + \text{terms with no } ds_1\Big) \\
&= \sum_{i=1}^{m}(-1)^{2p} \frac{\partial \alpha_1}{\partial y_i} \wedge dy_i + \text{terms in } K \\
&+ \sum_{i=1}^{m}(-1)^{2p+1} \frac{\partial \alpha_1}{\partial y_i} \wedge dy_i + \text{terms in } K
\end{aligned}$$

so $L_v\varphi$ lies in K and L_v induces a derivation in $\mathcal{A}^*_{X|S}$. Again by the Cartan homotopy formula we have that $dL_v\omega = di_v d\omega = L_v d\omega$ so L_v is a morphism of differential complexes and induces a derivation on $H^*(\mathcal{A}^*_{X|S}, d)$.

Let v, v' be two lifts of $\frac{\partial}{\partial s_1}$. The difference $u = v' - v$ contains no term $\frac{\partial}{\partial s_j}$, thus if $\omega \in K$, we have that $i_u\omega \in K$, and hence $L_u\omega = L_{v'}\omega - L_v\omega = di_u\omega + i_u d\omega \in K$. Therefore, the contraction i_u defines a homotopy between $L_v, L_{v'} : (\mathcal{A}^*_{X|S}, d) \to (\mathcal{A}^*_{X|S}, d)$.

(iii) is a local matter on S, so we may assume still that f is a trivial fibration $Y \times V \to V$, where Y is now the fiber of the original map f. Select coordinates $(s_1, \ldots, s_d)$ on V and local coordinates $(y_1, \ldots, y_m)$ on an open set of Y. As has been shown in (ii), if v_0 is a vector field on V and v, v' are lifts to $Y \times V$, the derivations $L_v, L_{v'}$ are homotopic endomorphisms of the sheaf complex $(\mathcal{A}^*_{X|S}, d)$. Therefore, the induced derivations on $H^p(\mathcal{A}^*_{X|S}, d) \cong \mathbb{R}^p f_* \mathcal{A}^*_{X|S}$ are the same. We will denote it as ∇_{v_0}.

For any $\omega \neq 0$ in $\mathcal{A}^*_{X|S}$ we may select a representative $\sum f_I dy_I$ in $\mathcal{A}^*_X$ with no term ds_j. Then

$$\begin{aligned} L_{\frac{\partial}{\partial s_i}} \omega &= i_{\frac{\partial}{\partial s_i}}(d\omega) + d(i_{\frac{\partial}{\partial s_i}} \omega) \\ &= \sum \frac{\partial f_I}{\partial s_i} dy_I \end{aligned}$$

Therefore, the horizontal sections of the connection given by $\nabla_v = L_v$ on $H^p(\mathcal{A}^*_{X|S}, d)$ are given locally on Y by the conditions

$$\frac{\partial f_I}{\partial s_1} = 0, \ldots, \frac{\partial f_I}{\partial s_d} = 0$$

i.e., the coefficients f_I must not depend on S. This holds under coordinate changes in Y, so we conclude that the horizontal sections of L_{v_0} are the cohomology classes in $H^*(Y, \mathbb{C})$, and this is the Gauss-Manin connection.

(iv) may be also shown locally. Given a local trivialization $Y \times V \to V$ as above, by linearity of the derivation we may assume that $u_0 = \frac{\partial}{\partial s_1}, v_0 = \frac{\partial}{\partial s_2}$, and select first as lifts to $Y \times V$ the fields $u = \frac{\partial}{\partial s_1}, v = \frac{\partial}{\partial s_2}$. A relative form ω admits a representative $\sum f_I(y_1, \ldots, y_m, s_1, \ldots, s_d) dy_I$, and by the Schwarz lemma one has

$$L_u(L_v \omega) = \sum \frac{\partial^2}{\partial s_1 \partial s_2} f_I dy_I = L_v(L_u \omega),$$

thus $[L_u, L_v] = 0$ with the selected liftings. To extend this case to arbitrary liftings it suffices to show that if we replace one of the derivations with a homotopic derivation, the resulting bracket is homotopic to the original one as a morphism of the complex $(\mathcal{A}^*_{X|S}, d)$. So let us choose another lifting $v' = v + w$ of the vector field v_0 and compute the bracket $[L_u, L_{v'}]$:

$$\begin{aligned} [L_u, L_{v'}] &= [L_u, L_v] + [L_u, L_w] \\ &= [L_u, L_v] + [di_u + i_u d, di_w + i_w d] \\ &= [L_u, L_v] + di_u di_w + i_u d di_v + di_u i_w d + i_u di_w d \\ &- di_w di_u - di_w i_u d - i_w d di_u - i_w di_u d \\ &= [L_u, L_v] + d(i_u di_w + i_u i_w d + di_w i_u - i_w di_u) \\ &+ (i_u di_w + i_u i_w d + di_w i_u - i_w di_u) d. \end{aligned}$$

The fact that v, v' are both liftings of v_0, i.e. the vector field w lies in $T_{X|S} = \ker(T_X \to T_S$, is essential so that the homotopy function $h = i_u di_w + i_u i_w d + di_w i_u - i_w di_u$ descends from $\mathcal{A}^*_X$ to the relative complex $\mathcal{A}^*_{X|S}$. □

Proposition 4.16 shows how the Gauss–Manin connection in the cohomology sheaves $\mathbb{R}^p f_*\mathcal{A}^*_{X|S}$ arises from parallel transport on the sheaves of forms $\mathcal{A}^*_{X|S}$, which is well–defined only up to homotopy. This indetermination can be overcome locally by choosing frames of horizontal fields:

DEFINITION 4.17. Let $V \subset S$ be an open subset, $v_1, \ldots, v_d$ vector fields such that they form a basis of the tangent space T_pS at every point $p \in V$, and $\tilde{v}_1, \ldots, \tilde{v}_d$ arbitrary lifts of $v_1, \ldots, v_d$ to $f^{-1}(V)$. The connection defined by setting

$$\nabla_{v_i}\omega = f_* L_{\tilde{v}_i}\omega$$

is *a local Gauss–Manin connection* on the complex of sheaves of forms $f_*\mathcal{A}^*_{X|S}$ restricted to V.

According to our definition, Gauss–Manin connections on the complex of forms $f_*\mathcal{A}^*_{X|S}$ are not unique, although homotopic, and defined only locally. This set of data forms a sheaf of connections up to homotopy on S, a *ho–connection* in the words of [**73**], §4, where such sheaves are introduced and studied. Although we will use only local Gauss–Manin connections, we will require a homotopic property of them:

DEFINITION 4.18 ([**73**]). A connection ∇ on a complex of sheaves $(\mathcal{A}^*, d)$ over a real analytic manifold S is *homotopically integrable* if for every pair of vector fields u, v defined on S the sheaf endomorphism

$$[\nabla_u, \nabla_v] - \nabla_{[u,v]} : \mathcal{A}^* \longrightarrow \mathcal{A}^*$$

is null–homotopic.

LEMMA 4.19. *Let $f : X \to S$ be a smooth locally trivial map between real analytic manifolds. Then every local Gauss–Manin connection defined on $V \subset S$ is homotopically integrable.*

PROOF. Our statement follows from Proposition 4.16 (iv). $\square$

Lemma 4.19 shows how the integrability of the Gauss–Manin connection is already present at the form level, the ultimate reason being the local integrability of parallel transport.

We return now from the real analytic to the complex analytic case. Let $f : X \to S$ be a smooth, topologically locally trivial map between complex manifolds. We have just established the relative Poincaré lemma and studied the Gauss–Manin connection for the underlying real analytic family. The additional complex structure in the cotangent bundles appears in the real analytic relative complex as a bigraduation, and the complex $\mathcal{A}^{*,*}_{X|S}$ is defined in this context as

$$0 \longrightarrow f^*\mathcal{A}^{1,0} \longrightarrow \mathcal{A}^{1,0}_X \longrightarrow \mathcal{A}^{1,0}_{X|S} \longrightarrow 0 \tag{13}$$

$$0 \longrightarrow f^*\mathcal{A}^{0,1} \longrightarrow \mathcal{A}^{0,1}_X \longrightarrow \mathcal{A}^{0,1}_{X|S} \longrightarrow 0 \tag{14}$$

for the degree one pieces, and $\mathcal{A}^{p,q}_{X|S} = \left(\bigwedge^p \mathcal{A}^{1,0}_{X|S}\right) \wedge \left(\bigwedge^q \mathcal{A}^{0,1}_{X|S}\right)$ in general. Consequently, any connection on a sheaf $\mathcal{A}$ over S decomposes as $\nabla^{1,0} + \nabla^{0,1}$, with

$$\nabla^{1,0} : \mathcal{A} \longrightarrow \mathcal{A}^{1,0}_S \otimes \mathcal{A}\,,$$

$$\nabla^{0,1} : \mathcal{A} \longrightarrow \mathcal{A}^{0,1}_S \otimes \mathcal{A}\,.$$

We will refer to the above summands of ∇ as its *complex structure components.*

Another consequence of the complex structure on cotangent bundles is the existence of a decreasing Hodge filtration $\mathcal{F}^\bullet$, given by

$$\mathcal{F}^p \mathcal{A}^{*,*}_{X|S} = \oplus_{r \geq p} \mathcal{A}^{r,*}_{X|S}\,.$$

This filtration is induced by its absolute counterpart in $\mathcal{A}^{*,*}_X$, and it is likewise preserved by the relative differentials $\partial_{X|S}, \bar{\partial}_{X|S}, d_{X|S} = \partial + \bar{\partial}$. Because of this, it induces a Hodge filtration on the derived sheaves $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}$, and as the homogeneous sheaves $\mathcal{F}^k \mathcal{A}^n_{X|S}$ are acyclic by Corollary 4.6, there are isomorphisms

$$\mathcal{F}^k \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S} \cong H^p(f_* \mathcal{F}^k \mathcal{A}^{*,*}_{X|S}, f_* d_{X|S})\,.$$

In the case of local Gauss–Manin connections on forms, and of the Gauss–Manin connection in cohomology, applying Corollary 4.6 to the exact sequence

$$0 \longrightarrow T^{1,0}_{X|S} \longrightarrow T^{1,0}_X \longrightarrow f^* T^{1,0}_S \longrightarrow 0$$

and to its conjugate, we find that any vector field v_0 of type $(1,0)$, resp. $(0,1)$, defined over an open set $V \subset S$ admits a lift v of the same complex type to $f^{-1}(V)$.

CONVENTION 4.20. We will assume henceforth that every local Gauss–Manin connection defined on $f_* \mathcal{A}^{*,*}_{X|S}$ has been defined by lifting a basis $v_1, \ldots, v_d$ of $T^{1,0}_S(V)$ to fields $\tilde{v}_1, \ldots, \tilde{v}_d \in T^{1,0}_X(f^{-1}(V))$, and lifting the conjugate fields $\bar{v}_i$ to the conjugates $(\bar{\tilde{v}}_i)$. Thanks to Corollary 4.6 we know that such a choice of liftings is always possible.

The behaviour of the $(1,0)$ and $(0,1)$ components of the Gauss–Manin connection with respect to the Hodge filtration $\mathcal{F}^\bullet$ is not difficult to observe:

PROPOSITION 4.21. *Let ∇ be a local Gauss–Manin connection on $\mathcal{A}^{*,*}_{X|S}$ defined over $V \subset S$. Its complex structure components $\nabla^{1,0}$, $\nabla^{0,1}$ verify that*

$$\nabla^{1,0}\mathcal{F}^p f_*\mathcal{A}^{*,*}_{X|S} \subset \mathcal{A}^{1,0}_S \otimes \mathcal{F}^{p-1} f_*\mathcal{A}^{*,*}_{X|S}$$
$$\nabla^{0,1}\mathcal{F}^p f_*\mathcal{A}^{*,*}_{X|S} \subset \mathcal{A}^{0,1}_S \otimes \mathcal{F}^{p} f_*\mathcal{A}^{*,*}_{X|S}$$

PROOF. This is a consequence of the Cartan homotopy formula

$$L_v\omega = di_v\omega + i_v d\omega$$

The exterior differential $d = \partial + \bar\partial$ preserves the Hodge filtration, while contraction along a vector field lowers it by one unit in the case of a type $(1,0)$ field, or also leaves it invariant if v has pure type $(0,1)$. □

REMARK 4.22. Due to their homotopic uniqueness, Lemma 4.21 is still true up to homotopy for local Gauss–Manin connections that do not respect our complex structure convention 4.20.

The consequence of Lemma 4.21 for the variation of cohomology is immediate and well–known:

COROLLARY 4.23 (Griffiths transversality). *The Gauss-Manin connection on $\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}$ verifies that*

$$\nabla^{1,0}\mathcal{F}^p\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S} \subset \mathcal{A}^{1,0}_S \otimes \mathcal{F}^{p-1}\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}$$
$$\nabla^{0,1}\mathcal{F}^p\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S} \subset \mathcal{A}^{0,1}_S \otimes \mathcal{F}^{p}\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}$$

Given a smooth topologically locally trivial morphism $f : X \to S$ between complex manifolds, we have developed so far a Gauss–Manin connection of the underlying real analytic map, defined on the cohomology real analytic bundles $(\mathbb{R}^p f_*\mathbb{C}_X) \otimes \mathcal{A}_S$, and defined already locally on the complex of forms $f_*\mathcal{A}^{*,*}_{X|S}$, and checked that it verifies Griffiths transversality. The following natural step will be to compare this connection with the holomorphic Gauss–Manin connection on the holomorphic bundles $(\mathbb{R}^p f_*\mathbb{C}_X) \otimes \mathcal{O}_S$ (see [**31**], [**48**], [**73**]). The switch from a holomorphic to a larger real analytic bundle requires the use of an intermediate complex:

DEFINITION 4.24. The complex of *fiberwise holomorphic forms* $(K^*_{X|S}, d)$ is formed by the $\mathcal{A}_X$–submodules

$$K^p_{X|S} = \ker\left(\mathcal{A}^{p,0}_{X|S} \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_{X|S}\right)$$

and the coboundary operator d of $\mathcal{A}^{*,*}_{X|S}$.

This definition is correct because, as $d\bar{\partial} = \bar{\partial}d$, the coboundary operator d of $\mathcal{A}^{*,*}_{X|S}$ preserves the kernel of $\mathcal{A}^{*,0}_{X|S} \xrightarrow{\bar{\partial}} \mathcal{A}^{*,1}_{X|S}$. Note also that its restriction to $K^*_{X|S}$ equals ∂. The complex of fiberwise holomorphic forms has a Hodge filtration $\mathcal{F}^\bullet$, induced by that of the relative Dolbeault complex.

The forms in the complex $K^*_{X|S}$ are indeed real analytic families over S of holomorphic forms ω_s defined on the fibers X_s. The complex of fiberwise holomorphic forms takes the place of the complex of holomorphic forms in the absolute case:

LEMMA 4.25. *The natural inclusion* $(K^*_{X|S}, \partial) \hookrightarrow (\mathcal{A}^{*,*}_{X|S}, d)$ *is a filtered quasi-isomorphism of complexes of sheaves.*

PROOF. Our statement is equivalent to the exactness of the sequences

$$0 \longrightarrow K^p_{X|S} \longrightarrow \mathcal{A}^{p,0}_{X|S} \xrightarrow{\bar{\partial}} \mathcal{A}^{p,1}_{X|S} \xrightarrow{\bar{\partial}} \cdots$$

for every $p \geq 0$. This is a local property on X, so we may assume that S is a ball, with a single holomorphic chart $(s_1, \ldots, s_d)$, $X = Y \times S$, with Y another ball with coordinates $(y_1, \ldots, y_m)$ and f the natural projection. The exactness of the sequence is obtained now by a verbatim repetition of the proof of the real analytic Dolbeault lemma, our Proposition 4.7, with the variables $s_1, \ldots, s_d$ as parameters which are unaffected by differentiation. □

The next step is to examine the relation between the complexes of holomorphic and fiberwise holomorphic relative forms. This will be done by means of local Gauss–Manin connections, defined already over the forms. As our Definition 4.17 shows, a local Gauss–Manin connection over $V \subset S$ is equivalent to a trivialization $f^{-1}(V) \cong X_s \times V$, in the sense that both result from choosing a frame of horizontal vector fields on $f^{-1}(V)$. Such a choice of a horizontal frame also corresponds to determining a local $\mathcal{A}_X$–linear section $\mathcal{A}^{*,*}_{X|S} \to \mathcal{A}^{*,*}_X$ to the natural projection morphism.

Thus the choice of a local Gauss–Manin connection is not uniquely determined. Even if it follows our convention 4.20, neither a local Gauss–Manin connection nor its complex structure components $\nabla^{1,0}$, $\nabla^{0,1}$ have to preserve the complex $f_*K^*_{X|S}$. However, it turns out that there exists a unique complex arising from any choice of ∇.

PROPOSITION 4.26. *Let ∇ be a local Gauss–Manin connection defined on $V \subset S$, and $\nabla^{0,1}$ its $(0,1)$–component. There is an exact sequence*

$$0 \longrightarrow f_*\Omega^p_{X|S} \longrightarrow f_*K^p_{X|S} \xrightarrow{\nabla^{0,1}} \mathcal{A}^{0,1}_S \otimes f_*\mathcal{A}^{p,0}_{X|S}$$

for every $p \geq 0$.

PROOF. Our statement is a local property, so we may assume as in previous proofs that V is a ball and $X = Y \times V$, with Y another ball. Moreover, we may select our local trivialization by using the horizontal frame of the local Gauss–Manin connection ∇. In this way, there are complex coordinates $(s_1, \dots, s_d)$ in V and $(y_1, \dots, y_m)$ in Y such that the section $\mathcal{A}^{*,*}_{X|S} \hookrightarrow \mathcal{A}^{*,*}_X$ consists of the forms

$$\omega = \sum f_{I,J}(s_1, \dots, \bar{s}_d, y_1, \dots, \bar{y}_m) dy_I \wedge d\bar{y}_J$$

and the local Gauss–Manin connection ∇ is given by

$$\nabla\omega = \sum_i ds_i \sum \frac{\partial}{\partial s_i} f_{I,J} dy_I \wedge d\bar{y}_J + \sum_j d\bar{s}_j \frac{\partial}{\partial \bar{s}_j} f_{I,J} dy_I \wedge d\bar{y}_J .$$

Let $\hat{\omega}$ be the class of the above form ω in $\mathcal{A}^{*,*}_{X|S}$. The section $\mathcal{A}^{*,*}_{X|S} \to \mathcal{A}^{*,*}_X$ determined by our local Gauss–Manin connection and the holomorphic charts adapted to its horizontal frame sends $\bar{\partial}\hat{\omega}$ to

$$\sum_j \sum \frac{\partial}{\partial \bar{y}_j} f_{I,J} d\bar{y}_j \wedge dy_I \wedge d\bar{y}_J .$$

Consequently, the fact that $\hat{\omega} \in K^p_{X|S}$ means that $J = \emptyset$ and the functions f_I are holomorphic in the variables $y_1, \dots, y_m$. Likewise, the vanishing of $\nabla^{0,1}\omega = \sum_j d\bar{s}_j \sum \frac{\partial}{\partial \bar{s}_j} f_I dy_I$ implies that the functions f_I are holomorphic also in the variables $s_1, \dots, s_d$ thus are holomorphic on $Y \times V$ and $\omega \in \Omega^p_X, \hat{\omega} \in \Omega^p_{X|S}$. □

Finally, let us describe the relation between the real analytic and holomorphic Gauss–Manin connections in cohomology.

PROPOSITION 4.27. *Let $f : X \to S$ be a smooth topologically trivial map between complex manifolds X, S, and let $\nabla = \nabla^{1,0} + \nabla^{0,1}$ be the Gauss–Manin connection on the derived sheaves $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}$. There is then a commutative diagram with exact rows and isomorphisms in the vertical arrows*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{R}^p f_* \mathbb{C}_X \otimes \mathcal{O}_S & \longrightarrow & \mathbb{R}^p f_* \mathbb{C}_X \otimes \mathcal{A}_S & \xrightarrow{\bar{\partial}_S} & \mathbb{R}^p f_* \mathbb{C}_X \otimes \mathcal{A}^{0,1}_S \\
 & & \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathbb{R}^p f_* \Omega^*_{X|S} & \longrightarrow & \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S} & \xrightarrow{\nabla^{0,1}} & \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S} \otimes \mathcal{A}^{0,1}_S
\end{array}$$

and $\nabla^{1,0}$ induces the holomorphic Gauss–Manin connection on $\mathbb{R}^p f_* \Omega^*_{X|S}$.

PROOF. The isomorphism $\mathbb{R}^p f_* \mathbb{C}_X \otimes \mathcal{O}_S \cong \mathbb{R}^p f_* \Omega^*_{X|S}$ is the complex Poincaré lemma, and the isomorphisms $\mathbb{R}^p f_* \mathbb{C}_X \otimes \mathcal{A}_S \cong \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}$, respectively $\otimes\mathcal{A}_S^{0,1}$ are our Corollary 4.14. These isomorphisms arise from a commutative diagram of inclusions

$$\begin{array}{ccccc} \mathbb{C}_X & \longrightarrow & f^{-1}\mathcal{O}_S & \longrightarrow & \Omega^*_{X|S} \\ & \searrow & \downarrow & & \downarrow \\ & & f^{-1}\mathcal{A}_S & \longrightarrow & \mathcal{A}^{*,*}_{X|S} \end{array}$$

and because of this the first square commutes.

The upper row is exact by the real analytic Dolbeault lemma, our Proposition 4.7. We have defined the Gauss–Manin connection in cohomology as having as horizontal sections the local system $\mathbb{R}^p f_*\mathbb{C}_X$. This property characterizes it uniquely, so ∂_S is its $(1,0)$ component, which by restriction to $\mathbb{R}^p f_*\mathbb{C}_X \otimes \mathcal{O}_S$ yields the holomorphic Gauss–Manin connection, and $\bar{\partial}_S$ is its $(0,1)$–component, so the second square is a commutative diagram. This completes our proof. □

2.4. The horizontal relative logarithmic Dolbeault lemma. Unless otherwise stated, we will work in this section under the following assumptions:

CONVENTION 4.28. Let X, S be complex analytic manifolds, $f : X \to S$ a smooth proper map between them, and $H \subset X$ a relative normal crossing divisor, i.e., H is a normal crossing divisor in X, it has smooth irreducible components $\{H_i\}$, and the restrictions $f_{|H_i} : H_i \to S$ are smooth.

Denote $U := X \setminus H$ and $j : U \hookrightarrow X$. The restriction $f : U \to S$ is smooth and topologically locally trivial. Thus there is a covering of X by coordinate sets $(V, z_1, \dots, z_n)$ such that $f(z_1, \dots, z_n) = (z_1, \dots, z_d)$ and H has a defining equation $z_{d+1} \dots z_r = 0$.

We have defined in 2.2 the real analytic logarithmic Dolbeault complex $\mathcal{A}_X^{*,*}(\log H)$. A relative logarithmic Dolbeault complex may be defined from the exact sequence

$$0 \longrightarrow f^*\mathcal{A}_S^{1,0} \longrightarrow \mathcal{A}_X^{1,0}(\log H) \longrightarrow \mathcal{A}_{X|S}^{1,0}(\log H) \longrightarrow 0\,,$$

its analogue (0,1) and the corresponding wedge products for $\mathcal{A}^{p,q}_{X|S}(\log H)$. There is a natural inclusion $\mathcal{A}^{*,*}_{X|S}(\log H) \hookrightarrow j_*\mathcal{A}^{*,*}_{U|S}$, induced already by the defining exact sequences, and it may be checked that, unlike its $\mathcal{C}^\infty$ analogue, in the above holomorphic coordinate sets V the sheaf $\mathcal{A}^{*,*}_{X|S}(\log H)$ is the free $\mathcal{A}_X$-commutative graded algebra generated by

$$\frac{dz_i}{z_i}, \quad \frac{d\bar{z}_i}{\bar{z}_i}, \quad \log|z_i|, \qquad d+1 \le i \le r$$

and

$$dz_i, \quad d\bar{z}_i, \qquad r+1 \le i \le n$$

and differential $d = \partial + \bar{\partial}$ induced by that of $\mathcal{A}_X^{*,*}(\log H)$.

We will call the complex $\mathcal{A}_{X|S}^{*,*}(\log H)$ the *horizontal logarithmic relative Dolbeault complex.* This complex has a weight filtration and a Hodge filtration induced by those of the absolute logarithmic Dolbeault complex of subsection 2.2:

DEFINITION 4.29. The *weight filtration* $W_\bullet$ on $\mathcal{A}_{X|S}^{*,*}(\log H)$ is the increasing multiplicative filtration defined by assigning weight one to the generators $\frac{dz_i}{z_i}, \frac{d\bar{z}_i}{\bar{z}_i}, \log|z_i|$ for $d+1 \leq i \leq r$, and zero to $dz_j, d\bar{z}_j$ for $r+1 \leq j \leq n$.

The *Hodge filtration* $\mathcal{F}^\bullet$ on $\mathcal{A}_{X|S}^{*,*}(\log H)$ is the decreasing filtration defined by

$$\mathcal{F}^p \mathcal{A}_{X|S}^{*,*}(\log H) = \bigoplus_{s \geq p} \mathcal{A}_{X|S}^{s,*}(\log H).$$

We compare the complexes of sheaves $\mathcal{A}_{X|S}^{*,*}(\log H)$, f_*-acyclic by Cor. 4.6, and $\mathcal{A}_{U|S}^{*,*}$, which has been studied in subsection 2.3:

PROPOSITION 4.30. *The natural inclusion induces quasi-isomorphisms*

$$\mathcal{A}_{X|S}^{*,*}(\log H) \xrightarrow{\cong} j_* \mathcal{A}_{U|S}^{*,*}$$

PROOF. This is a local question on X, so given a point $x \in X$ we may assume by restriction that X is a polydisk with holomorphic coordinates $(z_1, \ldots, z_d, \ldots, z_n)$, $x = (0, \ldots, 0)$, $Y \subset X$ has as a defining equation $z_{d+1} \cdots z_r = 0$, with $d < r \leq n$, and $f(z_1, \ldots, z_n) = (z_1, \ldots, z_d)$.

As the morphism f is now a trivial fibration with fiber $(\mathbb{D}^*)^{r-d} \times \mathbb{D}^{n-r}$, the integrable bundle formed by the cohomology classes of the fibers is

$$H^*(\mathcal{A}_{U|S}^{*,*}, d) \cong \bigwedge_{i=d+1}^{r} \mathcal{A}_S \frac{dz_i}{z_i}$$

The forms $\frac{dz_i}{z_i}$ are logarithmic, therefore the inclusion $\mathcal{A}_{X|S}^{*,*}(\log H) \hookrightarrow \mathcal{A}_{U|S}^{*,*}$ gives rise to a commutative diagram

$$\begin{array}{ccccc} & & H^*(\mathcal{A}_{X|S}^{*,*}(\log H), d) & & \\ & \nearrow & & \searrow & \\ \bigwedge^*(\bigoplus_{i=d+1}^r \mathcal{A}_S \frac{dz_i}{z_i}) & & \xrightarrow{\cong} & & H^*(\mathcal{A}_{U|S}^{*,*}, d) \end{array}$$

Thus it suffices to check that

$$\bigwedge^*(\bigoplus_{i=d+1}^{r} \mathcal{A}_S \frac{dz_i}{z_i}) \to H^*(\mathcal{A}_{X|S}^{*,*}(\log H), d) \tag{15}$$

is an isomorphism to show our statement. The morphism is onto by the previous commutative diagram, and it remains to check injectivity.

Define an ad hoc increasing non–holomorphic weight filtration on $\mathcal{A}^{*,*}_{X|S}(\log H)$ by setting weight one to the forms $\frac{d\bar{z}_i}{\bar{z}_i}$ and functions $\log|z_i|$ and weight zero to the forms $\frac{dz_i}{z_i}, dz_j, d\bar{z}_j$ for $d+1 \leq i \leq r$, $r+1 \leq j \leq n$. Extend the filtration multiplicatively to the $\mathcal{A}_X$–algebra $\mathcal{A}^{*,*}_{X|S}(\log H)$.

We will show that every class $\bar{\omega} \in H^*(\mathcal{A}^{*,*}_{X|S}(\log H), d)$ has a representing cocycle with non–holomorphic weight zero, i.e. without any factor $\frac{d\bar{z}_i}{\bar{z}_i}, \log|z_i|$ on its summands. This may be done by double induction on $i \in \{d+1, \ldots, r\}$ and the non–holomorphic weight on every z_i.

Let ω be a representing cocycle of the class $\bar{\omega}$. If ω has non–holomorphic weight zero on every z_i it is our sought cocycle. Otherwise, let $i \in \{d+1, \ldots, r\}$ be the highest coordinate with nonzero weight k. We may write the cocycle as

$$\omega = \log|z_i|^k \alpha + \log|z_i|^{k-1} \frac{d\bar{z}_i}{\bar{z}_i} \wedge \beta + \gamma\,,$$

where γ has non–holomorphic weight $< k$ in z_i and zero in $z_{i+1}, \ldots, z_r$. The fact that ω is a cocycle implies that

$$0 = d\omega = k \log|z_i|^{k-1} \frac{d\bar{z}_i}{\bar{z}_i} \wedge \alpha + \log|z_i|^k d\alpha - \log|z_i|^{k-1} \frac{d\bar{z}_i}{\bar{z}_i} \wedge d\beta + \gamma'\,,$$

with γ' again with top nonzero weight in z_i and lower than k. Grouping terms in the last equality we have that $d\alpha = 0, k\alpha - d\beta = 0$, thus $\alpha = \frac{1}{k} d\beta$, so the cocycle

$$\omega - d\left(\frac{1}{k} \log|z_i|^k \beta\right) = \gamma' - \log|z_i|^{k-1} \frac{dz_i}{z_i} \wedge \beta$$

has top nonzero weight in z_i and lower than k. Thus by induction we can find a cocycle representing $\bar{\omega}$ with no factors $\frac{d\bar{z}_i}{\bar{z}_i}, \log|z_i|$ in its summands.

Let now ω be an m–cocycle with non–holomorphic weight zero. It has an expression of the form

$$\omega = \sum \alpha_I \frac{dz_I}{z_I}\,,$$

where I ranges over the subsets of $\{d+1, \ldots, r\}$ with cardinal $|I| \leq m$, and the forms α_I contain only the differentials $dz_j, d\bar{z}_j$ with $r+1 \leq j \leq n$. As ω is a cocycle, there is an equality

$$0 = d\omega = \sum d\alpha_I \frac{dz_I}{z_I}\,.$$

By the freeness of the horizontal logarithmic relative complex in our local setting, this means that the forms α_I are cocycles. Since they contain no differential $\frac{dz_i}{z_i}, \frac{d\bar{z}_i}{\bar{z}_i}$, nor factor $\log|z_i|$, the cocycles α_I extend to X and by the relative real analytic Poincaré lemma, our Proposition 4.13 they are either exact or zero–cocycles.

By our previous induction computation, the cocycles $\varphi \in H^0(\mathcal{A}^{*,*}_{X|S}(\log H), d)$ may not have any factor $\log|z_i|$. Such cocycles lie therefore in $H^0(\mathcal{A}^{*,*}_{X|S}, d) \cong \mathcal{A}_S$, and the proof of our Proposition is now complete. □

The previous Proposition and Cor. 4.14 may be expressed as:

COROLLARY 4.31. *The natural map of sheaves* $\mathbb{C}_U \to \mathcal{A}^{*,*}_{X|S}(\log H)$ *induces isomorphisms*

$$\mathbb{R}^p f_* \mathbb{C}_U \underset{\mathbb{C}}{\otimes} \mathcal{A}_S \xrightarrow{\cong} (\mathcal{A}^{*,*}_{X|S}(\log H), f_* d_{X|S})$$

for all $p \geq 0$.

Therefore there is a Gauss-Manin connection on $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H)$, having the local systems $\{H^*(X_s \backslash H_s, \mathbb{C}) \mid s \in S\}$ as horizontal sections. This connection is actually the Gauss–Manin connection developed in the previous subsection, and it may also be locally defined over forms:

PROPOSITION 4.32. *Let* v_0 *be a real analytic tangent field defined on an open set* $V \subset S$. *Then:*

(i) *The field* v_0 *admits a real analytic lifting* v *defined over* $f^{-1}(V) \subset X$ *such that* v *is tangent to* H.
(ii) *The Lie derivative* $j_* L_v : j_* \mathcal{A}^{*,*}_{U|S} \to j_* \mathcal{A}^{*,*}_{U|S}$ *preserves the horizontal logarithmic subcomplex* $\mathcal{A}^{*,*}_X(\log H)$, *and induces a derivation in* $\mathcal{A}^{*,*}_{X|S}(\log H)$ *that depends only up to homotopy on the selected lifting* v.
(iii) *The induced derivation* $\nabla_{v_0} := \mathbb{R}^p f_* L_v$ *on* $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H)$ *depends only on* v_0, *and is the Gauss–Manin connection along* v_0.
(iv) *If* u_0, v_0 *are vector fields on* $V \subset S$ *such that* $[u_0, v_0] = 0$, *and* u, v *are relative tangent lifts to* $f^{-1}(V) \subset X$, *then the bracket* $[L_u, L_v]$ *of induced derivations in* $\mathcal{A}^{*,*}_{X|S}(\log H)$ *is null–homotopic.*

PROOF. (i) Let $T_{(X,H)}$ be the relative tangent sheaf, i.e. the sheaf of tangent fields on X which are also tangent to H. This is a coherent sheaf, and so is the relative tangent sheaf $T_{(X,H)|S}$, defined through the exact sequence

$$0 \longrightarrow T_{(X,H)|S} \longrightarrow T_{(X,H)} \longrightarrow f^* T_S \longrightarrow 0$$

As the sheaves in this sequence are acyclic, every vector field defined on $V \subset S$ lifts to $(f^{-1}(V), H \cap f^{-1}(V))$.
(ii),(iii) Choose local coordinates $(z_1, \ldots, z_n)$ as in the definition of the relative logarithmic complex, with $f(z_1, \ldots, z_n) = (z_1, \ldots, z_d)$ and H

defined by the equation $z_{d+1}\dots z_r = 0$. The relative tangent sheaf $T_{(X,H)}$ is the locally free sheaf with basis

$$\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j}, z_i\frac{\partial}{\partial z_i}, \bar{z}_i\frac{\partial}{\partial \bar{z}_i}$$

for $i \in \{d+1\dots,r\}$, $j \in \{r+1,\dots,n\}$. An immediate computation shows that derivation along these fields preserves the generators of the logarithmic complex $\mathcal{A}_X^{*,*}(\log H)$.

As has been seen in Proposition 4.16 for the complex $\mathcal{A}_{U|S}^{*,*}$, this derivation depends on the lifting of v_0 only up to homotopy, and given two lifts $v, v' \in T_{(X,H)}$ the homotopy between $L_v, L_{v'}$ is the contraction $h = i_{v'-v}$. This contraction along a vector field of the relative tangent field also preserves logarithmic forms, so the homotopy h restricts to the subcomplex $\mathcal{A}_{X|S}^{*,*}(\log H) \subset \mathcal{A}_{U|S}^{*,*}$. The induced derivation ∇_{v_0} : $\mathbb{R}^p f_*\mathcal{A}_{X|S}^{*,*}(\log H) \to \mathcal{A}_{X|S}^{*,*}(\log H)$ is by Propositions 4.16 and 4.30 the Gauss–Manin connection along v_0.
(iv) One may check in the same way that if u_0, v_0 are tangent vector fields in $V \subset S$ with $[u_0, v_0] = 0$ and u, v are relative tangent lifts, the homotopy in $\mathcal{A}_{U|S}^{*,*}$ between the bracket $[L_u, L_v]$ and the zero morphism defined in the proof of Prop. 4.16 (iv) preserves the horizontal logarithmic subcomplex. □

One may proceed now as in the relative case of subsection 2.3 to define local Gauss–Manin connections on forms and check the properties of the local and cohomology logarithmic connections:

DEFINITION 4.33. Let $V \subset S$ be an open subset, $v_1, \dots, v_d$ vector fields such that they form a basis of the tangent sheaf T_S on V, and $\tilde{v}_1, \dots, \tilde{v}_d$ relative tangent lifts to $(f^{-1}(V), H \cap f^{-1}(V))$. The connection defined by setting

$$\nabla_{v_i}\omega = f_* L_{\tilde{v}_i}\omega$$

is a *local Gauss–Manin connection* on the complex of forms $f_*\mathcal{A}_{X|S}^{*,*}(\log H)$ restricted to $f^{-1}(V)$.

Proposition 4.32 shows that the Gauss–Manin connection and its local analogues on forms for the relative logarithmic complex $\mathcal{A}_{X|S}^{*,*}(\log H)$ are induced by the local Gauss–Manin connections of the relative complex $f_*\mathcal{A}_{U|S}^{*,*}$. An immediate consequence of Proposition 4.32 (iv) is:

LEMMA 4.34. *Every local Gauss–Manin connection defined on $V \subset S$ is homotopically integrable.*

Another relevant property that local Gauss–Manin connections inherit from the complex $\mathcal{A}_{U|S}^{*,*}$ is Griffiths transversality:

PROPOSITION 4.35. *Let ∇ be a local Gauss–Manin connection on $\mathcal{A}^{*,*}_{X|S}(\log H)$ defined over $V \subset S$. Its complex structure components $\nabla^{1,0}, \nabla^{0,1}$ verify that*

$$\nabla^{1,0}\mathcal{F}^p f_*\mathcal{A}^{*,*}_{X|S} \subset \mathcal{A}^{1,0}_S \otimes \mathcal{F}^{p-1} f_*\mathcal{A}^{*,*}_{X|S}(\log H)$$
$$\nabla^{0,1}\mathcal{F}^p f_*\mathcal{A}^{*,*}_{X|S} \subset \mathcal{A}^{0,1}_S \otimes \mathcal{F}^{p} f_*\mathcal{A}^{*,*}_{X|S}(\log H)$$

PROOF. By Proposition 4.32 a local Gauss–Manin connection in $\mathcal{A}^{*,*}_{X|S}(\log H)$ is a particular case of a local Gauss–Manin connection in $\mathcal{A}^{*,*}_{U|S}$, thus our statement follows from Proposition 4.21. □

At the cohomology level, one has:

PROPOSITION 4.36. (i) *The Gauss–Manin connection ∇ on $H^p(j_*\mathcal{A}^{*,*}_{U|S}, j_*d)$ induces the relative logarithmic Gauss–Manin connection*

$$\nabla : \mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H) \longrightarrow \mathcal{A}^1_S \otimes \mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H)$$

(ii) *The Gauss–Manin connection $\nabla = \nabla^{1,0} + \nabla^{0,1}$ on $\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H)$ verifies the Griffiths transversality relations*

$$\nabla^{1,0}\mathcal{F}^k\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H) \subset \mathcal{A}^{1,0}_S \otimes \mathcal{F}^{k-1}\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H)$$
$$\nabla^{0,1}\mathcal{F}^k\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H) \subset \mathcal{A}^{0,1}_S \otimes \mathcal{F}^{k}\mathbb{R}^p f_*\mathcal{A}^{*,*}_{X|S}(\log H)$$

Another property specific to the logarithmic complex is:

PROPOSITION 4.37. *There exists a local Gauss–Manin connection on $f_*\mathcal{A}^{*,*}_{X|S}(\log H)$ defined over a neighbourhood $s \in V \subset S$ such that it preserves the weight filtration $W_\bullet$ in a possibly smaller neighbourhood $p \in V' \subset V$.*

PROOF. Given $s \in V$, we may take as V' a trivializing open subset for f, such that $f^{-1}(V')$ is real analytically isomorphic to $(X_s, H_s) \times V'$ over V'.

Take then a covering of (X_s, H_s) by coordinate charts $(y_1, \dots, y_m, \bar{y}_1, \dots, \bar{y}_m)$. This induces a covering of $(X_s, H_s) \times V'$ by coordinate charts $(y_1, \dots, \bar{y}_m, s_1, \dots, s_d, \bar{s}_1, \dots, \bar{s}_d)$, where the variables $s_i, \bar{s}_j$ come from V' and are invariant under change of coordinates.

A local Gauss–Manin connection ∇' is given now on the absolute logarithmic complex $\mathcal{A}_{X_s \times V'}(\log H_s \times V')$ by covariant derivation along the vector fields $\frac{\partial}{\partial s_1}, \dots, \frac{\partial}{\partial \bar{s}_d}$. The coboundary operator d preserves the weight filtration on the absolute logarithmic complex, and contraction along a field $\frac{\partial}{\partial s_j}, \frac{\partial}{\partial \bar{s}_j}$ does not affect the positive weight terms $\log|y_i|, \frac{dy_i}{y_i}, \frac{d\bar{y}_j}{\bar{y}_j}$. Therefore the weight filtration is preserved in the absolute complex, so also in its quotient $\mathcal{A}^{*,*}_{X_s \times V'|V'}(\log H_s \times V')$. □

The consequence of Proposition 4.37 at the derived level is

COROLLARY 4.38. *The Gauss–Manin connection* ∇ *on* $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H)$ *preserves the weight filtration* $W_\bullet$.

The comparison between the real analytic and the holomorphic Gauss–Manin connection parallels that of the relative complexes $\mathcal{A}^{*,*}_{X|S}$ discussed in the previous subsection. We will write the analogous statements and indicate the proof when it is not a consequence of its analogue in subsection 2.3:

DEFINITION 4.39. The complex of logarithmic fiberwise holomorphic forms $(K^{*,*}_{X|S}(\log H)$ is formed by the $\mathcal{A}_X$–submodules

$$K^p_{X|S}(\log H) = \ker\left(\mathcal{A}^{p,0}_{X|S}(\log H) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_{X|S}\right)$$

and the coboundary operator d of $\mathcal{A}^{*,*}_{X|S}(\log H)$.

By the natural inclusion $\mathcal{A}^{*,*}_{X|S}(\log H) \hookrightarrow j_*\mathcal{A}^{*,*}_{U|S}$, the complex $K^*_{X|S}(\log H)$ is a subcomplex of the fiberwise holomorphic complex $j_*K^*_{U|S}$ previously described. Again, the Hodge filtration on $\mathcal{A}^{*,*}_{X|S}(\log H)$ induces a Hodge filtration $\mathcal{F}^\bullet$ on $K^*_{X|S}(\log H)$. The relative Dolbeault lemma in this context has the same statement, but a more involved proof:

LEMMA 4.40 (relative Dolbeault lemma, horizontal logarithmic version). *The natural inclusion* $(K^*_{X|S}(\log H), \partial) \hookrightarrow (\mathcal{A}^{*,*}_{X|S}(\log H), d)$ *is a bifiltered quasi–isomorphism of complexes of sheaves.*

PROOF. Our statement is equivalent to the exactness of the sequences

$$0 \longrightarrow K^p_{X|S}(\log H) \longrightarrow \mathcal{A}^{p,0}_{X|S}(\log H) \xrightarrow{\bar\partial} \mathcal{A}^{p,1}_{X|S}(\log H) \xrightarrow{\bar\partial} \dots$$

for every $p \geq 0$. As in the relative case, this is a local question on X, so we may assume that X, S are polydisks, f has the form $f(z_1, \dots, z_n) = (z_1, \dots, z_d)$ and H has a defining equation $z_{d+1} \dots z_r = 0$.

The local question may now be proved following verbatim the proof of Theorem 8.8 in [**72**], which is the analogous absolute logarithmic case. The variables $z_1, \dots, z_d$ from S appear as additional parameters, but they do not show up in the exterior derivations because of our relative setting. □

On the other hand, the relation between the fiberwise holomorphic and the holomorphic complex are a straightforward consequence of Lemma 4.26:

LEMMA 4.41. *Let ∇ be a local Gauss–Manin connection defined on $\mathcal{A}^{*,*}_{X|S}(\log H)$ over an open set $V \subset S$, and $\nabla^{0,1}$ its $(0,1)$-component. There is an exact sequence*

$$0 \longrightarrow \Omega^p_{X|S}(\log H) \longrightarrow K^p_{X|S}(\log H) \xrightarrow{\nabla^{0,1}} \mathcal{A}^{0,1}_S \otimes \mathcal{A}^{p,0}_{X|S}$$

for every $p \geq 0$.

PROOF. As $K^p_{X|S}(\log H) \subset K^p_{U|S}$, and the local Gauss–Manin connection ∇ is actually defined on $\mathcal{A}^{*,*}_{U|S}$, by Lemma 4.26 the kernel of $\nabla^{0,1}_{|K^p_{X|S}(\log H)}$ is

$$\Omega^p_{U|S} \cap K^p_{X|S}(\log H) = \Omega^p_{X|S}(\log H)$$

□

We conclude this section with the commutative diagram formed by the holomorphic and real analytic cohomology connections. Its proof consists in applying Proposition 4.27 to the family $f : U \to S$ and then taking intersections with the logarithmic complexes $\mathcal{A}^{*,*}_{X|S}(\log H), \Omega^*_{X|S}(\log H)$ as in the previous proof.

PROPOSITION 4.42. *Let $f : (X, H) \to S$ be a smooth holomorphic mapping satisfying Convention 4.28, and let $\nabla = \nabla^{1,0} + \nabla^{0,1}$ be the Gauss–Manin connection on the derived sheaves $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H)$. There is then a commutative diagram with exact rows and isomorphisms in the vertical arrows*

$$\begin{array}{ccccccc}
0 & \longrightarrow & \mathbb{R}^p f_* \mathbb{C}_U \otimes \mathcal{O}_S & \longrightarrow & \mathbb{R}^p f_* \mathbb{C}_U \otimes \mathcal{A}_S & \xrightarrow{\bar\partial_S} & \mathcal{A}^{0,1}_S \otimes \mathbb{R}^p f_* \mathbb{C}_U \\
 & & \cong \downarrow & & \downarrow \cong & & \downarrow \cong \\
0 & \longrightarrow & \mathbb{R}^p f_* \Omega^p_{X|S}(\log H) & \longrightarrow & \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H) & \xrightarrow{\nabla^{0,1}} & \mathcal{A}^{0,1}_S \otimes \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H)
\end{array}$$

3. Real analytic variations of Hodge structure

When X is a compact Kähler manifold, the real analytic Dolbeault complex $\mathcal{A}^{*,*}_X$ satisfies a $\partial\bar\partial$-lemma, and the Hodge filtration $\mathcal{F}^\bullet$ of the complex induces the pure Hodge structures of the cohomology groups of X. Likewise, if $Y \subset X$ is a normal crossing divisor, the real analytic logarithmic Dolbeault complex induces Deligne's mixed Hodge structure on the cohomology groups $H^*(X \setminus Y)$ ([**72**]).

In the relative case, the cohomology of a holomorphic family of compact Kähler manifolds supports a variation of Hodge structures (see [**43**]). We recall its definition in order to compare it with our proposed definition for its real analytic analogue:

DEFINITION 4.43. A *variation of Hodge structure* of pure weight n over a complex manifold S consists of:

(i) a local system of $\mathbb{Z}$–free modules of finite rank $\mathbb{V}$ over S,
(ii) a holomorphic vector bundle $\mathcal{V}$ over S,
(iii) an integrable connection on $\mathcal{V}$, $\nabla : \mathcal{V} \to \Omega^1_S \otimes \mathcal{V}$,
(iv) a decreasing Hodge filtration of $\mathcal{V}$ by holomorphic subbundles

$$\mathcal{V} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^{n+1} = \{0\},$$

such that

- the local system formed by the horizontal sections of $(\mathcal{V}, \nabla)$ is isomorphic to $\mathbb{V} \otimes \mathbb{C}$,
- the fiber data $(\mathbb{V}_s, \mathcal{V}_s, \mathcal{F}^\bullet_s)$ defines a Hodge structure of pure weight n for every $s \in S$,
- (Griffiths transversality) the connection ∇ and the filtration $\mathcal{F}$ verify

$$\nabla(\mathcal{F}^p) \subset \Omega^1_S \otimes \mathcal{F}^{p-1}.$$

The corresponding concept that we have encountered using the real analytic structural sheaf is:

DEFINITION 4.44. A real analytic *variation of Hodge structure* of pure weight n over a complex manifold S consists of:

(i) a local system of $\mathbb{Z}$–free modules of finite rank $\mathbb{V}$ over S,
(ii) a real analytic vector bundle $\mathcal{W}$ over S,
(iii) an integrable connection on $\mathcal{W}$, $\nabla : \mathcal{W} \to \mathcal{A}^1_S \otimes \mathcal{W}$,
(iv) a decreasing Hodge filtration of $\mathcal{W}$ by real analytic subbundles

$$\mathcal{W} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^{n+1} = \{0\},$$

such that

- the local system formed by the horizontal sections of $(\mathcal{V}, \nabla)$ is isomorphic to $\mathbb{V} \otimes \mathbb{C}$,
- the fiber data $(\mathbb{V}_s, \mathcal{W}_s, \mathcal{F}^\bullet_s)$ defines a Hodge structure of pure weight n for every $s \in S$,
- (Griffiths transversality) the complex structure components $\nabla^{1,0}$, $\nabla^{0,1}$ of the connection ∇ and the filtration $\mathcal{F}$ verify

$$\nabla^{1,0}(\mathcal{F}^p) \subset \mathcal{A}^{1,0}_S \otimes \mathcal{F}^{p-1}$$
$$\nabla^{0,1}(\mathcal{F}^p) \subset \mathcal{A}^{0,1}_S \otimes \mathcal{F}^{p}$$

The concepts of real analytic and (complex) variation of Hodge structure are actually equivalent:

LEMMA 4.45. *Let $VHS(S)(n)$ and $\mathcal{A}-VHS(S)(n)$ be the categories of variations of Hodge structure, resp. real analytic variations of Hodge structure, of pure weight n over a complex manifold S. There exist functors*

$$\begin{aligned} .\otimes \mathcal{A}_S : VHS(S)(n) &\longrightarrow \mathcal{A}-VHS(S)(n) \\ (\mathbb{V},\mathcal{V},\nabla,\mathcal{F}^\bullet) &\longmapsto (\mathbb{V},\mathcal{V}\otimes_{\mathcal{O}_S}\mathcal{A}_S,\nabla_{an},\mathcal{F}^\bullet\otimes\mathcal{A}_S) \end{aligned}$$

and

$$\begin{aligned} \ker\nabla^{0,1} : \mathcal{A}-VHS(S)(n) &\longrightarrow VHS(S)(n) \\ (\mathbb{V},\mathcal{W},\nabla,\mathcal{F}^\bullet) &\longmapsto (\mathbb{V},\ker\nabla^{0,1},\nabla^{1,0},\mathcal{F}^\bullet\cap\ker\nabla^{0,1}) \end{aligned}$$

which induce an equivalence of categories.

PROOF. Given the underlying local system of free $\mathbb{Z}$–modules $\mathbb{V}$, by the correspondence between local systems and integrable bundles, the holomorphic vector sheaf of a variation of Hodge structure is isomorphic to $\mathbb{V}\otimes\mathcal{O}_S$, and the connection ∇ in this presentation is

$$\mathrm{Id}\otimes d : \mathbb{V}\otimes\mathcal{O}_S \longrightarrow \Omega^1_S\otimes\mathbb{V}.$$

In the same way, the vector bundle of a real analytic variation of Hodge structure is isomorphic to $\mathbb{V}\otimes\mathcal{A}_S$, and the connection ∇ becomes

$$\mathrm{Id}\otimes d : \mathbb{V}\otimes\mathcal{A}_S \longrightarrow \mathcal{A}^1_S\otimes\mathbb{V},$$

and the complex structure components of ∇ are $\nabla^{1,0}=\partial, \nabla^{0,1}=\bar\partial$.

Consequently, if $(\mathbb{V},\mathcal{V},\nabla,\mathcal{F}^\bullet)$ is a variation of Hodge structure of pure weight n, we extend to $\mathcal{W}=\mathcal{V}\otimes\mathcal{A}_S$ the connection ∇_{an} by imposing that $\nabla^{an}(\omega)=0$ for the holomorphic forms $\omega\in\mathcal{V}$ plus the Leibnitz rule, and the filtration $\mathcal{F}^\bullet$ by $\mathcal{A}_S$–linearity, the resulting data verifies:

- $\mathcal{W}\cong(\mathbb{V}\otimes\mathcal{O}_S)\otimes\mathcal{A}_S\cong\mathbb{V}\otimes\mathcal{A}_S$.
- There is an equality of Hodge structures $(\mathbb{V}_s,\mathcal{W}_s,\mathcal{F}_s)=(\mathbb{V}_s,\mathcal{V}_s,\mathcal{F}_s)$ for every $s\in S$.
- Let $\omega\in\mathcal{F}^p(V)$ be a local section of $\mathcal{V}$ over an open set $V\subset S$. By the Leibnitz rule, if $\varphi\in\mathcal{A}_S$, we have

$$\begin{aligned} \partial(\varphi\omega) &= (\partial\varphi)\omega+\varphi(\partial\omega)\in\mathcal{A}^{1,0}_S\otimes\mathcal{F}^p+\mathcal{A}^1_S\otimes\mathcal{F}^{p-1} \\ \bar\partial(\varphi\omega) &= (\bar\partial\varphi)\omega\in\mathcal{A}^{0,1}_S\otimes\mathcal{F}^p \end{aligned}$$

  Therefore the data $(\mathbb{V},\mathcal{V}\otimes\mathcal{A}_S,\nabla_{an},\mathcal{F}^\bullet\otimes\mathcal{A}_S)$ satisfies the real analytic Griffiths transversality condition.

Thus $(\mathbb{V}, \mathcal{V}, \nabla, \mathcal{F}^\bullet) \otimes \mathcal{A}_S$ is a real analytic variation of Hodge structure.

Conversely, if $(\mathbb{V}, \mathcal{W}, \nabla, \mathcal{F}^\bullet)$ is a real analytic variation of Hodge structure, by the isomorphism $\mathcal{W} \otimes \mathbb{V} \otimes \mathcal{A}_S$, sending $\nabla^{1,0}, \nabla^{0,1}$ to $\partial, \bar{\partial}$ we have:

- $\ker \nabla^{0,1} \cong \ker(\bar{\partial} : \mathcal{A}_S \otimes \mathbb{V} \to \mathcal{A}_S^{0,1} \otimes \mathbb{V}) = \mathbb{V} \otimes \mathcal{O}_S$, by the Dolbeault lemma.
- By the above isomorphism, there is an equality of Hodge structures $(\mathbb{V}_s, \mathcal{W}_s, \mathcal{F}_s^\bullet) = (\mathbb{V}_s, \ker \nabla_s^{0,1}, \mathcal{F}_s^\bullet \cap \ker \nabla_s^{0,1})$ for every $s \in S$.
- If $\omega \in \ker \nabla^{0,1} \cap \mathcal{F}^p(V)$ is a section defined on an open set $V \subset S$, $\nabla^{1,0}(\omega) \in \mathcal{A}^{1,0} \otimes (\ker \nabla^{0,1} \cap \mathcal{F}^{p-1}$ because $\nabla^{1,0}, \nabla^{0,1}$ commute and real analytic Griffiths transversality. Moreover, as $\ker \nabla^{0,1} \cong \mathbb{V} \otimes \mathcal{O}_S$, and $\nabla^{1,0} \cong \partial$, it turns out that $\omega = \varphi \otimes w$, with $w \in \mathbb{V}(V)$ a section of the underlying local system, and φ a holomorphic function. Hence follows that $\partial\omega = \partial\varphi \otimes w \in \Omega_S^1 \otimes \mathbb{V}$, and holomorphic Griffiths transversality.

We conclude that the functors $\otimes \mathcal{A}_S, \ker \nabla^{0,1}$ are well defined. Moreover, they are inverse functors by the isomorphisms

$$\ker(\bar{\partial} : \mathcal{A}_S \otimes \mathbb{V} \to \mathcal{A}_S^{0,1} \otimes \mathbb{V}) \cong \mathbb{V} \otimes \mathcal{O}_S$$
$$\mathcal{A}_S \otimes (\mathcal{O}_S \otimes \mathbb{V}) \cong \mathcal{A}_S \otimes \mathbb{V}$$

□

REMARK 4.46. If we let $\mathbb{V}$ be a local system of $\mathbb{Q}$–vector spaces in our definitions of variation of Hodge structure, the functors $\otimes \mathcal{A}_S, \ker \nabla^{0,1}$ actually define an equivalence of tensor categories.

The equivalence between complex variations of Hodge structure and real analytic variations also holds for variations with any coefficient ring. For instance, a real variation of Hodge structure $(\mathbb{R} - VHS)$ over a complex manifold S is defined in the same way as in Definition 4.43, except that the underlying local system $\mathbb{V}$ is formed by $\mathbb{R}$–vector spaces. The same change in Definition 4.44 yields a real analytic $\mathbb{R}$–variation of Hodge structure $(\mathcal{A} - \mathbb{R} - VHS)$, and the functors $\otimes \mathcal{A}_S, \ker \nabla^{0,1}$ extend to real variations and induce an equivalence between the categories $\mathbb{R} - VHS(S)(n)$ and $\mathcal{A} - \mathbb{R} - VHS(S)(n)$.

The relative Dolbeault complexes in this chapter provide a natural example of real analytic variations of Hodge structure, and its relation to the relative holomorphic de Rham complex may be explained by the correspondence between real analytic and complex variations:

PROPOSITION 4.47. *Let $f : X \to S$ be a smooth proper morphism between complex manifolds X, S, such that the fibers X_s form a family of compact Kähler manifolds, and let $\mathcal{A}_{X|S}^{*,*}$ be the relative Dolbeault* *complex defined in the previous section, with its Hodge filtration* $\mathcal{F}^\bullet$ *and its Gauss–Manin connection* ∇ *on* $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}$*. The set of data*

$$(\mathbb{R}^p f_* \mathbb{R}_X, \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}, \nabla, \mathcal{F}^\bullet)$$

defines a real analytic $\mathbb{R}$*–variation of Hodge structure of pure weight* p *over* S*. The complex* $\mathbb{R}$*–variation* $\ker \nabla^{0,1}(\mathbb{R}^p f_* \mathbb{R}_X, \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}, \nabla, \mathcal{F}^\bullet)$ *is isomorphic to the variation induced by the relative holomorphic de Rham complex*

$$(\mathbb{R}^p f_* \mathbb{R}_X, \mathbb{R}^p f_* \Omega^*_{X|S}, \nabla, \mathcal{F}^\bullet).$$

PROOF. By our Corollary 4.14, $\mathbb{R}^p f_* \mathbb{R}_X \otimes \mathcal{A}_S \cong \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}$, and $(\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S})_s \cong H^p(X_s, \mathbb{C})$. Moreover $(\mathcal{A}^{*,*}_{X|S})_{|X_s} \cong \mathcal{A}^{*,*}_{X_s}$, so by the real analytic $\partial\bar{\partial}$–lemma the data $(H^p(X_s, \mathbb{R}), (\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S})_s, \mathcal{F}^\bullet_s)$ is the pure Hodge structure of the fiber X_s (Lemma 4.8 and Proposition 4.9). Griffiths transversality is established in Proposition 4.23.

The comparison with the complex variation $(\mathbb{R}^p f_* \mathbb{R}_X, \mathbb{R}^p f_* \Omega^*_{X|S}, \nabla, \mathcal{F}^\bullet)$ follows from Proposition 4.27. □

We have shown so far the equivalence between real analytic and complex variations of pure Hodge structure, but the concepts and proof techniques involved carry into variations of mixed Hodge structure. The only new addition that is required is the fact that the Gauss–Manin connection preserves the weight filtration, which is our Proposition 4.36 (ii) in the horizontal logarithmic setting. The definitions and statements are:

DEFINITION 4.48. A *real analytic variation of mixed Hodge structure* over a complex manifold S consists of:

(i) a local system of free $\mathbb{Z}$–modules of finite rank $\mathbb{V}$ over S,
(ii) an increasing weight filtration $W_\bullet$ on $\mathbb{V} \otimes \mathbb{Q}$,
(iii) a real analytic vector bundle $\mathcal{W}$ over S,
(iv) an integrable connection on $\mathcal{W}$, $\nabla : \mathcal{W} \to \mathcal{A}^1_S \otimes \mathcal{W}$,
(v) a decreasing Hodge filtration of $\mathcal{W}$ by real analytic subbundles $\mathcal{W} = \mathcal{F}^0 \supseteq \mathcal{F}^1 \supseteq \cdots \supseteq \mathcal{F}^N = \{0\}$,

such that

- the local system formed by the horizontal sections of $(\mathcal{W}, \nabla)$ is isomorphic to $\mathbb{V} \otimes \mathbb{C}$,
- the fiber data $(\mathbb{V}_s, \mathcal{W}_s, (W_\bullet)_s, \mathcal{F}^\bullet_s)$ defines a mixed Hodge structure for every $s \in S$,

- (Griffiths transversality) the complex structure components $\nabla^{1,0}$, $\nabla^{0,1}$ of the connection ∇ and the Hodge filtration $\mathcal{F}$ verify

$$\nabla^{1,0}(\mathcal{F}^p) \subset \mathcal{A}_S^{1,0} \otimes \mathcal{F}^{p-1}$$
$$\nabla^{0,1}(\mathcal{F}^p) \subset \mathcal{A}_S^{0,1} \otimes \mathcal{F}^p$$

- the quotients $W_l/W_{l-1}(\mathbb{V}, \mathcal{W}, \nabla, \mathcal{F}^\bullet)$ are real analytic variations of pure Hodge structure of weight l.

REMARK 4.49. Given the isomorphism $\mathbb{V}\otimes\mathbb{C} \cong \mathcal{W}^\nabla$, the existence of a weight filtration $W_\bullet$ on $\mathbb{V}\otimes\mathbb{C}$ by sub–local systems is equivalent to the existence of a weight filtration on the vector bundle $\mathcal{W}$ such that it is preserved by the connection ∇, i.e. $\nabla W_k(\mathcal{W}) \subset \mathcal{A}_S^1 \otimes W_k(\mathcal{W})$. The additional condition in the definition of a variation of mixed Hodge structures is that this filtration $W_\bullet$ on $\mathcal{W}$ must induce a filtration in $\mathbb{V} \otimes \mathbb{C}$ defined already over $\mathbb{V} \otimes \mathbb{Q}$.

LEMMA 4.50. *Let $VMHS(S)$ and $\mathcal{A}{-}VMHS(S)$ be the categories of variations of mixed Hodge strucure, resp. real analytic variations of mixed Hodge structure over a complex manifold S. There are functors*

$$\begin{aligned}\otimes\mathcal{A}_S : VMHS(S) &\longrightarrow \mathcal{A} - VMHS(S)\\ (\mathbb{V}, \mathcal{V}, \nabla, W_\bullet, \mathcal{F}^\bullet) &\longmapsto (\mathbb{V}, \mathcal{V} \otimes \mathcal{A}_S, \nabla_{an}, W_\bullet, \mathcal{F}^\bullet \otimes \mathcal{A}_S)\end{aligned}$$

where ∇_{an} is the connection obtained by letting $\nabla^{1,0}_{|\mathcal{V}} = \nabla$, $\nabla^{0,1}_{|\mathcal{V}} = 0$ and the filtration $\mathcal{F}^\bullet$ has been extended by $\mathcal{A}_S$ linearity, and

$$\begin{aligned}\ker \nabla^{0,1} : \mathcal{A} - VMHS(S) &\longrightarrow VMHS(S)\\ (\mathbb{V}, \mathcal{W}, \nabla, W_\bullet, \mathcal{F}^\bullet) &\longmapsto (\mathbb{V}, \ker \nabla^{0,1}, \nabla^{1,0}, W_\bullet, \mathcal{F}^\bullet \cap (\ker \nabla^{0,1}))\end{aligned}$$

These two functors are inverse of each other, and they induce an equivalence of abelian categories.

As in the pure case, one may repeat the definitions and categoric equivalence functors for variations of mixed Hodge structure with coefficients, where $\mathbb{V}$ is a local system of $\mathbb{R}, \mathbb{C}, \dots$–linear spaces. The functors $\otimes\mathcal{A}_S, \ker \nabla^{0,1}$ induce equivalences of tensor categories in these cases. The final example is:

PROPOSITION 4.51. *Let X, S be complex algebraic manifolds, $f : X \to S$ a smooth, proper map, and $H \subset X$ a relative normal crossing divisor. Denote $U = X \setminus H$. Let $\mathcal{A}^{*;*}_{X|S}(\log H)$ be the relative horizontal logarithmic Dolbeault complex introduced in the previous section, $\mathcal{F}^\bullet$ its Hodge filtration, ∇ its Gauss–Manin connection in the derived sheaves, and $W_\bullet$ its weight filtration.*

The weight filtration induced by $W_\bullet$ *in* $\mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H) \cong \mathbb{R}^p f_* \mathbb{R}_U \otimes \mathcal{A}_S$ *is actually defined in the local systems* $\mathbb{R}^p f_* \mathbb{R}_U$*, and the set of data*

$$(\mathbb{R}^p f_* \mathbb{R}_U, \mathbb{R}^p f_* \mathcal{A}^{*,*}_{X|S}(\log H), \nabla, W_\bullet, \mathcal{F}^\bullet)$$

defines a real analytic $\mathbb{R}$*–variation of mixed Hodge structure on* S*. The image of this variation by the functor* $\ker \nabla^{0,1}$ *is the complex* $\mathbb{R}$*–variation of mixed Hodge structure*

$$(\mathbb{R}^p f_* \mathbb{R}_U, \mathbb{R}^p f_* \Omega^{*,*}_{X|S}(\log H), \nabla, W_\bullet, \mathcal{F}^\bullet)\,.$$

CHAPTER 5

de Rham realization

The Malcev algebra and rational homotopy groups of a suitable topological space may be studied by means of Sullivan's 1–minimal models, as has been described in Section 5 of Chapter I and applied in Chapter II, or equivalently through the iterated integrals developed by K.T. Chen (see [**26**]). In the case of a complex algebraic manifold X, both approaches may be applied to suitable holomorphic and Dolbeault complexes yielding information on the Hodge structures of the Malcev algebras $\mathcal{L}\pi_1(X,x)$, or of the homotopy algebras $\pi_*(X) \otimes \mathbb{Q}$ if X is simply connected ([**70**],[**47**], see also [**72**]).

This study of rational homotopy may be carried out in a relative case as well as in the absolute context. In the case of a real or complex analytic family of smooth manifolds $f : X \to S$, with a basepoint section $\sigma : S \to X$, there exist locally real analytic basepoint preserving parallel transports on the total space $(X, \sigma(S))$. These parallel transports, which we have studied in the previous chapter as local Gauss–Manin connections on forms, define a principal bundle of fundamental groups $\{\pi_1(X_s, \sigma(s)\}_{s\in S}$, and associated local systems formed by the homogeneous bracket spaces $\Gamma_n/\Gamma_{n+1} \otimes \mathbb{Q}$, which may be defined even without a basepoint section σ, and of Malcev algebras $\mathcal{L}_n(\Gamma)$.

In the case of smooth proper maps $f : X \to S$ between complex algebraic manifolds these local systems with complex coefficients become the horizontal sections of holomorphic flat bundles $\{\mathcal{L}(\pi_1(X_s, \sigma(s)), \mathbb{C})\}_{s\in S}$; they underlie variations of Hodge structure ([**48**]) and are endowed with a Gauss–Manin connection which is algebraic and singular–regular, as its cohomological counterpart ([**73**]).

The purpose of this Chapter is to study the variation of Malcev algebras in families of affine curves. We wish to compare the case of rational curves, which is the case covered by Deligne in [**34**], with the case of curves whose completion has a nonzero first Betti number. In order to do this, we will explicitly compute these variations in the case of the Legendre family of affine cubics, and also for families of punctured rational curves over $\mathbb{P}^1_{\mathbb{C}}$. The computation will be performed by applying the techniques and results of [**73**], so we will begin in Section 1 with a description of them in the case of Malcev algebras.

The reader is referred to [**73**] for the analogous results on the rational homotopy of simply connected and nilpotent manifolds.

The Gauss–Manin connection of the above holomorphic bundles is determined by the algebraic differential equation satisfied by its horizontal sections. The solutions are called the non-abelian periods of the family, in analogy to the abelian periods determined by the local systems of the cohomology of the fibers.

To every differential equation we can associate its differential Galois group (see [**57**] or [**14**]), which gives qualitative information on the solutions of the equation (see [**94**]). We will study in the final section of this chapter the differential Galois groups of the non–abelian periods of Malcev algebras in algebraic families. The basic tools that we use for this purpose are the theorem by Schlesinger characterising the differential Galois group of a Fuchsian equation as the Zariski closure of its monodromy (see [**95**],[**76**]), and the group–theoretic relations between the first homology group, the brackect quotients Γ_n/Γ_{n+1} and the Malcev algebras described in Section 7 of Chapter I.

1. The Gauss–Manin connection in the Malcev algebra

Let $k \hookrightarrow \mathbb{C}$ be a subfield of the complex numbers, X, S smooth k–schemes of finite type with S affine, $f : X \to S$ a smooth algebraic morphism such that the underlying holomorphic morpism $f^{an} : X^{an} \to S^{an}$ is topologically locally trivial, and $\sigma : S \to X$ a smooth section, transversal to the fibers of f.

We have seen in Chapter 4 how the parallel transport in the underlying real analytic family $f^{an} : X^{an} \to S^{an}$ is well–defined up to homotopy, and determines a real analytic Gauss–Manin connection, which may be defined locally on the relative Dolbeault complex $\mathcal{A}^{*,*}_{X^{an}|S^{an}}$.

Katz and Oda showed in [**58**] that the Gauss–Manin connection is of algebraic origin. The algebraic connection is not hard to define when the basis scheme S is affine: every algebraic vector field v on S lifts to a k–algebraic vector field $\tilde{v}$ on X, and the fiberwise transversality of the base point section $\sigma(S) \subset X$ assures the existence of lifts $\tilde{v}$ tangent to it. Consider the relative de Rham complex $\Omega^*_{X|S}$ over k. Given a relative form $\omega \in \Omega^*_{X|S}$, and a lift $\tilde{\omega} \in \Omega^*_X$, the Gauss–Manin connection of ω over v is the image in $f_*\Omega^*_{X|S}$ of the covariant derivative

$$\nabla_v(\omega) = L_{\tilde{v}}\tilde{\omega} \in \Omega^*_{X|S}\,.$$

Both selection of a different lift of the vector field v to X or of a different section of $\Omega^*_X \to \Omega_{X|S}$ result in a homotopic endomorphism of the complex of $\mathcal{O}_S$–sheaves

$$\nabla'_v : \left(f_*\Omega^*_{X|S}, d\right) \longrightarrow \left(f_*\Omega^*_{X|S}, d\right) .$$

Thus our ∇ is so far a connection up to homotopy. If v_1, v_2 are algebraic vector fields on S and a parallel transport over them is chosen by fixing liftings $\tilde{v}_1, \tilde{v}_2$ and a linear section $\Omega^*_{X|S} \to \Omega^*_X$, the morphisms $[\nabla_{v_1}, \nabla_{v_2}]$ and $\nabla_{[v_1,v_2]}$ are also homotopic endomorphisms of the complex of $\mathcal{O}_S$–sheaves $f_*\Omega^*_{X|S}$.

Therefore, by choosing basepoint preserving lifts of a basis of the tangent bundle Θ_S and a $\mathcal{O}_S$–linear section $\Omega^*_{X|S} \to \Omega^*_X$, the above process yields a connection on the relative differential bundle $\left(\Omega^*_{X|S}, d\right)$. This connection depends on the selected liftings and section only up to homotopy, and is homotopically integrable (see our Definition 4.18).

We will describe now after [**73**] the 1–minimal model of the algebraic Gauss–Manin connection over an affine base scheme S: Let $\mathcal{A}^*$ be a sheaf of quasi–coherent $\mathcal{O}_S$–CDGAs such that its cohomology sheaves are coherent, and set $A^* = \mathcal{A}^*(S)$ the CDGA of global sections. As S is affine, the complex of sheaves $\mathcal{A}^*$ is the sheafification of A^*.

We have described in Section 5 of Chapter I the construction of 1–minimal models of CDGAs, and stated Sullivan's theorem showing that if A^* has finite–dimensional cohomology and a basepoint morphism $A^* \to k$, then it has a basepointed minimal model $\rho : M(2,0)_{A^*} \to A^*$, obtained as an inductive limit of $(1,q)$–minimal models by succesive Hirsch extensions

$$\begin{aligned} M(1,q) &= M(1,q-1) \otimes \wedge^*(V^{1,q}) \\ V^{1,q} &\cong \ker\left(\rho^*_{1,q-1} : H^2(M(1,q-1)) \longrightarrow H^2(A^*)\right) . \end{aligned}$$

The 1–minimal model is functorial up to homotopy, i.e. given a morphism $f : A^* \to B^*$ and a choice of 1–minimal models $M(2,0)_{A^*}, M(2,0)_{B^*}$, there exists a lift $M(f)$ such that $\rho_B \circ M(f)$ and $f \circ \rho_A$ are homotopic.

A connection may be seen as a linearly varying derivation, so we will recall the definition of derivations in CDGAs: Let R be a k–algebra with $k \subset \mathbb{C}$ as before, A^*, B^* two R–CDGAs, $\varphi : A^* \to B^*$ be a R–CDGA morphism and v a derivation in R. A (v,φ)–derivation from A^* to B^* is a k–linear morphism $\delta : A^* \to B^*$ such that it has even degree and satisfies the Leibnitz identities

$$\begin{aligned} \delta(rx) &= v(r)\varphi(x) + r\delta(x)\,, & r \in R\,,\ x \in A^* \\ \delta(x \wedge y) &= \delta(x) \wedge \varphi(y) + \varphi(x) \wedge \delta(y)\,, & x, y \in A^* \end{aligned}$$

If $A^* = B^*$ and $\varphi = \mathrm{Id}$ we call δ a v–derivation. A (v, φ)–antiderivation $\lambda : A^* \to B^*$ is defined in the same way as a derivation, except that λ has odd degree and the second Leibnitz relation is

$$\lambda(x \wedge y) = \lambda(x) \wedge \varphi(y) + (-1)^{\deg(x)} \varphi(x) \wedge \lambda(y) .$$

The 1–minimal model of a v–derivation of degree 0 on a R–CDGA A^* may be built as an inductive system of $(1, q)$–minimal models for every $q \in \mathbb{N}$. The $(1,0)$–minimal model is the commutative square

$$\begin{array}{ccc} R & \xrightarrow{v} & R \\ \downarrow & & \downarrow \\ A^* & \xrightarrow{\delta} & A^* \end{array}$$

Commutativity of the square is usually possible only up to homotopy after the $(1,0)$–step, therefore the $(1, q)$–minimal model for $q > 0$ is defined as a square

$$\begin{array}{ccc} M(1,q) & \xrightarrow{\delta_{(1,q)}} & M(1,q) \\ \rho_{(1,q)} \downarrow & \searrow \lambda_{(1,q)} & \downarrow \rho_{(1,q)} \\ A^* & \xrightarrow[\delta]{} & A^* \end{array}$$

where $\delta_{(1,q)}$ is a v–derivation in the minimal model $M(1,q)$, and $\lambda_{(1,q)}$ is a $(v, \rho_{(1,q)})$–antiderivation of degree -1 such that

$$\delta \rho_{(1,q)} - \rho_{(1,q)} \delta_{(1,q)} = d\lambda_{(1,q)} + \lambda_{(1,q)} d .$$

If A^*, B^* are augmented R–CDGAs with augmentations $\varepsilon_{A^*}, \varepsilon_{B^*}$ respectively, a (v, φ)–derivation δ is augmented, or basepointed, when $\varepsilon_{B^*}\delta = v\varepsilon_{A^*}$. A homotopy λ between basepointed derivations is basepointed when $\varepsilon_{B^*}\lambda = 0$. Thus a basepointed $(1, q)$–minimal model of a derivation is a $(1, q)$–minimal model with $\delta_{(1,q)}, \lambda_{(1,q)}$ basepointed.

Assume now that the (basepointed) $(1, q-1)$–minimal model of a derivation $\delta : A^* \to A^*$ is known. As $M(1,q) = M(1,q-1) \otimes \wedge^*(V^{1,q})$, in order to construct the $(1, q)$–minimal model of δ it suffices to extend $\delta_{1,q-1}, \lambda_{1,q-1}$ to the space of indecomposables $V^{1,q}$ defining the Hirsch extension, provided that this extension is compatible with $\rho_{(1,q)}$, the boundary d of $M(1,q)$ and the augmentation. The obstruction to the existence of $(\delta_{(1,q)}(e), \lambda_{(1,q)}(e))$ for $e \in V^{1,q}$ is the relative cocycle ([**73**], Lemma 3.6)

$$\tilde{o}(e) = \left(\delta_{(1,q-1)}(de), \delta(\rho_{(1,q)}(e) + \lambda_{(1,q-1)}(de)\right) \in Z^2(M(1,q), A^*) .$$

Thus the $(1, q-1)$–minimal model of a derivation δ extends to a $(1, q)$–minimal model if and only if the relative cocycles $\tilde{o}(e)$ are exact for every $e \in V^{1,q}$. The derivations $\delta_{(1,q)}, \lambda_{(1,q)}$ may be defined on $V^{1,q}$

by setting a primitive to $\tilde{o}(e)$ $(\delta_{(1,q)}(e), \lambda_{(1,q)}(e))$ such that it varies k–linearly on e. A basepointed extension of the minimal model is achieved by a suitable choice of the primitive to $\tilde{o}(e)$.

The obstruction to the existence of a $(1,q)$–minimal model of a (basepointed) derivation has been presented here because it provides an effective algorithm to compute the 1–minimal model. On the other hand, Navarro Aznar has shown that this obstruction is trivial in our setting:

PROPOSITION 5.1. ([**73**], 3.8 and 3.11) *Let A^* be an augmented R–CDGA with finitely generated cohomology, $\delta : A^* \to A^*$ a basepointed v–derivation, and $M(2,0) = \lim M(1,q)$ a basepointed 1–minimal model of A^*. The basepointed 1–minimal model of δ exists and is unique as the limit of an inductive system of $(1,q)$–minimal models $\delta_{(1,q)}, \lambda_{(1,q)} : M(1,q) \to A^*$.*

Proposition 5.1 may be applied to the k–algebra $R = \mathcal{O}_S$, the $\mathcal{O}_S$–CDGA $A^* = \Omega^*_{X|S}(S)$ and the derivation ∇_v induced on $\Omega^*_{X|S}(S)$ by the Gauss–Manin connection along some algebraic vector field v on S. Due to the uniqueness of the basepointed $(1,q)$–minimal models of derivations, we may choose any basis $v_1, \dots, v_k$ of algebraic vector fields on S, and the $(1,q)$–minimal models of the derivations $\nabla_{v_1}, \dots, \nabla_{v_k}$ fit together to define a connection $\nabla_{(1,q)}$ on $M(1,q)$, and a homotopy between $\rho_{(1,q)}\nabla_{(1,q)}$ and $\nabla\rho_{(1,q)}$. This is the minimal model of the Gauss–Manin connection. As S is affine, the sheafification $\left(\mathcal{M}(1,q), \nabla_{(1,q)}, \lambda_{(1,q)}\right)$ of this 1–minimal model provides the unique 1–minimal model of the restriction to any open set $U \subset S$, and analogously to the absolute case, this 1–minimal model computes the variation of the Malcev algebras:

THEOREM 5.2. (Navarro Aznar, [**73**] 6.11) *Let $k \subset \mathbb{C}$, $f : X \to S$ and $\sigma : S \to X$ be as at the beginning of the section. Then for every $q > 0$:*

(i) *The algebraic Gauss–Manin connection $\nabla : \Omega^*_{X|S} \to \Omega^*_{X|S} \otimes \Omega^1_S$ with the augmentation given by the section σ has a basepointed $(1,q)$–minimal model $\left(\mathcal{M}(1,q), \nabla_{(1,q)}, \lambda_{(1,q)}\right)$, which is integrable and unique up to isomorphism.*

(ii) *The homogeneous component of degree 1 $\mathcal{M}(1,q)^1 \subset \mathcal{M}(1,q)$ is a finitely generated locally free $\mathcal{O}_S$–module, and the holomorphic flat bundle $\left(\mathcal{M}(1,q)^1, \nabla_{(1,q)}\right)^{an}$ is dual to the flat bundle of Malcev algebras $\mathcal{L}_q(\pi_1(X_s, \sigma(s)) \otimes \mathcal{O}_{S^{an}}$.*

Theorem 5.2 is proved in [**73**] without the affine base restriction, taking a cover of S by affine open sets, but the glueing process for the

Gauss–Manin connection and its minimal model is more delicate than what has been sketched here.

2. Families of curves

The purpose of this section is to study the Gauss–Manin connection in the Malcev algebras of some families of affine curves over the projective line, applying the algorithm of [**73**] described in the previous section. The examples presented here are on one hand the Legendre family of affine cubics, and on the other hand some families of punctured rational curves. While the fundamental group and Malcev algebra of an elliptic curve minus one point are isomorphic to those of a rational curve minus three points, the relative Malcev algebras of the examples presented here show a marked contrast. This contrast seems to arise from the fact that the first comomology group of a Legendre affine cubic E_t is pure of weight one, while $H^1(\mathbb{P}^1_{\mathbb{C}} \setminus \{p_1, p_2, p_3\})$ has pure weight two.

We will study smooth algebraic families over field $k \subset \mathbb{C}$ of the form $f : X \to S$, with $S \subset \mathbb{P}^1_k$, X a smooth surface, and f^{an} topologically locally trivial. As $S \subset \mathbb{P}^1_k$, if we select a uniformizing parameter s, the derivation $\frac{\partial}{\partial s}$ is a global generator of the algebraic tangent sheaf Θ_S, and the Gauss–Manin connection on the sheaves of Malcev algebras is determined by the parallel transport along $\frac{\partial}{\partial s}$ plus the Leibnitz identities. Hence the $(1,q)$–minimal model of this derivation is actually the $(1,q)$–minimal model of the Gauss–Manin connection.

The correspondence between holomorphic vector bundles over $S \subset \mathbb{P}^1_{\mathbb{C}}$ with a flat connection and Fuchsian linear differential equations over S, which is bijective up to isomorphism of bundles/equivalence of equations, allows us to present the holomorphic flat bundles $\{\mathcal{L}_n(\pi_1(X_s, \sigma(s)))\otimes \mathbb{C}\}_{s\in S} \otimes \mathcal{O}_S$ as the bundles of solutions of the differential equation satisfied by its horizontal sections. There exist a holomorphic and a k–algebraic version of this correspondence (see [**31**] I.2).

2.1. The Legendre family of affine cubics. The monodromy of the Legendre family of affine cubics

$$\begin{aligned} f : E = \{(x,y,t) \in \mathbb{C}^3 \mid y^2 = x(x-1)(x-t)\} &\longrightarrow S = \mathbb{C} \setminus \{0,1\} \\ (x,y,t) &\longmapsto t \end{aligned}$$

in the fundamental group and in the Malcev algebras up to $\mathcal{L}_3(\pi_1(E_t, *))$ has been computed in Propositions 3.46 and 3.76 respectively, using a topologic basepoint section. This family has algebraic origin, it arises from a morphism of $\mathbb{Z}$–schemes, thus we will regard it as an algebraic family of $\mathbb{Q}$–schemes in order to compute its minimal model.

Let us now compute the (1,3)-minimal model of the Gauss–Manin connection $(\Omega^*_{E|S}, \partial_t = \frac{\partial}{\partial t})$, which corresponds to the local system of duals of the Malcev algebras $\mathcal{L}_3\pi_1 \otimes \mathbb{C}$. For computational convenience we will choose the section $(0,0,t)$ as basepoint, instead of a section approaching the topologic basepoint section used in Chapter III. Thus the augmentation ε of the dga $\Omega^*_{E/S}$ will be the evaluation in $(x,y) = (0,0)$, and in the computations that follow we will select our obstructions and homotopies so that it is always verified that $\varepsilon \circ \lambda_{(1,i)} = 0$.

We start by directly finding the (1,1)-minimal model of ∂_t (cf. **[29]** 2.10).

A (1,1)-minimal model of $\Omega^*_{E/S}$ is given by

$$M(1,1) = \Lambda(\alpha, \beta), \quad |\alpha| = |\beta| = 1, \quad d\alpha = d\beta = 0$$

and

$$\begin{aligned} \rho_{(1,1)} : M(1,1) &\longrightarrow \Omega^*_{E/S} \\ \alpha &\longrightarrow w_1 = \frac{dx}{y} \\ \beta &\longrightarrow w_2 = \frac{xdx}{y} \end{aligned}$$

We have

$$\begin{aligned} \partial_t(w_1) &= -\partial_t y \frac{dx}{y^2} = \frac{1}{2}\frac{dx}{y(x-t)} \\ \partial_t(w_2) &= \frac{1}{2}x\frac{dx}{y(x-t)} \end{aligned}$$

Thus

$$\partial_t w_2 - t\partial_t w_1 = \frac{1}{2}w_1 \tag{16}$$

$$\partial_t(\frac{x^2dx}{y}) - t\partial_t w_2 = \frac{1}{2}w_2 \tag{17}$$

We now need an expresion of $\frac{x^2dx}{y}$ in terms of the cohomology basis w_1, w_2 and coboundaries: The identity

$$y^2 = x(x-1)(x-t) = x^3 - (t+1)x^2 + tx$$

yields

$$2ydy = (3x^2 - 2(t+1)x + t)dx \tag{18}$$

$$\frac{x^2dx}{y} = -\frac{t}{3}\frac{dx}{y} + \frac{2}{3}(t+1)\frac{xdx}{y} + \frac{2}{3}dy \tag{19}$$

Applying the connection to both sides we obtain

$$\partial_t\left(\frac{x^2dx}{y}\right) = -\frac{1}{3}w_1 - \frac{t}{3}\partial_t w_1 + \frac{2}{3}w_2 + \frac{2}{3}\partial_t w_2 + \frac{2}{3}d\partial_t y \tag{20}$$

Putting together (17) and (20) we arrive at

$$\partial_t w_1 = -\frac{1}{2(t-1)}w_1 + \frac{1}{2t(t-1)}w_2 + \frac{2}{t(t-1)}d(\partial_t y) \tag{21}$$

$$\partial_t w_2 = -\frac{1}{2(t-1)}w_1 + \frac{1}{2(t-1)}w_2 + \frac{2}{t-1}d(\partial_t y) \tag{22}$$

The above system (22) shows that a (1,1)-minimal model of ∂_t may be defined as

$$\begin{array}{ccc} M(1,1) & \overset{\rho(1,1)}{\longrightarrow} & \Omega^*_{E/S} \\ \delta\downarrow & & \downarrow\partial_t \\ M(1,1) & \underset{\rho(1,1)}{\longrightarrow} & \Omega^*_{E/S} \end{array}$$

where

$$\delta_{(1,1)}\alpha = -\frac{\alpha}{2(t-1)} + \frac{\beta}{2t(t-1)}$$

$$\delta_{(1,1)}\beta = -\frac{\alpha}{2(t-1)} + \frac{\beta}{2(t-1)}$$

and the pointed homotopy between $\rho_{(1,1)} \circ \delta_{(1,1)}$ and $\partial_t \circ \rho_{(1,1)}$ is

$$\lambda_{(1,1)}\alpha = \frac{2}{t(t-1)}\partial_t y$$

$$\lambda_{(1,1)}\beta = \frac{2}{t-1}\partial_t y$$

since the identities $\partial_t\rho_{(1,1)} - \rho_{(1,1)}\delta_{(1,1)} = d\lambda_{(1,1)} + \lambda_{(1,1)}d$ and $\varepsilon\,\lambda_{(1,1)}$ may be easily checked on the generators α, β.

We shall now switch to the method described in [**73**] to compute the (1,2)- and (1,3)-minimal models of ∂_t.

A (1,2)-minimal model of $\Omega^*_{E/S}$ is the Hirsch extension

$$M(1,2) = M(1,1) \otimes \Lambda(\eta), \quad |\eta| = 1, \quad d\eta = \alpha \wedge \beta$$

with the map induced by

$$\rho_{(1,2)}\eta = 0$$

The obstruction in $\left(M(1,2) \to \Omega^*_{E/S}\right)$ to the existence of the (1,2)-minimal model of ∂_t is given by

$$\begin{aligned}
\tilde{o}(\eta) &= (\delta_{(1,1)}(d\eta), \partial_t(\rho_{(1,2)}\eta) + \lambda_{(1,1)}(d\eta)) \\
&= (\delta_{(1,1)}(\alpha \wedge \beta), \lambda_{(1,1)}(\alpha \wedge \beta)) \\
&= \left(-\frac{1}{2(t-1)}\alpha \wedge \beta + \frac{1}{2(t-1)}\alpha \wedge \beta, \frac{2}{t(t-1)}(\partial_t y)w_2 - \frac{2}{t-1}(\partial_t y)w_1\right) \\
&= \left(0, -\frac{dx}{t(t-1)}\right) \\
&= d\left(0, \frac{x}{t(t-1)}\right)
\end{aligned}$$

Thus setting

$$\begin{aligned}
\delta_{(1,2)}\eta &= 0 \\
\lambda_{(1,2)}\eta &= \frac{x}{t(t-1)}
\end{aligned}$$

the induced $\delta_{(1,2)} : M(1,2) \to M(1,2)$, $\lambda_{(1,2)} : M(1,2) \to \Omega^*_{E/S}[-1]$ constitute a basepointed (1,2)-minimal model of ∂_t.

The computation of the pointed (1,3)-minimal model of ∂_t is wholly identical. We begin with a (1,3)-minimal model of $\Omega^*_{E/S}$ given by the Hirsch extension

$$\begin{aligned}
&M(1,3) = M(1,2) \otimes \Lambda(\gamma_1, \gamma_2) \\
&|\gamma_1| = |\gamma_2| = 1 \\
&d\gamma_1 = \alpha \wedge \eta, \quad d\gamma_2 = \beta \wedge \eta
\end{aligned}$$

and the map determined by

$$\rho_{(1,3)}\gamma_1 = \rho_{(1,3)}\gamma_2 = 0$$

The obstructions to building $\delta_{(1,3)}$ and $\lambda_{(1,3)}$ are

$$\begin{aligned}
\tilde{o}(\gamma_1) &= (\delta_{(1,2)}(d\gamma_1), \partial_t(\rho_{(1,3)}\gamma_1) + \lambda_{(1,2)}(d\gamma_1)) \\
&= \left(-\frac{1}{2(t-1)}\alpha \wedge \eta + \frac{1}{2t(t-1)}\beta \wedge \eta, -\frac{1}{t(t-1)}w_2\right) \\
&= d\left(-\frac{1}{2(t-1)}\gamma_1 + \frac{1}{2t(t-1)}\gamma_2 - \frac{1}{t(t-1)}\beta, 0\right)
\end{aligned}$$

$$\begin{aligned}
\tilde{o}(\gamma_2) &= \left(-\frac{1}{2(t-1)}\alpha \wedge \eta + \frac{1}{2(t-1)}\beta \wedge \eta, -\frac{1}{t(t-1)}\frac{x^2dx}{y}\right) \\
&= d\left(-\frac{1}{2(t-1)}\gamma_1 + \frac{1}{2(t-1)}\gamma_2 + \frac{1}{3(t-1)}\alpha - \frac{2(t+1)}{3t(t-1)}\beta, \frac{2}{3t(t-1)}y\right)
\end{aligned}$$

((19) should be used in the last step).
These computations show that the (1,3)-minimal model of ∂_t is induced by setting

$$\begin{aligned}
\delta_{(1,3)}\gamma_1 &= -\frac{1}{t(t-1)}\beta - \frac{1}{2(t-1)}\gamma_1 + \frac{1}{2t(t-1)}\gamma_2 \\
\delta_{(1,3)}\gamma_2 &= \frac{1}{3(t-1)}\alpha - \frac{2(t+1)}{3t(t-1)}\beta - \frac{1}{2(t-1)}\gamma_1 + \frac{1}{2(t-1)}\gamma_2
\end{aligned}$$

and

$$\begin{aligned}
\lambda_{(1,3)}\gamma_1 &= 0 \\
\lambda_{(1,3)}\gamma_2 &= \frac{2}{3t(t-1)}y
\end{aligned}$$

The differential equation of the horizontal sections of $\delta_{(1,3)}$ is

$$\partial\begin{pmatrix}\alpha\\ \beta\\ \eta\\ \gamma_1\\ \gamma_2\end{pmatrix} = \frac{1}{t-1}\begin{pmatrix} -\frac{1}{2} & \frac{1}{2t} & 0 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{t} & 0 & -\frac{1}{2} & \frac{1}{2t} \\ \frac{1}{3} & -\frac{2(t+1)}{3t} & 0 & -\frac{1}{2} & \frac{1}{2} \end{pmatrix}\begin{pmatrix}\alpha\\ \beta\\ \eta\\ \gamma_1\\ \gamma_2\end{pmatrix} \qquad (23)$$

By the isomorphism of [**73**] Thm. 6.10 it is also the differential equation that corresponds to the local system formed by the $\mathcal{L}_3\pi_1 E_t \otimes \mathbb{C}$.

REMARK 5.3. As in the case of their monodromy matrices explained in Section 5 of Chapter 3, we can obtain the differential equations corresponding to the local systems $\mathcal{L}_3$, Γ_1/Γ_2, Γ_2/Γ_3, $\Gamma_3/\Gamma_4 \otimes \mathbb{C}$ by restriction to the adequate minors of equation (23) as shown in Figure F5.1.

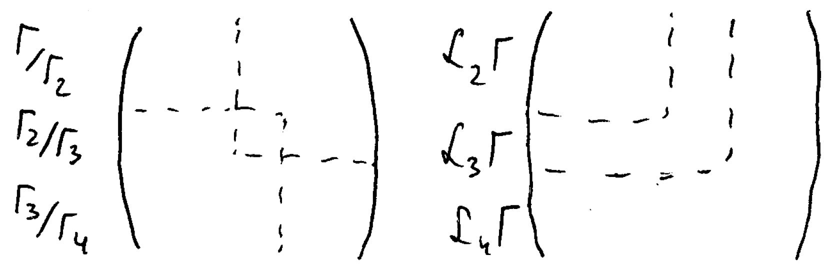

Figure F5.1 Minors of $A(n)$.

The equation obtained in the case of the abelianised of the fundamental group, $\pi_1 E_t/(\pi_1 E_t)_2 \otimes \mathbb{C} \cong H_1(E_t; \mathbb{C})$ is a linear system equivalent to the Legendre hypergeometric equation, and it is the differential equation satisfied by the abelian periods of the matrix (see [**29**],[**19**]).

REMARK 5.4. The equations given by the minors corresponding to $\Gamma_i/\Gamma_{i+1}\pi_1 E_t \otimes \mathbb{C}$ are the equations satisfied by the *non-abelian periods* of the affine Legendre family. These period maps may be realized as iterated integrals

$$\int_\gamma \omega_{i_1} \dots \omega_{i_k} = \int \cdots \int_{0 \le t_1 \le t_2 \le \cdots \le t_k \le 1} f_{i_1}(t_1) \dots f_{i_k}(t_k)\, dt_1 \dots dt_k$$

where $f_{i_j}(t)\, dt = \gamma^* \omega_{i_j}$ and are dual to the Γ_i/Γ_{i+1} as the ordinary periods to the H_1 (see [**49**] or [**25**]). They are often referred to as the quadratic, cubic, quartic, ... periods.

REMARK 5.5. The differential equation for $(\pi_1 E_t)_2/(\pi_1 E_t)_3 \otimes \mathbb{C}$ is zero. This means that the only quadratic period of the Legendre family is constant. The quadratic period map sends $s \in S$ to

$$\int_{[b,a]} \omega_1 \omega_2 = \begin{vmatrix} \int_b \omega_1 & \int_a \omega_1 \\ \int_b \omega_2 & \int_a \omega_2 \end{vmatrix}$$

where a, b are the paths used in Proposition 3.46. This determinant is known to be $2\pi i$ by the Jacobi relation between elliptic integrals. The affine Legendre family has infinitely many non-abelian periods algebraic over $\mathbb{C}$: they are just identities derived from the Jacobi relation.

Rather than carry on the full computation of the minimal model of $(\Omega^*_{E/S}, \partial_t)$ and its associated differential equation, we will inquire about the format of the matrix $A(n)$ of the differential equation associated to the $(1,n)$–minimal model and the ring of coefficients over which it is defined.

We will use some properties of 1–minimal models arising from the lower central series filtration in the Malcev algebra, so we start by showing them.

The Malcev algebra of a group admits a decreasing lower central series filtration $(\mathcal{L}\Gamma)_1 = \mathcal{L}\Gamma, (\mathcal{L}\Gamma)_n = [(\mathcal{L}\Gamma)_{n-1}, \mathcal{L}\Gamma]$, whose relation to the lower central series of the group Γ has been explained in Section 2 of Chapter I. Consider the $(1,n)$–minimal model of a manifold with fundamental group Γ, we will denote it as $M(1,n)_\Gamma$ as it depends solely on Γ. By Sullivan's theory of 1–minimal models the CDGA $M(1,n)_\Gamma$ is freely generated by a linear space of indecomposable elements $V_n = V^{1,1} \oplus V^{1,2} \oplus \cdots \oplus V^{1,n}$, and the spaces V_n form an inductive system. The duality theorem 1.43 of Sullivan states that the inductive system formed by the spaces V_n is dual to the projective system of Malcev algebras $\mathcal{L}_n\Gamma$, and that the algebra brackets $[.,.] : \mathcal{L}_n \wedge \mathcal{L}_n \to \mathcal{L}_n$ are dual to the coboundary operators $d : V_n \to V_n \wedge V_n$. Therefore the lower central series filtration of the Malcev algebras originates an increasing filtration in the inductive system of spaces V_n:

$$W_l(V_n) = V^{1,1} \oplus \cdots \oplus V^{1,l},$$

and as the Malcev algebra bracket preserves the lower central series filtration, the coboundary operator $d : V_n \to V_n \wedge V_n$ also preserves this filtration $W_\bullet$, multiplicatively extended to $V_n \wedge V_n$. As the $(1,n)$–minimal model is generated by V_n, the conclusion is:

LEMMA 5.6. *Let $M(1,n)_\Gamma = \wedge^*(V_n)$ be the $(1,n)$–minimal model of a finitely presented group Γ. There exists an increasing multiplicative filtration in the inductive system of $(1,n)$–minimal models $\cdots \hookrightarrow M(1,n)_\Gamma \hookrightarrow M(1,n+1)_\Gamma \hookrightarrow \ldots$ induced by the filtration of the spaces of indecomposable generators $W_l(V_n) = V^{1,1} \oplus \cdots \oplus V^{1,l}$. This filtration on $M(1,n)_\Gamma$ is preserved by the coboundary operator d and is dual to the bracket of the Malcev algebras.*

We will refer to the filtration $W_\bullet$ that we have just introduced as the *weight filtration* in the 1–minimal models. In the case of Kähler groups Γ, this filtration is indeed the weight filtration of the MHS of the 1–minimal models.

As the coboundary operator d preserves the weight filtration, its restriction to the spaces of indecomposable generators has the form

$$d : V^{1,n} \longrightarrow \oplus_{\substack{i+j=n \\ i<j}} V^{1,i} \wedge V^{1,j}.$$

We will require a further consequence of the inductive construction of 1–minimal models:

LEMMA 5.7. *Let* $\pi : \oplus_{\substack{i+j=n \\ i<j}} V^{1,i} \wedge V^{1,j} \to V^{1,1} \wedge V^{1,n-1}$ *be the natural projection. The linear map* $\pi \circ d : V^{1,n} \to V^{1,1} \wedge V^{1,n-1}$ *is one–to–one.*

PROOF. For any nonzero indecomposable element $v \in V^{1,n}$, its boundary $u = dv \in V^{1,1} \wedge V^{1,n-1} \oplus V^{1,2} \wedge V^{1,n-2} \oplus \dots$ is a cocycle in a nontrivial cohomology class of $\ker(H^2(M(1,n-1)) \to H^2(\Gamma))$. If this boundary u had no component in $V^{1,1} \wedge V^{1,n-1}$ then it belongs to $M(1,n-2)$, and as $u \in \ker(H^2(M(1,n-2)) \to H^2(\Gamma))$, its cohomology class is the boundary for some indecomposable $z \in V^{1,n-1}$ by the construction process of 1–minimal models, hence the cohomology class of u would be trivial in $H^2(M(1,n-1))$ contradicting our assumption. □

We are ready now to continue our study of the Legendre affine family. Let us fix some notation first: the ring $\mathbb{Z}[\frac{1}{n!}]$ is the subring of $\mathbb{Q}$ obtained by inverting the elements $2, \dots, n$. As the Legendre family of affine curves is defined over $\mathbb{Z}$, we will denote by $S|\mathbb{Z}[\frac{1}{n!}]$ the scheme defined by the corresponding restriction of scalars. The 1–minimal model constructions of [**73**] that we perform in this section are algebraic; we have considered S, E as $\mathbb{Q}$–schemes and computed the Gauss–Manin connection on $M(1,3)$ with coefficients in $\mathcal{O}_{S|\mathbb{Q}}$. The Gauss–Manin connection in the cohomology of the family is defined over $\mathbb{Z}[\frac{1}{2}]$, i.e. the flat coherent sheaf $\{H^1(E_t,\mathbb{Q})\}_{t\in S} \otimes \mathcal{O}_{S|\mathbb{Q}}$ arises by extension of scalars from its $\mathbb{Z}[\frac{1}{2}]$ analogue $\{H^1(E_t,\mathbb{Z}[\frac{1}{2}])\}_{t\in S} \otimes \mathcal{O}_{S|\mathbb{Z}[\frac{1}{2}]}$. Our goal in the sequel will be to determine whether all the sheaves of Malcev algebras $\mathcal{L}_q\pi_1(E_t,(0,0,t))$ are already defined over a scalar ring $\mathbb{Z}[\frac{1}{n!}]$ obtained by inverting finitely many primes. We start with some computations:

LEMMA 5.8. *Let* $\frac{x^n dx}{y} \in \Omega^*_{E/S}$*, with* $n \geq 2$*. Then we have*

(i) $\frac{x^n dx}{y} = p_n(t)\frac{dx}{y} + q_n(t)\frac{x dx}{y} + d(r_n(x,t)y)$ *where* $p_n(t), q_n(t) \in \mathbb{Z}[\frac{1}{(2n-1)!}][t]$ *and* $r_n(x,t) \in \left(\mathbb{Z}[\frac{1}{(2n-1)!}][t]\right)[x]$.

(ii) $p_n(t) = \frac{2n-2}{2n-1}(1+t)p_{n-1}(t) - \frac{2n-3}{2n-1}tp_{n-2}(t)$.

(iii) $q_n(t) = \frac{2n-2}{2n-1}(1+t)q_{n-1}(t) - \frac{2n-3}{2n-1}tq_{n-2}(t)$.

(iv) $r_n(x,t) = \frac{2}{2n-1}x^{n-2} + \frac{2n-2}{2n-1}(1+t)r_{n-1}(x,t) - \frac{2n-3}{2n-1}tr_{n-2}(x,t)$.

(v) $\deg p_n(t) = n-1$*, with leading coefficient* $p_{n\,n-1} = \frac{2\cdot4\cdots(2n-2)}{3\cdot5\cdots(2n-1)}$

(vi) $\deg q_n(t) = n-1$*, with leading coefficient* $q_{n\,n-1} = -\frac{4\cdot6\cdots(2n-2)}{3\cdot5\cdots(2n-1)}$.

(vii) $\deg_x r_n(x,t) = n-2$*, with leading coefficient* $r_{n\,n-2}(t) = \frac{2}{2n-1}$. *Moreover,* $r_n(0,t) = 0$.

PROOF. (i) is an immediate computation, and the other statements are obtained from (i) by an easy induction computation. For starting values, use $\frac{dx}{y} = \frac{dx}{y} + 0 \cdot \frac{xdx}{y} + d(0)$ and the analogous result for $\frac{xdx}{y}$. □

REMARK 5.9. Lemma 5.8 is just a step in the algorithm for the reduction of arbitrary elliptic integrals to the canonical first and second kind ones.

We are now ready to study the $(1,n)$-minimal model of $(\Omega^*_{E|S}, \partial_t)$ and the matrix $A(n)$ of its differential equation.

PROPOSITION 5.10. (i) *The ring spanned by the coefficients of* $A(n)$ *is* $\mathcal{O}_{S/\mathbb{Z}[\frac{1}{n!}]}$.

(ii) $\delta V^{1,n} \subset \bigoplus_{\substack{1\leq j\leq n \\ j\equiv n \mod 2}} V^{1,j}$.

(iii) $\lambda V^{1,n} \neq 0$ *for all* n.

PROOF. The proof is a straightforward, if cumbersome, argument on induction. We will prove a slightly more precise result implying our statement.

The 1-minimal model of $(\Omega^*_{E/S}, \partial_t)$ contains a sequence of indecomposables $\{\psi_n \in V^{1,n}\ ,\ n \in \mathbb{N}\}$ defined as follows:

$$\begin{array}{ll} \psi_1 = \beta \in V^{1,1} & \\ \psi_2 = \eta \in V^{1,2} & \\ \psi_3 = \gamma_2 \in V^{1,3} & d\psi_3 = \beta \wedge \psi_2 \\ \cdots\cdots\cdots\cdots & \\ \psi_n \in V^{1,n} & d\psi_n = \beta \wedge \psi_{n-1} \in H^2(M(1,n-1)) \end{array}$$

What may be directly proved by induction is:

1. $\lambda(V^{1,n})$ is formed by polynomials $p(x) \in \mathcal{O}_{S/\mathbb{Z}[\frac{1}{n!}]}[x]$ if n is even, and by $p(x)\cdot y$ if n is odd.

2. The polynomial with highest degree (hence nonzero) among the λv, $v \in V^{1,n}$ is $\lambda\psi_n$. $\lambda\psi_{2k-1} = \frac{1}{t(t-1)} s_k(x) y$ with $s_k(x) \in \mathbb{Z}[\frac{1}{(2k-1)!}][x]$ of degree $k-2$, and $\lambda\psi_{2k} = \frac{1}{t(t-1)} u_k(x)$ with $u_k(x) \in \mathbb{Z}[\frac{1}{(2k-1)!}][x]$ of degree k.

3. The leading coefficients of $s_k(x)$,$u_k(x)$, $s_{k\,k-2}$,$u_{k\,k}$ respectively satisfy the recurrence

$$\begin{aligned} u_{k\,k} =& \frac{1}{k-1} s_{k\,k-2} \\ s_{k+1\,k-1} =& \frac{2}{2k+1} u_{k\,k} \end{aligned}$$

For the last two steps of our induction hypothesis we use the method of [**73**] to determine λ, $\delta(v)$ of $v \in V^{1,n}$ as the primitive of $\tilde{\delta}(v) = (\delta(dv), \partial_t(\rho(v)) + \lambda(dv))$ with the correct basepoint.

4. $d^{-1}(\delta dv) \subset \bigoplus_{\substack{2 \le j \le n \\ j \equiv n+1 \mod 2}} V^{1,j}$.

5. If n is even, $\delta(V^{1,n})$ does not contain terms in $V^{1,1}$. If $n = 2k+1$, then

$$\delta\psi_n = p_{k+1\,k} u_{k\,k} \frac{t^k}{t(t-1)} \alpha + \left\{ \begin{smallmatrix} \textit{terms of lower} \\ \textit{degree in } t \end{smallmatrix} \right\} \alpha$$
$$+ q_{k+1\,k} u_{k\,k} \frac{t^k}{t(t-1)} \beta + \left\{ \begin{smallmatrix} \textit{terms of lower} \\ \textit{degree in } t \end{smallmatrix} \right\} \beta + \left\{ \begin{smallmatrix} \textit{terms in} \\ V^{1,3} \oplus \cdots \oplus V^{1,n} \end{smallmatrix} \right\}$$

The previous computation of the $(1,3)$–minimal model of the Legendre affine family confirms hypothesis 1-5 and therefore (i)-(iii) up to $n = 3$. Suppose them true up to n. Then, if $v_1, \dots, v_N = \psi_{n+1}$ form a k-basis for $V^{1,n+1}$, $\delta, \lambda(v_i)$ are obtained as the primitive of $\tilde{\delta}(v_i)$ with correct basepoint. Now, by Lemma 5.6 d respects the weights on $M(1, n+1)$ so $dv_i \in \bigoplus_{\substack{1 \le k < l \le n \\ k+l=n+1}} V^{1,k} \wedge V^{1,l}$. By induction hypothesis (ii) we have then

$$\delta(dv_i) \in \bigoplus_{\substack{1 \le h,d \le n \\ h \equiv k \mod 2 \\ d \equiv l \mod 2}} V^{1,h} \wedge V^{1,d} \subset \bigoplus_{\substack{1 \le h < d \le n \\ h+d \equiv n+1 \mod 2}} V^{1,h} \wedge V^{1,d} \tag{24}$$

Again because d respects the weight graduation, $d^{-1}(\delta dv_i) \subset \bigoplus_{\substack{2 \le j \le n \\ j \equiv n+1 \mod 2}} V^{1,j}$. Note that $j \ge 2$ because the elements of δdv_i have weight at least two.

The rest of the proof is just a cumbersome verification that hypothesis 1-5 up to n together with what we have just explained imply 1-5 for $n+1$. □

The conclusion that may be drawn from Proposition 5.10 is that using the algorithm of [**73**] and basepoint section $(0,0,t)$ the Gauss–Manin connection in the Malcev algebras $\mathcal{L}\pi_1(E_t, (0,0,t))$ cannot be defined on a subring of $\mathbb{Q}$.

2.2. Families of punctured rational curves. The Gauss-Manin connection has a simpler 1–minimal model in the surveyed families of rational curves, as the following computations in the case of the complement of a plane curve in the projective plane illustrate.

Let us consider

$$p(x,t) = x^n + a_{n-1}x^{n-1} + \cdots + a_0(t) \in \mathbb{C}(t)[x]$$

and let $\Delta(t)$ be its discriminant. Define $S = \{t \in \mathbb{C} \mid a_i(t) \ne \infty \forall i,\ \ \Delta(t) \ne 0\}$, $X = \{(x,t) \in \mathbb{C}^2 \mid t \in S,\ \ p(x,t) \ne 0\}$ and the projection $f : X \to S$ sending (x,t) to t. This is an algebraic family of rational curves with n punctures over an affine base. We will consider two cases:

CASE 1. : Line arrangements.

In this case $p(x,t) = (x - f_1(t)) \dots (x - f_n(t))$ with $f_i(t) \in \mathcal{O}_S$. The global forms in $\Omega^1_{X|S}$

$$\omega_1 = \frac{dx}{x - f_1(t)}, \quad \omega_2 = \frac{dx}{x - f_2(t)}, \dots, \omega_n = \frac{dx}{x - f_n(t)}$$

restrict to a basis of $H^1_{DR}(X_t)$ for all $t \in S$. The connection ∂_t in Ω^*_S admits a lifting to X satisfying $\partial_t x = 0$. This lifting applied to the ω_i yields

$$\partial_t \omega_i = \partial_t \left(\frac{dx}{x - f_i(t)} \right) = \frac{f_i'(t) dx}{(x - f_i(t))^2} = d \left(-f_i'(t) \frac{1}{x - f_i(t)} \right)$$

Thus fixing a basepoint section $\sigma : S \to X$ a $(1,1)$-minimal model is given by

$$\begin{aligned} M(1,1) = \bigwedge(\alpha_1, \dots, \alpha_n) \qquad & d\alpha_i = 0 \qquad \rho(\alpha_i) = \omega_i = \frac{dx}{x - f_i(t)} \\ \delta(\alpha_i) = 0 \qquad & \lambda(\alpha_i) = -f_i'(t) \frac{1}{x - f_i(t)} + \frac{f_i'(t)}{\sigma(t) - f_i(t)} \\ & i = 1, \dots, n \end{aligned}$$

Now, the $(1,2)$-minimal model of the fibre is given by

$$M(1,2) = M(1,1) \otimes \bigwedge(\langle \eta_{ij}\,,\ 1 \le i < j \le n \rangle)$$

where $\rho\eta_{ij} = 0$, $d\eta_{ij} = \alpha_i \wedge \alpha_j$. We determine $\lambda, \delta(\eta_{ij})$:

$$\begin{aligned} \tilde{o}(\eta_{ij}) &= (\delta(d\eta_{ij}), \partial_t \rho(\eta_{ij}) + \lambda(d\eta_{ij})) \\ &= (0, \lambda(\alpha_i) \cdot \rho(\alpha_j) - \rho(\alpha_i) \cdot \lambda(\alpha_j)) \\ &= d \left(\left(-\frac{f_j'(t) - f_i'(t)}{f_j(t) - f_i(t)} - \frac{f_j'(t)}{\sigma(t) - f_j(t)} \right) \alpha_i \right. \\ &\quad \left. + \left(\frac{f_j'(t) - f_i'(t)}{f_j(t) - f_i(t)} + \frac{f_i'(t)}{\sigma(t) - f_i(t)} \right) \alpha_j, 0 \right) \end{aligned}$$

Therefore $\lambda(V^{1,2}) = 0$. A simple induction computation shows that, as $\rho(V^{1,n}) = 0$, also $\lambda(V^{1,n}) = 0$ for $n \ge 2$.

CASE 2. : Generic curves.

There is an algebraic field extension $\mathcal{K}(C)|\mathcal{K}(S)$ where p splits in linear factors $x - f_1, \dots, x - f_n$, i.e., there is a curve C over S defined by the multivalued algebraic functions on S given by the zeros of p.

The pullback of X over C

$$\begin{array}{ccc} g^*X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ C & \underset{g}{\longrightarrow} & S \end{array}$$

has a cohomology basis for its fibres $\omega_1 = \frac{dx}{x-f_1}, \dots, \omega_n$ as in Case 1, and the computation performed there holds verbatim. An analogous basis for the family $g^*X \to C$ coming from $X \to S$ is given by the forms

$$\psi_1 = \frac{dx}{p},\ \psi_2 = \frac{xdx}{p}, \dots,\ \psi_n = \frac{x^{n-1}dx}{p}$$

The minimal model of $(\Omega^*_{g^*X|C}, d_{g^*X|C})$ computed with this basis is the pushout of the minimal model of $(\Omega^*_{X|S}, d_{X|S})$, i.e., the Hirsch extensions $V^{1,n}$ and the maps δ, λ are the induced by the ring extension $\mathcal{O}_S \hookrightarrow \mathcal{O}_C$. The $(1,1)$-minimal model is given by $M(1,1) = \bigwedge(\beta_1, \dots, \beta_n)$, $\rho(\beta_i) = \psi_i$, $d\beta_i = 0$, and $M(1,2) = M(1,1) \otimes \bigwedge(\langle \theta_{ij},\ 1 \leq i < j \leq n \rangle)$ with $\rho(\theta_{ij}) = 0$, $d\theta_{ij} = \beta_i \wedge \beta_j$. As $\psi_i = \sum b_{ij}\omega_j$ for all i, where $b_{ij} \in \mathcal{O}_C$, we have that $\beta_i = \sum b_{ij}\alpha_j$ and $\theta_{ij} = \sum_{k<l}(b_{ik}b_{jl} - b_{il}b_{jk})\eta_{kl}$. These linear relations allow us to compute δ, λ for $f : X \to S$. We just want to observe that as $\rho(\eta_{ij}) = 0$ and λ is an antiderivation, it turns out that $\lambda\theta_{ij} = 0$ for all i, j. Therefore, as in Case 1, $\lambda(V^{1,n}) = 0$ for all $n \geq 2$.

The fact that in both cases $\lambda(V^{1,n}) = 0$ for $n \geq 2$ has important consequences, contrasting with the case of the Legendre affine family:
- First, the computation of the 1-minimal model of $(\Omega^*_{X|S}, d_{X|S})$ becomes *formal* after the $(1,1)$-minimal model, i.e., we do not need the original algebra and connection anymore.
- Second, it is easily seen studying the succesive obstructions

$$\tilde{o}(\xi) = (\delta(d\xi), \partial_t(\rho(\xi)) + \lambda(d\xi)) = (\delta(d\xi), 0)$$

that after the $(1,1)$-minimal model, no new primes appear in the denominators of the coefficients of $A(n)$. Thus the differential equation is defined over $\mathcal{O}_{S/\mathbb{Z}[\frac{1}{N!}]}$, where N is the least common multiple of the denominators in $A(1)$.
- Third, one may proof a simpler version of Proposition 5.10 in this case. The result is that the matrices $A(n)$ are block band matrices of block width two, where the only nonnull blocks are those on the diagonal and on the first subdiagonal (only in the first subdiagonal in Case 1).

3. The differential Galois groups of Malcev algebras

We have studied in the previous section the differential equation satisfied by the variation of Malcev algebras in algebraic families of curves. In order to do some qualitative analysis of these equations and of their solutions, the non–abelian periods of the family, we will study in this section the differential Galois groups of the Gauss–Manin connection in the rational homotopy of families of algebraic manifolds. Basically, we compare these with the differential Galois groups of the abelian periods of the cohomology of the family, and show that the Malcev algebra Galois groups are unipotent extensions of the cohomology Galois groups. This characterization is based on the group–theoretic results of Section 7 in Chapter 1 and on Schlesinger's theorem characterizing differential Galois groups of the connection as the Zariski closure of the monodromy of the local system (see [**76**]).

Finally, we go back to the specific case of the Legendre affine family, and analyze in more detail its Malcev algebra differential Galois groups.

Let us start by fixing some notation first: for every $n \geq 1$, we will denote by $\mathcal{M}_n$ the monodromy groups of the local systems formed by the bracket spaces $(\pi_1(X_s, \sigma(s))_n / \pi_1(X_s, \sigma(s))_{n+1}) \otimes k$, and by $\mathcal{M}_{1n}$ the monodromy groups of the local systems of k–Malcev algebras $\mathcal{L}(\pi_1(X_s, \sigma(s))$. The coefficient field k will always be clear in every context. In the case of $k = \mathbb{C}$, the corresponding differential Galois groups will be denoted G_n, G_{1n} respectively. The consequence of Lemma 1.54 in differential Galois theory is:

THEOREM 5.11. *Let $f : X \to S$ be an algebraic family as above defined. Fix a homogeneous basis for $\mathcal{L}_n \pi_1 X_0 \otimes k$ with $\operatorname{char} k = 0$, and let*

$$Aut_{Aut\pi_1 X_0} \mathcal{L}_n \pi_1 X_0 \otimes k \xrightarrow{\psi_n} Aut_{Aut\pi_1 X_0} \pi_1(X_0)/\pi_1(X_0)_2 \otimes k$$

be the natural map induced by projection on the first piece of the graduate. Then:

(i) *The monodromy representation $\rho_n : \pi_1 S \to Aut_{Aut\pi_1 X_0} \pi_1(X_0)_n / \pi_1(X_0)_{n+1} \otimes k$ factors as $\phi_n \circ \rho$, where ϕ_n is the map defined in Lemma 1.54.*

(ii) *When $k = \mathbb{C}$, the differential Galois group $\bar{\mathcal{M}}_n$ corresponding to the local system formed by the $\pi_1(X_s)_n / \pi_1(X_s)_{n+1} \otimes \mathbb{C}$ is $\phi_n(\bar{\mathcal{M}}_1)$.*

(iii) *There is an exact sequence*

$$1 \longrightarrow \ker \psi_n \longrightarrow Aut_{Aut\pi_1 X_0} \mathcal{L}_n \pi_1 X_0 \otimes k \xrightarrow{\psi_n} Aut_{Aut\pi_1 X_0} \pi_1(X_0)/\pi_1(X_0)_2 \otimes k \longrightarrow 1$$

with $\ker \psi_n \subset Aut_{Aut\pi_1 X_0} \mathcal{L}_n \pi_1 X_0 \otimes k \subset GL(N, k)$ a unipotent subgroup.

(iv) *For $k = \mathbb{C}$, there is an exact sequence of differential Galois groups*

$$1 \longrightarrow U_n \longrightarrow G_{1n} \xrightarrow{\bar{\psi}_n} G_1 \longrightarrow 1$$

with U_n unipotent.

PROOF. (i) is an immediate consequence of Lemma 1.54.

(ii) comes from (i) and the fact that the ϕ_n are algebraic group morphisms.

(iii) The map ψ_n is onto by its definition, and sends every matrix $A \in \mathrm{Aut}_{\mathrm{Aut}\,\pi_1 X_0} \mathcal{L}_n \pi_1 X_0 \otimes k$ to $A^{1,1}$. Therefore, $A \in \ker \psi_n$ implies $A^{1,1} = \mathrm{Id}$. But by (i), $A^{i,i} = \phi_i A^{1,1}$ for all i. Therefore, the $A \in \ker \psi_n$ are block lower triangular matrices with the identity in the diagonal blocks, and so $\ker \psi_n$ is a unipotent group.

(iv) comes from taking Zariski closures in (iii): $\bar{\psi}_n$ is onto because it is an algebraic group morphism and $G_1 = \bar{\mathcal{M}}_1$, and $\ker \bar{\psi}_n$ is still unipotent because the relations $A^{i,i} = \phi_i A^{1,1}$ are algebraic.

□

REMARK 5.12. Because of the block structure of the matrices, the nilpotence class of the kernel is nil $\ker \psi_n \leq n - 2$. Due to the isomorphisms of 2.2 the unipotent groups $\ker \psi_n$ and U_n form towers, and again $U_{n+1}/(U_{n+1})_{n-2} = U_n$.

REMARK 5.13. When f is a smooth proper morphism, by Deligne's Semisimplicity Theorem ([**32**]) the monodromy group $\mathcal{M}_1$ or equivalently $G_1 = \bar{\mathcal{M}}_1$, $\mathcal{L}(G_1)$ are semisimple. Then by Levi's theorem (see [**98**] Thm. 3.14.1) $\mathcal{L}(G_{1n})$ is an extension of $\mathcal{L}(G_1)$ by the nilpotent algebra $\mathcal{L}(U_n)$.

In the remainder of this section we go back to the affine Legendre family

$$f : E = \{(x, y, t) \in \mathbb{C}^3 \mid y^2 = x(x-1)(x-t)\} \longrightarrow \mathbb{C}$$
$$(x, y, t) \longmapsto t$$

which is our primary example, and study its differential Galois groups G_n, G_{1n}, U_n with more detail.

We begin by recalling the differential Galois group of its abelian periods, which may be retrieved from the monodromy computations of Section 5 in Chapter 3.

LEMMA 5.14. *The Zariski closure G_1 of the monodromy group $\mathcal{M}_1$ in $H_1(E_t; \mathbb{C})$ of the Legendre family is $G_1 = SL(2, \mathbb{C})$*

PROOF. As has been explained in Remark 3.77, the group $\mathcal{M}_1$ is generated by

$$\rho_0 = \begin{pmatrix} 1 & 0 \\ -2 & 1 \end{pmatrix}, \qquad \rho_1 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$$

It is obvious that

$$\overline{\langle \rho_0 \rangle} = \left\{ \begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \mid u \in \mathbb{C} \right\} \subset G_1 ,$$
$$\overline{\langle \rho_1 \rangle} = \left\{ \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \mid u \in \mathbb{C} \right\} \subset G_1 ,$$

and both are abelian. These subgroups are contained in $\mathrm{GL}\,(H_1(E_t;\mathbb{C}))$, therefore by functoriality of the exponential map the Lie algebra of G_1 contains

$$\log \rho_0 = \begin{pmatrix} 0 & 0 \\ -2 & 0 \end{pmatrix} = -2Y , \qquad \log \rho_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = 2X$$

and $[\log \rho_1, \log \rho_0] = -4H$. Hence the algebra is $sl(2,\mathbb{C})$, and $G_1 = \mathrm{SL}\,(2,\mathbb{C})$. □

The previous lemma and Theorem 5.11 allow us now to determine the differential Galois groups associated to the Γ_n/Γ_{n+1}:

PROPOSITION 5.15. *The differential Galois group of the Fuchsian equation satisfied by the* $\pi_1(E_t)_n/\pi_1(E_t)_{n+1} \otimes \mathbb{C}$ *of the affine Legendre family of cubics over* $\mathbb{C} \setminus \{0,1\}$ *is* $\{Id\}$ *if* $n = 2$ *and* $PSL(2,\mathbb{C})$ *if* $n > 2$.

PROOF. The result for $n = 1$ has been establised in Lemma 5.14. For $n > 1$ we use the fact that the monodromy representation

$$\rho : \pi_1(B, \frac{1}{2}) \cong F_2 \longrightarrow \mathrm{Aut}\ ((F_2)_n/(F_2)_{n+1})$$

factors by Theorem 5.11 through

$$\rho : \pi_1(B, \frac{1}{2}) \cong F_2 \longrightarrow \mathrm{Aut}\ (F_2/(F_2)_2)$$

Thus $\mathcal{M}_n = \phi_n(\mathcal{M}_1)$ and $\bar{\mathcal{M}}_n = \phi_n(\bar{\mathcal{M}}_1) = \phi_n(\mathrm{SL}(2,\mathbb{C}))$. By Proposition 1.61 this image is $\{\mathrm{Id}\}$ if $n = 2$ and $\mathrm{SL}(2,\mathbb{C})/\mu_n \mathrm{Id} = \mathrm{PSL}(2,\mathbb{C})$ if $n > 2$. □

It is worth remarking that all the differential Galois groups above computed are irreducible.

We will also require the Lie algebra version of Proposition 5.15.

COROLLARY 5.16. *Let ρ_0,ρ_1 be the images of $[\gamma_0]$, $[\gamma_1]$ in the monodromy on $\pi_1(E_t)_n/\pi_1(E_t)_{n+1} \otimes \mathbb{C}$ with $n \neq 2$. There is a Lie algebra monomorphism*

$$\phi_n^* : sl(2,\mathbb{C}) \longrightarrow gl(\pi_1(E_t)_n/\pi_1(E_t)_{n+1} \otimes \mathbb{C})$$

such that

$$\exp \phi_n^*(-2Y) = \rho_0$$
$$\exp \phi_n^*(2X) = \rho_1 .$$

Its image is the Lie algebra of the differential Galois group of $\pi_1(E_t)_n/\pi_1(E_t)_{n+1} \otimes \mathbb{C}$.

PROOF. As we have seen in Prop. 5.15, the monodromy representation in $\pi_1(E_t)_n/\pi_1(E_t)_{n+1} \otimes \mathbb{C}$ factors as

$$\pi_1(B,\frac{1}{2}) \xrightarrow{\rho} \mathrm{GL}\,(\pi_1(E_t)/\pi_1(E_t)_2{\otimes}\mathbb{C}) \xrightarrow{\phi_n} \mathrm{GL}\,(\pi_1(E_t)_n/\pi_1(E_t)_{n+1}{\otimes}\mathbb{C})$$

and $\bar{\mathcal{M}}_n = \phi_n \bar{\mathcal{M}}_1 = \phi_n \mathrm{SL}\,\pi_1(E_t)/\pi_1(E_t)_2 \otimes \mathbb{C}$. The Lie algebras of $\bar{\mathcal{M}}_1$, $\bar{\mathcal{M}}_n$ satisfy the same relation $\mathcal{L}\bar{\mathcal{M}}_n = \phi_n^* \mathcal{L}\bar{\mathcal{M}}_1 = \phi_n^* sl(2,\mathbb{C})$, the latter equality given by the isomorphism of Lemma 5.14

$$\mathcal{L}(\bar{\mathcal{M}}_1) \xrightarrow{\sim} sl(2,\mathbb{C})$$
$$\log \rho_0 \longrightarrow -2Y$$
$$\log \rho_1 \longrightarrow 2X$$

As ϕ_n is a finite map for $n \neq 2$ by Prop. 1.61, ϕ_n^* is injective. □

We are able now to study the unipotent extensions

$$1 \to U_n \to G_{1n} \to G_n \to 1$$

and the generators of the groups in our example. Let $A(n)$ be the matrix of the differential equation associated to the $(1,n)$–minimal model.

PROPOSITION 5.17. *The groups U_n of the affine Legendre family verify that*

$$nil\, U_n \leq n-3$$

PROOF. The monodromy of the affine Legendre family may be computed from the differential equations extending the system of (23), the maps ρ_0, ρ_1 being the matrices of $\mathrm{res}_0\, A(n)$, $\mathrm{res}_1\, A(n)$ respectively. These matrices are block lower triangular with the first subdiagonal made of zero blocks, which assures that the $(n-2)$-brackets are zero. □

REMARK 5.18. Looking at the second subdiagonal in the differential equation 23 of $\mathcal{L}_3(\pi_1(E_t,*))$ we find the block $A(n)^{3,1} = A(3)^{3,1}$ with nonzero residue matrices in 0,1. It seems likely that $\operatorname{nil} U_n = n-3$ exactly.

Now we establish a property hinted at in Section 5 of Chapter III.

PROPOSITION 5.19. *The monodromy matrices $\rho_0, \rho_1 \in Aut_{Aut\pi_1 E_s}\mathcal{L}_n\pi_1 E_s\otimes k$ are unipotent.*

PROOF. Equivalently, we will check that $\log\rho_0$,$\log\rho_1$ are nilpotent.

We know by Prop. 5.17 that both ρ_0,ρ_1 are block lower triangular, and so are $\log\rho_0$,$\log\rho_1$. It is easily checked that the diagonal blocks $(\log\rho_0)^{i,i}$,$(\log\rho_1)^{i,i}$ are the logarithms of $\rho_0^{i,i}$,$\rho_1^{i,i}$ respectively, which are the matrices of the monodromy automorphisms ρ_0,ρ_1 in $\{\pi_1(E_t,*)_i/\pi_1(E_t,*)_{i+1}\otimes\mathbb{C}\}_{t\in S}$.

By Corollary 5.16 there is a Lie algebra monomorphism $\phi_i^* : sl(2,\mathbb{C}) \to \mathrm{GL}\,(\pi_1(E_t)_i/\pi_1(E_t)_{i+1}\otimes\mathbb{C})$ such that

$$\phi_i^*(-2Y) = (\log\rho_0)^{i,i}$$
$$\phi_i^*(2X) = (\log\rho_1)^{i,i}$$

Since $X, Y \in sl(2,\mathbb{C})$ are nilpotent, so must be $(\log\rho_0)^{i,i}$, $(\log\rho_1)^{i,i}$ for all i.

The only condition imposed by Prop. 5.17 on a basis $\{e_i\}$ of $\mathcal{L}_n\pi_1(E_t)\otimes\mathbb{C}$ to produce block lower triangular matrices for ρ_0,ρ_1 was that it should consist of homogeneous elements belonging to the $F^i/F^{i+1} \cong \pi_1(E_t)_i/\pi_1(E_t)_{i+1}\otimes\mathbb{C}$. We can form one such basis with a basis for every $\pi_1(E_t)_i/\pi_1(E_t)_{i+1}\otimes\mathbb{C}$ in which $(\log\rho_0)^{i,i}$ has its canonical Jordan matrix, and in this basis the matrix $\log\rho_0$ will be strictly lower triangular. The union of the corresponding Jordan basis for $(\log\rho_1)^{i,i}$ will also make $\log\rho_1$ strictly lower triangular, and complete our proof. □

Finally, we establish the triviality of π_0, $\pi_1(G_{1n})$ for the affine Legendre family.

PROPOSITION 5.20. *G_{1n} is connected and simply connected for all n.*

PROOF. As we have seen in Thm. 5.11, there is an exact sequence of algebraic groups

$$1 \longrightarrow U_n \longrightarrow G_{1n} \longrightarrow \mathrm{SL}\,(2,\mathbb{C}) \longrightarrow 1 \tag{25}$$

using that $G_1 = \mathrm{SL}\,(2,\mathbb{C})$ in our case. The first term U_n is unipotent, hence connected and simply connected. So is $\mathrm{SL}\,(2,\mathbb{C})$. The maps of

(25) define a topological fibration, and therefore the homotopy groups of the spaces form a homotopy long exact sequence

$$\cdots \to \pi_1(U_n) \to \pi_1(G_{1n}) \to \pi_1(\mathrm{SL}\,(2,\mathbb{C}) \to \pi_0(U_n) \to \pi_0(G_{1n}) \to \pi_0(\mathrm{SL}\,(2,\mathbb{C}))$$

which establishes our assertion. □

Bibliography

[1] N. A'Campo, *La fonction zeta d'une monodromie*, Comment. Math. Helv. **50** (1975) 233–248.

[2] J. Amorós, *On the Malcev completion of Kähler groups*, Comment. Math. Helv. **71** (1996) 192–212.

[3] J. Amorós, M. Burger, K. Corlette, D. Kotschick, D. Toledo, *Fundamental Groups of Compact Kähler Manifolds*, Math. Surveys and Monogr. **44**, A.M.S. 1996.

[4] A. Andreotti, T. Frankel, *The second Lefschetz theorem on hyperplane sections*, in *Global Analysis. Papers in honor of K. Kodaira*, Princeton U.P., 1969.

[5] D. Arapura, P. Bressler and M. Ramachandran, *On the fundamental group of a compact Kähler manifold*, Duke Math. Jour. **68** (1992) 477–488.

[6] V.I. Arnol'd, *Remark on the branching of hyperelliptic integrals as functions of the parameters*, Functional Analysis **2** (1968) 187–189.

[7] V. Arnold, A. Varchenko, S. Goussein-Zadé, *Singularités des Applications Différentiables*, 2 vols., Editions Mir 1986.

[8] M. Asada, M. Matsumoto, T. Oda, *Local monodromy in the fundamental groups of algebraic curves along a degenerate stable curve*, J. Pure Appl. Algebra **103** (1995) 235–283.

[9] F. Bardelli, *-lectures on stable curves*, in M. Cornalba, X. Gómez-Mont, A. Verjovsky (Eds.), *Lectures on Riemann Surfaces*, World Scientific 1989.

[10] W. Barth, C. Peters and A. Van de Ven, *Compact Complex Surfaces*, Springer Verlag 1984.

[11] G. Baumslag, *More groups that are just about free*, Bull. Amer. Math. Soc. **74** (1968) 752–754.

[12] A. Beauville, *Surfaces algébriques complexes* (3^{me} Ed.), Astérisque **54**, S.M.F. 1978.

[13] A. Beauville, Appendix to [**24**].

[14] F. Beukers, *Differential Galois Theory*, Ch. 8 of Waldschmidt et altri (Eds.), *From Number Theory to Physics*, Springer–Verlag 1992.

[15] J. Birman, *Braids, Links and Mapping Class Groups*, Annals of Math. Studies **82**, Princeton U. P. 1975.

[16] J. Birman, *The Algebraic Structure of Surface Mapping Class Groups*, in W. Harvey (Ed.), *Discrete Groups and Automorphic Functions*, Academic Press 1977, 163–198.

[17] N. Bourbaki, *Lie Groups and Lie Algebras. Part I: Chapters 1-3*, Hermann, Addison Wesley 1975.

[18] A. Bousfield and V. Gugenheim, *On PL De Rham theory and rational homotopy type*, Mem. Amer. Math. Soc. **179**, Amer. Math. Soc., Providence, R.I. 1976.

[19] E. Brieskorn, H. Knörrer, *Plane Algebraic Curves*, Birkhäuser 1986.

[20] F. Campana, *Remarques sur les groupes de Kähler nilpotents*, C. R. Acad. Sci. Paris **317** (1993) 777–780.

[21] F. Campana, *Fundamental group and positivity of cotangent bundles of compact Kähler manifolds*, J. Alg. Geom. **4** (1995) 487–502.

[22] F. Campana, *Remarques sur les groupes de Kähler nilpotents*, Ann. Scient. Éc. Norm. Sup. **28** (1995) 307–316.

[23] J. A. Carlson and D. Toledo, *Quadratic presentations and nilpotent Kähler groups*, Jour. Geometric Analysis (to appear).

[24] F. Catanese, *Moduli and classification of irregular Kähler manifolds (and algebraic varieties) with Albanese general type fibrations*, Invent. Math. **104** (1991) 263–289.

[25] K.T. Chen, *Algebras of iterated path integrals and fundamental groups*, Trans. Amer. Math. Soc. **156** (1971) 359–379.

[26] K.T. Chen, *Iterated integrals of differential forms and loop space cohomology*, Ann. of Math. **97** (1973) 217–246.

[27] K.T. Chen, R. Fox, R. Lyndon, *Free Differential Calculus, IV. The quotient groups of the lower central series*, Annals of Math. **68** (1958) 81–95.

[28] S. Chen, *Examples of n–step nilpotent 1–formal 1–minimal models*, C.R. Acad. Sci. Paris **321**, Série I (1995) 223–228.

[29] C.H. Clemens, *A Scrapbook of Complex Curve Theory*, Plenum Press 1980.

[30] M. Dehn, *Über unendliche diskontinuierliche Gruppen*, Math. Ann. **71** (1912) 116–144.

[31] P. Deligne, *Equations Differentielles à Points Singuliers Réguliers*, Lect. N. in Math. **163**, Springer–Verlag 1970.

[32] P. Deligne, *Théorie de Hodge II*, Publ. Math. IHES **40** (1971) 5–58.

[33] P. Deligne, *Un théorème de finitude pour la monodromie*, In: R. Howe (ed.) *Discrete Groups and Analysis*, Birkhäuser Verlag 1987.

[34] P. Deligne, *Le Groupe Fondamental de la Droite Projective Moins Trois Points*, in Y. Ihara, K. Ribet, J.-P. Serre (Eds.) *Galois Groups over* $\mathbb{Q}$, MSRI Publ. **16**, Springer–Verlag 1989, 79–297.

[35] P. Deligne, P. Griffiths, J. W. Morgan and D. Sullivan, *Real homotopy theory of Kähler manifolds*, Invent. Math. **29** (1975) 245–274.

[36] A. Fathi, M. Shub, *Some dynamics of pseudo–Anosov diffeomorphisms*, in *Travaux de Thurston sur les surfaces (Séminaire Orsay 1977–1978)*, Astérisque **66-67**, S.M.F. 1979.

[37] E. Ghys, P. de la Harpe (Eds.), *Sur les Groupes Hyperboliques d'après Mikhael Gromov*, Progr. Math. **83**, Birkhäuser Verlag 1990.

[38] W. M. Goldman and J. J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, Publ. Math. IHES **67** (1988) 43–96.

[39] R.E. Gompf *A new construction of symplectic manifolds*, Ann. of Math. **142** (1995) 527–595.

[40] H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. Math. **68** (1958) 460–472.

[41] M. Green, R. Lazarsfeld, *Higher obstructions to deforming cohomology groups of line bundles*, J. Amer. Math. Soc. **4** (1991) 87–103.

[42] P. Griffiths, *Periods of integrals on algebraic manifolds*, Bull. Amer. Math. Soc. **76** (1970) 228–296.

[43] P. Griffiths (Ed.), *Topics in Transcendental Algebraic Geometry*, Ann. of Math. Studies **106**, Princeton U.P. 1984.

[44] P. Griffiths and J. Harris, *Principles of Algebraic Geometry*, John Wiley & Sons, New York 1978.

[45] P. Griffiths and J. W. Morgan, *Rational Homotopy Theory and Differential Forms*, Progr. Math. **16**, Birkhäuser Verlag 1981.

[46] M. Gromov, *Sur le groupe fondamental d'une variété kählérienne*, C.R. Acad. Sci. Paris **308** Sér. I (1989) 67–70.

[47] R. Hain, *The de Rham Homotopy Theory of Complex Algebraic Varieties*, K-theory **1** (1987) 271–324.

[48] R. Hain, *The de Rham Homotopy Theory of Complex Algebraic Varieties II*, K-Theory **1** (1987) 481–497.

[49] R. Hain, *The Geometry of the Mixed Hodge Structure on the Fundamental Group*, Proc. Symp. Pure Math. **46** (1987), American Mathematical Society, 247–282.

[50] R. Hain, *Completions of mapping class groups and the cycle $C - C^-$*, in C.-F. Bödigheimer, R. Hain (Eds.), *Mapping Class Groups and Moduli Spaces of Riemann Surfaces*, Contemporary Mathematics **150**, Amer. Math. Soc. 1993.

[51] P. Hilton, G. Mislin and J. Roitberg, *Localization of Nilpotent Groups and Spaces*, Math. Studies **15**, North Holland 1975.

[52] R.-P. Holzapfel, *Geometry and Arithmetic around Euler Partial Differential Equations*, D. Reidel Publ. Co. (Kluwer) 1986.

[53] Hua Loo Keng, *Introduction to Number Theory*, Springer–Verlag 1982.

[54] A. Hurwitz, *Über Riemannsche Flächen mit gegebenen Verzweigungspunkten*, Math. Ann. **39** (1891) 1–61.

[55] F. E. A. Johnson and E. G. Rees, *On the fundamental group of a complex algebraic manifold*, Bull. London Math. Soc. **19** (1987) 463–466.

[56] F. E. A. Johnson and E. G. Rees, *The fundamental group of an algebraic variety*, in S. Jackowsky et al.(Eds.) *Algebraic Topology Poznan 1989*, LNM **1474**, Springer Verlag 1991.

[57] I. Kaplansky, *An Introduction to Differential Algebra*, Act. Sci. et Ind. 1251, Hermann 1957

[58] N. Katz and T. Oda, *On the Differentiation of De Rham Cohomology Classes with respect to Parameters*, J. Math. Kyoto Univ. **8** (1968) 199–213.

[59] S. Kerckhoff, *The Nielsen realization problem*, Annals of Math. **117** (1983) 235–265.

[60] T. Kohno, *Série de Poincaré-Koszul associée aux groupes de tresses pures*, Invent. Math. **82** (1985) 57–75.

[61] J. Kollár, *Shafarevich Maps and Automorphic Forms*, Princeton University Press 1995.

[62] D. Kotschick, *All fundamental groups are almost complex*, Bull. London Math. Soc. **24** (1992) 377–378.

[63] K. Lamotke, *Die Homologie isolierten Singularitäten*, Math. Z. **143** (1975) 27–44.

[64] M. Lazard, *Sur les groupes nilpotents et les anneaux de Lie*, Ann. scient. Éc. Norm. Sup. **71** (1954) 101–190.

[65] W. Magnus, A. Karrass, D. Solitar, *Combinatorial Group Theory*, Pure and Applied Math. XIII, Interscience/John Wiley & Sons 1966.

[66] A.I. Malcev, *On a class of homogeneous spaces*, Amer. Math. Soc. Transl. **39** (1951).

[67] A. Manning, *Topological Entropy and the First Homology Group*, in A. Manning (Ed.), *Dynamical Systems–Warwick 1974*, Lect. Notes in Math. **468**, Springer–Verlag 1975.

[68] J. Milnor, *Singular Points of Complex Hypersurfaces*, Annals of Math. Studies **61**, Princeton U. P. 1968.

[69] J. Milnor, J. Moore, *On the structure of Hopf algebras*, Annals Math. **81** (1965) 211–264.

[70] J.W. Morgan, *The algebraic topology of smooth algebraic varieties*, Publ. Math. IHES **48** (1978) 137–204.

[71] J.W. Morgan, *Hodge theory for the algebraic topology of smooth algebraic varieties*, Proc. Sympos. Pure Math. **32** AMS 1978, 119–128.

[72] V. Navarro Aznar, *Sur la théorie de Hodge–Deligne*, Invent. Math. **90** (1987) 11–76.

[73] V. Navarro Aznar, *Sur la connexion de Gauss–Manin en Homotopie Rationelle*, Ann. Sci. ENS **26** (1993) 99–148.

[74] V. Navarro Aznar, *Sistemas locales y conexiones*, notes from a course given at the Universitat de Barcelona, 1992-3.

[75] P. Orlik, *Introduction to Arrangements*, CBMS Reg. Conference Series **72**, American Mathematical Society 1989.

[76] J. Plemejl, *Problems in the sense of Riemann and Klein*, John Wiley 1964.

[77] M. Pulte, *The fundamental group of a Riemann surface: mixed Hodge structures and algebraic cycles*, Ph.D. Thesis, University of Utah, Salt Lake City, Utah, 1985.

[78] D. Quillen, *On the Associated Graded Ring of a Group Ring*, J. Algebra **10** (1968) 411–418.

[79] D. Quillen, *Rational homotopy theory*, Ann. of Math. **90** (1969) 205–295.

[80] M. Raghunathan, *Discrete Subgroups of Lie Groups*, Ergebnisse der Math. **68**, Springer–Verlag 1972.

[81] R. Remmert, A. Van de Ven, *Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten*, Topology **2** (1963) 137–157.

[82] D. Rolfsen, *Knots and Links*, Math. Lecture Ser. **7**, Publish or Perish 1990.

[83] J.-P. Serre, *Lie algebras and Lie groups*, W.A. Benjamin 1965.

[84] J.-P. Serre, *Trees*, Springer Verlag 1980.

[85] C. Simpson, *Higgs bundles and local systems*, Publ. Math. IHES **75** (1992) 5–95.

[86] Y.-T. Siu, *Strong rigidity for Kähler manifolds and the construction of bounded holomorphic functions*, In: R. Howe (ed.) *Discrete Groups and Analysis*, Birkhäuser Verlag 1987.

[87] J. Stallings, *Homology and central series of groups*, J. Algebra **2** (1965) 170–181.

[88] J.R. Stallings, *Quotients of the powers of the augmentation ideals in a group ring*, in L.P. Neuwirth, *Knots, Groups and 3–Manifolds*, Ann. of Math. Stud. **84**, Princeton University Press 1975, 101–118.

[89] J. Steenbrink, *Limits of Hodge structures*, Invent. Math. **31** (1976) 229–257.

[90] J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Graduate Texts in Math. **72**, Springer–Verlag 1980.

[91] D. Sullivan, *Infinitesimal computations in Topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977) 269–331.

[92] C.H. Taubes, *The existence of anti-self-dual conformal structures*, J. Diff. Geom. **36** (1992) 163–253.

[93] D. Toledo, *Examples of fundamental groups of compact Kähler manifolds*, Bull. London Maths. Soc. **22** (1990) 339–343.

[94] C. Tretkoff, M. Tretkoff, *Solution of the inverse problem of Differential Galois Theory in the classical case*, Amer. J. of Math. **101** (1978) 1327–1332

[95] C. Tretkoff, M. Tretkoff, *Combinatorial Group Theory, Riemann Surfaces and Differential Equations*, Contemporary Mathematics **33** (1984) 467–519.

[96] A.J. Tromba, *Teichmüller Theory in Riemannian Geometry*, Lect. in Math. ETH Zürich, Birkhähuser Verlag 1992.

[97] K. Ueno, *Classification Theory of Algebraic Varieties and Compact Complex Spaces*, Lecture Notes in Math. **439**, Springer–Verlag 1975.

[98] V.S. Varadarajan, *Lie Groups, Lie Algebras and their Representations*, Grad. Texts in Math. **102**, Springer–Verlag 1984.

[99] C. Weibel, *An introduction to homological algebra*, Cambridge studies in adv. math. **38**, Cambridge Univ. Press 1994.

[100] R.O. Wells, *Comparison of De Rham and Dolbeault cohomology for proper surjective mappings*, Pacific J. Math. **53** (1974) 281–300.

[101] R.O. Wells, *Differential Analysis on Complex Manifolds*, Graduate Texts in Math. **65**, Springer–Verlag 1980.

[102] H. Whitney, F. Bruhat, *Quelques propiétés fondamentales des ensembles analytiques–réels*, Comm. Math. Helv. **33** (1959) 132–160.

[103] C.P. Rourke, B.J. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer–Verlag 1982.