PhD program in Applied Mathematics

$(1, \leq \ell)$-identifying codes in digraphs and graphs

Doctoral thesis by:

Berenice Martínez Barona

Thesis advisors:

Camino Balbuena Martínez

Cristina Dalfó Simó

Departament d'Enginyeria Civil i Ambiental,

Universitat Politècnica de Catalunya

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ABSTRACT

The main subject of this PhD thesis is the study of $(1,\leq \ell)$-identifying codes in digraphs. The results presented in this work are divided into three parts. The first one focusses on the structural properties of digraphs admitting a $(1,\leq \ell)$-identifying code for $\ell \geq 2$. In the second part, we deal with the study of $(1,\leq \ell)$-identifying codes in line digraphs. Finally, in the third part we approach the problem from an algebraic perspective.

A $(1,\leq \ell)$-identifying code in a digraph D is a dominating subset C of vertices of D such that all distinct subsets of vertices of cardinality at most ℓ have distinct closed in-neighbourhoods within C. In the first part of the results, we prove that if D is a digraph admitting a $(1,\leq \ell)$-identifying code, then $\ell \leq \hat{\delta}^-(D)+1$, where $\hat{\delta}^-(D)$ denotes the minimum in-degree among all the vertices with at least one out-neighbour. Once this upper bound is established, we give some sufficient conditions for a digraph D, with $\hat{\delta}^-(D) \geq 1$, to admit a $(1,\leq \ell)$-identifying code for $\ell \in \{\hat{\delta}^-(D), \hat{\delta}^-(D)+1\}$. As a corollary, a result by Laihonen [45] (that states that a k-regular graph with girth at least 7 admits a $(1,\leq k)$-identifying code) is extended to any graph of minimum degree $\delta = k \geq 2$ and girth at least 7. Moreover, we show that every 1-in-regular digraph has a $(1,\leq 2)$-identifying code if and only if the girth of the digraph is at least 5. We also characterise all the 2-in-regular digraphs admitting a $(1,\leq \ell)$-identifying code for $\ell = 2,3$.

In the second part, we prove that every line digraph of minimum in-degree 1 does not admit a $(1,\leq \ell)$-identifying code for $\ell \geq 3$. Then, we give a characterisation of a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree 1 admitting a $(1,\leq 2)$-identifying code. The identifying number of a digraph D, $\overrightarrow{\gamma}^{ID}(D)$, is the minimum size among all the identifying codes of D. We establish for digraphs without digons (symmetric arcs) with both vertices of in-degree 1 that $\overrightarrow{\gamma}^{ID}(LD)$ is lower bounded by the number of arcs of D minus the number of vertices with out-degree at least one. Thus, we show that $\overrightarrow{\gamma}^{ID}(LD)$ attains the equality for a digraph having a 1-factor with minimum in-degree 2 and without digons with both vertices of in-degree 2. We conclude by giving an algorithm to construct identifying codes in oriented digraphs with minimum in-degree at least 2 and minimum out-degree at least 1.

In the third part, we give some sufficient algebraic and combinatorial conditions for a 2-in-regular digraph to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{2, 3\}$ by combining the results of the first part of this thesis with some algebraic results. As far as we know, it is the first time that the spectral graph theory has been applied to the identifying codes. We present a new method to obtain an upper bound on ℓ for digraphs by considering the positive and negative entries of eigenvectors associated with a negative eigenvalue of the adjacency matrix of the digraph. Likewise, we analyse the possible scope of using eigenvalue zero for the same purpose. The results obtained in the directed case can also be applied to graphs.

Mathematics Subject Classifications: 05C69, 05C20, 05C50.

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CONTENTS

1

INTRODUCTION

In the last hundred years, Graph Theory has gained more attention from researches due, among other reasons, to its beauty and multiple applications in several areas such as computer science, biology, social science, and chemistry. In this work, we focus on the study of $(1, \leq \ell)$-identifying codes in digraphs, which is a generalisation of the concept of identifying codes, first introduced for graphs by Karpovsky, Chakrabarty, and Levitin [44].

In the first part of this introduction, we give a few definitions regarding identifying codes together with some previous results needed in this work. In Section 1.2, we give a summary of the previously existing literature regarding identifying codes in digraphs and describe the contribution of the work presented in this thesis. In Section 1.3, we explain the structure of this thesis.

1.1 PRELIMINARIES

Since the introduction of identifying codes in graphs, motivated by solving fault diagnosis problems in multiprocessor systems, this concept has been widely studied partly due to its role in modelling different situations, such as emergency sensor networks in facilities (see e.g. Ray, Ungrangsi, Pellegrini, et al. [54]), routing in networks (see e.g. Laifenfeld, Trachtenberg, Cohen, et al. [49]), and analysis of secondary RNA structures (see e.g. Haynes, Knisley, Seier, et al. [41]). Let us explain in a simple way how the modelling of the fault diagnosis in multiprocessor systems works based on [14, 44]. The purpose of fault diagnosis is to test the system and locate exactly where the problem of the network is, that is, to locate the fault processors. The idea is to select some processors (constituting the code) to carry out a diagnosis. A processor belonging to the code, called codeword, tests itself and all the processors linked to it. Whenever a codeword detects a fault, it sends an alarm signal. It is expected, for a good diagnosis, that once some alarms are activated, to be able to determine where exactly the fault is.

Given a multiprocessor system, we can associate it with a graph $G = (V, E)$ such that V represents the processors and E the links between the processors. If there is just one fault at the time, we will need an identifying code to determine where exactly the fault is once some alarms are activated. If there can be at most k fault processors at the time, then we will need what is called a $(1, \leq k)$-identifying code. In the literature a large and rapidly growing number of papers related to identifying codes in graphs, can be found (see, e.g., [2, 3, 4, 8, 12, 20, 21, 22, 23, 33, 35, 38, 40, 44, 45, 50, 52]). Moreover, an online bibliography, constantly updated, on topics regarding identifying codes and some other related concepts is maintained by Lobstein [48].

Given a digraph D, let $d(u, v)$ denote the distance from u to v in D, that is, the length of any shortest directed path from vertex u to vertex v, if v is reachable from u, and $d(u, v) = \infty$ otherwise. For any vertex $v \in V(D)$ and any integer $t \geq 0$, define $B_t^-(v) = \{u \in V(D) \mid d(u, v) \leq t\}$ and $B_t^+(v) = \{u \in V(D) \mid d(v, u) \leq t\}$. We call $B_t^-(v)$ and $B_t^+(v)$ the *in-* and *out-ball* of radius t centred at v, respectively. Analogously, we have the following for graphs. Let G be a graph and let $d(u, v)$ denote the number of edges in any shortest path between u and v, then for any vertex $v \in V(G)$ we define $B_t(v) = \{u \in V(D) \mid d(u, v) \leq t\}$, called the ball of radius r centred at v. We denote with $|X|$ the cardinality of the set X.

Definition 1.1.1. *Let D be a digraph. Given two integers $t, \ell \geq 1$, a vertex set $C \subseteq V(D)$ is a $(t, \leq \ell)$-*identifying code *in D if for all distinct subsets $X, Y \subseteq V(D)$, with $|X|, |Y| \leq \ell$, we have*

$$B_t^-(X) \cap C \neq B_t^-(Y) \cap C. \tag{1}$$

Given a digraph D, a *dominating set* is a subset of vertices $S \subseteq V(D)$ such that all the vertices of $V(D) \setminus S$ are dominated by a vertex of S, that is, if $N^+[S] = V(D)$.

Notice that, if C is a $(t, \leq \ell)$-identifying code in D, then $X = \emptyset$ satisfies $|X| \leq \ell$ and $N^-[X] \cap C = \emptyset$, therefore, C is a dominating set. As we mentioned before, in this thesis we focus on the study of $(1, \leq \ell)$-identifying codes in digraphs. Observe that for $t = 1$, $B_t^-(v)$ is simply the closed in-neighbourhood of v. Hence, in particular we have the following definition.

Definition 1.1.2. *Let D be a digraph and $\ell \geq 1$ an integer. A vertex set $C \subseteq V(D)$ is a $(1, \leq \ell)$-*identifying code *in D if C for all distinct subsets $X, Y \subseteq V(D)$, with $|X|, |Y| \leq \ell$, we have*

$$N^-[X] \cap C \neq N^-[Y] \cap C. \tag{2}$$

An identifying code is known as an *identifying code*. The definition of identifying code for graphs introduced by Karpovsky, Chakrabarty, and Levitin [44], is obtained by taking $\ell = 1$ and omitting the superscript sign minus in the neighbourhoods in (2). Hence, an identifying code of a (di)graph is a dominating set such that any two vertices of the graph have distinct closed (in-)neighbourhoods within this set. Honkala, Laihonen, and Ranto [39, 47] generalised the notion of identifying codes in graphs to be able to identify not just vertices but also sets of vertices. This generalisation is obtained by exchanging B_t^- by B_t in (1). Thus, the definition of $(t, \leq \ell)$-identifying codes in digraphs is a natural extension of the concept of $(t, \leq \ell)$-identifying code in graphs. Not all graphs admit $(1, \leq \ell)$-identifying codes. For instance, Laihonen [45] pointed out that a graph containing an isolated edge cannot admit a

$(1, \leq 1)$-identifying code, because clearly, if $uv \in E(G)$ is isolated, then $N[u] = \{u, v\} = N[v]$. In fact, a graph containing an isolated complete bipartite graph $K_{r,d}$, with $r \leq d$, cannot admit a $(1, \leq d)$-identifying code. Regarding digraphs, we also have that not all digraphs admit $(1, \leq \ell)$-identifying codes.

1.2 $(t, \leq \ell)$-IDENTIFYING CODES IN DIGRAPHS

The model explained in the previous section regarding fault diagnosis in multiprocessors systems can be considered for modelling some other situations. For instance, it can be used to model situations where the link relation is not necessarily symmetric, for example, a hierarchical system in a social network. This gives rise to consider modelling the network with digraphs. Identifying codes in digraphs have not been much studied, unlike the case for graphs. Charon, Gravier, Hudry, and Lobstein [15] extended the concept of $(t, \leq 1)$-identifying codes from graphs to digraphs and proved that the decision problem of the existence of a $(t, \leq 1)$-identifying code of size at most k is NP-complete for any $t \geq 1$, even when restricted to strongly connected, directed, asymmetric, bipartite graphs or to directed asymmetric, bipartite graphs without directed cycles. A few years later, Charon, Gravier, Hudry, et al., [16] gave a linear algorithm to find a minimum identifying code in oriented trees. N. S. V. Rao [52] proposed a model to the *alarm placement problem* using directed graphs. Following this line, Xu and Xiao [60] introduced an alternative definition of identifying codes for digraphs by considering what they call the *to-set* of a vertex, which is, for each vertex u, the set of all vertices reachable from u. Skaggs studied identifying codes in oriented graphs in his PhD thesis [57] under the name *differentiating-domination set*. About identifying codes, he focused on the minimum value of an identifying number among all the orientations of some special graphs. Foucaud, Naserasr, and Parreau [28] characterised extremal digraphs for identifying codes. In Section 4.3, we show their characterisation.

Considering an alternative definition of the identifying code in terms of its closed out-neighbourhood, Cohen and Havent [13] gave some bounds for the minimum size of an identifying code, in their meaning, over all orientations of a graph. Coupechoux, Moncel, and Touati [18] studied $(t, \leq 1)$-identifying codes in tournaments. More recently, Boutin, Goliber, and Pelto [10] dealt $(t, \leq 1)$-identifying codes on directed de Bruijn graphs. With our work presented in this thesis, we contribute to this line of research by focusing on the study of $(1, \leq \ell)$-identifying codes in digraphs varying the parameter ℓ since, as we realised, there was no previous work on it.

We begin by pointing out that if C is a $(1, \leq \ell)$-identifying code in a digraph D, then the whole set of vertices V also is. Thus, we have the following straightforward observation.

Lemma 1.2.1. *A digraph $D = (V, A)$ admits some $(1, \leq \ell)$-identifying code if and only if for all distinct subsets $X, Y \subseteq V$ with $|X|, |Y| \leq \ell$, we have*

$$N^-[X] \neq N^-[Y]. \tag{3}$$

Two distinct vertices u and v of D are called *twins* if $N^-[u] = N^-[v]$, and called *false twins* if $N^-(u) = N^-(v)$ but u and v are not adjacent. Hence, we get the following.

Remark 1.2.1. *A digraph D admits an identifying code if and only if D is twin-free.*

In this thesis, we consider only finite digraphs. Another important concept regarding identifying codes in finite digraphs is the identifying number.

Definition 1.2.1. *Let D be a twin-free digraph. Then, the identifying number of D, $\overrightarrow{\gamma}^{ID}(D)$, is the minimum size among all the identifying codes of D.*

1.3 STRUCTURE OF THE THESIS

With the intention that this work is as self-contained as possible, in Chapter 2, we provide the basic definitions used throughout this thesis. As we mentioned before, the work published by other authors about $(t, \leq \ell)$-identifying codes in digraphs has been focused in the study for the case where $t \geq 1$ and $\ell = 1$. In Chapter 3 we focus on the study when $t = 1$ and $\ell \geq 1$. Regarding graphs, Laihonen and Ranto [47] proved that, if G is a connected graph with at least three vertices admitting a $(1, \leq \ell)$-identifying code, then the minimum degree is $\delta(G) \geq \ell$. We provided a similar result or digraphs by giving an upper tight bound for ℓ. Then, we give some necessary conditions for a digraph to admit a $(1, \leq \ell)$-identifying code, for ℓ reaching the previous bounds or almost, that is, the bound minus one. Likewise, Theorem 3.3.1 gives sufficient conditions, by prohibiting some subdigraphs, for a digraph to admit a $(1, \leq \ell)$-identifying code, for ℓ reaching the previous bounds or almost. As a corollary of this theorem, we extend a result given by Laihonen [45], regarding k-regular graphs. We finish this chapter by characterising the k-in-regular digraphs admitting a $(1, \leq \ell)$-identifying code for $k \in \{1, 2\}$ and $\ell \in \{2, 3\}$.

In Chapter 4, we focus on the study of $(1, \leq \ell)$-identifying codes in line digraphs. We prove that in this case $\ell \in \{1, 2\}$ and characterise the line digraphs admitting a $(1, \leq \ell)$-identifying code for $\ell \in \{1, 2\}$. Regarding line graphs, Foucaud, Gravier, Naserasr, Parreau, and Valicov [27] studied identifying codes and Junnila and Laihonen [43] studied $(1, \leq \ell)$-identifying codes for $\ell \geq 2$. In both papers, they use the edge-identifying code concept to work with the identifying code in line graphs. We use the analogy of this notion for digraphs to provide a lower bound for the identifying number of a line digraph admitting an identifying code. For this, we characterise the arc-identifying codes in digraphs in such a way that it also allows us to provide a linear algorithm to construct identifying codes in oriented graphs with minimum in-degree at least two and minimum out-degree at least one. Moreover, with this algorithm, we compute the exact value of the identifying code of the line digraph of oriented graphs with minimum degree at least two.

Chapter 5 contains our algebraic results regarding $(1, \leq \ell)$-identifying codes in digraphs. Some of these results are also applied to graphs. In the first part of this chapter, we combine algebraic graph theory with our results regarding 2-in-regular digraphs given in Section 3.4.1. As a result, we give some sufficient algebraic and combinatorial conditions for a 2-in-regular digraph to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{2, 3\}$. In Section 5.3, maintaining our goal of providing upper bounds for ℓ, we give a new method to obtain an upper bound for ℓ based on the eigenvalues and eigenvectors of the adjacency matrix of the digraph or graph.

We conclude this thesis by presenting in Chapter 6 the general conclusion of this work together with some suggestions for future research. It is important to mention that, in this thesis, we consider simple digraphs without loops or multiple edges.

The majority of the results presented in this thesis are contained in the following articles:

- C. Balbuena, C. Dalfó, and B. Martínez-Barona, Sufficient conditions for a digraph to admit a $(1, \leq \ell)$-identifying code, *Discuss. Math. Graph Theory*, in press.

- C. Balbuena, C. Dalfó, and B. Martínez-Barona, Characterizing identifying codes from the spectrum of a graph or digraph, *Linear Algebra Appl.*, **570** (2019) 138–147.

- C. Balbuena, C. Dalfó, and B. Martínez-Barona, Identifying codes in line digraphs, *Appl. Math. Comput.*, **383** (2020), in press.

2

BASIC DEFINITIONS

In this chapter, we introduce the basic notation, concepts and terminology used in the thesis. In general, we follow the book by Bang-Jensen and Gutin [9] for terminology and definitions. Besides, regarding spectral graph theory, we use the notation of Godsil and Royle [32].

2.1 DIGRAPHS

A *directed graph* (or just digraph) D consists of a non-empty finite set $V(D)$ of elements called *vertices* and a finite set $A(D)$ of ordered pairs of distinct vertices called *arcs*. For simplicity, in general we will denote with $D = (V, A)$ the digraph with vertex set $V(D) = V$ and arc set $A(D) = A$. The *order* and *size* of D is the number of vertices and arcs in D, respectively. For an arc (u, v), the vertex u is its *tail* and the vertex v is its *head*. We say that an arc is *incident* to a vertex v if v is the head or tail of a. We will often denote an arc (x, y) by xy. If (u, v) is an arc, we also say that u *dominates* v (v is dominated by u) and denote it by $u \to v$. A vertex u is *adjacent to* a vertex v if $(u, v) \in A$. We say that two different vertices u, v are adjacent if u is adjacent to v or v is adjacent to u. If both arcs $(u, v), (v, u) \in A$, then we say that they form a *digon*. A *loop* is an arc such that both, its head and its tail, are the same vertex. We say that two arcs are *multiple* if they have the same tail and the same head. A digraph is *simple* if it has neither loops nor multiple arcs. A digraph is *symmetric* if $(u, v) \in A$

implies $(v,u) \in A$. Therefore, a digon is often called *symmetric arc* of D. A digraph is *transitive* if $(u,v),(v,w) \in A$ implies $(u,w) \in A$. An *orientation* of a graph G is a digraph obtained from G by replacing each edge $\{x,y\} \in E(G)$ by either (x,y) or (y,x). A digraph D is said to be *oriented graph* if D is the orientation of a graph. Equivalently, a digraph is oriented if has no digons.

A *path* in D is a sequence $P = (x_1, x_2 \dots, x_k)$ of vertices from D such that for every $1 < i \leq k$ we have $(x_{i-1}, x_i) \in A(D)$ and $x_i \neq x_j$ for every $1 < i < j \leq k$. If $P = (x_1, x_2 \dots, x_k)$ is a path, then its *length* is $k-1$ and we say that P is a (x_1, x_k)-path. We say that a vertex u is *reachable from* vertex v if there is a (u,v)-path in D. A *directed cycle* in D is a sequence $C = (x_1, x_2 \dots, x_k, x_1)$ of vertices from D such that $(x_1, \dots, x_k)$ is a path and $(x_k, x_1) \in A(D)$. The length of a directed cycle $C = (x_1, x_2 \dots, x_k, x_1)$ is k. We call a directed cycle of length k a *k-cycle*. The *girth* g of a digraph is the length of a shortest directed cycle. Hence, an oriented graph has girth $g \geq 3$. A digraph H is a *subdigraph* of a digraph D if $V(H) \subseteq V(D)$, and for every $(u,v) \in A(H)$ we have $(u,v) \in A(D)$. If $V(H) = V(D)$, we say that H is a *spanning subdigraph* of D. A pair of digraphs D and H are *isomorphic*, denoted $D \cong H$, if there exists a bijection $\varphi : V(D) \to V(H)$ such that $(u,v) \in A(D)$ if and only if $(\varphi(u), \varphi(v)) \in A(H)$.

Let D be a digraph and $\mathcal{H}$ a family of digraphs, we say that D is *$\mathcal{H}$-free* if D does not contain a subdigraph isomorphic to any digraph in $\mathcal{H}$. If the family has only one element, $\mathcal{H} = \{H\}$, then we denote it as H-free.

The *out-neighbourhood* of a vertex u is $N^+(u) = \{v \in V : (u,v) \in A\}$ and the *in-neighbourhood* of u is $N^-(u) = \{v \in V : (v,u) \in A\}$. The *closed in-neighbourhood* of u is $N^-[u] = \{u\} \cup N^-(u)$. Given a subset of vertices $X \subseteq V$, let $N^-[X] = \bigcup_{u \in X} N^-[u]$. We denote with $|X|$ the cardinality of the set X. Then, the *out-degree* of u is $d^+(u) = |N^+(u)|$ and its *in-degree* $d^-(u) = |N^-(u)|$. We denote by $\delta^+(D)$ the *minimum out-degree* of the vertices in D, that is $\delta^+(D) = \min\{d^+(u) \mid u \in V(D)\}$, and by $\delta^-(D)$ its *minimum in-degree*. Analogously, $\Delta^+(D)$ and $\Delta^-(D)$ denote the *maximum out-degree* and *maximum in-degree*, respectively. The *minimum degree* is $\delta(D) = \min\{\delta^+(D), \delta^-(D)\}$. A digraph D is said to be *r-in-regular* if $d^-(v) = r$ for all $v \in V$, and *r-regular* if $d^+(v) = d^-(v) = r$ for all $v \in V$.

Given a digraph D, and an integer $i \geq 1$, we denote $V^+_{\geq i}(D) = \{v \in V(D) : d^+(v) \geq i\}$, $V^-_{\geq i}(D) = \{v \in V(D) : d^-(v) \geq i\}$, $V^+_i(D) = \{v \in V(D) : d^+(v) = i\}$, $V^-_i(D) = \{v \in V(D) : d^-(v) = i\}$, and $\hat{\delta}^-(D) = \min\{d^-_D(u) \mid u \in V^+_{\geq 1}(D)\}$.

Given a digraph D and a set of vertices $X \subset V(D)$, we denote with $D - X$ the subdigraph obtained from D by removing the vertices of X. That is, $D - X = (V(D) \setminus X, A(D) \setminus \{(u,v) \in A(D) \mid \{u,v\} \cap X \neq \emptyset\})$. In particular, if X consists of only a vertex, $X = \{u\}$, then we denote with $D - v$ the subdigraph $D - X$.

Given two sets S, T we denote with $S \triangle T$ the *symmetric difference* that is, $S \triangle T = (S \setminus T) \cup (T \setminus S)$. We say that two digraphs are disjoint if they does not have common vertices. Given two disjoint digraphs D_1 and D_2, that is, two digraphs such that $V(D_1) \cap V(D_2) = \emptyset$, the *disjoint union* of D_1 and D_2, denoted $D_1 \oplus D_2$ is the digraph with vertex set $V(D_1) \cup V(D_2)$ and arc set $A(D_1) \cup A(D_2)$.

2.2 GRAPHS

An *undirected graph* (or a graph) $G = (V, E)$ consists of a non-empty finite set $V = V(G)$ of elements called *vertices* and a finite set $E = E(G)$ of unordered pairs of distinct vertices called *edges*. An edge $\{u,v\} \in E(G)$ is denoted in short by uv. It will be clear from the context when we are refereeing to a an arc instead of an edge and vice versa. A graph H is a *subgraph* of graph G if $V(H) \subseteq V(G)$, and for every $\{u,v\} \in E(H)$ we have $\{u,v\} \in E(G)$. If every edge of $E(G)$ with both vertices in $V(H)$ is in $E(H)$, we say that H is an *induced subgraph* of G. A *path* in G is a sequence $P = (x_1, x_2 \ldots, x_k)$ of vertices from G such that for every $1 < i \leq k$ we have $\{x_{i-1}, x_i\} \in E(G)$ and $x_i \neq x_j$ for every $1 < i < j \leq k$. If such a path exists we say that there is a path between x_1 and x_k. A graph G is *connected* if for any two vertices $u, v \in V(G)$, there is a path between u and v. A *connected component* of a graph G is a maximal induced subgraph of G which is connected. A *cycle* in G is a sequence $C = (x_1, x_2 \ldots, x_k, x_1)$ of vertices from G such that $\{x_i, x_{i+1}\} \in E(G)$ for every $1 \leq i < k$ and $\{x_k, x_1\} \in E(G)$. The length of the cycle $C = (x_1, x_2 \ldots, x_k, x_1)$ is k. The *girth* g of a graph is the length of a shortest cycle. The *neighbourhood* of a vertex u is $N(u) = \{v \in V : uv \in E\}$. The *closed neighbourhood* of u is $N[u] = \{u\} \cup N^-(u)$. Given a vertex subset $X \subseteq V$, let $N[X] = \bigcup_{u \in X} N[u]$. The *degree* of u is $d(u) = |N(u)|$. The *minimum degree* of G is $\delta(D) = \min\{d(u) \mid u \in V(G)\}$. A graph G is said to be *r-regular* if $d(v) = r$ for all $v \in V$. *A dominating set* is a subset of vertices $S \subseteq V$ such that $N[S] = V$. Observe that every graph G with vertex set V and edge set E can be seen as a symmetric digraph denoted by $\overset{\leftrightarrow}{G}$, replacing each edge $uv \in E$ by the digon (u, v) and (v, u). Likewise, every digraph D has an associated graph G, known as the *underlying* graph of D, which consist of replacing each arc (u, v) in D by the edge $\{u, v\}$.

A graph is *bipartite* if its vertex set can be partitioned into two subsets X and Y so that every edge has one end in X and one end in Y. If G is a bipartite graph, with partition X and Y such that every vertex in X is adjacent to every vertex in Y, then G is called *complete bipartite graph.* We denote by $K_{r,d}$ the complete bipartite graph with partitions of cardinality r and d, respectively. The *complete graph* on n vertices, denoted K_n, is the graph having the set of all pairs of vertices as its edge set. A special family of oriented graphs is the family of tournaments. A *tournament* of order n is an orientation of the complete graph K_n of order.

3

STRUCTURAL PROPERTIES ON IDENTIFYING CODES

This chapter consists mainly of the results contained in [5]. Nevertheless, there are some essential differences, which will be described below. In the first section of this chapter, we focus on the question of given a digraph D how large can ℓ be in such a way that D admits a $(1, \leq \ell)$-identifying code. In other words, we give some upper bounds for ℓ. We also provide some results to justify why we focus on digraphs with minimum in-degree at least 1. The last two results of this section are contained in [6], but since these are results regarding general digraphs and not just line digraphs, we decided to place them in this chapter. In Section 3.2, we give some necessary conditions for a digraph to admit a $(1, \leq \ell)$-identifying code. These results are not contained in any of our three published papers.

Regarding the content of Section 3.3, let us first explain what is behind one of the differences between our work in [5] and the one contained in this chapter. In [5], we prove that if D is a digraph admitting a $(1, \leq \ell)$-identifying code, then $\ell \leq \min\{d^-(u)+1 \mid u \in V(D) \text{ and } d^+(u) \geq 1\}$ (Corollary 5 in [5]). We also give some sufficient conditions for a digraph of minimum in-degree at least 1 to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{\delta^-, \delta^- + 1\}$ (Theorem 8 in [5]). Nevertheless, after working further on this topic, we realised that Theorem 8 could be improved. First, we notice that since $\delta^-(D)$ can be arbitrarily smaller than $\hat{\delta}^-(D) = \min\{d^-(u) \mid u \in V(D) \text{ and } d^+(u) \geq 1\}$, we could implement some conditions over Theorem 8 to guarantee that D admits an $(1, \leq \ell)$-identifying code with $\ell \in \{\hat{\delta}^-(D), \hat{\delta}^-(D)+1\}$, which we do in Theorem 3.3.1. The idea of the proof of Theorem 3.3.1 is very similar to the one of Theorem 8 [5]. Hence, in Section 3.4, we give some sufficient conditions for a digraph with $\hat{\delta}^-(D) \geq 1$ to admit a $(1, \leq \ell)$-identifying code for $\ell = \hat{\delta}^-(D), \hat{\delta}^-(D)+1$. Furthermore, there is another improvement of Theorem 8 (ii), (iii), and (iv) [5]. For instance, in Theorem 8 (iii) [5], we forbid 9 digraphs to guarantee the existence of a $(1, \leq \delta^-(D))$-identifying code, while in Theorem 3.3.1 (iii), we forbid 6 digraphs to guarantee the existence of a $(1, \leq \hat{\delta}^-(D))$-identifying code. In Theorem 3.3.1 (iv), we forbid 9 digraphs as subdigraphs to guarantee that the digraph admits a $(1, \leq \hat{\delta}^-(D)+1)$-identifying code instead of 11, which is the case in Theorem 8 (iv) [5]. Moreover, as a corollary of Theorem 3.3.1, we obtain Theorem 3.3.2. The last section of this chapter corresponds to Section 3 in [5]. In this section, we prove that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if the girth of the digraph is at least 5. We also characterise all the 2-in-regular digraphs admitting a $(1, \leq \ell)$-identifying code for $\ell = 2, 3$.

3.1 UPPER BOUNDS FOR ℓ

As already mentioned in the introduction, Laihonen and Ranto [47] proved that if G is a connected graph with at least three vertices admitting a $(1, \leq \ell)$-identifying code, then the minimum degree is $\delta(G) \geq \ell$. We present the following similar result for digraphs.

Proposition 3.1.1. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code and u a vertex such that $d^+(u) \geq 1$. Then, $\ell \leq d^-(u)+1$. Furthermore, if u belongs to a digon, then $\ell \leq d^-(u)$.*

Proof. Let $u \in V^+_{\geq 1}(D)$ and $v \in N^+(u)$. Then, both sets $X = N^-(u) \cup \{u, v\}$ and $Y = N^-(u) \cup \{v\}$ have the same closed in-neighbourhood. Consequently, $\ell \leq d^-(u)+1$. Furthermore, if $v \in N^-(u)$, then $X' =$

$N^-(u) \cup \{u\}$ and $Y' = N^-(u)$ have the same closed in-neighbourhood implying that $\ell \leq d^-(u)$. □

Recall that we denote $\hat{\delta}^-(D) = \min\{d_D^-(u) \mid u \in V_{\geq 1}^+(D)\}$. Hence, we have the following direct consequence of the above proposition.

Corollary 3.1.1. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. Then,*

$$\ell \leq \hat{\delta}^-(D) + 1.$$

Moreover, if $\ell = \hat{\delta}^-(D) + 1$, then any vertex u with $d^-(u) = \hat{\delta}^-$ does not lay on a digon.

Observe that by Corollary 3.1.1, we know that if D is a digraph admitting a $(1, \leq \ell)$-identifying code, then $\ell \leq \hat{\delta}^-(D) + 1$, but ℓ could be strictly larger than $\delta^-(D)$ if $\delta^+(D) = 0$, as we show with the following result.

Proposition 3.1.2. *Let D be a digraph with minimum in-degree $\delta^-(D) \geq 2$ admitting a $(1, \leq \ell)$-identifying code and let β with $1 \leq \beta < \delta^-(D)$ be an integer. Then, there is a digraph D' with minimum in-degree $\delta^-(D') = \beta$ and minimum out-degree $\delta^+(D') = 0$, admitting a $(1, \leq \ell)$-identifying code and having D as an induced subdigraph.*

Proof. Take two different vertices u and w such that $u \in V(D)$ and $w \notin V(D)$. Let $\{u_1, u_2, \ldots, u_\beta\} \subseteq N_D^-(u)$ be a set of β different in-neighbours of u in D. Consider the digraph D' with vertex set $V(D) \cup \{w\}$ and arc set $A(D') = A(D) \cup \{(u_i, w) | \ 1 \leq i \leq \beta\}$. It is clear that D is an induced subdigraph of D' and that the minimum in-degree and out-degree of D' are $\delta^-(D') = \beta$ and $\delta^+(D') = 0$, respectively. Then, for any subset of vertices $U \subseteq V(D')$ we have

$$N_{D'}^-[U] = \begin{cases} N_D^-[U] & \text{if } w \notin U, \\ \{u_1, \ldots, u_\beta, w\} & \text{if } U = \{w\} \\ N_D^-[U \setminus \{w\}] \cup \{u_1, \ldots, u_\beta, w\} & \text{if } w \in U \text{ and } U \setminus \{w\} \neq \emptyset. \end{cases} \tag{4}$$

Now, we prove that D' admits a $(1, \leq \ell)$-identifying code reasoning by contradiction. Suppose D admits a $(1, \leq \ell)$-identifying code, but

D' does not. Let $X, Y \subseteq V(D')$ be two different subsets such that $1 \leq |Y| \leq |X| \leq \ell$ and $N^-_{D'}[X] = N^-_{D'}[Y]$. By (4), since D admits a $(1, \leq \ell)$-identifying code and $w \notin V(D)$, it follows that $w \in X \cap Y$. Observe that this implies $\ell \geq 2$. Let $X' = X \setminus \{w\}$ and $Y' = Y \setminus \{w\}$. Then, we get

$$\begin{aligned} N^-_{D'}[X] &= N^-_D[X'] \cup \{u_1, \ldots, u_\beta, w\} \\ &= N^-_D[Y'] \cup \{u_1, \ldots, u_\beta, w\} = N^-_{D'}[Y]. \end{aligned}$$

Hence, if $Y' = \emptyset$ (and, then, $Y = \{w\}$), we have $N^-_{D'}[Y] = \{u_1, \ldots, u_\beta, w\}$, implying that $N^-_D[X'] \subseteq \{u_1, \ldots, u_\beta\}$. Observe that since $\beta < \delta^-(D)$, it follows that $u \notin X'$. Hence, in D we have $N^-_D[X' \cup \{u\}] = N^-_D[u]$, a contradiction with the fact that D admits a $(1, \leq \ell)$-identifying code. Therefore, $|Y'| \geq 1$. Now, since $N^-_D[X'] \neq N^-_D[Y']$, it follows that $N^-_D[X'] \triangle N^-_D[Y'] \subseteq \{u_1, \ldots, u_\beta\}$. Consider the following two sets of vertices of D, $\widehat{X} = X' \cup \{u\}$ and $\widehat{Y} = Y' \cup \{u\}$, then $2 \leq |\widehat{Y}| \leq |\widehat{X}| \leq \ell$ and $N^-_D[\widehat{X}] = N^-_D[\widehat{Y}]$, a contradiction. This completes the proof.

□

Hence, in particular, if D is a digraph with minimum in-degree $\delta^-(D) > 2$ admitting a $(1, \leq \delta^-(D)+1)$-identifying code and β an integer such that $1 \leq \beta < \delta^-(D) - 1$, then there is a digraph D' with $\delta^-(D') = \beta$ and $\delta^+(D') = 0$ admitting a $(1, \leq \ell)$-identifying code, with $\ell = \delta^-(D) + 1 > \beta + 1$.

Recall that a digraph is *weakly connected* if its underlying graph is connected. Let D be a digraph, and $D_1, \ldots, D_r$ its weakly connected components. Then, for every integer $i \in \{1, \ldots, r\}$ and any $X \subseteq V(D_i)$, we have $N^-_D[X] \cap V(D_i) = N^-_{D_i}[X] = N^-_D[X]$ and, for any $U \subseteq V(D)$, we have $N^-_D[U] = \cup_{i=1}^r N^-_{D_i}[U \cap V(D_i)]$. Hence, the following result holds.

Proposition 3.1.3. *A digraph D admits a $(1, \leq \ell)$-identifying code if and only if any weakly connected component of D admits a $(1, \leq \ell)$-identifying code.*

As a consequence of Proposition 3.1.1 and Proposition 3.1.3, we get the following result regarding digraphs with minimum in-degree 0.

Corollary 3.1.2. *Let D be a digraph with $\delta^-(D) = 0$. Then,*

- *if there is $v \in V(D)$ such that $d^-(v) = 0$ and $d^+(v) \geq 1$, then if D admits a $(1, \leq \ell)$-identifying code, then $\ell = 1$,*
- *otherwise, D admits a $(1, \leq \ell)$-identifying code if and only if $D - V_0^-(D)$ admits a $(1, \leq \ell)$-identifying code.*

Our goal in this chapter is to establish sufficient conditions for a digraph to admit a $(1, \leq \ell)$-identifying code with ℓ as large as possible. Hence, by the above corollary, we consider only digraphs with minimum in-degree at least one. Furthermore, suppose D is a digraph with minimum in-degree 0, then, with the following result, we show that the identifying number of D is upper bounded by the sum of the identifying number of $D - V_0^-(D)$ and the cardinality of $V_0^-(D)$.

Proposition 3.1.4. *Let D be a digraph with minimum in-degree $\delta^- = 0$ and C' an identifying code of $D - V_0^-(D)$. Then, $C' \cup V_0^-(D)$ is an identifying code of D.*

Proof. Let C' be an identifying code of $D' = D - V_0^-(D)$ and $C = C' \cup V_0^-(D)$. To prove that C is an identifying code of D, let $u, v \in V(D)$ be two vertices such that $N^-[u] = N^-[v]$. By hypothesis, $C' \cap V_0^-(D) = \emptyset$, then

$$\begin{aligned} N^-[u] &= (N^-[u] \cap C') \cup (N^-[u] \cap V_0^-) \\ &= (N^-[v] \cap C') \cup (N^-[v] \cap V_0^-) \\ &= N^-[v]. \end{aligned}$$

Hence, $N^-[u] \cap C' = N^-[v] \cap C'$ and $N^-[u] \cap V_0^- = N^-[v] \cap V_0^-$. This implies that if $u \in V_0^-$, then also $v \in V_0^-$, otherwise $N^-[u] \cap C' = \emptyset$ and $N^-[v] \cap C' \neq \emptyset$, a contradiction. Hence, there are two cases to be considered. If $u, v \in V_0^-$, then $N^-[u] \cap V_0^- = N^-[u] = N^-[v] = N^-[v] \cap V_0^-$, implying $u = v$. If $u, v \in V(D) \setminus V_0^-$, then $N_{D'}^-[u] \cap C' =$

$N^-[u] \cap C' = N^-[v] \cap C' = N^-_{D'}[v] \cap C'$, implying that $u = v$ because C' is an identifying code of D'. This completes the proof. □

Corollary 3.1.3. *Let D be a digraph with minimum in-degree $\delta^- = 0$. Then,*

$$\overrightarrow{\gamma}^{ID}(D) \leq \overrightarrow{\gamma}^{ID}(D - V_0^-(D)) + |V_0^-(D)|.$$

We finish this section with another upper bound for ℓ that will be used in Chapter 3.

Lemma 3.1.1. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. If there are two different vertices $x, y \in V(D)$ such that $d^+(y) \geq 1$, then $\ell < d^-(y) - |N^-(x) \cap N^-(y)| + 3$. Moreover, if $x \in N^+(y)$, then $\ell < d^-(y) - |N^-(x) \cap N^-(y)| + 2$.*

Proof. Let x, y be two distinct vertices satisfying the hypothesis of the lemma, and let $w \in N^+(y)$. First, assume that $w \neq x$. Consider the set $X = (N^-(y) \setminus N^-(x)) \cup \{w, x, y\}$. Since $y \in N^-(w)$ and $w \in X - y$, we can check that $N^-[y] \subset N^-[X - y]$, which implies that $N^-[X] = N^-[X - y]$. Then, $\ell < |X| \leq d^-(y) - |N^-(x) \cap N^-(y)| + 3$. Finally, if $w = x$, repeating the same reasoning, we obtain that $\ell < |X| \leq d^-(y) - |N^-(x) \cap N^-(y)| + 2$. This completes the proof. □

Corollary 3.1.4. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code. If there are two different vertices $x, y \in V(D)$ such that $d^+(y) \geq 1$ and $N^-(y) \subseteq N^-(x)$, then $\ell \leq 2$.*

As a consequence, for any twin-free digraph with minimum degree $\delta \geq 1$ admitting a $(1, \leq \ell)$-identifying code, if D contains two false twin vertices, then $\ell \leq 2$.

3.2 SOME NECESSARY CONDITIONS FOR A DIGRAPH TO ADMIT A $(1, \leq \ell)$-IDENTIFYING CODE

We recall that a *transitive tournament* of three vertices is denoted by TT_3, see Figure 1. Observe that if D is a digraph with two twin vertices, say u and v, of in-degree at least 2, then D contains a TT_3. Moreover, we have the following result.

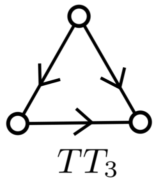


Figure 1. A transitive tournament on 3 vertices.

Proposition 3.2.1. *Let D be a digraph. If there is a vertex x with $d^+(x) \geq 1$ and $d^-(x) = \hat{\delta}^-(D)$, and it has an in-neighbour $u \in N^-(x)$ such that x and u lay in a TT_3, then D does not admit a $(1, \leq \hat{\delta}^- + 1)$-identifying code.*

Proof. Let x and u be as the hypothesis of the proposition. Observe that, by Corollary 3.1.1, we can assume x does not lay on a digon. Let $v \in V(D)$ such that $D[\{x, u, v\}] \cong TT_3$, then $uv \in A(D)$, and there are two cases to be considered: when $x \in N^-(v)$ and when $v \in N^-(x)$. In the first case, consider the sets of vertices $X = \{v\} \cup (N^-[x] \setminus \{u\})$ and $Y = X \setminus \{x\}$, then $N^-[X] = N^-[Y]$ implying that D does not admit a $(1, \leq \hat{\delta}^-(D) + 1)$-identifying code. Now suppose that $v \in N^-(x)$. Let $y \in N^+(x)$ and consider the sets of vertices $X = (N^-[x] \setminus \{u\}) \cup \{y\}$ and $Y = X \setminus \{x\}$, then $N^-[X] = N^-[Y]$ implying that D does not admit a $(1, \leq \hat{\delta}^-(D) + 1)$-identifying code. □

Let us construct from two disjoint digraphs admitting a $(1, \leq \ell)$-identifying code and with minimum degree at least 1, a digraph D admitting a $(1, \leq \ell)$-identifying code and containing a TT_3. The idea is to show the necessity of the conditions in Proposition 3.2.1.

Proposition 3.2.2. *Let D_1 and D_2 be two disjoint digraphs with minimum degree at least 1 and admitting a $(1, \leq \ell)$-identifying code. Then, there is a digraph D admitting a $(1, \leq \ell)$-identifying code and containing D_1, D_2, and TT_3 as subdigraphs.*

Proof. Let $u \in V(D_1)$ and $z, z' \in V(D_2)$ such that $zz' \in A(D)$. Consider the digraph D consisting of $V(D) = V(D_1) \cup V(D_2)$ and $A(D) = A(D_1) \cup A(D_2) \cup \{uz, uz'\}$.

We denote by $V_i = V(D_i)$ for $i \in \{1, 2\}$. Observe that for all $w \in V(D)$ we have:

$$N^-[w] = \begin{cases} N^-_{D_1}[w] & \text{if } w \in V_1, \\ N^-_{D_2}[w] & \text{if } w \in V_2 \setminus \{z, z'\}, \\ N^-_{D_2}[w] \cup \{u\} & \text{if } w \in \{z, z'\}. \end{cases}$$

Hence, for each set $W \subset V(D)$ we get:

$$N^-[W] = \begin{cases} N^-_{D_1}[W \cap V_1] \cup N^-_{D_2}[W \cap V_2] \cup \{u\} & \text{if } \{z, z'\} \cap W \neq \emptyset, \\ N^-_{D_1}[W \cap V_1] \cup N^-_{D_2}[W \cap V_2] & \text{otherwise.} \end{cases} \tag{5}$$

We prove that D admits a $(1, \leq \ell)$-identifying code by contradiction. Let $X, Y \subseteq V(D)$ be two different sets such that $1 \leq |X|, |Y|, \leq \ell$ and $N^-[X] = N^-[Y]$. Let $X_i = X \cap V_i$ and $Y_i = Y \cap V_i$, for $i \in \{1, 2\}$. By (5), it follows that $(X \cup Y) \cap \{z, z'\} \neq \emptyset$. Otherwise, $N^-_{D_i}[X_i] = N^-_{D_i}[Y_i]$ and $1 \leq |X_i|, |Y_i| \leq \ell$, implying since D_1 and D_2 admit a $(1, \leq \ell)$-identifying code, that $X = Y$, a contradiction. Without loss of generality, suppose that $X \cap \{z, z'\} \neq \emptyset$, then $X \cap \{z, z'\} \subseteq N^-_{D_2}[Y_2]$, implying that $Y_2 \neq \emptyset$. Hence, $1 \leq |X_2|, |Y_2| \leq \ell$ and $N^-_{D_2}[X_2] = N^-_{D_2}[Y_2]$ (by (5)). Then, by the hypothesis, $X_2 = Y_2$, implying $X_1 \neq Y_1$ and that

$$N^-_{D_1}[X_1] \cup \{u\} = N^-_{D_1}[Y_1] \cup \{u\}.$$

Moreover, $1 \leq |X_1|, |Y_1| < \ell$, since $X_2 = Y_2 \neq \emptyset$. Then, $N^-_{D_1}[X_1] \neq N^-_{D_1}[Y_1]$, implying that $N^-_{D_1}[X_1] \triangle N^-_{D_1}[Y_1] = \{u\}$. Thus, $N^-_{D_1}[X_1 \cup \{u\}] = N^-_{D_1}[Y_1 \cup \{u\}]$. Hence, by the hypothesis, $X_1 \cup \{u\} = Y_1 \cup \{u\}$. Without loss of generality, let us assume that $Y_1 = X_1 \cup \{u\}$. Let $w \in N^+(u) \setminus X_1$, which exists since $u \notin N^-[X_1]$ and $\delta^+(D_1) \geq 1$. Now, consider the sets $Y' = Y_1 \cup \{w\}$ and $X' = X_1 \cup \{w\}$. Then, $N^-_{D_1}[X'] = N^-_{D_1}[Y']$, a contradiction since $X' \neq Y'$ and $1 \leq |X'|, |Y'| \leq \ell$. This completes the proof. $\square$

Let us finish this section with a result regarding directed cycles.

Proposition 3.2.3. *Let D be a digraph and $C = (v_1, v_2, \dots, v_q, v_1)$ be a directed q-cycle of D, with $q \geq 2$.*

1. *If $q \in \{2, 3\}$ and there is at most one vertex $v_i \in V(C)$ such that $N^-[v_i] \setminus V(C) \neq \emptyset$, then D does not admit a $(1, \leq 2)$-identifying code.*

2. *If $q \geq 4$ and $N^-[V(C)] = V(C)$, then*

   - *D does not admit a $(1, \leq k)$-identifying code if $q = 2k$.*
   - *D does not admit a $(1, \leq k+1)$-identifying code if $q = 2k + 1$.*

Proof. 1. Without loss of generality, suppose that $N^-[v_i] \subseteq V(C)$ for any $i \neq 1$. Then, $N^-[\{v_1\}] = N[v_1] \cup \{v_1, v_2\} = N^-[\{v_1, v_2\}]$ if $q = 2$, and $N^-[\{v_1, v_2\}] = \{v_1, v_2, v_3\} \cup N^-[v_1] = N^-[\{v_1, v_3\}]$ if $q = 3$. In both cases, we get that D does not admit a $(1, \leq 2)$-identifying code.

2. Consider the sets $E_C = \{v_i \in C \mid i \text{ is an even number}\}$ and $O_C = \{v_i \in C \mid i \text{ is an odd number}\}$. Then, if $q = 2k$ for some $k \geq 2$, we have $N^-[E_C] = V(C) = N^-[O_C]$, implying that D does not admit a $(1, \leq k)$-identifying code. Analogously, if $q = 2k + 1$ for some $k \geq 2$, then $N^-[E_C \cup \{v_q\}] = V(C) = N^-[O_C]$, hence, D does not admit a $(1, \leq k+1)$-identifying code. □

3.3 SOME SUFFICIENT CONDITIONS FOR A DIGRAPH TO ADMIT A $(1, \leq \ell)$-IDENTIFYING CODE

We point out the following remark, which will be useful in the proof of the main result of this section.

Remark 3.3.1. *Let D be a TT_3-free digraph. Then, for every arc (x, y) of D, we have $N^-(x) \cap N^-(y) = \emptyset$ and $N^+(x) \cap N^+(y) = \emptyset$.*

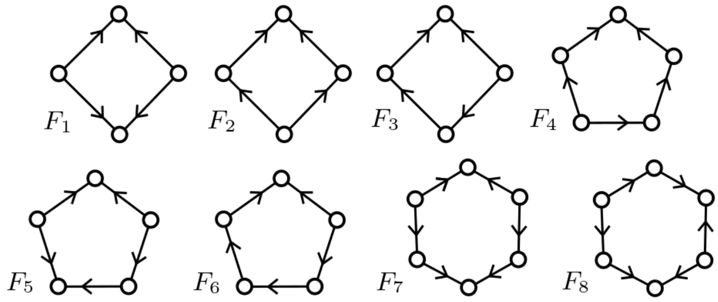


Figure 2. All the forbidden subdigraphs of Theorem 3.3.1.

We denote with $\mathcal{F}$ the family of digraphs F_1- F_8 of Figure 2. For the next theorem, we made reference to the different cases F_1-F_8 of Figure 2 without mentioning the figure.

Theorem 3.3.1. *Let D be a twin-free and TT_3-free digraph with $\hat{\delta}^-(D) \geq 1$.*

(*i*) *If $\hat{\delta}^- \geq 2$ and D is F_1-free, then D admits a $(1, \leq \hat{\delta}^- - 1)$-identifying code.*

(*ii*) *If the vertices of in-degree $\hat{\delta}^-$ does not lay on a digon and D is F_1-free, then D admits a $(1, \leq \hat{\delta}^-)$-identifying code.*

(*iii*) *If D is $\{F_1, F_2, F_5, F_7, F_8\}$-free, then D admits a $(1, \leq \hat{\delta}^-)$-identifying code.*

(*iv*) *Suppose that $\hat{\delta}^- \geq 2$ and the vertices of in-degree $\hat{\delta}^-$ do not lay on a digon. If D is $\mathcal{F}$-free, then D admits a $(1, \leq \hat{\delta}^- + 1)$-identifying code.*

(*v*) *Suppose that $\hat{\delta}^- = 1$ and the vertices of in-degree 1 do not lay on directed cycles of length less than five. If D is $\{F_2, F_3, F_4, F_5, F_8\}$-free, then D admits a $(1, \leq 2)$-identifying code.*

Proof. By Remark 1.2.1, D admits a $(1, \leq 1)$-identifying code because D is twin-free.

We start considering that D is F_1-free, which is the case in the cases from (i) to (iv). We reason by contradiction, that is, assuming that D does not admit a $(1, \leq \ell)$-identifying code with $\ell \in \{\hat{\delta}^- - 1, \hat{\delta}^-, \hat{\delta}^- + 1\}$.

Then, there are two different subsets X and Y with $1 \leq |Y| \leq |X| \leq \ell$ such that $N^-[X] = N^-[Y]$. Let $x \in X \setminus Y$ and $N^-(x) = \{v_1, \ldots, v_\tau\}$. Since $x \in N^-[X] = N^-[Y]$, there is $y \in Y$ such that $y \in N^+(x)$, then $d^+(x) \neq 0$ implying $d^-(x) = \tau \geq \hat{\delta}^-$. As $N^-(x) \subseteq N^-[X] = N^-[Y]$, for all v_i, with $i \in \{1, \ldots, \tau\}$, there exists a vertex $y_i \in Y$ such that $y_i \in N^+(v_i)$ or $y_i = v_i \in Y$ (See Figure 3). Moreover, for any two

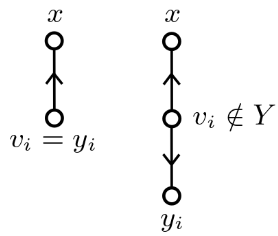


Figure 3. The three cases of the elements of Y.

different indices $1 \leq i < j \leq \tau$, we have $y_i \neq y_j$, otherwise D contains a TT_3 if $v_i \in Y$ and $v_j \notin Y$, or a F_1 if $v_i, v_j \notin Y$. Therefore,

$$\hat{\delta}^- \leq \tau \leq |Y| \leq |X| = \ell \leq \hat{\delta}^- + 1,$$

implying that $|Y| = \hat{\delta}^-$ if $y \notin N^-(x)$, and $|Y| = \hat{\delta}^- + 1$, otherwise. Hence, if there are two different sets of vertices with the same closed in-neighbourhood, its cardinality is at least $\hat{\delta}^-$ and the proof of (i) is completed. Moreover, in cases (ii) and (iii), since $\ell = \hat{\delta}^-$, we have that $y \in N^-(x)$ and $\tau = \hat{\delta}^-$, which is a contradiction in case (ii). Thus, the proof of (ii) is completed.

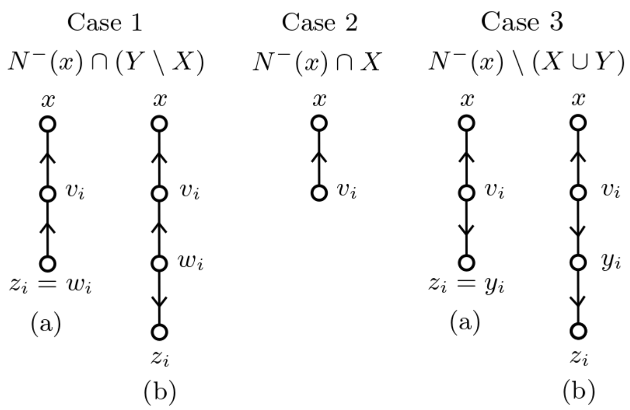


Figure 4. All the cases in the proof of (iii) and (iv) of Theorem 3.3.1.

Next, to prove (iii), we have $Y = \{y_1, y_2, \ldots, y_{\hat{\delta}^-}\}$ and we show that $|X| \geq \hat{\delta}^- + 1$. To do that, let us see that, for each $v_i \in N^-(x)$, one can associate to it a vertex $z_i \in X \setminus \{x\}$ in such a way that $z_i \neq z_j$ for all $i \neq j$. Notice that (iii) is true if $\hat{\delta}^- = 1$, so we may assume that $\hat{\delta}^- \geq 2$. Consider the following partition of $N^-(x)$: $N^-(x) \cap (Y \setminus X)$, $N^-(x) \cap X$, and $N^-(x) \setminus (X \cup Y)$. We have the following cases (see Figure 4):

Case 1: $v_i \in N^-(x) \cap (Y \setminus X)$. Since $\hat{\delta}^- \geq 2$, there is $w_i \in N^-(v_i) \setminus \{x\} \subseteq N^-[Y] \setminus \{x\} = N^-[X] \setminus \{x\}$. Hence: If $w_i \in X$, then $z_i = w_i$ and $z_i \neq x$; and if $w_i \notin X$, since $w_i \in N^-[Y] = N^-[X]$, there exists $z_i \in X$ such that $z_i \in N^+(w_i)$. In this case, we may assume that $z_i \neq x$, because D is TT_3-free.

Case 2: $v_i \in N^-(x) \cap X$. Then, $z_i = v_i$ and $z_i \neq x$.

Case 3: $v_i \in N^-(x) \setminus (X \cup Y) \subseteq N^-[X] \setminus (X \cup Y) = N^-[Y] \setminus (X \cup Y)$. If $y_i \in X$, then $z_i = y_i$, and $y_i \neq x$ because $x \in X \setminus Y$. If $y_i \in Y \setminus X$, then there exists $z_i \in X$ such that $z_i \in N^+(y_i)$. Observe that z_i is different from x, because D is TT_3-free.

Before showing that all the z_i are different, let us notice we have proved $d^-(x) = \hat{\delta}^-$ and that, for any $y \in N^+(x) \cap Y$, we have $y \in N^-(x)$ by assuming $x \in X \setminus Y$ and $\ell = \hat{\delta}^-$. Hence, in case (iii) for every vertex $y' \in Y \setminus X$, we have $d^-(y') = \hat{\delta}^-$ and, for all $x' \in N^+(y') \cap X$, we get $x' \in N^-(y')$. Using this, we can add to the three cases showed in Figure 4 the corresponding digons, as is shown in Figure 5.

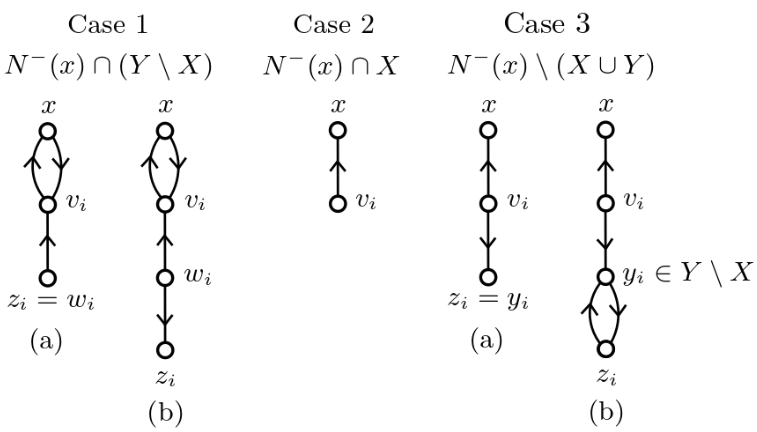


Figure 5. All the cases in the proof of (iii) of Theorem 3.3.1.

Now we prove that all z_i are different. For this, let $i, j \in \{1, \ldots, \tau\}$ such that $i \neq j$. If $v_i, v_j \in N^-(x) \cap (Y \setminus X)$ and $z_i = z_j$, then (see Figure 5 Case 1) it could be $w_j = z_j = z_i = w_i \in X$ (see Figure 6 (a)), and D would contain the subdigraph F_1, contradicting the hypothesis of (iii). It could be $z_j = z_i = w_i \in X$ and $w_j \notin X$ (see Figure 6 (b)), then D would contain the subdigraphs F_5, a contradiction. Finally, it could be $w_i, w_j \notin X$, $z_i = z_j$ and $z_i \in N^+(w_i) \cap N^+(w_j)$ (see Figure 6 (c)), then D would contain the subdigraph F_7, a contradiction. Therefore, all the z_i are different in Case 1.

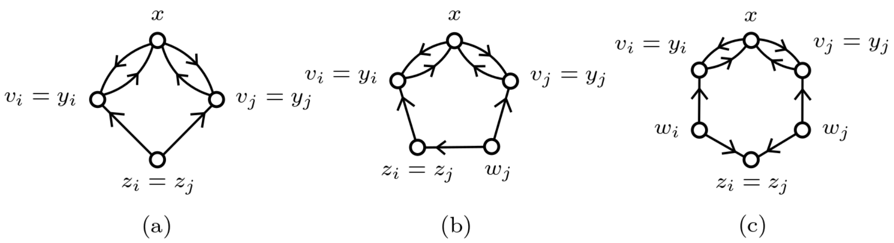


Figure 6. The three cases for z_i and z_j if $v_i, v_j \in N^-(x) \cap (Y \setminus X)$.

If $v_i, v_j \in N^-(x) \cap X$, it is clear that $z_i \neq z_j$ in Case 2. If $v_i, v_j \in N^-(x) \setminus (X \cup Y)$ and $z_i = z_j$ (see Figure 5 Case 3), then since we already know all y_i are different it could be $z_j = y_i \in X$, and D would contain the subdigraph F_5 (see Figure 7 (a)), a contradiction. Hence, $y_i, y_j \in Y \setminus X$ and D would contain the subdigraph F_7 (see Figure 7 (b)), a contradiction. Therefore, all the z_i are different in Case 3.

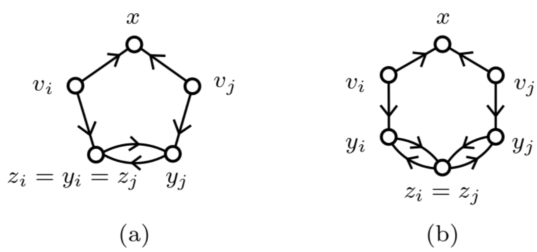


Figure 7. The three cases for z_i and z_j if $v_i, v_j \in N^-(x) \setminus (X \cup Y)$.

It remains to prove that, for all $i, j \in \{1, \ldots, \tau\}$, with $i \neq j$, $z_i \neq z_j$, when v_i and v_j are in different partite subsets of the considered partition of $N^-(x)$. Thus, if $z_i = z_j$ for some $i \neq j$, with $v_i \in N^-(x) \cap (Y \setminus X)$

and $v_j \in N^-(x) \cap X$ (see Figure 5 Cases 1 and 2), then D would contain one of the subdigraphs TT_3 or F_2, a contradiction (see Figure 8).

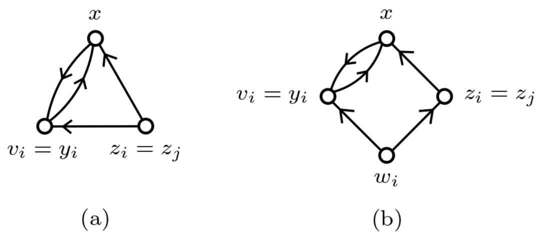


Figure 8. The two cases for z_i and z_j if $v_i \in N^-(x) \cap (Y \setminus X)$ and $v_j \in N^-(x) \cap X$.

If $v_i \in N^-(x) \cap (Y \setminus X)$ and $v_j \in N^-(x) \setminus (X \cup Y)$ (see Figure 5 Cases 1 and 3), then D would contain one of the subdigraphs F_2, F_5 or F_8, a contradiction (see Figure 9).

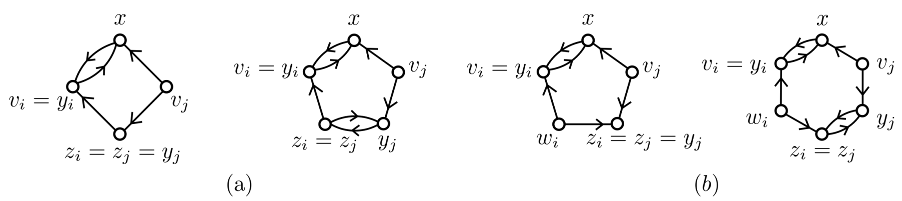


Figure 9. The four cases for z_i and z_j if $v_i \in N^-(x) \cap (Y \setminus X)$ and $v_j \in N^-(x) \setminus (X \cup Y)$.

Finally, if $v_i \in N^-(x) \cap X$ and $v_j \in N^-(x) \setminus (X \cup Y)$ (see Figure 5 Cases 2 and 3), then D would contain one of the subdigraphs TT_3 or F_1 (see Figure 10).

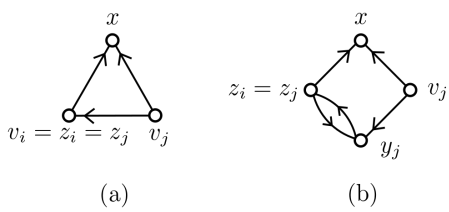


Figure 10. The two cases for z_i and z_j if $v_i \in N^-(x) \cap X$ and $v_j \in N^-(x) \setminus (X \cup Y)$.

In any case, we get a contradiction. Then, we can conclude that X has at least $\hat{\delta}^- + 1$ vertices, and the proof of (iii) is completed.

To prove (iv), we assume $\ell = \hat{\delta}^- + 1$. In this case, $\hat{\delta}^- + 1 \geq d^-(x) \geq \hat{\delta}^-$ and we do not necessarily have the digons between x and the vertices $v_i \in Y \setminus X$. Nevertheless, reasoning as before, we show that all the vertices $z_i \in X \setminus \{x\}$ associated with each vertex $v_i \in N^-(x)$, as we defined before, are different using the added fact that D does not contain any subdigraph as F_3, F_4, nor F_6. First, let

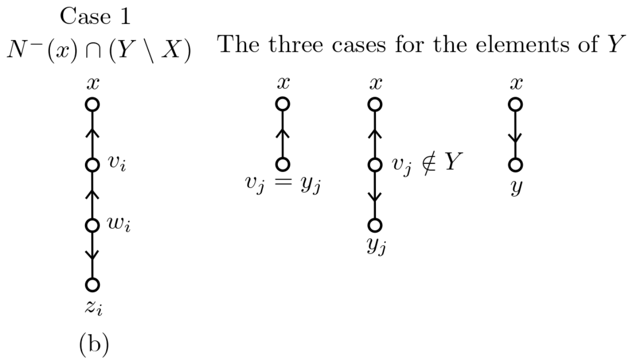


Figure 11. All the cases in the proof of (iv) of Theorem 3.3.1.

us show that, for each $v_i \in N^-(x) \cap (Y \setminus X)$, there is $z_i \in (X \setminus \{x\})$ such that $z_i \in N^-(v_i)$, that is, Case 1 (b) is not possible. Suppose the opposite, and let $v_i \in N^-(x) \cap (Y \setminus X)$ in Case 1 (b). Notice that $z_i \notin X \cap Y$ (see Figure 11), because if $z_i = y_j$ for some $j \neq i$, then D would contain a F_2 if $v_j = y_j$ (see Figure 12 (a)) or D would contain a F_5 if $y_j \in N^+(v_j)$ (see Figure 12 (b)), and if $z_i = y$, then D would contain a F_3 (see Figure 12 (c)). In any case, we reach a contradiction.

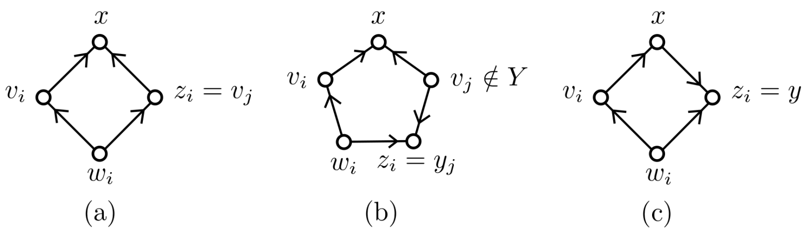


Figure 12. The three cases for $v_i \in N^-(x) \cap (Y \setminus X)$ and $z_i \in N^-(v_i) \cap (X \cap Y)$ in the proof of Theorem 3.3.1 (iv).

Hence, $z_i \in N^-(y_j)$ for some j or $z_i \in N^-(y)$. If $z_i \in N^-(y_j)$, then D would contain F_4 if $v_j = y_j$ (see Figure 13 (a)), or D would contain F_8 if $y_j \in N^+(v_j)$ (see Figure 13 (b)). If $z_i \in N^-(y)$, then D would contain F_4 (see Figure 13 (c)). In any case, we reach a contradiction, therefore there is no such v_i.

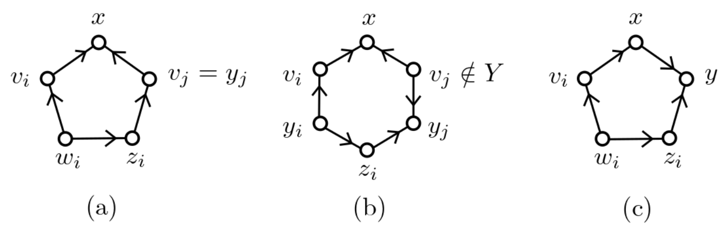


Figure 13. The three cases for $v_i \in N^-(x) \cap (Y \setminus X)$ and $z_i \in N^-(v_i) \cap (X \setminus Y)$ in the proof of Theorem 3.3.1 (iv).

Now let us see that all z_i are different in case (iv). For this, let $i, j \in \{1, \ldots, \tau\}$ be such that $i \neq j$. If $v_i \in N^-(x) \cap (Y \setminus X)$ (see Figure 4 Case 1 (a)) and $z_i = z_j$, there are four possibilities depending on the partite of $N^-(x)$ to which v_j belongs to. If $v_j \in N^-(x) \cap (Y \setminus X)$, then D would contain the subdigraph F_2 (see Figure 14 (a)); if $v_j \in N^-(x) \cap X$, then D would contain a TT_3 (see Figure 14 (b)); if $v_j \in N^-(x) \setminus (X \cup Y)$, then D would contain a F_3 or a F_6 (see Figure 14 (c) and (d)). In any case, we reach a contradiction. Therefore, $z_i \neq z_j$ if at least one of v_i and v_j belongs to Case 1.

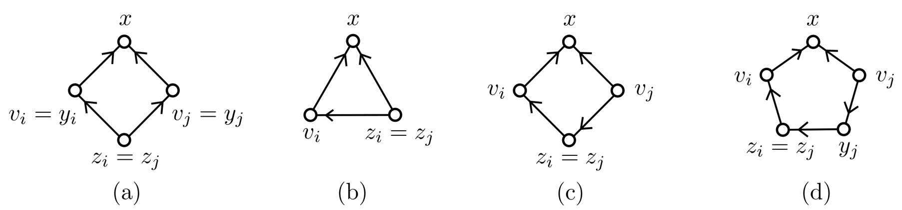
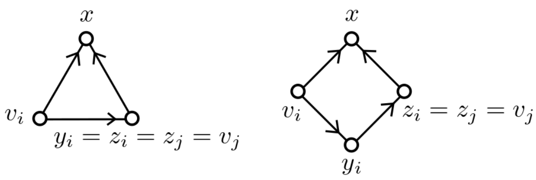


Figure 14. The four cases for $v_i \in N^-(x) \cap (Y \setminus X)$ and $z_i = z_j$ with $i \neq j$ in the proof of Theorem 3.3.1 (iv).

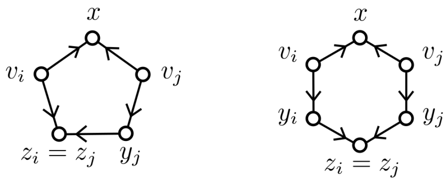
If $v_i \in N^-(x) \cap X$ (see Figure 4 Case 2), there are two cases to be considered: $v_j \in N^-(x) \cap X$ and $v_j \in N^-(x) \setminus (X \cup Y)$. If $v_j \in N^-(x) \cap X$, it is clear that $z_i \neq z_j$ in Case 2. If $v_j \in N^-(x) \setminus (X \cup Y)$ (see Figure 4 Case 3), then D would contain a TT_3 or a F_3 (see Figure 15). In any case, we reach a contradiction. Therefore, $z_i \neq z_j$ if at least one of v_i and v_j belongs to Case 2.



(a) (b)

Figure 15. The two remaining cases for $v_i \in N^-(x) \cap X$ and $z_i = z_j$ with $i \neq j$ in the proof of Theorem 3.3.1 (*iv*).

Finally, if $v_i, v_j \in N^-(x) \setminus (X \cup Y)$ and $z_i = z_j$, then D would contain a F_5 or a F_7 (see Figure 16), a contradiction. Therefore, all the z_i are different in Case 3.



(a) (b)

Figure 16. The two cases for $v_i, v_j \in N^-(x) \setminus (X \cup Y)$ and $z_i = z_j$ with $i \neq j$ in the proof of Theorem 3.3.1 (*iv*).

Therefore, all z_i are different, implying that $X = \{z_1, \ldots, z_\tau, x\}$, and $\tau = \hat{\delta}^-$. Hence, by the hypothesis, x does not belong to a digon, for instance $y \notin N^-(x)$, that is, $y \neq y_i$ for any $1 \leq i \leq \hat{\delta}^-$. Hence, $Y = \{y_1, y_2, \ldots, y_{\hat{\delta}^-}, y\}$.

Now, let us show that $y \in Y \cap X$. For this, suppose the opposite, that is, $y \in Y \setminus X$. Then, $d^-(y) = \hat{\delta}^-$. Since $\hat{\delta}^- \geq 2$, there is $z \in N^-(y) \setminus \{x\}$. Let us show that $z \notin X$. Otherwise, suppose $z \in X$, then $z = z_j$ for some $j = 1, \ldots, \hat{\delta}^-$. By Remark 3.3.1, $v_j \notin N^-(x) \cap X$. If $v_j \in N^-(x) \cap (Y \setminus X)$, then D would contain F_3 (see Figure 17 (a)); and if $v_j \in N^-(x) \setminus (X \cup Y)$, then D would contain F_2 or F_4 (see Figure 17 (b)). Therefore, $z \notin X$.

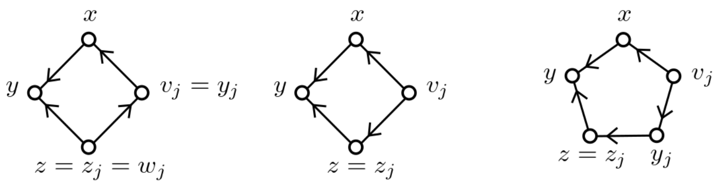


Figure 17. The three cases for $y \in Y \setminus X$ and $z \in N^-(y) \cap (X \setminus \{x\})$ in the proof of (iv).

Hence, $z \in N^-(z_i)$ for some $i \in \{1, \ldots, \hat{\delta}^-\}$. If $v_i \in N^-(x) \cap (Y \setminus X)$, then D would contain F_6 (see Figure 18 (a)); if $v_i \in N^-(x) \cap X$, then D would contain F_3 (see Figure 18 (b)); and if $v_i \in N^-(x) \setminus (X \cup Y)$, then D would contain F_5 or F_8 (see Figure 18 (c)), a contradiction.

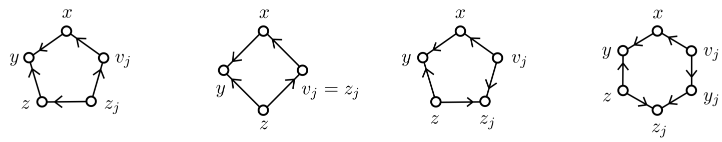


Figure 18. The four cases for $y \in Y \setminus X$ and $z \in N^-(y) \cap N^-(z_i)$ for some $i \in \{1, \ldots, \hat{\delta}^-\}$ in the proof of (iv).

This implies that $y \in X \cap Y$, as we claimed. So, $y = z_j$ for some $j \in \{1, \ldots, \hat{\delta}^-\}$. Notice that $v_j \in N^-(x) \cap (Y \setminus X)$, otherwise x would be contained in a digon, or D would contain a TT_3, a contradiction. Then, reasoning for v_j as for x, we obtain that every $t \in N^+(v_j) \cap X$ satisfies that $t \in X \cap Y$. However, $x \in N^+(v_j) \cap X$, but $x \notin Y$, which is a contradiction, and the proof of (iv) is complete.

To prove (v) we assume that $\hat{\delta}^- = 1$ and $|X| = 2$. Observe that, by Remark 3.3.1 and since there are no vertices of in-degree 1 laying on a digon, the following claim holds.

Claim 3.3.1. *Let $(u, v) \in A(D)$. Then, there is $w \in N^-(u) \setminus N^-[v]$.*

First, observe that if $|Y| = 1$, say $Y = \{y\}$, then $x \in N^-(y)$ and by Claim 3.3.1, there is $w \in N^-(x) \setminus N^-[y]$, implying that $N^-[X] \neq N^-[Y]$, a contradiction. Then, $|Y| = |X| = 2$.

Let $X = \{x, x'\}$, $x \in X \setminus Y$, and $Y = \{y, y'\}$ with $y \in N^+(x)$. Let us prove that the arc (x, y) is not on a digon. Otherwise, suppose that $(x, y), (y, x) \in A(D)$. By Claim 3.3.1, there exist $w, z \in V(D)$ such that $z \in N^-(x) \setminus N^-[y]$, and $w \in N^-(y) \setminus N^-[x]$. Hence, $z \in N^-[y']$ and $w \in N^-[x']$. If $z \notin Y$, then $z \neq y'$ and $z \in N^-(y')$. Moreover, since D is TT_3-free, $y' \in N^-[x'] \subset N^-[X]$. If $x' = y'$, then $w \neq x'$ because D is free of F_2, hence, $w \in N^-(x')$, implying that D would contain F_5, therefore, $x' \neq y'$ (and so $y' \in Y \setminus X$). Moreover, we can assume that $w \notin \{y', x'\}$, otherwise D would contain F_2 or F_4. Thus, $w \in N^-(x')$ implying that D would contain F_8, a contradiction. Hence, $y' = z$, therefore, $Y = \{y, z\}$, and analogously $x' = w$, that is, $X = \{x, w\}$. By Claim 3.3.1 and because $N^-[Y] = N^-[X]$, there is $u \in (N^-(y') \setminus N^-[x]) \cap N^-[x']$, then D would contain F_2 if $u = x'$ or F_4 if $u \in N^-(x')$. Therefore, the arc (x, y) is not on a digon.

Suppose that $X \cap Y \neq \emptyset$. Taking into account that $N^-[Y] = N^-[X]$, by Claim 3.3.1, there is $w \in N^-(x) \setminus N^-[y]$ and then $w \in N^-[y']$. First, assume that $X \cap Y = \{y'\}$, that is $x' = y'$. Since $N^-[Y] = N^-[X]$, we have $y \in N^-(x')$ because (x, y) is not on a digon. If $w = y'$, then $(xyx'x)$ is a 3-cycle in D and, by the hypothesis, there is $u \in N^-(x) \setminus \{x'\}$. By Remark 3.3.1, $u \notin N^-(y) \cup N^-(x')$, a contradiction. Then, $w \neq y'$, implying that D would contain F_3, a contradiction. Second, assume that $X \cap Y = \{y\}$, that is $x' = y$. If $w = y'$, there is $w' \in N^-(y') \setminus N^-[x]$ by Claim 3.3.1. Then, $w' \in N^-(y)$ implying that D would contain a F_3, a contradiction. Thus, $w \neq y'$ and $w \in N^-(y')$, and since $y' \in N^-(x) \cup N^-(y)$, D would contain a TT_3 or F_2, a contradiction.

Therefore, $X \cap Y = \emptyset$. Then, $y \in N^-(x')$, and since $y \in Y \setminus X$, reasoning for y as for x, the arc (y, x') is not lying on a digon. Then, $x' \in N^-(y')$ and, similarly, $y' \in N^-(x)$. By the hypothesis, there are no vertices of in-degree 1 lying on a 4-cycle, implying that there is $z \in N^-(x) \setminus \{y'\}$, but by Remark 3.3.1, $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$. Hence, $N^-[X] \neq N^-[Y]$, a contradiction. This completes the proof. □

Regarding identifying codes in graphs, Laihonen [45] proved the following result.

Theorem 3.3.2. *[45] Let $k \geq 2$ be an integer.*

1. *If a k-regular graph has girth $g \geq 7$, then it admits a $(1, \leq k)$-identifying code.*
2. *If a k-regular graph has girth $g \geq 5$, then it admits a $(1, \leq k-1)$-identifying code.*

If for each graph G, we consider its corresponding symmetric digraph $\overleftrightarrow{G}$. Then, we obtain the following corollary from Theorem 3.3.1.

Corollary 3.3.1. *Let G be a graph of girth g and minimum degree $\delta \geq 2$.*

1. *If $g \geq 7$, then G admits a $(1, \leq \delta)$-identifying code.*
2. *If $g \geq 5$, then G admits a $(1, \leq \delta - 1)$-identifying code.*

Observe that Theorem 3.3.2 by Laihonen is a consequence of Corollary 3.3.1.

3.4 r-IN-REGULAR DIGRAPHS

In this section, we point out two general results about in-regular digraphs regarding identifying codes. Besides, we give a characterisation of the r-in-regular digraphs admitting a $(1, \leq r)$-identifying code and a $(1, \leq r+1)$-identifying code for $r \in \{1, 2\}$.

Proposition 3.4.1. *Let D be an r-in-regular digraph admitting a $(1, \leq r+1)$-identifying code. Then,*

(i) D is an oriented graph,

(ii) D is TT_3-free, and

(iii) for every vertex $u \in V^+_{\geq 1}(D)$ and every vertex $v \in V(D) \setminus \{u\}$, we have $|N^-(u) \cap N^-(v)| \leq 1$.

Proof. (i) and (ii) are a direct consequence of Lemma 3.1.1 and Proposition 3.2.1, respectively. We prove (iii) by contrapositive. Let D be an r-in-regular digraph. Suppose that there is a vertex $u \in V^+_{\geq 1}(D)$ and a vertex $v \in V(D) \setminus \{u\}$ such that $|N^-(u) \cap N^-(v)| = k \geq 2$. Then, $v \notin N^-(u)$. Let $x \in N^+(u)$, $N^-(u) \cap N^-(v) = \{w_1, w_2, \ldots, w_k\}$, and $N^-(u) \setminus N^-(v) = \{u_1, \ldots, u_{r-k}\}$. Thus, $X = \{u, v, x, u_1, \ldots, u_{r-k}\}$ and $Y = X \setminus \{u\}$ are two different sets such that $N^-[X] = N^-[Y]$, with $|X| \leq r - k + 3 \leq r + 1$. Therefore, D does not admit a $(1, \leq r+1)$-identifying code. This completes the proof. □

Lemma 3.4.1. *Let D be an r-in-regular oriented TT_3-free graph, not containing F_1 of Figure 2 as subdigraph. Then,*

$$r \leq \max\{q \mid D \textit{ admits a } (1, \leq q)\textit{-identifying code}\} \leq r + 1.$$

Proof. Let $\ell = \max\{q \mid D$ admits a $(1, \leq q)$-identifying code$\}$. Observe that D is twin-free, and so D admits an identifying code, since it is oriented. By Proposition 3.1.1, we have that $\ell \leq r + 1$. And, by Theorem 3.3.1 (ii), we have that $\ell \geq r$. This completes the proof. □

3.4.1 *1-in-regular and 2-in-regular digraphs*

We start by giving a characterisation of 1-in-regular digraphs admitting a $(1, \leq 2)$-identifying code. Observe that every 1-in-regular digraph D admits an identifying code if and only if D does not contain digons.

Theorem 3.4.1. *Every 1-in-regular digraph D admits a $(1, \leq 2)$-identifying code if and only if the girth of D is at least 5.*

Proof. By Proposition 3.2.3, we have that if D admits a $(1, \leq 2)$-identifying code, then its girth is at least 5. Conversely, suppose that the girth of D is at least 5. Since D is 1-in-regular, it follows that D does not contain any subdigraph isomorphic to TT_3, F_2, F_3, F_4, F_5 nor F_8 of Figure 2. Then, by Theorem 3.3.1, D admits a $(1, \leq 2)$-identifying code. This completes the proof. □

The following result gives a complete characterisation of all 2-in-regular digraphs admitting an identifying code or a $(1, \leq 2)$-identifying code.

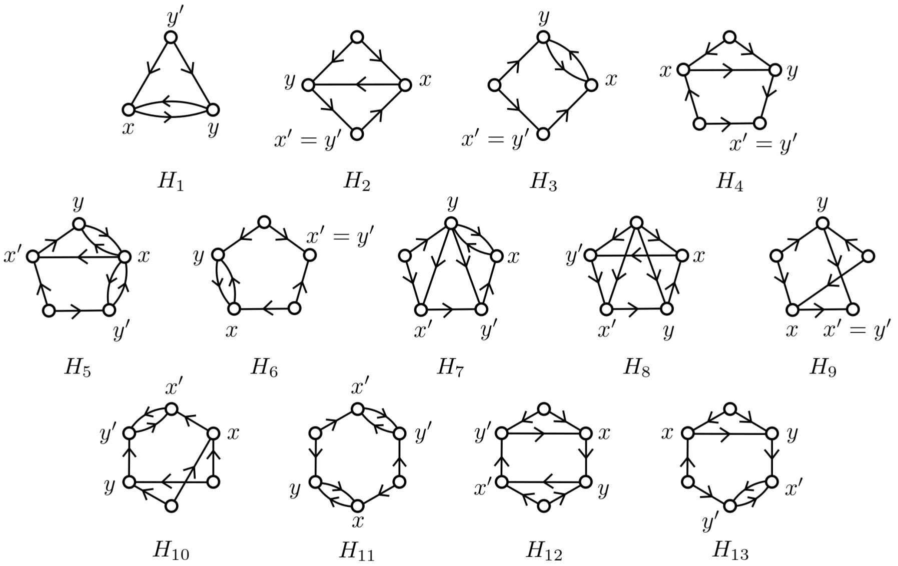


Figure 19. The forbidden subdigraphs in a 2-in-regular digraph admitting a $(1, \leq 2)$-identifying code.

We denote with $\mathcal{H}$ the family of digraphs H_1- H_{13} of Figure 19.

Theorem 3.4.2. *Let D be a 2-in-regular digraph.*

(i) *D admits an identifying code if and only if it is H_1-free.*

(ii) *D admits a $(1, \leq 2)$-identifying code if and only if it is $\mathcal{H}$-free.*

Proof. In what follows, for brevity, we made reference to the different cases H_1-H_{13} of Figure 19 without mentioning the figure. First, note that any digraph with twin vertices and minimum in-degree at least 2, necessarily contains H_1. Hence, the proof of (i) follows from Remark 1.2.1, because the vertices x, y of H_1 are twins. To prove (ii), first let $X = \{x, x'\}$ (or $X = \{x\}$) and $Y = \{y, y'\}$. It is direct to check that $N^-[X] = N^-[Y]$ in each one of the digraphs shown in Figure 19. For the converse, we assume that D does not contain any subdigraph isomorphic to the digraphs depicted in Figure 19, and $N^-[X] = N^-[Y]$ for $X \neq Y$ such that $1 \leq |Y| \leq |X| \leq 2$. According to (i), $|X| = 2$, consequently, $3 \leq |N^-[X]| \leq 6$. Notice that if $|Y| = 1$, then $|N^-[Y]| = 3$, and so $|N^-[X]| = 3$ yielding that D contains H_1. Therefore, we assume that $|Y| = |X| = 2$. Let $X = \{x, x'\}$ and $Y = \{y, y'\}$ with $x \in X \setminus Y$. Let $N^-(x) = \{v_1, v_2\}$ and $y \in Y$ such that $y \in N^+(x)$. As we did in the proof of Theorem 3.3.1, we consider the different cases according to the partition of $N^-(x)$: $N^-(x) \cap (Y \setminus X)$, $N^-(x) \cap X$ and $N^-(x) \setminus (X \cup Y)$.

Case 1: Suppose that $v_1, v_2 \in Y \setminus X$. Let $y = v_1$ and $y' = v_2$ and observe that, in this case, $x' \notin Y$. As D is H_1-free and H_3-free, $(N^-(y) \setminus \{x\}) \cap N^-[y'] = \emptyset$ and there is no arc between y' and $N^-(y) \setminus \{x\}$. Let $w \in N^-(y) \setminus \{x\}$ and $w' \in N^-(y') \setminus \{x\}$, then $w, w' \in N^-[x']$.

Subcase 1.1: Suppose that $\{w, w'\} \cap \{x'\} = \emptyset$. Hence, $N^-(x') = \{w, w'\}$. Since $x' \in N^-[Y]$, it follows that $x' \in N^-(y')$ implying that D would contain H_{13}, a contradiction.

Subcase 1.2: Suppose that $x' = w$. Hence, $w' \in N^-(x')$. If there is $z \in N^-(x') \setminus (X \cup Y \cup \{w'\})$, then $z \in N^-(y')$, implying that D would contain H_{10}, a contradiction. Therefore, $N^-[X] = X \cup Y \cup \{w'\}$, implying that $N^-(x') = \{w', x\}$ or $N^-(x') = \{w', y\}$. First, suppose that $N^-(x') = \{w', x\}$. If $x \in N^-(y')$, then D would contain H_5; and if $y \in N^-(y')$, then D would contain H_4, a contradiction. Therefore, $N^-(x') = \{w', y\}$. If $x \in N^-(y')$, then D would contain H_6; and if $y \in N^-(y')$, then D would contain H_5, a contradiction.

Subcase 1.3: Supposse that $x' = w'$. Hence, $w \in N^-(x')$. If there is $z \in N^-(x') \setminus (X \cup Y \cup \{w'\})$, then $z \in N^-(y')$, implying that D would contain H_{13}, a contradiction. Therefore, $N^-[X] = X \cup Y \cup \{w'\}$. Hence, $N^-(x') = \{w, x\}$ or $N^-(x') = \{w, y\}$. First suppose that $N^-(x') = \{w, x\}$. If $x \in N^-(y')$, then D contains H_4; and if $y \in N^-(y')$, then D would contain H_9, a contradiction. Therefore, $N^-(x') = \{w, y\}$. Hence, if $x \in N^-(y')$, then D would contain H_4; and if $y \in N^-(y')$, then D would contain H_7, a contradiction.

Case 2: Suppose that $v_1, v_2 \in X$. Since $|X| = 2$, this case is not possible.

Case 3: Suppose that $v_1, v_2 \notin (X \cup Y)$. Since $x \in N^-(y)$, then $|N^-(y) \cap \{v_1, v_2\}| \leq 1$ implying that $\{v_1, v_2\} \cap N^-(y') \neq \emptyset$. Without loss of generality, suppose that $v_1 \in N^-(y')$.

Subcase 3.1: If $y \in Y \setminus X$, then $y \in N^-(x')$. If $y' \in X \cap Y$, that is, $y' = x'$, then $v_2 \in N^-(y)$, implying that D would contain H_4. If $y' \in Y \setminus X$, then $N^-(x') = \{y, y'\}$ and $x' \in N^-(y) \cup N^-(y')$. If $x' \in N^-(y)$, then $v_2 \in N^-(y')$, implying that D would contain H_{10}. And, if $x' \in N^-(y')$, then $v_2 \in N^-(y)$, implying that D would contain H_{13}.

Subcase 3.2: If $y \in X \cap Y$ that is, $x' = y$, then $y' \in N^-(y)$ and $v_1, v_2 \in N^-(y')$, hence D would contain H_9, a contradiction. Therefore, the proof of Case 3 is finished.

Case 4: Suppose that $v_1 \in Y \setminus X$ and $v_2 \in X$, that is, $v_2 = x'$. Observe that if $v_1 \in N^+(x)$, since D is H_1-free, there is $w \in V(D) \setminus X$ such that $w \in N^-(v_1) \subset N^-[Y]$. Thus, $w \in N^-(x')$, implying that D would contain H_3, a contradiction. Then, $v_1 \notin N^+(x)$ and so, $v_1 = y'$ and $y \in N^-(x')$. If $x' \in N^+(x)$, then $N^-[X] = \{x, x', y, y'\}$, yielding that $y \in N^-(y')$, contradicting that D is H_3-free. Therefore, $N^+(x) \cap \{y', x'\} = \emptyset$, and recall that $y \in N^-(x')$. Moreover, reasoning for y as for x in Case 1, we get that $x' \notin N^-(y)$. Moreover, if $y' \in N^-(y)$, then D would contain H_2, a contradiction. Therefore, there is $w \in N^-(y) \setminus (X \cup Y)$. Hence, $w \in N^-(x')$, implying that D would contain H_2, a contradiction.

Case 5: Suppose that $v_1 \in Y \setminus X$ and $v_2 \notin (X \cup Y)$.

Subcase 5.1: Suppose that $v_1 \in N^+(x)$, then, we can assume that $v_1 = y$. Since D is H_1-free, $v_2 \in N^-(y')$ and there is $w \in V(D) \setminus \{x, v_2\}$ such that $N^-(y) = \{x, w\}$. Observe that since D is H_3-free, $v_2 \notin N^-(w)$, then $w \neq y'$. Moreover, since D is H_6-free, $w \notin N^-(y')$. Hence, $w \in N^-[x']$, implying that $x' \neq y'$. Observe that reasoning for y as for x in Case 1, we get that $w \neq x'$. Then, $w \in N^-(x')$ and, since $x', y' \in N^-[X] = N^-[Y]$, it follows that $x' \in N^-(y')$ and $y' \in N^-(x')$, therefore D would contain H_{11}, a contradiction.

Subcase 5.2: Suppose that $v_1 \notin N^+(x)$, then $v_1 = y'$ and $y \in N^-[x']$. First, suppose that $y = x'$. If $N^-(y') \subseteq X \cup \{v_2\}$, then $N^-(y') = \{x', v_2\}$ implying that D would contain H_2. Hence, there is $w \in N^-(y') \setminus (X \cup \{v_2\})$. Then, $w \in N^-(x')$ and $v_2 \in N^-(y')$, implying that D would contain H_4, a contradiction. Therefore, $y \neq x'$, implying that $y \in N^-(x')$. Reasoning for y as for x in Case 1 and Case 4, it follows that $N^-(y) \cap \{x', y'\} = \emptyset$. Then, $x' \in N^-(y')$. Moreover, since $v_2 \in N^-(x)$, $v_2 \in N^-(y) \cup N^-(y')$. Also, reasoning for x' as for x in Case 1 and Case 4, it follows that $N^-(x') \cap \{x, y'\} = \emptyset$. Hence, if $v_2 \in N^-(y) \cap N^-(y')$, then $N^-[Y] = X \cup Y \cup \{v_2\}$, implying that $v_2 \in N^-(x')$. Then, D would contain H_8, a contradiction. If $v_2 \in N^-(y') \setminus N^-(y)$, then there is $z \in N^-(y) \setminus (X \cup Y \cup \{v_2\})$, implying that $N^-(x') = \{y, z\}$ and D would contain H_{12}. Analogously, if $v_2 \in N^-(y) \setminus N^-(y')$. And the proof of this case is completed.

Case 6: Suppose that $v_1 \in X$ and $v_2 \notin (X \cup Y)$. That is, $v_1 = x'$. If $x' \in X \setminus Y$, then $y \in N^-(x')$. Since $y \in Y \setminus X$, reasoning for x' as for x in Cases 1, 4, and 5, we reach a contradiction. Hence, $x' \in X \cap Y$. If $x' = y$, then $y' \in N^-(x')$ and $v_2 \in N^-(y')$, implying that D would contain H_3. Therefore, $x' \neq y$ and, hence, $y \in Y \setminus X$. Since $x \in N^-(y)$, reasoning for y as for x in Cases 1, 4, and 5, we reach a contradiction. □

Corollary 3.4.1. *Every TT_3-free 2-in-regular oriented graph admits a $(1, \leq 2)$-identifying code if and only if it does not contain any subdigraph isomorphic to H_9 of Figure 19.*

Observe that Corollary 3.4.1 is an improvement of Theorem 3.3.1 *(ii)* for 2-in-regular oriented graphs. Now, the TT_3-free and 2-in-regular oriented graph can have two distinct vertices u, v with $|N^-(u) \cap N^-(v)| = 2$, that is, a subdigraph F_1 of Figure 2, but, in this case, there is no vertex $w \in V$ such that $u, v \in N^-(w)$.

In the following theorem, we characterise the 2-in-regular digraphs admitting a $(1, \leq 3)$-identifying code.

We denote with $\mathcal{J}$ the family of digraphs J_1-J_{15} of Figure 20.

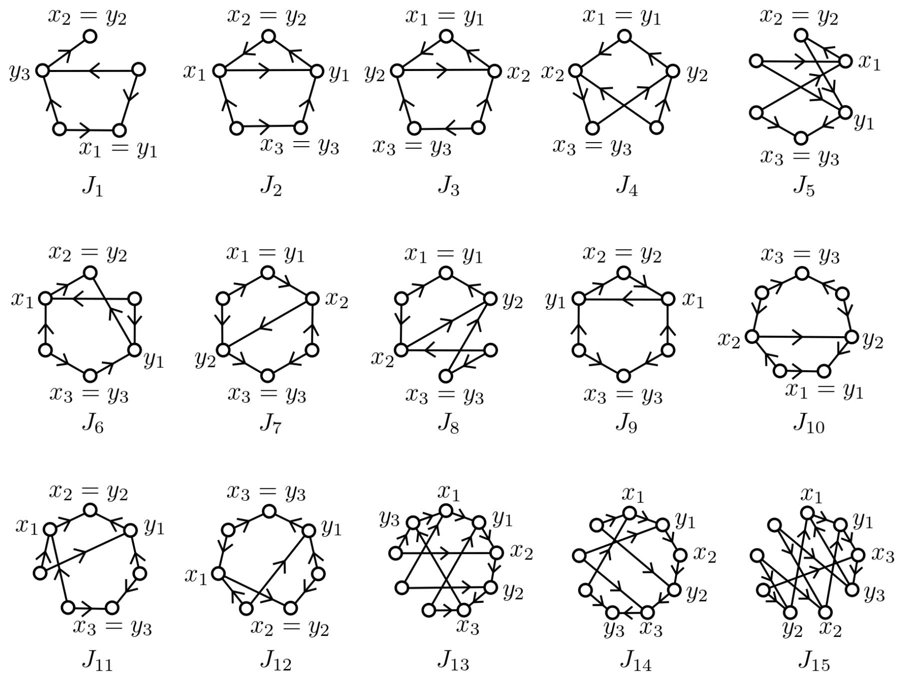


Figure 20. All the forbidden subdigraphs of Theorem 3.4.3.

Theorem 3.4.3. *Let D be a 2-in-regular digraph. Then, D has a $(1, \leq 3)$-identifying code if and only if it is $\{\{TT_3\} \cup \mathcal{J}\}$-free oriented graph.*

Proof. By Proposition 3.1.1 and Proposition 3.2.1, if D contains a digon or a TT_3, then D does not admit a $(1, \leq 3)$-identifying code. Furthermore, for every digraph shown in Figure 20, let $X = \{x_1, x_2, x_3\}$ (or $X = \{x_1, x_2\}$) and $Y = \{y_1, y_2, y_3\}$. It is direct to check that $N^-[X] = N^-[Y]$ in each case. To the converse, we reason by contradiction. Let D be a TT_3-free oriented graph without the subdigraphs of Figure 20. Let $X, Y \subseteq V(D)$, $X \neq Y$, with $N^-[X] = N^-[Y]$ and such that $1 \leq |X| \leq |Y| \leq 3$. Since D does not contain a subdigraph isomorphic to J_1 of Figure 20, then it does not contain a subdigraph H_9 of Figure 19. By Corollary 3.4.1, D admits a $(1, \leq 2)$-identifying code. Hence, $|Y| = 3$, $|N^-[Y]| \geq 6$, and $|X| \geq 2$. In what follows, for brevity, we always make reference to the different cases J_1-J_{15} of Figure 20 without mentioning the figure. Let us prove the following claim.

Claim 3.4.1. *Let $a, b \in V(D)$, with $a \neq b$, be such that $N^-(a) \subseteq N^-[b]$. Then, $N^-(a) = N^-(b)$ and $N^+(a) = N^+(b) = \emptyset$.*

Proof. If $b \in N^-(a)$, then D contains a TT_3, which is a contradiction. Hence, $N^-(a) = N^-(b)$ and $N^+(a) = N^+(b) = \emptyset$ because, otherwise, D would contain J_1. □

Suppose $X = \{x_1, x_2\}$, then $|N^-[X]| = 6$ (because $N^-[X] = N^-[Y]$) and $N^-[x_1] \cap N^-[x_2] = \emptyset$. Let $N^-(x_1) = \{u, v\}$ and $N^-(x_2) = \{z, t\}$, so that $N^-[X] = \{x_1, x_2, u, v, z, t\} = N^-[Y]$. Without loss of generality, we may assume that $u \in Y$. Since D has neither digon nor TT_3, $N^-(u) \subseteq N^-[x_2]$, which implies by Claim 3.4.1 that $N^-(u) = N^-(x_2)$ and $N^+(u) = \emptyset$, a contradiction. Therefore, $|X| = |Y| = 3$. Let us denote $X = \{x_1, x_2, x_3\}$. We prove the following claims.

Claim 3.4.2. *Let $a, b, c \in V(D)$. If $N^-[a] \subseteq N^-[b] \cup N^-[c]$, then $a \in \{b, c\}$.*

Proof. If $a \notin \{b, c\}$, then, without loss of generality, let us assume that $a \in N^-(b)$. Hence, by Remark 3.3.1, $N^-(a) \subseteq N^-[c]$, which contradicts Claim 3.4.1 because $N^+(a) \neq \emptyset$. □

Claim 3.4.3. *$N^-(x_i) \neq N^-(x_j)$ for all $1 \leq i < j \leq 3$.*

Proof. Suppose that $N^-(x_1) = N^-(x_2)$. Then, $N^+(x_1) = N^+(x_2) = \emptyset$, because D is J_1-free, which implies $x_1, x_2 \in Y$. Since $|N^-[X]| \geq 6$, there is $z \in N^-(x_3) \setminus (N^-[x_1] \cup N^-[x_2])$. Because $\{x_3, z\} \subseteq N^-[Y]$, D must contain a digon if $z = y_3 \in Y$, or a TT_3 if $\{x_3, z\} = N^-(y_3)$, which is a contradiction. Therefore, $N^-(x_1) \neq N^-(x_2)$. □

Claim 3.4.4. *If $7 \leq |N^-[X]| \leq 8$, $N^-(x_i) \cap N^-(x_j) = \{v\}$, $i \neq j$, and there are exactly two or no arc between the elements of X, then $|Y \cap \{x_i, x_j\}| \leq 1$.*

Proof. We proceed by contradiction. Assume $Y = \{x_1, x_2, y\}$. First, suppose that there is no arc between the elements of X. If $v \in N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)$, then according to Claim 3.4.3, $|N^-[X]| = 7$ and

$N^-[x_3] \subseteq N^-[x_1] \cup N^-[y]$, which contradicts Claim 3.4.2. Hence, $N^-(x_1) \cap N^-(x_2) \cap N^-(x_3) = \emptyset$. If $|N^-[X]| = 7$, let $N^-(x_1) = \{u, v\}$, $N^-(x_2) = \{v, z\}$, and $N^-(x_3) = \{z, w\}$. Since $N^-(v) \cap N^-[X] \subseteq \{x_3, w\}$, by Remark 3.3.1, $v \notin Y$, and analogously $z \notin Y$. Consequently, $N^-[x_3] \subseteq N^-[x_2] \cup N^-[y]$, which contradicts Claim 3.4.2. If $|N^-[X]| = 8$, then $N^-(x_3) \subseteq N^-[y]$, a contradiction to Claim 3.4.1 because $y \notin X$ and so $N^+(y) \neq \emptyset$. Finally assume that there are two arcs among the elements of X. Notice that, by Remark 3.3.1, both arcs between the elements of X are incident to x_3. Furthermore, since $7 \leq |N^-[X]| \leq 8$ and $N^-(x_1) \cap N^-(x_2) = \{v\}$, $v = x_3$ and $|N^-[X]| = 7$, we have $N^-(x_3) \subseteq N^-[y]$, a contradiction to Claim 3.4.1. □

Let $N^-(x_1) = \{u, v\}$. We distinguish the following cases according to the number of arcs between the vertices of X.

Case 1: First, let us assume that there are no arcs between the elements of X.

Subcase 1.1: Suppose $|N^-[X]| = 6$. Then, $N^-[X] = \{x_1, x_2, x_3, u, v, z\}$, so Claim 3.4.3 implies that $|N^-(x_i) \cap N^-(x_j)| = 1$ for all $i \neq j$. Let $N^-(x_2) = \{v, z\}$. Observe that $v \notin N^-(x_3)$, otherwise, $N^-(x_3) = N^-(x_i)$ for some $i \in \{1, 2\}$, contradicting Claim 3.4.3. Therefore $N^-(x_3) = \{u, z\}$. Let $y \in Y \setminus X$, then $y \in \{u, v, z\}$. We can check that $|N^-(y) \cap N^-[X]| \leq 1$ for all $y \in \{u, v, z\}$, because D is a TT_3-free oriented graph, which is a contradiction.

Subcase 1.2: Suppose $|N^-[X]| = 7$. Then, $N^-[X] = \{x_1, x_2, x_3, u, v, z, w\}$. By Claim 3.4.3, there are two cases to be considered, namely, $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 1$ and $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 0$.

Subsubcase 1.2.1: If $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 1$, without loss of generality, $N^-(x_2) = \{v, z\}$ and $N^-(x_3) = \{v, w\}$. Since D is an oriented graph and it does not contain TT_3, $N^-(v) \cap N^-[X] = \emptyset$, which means that $v \notin Y$ and $v \in N^-(Y)$. Since $N^+(v) \cap \{u, z, w\} = \emptyset$, it follows that $Y \cap X \neq \emptyset$. By Claim 3.4.4, $|X \cap Y| = 1$. Without loss of generality, suppose that $X \cap Y = \{x_1\}$. If $Y = \{x_1, z, w\}$, then $x_2 \in N^-(w)$ and $x_3 \in N^-(z)$, implying that D would contain

J_4. If $Y = \{x_1, u, z\}$, then $x_2 \in N^-(u)$ and $N^-(u) \subseteq \{x_2, x_3, w\}$. If $N^-(u) = \{x_2, x_3\}$, then $w \in N^-(z)$ and, hence, D would contain J_6. If $N^-(u) = \{x_2, w\}$, then $x_3 \in N^-(z)$, which implies that D would contain J_5.

Subsubcase 1.2.2: If $|N^-(x_1) \cap N^-(x_2) \cap N^-(x_3)| = 0$, without loss of generality, $N^-(x_2) = \{v, z\}$ and $N^-(x_3) = \{z, w\}$. By Claim 3.4.4, $|Y \cap \{x_1, x_2\}| \leq 1$ and $|Y \cap \{x_2, x_3\}| \leq 1$. Moreover, if $\{x_1, x_3\} \subseteq Y$, then since $x_2 \in N^-[Y]$, we have $\{u, w\} \cap Y \neq \emptyset$; without loss of generality, let us assume that $Y = \{x_1, x_3, u\}$. Then, $x_2 \in N^-(u)$ and $N^-(u) \subseteq \{x_2, x_3, w\}$. If $N^-(u) = \{x_2, x_3\}$, then D would contain J_8; and if $N^-(u) = \{x_2, w\}$, then D would contain J_{10}. Therefore, $|Y \cap X| \leq 1$. Suppose that $X \cap Y = \{x_1\}$ and let $Y = \{x_1, y, y'\}$, then $N^-[x_3] \subseteq N^-[y] \cup N^-[y']$, which contradicts Claim 3.4.2. Hence, $X \cap Y \neq \{x_1\}$, and similarly $X \cap Y \neq \{x_3\}$. Then, $X \cap Y = \{x_2\}$. If $v \in Y$, then there is $y \in Y \setminus \{x_2, v\}$, such that $N^-(y) = \{x_1, u\}$ contradicting Remark 3.3.1. Hence, $v \notin Y$, and analogously $z \notin Y$. Therefore, $Y = \{x_2, u, w\}$ and, then, $x_1 \in N^-(w)$, implying that $N^-(u) = \{x_3, x_2\}$. Consequently, D would contain J_8. If $|Y \cap X| = 0$, by symmetry, we only have to consider the following two cases. If $Y = \{u, v, z\}$, then $x_2 \in N^-(u)$ and $x_1 \in N^-(z)$, implying that D would contain J_4. If $Y = \{u, z, w\}$, then $x_3 \in N^-(u)$ implying that $N^-(u) = \{x_2, x_3\}$, and D would contain J_8.

Subcase 1.3: Suppose $|N^-[X]| = 8$. Without loss of generality, $N^-(x_2) = \{v, z\}$, and $N^-(x_3) = \{t, w\}$. Observe that $v \notin Y$, otherwise, $N^-(v) \subseteq N^-[x_3]$ in contradiction to Claim 3.4.1. If $Y \cap X = \emptyset$, then we can assume that $t \in Y$ and $v \in N^-(t)$. Consequently, $\{u, z\} \cap N^-(t) = \emptyset$, otherwise, D would contain J_1, thus, $\{x_3, w\} \cap N^-(t) \neq \emptyset$, a contradiction. Therefore, $Y \cap X \neq \emptyset$. If $|Y \cap X| = 2$, then by Claim 3.4.4, $\{x_1, x_3\} \subseteq Y$ or $\{x_2, x_3\} \subseteq Y$. If $Y = \{y, x_2, x_3\}$, then $N^-[x_1] \subseteq N^-[x_2] \cup N^-[y]$, contradicting Claim 3.4.2. Then, $Y \neq \{y, x_2, x_3\}$, and similarly $Y \neq \{y, x_1, x_3\}$. Thus, $|Y \cap X| = 1$. If $Y = \{x_1, y, y'\}$ or $Y = \{x_3, y, y'\}$, then $N^-[x_3] \subseteq N^-[y] \cup N^-[y']$ or $N^-[x_1] \subseteq N^-[y] \cup N^-[y']$, respectively, which contradicts Claim 3.4.2.

Subcase 1.4: Suppose $|N^-[X]| = 9$. Hence, the in-neighbourhoods of the elements of X must be disjoint, and the same is true for Y. Let $N^-(x_i) = \{u_i, v_i\}$, for $i = 1, 2, 3$. Observe that if $1 \leq |X \cap Y| \leq 2$, then $N^-[x_i] \subseteq N^-[y] \cup N^-[y']$ for some $i \in \{1, 2, 3\}$ and $y, y' \in Y \setminus \{x_i\}$, in contradiction to Claim 3.4.2. Therefore, $X \cap Y = \emptyset$. Without loss of generality, there are two cases to be considered.

Subsubcase 1.4.1: If $Y = \{u_1, v_1, u_2\}$, then $x_1 \in N^-(u_2)$. If $x_3 \in N^-(u_1)$, then, without loss of generality, $u_3 \in N^-(v_1)$ and $v_3 \in N^-(u_2)$; moreover, $x_2 \in N^-(v_1)$ and $v_2 \in N^-(u_1)$ or $x_2 \in N^-(u_1)$ and $v_2 \in N^-(v_1)$, implying that D would contain J_{14} or J_{15}, respectively. If $x_3 \in N^-(u_2)$, then we may assume that $u_3 \in N^-(u_1)$ and $v_3 \in N^-(v_1)$, and so $x_2 \in N^-(u_1)$ and $v_2 \in N^-(v_1)$, implying that D would contain J_{15}.

Subsubcase 1.4.2: Let $Y = \{u_1, u_2, u_3\}$. Without loss of generality, suppose $x_2 \in N^-(u_1)$, then by Remark 3.3.1, $N^-(u_1) \setminus \{x_2\} \subseteq N^-[x_3]$. Since there is no arc between the elements of Y, there are two cases to be considered.

1.4.2.1: If $N^-(u_1) = \{x_2, x_3\}$, then $v_3 \in N^-(u_2)$ and $v_2 \in N^-(u_3)$. Hence, $x_1 \in N^-(u_2)$ and $v_1 \in N^-(u_3)$, or $v_1 \in N^-(u_2)$ and $x_1 \in N^-(u_3)$; in any case, D would contain J_{14}.

1.4.2.2: If $N^-(u_1) = \{x_2, v_3\}$, then $x_3 \in N^-(u_2)$, and $v_2 \in N^-(u_3)$. If $x_1 \in N^-(u_2)$, then $v_1 \in N^-(u_3)$, implying that D would contain J_{14}. Finally, if $x_1 \in N^-(u_3)$, then $v_1 \in N^-(u_2)$, implying that D would contain J_{13}.

Case 2: Suppose there is just one arc between the elements of X, say $(x_1, x_2) \in A(D)$. Then, $|N^-(X)| = 6, 7, 8$, and $N^-(x_1) \cap N^-(x_2) = \emptyset$ by Remark 3.3.1. Let $N^-(x_2) = \{x_1, z\}$, and let us distinguish the following cases.

Subcase 2.1: $|N^-[X]| = 6$. Hence, $N^-[X] = \{x_1, x_2, x_3, u, v, z\}$, and by Claim 3.4.3 let us assume, without loss of generality, that $N^-(x_3) =$

$\{v, z\}$. Moreover, since D is an oriented graph and it does not contain J_1, $N^-(z) \cap N^-[X] \subseteq \{u\}$, and $N^-(v) \cap N^-[X] \subseteq \{x_2\}$, therefore $z, v \notin Y$; hence, $u \in Y$. Since D is a TT_3-free oriented graph, $N^-(u) \subseteq \{x_2, x_3, z\}$. Moreover, by Remark 3.3.1, $z \notin N^-(u)$. Hence, $N^-(u) = \{x_2, x_3\}$, implying that D would contain J_2.

Subcase 2.2: $|N^-[X]| = 7$. In this case, there is $w \in N^-(x_3) \setminus (X \cup \{u, v, z\})$. By symmetry, $N^-(x_3) = \{z, w\}$ or $N^-(x_3) = \{v, w\}$. First, suppose that $N^-(x_3) = \{z, w\}$. Since D is a TT_3-free oriented graph, if $z \in Y$, then $N^-(z) = N^-(x_1)$, which is a contradiction by Claim 3.4.1. Hence $z \notin Y$. Analogously, if $w \in Y$ and $x_2 \in N^-(w)$, then $N^-(w) \subseteq \{x_2, u, v\}$, implying that D would contain J_7; and if $x_2 \notin N^-(w)$, then $N^-(w) \subseteq N^-[x_1]$, contradicting Claim 3.4.1. Thus, $w \notin Y$. If $v \in Y$, then $N^-(v) \subseteq (N^-[x_3] \cup \{x_2\})$. Hence, by Claim 3.4.1, $x_2 \in N^-(v)$, implying that $N^-(v) \subseteq \{x_2, x_3, w\}$, but if $N^-(v) = \{x_2, x_3\}$ or $N^-(v) = \{x_2, w\}$, then D would contain J_2 or J_9, respectively. Therefore, $v \notin Y$, and by symmetry, we can also conclude that $u \notin Y$, a contradiction.

Assume now that $N^-(x_3) = \{v, w\}$. Observe that $N^-(v) \cap N^-[X] \subseteq \{x_2, z\}$, then $v \notin Y$. If $u \in Y$, then $N^-(u) \subseteq \{x_2, x_3, z, w\}$, but it could be neither $\{x_2, z\}$ nor $\{x_3, w\}$ (by Remark 3.3.1). If $N^-(u) = \{x_2, x_3\}$, then D would contain J_3; if $N^-(u) = \{x_2, w\}$, then D would contain J_9; if $N^-(u) = \{x_3, z\}$, then D would contain J_7; and if $N^-(u) = \{z, w\}$, then D would contain J_{10}. Therefore, $u \notin Y$. If $w \in Y$, then $N^-(w) \subseteq \{x_1, x_2, u, z\}$. Hence, by Remark 3.3.1 and Claim 3.4.1, $N^-(w) = \{u, z\}$ or $N^-(w) = \{u, x_2\}$, implying that D would contain J_{12} or J_5, respectively. Therefore, $w \notin Y$. If $z \in Y$, then $N^-(z) \subseteq (N^-[x_3] \cup N^-(x_1))$. Hence, by Claim 3.4.1 and Remark 3.3.1, $N^-(z) = \{u, w\}$ or $N^-(z) = \{u, x_3\}$, yielding that D would contain J_{11} or J_6, respectively. Hence, $z \notin Y$, a contradiction.

Subcase 2.3: $|N^-[X]| = 8$. In this case, $N^-(x_3) = \{t, w\}$ for $t, w \notin N^-[x_1] \cup N^-[x_2]$. First, observe that if $Y \cap \{x_1, x_2\} = \emptyset$, then, without loss of generality, $t \in Y$, $x_1 \in N^-(t)$, yielding that $N^-(t) = N^-(x_2)$, a contradiction to Claim 3.4.1. Therefore $Y \cap \{x_1, x_2\} \neq \emptyset$. Hence, since $N^-[x_3] \cap (N^-[x_1] \cup N^-[x_2]) = \emptyset$, it follows that $N^-[x_3] \subseteq N^-[y] \cup N^-[y']$, with $y, y' \in Y$, yielding by Claim 3.4.2 that $x_3 \in Y$. If $x_2 \notin Y$, then $Y = \{x_1, x_3, y\}$ and $\{x_2, z\} = N^-(y)$, which is a contradiction to Remark 3.3.1. Therefore, $Y = \{x_2, x_3, y\}$, yielding that $N^-(x_1) \subseteq N^-[y]$, contradicting Claim 3.4.2.

Case 3: Suppose there are exactly two arcs between the elements of X. Then, $|N^-[X]| = 6, 7$. Let us distinguish the following cases.

Subcase 3.1: First, assume that (x_1, x_2, x_3) is a path of D. Then, $N^-(x_2) \cap N^-(x_3) = N^-(x_2) \cap N^-(x_1) = \emptyset$ by Remark 3.3.1. Hence, $N^-(x_2) = \{z, x_1\}$.

Subsubcase 3.1.1: $|N^-[X]| = 6$. Without loss of generality, we may assume that $N^-(x_3) = \{x_2, u\}$. Observe that if $u \in Y$, then $N^-(u) = \{x_2, z\}$, a contradiction to Remark 3.3.1, and then $u \notin Y$. If $v \in Y$, then $x_2 \notin N^-(v)$ again by Remark 3.3.1. Hence, if $v \in Y$, then $N^-(v) = \{x_3, z\}$, yielding that D would contain J_4. Therefore, $z \in Y$ and $|Y \cap X| = 2$. By Remark 3.3.1 and Claim 3.4.1, $N^-(z) = \{x_3, v\}$, implying that D would contain J_3.

Subsubcase 3.1.2: $|N^-[X]| = 7$. Then, $N^-(x_3) = \{x_2, w\}$ for some $w \notin N^-[x_1] \cup N^-[x_2]$. If $w \in Y$, then $N^-(w) \subseteq (N^-[x_1] \cup \{z\})$ and, by Claim 3.4.1 and Remark 3.3.1, $z \in N^-(w)$ and $N^-(w) \subseteq \{u, v, z\}$. This implies that D would contain J_6. Therefore, $w \notin Y$. If $z \in Y$, then $N^-(z) \subseteq N^-(x_1) \cup \{x_3, w\}$. Hence, by Claim 3.4.1 and Remark 3.3.1, without loss of generality, $N^-(z) = \{v, w\}$ or $N^-(z) = \{v, x_3\}$, implying that D would contain J_8 or J_2, respectively. Therefore, $z \notin Y$. If $u \in Y$, then $N^-(u) \subseteq N^-[x_3] \cup \{z\}$. By Claim 3.4.1 and Remark 3.3.1, $N^-(u) = \{z, x_3\}$ or $N^-(u) = \{z, w\}$, yielding that D would contain J_4 or J_5, respectively. Therefore, $u \notin Y$ and, by symmetry, $v \notin Y$, hence, $Y \setminus X = \emptyset$, a contradiction.

Subcase 3.2: Let us assume that $N^-(x_2) = \{x_1, x_3\}$. If $|N^-[X]| = 6$, then, without loss of generality, suppose that $N^-(x_3) = \{v, z\}$. Observe that $v \notin Y$, otherwise, $N^-(v) = \{x_2\}$. If $z \in Y$, then $N^-(z) \subseteq$

$\{x_1, x_2, u\}$ and, by Remark 3.3.1, $N^-(z) = \{x_2, u\}$, yielding that D would contain J_3. Hence, $z \notin Y$, and reasoning similarly, $u \notin Y$, a contradiction. If $|N^-[X]| = 7$, then $N^-(x_3) = \{z, w\}$ for some $w \notin \{x_1, x_2, x_3, u, v, z\}$. If $u \in Y$, then $N^-(u) \subseteq N^-[x_3] \cup \{x_2\}$, and by Claim 3.4.1 and Remark 3.3.1, $x_2 \in N^-(u)$ and $N^-(u) \subseteq \{x_2, z, w\}$, implying that D would contain J_3. Therefore, $u \notin Y$. Analogously, $v, z, w \notin Y$, yielding that $Y \setminus X = \emptyset$, a contradiction.

Subcase 3.3: Without loss of generality, let us assume that $(x_1, x_2), (x_1, x_3) \in A$. If $|N^-[X]| = 6$, then $N^-(x_2) = \{x_1, z\} = N^-(x_3)$, which contradicts Claim 3.4.3. Hence, $|N^-[X]| = 7$. Let $N^-(x_2) = \{x_1, z\}$ and $N^-(x_3) = \{x_1, w\}$. Observe that we also may assume that there are exactly two arcs between the elements of Y, and there is some $y \in Y$ satisfying the same as x_1, that is, $N^+[y] \cap Y = Y - y$. Therefore, if $x_1 \in Y$, we can assume that $Y = \{x_1, u, w\}$ and $N^+(u) \cap Y = \{x_1, w\}$. Then, $N^-(u) \subseteq \{x_3, x_2, z\}$ and, by Remark 3.3.1, $x_3 \in N^-(u)$, implying that $N^-(u) = \{x_3, x_2\}$ or $N^-(u) = \{x_3, z\}$ yielding that D would contain J_2 or J_9, respectively. Moreover, since $N^+(x_1) \cap N^-[X] = \{x_2, x_3\}$, it follows that $Y \cap \{x_2, x_3\} \neq \emptyset$. Furthermore, by Claim 3.4.4, $|Y \cap \{x_2, x_3\}| = 1$. Without loss of generality, suppose $Y \cap X = \{x_2\}$.

If $Y = \{x_2, z, u\}$, then $u \in N^+(z)$, yielding that $N^-(z) \subseteq \{v, x_3, w\}$. By Remark 3.3.1, $N^-(z) = \{v, w\}$ or $N^-(z) = \{v, x_3\}$, implying that D would contain J_5 or J_4, respectively. If $Y = \{x_2, z, w\}$, then $z \in N^-(w)$ and $x_3 \in N^-(z)$. Thus, without loss of generality, $u \in N^-(z)$ yielding that D would contain J_3. Therefore, $z \notin Y$. If $Y = \{x_2, u, v\}$, then, without loss of generality, $x_3 \in N^-(u)$ and $w \in N^-(v)$, implying that D contains J_3. Finally, if $Y = \{x_2, u, w\}$, then $x_3 \in N^-(u)$ and $v \in N^-(w)$, yielding that D would contain J_2.

Case 4: Suppose there are three arcs between the elements of X. Hence, $|N^-[X]| = 6$ and since D is TT_3-free, we may assume that $(x_1 x_2 x_3 x_1)$ is a directed triangle. Then, $N^-(x_i) \cap N^-(x_j) = \emptyset$, for all $i \neq j$. Let $N^-(x_1) = \{x_2, u\}$, $N^-(x_2) = \{x_3, v\}$ and $N^-(x_3) = \{x_1, z\}$. Notice that if $z \in N^-(u)$ or $v \in N^-(u)$, then D would contain J_2 or J_3, respectively. Therefore, since D is a TT_3-free oriented graph, $N^-(u) \cap N^-[X] \subseteq \{x_2\}$ and $u \notin Y$. Observe that, by symmetry, we can conclude the same for v and z, obtaining a contradiction again. $\square$

4

IDENTIFYING CODES IN LINE DIGRAPHS

In this chapter, which consists mainly of the results published in [7], we focus on the study of $(1, \leq \ell)$-identifying codes in line digraphs. It is organised as follows. In Section 4.1, we present some preliminary definitions and results needed for the rest of this chapter. In Section 4.2 we prove that a line digraph of minimum in-degree one does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$. Then, we give a characterisation so that a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree one admits a $(1, \leq 2)$-identifying code. As a direct consequence, we obtain that a Kautz digraph $K(d,k)$ with $d \geq 3$ admits a $(1, \leq 2)$-identifying code. In [27], Foucaud, Gravier, et al. introduced the notion of *edge-identifying code* of a graph to study the identifying codes of line graphs. In Section 4.3, we use the analogous of this notion for digraphs to establish, for digraphs without digons with both vertices of in-degree one, that $\overrightarrow{\gamma}^{ID}(LD) \geq |A(D)| - |V^{+}_{\geq 1}(D)|$. As a consequence, we get that a digraph having a 1-factor with minimum in-degree two and without digons with both vertices of in-degree two satisfies that $\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|$. We also provide an algorithm to construct identifying codes in oriented graphs with minimum in-degree at least two and minimum out-degree at least one. This algorithm allows us to prove that an oriented graph with minimum in-degree and out-degree at least two satisfies that $\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|$.

4.1 PRELIMINARIES

Given a digraph D, the *line digraph* $L(D)$, also denoted LD, is the digraph with vertex set $A(D)$ and such that for any two different vertices $a, b \in V(LD)$, $(a, b) \in A(LD)$ if and only if the head of a coincides with the tail of b. A directed digraph H is a line digraph if there is a directed digraph D such that $H = L(D)$. For any integer $k \geq 1$, the *k-iterated line digraph* $L^k D$ is defined recursively by $L^k D = LL^{k-1}D$, where $L^0 D = D$. From the definition, it is evident that the order of LD equals the size of D, that is, $|V(LD)| = |A(D)|$. For each vertex $v \in V(D)$, we denote $\Omega^-(v) = \{(u, v) \in A(D)\}$ and $\Omega^+(v) = \{(v, u) \in A(D)\}$. Hence, for each vertex $v \in V(D)$, the set of arcs $\Omega^+(v)$ (or $\Omega^-(v)$) in D corresponds to a set of vertices in LD. Moreover, $d^+(v) = |\Omega^+(v)|$ and $d^-(v) = |\Omega^-(v)|$, so if D has minimum degree δ, then the iterated line digraph $L^k D$ has minimum degree δ as well. Other properties of line digraphs can be seen in Aigner [1], Fiol, Yebra, and Alegre [26], and Reddy, Kuhl, Hosseini, and Lee [53].

Line digraphs were characterised by Heuchenne [37] with the following property.

Lemma 4.1.1 ([37]). *A digraph D is a line digraph if and only if it has no multiple arcs and, for any pair of vertices u and v, either $N^-(u) \cap N^-(v) = \emptyset$ or $N^-(u) = N^-(v)$. (A similar result is obtained replacing N^- by N^+.)*

The following theorem is another useful characterisation of line digraphs given by Beineke and Zamfirescu [11].

Theorem 4.1.1 ([11]). *A (simple) digraph D is a line digraph if and only if D is TT_3-free, the paths of length two are unique, there are no two digons incident to the same vertex, and if there are two vertices u, v such that $N^+(u) \cap N^+(v) \neq \emptyset$, then $N^+(u) = N^+(v)$.*

4.2 A CHARACTERISATION OF LINE DIGRAPHS ADMITTING A $(1, \leq \ell)$-IDENTIFYING CODE

As mentioned, in this section, we present a characterisation of line digraphs admitting a $(1, \leq \ell)$-identifying code. First, we consider the case $\ell = 1$ with the following result.

Proposition 4.2.1. *Let D be a digraph. Then, its line digraph admits an identifying code if and only if there is no digon with both vertices of in-degree 1 in D.*

Proof. We know that a digraph F admits an identifying code if and only if for any two different vertices $x, y \in V(F)$ we have $N_F^-[x] \neq N_F^-[y]$. We reason by contraposition. First, let LD be the line digraph of a digraph, and suppose that LD does not admit an identifying code. This is equivalent to have two different vertices $x, y \in V(LD)$ such that $N^-[x] = N^-[y]$, which implies that x and y form a digon. By Theorem 4.1.1, LD is TT_3-free, and by Remark 3.3.1, $N^-(x) \cap N^-(y) = \emptyset$, yielding that $d^-(x) = d^-(y) = 1$. Hence, LD contains a digon with both vertices of in-degree 1, thus, D contains a digon with both vertices of in-degree 1. Conversely, suppose that there is a digon $uv, vu \in A(D)$ with $d^-(u) = d^-(v) = 1$, then $uv, vu \in V(LD)$ form a digon in LD, and these vertices are twins since $N_{LD}^-[uv] = \{uv, vu\} = N_{LD}^-[vu]$, implying that LD does not admit an identifying code. □

Corollary 4.2.1. *Let D be a digraph with minimum in-degree $\delta^- \geq 2$. Then, its line digraph admits an identifying code.*

Next, we establish that if a line digraph admits a $(1, \leq \ell)$-identifying code, then $\ell \leq 2$. To this end, we need to prove the following preliminary result.

Lemma 4.2.1. *Let D be a digraph with minimum in-degree $\delta^- \geq 2$. Then, there exists a vertex $u \in V(D)$ with $d^+(u) \geq 2$ and such that there are at least two out-neighbours $x, y \in N^+(u)$ such that $d^+(x), d^+(y) \geq 1$.*

Proof. Let D be a digraph with minimum in-degree $\delta^- \geq 2$, and consider the subdigraph $D' = D - \{w \in V(D) \mid d^+(w) = 0\}$. Then, $\delta^-(D') \geq 2$. If $d^+_{D'}(u) < 2$ for all $u \in V(D')$, then we would reach the contradiction:

$$2|V(D')| \leq \sum_{v \in V(D')} d^-_{D'}(v) = \sum_{v \in V(D')} d^+_{D'}(v) \leq |V(D')|.$$

Hence, there is $u \in V(D')$ such that $d^+_{D'}(u) \geq 2$ and therefore, $d^+(u) \geq 2$. Since for any $v \in N^+_{D'}(u) \subseteq N^+(u)$ we have $d^+(v) \geq 1$, the proof is completed. □

Proposition 4.2.2. *Let LD be a line digraph of a digraph D with minimum in-degree $\delta^- \geq 1$, then LD does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$.*

Proof. Note that $\delta^-(LD) = \delta^-(D) = \delta^-$. If $\delta^- \geq 2$, by Lemma 4.2.1, there exists a vertex v in LD with $d^+(v) \geq 2$ and two vertices $x, y \in N^+(v)$ such that $d^+(x), d^+(y) \geq 1$. By Lemma 4.1.1, we have $N^-(x) = N^-(y)$. Hence, by Corollary 3.1.4, if LD admits a $(1, \leq \ell)$-identifying code, then $\ell \leq 2$, and the result is valid. Suppose that $\delta^- = 1$. Take a vertex u with $d^-(u) = 1$. If $d^+(u) \geq 1$, then by Proposition 3.1.1, we get that $\ell \leq 2$ and we obtain the result. Therefore we assume that every vertex with in-degree one has out-degree zero. Let F be the digraph obtained from LD by removing all the vertices of in-degree one. Observe that $\delta^-(F) \geq 2$, then reasoning as in the first part of the proof, F does not admit a $(1, \leq 3)$-identifying code. This means that there are two different sets $X, Y \subseteq F \subset V(LD)$ such that $1 \leq |X| \leq |Y| \leq 3$ and $N^-_F[X] = N^-_F[Y]$. Since for any vertex $u \in V(F)$, $N^-_F[u] = N^-_{LD}[u]$, it follows that $N^-_{LD}[X] = N^-_{LD}[Y]$. Hence, LD does not admit a $(1, \leq 3)$-identifying code. □

Remember that, according to Proposition 3.1.1, if D is a digraph admitting a $(1, \leq 2)$-identifying code, then there is no vertex of in-degree 1 belonging to a digon. In the following result, we give sufficient and necessary conditions for a line digraph to admit a $(1, \leq 2)$-identifying code. To do that, we use the following result, which follows from the fact that, in a line digraph, the paths of length two are unique by Theorem 4.1.1.

Corollary 4.2.2. *Let LD be a line digraph. If $u, v \in V(LD)$ are two different vertices such that $N^+(u) \cap N^+(v) \neq \emptyset$, then $N^-(u) \cap N^-(v) = \emptyset$.*

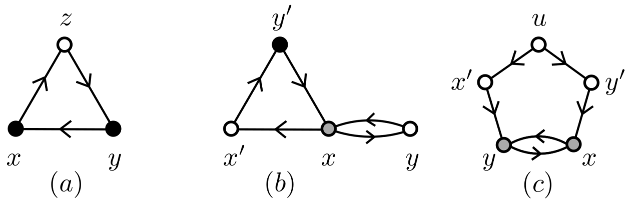


Figure 21. The forbidden subdigraphs of Theorem 4.2.1 and Corollary 4.2.3, where the vertices of in-degree one are indicated in black colour and the vertices of in-degree two in grey colour.

Theorem 4.2.1. *Let LD be a line digraph with minimum in-degree $\delta^- \geq 1$, such that the vertices of in-degree 1 (if any) do not lie on a digon. Then, LD admits a $(1, \leq 2)$-identifying code if and only if LD satisfies the following conditions:*

- (i) *There are no directed 3-cycles with at least 2 vertices of in-degree 1 (see Figure 21 (a)).*
- (ii) *There do not exist four vertices x, x', y and y' such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$, and $x \in N^-(x') \cap N^-(y)$ (see Figure 21 (b)).*
- (iii) *There do not exist four vertices x, x', y, and y' in $V(LD)$ such that $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$, and $N^-(x') \cap N^-(y') \neq \emptyset$ (see Figure 21 (c)).*
- (iv) *There is no directed 4-cycle with the four vertices of in-degree 1.*

Proof. First, suppose that LD admits a $(1, \leq 2)$-identifying code and let us show that LD satisfies all the conditions (i)-(iv).

(i) Suppose that LD does not satisfy (i). Hence, let (z, y, x, z) be a directed 3-cycle in LD such that $d^-(x) = 1 = d^-(y)$ (see Figure 21 (a)). Then, $N^-[\{x,z\}] = \{x,z\} \cup \{y\} \cup N^-(z) = \{y\} \cup N^-[z]$, and $N^-[\{y,z\}] = \{y,z\} \cup N^-(z) = \{y\} \cup N^-[z]$, implying that LD does not admit a $(1, \leq 2)$-identifying code, which is a contradiction.

(ii) Suppose that LD does not satisfy (ii). Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where x, x', y, y' are four different vertices of LD such that $N^-(x) = \{y, y'\}$, $N^-(y') = \{x'\}$, and $x \in N^-(x') \cap N^-(y)$ (see Figure 21 (b)). Hence, by Lemma 4.1.1, we get $N^-(x') = N^-(y)$, and it follows that

$$\begin{aligned} N^-[X] &= N^-(x) \cup N^-(x') \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y) \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y) \cup \{x'\} \\ &= \{y, y'\} \cup N^-(y) \cup N^-(y') \\ &= N^-[Y]. \end{aligned}$$

Therefore, LD does not admit a $(1, \leq 2)$-identifying code, which is a contradiction.

(iii) Suppose that LD does not satisfy (iii). Let $X = \{x, x'\}$ and $Y = \{y, y'\}$, where $N^-(x) = \{y, y'\}$, $N^-(y) = \{x, x'\}$, and $N^-(x') \cap N^-(y') \neq \emptyset$ (see Figure 21 (c)). Since, by Lemma 4.1.1, $N^-(x') = N^-(y')$, it follows that

$$\begin{aligned} N^-[X] &= N^-(x) \cup N^-(x') \cup \{x, x'\} \\ &= \{y, y'\} \cup N^-(y') \cup N^-(y) \\ &= N^-[Y]. \end{aligned}$$

Therefore, LD does not admit a $(1, \leq 2)$-identifying code, which is a contradiction.

(iv) Suppose that LD does not satisfy (iv). Let $(u_1, u_2, u_3, u_4, u_1)$ be a 4-cycle of LD such that $d^-(u_i) = 1$ for all $i \in \{1, 2, 3, 4\}$. Then, $N^-[\{u_1, u_3\}] = N^-[\{u_2, u_4\}]$, implying that LD does not admit a $(1, \leq 2)$-identifying code which is a contradiction.

For the converse, suppose that LD satisfies all the conditions (i)-(iv), and that it does not admit a $(1, \leq 2)$-identifying code. Let $X, Y \subseteq V(LD)$ be two different subsets such that $1 \leq |X| \leq |Y| \leq 2$

and $N^-[X] = N^-[Y]$. Since the vertices of in-degree one do not lie on a digon, by Proposition 4.2.1, $|Y| = 2$. If $|X| = 1$, say $X = \{x\}$, then $N^-[Y] = N^-[X] = N^-[x]$. It follows that $N^-[y] \subseteq N^-(x)$ for all $y \in Y \setminus X$, hence $y \in N^-(x)$. By Theorem 4.1.1, LD is TT_3-free, which allows us to apply Remark 3.3.1, so $N^-(y) \cap N^-(x) = \emptyset$, then $N^-(y) = \emptyset$, which contradicts that $\delta^- \geq 1$.

Suppose $|X| = 2$ and consider two cases according to $X \cap Y \neq \emptyset$ or if $X \cap Y = \emptyset$.

(a) Suppose that $X \cap Y \neq \emptyset$. Let $X = \{x, z\}$ and $Y = \{y, z\}$. We will consider two cases, when there is at least one arc between x and y, and the case when there is no arc between x and y.

$(a.1)$ If there is an arc between x and y, say $yx \in A(LD)$, then by Remark 3.3.1, $N^-(x) \cap N^-(y) = \emptyset$. Then, $N^-(y) \subseteq N^-[z] \cup \{x\}$ and $N^-(x) \subseteq N^-[z] \cup \{y\}$. First, suppose that $d^-(x) \geq 2$ and let $u \in N^-(x) \setminus \{y\}$. Hence, $u \in N^-[z]$. If $u = z$, then $N^-(x) \cap N^-(z) = \emptyset$ by Remark 3.3.1, and $N^-(y) \cap N^-(z) = \emptyset$. Hence, $N^-(x) = \{y, z\}$ and $N^-(y) = \{x\}$ (since $\delta^-(LD) \geq 1$). Then y is a vertex of in-degree 1 lying on a digon, a contradiction to the hypothesis. Therefore, $u \neq z$, that is, $u \in N^-(z) \cap N^-(x)$ implying, by Lemma 4.1.1, that $N^-(z) = N^-(x)$, hence $y \in N^-(z)$, implying that $N^-(y) \cap N^-(z) = \emptyset$ by Remark 3.3.1. Then $N^-(y) \subseteq \{x, z\}$. Since $\delta^-(LD) \geq 1$, it follows that $N^-(y) = \{x\}$, $N^-(y) = \{z\}$ or $N^-(y) = \{x, z\}$. The first two cases are not possible because vertices of degree one do not lie on digons, and the third case is not possible because, by Theorem 4.1.1, LD does not contain two digons incident to the same vertex. Second, suppose that $d^-(x) = 1$, then $N^-(x) = \{y\}$. Since $x \in N^-[Y]$ and x does not lie on a digon, $x \in N^-(z)$. Since, $x \notin N^-(y)$, $N^-(y) \cap N^-(z) = \emptyset$ by Lemma 4.1.1, implying that $N^-(y) = \{z\}$ because $N^-(y) \subseteq N^-[z]$. Therefore, (x, z, y, x) is a directed 3-cycle of LD with two vertices of in-degree 1, implying that LD does not satisfy (i).

$(a.2)$ Now, suppose that there is no arc between x and y. Since $x \in N^-[Y]$ and $y \in N^-[X]$, it follows that $x, y \in N^-(z)$. Since LD is

TT_3-free, $y \notin N^-(x)$, by Remark 3.3.1, $N^-(x) \cap N^-(z) = \emptyset$, and by Corollary 4.2.2, $N^-(x) \cap N^-(y) = \emptyset$ implying that $N^-(x) = \{z\}$ and x, z form a digon, a contradiction since there are no vertices of in-degree 1 lying on a digon.

(b) Suppose $X \cap Y = \emptyset$, with $X = \{x, x'\}$ and $Y = \{y, y'\}$. Notice that we can assume $y \in N^-(x)$, that is, $yx \in A(LD)$. Then by Remark 3.3.1, $N^-(x) \cap N^-(y) = \emptyset$ implying that $N^-(y) \subseteq N^-(x') \cup \{x, x'\}$. Since $x \in N^-[Y]$, there are two cases to be considered.

$(b.1)$ Suppose that $x \in N^-(y)$. Then $d^-(x), d^-(y) \geq 2$, since both vertices lie on a digon. If there is $u \in N^-(y) \setminus (X \cup Y)$, then $u \in N^-(x')$, and by Lemma 4.1.1, $N^-(y) = N^-(x')$ implying that $x \in N^-(x')$. Hence, since $x' \in N^-[Y]$ and $N^-(x') = N^-(y)$, it follows that $x' \in N^-(y')$. Furthermore, $y' \in N^-(x') \cup N^-(x)$. If $y' \in N^-(x')$, then by Remark 3.3.1, $N^-(x') \cap N^-(y') = \emptyset$, and by Corollary 4.2.2, $N^-(x) \cap N^-(y') = \emptyset$, because $x' \in N^+(x) \cap N^+(y')$. Moreover, by Theorem 4.1.1, $x \notin N^-(y')$ because LD is TT_3-free, and $y \notin N^-(y')$ because, otherwise LD would have two digons incident to the vertex y. This implies that $N^-(y') = \{x'\}$, that is, $d^-(y') = 1$, a contradiction because y' lies on a digon. Then $y' \in N^-(x)$ and by Remark 3.3.1, $N^-(y') \cap N^-(x) = \emptyset$. Moreover, $x \notin N^-(y')$ because otherwise LD would have two digons incident to vertex x. If $N^-(y') \cap N^-(x') \neq \emptyset$ by Lemma 4.1.1, $N^-(y') = N^-(x')$ implying that $x \in N^-(y')$, which is a contradiction. Therefore, $N^-(y') \cap N^-(x') = \emptyset$ and so $N^-(y') = \{x'\}$ and $N^-(x) = \{y, y'\}$. Therefore, LD does not satisfy (ii). Thus, we have proved that $N^-(y) \subseteq X \cup Y$. Reasoning similarly for x as we did for y, this time considering the arc xy, we get that $N^-(x) \subseteq X \cup Y$.

If $x' \in N^-(x)$, then $x' \in N^-(y')$. Since LD is TT_3-free, $x' \notin N^-(y)$ and by Corollary 4.2.2, $N^+(x) \cap N^+(y') = \emptyset$ implying $y \notin N^+(y')$, therefore $|N^-(y) \cap \{x', y'\}| = 0$ implying that $d^-(y) = 1$, a contradiction. Therefore, $x' \in N^-(y)$ and $y' \in N^-(x)$. Since $x \in N^+(y) \cap N^+(y')$, by Corollary 4.2.2, $N^-(y) \cap N^-(y') = \emptyset$, that is, $x' \notin N^-(y')$. Moreover, since LD is TT_3-free, $y \notin N^-(y')$. And since there are no two digons incident to the same vertex $x \notin N^-(y')$. Therefore there is a vertex $u \in N^-(y') \setminus (X \cup Y)$ and since $N^-(y') \cap N^-(x) = \emptyset$, $u \in N^-(x')$. Hence, LD does not satisfy (iii).

$(b.2)$ Suppose that $x \in N^-(y') \setminus N^-(y)$. Then $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$ by Remark 3.3.1, implying that $N^-(x) \subseteq \{y, y'\}$.

$(b.2.1)$ If $N^-(x) = \{y\}$, then $y' \in N^-(x')$, implying that $N^-(y') \cap (N^-(x') \cup N^-(x)) = \emptyset$, and consequently $N^-(y') \subseteq \{x, x'\}$. If $x' \in N^-(y)$, then $N^-(x') \cap (N^-(y) \cup N^-(y')) = \emptyset$, implying that $N^-(x') \subseteq \{y, y'\}$. Observe that if $y \in N^-(x')$, then by Lemma 4.1.1, $N^-(x) = N^-(x') = \{y, y'\}$, a contradiction with the assumption that $N^-(x) = \{y\}$. Hence, $N^-(x') = \{y'\}$. Moreover, $N^-(y') \subseteq \{x, x'\}$ and $N^-(y') = \{x\}$ because otherwise the vertices x', y' form a digon with vertex x' of degree one contradicting the hypothesis. Also, $N^-(y) \subseteq \{x', y'\}$ and since $N^-(x') \cap N^-(y) = \emptyset$, $y' \notin N^-(y)$, we have $N^-(y) = \{x'\}$. Therefore, (x, y', x', y, x) is a directed 4-cycle in LD with four vertices of in-degree one, and LD does not satisfy (iv).

$(b.2.2)$ If $N^-(x) = \{y, y'\}$, we have a digon formed by vertices x and y', also $N^-(x) \cap (N^-(y) \cup N^-(y')) = \emptyset$, and consequently $N^-(y) \subseteq \{x'\} \cup N^-(x')$ (recall that we are assuming that $x \in N^-(y') \setminus N^-(y)$). First, suppose that $x' \in N^-(y)$. Then $N^-(y) \cap N^-(x') = \emptyset$ and so $N^-(y) = \{x'\}$, and therefore $y \notin N^-(x')$ because LD has no digons consisting of vertices of in-degree one. Hence, by Lemma 4.1.1, we have $N^-(x) \cap N^-(x') = \emptyset$. Also $x \notin N^-(x')$ because otherwise LD does not satisfy (ii), a contradiction, and then $N^-(x') \cap N^-(y') = \emptyset$, concluding that $N^-(x') = \emptyset$, which is a contradiction. Therefore, suppose that $x' \in N^-(y') \setminus N^-(y)$. By Theorem 4.1.1, $x, y' \notin N^-(x')$, implying by Lemma 4.1.1 that $N^-(x') \cap N^-(x) = \emptyset$, $N^-(x') \cap N^-(y') = \emptyset$. Hence, $N^-(y') = \{x, x'\}$ and $N^-(x') \subseteq N^-(y)$ and, since $\delta^-(LD) \geq 1$, there is $u \in N^-(x') \setminus (X \cup Y)$. Therefore, LD does not satisfy (iii), a contradiction. This completes the proof. □

Notice that, according to the above theorem, if a line digraph with minimum in-degree $\delta^-_{LD} \geq 2$ does not admit a $(1, \leq 2)$-identifying code, then $\delta^-_{LD} = 2$. In the following corollary, we give some sufficient conditions for a line digraph with minimum in-degree at least two, to admit a $(1, \leq 2)$-identifying code.

Corollary 4.2.3. *Let D be a digraph with minimum in-degree $\delta^-(D) \geq 2$. Then the following assertions hold.*

(i) The line digraph LD admits a $(1, \leq 2)$-identifying code if and only if LD is F-free, where F is the digraph (c) of Figure 21.

(ii) If $\delta^- \geq 3$, then LD admits a $(1, \leq 2)$-identifying code.

(iii) If $k \geq 2$, then $L^k D$ admits a $(1, \leq 2)$-identifying code.

Proof. Let D be a digraph with minimum in-degree $\delta^-(D) \geq 2$. Items (i) and (ii) follow directly from Theorem 4.2.1. To prove (iii) observe that if $k \geq 2$, then $L^k D$ does not contain the subdigraph of Figure 21 (c), otherwise $L^{k-1}D$ would contain a TT_3, a contradiction to Theorem 4.1.1. More precisely, suppose that $u, x, x', y, y' \in V(L^k D)$ are five different vertices such that $L^k D[\{u, x, x', y, y'\}]$ is isomorphic to Figure 21 (c). Let $u = (u_1, u_2)$ with $u_1, u_2 \in V(L^{k-1}D)$. Then $x' = (u_2, x'_2)$ and $y' = (u_2, y'_2)$ for some two different vertices $x'_2, y'_2 \in V(L^{k-1}D)$, $x = (y'_2, x'_2)$ and $y = (x'_2, y'_2)$. Therefore, $L^{k-1}D[\{u_2, x'_2, y'_2\}] \cong TT_3$. □

A large known family of digraphs obtained with the line digraph technique is the family of Kautz digraphs. The *Kautz digraph* of degree d and diameter k is defined as the $(k-1)$-iterated line digraph of the symmetric complete digraph of $d+1$ vertices $\overleftrightarrow{K}_{d+1}$, that is, $K(d, k) \cong L^{k-1}\overleftrightarrow{K}_{d+1}$. For instance, the Kautz digraph $K(2, 2)$ shown in Figure 22, is the line digraph of the symmetric complete digraph on three vertices.

Corollary 4.2.4. *For each $d \geq 3$, the Kautz digraph $K(d, 2) \cong L\overleftrightarrow{K}_{d+1}$ admits a $(1, \leq 2)$-identifying code.*

By Corollary 4.2.3 (iii), the Kautz digraph $K(2, 2) = L\overleftrightarrow{K}_3$ (see Figure 22) does not admit a $(1, \leq 2)$-identifying code. Therefore, the condition $k \geq 2$ in Corollary 4.2.3 (iii) is necessary.

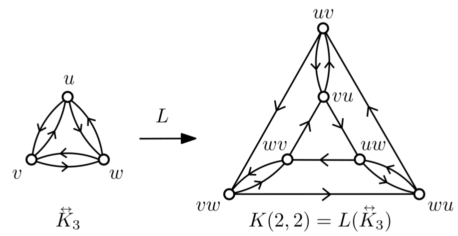


Figure 22. The Kautz digraph $K(2,2)$ as the line digraph of the symmetric complete digraph $\overleftrightarrow{K}_3$.

4.3 ARC-IDENTIFYING CODES

Foucaud, Naserasr, and Parreau [28] characterised the digraphs that only admit as identifying code the whole set of vertices. This allows us to have a first upper bound for the identifying number of a line digraph. Let us introduce the terminology they used for this characterisation.

Given a digraph D and a vertex $x \notin V(D)$, $x\overrightarrow{\triangleleft}(D)$ is the digraph with vertex set $V(D) \cup \{x\}$, and whose arcs are the arcs of D together with each arc (x, v) for every $v \in V(D)$.

Definition 4.3.1. *$(K_1, \oplus, \overrightarrow{\triangleleft})$ is the* closure *of the one-vertex graph K_1 with respect to the operations $\oplus$ and $\overrightarrow{\triangleleft}$. Namely, the class of all digraphs that can be built from K_1 by repeated applications of $\oplus$ and $\overrightarrow{\triangleleft}$.*

Foucaud, Naserasr, et al. [28] proved that for any digraph D, $\overrightarrow{\gamma}^{ID}(D) = |V(D)|$ if and only if $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$. Notice that, if $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$ is a digraph with at least one arc, then the digraph $D' = x\overrightarrow{\triangleleft}(D)$ is not TT_3-free and hence, by Theorem 4.1.1, it is not a line digraph. Let r be a positive integer, we denote by $\overrightarrow{K}_{1,r}$ the digraph obtained from the star $K_{1,q}$ by orienting the edges from the vertex with degree q to each neighbour. Now, consider $\mathcal{F}$, the family of digraphs defined recursively with the following three rules.

1. $K_1 \in \mathcal{F}$;
2. $\overrightarrow{K}_{1,q} \in \mathcal{F}$, for every $q \geq 1$;
3. if $D_1, D_2 \in \mathcal{F}$, then $D_1 \oplus D_2 \in \mathcal{F}$.

Thus, we have the following result.

Corollary 4.3.1. *Let LD be a line digraph. Then, $\overrightarrow{\gamma}^{ID}(LD) = |V(LD)|$ if and only if $LD \in \mathcal{F}$.*

Before continuing with the study of line digraphs, let us point out another consequence of the characterisation mentioned above, a consequence for digraphs in general. Since for any $D \in (K_1, \oplus, \overrightarrow{\triangleleft})$ with at least one arc we get $\hat{\delta}^-(D) = 0$, by Corollary 3.1.1 we have the following result.

Corollary 4.3.2. *Let D be a digraph admitting a $(1, \leq \ell)$-identifying code and such that $\overrightarrow{\gamma}^{ID}(LD) = |V(LD)|$. Then, $\ell = 1$.*

Observe that, in particular, if LD is a line digraph with minimum in-degree $\delta^- \geq 2$, then $\overrightarrow{\gamma}^{ID}(LD) \leq |V(LD)| - 1$. Next, we give a lower bound on $\overrightarrow{\gamma}^{ID}(LD)$.

With this goal, we define the relation $\sim$ over the set of vertices $V(LD)$ as follows. For all $u, v \in V(LD)$, $u \sim v$ if and only if $N^-(u) = N^-(v)$. Clearly, $\sim$ is an equivalence relation. For any $u \in V(LD)$, let $[u]_\sim = \{v \in V(LD) : v \sim u\}$.

Lemma 4.3.1. *Let C be an identifying code of a line digraph LD. Then, for any vertex $w \in V(LD)$,*

$$|[w]_\sim \setminus C| \leq 1.$$

Proof. Let $w \in V(LD)$ and $u, v \in [w]_\sim \setminus C$. Then, $N^-(u) = N^-(v)$ and, since $u, v \notin C$, it follows that $N^-[u] \cap C = N^-(u) \cap C = N^-(v) \cap C = N^-[v] \cap C$, which is a contradiction if $u \neq v$. □

Definition 4.3.2. *Given a digraph D, a subset $\widetilde{C}$ of $A(D)$ is an* arc-identifying code *of D if $\widetilde{C}$ is both:*

- *an arc-dominating set of D, that is, for each arc $uv \in A(D)$, $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} \neq \emptyset$, and*
- *an arc-separating set of D, that is, for each pair $uv, wz \in A(D)$ (with $uv \neq wz$), $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} \neq (\{wz\} \cup \Omega^-(w)) \cap \widetilde{C}$.*

Hence, an arc-identifying code of D is an identifying code of its line digraph LD. As a consequence, given a digraph D, the minimum size of an identifying code of its line digraph, $\overrightarrow{\gamma}^{ID}(LD)$, is equivalent to the minimum size of an arc-identifying code of D.

With the following result, we characterize the arc-identifying codes.

Theorem 4.3.1. *Let D be a digraph and $\widetilde{C} \subseteq A(D)$. Then, $\widetilde{C}$ is an arc-identifying code of D if and only if $\widetilde{C}$ satisfies the following two conditions:*

(i) For all $v \in V(D)$, $|\Omega^+(v) \setminus \widetilde{C}| \leq 1$, and if $|\Omega^+(v) \setminus \widetilde{C}| = 1$, then $\Omega^-(v) \cap \widetilde{C} \neq \emptyset$;

(ii) for all $uv \in \widetilde{C}$, if $vu \in \widetilde{C}$ or $|\Omega^+(v) \setminus \widetilde{C}| = 1$, then $((\Omega^-(u) \cup \Omega^-(v)) \setminus \{uv, vu\}) \cap \widetilde{C} \neq \emptyset$.

Proof. Suppose that $\widetilde{C}$ is an arc-identifying code of D. Hence, $\widetilde{C}$ is an identifying code of LD, and by Lemma 4.3.1, we have for all $vw \in V(LD)$, $|[vw]_\sim \setminus \widetilde{C}| \leq 1$. Observe that $rs \in [vw]_\sim$ if and only if $N^-_{LD}(rs) = N^-_{LD}(vw)$, which only can occur if and only if $r = v$. Therefore, we get that for all $v \in V(D)$, $|\Omega^+(v) \setminus \widetilde{C}| \leq 1$ holds. Moreover, let $v \in V(D)$ be such that $|\Omega^+(v) \setminus \widetilde{C}| = 1$ and let $vx \in \Omega^+(v) \setminus \widetilde{C}$. Hence, $(\{vx\} \cup \Omega^-(v)) \cap \widetilde{C} = \Omega^-(v) \cap \widetilde{C}$. Since $\widetilde{C}$ is an arc-identifying code, $(\{vx\} \cup \Omega^-(v)) \cap \widetilde{C} \neq \emptyset$, hence $\widetilde{C}$ satisfies (i). To prove (ii), let $uv \in \widetilde{C}$ be such that $((\Omega^-(u) \cup \Omega^-(v)) \setminus \{vu, uv\}) \cap \widetilde{C} = \emptyset$. If $vu \in \widetilde{C}$, then $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} = \{uv, vu\} = (\{vu\} \cup \Omega^-(v)) \cap \widetilde{C}$, contradicting that $\widetilde{C}$ is an arc-identifying code. Hence, $vu \notin \widetilde{C}$. If $|\Omega^+(v) \setminus \widetilde{C}| = 1$, say $\Omega^+(v) \setminus \widetilde{C} = \{vx\}$, then $(\{uv\} \cup \Omega^-(u)) \cap \widetilde{C} = \{uv\} = (\{vx\} \cup \Omega^-(v)) \cap \widetilde{C}$, a contradiction. Therefore, $\widetilde{C}$ satisfies (ii).

Now, suppose that $\widetilde{C}$ is a set of arcs of D satisfying (i) and (ii), and let us show that $\widetilde{C}$ is an arc-identifying code. To see that $\widetilde{C}$ is an arc-dominating set of D, let $ab \in A(D)$. By (i), $\Omega^+(a) \subseteq \widetilde{C}$ or $\Omega^-(a) \cap \widetilde{C} \neq \emptyset$, implying that $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} \neq \emptyset$. Therefore, $\widetilde{C}$ is an arc-dominating set of D. Next, let us prove that $\widetilde{C}$ is an arc-separating set of D. On the contrary, suppose that there are two different arcs ab and cd, such that $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} = (\{cd\} \cup \Omega^-(c)) \cap \widetilde{C}$. First, let us assume that $ab, cd \notin \widetilde{C}$ and take an arc $uv \in (\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} = (\{cd\} \cup \Omega^-(c)) \cap \widetilde{C}$. Then $v = a = c$, implying that $ab, cd \in \Omega^+(v) \setminus \widetilde{C}$, contradicting (i). Second, assume that $ab \in \widetilde{C}$, hence, $ab \in \Omega^-(c)$, implying that $c = b$. If $bd \notin \widetilde{C}$, then $|\Omega^+(b) \setminus \widetilde{C}| = 1$ and by (ii), $((\Omega^-(a) \cup \Omega^-(b)) \setminus \{ba, ab\}) \cap \widetilde{C} \neq \emptyset$. Then $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} \neq (\{bd\} \cup \Omega^-(b)) \cap \widetilde{C}$, a contradiction with our assumption. Therefore, $bd \in \widetilde{C}$ implying that $bd \in \Omega^-(a)$ and $d = a$. Again by (ii), $((\Omega^-(a) \cup \Omega^-(b)) \setminus \{ab, ba\}) \cap \widetilde{C} \neq \emptyset$, yielding that $(\{ab\} \cup \Omega^-(a)) \cap \widetilde{C} \neq (\{ba\} \cup \Omega^-(b)) \cap \widetilde{C}$, a contradiction. Therefore, $\widetilde{C}$ is an arc-separating set. This completes the proof. □

Recall that, by Proposition 4.2.1, a digraph D admits an arc-identifying code if and only if there is no digon with both vertices of in-degree one.

Theorem 4.3.2. *Let D be a digraph without digons with both vertices of in-degree 1. Then,*

$$\overrightarrow{\gamma}^{ID}(LD) \geq |A(D)| - |V^+_{\geq 1}(D)|.$$

Proof. By Proposition 4.2.1, LD admits a $(1, \leq 1)$-identifying code. Let $\widetilde{C}$ be an arc-identifying code of D. Then, by Theorem 4.3.1,

$$\begin{aligned} |\widetilde{C}| &\geq \sum_{u \in V^+_{\geq 1}(D)} (d^+_D(u) - 1) \\ &= \sum_{u \in V^+_{\geq 1}(D)} d^+_D(u) - |V^+_{\geq 1}(D)| \\ &= |A(D)| - |V^+_{\geq 1}(D)|. \end{aligned}$$

□

Notice that by the proof of Theorem 4.3.2, we have $\gamma^{ID}(LD) = |A(D)| - |V^+_{\geq 1}(D)|$ if and only if $|\Omega^+(v) \setminus \widetilde{C}| = 1$ for each vertex $v \in V^+_{\geq 1}(D)$. In particular, if $d^+(v) = 1$ and the lower bound is reached, then $\Omega^+(v) \cap \widetilde{C} = \emptyset$.

Recall that a *1-factor* of a digraph is a 1-regular spanning subdigraph. Next, we show that some digraphs with a 1-factor have an identifying number that attains the equality in Theorem 4.3.2.

Theorem 4.3.3. *Let D be a digraph having a 1-factor, with minimum in-degree $\delta^- \geq 2$, and without digons with both vertices of in-degree two. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.*

Proof. Let F denote a 1-factor in D. Let $\widetilde{C} = A(D) \setminus A(F)$. Let us show that $\widetilde{C}$ satisfies the requirements of Theorem 4.3.1. By definition of $\widetilde{C}$, $|\Omega^+(v) \setminus \widetilde{C}| = 1$ and $|\Omega^-(v) \setminus \widetilde{C}| = 1$ for each vertex $v \in V(D)$. Hence, Theorem 4.3.1 (i) holds because $|\Omega^+(v) \setminus \widetilde{C}| = 1$ and $\Omega^-(v) \cap \widetilde{C} \neq \emptyset$, since $\delta^- \geq 2$. Moreover, for any arc $uv \in \widetilde{C}$ not in a digon we have $((\Omega^-(v) \cup \Omega^-(u)) \setminus \{uv\}) \cap \widetilde{C} \neq \emptyset$ because $|\Omega^-(v) \setminus \widetilde{C}| = 1$, which implies that $|\Omega^-(v) \cap \widetilde{C}| = d^-(v) - 1 \geq 1$. And if $uv \in \widetilde{C}$ belongs to a digon, since one of the two vertices, say v, must have $d^-(v) \geq 3$ we have $((\Omega^-(v) \cup \Omega^-(u)) \setminus \{uv, vu\}) \cap \widetilde{C} \neq \emptyset$ because $|\Omega^-(v) \cap \widetilde{C}| = d^-(v) - 1 \geq 2$. In either case Theorem 4.3.1 (ii) holds. Therefore, $\widetilde{C}$ is an arc-identifying code of D and $\gamma^{ID}(LD) \leq |A(D)| - |V(D)|$. Furthermore, by Theorem 4.3.2, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$. □

Recall that a digraph D is *Hamiltonian* if D contains a directed cycle C such that $V(C) = V(D)$, and this cycle is called *Hamiltonian cycle*. Clearly, a Hamiltonian digraph has a 1-factor consisting of a directed cycle W such that $V(W) = V(D)$. The following result is an immediate consequence of Theorem 4.3.3.

Corollary 4.3.3. *Let D be a Hamiltonian digraph with minimum in-degree $\delta^- \geq 2$ and without digons with both vertices of in-degree two. Then, $\gamma^{ID}(LD) = |A(D)| - |V(D)|$.*

Corollary 4.3.4. *The identifying number of a Kautz digraph $K(d, k)$ is $\gamma^{ID}(K(d, k)) = d^k - d^{k-2}$ for $d \geq 3$ and $k \geq 2$.*

Proof. Note that $K(d, 2) = LK_{d+1}$. Since K_{d+1} is Hamiltonian and $d \geq 3$, by Corollary 4.3.3, $\gamma^{ID}(K(d, 2)) = \gamma^{ID}(LK_{d+1}) = |A(K_{d+1})| -$

$|V(K_{d+1})| = d(d+1) - (d+1) = d^2 - 1$, and the result holds for $k = 2$. For any $k \geq 3$, the Kautz digraph $K(d,k) = L^{k-1}K_{d+1} = LL^{k-2}K_{d+1} = LK(d,k-1)$. Since $K(d,k-1)$ is a Hamiltonian digraph and $d \geq 3$, by Corollary 4.3.3, $\gamma^{ID}(K(d,k)) = \gamma^{ID}(LK(d,k-1)) = d^k + d^{k-1} - (d^{k-1} + d^{k-2}) = d^k - d^{k-2}$, and the result holds. $\square$

To extend Corollary 4.3.4 to $K(2,k)$, we use the 1-factorization of Kautz digraphs obtained by Tvrdík [58]. This 1-factorization uses the following operation.

Definition 4.3.3. *[58] If $x = x_1 \ldots x_k \in V(K(d,k))$, then*

- $\sigma_1(x) = x_2 \ldots x_{k-1}x_k x_1$ *if* $x_1 \neq x_k$.
- $\sigma_1(x) = x_2 \ldots x_{k-1}x_k x_2$ *if* $x_1 = x_k$.

Let $Inc : V(K(d,k)) \times \mathbb{Z}_d \to V(K(d,k))$ denote a binary operation such that

$$Inc(x_1 \ldots x_{k-1}x_k, i) = x_1 \ldots x_{k-1}x'_k,$$

where

$$x'_k = \begin{cases} x_k + i \mod (d+1) & \text{if } x_{k-1} > x_k \text{ and } x_{k-1} > x_k + i, \text{ or} \\ & x_{k-1} < x_k \text{ and } x_{k-1} + d + 1 > x_k + i; \\ x_k + i + 1 \mod (d+1) & \text{otherwise.} \end{cases}$$

Then, the generalized K-shift operation is defined as follows:

$$\begin{aligned} \sigma_1^{+i}(x) &= Inc(\sigma_1(x), i), \\ \sigma_k^{+i}(x) &= \sigma_1^{+i}(\sigma_{k-1}^{+i}(x)). \end{aligned}$$

Theorem 4.3.4. *[58] The arc set of $K(d,k)$ can be partitioned into d 1-factors $\mathcal{F}_0, \ldots, \mathcal{F}_{d-1}$ such that the cycles of $\mathcal{F}_i$ are closed under the operation σ_1^{+i}.*

Theorem 4.3.5. *The identifying number of a Kautz digraph $K(2,k)$ is $\gamma^{ID}(K(2,k)) = 2^k - 2^{k-2}$ for $k \geq 2$.*

Proof. We can check in Figure 22 that $\widetilde{C} = \{uv, vw, wu\}$ is an identifying code of $K(2,2)$, then $\gamma^{ID}(K(2,2)) = 3$, and the theorem holds

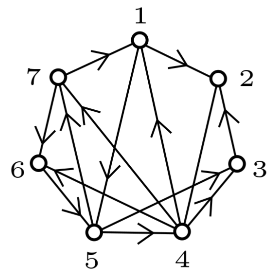


Figure 23. A digraph to illustrate Algorithm 1.

for $k = 2$. Suppose that $k \geq 3$ and let us consider the Kautz digraph $K(2, k-1)$. By Theorem 4.3.4, we can take a partition of the arcs of $K(2, k-1)$ into two 1-factors $\mathcal{F}_0$ and $\mathcal{F}_1$, such that the cycles of $\mathcal{F}_i$ are closed under the operation called σ_1^{+i}, given in Definition 4.3.3. In particular the relation σ_1^{+0} preserves digons, implying that all the digons of $K(2, k-1)$ belong to the family $\mathcal{F}_0$. Hence, since $\mathcal{F}_1$ is a 1-factor of $K(2, k-1)$, the set of arcs in $\mathcal{F}_1$, say A_1, satisfies the conditions of Theorem 4.3.1. Therefore, A_1 is an arc-identifying code of $K(2, k-1)$, that is, an identifying code of $K(2, k)$ and, $\gamma^{ID}(K(2, k)) = |A_1| = |V(K(2, k-1)| = 3 \cdot 2^{k-2} = 2^k - 2^{k-2}$. □

4.3.1 *Arc-identifying codes in oriented graphs*

Now, we present an algorithm for constructing an arc-identifying code $\widetilde{C}$ of an oriented graph D with minimum in-degree $\delta^- \geq 2$ and minimum out-degree $\delta^+ \geq 1$. The idea of this algorithm is to add to $\widetilde{C}$ all the arcs but one from $\Omega^+(v)$, for each vertex $v \in V(D)$ trying to reach an arc-identifying code of order $|A(D)| - |V(D)|$. Notice that in particular, for each vertex $v \in V_1^+(D)$ we have $\Omega^+(v) \cap \widetilde{C} = \emptyset$ or $\Omega^+(v) \subset \widetilde{C}$, and in the latter case the obtained arc-identifying code has order strictly greater than $|A(D)| - |V(D)|$.

Algorithm 1 Arc-identifying code

Input: An oriented graph D with minimum out-degree $\delta^+ \geq 1$ and minimum in-degree $\delta^- \geq 2$.

Output: An arc-identifying code of D.

1: let $X := \emptyset$, $Y := \emptyset$, and $\widetilde{C} := \emptyset$

2: **while** $Y \neq V(D)$ **do**

3:     **take** $xy \in A(D)$ such that $y \in V(D) \setminus Y$

4:     $X := X \cup \{x\}$, $Y := Y \cup (N^+(x) \setminus \{y\})$, and $\widetilde{C} := \widetilde{C} \cup (\Omega^+(x) \setminus \{xy\})$

5:     **if** $N^-(y) \setminus X \neq \emptyset$ **then**

6:         **if** there is $t \in N^-(y) \setminus X$ such that $t \in V^+_{\geq 2}(D)$ **then**

7:             **take** $tz \in A(D)$ such that $z \neq y$

8:             **let** $x := t$ and $y := z$

9:             **return** to 4

10:         **else**

11:             **take** $t \in N^-(y) \setminus X$

12:             $X := X \cup \{t\}$, $Y := Y \cup \{y\}$, and $\widetilde{C} := \widetilde{C} \cup \{ty\}$

13:             **return** to 3

14:         **end if**

15:     **else**

16:         **return** to 3

17:     **end if**

18: **end while**

19: **if** $Y = V(D)$ **then**

20:     **while** $X \neq V(D)$ **do**

21:         **take** $uv \in A(D)$ such that $u \notin X$

22:         $X := X \cup \{u\}$ and $\widetilde{C} := \widetilde{C} \cup (\Omega^+(u) \setminus \{uv\})$

23:     **end while**

24:     **if** $X = V(D)$ **then**

25:         **return** $\widetilde{C}$

26:     **end if**

27: **end if**

To illustrate the Algorithm 1, as an example we run the algorithm for the oriented graph of Figure 23 with 7 vertices and 14 arcs.

Input: Oriented graph depicted in Figure 23.

Steps 1-4: Start with the arc $(1,5)$, then $X := \{1\}$, $Y := \{2\}$, and $\widetilde{C} := \{(1,2)\}$.

Step 5: $N^-(5) \setminus X = \{6\} \subseteq V_1^+(D)$, then we go to step 11.

Steps 11-13: $X = \{1,6\}$, $Y := \{2,5\}$, and $\widetilde{C} := \{(1,2),(6,5)\}$, go to step 3.

Steps 3-4: Take the arc $(4,3)$, then $X = \{1,6,4\}$, $Y := \{2,5,1,6,7\}$, and $\widetilde{C} := \{(1,2),(6,5),(4,1),(4,7),(4,6)\}$.

Step 5: $N^-(3) \setminus X = \{5\} \subseteq V_{\geq 2}^+(D)$.

Steps 6-9: Take the arc $(5,7)$, that is $t = 5$ and $z = 7$.

Step 4: $X = \{1,4,6,5\}$, $Y := \{1,2,5,6,7,3,4\}$, and $\widetilde{C} := \{(1,2),(6,5),(4,1),(4,7),(4,6),(5,4),(5,3)\}$.

Step 5: $N^-(7) \setminus X = \emptyset$, then go to step 3 but since $Y = V(D)$, go to step 19.

Step 20: Since $X \neq V(D)$, start steps 21 to 22 until $X = V(D)$.

Steps 21-22: Take the arc $(7,6)$, then $X = \{1,4,6,5,7\}$, and $\widetilde{C} := \{(1,2),(4,1),(4,7),(4,6),(5,3),(5,4),(6,5),(7,1)\}$.

Steps 20-22: Take the arc $(3,2)$, then $X = \{1,4,6,5,7,3\}$, and $\widetilde{C} := \widetilde{C}$.

Steps 20-22: Take the arc $(2,4)$, then $X = \{1,2,3,4,5,6,7\}$, and $\widetilde{C} := \widetilde{C}$

Step 25: **Output** $\widetilde{C} := \{(1,2),(4,1),(4,7),(4,6),(5,3),(5,4),(6,5),(7,1)\}$.

Theorem 4.3.6. *Let D be an oriented graph with minimum out-degree $\delta^+ \geq 1$ and minimum in-degree $\delta^- \geq 2$. Then, the Algorithm 1 produces an arc-identifying code of D.*

Proof. By construction, the Algorithm 1 produces two sets of vertices X and Y such that every vertex $v \in V(D)$ satisfies that $v \in X$ at a certain step of the algorithm (steps: 4, 12 or 22); and that $v \in Y$ at a certain step of the algorithm (steps: 4 or 12). If v was added to X at step 4, then $|\Omega^+(v) \setminus \widetilde{C}| = 1$. If v was added to X at step 12, then $\Omega^+(v) \subseteq \widetilde{C}$. And, if v was added to X at step 22, then $|\Omega^+(v) \setminus \widetilde{C}| = 1$. In any case, $|\Omega^+(v) \setminus \widetilde{C}| \leq 1$. Hence, $\widetilde{C}$ satisfies the first part of Theorem 4.3.1 (i). Now, we analyse the filling process of set Y. If v was added to Y at step 4, then $v \in N^+(x) \setminus \{y\}$ for certain x and y in the algorithm such that $xv \in \Omega^+(x) \setminus \{xy\} \subset \widetilde{C}$. Otherwise, at some point, due to step 12, there is a vertex $t \in N^-(y)$ such that $t \in X$ and $ty \in \widetilde{C}$. In any case, $\Omega^-(v) \cap \widetilde{C} \neq \emptyset$ and Theorem 4.3.1 (i) holds. Finally, since D is oriented, for all $uv \in \widetilde{C}$, clearly $vu \notin A(D)$, and we have $|(\Omega^-(u) \cup (\Omega^-(v) \setminus \{uv\})) \cap \widetilde{C}| \geq 1$ because $\Omega^-(u) \cap \widetilde{C} \neq \emptyset$. Hence, Theorem 4.3.1 (ii) also holds. Therefore, $\widetilde{C}$ is an arc-identifying code of D. □

Now, consider an oriented graph D with minimum out-degree $\delta^+ \geq 1$ and minimum in-degree $\delta^- \geq 2$. As mentioned in the above proof, if we run the Algorithm 1, for each vertex $v \in V(D)$, we have that $v \in X$ at a certain step of the algorithm (steps: 4, 12 or 22). And, that $|\Omega^+(v) \setminus \widetilde{C}| \neq 1$ if and only if v was added to X at step 12, where $\Omega^+(v) \subseteq \widetilde{C}$. Moreover, due to step 12, it follows that $d^+(v) = 1$. Hence, we have the following result.

Corollary 4.3.5. *Let D be an oriented graph with minimum degree $\delta = \min\{\delta^+, \delta^-\} \geq 2$. Then, the Algorithm 1 produces an arc-identifying code $\widetilde{C} \subset A(D)$ of size*

$$|\widetilde{C}| = |A(D)| - |V(D)|.$$

As a consequence of Theorem 4.3.2 and Corollary 4.3.5, we conclude the following.

Corollary 4.3.6. *Let D be an oriented graph D with minimum degree $\delta \geq 2$. Then,*

$$\overrightarrow{\gamma}^{ID}(LD) = |A(D)| - |V(D)|.$$

5

SPECTRAL RESULTS ON IDENTIFYING CODES FROM GRAPHS AND DIGRAPHS

This chapter consists mainly of the results published in [6]. As far as we know, this is the first time the spectral graph theory has been applied to identifying codes. We begin this chapter with some preliminary definitions. In the Section 5.2, we give some sufficient algebraic and combinatorial conditions for a 2-in-regular digraph to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{2, 3\}$. In Section 5.3, we provide a new method to obtain an upper bound for ℓ based on the eigenvalues and eigenvectors of the adjacency matrix of the digraph. The results obtained in this last section can also be applied to graphs, as it is mentioned at the end of this chapter. We finish this section with a discussion motivating future research.

5.1 PRELIMINARIES

Let D be a (di)graph. The *adjacency matrix* of D is the binary matrix $M = (a_{uv})$ indexed by the vertex set $V(D)$, where $a_{xy} = 1$ if and only if $xy \in A(D)$ ($xy \in E(D)$ in the case of graphs), and $a_{xy} = 0$ otherwise.

Recall that a digraph with adjacency matrix $M = (a_{uv})$ has eigenvalue λ and eigenvector $\boldsymbol{x} = (x_u)$ if and only if

$$M\boldsymbol{x} = \lambda\boldsymbol{x} \quad \Leftrightarrow \quad \sum_{v \in V} a_{uv} x_v = \sum_{v \in N^+(u)} x_v = \lambda x_u \qquad \text{for all } u \in V. \tag{6}$$

The *spectrum* of the adjacency matrix M of a (di)graph is denoted by $\mathrm{sp}(M) = \{\lambda_0^{m_0}, \lambda_1^{m_1}, \ldots, \lambda_d^{m_d}\}$, where λ_i are the different eigenvalues and the superscripts stand for their (algebraic) multiplicities $m_i = m(\lambda_i)$, whereas $ev(A)$ is the set of the different eigenvalues (without their multiplicities).

5.2 SUFFICIENT CONDITIONS

We begin with a result that gives a sufficient (spectral) condition for a digraph to admit an identifying code.

Lemma 5.2.1. *Let D be a digraph with adjacency matrix M and with a set of eigenvalues denoted by* $\mathrm{ev}(M)$*. If $-1 \notin \mathrm{ev}(M)$, then D admits an identifying code.*

Proof. We reason by contraposition. The digraph D does not admit an identifying code if and only if there exists a pair of twin vertices u and v. So,

$$(M + I)\boldsymbol{e_u} = (M + I)\boldsymbol{e_v},$$

where $\boldsymbol{e_u}$ and $\boldsymbol{e_v}$ are the unitary characteristic vectors corresponding to vertices u and v, respectively, and I is the identity matrix. Then, $(M + I)\boldsymbol{x} = \boldsymbol{0}$ with $\boldsymbol{x} = \boldsymbol{e_u} - \boldsymbol{e_v}$, whence $M\boldsymbol{x} = -\boldsymbol{x}$ and $-1 \in ev(M)$. □

Observe that if D is a digraph with $-1 \in ev(M)$, then this does not imply that some of its corresponding eigenvectors are of the form $\boldsymbol{e_i} - \boldsymbol{e_j}$. Hence, the converse of Lemma 5.2.1 it is not true. For example, consider the forbidden subdigraph H_5 of Theorem 3.4.2 shown in Figure 24. This digraph has eigenvalue -1, but it does admit an identifying code.

Now, we give an algebraic-combinatorial sufficient condition for a 2-in-regular digraph to admit a $(1, \leq 2)$-, or $(1, \leq 3)$-identifying code. The goal is to use to the eigenvalues of the digraph to reduce the number of forbidden subdigraphs considered in Theorem 3.4.2 and Theorem 3.4.3. First, we prove the following lemma.

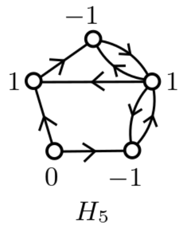


Figure 24. The forbidden subdigraph H_5 from Theorem 3.4.2. The numbering represents the entries of the eigenvector corresponding to eigenvalue -1.

Lemma 5.2.2. *Let D' be a digraph with maximum in-degree Δ^- having an eigenvalue λ with eigenvector $\boldsymbol{x}' = (x'_u)$, such that $x'_v = 0$ for any vertex $v \in V(D')$ with $d^-(v) < \Delta^-$. Then, any digraph D with maximum in-degree Δ^- containing D' as a subdigraph has also the eigenvalue λ.*

Proof. Let M' be the adjacency matrix of D'. We know that $M'\boldsymbol{x}' = \lambda\boldsymbol{x}'$. Let M be the adjacency matrix of D containing D' as a subdigraph. Now let us show that λ also is an eigenvalue of M. To see this, it is enough to check that vector $\boldsymbol{x}$, obtained from $\boldsymbol{x}'$ by adding zeros to the entries of D corresponding to the vertices that are not in D', is an eigenvector of M with eigenvalue λ. Indeed, from (6), for all $u \in V$ we get

$$\sum_{v \in N_D^+(u)} x_v = \sum_{\substack{v \in N_D^+(u) \\ v \in V'}} x_v + \sum_{\substack{v \in N_D^+(u) \\ v \notin V(D')}} x_v = \sum_{\substack{v \in N_D^+(u) \\ v \in V(D')}} x_v, \tag{7}$$

because by the construction of vector $\boldsymbol{x}$, the sum when $v \notin V(D')$ is zero. Then, we get the following:

- If $u \in V(D')$, equality (7) gives

$$\sum_{v \in N_D^+(u)} x_v = \sum_{\substack{v \in N_D^+(u) \cap V(D') \\ d_D^-(v) < \Delta^-}} x_v + \sum_{\substack{v \in N_D^+(u) \cap V(D') \\ d_D^-(v) = \Delta^-}} x_v$$

  Observe that if $v \in N_D^+(u) \cap V(D')$, then $v \in N_{D'}^+(u)$ or $v \in (N_D^+(u) \setminus N_{D'}^+(u))$. If $v \in (N_D^+(u) \setminus N_{D'}^+(u)) \cap V(D')$, then

$d^-_{D'}(v) < \Delta^-$ implying, by hypothesis, that $x_v = 0$. Hence, we have the following:

$$\begin{aligned}\sum_{v\in N^+_D(u)} x_v &= \sum_{\substack{v\in N^+_D(u)\cap V(D')\\ d^-_D(v)<\Delta^-}} x_v + \sum_{\substack{v\in N^+_D(u)\cap V(D')\\ d^-_D(v)=\Delta^-}} x_v\\ &= \sum_{\substack{v\in N^+_{D'}(u)\\ d^-_{D'}(v)<\Delta^-}} x'_v + \sum_{\substack{v\in N^+_{D'}(u)\\ d^-_{D'}(v)=\Delta^-}} x'_v\\ &= \sum_{v\in N^+_{D'}(u)} x'_v = \lambda x'_u = \lambda x_u.\end{aligned}$$

- If $u \notin V(D')$, then (7) provides

$$\begin{aligned}\sum_{v\in N^+_D(u)} x_v &= \sum_{v\in N^+_D(u)\cap V(D')} x'_v\\ &= \sum_{\substack{v\in N^+_D(u)\\ d^-_{D'}(v)<\Delta^-}} x'_v = 0 = \lambda x_u.\end{aligned}$$

□

Recall that we denote by $\mathcal{H}$ the family of digraphs H_1-H_{13} of Figure 19, and with $\mathcal{J}$ the family of digraphs J_1-J_{15} of Figure 20. That is, $\mathcal{H}$ and $\mathcal{J}$ are the the families of all forbidden digraphs of Theorems 3.4.2, and 3.4.3, respectively. Notice that from all the digraphs in $\mathcal{H} \cup \mathcal{J}$, H_1, H_5, H_7 to H_{13}, J_1, and J_{13} to J_{15}, have maximum in-degree 2 and an eigenvalue λ with eigenvector $\boldsymbol{x'} = (x'_u)$, such that $x'_v = 0$ for any vertex v such that $d^-(v) < 2$ (see Figures 25 and 26). Hence, we have the following result.

Hence, we have the following result.

Theorem 5.2.1. *Let D be a 2-in-regular digraph with adjacency matrix M.*

(*i*) *If $-1, 0 \notin \mathrm{ev}(M)$ and D is $\{H_2, H_3, H_4, H_6\}$-free, then D admits a $(1, \leq 2)$-identifying code.*

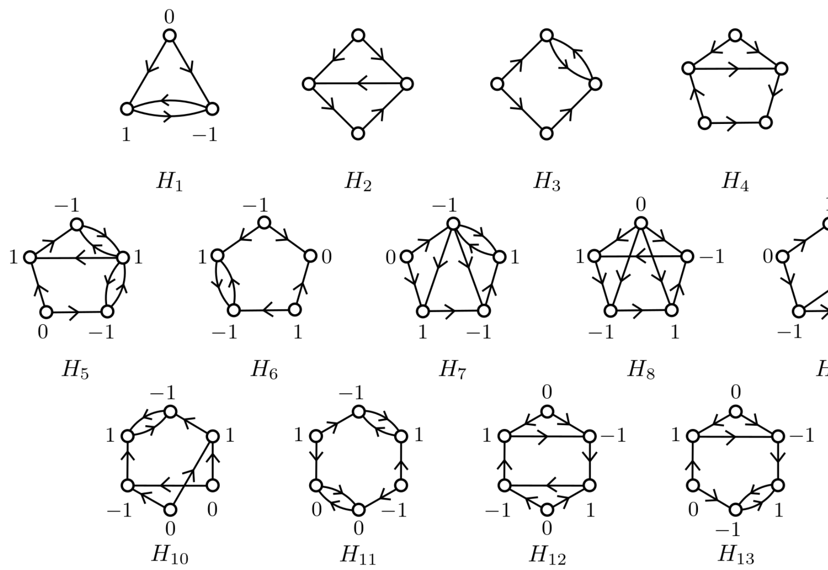


Figure 25. The forbidden subdigraphs of Theorem 3.4.2. In the subdigraphs with eigenvalue -1, it is show the entries of an eigenvector corresponding to this eigenvalue in the exterior, except for H_9 where it is show the entries of an eigenvector corresponding to 0.

(*ii*) *If* $-1, 0 \notin \text{ev}(M)$ *and* D *is* $\{J_i \in \mathcal{J} \mid 2 \leq i \leq 12\}$*-free, then* D *admits a* $(1, \leq 3)$*-identifying code.*

Proof. To prove (i), by Theorem 3.4.2 (ii), we know that if a 2-in-regular digraph D does not contain any of the subdigraphs of Figure 25, then D admits a $(1, \leq 2)$-identifying code. The subdigraphs H_1, H_5, H_7, H_8, and $H_{10} - H_{13}$ satisfy Lemma 5.2.2 for $\lambda = -1$, and H_9 satisfies Lemma 5.2.2 for $\lambda = 0$. Then, we only need to forbid the subdigraphs H_2, H_3, H_4, and H_6 to obtain the result.

To prove (ii), reasoning similarly, we get that the subdigraph J_1 of Figure 26 satisfies Lemma 5.2.2 for $\lambda = 0$, and subdigraphs J_{13}-J_{15} satisfy Lemma 5.2.2 for $\lambda = -1$, so we only need to forbid the rest of the subdigraphs in this figure. □

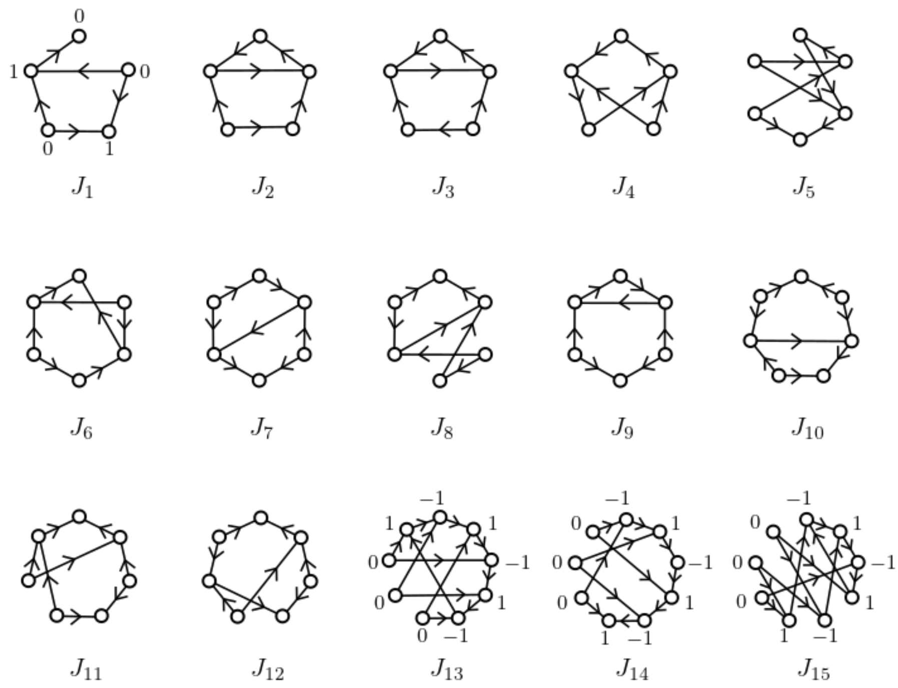


Figure 26. The forbidden subdigraphs of Theorem 3.4.3. The subdigraphs with eigenvalue -1 show the entries of an eigenvector corresponding to eigenvalue -1 in the exterior, except for J_1 that shows the entries of an eigenvector corresponding to eigenvalue 0.

5.3 ALGEBRAIC UPPER BOUND FOR ℓ

We provide some necessary notation introduced by Powers [51] and referenced in the book by Cvetković, Rowlinson, and Simić [19]. Let $\boldsymbol{x} = (x_i)$ be an eigenvector associated with an eigenvalue λ. We denote by $\mathcal{P}(\boldsymbol{x})$, $\mathcal{N}(\boldsymbol{x})$, and $O(\boldsymbol{x})$ the set of its positive, negative, and zero entries, respectively. That is, $\mathcal{P}(\boldsymbol{x}) = \{i : x_i > 0\}$, $\mathcal{N}(\boldsymbol{x}) = \{i : x_i < 0\}$, and $\mathcal{O}(\boldsymbol{x}) = \{i : x_i = 0\}$.

Now, we show how, in some cases, we can use these sets to construct two different sets of vertices having the same closed in-neighbourhood, providing a bound for ℓ.

Proposition 5.3.1. *Let $D = (V, E)$ be a digraph with adjacency matrix M having some real eigenvalue, say $\lambda \in \mathrm{ev}(M)$, with an associated* *eigenvector* $\boldsymbol{x} = (x_u)_{u\in V}$ *such that* $X = \mathcal{P}(\boldsymbol{x}) \neq \emptyset$ *and* $Y = \mathcal{N}(\boldsymbol{x}) \neq \emptyset$. *Then, depending on the sign of* λ, *the following holds:*

(a) *If* $\lambda < 0$, *then* $X \cup N^-(X) = Y \cup N^-(Y) \quad (\Leftrightarrow N^-[X] = N^-[Y])$.

(b) *If* $\lambda > 0$, *then* $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) *If* $\lambda = 0$, *then* $N^-(X) = N^-(Y)$.

Proof. Let $\mathbf{e}_u$ be the unitary characteristic vector of the vertex u. Then, $M\mathbf{e}_u$ is the characteristic vector of the open in-neighbourhood of vertex u. Let $\boldsymbol{x}^+$ be the vector obtained from a vector $\boldsymbol{x}$ by changing all its negative components to zero. Similarly, $\boldsymbol{x}^-$ is obtained from $\boldsymbol{x}$ by changing all its positive components to zero. Then, $\boldsymbol{x} = \boldsymbol{x}^+ + \boldsymbol{x}^-$. Since $\boldsymbol{x}$ is a λ-eigenvector of M, we get $M\boldsymbol{x} = \lambda\boldsymbol{x}$ or $M\boldsymbol{x} - \lambda\boldsymbol{x} = \mathbf{0}$. Now we distinguish the possible cases according to the sign of λ:

(a) and (b): If $\lambda < 0$ or $\lambda > 0$, we have

$$\sum_{u\in X} x_u M\mathbf{e}_u - \lambda\boldsymbol{x}^+ + \sum_{v\in Y} x_v M\mathbf{e}_v - \lambda\boldsymbol{x}^- = \mathbf{0}. \tag{8}$$

For the case $\lambda < 0$ we have that $\sum_{u\in X} x_u M\mathbf{e}_u$ and $-\lambda x^+$ are positive, whereas $\sum_{v\in Y} x_v M\mathbf{e}_v$ and $-\lambda x^-$ are negative. Let $\hat{x}^+ = \sum_{u\in X} x_u M\mathbf{e}_u - \lambda x^+ = (\hat{x}_i^+)$, and $\hat{x}^- = \sum_{v\in X} x_v M\mathbf{e}_v - \lambda x^- = (\hat{x}_i^-)$. Hence, for every $w \in V(D)$ we have $\hat{x}_w^+ = \hat{x}_w^-$. Let $w \in N^-[X]$, then $\hat{x}_w^+ \neq 0$, implying that $\hat{x}_w^- \neq 0$. Hence, $w \in Y$ or there is $v \in Y$ such that $w \in N^-(v)$. By the same argument, we conclude that $N^-[X] = N^-[Y]$.

For the case $\lambda > 0$, $\sum_{u\in X} x_u M\mathbf{e}_u$ and $-\lambda x^-$ are positive, whereas $-\lambda x^+$ and $\sum_{v\in Y} x_v M\mathbf{e}_v$ are negative. Then, the global sum is 0 if and only if $X \cup N^-(Y) = Y \cup N^-(X)$.

(c) Finally, if $\lambda = 0$, the vector equality

$$\sum_{u\in X} x_u M\mathbf{e}_u + \sum_{v\in Y} x_v M\mathbf{e}_v = \mathbf{0}$$

gives the result. □

Regarding identifying codes, by the above proposition, we can use this result to obtain an upper bound for ℓ when the adjacency matrix of the digraph we are working with has a negative eigenvalue with an associated eigenvector having positive and negative entries. Let us see that this is always the case when there is a negative eigenvalue. For this, we use the same notation as in the proof of Proposition 5.3.1. Let $D = (V, E)$ be a digraph with adjacency matrix M having some negative eigenvalue $\lambda \in ev(M)$, with an associated eigenvector $\boldsymbol{x} = (x_u)_{u \in V}$. Hence, from equation (8), if $\mathcal{P}(\boldsymbol{x}) = \emptyset$, then

$$\sum_{v \in \mathcal{N}(\boldsymbol{x})} x_v M \mathbf{e}_v - \lambda \boldsymbol{x}^- = \mathbf{0}.$$

This is clearly a contradiction because all the non-zero entries in the left side of this equality are non-positive. We have the analogous result for the case $\mathcal{N}(\boldsymbol{x}) = \emptyset$. Therefore, if $\lambda < 0$ and $\boldsymbol{x}$ an eigenvector associated with it, then $\mathcal{P}(\boldsymbol{x}) \neq \emptyset$ and $\mathcal{N}(\boldsymbol{x}) \neq \emptyset$. As a consequence we obtain the following result, which gives an upper bound for ℓ in a digraph D having a $(1, \leq \ell)$-identifying code. Moreover, by changing $N^-(X)$ and $N^-(Y)$ by $N(X)$ and $N(Y)$, respectively, we obtain an analogous result for graphs. Let $E_\lambda(M)$ denote the set of eigenvectors of a matrix A associated with λ.

Corollary 5.3.1. *Let D be a graph or a digraph admitting a $(1, \leq \ell)$-identifying code. Let M be its adjacency matrix having at least one negative eigenvalue λ. Then, $\ell < \min_{\boldsymbol{x} \in E_\lambda(M)} \max\{|\mathcal{P}(\boldsymbol{x})|, |\mathcal{N}(\boldsymbol{x})|\}$.*

Hence, to find a good upper bound for ℓ using the eigenvectors associated with the negative eigenvalues of the adjacency matrix, the desirable option is to use eigenvectors such that they have as many zeros as possible and that the difference between the cardinalities of $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{N}(\boldsymbol{x})$ is minimal. Recall that if α is an eigenvalue with geometric multiplicity m, a corresponding eigenvector with at least $m - 1$ zeros can be constructed. Hence, a good eigenvector to be considered for our purpose of finding a good upper bound for ℓ would be a negative eigenvector with large multiplicity.

Now, we show some examples of how to use Proposition 5.3.1.

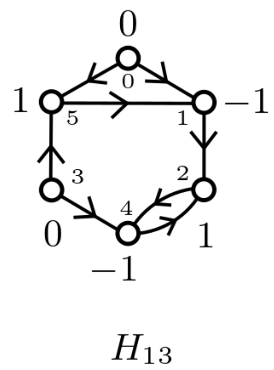


Figure 27. The labels of the vertices are shown in the interior, and the entries of the eigenvector corresponding to eigenvalue -1 in the exterior.

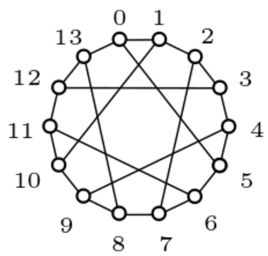


Figure 28. The Heawood graph.

Consider, the digraph H_{13} (see Figure 27). Its spectrum is $\{0^4, 1^1, -1^1\}$ (we write multiplicities as exponents). An eigenvector corresponding to the eigenvalue -1 is the vector $(0, -1, 1, -1, 0, 1)^t$. The positions of the positive entries of this eigenvector give us vertex subset $X = \{2, 5\}$, and the positions of the negatives entries give $Y = \{1, 3\}$. We can check that $N^-[X] = N^-[Y] = \{0, 1, 2, 3, 4, 5\}$. Then, this digraph does not admit a $(1, \leq 2)$-identifying code. Since the digraph is clearly twin-free, it does admit an identifying code, then in this case the upper bound is tight.

Let us see an example on a graph. Consider the Heawood graph (see Figure 28). Its spectrum is $\{3^1, \sqrt{2}^6, -\sqrt{2}^6, -3^1\}$. The eigenvector corresponding to -3 has no zero entries while the 6 eigenvectors forming a basis of $E_{-\sqrt{2}}(M)$ have all 8 zero entries, these are:

$$(-1,0,1,0,-1,\sqrt{2},0,-\sqrt{2},1,0,0,0,0,0)^t,$$
$$(-1,0,1,-\sqrt{2},0,\sqrt{2},-1,0,0,0,0,0,1,0)^t,$$
$$(0,-1,\sqrt{2},-1,0,1,-\sqrt{2},0,0,0,0,1,0,0)^t,$$
$$(0,-1,\sqrt{2},0,-\sqrt{2},1,0,-1,0,1,0,0,0,0)^t,$$
$$(0,-\sqrt{2},1,0,-1,\sqrt{2},-1,0,0,0,1,0,0,0)^t,$$
$$(-\sqrt{2},0,\sqrt{2},-1,0,1,0,-1,0,0,0,0,0,1)^t.$$

Notice that, for each of these six eigenvectors, we have that the number of positive and negative entries is 3. If we consider, for instance, the last one of these six eigenvectors, the positions of its positive entries give us the set of vertices $X = \{2,5,13\}$. The positions of its negatives entries provide us the set $Y = \{0,3,7\}$. We can check that $N[X] = N[Y] = \{0,1,2,3,4,5,6,7,8,12,13\}$. Then, the Heawood graph does not admit a $(1,\leq 3)$-identifying code. By Theorem 3.3.2 we know that the Heawood graph, which is 3-regular and it has girth 6, admits a $(1,\leq 2)$-identifying code. In this case, the upper bound is again tight.

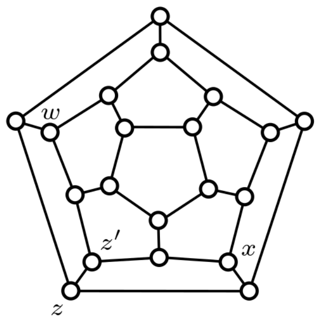


Figure 29. Dodecahedron graph

Now, consider the Dodecahedron graph (see Figure 29). Its spectrum is $\{3^1, \sqrt{5}^3, 1^1, 0^4, -2^4, -\sqrt{5}^3\}$. The eigenvectors corresponding to $-\sqrt{5}$ have four zero entries while the eigenvectors corresponding to

-2 have 8 zero entries and the same number of positive and negative entries, these are:

$$(0,0,0,-1,1,-1,0,1,1,-1,1,-1,-1,0,1,-1,1,0,0,0)^t,$$
$$(1,0,0,-1,0,-2,-1,1,2,0,2,0,-2,-1,1,-1,0,1,0,0)^t,$$
$$(0,1,0,-1,0,-1,-2,0,2,1,2,1,-1,-2,0,-1,0,0,1,0)^t,$$
$$(0,0,1,-1,0,0,-1,-1,1,1,1,1,0,-1,-1,-1,0,0,0,1)^t.$$

Hence, by Corollary 5.3.1 we get that the Dodecahedron admits at most a $(1, \leq 6)$-identifying code. Nevertheless, by Theorem 3.3.2, we know that the Dodecahedron graph admits a $(1, \leq 2)$-identifying code and, by Laihonen and Ranto [47], we know that it does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 4$. Is easy to see that the Dodecahedron does not admit a $(1, \leq 3)$-identifying code. For instance, consider the vertices w, x, z, z' shown in Figure 29. Then, $N[\{x, w, z\}] = N[\{x, w, z'\}]$. This example shows that the bound provided by Corollary 5.3.1 is not always so good.

In the light of Corollary 3.1.1, it is natural point of future research to find sufficient or necessary conditions for a digraph or graph, to have an eigenvector, let say $\boldsymbol{x}$, associated with a negative eigenvalue and such that $\min\{|\mathcal{P}(\boldsymbol{x})|, |\mathcal{N}(\boldsymbol{x})|\} \leq \hat{\delta}^-$. Another point of future research is to have the possibility that the two sets with the same closed in-neighbourhood constructed based on the eigenvectors associated with the negative eigenvalues, can have common vertices, unlike the two sets constructed in Proposition 5.3.1.

Now, following the line of future research, let us see what we can say when zero is an eigenvalue of the digraph. For this, let us show the following result for digraphs with minimum degree. This is a first result, which may be the starting point for future research in the case where the digraph has zero as an eigenvalue.

Corollary 5.3.2. *Let $D=(V,E)$ be a digraph admitting a $(1,\leq \ell)$-identifying code, with $\delta(D)\geq 1$. Let M be its adjacency matrix. If $0\in \mathrm{ev}(M)$, then*

$$\ell < \min_{\boldsymbol{x}\in E_0(M)} \{\max\{|\mathcal{P}(\boldsymbol{x})|, |\mathcal{N}(\boldsymbol{x})|\} + |\mathcal{O}(\boldsymbol{x})|\}.$$

Proof. Let $\boldsymbol{x}\in E_0(M)$. Suppose that $\mathcal{P}(\boldsymbol{x})=\emptyset$. Then,

$$\sum_{v\in V(D)} x_v M\mathbf{e}_v = \sum_{v\in \mathcal{N}(\boldsymbol{x})} x_v M\mathbf{e}_v = 0.$$

This implies that, for every $v\in\mathcal{N}(\boldsymbol{x})$, $N^-(v)=\emptyset$, a contradiction since $\delta^-\geq 1$. We have the analogous result for the case $\mathcal{N}(\boldsymbol{x})=\emptyset$. Therefore, $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{N}(\boldsymbol{x})$ are both nonempty. Hence, from Proposition 5.3.1, we have $N^-(X)=N^-(Y)$, where $X=\mathcal{P}(\boldsymbol{x})$ and $Y=\mathcal{N}(\boldsymbol{x})$. Let $Z=\mathcal{O}(\boldsymbol{x})$. Notice that, since $\delta^+\geq 1$, for each vertex $x\in X\setminus N^-(Y)$, there is $z_x\in Z$ such that $x\in N^-(z_x)$. Analogously, for every $y\in Y\setminus N^-(X)$, there is $z_y\in Z$ such that $y\in N^-(z_y)$. Let $Z'=\{z_x\in Z\mid x\in X\setminus N^-(Y)\}\cup\{z_y\mid y\in Y\setminus N^-(X)\}$. Hence, since $N^-(X)=N^-(Y)$, we have $N^-[X\cup Z']=N^-[Y\cup Z']$. Therefore, $\ell<\max\{|X|,|Y|\}+|Z'|$. This completes the proof.

□

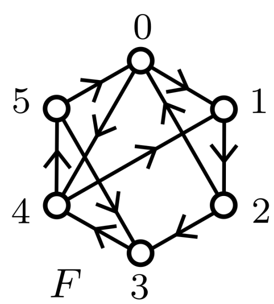


Figure 30.

Consider the digraph F shown in Figure 30. This is a digraph with $\delta(F)=1$ and spectrum $\{\frac{1}{2}(\sqrt{5}+1), 0^2, \frac{1}{2}(-\sqrt{5}+1), \frac{1}{2}(-1+i\sqrt{7}), \frac{1}{2}(-1-i\sqrt{7})\}$. The vectors $(1,\frac{1}{2}(-\sqrt{5}-1),1,\frac{1}{2}(-\sqrt{5}-1),1,1)^t$ and $(-1,0,0,1,0,0)^t$ are the only eigenvectors associated with $\frac{1}{2}(-\sqrt{5}+1)$ and 0 (up to scalar product), respectively. By Proposition 5.3.1, with the negative eigenvector $\frac{1}{2}(-\sqrt{5}+1)$, we obtain that $N^-[\{0,2,4,5\}] = N^-[\{1,3\}]$, implying that F does not admit a $(1,\leq 4)$-identifying code. Nevertheless, by Corollary 3.1.1 we know that F does not admit a $(1,\leq \ell)$-identifying code for any $\ell \geq 3$.

Let us denote the eigenvector $\boldsymbol{x} = (-1,0,0,1,0,0)^t$. By Corollary 5.3.2, we have that the sets $X = \mathcal{P}(\boldsymbol{x}) \cup \mathcal{O}(\boldsymbol{x}) = \{1,2,3,4,5\}$ and $Y = \mathcal{N}(\boldsymbol{x}) \cup \mathcal{O}(\boldsymbol{x}) = \{0,1,2,4,5\}$ satisfies that $N^-[X] = N^-[Y]$. Nevertheless, according to the proof of Corollary 5.3.2, it is not necessary to add to $\mathcal{P}(\boldsymbol{x})$ and $\mathcal{N}(\boldsymbol{x})$ all the elements of $\mathcal{O}(\boldsymbol{x})$. Let us use the same notation as in the proof of Corollary 5.3.2. Then, for the vertex $0 \in \mathcal{N}(\boldsymbol{x})$, we have that z_0 could be 1 or 4, since z_0 is such that $z_0 \in \mathcal{O}(\boldsymbol{x})$ and $0 \in N^-(z_0)$. For the vertex 3, we have that z_3 could be just 4. Hence, the set $Z' = \{4\}$ satisfies that $N^-[\mathcal{P}(\boldsymbol{x}) \cup Z'] = N^-[\{3,4\}] = N^-[\{0,4\}] = N^-[\mathcal{N}(\boldsymbol{x}) \cup Z']$. Therefore, F does not admit a $(1,\leq 2)$-identifying code. Hence, in this case, with eigenvalue 0, we could find a better upper bound for ℓ.

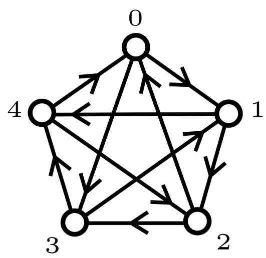


Figure 31.

Now, consider the tournament shown in Figure 31. Its spectrum is $\{2, \frac{-1+i\sqrt{5+2\sqrt{5}}}{2}, \frac{-1-i\sqrt{5+2\sqrt{5}}}{2}, \frac{-1+i\sqrt{5-2\sqrt{5}}}{2}, \frac{-1-i\sqrt{5-2\sqrt{5}}}{2}\}$. We can not apply neither Corollary 5.3.1 nor Corollary 5.3.2 to obtain an upper bound for ℓ because all its eigenvalues are positive or complex. Nevertheless, in this case, it is easy to find two different subsets of vertices with the same closed in-neigbourhood, for instance, $N^-[\{0,1\}] = N^-[\{1,2\}]$. Hence, since there are no twin vertices, this tournament admits an identifying code but does not admit a $(1,\leq \ell)$-identifying code for any

$\ell \geq 2$. An interesting possible direction for future research would be to use the complex eigenvalues in order to find a bound for ℓ.

6

CONCLUSIONS

In conclusion, we summarise the main results contained in this thesis, and also we give some suggestions for future research.

In Chapter 3, we show that if D is a digraph admitting a $(1, \leq \ell)$-identifying code, then $\ell \leq \hat{\delta}(D)+1$, where $\hat{\delta}(D) = \min\{d^-(u) \mid u \in V(D) \text{ and } d^+(u) \geq 1\}$. We have focused our study on boundary cases when ℓ achieves its upper bound. More precisely, we have stablished some conditions to guarantee that a digraph admits a $(1, \leq \ell)$-identifying code, when $\ell \in \{\hat{\delta}(D), \hat{\delta}(D)+1\}$. These sufficient conditions are presented in Theorem 3.3.1. As a corollary, a result by Laihonen [45], that states that a k-regular graph with girth at least 7 admits a $(1, \leq k)$-identifying code, is extended to any graph of minimum degree $\delta = k \geq 2$ and girth at least 7. Moreover, we show that every 1-in-regular digraph has a $(1, \leq 2)$-identifying code if and only if the girth of the digraph is at least 5. We also characterise all the 2-in-regular digraphs admitting a $(1, \leq \ell)$-identifying code for $\ell = 2, 3$.

In Chapter 4, we are concerned about the study of $(1, \leq \ell)$-identifying codes in line digraphs. We prove that a line digraph of minimum in-degree one does not admit a $(1, \leq \ell)$-identifying code for $\ell \geq 3$. Then, we give a characterisation for a line digraph of a digraph different from a directed cycle of length 4 and minimum in-degree one, to admit a $(1, \leq 2)$-identifying code. As a direct consequence, we obtain that a Kautz digraph $K(d, k)$ with $d \geq 3$ admits a $(1, \leq 2)$-identifying code. For a digraph without digons with both vertices of in-degree one, we also prove that the identifying number of its line digraph is at least the size of the original digraph minus the number of vertices with out-degree at least one. As a consequence, the equality is attained by a digraph having a 1-factor with minimum in-degree two and without digons with both vertices of in-degree two. We also prove this equality in oriented graphs with minimum in-degree and out-degree at least two by means of a linear algorithm that provides identifying codes in this kind of oriented graphs.

In Chapter 5, we give some sufficient algebraic and combinatorial conditions for a 2-in-regular digraph to admit a $(1, \leq \ell)$-identifying code for $\ell \in \{2, 3\}$. We prove that if -1 is not an eigenvalue of the adjacency matrix of the digraph, then D admits an identifying code. Moreover, we provide a new method to obtain two different sets with the same closed in-neighbourhood based on the eigenvalues and eigenvectors of the adjacency matrix of the digraph. As a consequence, we obtain an upper bound for ℓ. These results are also applied to graphs.

As we mentioned in the Introduction, the study of $(1, \leq \ell)$-identifying codes in digraphs has not been so much studied for $\ell \geq 2$. The work presented in this thesis is our contribution to the study on this subject. While doing this thesis, potential lines of future research have raised. Let us present below some of them.

- It would be very interesting to improve the results included in Chapter 3, applying a similar algebraic and combinatorial method used in Chapter 5.

- As we mentioned in the introduction, Coupechoux, Moncel, and Touati [18] studied $(t, \leq 1)$-identifying codes in tournaments. Among other things, they proved that a tournament admitting a $(t, \leq 1)$-identifying code with $t \geq 2$ is a transitive tournament. One of our goals for the future is to study $(1, \leq \ell)$-identifying codes in tournaments, for $\ell \geq 2$.

- Regarding the applications, another interesting family of digraphs is the de Bruijn digraphs. Boutin, Goliber, and Pelto [10] studied $(t, \leq 1)$-identifying codes in de Bruijn digraphs, therefore another of our goals for the future is to study $(1, \leq \ell)$-identifying codes in de Bruijn graphs, for $\ell \geq 2$.

- Another contribution for the future would be to work on $(t, \leq \ell)$-identifying codes in digraphs for some related values of t and ℓ, for instance, when t and ℓ are lineal related ($t = m\ell + b$ or $\ell = mt + b$).

- Regarding identifying codes in line digraphs, in Chapter 4, we give tight upper bounds for the minimum cardinality of every identifying code of a line digraph. Then, our objective is to provide bounds for the minimum cardinality of every $(1, \leq 2)$-identifying code of a line digraph. Moreover, we also want to find better bounds for the identifying number of a line digraph when it is not an oriented graph.

- To study the existence of $(1, \leq \ell)$-identifying codes of Cartesian product of digraphs. One starting point could be the work regarding identifying codes in the product of graphs by Laihonen and Moncel [46], Gravier, Moncel, and Semri [34], Wash [59], Rall and Wash [55], Hedetniemi [36], and Lu, Xu, and Zhang [50]. Furthermore, other products could also be very interesting for studying the existence of $(1, \leq \ell)$-identifying codes.

- Regarding the spectral line of research, a point for future study is to improve the constructions of sets having the same closed in-neighbourhood, allowing that these new sets share one or more vertices. Moreover, we consider that the ideas presented after Corollary 5.3.2 are a starting point for finding bounds for ℓ using eigenvalue 0 of the adjacency matrix of the digraph. Similarly, we want to address the problem of finding the way of using the complex eigenvalues to calculate a bound for ℓ.

- While doing this thesis I made an academic stay in Primorska University in Slovenia. As a result we have joint paper [17] with Chiarelli, Milanič, Monnot, and Muršič. In this joint work we studied strong cliques in diamond-free graphs. A natural line of research for the future is to look for relationships between strong cliques and identifying codes in diamond-free graphs.

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