Universitat Autònoma de Barcelona

WEAK APPROXIMATION OF THE COMPLEX BROWNIAN SHEET AND APPLICATIONS TO SPDES

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Introduction

It is usual that in works related to stochastic processes, the Poisson process or the Brownian motion are mentioned. This work is not an exception, we will not deal with them directly but they will do appear. It is something normal due to the fact that these processes are the most important in the stochastic processes theory, they both are Lévy processes and they both enjoy a rich theory.

The Poisson process generates point patterns in a purely random way, it is the simplest counting points processes but it is quite accurate. The Poisson process owes its name to the fact that the number of points in a region of finite size is a Poisson random variable. The term was coined in 1940 by William Feller (see [27]). It has a lot of applications as the number of car accidents, the number of calls arriving when the user is on the phone, the number of customers arriving to a bank attention line or queuing theory problems. It is a discontinuous jumping process.

The Brownian motion is the random motion of pollen suspended in a water resulting from their collision with the fast-moving molecules in the water. It was first observed by Robert Brown in 1828. It was studied by Albert Einstein with a physicist model in 1905 (see [26]). Norbert Wiener started a rigorous mathematical study in 1923 (see [44] and [45]), he also proved its existence and the non-differentiability of the paths. But it was Paul Lévy the one who proved the most useful results in this initial stage (see [35] and [36]). One of those results is the characterization of the Brownian motion as a martingale. This process has applications in the stock exchange, in the noise of an electric circuit study, in the limit behavior of queuing systems and in random perturbations studied in many other physics, biology and economy systems.

The Brownian motion is a canonical example of two of the most fundamental concepts in the stochastic processes theory: the Markov property and the martingale property. The Poisson process only holds the Markov property. The martingale concept was introduced by J. Vile in 1939 (see [40]). But Paul Lévy had already created it in 1934, when he tried to extend the Kolmogorov inequality and the strong law of large numbers further than the independence case.

In this work we will prove a result in weak convergence and it's application to SPDE, in particular to the stochastic heat equation. We proceed to explain the results that have motivated our work.

In 1982, Stroock proved in [38] that the following family of processes,

$$y_\varepsilon(t) := \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(s)} ds, \ \varepsilon > 0, \tag{1}$$

where N denotes a standard Poisson process, converges in law, in the space of continuous functions, to a standard Brownian motion. Note that this kind of processes had already been used by Kac in 1956 in order to express the solution of the telegrapher's equation in terms of a Poisson process (see [33]). Specially, if we consider the equation

$$\frac{1}{v}\frac{\partial^2 F}{\partial t^2} = v\frac{\partial^2 F}{\partial x^2} - \frac{2a}{v}\frac{\partial F}{\partial t}, \tag{2}$$

with $a, v > 0$, $F(x, 0) = \varphi(x)$, where φ is a smooth enough function and $\left(\frac{\partial F}{\partial t}\right)_{t=0} = 0$. Let us also consider

$$x(t) = v \int_0^t (-1)^{N_a(r)} dr,$$

where $N_a = \{N_a(t), t \geq 0\}$ is a Poisson process of intensity a. Then we can write the solution of the equation as:

$$F(x,t) = \frac{1}{2}\mathbb{E}[\varphi(x + x(t)) + \varphi(x - x(t))].$$

Kac did not prove nor mention that the processes $x(t)$ converge in certain sense to the Brownian motion. Nonetheless this convergence appears in his work in an implicit way. He notices that (2) converges to the heat equation:

$$\frac{1}{D}\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}, \tag{3}$$

if a and v tend to infinity with $\frac{2a}{v^2} := \frac{1}{D}$ constant.

We can rewrite the process (1) and we have

$$\begin{aligned} x_n(t) =& \frac{1}{\sqrt{n}} \int_0^{nt} (-1)^{N(u)} du = \sqrt{n} \int_0^t (-1)^{N(ns)} ds \\ =& \sqrt{n} \int_0^t (-1)^{N_n(s)} ds. \end{aligned}$$

We observe that these are the same processes considered by Kac with $a = n$ and $v = \sqrt{n}$, and it is clear that $\frac{2a}{v^2} = 2$ holds, so $D = 2$. It is well known that, the solution of the heat equation given in (3) with $D = 2$ can be written as

$$F(x,t) = \mathbb{E}[\varphi(W_t)|W_0 = x],$$

where W is a Brownian motion.

We can find in the literature a lot of generalizations of Stroock's result in three directions:
The first of them consists in modifying the process $y_\varepsilon(t)$ in order to obtain approximations of other Gaussian processes.

One first example of an extension of the above result is also due to Stroock, who modified the processes x_n in order to obtain approximations of stochastic differential equations:

$$Y_n(t) = x + \frac{1}{\sqrt{n}} \int_0^{nt} (-1)^{N(u)} \sigma(Y_n(s)) ds + \int_0^t b(Y_n(s)) ds,$$

where $t \in [0, T]$. Assume that σ and b belong to the space $\mathcal{C}_b^2$ of the continuous function with bounded first an second orde derivatives. Let $\{\mathbb{P}^n\}_{n\geq 1}$ be the laws of the processes Y_n in the space of continuous functions $\mathcal{C}([0,T])$. Then, $\{\mathbb{P}^n\}_{n\geq 1}$ converges weakly, when n tends to infinity, to $\mathbb{P}$, where $\mathbb{P}$ is the law, in the space $\mathcal{C}([0,T])$, of the unique solution of the stochastic equation

$$Y_t = x + \int_0^t \sigma(Y_s) \circ dW_s + \int_0^t b(Y_s) ds,$$

where $W = \{W_t, t \in [0,T]\}$ is a standard Brownian motion and the integral in the sum's second term is in the Stratonovich sense.

Another extension in this direction is the following one given by Delgado and Jolis for a class of Gaussian processes that include the fractional Brownian motion (see [22]). Let Y be a stochastic process that supports this representation:

$$Y := \left\{ Y_t = \int_0^1 K(t,r) dW_r, t \in [0,1] \right\},$$

where W is a standard Brownian motion, the kernel $K : [0,1] \times [0,1] \to \mathbb{R}$ satisfies that K is mesurable and $K(0,r) = 0$ for all $r \in [0,1]$, and that for all $0 \leq s \leq t \leq 1$,

$$\int_0^t (K(t,r) - K(s,r))^2 dr \leq (G(t) - G(s))^\alpha,$$

where $\alpha > 0$ and $G : [0,1] \to \mathbb{R}$ is a continuous and increasing function. Let us consider

$$Y_t^n := \sqrt{n} \int_0^1 K(t,r)(-1)^{N(nr)} dr.$$

Then

$$Y^n \overset{\mathcal{L}}{\Rightarrow} Y, \quad \text{in} \quad \mathcal{C}([0,1]).$$

i.e., the family of laws of $\{Y^n\}_{n\geq 1}$ converges to the law of Y, as n tends to infinity.

The following result is another extension in the same direction. Let us consider $\{N(x,y);\, x,y \geq 0\}$ to be a Poisson process in the plane and $S, T > 0$. For any $\varepsilon > 0$, define the following random field:

$$x_\varepsilon(s,t) := \varepsilon \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \sqrt{xy}\,(-1)^{N(x,y)} dx dy, \qquad (s,t) \in [0,S] \times [0,T]. \tag{4}$$

Then, in [4] (see Theorem 1.1 therein), Bardina and Jolis proved that, as ε tends to zero, x_ε converges in law, in the Banach space $\mathcal{C}([0,S]\times[0,T])$ of continuous functions, to the Brownian sheet on $[0,S]\times[0,T]$. That is, if we consider $\{\mathbb{P}^\varepsilon\}_{\varepsilon>0}$ the image law of the process x_ε in the Banach space $\mathcal{C}([0,S]\times[0,T])$ of continuos functions on $[0,S]\times[0,T]$, then $\{\mathbb{P}^\varepsilon\}_{\varepsilon>0}$ converges weakly, when ε tends to zero, towards to the image law of a Brownian sheet. It is worth to mention that this result motivates our main result.

And, more generally, Bardina, Jolis and Rovira proved in [10] that if we consider

$$x_n(s_1,\ldots,s_d) := \frac{1}{\sqrt{n}} \int_0^{n^{1/d}s_d} \cdots \int_0^{n^{1/d}s_1} \left(\prod_{i=1}^d x_i\right)^{\frac{d-1}{2}} (-1)^{N(x_1,\ldots,x_d)} dx_1,\ldots,dx_d,$$

then the laws of these processes converge weakly, in the space $\mathcal{C}(\prod_{i=1}^d [0,S_i])$ to the law of a d-parametric Wiener process.

Li and Dai in [20] obtained an alternative approach to the fractional Brownian motion using the processes constructed in [4] for the Brownian sheet. They proved that, for $H \in (\frac{1}{2},1)$, the laws of the processes

$$Y_n(t) = n\sqrt{2C_H} \int_0^n \int_{-n}^0 g_s(t,x,y) x^{H-2} \sqrt{x|y|}(-1)^{N(\sqrt{n}x,\sqrt{n}y)} dx dy,$$

where $C_H = H(2H-1)(1-H)(3-2H)$ and

$$g_s(t,x,y) = \int_{-t}^0 I_{[0,x]}(u-y-s) du,$$

converge weakly to the law of a fractional Brownian motion.

We can also find results of weak converge towards a:

Complex Brownian motion. In [13], Bardina and Rovira show an approximation in law of the complex Brownian motion by processes constructed from a Lévy process. Bardina also showed an approximation in law of the complex Brownian motion by processes constructed from an standard Poisson process (see [1]).

Stratonovich multiple integrals. In [5], Bardina and Jolis considered the problem of the weak converge of the multiple integral processes

$$\left\{ \int_0^t \cdots \int_0^t f(t_1,\ldots,t_n) d\eta_\varepsilon(t_1)\ldots\eta_\varepsilon(t_n), \quad t \in [0,T] \right\}$$

as ε goes to zero in the space $\mathcal{C}_0([0,T])$, where $f \in L^2([0,T]^n)$ is a given function, and $\{\eta_\varepsilon(t)\}_{\varepsilon>0}$ is a family of stochastic processes with absolutely continuous paths that converges weakly to the Brownian motion. They have obtained the existence of the limit for any $\{\eta_\varepsilon\}_{\varepsilon>0}$, when f is given by a multimeasure, and under some conditions on $\{\eta_\varepsilon\}_{\varepsilon>0}$ if f is continuous and when $f(t_1,\ldots,t_n) = f_1(t_1)\cdots f_n(t_n) I_{\{t_1,\ldots t_n\}}$, with $f_i \in L^2([0,T])$ for any $i = 1,\ldots,n$. In all the cases the limit process is the multiple Stratonovich integral of the function f.

Fractional multiple integrals. Bardina, Jolis and Tudor in [6] studied the problem of the weak converge of the multiple integral processes

$$\left\{ \int_0^t \dots \int_0^t f(t_1, \dots, t_n) d\eta_\varepsilon(t_1) \dots \eta_\varepsilon(t_n), \quad t \in [0,1] \right\}$$

as ε goes to zero in the space $\mathcal{C}_0([0,1])$, where $f \in L^2([0,1]^n)$ is a given function, and $\{\eta_\varepsilon(t)\}_{\varepsilon>0}$ is a family of stochastic processes with absolutely continuous paths that converges weakly to the fractional Brownian motion with Hurst parameter $H > \frac{1}{2}$. They have obtained the existence of the limit for any $\{\eta_\varepsilon(t)\}_{\varepsilon>0}$, when f is given by a multimeasure for any $\{\eta_\varepsilon\}_{\varepsilon>0}$ with trajectories absolutely continuous whose derivatives are in $L^2([0,1])$, they proved that limit is the multiple fractional integral of f. Bardina, Es-Sebaiy and Tudor in [2] gave us another example. They constructed a family of continuous stochastic processes that converges in the sense of finite dimentional distributions to a multiple Wiener-Itô integral with respect to the fractional Brownian motion. They assumed $H > \frac{1}{2}$ and they proved their approximation result for the integrands f in a rather general class.

Fractional Brownian sheet. Bardina, Jolis and Tudor showed an approximation in law, in the space of continuous function in $[0,1]^2$, of Gaussian two-parameter processes that can be represented in law as a certain Wiener-type integral. The approximations are constructed from a Poisson process in the plane (see [8]).

Multiple Wiener-Itô integral. Bardina, Jolis and Tudor [7] studied the convergence to the multiple Wiener-Itô integral from processes with absolutely continuous paths. They considered a family of processes, with paths in the Cameron-Martin space, that converges weakly to a standard Brownian motion in $\mathcal{C}_0([0,T])$. Using these processes, they constructed a family that converges weakly, in the sense of finite dimentional distributions, to the multiple Wiener-Itô integral process of a function $f \in L^2([0,T]^n)$.

Fractional SDE. Bardina, Nourdin, Rovira and Tindel in [11] showed a diffusion approximation result for stochastic differential equations driven by a (Liouville) fractional Brownian motion B with Hurst parameter $H \in (1/3, 1/2)$. They resorted to the Kac-Strook type approximation using a Poisson process studied in [6] and in [22], and their method proof relies on the algebraic integration theory introduced by Gubinelli in [31].

The second direction for extending Kac-Stroock's result is by proving convergence in a stronger sense than convergence in law in the space of continuous functions. We also find papers where realizations of the processes that converge almost surely to the Brownian motion, uniformly on the unit time interval, are constructed (see [32], [30] and [19]). In order to obtain this convergence, the authors modify the processes to avoid that they always start in an increasing way. The modified process is the following:

$$X_n(t) = \frac{1}{\sqrt{n}}(-1)^A \int_0^{nt} (-1)^{N(u)} du,$$

where $A \sim \text{Bern}(\frac{1}{2})$ is independent of the Poisson process N. These processes are usually called in the literature as uniform transport processes. The weak convergence of these processes to the Wiener integral can be deduced from results of Pinksy in [37] and Watanabe in [43] previous to the Stroock's result.

Griego, Heath and Ruiz-Moncayo in [32] showed the strong and uniform convergences to the Brownian motion on bounded time intervals of these processes. Later, Gorostiza and Griego in [30] and also Csörgő and Horváth in [19] obtained a rate of convergence.

Garzón, Gorostiza and León in [29] constructed a sequence of processes that converges strongly to fractional Brownian motion uniformly on bounded intervals, for any Hurst parameter H. They also obtained a rate of convergence.

More recently Bardina, Ferrante and Rovira proved strong approximation to the Brownian sheet using uniform transport processes (see [3]). They extended the results given by Griego, Heath and Ruiz-Mocayo in [32] to the multiparameter case. They constructed a family of processes, starting from a set of independent standard Poisson processes, that has realizations that converge almost surely to the Brownian sheet, uniformly on the unit square. At the end they presented an extension to the d-parameter Wiener processes.

The third direction to extend Kac-Stroock's result consist in weakening the conditions of the approximating processes in order to find generalizations of the processes $(-1)^{N(u)}$ that also converge to the Brownian motion. We observe that

$$(-1)^{N(u)} = \cos(\pi N(u)).$$

A first question was if the convergence results hold with other angles. Bardina in [1] showed that if we consider

$$z_\theta^n(t) = \frac{1}{\sqrt{n}} \int_0^{2nt} \cos(\theta N_s) ds$$

and

$$y_\theta^n(t) = \frac{1}{\sqrt{n}} \int_0^{2nt} \sin(\theta N_s) ds,$$

where $\theta \neq 0, \pi$, the laws of the processes converge weakly towards the law of two independent Brownian motions. This result is interesting because in the limit two independent processes are obtained in spite of the fact that the approximating processes are functionally dependent.
In [12] Bardina and Rovira showed that if they consider different angles θ_i the corresponding processes converge towards the law of independent Brownian motions despite they only use one Poisson process and in [13] they also showed that if instead of using a Poisson process they consider a Lévy process, they obtain also the convergence towards the law of two independent Brownian motions.

In the present work, we aim to extend the above result of [4] to the case where the Poisson process is replaced by a Lévy sheet $\{L(x, y);\, x, y \geq 0\}$ (see Chapter 2 for the precise definition). Indeed, note that expression $(-1)^{N(x,y)}$ can be written in terms of the complex exponential as $e^{i\pi N(x,y)}$. Hence, when replacing N by L, we will use the form $e^{i\pi L(x,y)} = \cos(\pi L(x, y)) + i \sin(\pi L(x, y))$ since the expression $(-1)^{L(x,y)}$ may not be well-defined in $\mathbb{R}$. On the other hand, we will replace π by an arbitrary angle $\theta \in (0, 2\pi)$. The main result of this work is the following:

Theorem 0.1. *Let $\{L(x, y);\, x, y \geq 0\}$ be a Lévy sheet and $\Psi(\xi) := a(\xi)+ib(\xi)$, $\xi \in \mathbb{R}$, its Lévy exponent. Let $\theta \in (0, 2\pi)$ and $S, T > 0$, and define, for any $\varepsilon > 0$ and $(s, t) \in [0, S] \times [0, T]$,*

$$X_\varepsilon(s, t) := \varepsilon K \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \sqrt{xy}\, \{\cos(\theta L(x, y)) + i \sin(\theta L(x, y))\} dx dy, \tag{5}$$

where the constant K is given by

$$K = \frac{a(\theta)^2 + b(\theta)^2}{\sqrt{2}a(\theta)}. \tag{6}$$

Assume that $a(\theta)a(2\theta) \neq 0$. Then, as ε tends to zero, X_ε converges in law, in the space of complex-valued continuous functions $\mathcal{C}([0, S] \times [0, T]; \mathbb{C})$, to a complex Brownian sheet.

We recall that, by definition, a complex Brownian sheet is a complex random field whose real and imaginary parts are independent Brownian sheets. Hence, in view of the above theorem, we observe that the real and imaginary parts of X_ε are clearly not independent, for any $\varepsilon > 0$, while in the limit they are. As we already mentioned, this phenomenon is not new, for it already appeared in the study of analogous problems in the one-parameter setting (see, e.g., [1, 13]). Indeed, we recall that in [1] a family of processes that converges in law to a complex Brownian motion was constructed from a unique Poisson process. This result was generalized in [13], where the Poisson process was replaced by processes with independent increments whose characteristic functions satisfy some properties. Lévy processes are one of the examples where the latter results may be applied.

The main strategy in order to prove the kind of weak convergence stated in Theorem 0.1 consists in proving that the underlying family of laws is relatively compact in the space of continuous functions (with the usual topology). By Prokhorov's theorem, this is equivalent to proving the tightness property of this family of laws. Next, we will check that every weakly convergent partial sequence converges to the limit law that we want to obtain.

In the last chapter of this work, we consider the following semilinear stochastic heat equation driven by the space-time white noise:

$$\frac{\partial U}{\partial t}(t, x) - \frac{\partial^2 U}{\partial x^2}(t, x) = b(U(t, x)) + \dot{W}(t, x), \quad (t, x) \in [0, T] \times [0, 1], \tag{7}$$

where $T > 0$ and b is a globally Lipschitz function. We impose some initial datum and Dirichlet boundary conditions. In Theorem 4.1 below, we will prove that the random field solution U of (7) can be approximated in law, in the space of continuous functions, by a sequence of random fields $\{U_\varepsilon\}_\varepsilon$, where U_ε is the mild solution to a stochastic heat equation like (7) but driven by either the real or imaginary part of the noise X_ε. This result provides an example of a kind of weak continuity phenomenon in the path space, where convergence in law of the noisy inputs implies convergence in law of the corresponding solutions. Another example of this fact was provided by Walsh in [41], where a parabolic stochastic partial differential equation was used to model a discontinuous neurophysiological phenomenon. Other two examples that involve weak convergence with the stochastic heat equation are:

The stochastic heat equation driven by Gaussian white noise. In [9], Bardina, Jolis and Quer-Sardanyons consider a quasi-linear stochastic heat equation on $[0,1]$, with Dirichlet boundary conditions and controlled by space-time white noise. They replace the random perturbation by a family of noisy imputs depending in a parameter $n \in \mathbb{N}$ that approximate the white noise in some sense. Then, they provide sufficient conditions ensuring that the real-valued mild solution of the SPDE perturbed by this family of noises converges in law in the space of continuous functions, to the solution of the white noise driven SPDE.

The Stratonovich heat equation. Deya, Jolis and Quer-Sardanyons in [23] they considered a Stratonovich heat equation in $(0,1)$ with a nonlinear multiplicative noise driven by a trace-class Wiener process. First, the equation is shown to have a unique mild solution. Secondly, convolutional rough paths techniques are used to provide an almost sure continuity result for the solution with respect to the solution of the *smooth* equation obtained by replacing the noise with an absolutely continuous process. This continuity result is then exploited to prove weak convergence results based on Donsker and Kac-Stroock type approximations of the noise.

The stochastic heat equation driven by a fractional noise. El Mellali and Ouknine in [25] considered a quasi-linear stochastic heat equation in one dimension on $[0,1]$, with Dirichlet boundary conditions driven by an additive fractional white noise. They replace the random perturbation by a family of noisy inputs depending on a parameter $n \in \mathbb{N}$ which can approximate the fractional noise in some sense. Then, they provide sufficient conditions ensuring that the real-valued mild solution of the SPDE perturbed by this family of noises converges in law, in the space $\mathcal{C}([0,T]\times[0,1])$ of continuous functions, to the solution of the fractional noise driven SPDE.

The proof of Theorem 4.1 will follow from [9, Thm. 1.4], where there are established sufficient conditions on a family of random fields that approximate the Brownian sheet (in some sense) under which the solutions of (7) driven by this family converges in law, in the space of continuous functions, to the random field U. We refer to Chapter 4 for the precise statement of the above-mentioned conditions. In [9], the authors apply their main result to two important families of random fields that approximate the Brownian sheet. Fisrt, the Donsker kernels, which are defined, for $n \geq 1$ and $(t,x) \in [0,T]\times[0,1]$, by

$$\theta_n(t,x) := n \sum_{k=(k^1,k^2)\in\mathbb{N}^2} Z_k I_{[k^1-1,k^1)\times[k^2-1,k^2)}(tn, xn),$$

where Z_k, $k \in \mathbb{N}^2$, is an independent family of identically distributed and centered random variables, with $\mathbb{E}[Z_k^2] = 1$ for all $k \in \mathbb{N}^2$, such that $\mathbb{E}[|Z_k|^m] < +\infty$ for all $k \in \mathbb{N}^2$ and some even number $m \geq 10$.

The other one are the Kac-Stroock processes defined by

$$\theta_n(t,x) := n\sqrt{tx}\,(-1)^{N\left(\sqrt{n}\,t,\sqrt{n}\,x\right)},$$

where N denotes a standard Poisson process in the plane (indeed, this case corresponds to (4)). As it will be exhibited in Chapter 4, the proof of Theorem 4.1 is strongly based on the treatment of the Kac-Stroock processes in [9] (see Chapter 4 therein), and also on some technical estimates contained in the proof of the tightness result given in Proposition 2.1 of the present work.

Eventually, we note that the kind of convergence results that are obtained in the present work assure that the limit processes, which in our case correspond to the complex Brownian sheet and the solution to the stochastic heat equation, are robust when used as models in practical situations. Moreover, the obtained results provide expressions that can be useful to study simulations of these limit processes.

This work is organized as follows. Chapter 1 contains some preliminaries on two-parameter random fields, the definition of Lévy sheet and some weak convergence concepts and related results. Chapter 2 is devoted to prove that the family of laws of $(X_\varepsilon)_{\varepsilon>0}$ is tight in the space of complex-valued continuous functions. The limit identification is addressed in Chapter 3. Finally, the result on weak convergence for the stochastic heat equation is obtained in Chapter 4.

Chapter 1

Preliminaries

In this chapter, we recall the results and concepts that we will use throughout our work. The contents in this chapter will be used in order to achieve our results. It is divided in three sections: in the first one we give the definition of the Brownian sheet and the Lévy sheet, the second one is a brief account on weak convergence of probability laws and the third one is devoted to the Wiener integral with respect to Brownian sheet.

1.1 The Lévy sheet

In this section we will introduce the Lévy sheet, the process that we use for constructing X_ε in (5).

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. We will use some notation introduced by Cairoli and Walsh in [18]. Namely, let $\{\mathcal{F}_{s,t};\ (s,t) \in [0,S] \times [0,T]\}$ be a family of sub-σ-algebras of $\mathcal{F}$ satisfying:

(i) $\mathcal{F}_{s,t} \subset \mathcal{F}_{s',t'}$, for all $s \leq s'$ and $t \leq t'$.

(ii) All zero sets of $\mathcal{F}$ are contained in $\mathcal{F}_{0,0}$.

(iii) For any $z \in [0,S] \times [0,T]$, $\mathcal{F}_z = \cap_{z<z'} \mathcal{F}_{z'}$, where $z = (s,t) < z' = (s',t')$ denotes the partial order in $[0,S] \times [0,T]$, which means that $s < s'$ and $t < t'$.

If $(s,t) < (s',t')$ and Y denotes any random field defined in $[0,S] \times [0,T]$, the increment of Y on the rectangle $[s,s'] \times [t,t']$ is defined by

$$\Delta_{s,t} Y(s',t') := Y(s',t') - Y(s,t') - Y(s',t) + Y(s,t).$$

An adapted process $\{Y(s,t);\ (s,t) \in [0,S] \times [0,T]\}$ with respect to the filtration $\{\mathcal{F}_{s,t};\ (s,t) \in [0,S] \times [0,T]\}$ is called a martingale if $\mathbb{E}[|Y(s,t)|] < \infty$ for all $(s,t) \in [0,S] \times [0,T]$ and

$$\mathbb{E}[\Delta_{s,t} Y(s',t') | \mathcal{F}_{s,t}] = 0, \quad \text{for all} \quad (s,t) < (s',t').$$

It will be called a strong martingale if $\mathbb{E}[|Y(s,t)|] < \infty$ for all $(s,t) \in [0,S] \times [0,T]$, $Y(s,0) = Y(0,t) = 0$ for all s,t and

$$\mathbb{E}[\Delta_{s,t} Y(s',t') | \mathcal{F}_{S,t} \vee \mathcal{F}_{s,T}] = 0, \quad \text{for all} \quad (s,t) < (s',t').$$

We recall that a Brownian sheet is an adapted process $\{W(s,t);\ (s,t) \in [0,S] \times [0,T]\}$ such that $W(s,0) = W(0,t) = 0$ $\mathbb{P}$-a.s., the increment $\Delta_{s,t} W(s',t')$ is independent of $\mathcal{F}_{S,t} \vee \mathcal{F}_{s,T}$, for all $(s,t) < (s',t')$, and it is normally distributed with mean zero and variance $(s'-s)(t'-t)$. If no filtration is specified, we will consider the one generated by the process itself, namely $\mathcal{F}^W_{s,t} := \sigma\{W(r,z);\ (r,z) \in [0,s] \times [0,t]\}$ (conveniently completed).

A Lévy sheet is defined as follows. In general, if Q is any rectangle in $\mathbb{R}^2_+$ and Y any random field in $\mathbb{R}^2_+$, we will also denote by $\Delta_Q Y$ the increment of Y on Q. It is well-known that, for any negative definite function Ψ in $\mathbb{R}$, there exists a real-valued random field $L = \{L(s,t);\ s,t \geq 0\}$ such that:

(i) For any family of disjoint rectangles $Q_1, \ldots, Q_n$ in $\mathbb{R}^2_+$, the increments $\Delta_{Q_1} L, \ldots, \Delta_{Q_n} L$ are independent random variables.

(ii) For any rectangle Q in $\mathbb{R}^2_+$, the characteristic function of the increment $\Delta_Q L$ is given by

$$\mathbb{E}\left[e^{i\xi \Delta_Q L}\right] = e^{-\lambda(Q)\Psi(\xi)}, \quad \xi \in \mathbb{R}, \tag{1.1}$$

where λ denotes the Lebesgue measure on $\mathbb{R}^2_+$.

A random field $L = \{L(s,t);\, s,t \geq 0\}$ taking values in $\mathbb{R}$ that is continuous in probability and satisfies the above conditions (i) and (ii) is called a Lévy sheet with exponent Ψ.
By the Lévy-Khintchine formula, we have

$$\Psi(\xi) = ia\xi + \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} \left[1 - e^{i\xi x} + \frac{ix\xi}{1+|x|^2}\right] \eta(dx), \quad \xi \in \mathbb{R},$$

where $a \in \mathbb{R}$, $\sigma \geq 0$ and η is the corresponding Lévy measure, that is a Borel measure on $\mathbb{R} \setminus \{0\}$ that satisfies

$$\int_{\mathbb{R}} \frac{|x|^2}{1+|x|^2} \eta(dx) < \infty.$$

We write $\Psi(\xi) = a(\xi) + ib(\xi)$, where

$$a(\xi) := \frac{1}{2}\sigma^2\xi^2 + \int_{\mathbb{R}} [1 - \cos(\xi x)]\eta(dx),$$

and

$$b(\xi) := a\xi + \int_{\mathbb{R}} \left[\frac{x\xi}{1+|x|^2} - \sin(\xi x)\right] \eta(dx).$$

Observe that $a(\xi) \geq 0$ and, if $\xi \neq 0$, $a(\xi) > 0$ whenever $\sigma > 0$ and/or η is nontrivial.

1.2 Weak convergence and tightness

In this section we will recall some definitions and properties related to weak convergence and tightness.

Let us consider G a separable and complete metric space, and let us consider $\mathcal{E}$ its Borel σ-field. Also, let us denote by $\mathcal{P}(G)$ the space of probability measures on $(G, \mathcal{E})$, and we consider in this space the weak topology, i.e., the smallest topology for which the application

$$\mu \to \mu(f) := \int_G f d\mu, \quad \mu \in \mathcal{P}(G)$$

is continuous, for all bounded continuous function f on G. There exists different compatible metrics with this topology that make $\mathcal{P}(G)$ a separable and complete space too, an example of it is the Prokhorov metric (see [16]).

Definition 1.1. *We say that a sequence of probability measures in $(G, \mathcal{E})$, $\{\mu_n, n \geq 0\}$, converges weakly to a probability measure μ and we denote it as*

$$\mu_n \overset{w}{\Rightarrow} \mu$$

if

$$\mu_n(f) \longrightarrow \mu(f)$$

for all bounded and continuous $f : G \longrightarrow \mathbb{R}$. In other words, if there is a convergence of μ_n towards μ in the topology mentioned above.

Definition 1.2. *A set $A \subset \mathcal{P}(G)$ is relatively compact if any sequence in A has a weakly convergent subsequence.*

Definition 1.3. *A set $A \subset \mathcal{P}(G)$ is tight if, for all $\varepsilon > 0$, there exists a compact set K in G such that, for all $\mu \in A$, $\mu(G \setminus K) \leq \varepsilon$.*

In a metric space, a usual way of proving the convergence of a sequence is by proving that the sequence is relatively compact and then, it is enough to prove that every convergent subsequence converges to the same limit. Prokhorov Theorem describes which are the relatively compact sets of $\mathcal{P}(G)$ (see [16, Thm. 5.1]).

Theorem 1.1. *(Prokhorov Theorem) A set $A \subset \mathcal{P}(G)$ is relatively compact (with respect to its weak topology), if and only if it is tight.*

Suppose that we have, in a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, a random variable X with values in G which is a complete, separable and bounded space. We will call the law or distribution of X to the image law $\mathbb{P} \circ X^{-1}$ in $(G, \mathcal{B}(G))$, where $\mathcal{B}(G)$ is the Borel σ-field. We will refer to this probability measure as $\mathcal{L}(X)$, which defines a probability law in $(G, \mathcal{B}(G))$.

Definition 1.4. *Let $\{X^n, n \geq 0\}$ be a sequence of random variables with values in G, defined in some probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$, respectively, with $\mathcal{L}(X^n)$ as their law. We will say that $\{X^n, n \geq 0\}$ converges in law to a random variable X in G with law $\mathcal{L}(X)$, and we will denote it by*

$$X^n \overset{\mathcal{L}}{\Rightarrow} X,$$

if $\mathcal{L}(X^n) \overset{w}{\Rightarrow} \mathcal{L}(X)$.

If we denote the mathematical expectation with respect the probability Q by $\mathbb{E}_Q$, this definition is equivalent to say that, for all bounded and continuous functions $f : G \longrightarrow \mathbb{R}$, we have

$$\mathbb{E}_{\mathbb{P}^n}[f(X^n)] \longrightarrow \mathbb{E}_{\mathbb{P}}[f(X)], \quad \text{as} \quad n \to \infty.$$

Let $\{X^n(t), t \in T, n \geq 0\}$ be a sequence of stochastic processes with values in $\mathbb{R}$ parameterized by a compact metric space T, with continuous paths. We can consider the processes X^n as random variables in the Banach space of the continuous functions $\mathcal{C}(T)$. In order to prove the convergence in law of the processes X^n to a certain process X, in $\mathcal{C}(T)$, we can follow the following steps:

1. Prove that the family of laws $\{\mathcal{L}(X^n)\}_{n \geq 1}$ is relatively compact in $\mathcal{P}(\mathcal{C}(T))$ or, thanks to Prokhorov Theorem, proving that this family is tight.
2. Prove that any weakly convergent subsequence $\{\mathcal{L}(X^{n_k})\}_{k \geq 1}$ converges to the same limit, that is, to the law $\mathcal{L}(X)$.

   This second condition can be replaced by the following one:

2'. Prove that for all $k \geq 1$ and for all $t_1, \ldots, t_k \in T$,

$$(X^n(t_1), \ldots, X^n(t_k)) \overset{\mathcal{L}}{\Rightarrow} (X(t_1), \ldots, X(t_k))$$

in $\mathbb{R}^k$. This corresponds to the convergence of the finite dimentional distributions.

For proving that, it is common to use some criteria based on the characterization of the compact spaces given by the Arzelà-Ascoli theorem.

1.3 The Wiener integral with respect to the Brownian sheet

In this section, we define the Wiener integral with respect to the Brownian sheet (we follow Khoshnevisan's course in [21]). In order to define it, we will first recall the definition of white noise and its relationship with the Brownian motion and the Brownian sheet. These last two processes were defined in the first section of this chapter.

Definition 1.5. *A white noise in $\mathbb{R}^N$ is a family of random variables $\{\dot{W}(A), A \in \mathcal{B}(\mathbb{R}^N)\}$, defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that:*

1. *For all* $B_1, B_2, \ldots, B_n \in \mathcal{B}_0(\mathbb{R}^N)$ *(bounded borel sets), the vector* $(\dot{W}(B_1), \dot{W}(B_2), \ldots, \dot{W}(B_n))$ *is Gaussian.*
2. $\mathbb{E}[\dot{W}(A)] = 0$ *and* $\mathbb{E}[\dot{W}(A)\dot{W}(B)] = \lambda(A \cap B)$, *for* $A, B \in \mathcal{B}(\mathbb{R}^N)$.

In particular, if $N \geq 1$, we consider the Gaussian process $\{W_t, t \in \mathbb{R}_+^N\}$ given by

$$W_t := \dot{W}([0,t_1]\times, \cdots \times [0,t_N]), \qquad t = (t_1, t_2, \ldots, t_N).$$

This process is a Brownian sheet, namely it has mean zero and $\mathbb{E}[W_s W_t] = \prod_{n=1}^N (s_n \wedge t_n)$.

If we fix N=1, the process W_t is a standard Brownian motion with.

We can observe that Brownian sheet generalizes Brownian motion to an N-parameter random field. It is also possible to introduce the d-dimensional, N-parameter Brownian sheet as the d-dimensional process whose coordinates are independent, (one dimensional) N-parameter Brownian sheets.

Let us consider $\dot{W}$ a white noise in $\mathbb{R}^N$. First, let $\dot{W}(\mathbf{1}_A) := \dot{W}(A)$, for any $A \in \mathcal{B}(\mathbb{R}^N)$. Now, we define, for all disjoint $B_1, B_2, \ldots, B_n \in \mathcal{B}(\mathbb{R}^N)$ and $a_1, a_2, \ldots, a_n \in \mathbb{R}$,

$$\dot{W}\left(\sum_{j=1}^n a_j I_{B_j}\right) := \sum_{j=1}^n a_j \dot{W}(B_j).$$

Note that, by definition, the random variables $\dot{W}(B_1), \dot{W}(B_2), \ldots, \dot{W}(B_n)$ are independent. Therefore,

$$\mathbb{E}\left[\left|\dot{W}\left(\sum_{j=1}^n a_j I_{B_j}\right)\right|^2\right] = \sum_{j=1}^n a_j^2 I_{B_j}| = \left\|\sum_{j=1}^n a_j I_{B_j}\right\|^2_{L^2(\mathbb{R}^N)}. \tag{1.2}$$

We can infer that, for all $h \in L^2(\mathbb{R}^N)$ there exists $\{h_m\}_{m\geq 1}$ of the form $\sum_{j=1}^{n(m)} a_j^m I_{B_j^m}$ such that $B_1^m, B_2^m, \ldots,\ B_n^m \in \mathbb{B}(\mathbb{R}^N)$ are disjoint and $\|h - h_m\|_{L^2(\mathbb{R}^N)} \to 0$ as $m \to \infty$. This fact and (1.2) tell us that $\{\dot{W}(h_m)\}_{m\geq 1}$ is a Cauchy sequence in $L^2(\Omega)$. Let us call $\dot{W}(h)$ to its limit. This is the Wiener integral of $h \in L^2(\mathbb{R}^N)$, and we write

$$\dot{W}(h) := \int h dW = \int_{\mathbb{R}^N} h(x) W(dx).$$

Its main property is

$$\|\dot{W}(h)\|_{L^2(\Omega)} = \|h\|_{L^2(\mathbb{R}^N)}. \tag{1.3}$$

Thus, $\dot{W} : L^2(\mathbb{R}^N) \to L^2(\Omega)$ is an isometry. (1.3) is called the Wiener's isometry ([46]). We observe that the Wiener integral of $h \in L^2(\mathbb{R}^N)$ is only for a nonrandom h. The process $\{\dot{W}(h)\}_{h\in L^2(\mathbb{R}^N)}$ is called the isonormal process (see [24]) and it is a Gaussian process with mean zero and covariance functional given by the inner product:

$$\langle h, g \rangle_{L^2(\mathbb{R}^N)} = \int_{\mathbb{R}^N} h(x) g(x) dx,$$

for all $h, g \in L^2(\mathbb{R}^N)$. We will use this integral when we define the mild solution of (4.1) in Chapter 4.

Chapter 2

Tightness

In order to prove Theorem 0.1, we must show that the family $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ is tight. Moreover, we need to prove that any subsequence of this family that converges weakly, it should do it to a complex-valued Brownian sheet. In this section we will prove that $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ is tight. Instead of using the tightness definition, we will use the tightness criterion given by Centsov [17] (see also Bickel and Wichura [15, Theorem 3]), and the fact that X_ε vanishes on both axes. Due to this, it is enough to prove the next proposition.

Proposition 2.1. *Let $\{X_\varepsilon\}_{\varepsilon>0}$ be the family of random fields defined by* (4). *There exists a positive constant C such that, for all* $(0,0) \le (s,t) < (s',t') \le (S,T)$,

$$\sup_{\varepsilon>0} \mathbb{E}\left[|\Delta_{s,t} X_\varepsilon(s',t')|^4\right] \le C(s'-s)^2(t'-t)^2.$$

This implies that the the family of probability laws of $(X_\varepsilon)_{\varepsilon>0}$ is tight in $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$.

Proof. By definition of X_ε and the properties of the modulus $|\cdot|$, we have

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t} X_\varepsilon(s',t')|^4\right] \\
&= \varepsilon^4 K^4 \mathbb{E}\left[\left|\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\,\{\cos(\theta L(x,y)) + i\sin(\theta L(x,y))\} dxdy\right|^4\right] \\
&= \varepsilon^4 K^4 \mathbb{E}\left[\left|\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\, e^{i\theta L(x,y)} dxdy\right|^4\right] \\
&= \varepsilon^4 K^4 \mathbb{E}\left[\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1y_1}\, e^{i\theta L(x_1,y_1)} dx_1dy_1 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_2y_2}\, e^{-i\theta L(x_2,y_2)} dx_2dy_2\right)^2\right] \\
&= \varepsilon^4 K^4 \mathbb{E}\left[\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2y_1y_2}\, e^{i\theta(L(x_2,y_2)-L(x_1,y_1))} dx_1dx_2dy_1dy_2\right)^2\right] \\
&= \varepsilon^4 K^4 \mathbb{E}\Bigg[\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\qquad\qquad \times e^{i\theta(L(x_4,y_4)-L(x_3,y_3)+L(x_2,y_2)-L(x_1,y_1))} dx_1\dots dx_4dy_1\dots dy_4\Bigg].
\end{aligned}$$

We observe that $e^{i\theta\sum_{j=1}^4(-1)^j L(x_j,y_j)} = e^{i\theta\sum_{j=1}^4(-1)^j \Delta_{0,0}L(x_j,y_j)}$, where in the latter sum we have the increments of L from $(0,0)$ and (x_i,y_i), for $i=1,\dots,4$ multiplied by $(-1)^j$. If we apply this fact and

Fubini's theorem, we obtain

$$\begin{aligned}
\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^4\right] &= \varepsilon^4K^4\mathbb{E}\Bigg[\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\dots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\qquad\times e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}dx_1\dots dx_4dy_1\dots dy_4\Bigg] \\
&= \varepsilon^4K^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\dots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\qquad\times \mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right]dx_1\dots dx_4dy_1\dots dy_4.
\end{aligned}$$

Since we have 24 possible orders for the x-variables and another 24 possible orders for the y-variables, we have that

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^4\right] \\
&= \sum_{\sigma\in\mathcal{P}_4}\sum_{\beta\in\mathcal{P}_4}\varepsilon^4K^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\cdots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right] \\
&\quad\times I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}}I_{\{y_{\beta(1)}<y_{\beta(2)}<y_{\beta(3)}<y_{\beta(4)}\}}dx_1\dots dx_4dy_1\dots dy_4 \\
&\le \Bigg|\sum_{\sigma\in\mathcal{P}_4}\sum_{\beta\in\mathcal{P}_4}\varepsilon^4K^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\cdots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right] \\
&\quad\times I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}}I_{\{y_{\beta(1)}<y_{\beta(2)}<y_{\beta(3)}<y_{\beta(4)}\}}dx_1\dots dx_4dy_1\dots dy_4\Bigg| \\
&\le \sum_{\sigma\in\mathcal{P}_4}\sum_{\beta\in\mathcal{P}_4}\varepsilon^4K^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\cdots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right]\right| \\
&\quad\times I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}}I_{\{y_{\beta(1)}<y_{\beta(2)}<y_{\beta(3)}<y_{\beta(4)}\}}dx_1\dots dx_4dy_1\dots dy_4,
\end{aligned}\tag{2.1}$$

where $\mathcal{P}_4$ is the group of permutations of order 4. This give us 576 integrals. Now, thanks to the geometrical structure of the increments of L in the expression $\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)$, we observe that we can reduce this number of cases to only 72, since the inequalities $y_2\le y_4$, $y_1\le y_3$ and $y_3\le y_4$ generate the same cases as the inequalities $y_4\le y_2$, $y_3\le y_1$ and $y_4\le y_3$. So we will have 24 times the cases where

$$\begin{aligned}
&y_1\le y_2\le y_3\le y_4,\\
&y_2\le y_1\le y_3\le y_4,\\
&y_1\le y_3\le y_2\le y_4.
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
&\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^4\right] \\
&= 8\varepsilon^4\ K^4\sum_{\ell=1}^3\sum_{\sigma\in\mathcal{P}_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\cdots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)}\right]\right| \\
&\quad\times I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}}I_{Y_\ell}dx_1\dots dx_4dy_1\dots dy_4,
\end{aligned}$$

where $Y_1=\{\frac{t}{\varepsilon}\le y_1\le y_2\le y_3\le y_4\le\frac{t'}{\varepsilon}\}$, $Y_2=\{\frac{t}{\varepsilon}\le y_2\le y_1\le y_3\le y_4\le\frac{t'}{\varepsilon}\}$ and $Y_3=\{\frac{t}{\varepsilon}\le y_1\le y_3\le y_2\le y_4\le\frac{t'}{\varepsilon}\}$.

By now we only have 72 integral cases. We use once more the geometrical structure of the increments of L in the expression $\sum_{j=1}^4(-1)^j\Delta_{0,0}L(x_j,y_j)$ to reduce our number of integrals to 24. This is due to the fact that the structure of our 72 integrals corresponds to one of the 24 cases appearing in Figure 2.1; we notice that the 24 cases of Figure 2.1 correspond to the the cases where $y_1 \le y_2 \le y_3 \le y_4$. In each one of this 24 geometrical structures, we will have that the corresponding increments of L will be multiplied by $c_1 \in \{-1, 1\}$ in the black regions or by $c_2 \in \{-2, 0, 2\}$ in the white regions.

Now, let us fix $\sigma \in \mathcal{P}_4$ and Y_ℓ. We analize the term

$$\left|\mathbb{E}\left[e^{i\theta \sum_{j=1}^{4}(-1)^j \Delta_{0,0} L(x_j, y_j)}\right]\right| I_{\{x_{\sigma(1)}<x_{\sigma(2)}<x_{\sigma(3)}<x_{\sigma(4)}\}} I_{Y_\ell}. \tag{2.2}$$

Here, we perform some suitable changes of variables and a harmless abuse of notation (using the same variables for simplicity's sake), in order to have $x_1 < x_2 < x_3 < x_4$ and $y_1 < y_2 < y_3 < y_4$.

Let us denote by Q the area in Figure 2.1 corresponding to the fixed variables order we just mentioned. We can divide Q as a union of black and white disjoint rectangles. In a precise way, we will write it as

$$Q = \big(\cup_k \bar{Q}_k\big) \cup \big(\cup_l \tilde{Q}_l\big),$$

where the increments $\Delta_{\bar{Q}_k} L$ are multiplied by $c_1^k \in \{-1, 1\}$ and $\Delta_{\tilde{Q}_l} L$ are multiplied by $c_2^l \in \{-2, 0, 2\}$. Then, we have that (2.2) equals

$$\begin{aligned}
&\left|\mathbb{E}\left[e^{i\theta(\sum_k c_1^k \Delta_{\bar{Q}_k} L + \sum_l c_2^l \Delta_{\tilde{Q}_l} L)}\right]\right| I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} \\
&= \left|\mathbb{E}\left[e^{i\theta \sum_k c_1^k \Delta_{\bar{Q}_k} L}\right] \mathbb{E}\left[e^{i\theta \sum_l c_2^l \Delta_{\tilde{Q}_l} L}\right]\right| I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} \\
&= \left|\mathbb{E}\left[e^{i\theta \sum_k c_1^k \Delta_{\bar{Q}_k} L}\right]\right| \times \left|\mathbb{E}\left[e^{i\theta \sum_l c_2^l \Delta_{\tilde{Q}_l} L}\right]\right| I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} \\
&= \prod_k \left|e^{-\lambda(\bar{Q}_k)\Psi(c_1^k\theta)}\right| \prod_l \left|e^{-\lambda(\tilde{Q}_l)\Psi(c_2^l\theta)}\right| I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} \\
&= \prod_k e^{-\lambda(\bar{Q}_k)a(c_1^k\theta)} \prod_l e^{-\lambda(\tilde{Q}_l)a(c_2^l\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} \\
&\le \prod_k e^{-\lambda(\bar{Q}_k)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} \\
&= e^{-\lambda(\bar{Q})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}},
\end{aligned} \tag{2.3}$$

where $\bar{Q} := \cup_k \bar{Q}_k$. Here we recall that λ is the Lebesgue measure in $\mathbb{R}^2$ and that $\Psi(\theta) = a(\theta) + ib(\theta)$ satisfies $a(\theta) = a(-\theta)$. The importance in this calculations is that, independently of the constants c_1^k and c_2^l, we got an estimation of (2.2) which only involves the black regions multiplied by 1. This reduces our number of cases to the study of only 24 of them, the ones in Figure 2.1. Now, we will deal with these 24 integrals and we will try to reduce our analysis into just 4 integrals. For this, we will display the 24 cases of Figure 1, line by line and in a left to right order. Displaying all of the integrals is going to be repetitive and tedious but we need to do it in order to show that they all can be bound by some integral of a set of integrals that we will give explicitly. Let's call this set $BI := \{I_1', I_2', I_3', I_4'\}$, where I_i' with $i = 1, 2, 3, 4$ will be defined as they appear when we deal with the 24 integrals we have by now.

In order to simplify the calculations in the 24 integrals, we will divide the cases in Figure 2.1 in convenient disjoints rectangles in each integral.

I. First case. The black area Q^1 of the first case in Figure 2.1 can be divided in four disjoint rectangles. Set Q_i^1, $i = 1, 2, 3, 4$, the disjoint rectangles in the first case of Figure 2.1, whose union is equal to the

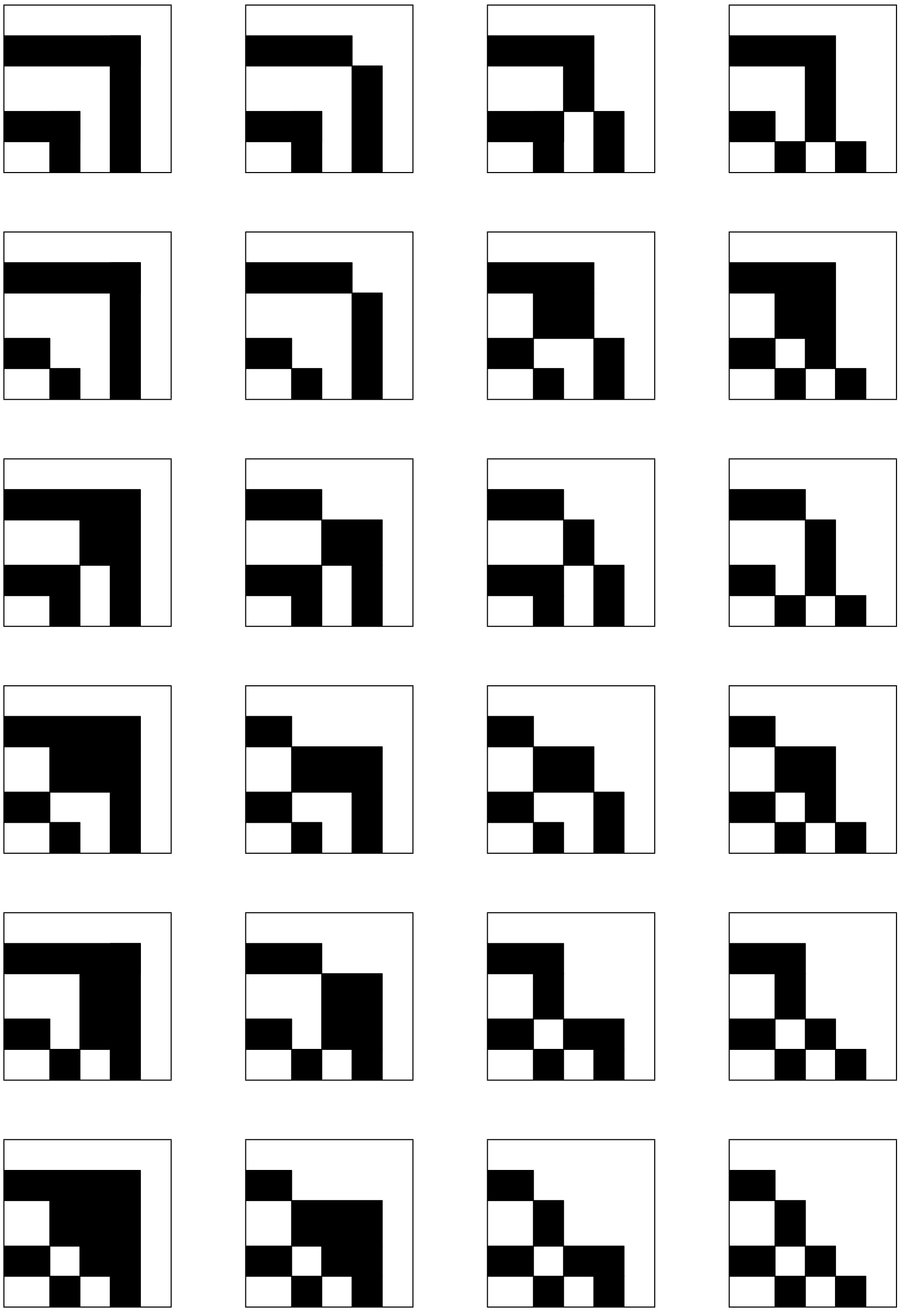

Figure 2.1: Each square represents the rectangle $(s,t) \times (s',t')$. Regions corresponding to $\sum_{j=1}^{4}(-1)^j \Delta_{0,0} L(x_j, y_j)$, for all possible 24 orders of the x-variables and $y_1 < y_2 < y_3 < y_4$, are drawn in each square. Black areas are regions where the corresponding increment of L appears an odd number of times. Note that, indeed, all areas are extended up to the plane axes.

black area Q^1, i.e., $Q^1 := \cup_{i=1}^4 Q_i^1$. By the above arguments and computing (2.1) and (2.3), we have that

$$
\begin{aligned}
I_1 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^1)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^4 \lambda(Q_i^1)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_4+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_3+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 := I_1'.
\end{aligned}
$$

II. Second case. Set Q_i^2, $i = 1, 2, 3, 4$, the disjoint rectangles in the second case of Figure 2.1, whose union is equal to the black area Q^2, i.e., $Q^2 := \cup_{i=1}^4 Q_i^2$ and again, by the above arguments and computing (2.1) and (2.3), we have that

$$
\begin{aligned}
I_2 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^2)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^4 \lambda(Q_i^2)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_3+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_3+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_1'.
\end{aligned}
$$

III. Third case. Set Q_i^3, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the third case of Figure 2.1, whose union is equal to the black area Q^3, i.e., $Q^3 := \cup_{i=1}^5 Q_i^3$ and once more, by the above arguments and computing (2.1) and (2.3), we have that

$$
\begin{aligned}
I_3 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^3)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{5}\lambda(Q_i^3)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_4-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 := I_4'.
\end{aligned}
$$

IV. Fourth case. Set Q_i^4, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the fourth case of Figure 2.1, whose union is equal to the black area Q^4, i.e., $Q^4 := \cup_{i=1}^{5} Q_i^4$ and by the above arguments and calculating (2.1) and (2.3), we have that

$$
\begin{aligned}
I_4 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^4)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{5}\lambda(Q_i^4)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_4-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 := I_3'.
\end{aligned}
$$

V. Fifth case. Set Q_i^5, $i = 1, 2, 3, 4$, the disjoint rectangles in the fifth case of Figure 2.1, whose union is equal to the black area Q^5, i.e., $Q^5 := \cup_{i=1}^{4} Q_i^5$ and again, by the above arguments and calculating (2.1) and (2.3), we have that

$$\begin{aligned}
I_5 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^5)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^4 \lambda(Q_i^5)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_4+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_3+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_1'.
\end{aligned}$$

VI. Sixth case. Set Q_i^6, $i = 1, 2, 3, 4$, the disjoint rectangles in the sixth case of Figure 2.1, whose union is equal to the black area Q^6, i.e., $Q^6 := \cup_{i=1}^4 Q_i^6$. So,

$$\begin{aligned}
I_6 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^6)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^4 \lambda(Q_i^6)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_3+(x_4-x_3)y_3+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_1'.
\end{aligned}$$

VII. Seventh case. Set Q_i^7, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the seventh case of Figure 2.1, whose union is equal to the black area Q^7, i.e., $Q^7 := \cup_{i=1}^{5} Q_i^7$. Then,

$$
\begin{aligned}
I_7 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
& \times e^{-\lambda(Q^7)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
& \times e^{-\sum_{i=1}^{5} \lambda(Q_i^7)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(\theta)(x_3-x_1)(y_4-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(\theta)(x_3-x_1)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \cdots \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4.
\end{aligned}
$$

In the last line we did some convenient changes of variables swapping the x-variables with the y-variables. They change the role of s, s', t and t' but at the end the integral is still equal to I_3'.

VIII. Eighth case. Set Q_i^8, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the eighth case of Figure 2.1, whose union is equal to the black area Q^8, i.e., $Q^8 := \cup_{i=1}^{6} Q_i^8$. Thus,

$$
\begin{aligned}
I_8 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
& \times e^{-\lambda(Q^8)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
& \times e^{-\sum_{i=1}^{6} \lambda(Q_i^8)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(\theta)\big((x_2-x_1)(y_4-y_2)+(x_3-x_2)(y_4-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(\theta)\big((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 := I_2'.
\end{aligned}
$$

IX. Nineth case. Set Q_i^9, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the ninth case of Figure 2.1, whose union is equal to the black area Q^9, i.e., $Q^9 := \cup_{i=1}^5 Q_i^9$. Therefore,

$$\begin{aligned}
I_9 :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^9)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^5 \lambda(Q_i^9)a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_4-x_2)(y_4-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_4'.
\end{aligned}$$

X. Tenth case. Set Q_i^{10}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the tenth case of Figure 2.1, whose union is equal to the black area Q^{10}, i.e., $Q^{10} := \cup_{i=1}^5 Q_i^{10}$. Thus,

$$\begin{aligned}
I_{10} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{10})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^5 \lambda(Q_i^{10})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_4-x_2)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_4'.
\end{aligned}$$

XI. Eleventh case. Set Q_i^{11}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the eleventh case of Figure 2.1, whose union is equal to the black area Q^{11}, i.e., $Q^{11} := \cup_{i=1}^5 Q_i^{11}$. Thus, we have that

$$\begin{aligned}
I_{11} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{11})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^5 \lambda(Q_i^{11})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_4'.
\end{aligned}$$

XII. Twelfth case. Set Q_i^{12}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the twelfth case of Figure 2.1, whose union is equal to the black area Q^{12}, i.e., $Q^{12} := \cup_{i=1}^{5} Q_i^{12}$. Then, we have that

$$
\begin{aligned}
I_{12} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
& \times e^{-\lambda(Q^{12}) a(\theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
& \times e^{-\sum_{i=1}^{5} \lambda(Q_i^{12}) a(\theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\big((y_4 - y_3)x_2 + (x_4 - x_3)y_1 + (y_2 - y_1)x_1 + (x_2 - x_1)y_1\big)} \\
& \times e^{-a(\theta)(x_3 - x_2)(y_3 - y_1)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_3'.
\end{aligned}
$$

XIII. Thirteenth case. Set Q_i^{13}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the thirteenth case of Figure 2.1, whose union is equal to the black area Q^{13}, i.e., $Q^{13} := \cup_{i=1}^{6} Q_i^{13}$. Hence, we have that

$$
\begin{aligned}
I_{13} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
& \times e^{-\lambda(Q^{13}) a(\theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
& \times e^{-\sum_{i=1}^{6} \lambda(Q_i^{13}) a(\theta)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\big((y_4 - y_3)x_3 + (x_4 - x_3)y_3 + (y_2 - y_1)x_1 + (x_2 - x_1)y_1\big)} \\
& \times e^{-a(\theta)\big((x_4 - x_3)(y_4 - y_3) + (x_3 - x_2)(y_3 - y_2)\big)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\big((y_4 - y_3)x_3 + (x_4 - x_3)y_3 + (y_2 - y_1)x_1 + (x_2 - x_1)y_1\big)} \\
& \times I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_1'.
\end{aligned}
$$

XIV. Fourteenth case. Set Q_i^{14}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the fourteenth case of Figure

2.1, whose union is equal to the black area Q^{14}, i.e., $Q^{14} := \cup_{i=1}^5 Q_i^{14}$. Therefore, we have

$$
\begin{aligned}
I_{14} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{14})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^5 \lambda(Q_i^{14})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_4-x_1)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\le& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_1)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \cdots \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4.
\end{aligned}
$$

Here we did changes of variable as in the seventh case. They change the role of s, s', t and t' but at the end the integral is still equal to I_3'.

XV. Fifteenth case. Set Q_i^{15}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the fifteenth case of Figure 2.1, whose union is equal to the black area Q^{15}, i.e., $Q^{15} := \cup_{i=1}^5 Q_i^{15}$. Then,

$$
\begin{aligned}
I_{15} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{14})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^5 \lambda(Q_i^{14})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_1)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \cdots \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_3'.
\end{aligned}
$$

Here we use the same arguments as in seventh and fourteenth cases, in order to get the last equality.

XVI. Sixteenth case. Set Q_i^{16}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the sixteenth case of Figure

2.1, whose union is equal to the black area Q^{16}, i.e., $Q^{16} := \cup_{i=1}^{6} Q_i^{16}$. So,

$$
\begin{aligned}
I_{16} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
&\times e^{-\lambda(Q^{16}) a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{16}) a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_3-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 = I_2'.
\end{aligned}
$$

XVII. Seventeenth case. Set Q_i^{17}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the seventeenth case of Figure 2.1, whose union is equal to the black area Q^{17}, i.e., $Q^{17} := \cup_{i=1}^{5} Q_i^{17}$. Therefore,

$$
\begin{aligned}
I_{17} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
&\times e^{-\lambda(Q^{17}) a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
&\times e^{-\sum_{i=1}^{5} \lambda(Q_i^{17}) a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\left((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)(x_4-x_2)(y_4-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\left((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 = I_3'.
\end{aligned}
$$

XVIII. Eighteenth case. Set Q_i^{18}, $i = 1, 2, 3, 4, 5$, the disjoint rectangles in the eighteenth case of

Figure 2.1, whose union is equal to the black area Q^{18}, i.e., $Q^{18} := \cup_{i=1}^{5} Q_i^{18}$. Hence, we have that

$$\begin{aligned}
I_{18} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{18})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{5} \lambda(Q_i^{18})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_4-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 = I_3'.
\end{aligned}$$

XIX. Nineteenth case. Set Q_i^{19}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the nineteenth case of Figure 2.1, whose union is equal to the black area Q^{19}, i.e., $Q^{19} := \cup_{i=1}^{6} Q_i^{19}$. Thus,

$$\begin{aligned}
I_{19} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{19})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{19})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)\big((x_2-x_1)(y_4-y_2)+(x_4-x_2)(y_2-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)\big((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 = I_2'.
\end{aligned}$$

XX. Twentieth case. Set Q_i^{20}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the twentieth case of Figure

2.1, whose union is equal to the black area Q^{20}, i.e., $Q^{20} := \cup_{i=1}^{6} Q_i^{20}$. Then,

$$\begin{aligned}
I_{20} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{20})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{20})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)\big((x_2-x_1)(y_4-y_2)+(x_3-x_2)(y_2-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)\big((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_2'.
\end{aligned}$$

XXI. Twenty-first case. Set Q_i^{21}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the twenty-first case of Figure 2.1, whose union is equal to the black area Q^{21}, i.e., $Q^{21} := \cup_{i=1}^{6} Q_i^{21}$. Therefore, we have

$$\begin{aligned}
I_{21} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{21})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{21})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)\big((x_2-x_1)(y_4-y_2)+(x_4-x_2)(y_4-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)\big((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\big)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_2'.
\end{aligned}$$

XXII. Twenty-second case. Set Q_i^{22}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the twenty-second case of Figure 2.1, whose union is equal to the black area Q^{22}, i.e., $Q^{22} := \cup_{i=1}^{6} Q_i^{22}$. Thus, we have

$$
\begin{aligned}
I_{22} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{22})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{22})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_4-x_2)(y_3-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_2'.
\end{aligned}
$$

XXIII. Twenty-third case. Set Q_i^{23}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the twenty-third case of Figure 2.1, whose union is equal to the black area Q^{23}, i.e., $Q^{23} := \cup_{i=1}^{6} Q_i^{23}$. Hence,

$$
\begin{aligned}
I_{23} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{23})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{23})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_4-x_2)(y_2-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
\leq& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_2'.
\end{aligned}
$$

XXIV. Tweny-fourth case. Set Q_i^{24}, $i = 1, 2, 3, 4, 5, 6$, the disjoint rectangles in the twenty-fourth case of Figure 2.1, whose union is equal to the black area Q^{24}, i.e., $Q^{24} = \cup_{i=1}^{6} Q_i^{24}$. Therefore,

$$
\begin{aligned}
I_{24} :=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\lambda(Q^{24})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{-\sum_{i=1}^{6} \lambda(Q_i^{24})a(\theta)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
=& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\left((y_4-y_3)x_1+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)(y_3-y_2)+(x_3-x_2)(y_2-y_1)\right)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 = I_2'.
\end{aligned}
$$

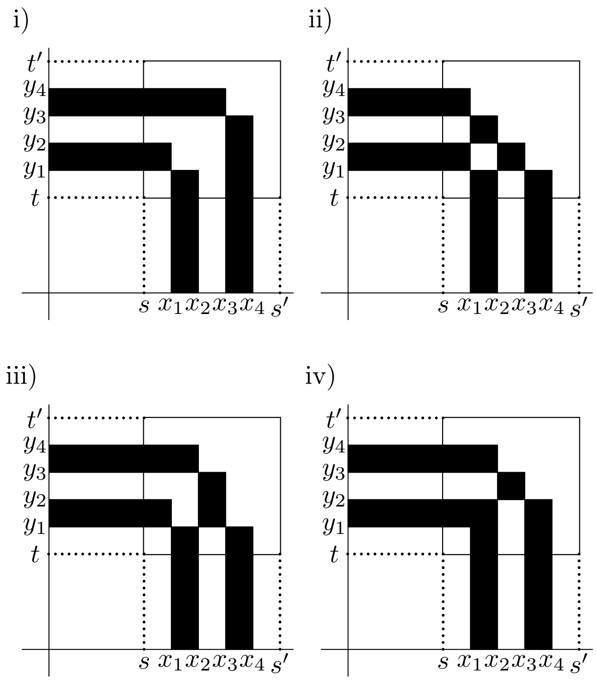

Figure 2.2: The four relevant cases of Figure 2.1

Hence, we only have to find some suitable bounds for I'_1, I'_2, I'_3 and I'_4, in order to get our proof finished. This integrals are the cases on Figure 2.2.

Let us tackle first I'_1, the case i) in Figure 2.2. We will first bound $\sqrt{x_4}$ and $\sqrt{y_4}$ by $\frac{s'}{\varepsilon}$ and $\frac{t'}{\varepsilon}$, respectively. After that, we will integrate with respect x_4 and y_4 and then we bound the exponential functions given in both integrals by 1. Then, we have

$$
\begin{aligned}
I'_1 =& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_4} \int_{\frac{s}{\varepsilon}}^{x_3} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 x_3 x_4} \sqrt{y_1 y_2 y_3 y_4} \\
&\times e^{-a(\theta)\left((y_4-y_3)x_3+(x_4-x_3)y_3+(y_2-y_1)x_1+(x_2-x_1)y_1\right)} dx_1 dx_2 dx_3 dx_4 dy_1 dy_2 dy_3 dy_4 \qquad (2.4) \\
\leq& C\sqrt{s'}\sqrt{t'}\varepsilon^3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_3} \int_{\frac{s}{\varepsilon}}^{x_2} \frac{\sqrt{x_1 x_2 x_3}\sqrt{y_1 y_2 y_3}}{x_3 y_3} \\
&\times e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_1\right)} dx_1 dx_2 dx_3 dy_1 dy_2 dy_3.
\end{aligned}
$$

Now, we bound the same way $\sqrt{x_2}$ and $\sqrt{y_2}$, we integrate with respect x_2 and y_2, and we bound the result as before. Hence

$$
\begin{aligned}
I_1' \leq & C s' t' \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_3} \frac{\sqrt{x_1 x_3}\sqrt{y_1 y_3}}{x_1 x_3 y_1 y_3} dx_1 dx_3 dy_1 dy_3 \\
\leq & C s' t' \varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1 x_3}\sqrt{y_1 y_3}} dx_1 dx_3 dy_1 dy_3 \\
= & C s' t' \varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1}} dx_1 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_3}} dx_3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_1}} dy_1 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_3}} dy_3 \\
= & C \big[\sqrt{s'}(\sqrt{s'} - \sqrt{s})\big]^2 \big[\sqrt{t'}(\sqrt{t'} - \sqrt{t})\big]^2 \\
\leq & C (s' - s)^2 (t' - t)^2,
\end{aligned}
$$

where C is a positive constant.

We have finished the analysis of i) in Figure 2.2.

The other three cases are a little bit more difficult. Hence, we will add some small area in the corresponding drawing of Figure 2.2 in order to get the same or similar situation than in case i). We must mention that some of the bounds that we will obtain for the integrands will be satisfied everywhere except of a zero Lebesgue measure set in $\mathbb{R}^8$

Let us proceed with the analysis of ii) in Figure 2.2. Recall that

$$
\begin{aligned}
I_2' = & \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 x_2 x_3 x_4}\sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)\big((y_4 - y_3)x_1 + (x_4 - x_3)y_1 + (y_2 - y_1)x_1 + (x_2 - x_1)y_1\big)} \\
& \times e^{-a(\theta)\big((x_2 - x_1)(y_3 - y_2) + (x_3 - x_2)(y_2 - y_1)\big)} I_{\{x_1 < x_2 < x_3 < x_4\}} I_{\{y_1 < y_2 < y_3 < y_4\}} dx_1 \dots dx_4 dy_1 \dots dy_4 \\
= & \varepsilon^4 \int_I \int_J \sqrt{x_1 x_2 x_3 x_4}\sqrt{y_1 y_2 y_3 y_4}\, e^{-\lambda(\bar{Q}_2)a(\theta)} dx_1 ... dx_4 dy_1 ... dy_4,
\end{aligned}
$$

where $J = \{\frac{s}{\varepsilon} \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq \frac{s'}{\varepsilon}\}$, $I = \{\frac{t}{\varepsilon} \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq \frac{t'}{\varepsilon}\}$, and $\bar{Q}_2$ is the union of black rectangles corresponding to the case ii). Note that $A := \lambda(\bar{Q}_2)$ is given by

$$
A = (x_4 - x_3)y_1 + (y_4 - y_3)x_1 + (x_2 - x_1)y_1 + (y_2 - y_1)x_1 + (x_3 - x_2)(y_2 - y_1) + (y_3 - y_2)(x_2 - x_1).
$$

We split the last integral into two terms. Thus,

$$
\begin{aligned}
I_2' = & \varepsilon^4 \int_I \int_J \sqrt{x_1 x_2 x_3 x_4}\sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)A}\, I_{\{A \geq 2(x_2 - x_1)(y_2 - y_1)\}} dx_1 ... dx_4 dy_1 ... dy_4 \\
& + \varepsilon^4 \int_I \int_J \sqrt{x_1 x_2 x_3 x_4}\sqrt{y_1 y_2 y_3 y_4}\, e^{-a(\theta)A}\, I_{\{A < 2(x_2 - x_1)(y_2 - y_1)\}} dx_1 ... dx_4 dy_1 ... dy_4. \qquad (2.5)
\end{aligned}
$$

When $A \geq 2(x_2 - x_1)(y_2 - y_1)$, we observe that $\frac{A}{2} \geq (x_2 - x_1)(y_2 - y_1)$. Then,

$$
\begin{aligned}
-a(\theta)A \quad \leq & \quad -\frac{a(\theta)}{2} A - a(\theta)(x_2 - x_1)(y_2 - y_1) \\
= & \quad -\frac{a(\theta)}{2}[(x_4 - x_3)y_1 + (y_4 - y_3)x_1 + (x_2 - x_1)y_3 + (y_2 - y_1)x_3].
\end{aligned}
$$

Therefore, the first term in the right-side of (2.5) is less or equal than

$$\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\ \exp\left\{-\frac{a(\theta)}{2}[(x_4-x_3)y_1+(y_4-y_3)x_1\right.\\
&\quad\left.+(x_2-x_1)y_3+(y_2-y_1)x_3]\right\} dx_1...dx_4dy_1...dy_4\\
&=\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\ \exp\left\{-a'(\theta)[(x_4-x_3)y_1+(y_4-y_3)x_1\right.\\
&\quad\left.+(x_2-x_1)y_3+(y_2-y_1)x_3]\right\} dx_1...dx_4dy_1...dy_4,
\end{aligned}$$

where $a'(\theta)=\frac{a(\theta)}{2}$, then, it has the same form of the (2.4). Hence, we infer that this term can be bounded by $(s'-s)^2(t'-t)^2$ multiplied by some positive constant just by following the same arguments used in the case i).

Let us deal with the second integral in (2.5). Note that

$$\begin{aligned}
&\{A<2(x_2-x_1)(y_2-y_1)\}\\
&\quad=\{(x_4-x_3)y_1+(y_4-y_3)x_1+(y_2-y_1)x_3+(x_2-x_1)y_3<4(x_2-x_1)(y_2-y_1)\}.
\end{aligned}$$

From this, we can infer that

$$\frac{1}{4}y_3<(y_2-y_1)\quad\text{and}\quad\frac{1}{4}x_3<(x_2-x_1),$$

Thus,

$$\begin{aligned}
A &\geq (x_4-x_3)y_1+(y_4-y_3)x_1+\frac{1}{4}x_3y_1+\frac{1}{4}y_3x_1+\frac{1}{4}(x_3-x_2)y_3+\frac{1}{4}(y_3-y_2)x_3\\
&\geq \frac{1}{4}\big[x_4y_1+y_4x_1+(x_3-x_2)y_3+(y_3-y_2)x_3\big].
\end{aligned}$$

Therefore, the second term of (2.4) can be bounded by

$$\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\quad\times\exp\left\{-\frac{a(\theta)}{4}[x_4y_1+y_4x_1+(x_3-x_2)y_3+(y_3-y_2)x_3]\right\} dx_1...dx_4dy_1...dy_4\\
&=\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\quad\times\exp\left\{-a''(\theta)[x_4y_1+y_4x_1+(x_3-x_2)y_3+(y_3-y_2)x_3]\right\} dx_1...dx_4dy_1...dy_4,
\end{aligned}$$

where $a''(\theta)=\frac{a(\theta)}{4}$. Here again the arguments of the case i) may be applied, in order to yield an estimate of the form $(s'-s)^2(t'-t)^2$, up to some positive constant. For doing this we will first bound the square roots of x_4 and y_4 by the upper integral limits, then we will integrate with respect x_4 and y_4 and bound the exponencial functions (we will bound them by 1). Hence

$$\begin{aligned}
&\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{t}{\varepsilon}}^{y_3}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{s}{\varepsilon}}^{x_3}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\quad\times e^{-a''(\theta)\left(x_4y_1+y_4x_1+(x_3-x_2)y_3+(y_3-y_2)x_3\right)}dx_1dx_2dx_3dx_4dy_1dy_2dy_3dy_4\\
&\leq C\sqrt{s'}\sqrt{t'}\varepsilon^3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_3}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_3}\int_{\frac{s}{\varepsilon}}^{x_2} \frac{\sqrt{x_1x_2x_3}\sqrt{y_1y_2y_3}}{x_1y_1}\\
&\quad\times e^{-a''(\theta)\left((x_3-x_2)y_3+(y_3-y_2)x_3\right)}dx_1dx_2dx_3dy_1dy_2dy_3.
\end{aligned}$$

Now, we will do the same for x_2 and y_2. Thus, we have that the last expression is less equal to

$$\begin{aligned}
&Cs't'\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_3} \frac{\sqrt{x_1 x_3}\sqrt{y_1 y_3}}{x_1 x_3 y_1 y_3}\, dx_1 dx_3 dy_1 dy_3 \\
&\le Cs't'\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1 x_3}\sqrt{y_1 y_3}}\, dx_1 dx_3 dy_1 dy_3 \\
&= Cs't'\varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1}}\, dx_1 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_3}}\, dx_3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_1}}\, dy_1 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_3}}\, dy_3 \\
&= C\big[\sqrt{s'}(\sqrt{s'} - \sqrt{s})\big]^2 \big[\sqrt{t'}(\sqrt{t'} - \sqrt{t})\big]^2 \\
&\le C(s'-s)^2(t'-t)^2.
\end{aligned}$$

The analysis of case ii) is done, let us continue with the case iii).

$$\begin{aligned}
I_3' =& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \cdots \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_1+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_1)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1 \ldots dx_4 dy_1 \ldots dy_4 \\
=& \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-\lambda(\bar{Q}_3)a(\theta)} dx_1 ... dx_4 dy_1 ... dy_4,
\end{aligned}$$

where $J = \{\frac{s}{\varepsilon} \le x_1 \le x_2 \le x_3 \le x_4 \le \frac{s'}{\varepsilon}\}$, $I = \{\frac{t}{\varepsilon} \le y_1 \le y_2 \le y_3 \le y_4 \le \frac{t'}{\varepsilon}\}$, and $\bar{Q}_3$ is the union of black rectangles corresponding to the case iii). Note that $A := \lambda(\bar{Q}_3)$ is given by

$$A = (x_4 - x_3)y_1 + (x_2 - x_1)y_1 + (x_3 - x_2)(y_3 - y_1) + (y_4 - y_3)x_2 + (y_2 - y_1)x_1,$$

and here we will split the above integral taking into account the regions $\{A \ge 2(x_3 - x_2)y_1\}$ and $\{A < 2(x_3 - x_2)y_1\}$. Then, we have

$$\begin{aligned}
I_3' =& \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A \ge 2(x_3-x_2)y_1\}} dx_1 ... dx_4 dy_1 ... dy_4 \\
&+ \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A < 2(x_3-x_2)y_1\}} dx_1 ... dx_4 dy_1 ... dy_4. \qquad (2.6)
\end{aligned}$$

Concerning to the first integral in (2.6), we observe that

$$\begin{aligned}
-a(\theta)A &\le -\frac{a(\theta)}{2}A - a(\theta)(x_3 - x_2)y_1 \\
&= -\frac{a(\theta)}{2}\big((x_4 - x_3)y_1 + (x_2 - x_1)y_1 + (x_3 - x_2)(y_3 - y_1) + (y_4 - y_3)x_2 + (y_2 - y_1)x_1\big) \\
&\quad -\frac{2a(\theta)}{2}(x_3 - x_2)y_1 \\
&= -\frac{a(\theta)}{2}\big(x_4y_1 - x_3y_1 + x_2y_1 - x_1y_1 + x_3y_3 - x_2y_3 - x_3y_1 + x_2y_1 + y_4x_2 - y_3x_2\big) \\
&\quad -\frac{a(\theta)}{2}(y_2x_1 - y_1x_1) - \frac{2a(\theta)}{2}(x_3y_1 - x_2y_1) \\
&= -\frac{a(\theta)}{2}\big(x_4y_1 - x_1y_1 + x_3y_3 - x_2y_3 + y_4x_2 - y_3x_2 + y_2x_1 - y_1x_1\big) \\
&= -\frac{a(\theta)}{2}[(x_4 - x_1)y_1 + (y_2 - y_1)x_1 + (x_3 - x_2)y_3 + (y_4 - y_3)x_2].
\end{aligned}$$

Thus,

$$\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A\geq 2(x_3-x_2)y_1\}} dx_1...dx_4dy_1...dy_4 \\
&\leq \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a'(\theta)[(x_4-x_1)y_1+(y_2-y_1)x_1+(x_3-x_2)y_3+(y_4-y_3)x_2]}\, dx_1...dx_4dy_1...dy_4,
\end{aligned}$$

where $a'(\theta) = \frac{a(\theta)}{2}$. Now, we bound the square roots of x_4, y_4, y_2 and x_3 by their upper integral limits (respectively) and we will continue integrating with respect of these variables in the order mentioned before. Therefore, we can bound the latter expression by

$$\begin{aligned}
&Cs't'\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \frac{\sqrt{x_1x_2}\sqrt{y_1y_3}}{x_1x_2y_1y_3}\, dx_1dx_2dy_1dy_3 \\
&\leq Cs't'\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1x_2}\sqrt{y_1y_3}}\, dx_1dx_2dy_1dy_3 \\
&= Cs't'\varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1}}\, dx_1 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_2}}\, dx_2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_1}}\, dy_1 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_3}}\, dy_3 \\
&= C\big[\sqrt{s'}(\sqrt{s'}-\sqrt{s})\big]^2\big[\sqrt{t'}(\sqrt{t'}-\sqrt{t})\big]^2 \\
&\leq C(s'-s)^2(t'-t)^2.
\end{aligned}$$

On the other hand, for the second integral of (2.6) we observe that

$$\{A < 2(x_3-x_2)y_1\} = \{(x_4-x_1)y_1 + (y_2-y_1)x_1 + (x_3-x_2)y_3 + (y_4-y_3)x_2 < 4(x_3-x_2)y_1\},$$

where we have done some analogue calculations as the above part. From here we can deduce that

$$\frac{1}{4}y_3 < y_1 \quad \text{and} \quad \frac{1}{4}(x_4-x_1) < (x_3-x_2).$$

Hence,

$$\begin{aligned}
A &\geq \frac{1}{4}\big[(x_4-x_3)y_3 + (x_2-x_1)y_3 + (x_4-x_1)(y_3-y_1) + (y_4-y_3)x_2 + (y_2-y_1)x_1\big] \\
&\geq \frac{1}{4}\big[(y_4-y_3)x_1 + (x_2-x_1)y_3 + (x_4-x_3)y_1 + (y_2-y_1)x_3\big].
\end{aligned}$$

Then, once more, we repeat the same kind of arguments as in case i):

$$\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A<2(x_3-x_2)y_1\}} dx_1...dx_4dy_1...dy_4 \\
&\leq \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a''(\theta)[(y_4-y_3)x_1+(x_2-x_1)y_3+(x_4-x_3)y_1+(y_2-y_1)x_3]}\, dx_1...dx_4dy_1...dy_4
\end{aligned}$$

where $a''(\theta) = \frac{a(\theta)}{4}$. Now, again, we bound the square roots of x_4, y_4, x_2 and y_2 by their upper integral limits (respectively) and, we will continue integrating with respect of these variables in the order we mentioned them. So we can bound the latter expression by

$$
\begin{aligned}
Cs't'\varepsilon^2 &\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_3}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_3} \frac{\sqrt{x_1x_3}\sqrt{y_1y_3}}{x_1x_3y_1y_3}\,dx_1dx_3dy_1dy_3 \\
\leq & Cs't'\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1x_3}\sqrt{y_1y_3}}\,dx_1dx_3dy_1dy_3 \\
= & Cs't'\varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_1}}\,dx_1 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \frac{1}{\sqrt{x_3}}\,dx_3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_1}}\,dy_1 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{\sqrt{y_3}}\,dy_3 \\
= & C\big[\sqrt{s'}(\sqrt{s'}-\sqrt{s})\big]^2\big[\sqrt{t'}(\sqrt{t'}-\sqrt{t})\big]^2 \\
\leq & C(s'-s)^2(t'-t)^2.
\end{aligned}
$$

Finally, it only remains to tackle case iv) of Figure 2.2. For this, we have that

$$
\begin{aligned}
I_4' =& \varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\cdots\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)\big((y_4-y_3)x_2+(x_4-x_3)y_2+(y_2-y_1)x_1+(x_2-x_1)y_2\big)} \\
&\times e^{-a(\theta)(x_3-x_2)(y_3-y_2)} I_{\{x_1<x_2<x_3<x_4\}} I_{\{y_1<y_2<y_3<y_4\}} dx_1\ldots dx_4dy_1\ldots dy_4 \\
=& \varepsilon^4 \int_I\int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-\lambda(\bar{Q}_4)a(\theta)} dx_1...dx_4dy_1...dy_4,
\end{aligned}
$$

where $J = \{\frac{s}{\varepsilon} \leq x_1 \leq x_2 \leq x_3 \leq x_4 \leq \frac{s'}{\varepsilon}\}$, $I = \{\frac{t}{\varepsilon} \leq y_1 \leq y_2 \leq y_3 \leq y_4 \leq \frac{t'}{\varepsilon}\}$, and $\bar{Q}_4$ is the union of black rectangles corresponding to the case iv). Let $A := \lambda(\bar{Q}_4)$ be given by

$$
A = (x_4-x_3)y_2 + (x_2-x_1)y_2 + (y_4-y_3)x_2 + (y_2-y_1)x_1 + (x_3-x_2)(y_3-y_2).
$$

This case gets a little bit longer than the others due to the splitting regions that we will use. These ones are given by $\{A \geq 2(x_3-x_2)y_2\} \cup \{A \geq 2(y_3-y_2)x_2\}$ and its corresponding complement, $\{A < 2(x_3-x_2)y_2\} \cap \{A < 2(y_3-y_2)x_2\}$. Thus,

$$
\begin{aligned}
I_4' =& \varepsilon^4 \int_I\int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A\geq 2(x_3-x_2)y_2\}\cup\{A\geq 2(y_3-y_2)x_2\}} dx_1...dx_4dy_1...dy_4 \\
&+ \varepsilon^4 \int_I\int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A<2(x_3-x_2)y_2\}\cap\{A<2(y_3-y_2)x_2\}} dx_1...dx_4dy_1...dy_4. \qquad (2.7)
\end{aligned}
$$

In the first region, we have that for $\{A \geq 2(x_3-x_2)y_2\}$,

$$
\begin{aligned}
-a(\theta)A \leq& -\frac{a(\theta)}{2}A - a(\theta)2(x_3-x_2)y_2 \\
=& -\frac{a(\theta)}{2}[(x_3-x_2)y_3 + (y_4-y_3)x_2 + (y_2-y_1)x_1 + (x_4-x_1)y_1],
\end{aligned}
$$

where the calculations are analogue as in the other cases. Now, we observe that, for $\{A \geq 2(y_3-y_2)x_2\}$ (once more, by doing analogue calculations as in the above cases),

$$
\begin{aligned}
-a(\theta)A \quad \leq& \quad -\frac{a(\theta)}{2}A - a(\theta)(y_3-y_2)x_2 \\
=& \quad -\frac{a(\theta)}{2}[(x_4-x_1)y_2 + (y_4-y_3)x_2 + (y_2-y_1)x_1 + (x_3-x_2)y_3].
\end{aligned}
$$

Therefore, the first integral in (2.7) is bounded by

$$\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a'(\theta)[(x_3-x_2)y_3+(y_4-y_3)x_2+(y_2-y_1)x_1+(x_4-x_1)y_1]}\, dx_1...dx_4dy_1...dy_4$$
$$+ \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a'(\theta)[(x_4-x_1)y_2+(y_4-y_3)x_2+(y_2-y_1)x_1+(x_3-x_2)y_3]}\, dx_1...dx_4dy_1...dy_4, \tag{2.8}$$

where $a'(\theta) = \frac{a(\theta)}{2}$. In here, we proceed with the same arguments as before. In the first and second integrals of (2.8), we will proceed with the following order: x_4, y_4, x_2 and y_2, always bounding their square roots with the upper integral limits before integrating with respect of them.
By doing this, we will find a bound of the form $(s'-s)^2(t'-t)^2$, up to a some positive constant.

Regarding the second integral in (2.7), we have that

$$\begin{aligned}
&\{A < 2(x_3-x_2)y_2\} \cap \{A < 2(y_3-y_2)x_2\} \\
&\quad = \{(x_4-x_1)y_2 + (y_4-y_3)x_2 + (y_2-y_1)x_1 + (x_3-x_2)y_3 \le 4(x_3-x_2)y_2\} \\
&\qquad \cap \{(y_4-y_1)x_2 + (x_4-x_3)y_2 + (x_2-x_1)y_1 + (y_3-y_2)x_3 \le 4(y_3-y_2)x_2\},
\end{aligned}$$

from where we can infer that

$$\frac{1}{4}y_3 \le y_2 \quad \text{and} \quad \frac{1}{4}x_3 \le x_2,$$

which implies

$$\begin{aligned}
A &= (x_4-x_3)y_2 + (x_2-x_1)y_2 + (y_4-y_3)x_2 + (y_2-y_1)x_1 + (x_3-x_2)(y_3-y_2) \\
&\ge \frac{1}{4}\big[(x_4-x_3)y_3 + (x_2-x_1)y_3 + (y_4-y_3)x_3 + (y_2-y_1)x_1\big] \\
&\ge \frac{1}{4}\big[(x_4-x_3)y_3 + (y_4-y_3)x_3 + (x_2-x_1)y_1 + (y_2-y_1)x_1\big].
\end{aligned}$$

Now, for the second integral in (2.7) we have that

$$\begin{aligned}
&\varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a(\theta)A}\, I_{\{A<2(x_3-x_2)y_2\}\cap\{A<2(y_3-y_2)x_2\}} dx_1...dx_4dy_1...dy_4 \\
&\le \varepsilon^4 \int_I \int_J \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, e^{-a''(\theta)[(x_4-x_3)y_3+(y_4-y_3)x_3+(x_2-x_1)y_1+(y_2-y_1)x_1]}\, dx_1...dx_4dy_1...dy_4,
\end{aligned}$$

where $a''(\theta) = \frac{a(\theta)}{4}$. We observe that it has the same form of (2.4). We can infer, once again, that this term can be bounded by $(s'-s)^2(t'-t)^2$ multiplied by some positive constant, just by following the same arguments used in the case i).
With this last case of Figure 2.2, we have concluded our tightness result.

□

Chapter 3

Limit law identification

In the previous chapter we proved the tightness of the family $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$. Recall that $\mathbb{P}_\varepsilon$ is the law of X_ε defined in (5). In this chapter, in order to prove Theorem 0.1, we should prove that the law of any possible weak limit is the law of a complex process whose real and imaginary parts are independent Brownian sheets. At the end of this chapter we will complete the proof of Theorem 0.1.

As a consequence of Proposition 2.1, there exists a subsequence $\{\mathbb{P}_{\varepsilon_n}\}_{n\geq 1}$ of $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ converging, in the weak sense in the space $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$, to some probability measure $\mathbb{P}$. More precisely, we want to prove that the canonical process $\{X_{s,t}(\omega):=\omega(s,t)\}$ is a complex process that its real and imaginary parts are a Brownian sheets under the probability $\mathbb{P}$ and, moreover, they are independent.

In order to achieve what we want, we will use a characterization of the Brownian sheet. Fortunately, there are several characterizations of it and we can find some of them in [39, Theorem 6] and [28, Theorem 2.2]. In particular, we will use the one given by [28, Theorem 2.2]. There, the necessary and sufficient conditions for a process to be a Brownian sheet, with respect an arbitrary filtration, have been proved. In our case, considering that the underlaying filtration is the natural one, we can weaken the hypothesis of [28, Theorem 2.2] and we get us to the following theorem, which is a quotation of [4, Theorem 4.1].

Theorem 3.1. *Let $\{Y(s,t)(x);(s,t)\in[0,S]\times[0,T]\}$ be continuous process, such that $Y(s,0)=Y(0,t)=0$ for all $s\in[0,S]$ and $t\in[0,T]$. Let be the natural filtration associated to Y. Then, the following statements are equivalents:*

i) Y is a Brownian sheet.

ii) Y is a strong martingale with

$$\mathbb{E}\left[\left(\Delta_{s,t}Y(s',t')\right)^2|\mathcal{F}_{s,T}\right]=(s'-s)(t'-t),$$

for all $0\leq s\leq s'\leq S$, $0<t\leq t'\leq T$.

Owing to Theorem 3.1 and Proposition 2.1, the following two propositions will guarantee the validity of (almost all) the statement of Theorem 0.1.

Proposition 3.1. *Let us assume that $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$ are the laws in $\mathcal{C}([0,S]\times[0,T])$ of the processes $\{X_\varepsilon\}_{\varepsilon>0}$ defined by (1) and let us assume also that $\{\mathbb{P}_{\varepsilon_n}\}_{n\in\mathbb{N}}$ is a subsequence of $\{\mathbb{P}_\varepsilon\}_{\varepsilon>1}$ that weakly converges to $\mathbb{P}$. Let X be the canonical process and let $\{\mathcal{F}_{s,t};(s,t)\in[0,S]\times[0,T]\}$ be its natural filtration. Then, the real and imaginary parts of $\{X(s,t);s,t\in[0,S]\times[0,T]\}$, are a $\mathcal{F}_{s,t}$- strong martingales under $\mathbb{P}$.*

Proposition 3.2. *Under the same hypotheses than the previous proposition, we have that*

$$\mathbb{E}_{\mathbb{P}}\left[\left(\Delta_{s,t}\mathsf{Re}(X)(s',t')\right)^2|\mathcal{F}_{s,T}\right]=(s'-s)(t'-t),$$

and

$$\mathbb{E}_{\mathbb{P}}\left[\left(\Delta_{s,t}\mathsf{Im}(X)(s',t')\right)^2|\mathcal{F}_{s,T}\right]=(s'-s)(t'-t),$$

for any $0<s\leq s'$, $0<t\leq t'$.

3.1 Proof of Proposition 3.1

In this part, we will need to calculate some limits and we will need l'Hôpital's rule. It can get complicated to verify l'Hôpital's hypotheses. So, in order to avoid any complication we will state the next lemma that saves us from those verifications.

Lemma 3.1. *Suppose that $f : [M, \infty) \to \mathbb{R}$ is a differentiable function, such that f' is continuous on $[M, \infty)$, $M \geq 0$, and let us assume*

$$\lim_{u \to \infty} f'(u) = a < \infty.$$

Thus,

$$\lim_{u \to \infty} \frac{f(u)}{u} = a.$$

This lemma can be proved by using the mean value theorem.

In order to prove Proposition 3.1, we will first prove the next lemma.

Lemma 3.2. *If X_ε is defined by* (5)*, then, for all* $0 < (s,t) \leq (s',t') \leq (S,T)$,

$$\mathbb{E}\left[\Delta_{s,t} X_\varepsilon(s',t') | \mathcal{F}^\varepsilon_{S,t} \vee \mathcal{F}^\varepsilon_{s,T}\right] \stackrel{L^2(\Omega)}{\longrightarrow} 0 \qquad \textit{if} \quad \varepsilon \to 0,$$

where $(\mathcal{F}^\varepsilon_{s,t})$ is the natural filtration generated by the process X_ε.

Proof. Let us define Y_ε as the expectation that appears in our lemma's statement:

$$\begin{aligned} Y_\varepsilon &:= \mathbb{E}\left[\Delta_{s,t} X_\varepsilon(s',t') | \mathcal{F}^\varepsilon_{S,t} \vee \mathcal{F}^\varepsilon_{s,T}\right] \\ &= \mathbb{E}\left[\varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\big(\cos(\theta L(x,y)) + i \sin(\theta L(x,y))\big) dxdy | \mathcal{F}^\varepsilon_{S,t} \vee \mathcal{F}^\varepsilon_{s,T}\right] \\ &= \mathbb{E}\left[\varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy} e^{i\theta L(x,y)} dxdy | \mathcal{F}^\varepsilon_{S,t} \vee \mathcal{F}^\varepsilon_{s,T}\right] \\ &= \mathbb{E}\left[\varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy} e^{i\theta\left(L(\frac{s}{\varepsilon},y) + L(x,\frac{t}{\varepsilon}) - L(\frac{s}{\varepsilon},\frac{t}{\varepsilon}) + \Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}} L(x,y)\right)} dxdy | \mathcal{F}^\varepsilon_{S,t} \vee \mathcal{F}^\varepsilon_{s,T}\right]. \end{aligned}$$

Notice that $L(\frac{s}{\varepsilon}, y) + L(x, \frac{t}{\varepsilon}) - L(\frac{s}{\varepsilon}, \frac{t}{\varepsilon})$ is mesurable with respect to the conditioning σ-field and that $\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}} L_{x,y}$ is independent of this σ-field. Also, if we apply Fubini theorem, we get that the last expression is equal to

$$\begin{aligned} &\varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy} e^{i\theta\left(L(\frac{s}{\varepsilon},y) + L(x,\frac{t}{\varepsilon}) + L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} \mathbb{E}\left[e^{i\theta \Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}} L_{x,y}}\right] dxdy \\ &= \varepsilon K \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy} e^{i\theta\left(L(\frac{s}{\varepsilon},y) + L(x,\frac{t}{\varepsilon}) + L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} e^{-\Psi(\theta)\left(x - \frac{s}{\varepsilon}\right)\left(y - \frac{t}{\varepsilon}\right)} dxdy. \end{aligned}$$

Hence,

$$
\begin{aligned}
\mathbb{E}\left[Y_\varepsilon^2\right] =&\mathbb{E}\left[\varepsilon^2 K^2 \left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy} e^{i\theta\left(L(\frac{s}{\varepsilon},y)+L(x,\frac{t}{\varepsilon})+L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} e^{-\Psi(\theta)\left(x-\frac{s}{\varepsilon}\right)\left(y-\frac{t}{\varepsilon}\right)} dxdy\right)^2\right]\\
=&\varepsilon^2 K^2 \mathbb{E}\left[\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1 y_1} e^{i\theta\left(L(\frac{s}{\varepsilon},y_1)+L(x_1,\frac{t}{\varepsilon})+L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} e^{-\Psi(\theta)\left(x_1-\frac{s}{\varepsilon}\right)\left(y_1-\frac{t}{\varepsilon}\right)} dx_1dy_1\right)\right.\\
&\times\left.\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_2 y_2} e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)+L(x_2,\frac{t}{\varepsilon})+L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)} e^{-\Psi(\theta)\left(x_2-\frac{s}{\varepsilon}\right)\left(y_2-\frac{t}{\varepsilon}\right)} dx_2dy_2\right)\right]\\
=&\varepsilon^2 K^2 \mathbb{E}\left[\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1 y_2} e^{i\theta\left(L(\frac{s}{\varepsilon},y_1)+L(\frac{s}{\varepsilon},y_2)+L(x_1,\frac{t}{\varepsilon})+L(x_2,\frac{t}{\varepsilon})+2L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right.\\
&\times\left. e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)} dx_1dx_2dy_1dy_2\right]\\
=&\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_1)+L(\frac{s}{\varepsilon},y_2)+L(x_1,\frac{t}{\varepsilon})+L(x_2,\frac{t}{\varepsilon})+2L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right] dx_1dx_2dy_1dy_2,
\end{aligned}
$$

where we applied Fubini theorem once again. Now, we must consider the 4 possible orders of (x_1, y_1) and (x_2, y_2), respectively. By applying some changes of variable and some arrangements, we have that the above expression is equal to

$$
\begin{aligned}
&\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times \mathbb{E}\left[e^{i\theta\left(\Delta_{x_1,0}L(x_2,\frac{t}{\varepsilon})+2\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})+\Delta_{0,y_1}L(\frac{s}{\varepsilon},y_2)+2\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+2\Delta_{0,0}L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right] dx_1dx_2dy_1dy_2\\
&+\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_1} \sqrt{x_1x_2}\sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times \mathbb{E}\left[e^{i\theta\left(\Delta_{x_2,0}L(x_1,\frac{t}{\varepsilon})+2\Delta_{\frac{s}{\varepsilon},0}L(x_2,\frac{t}{\varepsilon})+\Delta_{0,y_1}L(\frac{s}{\varepsilon},y_2)+2\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+2\Delta_{0,0}L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right] dx_2dx_1dy_1dy_2\\
&+\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_1} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times \mathbb{E}\left[e^{i\theta\left(\Delta_{x_1,0}L(x_2,\frac{t}{\varepsilon})+2\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})+\Delta_{0,y_2}L(\frac{s}{\varepsilon},y_1)+2\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_2)+2\Delta_{0,0}L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right] dx_1dx_2dy_2dy_1\\
&+\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_1} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_1} \sqrt{x_1x_2}\sqrt{x_1x_2}\sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times \mathbb{E}\left[e^{i\theta\left(\Delta_{x_2,0}L(x_1,\frac{t}{\varepsilon})+2\Delta_{\frac{s}{\varepsilon},0}L(x_2,\frac{t}{\varepsilon})+\Delta_{0,y_2}L(\frac{s}{\varepsilon},y_1)+2\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_2)+2\Delta_{0,0}L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\right)}\right] dx_2dx_1dy_2dy_1
\end{aligned}
$$

$$
\begin{aligned}
=&\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2\\
&+\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_1} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_1-x_2)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_2-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_2dx_1dy_1dy_2\\
&+\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_1} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_1-y_2)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_2dy_1\\
&+\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_1} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_1} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_1-x_2)\frac{t}{\varepsilon}+(y_1-y_2)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_2-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_2dx_1dy_2dy_1\\
=&2\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2\\
&+2\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2.
\end{aligned}
$$

Then,

$$\mathbb{E}\left[Y_\varepsilon^2\right] = 2(I_1 + I_2),$$

where

$$
\begin{aligned}
I_1 :=&\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2
\end{aligned}
$$

and

$$
\begin{aligned}
I_2 :=&\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2}\sqrt{y_1y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)}\\
&\times e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1dx_2dy_1dy_2.
\end{aligned}
$$

If $|I_1|$ and $|I_2|$ converge to zero as ε tends to zero, that will be enough for proving that $\mathbb{E}[Y_\varepsilon^2]$ also converges to zero. We have

$$
\begin{aligned}
|I_1| =& \varepsilon^2 K^2 \left| \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)} \right. \\
&\times \left. e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1 dx_2 dy_1 dy_2 \right| \\
\leq& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} \left| e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)} \right. \\
&\times \left. e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} \right| dx_1 dx_2 dy_1 dy_2 \\
=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})\right)} \\
&\times e^{-a(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1 dx_2 dy_1 dy_2 := I_1'
\end{aligned}
$$

and

$$
\begin{aligned}
|I_2| =& \varepsilon^2 K^2 \left| \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)} \right. \\
&\times \left. e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1 dx_2 dy_1 dy_2 \right| \\
\leq& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} \left| e^{-\Psi(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)} \right. \\
&\times \left. e^{-\Psi(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} \right| dx_1 dx_2 dy_1 dy_2 \\
=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)} \\
&\times e^{-a(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1 dx_2 dy_1 dy_2 := I_2'.
\end{aligned}
$$

Thus

$$
\mathbb{E}\left[Y_\varepsilon^2\right] = 2(I_1 + I_2) \leq 2(I_1' + I_2') \leq 4I_2'.
$$

This last inequality is a consequence of the properties of the exponential function (notice that the exponent is negative). So it is enough to prove that I_2' converges to zero in order to get that $\mathbb{E}\left[Y_\varepsilon^2\right]$ also converges to zero. We have

$$
\begin{aligned}
I_2' =& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})(y_2-\frac{t}{\varepsilon})+(x_2-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})\right)} \\
&\times e^{-a(2\theta)\left((x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+\frac{st}{\varepsilon^2}\right)} dx_1 dx_2 dy_1 dy_2 \\
\leq& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}\right)} \\
&\times e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_1 dx_2 dy_1 dy_2 \\
=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2}\sqrt{y_1 y_2} e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}\right)} \\
&\times e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_1 dx_2 dy_2 dy_1.
\end{aligned}
$$

Remember that s and t are not equal to zero. We can bound $\sqrt{x_1}$ by $\sqrt{x_2}$ and $\sqrt{y_2}$ by $\sqrt{\frac{t'}{\varepsilon}}$. Then, we obtain

$$\begin{aligned} I_2' \leq & C\varepsilon\sqrt{\varepsilon}K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} x_2\sqrt{y_1} e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}\right)} \\ & \times e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_1 dx_2 dy_2 dy_1 \\ \leq & C\varepsilon^2\sqrt{\varepsilon}K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} x_2\sqrt{y_1} e^{-a(\theta)(y_2-y_1)\frac{s}{\varepsilon}} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_2 dy_2 dy_1, \end{aligned}$$

where we have integrated with respect to x_1, bounded the exponential by 1 and C is a positive constant that changes at each step we integrate. Let us integrate with respect to y_2 and we obtain

$$\begin{aligned} I_2' &\leq C\varepsilon^3\sqrt{\varepsilon}K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} x_2\sqrt{y_1} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_2 dy_1 \\ &\leq C\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_2 dy_1 \\ &\leq C\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{s'-s}{\varepsilon} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dy_1 \\ &= C\varepsilon K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} e^{-a(2\theta)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dy_1 \end{aligned}$$

Here, we have bounded x_2 and y_1 by $\frac{s'}{\varepsilon}$ and $\frac{t'}{\varepsilon}$ respectively, then we have integrated with respect to x_2 and after that we have simplified the expression. Finally we integrate with respect to y_1 and we get that

$$I_2' \leq C\varepsilon^2 K^2 \left(1 - e^{-a(2\theta)(\frac{t'}{\varepsilon}-\frac{t}{\varepsilon})\frac{s}{\varepsilon}}\right).$$

This latter expression converges to zero as the ε tends to zero.

□

At last, we can give a proof of the Proposition 3.1.

Proof of Proposition 3.1. It is very similar that of [4, Proposition 4.2]. Set $(0,0) < (s,t) \leq (s',t') \leq (S,T)$. We want to prove that, for any $(s_1,t_1),\dots,(s_n,t_n)$ with $s_i \leq S$, $t_i \leq t$, or $s_i \leq s$, $t_i \leq T$, and for any bounded continuous function $\varphi : \mathbb{C}^n \to \mathbb{R}$, it holds

$$\left|\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\Delta_{s,t} Re[X(s',t')]\right)\right]\right| = 0$$

and

$$\left|\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\Delta_{s,t} Im[X(s',t')]\right)\right]\right| = 0.$$

We recall that the notation $|z|$ stands for the modulus of $z \in \mathbb{C}$. Hence, we only have to prove that for any $(s_1,t_1),\dots,(s_n,t_n)$ with $s_i \leq S$, $t_i \leq t$, or $s_i \leq s$, $t_i \leq T$, and for any bounded continuous function $\varphi : \mathbb{C}^n \to \mathbb{R}$, it holds

$$\left|\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\Delta_{s,t} X(s',t')\right)\right]\right| = 0.$$

Without any loss of generality, the converging subsequence of probability measures to $\mathbb{P}$ will be simply denoted by $\{\mathbb{P}_\varepsilon\}_{\varepsilon>0}$. Since $\mathbb{P}_\varepsilon \stackrel{w}{\Rightarrow} \mathbb{P}$, and taking into account Proposition 2.1, we have that

$$\begin{aligned} &\lim_{\varepsilon\to 0} \mathbb{E}_{\mathbb{P}_\varepsilon}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\Delta_{s,t} X(s',t')\right)\right] \\ &\quad = \mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\Delta_{s,t} X(s',t')\right)\right]. \end{aligned}$$

So, it is enough to prove that

$$\lim_{\varepsilon\to 0}\left|\mathbb{E}_{\mathbb{P}_\varepsilon}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\big(\Delta_{s,t}X(s',t')\big)\right]\right| = 0.$$

We have that

$$\begin{aligned}
&\left|\mathbb{E}_{\mathbb{P}_\varepsilon}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\big(\Delta_{s,t}X(s',t')\big)\right]\right| \\
&=\left|\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\big(\Delta_{s,t}X_\varepsilon(s',t')\big)\right]\right| \\
&\le\left|\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\right]\right|\left|\mathbb{E}\left[\mathbb{E}\left[\big(\Delta_{s,t}X_\varepsilon(s',t')\big)|\mathcal{F}^\varepsilon_{S,t}\vee\mathcal{F}^\varepsilon_{s,T}\right]\right]\right| \\
&\le\left|\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\right]\right|\left|\mathbb{E}\left[Y_\varepsilon\right]\right| \\
&\le\left(\mathbb{E}\left[\varphi^2\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\right]\right)^{\frac12}\left(\mathbb{E}\left[\left|Y_\varepsilon\right|^2\right]\right)^{\frac12}\le K\left(\mathbb{E}\left[|Y_\varepsilon|^2\right]\right)^{\frac12},
\end{aligned}$$

where K is a positive constant, and this last expression converges to zero as a consequence of the Lemma 3.2. □

3.2 Proof of Proposition 3.2

In this section we will finish to validate Theorem 3.1. In order to do so, we will prove that, for every $s_1<\dots<s_n\le S$, $t_1<\dots<t_n\le T$ and for every bounded continuous function $\varphi:\mathbb{R}^n\to\mathbb{R}$, we have

$$\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\big(\Delta_{s,t}\mathsf{Re}[X(s',t')]\big)^2-(s'-s)(t'-t)\right)\right]=0$$

and

$$\mathbb{E}_{\mathbb{P}}\left[\varphi(X(s_1,t_1),\dots,X(s_n,t_n))\left(\big(\Delta_{s,t}\mathsf{Im}[X(s',t')]\big)^2-(s'-s)(t'-t)\right)\right]=0,$$

for all $0<(s,t)\le(s',t')$.

Since $\mathbb{P}_\varepsilon$ converges weakly to $\mathbb{P}$, it is enough to prove that

$$\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\big(\Delta_{s,t}\mathsf{Re}[X_\varepsilon(s',t')]\big)^2-(s'-s)(t'-t)\right)\right]$$

and

$$\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\big(\Delta_{s,t}\mathsf{Im}[X_\varepsilon(s',t')]\big)^2-(s'-s)(t'-t)\right)\right]$$

converge to zero when ε tends to zero.

In order to prove Proposition 3.2 we will use the same idea used in the proof of [13, Theorem 3.1].

Let A_ε and B_ε be

$$A_\varepsilon:=\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\big(\Delta_{s,t}\mathsf{Re}[X_\varepsilon(s',t')]\big)^2-(s'-s)(t'-t)\right)\right]$$

and

$$B_\varepsilon:=\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\big(\Delta_{s,t}\mathsf{Im}[X_\varepsilon(s',t')]\big)^2-(s'-s)(t'-t)\right)\right].$$

Then, we should prove that $\lim_{\varepsilon\to0}A_\varepsilon=0$ and that $\lim_{\varepsilon\to0}B_\varepsilon=0$. In order to do it, it will be sufficient to prove that $\lim_{\varepsilon\to0}(A_\varepsilon+B_\varepsilon)=0$ and $\lim_{\varepsilon\to0}(A_\varepsilon-B_\varepsilon)=0$, where we have

$$A_\varepsilon+B_\varepsilon=\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\left|\Delta_{s,t}X_\varepsilon(s',t')\right|^2-2(s'-s)(t'-t)\right)\right]$$

and

$$A_\varepsilon-B_\varepsilon=\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\big(\Delta_{s,t}\mathsf{Re}[X_\varepsilon(s',t')]\big)^2-\big(\Delta_{s,t}\mathsf{Im}[X_\varepsilon(s',t')]\big)^2\right)\right].$$

We deal with the sum limit in the following way:

$$\begin{aligned}A_\varepsilon + B_\varepsilon =& \mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)\big)\left(\big|\Delta_{s,t}X_\varepsilon(s',t')\big|^2 - 2(s'-s)(t'-t)\right)\right]\\ =& \mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)\big)\left(\mathbb{E}\left[\big|\Delta_{s,t}X_\varepsilon(s',t')\big|^2|\mathcal{F}^\varepsilon_{s,T}\right] - 2(s'-s)(t'-t)\right)\right].\end{aligned}$$

In order to prove that this last expression converges to zero as ε tends to zero, it is enough to prove that

$$\mathbb{E}\left[\big|\Delta_{s,t}X_\varepsilon(s',t')\big|^2|\mathcal{F}^\varepsilon_{s,T}\right] \stackrel{L^2(\Omega)}{\longrightarrow} 2(s'-s)(t'-t) \qquad \text{as} \quad \varepsilon \to 0. \tag{3.1}$$

This convergence can be obtained thanks to the following facts:

1) $\mathbb{E}\left[\mathbb{E}\left[\big|\Delta_{s,t}X_\varepsilon(s',t')\big|^2|\mathcal{F}^\varepsilon_{s,T}\right]\right] = \mathbb{E}\left[\big|\Delta_{s,t}X_\varepsilon(s',t')\big|^2\right] \longrightarrow 2(s'-s)(t'-t)$ as $\varepsilon \to 0$. .
   This result is proved in Lemma 3.3.
2) There exist a sequence $\{C_\varepsilon\}_{\varepsilon>0}$ of positive constants, that converges to $4(s'-s)^2(t'-t)^2$, when ε tends to zero, such that
   $$\mathbb{E}\left[\mathbb{E}\left[\big|\Delta_{s,t}X_\varepsilon(s',t')\big|^2|\mathcal{F}^\varepsilon_{s,T}\right]\right]^2 \le C_\varepsilon.$$
   This result is proved in Lemma 3.4.

Lemma 3.3. *For any $(0,0) \le (s,t) \le (s',t') \le (S,T)$, it holds:*

$$\lim_{\varepsilon\to 0} \mathbb{E}\big[|\Delta_{s,t}X_\varepsilon(s',t')|^2\big] = 2(s'-s)(t'-t).$$

Proof. We split the proof in three steps.

Step 1. Owing to the definition of X_ε (see (5)) and Fubini theorem, we can make the following calculations:

$$\begin{aligned}&\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^2\right] = \varepsilon^2K^2\mathbb{E}\left[\left|\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}e^{i\theta L(x,y)}dxdy\right|^2\right]\\ =&\varepsilon^2K^2\mathbb{E}\left[\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1y_1}e^{-i\theta L(x_1,y_1)}dx_1dy_1\right)\left(\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_2y_2}e^{i\theta L(x_2,y_2)}dx_2dy_2\right)\right]\\ =&\varepsilon^2K^2\mathbb{E}\left[\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1y_2}e^{i\theta\left(L(x_2,y_2)-L(x_1,y_1)\right)}dx_1dx_2dy_1dy_2\right]\\ =&\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1y_2}\ \mathbb{E}\left[e^{i\theta\left(L(x_2,y_2)-L(x_1,y_1)\right)}\right]dx_1dx_2dy_1dy_2\\ =&\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2}\sqrt{y_1y_2}\ \mathbb{E}\left[e^{i\theta\left(\Delta_{0,0}L(x_2,y_2)-\Delta_{0,0}L(x_1,y_1)\right)}\right]dx_1dx_2dy_1dy_2.\end{aligned}$$

As we did in Lemma 3.2, we should also take in account the 4 possible orders of the points (x_1,y_1) and (x_2,y_2) in the plane. By doing this, we have that the latter expression is equal to

$$\begin{aligned}&\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}e^{-\Psi(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)}dx_1dx_2dy_1dy_2\\ &+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{y_2}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}e^{-\Psi(\theta)(x_2-x_1)y_2}e^{-\Psi(-\theta)(y_1-y_2)x_1}dx_1dx_2dy_1dy_2\\ &+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{x_2}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2y_1y_2}e^{-\Psi(-\theta)(x_1-x_2)y_1}e^{-\Psi(\theta)(y_2-y_1)x_2}dx_1dx_2dy_1dy_2\\ &+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{y_2}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{x_2}^{\frac{s'}{\varepsilon}} \sqrt{x_1x_2y_1y_2}e^{-\Psi(-\theta)\left((y_1-y_2)x_2+(x_1-x_2)y_1\right)}dx_1dx_2dy_1dy_2.\end{aligned}$$

Now, if we apply some suitable changes of variable, we get that our sum of integrals is equal to

$$\begin{aligned}
&\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-\Psi(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} dx_1 dx_2 dy_1 dy_2 \\
&\quad + \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)(y_2-y_1)x_1} dx_1 dx_2 dy_1 dy_2 \\
&\quad + \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-\Psi(-\theta)(x_2-x_1)y_1} e^{-\Psi(\theta)(y_2-y_1)x_1} dx_1 dx_2 dy_1 dy_2 \\
&\quad + \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-\Psi(-\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} dx_1 dx_2 dy_1 dy_2. \qquad (3.2)
\end{aligned}$$

Recall that $\Psi(\theta) = a(\theta) + ib(\theta)$, where $a(\theta) = a(-\theta)$ and $-b(\theta) = b(-\theta)$. Let us observe the following arrangements:

$$\begin{aligned}
& e^{-\Psi(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} + e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)(y_2-y_1)x_1} \\
&\quad + e^{-\Psi(-\theta)(x_2-x_1)y_1} e^{-\Psi(\theta)(y_2-y_1)x_1} + e^{-\Psi(-\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} \\
&= e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} e^{-ib(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} \\
&\quad + e^{-a(\theta)(x_2-x_1)y_1} e^{-ib(\theta)(x_2-x_1)y_1} e^{-a(\theta)(y_2-y_1)x_1} e^{ib(\theta)(y_2-y_1)x_1} \\
&\quad + e^{-a(\theta)(x_2-x_1)y_1} e^{ib(\theta)(x_2-x_1)y_1} e^{-a(\theta)(y_2-y_1)x_1} e^{-ib(\theta)(y_2-y_1)x_1} \\
&\quad + e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} e^{ib(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} \\
&= e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} \left(e^{ib(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} + e^{-ib(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_2\right)} \right) \\
&\quad + e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_1\right)} \left(e^{ib(\theta)\left((y_2-y_1)x_1-(x_2-x_1)y_1\right)} + e^{-ib(\theta)\left((y_2-y_1)x_1-(x_2-x_1)y_1\right)} \right) \\
&= e^{-a(\theta)\left(x_2y_2-x_1y_1\right)} \left(e^{ib(\theta)\left(x_2y_2-x_1y_1\right)} + e^{-ib(\theta)\left(x_2y_2-x_1y_1\right)} \right) \\
&\quad + e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_1\right)} \left(e^{ib(\theta)\left((y_2-y_1)x_1-(x_2-x_1)y_1\right)} + e^{-ib(\theta)\left((y_2-y_1)x_1-(x_2-x_1)y_1\right)} \right) \\
&= e^{-a(\theta)\left(x_2y_2-x_1y_1\right)} 2\cos\left(b(\theta)(x_2y_2 - x_1y_1)\right) \\
&\quad + e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_1\right)} 2\cos\left(b(\theta)\left((y_2-y_1)x_1 - (x_2-x_1)y_1\right)\right).
\end{aligned}$$

The last sum is due to the definition of the exponential function. From here, we can infer that

$$\mathbb{E}\left[|\Delta_{s,t} X_\varepsilon(s',t')|^2\right] = 2(I_1^\varepsilon + I_2^\varepsilon), \qquad (3.3)$$

where

$$\begin{aligned}
I_1^\varepsilon :=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-a(\theta)\left(x_2y_2-x_1y_1\right)} \\
& \times \cos\left(b(\theta)(x_2y_2 - x_1y_1)\right) dx_1 dx_2 dy_1 dy_2
\end{aligned}$$

and

$$\begin{aligned}
I_2^\varepsilon :=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-a(\theta)\left((y_2-y_1)x_1+(x_2-x_1)y_1\right)} \\
& \times \cos\left(b(\theta)\left((y_2-y_1)x_1 - (x_2-x_1)y_1\right)\right) dx_1 dx_2 dy_1 dy_2.
\end{aligned}$$

Hence, we must prove that $\lim_{\varepsilon\to 0} 2(I_1^\varepsilon + I_2^\varepsilon) = 2(s'-s)(t'-t)$.

Step 2. Let us start with the case where $s = t = 0$. If in I_1^ε we apply the changes of variable $z_i := x_i y_i$ and $v_i := \frac{\varepsilon}{s'} x_i$, $i = 1, 2$ and we define $u := \frac{s't'}{\varepsilon^2}$, we have

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_1^\varepsilon &= \lim_{\varepsilon\to 0} \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\left(x_2y_2 - x_1y_1\right)} \cos\big(b(\theta)(x_2y_2 - x_1y_1)\big) dx_1 dy_1 dx_2 dy_2 \\
&= \lim_{u\to\infty} \frac{s't'}{u} K^2 \int_0^1 \int_0^{uv_2} \int_0^{v_2} \int_0^{\frac{z_2 v_1}{v_2}} \frac{\sqrt{z_1 z_2}}{v_1 v_2} e^{-a(\theta)z_2 + a(\pi)z_1} \cos\big(b(\theta)(z_2 - z_1)\big) dz_1 dv_1 dz_2 dv_2 \\
&= \lim_{u\to\infty} s't'K^2 \int_0^1 \int_0^{v_2} \int_0^{uv_1} \frac{\sqrt{z_1 u v_2}}{v_1} e^{-a(\theta)uv_2 + a(\theta)z_1} \cos\big(b(\theta)(uv_2 - z_1)\big) dz_1 dv_1 dv_2,
\end{aligned}$$

where in the last line we have applied Lemma 3.1. Applying now the variable changes $v_2' := uv_2$ and $v_1' := uv_1$ and Lemma 3.1 again, we have that the last expression is equal to

$$\begin{aligned}
&\lim_{u\to\infty} \frac{s't'}{u} K^2 \int_0^u \int_0^{v_2'} \int_0^{v_1'} \frac{\sqrt{z_1 v_2'}}{v_1'} e^{-a(\theta)v_2' + a(\theta)z_1} \cos\big(b(\theta)(v_2' - z_1)\big) dz_1 dv_1' dv_2' \\
&= \lim_{u\to\infty} s't'K^2 \int_0^u \int_0^{v_1'} \frac{\sqrt{z_1 u}}{v_1'} e^{-a(\theta)u + a(\theta)z_1} \cos\big(b(\theta)(u - z_1)\big) dz_1 dv_1'. \qquad (3.4)
\end{aligned}$$

Here we will use the formula $\cos(\theta) = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$. Therefore, expression (3.4) can be written as $\frac{1}{2}(A_u + B_u)$, where

$$A_u := s't'K^2 \int_0^u \int_0^{v_1'} \frac{\sqrt{z_1 u}}{v_1'} e^{-a(\theta)u + a(\theta)z_1} e^{ib(\theta)(u - z_1)} dz_1 dv_1'$$

and

$$B_u := s't'K^2 \int_0^u \int_0^{v_1'} \frac{\sqrt{z_1 u}}{v_1'} e^{-a(\theta)u + a(\theta)z_1} e^{-ib(\theta)(u - z_1)} dz_1 dv_1'.$$

Let us rewrite these integrals in a simpler form:

$$A_u = s't'K^2 \int_0^u \int_0^v \frac{\sqrt{zu}}{v} e^{-a(\theta)u + a(\theta)z} e^{ib(\theta)(u - z)} dz dv,$$

$$B_u = s't'K^2 \int_0^u \int_0^{v'} \frac{\sqrt{zu}}{v} e^{-a(\theta)u + a(\theta)z} e^{-ib(\theta)(u - z)} dz dv.$$

First, let us deal with A_u, for which we rewrite it as follows:

$$\begin{aligned}
\lim_{u\to\infty} A_u &= \lim_{u\to\infty} s't'K^2 \int_0^u \int_0^v \frac{\sqrt{zu}}{v} e^{-a(\theta)u + a(\theta)z} e^{ib(\theta)(u - z)} dz dv \\
&= \lim_{u\to\infty} s't'K^2 \frac{\int_0^u \int_0^v \frac{\sqrt{z}}{v} e^{a(\theta)z} e^{-ib(\theta)z} dz dv}{\frac{1}{\sqrt{u}} e^{a(\theta)u - ib(\theta)u}}.
\end{aligned}$$

If we apply l'Hôpital's rule, we have that the latter expression is equal to

$$\begin{aligned}
&\lim_{u\to\infty} s't'K^2 \frac{\int_0^u \frac{\sqrt{z}}{u} e^{a(\theta)z} e^{-ib(\theta)z} dz}{\frac{\sqrt{u}(a(\theta) - ib(\theta))e^{a(\theta)u - ib(\theta)u}}{u} - \frac{e^{a(\theta)u - ib(\theta)u}}{2u^{\frac{3}{2}}}} \\
&= \lim_{u\to\infty} s't'K^2 \frac{\int_0^u \sqrt{z} e^{a(\theta)z} e^{-ib(\theta)z} dz}{\sqrt{u}(a(\theta) - ib(\theta))e^{a(\theta)u - ib(\theta)u} - \frac{e^{a(\theta)u - ib(\theta)u}}{2\sqrt{u}}}.
\end{aligned}$$

We apply l'Hôpital's rule once more and we get that our limit above is equal to

$$\begin{aligned}
&\lim_{u\to\infty} s't'K^2 \frac{\sqrt{u}}{\sqrt{u}(a(\theta) - ib(\theta))^2 + \frac{1}{4u^{\frac{3}{2}}}} \\
&= \lim_{u\to\infty} s't'K^2 \frac{1}{(a(\theta) - ib(\theta))^2 + \frac{1}{4u^2}} \\
&= s't'K^2 \frac{1}{(a(\theta) - ib(\theta))^2}.
\end{aligned}$$

Now, we will deal with B_u in the same way, the difference will only be a couple of signs:

$$\begin{aligned}\lim_{u\to\infty} B_u &= \lim_{u\to\infty} s't'K^2 \int_0^u \int_0^v \frac{\sqrt{zu}}{v} e^{-a(\theta)u+a(\theta)z} e^{-ib(\theta)(u-z)} dzdv \\ &= \lim_{u\to\infty} s't'K^2 \frac{\int_0^u \int_0^v \frac{\sqrt{z}}{v} e^{a(\theta)z} e^{ib(\theta)z} dzdv}{\frac{1}{\sqrt{u}} e^{a(\theta)u+ib(\theta)u}}.\end{aligned}$$

Then we apply l'Hôpital's rule for the first time and we obtain that the latter limit is equal to

$$\begin{aligned}&\lim_{u\to\infty} s't'K^2 \frac{\int_0^u \frac{\sqrt{z}}{u} e^{a(\theta)z} e^{ib(\theta)z} dz}{\frac{\sqrt{u}(a(\theta)+ib(\theta))e^{a(\theta)u+ib(\theta)u}}{u} - \frac{e^{a(\theta)u+ib(\theta)u}}{2u^{\frac{3}{2}}}} \\ &= \lim_{u\to\infty} s't'K^2 \frac{\int_0^u \sqrt{z} e^{a(\theta)z} e^{ib(\theta)z} dz}{\sqrt{u}(a(\theta)+ib(\theta))e^{a(\theta)u+ib(\theta)u} - \frac{e^{a(\theta)u+ib(\theta)u}}{2\sqrt{u}}}.\end{aligned}$$

Applying l'Hôpital's rule for a second time, we get that our limit above is equal to

$$\begin{aligned}&\lim_{u\to\infty} s't'K^2 \frac{\sqrt{u}}{\sqrt{u}(a(\theta)+ib(\theta))^2 + \frac{1}{4u^{\frac{3}{2}}}} \\ &= \lim_{u\to\infty} s't'K^2 \frac{1}{(a(\theta)+ib(\theta))^2 + \frac{1}{4u^2}} \\ &= s't'K^2 \frac{1}{(a(\theta)+ib(\theta))^2}.\end{aligned}$$

Hence, we have that

$$\begin{aligned}\lim_{\varepsilon\to 0} I_1^\varepsilon &= \frac{s't'K^2}{2}\left(\frac{1}{(a(\theta)-ib(\theta))^2} + \frac{1}{(a(\theta)+ib(\theta))^2}\right) \\ &= \frac{s't'K^2}{2} \frac{a(\theta)^2 + 2ia(\theta)b(\theta) - b(\theta)^2 + a(\theta)^2 - 2ia(\theta)b(\theta) - b(\theta)^2}{(a(\theta)^2 + ia(\theta)b(\theta) - ia(\theta)b(\theta) + b(\theta)^2)^2} \\ &= s't'K^2 \frac{(a(\theta)^2 - b(\theta)^2)}{(a(\theta)^2 + b(\theta)^2)^2}.\end{aligned}$$

Now, we will deal with the limit of I_2^ε. We cannot apply the same strategy that we used with I_1^ε. Moreover, we could not obtain the limit of I_2^ε in a direct way. Due to this situation, we will introduce an auxiliary term which converges to some value and then we will prove that the remainder converges to zero.

We will start computing the limit of I_2^2 applying the same changes of variable that we used for I_1^ε. We set $z_i := x_i y_i$ and $v_i := \frac{\varepsilon}{s'} x_i$, for $i = 1, 2$ and we define $u := \frac{s't'}{\varepsilon^2}$. Then, we have that

$$\begin{aligned}\lim_{\varepsilon\to 0} I_2^\varepsilon &= \lim_{\varepsilon\to 0} \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2} e^{-a(\theta)\big((y_2-y_1)x_1 + (x_2-x_1)y_1\big)} \\ &\quad \times \cos\big(b(\theta)\big((y_2-y_1)x_1 - (x_2-x_1)y_1\big)\big) dx_1 dx_2 dy_1 dy_2 \\ &= \lim_{u\to\infty} \frac{s't'}{u} K^2 \int_0^1 \int_0^{uv_2} \int_0^{v_2} \int_0^{\frac{z_2 v_1}{v_2}} \frac{\sqrt{z_1 z_2}}{v_1 v_2} e^{-a(\theta)\left(v_1\left(\frac{z_2}{v_2} - \frac{z_1}{v_1}\right) + \frac{z_1}{v_1}(v_2 - v_1)\right)} \\ &\quad \times \cos\left(b(\theta)\left(v_1\left(\frac{z_2}{v_2} - \frac{z_1}{v_1}\right) - \frac{z_1}{v_1}(v_2 - v_1)\right)\right) dz_1 dv_1 dz_2 dv_2 \\ &= \lim_{u\to\infty} \frac{s't'}{u} K^2 \int_0^1 \int_0^{uv_2} \int_0^{v_2} \int_0^{\frac{z_2 v_1}{v_2}} \frac{\sqrt{z_1 z_2}}{v_1 v_2} e^{-a(\theta)\left(\frac{v_1 z_2}{v_2} - z_1 - \frac{z_1 v_2}{v_1} + z_1\right)} \\ &\quad \times \cos\left(b(\theta)\left(\frac{v_1 z_2}{v_2} - z_1 - \frac{z_1 v_2}{v_1} + z_1\right)\right) dz_1 dv_1 dz_2 dv_2 \\ &= \lim_{u\to\infty} \frac{s't'}{u} K^2 \int_0^1 \int_0^{uv_2} \int_0^{v_2} \int_0^{\frac{z_2 v_1}{v_2}} \frac{\sqrt{z_1 z_2}}{v_1 v_2} e^{a(\theta)\left(2z_1 - \frac{v_1 z_2}{v_2} - \frac{z_1 v_2}{v_1}\right)} \\ &\quad \times \cos\left(b(\theta)\left(\frac{v_1 z_2}{v_2} - \frac{z_1 v_2}{v_1}\right)\right) dz_1 dv_1 dz_2 dv_2.\end{aligned}$$

Applying l'Hôpital's rule we obtain that the last expression is equal to

$$\lim_{u\to\infty} s't'K^2 \int_0^1 \int_0^{v_2} \int_0^{uv_1} \sqrt{uv_2}\frac{\sqrt{z_1}}{v_1} e^{a(\theta)\left(2z_1-uv_1-\frac{z_1v_2}{v_1}\right)} \times \cos\left(b(\theta)\left(uv_1-\frac{z_1v_2}{v_1}\right)\right) dz_1dv_1dv_2.$$

In order to apply l'Hôpital's rule once again, we will make another changes of variable. Set $\bar{v}_1 := uv_1$ and $\bar{v}_2 := uv_2$ and we have that the last limit is equal to

$$\begin{aligned}
&\lim_{u\to\infty} \frac{s't'}{u}K^2 \int_0^u \int_0^{\bar{v}_2} \int_0^{\bar{v}_1} \frac{\sqrt{z_1}\sqrt{\bar{v}_2}}{\bar{v}_1} e^{a(\theta)\left(2z_1-\bar{v}_1-\frac{z_1\bar{v}_2}{\bar{v}_1}\right)} \cos\left(b(\pi)\left(\bar{v}_1-\frac{z_1\bar{v}_2}{\bar{v}_1}\right)\right) dz_1d\bar{v}_1d\bar{v}_2\\
&=\lim_{u\to\infty} s't'K^2\sqrt{u} \int_0^u \int_0^{\bar{v}_1} \frac{\sqrt{z_1}}{\bar{v}_1} e^{a(\theta)\left(2z_1-\bar{v}_1-\frac{z_1u}{\bar{v}_1}\right)} \cos\left(b(\pi)\left(\bar{v}_1-\frac{z_1u}{\bar{v}_1}\right)\right) dz_1d\bar{v}_1,
\end{aligned}$$

where we applied l'Hôpital's rule in the last line. Now, we perform the changes $x := \frac{z_1}{\bar{v}_1}$ and $y := \frac{\bar{v}_1}{u}$, so we obtain that

$$\begin{aligned}
\lim_{\varepsilon\to 0} I_2^\varepsilon &= \lim_{u\to\infty} s't'K^2u^2 \int_0^1 \int_0^1 \sqrt{xy}e^{a(\theta)(2xy-y-x)u} \cos\left(b(\pi)(y-x)u\right) dxdy \qquad (3.5)\\
&=: C_u.
\end{aligned}$$

This is the point where we introduce the auxiliary term mentioned above:

$$\tilde{C}_u := K^2s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{a(\theta)(2xy-y-x)u} \cos(b(\theta)(y-x)u)dxdy, \qquad (3.6)$$

where we observe that we have replaced $\sqrt{xy}$ on the right hand-side of (3.5) by $\sqrt{y}$. We will asume, for now, that $\lim_{u\to\infty}(C_u - \tilde{C}_u) = 0$. We will prove this limit later, and we will first compute the limit of $\tilde{C}_u$. Once again, we will use the fact that $\cos(\theta) = \frac{1}{2}(e^{i\theta}+e^{-i\theta})$ as we did it in the analysis of I_1^ε. Thus, we have that

$$\begin{aligned}
\tilde{C}_u =&K^2s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{a(\theta)(2xy-y-x)u} \cos(b(\theta)(y-x)u)dxdy\\
=&\frac{1}{2}\Bigg(K^2s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{u(2xya(\theta)-y(a(\theta)-ib(\theta))-x(a(\theta)+ib(\theta)))}dxdy\\
&+K^2s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{u(2xya(\theta)-y(a(\theta)+ib(\theta))-x(a(\theta)-ib(\theta)))}dxdy\Bigg)\\
:=&\frac{1}{2}(J_1^u + J_2^u).
\end{aligned}$$

Let us compute the limit for J_1^u. First, we will integrate with respect to x, then we will rewrite it, and after that we will apply l'Hôpital's rule:

$$\begin{aligned}
\lim_{u\to\infty} J_1^u &= \lim_{u\to\infty} K^2s't'u^2 \int_0^1 \int_0^1 \sqrt{y}\, e^{u(2xya(\theta)-y(a(\theta)-ib(\theta))-x(a(\theta)+ib(\theta)))}dxdy\\
&= \lim_{u\to\infty} K^2s't'u \int_0^1 \frac{\sqrt{y}}{2ya(\theta)-(a(\theta)+ib(\theta))} \left(e^{u(y(a(\theta)+ib(\theta))-(a(\theta)+ib(\theta)))} - e^{-uy(a(\theta)+ib(\theta))}\right) dy\\
&= \lim_{u\to\infty} K^2s't'u \int_0^1 \frac{\sqrt{y}}{2ya(\theta)-(a(\theta)+ib(\theta))} e^{u(y(a(\theta)+ib(\theta))-(a(\theta)+ib(\theta)))}dy + \delta_u\\
&= \lim_{u\to\infty} K^2s't'u\frac{a(\theta)+ib(\theta)}{a(\theta)} \int_0^1 \frac{\sqrt{y}}{2y(a(\theta)+ib(\theta)) - \frac{(a(\theta)+ib(\theta))^2}{a(\theta)}} e^{u(y(a(\theta)+ib(\theta))-(a(\theta)+ib(\theta)))}dy + \delta_u.
\end{aligned}$$

In order to avoid complicated calculations, we will only write the terms that affect the limit. From now on we will call δ_u to all the terms such that $\lim_{u\to\infty}\delta_u = 0$ and we will call ς_u to the terms in the denominator that does not affect the limit. This terms may not be same from one line to another. Thus, the latter expresion is equal to

$$
\lim_{u\to\infty} K^2 s' t' \frac{a(\theta)+ib(\theta)}{a(\theta)} \frac{\int_0^1 \frac{\sqrt{y}}{2y(a(\theta)+ib(\theta))-\frac{(a(\theta)+ib(\theta))^2}{a(\theta)}} e^{u\left(y(a(\theta)+ib(\theta))-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)} dy}{\frac{e^{u\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)}}{u}}
$$

$$
= \lim_{u\to\infty} K^2 s' t' \frac{a(\theta)+ib(\theta)}{2a(\theta)} \frac{\int_0^1 \sqrt{y} e^{u\left(y(a(\theta)+ib(\theta))-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)} dy}{\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right) \frac{e^{u\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)}}{u} + \varsigma_u}
$$

$$
= \lim_{u\to\infty} K^2 s' t' \frac{a(\theta)+ib(\theta)}{2a(\theta)} \frac{\int_0^1 \sqrt{y} e^{u(y(a(\theta)+ib(\theta)))} dy}{\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right) \frac{e^{u(a(\theta)+ib(\theta))}}{u} + \varsigma_u},
$$

where we have applied l'Hôpital's rule. Set $y' := uy$ and we have that the latter limit is equal to

$$
\lim_{u\to\infty} K^2 s' t' \frac{a(\theta)+ib(\theta)}{2a(\theta)} \frac{\int_0^u \sqrt{y'} e^{y'(a(\theta)+ib(\theta))} dy'}{\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right) \sqrt{u} e^{u(a(\theta)+ib(\theta))} + \varsigma_u}
$$

$$
= \lim_{u\to\infty} \frac{K^2 s' t' (a(\theta)+ib(\theta)) \sqrt{u} e^{u(a(\theta)+ib(\theta))}}{2a(\theta)\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)\left((a(\theta)+ib(\theta))\sqrt{u} e^{u(a(\theta)+ib(\theta))} - \frac{e^{u(a(\theta)+ib(\theta))}}{u}\right) + \varsigma_u}.
$$

Here, we have applied l'Hôpital's rule again. Hence, the last expression equals

$$
\lim_{u\to\infty} K^2 s' t' \frac{a(\theta)+ib(\theta)}{2a(\theta)} \frac{\sqrt{u}}{\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)\left((a(\theta)+ib(\theta))\sqrt{u}-\frac{1}{u}\right)}
$$

$$
= \lim_{u\to\infty} K^2 s' t' \frac{a(\theta)+ib(\theta)}{2a(\theta)} \frac{\sqrt{u}}{\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)\left((a(\theta)+ib(\theta))\sqrt{u}\right)}
$$

$$
= K^2 s' t' \frac{1}{2a(\theta)} \frac{1}{\left(a(\theta)+ib(\theta)-\frac{(a(\theta)+ib(\theta))^2}{2a(\theta)}\right)}
$$

$$
= K^2 s' t' \frac{1}{2a(\theta)} \frac{1}{\left(a(\theta)+ib(\theta)\right)\left(1-\frac{a(\theta)+ib(\theta)}{2a(\theta)}\right)}
$$

$$
= K^2 s' t' \frac{1}{\left(a(\theta)+ib(\theta)\right)\left(a(\theta)-ib(\theta)\right)} = K^2 s' t' \frac{1}{a(\theta)^2+b(\theta)^2}.
$$

Next, we follow the same steps for computing the limit of J_2^u, we integrate with respect of x and then we apply l'Hôpital's rule:

$$
\begin{aligned}
\lim_{u\to\infty} J_2^u &= \lim_{u\to\infty} K^2 s't'u^2 \int_0^1\int_0^1 \sqrt{y}\, e^{u(2xya(\theta)-y(a(\theta)+ib(\theta))-x(a(\theta)-ib(\theta)))} dxdy \\
&= \lim_{u\to\infty} K^2 s't'u \int_0^1 \frac{\sqrt{y}}{2ya(\theta)-(a(\theta)-ib(\theta))}\left(e^{u(y(a(\theta)-ib(\theta))-(a(\theta)-ib(\theta)))} - e^{-uy(a(\theta)+ib(\theta))}\right) dy \\
&= \lim_{u\to\infty} K^2 s't'u \int_0^1 \frac{\sqrt{y}}{2ya(\theta)-(a(\theta)-ib(\theta))} e^{u(y(a(\theta)-ib(\theta))-(a(\theta)-ib(\theta)))} dy + \delta_u \\
&= \lim_{u\to\infty} K^2 s't'u \frac{a(\theta)-ib(\theta)}{a(\theta)} \int_0^1 \frac{\sqrt{y}}{2y(a(\theta)-ib(\theta))-\frac{(a(\theta)-ib(\theta))^2}{a(\theta)}} e^{u(y(a(\theta)-ib(\theta))-(a(\theta)-ib(\theta)))} dy + \delta_u.
\end{aligned}
$$

We apply l'Ĥopital's rule once more and we have the latter limit equals

$$
\begin{aligned}
&\lim_{u\to\infty} K^2 s't' \frac{a(\theta)-ib(\theta)}{a(\theta)} \frac{\int_0^1 \frac{\sqrt{y}}{2y(a(\theta)-ib(\theta))-\frac{(a(\theta)-ib(\theta))^2}{a(\theta)}} e^{u\left(y(a(\theta)-ib(\theta))-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)} dy}{\frac{e^{u\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)}}{u}} \\
&\quad= \lim_{u\to\infty} K^2 s't' \frac{a(\theta)-ib(\theta)}{2a(\theta)} \frac{\int_0^1 \sqrt{y} e^{u\left(y(a(\theta)-ib(\theta))-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)} dy}{\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)\frac{e^{u\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)}}{u} + \varsigma_u} \\
&\quad= \lim_{u\to\infty} K^2 s't' \frac{a(\theta)-ib(\theta)}{2a(\theta)} \frac{\int_0^1 \sqrt{y} e^{u(y(a(\theta)-ib(\theta)))} dy}{\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)\frac{e^{u(a(\theta)-ib(\theta))}}{u} + \varsigma_u}.
\end{aligned}
$$

Set $y' := uy$ and we have that the latter expression is equal to

$$
\begin{aligned}
&\lim_{u\to\infty} K^2 s't' \frac{a(\theta)-ib(\theta)}{2a(\theta)} \frac{\int_0^u \sqrt{y'} e^{y'(a(\theta)-ib(\theta))} dy'}{\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)\sqrt{u} e^{u(a(\theta)-ib(\theta))} + \varsigma_u} \\
&= \lim_{u\to\infty} \frac{K^2 s't'(a(\theta)-ib(\theta))\sqrt{u} e^{u(a(\theta)-ib(\theta))}}{2a(\theta)\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)\left((a(\theta)-ib(\theta))\sqrt{u} e^{u(a(\theta)-ib(\theta))} - \frac{e^{u(a(\theta)-ib(\theta))}}{u}\right) + \varsigma_u},
\end{aligned}
$$

where we have applied l'Hôpital's rule again. Hence, the last expression equals

$$
\begin{aligned}
&\lim_{u\to\infty} K^2 s't' \frac{a(\theta)-ib(\theta)}{2a(\theta)} \frac{\sqrt{u}}{\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)\left((a(\theta)-ib(\theta))\sqrt{u}-\frac{1}{u}\right)} \\
&\quad= \lim_{u\to\infty} K^2 s't' \frac{a(\theta)-ib(\theta)}{2a(\theta)} \frac{\sqrt{u}}{\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)\left((a(\theta)-ib(\theta))\sqrt{u}\right)} \\
&\quad= K^2 s't' \frac{1}{2a(\theta)} \frac{1}{\left(a(\theta)-ib(\theta)-\frac{(a(\theta)-ib(\theta))^2}{2a(\theta)}\right)} \\
&\quad= K^2 s't' \frac{1}{2a(\theta)} \frac{1}{\left(a(\theta)-ib(\theta)\right)\left(1-\frac{a(\theta)-ib(\theta)}{2a(\theta)}\right)} \\
&\quad= K^2 s't' \frac{1}{\left(a(\theta)-ib(\theta)\right)\left(a(\theta)+ib(\theta)\right)} = K^2 s't' \frac{1}{a(\theta)^2+b(\theta)^2}.
\end{aligned}
$$

Thus,

$$\lim_{u\to\infty} \tilde{C}_u = \frac{K^2 s't'}{2}\left(\frac{1}{a(\theta)^2+b(\theta)^2}+\frac{1}{a(\theta)^2+b(\theta)^2}\right)$$
$$= K^2 s't' \frac{1}{a(\theta)^2+b(\theta)^2}.$$

Hence, due to the definition of K in (6) and to (3.3), we have that Lemma 3.3 holds in the case $s=t=0$.

In order to finish this part of the proof, we only need to check that $\lim_{u\to\infty}(C_u-\tilde{C}_u)=0$. We have to verify that

$$\lim_{\varepsilon\to 0} I_2^\varepsilon = \lim_{u\to\infty} s't'K^2u^2\int_0^1\int_0^1(\sqrt{xy}-\sqrt{y})e^{a(\theta)(2xy-y-x)u}\cos\left(b(\pi)(y-x)u\right)dxdy=0.$$

Let us define

$$D_u := s't'K^2u^2\int_0^1\int_0^1(\sqrt{xy}-\sqrt{y})e^{a(\theta)(2xy-y-x)u}dxdy.$$

Then, we clearly have that

$$-D_u \le C_u-\tilde{C}_u \le D_u.$$

We will show that both $-D_u$ and D_u converge to zero as u tends to infinity. We will only deal with D_u, since analysis is analogous for $-D_u$. Observe that $D_u = D_u^1 - D_u^2$, with

$$D_u^1 := u^2\int_0^1\int_0^1\sqrt{xy}\,e^{a(\theta)(2xy-y-x)u}dxdy \quad\text{and}\quad D_u^2 := u^2\int_0^1\int_0^1\sqrt{y}\,e^{a(\theta)(2xy-y-x)u}dxdy.$$

Observing D_u^2, we notice that the integral in x can be computed in an analogous way as we did it for J_1^u and J_2^u. We first integrate with respect of x and then we apply l'Hôpital's rule, and it follows:

$$\lim_{u\to\infty} D_u^2 = \lim_{u\to\infty} u^2\int_0^1\int_0^1\sqrt{y}\,e^{a(\theta)(2xy-y-x)u}dxdy$$
$$= \lim_{u\to\infty} u\int_0^1\frac{\sqrt{y}}{a(\theta)(2y-1)}\left(e^{a(\theta)(y-1)u}-e^{-a(\theta)yu}\right)dy$$
$$= \lim_{u\to\infty}\frac{1}{a(\theta)}\frac{\int_0^1\frac{\sqrt{y}}{(2y-1)}\left(e^{a(\theta)\left(y-\frac{1}{2}\right)u}-e^{-a(\theta)\left(y-\frac{1}{2}\right)u}\right)dy}{\frac{e^{u\frac{a(\theta)}{2}}}{u}}$$
$$= \lim_{u\to\infty}\frac{\int_0^1\sqrt{y}\left(e^{a(\theta)\left(y-\frac{1}{2}\right)u}+e^{-a(\theta)\left(y-\frac{1}{2}\right)u}\right)dy}{\frac{a(\theta)e^{u\frac{a(\theta)}{2}}}{u}-\frac{2e^{u\frac{a(\theta)}{2}}}{u^2}}.$$

By setting $y' := uy$, we have that the latter limit is equal to

$$\lim_{u\to\infty}\left\{\frac{\int_0^u\sqrt{y'}e^{y'a(\theta)}dy'}{\sqrt{u}a(\theta)e^{ua(\theta)}-\frac{2e^{ua(\theta)}}{\sqrt{u}}}+\frac{\int_0^u\sqrt{y'}e^{-y'a(\theta)}dy'}{\sqrt{u}-\frac{1}{\sqrt{u}}}\right\} \tag{3.7}$$
$$= \lim_{u\to\infty}\frac{\sqrt{u}e^{ua(\theta)}}{\frac{a(\theta)e^{a(\theta)u}}{2\sqrt{u}}+\sqrt{u}a(\theta)^2e^{a(\theta)u}-\frac{2a(\theta)e^{a(\theta)u}}{\sqrt{u}}+\frac{e^{a(\theta)u}}{u^{\frac{3}{2}}}}$$
$$= \lim_{u\to\infty}\frac{1}{\frac{a(\theta)}{2u}+a(\theta)^2-\frac{2a(\theta)}{u}+\frac{1}{u^2}} = \frac{1}{a(\theta)^2}.$$

We observe that in (3.7), the second term inside the braces clearly converges to zero as $u\to\infty$ and we applied l'Hôpital's rule to the second one. After that, we just computed the limit. Then we have that $\lim_{u\to\infty} D_u^2 = \frac{1}{a(\theta)^2}$.

In order to deal with D_u^1, we will also use a kind of sandwich argument. We observe that $D_u^1 \le D_u^2$, because $\sqrt{xy} \le \sqrt{y}$. Now, set $v := uy$ and $z := \frac{xv}{u}$, so we obtain that, for all $x, y \in [0,1]$,

$$\begin{aligned}\lim_{u\to\infty} D_u^1 &= u^2 \int_0^1 \int_0^1 \sqrt{xy}\, e^{a(\theta)(2xy-y-x)u} dxdy \\ &= \lim_{u\to\infty} \sqrt{u} \int_0^u \int_0^v \frac{\sqrt{z}}{v}\, e^{-a(\theta)(v+\frac{zu}{v}-2z)} dzdv \\ &\ge \lim_{u\to\infty} \sqrt{u} \int_0^u \int_0^v \frac{\sqrt{z}}{v}\, e^{-a(\theta)(u-z)} dzdv \\ &= \lim_{u\to\infty} \frac{\int_0^u \int_0^v \frac{\sqrt{z}}{v} e^{a(\theta)z} dzdv}{\frac{e^{a(\theta)u}}{\sqrt{u}}}.\end{aligned}$$

At this point we apply once more l'Hôpital's rule in order to get

$$\begin{aligned}\lim_{u\to\infty} D_u^1 &\ge \lim_{u\to\infty} \frac{\int_0^u \frac{\sqrt{z}}{u} e^{a(\theta)z} dz}{\frac{\sqrt{u}a(\theta)e^{a(\theta)u}}{u} - \frac{e^{a(\theta)u}}{2u^{\frac{3}{2}}}} \\ &= \lim_{u\to\infty} \frac{\int_0^u \sqrt{z} e^{a(\theta)z} dz}{\sqrt{u}a(\theta)e^{a(\theta)u} - \frac{e^{a(\theta)u}}{2\sqrt{u}}}.\end{aligned}$$

Applying l'Hôpital's rule for another time, we get that our limit above is equal to

$$\begin{aligned}&\lim_{u\to\infty} \frac{\sqrt{u}e^{a(\theta)u}}{\frac{a(\theta)e^{a(\theta)u}}{2\sqrt{u}} + \sqrt{u}a(\theta)^2 e^{a(\theta)u} - \frac{a(\theta)e^{a(\theta)u}}{2\sqrt{u}} + \frac{e^{a(\theta)u}}{4u^{\frac{3}{2}}}} \\ &= \lim_{u\to\infty} \frac{\sqrt{u}}{\sqrt{u}a(\theta)^2 + \frac{1}{4u^{\frac{3}{2}}}} = \lim_{u\to\infty} \frac{1}{a(\theta)^2 + \frac{1}{4u^2}} = \frac{1}{a(\theta)^2}.\end{aligned}$$

Thus,

$$\frac{1}{a(\theta)^2} \le \lim_{u\to\infty} D_u^1 \le \lim_{u\to\infty} D_u^2 = \frac{1}{a(\theta)^2},$$

therefore $\lim_{u\to\infty} D_u = 0$ and then we end up with $\lim_{u\to\infty}(C_u - \tilde{C}_u) = 0$.

Step 3. Now we will prove the general case, Let us assume that either $s \neq 0$ or $t \neq 0$. By (3.3), we recall that

$$\mathbb{E}\left[|\Delta_{s,t} X_\varepsilon(s',t')|^2\right] = 2(I_1^\varepsilon + I_2^\varepsilon), \tag{3.8}$$

where

$$\begin{aligned}I_1^\varepsilon :=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2} e^{-a(\theta)\big(x_2y_2 - x_1y_1\big)} \\ &\times \cos\big(b(\theta)(x_2y_2 - x_1y_1)\big) dx_1dx_2dy_1dy_2\end{aligned}$$

and

$$\begin{aligned}I_2^\varepsilon :=& \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2} e^{-a(\theta)\big((y_2-y_1)x_1 + (x_2-x_1)y_1\big)} \\ &\times \cos\big(b(\theta)\big((y_2 - y_1)x_1 - (x_2 - x_1)y_1\big)\big) dx_1dx_2dy_1dy_2.\end{aligned}$$

Set

$$F^\varepsilon(s,t) := \varepsilon^2 K^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2, y_1\le y_2\}} dx_1dy_1dx_2dy_2,$$

where $f(x_1,y_1,x_2,y_2) = \sqrt{x_1y_1x_2y_2} e^{-a(\theta)(x_2y_2 - x_1y_1)} \cos\big(b(\theta)(x_2y_2 - x_1y_1)\big)$, and

$$G^\varepsilon(s,t) := \varepsilon^2 K^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} g(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2, y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2,$$

where $g(x_1,y_1,x_2,y_2) = \sqrt{x_1 y_1 x_2 y_2} e^{-a(\theta)(y_2-y_1)x_1 - a(\theta)(x_2-x_1)y_1} \cos\big(b(\theta)((y_2-y_1)x_1 - (x_2-x_1)y_1)\big)$. Moreover, we notice that

$$\begin{aligned}
\Delta_{s,t} F^\varepsilon(s',t') =& \varepsilon^2 K^2 \int_0^{\frac{t'}{\varepsilon}} \int_0^{\frac{s'}{\varepsilon}} \int_0^{\frac{t'}{\varepsilon}} \int_0^{\frac{s'}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2, y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2 \\
& - \varepsilon^2 K^2 \int_0^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2, y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2 \\
& - \varepsilon^2 K^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s'}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s'}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2, y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2 \\
& + \varepsilon^2 K^2 \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2, y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2.
\end{aligned}$$

Then,

$$\begin{aligned}
I_1^\varepsilon =& \Delta_{s,t} F^\varepsilon(s',t') - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2 \\
& - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} f(x_1,y_1,x_2,y_2) dx_1 dy_1 dx_2 dy_2 \\
& - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} f(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2\}} dx_1 dy_1 dx_2 dy_2 \\
:=& \Delta_{s,t} F^\varepsilon(s,t) - I_{11}^\varepsilon - I_{12}^\varepsilon - I_{13}^\varepsilon.
\end{aligned}$$

Later we will prove that I_{11}^ε, I_{12}^ε i I_{13}^ε converge to zero. But first, we observe that, as a consequence of the case where $s = t = 0$, we have that

$$\begin{aligned}
\lim_{\varepsilon\to 0} \Delta_{s,t} F^\varepsilon(s',t') =& \frac{a(\theta)^2 - b(\theta)^2}{\big(a(\theta)^2 + b(\theta)^2\big)^2} K^2 t's' - \frac{a(\theta)^2 - b(\theta)^2}{\big(a(\theta)^2 + b(\theta)^2\big)^2} K^2 t's \\
& - \frac{a(\theta)^2 - b(\theta)^2}{\big(a(\theta)^2 + b(\theta)^2\big)^2} K^2 ts' + \frac{a(\theta)^2 - b(\theta)^2}{\big(a(\theta)^2 + b(\theta)^2\big)^2} K^2 ts \\
=& \frac{a(\theta)^2 - b(\theta)^2}{\big(a(\theta)^2 + b(\theta)^2\big)^2} K^2 (s'-s)(t'-t).
\end{aligned}$$

Following analogue arguments we have that

$$\begin{aligned}
I_2^\varepsilon =& \Delta_{s,t} G^\varepsilon(s',t') - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} g(x_1,y_1,x_2,y_2) I_{\{y_1\le y_2\}} dx_1 dy_1 dx_2 dy_2 \\
& - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{s}{\varepsilon}} g(x_1,y_1,x_2,y_2) dx_1 dy_1 dx_2 dy_2 \\
& - \varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_0^{\frac{t}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} g(x_1,y_1,x_2,y_2) I_{\{x_1\le x_2\}} dx_1 dy_1 dx_2 dy_2 \\
:=& \Delta_{s,t} G^\varepsilon(s,t) - I_{21}^\varepsilon - I_{22}^\varepsilon - I_{23}^\varepsilon.
\end{aligned}$$

We will also prove that I_{21}^ε, I_{22}^ε and I_{23}^ε converge to zero. First, we observe that, thanks to the case where

$s = t = 0$, we have that

$$\lim_{\varepsilon\to 0}\Delta_{s,t}G^{\varepsilon}(s',t') = \frac{1}{a(\theta)^2+b(\theta)^2}K^2t's' - \frac{1}{a(\theta)^2+b(\theta)^2}K^2t's \\ - \frac{1}{a(\theta)^2+b(\theta)^2}K^2ts' + \frac{1}{a(\theta)^2+b(\theta)^2}K^2ts \\ = \frac{1}{a(\theta)^2+b(\theta)^2}K^2(s'-s)(t'-t).$$

Therefore,

$$\begin{aligned}\lim_{\varepsilon\to\infty}\mathbb{E}\left[|\Delta_{s,t}X_\varepsilon(s',t')|^2\right] &= \lim_{\varepsilon\to\infty}2(I_1^\varepsilon+I_2^\varepsilon)\\ &= 2\left(\frac{a(\theta)^2-b(\theta)^2}{\left(a(\theta)^2+b(\theta)^2\right)^2}K^2(s'-s)(t'-t)+\frac{1}{a(\theta)^2+b(\theta)^2}K^2(s'-s)(t'-t)\right)\\ &= 2K^2(s'-s)(t'-t)\left(\frac{a(\theta)^2-b(\theta)^2}{\left(a(\theta)^2+b(\theta)^2\right)^2}+\frac{1}{a(\theta)^2+b(\theta)^2}\right)\\ &= 2K^2(s'-s)(t'-t)\left(\frac{2a(\theta)^2}{\left(a(\theta)^2+b(\theta)^2\right)^2}\right) = 2(s'-s)(t'-t).\end{aligned}$$

Now, we will prove that I_{ji}^ε converges to zero, as ε tends to zero, for $i = 1,2,3$ and $j = 1,2$. Let us analyze this 6 integrals:

$$0 \le |I_{11}^\varepsilon| \le \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_0^{\frac{s}{\varepsilon}}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_1dy_1dx_2dy_2 := \tilde{I}_{11}^\varepsilon$$

$$0 \le |I_{12}^\varepsilon| \le \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}}\int_0^{\frac{s}{\varepsilon}}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_1dy_1dx_2dy_2 := \tilde{I}_{12}^\varepsilon$$

$$0 \le |I_{13}^\varepsilon| \le \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_1dy_1dx_2dy_2 := \tilde{I}_{13}^\varepsilon$$

$$0 \le |I_{21}^\varepsilon| \le \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_0^{\frac{s}{\varepsilon}}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_1dy_1dx_2dy_2 := \tilde{I}_{21}^\varepsilon$$

$$0 \le |I_{22}^\varepsilon| \le \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}}\int_0^{\frac{s}{\varepsilon}}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_1dy_1dx_2dy_2 := \tilde{I}_{22}^\varepsilon$$

$$0 \le |I_{23}^\varepsilon| \le \varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_0^{\frac{t}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_1dy_1dx_2dy_2 := \tilde{I}_{23}^\varepsilon.$$

Notice that $\tilde{I}_{1i}^\varepsilon \le \tilde{I}_{2i}^\varepsilon$ for all $i = 1,2,3$. So, it is enough to prove that $\lim_{\varepsilon\to 0}\tilde{I}_{2i}^\varepsilon = 0$. $\tilde{I}_{23}^\varepsilon$ is analogous to $\tilde{I}_{21}^\varepsilon$, since it is just a change of roles between x and y. Moreover, $\tilde{I}_{21}^\varepsilon$ and $\tilde{I}_{22}^\varepsilon$ are bounded by

$$\begin{aligned}&\varepsilon^2K^2\int_0^{\frac{t'}{\varepsilon}}\int_0^{\frac{s}{\varepsilon}}\int_0^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}g(x_1,y_1,x_2,y_2)dx_2dy_1dx_1dy_2\\ &=\varepsilon^2K^2\int_0^{\frac{t'}{\varepsilon}}\int_0^{\frac{s}{\varepsilon}}\int_0^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1y_1x_2y_2}e^{-a(\theta)(y_2-y_1)x_1-a(\theta)(x_2-x_1)y_1}dx_2dy_1dx_1dy_2\\ &=u^2K^2\int_0^{t'}\int_0^{s}\int_0^{y_2'}\int_s^{s'}\sqrt{x_1'y_1'x_2'y_2'}e^{-a(\theta)u(y_2'-y_1')x_1'-a(\theta)u(x_2'-x_1')y_1'}dx_2'dy_1'dx_1'dy_2',\end{aligned}$$

where we have used the changes of variables $x_i' = \varepsilon x_i$ and $y_i' = \varepsilon y_i$, $i = 1, 2$ with $u = \frac{1}{\varepsilon^2}$. For simplicity's sake, we will still use x_1, x_2, x_3 and x_4 instead of x_1', x_2', x_3' and x_4'. As $x_2 \leq s'$, we have that the above expression is bounded by

$$\begin{aligned} u^2K^2 \int_0^{t'} \int_0^{s} \int_0^{y_2} \int_s^{s'} & \sqrt{x_1 y_1 s' y_2} e^{-a(\theta)u(y_2-y_1)x_1 - a(\theta)u(x_2-x_1)y_1} dx_2 dy_1 dx_1 dy_2 \\ & \leq CuK^2 \int_0^{t'} \int_0^{s} \int_0^{y_2} \frac{\sqrt{x_1 y_2}}{\sqrt{y_1}} e^{-a(\theta)u(y_2-y_1)x_1 - a(\theta)u(s-x_1)y_1} dy_1 dx_1 dy_2. \end{aligned}$$

Here, we have integrated with respect of x_2. Moreover, if we bound y_2 by t', the last expression can be estimated by

$$\begin{aligned} CuK^2 & \int_0^{t'} \int_0^{s} \int_0^{y_2} \frac{\sqrt{x_1}}{\sqrt{y_1}} e^{-a(\theta)u(y_2-y_1)x_1 - a(\theta)u(s-x_1)y_1} dy_1 dx_1 dy_2 \\ & \leq CK^2 \int_0^{s} \int_0^{t'} \frac{1}{\sqrt{x_1 y_1}} e^{-a(\theta)u(s-x_1)y_1} dy_1 dx_1. \end{aligned}$$

Let $\{u_n\}_{n\geq 0}$ be a sequence such that $u_n \to \infty$ and $u_n \leq u_m$, for any $n \leq m$. We get that

$$f_n(x_1, y_1) := \frac{1}{\sqrt{x_1 y_1}} e^{-a(\theta)u_n(s-x_1)y_1}$$

is a monotone decreasing sequence and also its limit is 0, when n tends to infinity. We can also notice that

$$\begin{aligned} \int_0^s \int_0^{t'} f_1 dy_1 dx_1 &= \int_0^s \int_0^{t'} \frac{1}{\sqrt{x_1 y_1}} e^{-a(\theta)u_1(s-x_1)y_1} dy_1 dx_1 \leq \int_0^s \int_0^{t'} \frac{1}{\sqrt{x_1 y_1}} dy_1 dx_1 \\ &\leq 2 \int_0^s \frac{\sqrt{t'}}{\sqrt{x_1}} dx_1 = 4\sqrt{st'} < \infty. \end{aligned}$$

Thus, thanks to the Dominated convergence theorem, we have

$$\lim_{u\to\infty} CK^2 \int_0^s \int_0^{t'} \frac{1}{\sqrt{x_1 y_1}} e^{-a(\theta)u(s-x_1)y_1} dy_1 dx_1 = 0.$$

□

Lemma 3.4. *Let $(0,0) \leq (s,t) \leq (s',t') \leq (S,T)$. Then, there exists a sequence $\{C_\varepsilon\}_{\varepsilon>0}$ such that $\lim_{\varepsilon\to 0} C_\varepsilon = 4(s'-s)^2(t'-t)^2$ and*

$$\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t} X^\varepsilon(s',t')|^2 \,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] \leq C_\varepsilon.$$

Proof. We split the proof in 4 steps.

Step 1. Owing to the definition of the random field X^ε, first we can observe that

$$
\begin{aligned}
\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2|\mathcal{F}^\varepsilon_{s,T}\right] =&\mathbb{E}\left[\left|K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\, e^{i\theta L(x,y)}dxdy\right|^2\middle|\mathcal{F}^\varepsilon_{s,T}\right]\\
=&\mathbb{E}\left[\left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1y_1}\, e^{-i\theta L(x_1,y_1)}dx_1dy_1\right)\right.\\
&\times\left.\left(K\varepsilon\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_2y_2}\, e^{i\theta L(x_2,y_2)}dx_2dy_2\right)\middle|\mathcal{F}^\varepsilon_{s,T}\right]\\
=&K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\\
&\times\mathbb{E}\left[e^{i\theta\left(L(x_2,y_2)-L(x_1,y_1)\right)}\middle|\mathcal{F}^\varepsilon_{s,T}\right]dx_1dy_1dx_2dy_2.
\end{aligned}
$$

We must consider the 4 possible orders of two points in the plane. Thus, we have that

$$
\begin{aligned}
\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2|\mathcal{F}^\varepsilon_{s,T}\right] =&K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{y_2}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_1)-L(\frac{s}{\varepsilon},y_2)\right)}\\
&\times e^{-\Psi(\theta)(x_2-x_1)y_2}e^{-\Psi(-\theta)(y_1-y_2)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{x_2}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times e^{-\Psi(-\theta)(x_1-x_2)y_1}e^{-\Psi(\theta)(y_2-y_1)(x_2-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{y_2}^{\frac{t'}{\varepsilon}}\int_{x_2}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_1)-L(\frac{s}{\varepsilon},y_2)\right)}\\
&\times e^{-\Psi(-\theta)\left((x_1-x_2)y_1+(y_1-y_2)(x_2-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2.
\end{aligned}
$$

We apply Fubini theorem and we make some suitable changes of variable to get that the last expression is equal to

$$
\begin{aligned}
&K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times e^{-\Psi(\theta)(x_2-x_1)y_1}e^{-\Psi(-\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2,
\end{aligned}
$$

where we have applied some suitable changes of variable in order to have $x_1 \leq x_2$ and $y_1 \leq y_2$. Then,

$$
\begin{aligned}
\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] = &\,\mathbb{E}\Bigg[\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times\ e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2 \\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times e^{-\Psi(\theta)(x_2-x_1)y_1}e^{-\Psi(-\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2 \\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2 \\
&+K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times\ e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\Bigg)^2\Bigg] \\
:= &\,\mathbb{E}\left[\left(A_1^\varepsilon+A_2^\varepsilon+A_3^\varepsilon+A_4^\varepsilon\right)^2\right].
\end{aligned}
$$

If we develop the square of the last expression, we will have an expresion of the form

$$
\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] = \sum_{i,j=1}^{4}\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right].
$$

So,

$$
\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] = \sum_{i=1}^{4}\mathbb{E}\left[(A_i^\varepsilon)^2\right]+\sum_{i,j=1,i\neq j}^{4}\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right]. \tag{3.9}
$$

Next, we analize each $\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right]$ (this part may be a bit tedious). We apply Fubini theorem a couple of times to get:

$$\begin{aligned}
\mathbb{E}\left[(A_1^\varepsilon)^2\right] =&\mathbb{E}\left[\left(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.\right.\\
&\left.\left.\times\ e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\right)^2\right]\\
=&\mathbb{E}\left[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.\\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\left.dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\right]\\
=&K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}\tag{3.10}$$

$$\begin{aligned}
\mathbb{E}\left[(A_2^\varepsilon)^2\right] =&\mathbb{E}\left[\left(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.\right.\\
&\left.\left.\times\ e^{-\Psi(\theta)(x_2-x_1)y_1}e^{-\Psi(-\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\right)^2\right]\\
=&\mathbb{E}\left[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.\\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)}e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\left.dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\right]\\
=&K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)}e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}$$

$$
\begin{aligned}
\mathbb{E}\left[(A_3^\varepsilon)^2\right] =&\mathbb{E}\left[\left(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.\right.\\
&\left.\left.\times\ e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\right)^2\right]\\
=&\mathbb{E}\left[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.\\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)}e^{-\Psi(\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\left.\times\ dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\right]\\
=&K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)}e^{-\Psi(\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[(A_4^\varepsilon)^2\right] =&\mathbb{E}\left[\left(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.\right.\\
&\left.\left.\times\ e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\right)^2\right]\\
=&\mathbb{E}\left[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.\\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\left.\times\ dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\right]\\
=&K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4
\end{aligned}
$$

The following 6 expectations are the ones that appear twice in (3.9).

$$\begin{aligned}\mathbb{E}\left[A_1^\varepsilon A_2^\varepsilon\right] =&\mathbb{E}\Bigg[\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times\ e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\Bigg)\\
&\times\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times\ e^{-\Psi(\theta)(x_2-x_1)y_1}e^{-\Psi(-\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\Bigg)\Bigg]\\
=&\mathbb{E}\Bigg[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_3}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}e^{-\Psi(-\theta)(y_4-y_3)(x_3-\frac{s}{\varepsilon})}\\
&\times\ e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)}dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\Bigg]\\
=&K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_3}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]e^{-\Psi(-\theta)(y_4-y_3)(x_3-\frac{s}{\varepsilon})}\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)}dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[A_1^\varepsilon A_3^\varepsilon\right] =&\mathbb{E}\Bigg[\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times\ e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\Bigg)\\
&\times\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times\ e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\Bigg)\Bigg]\\
=&\mathbb{E}\Bigg[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_3}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}e^{-\Psi(-\theta)(x_4-x_3)y_3}\\
&\times\ dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\Bigg]\end{aligned}$$

$$
\begin{aligned}
=&K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_3} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right] \\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} e^{-\Psi(-\theta)(x_4-x_3)y_3} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[A_1^\varepsilon A_4^\varepsilon\right] =&\mathbb{E}\Bigg[\Bigg(K^2\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times\ e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} dx_1dy_1dx_2dy_2\Bigg) \\
&\times\ \Bigg(K^2\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times\ e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} dx_1dy_1dx_2dy_2\Bigg)\Bigg] \\
=&\mathbb{E}\Bigg[K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_4} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} \\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} e^{-\Psi(-\theta)\left((x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\times\ dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\Bigg] \\
=&K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_4} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4} \\
&\times \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right] \\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} e^{-\Psi(-\theta)\left((x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[A_2^\varepsilon A_3^\varepsilon\right] =&\mathbb{E}\Bigg[\Bigg(K^2\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} \\
&\times\ e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})} dx_1dy_1dx_2dy_2\Bigg)
\end{aligned}
$$

$$\times \left(K^2\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.$$

$$\left.\left. \times\ e^{-\Psi(-\theta)(x_2-x_1)y_1} e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})} dx_1dy_1dx_2dy_2 \right)\right]$$

$$=\mathbb{E}\left[K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_3} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.$$

$$\times\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_1+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}$$

$$\left.\times\ e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)} dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\right]$$

$$=K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_3} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}$$

$$\times\, \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right] e^{-\Psi(\theta)\left((x_2-x_1)y_1+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}$$

$$\times\, e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)} dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,$$

$$\mathbb{E}\left[A_2^\varepsilon A_4^\varepsilon\right] =\mathbb{E}\left[\left(K^2\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.\right.$$

$$\left.\times\ e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})} dx_1dy_1dx_2dy_2 \right)$$

$$\times \left(K^2\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\right.$$

$$\left.\left.\times\ e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} dx_1dy_1dx_2dy_2 \right)\right]$$

$$=\mathbb{E}\left[K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_4} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\right.$$

$$\times\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} e^{-\Psi(\theta)(x_2-x_1)y_1}$$

$$\left.\times\ e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\right]$$

$$=K^4\varepsilon^4 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{s}{\varepsilon}}^{x_4} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}$$

$$\times\, \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right] e^{-\Psi(\theta)(x_2-x_1)y_1}$$

$$\times\, e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,$$

$$
\begin{aligned}
\mathbb{E}\left[A_3^\varepsilon A_4^\varepsilon\right] =&\mathbb{E}\Bigg[\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times\, e^{-\Psi(-\theta)(x_2-x_1)y_1}e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}dx_1dy_1dx_2dy_2\Bigg)\\
&\times\Bigg(K^2\varepsilon^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}\\
&\times\, e^{-\Psi(-\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}dx_1dy_1dx_2dy_2\Bigg)\Bigg]\\
=&\mathbb{E}\Bigg[K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\, e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\\
&\times\, e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\Bigg]\\
=&K^4\varepsilon^4\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{s}{\varepsilon}}^{x_4}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\\
&\times\,\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\\
&\times\, e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})}e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\
&\times\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}
$$

Observe that in all the above integrals we have that $y_1 \leq y_2$ and $y_3 \leq y_4$. If we want to compute the expectations inside of them, we must consider all the orders of y_1, y_2, y_3, y_4, with the restrictions of $y_1 \leq y_2$ and $y_3 \leq y_4$. This leads us to 6 different possible orders, which we will divide in two groups:

(i) $y_1 \leq y_2 \leq y_3 \leq y_4$ and $y_3 \leq y_4 \leq y_1 \leq y_2$,

(ii) $y_1 \leq y_3 \leq y_2 \leq y_4$, $y_1 \leq y_3 \leq y_4 \leq y_2$, $y_3 \leq y_1 \leq y_4 \leq y_2$ and $y_3 \leq y_1 \leq y_2 \leq y_4$.

Then,

$$
\mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right] = \sum_{k=1}^{6} B_k^\varepsilon(i,j), \tag{3.11}
$$

for $i, j \in \{1,2,3,4\}$, where $B_1^\varepsilon(i,j)$ and $B_2^\varepsilon(i,j)$ correspond to (3.10) with the orders of (i), respectively, and $B_k^\varepsilon(i,j)$, $k = 3,4,5,6$, correspond to the above integral with orders of (ii), respectively. Hence,

$$
\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}_{s,T}^\varepsilon\right]\right)^2\right] = \sum_{i,j=1}^{4}\sum_{k=1}^{6} B_k^\varepsilon(i,j). \tag{3.12}
$$

Step 2. Now we will analyze the terms in (3.11) and after that we will go back to deal with (3.12). Let us start this analysis with the terms in $\mathbb{E}\left[(A_1^\varepsilon)^2\right]$. Recall that

$$
\mathbb{E}\left[(A_1^\varepsilon)^2\right] = \sum_{k=1}^{6} B_k^\varepsilon(1,1).
$$

For now, we will focus only on the terms $B_k^\varepsilon(1,1)$ with $k = 3,4,5,6$.

Let us compute $\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]$ for $B_3^\varepsilon(1,1)$, where $y_1 \leq y_3 \leq y_2 \leq y_4$:

$$\begin{aligned}&\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}I_{\{y_1\leq y_3\leq y_2\leq y_4\}}\right]\\ =&e^{-\Psi(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_2-y_3)\frac{s}{\varepsilon}\right)}.\end{aligned} \tag{3.13}$$

Plugging the expectation inside the integral, we get that

$$\begin{aligned}B_3^\varepsilon(1,1)=&K^4\varepsilon^4\int_D\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,I\{x_1\leq x_2\}\,I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_3\leq y_2\leq y_4\}}\\ &\times e^{-\Psi(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_2-y_3)\frac{s}{\varepsilon}\right)}\\ &\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,\end{aligned}$$

where $D:=[\frac{s}{\varepsilon},\frac{s'}{\varepsilon}]^4\times[\frac{t}{\varepsilon},\frac{t'}{\varepsilon}]^4$. We recall that $a(\theta)$ is the real part of $\Psi(\theta)$ and that $\frac{t}{\varepsilon}\leq y_i$ for $i=1,2,3,4$. Hence, by shifting the modulus inside the integral, we have

$$\begin{aligned}|B_3^\varepsilon(1,1)|\leq&K^4\varepsilon^4\int_D\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_3\leq y_2\leq y_4\}}\\ &\times e^{-a(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_2-y_3)\frac{s}{\varepsilon}\right)}\\ &\times e^{-a(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\\ \leq&K^4\varepsilon^4\int_D\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_3\leq y_2\leq y_4\}}\\ &\times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)}\\ &\times e^{-a(\theta)\left(2(y_2-y_3)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

Applying some suitable changes of variable, we have that

$$\begin{aligned}|B_3^\varepsilon(1,1)|\leq&K^4\varepsilon^4\int_D\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_2\leq y_3\leq y_4\}}\\ &\times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_1)(x_1-\frac{s}{\varepsilon})\right)}\\ &\times e^{-a(\theta)\left(2(y_3-y_2)\frac{s}{\varepsilon}+(y_4-y_2)(x_3-\frac{s}{\varepsilon})\right)}\\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4\\ \leq&K^4\varepsilon^4\int_D\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_2\leq y_3\leq y_4\}}\\ &\times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_1)(x_1-\frac{s}{\varepsilon})\right)}\\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

As $y_1\leq y_2$, notice that $(y_3-y_1)(x_1-\frac{s}{\varepsilon})\geq(y_3-y_2)(x_1-\frac{s}{\varepsilon})$. Thus,

$$\begin{aligned}|B_3^\varepsilon(1,1)|\leq&K^4\varepsilon^4\int_D\sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\,I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_2\leq y_3\leq y_4\}}\\ &\times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)}\\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

Step 3. Here we deal with $|B^\varepsilon_k(i,j)|$, which are defined in (3.11). Set

$$\begin{aligned}\beta^\varepsilon :=& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4. \end{aligned} \tag{3.14}$$

Let us deal with $\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]$ for $B^\varepsilon_4(1,1)$, where $y_1\le y_3\le y_4\le y_2$:

$$\begin{aligned}&\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}I_{\{y_1\le y_3\le y_2\le y_4\}}\right] \\ =& e^{-\Psi(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_4-y_3)\frac{s}{\varepsilon}\right)}. \end{aligned} \tag{3.15}$$

So,

$$\begin{aligned}B^\varepsilon_4(1,1) =& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_3\le y_4\le y_2\}} \\ & \times e^{-\Psi(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_4-y_3)\frac{s}{\varepsilon}\right)} \\ & \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4. \end{aligned}$$

Then,

$$\begin{aligned}|B^\varepsilon_4(1,1)| \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_3\le y_4\le y_2\}} \\ & \times e^{-a(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_4-y_3)\frac{s}{\varepsilon}\right)} \\ & \times e^{-a(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_3\le y_4\le y_2\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left(2(y_4-y_3)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4. \end{aligned}$$

Again, applying suitable changes of variable, we can rewrite the latter expression as

$$\begin{aligned}& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left(2(y_3-y_2)\frac{s}{\varepsilon}+(y_3-y_2)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4. \end{aligned}$$

We notice that if $y_1 \le y_2 \le y_3 \le y_4$ then $(y_4 - y_1)(x_1 - \frac{s}{\varepsilon}) \ge (y_3 - y_2)(x_1 - \frac{s}{\varepsilon})$. Thus,

$$\begin{aligned}|B_4^\varepsilon(1,1)| \le & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ = & \beta^\varepsilon,\end{aligned}$$

where β^ε has been defined in (3.14).
Next, we compute $\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]$ for $B_5^\varepsilon(1,1)$, where $y_3 \le y_1 \le y_4 \le y_2$:

$$\begin{aligned}&\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} I_{\{y_3\le y_1\le y_4\le y_2\}}\right] \\ &\quad = e^{-\Psi(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_4-y_1)\frac{s}{\varepsilon}\right)}. \qquad (3.16)\end{aligned}$$

We have that

$$\begin{aligned}B_5^\varepsilon(1,1) = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_1\le y_4\le y_2\}} \\ & \times e^{-\Psi(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_4-y_1)\frac{s}{\varepsilon}\right)} \\ & \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

We can bound the modulus of $B_5^\varepsilon(1,1)$ as follows:

$$\begin{aligned}|B_5^\varepsilon(1,1)| \le & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_1\le y_4\le y_2\}} \\ & \times e^{-a(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_4-y_1)\frac{s}{\varepsilon}\right)} \\ & \times e^{-a(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \le & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_1\le y_4\le y_2\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left(2(y_4-y_1)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

By doing the proper changes of variable, we have that the last expression equals

$$\begin{aligned}& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \quad \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \quad \times e^{-a(\theta)\left(2(y_3-y_2)\frac{s}{\varepsilon}+(y_3-y_1)(x_3-\frac{s}{\varepsilon})\right)} \\ & \quad \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4. \\ & \le K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ & \quad \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \quad \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,\end{aligned}$$

We have $y_3 \leq y_4$, so we have that $(y_4 - y_2)(x_1 - \frac{s}{\varepsilon}) \geq (y_3 - y_2)(x_1 - \frac{s}{\varepsilon})$. Then,

$$
\begin{aligned}
|B_5^\varepsilon(1,1)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\
& \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
= & \beta^\varepsilon
\end{aligned}
$$

where β^ε has been defined in (3.14).

Let us compute $\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]$ for $B_6^\varepsilon(1,1)$, where $y_3 \leq y_1 \leq y_2 \leq y_4$:

$$
\begin{aligned}
& \mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} I_{\{y_3\leq y_1\leq y_2\leq y_4\}}\right] \\
& \quad = e^{-\Psi(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_2-y_1)\frac{s}{\varepsilon}\right)}. \qquad (3.17)
\end{aligned}
$$

Thus,

$$
\begin{aligned}
B_6^\varepsilon(1,1) = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_3\leq y_1\leq y_2\leq y_4\}} \\
& \times e^{-\Psi(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_2-y_1)\frac{s}{\varepsilon}\right)} \\
& \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}
$$

Now, we bound the modulus of $B_6^\varepsilon(1,1)$ in the following way:

$$
\begin{aligned}
|B_6^\varepsilon(1,1)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_3\leq y_1\leq y_2\leq y_4\}} \\
& \times e^{-a(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_2-y_1)\frac{s}{\varepsilon}\right)} \\
& \times e^{-a(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_3\leq y_1\leq y_2\leq y_4\}} \\
& \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times e^{-a(\theta)\left(2(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}
$$

Again, we do the proper variable changes to obtain that the latter integral is equal to

$$
\begin{aligned}
& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\
& \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times e^{-a(\theta)\left(2(y_3-y_2)\frac{s}{\varepsilon}+(y_4-y_1)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\
& \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
= & \beta^\varepsilon,
\end{aligned}
$$

where β^ε has been defined in (3.14).
Then, we have that for $k = 3, 4, 5, 6$

$$|B_k^\varepsilon(1,1)| \leq \beta^\varepsilon.$$

We will now do the analysis for the terms in $\mathbb{E}\left[(A_2^\varepsilon)^2\right]$. Recall that

$$\mathbb{E}\left[(A_2^\varepsilon)^2\right] = \sum_{k=1}^{6} B_k^\varepsilon(2,2).$$

For now, we will focus on the terms $B_k^\varepsilon(2,2)$ with $k = 3, 4, 5, 6$.
Due to (3.13), we have that

$$\begin{aligned}
B_3^\varepsilon(2,2) =& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_2\leq y_4\}} \\
&\times e^{-\Psi(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_2-y_3)\frac{s}{\varepsilon}\right)} \\
&\times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)} e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}$$

We recall that $a(\theta) = a(-\theta)$. Thus

$$\begin{aligned}
|B_3^\varepsilon(2,2|) \leq& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_2\leq y_4\}} \\
&\times e^{-a(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_2-y_3)\frac{s}{\varepsilon}\right)} \\
&\times e^{-a(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\leq& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_2\leq y_4\}} \\
&\times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_2)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\
&\times e^{-a(\theta)\left(2(y_2-y_3)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}$$

By doing suitable changes of variable we get that the latter expression equals

$$\begin{aligned}
K^4\varepsilon^4 &\int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\
&\times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\
&\times e^{-a(\theta)\left(2(y_3-y_2)\frac{s}{\varepsilon}+(y_4-y_2)(x_3-\frac{s}{\varepsilon})\right)} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\leq& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\
&\times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\
&\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}$$

Since $y_1 \leq y_2$, we observe that $(y_3 - y_1)(x_1 - \frac{s}{\varepsilon}) \geq (y_3 - y_2)(x_1 - \frac{s}{\varepsilon})$. Hence

$$\begin{aligned}|B_3^\varepsilon(2,2)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ = & \beta^\varepsilon.\end{aligned}$$

Let us analize the case of $B_4^\varepsilon(2,2)$. By (3.15) we have

$$\begin{aligned}B_4^\varepsilon(2,2) = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_3y_4y_2}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_4\leq y_2\}} \\ & \times e^{-\Psi(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_4-y_3)\frac{s}{\varepsilon}\right)} \\ & \times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)} e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

Then we have

$$\begin{aligned}|B_4^\varepsilon(2,2)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_4\leq y_2\}} \\ & \times e^{-a(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+2(y_4-y_3)\frac{s}{\varepsilon}\right)} \\ & \times e^{-a(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_4\leq y_2\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_2-y_4)\frac{s}{\varepsilon}+(y_3-y_1)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left((2(y_4-y_3)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

Making the proper variable changes we get that the last integral equals

$$\begin{aligned} & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left((2(y_3-y_2)\frac{s}{\varepsilon}+(y_3-y_2)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

We notice that, if $y_1 \le y_2 \le y_3 \le y_4$, then $(y_4 - y_1)(x_1 - \frac{s}{\varepsilon}) \ge (y_3 - y_2)(x_1 - \frac{s}{\varepsilon})$. Thus,

$$\begin{aligned}|B_4^\varepsilon(2,2)| \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ &\times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ =&\beta^\varepsilon.\end{aligned}$$

We continue with the $B_5^\varepsilon(2,2)$ case, where thanks to (3.16) we have

$$\begin{aligned}B_5^\varepsilon(2,2) =& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_3y_1y_4y_2}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_3\le y_4\le y_2\}} \\ &\times e^{-\Psi(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_4-y_1)\frac{s}{\varepsilon}\right)} \\ &\times e^{-\Psi(\theta)\Big((x_2-x_1)y_1+(x_4-x_3)y_3\Big)} e^{-\Psi(-\theta)\Big((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\Big)} \\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

So,

$$\begin{aligned}|B_5^\varepsilon(2,2)| \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_1\le y_4\le y_2\}} \\ &\times e^{-a(\theta)\left((y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_4-y_1)\frac{s}{\varepsilon}\right)} \\ &\times e^{-a(\theta)\Big((x_2-x_1)y_1+(x_4-x_3)y_3+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\Big)} \\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_1\le y_4\le y_2\}} \\ &\times e^{-a(\theta)\Big((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_2-y_4)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\Big)} \\ &\times e^{-a(\theta)\left((2(y_4-y_1)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

We apply some changes of variable and we get that the last expression is equal to

$$\begin{aligned}& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ &\times e^{-a(\theta)\Big((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_2)(x_1-\frac{s}{\varepsilon})\Big)} \\ &\times e^{-a(\theta)\left((2(y_3-y_2)\frac{s}{\varepsilon}+(y_3-y_1)(x_3-\frac{s}{\varepsilon})\right)} \\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\ &\times e^{-a(\theta)\Big((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_2)(x_1-\frac{s}{\varepsilon})\Big)} \\ &\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

We notice that $(y_4-y_2)(x_1-\frac{s}{\varepsilon}) \geq (y_3-y_2)(x_1-\frac{s}{\varepsilon})$, since $y_3 \leq y_4$. Thus,

$$\begin{aligned}|B_5^\varepsilon(2,2)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ = & \beta^\varepsilon.\end{aligned}$$

Let us compute the same for $B_6^\varepsilon(2,2)$. Thanks to (3.17) we have

$$\begin{aligned}B_6^\varepsilon(2,2) = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_3y_1y_4y_2}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_3\leq y_4\leq y_2\}} \\ & \times e^{-\Psi(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_2-y_1)\frac{s}{\varepsilon}\right)} \\ & \times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)} e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

Then we can bound the modulus of $B_6^\varepsilon(2,2)$ in the following way:

$$\begin{aligned}|B_6^\varepsilon(2,2)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_3\leq y_1\leq y_2\leq y_4\}} \\ & \times e^{-a(\theta)\left((y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+2(y_2-y_1)\frac{s}{\varepsilon}\right)} \\ & \times e^{-a(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_3\leq y_1\leq y_2\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_2)\frac{s}{\varepsilon}+(y_1-y_3)\frac{s}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left((2(y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.\end{aligned}$$

We apply some changes of variable and we get that the last integral equals

$$\begin{aligned}& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times e^{-a(\theta)\left((2(y_3-y_2)\frac{s}{\varepsilon}+(y_4-y_1)(x_3-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}} I_{\{x_3\leq x_4\}} I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\ & \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\ & \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\ = & \beta^\varepsilon.\end{aligned}$$

Therefore, we have that, for $k = 3, 4, 5, 6$,

$$|B_k^\varepsilon(2,2)| \leq \beta^\varepsilon.$$

We have made ourselves a clear idea of the forms of $B_k^\varepsilon(i,j)$ and for our own convenience, we can show in a general way that

$$|B_k^\varepsilon(i,j)| \leq \beta^\varepsilon,$$

for $k = 3,4,5,6$ and $i,j = 1,2,3,4$.

Notice that $\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]$ appears in all $B_k^\varepsilon(i,j)$, no matter the value of k, i or j, for $k = 1,2,3,4,5,6$ and $i,j = 1,2,3,4$. In particular, its values are the same ones given by (3.13), (3.15), (3.16) and (3.17) for each of the orders in (ii), respectively. We have that

$$
\begin{aligned}
|B_k^\varepsilon(i,j)| \leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{O_k} \\
& \times \left|\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\right| \\
& \times e^{-a(\theta)\left((x_2-x_1)y_r+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_s+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\leq & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{O_k} \\
& \times \left|\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)}e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]\right| \\
& \times e^{-a(\theta)\left((x_2-x_1)\frac{t}{\varepsilon}+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)\frac{t}{\varepsilon}+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

for $k = 3,4,5,6$, $i = 1,2$, $j = 3,4$ and where O_k is the k-order given in (ii). We notice that the exponential function that appears in the third line of the second term is the same than the one appearing in all B_k^ε after shifting the modulus inside the integral. So we can infer that

$$
|B_k^\varepsilon(i,j)| \leq \beta^\varepsilon.
$$

for $k = 3,4,5,6$ and $i,j = 1,2,3,4$.

We want to verify that $\lim_{\varepsilon\to 0}\beta^\varepsilon = 0$. Recall that $s,t > 0$. Hence,

$$
\begin{aligned}
\beta^\varepsilon = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\; I_{\{x_1\leq x_2\}}I_{\{x_3\leq x_4\}}I_{\{y_1\leq y_2\leq y_3\leq y_4\}} \\
& \times e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\leq & K^4(t's')^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{x_3}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_3}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2} e^{-a(\theta)\left((x_4-x_3)\frac{t}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}\right)} \\
& \times e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dx_4dy_3dx_3dy_4 \\
= & K^4\frac{(t's')^2\varepsilon}{a(\theta)t} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_4}\int_{\frac{t}{\varepsilon}}^{y_3}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2} e^{-a(\theta)(x_2-x_1)\frac{t}{\varepsilon}}\left(1-e^{-a(\theta)(\frac{s'}{\varepsilon}-x_3)\frac{t}{\varepsilon}}\right) \\
& \times e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \\
& \times dx_1dy_1dx_2dy_2dy_3dx_3dy_4,
\end{aligned}
$$

where at first place we bounded the square roots by the upper limits of x_i and y_i, and then we integrated with respect of x_4. Now, we bound the parenthesis inside the integral by 1 and then integrate with respect to x_3 to get that the latter expression is bounded by

$$K^4 \frac{(t's')^2(s'-s)}{a(\theta)t} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{x_1}^{\frac{s'}{\varepsilon}} e^{-a(\theta)(x_2-x_1)\frac{t}{\varepsilon}}$$
$$\times e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} dx_2dy_1dx_1dy_2dy_3dy_4$$
$$=K^4 \frac{(t's')^2(s'-s)\varepsilon}{a(\theta)^2t} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_4} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \left(1-e^{-a(\theta)(\frac{s'}{\varepsilon}-x_1)\frac{t}{\varepsilon}}\right)$$
$$\times e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} dy_1dx_1dy_2dy_3dy_4.$$

Here, we have integrated with respect of x_2. Let us denote $K^4\frac{(t's')^2(s'-s)}{a(\theta)^2t}$ by C, and from now we will make a notation abuse by denoting all the constant terms C, no matter how they change through the integration process.
So, if we apply to the parenthesis the same bound as before, we have that

$$\beta^\varepsilon \le C\varepsilon \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{y_3}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_2-y_1)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} dy_1dy_4dy_2dy_3dx_1.$$

By integrating with respect of y_1, we get

$$\beta^\varepsilon \le C\varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{y_3}^{\frac{t'}{\varepsilon}} e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} \left(1-e^{-a(\theta)(y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}}\right) dy_4dy_2dy_3dx_1$$
$$\le C\varepsilon^2 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} \int_{y_3}^{\frac{t'}{\varepsilon}} e^{-a(\theta)\left((y_4-y_3)\frac{s}{\varepsilon}+(y_3-y_2)(x_1-\frac{s}{\varepsilon})\right)} dy_4dy_2dy_3dx_1.$$

At this point, we integrate with respect of y_4 and then we will make the change of variable $x = x_1 - \frac{s}{\varepsilon}$, thus

$$\beta^\varepsilon \le C\varepsilon^3 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)(x_1-\frac{s}{\varepsilon})} \left(1-e^{-a(\theta)(\frac{t'}{\varepsilon}-y_3)\frac{s}{\varepsilon}}\right) dy_2dy_3dx_1$$
$$\le C\varepsilon^3 \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)(x_1-\frac{s}{\varepsilon})} dy_2dy_3dx_1$$
$$= C\varepsilon^3 \int_0^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x} dy_2dy_3dx$$
$$= C\varepsilon^3 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x} dy_2dy_3dx$$
$$+ C\varepsilon^3 \int_0^{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x} dy_2dy_3dx.$$

Let us tackle first, the second term of the sum:

$$C\varepsilon^3 \int_0^{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x} dy_2dy_3dx \le C\varepsilon^3 \int_0^{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} dy_2dy_3dx$$
$$\le C\varepsilon^3 \int_0^{\varepsilon} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} y_3\, dy_3dx$$
$$= C\varepsilon \int_0^{\varepsilon} dx = C\varepsilon^2.$$

It is clear that it vanishes as ε tends to zero.
For the first term, we have that

$$\begin{aligned}
&C\varepsilon^3 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_3} e^{-a(\theta)(y_3-y_2)x}\, dy_2 dy_3 dx \\
&= C\varepsilon^3 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} e^{-a(\theta)y_3 x} \int_{\frac{t}{\varepsilon}}^{y_3} e^{a(\theta)(y_2)x}\, dy_2 dy_3 dx \\
&= C\varepsilon^3 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{x}\left(1-e^{-a(\theta)(y_3-\frac{t}{\varepsilon})x}\right) dy_3 dx \\
&\leq C\varepsilon^3 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \frac{1}{x} dy_3 dx \leq\ C\varepsilon^2 \int_{\varepsilon}^{\frac{s'-s}{\varepsilon}} \frac{1}{x}\, dx \\
&= C\varepsilon^2 \left(\ln(s'-s) - 2\ln(\varepsilon)\right) =\ C\varepsilon^2 \ln(s'-s) - 2C\varepsilon^2 \ln(\varepsilon).
\end{aligned}$$

And it also vanishes as ε tends to zero, since we notice that

$$\lim_{x\to 0} x\ln(x) = \lim_{x\to 0} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x\to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x\to 0} -x = 0,$$

where we have applied l'Hôpital's rule. Hence, $\lim_{\varepsilon\to 0} \beta^{\varepsilon} = 0$.

We can infer that

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^{\varepsilon}(s',t')|^2\,|\mathcal{F}^{\varepsilon}_{s,T}\right]\right)^2\right] &= \sum_{i,j=1}^{4}\sum_{k=1}^{6} B_k^{\varepsilon}(i,j) \\
&= \sum_{i,j=1}^{4}\sum_{k=1}^{2} B_k^{\varepsilon}(i,j) + \rho_{\varepsilon}, \qquad (3.18)
\end{aligned}$$

where we recall that $B_1^{\varepsilon}(i,j)$ and $B_2^{\varepsilon}(i,j)$ are the terms in the decomposition of $\mathbb{E}\left[A_i^{\varepsilon}A_j^{\varepsilon}\right]$ with the orders of (i), respectively, and $\lim_{\varepsilon\to 0}\rho_{\varepsilon} = 0$.

Let us tackle $\sum_{k=1}^{2} B_k^{\varepsilon}(i,j)$, but in order to do so we will first compute

$$\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)}\right]$$

in $B_1^{\varepsilon}(i,j)$, where $y_1 \leq y_2 \leq y_3 \leq y_4$, and also we must compute it for $B_2^{\varepsilon}(i,j)$, where $y_3 \leq y_4 \leq y_1 \leq y_2$. Thus,

$$\begin{aligned}
&\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} I_{\{y_1\leq y_2\leq y_3\leq y_4\}}\right] \\
&\quad = e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} \qquad (3.19)
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[e^{i\theta\left(L(\frac{s}{\varepsilon},y_2)-L(\frac{s}{\varepsilon},y_1)\right)} e^{i\theta\left(L(\frac{s}{\varepsilon},y_4)-L(\frac{s}{\varepsilon},y_3)\right)} I_{\{y_3\leq y_4\leq y_1\leq y_2\}}\right] \\
&\quad = e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)}. \qquad (3.20)
\end{aligned}$$

We notice that

$$\sum_{k=1}^{2} B_k^{\varepsilon}(1,1) = K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}}$$
$$\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)}$$
$$\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}$$
$$\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4$$
$$+ K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}}$$
$$\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)}$$
$$\times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}$$
$$\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4$$
$$= K^4\varepsilon^2 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}\}}$$
$$\times e^{-\Psi(\theta)((x_2-x_1)y_2+(x_4-x_3)y_4+(y_2-y_1)x_1+(y_4-y_3)x_3)}$$
$$\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,$$

where we recall that $D := [\frac{s}{\varepsilon}, \frac{s'}{\varepsilon}]^4 \times [\frac{t}{\varepsilon}, \frac{t'}{\varepsilon}]^4$ and that

$$I_{\{y_1\le y_2\le y_3\le y_4\}} + I_{\{y_3\le y_4\le y_1\le y_2\}} = I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}},$$

since they are indicator functions of disjoint sets. Observing that

$$I_{\{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}\}} \le I_{\{y_1\le y_2\}} I_{\{y_3\le y_4\}},$$

we get that

$$\sum_{k=1}^{2} B_k^{\varepsilon}(1,1) \le K^4\varepsilon^2 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\}} I_{\{y_3\le y_4\}}$$
$$\times e^{-\Psi(\theta)((x_2-x_1)y_2+(x_4-x_3)y_4+(y_2-y_1)x_1+(y_4-y_3)x_3)}$$
$$\times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.$$

Let us deal with the rest of the terms $\sum_{k=1}^{2} B_k^{\varepsilon}(i,j)$ by using the same arguments. We observe that

$$\sum_{k=1}^{2} B_k^{\varepsilon}(2,2) = K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}}$$
$$\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}$$
$$\times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4$$
$$+ K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}}$$
$$\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}$$
$$\times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4$$
$$\le K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}}$$
$$\times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_3\right)} e^{-\Psi(-\theta)\left((y_2-y_1)x_1+(y_4-y_3)x_3\right)}$$
$$\times e^{-2ib(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.$$

In the same way, we have

$$\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(3,3) &= K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
&\quad \times e^{-\Psi(\theta)\big((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\big)} e^{-\Psi(\theta)\Big((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\Big)} \\
&\quad \times e^{-\Psi(-\theta)\Big((x_2-x_1)y_1+(x_4-x_3)y_3\Big)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\quad +\ K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(\theta)\big((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\big)} e^{-\Psi(\theta)\Big((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\Big)} \\
&\quad \times e^{-\Psi(-\theta)\Big((x_2-x_1)y_1+(x_4-x_3)y_3\Big)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\le K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(-\theta)\Big((x_2-x_1)y_1+(x_4-x_3)y_3\Big)} e^{-\Psi(\theta)\Big((y_2-y_1)x_1+(y_4-y_3)x_3\Big)} \\
&\quad \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}$$

We list the rest of them:

$$\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(4,4) &= K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
&\quad \times e^{-\Psi(\theta)\big((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\big)} \\
&\quad \times e^{-\Psi(-\theta)\Big((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\Big)} \\
&\quad \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\quad +\ K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(\theta)\big((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\big)} \\
&\quad \times e^{-\Psi(-\theta)\Big((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\Big)} \\
&\quad \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\le K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(-\theta)\Big((x_2-x_1)y_2+(y_2-y_1)x_1+(x_4-x_3)y_4+(y_4-y_3)x_3\Big)} \\
&\quad \times e^{-2ib(\theta)\big((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\big)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}$$

$$
\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(1,2) = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
& \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(-\theta)(y_4-y_3)(x_3-\frac{s}{\varepsilon})} \\
& \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
& + K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
& \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(-\theta)(y_4-y_3)(x_3-\frac{s}{\varepsilon})} \\
& \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\le & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
& \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)x_1+(x_4-x_3)y_3\right)} e^{-\Psi(-\theta)(y_4-y_3)x_3} \\
& \times e^{-2ib(\theta)(y_4-y_3)\frac{s}{\varepsilon}}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(1,3) = & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
& \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times e^{-\Psi(-\theta)(x_4-x_3)y_3}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
& + K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
& \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
& \times e^{-\Psi(-\theta)(x_4-x_3)y_3}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\le & K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
& \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)x_1+(y_4-y_3)x_3\right)} e^{-\Psi(-\theta)(x_4-x_3)y_3} \\
& \times dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(1,4) &= K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
&\quad \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\
&\quad \times e^{-\Psi(-\theta)\left((x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\quad + K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)(x_1-\frac{s}{\varepsilon})\right)} \\
&\quad \times e^{-\Psi(-\theta)\left((x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\le K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(\theta)\left((x_2-x_1)y_2+(y_2-y_1)x_1\right)} e^{-\Psi(-\theta)\left((x_4-x_3)y_4+(y_4-y_3)x_3\right)} \\
&\quad \times e^{-2ib(\theta)(y_4-y_3)\frac{s}{\varepsilon}} dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(2,3) &= K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
&\quad \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_1+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\quad \times e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\quad + K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)\left((x_2-x_1)y_1+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)} \\
&\quad \times e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_3\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&\le K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
&\quad \times e^{-\Psi(\theta)\left((x_2-x_1)y_1+(y_4-y_3)x_3\right)} e^{-\Psi(-\theta)\left((y_2-y_1)x_1+(x_4-x_3)y_3\right)} \\
&\quad \times e^{-2ib(\theta)(y_2-y_1)\frac{s}{\varepsilon}}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(2,4) =& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
&\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)(x_2-x_1)y_1} \\
&\times e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&+\; K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
&\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)(x_2-x_1)y_1} \\
&\times e^{-\Psi(-\theta)\left((y_2-y_1)(x_1-\frac{s}{\varepsilon})+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
&\times e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)\left((y_2-y_1)x_1+(x_4-x_3)y_4+(y_4-y_3)x_3\right)} \\
&\times e^{-2ib(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4,
\end{aligned}
$$

$$
\begin{aligned}
\sum_{k=1}^{2} B_k^{\varepsilon}(3,4) =& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}} \\
&\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})} \\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
&+\; K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_3\le y_4\le y_1\le y_2\}} \\
&\times e^{-\Psi(\theta)\left((y_2-y_1)\frac{s}{\varepsilon}+(y_4-y_3)\frac{s}{\varepsilon}\right)} e^{-\Psi(\theta)(y_2-y_1)(x_1-\frac{s}{\varepsilon})} \\
&\times e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_4+(y_4-y_3)(x_3-\frac{s}{\varepsilon})\right)}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4 \\
\le& K^4\varepsilon^4 \int_D \sqrt{x_1x_2x_3x_4}\sqrt{y_1y_2y_3y_4}\, I_{\{x_1\le x_2\}} I_{\{x_3\le x_4\}} I_{\{y_1\le y_2\le y_3\le y_4\}\cup\{y_3\le y_4\le y_1\le y_2\}} \\
&\times e^{-\Psi(\theta)(y_2-y_1)x_1} e^{-\Psi(-\theta)\left((x_2-x_1)y_1+(x_4-x_3)y_4+(y_4-y_3)x_3\right)} \\
&\times e^{-2ib(\theta)(y_4-y_3)\frac{s}{\varepsilon}}\, dx_1dy_1dx_2dy_2dx_3dy_3dx_4dy_4.
\end{aligned}
$$

Step4. We recall, from (3.12), that

$$\begin{aligned}
\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] &= \sum_{i=1}^4 \mathbb{E}\left[(A_i^\varepsilon)^2\right] + \sum_{i,j=1,i\neq j}^4 \mathbb{E}\left[A_i^\varepsilon A_j^\varepsilon\right] \\
&= \sum_{i=1}^4 \mathbb{E}\left[(A_i^\varepsilon)^2\right] + 2\mathbb{E}\left[A_1^\varepsilon A_2^\varepsilon\right] + 2\mathbb{E}\left[A_1^\varepsilon A_3^\varepsilon\right] + 2\mathbb{E}\left[A_1^\varepsilon A_4^\varepsilon\right] \\
&\quad + 2\mathbb{E}\left[A_2^\varepsilon A_3^\varepsilon\right] + 2\mathbb{E}\left[A_2^\varepsilon A_4^\varepsilon\right] + 2\mathbb{E}\left[A_3^\varepsilon A_4^\varepsilon\right] \\
&= \sum_{k=1}^2 B_k^\varepsilon(1,1) + \sum_{k=1}^2 B_k^\varepsilon(2,2) + \sum_{k=1}^2 B_k^\varepsilon(3,3) + \sum_{k=1}^2 B_k^\varepsilon(4,4) \\
&\quad + 2\sum_{k=1}^2 B_k^\varepsilon(1,2) + 2\sum_{k=1}^2 B_k^\varepsilon(1,3) + 2\sum_{k=1}^2 B_k^\varepsilon(1,4) \\
&\quad + 2\sum_{k=1}^2 B_k^\varepsilon(2,3) + 2\sum_{k=1}^2 B_k^\varepsilon(2,4) + 2\sum_{k=1}^2 B_k^\varepsilon(3,4) + \rho_\varepsilon,
\end{aligned}$$

where ρ_ε is defined in (3.18). And gathering all the above resulting bounds together, we end up with

$$\mathbb{E}\left[\left(\mathbb{E}\left[|\Delta_{s,t}X^\varepsilon(s',t')|^2\,|\mathcal{F}^\varepsilon_{s,T}\right]\right)^2\right] \leq \bar{\Theta}_\varepsilon^2 + \rho_\varepsilon, \tag{3.21}$$

where

$$\begin{aligned}
\bar{\Theta}_\varepsilon = K^2\varepsilon^2 &\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{-\Psi(\theta)((x_2-x_1)y_2+(y_2-y_1)x_1)} dx_1dx_2dy_1dy_2 \\
+ K^2\varepsilon^2 &\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{-\Psi(\theta)(x_2-x_1)y_1} e^{-\Psi(-\theta)(y_2-y_1)x_1} e^{-2ib(\theta)(y_2-y_1)} dx_1dx_2dy_1dy_2 \\
+ K^2\varepsilon^2 &\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{-\Psi(-\theta)(x_2-x_1)y_1} e^{-\Psi(\theta)(y_2-y_1)x_1} dx_1dx_2dy_1dy_2 \\
+ K^2\varepsilon^2 &\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1x_2y_1y_2}\, e^{-\Psi(-\theta)((x_2-x_1)y_2+(y_2-y_1)x_1)} e^{-2ib(\theta)(y_2-y_1)} dx_1dx_2dy_1dy_2.
\end{aligned}$$

We observe that Θ_ε coincides with the right hand-side of equality (3.2) in the proof of Lemma 3.3, where it was proved that

$$\lim_{\varepsilon\to 0}\Theta_\varepsilon = 2(t'-t)(s'-s).$$

Thus, thanks to (3.21) and recalling that $\lim_{\varepsilon\to 0}\rho_\varepsilon = 0$, we finish the proof by choosing $C_\varepsilon := \Theta_\varepsilon^2 + \rho_\varepsilon$. □

Now we can prove Proposition 3.2 .

Proof of Proposition 3.2. We recall from the beginning of this section that

$$A_\varepsilon + B_\varepsilon = \mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)\big)\left(\mathbb{E}\left[\left|\Delta_{s,t}X_\varepsilon(s',t')\right|^2|\mathcal{F}^\varepsilon_{s,T}\right] - 2(s'-s)(t'-t)\right)\right].$$

In order to prove that $A_\varepsilon + B_\varepsilon$ converges to zero as ε tends to zero, it is enought to verify that $\mathbb{E}\left[\left|\Delta_{s,t}X_\varepsilon(s',t')\right|^2|\mathcal{F}^\varepsilon_{s,T}\right]$ converges to $2(s'-s)(t'-t)$ in $L^2(\Omega)$, as $\varepsilon \to 0$. By Lemma 3.4, we have that

$$\begin{aligned}
0 &\leq \mathbb{E}\left[\left(\mathbb{E}\left[\left|\Delta_{s,t}X_\varepsilon(s',t')\right|^2|\mathcal{F}^\varepsilon_{s,T}\right] - 2(s'-s)(t'-t)\right)^2\right] \\
&\leq C_\varepsilon - 4(s'-s)(t'-t)\mathbb{E}\left[\left|\Delta_{s,t}X_\varepsilon(s',t')\right|^2\right] + 4(s'-s)^2(t'-t)^2,
\end{aligned}$$

where $\lim_{\varepsilon \to 0} C_\varepsilon = 4(s'-s)^2(t'-t)^2$. Therefore, by Lemma 3.3, the right hand-side of the above inequality vanishes as $\varepsilon \to 0$.

Now, we will deal with the limit of $A_\varepsilon - B_\varepsilon$. Let us recall that

$$A_\varepsilon - B_\varepsilon = \mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\left|\Delta_{s,t}\mathsf{Re}[X_\varepsilon(s',t')]\right|^2 - \left|\Delta_{s,t}\mathsf{Im}[X_\varepsilon(s',t')]\right|^2\right)\right].$$

Then, we have that

$$\begin{aligned}A_\varepsilon - B_\varepsilon =&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\Bigg(\left(\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\big(\cos(\theta L(x,y))\big)dxdy\right)^2\\&-\left(\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\big(\sin(\theta L(x,y))\big)dxdy\right)^2\Bigg)\Bigg].\end{aligned}$$

Since $2(a^2-b^2) = (a+ib)^2 + (a-ib)^2$, we infer that

$$\begin{aligned}A_\varepsilon - B_\varepsilon =&\frac{1}{2}\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\Bigg(\Bigg\{\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\Bigg)^2\\&+\left(\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{-i\theta L(x,y)}dxdy\right)^2\Bigg\}\Bigg].\end{aligned}$$

Thus, it is enought to prove

$$\lim_{\varepsilon\to 0}\frac{1}{2}\Lambda_\varepsilon^1 = 0 \qquad \text{and} \qquad \lim_{\varepsilon\to 0}\frac{1}{2}\Lambda_\varepsilon^2 = 0,$$

where

$$\Lambda_\varepsilon^1 := \mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\right)^2\right] \tag{3.22}$$

and

$$\Lambda_\varepsilon^2 := \mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{-i\theta L(x,y)}dxdy\right)^2\right]. \tag{3.23}$$

Let us tackle first Λ^1_ε. In order to do this, we will expand its squared integral, so that we have

$$\begin{aligned}
\Lambda^1_\varepsilon =&\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\left(\varepsilon K\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{xy}\,e^{i\theta L(x,y)}dxdy\right)^2\right]\\
=&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\\
&\times\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\Bigg]\\
=&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\\
&\times\Bigg(\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\\
&+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{x_2}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\\
&+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{y_2}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\\
&+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{y_2}^{\frac{t'}{\varepsilon}}\int_{x_2}^{\frac{s'}{\varepsilon}}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\Bigg)\Bigg].
\end{aligned}$$

If we put a different integration order and if then we apply the required changes of variables, we obtain

$$\begin{aligned}
\Lambda^1_\varepsilon =&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\\
&\times\Bigg(\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\\
&+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_2,y_1)+L(x_1,y_2))}dx_1dy_1dx_2dy_2\\
&+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_2,y_1)+L(x_1,y_2))}dx_1dy_1dx_2dy_2\\
&+\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{t}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\Bigg)\Bigg]\\
=&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\\
&\times\Bigg(2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_1,y_1)+L(x_2,y_2))}dx_1dy_1dx_2dy_2\\
&+2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta(L(x_2,y_1)+L(x_1,y_2))}dx_1dy_1dx_2dy_2\Bigg)\Bigg].
\end{aligned}$$

At this point we will set $L(x_1,y_1)+L(x_2,y_2)$ and $L(x_2,y_1)+L(x_1,y_2)$ as sum of disjoint rectangular increments of L. Hence,

$$\begin{aligned}\Lambda^1_\varepsilon =&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)\\
&\times\Bigg(2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\big(\Delta_{x_1,y_1}L(x_2,y_2)+\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\\
&\times e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})+\Delta_{0,0}L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\big)}dx_1dy_1dx_2dy_2\\
&+2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\big(\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\\
&\times\ e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})+\Delta_{0,0}L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})\big)}dx_1dy_1dx_2dy_2\Bigg)\Bigg]\\
=&\mathbb{E}\Bigg[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)e^{i2\theta L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})}\\
&\times\Bigg(2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\big(\Delta_{x_1,y_1}L(x_2,y_2)+\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\\
&\times e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\big)}dx_1dy_1dx_2dy_2\\
&+2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\big(\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\\
&\times\ e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon),y_1}+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\big)}dx_1dy_1dx_2dy_2\Bigg)\Bigg].\end{aligned}$$

Next, we split the expectations and inmediately apply Fubini's theorem. Hence,

$$\begin{aligned}\Lambda^1_\varepsilon =&\mathbb{E}\Big[\varphi\big(X_\varepsilon(s_1,t_1),\dots,X_\varepsilon(s_n,t_n)\big)e^{i2\theta L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})}\Big]\\
&\times\mathbb{E}\Bigg[\Bigg(2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\big(\Delta_{x_1,y_1}L(x_2,y_2)+\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\\
&\times e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\big)}dx_1dy_1dx_2dy_2\\
&+2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{i\theta\big(\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\\
&\times\ e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\big)}dx_1dy_1dx_2dy_2\Bigg)\Bigg]\end{aligned}$$

$$
\begin{aligned}
=&\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)\big)e^{i2\theta L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})}\right]\\
&\times\Bigg(2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,\mathbb{E}\left[e^{i\theta\big(\Delta_{x_1,y_1}L(x_2,y_2)+\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\right]\\
&\times\mathbb{E}\left[e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\big)}\right]dx_1dy_1dx_2dy_2\\
&+2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,\mathbb{E}\left[e^{i\theta\big(\Delta_{0,y_1}L(x_1,y_2)+\Delta_{x_1,0}L(x_2,y_1)\big)}\right]\\
&\times\mathbb{E}\left[e^{2i\theta\big(\Delta_{\frac{s}{\varepsilon},\frac{t}{\varepsilon}}L(x_1,y_1)+\Delta_{0,\frac{t}{\varepsilon}}L(\frac{s}{\varepsilon},y_1)+\Delta_{\frac{s}{\varepsilon},0}L(x_1,\frac{t}{\varepsilon})\big)}\right]dx_1dy_1dx_2dy_2\Bigg).
\end{aligned}
$$

Here, we observe that φ is a bounded function. So, there exists a positive constant C, such that

$$
\mathbb{E}\left[\varphi\big(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)\big)e^{i2\theta L(\frac{s}{\varepsilon},\frac{t}{\varepsilon})}\right]\leq C.
$$

Then,

$$
\begin{aligned}
\Lambda_\varepsilon^1\leq C&\Bigg(2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{\Psi(\theta)\big((x_2-x_1)(y_2-y_1)+(y_2-y_1)x_1+(x_2-x_1)y_1\big)}\\
&\times e^{\Psi(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)}dx_1dy_1dx_2dy_2\\
&+2\varepsilon^2K^2\int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}}\int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}}\int_{\frac{t}{\varepsilon}}^{y_2}\int_{\frac{s}{\varepsilon}}^{x_2}\sqrt{x_1x_2y_1y_2}\,e^{\Psi(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)}\\
&\times e^{\Psi(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)}dx_1dy_1dx_2dy_2\Bigg).
\end{aligned}
$$

By taking modulus and shifting it inside the integrals, we end up with

$$
\begin{aligned}
|\Lambda_\varepsilon^1| \le C \Bigg(& 2\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((x_2-x_1)(y_2-y_1)+(y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \\
& + 2\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \Bigg) \\
\le C \Bigg(& 2\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \\
& + 2\varepsilon^2 K^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \Bigg) \\
\le C\varepsilon^2 K^2 & \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2.
\end{aligned}
$$

Set

$$
\begin{aligned}
\tilde{\Lambda}_\varepsilon^1 := \varepsilon^2 K^2 & \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2.
\end{aligned}
$$

As $|\Lambda_\varepsilon^1| \le C\tilde{\Lambda}_\varepsilon^1$, it is enough to show that $\lim_{\varepsilon\to 0} \tilde{\Lambda}_\varepsilon^1 = 0$ to get our result finished.
We bound the square roots using the upper limit in the integrals and we use at our convinience the limits of the integrals for bounding the exponential function in a suitable way, in particular we use that $\frac{s}{\varepsilon} \le x_1$ and that $\frac{t}{\varepsilon} \le y_1$. Thus,

$$
\begin{aligned}
\tilde{\Lambda}_\varepsilon^1 = \varepsilon^2 K^2 & \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} \sqrt{x_1 x_2 y_1 y_2}\, e^{-a(\theta)\big((y_2-y_1)x_1+(x_2-x_1)y_1\big)} \\
& \times e^{-a(2\theta)\big((x_1-\frac{s}{\varepsilon})(y_1-\frac{t}{\varepsilon})+(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \\
\le C & \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} e^{-a(\theta)\big((y_2-y_1)\frac{s}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}\big)} \\
& \times e^{-a(2\theta)\big((y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \\
\le C & \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{\frac{t}{\varepsilon}}^{y_2} \int_{\frac{s}{\varepsilon}}^{x_2} e^{-\min\big(a(\theta),a(2\theta)\big)\big((y_2-y_1)\frac{s}{\varepsilon}+(x_2-x_1)\frac{t}{\varepsilon}\big)} \\
& \times e^{-\min\big(a(\theta),a(2\theta)\big)\big((y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_1 dy_1 dx_2 dy_2 \\
= C & \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{x_1}^{\frac{s'}{\varepsilon}} e^{-\min\big(a(\theta),a(2\theta)\big)\big((y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_2-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\big)} dx_2 dy_2 dx_1 dy_1,
\end{aligned}
$$

where we have applied Fubini's theorem.
Let us now integrate first with respect of x_2 and y_2 and then with respect of x_1 and y_1, hence

$$
\begin{aligned}
\tilde{\Lambda}_\varepsilon^1 \leq & C \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} \int_{x_1}^{\frac{s'}{\varepsilon}} e^{-\min\left(a(\theta),a(2\theta)\right)\left((y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}+(x_2-\frac{s}{\varepsilon})\frac{t}{\varepsilon}\right)} dx_2 dy_2 dx_1 dy_1 \\
= & C\varepsilon \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \int_{y_1}^{\frac{t'}{\varepsilon}} e^{-\min\left(a(\theta),a(2\theta)\right)(y_2-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} \\
& \times \left(e^{-\min\left(a(\theta),a(2\theta)\right)(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}} - e^{-\min\left(a(\theta),a(2\theta)\right)(\frac{s'}{\varepsilon}-\frac{s}{\varepsilon})\frac{t}{\varepsilon}} \right) dy_2 dx_1 dy_1 \\
\leq & C\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} e^{-\min\left(a(\theta),a(2\theta)\right)(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}} \\
& \times \left(e^{-\min\left(a(\theta),a(2\theta)\right)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} - e^{-\min\left(a(\theta),a(2\theta)\right)(\frac{t'}{\varepsilon}-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} \right) dx_1 dy_1 \\
\leq & C\varepsilon^2 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} e^{-\min\left(a(\theta),a(2\theta)\right)(x_1-\frac{s}{\varepsilon})\frac{t}{\varepsilon}} e^{-\min\left(a(\theta),a(2\theta)\right)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dx_1 dy_1 \\
\leq & C\varepsilon^3 \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} e^{-\min\left(a(\theta),a(2\theta)\right)(y_1-\frac{t}{\varepsilon})\frac{s}{\varepsilon}} dy_1 \\
\leq & C\varepsilon^4.
\end{aligned}
$$

Therefore, we can infer that $\lim_{\varepsilon\to 0} \tilde{\Lambda}_\varepsilon^1 = 0$, which implies that $\lim_{\varepsilon\to 0} \Lambda_\varepsilon^1 = 0$. The case for Λ_ε^2 is analogous using the same arguments and considering two facts. The first one is that $a(-\theta) = a(\theta)$ and the second one is that, if the modulus of a complex number converges to zero, its conjugate should do it too. Thus, $\lim_{\varepsilon\to 0} A_\varepsilon - B_\varepsilon = 0$. And this completes the proof of Proposition 3.2. □

Now we can prove the main theorem of the thesis:

Proof of Theorem 0.1. The tightness result given by Proposition 2.1, and Propositions 3.1 and 3.2 imply, by Theorem 3.1, that X converges in law, as ε tends to zero and in a space $\mathcal{C}([0,S]\times[0,T];\mathbb{C})$, to a complex random field $X = \{X(s,t);\, (s,t) \in [0,S]\times[0,T]\}$ whose real and imaginary parts are (real-valued) Brownian sheets. We only have to verify that those real and imaginary parts are independent, for which it is enough to prove that the corresponding covariance vanishes. In order to do this, we will make use of the approximation sequence $(X_\varepsilon)_{\varepsilon>0}$, in the following way:
We observe that $\mathsf{Re}(X)$ and $\mathsf{Im}(X)$ are independent if, for any $(0,0) \leq (s,t) \leq (s',t') \leq (S,T)$, $0 \leq s_1 < \cdots < s_n \leq s$ and $0 \leq t_1 < \cdots < t_n \leq t$, and any continuous bounded function $\varphi : \mathbb{C}^n \to \mathbb{R}$, we have

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\,(\Delta_{s,t}\mathsf{Re}(X_\varepsilon)(s',t'))\,(\Delta_{s,t}\mathsf{Im}(X_\varepsilon)(s',t'))\right] = 0.$$

Using the equality $\alpha\beta = \frac{i}{4}\left\{(\alpha - i\beta)^2 - (\alpha + i\beta)^2\right\}$, we obtain that

$$
\begin{aligned}
& \mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n))\,(\Delta_{s,t}\mathsf{Re}(X_\varepsilon)(s',t'))\,(\Delta_{s,t}\mathsf{Im}(X_\varepsilon)(s',t'))\right] \\
= & \frac{i}{4}\,\mathbb{E}\left[\varphi(X_\varepsilon(s_1,t_1),\ldots,X_\varepsilon(s_n,t_n)) \left\{ \left(K\varepsilon \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\, e^{-i\theta L(x,y)} dx dy \right)^2 \right.\right. \\
& \left.\left. - \left(K\varepsilon \int_{\frac{t}{\varepsilon}}^{\frac{t'}{\varepsilon}} \int_{\frac{s}{\varepsilon}}^{\frac{s'}{\varepsilon}} \sqrt{xy}\, e^{i\theta L(x,y)} dx dy \right)^2 \right\}\right] \\
= & \Lambda_\varepsilon^1 - \Lambda_\varepsilon^2,
\end{aligned}
$$

where Λ_ε^1 and Λ_ε^2 have been defined in (3.22) and (3.23), repectively. As a consequence of the proof of Proposition 3.2, the latter expression converges to zero as $\varepsilon \to 0$. So the proof has been completed. □

Chapter 4

Weak convergence to the stochastic heat equation

Let us consider the following one-dimensional quasi-linear stochastic heat equation:

$$\frac{\partial U}{\partial t}(t,x) - \frac{\partial^2 U}{\partial x^2}(t,x) = b(U(t,x)) + \dot{W}(t,x), \qquad (t,x) \in [0,T] \times [0,1], \tag{4.1}$$

where $T > 0$ is a fixed time horizon, $b : \mathbb{R} \to \mathbb{R}$ is a globally Lipschitz function and $\dot{W}(t,x)$ is the space-time white noise. We impose the initial condition $U(0,x) = u_0(x)$, $x \in [0,1]$, where $u_0 : [0,1] \to \mathbb{R}$ is a continuous function, with Dirichlet boundary conditions:

$$U(t,0) = U(t,1) = 0, \qquad t \in [0,T].$$

For making the calculations simpler, we will assume $T = 1$ throughout all the chapter. All results can be extended to a general $T > 0$, since the calculations are analogous.

The solution of equation (4.1) has a mild interpretation, as follows. Let $\{W(t,x); (t,x) \in [0,1]^2\}$ be a Brownian sheet defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with $\{\mathcal{F}_{t,x}; (t,x) \in [0,1]^2\}$ its natural filtration. A jointly measurable and adapted random field $U = \{U(t,x); (t,x) \in [0,1]^2\}$ is a solution of (4.1) if it satisfies

$$\begin{aligned} U(t,x) = \int_0^1 G_t(x,y) u_0(y) dy &+ \int_0^t \int_0^1 G_{t-s}(x,y) b(U(s,y)) dy ds \\ &+ \int_0^t \int_0^1 G_{t-s}(x,y) W(ds,dy), \qquad \text{a.s.} \end{aligned} \tag{4.2}$$

for all $(x,t) \in (0,1] \times (0,1)$, where G is the Green function associated to the heat equation in $[0,1]$ with Dirichlet boundary conditions. The existence, uniqueness and pathwise continuity of the solution to (4.2) are a consequence of [42, Theorem 3.5].

Let us recall that the Green function G is given by

$$G_t(x,y) = 2 \sum_{n=1}^{\infty} \sin(n\pi x) \sin(n\pi y) e^{n^2\pi^2 t}.$$

Notice that the last integral in (4.2) is the Wiener integral that we defined in Section 1.3.

In this section, we aim to apply [9, Theorem 1.4] in order to deduce that the solution U given above can be approximated in law, in the space $\mathcal{C}([0,1]^2)$ of continuous functions, by a family of mild solutions $\{U_n\}_{n \geq 0}$, where U_n is the solution of a stochastic heat equation perturbed by the formal derivative of either the real or imaginary part of the family introduced in (5):

$$X_\varepsilon(t,x) := \varepsilon K \int_0^{\frac{t}{\varepsilon}} \int_0^{\frac{x}{\varepsilon}} \sqrt{sy} \big(\cos(\theta L(s,y)) + i \sin(\theta L(s,y)) \big) dy ds,$$

where $\{L(s,y); s,y \geq 0\}$ denotes a Lévy sheet and its Lévy exponent is given by $\Psi(\xi) = a(\xi) + b(\xi)$. The constant K is given in (6) and $\theta \in (0, 2\pi)$ satisfies $a(\theta) \cdot a(2\theta) \neq 0$. We observe that, compared to the expression given in (5), in the above expression of X_ε we have modified the variables' notation in order to properly match with the framework of stochastic partial differential equations.

Let us be more precise about the above statement. First, we rewrite X_ε considering the following changes of variable:

$$s' = \varepsilon s, \quad y' = \varepsilon y \quad \text{and with} \quad n = \frac{1}{\varepsilon^2}.$$

Then, we have that

$$X_\varepsilon(t,x) = nK \int_0^t \int_0^x \sqrt{s'y'}\big(\cos(\theta L(\sqrt{n}s', \sqrt{n}y')) + i\sin(\theta L(\sqrt{n}s', \sqrt{n}y'))\big) dy'ds'.$$

In order to simplify this expression, we will make a notation abuse and we will rewrite it as:

$$X_\varepsilon(t,x) = nK \int_0^t \int_0^x \sqrt{sy}\big(\cos(\theta L_n(s,y)) + i\sin(\theta L_n(s,y))\big) dyds,$$

where $L_n(s,y) = L(\sqrt{n}s, \sqrt{n}y)$. Set

$$\theta_n^1(s,y) := nK\sqrt{sy}\cos(\theta L_n(s,y)) \quad \text{and} \quad \theta_n^2(s,y) := nK\sqrt{sy}\sin(\theta L_n(s,y)).$$

Let $i \in \{1,2\}$ and consider the stochastic heat equation

$$\frac{\partial U_n^i}{\partial t}(t,x) - \frac{\partial^2 U_n^i}{\partial x^2}(t,x) = b(U_n^i(t,x)) + \theta_n^i(t,x), \qquad (t,x) \in [0,1]^2, \tag{4.3}$$

with initial condition u_0 and Dirichlet boundary conditions. The mild form of the equation is given by

$$\begin{aligned} U_n^i(t,x) = \int_0^1 G_t(x,y)u_0(y)dy + \int_0^t \int_0^1 G_{t-s}(x,y) b(U_n^i(s,y)) dyds \\ + \int_0^t \int_0^1 G_{t-s}(x,y)\theta_n^i(s,y) dyds. \end{aligned} \tag{4.4}$$

Thanks to Section 3 of [9], we know that (4.4) admits only a unique solution U_n^i with almost surely continuous paths. Now, we state the main result of this chapter:

Theorem 4.1. *For any $i \in \{1,2\}$, the sequence $\{U_n^i\}_{n\geq 1}$ converges in law, as $n \to \infty$ and in the space $\mathcal{C}([0,1]^2)$, to the solution U of (4.2).*

The proof of this theorem is based on Theorem 1.4 de [9], where sufficient conditions on a family of random fields $\{\theta_n\}_{n\geq 1}$ have been established such that the sequence of solutions to the stochastic heat equation driven by θ_n converges in law to U, in the space of continuous function. The first requirement is that $\theta_n \in L^2([0,1]^2)$ a.s., and then there are the following conditions (these are the hypotheses 1.1, 1.2 i 1.3 in [9]):

i) The finite dimensional distributions of the process

$$\zeta_n(t,x) := \int_0^t \int_0^x \theta_n(s,y) dyds, \qquad (t,x) \in [0,1]^2,$$

converges in law to those of the Brownian sheet.

ii) For some $q \in [2,3)$, there exists a positive constant C_q such that, for any $f \in L^p([0,1]^2)$, it holds:

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^1 \int_0^1 f(t,x)\theta_n(t,x) dxdt\right)^2\right] \leq C_q \left(\int_0^1 \int_0^1 |f(t,x)|^q dxdt\right)^{\frac{2}{q}}.$$

iii) There exists m even and a positive constant C such that the following is satisfied: for all $s_0, s_0' \in [0,1]$ and $x_0, x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, and for any $f \in L^2([0,1]^2)$, it holds:

$$\sup_{n\geq 1} \mathbb{E}\left[\left|\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)\theta_n(s,y)dyds\right|^m\right] \leq C_q \left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)^2 dyds\right)^{\frac{m}{2}}.$$

Therefore, in the proof of Theorem 4.1 we will prove the validity all above conditions in the case where θ_n is given by θ_n^i for any $i \in \{1,2\}$. In fact, as we explain below, we will use similar arguments as those used in one of the applications in [9], namely the case where θ_n are given by the Kac-Stroock processes on the plane:

$$\theta_n(t,x) = n\sqrt{tx}(-1)^{N_n(t,x)},$$

where $N_n(t,x) := N(\sqrt{n}t, \sqrt{n}x)$, and $\{N(t,x); (t,x) \in [0,1]^2\}$ is a standard Poisson process in the plane.

4.1 Proof of Theorem 4.1

Before we start the proof of the Theorem 4.1, we observe that condition i) has been proved in Theorem 0.1. Now, we will prove the following technical lemma, which is analogous of [9, Lemma 4.2]:

Lemma 4.1. *Let $f \in L^2([0,1]^2)$ and $\alpha \geq 1$. Then, for any $u, u' \in (0,1)$ such that $0 < u < u' < 2^\alpha u$, we have*

$$\sup_{n\geq 1} \mathbb{E}\left[\left(\int_0^1\int_u^{u'} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] \leq \frac{3(2^{\alpha+1}-1)K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t,x)^2 dxdt,$$

for any $i \in \{1,2\}$.

Proof. We will only deal with the case involving θ_n^1, because the first inequality below holds in both cases, and so the result for θ_n^2 is identical. We have

$$\begin{aligned}
\mathbb{E}\left[\left(\int_0^1\int_u^{u'} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] &\leq \mathbb{E}\left[\left(\int_0^1\int_u^{u'} f(t,x)\theta_n^1(t,x)dxdt\right)^2 + \left(\int_0^1\int_u^{u'} f(t,x)\theta_n^2(t,x)dxdt\right)^2\right] \\
&= \mathbb{E}\left[\left|nK\int_0^1\int_u^{u'} \sqrt{tx}\, e^{i\theta L_n(t,x)} f(t,x)dxdt\right|^2\right].
\end{aligned}$$

We can also note that the latter term is equal to

$$\begin{aligned}
&n^2K^2\mathbb{E}\left[\left(\int_0^1\int_u^{u'} \sqrt{t_1x_1}\, e^{-i\theta L_n(t_1,x_1)} f(t_1,x_1)dx_1dt_1\right)\left(\int_0^1\int_u^{u'} \sqrt{t_2x_2}\, e^{i\theta L_n(t_2,x_2)} f(t_2,x_2)dx_2dt_2\right)\right] \\
&= n^2K^2\mathbb{E}\left[\int_0^1\int_u^{u'}\int_0^1\int_u^{u'} \sqrt{t_1t_2}\sqrt{x_1x_2}\, e^{i\theta\left(L_n(t_2,x_2)-L_n(t_1,x_1)\right)} f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2\right] \\
&= n^2K^2\int_0^1\int_u^{u'}\int_0^1\int_u^{u'} \sqrt{t_1t_2}\sqrt{x_1x_2}\, \mathbb{E}\left[e^{i\theta\left(L_n(t_2,x_2)-L_n(t_1,x_1)\right)}\right] f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2 \\
&= n^2K^2\int_0^1\int_u^{u'}\int_0^1\int_u^{u'} \sqrt{t_1t_2}\sqrt{x_1x_2}\, \mathbb{E}\left[e^{i\theta\left(\Delta_{0,0}L_n(t_2,x_2)-\Delta_{0,0}L_n(t_1x_1)\right)}\right] f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2.
\end{aligned}$$

Due to the 4 order possibilities of two dots in the plane, this last expression is equal to

$$n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2$$
$$+n^2K^2\int_0^1\int_u^{u'}\int_{t_2}^1\int_u^{x_2}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(\theta)n(x_2-x_1)t_2}e^{-\Psi(-\theta)n(t_1-t_2)x_1}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2$$
$$+n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_{x_2}^{u'}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(-\theta)n(x_1-x_2)t_1}e^{-\Psi(\theta)n(t_2-t_1)x_2}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2$$
$$+n^2K^2\int_0^1\int_u^{u'}\int_{t_2}^1\int_{x_2}^{u'}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(-\theta)n\big((t_1-t_2)x_2+(x_1-x_2)t_1\big)}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2.$$

If we apply some proper changes of variable we obtain that this sum of integrals is equal to

$$n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2$$
$$+n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(\theta)n(x_2-x_1)t_1}e^{-\Psi(-\theta)n(t_2-t_1)x_1}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2$$
$$+n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(-\theta)n(x_2-x_1)t_1}e^{-\Psi(\theta)n(t_2-t_1)x_1}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2$$
$$+n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2}\sqrt{t_1t_2x_1x_2}\,e^{-\Psi(-\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}f(t_1,x_1)f(t_2,x_2)dx_1dt_1dx_2dt_2.$$

Recall that $\Psi(\theta)=a(\theta)+ib(\theta)$, where $a(\theta)=a(-\theta)$ and $-b(\theta)=b(-\theta)$. Let us observe the following arrangement

$$\begin{aligned}
&e^{-\Psi(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}+e^{-\Psi(\theta)n(x_2-x_1)t_1}e^{-\Psi(-\theta)n(t_2-t_1)x_1}\\
&\quad+e^{-\Psi(-\theta)n(x_2-x_1)t_1}e^{-\Psi(\theta)n(t_2-t_1)x_1}+e^{-\Psi(-\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}\\
=&e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}e^{-ib(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}\\
&\quad+e^{-a(\theta)n(x_2-x_1)t_1}e^{-ib(\theta)n(x_2-x_1)t_1}e^{-a(\theta)n(t_2-t_1)x_1}e^{ib(\theta)n(t_2-t_1)x_1}\\
&\quad+e^{-a(\theta)n(x_2-x_1)t_1}e^{ib(\theta)n(x_2-x_1)t_1}e^{-a(\theta)n(t_2-t_1)x_1}e^{-ib(\theta)n(t_2-t_1)x_1}\\
&\quad+e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}e^{ib(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}\\
=&e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}\left(e^{ib(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}+e^{-ib(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_2\big)}\right)\\
&\quad+e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_1\big)}\left(e^{ib(\theta)n\big((t_2-t_1)x_1-(x_2-x_1)t_1\big)}+e^{-ib(\theta)n\big((t_2-t_1)x_1-(x_2-x_1)t_1\big)}\right)\\
=&e^{-a(\theta)n\big(x_2t_2-x_1t_1\big)}\left(e^{ib(\theta)n\big(x_2t_2-x_1t_1\big)}+e^{-ib(\theta)n\big(x_2t_2-x_1t_1\big)}\right)\\
&\quad+e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_1\big)}\left(e^{ib(\theta)n\big((t_2-t_1)x_1-(x_2-x_1)t_1\big)}+e^{-ib(\theta)n\big((t_2-t_1)x_1-(x_2-x_1)t_1\big)}\right)\\
=&e^{-a(\theta)n\big(x_2t_2-x_1t_1\big)}2\cos\big(b(\theta)n(x_2t_2-x_1t_1)\big)\\
&\quad+e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_1\big)}2\cos\big(b(\theta)n\big((t_2-t_1)x_1-(x_2-x_1)t_1\big)\big),
\end{aligned}$$

where the last expresion is a consequence of the definition of the exponential function. Therefore,

$$\mathbb{E}\left[\left(\int_0^1\int_u^{u'}f(t,x)\theta_n^i(t,x)dxdt\right)^2\right]\leq 2\left(I_1^n+I_2^n\right), \tag{4.5}$$

where

$$
\begin{aligned}
I_1^n &= n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} f(t_1,x_1)f(t_2,x_2)\sqrt{t_1t_2x_1x_2}\, e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)} \\
&\quad \times \cos\big(b(\theta)n(x_2t_2-x_1t_1)\big)dx_1dt_1dx_2dt_2
\end{aligned}
$$

and

$$
\begin{aligned}
I_2^n &= n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} f(t_1,x_1)f(t_2,x_2)\sqrt{t_1t_2x_1x_2}\, e^{-a(\theta)n\left((t_2-t_1)x_1+(x_2-x_1)t_1\right)} \\
&\quad \times \cos\big(b(\theta)n\big((t_2-t_1)x_1-(x_2-x_1)t_1\big)\big)dx_1dt_1dx_2dt_2.
\end{aligned}
$$

Let us apply the inequality $zw \le \frac{1}{2}\left(z^2+w^2\right)$:

$$
f(t_1,x_1)f(t_2,x_2)\sqrt{t_1t_2x_1x_2} \le \frac{1}{2}\left(t_1x_1f(t_1,x_1)^2+t_2x_2f(t_2,x_2)^2\right).
$$

If we bound the cosines by 1, we have

$$
\begin{aligned}
I_1^n \le& \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} \left(t_1x_1f(t_1,x_1)^2+t_2x_2f(t_2,x_2)^2\right)e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)}dx_1dt_1dx_2dt_2 \\
=& \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_1x_1f(t_1,x_1)^2e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)}dx_1dt_1dx_2dt_2 \\
&+ \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_2x_2f(t_2,x_2)^2e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)}dx_1dt_1dx_2dt_2 \\
:=& \ I_{1,1}^n + I_{1,2}^n,
\end{aligned}
$$

and

$$
\begin{aligned}
I_2^n \le& \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} \left(t_1x_1f(t_1,x_1)^2+t_2x_2f(t_2,x_2)^2\right)e^{-a(\theta)n\left((t_2-t_1)x_1+(x_2-x_1)t_1\right)}dx_1dt_1dx_2dt_2 \\
=& \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_1x_1f(t_1,x_1)^2e^{-a(\theta)n\left((t_2-t_1)x_1+(x_2-x_1)t_1\right)}dx_1dt_1dx_2dt_2 \\
&+ \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_2x_2f(t_2,x_2)^2e^{-a(\theta)n\left((t_2-t_1)x_1+(x_2-x_1)t_1\right)}dx_1dt_1dx_2dt_2 \\
:=& \ I_{2,1}^n + I_{2,2}^n.
\end{aligned}
$$

Next we will estimate these integrals in a convenient way. These calculations are analogues to the ones shown in the proof of Lemma 4.2 in [9].

For $I_{1,1}^n$, we will first apply Fubini's theorem and, after integrating with respect x_2 and t_2, we will bound the exponential function by 1. In the fourth line we use the fact that $t_1 \le t_2$:

$$
\begin{aligned}
I^n_{1,1} &= \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_1x_1f(t_1,x_1)^2e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)}dx_1dt_1dx_2dt_2\\
&\leq \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_{t_1}^{1}\int_{x_1}^{u'} t_1x_1f(t_1,x_1)^2e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)}dx_2dt_2dx_1dt_1\\
&\leq \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_{t_1}^{1} t_1x_1f(t_1,x_1)^2e^{-a(\theta)n(-x_1t_1)}\left(\frac{-1}{a(\theta)nt_2}\right)\left(e^{-a(\theta)nt_2tu}-e^{-a(\theta)nt_2x_1}\right)dt_2dx_1dt_1\\
&\leq \frac{1}{2}\frac{nK^2}{a(\theta)}\int_0^1\int_u^{u'}\int_{t_1}^{1} x_1f(t_1,x_1)^2e^{-a(\theta)n(-x_1t_1)}e^{-a(\theta)nt_2x_1}dt_2dx_1dt_1\\
&= \frac{1}{2}\frac{nK^2}{a(\theta)}\int_0^1\int_u^{u'} x_1f(t_1,x_1)^2e^{-a(\theta)n(-x_1t_1)}\left(\frac{-1}{a(\theta)nx_1}\right)\left(e^{-a(\theta)nx_1}-e^{-a(\theta)nt_1x_1}\right)dx_1dt_1\\
&\leq \frac{1}{2}\frac{K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_1,x_1)^2e^{-a(\theta)n(-x_1t_1)}e^{-a(\theta)nt_1x_1}dx_1dt_1\\
&= \frac{1}{2}\frac{K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_1,x_1)^2dx_1dt_1.
\end{aligned}
$$

For $I^n_{1,2}$, first we integrate with respect t_1, then we bound the exponential by 1 and in the second line we use the fact that $x_2 \leq 2^\alpha x_1$, since both x_1 and x_2 are in (u,u'):

$$
\begin{aligned}
I^n_{1,2} &= \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_2x_2f(t_2,x_2)^2e^{-a(\theta)n\left(x_2t_2-x_1t_1\right)}dx_1dt_1dx_2dt_2\\
&\leq \frac{1}{2}n^2K^22^\alpha\int_0^1\int_u^{u'}\int_u^{x_2} t_2x_1f(t_2,x_2)^2e^{-a(\theta)nx_2t_2}\left(\frac{1}{a(\theta)nx_1}\right)\left(e^{a(\theta)nx_1t_2}-1\right)dx_1dx_2dt_2\\
&\leq \frac{1}{2}\frac{nK^22^\alpha}{a(\theta)}\int_0^1\int_u^{u'}\int_u^{x_2} t_2f(t_2,x_2)^2e^{-a(\theta)nx_2t_2}e^{a(\theta)nx_1t_2}dx_1dx_2dt_2\\
&= \frac{1}{2}\frac{nK^22^\alpha}{a(\theta)}\int_0^1\int_u^{u'} t_2f(t_2,x_2)^2e^{-a(\theta)nx_2t_2}\left(\frac{1}{a(\theta)nt_2}\right)\left(e^{a(\theta)nx_2t_2}-1\right)dx_2dt_2\\
&\leq \frac{1}{2}\frac{K^22^\alpha}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_2,x_2)^2e^{-a(\theta)nx_2t_2}e^{a(\theta)nt_2x_2}dx_2dt_2\\
&= \frac{1}{2}\frac{K^22^\alpha}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_2,x_2)^2dx_2dt_2.
\end{aligned}
$$

We will leave $I^n_{2,1}$ for the end because it deserves a little more care, and now we will deal with $I^n_{2,2}$ in a similar way as we did with $I^n_{1,2}$:

$$
\begin{aligned}
I^n_{2,2} &= \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_2x_2f(t_2,x_2)^2e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_1\big)}dx_1dt_1dx_2dt_2\\
&\le \frac{1}{2}n^2K^22^\alpha\int_0^1\int_u^{u'}\int_u^{x_2}\int_0^{t_2} t_2x_1f(t_2,x_2)^2e^{-a(\theta)n(t_2-t_1)x_1}dt_1dx_1dx_2dt_2\\
&\le \frac{1}{2}n^2K^22^\alpha\int_0^1\int_u^{u'}\int_u^{x_2} t_2x_1f(t_2,x_2)^2\left(\frac{-1}{a(\theta)nx_1}\right)\big(1-e^{-a(\theta)nx_1t_2}\big)dx_1dx_2dt_2\\
&\le \frac{1}{2}\frac{nK^22^\alpha}{a(\theta)}\int_0^1\int_u^{u'}\int_u^{x_2} t_2f(t_2,x_2)^2e^{-a(\theta)nx_1t_2}dx_1dx_2dt_2\\
&= \frac{1}{2}\frac{nK^22^\alpha}{a(\theta)}\int_0^1\int_u^{u'} t_2f(t_2,x_2)^2\left(\frac{-1}{a(\theta)nt_2}\right)\big(e^{-a(\theta)nx_2t_2}-e^{-a(\theta)nut_2}\big)dx_2dt_2\\
&\le \frac{1}{2}\frac{K^22^\alpha}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_2,x_2)^2dx_2dt_2.
\end{aligned}
$$

Before dealing with $I^n_{2,1}$, let us make the following observation about the integration region:

$$
\begin{aligned}
x_2\le 2^\alpha x_1 \quad &\text{implies} \quad x_2-x_1\le 2^\alpha x_1-x_1,\\
&\text{thus} \quad \frac{x_2-x_1}{2^\alpha-1}\le x_1.
\end{aligned}
$$

Moreover, as $\alpha\ge 1$,

$$
2^\alpha\ge 2 \qquad \text{implies} \qquad 2^\alpha-1\ge 1.
$$

Hence,

$$
\begin{aligned}
a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_1\big) &\ge \frac{a(\theta)n}{2}(t_2-t_1)x_1+\frac{a(\theta)n}{2}(t_2-t_1)x_1+\frac{a(\theta)n}{2}(x_2-x_1)t_1\\
&\ge \frac{a(\theta)n}{2}(t_2-t_1)x_1+\frac{a(\theta)n}{2(2^\alpha-1)}(t_2-t_1)(x_2-x_1)+\frac{a(\theta)n}{2(2^\alpha-1)}(x_2-x_1)t_1\\
&\ge \frac{a(\theta)n}{2}(t_2-t_1)x_1+\frac{a(\theta)n}{2(2^\alpha-1)}(x_2-x_1)t_2.
\end{aligned}
$$

Then

$$
\begin{aligned}
I^n_{2,1} &= \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_1x_1f(t_1,x_1)^2e^{-a(\theta)n\big((t_2-t_1)x_1+(x_2-x_1)t_1\big)}dx_1dt_1dx_2dt_2\\
&\le \frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_0^{t_2}\int_u^{x_2} t_1x_1f(t_1,x_1)^2e^{-\frac{a(\theta)n}{2}(t_2-t_1)x_1-\frac{a(\theta)n}{2(2^\alpha-1)}(x_2-x_1)t_2}dx_1dt_1dx_2dt_2.
\end{aligned}
$$

Lets us apply Fubini's theorem and then we integrate with respect x_2 and t_2. Thus, this last integral is equal to

$$
\begin{aligned}
&\frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_{t_1}^1\int_{x_1}^{u'} t_1x_1f(t_1,x_1)^2e^{-\frac{a(\theta)n}{2}(t_2-t_1)x_1-\frac{a(\theta)n}{2(2^\alpha-1)}(x_2-x_1)t_2}dx_2dt_2dx_1dt_1\\
&\quad=\frac{1}{2}n^2K^2\int_0^1\int_u^{u'}\int_{t_1}^1 t_1x_1f(t_1,x_1)^2e^{-\frac{a(\theta)n}{2}(t_2-t_1)x_1}\left(\frac{-2(2^\alpha-1)}{a(\theta)nt_2}\right)\left(e^{-\frac{a(\theta)n}{2(2^\alpha-1)}(u'-x_1)t_2}-1\right)dt_2dx_1dt_1.
\end{aligned}
$$

Considering that $t_1\le t_2$ and again bounding the exponential function by 1, the latter integral gets bounded by

$$\frac{1}{2}\frac{2(2^{\alpha}-1)nK^2}{a(\theta)}\int_0^1\int_u^{u'}\int_{t_1}^1 x_1 f(t_1,x_1)^2 e^{-\frac{a(\theta)n}{2}(t_2-t_1)x_1}dt_2dx_1dt_1$$
$$= \frac{1}{2}\frac{2(2^{\alpha}-1)nK^2}{a(\theta)}\int_0^1\int_u^{u'} x_1 f(t_1,x_1)^2\left(\frac{-2}{a(\theta)nx_1}\right)\left(e^{-\frac{a(\theta)n}{2}(1-t_1)x_1}-1\right)dx_1dt_1$$
$$= \frac{1}{2}\frac{4(2^{\alpha}-1)K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_1,x_1)^2dx_1dt_1.$$

Therefore

$$\mathbb{E}\left[\left(\int_0^1\int_u^{u'} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] \le 2\left(I_{1,1}^n+I_{1,2}^n+I_{2,1}^n+I_{2,2}^n\right)$$
$$\le \frac{K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_1,x_1)^2dx_1dt_1+\frac{K^2 2^{\alpha}}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_2,x_2)^2dx_2dt_2$$
$$+\frac{4(2^{\alpha}-1)K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_1,x_1)^2dx_1dt_1+\frac{K^2 2^{\alpha}}{a(\theta)^2}\int_0^1\int_u^{u'} f(t_2,x_2)^2dx_2dt_2.$$

Note that

$$1+2^{\alpha}+2^{\alpha}+4(2^{\alpha}-1)=2(2^{\alpha})+4(2^{\alpha})+3=6(2^{\alpha})+3=3(2^{\alpha+1}+1).$$

Thus,

$$\mathbb{E}\left[\left(\int_0^1\int_u^{u'} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] \le \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2}\int_0^1\int_u^{u'} f(t,x)^2dxdt.$$

□

Now we can prove the next proposition, this proof will give us the validity of condition ii) and it is quite similar to that of [9, Proposition 4.1].

Proposition 4.1. *Let $p>1$ and $f\in L^{2p}([0,1]^2)$. Then, there exists a positive constant C_p, which does not depends on f, such that*

$$\sup_{n\ge 1}\mathbb{E}\left[\left(\int_0^1\int_0^1 f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] \le C_p\left(\int_0^1\int_0^1 |f(t,x)|^{2p}dxdt\right)^{\frac{1}{p}},$$

for any $i\in\{1,2\}$.

Proof. Consider a diadic partition of $(0,1]$:

$$(0,1]=\bigcap_{k=0}^{\infty}(a_{k+1},a_k],$$

with $a_k=\frac{1}{2^{k\alpha}}$, for some $\alpha\ge 1$. Observe that $a_k-a_{k+1}=\frac{2^{\alpha}-1}{2^{(k+1)\alpha}}$ and now we apply Lemma 4.1 for all $k\ge 0$:

$$\mathbb{E}\left[\left(\int_0^1\int_{a_{k+1}}^{a_k} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] \le \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2}\int_0^1\int_{a_{k+1}}^{a_k} f(t,x)^2dxdt. \tag{4.6}$$

Then, we can make the following arrangements and calculations:

$$\mathbb{E}\left[\left(\int_0^1\int_0^1 f(t,x)\theta_n^i(t,x)dxdt\right)^2\right] = \mathbb{E}\left[\left(\sum_{k=0}^{\infty}\int_0^1\int_{a_{k+1}}^{a_k} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right]$$
$$\le \sum_{k=0}^{\infty}2^{k+1}\mathbb{E}\left[\left(\int_0^1\int_{a_{k+1}}^{a_k} f(t,x)\theta_n^i(t,x)dxdt\right)^2\right]$$
$$\le \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2}\sum_{k=0}^{\infty}2^{k+1}\int_0^1\int_{a_{k+1}}^{a_k} f(t,x)^2dxdt, \tag{4.7}$$

where in the second line we have applied the following inequation

$$\Big(\sum_{k=0}^{n} a_k\Big)^2 \leq \sum_{k=0}^{n} 2^{k+1} a_k^2,$$

and we applied (4.6) in the third line. Let $p, q > 1$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. By Hölder's inequality, the latter term in (4.7) can be bounded by

$$\begin{aligned}
&\frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2} \sum_{k=0}^{\infty} 2^{k+1} \left(\int_0^1 \int_{a_{k+1}}^{a_k} |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \left(\int_0^1 \int_{a_{k+1}}^{a_k} dx dt \right)^{\frac{1}{q}} \\
&= \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2} \sum_{k=0}^{\infty} 2^{k+1} \left(\int_0^1 \int_{a_{k+1}}^{a_k} |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \left(a_k - a_{k+1} \right)^{\frac{1}{q}} \\
&\leq \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2} \sum_{k=0}^{\infty} 2^{k+1} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \left(a_k - a_{k+1} \right)^{\frac{1}{q}} \\
&= \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} 2^{k+1} \left(a_k - a_{k+1} \right)^{\frac{1}{q}} \\
&= \frac{3(2^{\alpha+1}+1)K^2}{a(\theta)^2} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} 2^{k+1} \left(\frac{2^{\alpha}-1}{2^{(k+1)\alpha}} \right)^{\frac{1}{q}} \\
&= \frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} 2^{k+1} \frac{1}{2^{(k+1)\frac{\alpha}{q}}} \\
&= \frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} \frac{1}{2^{(k+1)(\frac{\alpha}{q}-1)}} \\
&= \frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2 2^{\frac{\alpha}{q}-1}} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} \frac{1}{2^{k(\frac{\alpha}{q}-1)}} \\
&= \frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2 2^{\frac{\alpha}{q}-1}} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \sum_{k=0}^{\infty} \left(\frac{1}{2^{(\frac{\alpha}{q}-1)}} \right)^k,
\end{aligned} \tag{4.8}$$

and the series in (4.8) converges if α is such that $\alpha > q$. Thus, (4.8) is equal to

$$\begin{aligned}
&\frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2 2^{\frac{\alpha}{q}-1}} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \frac{1}{1-2^{(\frac{\alpha}{q}-1)}} \\
&\qquad = \frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2 2^{\frac{\alpha}{q}-2}} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}} \frac{1}{2-2^{\frac{\alpha}{q}}} \\
&\qquad \leq \frac{3(2^{\alpha+1}+1)(2^{\alpha}-1)^{\frac{1}{q}}K^2}{a(\theta)^2 2^{\frac{\alpha}{q}-2}} \left(\int_0^1 \int_0^1 |f(t,x)|^{2p} dx dt \right)^{\frac{1}{p}}.
\end{aligned}$$

This last inequality finishes our proof. □

Now we will prove one last proposition in order to prove the validity of condition iii). The proof is barely the same as the one of [9, Proposition 4.4].

Proposition 4.2. *Let $m \in \mathbb{N}$ be an even number and $f \in L^{2p}([0,1]^2)$. Then, there exists a positive constant C_m that does not depend on f such that, for all $s_0, s_0', x_0, x_0' \in [0,1]$ satisfying $0 < s_0 < s_0' < 2s_0$ and $0 < x_0 < x_0' < 2x_0$, we have that*

$$\sup_{n \geq 1} \mathbb{E}\left[\left(\int_{s_0}^{s_0'} \int_{x_0}^{x_0'} f(s,y) \theta_n^i(s,y) dy ds \right)^m \right] \leq C_q \left(\int_{s_0}^{s_0'} \int_{x_0}^{x_0'} f(s,y)^2 dy ds \right)^{\frac{m}{2}},$$

for any $i \in \{1,2\}$.

Proof. Let $i \in \{1,2\}$. For any $s_0, x_0 \in [0,1]$, we define

$$Z_n^i(s_0,x_0) := \int_0^{s_0}\int_0^{x_0} f(s,y)\theta_n^i(s,y)dyds.$$

Note that, for all s_0, s_0', x_0, x_0' such that $0 \le s_0 < s_0' \le 1$ and $0 \le x_0 < x_0' \le 1$, we have that

$$\mathbb{E}\left[\left(\Delta_{s_0,x_0} Z_n^i(s_0',x_0')\right)^m\right] \le \mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right], \tag{4.9}$$

where the random field $\bar{Z}_n$, which does not depends on i, is complex-valued and it is defined in the following way:

$$\bar{Z}_n(s_0,x_0) := \int_0^{s_0}\int_0^{x_0} f(s,y)\left(\theta_n^1(s,y) + i\theta_n^2(s,y)\right)dyds.$$

In this last expression $i = \sqrt{-1}$. At this point, we will try to apply some analogous arguments like the ones that are used in Lemma 3.3 of [8]. In order to bound the right hand-side in (4.9), we can deal with it as we did in the first part of Proposition 2.1, when we proved the tightness:

$$\begin{aligned}
\mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right] &= n^m K^m \mathbb{E}\left[\left|\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s,y)\sqrt{sy}e^{i\theta L_n(s,y)}dyds\right|^{2\frac{m}{2}}\right] \\
&= n^m K^m \mathbb{E}\left[\left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s_1,y_1)\sqrt{s_1y_1}e^{-i\theta L_n(s_1,y_1)}dy_1ds_1 \times \int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s_2,y_2)\sqrt{s_2y_2}e^{i\theta L_n(s_2,y_2)}dy_2ds_2\right)^{\frac{m}{2}}\right] \\
&= n^m K^m \mathbb{E}\left[\left(\int_{s_0}^{s_0'}\int_{x_0}^{x_0'}\int_{s_0}^{s_0'}\int_{x_0}^{x_0'} f(s_1,y_1)f(s_2,y_2)\sqrt{s_1s_2y_1y_2}e^{i\theta\left(L_n(s_2,y_2)-L_n(s_1,y_1)\right)}dy_1ds_1dy_2ds_2\right)^{\frac{m}{2}}\right] \\
&= n^m K^m \mathbb{E}\left[\int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^m f(s_j,y_j)\sqrt{s_jy_j}e^{(-1)^j i\theta L_n(s_j,y_j)}\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)dy_1ds_1\dots dy_mds_m\right].
\end{aligned}$$

Applying Fubini's theorem we have that the latter term is equal to

$$\begin{aligned}
&n^m K^m \int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^m f(s_j,y_j)\sqrt{s_jy_j}\ \mathbb{E}\left[e^{\sum_{j=1}^m(-1)^j i\theta L_n(s_j,y_j)}\right]\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)dy_1ds_1\dots dy_mds_m \\
&= n^m K^m \int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^m f(s_j,y_j)\sqrt{s_jy_j}\ \mathbb{E}\left[e^{i\theta\sum_{j=1}^m(-1)^j\Delta_{0,0}L_n(s_j,y_j)}\right]\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)dy_1ds_1\dots dy_mds_m \\
&\le \left|n^m K^m \int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^m f(s_j,y_j)\sqrt{s_jy_j}\ \mathbb{E}\left[e^{i\theta\sum_{j=1}^m(-1)^j\Delta_{0,0}L_n(s_j,y_j)}\right]\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)dy_1ds_1\dots dy_mds_m\right| \\
&\le n^m K^m \int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^m f(s_j,y_j)\sqrt{s_jy_j}\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^m(-1)^j\Delta_{0,0}L_n(s_j,y_j)}\right]\right|\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)dy_1ds_1\dots dy_mds_m.
\end{aligned}$$

Now we can observe that

$$\begin{aligned}
\sum_{j=1}^m(-1)^j\Delta_{0,0}L_n(s_j,y_j) &= \sum_{j=1}^{\frac{m}{2}}\Delta_{0,0}L_n(s_{2j},y_{2j}) - \sum_{j=1}^{\frac{m}{2}}\Delta_{0,0}L_n(s_{2j-1},y_{2j-1}) \\
&= \sum_{j=1}^{\frac{m}{2}}\Delta_{s_0,t_0}L_n(s_{2j},y_{2j}) + \sum_{j=1}^{\frac{m}{2}}\Delta_{s_0,0}L_n(s_{2j},t_0) + \sum_{j=1}^{\frac{m}{2}}\Delta_{0,t_0}L_n(s_0,y_{2j}) + mL_n(s_0,t_0) \\
&\quad - \sum_{j=1}^{\frac{m}{2}}\Delta_{s_0,t_0}L_n(s_{2j-1},y_{2j-1}) - \sum_{j=1}^{\frac{m}{2}}\Delta_{s_0,0}L_n(s_{2j-1},t_0) - \sum_{j=1}^{\frac{m}{2}}\Delta_{0,t_0}L_n(s_0,y_{2j-1}) - mL_n(s_0,t_0) \\
&= \sum_{j=1}^m(-1)^j\Delta_{s_0,t_0}L_n(s_j,y_j) + \sum_{j=1}^m(-1)^j\Delta_{s_0,0}L_n(s_j,t_0) + \sum_{j=1}^m(-1)^j\Delta_{0,t_0}L_n(s_0,y_j).
\end{aligned}$$

Hence

$$\exp\left(\sum_{j=1}^{m}(-1)^j\Delta_{0,0}L_n(s_j,y_j)\right) = \exp\left(\sum_{j=1}^{m}(-1)^j\Delta_{s_0,t_0}L_n(s_j,y_j) + \sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)\right.$$
$$\left. + \sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)\right)$$
$$= \exp\left(\sum_{j=1}^{m}(-1)^j\Delta_{s_0,t_0}L_n(s_j,y_j)\right)\exp\left(\sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)\right)$$
$$\times \exp\left(\sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)\right).$$

We observe that the families $\left\{\Delta_{s_0,t_0}L_n(s_j,y_j)\right\}_{j=1}^{m}$, $\left\{\Delta_{s_0,0}L_n(s_j,t_0)\right\}_{j=1}^{m}$ and $\left\{\Delta_{0,t_0}L_n(s_0,y_j)\right\}_{j=1}^{m}$ are independent. Then

$$\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{0,0}L_n(s_j,y_j)}\right]\right| = \left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,t_0}L_n(s_j,y_j)}\right]\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)}\right]\right.$$
$$\left.\times\ \mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)}\right]\right|$$
$$\leq \left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,t_0}L_n(s_j,y_j)}\right]\right|\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)}\right]\right|$$
$$\times \left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)}\right]\right|$$
$$\leq \left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)}\right]\right|\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)}\right]\right|.$$

Thus,

$$\mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right] \leq n^mK^m\int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^{m}f(s_j,y_j)\sqrt{s_jy_j}\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)}\right]\right|$$
$$\times\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)}\right]\right|\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)dy_1ds_1\dots dy_mds_m.$$

Since $y_j < x_0' < 2x_0$ and $s_j < s_0' < s_0$, the last expression can be bounded by

$$2^m(s_0x_0)^{\frac{m}{2}}n^mK^m\int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^{m}\left(\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)f(s_j,y_j)\right)\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{s_0,0}L_n(s_j,t_0)}\right]\right|$$
$$\times\left|\mathbb{E}\left[e^{i\theta\sum_{j=1}^{m}(-1)^j\Delta_{0,t_0}L_n(s_0,y_j)}\right]\right|dy_1ds_1\dots dy_mds_m$$
$$= m!2^m(s_0x_0)^{\frac{m}{2}}n^mK^m\int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^{m}\left(\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)f(s_j,y_j)\right)$$
$$\times e^{-a(\theta)n\left((y_{(m)}-y_{(m-1)})s_{m-1}+\dots+(y_{(2)}-y_{(1)})s_1\right)}$$
$$\times e^{-a(\theta)n\left((s_m-s_{m-1})y_{(m-1)}+\dots+(s_2-s_1)y_{(1)}\right)}\mathbf{1}_{\{s_1\leq\dots\leq s_m\}}dy_1ds_1\dots dy_mds_m.$$

Here, we have dealt with the modulus of the expectations in the same way we did in Proposition 2.1 and where $y_{(1)},\dots,y_{(m)}$ are the variables $y_1,\dots,y_m$ in an increasing form. We also have considered all the possible orders for the s_j. Now, as $s_0 \leq s_j$ and $x_0 \leq y_{(j)}$ for all j, we will bound again the exponential function:

$$\begin{aligned}\mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right] \leq & m!2^m(s_0x_0)^{\frac{m}{2}}n^mK^m\int_{\left([0,1]\times[0,1]\right)^m}\prod_{j=1}^m\left(\mathbf{1}_{[s_0,s_0']}(s_j)\mathbf{1}_{[x_0,x_0']}(y_j)f(s_j,y_j)\right)\\ & \times e^{-a(\theta)ns_0\left((y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})\right)}\\ & \times e^{-a(\theta)nx_0\left((s_m-s_{m-1})+\cdots+(s_2-s_1)\right)}\\ & \times \mathbf{1}_{\{s_1\leq\cdots\leq s_m\}}dy_1ds_1\dots dy_mds_m. \qquad (4.10)\end{aligned}$$

Note that in equation (4.10) it was not possible to order $y_1,\dots,y_m$, because neither the function $(s,y)\mapsto f(s,y)$ factorizes nor $(y_1,\dots,y_m)\mapsto f(s_1,y_1)\dots f(s_m,y_m)$ is symmetric. The fact that the variables s_i appeared ordered determines $\frac{m}{2}$ couples $(s_1,s_2),(s_3,s_4),\dots,(s_{m-1},s_m)$, such that the second element in each couple is greater o equal than the first one. And for y_i, we also have $\frac{m}{2}$ couples $(y_{(1)},y_{(2)}),\dots,(y_{(m-1)},y_{(m)})$ with the same property.

The key point of this proof is to factorize the product inside the integral in (4.10) as two convenient products:

$$\prod_{j=1}^{\frac{m}{2}}\left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j})f(s_{i_j},y_{i_j})\right)\prod_{k=1}^{\frac{m}{2}}\left(\mathbf{1}_{[s_0,s_0']}(s_{r_k})\mathbf{1}_{[x_0,x_0']}(y_{r_k})f(s_{r_k},y_{r_k})\right),$$

where $\mathcal{J}=\{i_j,j=1,\dots,\frac{m}{2}\}$ and $\mathcal{R}=\{r_k,k=1,\dots,\frac{m}{2}\}$ are two disjoint subsequences of $\{1,2,\dots,m\}$. In particular, we have that $\mathcal{J}\biguplus\mathcal{R}=\{1,2,\dots,m\}$. We will cast these subsequences using the following rule: each couple (s_i,s_{i+1}) should be formed by one element of the form s_{i_j} and by another one of the form s_{r_k}, and each couple (y_i,y_{i+1}) should have one element of the form y_{i_j} and another one of the form y_{r_k}. To obtain this, we will split the m elements of $f(s_1,y_1),\dots,f(s_m,y_m)$ into two groups of $\frac{m}{2}$ elements:

$$\begin{aligned}F_1&=\{f(s_{i_1},y_{i_1}),\dots f(s_{i_{\frac{m}{2}}},y_{i_{\frac{m}{2}}})\},\\ F_2&=\{f(s_{r_1},y_{r_1}),\dots f(s_{r_{\frac{m}{2}}},y_{r_{\frac{m}{2}}})\}.\end{aligned}$$

In order to determine the elements of each group satisfying the above conditions, we will use an iterative method: let us start with an element of F_1 and we associate to it an element of F_2 satisfaying what we want; then, we will associate to the last element of F_2 a suitable element of F_1, and we will continue the same way on. More precisely, we will start with $f(s_i,y_i)=f(s_1,y_1)$. So, if at any step of the iteration we have an element $f(s_{i_j},y_{r_k})\in F_1$, we will associate to it an element $f(s_{r_k},y_{r_k})\in F_2$ such that $\{s_{i_j},s_{r_k}\}$ is one of the (s_i,s_{i+1}) couples (note that it does not matter the order between i_j and r_k), this will happen in the odd numbered steps. On the other hand, if at any step of the iteration we have an element $f(s_{r_k},y_{r_k})\in F_2$, in this case we will associate to it an element $f(s_{i_j},y_{s_j})\in F_1$ such that $\{y_{i_j},y_{r_k}\}$ is one of the $(y_{(2i-1)},y_{(2i)})$ couples with $i\geq 1$ (note that it does not matter the order between i_j and r_k), this will happen in the even numbered steps. There is one thing left to clear up and it is if we reach the case where at some step of the iteration we get to an element of F_1 or F_2 that had been already selected. In this case, we won't select the last element, but instead we will choose another element that had not been chosen before.
Let us illustrate our iterative method with the following examples:

- Set m=8 and $y_1,\dots,y_8$ such that

$$y_8<y_5<y_4<y_7<y_1<y_6<y_2<y_3,$$

therefore:

$$\begin{aligned}&y_{(1)}=y_8,y_{(2)}=y_5,y_{(3)}=y_4,y_{(4)}=y_7,\\ &y_{(5)}=y_1,y_{(6)}=y_6,y_{(7)}=y_2,y_{(8)}=y_3.\end{aligned}$$

Remember that $s_1\leq\cdots\leq s_8$. Let us start with $f(s_1,y_1)\in F_1$. Then, the iteration will associate to this element $f(s_2,y_2)\in F_2$, because $\{s_1,s_2\}$ determines (s_1,s_2). Note that with the s_i is easier as they are ordered. Now, we can see that $y_{(7)}=y_2$, so the iteration determines that the following element of our sequence is $f(s_3, y_3)$. This, since $\{y_2, y_3\}$ determines the $(y_{(7)}, y_{(8)})$. If we continue this way we have that the sequence generated by the iteration is:

$$\begin{aligned} f(s_1,y_1) &\to f(s_2,y_2) = f(s_2,y_{(7)}),\\ f(s_2,y_{(7)}) &\to f(s_3,y_{(8)}) = f(s_3,y_3),\\ f(s_3,y_3) &\to f(s_4,y_4) = f(s_4,y_{(3)}),\\ f(s_4,y_{(3)}) &\to f(s_7,y_{(4)}) = f(s_7,y_7),\\ f(s_7,y_7) &\to f(s_8,y_8) = f(s_8,y_{(1)}),\\ f(s_8,y_{(1)}) &\to f(s_5,y_{(2)}) = f(s_5,y_5),\\ f(s_5,y_5) &\to f(s_6,y_6). \end{aligned}$$

Then, $F_1 = \{f(x_1,y_1), f(x_3,y_3), f(x_7,y_7), f(x_5,y_5)\}$ and $F_2 = \{f(x_2,y_2), f(x_4,y_4), f(x_8,y_8), f(x_6,y_6)\}$. In particular, any couple (s_i, s_{i+1}) contains an s of F_1 and another one of F_2 and any couple $(y_{(2i-1)}, y_{(2i)})$ contains a y of F_1 and another one of F_2.

- Set m=10 and $y_1, \dots, y_{10}$ such that

$$y_7 < y_{10} < y_1 < y_8 < y_6 < y_2 < y_3 < y_5 < y_9 < y_4,$$

hence:

$$\begin{aligned} &y_{(1)} = y_7, y_{(2)} = y_{10}, y_{(3)} = y_1, y_{(4)} = y_8, y_{(5)} = y_6,\\ &y_{(6)} = y_2, y_{(7)} = y_3, y_{(8)} = y_5, y_{(9)} = y_9, y_{(10)} = y_4. \end{aligned}$$

So, we have that the sequence generated by the iteration is:

$$\begin{aligned} f(s_1,y_1) &\to f(s_2,y_2) = f(s_2,y_{(6)}),\\ f(s_2,y_{(6)}) &\to f(s_3,y_{(7)}) = f(s_3,y_3),\\ f(s_3,y_3) &\to f(s_4,y_4) = f(s_4,y_{(10)}),\\ f(s_4,y_{(10)}) &\to f(s_9,y_{(9)}) = f(s_9,y_9),\\ f(s_9,y_9) &\to f(s_{10},y_{10}) = f(s_{10},y_{(2)}),\\ f(s_{10},y_{(2)}) &\to f(s_7,y_{(1)}) = f(s_7,y_7),\\ f(s_7,y_7) &\to f(s_8,y_8) = f(s_8,y_{(4)}),\\ f(s_8,y_{(4)}) &\to f(s_6,y_{(5)}) = f(s_6,y_6),\\ f(s_6,y_6) &\to f(s_7,y_7). \end{aligned}$$

Then, $F_1 = \{f(s_1,y_1), f(x_3,y_3), f(x_9,y_9), f(x_7,y_7), f(x_6,y_6)\}$ and $F_2 = \{f(x_2,y_2), f(x_4,y_4), f(x_{10},y_{10}), f(x_8,y_8), f(x_7,y_7)\}$. In particular, any couple (s_i, s_{i+1}) contains a s of F_1 and another one of F_2 and any couple $(y_{(2i-1)}, y_{(2i)})$ contains a y of F_1 and another one of F_2.

Now, let us return to (4.10), where we can use the above iteration process and also the fact that $ab \leq \frac{1}{2}(a^2+b^2)$, in order to obtain

$$\mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right] \leq m!2^{m-1}(s_0x_0)^{\frac{m}{2}}K^m(J_1+J_2),$$

with

$$\begin{aligned} J_1 =& n^m \int_{\left([0,1]\times[0,1]\right)^m} \prod_{i_j\in\mathcal{J}} \left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j})f(s_{i_j},y_{i_j})^2\right)\\ &\times e^{-a(\theta)ns_0\left((y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})\right)}\\ &\times e^{-a(\theta)nx_0\left((s_m-s_{m-1})+\cdots+(s_2-s_1)\right)}\\ &\times \mathbf{1}_{\{s_1\leq\cdots\leq s_m\}}dy_1ds_1\dots dy_mds_m \end{aligned}$$

and

$$
\begin{aligned}
J_2 =& n^m \int_{\left([0,1]\times[0,1]\right)^m} \prod_{r_k\in\mathcal{R}} \left(\mathbf{1}_{[s_0,s_0']}(s_{r_k})\mathbf{1}_{[x_0,x_0']}(y_{r_k}) f(s_{r_k},y_{r_k})^2\right) \\
& \times e^{-a(\theta)ns_0\left((y_{(m)}-y_{(m-1)})+\cdots+(y_{(2)}-y_{(1)})\right)} \\
& \times e^{-a(\theta)nx_0\left((s_m-s_{m-1})+\cdots+(s_2-s_1)\right)} \\
& \times \mathbf{1}_{\{s_1\le\cdots\le s_m\}} dy_1 ds_1\dots dy_m ds_m.
\end{aligned}
$$

We will only deal with the bound for J_1, because for the bound of J_2 the arguments are the same. We integrate J_1 with respect to s_{r_k} and y_{r_k}, with $r_k\in\mathcal{R}$, for $k=1,\dots,\frac{m}{2}$. Recall that thanks to our iterative method, the variables s_{r_k} have been chosen in a way that they appear only once in the couples (s_i,s_{i+1}) (the same happens with y_{r_k} with respect of the couples $(y_{(2i-1)},y_{(2i)})$).
Notice that, for $k=1,\dots,\frac{m}{2}$,

$$
\int_{s_0}^{s_0'} \exp\{-a(\theta)nx_0(s_{r_k}-s_i)\}\mathbf{1}_{\{s_i\le s_{r_k}\}} ds_{r_k} \le (s_{r_k}-s_i)
$$

or

$$
\int_{s_0}^{s_0'} \exp\{-a(\theta)nx_0(s_i-s_{r_k})\}\mathbf{1}_{\{s_{r_k}\le s_i\}} ds_{r_k} \le (s_i-s_{r_k}),
$$

for some s_i and s_{i+1}, depending on the position that s_{r_k} has in its couple. For the integral with repect of y_{r_k} we can get the same kind of bound. Also, as $0<s_0\le s_0'\le 1$, we can find a constant C such that

$$
\int_{s_0}^{s_0'} \exp\{-a(\theta)nx_0(s_{r_k}-s_i)\}\mathbf{1}_{\{s_i\le s_{r_k}\}} ds_{r_k} \le C\frac{1}{n}
$$

or

$$
\int_{s_0}^{s_0'} \exp\{-a(\theta)nx_0(s_i-s_{r_k})\}\mathbf{1}_{\{s_{r_k}\le s_i\}} ds_{r_k} \le C\frac{1}{n}.
$$

Hence

$$
\begin{aligned}
J_1 \le C_m & \int_{\left([0,1]\times[0,1]\right)^{\frac{m}{2}}} \prod_{j=1}^{\frac{m}{2}} \left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j}) f(s_{i_j},y_{i_j})^2\right) \\
& \times e^{-a(\theta)ns_0\left((y_{(m)}-y_{(m-1)})+\cdots+(y_{(4)}-y_{(3)})+(y_{(2)}-y_{(1)})\right)} dy_{i_1}ds_{i_1}\dots dy_{i_{\frac{m}{2}}}ds_{i_{\frac{m}{2}}}.
\end{aligned}
$$

Let us notice that

$$
e^{-a(\theta)ns_0(y_{(2i)}-y_{(2i-1)})} \le 1,
$$

for all $1\le i\le\frac{m}{2}$. Then

$$
\begin{aligned}
J_1 &\le C_m \int_{\left([0,1]\times[0,1]\right)^{\frac{m}{2}}} \prod_{j=1}^{\frac{m}{2}} \left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j}) f(s_{i_j},y_{i_j})^2\right) dy_{i_1}ds_{i_1}\dots dy_{i_{\frac{m}{2}}}ds_{i_{\frac{m}{2}}} \\
&= C_m \prod_{j=1}^{\frac{m}{2}} \left(\int_0^1\int_0^1 \left(\mathbf{1}_{[s_0,s_0']}(s_{i_j})\mathbf{1}_{[x_0,x_0']}(y_{i_j}) f(s_{i_j},y_{i_j})^2\right) dy_{i_j}ds_{i_j}\right) \\
&= C_m \left(\int_0^1\int_0^1 \left(\mathbf{1}_{[s_0,s_0']}(s)\mathbf{1}_{[x_0,x_0']}(y) f(s,y)^2\right) dyds\right)^{\frac{m}{2}}.
\end{aligned}
$$

As we mentioned before, we can use the same arguments in order to obtain the same bound of J_2. Thus,

$$
\mathbb{E}\left[\left|\Delta_{s_0,x_0}\bar{Z}_n(s_0',x_0')\right|^m\right] \le C_m \left(\int_0^1\int_0^1 \left(\mathbf{1}_{[s_0,s_0']}(s)\mathbf{1}_{[x_0,x_0']}(y) f(s,y)^2\right) dyds\right)^{\frac{m}{2}}.
$$

□

Proof of Theorem 4.1. As we explained at the beginnig of this chapter, we need $\theta_n^i \in L^2([0,1]^2)$, a.s., which is clear due to the definition of the random fields θ_n^i, $i = 1, 2$, and conditions i), ii) i iii) to be satisfied.

Notice that the condition i) is a consequence of the Theorem 0.1. Next, Proposition 4.1 implies condition ii). And finally, Proposition 4.2 has as a consequence condition iii).
Thus, the proof is complete. □

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