APPENDIX A: DERIVATION OF THE GOVERNING EQUATION FOR SOLID DRYING WITH SHRINKAGE

Take a cube of a solid. The initial cube size is R_0. Let's introduce the following definition

$$\delta = \frac{R}{R_0} \tag{A.1}$$

In free shrinkage the actual solid density (kg/m^3) is:

for one-dimensional shrinkage

$$\rho_m = \frac{m_s}{R_0^2 R} = \frac{\rho_0 R_0^3}{R_0^2 R} = \rho_0 \frac{R_0}{R} = \frac{\rho_0}{\delta} \tag{A.2}$$

for two-dimensional shrinkage

$$\rho_m = \frac{m_s}{R_0 R^2} = \frac{\rho_0 R_0^3}{R_0 R^2} = \rho_0 \left(\frac{R_0}{R}\right)^2 = \frac{\rho_0}{\delta^2} \tag{A.3}$$

for three-dimensional shrinkage

$$\rho_m = \frac{m_s}{R^3} = \frac{\rho_0 R_0^3}{R^3} = \rho_0 \left(\frac{R_0}{R}\right)^3 = \frac{\rho_0}{\delta^3} \tag{A.4}$$

The flat plate geometry is shown in Figure A.1.

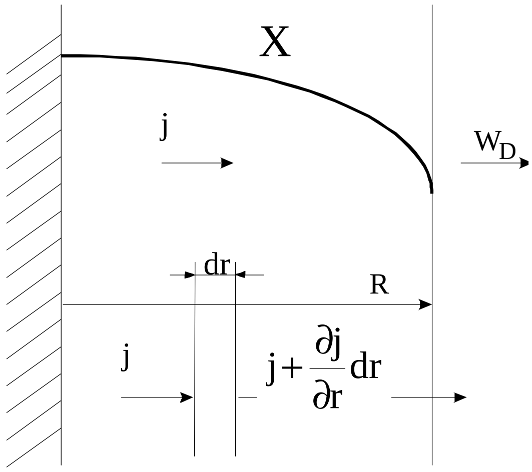

Figure A.1 Schematic of the flat slab drying with shrinkage

Mass balance of moisture for a slice dr thick is

$$jA - \left(jA + \frac{\partial (jA)}{\partial r} dr \right) = \frac{\partial (dr \rho_m XA)}{\partial \tau}$$

A.5

A is cross sectional area of the slice. It is large (infinite) compared to R and therefore can be reduced. Therefore one obtains

$$\frac{\partial j}{\partial r} dr = \rho_m X \frac{\partial dr}{\partial \tau} + dr \frac{\partial (\rho_m X)}{\partial \tau} \quad |: dr$$

A.6

$$\frac{\partial j}{\partial r} = \rho_m X \frac{\partial dr}{\partial \tau} + dr \frac{\partial (\rho_m X)}{\partial \tau}$$

A.7

For isometric shrinkage

$$\frac{1}{dr} \frac{\partial dr}{\partial \tau} = \frac{1}{dR} \frac{dR}{d\tau}$$

A.8

Introducing (A.7) in (A.8) and introducing the constitutive equation

$$j = -D\rho_m \frac{\partial X}{\partial r}$$

A.9

one obtains

$$D\frac{\partial \rho_m}{\partial r}\frac{\partial X}{\partial r} + D\rho_m \frac{\partial^2 X}{\partial r^2} = \rho_m \frac{X}{R}\frac{dR}{d\tau} + \rho_m \frac{\partial X}{\partial \tau} + X\frac{\partial \rho_m}{\partial \tau}$$

A.10

By neglecting first term of equation (A.10) (in isotropic shrinkage ρ_m is constant in space) and dividing by ρ_m one obtains

$$\boxed{D\frac{\partial^2 X}{\partial r^2} = \frac{X}{R}\frac{dR}{d\tau} + \frac{\partial X}{\partial t} + \frac{X}{\rho_m}\frac{\partial \rho_m}{\partial \tau}}$$

A.11

In (A.11) the value of dR/dt must be known. It can be evaluated from the overall mass balance for the plate

$$\frac{\partial(\varepsilon \rho_L V)}{\partial \tau} + Aw_D = 0$$

A.12

where

$$V = AR$$

and therefore

$$\frac{\partial(\varepsilon \rho_L R)}{\partial \tau} = -w_D$$

A.13

$$\varepsilon \rho_L \frac{dR}{d\tau} + R\rho_L \frac{d\varepsilon}{d\tau} + R\varepsilon \frac{d\rho_L}{d\tau} = -w_D$$

A.14

Finally

$$\boxed{\frac{dR}{d\tau} = -\frac{1}{\varepsilon\rho_L}\left[R\left(\rho_L\frac{d\varepsilon}{d\tau} + \varepsilon\frac{d\rho_L}{d\tau}\right) + w_D\right]}$$

A.15

In this equation dε/dt can be calculated as

for 3D shrinkage

$$\frac{d\varepsilon}{d\tau} = (1-\varepsilon_0)\frac{3}{\delta^4}\frac{d\delta}{d\tau}$$

A.16

for 2D shrinkage

$$\frac{d\varepsilon}{d\tau} = (1-\varepsilon_0)\frac{2}{\delta^3}\frac{d\delta}{d\tau}$$

A.17

for 1D shrinkage

$$\frac{d\varepsilon}{d\tau} = (1-\varepsilon_0)\frac{1}{\delta^2}\frac{d\delta}{d\tau}$$

A.18

For all three one-dimensional geometries (plate, cylinder, sphere) equation (A.11) can be generalised by introducing a proper expression for second order derivative and the formulas (A.2 –A.4) for solid density. In the result one obtains:

$$\frac{1}{r^n}\frac{\partial}{\partial r}\left(r^n D\frac{\partial X}{\partial r}\right) = (n+1)\frac{X}{\delta}\frac{d\delta}{dt} - m\frac{X}{\delta}\frac{d\delta}{dt} + \frac{dX}{dt}$$

A.19

By introducing dimensionless variables

$$Fo = \frac{D_0 d\tau}{R_0^2} \qquad \zeta = \frac{r}{R} \qquad \text{and} \qquad \Phi = \frac{X - X^*}{X_0 - X^*}$$

A.20

one obtains

$$\frac{1}{\zeta^n}\frac{\partial}{\partial\zeta}\left(\zeta^n\frac{D}{D_0}\frac{\partial\Phi}{\partial\zeta}\right)=(n+1-m)\left(\Phi+\frac{X^*}{X_0-X^*}\right)\delta\frac{d\delta}{dFo}+\delta^2\frac{\partial\Phi}{\partial Fo} \qquad \text{A.21}$$

where n – geometry index, m – shrinkage index

n=0 plate possible m=1 1D shrinkage

m=2 2D shrinkage

m=3 3D shrinkage

n=1 cylinder possible m=2

m=3

n=2 sphere possible m=3

When solving equation (A.21) dδ/dt can be calculated from the linear shrinkage formula

$$R = R_0\left(s_1\overline{X}+1\right) \qquad \text{A.22}$$

But, first of all one must derive a formula for space averaged X. It can be done by virtue of the overall moisture balance

$$\frac{d\left(\rho_m\overline{X}V\right)}{dt} - Aw_D = 0 \qquad \text{A.23}$$

The above equation can be easily converted to

$$\frac{dR}{d\tau} = -\frac{1}{m\overline{X}\rho_m}\left[R\left(\overline{X}\frac{\partial\rho_m}{\partial\tau} + \rho_m\frac{\partial\overline{X}}{\partial t}\right) - (n+1)w_D\right]$$

A.24

Introducing δ one obtains

$$\frac{d\overline{X}}{dt} = \delta^{m+1}\frac{(n+1)}{\rho_0 R_0} w_D$$

A.25

Boundary condition of the I type can be used as is, BC II must be derived for the conditions of shrinkage. It leads to the following equation:

$$Bi_0\delta^{m+1}\frac{D}{D_0}\frac{Y_i - Y}{X - X^*}\Phi + \frac{1}{X_0 - X^*}\frac{X_i\rho_m - Y_i\rho_g}{\rho_m}\frac{D_0}{D}\delta\frac{d\delta}{dt} + \frac{d\Phi}{d\zeta} = 0$$

A.26

where

$$Bi_0 = \frac{k_Y R_0}{\rho_{m0} D_0}$$

A.27