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PhD Thesis in Mathematics

Special values of the triple product p-adic L-function

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Introduction

This thesis collects the results of [GG20], [GGMR19] and [GR20], whose object of study are the special values of the triple product p-adic L-function constructed in [DR14]. In order to explain more precisely the results of the thesis, we need to settle some notation. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N_f, let $f \in S_2(N_f)$ be the corresponding weight two newform and let $V_f = T_pE \otimes \mathbb{Q}_p$ be the Galois representation attached to it as in §1.6.4.3, where T_pE denotes the Tate module of E. Let

$$g \in M_1(N_g, \chi), \quad h \in M_1(N_h, \chi^{-1})$$

be two newforms of weight one, mutually inverse nebentype characters and let L be the number field generated by their Fourier coefficients. Let V_g and V_h be the Galois representations attached to g and h respectively as in §1.6.4.5 and let

$$\rho := \rho_g \otimes \rho_h : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_L(V_{gh}^\circ) \cong \mathrm{GL}_4(L)$$

be the tensor product $V_{gh}^\circ := V_g^\circ \otimes V_h^\circ$ of the Artin representations attached to g and h, which factors through a finite Galois group $\mathrm{Gal}(H/\mathbb{Q})$. As explained in §1.7.8, there are (non-canonical) isomorphisms

$$V_g^\circ \otimes_L L_p \cong V_g, \quad \text{and} \quad V_h^\circ \otimes_L L_p \cong V_h,$$

where L_p is a fixed p-adic completion of L. Let

$$L(f \otimes g \otimes h, s) = L(E \otimes \rho, s) = L(V_f \otimes V_g \otimes V_h, s)$$

be the complex L-function attached to the Galois representation $V_f \otimes V_g \otimes V_h$. As explained in §1.12.3, by multiplying by an appropriate factor $L_\infty(E \otimes \rho, s)$, one can complete this L-function to an entire function $\Lambda(E \otimes \rho, s)$ which satisfies a functional equation of the form

$$\Lambda(E \otimes \rho, s) = \varepsilon \cdot \Lambda(E \otimes \rho, 2 - s). \tag{0.0.1}$$

If N denotes the prime-to-p part of $\mathrm{lcm}(N_f, N_g, N_h)$, then

$$\varepsilon = \prod_{v \nmid N} \varepsilon_v \in \{\pm 1\}$$

is a product of local signs, and we assume that

$$\varepsilon_v = +1 \quad \text{for all primes } v \mid N. \tag{0.0.2}$$

By the functional equation (0.0.1), since $L_\infty(E \otimes \rho, s)$ has no zeros nor poles at $s = 1$, the order of vanishing at the centre of symmetry $s = 1$ of the complex L-function $L(E \otimes \rho, s)$ is *even.* As we will recall in §1.14, the Galois-equivariant version of the Birch and Swinnerton–Dyer conjecture predicts that

$$\mathrm{ord}_{s=1} L(E \otimes \rho, s) \stackrel{?}{=} \text{ rank } E(H)_L^\rho,$$

where

$$E(H)^{\rho}_L := \mathrm{Hom}_{G_{\mathbb{Q}}}(V^{\circ}_{gh}, E(H) \otimes L).$$

Fix a prime number p such that

$$p \nmid N_g N_h \ \text{ and } \ \mathrm{ord}_p(N_f) \leq 1.$$

Let α_g, β_g and α_h, β_h be the eigenvalues for the action of the Frobenius element Frob_p at p on the representations V_g and V_h. We will assume throughout the thesis that g and h are *regular* at p, i.e. that

$$\alpha_g \neq \beta_g \ \text{ and } \ \alpha_h \neq \beta_h.$$

Then g has two different ordinary p-stabilisations g_α, g_β (see Definition 1.57) and the same holds for h. Let α_f, β_f be the roots of the p-th Hecke polynomial attached to f labeled in such a way that $\mathrm{ord}_p(\alpha_f) = 0$. We fix the stabilisations $f_\alpha, g_\alpha, h_\alpha$ and let

$$\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]], \ \mathbf{g} \in \Lambda_{\mathbf{g}}[[q]], \ \mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$$

be the Hida families passing through $f_\alpha, g_\alpha, h_\alpha$ respectively, as explained in §1.7, where $\Lambda_{\mathbf{f}}, \Lambda_{\mathbf{g}}, \Lambda_{\mathbf{h}}$ are finite flat extensions of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[T]]$. Let $\mathcal{W}_{\mathbf{fgh}} := \mathrm{Spf}(\Lambda_{\mathbf{f}} \otimes_\Lambda \Lambda_{\mathbf{g}} \otimes_\Lambda \Lambda_{\mathbf{h}})$, and consider the triple product L-function

$$\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) : \mathcal{W}_{\mathbf{fgh}} \longrightarrow \mathbb{C}_p$$

attached as in [DR14] to a triple of Λ-adic *test vectors* $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ (see (1.15.3)). We now recall briefly the interpolation formula and the main properties of this function, which are explained in details in §1.15.3 and §1.16.4. We denote by $\mathcal{W}^{\circ}_{\mathbf{fgh}}$ the subset of *cristalline* points of $\mathcal{W}_{\mathbf{fgh}}$. In order to simplify the notation, in this introduction we will identify the points of $\mathcal{W}^{\circ}_{\mathbf{fgh}}$ with triples (k, ℓ, m) of positive integers. Define

$$\begin{aligned} \mathcal{W}^f_{\mathbf{fgh}} &:= \{(k,\ell,m) \in \mathcal{W}^{\circ}_{\mathbf{fgh}} \mid k \geq \ell + m\}, \quad \mathcal{W}^g_{\mathbf{fgh}} := \{(k,\ell,m) \in \mathcal{W}^{\circ}_{\mathbf{fgh}} \mid \ell \geq k + m\}, \\ \mathcal{W}^h_{\mathbf{fgh}} &:= \{(k,\ell,m) \in \mathcal{W}^{\circ}_{\mathbf{fgh}} \mid m \geq k + \ell\}, \quad \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}} := \mathcal{W}^{\circ}_{\mathbf{fgh}} \smallsetminus (\mathcal{W}^f_{\mathbf{fgh}} \cup \mathcal{W}^g_{\mathbf{fgh}} \cup \mathcal{W}^h_{\mathbf{fgh}}), \end{aligned}$$

so that $\mathcal{W}^{\circ}_{\mathbf{fgh}}$ is the disjoint union of the sets above. A triple of cristalline weights is called *balanced* if it belongs to $\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ and it is called *unbalanced* otherwise. If f_k denotes the weight k specialisation of $\mathbf{f}$ and we use a similar notation for g_ℓ and h_m, then we can consider, as (k, ℓ, m) varies in $\mathcal{W}^{\circ}_{\mathbf{fgh}}$, the complex L-function $L(f_k \otimes g_\ell \otimes h_m, s)$. Similarly as above, it satisfies a functional equation whose center of symmetry is

$$c := \frac{k + \ell + m - 2}{2}$$

and whose sign is

$$\varepsilon = \begin{cases} +1 & \text{if } (k,\ell,m) \text{ is unbalanced} \\ -1 & \text{if } (k,\ell,m) \text{ is balanced.} \end{cases} \tag{0.0.3}$$

The p-adic L-function $\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ interpolates the square roots of the complex values

$$L(f_k \otimes g_\ell \otimes h_m, c) \tag{0.0.4}$$

as (k, ℓ, m) varies in $\mathcal{W}^g_{\mathbf{fgh}}$. More precisely,

$$\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,\ell,m) = \mathcal{E} \cdot \sqrt{L^{\mathrm{alg}}(f_k \otimes g_\ell \otimes h_m, c)} \mod L^\times$$

for $(k, \ell, m) \in \mathcal{W}^g_{\mathbf{fgh}}$, where $\mathcal{E}$ is an *Euler factor* and $L^{\mathrm{alg}}(f_k \otimes g_\ell \otimes h_m, c)$ denotes the algebraic part of the complex L-value (0.0.4). Notice that we are looking at this p-adic L-value modulo

$L^\times$: indeed, different choices of Λ-adic test vectors yield to the same L-value up to multiplication by an element of $L^\times$. For the precise interpolation formula, see Theorem 1.96. By (0.0.3),

$$L(f_k \otimes g_\ell \otimes h_m, c) = 0 \quad \text{for all } (k,\ell,m) \in \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}. \tag{0.0.5}$$

As explained in §1.14, by a conjecture of Beilinson, (0.0.5) implies that there should be a nontorsion agebraic cycle in the Chow group

$$e_{f_k g_\ell h_m}\, \mathrm{CH}^c(W_{k-2} \times W_{\ell-2} \times W_{m-2}/\mathbb{Q})_0 \otimes L \tag{0.0.6}$$

attached to (f_k, g_ℓ, h_m), where $W_r/\mathbb{Q}$ denotes the *Kuga–Sato* variety of dimension $r+1$, whose definition is given in §1.6.4.3. As explained in §1.16.4, the main result of [DR14] claims that the values of the triple product p-adic L-function $\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at *balanced* weights (k,ℓ,m) are related to the so-called *generalised Kato classes* $\Delta_{k,\ell,m}$, which belong to (0.0.6). More explicitly,

$$\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,\ell,m) = \mathcal{E}' \cdot \mathrm{AJ}_p(\Delta_{k,\ell,m})(\omega_{f_k} \otimes \eta_{g_\ell} \otimes \omega_{h_m}) \mod L^\times,$$

where $\mathcal{E}'$ is an Euler factor,

$$\mathrm{AJ}_p : \mathrm{CH}^c(W_{k-2} \times W_{\ell-2} \times W_{m-2})_0 \longrightarrow \mathrm{Fil}^c\, \mathrm{H}^{2c+1}_{\mathrm{dR}}(W_{k-2} \times W_{\ell-2} \times W_{m-2})^\vee$$

is the p-adic Abel–Jacobi map defined in §1.11 and $\omega_{f_k} \otimes \eta_{g_\ell} \otimes \omega_{h_m}$ is an appropriate element of the de Rham cohomology. We are only left to understand the values of $\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at the points in $\mathcal{W}^f_{\mathbf{fgh}}$ and $\mathcal{W}^h_{\mathbf{fgh}}$, and by symmetry, we can restrict to set $\mathcal{W}^f_{\mathbf{fgh}}$. Note that the triple (f,g,h) introduced in the beginning, which corresponds to the pair (E,ρ), has weights

$$(2,1,1) \in \mathcal{W}^f_{\mathbf{fgh}}.$$

The conjectures and the results of [DLR15] and of this thesis suggest that the value $\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1)$ can be interpreted in terms of the arithmetic of the pair (E,ρ). The results on the value of $\mathcal{L}^g_p(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at $(2,1,1)$ (and, more in general, at triples in $\mathcal{W}^f_{\mathbf{fgh}}$), are different depending on the setting. In this thesis, we analyse the following three situations.

- The setting of [DLR15]:
$$p \nmid N_f, \qquad L(E \otimes \rho, 1) = 0, \tag{0.0.7}$$
  and, more in general, the setting of Chapter 2:
$$p \nmid N_f, \qquad L(f_k \otimes g_\ell \otimes h_m, c) = 0 \;\text{ for } (k,\ell,m) \in \mathcal{W}^f_{\mathbf{fgh}}. \tag{0.0.8}$$

- The setting of Chapter 3:
$$p \nmid N_f, \qquad L(E \otimes \rho, 1) \neq 0, \tag{0.0.9}$$
  and we also allow f_k to have non-rational coefficients and a nontrivial nebentype character χ_f, provided that $\chi_f \cdot \chi_g \cdot \chi_h = 1$.

- The setting of Chapter 4:
$$p \mid\mid N_f, \qquad L(E \otimes \rho, 1) \neq 0, \qquad g = \theta(\psi_g),\; h = \theta(\psi_h) \tag{0.0.10}$$
  i.e. g and h are the theta series of the Hecke characters $\psi_g, \psi_h : \mathbb{A}_K^\times \longrightarrow \mathbb{C}$, where K is an imaginary quadratic field in which p is *inert*.

We now explain briefly the results of this thesis in these three settings.

Chapter 2: a generalisation of the elliptic Stark conjecture.

As explained in §2.1, (0.0.7) is the setting of the *elliptic Stark conjecture* of [DLR15]. Recall that we are assuming condition (0.0.2). By the functional equation satisfied by $L(E \otimes \rho, s)$, under (0.0.7) we have

$$r_{\mathrm{an}}(E,\rho) := \mathrm{ord}_{s=1} L(E \otimes \rho, s) \geq 2 \text{ and even.}$$

Recall that the Birch and Swinnerton–Dyer conjecture predicts that

$$r_{\mathrm{an}}(E,\rho) \stackrel{?}{=} r(E,\rho) := \dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}^{\circ}, E(H) \otimes L) \qquad \text{(BSD).}$$

In [DLR15] the authors conjecture that, when these ranks are exactly 2, the value at $(2,1,1)$ of the triple product p-adic L-function $\mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ should be related to a *p-adic regulator* $\mathrm{Reg}_{g_\alpha}(E,\rho)$ made of points in the ρ-isotypical component of the Mordell–Weil group $E(H)$. More precisely, the elliptic Stark conjecture claims that, if $r(E,\rho) = 2$, then

$$\mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1) \stackrel{?}{=} \frac{\mathrm{Reg}_{g_\alpha}(E,\rho)}{\log_p(u_{g_\alpha})} \mod L^\times \qquad \text{(ESC).}$$

Here $u_{g_\alpha} \in \mathcal{O}_H^\times \otimes L$ is a *Stark unit* defined in [DLR15, 1.5], on which the Frobenius element at p acts with eigenvalue β_g/α_g, and $\log_p$ denotes the p-adic logarithm. Moreover, in loc. cit. the authors conjecture that $\mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ should vanish at the point $(2,1,1)$ if $r(E,\rho)$ is strictly greater than 2. For a precise statement of the elliptic Stark conjecture and of the objects appearing in (ESC), see §2.1. We describe now more precisely the definition of the regulator $\mathrm{Reg}_{g_\alpha}(E,\rho)$, in order to understand better its generalisation defined on Chapter 2. Let $E(H)_L := E(H) \otimes L$. There is an isomorphism

$$\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}^{\circ}, E(H)_L) \cong (E(H)_L \otimes V_{gh}^{\circ})^{G_{\mathbb{Q}}}.$$

Recall the eigenvalues for the action of Frob_p on V_g and V_h introduced at the beginning and fix an L-basis of Frobenius eigenvectors

$$\{v^{\alpha\alpha}, v^{\alpha\beta}, v^{\beta\alpha}, v^{\beta\beta}\} \tag{0.0.11}$$

for V_{gh}° with eigenvalues $\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h$ respectively and a basis

$$\{P, Q\} \text{ for } (E(H)_L \otimes V_{gh}^{\circ})^{G_{\mathbb{Q}}}.$$

Write

$$P = P^{1/\alpha\alpha} \otimes v^{\alpha\alpha} + P^{1/\alpha\beta} \otimes v^{\alpha\beta} + P^{1/\beta\alpha} \otimes v^{\beta\alpha} + P^{1/\beta\beta} \otimes v^{\beta\beta}$$

with $P^{1/\alpha\alpha}, \in E(H_p)^{1/\alpha\alpha}$ and analogously for the other local points, and use a similar notation for Q. With this notation,

$$\mathrm{Reg}_{g_\alpha}(E,\rho_{gh}) := \det \begin{pmatrix} \log_{E,p}(P^{1/\alpha\alpha}) & \log_{E,p}(P^{1/\alpha\beta}) \\ \log_{E,p}(Q^{1/\alpha\alpha}) & \log_{E,p}(Q^{1/\alpha\beta}), \end{pmatrix}$$

where $\log_{E,p} : E(H_p) \otimes L \to H_p \otimes L$ is the p-adic logarithm of the elliptic curve E and H_p is a (fixed) completion of H at p. In Chapter 2, we generalise (ESC) giving a conjecture for the values of $\mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at general weights in $\mathcal{W}_{\mathbf{fgh}}^f$. More precisely, we put ourselves in the setting of (0.0.8). As explained in §1.14, the algebraic object attached to the triple (f_k, g_ℓ, h_m) with $(k,\ell,m) \in \mathcal{W}_{\mathbf{fgh}}^f$ is now

$$\mathrm{CH}(M(f_k \otimes g_\ell \otimes h_m)) := \begin{cases} \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}^{\circ}, e_f\, \mathrm{CH}^{k/2}(W_{k-2}/H)_0 \otimes L) & \text{if } (k,\ell,m) = (k,1,1) \\ \mathrm{CH}^c(W_{k-2} \times W_{\ell-2} \times W_{m-2}/\mathbb{Q})_0 \otimes L & \text{if } k,\ell,m > 2. \end{cases}$$

Let

$$r(f_k, g_\ell, h_m) := \dim_L \mathrm{CH}(M(f_k \otimes g_\ell \otimes h_m)).$$

Conjecture 0.1 (cf. Conjectures 2.13 and 2.17)**.** Let $(k,\ell,m) \in \mathcal{W}^f_{\mathbf{fgh}}$. Then

- if $r(f_k, g_\ell, h_m) > 0$, then $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,\ell,m) = 0$;
- if $r(f_k, g_\alpha, h_\alpha) = 2$, then

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,1,1) = \frac{\mathrm{Reg}(f_k, g_\alpha, h_\alpha)}{\mathfrak{g}(\chi_f)\log_p(u_{g_\alpha})} \qquad \bmod L^\times; \tag{0.0.12}$$

- if $k,\ell,m > 2$ and $r(f_k, g_\ell, h_m) = 2$, then there is a finite extension L_0 of L such that

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,\ell,m) = \mathrm{Reg}(f_k, g_\ell, h_m) \qquad \bmod L_0^\times, \tag{0.0.13}$$

where $\mathfrak{g}(\chi_f)$ denotes the Gauss sum of the nebentype character of f_k and the regulator is defined as follows.

Fix a basis

$$\{\Phi_1, \Phi_2\} \quad \text{of } \mathrm{CH}(M(f_k \otimes g_\ell \otimes h_m)).$$

- If $(k,\ell,m) = (k,1,1)$, fix also an L-basis $\{v_1, v_2\}$ for $\{V_g^\alpha \otimes V_h^\circ\}$, where V_g^α denotes the Frob_p-eigenspace with eigenvalue α_g. Then

$$\mathrm{Reg}(f, g_\alpha, h_\alpha) := \det\begin{pmatrix} \mathrm{AJ}_p(\Phi_1(v_1))(\omega_f) & \mathrm{AJ}_p(\Phi_1(v_2))(\omega_f) \\ \mathrm{AJ}_p(\Phi_2(v_1))(\omega_f) & \mathrm{AJ}_p(\Phi_2(v_2))(\omega_f) \end{pmatrix}, \tag{0.0.14}$$

  where $\omega_f \in e_f \cdot \mathrm{Fil}^{k-2}\,\mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/H_p)$ is defined in §1.8.

- If $k,\ell,m > 2$, then

$$\mathrm{Reg}(f_k, g_\ell, h_m) := \det\begin{pmatrix} \mathrm{AJ}_p(\Phi_1)(\omega_f \wedge \eta_g \wedge \omega_h) & \mathrm{AJ}_p(\Phi_1)(\omega_f \wedge \eta_g \wedge \eta_h) \\ \mathrm{AJ}_p(\Phi_2)(\omega_f \wedge \eta_g \wedge \omega_h) & \mathrm{AJ}_p(\Phi_2)(\omega_f \wedge \eta_g \wedge \eta_h), \end{pmatrix} \tag{0.0.15}$$

  where the differentials η_g, η_h, ω_h are defined as in §1.8.2.

Remark 0.2. By definition, the regulator of the elliptic Stark conjecture is actually made up of points in $(E(H)_L \otimes V_g^\alpha \otimes V_h^\circ)^{G_\mathbb{Q}}$. Analogously, in order to construct (0.0.14) we look at

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_g^\alpha \otimes V_h^\circ, e_f\,\mathrm{CH}^{k/2}(W_{k-2}/H)_L)_0.$$

Finally, also in the case of (0.0.15) we are fixing an eigenspace of V_g°, this time with the choice of the differentials $\eta_g \wedge \omega_h$ and $\eta_g \wedge \eta_h$.

Remark 0.3. Recall the element $\log_p(u_{g_\alpha})$ which appears in the elliptic Stark conjecture and in (0.0.12), and notice that it does not appear in the conjectural formula (0.0.13). An explanation for this fact is that it only makes sense to define the Stark unit u_{g_α} if the modular form g has an *Artin representaion* attached to it, which is the case only if it has weight one. A more theoretical explanation of this fact can be found after Theorem 0.5.

Finally, in §2.4, we prove the conjectures (0.0.12) and (0.0.13) for special instances of triples (f_k, g_ℓ, h_m), generalising the proof of a special case of the elliptic Stark conjecture given in [DLR15, §3]. More precisely, if g_ℓ and h_m are theta series of Hecke characters of the same imaginary quadratic field K in which p *splits*, then the representation attached to f_k, g_ℓ, h_m factors as

$$V_{f_k} \otimes V_{g_\ell} \otimes V_{h_m} = (V_f \otimes V_{\psi_1}) \oplus (V_f \otimes V_{\psi_2}), \tag{0.0.16}$$

where, for $i = 1,2$, ψ_i is a ring class character of K and V_{ψ_i} denotes the $G_\mathbb{Q}$-representation induced from ψ_i.

This allows us to prove the following factorisation for the triple product p-adic L-function

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})^2(k,\ell,m)\cdot\mathcal{L}_p(K)(\Psi_g(\ell))^2 = \mathcal{L}_p(\mathbf{f},K)(k,\Psi_{gh}(k,\ell,m))^2\cdot\mathcal{L}_p(\mathbf{f},K)(k,\Psi_{gh'}(k,\ell,m))^2\cdot\mathfrak{f}(k,\ell,m),$$

where $\Psi_g, \Psi_{gh}, \Psi_{gh'}$ are appropriate characters, $\mathcal{L}_p(K)$ is the *Katz*'s p-adic L-function of [Kat76] described in §1.15.1 and $\mathcal{L}_p(\mathbf{f},K)$ is the p-adic L-function constructed by Bertolini, Darmon and Prasanna in [BDP13] (see §1.15.2). For the definition of the factor $\mathfrak{f}(k,\ell,m)$, see Theorem 2.18. Assume that

$$\mathrm{ord}_{s=k/2}L(f_k,\psi_i,s) = 1 \quad \text{for } i=1,2.$$

Under further assumptions on the levels and the finite types of the Hecke characters (see §2.4.2), the factorisation above together with the formulas for the special values of Katz's and BDP p-adic L-functions, allow us to prove, in this setting:

- (0.0.12) (cf. §2.4.2) under the additional assumptions that the relevant *generalised Heegner cycle* defined in [BDP13] generates the space $\mathrm{Hom}_{G_\mathbb{Q}}(V_{\psi_i}, e_f\,\mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L})$ (see Assumption 2.22).
- (0.0.13) (cf. §2.4.3) restricting to the case in which ψ_g,ψ_h are powers of the Hecke character attached to the same elliptic curve with complex multiplication by K and under the additional assumptions that K has class number 1 and that its discriminant satisfies Assumption 2.26. Moreover, we need to put ourselves in a *rank* $(1,1)$*-setting* (see Assumption 2.28), we need assume also Tate's conjecture (see Conjecture 1.71) and a compatibility between the Abel–Jacobi map and natural isomorphisms (Assumption 2.29).

Chapter 3: the rank 0 setting.

In Chapter 3 we return to the triple (f,g,h) of weight $(2,1,1)$ corresponding to the pair (E,ρ), and we put ourselves in the setting in which

$$p\nmid N_fN_gN_h \quad \text{and} \quad L(E\otimes\rho,1)\neq 0.$$

In this case, the Birch and Swinnerton Dyer conjecture predicts that

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, E(H)_L) = 0,$$

so we do not expect the value $\mathcal{L}_p^g(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})(2,1,1)$ to be related to (logarithms of) points. Nevertheless, there still are non-cristalline cohomology classes in the relaxed Selmer group

$$\mathrm{Sel}_{(p)}(E,\rho) := \mathrm{Sel}_{(p)}(\mathbb{Q}, V_f\otimes V_g\otimes V_h)$$

(see Definition 1.85) and the main result of Chapter 3 is a formula for the p-adic L-value $\mathcal{L}_p^g(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})(2,1,1)$ in terms of the Bloch–Kato logarithm of a canonical *non-cristalline class* along a certain *cristalline direction*. It is important to stress the fact that the main formula holds under the following assumption:

Assumption 0.4.

$$\mathrm{Sel}_p(E,\rho) = 0, \tag{0.0.17}$$

where $\mathrm{Sel}_p(E,\rho) := \mathrm{Sel}_p(\mathbb{Q}, V_f\otimes V_g\otimes V_h)$ denotes the Bloch–Kato Selmer group. Recall that, by [DR17], in this setting we know that $\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, E(H)_L)=0$, thus by the Shafarevich–Tate conjecture one also expects $\mathrm{Sel}_p(E,\rho)$ to be trivial. This conjecture is widely open, hence we need to assume (0.0.17).

In order to state more precisely the main result of Chapter 4, let $V := V_f\otimes V_g\otimes V_h$, and recall from the beginning of the introduction the eigenvalues for the action of Frob_p on V_g and

V_h. Let V_g^α, V_g^β be the eigenspaces of V_g relative to the eigenvalues α_g, β_g respectively, and use the analogous notation for V_h. Then

$$V = V^{\alpha\alpha} \oplus V^{\alpha\beta} \oplus V^{\beta\alpha} \oplus V^{\beta\beta}, \tag{0.0.18}$$

where $V^{\alpha\alpha} := V_f \otimes V_g^\alpha \otimes V_h^\alpha$, etc. and let

$$\partial_p : \mathrm{Sel}_{(p)}(E, \rho) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_p, V)$$

be the composition of the restriction to $\mathbb{Q}_p$ with the projection onto the singular quotient. As explained in §3.1, for $\heartsuit, \triangle \in \{\alpha, \beta\}$, there are elements $X_{\alpha\alpha}, X_{\alpha\beta}, ... \in \mathrm{H}^1_s(H_p, V_f)$ characterised by

$$\mathrm{Frob}_p X_{\alpha\alpha} = \frac{1}{\alpha_g \alpha_h} X_{\alpha\alpha} \quad \text{and} \quad \exp^*(X_{\alpha\alpha}) = 1,$$

and similarly for the remaining elements, where $\exp^*$ denotes the Bloch–Kato dual exponential (see §1.4 for the precise definition). Then $\mathrm{Sel}_{(p)}(E, \rho)$ admits a basis

$$\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$$

characterised by the fact that

$$\partial_p(\xi^{\heartsuit\triangle}) = X_{\heartsuit\triangle} \otimes v^{\heartsuit\triangle} \in (\mathrm{H}^1_s(H_p, V_f) \otimes V_g^\heartsuit \otimes V_h^\triangle)^{G_{\mathbb{Q}_p}} \cong \mathrm{H}^1_s(\mathbb{Q}_p, V^{\heartsuit\triangle}), \tag{0.0.19}$$

where $v^{\heartsuit\triangle}$ is the element of the basis (0.0.11). In particular, with respect to the decomposition (0.0.18), the only non-cristalline component of $\xi^{\heartsuit\triangle}$ is $\heartsuit\triangle$. Then, if

$$\pi_{\alpha\beta} : \mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V^{\alpha\beta})$$

denotes the projection, and

$$\mathrm{res}_p : \mathrm{H}^1(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V)$$

is the restriction to $G_{\mathbb{Q}_p}$, then the element $\pi_{\alpha\beta} \mathrm{res}_p \xi^{\beta\beta}$ belongs to $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$. This implies that there is a local point $P_{\alpha\beta} \in E(H_p)$ such that

$$\mathrm{Frob}_p P_{\alpha\beta} = \frac{1}{\alpha_g \beta_g} P_{\alpha\beta} \quad \text{and} \quad \pi_{\alpha\beta}\xi^{\beta\beta} = \delta_p P_{\alpha\beta} \otimes v^{\alpha\beta} \in (\mathrm{H}^1_f(H_p, V_f) \otimes V_g^\alpha \otimes V_h^\beta)^{G_{\mathbb{Q}_p}} \cong \mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta}), \tag{0.0.20}$$

where $\delta_p : E(H_p) \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{H}^1_f(H_p, V_f)$ is the Kummer map. The main result of Chapter 3 is the following.

Theorem 0.5 (cf. Theorem 3.10)**.** *If* $\mathrm{Sel}_p(E, \rho) = 0$*, then*

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2, 1, 1) = \frac{\mathcal{E}''}{\pi\langle f, f\rangle} \times \frac{\log_p(R_{\beta\alpha})}{\mathcal{L}_{g_\alpha}} \times \sqrt{L(E \otimes \rho_{gh}, 1)} \qquad \mod L^\times,$$

where $\mathcal{E}''$ *is an Euler factor,* $\langle f, f\rangle$ *denotes the Petersson product (see §1.5.2) and* $\mathcal{L}_{g_\alpha}$ *is defined as follows.*

Fix L-bases $\{v_g^\alpha, v_g^\beta\}$, $\{v_h^\alpha, v_h^\beta\}$ for V_g, V_h respectively such that the basis (0.0.11) is obtained as $v^{\alpha\alpha} = v_g^\alpha \otimes v_h^\alpha$, etc. In Definition 1.62 we introduce the differentials

$$\omega_{g_\alpha} \in \mathrm{D}_{\mathrm{dR}}(V_g^\alpha), \qquad \eta_{g_\alpha} \in \mathrm{D}_{\mathrm{dR}}(V_g^\beta),$$

which are the same elements that appear in (0.0.13) (see §1.2.2 for the definition of D_{dR}). There are elements

$$\Omega_{g_\alpha} \in H_p^{1/\alpha_g}, \qquad \Theta_{g_\alpha} \in H_p^{1/\beta_g}$$

characterised by

$$\Omega_{g_\alpha} \otimes v_g^\alpha = \omega_{g_\alpha} \in \mathrm{D}_{\mathrm{dR}}(V_g^\alpha), \qquad \Theta_{g_\alpha} \otimes v_g^\beta = \eta_{g_\alpha} \in \mathrm{D}_{\mathrm{dR}}(V_g^\beta).$$

We define

$$\mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}} \in H_p^{\beta_g/\alpha_g}.$$

This element was first defined in [DR16], and in loc. cit. the authors predicts that

$$\mathcal{L}_{g_\alpha} \overset{?}{=} \log_p(u_{g_\alpha}) \tag{0.0.21}$$

(see [DR16, Conjecture 2.1]), where $u_{g_\alpha} \in \mathcal{O}_H^\times \otimes L$ is the Stark unit which appears in (ESC) and (0.0.12). For further details, see Remark 2.7. The period $\mathcal{L}_{g_\alpha}$ arises from the comparison between the L-structure and the p-adic structure of the (adjoint) representation attached to g, thus also in this case, the definition only makes sense when g has weight one. Under (0.0.21), the formula of Theorem 0.5 resembles the conjectural formulas of the elliptic Stark conjecture and its generalisation at weights $(k, 1, 1)$ of Chapter 2.

The proof of Theorem 0.5 uses extensively the relation between the values

$$\mathcal{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1), \qquad \mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1)$$

and the *generalised Kato class*

$$\kappa \in \mathrm{Sel}_{(p)}(E, \rho) \tag{0.0.22}$$

introduced in §1.16.4.

In a similar setting as in the special cases studied in Chapter 2, we have the following corollary as a consequence of the main result.

Corollary 0.6 (cf. Theorem 3.12)**.** *Assume that g and h are theta series of the same imaginary quadratic field in which p splits. Then,*

$$L(E \otimes \rho, 1) \neq 0 \implies \mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1) = 0.$$

Finally, in §3.4 we report the results of the numerical computations that have been carried out by Marc Masdeu in [GGMR19], whome I would like to thank for letting me add these results in my thesis. They are obtained by calculating the so-called *p-adic iterated integrals* (see §2.1), using modifications of Lauder's algorithms [Lau14].

Chapter 4: rank 0 and Kolyvagin classes

In the last chapter of this thesis we explore the setting in which (f, g, h) satisfy what follows.

- p divides exactly N_f, so that the elliptic curve E has multiplicative reduction at p, and let $q_E \in p\mathbb{Z}_p$ be its Tate uniformiser (see §1.6.3.1) ;
- $g = \theta(\psi_g), h = \theta(\psi_h)$ where ψ_g, ψ_h are Hecke character of the same imaginary quadratic field K in which p is *inert*;
- $L(E \otimes \rho, 1) \neq 0$.

As in the special case of Chapter 2, the representation $V := V_f \otimes V_g \otimes V_h$ decomposes as (0.0.16), and this implies the following factorisations

$$L(E \otimes \rho, s) = L(E \otimes \psi_1, s) \cdot L(E \otimes \psi_2, s), \qquad \mathrm{Sel}_p(E, \rho) = \mathrm{Sel}_p(E \otimes \psi_1) \oplus \mathrm{Sel}_p(E \otimes \psi_2).$$

This fact, together with [BD97, Theorem B] implies that

$$L(E \otimes \rho) \neq 0 \implies \mathrm{Sel}_p(E, \rho) = 0,$$

thus we do not need to assume that the Selmer group vanishes, as we do in Chapter 3. Similarly as in Chapter 3, also in this case the value $\mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1)$ can be expressed in terms of the logarithm of a cristalline projection of a non-cristalline cohomology class in $\mathrm{Sel}_{(p)}(E,\rho)$, indeed we prove the following formula (cf. Theorem 4.19)

$$\mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1) = \frac{1}{\pi^2\langle f, f\rangle} \times \sqrt{L(E\otimes\rho,1)} \times \frac{\log_p(P_{\alpha\beta})}{\mathcal{L}_{g_\alpha}} \qquad \mod L^\times, \tag{0.0.23}$$

where $P_{\alpha\beta} \in E(K_p)^{1/\alpha_g\beta_g}$ is characterised exactly by the same relation as in (0.0.20). Here we are using the fact that, in this setting, $H_p = K_p$. The proof of (0.0.23) follows an argument which is similar to the one of the proof of Theorem 0.5, with the important difference that now we are in presence of a so-called *exceptional zero*. Indeed, if $\boldsymbol{\kappa} := \boldsymbol{\kappa}(\mathbf{f}, \mathbf{g}, \mathbf{h})$ denotes the Λ-adic generalised Kato class constructed in [BSV] and [DRa] (see §1.16.4), then the class (0.0.22) is the specialisation at weigh $(2,1,1)$ of $\boldsymbol{\kappa}$, which in this case is

$$\boldsymbol{\kappa}(2,1,1) = 0,$$

as explained in §1.16.4.3. In Chapter 4 we then use an *improved* cohomology class

$$\kappa_g^* \in \mathrm{Sel}_{(p)}(E,\rho)$$

costructed in [BSV], whose relation with the p-adic L-values

$$\mathscr{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1), \qquad \mathscr{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1)$$

are summarised in §1.16.4.3.

Finally, we prove a relation between the local point $P_{\alpha\beta}$ appearing in (0.0.23) and a projection along a cristalline direction of the restriction to G_{K_p} of the *Kolyvagin class*

$$\boldsymbol{K} \in \mathrm{H}^1(H, V_f) \tag{0.0.24}$$

constructed in [BD97], whose definition we recall in §1.16.2. Let

$$\alpha_f := a_p(E) = \begin{cases} +1 & \text{if } E \text{ has split multiplicative reduction at } p \\ -1 & \text{if } E \text{ has nonsplit multiplicative reduction at } p \end{cases}.$$

Even if the class (0.0.24) is not cristalline at p, its α_f-projection

$$\mathrm{res}_p(\mathbf{K})^{\alpha_f} := \mathrm{res}_p(\mathbf{K}) + \alpha_f(\mathrm{Frob}_p \mathbf{K})$$

lies in $\mathrm{H}^1_f(K_p, V_f)^{\alpha_f}$. Then there is a local point $Q_{\alpha_f} \in E(K_p)\otimes\mathbb{Q}_p$ such that

$$\mathrm{Frob}_p Q_{\alpha_f} = \alpha_f Q_{\alpha_f} \qquad \text{and} \qquad \delta_p(Q_{\alpha_f}) = \mathrm{res}_p(\mathbf{K})^{\alpha_f} \in \mathrm{H}^1_f(K_p, V_f).$$

The class $\boldsymbol{K}$ is constructed using a family of *Heegner points* $\{\alpha_m \in E(F_m)\}$, where F_m/H is the layer of degree p^m within the anticyclotomic $\mathbb{Z}_p$-extension of H, which can be taken to be a ring class field of K of conductor prime to p. We fix a prime $\mathfrak{p}$ of H above p, and we denote by $\bar{\alpha}_m$ the image of α_m in the p-primary part $\Phi_{m,\mathfrak{p}}$ of the group of connected components of the Néron model of E over the completion of F_m at the prime above $\mathfrak{p}$. Let

$$\bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty,\mathfrak{p}} := \varprojlim_m \Phi_{m,\mathfrak{p}}.$$

There is a canonical isomorphism

$$\varphi : \Phi_{\infty,\mathfrak{p}} \xrightarrow{\cong} \mathbb{Z}_p$$

Combining the results of [BD97] on $\mathbf{K}$, the results of [BSV] on κ_g^* and the fact that we can label the eigenvalues of V_g and V_h so that

$$\alpha_g \cdot \alpha_h = -\alpha_f,$$

we obtain the relation

$$\delta_p(P_{\alpha_f}) = \frac{p}{\mathrm{ord}_p(q_E)\varphi(\bar{\alpha})} \cdot \mathrm{res}_p(\mathbf{K})^{\alpha_f}. \tag{0.0.25}$$

Finally, combining (0.0.25) with (0.0.23) we obtain the main result of Chapter 4.

Theorem 0.7.

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2,1,1) = \frac{\sqrt{L(E \otimes \rho, 1)}}{\varphi(\bar{\alpha}) \cdot \mathcal{L}_{g_\alpha}} \times \log_p(Q_{\alpha_f}) \qquad \mod L^\times.$$

Notation

For a number field K, we denote $\mathcal{O}_K$ its ring of integers, $\bar{K}$ a fixed algebraic closure and $G_K := \mathrm{Gal}(\bar{K}/K)$. We denote K^{ur} the maximal unramified extension of K, K^{ab} the maximal abelian extension of K and $G_K^{\mathrm{ur}} := \mathrm{Gal}(K^{\mathrm{ur}}/K), G_K^{\mathrm{ab}} := \mathrm{Gal}(K^{\mathrm{ab}}/K)$. For a prime number p, we denote $\mathbb{Q}_p$ the field of p-adic numbers, $\mathbb{Z}_p := \mathcal{O}_{\mathbb{Q}_p}$ the ring of p-adic integers, $\mathbb{F}_p = \mathbb{Z}_p/p\mathbb{Z}_p \cong \mathbb{Z}/p\mathbb{Z}$ the residue field and $\mathbb{C}_p$ the p-adic completion of an integral closure $\bar{\mathbb{Q}}_p$ of $\mathbb{Q}_p$.

Chapter 1

Preliminaries

1.1 Algebraic de Rham cohomology

Let X be a smooth proper scheme defined over a field K of characteristic zero of dimension $d := \dim X \geq 1$. Let $\mathcal{O}_X$ be the structure sheaf of X and for all $k \geq 1$ let Ω_X^k be the sheaf of regular differential k-forms on X. They form a complex

$$\Omega_X^\bullet : 0 \longrightarrow \mathcal{O}_X \xrightarrow{d_0} \Omega_X^1 \xrightarrow{d_1} \Omega_X^2 \xrightarrow{d_2} \cdots ,$$

and the de Rham cohomology of X/K is defined as the hypercohomology of this complex. We describe now briefly its construction. Fix a covering of X by Zariski open sets, and consider the double complex given by the Čech resolution

$$\begin{array}{ccccccc}
\mathcal{C}^\bullet(\Omega_X^\bullet) : & & \mathcal{C}^q(\Omega_X^{p-1}) & \xrightarrow{c_{q,p-1}} & \mathcal{C}^{q+1}(\Omega_X^{p-1}) & \longrightarrow & \\
 & & \big\downarrow{\scriptstyle d_{p-1,q}} & & \big\downarrow{\scriptstyle d_{p-1,q+1}} & & \\
\mathcal{C}^{q-1}(\Omega_X^{p}) & \xrightarrow{c_{q-1,p}} & \mathcal{C}^q(\Omega_X^{p}) & \xrightarrow{c_{q,p}} & \mathcal{C}^{q+1}(\Omega_X^{p}) & \xrightarrow{c_{q+1,p}} & \\
\big\downarrow & & \big\downarrow{\scriptstyle d_{p,q}} & & \big\downarrow{\scriptstyle d_{p,q+1}} & & \\
\mathcal{C}^{q-1}(\Omega_X^{p+1}) & \longrightarrow & \mathcal{C}^q(\Omega_X^{p+1}) & \xrightarrow{c_{q,p+1}} & \mathcal{C}^{q+1}(\Omega_X^{p}) & \longrightarrow & \\
\big\downarrow & & \big\downarrow{\scriptstyle d_{p+1,q}} & & \big\downarrow & &
\end{array}$$

Definition 1.1. The *total complex* $(\mathrm{Tot}^\bullet \mathcal{C}^\bullet(\Omega_X^\bullet), \delta_\bullet)$ of $\mathcal{C}^\bullet(\Omega_X^\bullet)$ is given by

i) $\mathrm{Tot}_X^k := \mathrm{Tot}^k(\mathcal{C}^\bullet(\Omega_X^\bullet)) := \oplus_{p+q=k} \mathcal{C}^q(\Omega_X^p)$;

ii) $\delta_k := \sum_{p+q=k} d_{p,q} + (-1)^p c_{q,p} : \mathrm{Tot}_X^k \longrightarrow \mathrm{Tot}_X^{k+1}$.

Definition 1.2. The *k-th de Rham comology space of X/K* is

$$\mathrm{H}_{\mathrm{dR}}^k(X/K) := \mathrm{H}^k(\mathrm{Tot}^\bullet \mathcal{C}^\bullet(\Omega_X^\bullet)) = \ker(\delta_k)/\mathrm{image}(\delta_{k-1}).$$

The attached spectral sequence

$$E_1^{q,p} := \mathrm{H}^q(X/K, \Omega_X^p) \implies \mathrm{H}_{\mathrm{dR}}^{p+q}(X/K) \tag{1.1.1}$$

degenerates at the first page. The cohomology $\mathrm{H}^k(X/K)$ is a finite dimensional K-vector space, and it is equipped with a decreasing, exhaustive and separated filtration

$$\mathrm{H}_{\mathrm{dR}}^k(X/K) = \mathrm{Fil}^0 \mathrm{H}_{\mathrm{dR}}^k(X/K) \supseteq \mathrm{Fil}^1 \mathrm{H}_{\mathrm{dR}}^k(X/K) \supseteq \cdots \supseteq \mathrm{Fil}^{d+1} \mathrm{H}_{\mathrm{dR}}^k(X/K) = 0 \tag{1.1.2}$$

induced on cohomology by the following filtration on $\mathrm{Tot}_X^\bullet$:

$$\mathrm{Fil}^j\,\mathrm{Tot}_X^k := \oplus_{p+q=k,p\geq j}\mathcal{C}^q(\Omega_X^p).$$

In the case in which $K=\mathbb{C}$, this construction coincides with the classical de Rham cohomology. In particular, it is equipped with the *Hodge decomposition*

$$\mathrm{H}^k_{\mathrm{dR}}(X/\mathbb{C}) \cong \oplus_{p+q=k}\,\mathrm{H}^{p,q}(X/\mathbb{C}), \tag{1.1.3}$$

where $\mathrm{H}^{p,q}(X/\mathbb{C}) = \mathrm{H}^q(X,\Omega_X^p)$, which coincides with the subspace of the complex de Rham cohomology spanned by harmonic forms of type (p,q). Moreover, in this case the filtration (1.1.2) can be expressed as

$$\mathrm{Fil}^j\,\mathrm{H}^k_{\mathrm{dR}}(X/\mathbb{C}) \cong \oplus_{p+q=k,p\geq j}\,\mathrm{H}^{p,q}(X/\mathbb{C}).$$

Finally, there is a comparison isomorphism

$$\mathrm{H}^k_{\mathrm{dR}}(X/\mathbb{C}) \cong \mathrm{H}^k_{\mathrm{B}}(X(\mathbb{C}),\mathbb{Z})\otimes\mathbb{C} =: \mathrm{H}^k_{\mathrm{B}}(X(\mathbb{C}),\mathbb{C}), \tag{1.1.4}$$

which allows to define a Hodge decomposition on the Betti cohomology $\mathrm{H}^k_{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})$.

1.2 Basics of p-adic Hodge theory

Let X be a nice (e.g. smooth and proper) scheme defined over $\mathbb{Q}$. A certain number of cohomology spaces attached to X are endowed with extra structures: for instance, complex cohomology spaces over $\mathbb{C}$ as $\mathrm{H}^*_{\mathrm{dR}}(X/\mathbb{C})$ or $\mathrm{H}^*_{\mathrm{B}}(X(\mathbb{C}),\mathbb{C})$ are endowed with Hodge decomposition, and the ℓ-adic representations as $\mathrm{H}^*_{\mathrm{et}}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_\ell)$ are endowed with a well-behaved action of the (arithmetic) Frobenius element at $p\neq\ell$. Complications arise when we want to study $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules, such as $\mathrm{H}^*_{\mathrm{et}}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p)$ as a $G_{\mathbb{Q}_p}$-representation. One of the main motivation for p-adic Hodge theory is to deal with p-adic representations of $G_{\mathbb{Q}_p}$ (or of G_K with K a p-adic field). Let K be a finite extension of $\mathbb{Q}_p$ and let $\mathbb{C}_K$ be the completion of a fixed algebraic closure of K.

Definition 1.3. A *$\mathbb{C}_K$-representation of G_K* is a $\mathbb{C}_K$-vector space W endowed with a continuous and semilinear action of G_K, i.e. such that $\sigma(xw)=\sigma(x)\sigma(w)$ for all $\sigma\in G_K, x\in\mathbb{C}_K, w\in W$.

Roughly speaking, p-adic Hodge theory consists in the construction of various *rings of periods* B endowed with some additional structure (e.g. a grading, a filtration, the action of some operator). These rings allow to attach to any $\mathbb{Q}_p[G_K]$-module V, a $\mathbb{C}_K$-representation of G_K

$$\mathrm{D}(V) := (V\otimes_{\mathbb{Q}_p}\mathrm{B})^{G_K}.$$

with a natural inclusion

$$\mathrm{B}\otimes_{\mathrm{B}^{G_K}}\mathrm{D}(V)\hookrightarrow \mathrm{B}\otimes_{\mathbb{Q}_p}V. \tag{1.2.1}$$

When (1.2.1) is an isomorphism, then $D(V)$ inherits the extra structure from B.

1.2.1 Hodge–Tate representations and the ring B_{HT}

For an integer $q\in\mathbb{Z}$, let $\mathbb{C}_K(q) := \mathbb{C}_K\otimes_{\mathbb{Q}_p}\mathbb{Q}_p(q)$. Fontaine's ring of Hodge–Tate periods is

$$\mathrm{B}_{\mathrm{HT}} := \oplus_{q\in\mathbb{Z}}\mathbb{C}_K(q).$$

It is a graded $\mathbb{C}_K$-algebra such that $(\mathrm{B}_{\mathrm{HT}})^{G_K}=K$. More precisely, fix a compatible system $\epsilon := (\epsilon^{(m)})_m$ of p-power roots of unity, which defines a basis of $\mathbb{Z}_p(1)$ such that

$$\sigma(\epsilon) = \chi_{\mathrm{cyc}}(\sigma)\cdot\epsilon \qquad \forall\,\sigma\in G_K, \tag{1.2.2}$$

where $\chi_{\rm cyc}$ denotes the cyclotomic character of K. Then B_{HT} is isomorphic to the graded $\mathbb{C}_K$-algebra $\mathbb{C}_K[\epsilon, \epsilon^{-1}]$ of Laurent power series, where G_K acts via (1.2.2) and the grading is given by the monomials of fixed degrees. We denote t the element of $\mathbb{C}_K(1)$ corresponding to ϵ.

A motivation for the introduction of this ring in the following comparison theorem due to Faltings.

Theorem 1.4. *Let X be a smooth proper scheme over K. There is a canonical isomorphism of $\mathbb{C}_K$-representations of G_K*

$$\mathbb{C}_K \otimes_{\mathbb{Q}_p} \mathrm{H}^n_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p) \cong \oplus_{q\in\mathbb{Z}} \left(\mathbb{C}_K(-q) \otimes_K \mathrm{H}^{n-q}(X, \Omega^q_X)\right), \tag{1.2.3}$$

where $\mathbb{C}_K$ acts on the right hand side via its action on the various $\mathbb{C}_K(-q)$.

The isomorphism (1.2.3) has to be compared with the (complex) Hodge filtration of $\mathrm{H}^n_{\mathrm{dR}}(X/\mathbb{C})$ over $\mathbb{C}$ (cf. (1.1.3)).

Let V be a $\mathbb{Q}_p[G_K]$-module. We define

$$\mathrm{D}_{\mathrm{HT}}(V) := (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{HT}})^{G_K}.$$

We say that V is *Hodge–Tate* if the inclusion $\mathrm{B}_{\mathrm{HT}} \otimes_K \mathrm{D}_{\mathrm{HT}}(V) \hookrightarrow \mathrm{B}_{\mathrm{HT}} \otimes_{\mathbb{Q}_p} V$ is an isomorphism. If V is a Hodge–Tate representation, then $\mathrm{D}_{\mathrm{HT}}(V)$ inherits a grading defined by

$$\mathrm{D}_{\mathrm{HT}}(V)_j := (V \otimes_{\mathbb{Q}_p} \mathbb{C}_K(j))^{G_K}.$$

Definition 1.5. The *Hodge–Tate (HT) weights* of a Hodge–Tate representation V are those integers $j \in \mathbb{Z}$ such that $\mathrm{D}_{\mathrm{HT}}(V)_{-j} \neq 0$.

For instance, $\mathrm{D}_{\mathrm{HT}}(\mathbb{Q}_p) := (\mathbb{Q}_p \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{HT}})^{G_K} = K$, so $\mathbb{Q}_p$ has HT weight 0. More in general,

$$\mathrm{D}_{\mathrm{HT}}(\mathbb{Q}_p(n)) := (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} \oplus_q \mathbb{C}_K(q))^{G_K} = (\mathbb{Q}_p(n) \otimes_{\mathbb{Q}_p} \oplus_q \mathbb{C}_K(-n))^{G_K} = \mathrm{D}_{\mathrm{HT}}(\mathbb{Q}_p(n))_{-n}.$$

So $\mathbb{Q}_p(n)$ has Hodge–Tate weight $-n$.

1.2.2 de Rham representations and the ring B_{dR}

Let

$$\mathcal{R} := \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}.$$

It is a perfect $\mathbb{F}_p$-algebra, and there is an isomorphism

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_K} \xrightarrow{\cong} \mathcal{R}, \qquad (x^{(n)})_n \mapsto (x_n := x^{(n)} \mod p)_n.$$

Let $W(\mathcal{R})$ be the ring of Witt vectors of $\mathcal{R}$. Denote $[\,\cdot\,] : \mathcal{R} \longrightarrow W(\mathcal{R})$ the Teichmüller lift and recall that every element of $W(\mathcal{R})$ can be written as $\sum_n [c_n]p^n$, with $c_n \in \mathcal{R}$. Since $W(\mathcal{R})/pW(\mathcal{R}) = \mathcal{R}$, the universal property of Witt vectors assures that the map

$$\mathrm{Hom}(W(\mathcal{R}), \mathcal{O}_{\mathbb{C}_K}) \longrightarrow \mathrm{Hom}(\mathcal{R}, \mathcal{O}_{\mathbb{C}_K}) \tag{1.2.4}$$

given by reducing modulo p is bijective. Consider the homomorphism

$$\theta_0 : \mathcal{R} \longrightarrow \mathcal{O}_{\mathbb{C}_K}/p\mathcal{O}_{\mathbb{C}_K}, \qquad (x_n)_n \mapsto x_0.$$

It lifts uniquely to

$$\theta : W(\mathcal{R}) \longrightarrow \mathcal{O}_{\mathbb{C}_K}$$

via (1.2.4). It is given explicitly by

$$\theta(\sum_n [c_n]p^n) := \sum_n c_n^{(0)} p^n$$

and it is a G_K-equivariant surjective ring homomorphism. Moreover, it extends to

$$\theta[1/p] : W(\mathcal{R})[1/p] \longrightarrow \mathcal{O}_{\mathbb{C}_K}[1/p] = \mathbb{C}_K.$$

One can check that $\ker(\theta[1/p]) = \ker(\theta)[1/p]$ is a principal ideal, and define

$$\mathrm{B}_{\mathrm{dR}}^+ := \varprojlim_j W(\mathcal{R})[1/p]/(\ker\theta[1/p])^j$$

the $\ker\theta[1/p]$-adic completion of $W(\mathcal{R})[1/p]$.

Definition 1.6. The ring of *de Rham periods* is $\mathrm{B}_{\mathrm{dR}} := \mathrm{frac}(\mathrm{B}_{\mathrm{dR}}^+)$.

By definition, $\mathrm{B}_{\mathrm{dR}}^+$ is a complete local ring with principal maximal ideal, moreover the action of G_K on $W(\mathcal{R})$ is compatible with the quotient maps $W(\mathcal{R})/\ker\theta^j \twoheadrightarrow W(\mathcal{R})/\ker\theta^{j+1}$, so $\mathrm{B}_{\mathrm{dR}}^+$ and B_{dR} are naturally endowed with a G_K-action.

Recall the basis $\epsilon = (\epsilon^{(n)})$ of $\mathbb{Z}_p(1)$ we introduced in §1.2.1. It can be regarded as an element of $\varprojlim \mathcal{O}_K \cong \mathcal{R}$, and if $\log_p$ denotes the branch of the p-adic logarithm such that $\log_p(p) = 0$, then

$$t := \log_p([\epsilon]) \tag{1.2.5}$$

generates the maximal ideal of $\mathrm{B}_{\mathrm{dR}}^+$. This defines a filtration on B_{dR} by

$$\mathrm{Fil}^j \mathrm{B}_{\mathrm{dR}} := t^j \mathrm{B}_{\mathrm{dR}}^+,$$

which does not depend on the choice of the uniformiser t and it is stable by the action of G_K. Other important properies are:

1. $\bar{K} \subseteq \mathrm{B}_{\mathrm{dR}}^+ \subseteq \mathrm{B}_{\mathrm{dR}} \subseteq \mathrm{B}_{\mathrm{HT}}$;
2. $\left(\mathrm{B}_{\mathrm{dR}}^+\right)^{G_K} = (\mathrm{B}_{\mathrm{dR}})^{G_K} = K$;
3. for all $j \in \mathbb{Z}$, we have the filtration of B_{dR} is related to the grading of B_{HT} by

$$\mathrm{Fil}^j \mathrm{B}_{\mathrm{dR}}/\mathrm{Fil}^{j+1} \mathrm{B}_{\mathrm{dR}} = \mathbb{C}_K(j) = \mathrm{B}_{\mathrm{HT},j}. \tag{1.2.6}$$

For a $\mathbb{Q}_p[G_K]$-module V, we define

$$\mathrm{D}_{\mathrm{dR}}(V) := (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}})^{G_K}.$$

Analogously as in §1.2.1, we say that V is *de Rham* if the inclusion

$$\mathrm{B}_{\mathrm{dR}} \otimes_K \mathrm{D}_{\mathrm{dR}}(V) \hookrightarrow \mathrm{B}_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V$$

is an isomorphism. In this case, $\mathrm{D}_{\mathrm{dR}}(V)$ is a K-vector space endowed with an action of G_K and a stable filtration induced by the action and the filtration of B_{dR} and $\dim_K \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V$. Moreover, if V is de Rham, then it is also Hodge–Tate, and (1.2.6) implies that

$$\mathrm{D}_{\mathrm{HT}}(V)_j = \mathrm{Fil}^j \mathrm{D}_{\mathrm{dR}}(V)/\mathrm{Fil}^{j+1} \mathrm{D}_{\mathrm{dR}}(V).$$

Finally, the following comparison theorem gives an important example of de Rham representations.

Theorem 1.7. *Let X be a smooth proper scheme defined over K and let $V := \mathrm{H}^n_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_p)$. Then V is a de Rham G_K-representation. Moreover, there is an isomorphism of filtered K-vector spaces*

$$\mathrm{D}_{\mathrm{dR}}(V) \cong \mathrm{H}^n_{\mathrm{dR}}(X/K).$$

Moreover, if V is an unramified representation, then

$$\mathrm{D}_{\mathrm{dR}}(V) \cong (V \otimes \hat{\mathbb{Q}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}, \tag{1.2.7}$$

where $\hat{\mathbb{Q}}_p^{\mathrm{ur}}$ is the p-adic completion of the maximal unramified extension of $\mathbb{Q}_p$.

1.2.3 Cristalline representations and the ring $\mathrm{B}_{\mathrm{cris}}$

Another Fontaine's ring that we will use in this thesis is the ring of *cristalline periods* $\mathrm{B}_{\mathrm{cris}}$. We do not give here the explicit definition, but we list the most important properties.

1. $\mathrm{B}_{\mathrm{cris}}$ is a subring of B_{dR}, which induces on $\mathrm{B}_{\mathrm{cris}}$ a G_K-action and a filtration;
2. the basis t of the maximal ideal $\mathrm{Fil}^1 \mathrm{B}_{\mathrm{dR}}$ of $\mathrm{B}_{\mathrm{dR}}^+$ belongs to $\mathrm{B}_{\mathrm{cris}}$;
3. if we denote $\mathrm{B}_{\mathrm{cris}}^+ := \mathrm{B}_{\mathrm{dR}}^+ \cap \mathrm{B}_{\mathrm{cris}}$, then $\mathrm{B}_{\mathrm{cris}} = \mathrm{B}_{\mathrm{cris}}^+[1/t]$;
4. $\mathbb{Q}_p^{\mathrm{ur}} \subseteq \mathrm{B}_{\mathrm{cris}}$ and $(\mathrm{B}_{\mathrm{cris}})^{G_K} = K \cap \mathbb{Q}_p^{\mathrm{ur}} =: K_0$;
5. it is endowed with an operator
$$\varphi : \mathrm{B}_{\mathrm{cris}} \longrightarrow \mathrm{B}_{\mathrm{cris}}$$
such that $\varphi_{|(\mathrm{B}_{\mathrm{cris}})^{G_K}}$ coincides with the Frobenius element acting on $K \cap \mathbb{Q}_p^{\mathrm{ur}}$ and such that $\varphi(t) = pt$. The operator φ is called *Frobenius*.

For a $\mathbb{Q}_p[G_K]$-module, we denote

$$\mathrm{D}_{\mathrm{cris}}(V) := (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}})^{G_K},$$

and we say that V is *cristalline* if if the inclusion $\mathrm{B}_{\mathrm{cris}} \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) \hookrightarrow \mathrm{B}_{\mathrm{cris}} \otimes_{\mathbb{Q}_p} V$ is an isomorphism. If V is cristalline, then it is de Rham (and then Hodge–Tate) and

$$\dim_{K_0} \mathrm{D}_{\mathrm{cris}}(V) = \dim_K \mathrm{D}_{\mathrm{dR}}(V) = \dim_{\mathbb{Q}_p} V.$$

Moreover, if V is cristalline, then the K_0-vector space $\mathrm{D}_{\mathrm{cris}}(V)$ inherits an action of G_K, a filtration and a semilinear Frobenius, and there is an isomorphism of filtered K-vector spaces

$$K \otimes_{K_0} \mathrm{D}_{\mathrm{cris}}(V) \cong \mathrm{D}_{\mathrm{dR}}(V).$$

The following theorem shows a large class of cristalline representations.

Theorem 1.8. *Let k be a perfect field of characteristic p, let $\mathcal{O}_K := W(k)$ and let $K :=$ frac$(\mathcal{O}_K)$. Let X be a smooth proper scheme defined over $\mathcal{O}_K$ and geometrically irreducible. Then $V := \mathrm{H}^n_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_p)$ is a cristalline G_K-representation. Moreover, there in an isomoprhism of filtered G_K-modules endowed with a semilinear Frobenius*

$$V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}} \cong \mathrm{H}^n_{\mathrm{cris}}(X_k/\mathcal{O}_K) \otimes_{\mathcal{O}_K} \mathrm{B}_{\mathrm{cris}}. \tag{1.2.8}$$

Via comparison between cristalline and de Rham cohomology, (1.2.8) *induces an isomorphism of filtered G_F-modules*

$$\mathrm{D}_{\mathrm{cris}}(V) \cong \mathrm{H}^n_{\mathrm{dR}}(X/K).$$

1.3 Exponential, Bloch–Kato and geometric subspaces

Let K be a finite extension of $\mathbb{Q}_p$ and let V be a $\mathbb{Q}_p[G_K]$-module. Using Fontaine's period rings we define distinguished subspaces of the Galois cohomology space $\mathrm{H}^1(K, V)$. Let $\mathrm{B}_{\mathrm{cris}}^{\varphi=1}$ be the subspace of $\mathrm{B}_{\mathrm{cris}}$ on which φ is trivial.

Definition 1.9. *i)* $\mathrm{H}^1_e(K, V) := \ker \left(\mathrm{H}^1(K, V) \longrightarrow \mathrm{H}^1(K, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}}^{\varphi=1})\right)$ the *exponential* subspace;

ii) $\mathrm{H}^1_f(K, V) := \ker \left(\mathrm{H}^1(K, V) \longrightarrow \mathrm{H}^1(K, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{cris}})\right)$ the *Bloch–Kato* subspace;

iii) $\mathrm{H}^1_g(K,V) := \ker\left(\mathrm{H}^1(K,V) \longrightarrow \mathrm{H}^1(K, V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}})\right)$ the *geometric* subspace.

Since $\mathrm{B}_{\mathrm{cris}}^{\varphi=1} \subseteq \mathrm{B}_{\mathrm{cris}} \subseteq \mathrm{B}_{\mathrm{dR}}$, we have the inclusions

$$\mathrm{H}^1_e(K,V) \subseteq \mathrm{H}^1_f(K,V) \subseteq \mathrm{H}^1_g(K,V).$$

Remark 1.10. Recall that the Galois cohomology $\mathrm{H}^1(K,V)$ is in bijection with the set of G_K-extensions of V by $\mathbb{Q}_p$, i.e. $\mathbb{Q}_p[G_K]$-modules W fitting into exact sequences

$$0 \longrightarrow V \longrightarrow W \longrightarrow \mathbb{Q}_p \longrightarrow 0. \tag{1.3.1}$$

More precisely, given such an extension, consider the long exact sequence induced in G_K-cohomology

$$0 \longrightarrow V^{G_K} \longrightarrow W^{G_K} \longrightarrow \mathbb{Q}_p \xrightarrow{\delta} \mathrm{H}^1(K,V).$$

Then (1.3.1) corresponds to the cohomology class $\delta(1) \in \mathrm{H}^1(K,V)$.

Via this identification,

1. if V is de Rham, then $\mathrm{H}^1_g(K,V)$ consists of the extensions W which are de Rham;

2. if V is de cristalline, then $\mathrm{H}^1_g(K,V)$ consists of the extensions W which are cristalline.

1.3.1 Examples

Let $\mathfrak{p} = (\varpi)$ be the maximal ideal of $\mathcal{O}_K$ with a choice of uniformiser. Recall that the group of units of $K^\times$ decomposes as

$$K^\times \cong \varpi^{\mathbb{Z}} \oplus (1 + \varpi\mathcal{O}_K) \oplus (\mathcal{O}_K/\varpi\mathcal{O}_K)^\times. \tag{1.3.2}$$

We give now some examples of one-dimensional representations, for more details see, e.g. [BK90, §3].

1. The trivial $\mathbb{Q}_p[G_K]$-module $\mathbb{Q}_p$:

$$\mathrm{H}^1(K,\mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G_K^{\mathrm{ab}}, \mathbb{Q}_p) \cong \mathrm{Hom}(K^\times, \mathbb{Q}_p).$$

The last isomorphism is given by the Artin map

$$\mathrm{Art} : K^\times \longrightarrow G_K^{\mathrm{ab}} \tag{1.3.3}$$

of local Class Field Theory, which gives an isomorphism once extended to the profinite completion $\widehat{K^\times}$ of $K^\times$. The decomposition (1.3.2) shows that $\mathrm{H}^1(K,\mathbb{Q}_p)$ has $\mathbb{Q}_p$-dimension $[K:\mathbb{Q}_p]+1$. Moreover, $\mathrm{H}^1_e(K,\mathbb{Q}_p) = 0$ and

$$\mathrm{H}^1_f(K,\mathbb{Q}_p) = \mathrm{H}^1_g(K,\mathbb{Q}_p) = \mathrm{H}^1_{\mathrm{ur}}(K,\mathbb{Q}_p) = \mathrm{Hom}(G_K^{\mathrm{ur}}, \mathbb{Q}_p) = \mathrm{Hom}(\varpi^{\mathbb{Z}}, \mathbb{Q}_p)$$

of dimension 1. Here the last equality is given by the fact that the image $\mathrm{Art}(\varpi)$ of the uniformiser generates topologically G_K^{ur}. We describe more in detail the situation in the setting that we will need in Chapter 4. Assume that K is the completion at p of an imaginary quadratic field in which p is inert. In this case,

$$K^\times = p^{\mathbb{Z}} \oplus (1 + p\mathcal{O}_K) \oplus (\mathcal{O}_K/p\mathcal{O}_K)^\times. \tag{1.3.4}$$

The decomposition (1.3.4) corresponds via (1.3.3) to

$$G_K^{\mathrm{ab}} \cong \mathrm{Gal}(K^{\mathrm{ur}}/K) \times \mathrm{Gal}(K_\infty/K) \times \mathrm{Gal}(K'/K),$$

where K_∞/K is a $\mathbb{Z}_p^2$-extension and K'/K is finite. More precisely,

$$K_\infty = K_{\mathrm{cyc}} \cdot K_{\mathrm{ant}}$$

is the composition of the so-called *cyclotomic* and *anticyclotomic* $\mathbb{Z}_p$-extension of K, which are explicitly constructed as in §4.1.2. Let $\log_p : 1 + p\mathbb{Z}_p \longrightarrow p\mathbb{Z}_p$ be the usual p-adic logarithm, $\exp_p$ denote its inverse and set $u := \exp_p(p) \in 1 + p\mathbb{Z}_p$. Fix an element $u_\star \in 1 + p\mathcal{O}_K$ such that $\mathrm{Frob}_p\, u_\star = -u_\star$ and $\{u, u_\star\}$ is a $\mathbb{Z}_p$-basis of $1 + p\mathcal{O}_{K_p}$. Then

$$\mathrm{Gal}(K_\infty/K) \cong \Gamma_{\mathrm{cyc}} \times \Gamma_{\mathrm{ant}},$$

where

(a) $\Gamma_{\mathrm{cyc}} := \mathrm{Gal}(K_{\mathrm{cyc}}/K)$ is generated topologically by $\sigma_{\mathrm{cyc}} := \mathrm{Art}(u)$;

(b) $\Gamma_{\mathrm{ant}} := \mathrm{Gal}(K_{\mathrm{ant}}/K)$ is generated topologically by $\sigma_{\mathrm{ant}} := \mathrm{Art}(u_\star)$.

Set also $\Gamma_{\mathrm{ur}} := \mathrm{Gal}(K^{\mathrm{ur}}/K)$. Recall the cyclotomic character $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^\times$; by an abuse of notation, we continue to denote $\chi_{\mathrm{cyc}} : \Gamma_{\mathrm{cyc}} \longrightarrow 1 + p\mathbb{Z}_p$ its restriction to Γ_{cyc}. Let also χ_{ant} and χ_{ur} denote the characters of Γ_{ant} and Γ_{ur} such that

$$\chi_{\mathrm{ant}}(\sigma_{\mathrm{ant}}) = u, \qquad \chi_{\mathrm{ur}}(\mathrm{Frob}_p) = u$$

respectively. Then

$$\{\xi_{\mathrm{ur}} := \log_p(\chi_{\mathrm{ur}}),\ \xi_{\mathrm{cyc}} := \log_p(\chi_{\mathrm{cyc}}),\ \xi_{\mathrm{ant}} := \log_p(\chi_{\mathrm{ant}})\}$$

is a $\mathbb{Q}_p$-basis of $\mathrm{H}^1(K, \mathbb{Q}_p)$ and $\{\xi_{\mathrm{ur}}\}$ is a $\mathbb{Q}_p$-basis of $\mathrm{H}^1_f(K, \mathbb{Q}_p) = \mathrm{Hom}(\Gamma_{\mathrm{ur}}, \mathbb{Q}_p)$.

2. The cyclotomic $\mathbb{Q}_p[G_K]$-module $\mathbb{Q}_p(1)$. Kummer map gives an isomorphism

$$\delta_p : K^\times \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{H}^1(K, \mathbb{Q}_p(1)) = \mathrm{H}^1_g(K, \mathbb{Q}_p(1)),$$

of dimension $[K : \mathbb{Q}_p] + 1$ by (1.3.2). Moreover, δ_p restricts to

$$\delta_p : \mathcal{O}_K^\times \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{H}^1_f(K, \mathbb{Q}_p(1)) = \mathrm{H}^1_e(K, \mathbb{Q}_p(1)). \qquad (1.3.5)$$

Since $\mathcal{O}_K^\times \otimes \mathbb{Q}_p = 1 + \varpi\mathcal{O}_K$, the Bloch–Kato (and the exponential) subgroup of $\mathbb{Q}_p(1)$ has dimension $[K : \mathbb{Q}_p]$.

3. Unramified one dimensional $\mathbb{Q}_p[G_K]$-modules. Let ψ be a nontrivial unramified character of G_K and let $\mathbb{Q}_p(\psi)$ be the representation given by ψ. Then

$$\mathrm{H}^1_e(K, \mathbb{Q}_p(\psi)) = \mathrm{H}^1_f(K, \mathbb{Q}_p(\psi)) = \mathrm{H}^1_g(K, \mathbb{Q}_p(\psi)) = 0$$

and $\mathrm{H}^1(K, \mathbb{Q}_p(\psi))$ has dimension $[K : \mathbb{Q}_p]$.

4. If $V = \mathbb{Q}_p(\psi)(1)$ with ψ is a nontrivial unramified character of G_K, then

$$\mathrm{H}^1_e(K, V) = \mathrm{H}^1_f(K, V) = \mathrm{H}^1_g(K, V) = \mathrm{H}^1(K, V).$$

5. The Tate module of an abelian variety. Let A be an abelian variety defined over K, let $T_pA := \varprojlim_n A[n]$ be the Tate module of A and denote $V_A := T_pA \otimes \mathbb{Q}_p$. Then the Kummer map gives an isomorphism

$$\delta_p : A(K) \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{H}^1_e(K, V) = \mathrm{H}^1_f(K, V) = \mathrm{H}^1_g(K, V). \qquad (1.3.6)$$

6. More in general, if V is de Rham, then

$$\mathrm{H}^1_e(K, V) = \mathrm{H}^1_f(K, V).$$

1.4 Bloch–Kato logarithm and dual exponential

Let V be a de Rham $\mathbb{Q}_p[G_K]$-module, where, as above, $K/\mathbb{Q}_p$ is a p-adic field. Consider the short exact sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow \mathrm{B}_{\mathrm{cris}}^{\varphi=1} \xrightarrow{\pi} \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+} \longrightarrow 0,$$

where π is given by $\mathrm{B}_{\mathrm{cris}}^{\varphi=1} \hookrightarrow \mathrm{B}_{\mathrm{cris}} \hookrightarrow \mathrm{B}_{\mathrm{dR}} \twoheadrightarrow \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+}$. Tensoring by V and taking G_K-invariants, we get the long exact sequence in cohomology

$$0 \longrightarrow V^{G_K} \longrightarrow \mathrm{D}_{\mathrm{cris}}(V)^{\varphi=1} \longrightarrow (V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} \longrightarrow \mathrm{H}^1_e(K,V) \longrightarrow 0 \tag{1.4.1}$$

Since V is de Rham,

$$(V \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}/\mathrm{B}_{\mathrm{dR}}^{+})^{G_K} = \mathrm{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V), \tag{1.4.2}$$

Definition 1.11. The *Bloch–Kato exponential* attached to V is the isomorphism

$$\exp : \frac{\mathrm{D}_{\mathrm{dR}}(V)}{\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) + \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(V)} \xrightarrow{\cong} \mathrm{H}^1_e(K,V)$$

induced by the connecting isomorphism of (1.4.1) combined with (1.4.2).

The terminology is due to the fact that exp is a generalisation of the exponential attached to formal Lie groups. In particular, if E is an elliptic curve defined over K, then $V := T_pE \otimes \mathbb{Q}_p$ is a de Rham $\mathbb{Q}_p[G_K]$-module. By 4. of §1.3.1, the domain of exp is $\mathrm{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)$, which can be identified with the tangent space of E, and there is a commutative diagram

$$\begin{array}{ccc} \tan(E) & \xrightarrow{\exp} & E(K) \otimes \mathbb{Q}_p \\ \downarrow{\cong} & & \downarrow{\delta_p} \\ \mathrm{D}_{\mathrm{dR}}(V)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) & \xrightarrow{\exp} & \mathrm{H}^1_f(K,V). \end{array}$$

Definition 1.12. The *Block–Kato logarithm* attached to V is

$$\log := \exp^{-1} : \mathrm{H}^1_e(K,V) \longrightarrow \frac{\mathrm{D}_{\mathrm{dR}}(V)}{\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V) + \mathrm{D}_{\mathrm{cris}}^{\varphi=1}(V)}.$$

Let V be a de Rham $\mathbb{Q}_p[G_K]$-module, and consider the Bloch–Kato exponential

$$\exp : \mathrm{H}^1_f(K,V^*) \longrightarrow \mathrm{D}_{\mathrm{cris}}(V^*)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*) \tag{1.4.3}$$

attached to the *Kummer dual* $V^* := V^\vee(1) = \operatorname{Hom}(V,\mathbb{Q}_p(1))$ of V. Notice that there is a natural pairing

$$V \otimes V^* \longrightarrow \mathbb{Q}_p(1), \qquad \langle v, v^* \rangle = v^*(v). \tag{1.4.4}$$

Recall that, for a fixed basis ϵ of $\mathbb{Q}_p(1)$ we have defined a uniformiser t for $\operatorname{Fil}^1 \mathrm{B}_{\mathrm{dR}}$ as in (1.2.5). Then $\varepsilon \otimes t^{-1}$ gives a basis for $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)) = (\mathbb{Q}_p(1) \otimes \mathrm{B}_{\mathrm{dR}})^{G_K}$ which is independent of ϵ. Then (1.4.4) induces a pairing

$$\mathrm{D}_{\mathrm{dR}}(V) \otimes \mathrm{D}_{\mathrm{dR}}(V^*) \longrightarrow \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)) \cong K \xrightarrow{\mathrm{tr}_{K/\mathbb{Q}_p}} \mathbb{Q}_p, \tag{1.4.5}$$

where the isomorphism $\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(1)) \cong K$ sends $\epsilon \otimes t^{-1}$ to 1. Via (1.4.5) we have $\mathrm{D}_{\mathrm{dR}}(V^*) \cong \mathrm{D}_{\mathrm{dR}}(V)^\vee = \operatorname{Hom}(V,\mathbb{Q}_p)$ and

$$\mathrm{D}_{\mathrm{dR}}(V^*)/\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V^*) \cong \left(\operatorname{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V)\right)^\vee. \tag{1.4.6}$$

On the other hand, (1.4.4) and cup product also induce a pairing on cohomology

$$\mathrm{H}^1(K,V)\otimes \mathrm{H}^1(K,V^*) \longrightarrow \mathrm{H}^2(K,V\otimes V^*) \longrightarrow \mathrm{H}^2(K,\mathbb{Q}_p(1)) \cong \mathbb{Q}_p. \tag{1.4.7}$$

So $\mathrm{H}^1(K,V^*) \cong \mathrm{H}^1(K,V)^\vee$ and

$$\mathrm{H}^1_f(K,V^*) \cong \left(\mathrm{H}^1(K,V)/\,\mathrm{H}^1_f(K,V)\right)^\vee. \tag{1.4.8}$$

Definition 1.13. Denote

$$\mathrm{H}^1_s(K,V) := \mathrm{H}^1(K,V)/\,\mathrm{H}^1_f(K,V).$$

The *Bloch–Kato dual exponential* attached to V is the isomorphism

$$\exp^* : \mathrm{H}^1_s(K,V) \longrightarrow \mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(V)$$

obtained by taking the dual of (1.4.3) and using (1.4.6) and (1.4.8).

In Chapter 4 we will need the following explicit formula for $\exp^*$, which is due to Kato (see [Kat93]). Let

$$\gamma : \mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(V) = (V\otimes \mathrm{B}^+_{\mathrm{dR}})^{G_K} \longrightarrow \mathrm{H}^1(K,V\otimes \mathrm{B}^+_{\mathrm{dR}})$$

be the isomorphism defined by $x \mapsto \gamma(x)$, where $\gamma(x)$ is the cohomology class of the cocycle

$$\sigma \mapsto x\cdot \log_p(\chi_{\mathrm{cyc}}(\sigma)).$$

The inclusion $\mathrm{B}^+_{\mathrm{cris}} \subseteq \mathrm{B}_{\mathrm{cris}}$ induces a map $\mathrm{H}^1(K,V\otimes \mathrm{B}^+_{\mathrm{dR}}) \longrightarrow \mathrm{H}^1(K,V\otimes \mathrm{B}_{\mathrm{dR}})$ which in injective, so if $\alpha : V \longrightarrow V\otimes \mathrm{B}^+_{\mathrm{dR}}$ denotes the inclusion, then

$$\mathrm{H}^1_f(K,V) = \ker(\mathrm{H}^1(K,V) \xrightarrow{\alpha_*} \mathrm{H}^1(K,V\otimes \mathrm{B}^+_{\mathrm{dR}}).$$

Proposition 1.14. *The dual exponential attached to V is the isomorphism given by the composition*

$$\mathrm{H}^1_s(K,V) \xrightarrow{\alpha_*} \mathrm{H}^1(K,V\otimes \mathrm{B}^+_{\mathrm{dR}}) \xrightarrow{\gamma^{-1}} \mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(V).$$

1.4.1 examples

We go back to the examples of §1.3.1 and we describe more explicitly Bloch–Kato logarithm and dual exponential in these settings.

1. The trivial representation. We put ourselves in the setting in which K is the completion at p of an imaginary quadratic field in which p is inert. In this case it only makes sense to consider the dual exponential $\mathrm{H}^1_s(K,\mathbb{Q}_p) \longrightarrow \mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p)$. Since $\mathrm{H}^1(K,\mathbb{Q}_p) \cong \mathrm{Hom}(K^\times,\mathbb{Q}_p)$ and $\mathrm{H}^1_f(K,\mathbb{Q}_p) \cong \mathrm{Hom}(\varpi^{\mathbb{Z}},\mathbb{Q}_p)$, local class field theory gives an isomorphism

   $$\mathrm{H}^1_s(K,\mathbb{Q}_p) \cong \mathrm{Hom}((K^{\mathrm{ur}})^\times,\mathbb{Q}_p).$$

   Moreover, $\mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p) = K$, so we have

   $$\exp^* : \mathrm{Hom}((K^{\mathrm{ur}})^\times,\mathbb{Q}_p) \longrightarrow K.$$

   Assume now that K is the completion at p of an imaginary quadratic field in which p is inert. The description in §1.3.1 shows that

   $$\mathrm{H}^1_s(K,\mathbb{Q}_p) \cong \mathrm{Hom}(\Gamma_{\mathrm{cyc}},\mathbb{Q}_p) \oplus \mathrm{Hom}(\Gamma_{\mathrm{ant}},\mathbb{Q}_p)$$

   with $\mathbb{Q}_p$-basis $\{\xi_{\mathrm{cyc}},\xi_{\mathrm{ant}}\}$. Moreover, we have the following description of the dual exponential of the basis above, that we will use in Chapter 4.

Proposition 1.15. *We have*

$$\exp^*(\xi_{\mathrm{cyc}}) = 1 \quad \textit{and} \quad z := \exp^*(\xi_{\mathrm{ant}}) \in K^-$$

where K^- denotes the $\mathbb{Q}_p$-subspace of K on which Frob_p *acts as multiplication by* -1.

Proof. The action of $\mathrm{Frob}_p \in G_{\mathbb{Q}_p} \smallsetminus G_K$ on $\xi \in \mathrm{H}^1_s(K, \mathbb{Q}_p) \cong \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p) \oplus \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p)$ is given by

$$\xi^{\mathrm{Frob}_p}(\sigma) = \mathrm{Frob}_p\, \xi(\mathrm{Frob}_p\, \sigma\, \mathrm{Frob}_p^{-1}) = \xi(\mathrm{Frob}_p\, \sigma\, \mathrm{Frob}_p^{-1}) = \begin{cases} \xi(\sigma) & \sigma \in \Gamma_{\mathrm{cyc}} \\ \xi(\sigma^{-1}) = -\xi(\sigma) & \sigma \in \Gamma_{\mathrm{ant}}. \end{cases}$$

In other words,

$$\mathrm{H}^1_s(K, \mathbb{Q}_p)^+ = \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p), \quad \text{and} \quad \mathrm{H}^1_s(K, \mathbb{Q}_p)^- = \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p).$$

By the $G_{\mathbb{Q}_p}$-equivariance of the dual exponential, it decomposes then in

$$\exp^* : \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p) \longrightarrow K^+ = \mathbb{Q}_p, \qquad \exp^* : \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p) \longrightarrow K^-.$$

It follows that $\exp^*(\xi_{\mathrm{cyc}}) \in \mathbb{Q}_p$ and $z := \exp^*(\xi_{\mathrm{ant}}) \in K^-$. In order to compute explicitly the dual exponential of ξ_{cyc}, recall that $\exp^*$ is given by the composition

$$\mathrm{Hom}(\Gamma_{\mathrm{cyc}} \oplus \Gamma_{\mathrm{ant}}, \mathbb{Q}_p) \xrightarrow{\alpha_*} \mathrm{H}^1(K, \mathbb{Q}_p \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}}) \xrightarrow{\gamma^{-1}} (\mathbb{Q}_p \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}})^{G_K} = K$$

where α_* is the composition of an homomorphism with the inclusion $\mathbb{Q}_p \hookrightarrow \mathrm{B}^+_{\mathrm{dR}}$. All the maps above are isomorphisms. Then

$$y := \exp^*(\xi_{\mathrm{cyc}}) \in (\mathrm{B}^+_{\mathrm{dR}})^{G_K}$$

is the element such that

$$\gamma(y) = \alpha_*(\xi_{\mathrm{cyc}}) \ \text{ in } \ \mathrm{H}^1(K, \mathrm{B}^+_{\mathrm{dR}}). \tag{1.4.9}$$

By definition, $\gamma(x) : \sigma \mapsto x \cdot \log_p(\chi_{\mathrm{cyc}}(\sigma)) \in \mathrm{B}^+_{\mathrm{dR}}$, so (1.4.9) assures that, for all $\sigma \in G_K$,

$$y \cdot \log_p(\chi_{\mathrm{cyc}}(\sigma)) = \xi_{\mathrm{cyc}}(\sigma) \in \mathbb{Q}_p(\chi_E) \otimes_{\mathbb{Q}_p} \mathrm{B}^+_{\mathrm{dR}}.$$

Comparing it with the very definition of ξ_{cyc} we conclude that $y = 1$. □

2. The cyclotomic representation. Via the Kummer map isomorphism (1.3.5), the Bloch–Kato logarithm on $\mathrm{H}^1_f(K, \mathbb{Q}_p(1))$ coincides with the p-adic logarithm

$$\log_p : \mathcal{O}_K^\times \otimes \mathbb{Q}_p = (1 + \varpi\mathcal{O}_K) \otimes \mathbb{Q}_p \longrightarrow K.$$

3. Unramified 1-dimensional representation $V = \mathbb{Q}_p(\psi)$ with ψ nontrivial unramified character of G_K. In this case, it only makes sense to consider the dual exponential

$$\exp^* : \mathrm{H}^1_s(K, \mathbb{Q}_p(\psi)) = \mathrm{H}^1(K, \mathbb{Q}_p(\psi)) \longrightarrow \mathrm{Fil}^0\, \mathrm{D}_{\mathrm{dR}}(\mathbb{Q}_p(\psi)) = (\mathbb{Q}_p(\psi) \otimes \hat{\mathbb{Q}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}},$$

where the last equality is (1.2.7).

4. If $V = \mathbb{Q}_p(\psi)(1)$ with ψ is a nontrivial unramified character of G_K, then it only makes sense to consider the Bloch–Kato logarithm

$$\log : \mathrm{H}^1_e(K, V) = \mathrm{H}^1(K, V) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V)/\,\mathrm{Fil}^0\, \mathrm{D}_{\mathrm{dR}}(V) = \mathrm{D}_{\mathrm{dR}}(V).$$

5. If E/K is an elliptic curve. The Bloch–Kato logarithm attached to $\mathrm{H}^1_f(K, V_E) = \mathrm{H}^1_e(K, V_E)$ coincides via the Kummer isomorphism (1.3.6) to the formal group logarithm

$$\log_{E,p} : E(K) \longrightarrow K$$

corresponding to the Néron differential of E. There is also a dual exponential

$$\exp^*_{E,p} : \mathrm{H}^1_s(K, V_E) \longrightarrow \mathrm{Fil}^0\, \mathrm{D}_{\mathrm{dR}}(V_E)$$

1.5 Classical modular forms

In this section we recall the anaytic and the cohomological definition of classical modular forms and we set some notation that we will need throughout the thesis. More precisely, in §1.5.2 we recall the classical analytic definition of modular forms, in §1.5.5 we explain their description as global sections of sheaves on the modular curve and in §1.5.7 we report the geometric description of modular forms introduced by Katz.

1.5.1 Modular curves and universal elliptic curves

In this section, we recall the definition of the modular curve attached to suitable arithmetic groups and we set some notation. We focus on the modular curve $X_1(N)$, we recall the moduli interpretation of this curve and we introduce the *universal elliptic curve* attached to the relevant moduli problem. The universal elliptic curve is a crucial ingredient in the definition of the *Kuga–Sato variety* introduced in §1.6.4.3 and that will be central in Chapter 2.

Let $N \geq 3$ be an integer and denote

1. $\Gamma_1(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \mod N, \text{ for some } x \in \mathbb{Z} \right\}$;
2. $\Gamma_0(N) := \left\{ \gamma \in \mathrm{SL}_2(\mathbb{Z}) \;\middle|\; \gamma \equiv \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mod N, \text{ for some } a, b, d \in \mathbb{Z} \right\}$.

Note that $\Gamma_1(N)$ is a normal subgroup of $\Gamma_0(N)$ and that the map

$$\Gamma_0(N)/\Gamma_1(N) \longrightarrow (\mathbb{Z}/N\mathbb{Z})^{\times}, \; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto d \tag{1.5.1}$$

is a group isomorphism. Let

$$\mathcal{H} := \{z \in \mathbb{C} \mid \mathrm{im}(z) > 0\}$$

be the complex upper half plane and denote $\mathcal{H}^* := \mathcal{H} \cup \mathbb{Q} \cup \{\infty\}$. The group $\mathrm{SL}_2(\mathbb{Z})$ acts transitively on $\mathcal{H}^*$ via Moebius transformations, i.e.

$$\gamma(z) := \frac{az+b}{cz+d}, \quad \gamma(\infty) := \frac{a}{b}$$

for all $z \in \mathcal{H}^*$ and $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. We can endow $\mathcal{H}^*$ with a suitable topology which restricts to the euclidean topology on $\mathcal{H}$ and so $\Gamma_1(N)$ acts continuously on $\mathcal{H}^*$ (for details, see e.g. [Kna92, XI §2]).

Using the action of $\Gamma_1(N)$ on $\mathcal{H}$ described at the beginning of the section, define

$$Y_0(N) := \Gamma_0(N)\backslash\mathcal{H}, \quad X_1(N) := \Gamma_0(N)\backslash\mathcal{H}^*, \quad Y_1(N) := \Gamma_1(N)\backslash\mathcal{H}, \quad X_1(N) := \Gamma_1(N)\backslash\mathcal{H}^*.$$

In this thesis we will only use the spaces relative to the group $\Gamma_1(N)$, so we now focus on the properties of $Y_1(N)$ and $X_1(N)$. They are Riemann surfaces; $Y_1(N)$ is an open and dense subvariety of $X_1(N)$. The variety $X_1(N)$ is compact and it is obtained by adjoining to $Y_1(N)$ the finite set of points $\Gamma_1(N)\backslash\mathbb{P}^1(\mathbb{Q})$, which are called *cusps*. It can be shown that $\Gamma_1(N)\backslash\mathcal{H}^*$ (resp. $\Gamma_1(N)\backslash\mathcal{H}$) is the set of $\mathbb{C}$-points of an algebraic variety defined over $\mathbb{Q}$, that we still denote by $X_1(N)$ (resp. $Y_1(N)$) and that is called the *modular curve* (resp. the *open modular curve*) for $\Gamma_1(N)$. These curves admit smooth proper models (which we still call $Y_1(N)$ and $X_1(N)$) over $\mathbb{Z}[1/N]$ and they are fine moduli space. In order to explain it more precisely we need some definitions.

Definition 1.16. Let S be a scheme. An *elliptic curve over* S is an S-scheme $\pi : E \longrightarrow S$ together with a section $s_0 : S \longrightarrow E$ such that π is a flat and proper morphism whose fibres are smooth genus one curves.

The open curve $Y_1(N)$ is a scheme over $S_N := \mathrm{Spec}(\mathbb{Z}[1/N])$, so it can be interpreted as a functor $Y_1(N) : S_N - \mathcal{S}\mathrm{chemes} \longrightarrow \mathcal{S}\mathrm{ets}$ given by

$$Z \mapsto Y_1(N)(Z) := \mathrm{Hom}(Z, Y_1(N)).$$

Consider the functor

$$\mathrm{Ell}_N : S_N - \mathcal{S}chemes \longrightarrow \mathcal{S}\mathrm{ets}$$

such that, for a S_N-scheme S, $\mathrm{Ell}_N(S)$ is the set of isomorphism classes of pairs (E, P) where E is an elliptic curve over S and $P \in E$ is a point of exact order N. Such a point is called a $\Gamma_1(N)$*-structure* for E. Then $Y_1(N) \cong \mathrm{Ell}_N$. In other words, for any S_N-scheme S, there is a bijection

$$Y_1(N)(S) \xrightarrow{\cong} \mathrm{Ell}_N(S), \tag{1.5.2}$$

which is compatible with homomorphisms $S \longrightarrow S'$ of S_N-schemes. For $S = \mathrm{Spec}(\mathbb{C})$, (1.5.2) coincides with the map

$$\Gamma_1(N)\backslash\mathcal{H} \ni \tau \mapsto \left(\mathbb{C}/\Lambda_\tau, \frac{1}{N}\right), \qquad \Lambda_\tau := \mathbb{Z} \oplus \tau\mathbb{Z}. \tag{1.5.3}$$

As a consequence of this fact, there is a *universal elliptic curve*

$$\pi : \mathcal{E} \longrightarrow Y_1(N)$$

over $Y_1(N)$ (with a universal $\Gamma_1(N)$-structure $\mathcal{P}$) satisfying the following universal property. For every S_N-scheme S and every class $[(E, P)] \in \mathrm{Ell}_N(S)$ there exists a unique morphism $\varphi : S \longrightarrow Y_1(N)$ such that E is the fiber product

$$\begin{array}{ccc} E & \longrightarrow & \mathcal{E} \\ \downarrow & & \downarrow \pi \\ S & \xrightarrow{\varphi} & Y_1(N), \end{array} \tag{1.5.4}$$

where the top arrow sends P to $\mathcal{P}$. Similarly, the compact modular curve $X_1(N)$ is the moduli space parametrizing *generalised elliptic curves* with $\Gamma_1(N)$ structure (see [DR73]). So it also atmits a universal (generalised) elliptic curve, which we still denote $\pi : \mathcal{E} \longrightarrow X_1(N)$ with its *identity section* $s_0 : X_1(N) \longrightarrow \mathcal{E}$.

1.5.2 Modular forms for $\Gamma_1(N)$, I

We recall the definition and we fix the notation for modular and cusp forms for the group $\Gamma_1(N)$. We also recall the definition of the Petersson scalar product on the space of cuspforms of a fixed weight.

Let $N \geq 3$ be an integer. For any positive integer k, there is a *weight-k-action* of $\mathrm{SL}_2(\mathbb{Z})$ (and therefore of $\Gamma_0(N)$ and $\Gamma_1(N)$) on the set of holomorphic functions $f : \mathcal{H} \longrightarrow \mathbb{C}$ defined by

$$f|_k\gamma(z) := (cz + d)^{-k} f(\gamma(z))$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ and all $z \in \mathcal{H}$. Let f is a holomorphic function of $\mathcal{H}$ invariant with respect to the action of $\Gamma_1(N)$ of some weight. Since the matrix $\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ belongs to $\Gamma_1(N)$, the function f is $\mathbb{Z}$-periodic. We say that f is *holomorphic at* ∞ if it admits a Fourier expansion

$$f(z) = \sum_{n=0}^{\infty} a_n(f) q^n \tag{1.5.5}$$

with $q = e^{2\pi i z}$. In other words, if D denotes the complex open disk centered in 0 of radius 1, then the periodicity of f implies the existence of a holomorphic function $g\colon D \smallsetminus \{0\} \longrightarrow \mathbb{C}$ such that $f(z) = g(e^{2\pi i z})$, and f is holomorphic at ∞ if and only if g can be extended to a holomorphic function on D. In this case, (1.5.5) is the Laurent expansion of g at $q = 0$, and it is called the *q-expansion* of f. For $t \in \mathbb{Q} \cup \{\infty\}$, let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be a matrix such that $t = \gamma(\infty)$. If f is invariant for the action of $\Gamma_1(N)$, then the function $f|_k\gamma$ is invariant with respect to the subgroup $\gamma^{-1}\Gamma_1(N)\gamma$, and it is $h\mathbb{Z}$-periodic for some integer h, which can be choosen to be minimal. We say that f is *holomorphic at* $t \in \mathbb{Q}$ if $f|_k\gamma$ admits a Fourier expansion (1.5.5) with $q = e^{2\pi i/h}$.

Definition 1.17. A *modular form* of *level* N and *weight* k is a holomorphic function

$$f\colon \mathcal{H} \longrightarrow \mathbb{C}$$

such that

i) $f|_k\gamma = f$ for all $\gamma \in \Gamma_1(N)$;

ii) f is holomorphic at all $t \in \mathbb{Q} \cup \{\infty\}$.

A modular form f is *cuspidal* (or is a *cuspform*) if $a_0(f|_k\gamma) = 0$ in the q-expansion (1.5.5), for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

We denote $M_k(N)$ (respectively $S_k(N)$) the space of modular forms (respectively cuspforms) for $\Gamma_1(N)$ of weight k. They are finite-dimensional $\mathbb{C}$-vector spaces. Moreover, $S_k(N)$ is endowed with a hermitian inner product called *Petersson inner product*. It is defined by

$$\langle f, g\rangle := \int_{\Gamma_1(N)\backslash\mathcal{H}} f(z)\overline{g(z)}y^k \frac{dxdy}{y^2}$$

for all $f, g \in S_k(N)$, where $z = x + iy \in \mathcal{H}$.

1.5.3 Hecke operators

Fix a pair of positive integers $N \geq 3, k$. We recall now the definition and the main properties of the Hecke operators acting on $M_k(N)$ and $S_k(N)$ and we set some notation on the Hecke algebras. We also recall the definition of *Nebentype* character, *oldforms* and *newforms*.

Given an integer d such that $(d, N) = 1$, and let $\gamma \in \Gamma_0(N)$ any matrix of the form $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. The diamond operator is

$$\langle d\rangle : M_k(N) \longrightarrow M_k(N), \qquad \langle d\rangle f := f|_k\gamma.$$

The operator $\langle d\rangle$ is well defined and only depends on the class of d in $(\mathbb{Z}/N\mathbb{Z})^\times$. It follows from the isomorphism (1.5.1) and from the invariance of f with respect to the weight-k-action of $\Gamma_1(N)$.

For a prime p, define

$$U_p f := \sum_{i=0}^{p-1} f|_k \left(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}\right) \qquad \text{if } p \mid N;$$

$$T_p f := \sum_{i=0}^{p-1} f|_k \left(\begin{smallmatrix} 1 & i \\ 0 & p \end{smallmatrix}\right) + f|_k \left(\begin{smallmatrix} a & b \\ N & p \end{smallmatrix}\right) \cdot \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right), \text{ where } ap - bN = 1, \qquad \text{if } p \nmid N.$$

We extend the definition above setting $T_1 := 1$, $T_{p^r} := T_pT_{p^{r-1}} - p^{k-1}\langle p\rangle T_{p^{r-2}}$ for $p \nmid N$, $r \geq 2$ and $T_n := \prod T_{p_i^{e_i}}$ if $n = \prod p_i^{e_i}$ is coprime to N.

For a character $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}^\times$, we denote

$$M_k(N,\chi) := \left\{ f \in M_k(N) \;\middle|\; f|_k\gamma = \chi(d)f \text{ for all } \gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma_0(N) \right\}.$$

There is a decomposition

$$M_k(N) = \oplus_\chi M_k(N,\chi),$$

where the sum runs over all Dirichlet characters modulo N, and the character χ is called the *Nebentype character* of all $f \in M_k(N,\chi)$. An analogous decomposition also holds for $S_k(N)$.

For $f \in M_k(N,\chi)$, there is an explicit espression for the q-expansion of T_pf. If $p \nmid N$, then

$$a_n(T_pf) = a_{np}(f) + \chi(p)p^{k-1}a_{n/p}(f), \tag{1.5.6}$$

where $a_{n/p}(f) := 0$ if $p \nmid n$. If $p \mid N$, then the Fourier coefficient $a_n(U_pf)$ have the same expression as (1.5.6).

The *Hecke algebra* $\mathfrak{h}_k(N)$ is the subalgebra of $\mathrm{End}(S_k(N))$ generated by the *Hecke operators* T_n and $\langle n \rangle$ for $(n,N) = 1$. A modular form $f \in S_k(N)$ is an *eigenform* if it is a simultaneous eigenvector for the elements of $\mathfrak{h}_k(N)$. The operators T_n and $\langle n \rangle$ for $(n,N) = 1$ commute and are normal with respect to the Petersson inner product, so $S_k(N)$ admits a basis of eigenforms.

For any $M \mid N$, the space $S_k(M)$ can be embedded into $S_k(N)$ in various way. Indeed, for any divisor t of N/M, there is an injection $S_k(M) \hookrightarrow S_k(N)$ given by $f \mapsto f|_k \left(\begin{smallmatrix} t & 0 \\ 0 & 1 \end{smallmatrix}\right)$. Define, for any $M \mid N$ and any prime $p \mid \frac{N}{M}$,

$$\iota_{p,M} : S_k(M)^2 \longrightarrow S_k(N), \qquad (f,g) \mapsto f + g|_k \left(\begin{smallmatrix} p & 0 \\ 0 & 1 \end{smallmatrix}\right).$$

Define the following subspaces of respectively old and new modular forms of $S_k(N)$

$$S_k(N)^{\mathrm{old}} := \sum_{p \mid N} \iota_{p,N/p}\left(S_k(N/p)^2\right), \quad \text{and} \quad S_k(N)^{\mathrm{new}} := \left(S_k(N)^{\mathrm{old}}\right)^\perp.$$

Definition 1.18. *i)* An eigenform f is *normalised* if $a_1(f) = 1$.

ii) A cuspform $f \in S_k(N)$ is a *newform* if it is an eigenform and belongs to $S_k(N)^{\mathrm{new}}$.

iii) A normalised eigenform f of level N is *ordinary* at a prime p if $a_p(f)$ is a p-adic unit.

If f is a normalised newform of level N, then

$$T_nf = a_n(f) \cdot f, \quad \text{and} \quad U_pf = a_p(f) \cdot f$$

for all n coprime with N and all prime $p \mid N$. Moreover, the field

$$K_f := \mathbb{Q}(\{a_n(f) \mid n \geq 1\})$$

is a number field. Using the moduli interpretation, one can define an action of the Hecke algebra on the modular curve as follows. Using the notation of (1.5.3), let $[\tau] = [(E,P)] \in Y_1(N)$ be the class of the pair (E,P), where $\tau \in \mathcal{H}$, $E(\mathbb{C}) = \mathbb{C}/\Lambda_\tau$ and $P = 1/N$. Then

$$\begin{aligned}
\langle d \rangle[(E,P)] &= [(E,dP)] && \text{for } (d,N) = 1; \\
T_p[(E,P)] &= \sum_{i=0}^{p-1}[(\mathbb{C}/\Lambda_{\frac{\tau+i}{p}}, 1/N)] + [(\mathbb{C}/\Lambda_{p\tau}, 1/N)] && \text{for } p \nmid N; \\
U_p[(E,P)] &= \sum_C [(E/C, P \mod C)] && \text{for } p \mid N,
\end{aligned}$$

where the last sum runs over the subgroups of E of order p.

1.5.4 Hecke characters of imaginary quadratic fields and theta series

In this section we give the definition of a special case of modular forms, which arise as *theta series* from *Hecke characters* of imaginary quadratic fields. These modular forms can be considered simpler then general modular forms, in a sense that will be clear in §1.6.4: the representation attached to such modular forms are induced by 1-dimensional representations. This peculiarity will allow us to prove some special cases of the conjectures stated in Chapter 2.

Let K be an imaginary quadratic field of discriminant $-D_K$. We denote $\mathbb{A}_K$ the adele ring and $\mathbb{A}_K^\times$ the idele group.

Definition 1.19. A *Hecke character of* K of *infinity type* $(\ell_1, \ell_2) \in \mathbb{Z}^2$ is a continuous homomorphism $\psi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ such that

$$\psi(\alpha \cdot x \cdot x_\infty) = \psi(x) x_\infty^{-\ell_1} \bar{x}_\infty^{-\ell_2}, \text{ for all } \alpha \in K^\times,\ x \in \mathbb{A}_K^\times,\ x_\infty \in \mathbb{C}^\times.$$

In the definition above, we are identifying $(K \otimes \mathbb{R})^\times = \mathbb{C}^\times$. For any place v of K, we denote K_v the completion of K at v. We write any element $x \in \mathbb{A}_K^\times$ as $x = (x_v)_v$, where v runs over the places of K and $x_v \in K_v^\times$. Given a Hecke character ψ of K, we denote

$$\psi_v : K_v^\times \longrightarrow \mathbb{C}^\times$$

the restriction of ψ to $K_v^\times$ with respect to the canonical inclusion $K_v^\times \hookrightarrow \mathbb{A}_K^\times$ sending $x_v \mapsto (x_w)_w$ with $x_w = 1$ for all $w \neq v$. For all but finitely many prime ideals $\mathfrak{p}$ of K, the character ψ is *unramified at* $\mathfrak{p}$, meaning that $\psi_\mathfrak{p}$ restricted to $\mathcal{O}_{K_\mathfrak{p}}^\times$ is trivial. If $\mathfrak{p}$ is a prime ideal at which ψ is ramified, then there is an integer $c_\mathfrak{p} > 0$ such that $\psi_\mathfrak{p}$ becomes trivial when restricted to $1 + \mathfrak{p}^{c_\mathfrak{p}} \mathcal{O}_{K_\mathfrak{p}}$.

Definition 1.20. The *conductor* of ψ is the largest ideal $\mathfrak{c}_\psi$ of K such that

$$\psi_{\mathfrak{p}|1+\mathfrak{c}_\psi \mathcal{O}_{K_\mathfrak{p}}} = 1$$

for all prime $\mathfrak{p}$ of K.

The restriction of ψ to the adele group of $\mathbb{Q}$ is of the form

$$\psi_{|\mathbb{A}_\mathbb{Q}^\times} = \varepsilon_\psi \cdot \mathrm{N}^{\ell_1+\ell_2},$$

where N stands for the norm character, and ε_ψ is a Dirichlet character called the *central character* of ψ. Let ψ be a Hecke character of conductor $\mathfrak{c}_\psi$, and denote

$$I(\mathfrak{c}_\psi) := \{\text{ideals of } K \text{ prime to } \mathfrak{c}_\psi\}.$$

We can look at ψ as a character of $I(\mathfrak{c}_\psi)$ as follows. For each prime ideal $\mathfrak{p}$ of K choose a uniformiser $\pi_\mathfrak{p}$ for $\mathfrak{p}\mathcal{O}_{K_\mathfrak{p}}$. Then we can consider

$$\psi : I(\mathfrak{c}_\psi) \longrightarrow \mathbb{C}^\times$$

given by $\psi(\mathfrak{a}) = \prod_{\mathfrak{p}|\mathfrak{a}} \psi_\mathfrak{p}(\pi_\mathfrak{p})^{\mathrm{ord}_\mathfrak{p}(\mathfrak{a})}$ for all $\mathfrak{a} \in I(\mathfrak{c}_\psi)$. Here $\mathrm{ord}_\mathfrak{p}$ is the valuation at $\mathfrak{p}$ normalised so that $\mathrm{ord}_\mathfrak{p}(\mathfrak{p}) = 1$.

Starting from a Hecke character of K, we can construct a modular form as follows. Let $\psi : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$ be a Hecke character of K of infinity type $(0, \ell)$ or $(\ell, 0)$. The *theta series* of ψ is

$$\theta(\psi) := \sum_{\mathfrak{a} \in I(\mathfrak{c}_\psi)} \psi(\mathfrak{a}) q^{\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a})},$$

where $q = e^{2\pi i}$ and $\mathrm{N}_{K/\mathbb{Q}}$ denotes the ideal norm of K. More explicitly,

$$\theta(\psi) = \sum_{n\geq 0} a_n(\theta(\psi))q^n, \qquad \text{where } a_n(\theta(\psi)) = \sum_{\mathfrak{a}\in I(\mathfrak{c}_\psi)^n} \psi(\mathfrak{a}) \tag{1.5.7}$$

and $I(\mathfrak{c}_\psi)^n := \{\mathfrak{a} \in I(\mathfrak{c}_\psi) \mid \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{a}) = n\}$. The expression (1.5.7) is the q-expansion of a modular form $\theta(\psi) \in M_\ell(N,\chi)$ of weight ℓ with level and Nebentype character

$$N = D_K\, \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c}_\psi) \qquad \text{and} \qquad \chi := \varepsilon_\psi\chi_K,$$

where ε_ψ is the central character of ψ and χ_K is the Dirichlet character attached to the imaginary quadratic field K. Moreover, if $\ell > 1$, then $\theta(\psi)$ is a cuspform.

1.5.5 Modular forms as global sections

In this section we give an algebraic description of modular forms, which is equivalent to the analytic definition given in §1.5.2. It uses locally free sheaves on $X_1(N)$ and on the universal elliptic curve $\mathcal{E}$. One of the main motivations is that via this description the space of modular forms of level N and of fixed weight has a natural structure over $\mathbb{Z}[1/N]$. This interpretation of the space $M_k(N)$ is very simple to state (see (1.5.8)), while the description of $S_k(N)$ requires a number of cohomological objects such as the *Gauss–Manin connection*, which we briefly describe in §1.5.5.1.

Fix the pair of integers $k \geq 1, N \geq 3$. Recall that the modular curve $X_1(N)$ is a smooth proper scheme over $\mathbb{Z}[1/N]$. In §1.5.1 we introduced the universal elliptic curve $\pi : \mathcal{E} \longrightarrow X_1(N)$ endowed with the identity section $s_0 : X_1(N) \longrightarrow \mathcal{E}$. Let $\Omega_{X_1(N)}$ be the sheaf of differentials of $X_1(N)$ and denote $\Omega_{\mathcal{E}/X_1(N)}$ the sheaf of $\mathcal{E}$ of differentials over $X_1(N)$ with respect to the morphism π. Define

$$\underline{\omega} := s_0^*(\Omega_{\mathcal{E}/X_1(N)}).$$

It is a locally free sheaf of $X_1(N)$ of rank 1. Let $X_1(N)_\mathbb{C} := X_1(N) \times_{\mathbb{Z}[1/N]} \mathbb{C}$ and denote, for all integer k,

$$\underline{\omega}^k := \underline{\omega}^{\otimes k}.$$

There is an isomorphism

$$M_k(N)_\mathbb{C} \xrightarrow{\cong} \mathrm{H}^0(X_1(N)_\mathbb{C}, \underline{\omega}^k_\mathbb{C}), \qquad f \mapsto \omega_f. \tag{1.5.8}$$

1.5.5.1 The sheaf $\mathcal{L}$, the Gauss–Manin connection and the Kodaira–Spencer isomorphism

We give a brief description of the objects needed to define the space of cuspforms as a space of global section of a sheaf on $X_1(N)$. For more details, we refer to [DR14, §2] and to the references given there.

Let $\Omega_{\mathcal{E}/X_1(N)}$ be the sheaf of relative 1-differentials on $\mathcal{E}$ with respect to the projection $\pi : \mathcal{E} \longrightarrow X_1(N)$. Then $\pi_*\Omega_{\mathcal{E}/X_1(N)}$ is a sheaf on $X_1(N)$ defined by

$$\pi_*\Omega_{\mathcal{E}/X_1(N)}(U) := \Omega_{\mathcal{E}/X_1(N)}(\pi^{-1}(U))$$

for all open subsets U of $X_1(N)$. Define

$$\mathcal{L} := \mathrm{R}^1\pi_*\Omega_{\mathcal{E}/X_1(N)}$$

the first right derived functor of $\Gamma(\,-\,, \pi_*\Omega_{\mathcal{E}/X_1(N)})$. It fits into an exact sequence

$$0 \longrightarrow \underline{\omega} \longrightarrow \mathcal{L} \longrightarrow \underline{\omega}^{-1} \longrightarrow 0 \tag{1.5.9}$$

induced by Hodge filtration on fibres. For $r \in \mathbb{Z}_{\geq 0}$ let

$$\mathcal{L}_r := \mathrm{Sym}^r \mathcal{L},$$

equipped with the filtration

$$\mathcal{L}_r = \mathrm{Fil}^0 \mathcal{L}_r \supseteq \mathrm{Fil}^1 \mathcal{L}_r \supseteq \cdots \supseteq \mathrm{Fil}^r \mathcal{L}_r = \underline{\omega}^r$$

induced by (1.5.9) and satisfying, for $0 \leq j \leq r$,

$$\mathrm{Fil}^{r-j} \mathcal{L}_r / \mathrm{Fil}^{r-j+1} \mathcal{L}_r \cong \underline{\omega}^{r-2j}.$$

The sheaf $\mathcal{L}_r$ is equipped with the so-called *Gauss–Manin connection*

$$\nabla : \mathcal{L}_r \longrightarrow \mathcal{L}_r \otimes \Omega_{X_1(N)}(\log),$$

where $\Omega_{X_1(N)}(\log)$ denotes the sheaf of differentials with at most logarithmic poles at the cusps. The relation between the connection ∇ and the filtration is the so-called *Griffith transversality*:

$$\nabla \mathrm{Fil}^{r-j} \mathcal{L}_r \subseteq \mathrm{Fil}^{r-j-1} \mathcal{L}_r \otimes \Omega_{X_1(N)}(\log),$$

which induces isomorphisms

$$\nabla : \frac{\mathrm{Fil}^{r-j} \mathcal{L}_r}{\mathrm{Fil}^{r-j+1} \mathcal{L}_r} \cong \frac{\mathrm{Fil}^{r-j-1} \mathcal{L}_r}{\mathrm{Fil}^{r-j} \mathcal{L}_r} \otimes \Omega_{X_1(N)}(\log).$$

In particular, for $r = 2$ and $j = 0$ we get the *Kodaira–Spencer* isomorphism

$$\mathrm{KS} : \underline{\omega}^2 \xrightarrow{\cong} \Omega_{X_1(N)}(\log). \tag{1.5.10}$$

1.5.5.2 Modular forms for $\Gamma_1(N)$, II

When k is even, then the isomorphism (1.5.8) coincides with the following composition

$$\begin{aligned} M_k(N) &\longrightarrow \Omega_{X_1(N)}^{\otimes k/2}(X_1(N)) \xrightarrow{\mathrm{KS}_*^{\otimes k/2}} \underline{\omega}^k(X_1(N)) \\ f &\longmapsto f(\tau)(\mathrm{d}\,\tau)^{k/2}. \end{aligned}$$

For any k, we will always identify

$$M_k(N)_{\mathbb{C}} \cong \mathrm{H}^0(X_1(N), \underline{\omega}^k) \cong \mathrm{H}^0(X_1(N), \underline{\omega}^{k-2} \otimes \Omega_{X_1(N)}(\log))$$

via (1.5.8) and (1.5.10), since all these isomorphisms are compatible with the action of the Hecke algebra. Moreover, we have

$$S_k(N)_{\mathbb{C}} \cong \mathrm{H}^0(X_1(N), \underline{\omega}^{k-2} \otimes \Omega_{X_1(N)}). \tag{1.5.11}$$

For any $\mathbb{Z}[1/N]$-algebra R, denote $X_1(N)_R := X_1(N) \times_{\mathbb{Z}[1/N]} R$ the change of basis of $X_1(N)$ to R. The identifications above allow us to define modular forms with coefficients on any $\mathbb{Z}[1/N]$-algebra R in a natural way.

Definition 1.21. The space of (classical) modular forms of weight k, level N and coefficients in R is

$$M_k(N)_R := \mathrm{H}^0(X_1(N)_R, \underline{\omega}_R^k) = \mathrm{H}^0(X_1(N)_R, \underline{\omega}_R^{k-2} \otimes \Omega_{X_1(N)/R}(\log)).$$

The corresponding space of cuspforms is

$$S_k(N)_R = \mathrm{H}^0(X_1(N)_R, \underline{\omega}_R^{k-2} \otimes \Omega_{X_1(N)/R}).$$

Notice that, in particular,

$$S_2(N) = \mathrm{H}^0(X_1(N), \Omega_{X_1(N)}). \tag{1.5.12}$$

1.5.6 The Eichler–Shimura isomorphism

Via the description of §1.5.5 and using comparison isomorphisms between cohomologies, the space of cuspforms can be related to the *first* cohomology of a sheaf on $X_1(N)$. This relation is given by the *Eichler–Shimura* isomorphism, which we briefly recall here. For more details, we refer to [DI95, §12.2].

Let $\mathrm{H}^1_{\mathrm{dR}}(X_1(N)/\mathbb{C})$ be the (first) de Rham cohomology group. Recall that, if $\mathcal{C}^j_{X_1(N)}$ denotes the sheaf of smooth complex-valued j-differentials of $X_1(N)$, then $\mathrm{H}^1_{\mathrm{dR}}(X_1(N)/\mathbb{C})$ is the first cohomology of the complex

$$0 \longrightarrow \mathcal{C}^0_{X_1(N)} \xrightarrow{\mathrm{d}} \mathcal{C}^1_{X_1(N)} \xrightarrow{\mathrm{d}} \mathcal{C}^2_{X_1(N)} \longrightarrow 0.$$

By Hodge decomposition, the de first Rham cohomology of $X_1(N)$ over $\mathbb{C}$ is the direct sum

$$\mathrm{H}^1_{\mathrm{dR}}(X_1(N)/\mathbb{C}) \cong \mathrm{H}^{1,0}(X_1(N)/\mathbb{C}) \oplus \mathrm{H}^{0,1}(X_1(N)/\mathbb{C}) \tag{1.5.13}$$

of the space of holomorphic and anti-holomorphic 1-forms. The complex conjugation gives a $\mathbb{C}$-linear isomorphism

$$\mathrm{H}^{1,0}(X_1(N)/\mathbb{C}) \xrightarrow{\cong} \mathrm{H}^{0,1}(X_1(N)/\mathbb{C}), \qquad f(z)\,\mathrm{d}\,z \mapsto \overline{f(z)}\,\mathrm{d}\,\bar{z}.$$

By (1.5.12) we have isomorphisms

$$S_2(N) \xrightarrow{\cong} \mathrm{H}^0(X_1(N), \Omega_{X_1(N)}) = \mathrm{H}^{1,0}(X_1(N)/\mathbb{C}), \tag{1.5.14}$$

and

$$\overline{S_2(N)} := \mathbb{C} \otimes_{\mathbb{C}} S_2(N) \xrightarrow{\cong} \mathrm{H}^{0,1}(X_1(N)/\mathbb{C}),$$

where the tensor product is taken with respect to the complex conjugation. Finally, by de Rham Theorem, there is an isomorphism

$$\mathrm{H}^1(X_1(N), \mathbb{C}) \cong \mathrm{H}^1_{\mathrm{dR}}(X_1(N)/\mathbb{C}),$$

between the de Rham cohomology and the singular cohomology of $X_1(N)$, which also coincides with the sheaf cohomology of the modular curve with respect to the constant sheaf $\mathbb{C}$.

Summarising, we get the *Eichler–Shimura* isomorphism

$$S_2(N) \oplus \overline{S_2(N)} \xrightarrow{\cong} \mathrm{H}^1(X_1(N), \mathbb{C}), \tag{1.5.15}$$

which respects the action of the Hecke operators. More in general, consider the universal elliptic curve $\pi : \mathcal{E} \longrightarrow Y_1(N)$ described in §1.5.1 and let $\iota : Y_1(N) \hookrightarrow X_1(N)$ be the immersion of the open modular curve into its compactification. For $k \geq 2$, the isomorphism (1.5.15) generalises as

$$S_k(N) \oplus \overline{S_k(N)} \xrightarrow{\cong} \mathrm{H}^1(X_1(N), \mathcal{F}_{k-2}) \otimes \mathbb{C}, \tag{1.5.16}$$

where the sheaf $\mathcal{F}_{k-2}$ on $X_1(N)$ is defined as

$$\mathcal{F}_{k-2} := \iota_* \mathrm{Sym}^{k-2} \mathrm{R}^1 \pi_* \mathbb{Q}.$$

1.5.7 Modular forms à la Katz

Let $N \geq 3$ and let R be a $\mathbb{Z}[1/N]$-algebra. In [Kat76], Katz gives a geometric description of modular forms in $M_k(N)_R$ using the moduli interpretation of $X_1(N)$, that we describe briefly in this section.

For an elliptic curve E defined over a ring A, let $\underline{\omega}_{E/A}$ be the sheaf of invariant differentials of E over A. Recall that it is free of rank one.

Definition 1.22. A *meromorphic* modular form of level N, weight k and coefficients in R *à la Katz* is a rule f which assigns to every triple $(E/R', P_N, \omega)$ where R' is a R-algebra, E/R' is an elliptic curve defined over R, P_N is a level $\Gamma_1(N)$-structure on E and ω is a R'-basis of $\underline{\omega}_{E/R'}$, an element

$$f(E/R', P_N, \omega) \in R'$$

satisfying the following

i) $f(E/R', P_N, \omega)$ only depends on the isomorphism class of the triple $(E/R', P_N, \omega)$;

ii) f commutes with the base changes: if $\alpha : R' \longrightarrow R''$ is a morphism and $(E/R'', \tilde{P}_N, \tilde{\omega})$ is the triple obtained by base changing $(E/R', P_N, \omega)$ via α, then $f(E/R'', \tilde{P}_N, \tilde{\omega}) = \alpha(f(E/R', P_N, \omega))$;

iii) for all $(E/R', P_N, \omega)$ and all $a \in (R')^\times$,

$$f(E/R', P_N, a\omega) = a^k f(E/R', P_N, \omega).$$

We denote $M_k^{\text{mer}}(N)_R$ the space of such forms.

In order to define modular forms *à la Katz* we need a geometric interpretation of q-expansions and of holomorphicity at the cusps. Let $f \in M_k^{\text{mer}}(N)_R$, let q be a variable, let $R_0 := R(q^{1/N})$ and consider the *Tate curve* $\text{Tate}(q)/R_0$. Recall that $\text{Tate}(q) = \mathbb{G}_m/q^{\mathbb{Z}}$ is an elliptic curve over R_0, and denote $\varphi : \mathbb{G}_m/R_0 \longrightarrow \text{Tate}(\rho)/R_0$ the projection. Denote P_N a level $\Gamma_1(N)$-structure on $\text{Tate}(q)$ (for instance, $q^{1/N} \in R_0^\times/q^{\mathbb{Z}} = \text{Tate}(q)(R_0)$ is a point of exact order N). There is a differential form $\omega_{\text{can}} \in \underline{\omega}_{\text{Tate}(q)/R_0}$ uniquely determined by the equality $\varphi^*\omega_{\text{can}} = dt/t$. Then

$$f(\text{Tate}(q)/R_0, P_N, \omega_{\text{can}}) \in R_0 = R(q^{1/N}).$$

Definition 1.23. For each P_N on $\text{Tate}(q)/R_0$ the *q-expansion* at P_N is the morphism

$$q\text{-exp} : M_k^{\text{mer}}(N)_R \longrightarrow R_0 \qquad f \mapsto f(\text{Tate}(q)/R_0, P_N, \omega_{\text{can}}).$$

Consider the inclusion $R[[q^{1/N}]] \subseteq R_0$.

Definition 1.24. A (classical) of modular form of level N, weight k and coefficients in R *à la Katz* is a meromorphic form $f \in M_k^{\text{mer}}(N)_R$ whose q-expansion

$$f(q) := f(\text{Tate}(q)/R_0, P_N, \omega_{\text{can}}) = \sum_{n=0}^{\infty} a_n q^{n/N}$$

belongs to $R[[q^{1/N}]]$ for all P_N.

A modular form $\omega_f \in H^0(X_1(N), \omega^k) = M_k(N)$ corresponds uniquely to a Katz modular form f with coefficients in $\mathbb{C}$ defined as follows. Let $(E/\mathbb{C}, P_N, \omega)$ be a triple as in Definition 1.22 and let $\psi : S = \text{Spec}(\mathbb{C}) \longrightarrow X_1(N)$ be the morphism associated to $(E/\mathbb{C}, P_N)$ and $\mathcal{E}$ of (1.5.4) and consider $\varphi^* : \underline{\omega}^k \longrightarrow \underline{\omega}_{E/\mathbb{C}}^k$. Then $\varphi^*(\omega_f) = \alpha \cdot \omega^k$ and

$$f(E/\mathbb{C}, P_N, \omega) = \alpha \in \mathbb{C}.$$

We will identify the two spaces. Via this identification, a cuspform f is a modular form whose q-expansions lie in $q^{1/N} \cdot \mathbb{C}[[q^{1/N}]]$.

1.6 Galois representations

In this section we recall the basic definitions and the main properties of Galois representations. In particular, in §1.6.4 we introduce some of the Galois modules that will appear in this thesis: the compatible system of representations attached to classical modular forms of weight ≥ 2 and the Artin representation attached to weight one modular forms. In particular, in §1.6.3.1 we will focus on the p-adic representation attached to elliptic curves with multiplicative reduction at p, that we will need in Chapter 4. Moreover, in all chapters 2, 3 and 4 we will make an extensive use of the representations of $G_{\mathbb{Q}}$ induced by Hecke characters on G_K for some K imaginary quadratic field. These are the representations attached to modular forms which are theta series (cf. §1.5.4), and we introduce these notions in §1.6.2 and §1.6.4.5.

Definition 1.25. A *Galois representation* is a continuous homomorphism

$$\rho : G_K \longrightarrow \mathrm{GL}(V),$$

where K is a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_p$ for some p, and V is a finite-dimensional vector space over some field L.

i) The *dimension* of ρ is defined as the dimension of V over L.

ii) If $K/\mathbb{Q}_p$ is a local field with prime $\mathfrak{p}$ over p, we say that we say that ρ is *unramified* if it is trivial when restricted to the inertia group I_K of G_K.

iii) If $K/\mathbb{Q}$ is a number field and $\mathfrak{p}$ is a prime ideal of K, we say that ρ is *unramified at* $\mathfrak{p}$ if ρ is trivial when restricted to the inertia group $I_{\mathfrak{p}} \hookrightarrow G_{K_{\mathfrak{p}}} \hookrightarrow G_K$ at $\mathfrak{p}$. In other words, if $\rho_{|G_{K_{\mathfrak{p}}}}$ is unramified.

iv) For a rational prime p, we say that ρ is *unramified at* p if it is unramified at all primes of K lying above p.

Notice that, if a Galois representation ρ is unramified at a prime $\mathfrak{p}$ of K, then it is well defined the linear transformation $\rho(\mathrm{Frob}_{\mathfrak{p}})$. More in general, we can always consider the action of $\mathrm{Frob}_{\mathfrak{p}}$ restricted to the elements of V fixed by the inertia group at $\mathfrak{p}$, i.e. the linear transformation $\rho(\mathrm{Frob}_{\mathfrak{p}})|V^{I_{\mathfrak{p}}}$. We will be dealing mainly with two kinds of Galois representations, namely

1. *Artin representations*:
$$\rho : G_K \longrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C}),$$
where V is a $\mathbb{C}$-vector space of dimension n. A direct consequence of the continuity of ρ is the fact that $\rho(G_K) \cong G_K/\ker\rho$ is finite, so it factors through $\mathrm{Gal}(H/K)$, where $H := \left(\bar{K}\right)^{\ker\rho}$ is a finite extension of $\mathbb{Q}$. Moreover, ρ is unramified at almost all primes and it is semisimple.

2. *ℓ-adic representations* (for a prime ℓ):
$$\rho : G_K \longrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n(L),$$
where L a finite extension of $\mathbb{Q}_\ell$ and V is a L-vector space of dimension n. An ℓ-adic representation may ramify at infinitely many primes, and it may be not semisimple. If λ is the prime ideal of L above ℓ, then ρ is also called *λ-adic representation.*

Both Artin and semisimple ℓ-adic representations ρ are uniquely determined by the set

$$\{\mathrm{Tr}(\rho(\mathrm{Frob}_p)) \mid p \notin \Sigma\}$$

where Σ is a finite set of primes containing the primes at which ρ is ramified.

Definition 1.26. A *compatible system of λ-adic representations* is a collection

$$\{\rho_\lambda : G_K \longrightarrow \mathrm{GL}(V_\lambda)\}_\lambda$$

of λ-adic representations, where

i) K and L are number fields;

ii) λ varies over all prime ideals of L;

iii) V_λ is a vector space over the completion L_λ of L at λ,

satisfying the following conditions. For a prime λ of L with residue characteristic ℓ, let S_λ be the set of primes of K lying above ℓ.

i) There exists a finite set of primes S of K such that, for all λ, the representation ρ_λ is unramified outside $S \cup S_\lambda$;

ii) for every prime ideal $\mathfrak{p}$ of K not belonging to S, the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_\mathfrak{p})|V_\lambda$ belongs to $L[x]$ and depends only on $\mathfrak{p}$ and not on λ, as λ varies over all primes of L such that $\mathfrak{p} \notin S_\lambda$.

The set S above is called the *ramification set* for $\{\rho_\lambda\}_\lambda$.

Remark 1.27. In particular, if we fix $\mathfrak{p} \notin S$, the degree of the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_\mathfrak{p})$ is constant as λ varies. In other words, $\dim_{L_\lambda} V_\lambda$ does not depend on λ.

For a prime $\mathfrak{p}$ of K, denote $q_\mathfrak{p} := \mathrm{N}_K(\mathfrak{p}) := \#(\mathcal{O}_K/\mathfrak{p})$.

Definition 1.28. A compatible system of λ-adic representation is *pure of weight w* if, for all $\mathfrak{p} \notin S$, all the roots of the characteristic polynomial of $\rho_\lambda(\mathrm{Frob}_\mathfrak{p})$ have complex absolute value $q_\mathfrak{p}^{w/2}$.

All compatible systems of representations we will consider in this thesis (e.g. *Tate modules* of abelian varieties, representations attached to modular forms, étale cohomology of some variety, Artin representations) are pure of some weight, and this condition implies the convergence of the complex L-function attached to $\{V_\lambda\}_\lambda$ in some right half-plane of $\mathbb{C}$.

Remark 1.29. Every Artin representation

$$\rho : G_K \longrightarrow \mathrm{GL}(V)$$

can be regarded as a compatible system of λ-adic representations as follows. As explained above, there exist number fields H and L such that

$$\rho : \mathrm{Gal}(H/K) \longrightarrow \mathrm{GL}(V) \cong \mathrm{GL}_n(L).$$

For a prime ideal λ of L, let L_λ be the corresponding completion and use the inclusions $L \hookrightarrow L_\lambda$ to construct

$$\{\rho_\lambda : G_K \longrightarrow \mathrm{GL}(V \otimes L_\lambda) \cong \mathrm{GL}_n(L_\lambda)\}_\lambda,$$

which is a compatible system of λ-adic representations.

1.6.1 The cyclotomic character and Tate twists

An example of ℓ-adic representation of dimension 1 is the ℓ-adic *cyclotomic character*, defined as follows. For all integer n, denote by $\mu_{\ell^n} \subseteq \bar{\mathbb{Q}}$ the group of ℓ^n-th roots of unity. Choose a compatible system of primitive ℓ-roots of unity, i.e. a generator ξ_{ℓ^n} of μ_{ℓ^n} for every n, such that $(\xi_{\ell^n})^{\ell} = \xi_{\ell^{n-1}}$. Recall that, for any n, there is an isomorphism

$$\chi_n : \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q}) \longrightarrow (\mathbb{Z}/\ell^n\mathbb{Z})^{\times} ,$$

where for every $\sigma \in \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^n})/\mathbb{Q})$, the element $\chi_n(\sigma)$ is defined by the equality

$$\sigma(\xi_{\ell^n}) = (\xi_{\ell^n})^{\chi_n(\sigma)} .$$

These $(\chi_n)_n$ are compatible, and the *cyclotomic character* $\chi_{\mathrm{cyc},\ell}$ is defined as the projective limit

$$\chi_{\mathrm{cyc},\ell} : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_{\ell^\infty})/\mathbb{Q}) \overset{\varprojlim \chi_n}{\longrightarrow} \mathbb{Z}_\ell^{\times} \hookrightarrow \mathbb{Q}_\ell^{\times} = \mathrm{GL}_1(\mathbb{Q}_\ell),$$

where $\mu_{\ell^\infty} := \cup_n \mu_{\ell^n}$. We denote $\mathbb{Q}_\ell(1)$ the one dimensional ℓ-adic representation on which $G_{\mathbb{Q}}$ acts via $\chi_{\mathrm{cyc},\ell}$. More in general, $\mathbb{Q}_\ell(n)$ will be the ℓ-adic representation given by $\chi^n_{\mathrm{cyc},\ell}$, and for any ℓ-adic representation V, we will denote the *Tate twist*

$$V(n) := V \otimes_{\mathbb{Q}_\ell} \mathbb{Q}_\ell(n). \tag{1.6.1}$$

Notice that $\{\mathbb{Q}_\ell(n)\}_\ell$ is a compatible system of ℓ-adic representations with set of ramification $S = \emptyset$. Finally, if $V = \{V_\ell\}_\ell$ is a compatible system, we denote

$$V(n) := \{V_\ell(n)\}_\ell,$$

which is a compatible system of ℓ-adic representations with the same ramification set of primes than V.

1.6.2 Induced representations

Let $F \subseteq K$ be number fields. We now recall how a G_K-representation *induces* a representation of G_F. Moreover, we recall Shapiro's Lemma (stated as Proposition 1.31), which describes the relation between the cohomology of the two representations, and that we will use in Chapter 4.

Let L be a number field and let

$$\rho : G_K \longrightarrow \mathrm{Aut}(V) \cong \mathrm{GL}_n(L)$$

be a Galois representation. Denote $m := [K : F]$.

Definition 1.30. The G_F-representation induced by ρ

$$\mathrm{Ind}_K^F(\rho) : G_F \longrightarrow \mathrm{Aut}(\mathrm{Ind}_K^F(V)) \cong \mathrm{GL}_{nm}(L)$$

is given, as a L-vector space, by

$$\mathrm{Ind}_K^F(V) := \{x : G_F \longrightarrow V \mid x(\sigma\tau) = \rho(\sigma)(x(\tau)) \ \forall \sigma \in G_K, \tau \in G_F\}.$$

The action of G_F on $\mathrm{Ind}_K^F(V)$ is defined as follows. If $x \in \mathrm{Ind}_K^F(V)$ and $g \in G_F$, then

$$x^g := \mathrm{Ind}_K^F(\rho)(g)(x) \in \mathrm{Ind}_K^F(V)$$

is defined by

$$x^g(\tau) := x(\tau g) \ \ \forall \sigma \in G_K.$$

Note that there is a natural map

$$\pi : \mathrm{Ind}_K^F(V) \longrightarrow V, \qquad x \mapsto x(1).$$

It is a homomorphism of $L[G_K]$-modules and it induces an isomorphism between V and

$$\{x \in \mathrm{Ind}_K^F(V) \mid x(\tau) = 0 \ \forall \tau \in G_F \smallsetminus G_K\}.$$

For a G_K-representation V and a G_F representation W, there is an isomorphism

$$\mathrm{Hom}_{G_F}(\mathrm{Ind}_K^F(V), W) \cong \mathrm{Hom}_{G_K}(V, W_{|G_K}) \tag{1.6.2}$$

called *Frobenius reciprocity*, where $W_{|G_K}$ denotes W regarded as a G_K-representation by restricting the action.

Proposition 1.31 (Shapiro's Lemma)**.** *There is an isomorphism*

$$\mathrm{Sh} : \mathrm{H}^1(F, \mathrm{Ind}_K^F(V)) \xrightarrow{\cong} \mathrm{H}^1(K, V).$$

Proof. See [Neu99, §8.1]. □

Explicitly, Shapiro's isomorphism is defined as follows. Let $[\xi] \in \mathrm{H}^1(F, \mathrm{Ind}_K^F(V))$ be represented by the cocycle

$$\xi : G_F \longrightarrow \mathrm{Ind}_K^F(V).$$

Then $\mathrm{Sh}([\xi]) \in \mathrm{H}^1(K, V)$ is represented by

$$\mathrm{Sh}(\xi) : G_K \longrightarrow V,$$

which is defined by

$$\mathrm{Sh}(\xi)(\sigma) := \xi(\sigma)(1) = \pi(\xi(\sigma)) \in V \qquad \forall \sigma \in G_K.$$

We describe now an equivalent description of the induced representation which will also be useful in this thesis. Choose a complete set of representatives $\{g_1, \ldots, g_m\}$ of the left cosets of G_K inside G_F, and consider the formal

$$\bigoplus_{i=1}^m g_i V = \left\{ \sum_{i=0}^m g_i v_i \mid v_i \in V \right\},$$

on which we define an action of G_F as follows. For $g \in G_F$ and for $i \in \{1, \ldots, m\}$, let $h_i \in G_K$ such that

$$g \cdot g_i = g_{j(i)} h_i,$$

where $j(i) \in \{1, \ldots, m\}$. Then we define

$$g \cdot \sum_{i=0}^m g_i v_i = \sum_{i=0}^m g_{j(i)} \cdot \rho(h_i)(v_i).$$

Then there is an isomorphism of $L[G_F]$-modules

$$\mathrm{Ind}_K^F(V) \cong \bigoplus_{i=1}^m g_i V.$$

1.6.3 Representations attached to elliptic curves: Tate modules

We recall the definition and the main properties of the ℓ-adic representation given by the Tate module of an elliptic curve. Moreover, we focus on elliptic curves with multiplicative reduction as it is the setting of Chapter 4.

Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$ of conductor N. For a prime ℓ and an integer n, denote $E[\ell^n] := E(\bar{\mathbb{Q}})[\ell^n]$ the set of ℓ^n-torsion points of E, which is a group isomorphic to $\mathbb{Z}/\ell^n\mathbb{Z}$.

Definition 1.32. The *ℓ-adic Tate module* of E is

$$T_\ell E := \varprojlim_n E[\ell^n] \cong \mathbb{Z}_\ell^2.$$

The action of $G_\mathbb{Q}$ on $E(\bar{\mathbb{Q}})$ extends to an action on $T_\ell E$, so it defines an ℓ-adic representation

$$\rho_{E,\ell} : G_\mathbb{Q} \longrightarrow \mathrm{Aut}(T_\ell E) \cong \mathrm{GL}_2(\mathbb{Z}_\ell) \subseteq \mathrm{GL}_2(\mathbb{Q}_\ell)$$

which is irreducible, is unramified at all primes $p \nmid N\ell$, and for these primes, $\rho_{E,\ell}(\mathrm{Frob}_p)$ has characteristic polynomial

$$x^2 - a_p(E)x + p, \tag{1.6.3}$$

where $a_p(E) := 1 + p - E_{\mathbb{F}_p}(\mathbb{F}_p)$. Here we are using the fact that E has an integral model over $\mathbb{Z}_p$ (given by a minimal Weierstrass equation at p), so that we can consider the base change

$$E_{\mathbb{F}_p} := E \times_{\mathbb{Z}_p} \mathbb{F}_p.$$

In particular, for every $p \nmid N\ell$, we have $\mathrm{Tr}\,\rho_{E,\ell}(\mathrm{Frob}_p) = a_p(E)$.

Notice that the polynomial (1.6.3) belongs to $\mathbb{Z}[x]$ and it is independent on ℓ. In other words, $\{T_\ell E\}_\ell$ form a compatible system of ℓ-adic representations with, in the notation of Definition 1.26, $S = \{p \mid N\}$.

Definition 1.33. An elliptic curve $E/\mathbb{Q}$ is *ordinary* at a at a prime p if the reduction $E_{\mathbb{F}_p}$ is ordinary, i.e. if $E_{\mathbb{F}_p}[p] \cong \mathbb{F}_p$, which is equivalent to the fact that $p \nmid a_p(E)$. Otherwise, we say that E is *supersingular* at p, in which case $E_{\mathbb{F}_p}[p] = 0$ (and $p \mid a_p(E)$).

Remark 1.34.
1. The terminology in the above definition is equivalent to the one introduced in Definition 1.18 via modularity Theorem 1.40.
2. Let $p \nmid N\ell$ and factor the polynomial (1.6.3) in $\bar{\mathbb{Q}}_p[x]$ as

$$x^2 - a_p(E)x + p = (x-\alpha)(x-\beta).$$

Then E is ordinary at p if and only if one of the roots is a p-adic unit.

1.6.3.1 The case of elliptic curves of multiplicative reduction

Let $E/\mathbb{Q}_p$ be an elliptic curve of multiplicative reduction at a prime p. Let

$$\chi_E : G_{\mathbb{Q}_p} \longrightarrow \{\pm 1\}$$

be the trivial character if E has split multiplicative reduction over $\mathbb{Q}_p$, and the quadratic unramified character if E has non-split multiplicative reduction. For any $G_{\mathbb{Q}_p}$-module M, let $M(\chi_E)$ denote the twist of M by χ_E.

Tate's uniformisation provides a $G_{\mathbb{Q}_p}$-equivariant isomorphism

$$\varphi_{\mathrm{Tate}} : \bar{\mathbb{Q}}_p^\times(\chi_E)/q_E^\mathbb{Z} \xrightarrow{\cong} E(\bar{\mathbb{Q}}_p) \tag{1.6.4}$$

for some $q_E \in p\mathbb{Z}_p$. As in (4.0.4), assume throughout that $p \nmid n := \mathrm{ord}_p(q_E)$.

Let $T_E = T_pE := \varprojlim E[p^m]$ denote the Tate module associated to E and set $V_E := T_E \otimes \mathbb{Q}_p$. Let $E_0(\bar{\mathbb{Q}}_p)$ denote the set of points in $E(\bar{\mathbb{Q}}_p)$ that stay nonsingular in the special fiber of E at p. The module

$$T_E^+ := \varprojlim E_0[p^m]$$

fits in an exact sequence of $\mathbb{Z}_p[G_{\mathbb{Q}_p}]$-modules

$$0 \longrightarrow T_E^+ \xrightarrow{\iota} T_E \xrightarrow{\pi} T_E^- \longrightarrow 0, \tag{1.6.5}$$

where $T_E^- := T_E/T_E^+$. The modules T_E^+ and T_E^- are free over $\mathbb{Z}_p$ of rank one, and Tate's uniformization (1.6.4) induces $G_{\mathbb{Q}_p}$-equivariant isomorphisms

$$\mathbb{Z}_p(\chi_E)(1) \xrightarrow{\cong} T_E^+, \qquad \mathbb{Z}_p(\chi_E) = \varprojlim_m (\bar{\mathbb{Q}}_p^\times(\chi_E)/q_E^{\mathbb{Z}})[p^m]/\mu_{p^m}(\chi_E) \xrightarrow{\cong} T_E^-. \tag{1.6.6}$$

More precisely, if we fix compatible systems

$$\tilde{\varepsilon} := (\tilde{\varepsilon}^{(m)})_m \in \mathbb{Z}_p(\chi_E)(1), \qquad \tilde{q} := (q_E^{1/p^m})_m \in \varprojlim \left(\bar{\mathbb{Q}}_p^\times/q_E^{\mathbb{Z}}\right)[p^m]$$

then

$$\varepsilon := \varphi_{\mathrm{Tate}}(\tilde{\varepsilon}),\ q := \varphi_{\mathrm{Tate}}(\tilde{q})$$

form a $\mathbb{Z}_p$-basis of T. Moreover, if $\chi_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \mathbb{Z}_p^\times$ denotes the cyclotomic character, then

1. ε is a basis of T_E^+ on which $G_{\mathbb{Q}_p}$ acts as the character $\chi_E\chi_{\mathrm{cyc}}$;
2. $\bar{q} := \pi(q)$ is a basis of T_E^- on which $G_{\mathbb{Q}_p}$ acts via χ_E.

Let K be an imaginary quadratic field in which p is inert and let K_p denote the completion of K at p. Notice that $\chi_{E|G_{K_p}} = 1$ and thus (1.6.5) gives an exact sequence of $\mathbb{Q}_p[G_{K_p}]$-modules

$$0 \longrightarrow V_E^+ \xrightarrow{\iota} V_E \xrightarrow{\pi} V_E^- \longrightarrow 0 \tag{1.6.7}$$

such that $\dim_{\mathbb{Q}_p} V_E^+ = \dim_{\mathbb{Q}_p} V_E^- = 1$, G_{K_p} acts on V_E^+ via χ_{cyc}, and G_{K_p} acts trivially on V_E^-.

1.6.4 Representations attached to modular forms

The Galois representation attached to a normalised newform of weight k is described by the following theorem due to Eichler and Shimura for $k = 2$, Deligne for $k \geq 2$ and Deligne and Serre for $k = 1$.

Theorem 1.35. *Let $f \in S_k(N, \chi)$ be a normalised newform with q-expansion $f = \sum_{n\geq 1} a_n(f)q^n$ and let $K_f := \mathbb{Q}(\{a_n(f)\}_n)$ be the number field generated by the Fourier coefficients of f. For any prime number ℓ, there is an irreducible 2-dimensional odd representation*

$$\rho_{f,\ell} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell \otimes K_f)$$

which is unramified outside $N\ell$, and such that, for every prime $p \nmid N\ell$, the characteristic polynomial of $\rho_{f,\ell}(\mathrm{Frob}_p)$ is

$$x^2 - a_p(f)x + \chi(p)p^{k-1}. \tag{1.6.8}$$

Since

$$K_f \otimes \mathbb{Q}_\ell = \prod_{\lambda|\ell} K_{f,\lambda}, \tag{1.6.9}$$

for each prime ideal λ of K_f dividing ℓ, we can attach to f a 2-dimensional ℓ-adic representation

$$\rho_{f,\lambda} : G_\mathbb{Q} \longrightarrow \mathrm{Aut}_{K_{f,\lambda}}(V_{f,\lambda}) \cong \mathrm{GL}_2(K_{f,\lambda})$$

which is irreducible, unramified outside $N\ell$ and such that the characteristic polynomial $\rho_{f,\lambda}(\mathrm{Frob}_p)$ is the image of (1.6.8) under the map $K_f \longrightarrow K_{f,\lambda}$ obtained by (1.6.9). We call $\rho_{f,\ell}$ and $\rho_{f,\lambda}$ respectively the *ℓ-adic and the λ-adic representation attached to f* and we call (1.6.8) the p-th Hecke polynomial of f. In particular, Theorem 1.35 shows that

$$V_f = \{V_{f,\lambda}\}_\lambda$$

form a compatible system of λ-adic representations with ramification in the set of primes dividing the level of f.

If f has weight 2, then Shimura's construction attaches to f an abelian variety $A_f/\mathbb{Q}$ such that $\rho_{f,\lambda}$ is defined as the Tate module of f. In particular, when $K_f = \mathbb{Q}$, then A_f is an elliptic curve and this construction coincides with §1.6.3. Moreover, by modularity theorem (cf. Theorem 1.40) the Tate module of any elliptic curve over $\mathbb{Q}$ is equivalent to V_f for some weight 2 cuspform f. We recall these facts in §1.6.4.2. As a preliminary, we introduce the Jacobian of the modular curve and set some notation that we will need in the following. In §1.6.4.3 we recall how to describe the representation attached to cuspforms of weight ≥ 2 as (a subquotient of) the étale cohomology of the so-called Kuga–Sato variety. Finally, in §1.6.4.5 we state the results characterising the Artin representation attached to a modular form of weight one, that we can regard as a compatible system of λ-adic representations as described in Remark 1.6.3.

1.6.4.1 The Jacobian of the modular curve

Let $X := X_1(N)$ be the modular curve (although most of the definitions and properties stated in this section also hold for more general irreducible projective curves), regarded as a smooth scheme over $\mathbb{Z}[1/N]$. The group of *divisors* on X is the abelian free group

$$\mathrm{Div}(X) := \oplus_{P\in X}\mathbb{Z} \cdot P$$

generated by the points of X. For a rational function f on X, the *divisor of f* is

$$\mathrm{div}(f) := \sum_P \mathrm{ord}_P(f) \cdot P \in \mathrm{Div}(X),$$

where $\mathrm{ord}_P(f)$ denotes the order of vanishing of f at P. Such element is called a *principal* divisor. There is a *degree* map

$$\deg : \mathrm{Div}(X) \longrightarrow \mathbb{Z}, \qquad \sum_n n_P \cdot P \mapsto \sum_n n_P.$$

We denote $\mathrm{Div}(X)^0$ the kernel of this map, i.e. the subgroup of degree zero divisors of $\mathrm{Div}(X)$. Since X is a compact Riemann surface, every principal divisor has degree 0 so

$$\mathrm{Princ}(X) := \{\text{principal divisors}\}$$

is a subgroup of $\mathrm{Div}(X)^0$.

Definition 1.36. The *Picard* group of X is

$$\mathrm{Pic}(X) := \mathrm{Div}(X)/\mathrm{Princ}(X).$$

Moreover, we denote

$$\mathrm{Pic}^0(X) := \mathrm{Div}(X)^0/\mathrm{Princ}(X).$$

Definition 1.37. The *Jacobian of* $X = X_1(N)$ is

$$J(X) := J_1(N) := \mathrm{H}^0(X_1(N), \Omega_{X_1(N)})^\vee / \mathrm{H}_1(X_1(N), \mathbb{Z}).$$

By (1.5.12), the Jacobian of $X_1(N)$ is strictly related to the space of weight 2 cuspforms of level N. We describe it explicitly, and we also describe the relation between $J_1(N)$ and the Picard group of $X_1(N)$. Let g be the genus of $X_1(N)$, choose a basis $\{f_1, \dots, f_g\}$ for $S_2(N)$ made of normalised cuspforms with Fourier coefficients in $\mathbb{Z}$, and denote $\{\omega_1, \dots, \omega_g\}$ the corresponding dual basis of

$$\mathrm{Hom}_{\mathbb{C}}(S_2(N), \mathbb{C}) = \mathrm{H}^0(X_1(N), \Omega_{X_1(N)})^\vee \cong \mathbb{C}^g,$$

where we are using the isomorphism (1.5.12). On the other hand,

$$\mathrm{H}_1(X_1(N), \mathbb{Z}) \hookrightarrow \mathrm{H}^0(X_1(N), \Omega_{X_1(N)})^\vee \cong \mathbb{C}^g$$

is a lattice of rank $2g$. Fixing a basis $\{c_1, \dots, c_{2g}\}$ for $\mathrm{H}_1(X_1(N), \mathbb{Z})$, the image in $\mathbb{C}^g$ is the lattice $\Lambda \subseteq \mathbb{C}^g$ generated by $\{\left(\int_{c_1} \omega_1, \dots, \int_{c_1} \omega_g\right), \dots, \left(\int_{c_{2g}} \omega_1, \dots, \int_{c_{2g}} \omega_g\right)\}$. Via these assignations, we have

$$J(X) \cong \mathbb{C}^g/\Lambda. \tag{1.6.10}$$

Finally, fixing $\tau_0 \in X_1(N)$, there is an isomorphism

$$\Phi : \mathrm{Pic}^0(X_1(N)) \xrightarrow{\cong} J_1(N) \cong \mathbb{C}^g/\Lambda, \qquad \tau - \tau_0 \mapsto \left(\int_{\tau_0}^{\tau} \omega_1, \dots, \int_{\tau_0}^{\tau} \omega_g\right). \tag{1.6.11}$$

As the modular curve, also $\mathrm{Pic}^0(X_1(N))$, via Φ, has a model defined over $\mathbb{Q}$, and if we choose $\tau_0 \in X_1(N)(\mathbb{Q})$, the morphism

$$X_1(N) \longrightarrow \mathrm{Pic}^0(X_1(N)), \qquad \tau \mapsto \tau - \tau_0$$

is an embedding defined over $\mathbb{Q}$. Notice that $X_1(N)$ does not have group structure, while $\mathrm{Pic}^0(X_1(N)) \cong J_1(N)$ is an abelian variety. Moreover, it has good reduction outside N. Recall that, as described in §1.5.3, the Hecke algebra $\mathfrak{h}(N)$ acts on $X_1(N)$. This action induces, by linearity, an action on the Jacobian $J_1(N)$,

1.6.4.2 Shimura's construction and the modularity theorem

Let $f \in S_2(N, \chi)$ be a normalised newform with q-expansion $f(q) = \sum_n a_n(f) q^n$. Notice that χ is a Dirichlet character modulo N, but it always can be regarded as a character

$$\chi : G_{\mathbb{Q}} \twoheadrightarrow \mathrm{Gal}(\mathbb{Q}(\mu_N)/\mathbb{Q}) \cong (\mathbb{Z}/N\mathbb{Z})^\times \xrightarrow{\chi} \mathbb{C}^\times$$

such that $\chi(\mathrm{Frob}_p) = \chi(p)$ for all $p \nmid N$, and $\chi(\text{complex conjugation}) = \chi(-1) = 1$. Denote by K_f the finite extension of $\mathbb{Q}$ generated by the Fourier coefficients of f. Let $\mathfrak{h}(N)$ be the

$\mathbb{Z}$-algebra generated by the Hecke operators acting on cuspforms of weight 2 and level N, and for any ring A denote $\mathfrak{h}(N)_A := \mathfrak{h}_k(N) \otimes A$. The cuspform f defines an homomorphism

$$\lambda_f : \mathfrak{h}(N)_{\mathbb{Q}} \longrightarrow K_f \tag{1.6.12}$$

given by $\lambda_f(T_n) = a_n(f)$, $\lambda_f(\langle d \rangle) = \chi(d)$ for $(n, N) = 1$ and $\lambda_f(U_p) = a_p(f)$ for $p \mid N$. Define the following ideal of $\mathfrak{h}(N)$

$$I_f := \ker(\lambda_f) \cap \mathfrak{h}(N).$$

Recall that $\mathfrak{h}(N)$ acts on the Jacobian $J_1(N)$. Then we can consider $I_f J_1(N)$, which is an abelian subvariety of $J_1(N)$ defined over $\mathbb{Q}$.

Definition 1.38. The abelian variety attached to f is

$$A_f := J_1(N)/I_f J_1(N).$$

It is an abelian variety of dimension $[K_f : \mathbb{Q}]$. For any embedding $\sigma : K_f \hookrightarrow \mathbb{C}$, let $f^\sigma := \sum_n \sigma(a_n(f))q^n$. It is a normalised newform of level N, weight 2 and Nebentype character χ^σ. The abelian variety A_f only depends on the conjugacy class

$$\underline{f} := \{f^\sigma \mid \sigma : K_f \hookrightarrow \mathbb{C}\}.$$

Remark 1.39. Notice that, if the newform f has trivial Nebentype character and rational Fourier coefficients, then A_f is an elliptic curve.

The representation $V_{f,\ell}$ is then given by the $\mathbb{Q}_\ell$-dual of $V_{A_f} := T_\ell A_f \otimes \mathbb{Q}_\ell$, where $T_\ell A_f := \varprojlim_n A[\ell^n]$ is the Tate module of A_f. This produces the required representation

$$\rho_{f,\lambda} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\mathbb{Q}_\ell \otimes K_f).$$

More precisely, the compatibility with the action of the Hecke algebra on A_f (and then the expression (1.6.8) for the Hecke polynomial) is given by the so-called Eichler–Shimura relation and by the isomorphism

$$\lambda_f : \mathfrak{h}(N)_{\underline{f}} \otimes \mathbb{Q}_\ell \xrightarrow{\cong} K_f \otimes \mathbb{Q}_\ell$$

induced by (1.6.12), where $\mathfrak{h}(N)_{\underline{f}}$ is the quotient of $\mathfrak{h}(N)$ determined by the isomorphism above.

Conversely, we have the following modularity theorem, which was conjectured by Shimura and Taniyama in 1957 and has been proved by Breuil, Conrad, Diamond, Taylor and Wiles building on the techniques introduced in [Wil95] and [TW95].

Theorem 1.40. *Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$ of conductor N. There exists a normalised newform $f \in S_2(N)$ with trivial Nebentype character and rational Fourier coefficients such that*

$$\rho_{E,\ell} \sim \rho_{f,\ell}$$

for all ℓ.

Here the symbol $\sim$ denotes equivalence of representations: explicitly, the statement above says that for all prime ℓ there exists a matrix $M_\ell \in \mathrm{GL}_2(\mathbb{Q}_\ell)$ such that

$$M_\ell^{-1} \rho_{E,\ell}(\sigma) M_\ell = \rho_{f,\ell}(\sigma) \qquad \text{for all } \sigma \in G_{\mathbb{Q}}.$$

In general it is not always true that all abelian varieties defined over $\mathbb{Q}$ are *modular*: it has been proved by Ribet and by Khare and Wintenberger that the abelian varieties which are modular are exactly those of GL_2-*type*, i.e. those varieties $A/\mathbb{Q}$ for which $\mathrm{End}_{\mathbb{Q}}(A)$ is a number field of degree equal to the dimension of A.

1.6.4.3 Representations attached to modular forms of weight $k \geq 2$: étale cohomology

The Galois representation attached to the newform $f \in S_2(N,\chi)$ described in the previous section, can be reinterpreted by using étale cohomology. Indeed, from the definition of the Jacobian variety one obtains an Hecke-equivariant isomorphism $T_\ell J_1(N) = \varprojlim_n J_1(N)[\ell^n] \cong \mathrm{H}_1(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Z}_\ell)$, which induces the isomorphism

$$(T_\ell J_1(N) \otimes \mathbb{Q}_\ell)^\vee \cong \mathrm{H}^1(X_1(N), \mathbb{Q}_\ell) \cong \mathrm{H}^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell). \tag{1.6.13}$$

Here the ℓ-adic étale cohomology of $X_1(N)_{\bar{\mathbb{Q}}}$ can be endowed with a Hecke action, which is compatible the isomorphism above. For a newform $f \in S_2(N,\chi)$, the isomorphism (1.6.13) in turn induces

$$V_f = V_{A_f}^\vee \cong e_f \cdot \mathrm{H}^1(X_1(N), \mathbb{Q}_\ell) \cong e_f \cdot \mathrm{H}^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell),$$

where e_f is the projector onto the Hecke-eigenspace of f. This projector belongs to the ring of correspondences of $X_1(N)$ and it is constructed from Hecke correspondences as in [Sch90].

This cohomological interpretation generalises and permits to construct the ℓ-adic representation attached to newforms of more general weight. Let $k \geq 2$ be an integer and let ℓ be a prime number. Similarly as in (1.5.16), we define

$$\mathcal{F}_{k-2,\ell} := \iota_* \mathrm{Sym}^{k-2} \mathrm{R}^1 \pi_* \mathbb{Q}_\ell$$

with respect to the maps $\mathcal{E} \xrightarrow{\pi} Y_1(N) \xhookrightarrow{\iota} X_1(N)$, which are defined over $\mathbb{Q}$. Then $\mathfrak{h}(N)_{\mathbb{Q}_\ell}$ acts on $\mathrm{H}^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathcal{F}_{k-2,\ell})$ which is a $\mathfrak{h}(N)_{\mathbb{Q}_\ell}$-module of rank 2. Notice that for $k = 2$ we recover (1.6.13). Similarly to the weight 2-case, for a normalised cuspform f of weight $k \geq 2$ and level N, we recover its system of $\mathfrak{h}(N)$-eigenvalues in the representation $\mathrm{H}^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathcal{F}_{k-2,\ell})$, and if e_f is the projector onto the corresponding eigenspace, then the ℓ-adic representation attached to f is

$$V_f := e_f \cdot \mathrm{H}^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathcal{F}_{k-2,\ell}).$$

In Chapter 2, we will use an alternative description of V_f in terms of the étale cohomology of a higher dimensional scheme fibered over $X_1(N)$, which we now recall.

Definition 1.41. For an integer $r \geq 0$, the *Kuga–Sato variety* W_r is the canonical desingularisation of the r-th fibered product $\mathcal{E} \times_{X_1(N)} \overset{(r)}{\cdots} \times_{X_1(N)} \mathcal{E}$.

Recall from §1.5.1 that the fibers of $\pi : \mathcal{E} \to X_1(N)$ are elliptic curves. By definition, the generic point of W_r is of the form $(x; P_1, \ldots P_r)$ where $x \in X_1(N)$ and $P_1, \ldots, P_r$ belongs to the fiber $\pi^{-1}(x)$. The variety W_r has dimension $r+1$ and it is defined over $\mathbb{Q}$; for a detailed description see the appendix of [BDP13]. As explained in [Sch90, §4], there is an isomorphism

$$\mathrm{H}^{k-1}_{\mathrm{et}}(W_{k-2,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(\varepsilon_k) \xrightarrow{\cong} \mathrm{H}^1_{\mathrm{et}}(X_1(N)_{\bar{\mathbb{Q}}}, \mathcal{F}_{k-2,\ell}) \tag{1.6.14}$$

of $\mathfrak{h}(N)_{\mathbb{Q}_\ell}$-modules, where ε_k is a quadratic character. More precisely, for $p \nmid N$, the Hecke operator T_p acts on the left hand side of (1.6.14) as described in [Sch90, 4.0.2], so the projector e_f can be regarded as a correspondence of W_{k-2}, and

$$V_f = e_f \cdot \mathrm{H}^{k-1}_{\mathrm{et}}(W_{k-2,\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)(\varepsilon_k).$$

1.6.4.4 The filtration of the representation attached to an ordinary modular form

Let $f \in S_k(N, \chi)$ be a newform of weight $k \geq 2$, level N and Nebentype character χ, and denote by V_f the Galois representation attached to f. Fix a prime p be a prime not dividing the level N and assume that f is ordinary at p. Let α_f, β_f be the eigenvalues for the action of Frob_p on V_f. Since f is ordinary, one of the eigenvalues, say α_f, is a p-adic unit. As a $G_{\mathbb{Q}_p}$-representation, V_f is equipped with a decreasing filtration

$$\mathrm{Fil}^2(V_f) = 0 \subseteq \mathrm{Fil}^1(V_f) \subseteq V_f = \mathrm{Fil}^0(V_f). \tag{1.6.15}$$

Let

$$\psi_f : G_{\mathbb{Q}_p} \longrightarrow L_p^\times$$

be the unramified character such that $\psi_f(\mathrm{Frob}_p) = \alpha_f$, where L_p is a suitable finite extension of $\mathbb{Q}_p$ containing the Fourier coefficients of f. The filtration above satisfies the following properties:

1. $V_f^+ := \mathrm{Fil}^1(V_f)$ and $V_f^- := V_f/V_f^+$ are $\mathbb{Q}_p$-vector spaces of dimension one;
2. $G_{\mathbb{Q}_p}$ acts on V_f^+ via $\chi_{\mathrm{cyc}}^{k-1}\chi\psi_f^{-1}$;
3. $G_{\mathbb{Q}_p}$ acts on V_f^- via ψ_f. In particular, it is an unramified $G_{\mathbb{Q}_p}$-module.

1.6.4.5 Artin representations attached to weight one modular forms

The case of modular forms of weight one is degenerate with respect to what we have described in the previous sections, in the sense that the Galois representation attached to a weight one modular forms does not arise in the étale cohomology of some variety defined over $\mathbb{Q}$ as in the case of weight greater than one. Despite this, Deligne and Serre proved in [DS74] that one can attach to a modular form of weight one an Artin representation.

Theorem 1.42. *Let $f \in M_1(N, \chi)$ be a weight one eigenform with q-expansion $f = \sum a_n(f)q^n$. Then there exists a semisimple Artin representation $\rho_f : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\mathbb{C}}(V_f^\circ) \cong \mathrm{GL}_2(\mathbb{C})$ unramified outside N and such that, for all prime $p \nmid N$, the characteristic polynomial of $\rho_f(\mathrm{Frob}_p)$ is*

$$x^2 - a_p(f)x + \chi(p).$$

Moreover, ρ_f is irreducible if and only if f is cuspidal.

By Remark 1.29 we can regard this Artin representation as a compatible system of λ-adic representations. Moreover, the λ-adic representation attached to a weight one modular form can also be described by using the theory of Hida families introduced in §1.7, as explained in §1.7.8.

The case of theta series The construction of the Galois representation attached to a weight one modular form is not as explicit as in the case of modular forms of higher weights. Nevertheless, in the special setting of the Theta series of a Hecke character, one can describe its Artin representation in a simple way, that will be crucially exploited in chapters 2, 3 and 4. Using the notation of §1.5.4, let ψ be a finite order Hecke character of an imaginary quadratic field K and let $f := \theta(\psi) \in M_1(N, \chi)$. Then the Artin representation attached to f is

$$V_f^\circ = \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$$

where $\mathrm{Ind}_K^{\mathbb{Q}}(\psi)$ denotes the representation induced from G_K to $G_{\mathbb{Q}}$ by ψ described in §1.6.2.

1.7 Hida families

The theory of Hida families was introduced by Hida in [Hid86] and [Hid93] and concerns the p-adic interpolation of *ordinary* modular forms. Roughly speaking, a *Hida family* is a continuous collection $\mathbf{f} = (\mathcal{W} \ni k \mapsto \mathbf{f}_k)$ of formal power series, where $\mathcal{W}$ is a suitable p-adic space called *weight space*, which contains $\mathbb{Z}$ as a dense subset and such that, for a dense set of integers k, $\mathbf{f}_k$ is the q-expansion of a (classical) eigenform of weight k. At non-integer weights $k \in \mathcal{W}$, we can still consider $\mathbf{f}_k$ as a *p-adic* modular form, whose definition is given in §1.7.2. In particular, we will be dealing with a subclass of p-adic modular form which are called *overconvergent* modular forms, which we introduce in §1.7.4.

1.7.1 Weight spaces

Let p be an odd prime and let $\Gamma := 1 + p\mathbb{Z}_p$.

Definition 1.43. The *Iwasawa algebra* is $\Lambda := \mathbb{Z}_p[[\Gamma]] := \varprojlim_n \mathbb{Z}_p[\Gamma/\Gamma^n]$.

Let $x = x_0 + x_1 p + x_2 p^2 + \cdots \in \mathbb{Z}_p$, with $a_i \in \{0\} \cup \mu_{p-1}$ for all $i \geq 0$. Recall that $x \in \mathbb{Z}_p^\times$ if and only if $x_0 \neq 0$ so there is an isomorphism

$$\mu_{p-1} \oplus (1 + p\mathbb{Z}_p) \xrightarrow{\cong} \mathbb{Z}_p^\times \qquad (a, b) \mapsto ab. \tag{1.7.1}$$

The *Teichmuller character* is the homomorphism

$$\omega : \mathbb{Z}_p^\times \longrightarrow \mu_{p-1}$$

given by the composition of the inverse of the isomorphism (1.7.1) and the projection onto the first component.

Definition 1.44. The *weight space* (attached to Λ) is the formal scheme $\mathcal{W} := \mathrm{Spf}(\Lambda)$.

The weight space $\mathcal{W}$ is characterised by the property that, for any p-adic ring A, the set of A-rational points of $\mathcal{W}$ is

$$\mathcal{W}(A) = \mathrm{Hom}_{\mathbb{Z}_p-\mathrm{alg}}(\Lambda, A) \cong \mathrm{Hom}_{\mathrm{cont}}(\Gamma, A^\times).$$

Given an integer $k \geq 2$ and a Dirichlet character ϵ of conductor a power of p, we can construct a point $\nu_{k,\epsilon} \in \mathcal{W}(\mathbb{C}_p)$ by

$$\nu_{k,\epsilon}(x) := \epsilon(x) x^k \ \text{ for all } x \in \Lambda.$$

In particular, we can embed the set $\mathbb{Z}_{\geq 2}$ in $\mathcal{W}(\mathbb{Z}_p)$ via $k \mapsto \nu_{k,1}$, and we will identify k with $\nu_{k,1}$. Moreover, this set is dense inside $\mathcal{W}(\mathbb{Z}_p)$.

Definition 1.45. *i)* A point $\nu \in \mathcal{W}(\mathbb{C}_p)$ is called *classical* if there exist a Dirichlet character ϵ and an integer $k \geq 2$ such that $\nu = \nu_{k,\epsilon}$. In this case, the integer k is called the *weight* of ν.

ii) A classical point is called *cristalline* if it is of the form ν_{k,ω^k}.

We will denote respectively $\mathcal{W}^\circ$ and $\mathcal{W}^{\mathrm{cl}}$ the subsets of cristalline and classical points of $\mathcal{W}$.

We can attach a weight space to any finite flat extension Λ_0 of Λ by defining $\mathcal{W}_0 := \mathrm{Spf}(\Lambda_0)$. The inclusion $\Lambda \subseteq \Lambda_0$ induces by functoriality a *weight map*

$$\kappa : \mathcal{W}_0 \longrightarrow \mathcal{W}.$$

Definition 1.46. A point $\nu \in \mathcal{W}_0(\mathbb{C}_p)$ is called *classical* (or *cristalline*) if $\kappa(\nu)$ is a classical (or cristalline) point of $\mathcal{W}$. In this case, the *weight* of ν is defined as the weight of $\kappa(\nu)$.

We will use the notation $\mathcal{W}_0^\circ \subseteq \mathcal{W}_0^{\mathrm{cl}} \subseteq \mathcal{W}_0$ for the sets of cristalline and classical points of $\mathcal{W}$.

1.7.2 p-adic modular forms: Serre's definition

Fix and integer $N \geq 3$, a prime number $p \geq 3$ such that $p \nmid N$ and a cusp of $X_1(N)$. For a modular form $f \in M_k(N)_{\mathbb{Z}_p}$, denote $f(q) \in \mathbb{Z}_p[[q^{1/N}]]$ the q-expansion of f at the fixed cusp. The ring $\mathbb{Z}_p[[q^{1/N}]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a Banach space with respect to the norm $|g| := \inf\{n \mid p^n g \in \mathbb{Z}_p[[q^{1/N}]]\}$.

Theorem 1.47 (Serre). *Let $(k_n)_{n\geq 0}$ be a sequence of non-negative integers; for each $n \geq 0$, let $f_n \in M_{k_n}(N)_{\mathbb{Z}_p}$ such that*

$$f_{n+1}(q) \equiv f_n(q) \mod p^n, \quad \textit{and} \quad f_n(q) \not\equiv 0 \mod p^n.$$

Then the sequence $(k_n)_n$ seen in $\mathcal{W}(\mathbb{Z}_p)$ converges to an element $k \in \mathcal{W}(\mathbb{Z}_p)$. Moreover, k is independent of $(f_n)_n$ and only depends on the limit

$$f(q) := \lim_{n\to\infty} f_n(q) \in \mathbb{Z}_p[[q^{1/N}]] \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

Definition 1.48. A *p-adic modular form* of level N is a power series obtained as the limit of classical modular forms as in the previous theorem. In the notation of Theorem 1.47, the point $k \in \mathcal{W}(\mathbb{Z}_p)$ is called the *weight* of the p-adic modular form.

We denote $M_k^{(p)}(N)_{\mathbb{Q}_p}$ the space of such forms. The action of Hecke operators extends to an action on $M_k^{(p)}(N)_{\mathbb{Q}_p}$ and on $S_k^{(p)}(N)_{\mathbb{Q}_p}$, whose action on q-expansions coincides with (1.5.6).

1.7.3 p-adic modular forms as sections

Analogously to what explained in §1.5.5 for classical modular forms, also the space of p-adic modular forms can be interpreted as a cohomology space, as we now explain.

Recall that $X_1(N)$ is a scheme over $\mathbb{Z}[1/N]$ and that $p \nmid N$. Then we can consider the base change $X_1(N)_{\mathbb{Z}_p}$, which is proper, so there is a reduction map

$$\mathrm{red} : X_1(N)_{\mathbb{Z}_p}(\mathbb{C}_p) \longrightarrow X_1(N)_{\mathbb{F}_p}(\overline{\mathbb{F}}_p).$$

Recall from §1.6.3 that an elliptic curve E defined over $\overline{\mathbb{F}}_p$ is *ordinary* if $p \nmid a_p(E) := 1 + p - \#E(\mathbb{F}_p)$ and it is *supersingular* otherwise. Using the moduli interpretation of the modular curve, let Z be the subset of $X_1(N)_{\mathbb{F}_p}(\bar{\mathbb{F}}_p)$ of points representing supersingular elliptic curves. It is a finite set of closed points of $X_1(N)_{\mathbb{F}_p}(\bar{\mathbb{F}}_p)$, and the inverse image $\mathrm{red}^{-1}(Z)$ is a finite union of rigid analytic open disks.

Definition 1.49. The *ordinary locus* of $X_1(N)$ is $X_1(N)^{\mathrm{ord}} := X_1(N) \smallsetminus \mathrm{red}^{-1}(Z)$.

In the spirit of §1.5.5, we have identifications

$$M_k^{(p)}(N)_{\mathbb{C}_p} = \mathrm{H}^0(X_1(N)^{\mathrm{ord}}_{\mathbb{C}_p}, \underline{\omega}^k) \cong \mathrm{H}^0(X_1(N)^{\mathrm{ord}}_{\mathbb{C}_p}, \underline{\omega}^{k-2} \otimes \Omega_{X_1(N)/\mathbb{C}_p}(\log))$$

and

$$S_k^{(p)}(N)_{\mathbb{C}_p} = \mathrm{H}^0(X_1(N)^{\mathrm{ord}}_{\mathbb{C}_p}, \underline{\omega}^{k-2} \otimes \Omega_{X_1(N)/\mathbb{C}_p})$$

for all $k \in \mathbb{Z}$. These spaces are infinite-dimensional, so they do not have a nice spectral theory for the Hecke operators. Recall in particular the operators U_p and V_p on $S_k^{(p)}(N)$. They act on q-expansions $f(q) = \sum a_n(f)q^n$ as

$$U_p f(q) = \sum_{n\geq 1} a_{pn}(f)q^n, \qquad V_p f(q) = \sum_{n\geq 1} a_p(f)q^{pn}.$$

They satisfy

$$U_p V_p f = f, \qquad V_p U_p f(q) = \sum_{n\geq 1} a_{np}(f)q^{pn}.$$

In terms of these operators we define the *p-depletion* $f^{[p]}$ of f as

$$f^{[p]}(q) := (1 - V_p U_p) f(q) = \sum_{p\nmid n} a_n(f)q^n. \tag{1.7.2}$$

1.7.4 Overconvergent modular forms

As mentioned in the previous section, the space of p-adic modular forms of a given weight is infinite dimensional. Taking sections of suitable open neighborhoods of the ordinary locus, one obtains a better behaved subspace of $M_k^{(p)}(N)$ which is still infinite dimensional but on which U_p acts as a compact operator. This is the subspace $M_k^{\mathrm{oc}}(N)$ of *overconvergent modular forms*, and the operator U_p has then a nice spectral theory on it. In particular, $M_k^{\mathrm{oc}}(N)$ is equipped with an *ordinary projector* cutting out the subspace on which U_p acts as a p-adic unit. This *ordinary* subspace is finite dimensional, and its elements are the modular forms used to interpolate p-adically ordinary eigenforms in Hida families.

In order to define the neighborhoods of $X_1(N)^{\mathrm{ord}}$ we are interested in, we need to introduce the so-called *Hasse invariant*. Let R be a ring of characteristic $p \geq 3$, let E be an elliptic curve defined over R and let $F : E \longrightarrow E$ be the absolute Frobenius, defined as $f \mapsto f^p$ on affine open subsets of E. Given an R-basis ω of $\mathrm{H}^0(E, \Omega_{E/R})$, we get a dual basis $\omega^\vee$ of $\mathrm{H}^1(E, \mathcal{O}_E)$ via Serre's duality.

Definition 1.50. The *Hasse invariant* of (E, ω) is the scalar $\mathrm{Ha}(E/R, \omega) \in R$ such that

$$F^*(\omega^\vee) = \mathrm{Ha}(E/R, \omega) \cdot \omega^\vee.$$

As explained in §1.5.7, the assignment $(E/R, \omega) \mapsto \mathrm{Ha}(E/R, \omega)$ defines a modular form

$$\mathrm{Ha} \in M_{p-1}(1)_{\mathbb{F}_p}$$

whose q-expansion is $\mathrm{Ha}(q) = 1$ and such that $\mathrm{Ha}(E/R, \omega) = 0$ if and only if E is supersingular. Given $N \geq 3$ such that $p \nmid N$, the Hasse invariant can be regarded as a modular form

$$\mathrm{Ha} \in M_{p-1}(N)_{\mathbb{F}_p} = \mathrm{H}^0(X_1(N)_{\mathbb{F}_p}, \underline{\omega}_{\mathbb{F}_p}^{p-1})$$

of level N, wieight $p-1$ and coefficients in $\mathbb{F}_p$, and it is characterised by its q-expansion $\mathrm{Ha}(q) = 1$. Moreover, it admits a characteristic zero lift

$$\widetilde{\mathrm{Ha}} \in M_{p-1}^{(p)}(N) = \mathrm{H}^0(X_1(N)^{\mathrm{ord}}, \underline{\omega}^k),$$

which is a modular form with coefficients in $\mathbb{Z}_p$ such that $\widetilde{\mathrm{Ha}}(q) \equiv 1 \mod p$. Since $\widetilde{\mathrm{Ha}}$ only vanishes on elliptic curves with supersingular reduction, $\widetilde{\mathrm{Ha}}$ is invertible as a p-adic modular form. If $p > 3$, we can choose the lift of the Hasse invariant to be the Eisenstein series $\widetilde{\mathrm{Ha}} = E_{p-1}$.

Definition 1.51. For $0 \leq r \leq 1$, define $X_1(N)^{\mathrm{ord}} \subseteq X_1(N)^r \subseteq X_1(N)$ such that

$$X_1(N)^r(\mathbb{C}_p) := \{x \in X_1(N)(\mathbb{C}_p) \mid \mathrm{ord}_p(\widetilde{\mathrm{Ha}}(x)) \leq r\}.$$

Notice that $X_1(N)^0 = X_1(N)^{\mathrm{ord}}$.

Definition 1.52. Let k be a positive integer. The space of *overconvergent modular forms* of level N, weight k and *degree of overconvergence* r is

$$M_k^{\mathrm{oc}}(N, r) := \mathrm{H}^0(X_1(N)^r, \underline{\omega}^k) = \mathrm{H}^0(X_1(N)^r, \underline{\omega}^{k-2} \otimes \Omega_{X_1(N)}(\log)).$$

The corresponding space of cuspforms is

$$S_k^{\mathrm{oc}}(N, r) := \mathrm{H}^0(X_1(N)^r, \underline{\omega}^{k-2} \otimes \Omega_{X_1(N)}).$$

For every $0 \leq r \leq 1$ we have the inclusions

$$M_k^{\mathrm{oc}}(N, r) \subseteq M_k^{(p)}(N), \qquad S_k^{\mathrm{oc}}(N, r) \subseteq S_k^{(p)}(N)$$

All these spaces are infinite-dimensional vector spaces, and $M_k^{\mathrm{oc}}(N, r)$ and $M_k^{\mathrm{oc}}(N, r)$ are $\mathbb{C}_p$-Banach spaces with respect to a suitable norm. They inherit an action of the Hecke operators T_ℓ and U_q for every $\ell \nmid Np,\ q \mid N$, but the action of U_p is not preserved. More precisely,

$$U_p : M_k^{\mathrm{oc}}(N, r) \longrightarrow M_k^{\mathrm{oc}}(N, rp).$$

Notice that, if $r < R$, then $M_k^{\mathrm{oc}}(N, r) \subseteq M_k^{\mathrm{oc}}(N, R)$.

Definition 1.53. Define the space of *overconvergent* modular forms and cuspforms of weight k and (tame) level N as

$$M_k^{\mathrm{oc}}(N) := \varinjlim_{r>0} M_k^{\mathrm{oc}}(N, r), \quad \text{and} \quad S_k^{\mathrm{oc}}(N) := \varinjlim_{r>0} S_k^{\mathrm{oc}}(N, r).$$

These are infinite-dimensional $\mathbb{C}_p$-Banach spaces, but U_p acts as a compact operator on both spaces, so it admits a discrete spectrum of non-zero eigenvalues with their corresponding generalised eigenvectors. Hida defined the operator on $M_k^{\mathrm{oc}}(N)$

$$e_{\mathrm{ord}} := \lim_{n \to \infty} U_p^{n!}$$

called the *ordinary projector* such that

$$e_{\mathrm{ord}} \cdot M_k^{\mathrm{oc}}(N) := M_k^{\mathrm{oc,ord}}(N) \subseteq M_k^{\mathrm{oc}}(N)$$

is the subspace generated by ordinary overconvergent eigenforms, i.e. (generalised) eigenforms whose U_p-eigenvalue is a p-adic unit.

Definition 1.54. We call the *slope* of an eigenform $f \in M_k^{\mathrm{oc}}(N)$ the p-adic valuation of its U_p-eigenvalue.

An ordinary eigenform is then an eigenform of slope 0, and there is an embedding

$$M_k(Np)_{\mathbb{C}_p} \hookrightarrow M_k^{\mathrm{oc}}(N).$$

One can also apply the ordinary projection to the space of classical modular forms, obtaining

$$e_{\mathrm{ord}} \cdot M_k(Np) =: M_k^{\mathrm{ord}}(Np) \subseteq M_k(Np)$$

the subspace of ordinary modular forms.

Theorem 1.55 (Hida, Coleman)**.** *The space $M_k^{\mathrm{oc,ord}}(N)$ is finite dimensional. Moreover, for $k \geq 2$, any eigenform $f \in M_k^{\mathrm{oc}}(N)$ of slope $< k-1$ is classical. In particular, if $k \geq 2$, then*

$$M_k^{\mathrm{oc,ord}}(N) = M_k^{\mathrm{ord}}(Np)_{\mathbb{C}_p}.$$

Moreover, for $k > 2$, the dimension of $S_k^{\mathrm{ord}}(Np)$ is independent on k.

1.7.4.1 Serre's differential operator and overconvergent primitives

On the space of p-adic modular forms acts the so-called *Serre's operator*

$$\mathrm{d} : M_k^{(p)}(N) \longrightarrow M_{k+2}^{(p)}(N)$$

which acts on q-expansion as $\mathrm{d} := q\frac{\mathrm{d}}{\mathrm{d}\,q}$. In general, it does not conserve overconvergence, but it does respect so-called *nearly overconvergent* modular forms, which we do not define here since we will not need them in this thesis. The reader can find the definition of nearly overconvergent modular forms and proofs of the facts stated in this section in [DR14, §2.4].

Although in general Serre's operator does not respect overconvergence, one can show that $\mathrm{d} : S_0^{\mathrm{oc}}(N) \longrightarrow S_2^{\mathrm{oc}}(N)$. More in general, for all $k \in \mathbb{Z}_{\geq 0}$, Serre's operator gives a well-defined map

$$\mathrm{d}^{k+1} : S_{-k}^{\mathrm{oc}}(N) \longrightarrow S_{k+2}^{\mathrm{oc}}(N).$$

Let $f \in S_2(N)_{\mathbb{Q}}$ be a weight 2 cuspform with rational Fourier coefficients and recall its p-depletion $f^{[p]}(q)$ defined in §1.7.3. There exists an overconvergent weight zero cuspform $F \in S_0^{\mathrm{oc}}(N)_{\mathbb{Q}}$ such that

$$\mathrm{d}\, F = f.$$

The cuspform F is called the *overconvergent primitive of* f and has q-expansion

$$F(q) = \sum_{p \nmid n} \frac{a_n(f)}{n} q^n.$$

The existence of the overconvergent primitive of f follows from the fact that it is possible to interpolate p-adically the integral powers of d, so

$$F = \mathrm{d}^{-1} f^{[p]} = \lim_{t \to -1} \mathrm{d}^t f^{[p]}.$$

1.7.5 Hida families: definition and main properties

Let N be a positive integer such that $p \nmid N$ and let $\chi : (\mathbb{Z}/N\mathbb{Z})^\times \longrightarrow \mathbb{C}_p^\times$ be a Dirichlet character.

Definition 1.56. A *Hida family* of tame level N and tame Nebentype character χ is a triple $\mathbf{f} = (\Lambda_{\mathbf{f}}, \mathcal{W}_{\mathbf{f}}, \mathbf{f})$, where

i) $\Lambda_{\mathbf{f}}$ is a finite flat extension of Λ;

ii) $\mathcal{W}_{\mathbf{f}}$ is a rigid analytic open subvariety of $\mathrm{Spf}(\Lambda_{\mathbf{f}})$;

iii) $\mathbf{f} = \sum a_n(\mathbf{f}) q^n \in \Lambda_{\mathbf{f}}[[q]]$ is a formal series such that, for each $\nu \in \mathcal{W}_{\mathbf{f}}^{\mathrm{cl}}$ with $\kappa(\nu) = \nu_{k,\epsilon}$ for some ϵ of conductor p^r, the *specialisation at* ν

$$\mathbf{f}_\nu := \sum_{n=1}^{\infty} \nu(a_n(\mathbf{f})) q^n$$

is the q-expansion of a classical p-ordinary eigenform of weight k, level Np^r and Nebentype character $\chi\varepsilon\omega^{-k}$. We call $\mathbf{f}_\nu$ the *specialisation of* $\mathbf{f}$ *at* ν and we say that $\mathbf{f}$ *passes through* $\mathbf{f}_\nu$.

We will denote by $S_{\Lambda_{\mathbf{f}}}^{\mathrm{ord}}(N, \chi)$ the set of such Hida families.

Note that if we restrict to the set of cristalline points $\mathcal{W}_{\mathbf{f}}^{\circ}$, then all the specialisations of a $\mathbf{f}$ have Nebentype character χ.

Let $f \in S_k(N, \chi)$ be a normalised newform. Since $p \nmid N$ the characteristic polynomial for the Frobenius at p acting on the representation V_f is

$$x^2 - a_p(f)x + \chi(p)p^{k-1} = (x - \alpha_f)(x - \beta_f) \in \bar{\mathbb{Q}}[x].$$

We will always label these roots so that $\mathrm{ord}_p(\alpha_f) \leq \mathrm{ord}_p(\beta_f)$.

Definition 1.57. The modular forms $f_\alpha, f_\beta \in S_k(Np, \chi)$ defined by

$$f_\alpha(z) = f(z) - \beta_f f(pz), \qquad f_\beta(z) = f(z) - \alpha_f f(pz)$$

are called *p-stabilisations of f*.

The stabilisations above satisfy $U_p f_\alpha = \alpha_f \cdot f_\alpha$ and $U_p f_\beta = \beta_f \cdot f_\beta$. Notice that, if f is ordinary at p, then α_f is a p-adic unit, and then f_α is ordinary. In the case in which f has weight one, then both α_f and β_f are p-adic units and f has two ordinary p-stabilisations.

Remark 1.58. In the notation of Definition 1.56, let ν be a cristalline point of $\mathcal{W}_{\mathbf{f}}$ of weight $k \geq 2$. Since p does not divide the level of χ, if ν has weight $k > 2$, then by [How07, Lemma 2.1.5], the specialisation $\mathbf{f}_\nu$ is olt at p; we will denote $f_\nu \in S_k(N, \chi)$ the newform whose p-stabilisation if $\mathbf{f}_\nu$. If $k = 2$, then $\mathbf{f}_\nu$ can be either old or new. In this case we denote $f_\nu := \mathbf{f}_\nu$ if it is new, while, if $\mathbf{f}_\nu$ is old at p, we denote f_ν the newform whose p-stabilisation is $\mathbf{f}_\nu$.

Given a p-ordinary eigenform, it is natural to wonder if there is a Hida family passing through it and whether this family is unique. We collect in the following theorem some results in this direction.

Theorem 1.59. *Let $f \in S_k(Np, \chi)$ be a p-ordinary, p-stabilised newform of level Np such that $p \nmid N$, then there exists a Hida family $\mathbf{f}$ of tame level N whose specialization at a cristalline point of weight k is f. Moreover,*

i) if f has weight $k \geq 2$, or

ii) if f has weight one and is regular (cf. Assumption 2.3),

then the Hida family passing through f is unique.

Proof. For weights $k \geq 2$, Hida proved the existence and uniquess of $\mathbf{f}$ in [Hid86]. The existence of the Hida family in the weight one case is proved in [Wil88]. Finally, the uniqueness of $\mathbf{f}$ in case *ii*) is a theorem of [BD16]. □

We will need the following more general definition of p-adic families.

Definition 1.60. A Λ-adic modular form of tame level N is a triple $\mathbf{f} = (\Lambda_{\mathbf{f}}, \mathcal{W}_{\mathbf{f}}, \mathbf{f})$, where

i) $\Lambda_{\mathbf{f}}$ is a complete, finitely generated and flat extension of Λ;

ii) $\mathcal{W}_{\mathbf{f}}$ is a rigid analytic open subvariety of $\mathrm{Spf}(\Lambda_{\mathbf{f}})$;

iii) $\mathbf{f} = \sum a_n(\mathbf{f})q^n \in \Lambda_{\mathbf{f}}[[q]]$ is a formal series such that, for all $\nu \in \mathcal{W}_{\mathbf{f}}^{\mathrm{cl}}$, the specialisation at ν

$$\mathbf{f}_\nu := \sum_{n \geq 1} \nu(a_n(\mathbf{f}))q^n \in \mathbb{C}_p[[q]]$$

is the q-expansion of a classical ordinary cuspform in $S_{\kappa(\nu)}(\Gamma_1(N) \cap \Gamma_0(p))_{\mathbb{C}_p}$ (not necessarily an eigenform).

1.7.6 Hida families of theta series

Let K be an imagirary quadratic field and let ψ be a Hecke character of K of infinity type $(0, \ell_0 - 1)$, with $\ell_0 \geq 1$, and conductor $\mathfrak{c}$. In §1.5.4 we explained how construct the theta series $g := \theta(\psi) \in S_{\ell_0}(N, \chi)$. Fix a prime number p not dividing N and assume that it splits in K as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. Recall that we call α_g a fixed p-unit root of the p-th Hecke polynomial of g (which is unique if $\ell_0 \geq 2$), and g_α the corresponding p-stabilisation. There is a Hida family $\mathbf{g}$ of theta series passing through g_α, whose construction can be found in [Gha05, §5]. We next recall the specialisations at integer weights of this family.

Fix a Hecke character λ of K with infinity type $(0,1)$ and conductor $\bar{\mathfrak{p}}$. Let $\mathbb{Q}_p(\lambda)$ be the field obtained by adjoining to $\mathbb{Q}$ the values of λ and taking the p-adic completion. Consider the factorisation of its group of units

$$\langle \cdot \rangle : \mathcal{O}_{\mathbb{Q}_p(\lambda)}^\times \longrightarrow W.$$

For each $\ell \in \mathbb{Z}_{\geq 0}$ such that $\ell \equiv \ell_0 \mod (p-1)$, define

$$\psi_{\ell-1}^{(p)} := \psi \langle \lambda \rangle^{\ell - \ell_0} \text{ and } \psi_{\ell-1}(\mathfrak{q}) := \begin{cases} \psi_{\ell-1}^{(p)}(\mathfrak{q}) & \mathfrak{q} \neq \bar{\mathfrak{p}}; \\ \chi(p) p^{\ell-1} / \psi_{\ell-1}^{(p)}(\mathfrak{p}) & \mathfrak{q} = \bar{\mathfrak{p}}. \end{cases}$$

Then $\psi_{\ell-1}$ is a Hecke character of infinity type $(0, \ell-1)$. Define

$$g_\ell := \theta(\psi_{\ell-1}) \in S_\ell(N, \chi).$$

Then g_ℓ is the newform whose p-stabilisation is the theta series

$$\mathbf{g}_\ell = \theta(\psi_{\ell-1}^{(p)}) \in S_\ell(Np, \chi),$$

which is the weight ℓ specialisation of the Hida family $\mathbf{g}$.

1.7.7 Representations attached to Hida families

Let $\mathbf{f} = (\Lambda_{\mathbf{f}}, \mathcal{W}_{\mathbf{f}}, \mathbf{f})$ be a Hida family of tame level N and tame character χ. There is a Galois representation

$$\rho_{\mathbf{f}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}_{\Lambda_{\mathbf{f}}}(V_{\mathbf{f}})$$

where $V_{\mathbf{f}}$ is a locally free $\Lambda_{\mathbf{f}}$-module of rank 2 such that, for any cristalline point $\nu \in \mathcal{W}_{\mathbf{f}}^\circ$ of weight $k \geq 2$,

$$V_{\mathbf{f}} \otimes_{\Lambda_{\mathbf{f}}, \nu} L_p \cong V_{f_\nu},$$

where L_p is a the finite extension of $\mathbb{Q}_p$ generated by the Fourier coefficients of f_ν. Moreover, as $G_{\mathbb{Q}_p}$-module, $V_{\mathbf{f}}$ is equipped with a decreasing filtration

$$\cdots = \mathrm{Fil}^3(V_{\mathbf{f}}) = \mathrm{Fil}^2(V_{\mathbf{f}}) = 0 \subseteq \mathrm{Fil}^1(V_{\mathbf{f}}) \subseteq V_{\mathbf{f}} = \mathrm{Fil}^0(V_{\mathbf{f}}) = \mathrm{Fil}^{-1}(V_{\mathbf{f}}) = \cdots$$

such that $V_{\mathbf{f}}^+ := \mathrm{Fil}^1(V_{\mathbf{f}})$ and $V_{\mathbf{f}}^- := V_{\mathbf{f}}/V_{\mathbf{f}}^+$ are locally free $\Lambda_{\mathbf{f}}$-modules of rank one and $V_{\mathbf{f}}^-$ is unramified. More precisely, $G_{\mathbb{Q}_p}$ acts on $V_{\mathbf{f}}^-$ via the unramified character

$$\psi_{\mathbf{f}} : G_{\mathbb{Q}_p} \longrightarrow \Lambda_{\mathbf{f}}^\times$$

characterised by $\psi_{\mathbf{f}}(\mathrm{Frob}_p) = a_p(\mathbf{f})$. Moreover, let

$$\boldsymbol{\psi}_{\mathrm{cyc}} : G_{\mathbb{Q}} \longrightarrow \Lambda^\times$$

be the Λ-adic character characterised by $\nu_{k,\epsilon} \circ \boldsymbol{\psi}_{\mathrm{cyc}} = \epsilon \cdot \chi_{\mathrm{cyc}}^{k-1} \cdot \omega^{-k}$ for all cristalline weight $\nu_{k,\epsilon} \in \mathcal{W}^\circ$. Then $G_{\mathbb{Q}_p}$ acts on $V_{\mathbf{f}}^+$ via the character $\psi_{\mathbf{f}}^{-1} \cdot \chi \cdot \boldsymbol{\psi}_{\mathrm{cyc}}$. The representation $V_{\mathbf{f}}$ is constructed by Hida in [Hid86] packing together the Galois representations of the classical specialisations of $\mathbf{f}$.

1.7.8 p-adic representations attached to weight one modular forms

Let f be a weight one normalised newform of level N and Nebentype character χ and let

$$\rho_f : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_f^\circ) \cong \mathrm{GL}_2(L)$$

be the Artin representation attached to it as described in §1.6.4.5, where H and L are number fields. Let p be a prime number such that $p \nmid N$ and assume that f is ordinary at p. Fix once and for all a prime $\mathfrak{p}$ of H and a prime $\mathfrak{P}$ of L above p and denote the corresponding completion by $H_p := H_\mathfrak{p}$ and $L_p := L_\mathfrak{P}$. Let α_f, β_f be the eigenvalues for the action of Frob_p on V_f°, which are p-adic units. Fix a p-stabilisation f_α and let $\mathbf{f}$ be the Hida family passing through it. This can be regarded a s a power series $\mathbf{f} \in \Lambda_\mathbf{f}[[q]]$, where $\Lambda_\mathbf{f}$ is a finite flat extension of the Iwasawa algebra Λ with the property that, if we denote by $y_f : \Lambda_\mathbf{f} \longrightarrow L_p$ the weight corresponding to f_α then $y_f(\mathbf{f}) = f_\alpha$. Let $\rho_\mathbf{f} : G_\mathbb{Q} \longrightarrow \mathrm{Aut}(V_\mathbf{f}) \cong \mathrm{GL}_2(\Lambda_\mathbf{f})$ be the repesentation attached to $\mathbf{f}$. The *p-adic representation attached to* f is defined as

$$V_f := V_\mathbf{f} \otimes_{\Lambda_\mathbf{f}, y_f} L_p.$$

There is a non-canonical isomorphism

$$V_f^\circ \otimes_L L_p \xrightarrow{\cong} V_f. \tag{1.7.3}$$

1.8 Filtrations and differentials attached to Galois representations of modular forms

We now define two differentials attached to each modular form f, that we will use in chapters 2, 3 and 4.

1.8.1 Differentials for modular forms of weight $k \geq 2$

Let $f \in S_k(N, \chi)$ be a normalised newform of weight $k \geq 2$. Fix a prime p at which f is ordinary and assume that p does not divide N, unless f is the newform attached to an elliptic curve, in which case we allow p to divide N at most once. Let L be the number field generated by the Fourier coefficients of f and fix a completion L_p of L at a prime above p. Recall the p-adic Galois respresentation

$$V_f = e_f \cdot \mathrm{H}^{k-1}_{\mathrm{et}}((W_{k-2})_{\bar{\mathbb{Q}}}, \mathbb{Q}_p) \otimes L_p$$

attached to f as in §1.6.4.3. Let α_f, β_f be the eigenvalues for the action of Frob_p on V_f, labeled so that α_f is a p-unit. Recall that, if $p \nmid N$, there is an exact sequence of $L_p[G_{\mathbb{Q}_p}]$-modules

$$0 \longrightarrow V_f^+ \longrightarrow V_f \longrightarrow V_f^- \longrightarrow 0. \tag{1.8.1}$$

where $\dim_{L_p} V_f^\pm = 1$ and $G_{\mathbb{Q}_p}$ acts on the unramified quotient V_f^- as multiplication by α_f. Finally, in the case in which f is attached to an elliptic curve with multiplicative reduction at p, we also have a sequence of the form (1.8.1) as explained in §1.6.3.1.

Recall that the de Rham realisation

$$e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p) = (M_f)_{\mathrm{dR}}$$

is equipped with a 3-steps Hodge filtration

$$e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p) = \mathrm{Fil}^0(M_f)_{\mathrm{dR}} \supset \mathrm{Fil}^1(M_f)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^{k-1}(M_f)_{\mathrm{dR}} \supset \mathrm{Fil}^k(M_f)_{\mathrm{dR}} = 0. \tag{1.8.2}$$

Moreover, there are isomorphisms

$$\mathrm{Fil}^1(M_f)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^{k-1}(M_f)_{\mathrm{dR}} \cong e_f \cdot S_k(N)_{L_p}. \tag{1.8.3}$$

The filtration (1.8.2) has a splitting given by the *unit-root subspace*

$$e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)^{\mathrm{u-r}} \subseteq e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p),$$

i.e. the subspace on which Frob_p acts as the multiplication by the p-adic unit α_f. Notice that this is the p-adic analogous of the complex Hodge splitting of (1.5.14), (1.5.13).

Definition 1.61. Define

$$\omega_f \in \mathrm{Fil}^{k-1}\, e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)_{L_p} \tag{1.8.4}$$

as the element of $(M_f)_{\mathrm{dR}}$ corresponding to f via the isomorphism (1.8.3), and

$$\eta_f \in e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)^{\mathrm{u-r}}_{L_p} \tag{1.8.5}$$

the element of $(M_f)_{\mathrm{dR}}$ such that $\langle \eta_f, \omega_f \rangle = 1$.

The pairing we are considering in the definition above is the Poincaré pairing between $\mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)$ and $\mathrm{H}^{2(k-2)-(k-1)}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p) = \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)$, which induces

$$\langle\ ,\ \rangle : e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)^{\mathrm{u-r}}_{L_p} \times \mathrm{Fil}^{k-1}\, e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)_{L_p} \longrightarrow L_p.$$

It will be useful in the next chapter to regard the differentials ω_f and η_f as elements in $\mathrm{D}_{\mathrm{dR}}(V_f^+)$ and $\mathrm{D}_{\mathrm{dR}}(V_f^-)$ as we now explain. Recall that, by Theorem 1.7, there is an isomorphism

$$\mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p) \cong \mathrm{D}_{\mathrm{dR}}(\mathrm{H}^{k-1}_{\mathrm{et}}((W_{k-2})_{\bar{\mathbb{Q}}}, \mathbb{Q}_p))$$

compatible with the Hodge filtration, which induces

$$(M_f)_{\mathrm{dR}} = e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)_{L_p} \cong \mathrm{D}_{\mathrm{dR}}(V_f). \tag{1.8.6}$$

More precisely, via (1.8.6), we have

$$\mathrm{Fil}^{k-1}\, e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/\mathbb{Q}_p)_{L_p} \cong \mathrm{D}_{\mathrm{dR}}(V_f^-), \tag{1.8.7}$$

and we will still call ω_f the image of (1.8.4) in $\mathrm{D}_{\mathrm{dR}}(V_f^-)$. Denote $f^* := f \otimes \chi^{-1} \in S_k(N, \chi^{-1})$. As explained in §1.4, we have a pairing

$$\langle\ ,\ \rangle : V_f \otimes V_{f^*} \longrightarrow L_p(1) \tag{1.8.8}$$

which in turn induces perfect pairings

$$\langle\ ,\ \rangle : \mathrm{D}_{\mathrm{dR}}(V_f) \times \mathrm{D}_{\mathrm{dR}}(V_{f^*}) \longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p) = L_p$$

and

$$\langle\ ,\ \rangle : \mathrm{D}_{\mathrm{dR}}(V_f^+) \times e_{f^*} \cdot S_k(N)_{L_p} \longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p) = L_p \tag{1.8.9}$$

(cf. (1.4.4) and (1.4.5)). Here the second pairing is obtained from the first one combined with the isomorphisms (1.8.7), (1.8.6) and (1.8.3). The differential (1.8.5) then can be regarded via (1.8.6) as the element

$$\eta_f \in \mathrm{D}_{\mathrm{dR}}(V_f^+) \subseteq \mathrm{D}_{\mathrm{dR}}(V_f)$$

characterised by

$$\langle \eta_f, \omega_{f^*} \rangle = 1$$

via the pairing (1.8.9).

1.8.2 Differentials for modular forms of weight 1

Now let g be a weight one modular form of level N_g and Nebentype character χ_g and assume that $p \nmid N_g$. Let L be a number field containing the Fourier coefficients of g and fix a completion L_p at a prime ideal lying above p. The p-adic representation V_g attached to g as in §1.7.8 is then unramified at p and we assume from now on that Frob_p acts on it with distinct eigenvalues α_g and β_g. Denote V_g^α, V_g^β be the corresponding eigenspaces and fix the p-stabilisation g_α of g such that $U_p g_\alpha = \alpha_g \cdot g_\alpha$. As explained in §1.7.5, the theory of Hida families ensures the existence of a Hida family $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, where $\Lambda_{\mathbf{g}}$ is a finite flat extension of the Iwasawa algebra $\Lambda := \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$, with the property that, if we denote by $y_g : \Lambda_{\mathbf{g}} \longrightarrow L_p$ the weight corresponding to g, then $y_g(\mathbf{g}) = g_\alpha$. Let $\rho_{\mathbf{g}} : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V_{\mathbf{g}}) \cong \mathrm{GL}_2(\Lambda_{\mathbf{g}})$ be the Λ-adic representation attached to $\mathbf{g}$ as in §1.7.7. Recall that, as a $G_{\mathbb{Q}_p}$-representation, $V_{\mathbf{g}}$ is equipped with a filtration of $\Lambda_{\mathbf{g}}[G_{\mathbb{Q}_p}]$-modules

$$0 \longrightarrow V_{\mathbf{g}}^+ \longrightarrow V_{\mathbf{g}} \longrightarrow V_{\mathbf{g}}^- \longrightarrow 0, \tag{1.8.10}$$

where $V_{\mathbf{g}}^+$ and $V_{\mathbf{g}}^-$ are locally free of rank one and the action of $G_{\mathbb{Q}_p}$ on $V_{\mathbf{g}}^-$ is unramified, with Frob_p acting as multiplication by the p-th Fourier coefficient of $\mathbf{g}$. There is a perfect Galois equivariant pairing

$$\langle\ ,\ \rangle : V_{\mathbf{g}}^- \times V_{\mathbf{g}}^+ \longrightarrow \Lambda_{\mathbf{g}}(\det(\rho_{\mathbf{g}})). \tag{1.8.11}$$

In [Oht95], Ohta constructed a canonical period

$$\omega_{\mathbf{g}} \in \mathrm{D}_{\mathrm{dR}}(V_{\mathbf{g}}^-)$$

associated with $\mathbf{g}$. As explained in §1.7.8, the specialisation via y_g produces the L_p-vector space

$$V_g = y_g(V_{\mathbf{g}}) = V_{\mathbf{g}} \otimes_{\Lambda_{\mathbf{g}}, y_g} L_p.$$

Using the functoriality of D_{dR} and the identification

$$y_g(V_{\mathbf{g}}^+) = V_g^\beta, \qquad y_g(V_{\mathbf{g}}^-) = V_g^\alpha \tag{1.8.12}$$

we obtain a pairing

$$\langle\ ,\ \rangle : \mathrm{D}_{\mathrm{dR}}(V_g^\alpha) \times \mathrm{D}_{\mathrm{dR}}(V_g^\beta) \longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p(\chi_g)) = (H_p \otimes L_p(\chi_g))^{G_{\mathbb{Q}_p}}. \tag{1.8.13}$$

Here H_p is a fixed completion above p of the field cut out by the Artin representation attached to g.

Definition 1.62. Define

$$\omega_{g_\alpha} := y_g(\omega_{\mathbf{g}}) \in \mathrm{D}_{\mathrm{dR}}(V_g^\alpha),$$

and let

$$\eta_{g_\alpha} \in \mathrm{D}_{\mathrm{dR}}(V_g^\beta)$$

be the element characterised by the equality

$$\langle \omega_{g_\alpha}, \eta_{g_\alpha} \rangle = \mathfrak{g}(\chi_g) \otimes 1 \in \mathrm{D}_{\mathrm{dR}}(L_p(\chi_g)), \tag{1.8.14}$$

where $\mathfrak{g}(\chi_g)$ denotes the Gauss sum of χ_g viewed as an element of H_p.

Using the isomorphism (1.7.3), let

$$j_g : V_g^\circ \longrightarrow V_g^\circ \otimes_L L_p \xrightarrow{\cong} V_g \tag{1.8.15}$$

and define an L-structure on V_g by

$$V_g^L := j_g(V_g^\circ).$$

Let v_g^α (resp. v_g^β) be an L-basis of $V_g^L \cap V_g^\alpha$ (resp. of $V_g^L \cap V_g^\beta$).

Definition 1.63. Define

$$\Omega_{g_\alpha} \in H_p^{1/\alpha_g}, \qquad \Theta_{g_\alpha} \in H_p^{1/\beta_g}$$

to be the elements such that

$$\Omega_{g_\alpha} \otimes v_g^\alpha = \omega_{g_\alpha} \in \mathrm{D_{dR}}(V_g^\alpha), \qquad \Theta_{g_\alpha} \otimes v_g^\beta = \eta_{g_\alpha} \in \mathrm{D_{dR}}(V_g^\beta).$$

Here H_p^{1/α_g} denotes the set of elements of H_p on which Frob_p acts as multiplication by $1/\alpha_g$ and analogously we denote H_p^{1/β_g}.

1.9 Chow groups

Let X be a nonsingular irreducible variety of dimension $d \geq 1$ over a field K of characteristic 0. In this section we attach to X a number of arithmetic groups called *Chow groups*, whose elements are (equivalence classes of) algebraic cycles on X. The construction of these groups generalise those of Picard groups, described in §1.6.4.1.

For $c \in \{1, \ldots, d\}$, let X^c be the set of irreducible subvarieties of X of codimension c. We define

$$Z^c(X) := \bigoplus_{Z \in X^c} Z \cdot \mathbb{Z} = \left\{ \sum_Z n_Z \cdot Z \mid n_Z \in \mathbb{Z}, \text{ almost all } 0 \right\}$$

be the free abelian group generated by X^c. The elements of $Z^c(X)$ are called *codimension c algebraic cycles* of X.

Let $f \in K(V)$ be a rational function on a variety V. For any irreducible codimension 1 subvariety Z of V, let $\mathrm{ord}_Z(f)$ be the *order of vanishing of f along Z*. Recall that, if t is a uniformiser for the maximal ideal of the local ring $\mathcal{O}_{V,Z}$, then $\mathrm{ord}_Z(f)$ is the integer characterised by the equality

$$f = u \cdot t^{\mathrm{ord}_Z(f)}, \qquad u \in \mathcal{O}_{V,Z}^\times$$

and it does not depend on the choice of t. Recall the variety X fixed at the outset. The assignation above defines, for $V \in X^{c-1}$ and $Z \in V^1$, a ring homomorphism

$$\mathrm{ord}_Z : K(V)^\times \longrightarrow \mathbb{Z}.$$

Fix a subvariety $V \in X^{c-1}$. Notice that every $X \in V^1$ is a subvariety of X of codimension c. We attach a cycle in $Z^c(X)$ to each rational function $f \in K(V)^\times$ by defining

$$\mathrm{div}(f) := \sum_{Z \in V^1} \mathrm{ord}_Z(f) \cdot Z \in Z^c(X).$$

Let $B^c(X)$ be the subgroup of $Z^c(X)$ generated by cycles of the form $\mathrm{div}(f)$ for rational functions f on subvarieties of X of codimension $c-1$.

Definition 1.64. The *c-th Chow group of X* is

$$\mathrm{CH}^c(X) := Z^c(X)/B^c(X).$$

For any extension F/K, we denote

$$\mathrm{CH}^c(X/F) := \left(Z^c(X)/B^c(X)\right)^{G_F}.$$

Finally, for a number field L, we denote

$$\mathrm{CH}^c(X)_L := \mathrm{CH}^c(X) \otimes_{\mathbb{Q}} L.$$

Assume now that X has a structure of (smooth) complex variety (induced by choosing an embedding $K \subseteq \mathbb{C}$). There is a so-called *cycle class map*

$$\mathrm{cl} : \mathrm{CH}^c(X) \longrightarrow \mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z})(c) \tag{1.9.1}$$

defined by sending an irreducible subvariety $Z \in X^c$ to its homology class $[Z] \in \mathrm{H}_{2d-2c}(X(\mathbb{C}), \mathbb{Q})$ and then using the isomorphism $\mathrm{H}_{2d-2c}(X(\mathbb{C}), \mathbb{Q}) \cong \mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Q})(c)$ given by Poincaré duality.

A natural question in this setting is: do all cohomology classes in $\mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z})(c)$ arise from algebraic cycles of X? The answer of this question is negative: as explained in §1.1, the complex Betti cohomology $\mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{C}) = \mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Z}) \otimes \mathbb{C}$ is endowed with a Hodge decomposition

$$\mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{C}) = \oplus_{p+q=2c} \mathrm{H}^{p,q}(X(\mathbb{C}), \mathbb{C}),$$

and

$$\mathrm{cl}(\mathrm{CH}^c(X)) \subseteq \mathrm{H}^{c,c}(X(\mathbb{C}), \mathbb{C}).$$

Hodge conjecture gives a refined (conjectural) answer to the question.

Conjecture 1.65 (Hodge conjecture)**.** Every class in $\mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}), \mathbb{Q}) \cap \mathrm{H}^{c,c}(X(\mathbb{C}), \mathbb{C})$ is a rational linear combination of elements in the image of the cycle class map (1.9.1).

We denote

$$\mathrm{CH}^c(X)_0 := \ker(\mathrm{cl})$$

the subgroup of *null-homologous* cycles of $\mathrm{CH}^c(X)$. Coherently as above, we denote $\mathrm{CH}^c(X/F)_0$ the subgroup of null-homologous cycles defined over F, and $\mathrm{CH}^c(X)_{0,L} = \mathrm{CH}^c(X)_0 \otimes L$.

Remark 1.66. If X is a curve, a codimension one algebraic cycle on X is simply a divisor on X, and so $\mathrm{CH}^1(X) = \mathrm{Pic}(X) := \mathrm{Div}(X)/\mathrm{Princ}(X)$ is the Picard group defined in §1.6.4.1. Moreover, the cycle class map $\mathrm{cl} : \mathrm{Pic}(X) \longrightarrow \mathrm{H}_0(X(\mathbb{C}), \mathbb{Q})$ is given on divisors by

$$D = \sum_P n_P P \mapsto \sum_P n_P = \deg(D),$$

whose image is contained in $\mathbb{Z} = \mathrm{H}_0(X(\mathbb{C}), \mathbb{Z}) \subseteq \mathrm{H}_0(X(\mathbb{C}), \mathbb{Q}) = \mathbb{Q}$. Then $\mathrm{CH}^1(X)_0 = \mathrm{Pic}^0(X)$ is the subgroup of degree 0 divisors.

Remark 1.67. More in general, one can define on the group $Z^c(X)$ different equivalence relations.

1. We say that $Z_1, Z_2 \in Z^c(X)$ are *rationally equivalent*, and we write $Z_1 \sim_{\mathrm{rat}} Z_2$ if there are subvarieties $V_j \in X^{c-1}$ and rational functions $f_j \in K(V_j)^\times$ such that $Z_1 - Z_2 = \sum_j \mathrm{div}(f_j)$. Then $\mathrm{CH}^c(X) = Z^c(X)/\sim_{\mathrm{rat}}$.

2. We say that $Z_1, Z_2 \in Z^c(X)$ are *homologically equivalent*, and we write $Z_1 \sim_{\mathrm{hom}} Z_2$ if $\mathrm{cl}(Z_1) = \mathrm{cl}(Z_2)$. We define $\mathrm{CH}^c(X)_{\mathrm{hom}} := Z^c(X)/\sim_{\mathrm{hom}}$. This group will be related to Grothendieck motives.

1.9.1 Algebraic correspondences

For $p, q \in \{1, \dots, d\}$, there is a product

$$\mathrm{CH}^p(X) \otimes_{\mathbb{Z}} \mathrm{CH}^q(X) \longrightarrow \mathrm{CH}^{p+q}(X), \qquad \alpha \otimes \beta \mapsto \alpha \cdot \beta \tag{1.9.2}$$

given by intersection of cycles. The definition of $\alpha \cdot \beta$ is rather involved, so we describe it in a special case. More precisely, we say that α and β *intersect properly* if $\alpha = \sum a_i [V_i]$, $\beta = \sum b_j [W_j]$ with $V_i \in X^p, W_j \in X^q$ and $\mathrm{codim}_X(V_i \cap W_j) = p + q$. In this case,

$$\alpha \cdot \beta := \sum a_i b_j [V_i \cap W_j] \in \mathrm{CH}^{p+q}(X).$$

Let X, Y be smooth projective varieties.

Definition 1.68. The group of *r-correspondences* from X to Y is

$$\mathrm{Corr}^r(X,Y) := \mathrm{CH}^{\dim X + r}(X \times Y).$$

An element $\alpha \in \mathrm{Corr}^r(X,Y)$ will be denoted $\alpha : X \rightsquigarrow Y$. For a number field L, we denote $\mathrm{Corr}^r(X,Y)_L := \mathrm{Corr}^r(X,Y) \otimes L$. Similarly, we define

$$\mathrm{Corr}^r_{\mathrm{hom}}(X,Y) := \mathrm{CH}^{\dim X + r}(X \times Y)_{\mathrm{hom}}.$$

The product (1.9.2) can be used to define a composition of correspondences as follows. Consider the product $X_1 \times X_2 \times X_3$ of three smooth projective varieties, and let $\pi_{ij} : X_1 \times X_2 \times X_3 \longrightarrow X_i \times X_j$ for $1 \leq i < j \leq 3$ be the projections. If $\alpha : X_1 \rightsquigarrow X_2$ and $\alpha : X_2 \rightsquigarrow X_3$ are correspondences, then

$$\beta \circ \alpha := p_{13*}(p_{12}^*(\alpha) \cdot p_{23}^*(\beta)) : X_1 \rightsquigarrow X_3.$$

For a correspondence $\alpha \in \mathrm{Corr}^r(X,Y)$ the *transpose* of α is the correspondence $\alpha^t \in \mathrm{Corr}^r(Y,X)$ obtained via the isomorphism $X \times Y \cong Y \times X$ given by $(x,y) \mapsto (y,x)$.

Finally, if $\pi_X : X \times Y \longrightarrow X$, $\pi_Y : X \times Y \longrightarrow Y$ are the projections, every correspondence $\alpha \in \mathrm{Corr}^r(X,Y) = \mathrm{CH}^{\dim X + r}$ induces maps between Chow groups

$$\alpha_* : \mathrm{CH}^s(X) \longrightarrow \mathrm{CH}^{s+r}(Y), \qquad \alpha_*(Z) := \pi_{Y*}(\alpha \cdot \pi_X^*(Z))$$

and

$$\alpha^* : \mathrm{CH}^s(Y) \longrightarrow \mathrm{CH}^{s+r}(X), \qquad \alpha^*(Z) := \pi_{X*}(\alpha \cdot \pi_Y^*(Z)).$$

Remark 1.69. We can regard correspondences as a generalisation of regular maps. Indeed, if $f : X \longrightarrow Y$ is a regular map of schemes, then the class Γ_f of the graph of f in the Chow group of $X \times Y$ is a correspondence $\Gamma_f : X \rightsquigarrow Y$. Moreover, $f : X \longrightarrow Y$ induces maps on Chow groups and we have

$$f_* = \Gamma_{f*}, \quad f^* = \Gamma_f^*.$$

1.10 Chow motives and Grothendieck motives

In this section we introduce the notion of (Chow and Grothendieck) motives over K with coefficients in L, where K and L are number fields and we recall the definition of the main operations with motives, such as direct sums, tensor products, duals, restriction of scalars, and we explain how to attach a Chow group to a motive. Finally, in §1.10.4 we recall the motive attached to modular forms by Scholl in [Sch90]. Since we will need it in Chapter 2, we focus in §1.10.6 on the motive attached to Hecke characters and to CM modular forms.

Let K and L be two number fields.

Definition 1.70. The category of *Chow motives* $\mathcal{M}(K)_L$ over K with coefficients in L is defined by the following data.

i) The objects of $\mathcal{M}(K)_L$ are triples (X, e, m) where X is a smooth projective scheme over K, $e = e^2 \in \mathrm{Corr}^0(X,X)_L$ is a projector and m is an integer;

ii) For $i = 1, 2$, let $M_i := (X_i, e_i, m_i)$ be an object of $\mathcal{M}(K)_L$ and assume that X_1 is of pure dimension d_1; the morphisms from M_1 to M_2 are defined in terms of correspondences between the underlying varieties as

$$\mathrm{Hom}(M_1, M_2) := e_1 \circ \mathrm{Corr}^{m_2 - m_1}(X_1, X_2)_L \circ e_2.$$

The category of *Grothendieck motives* $\mathcal{M}(K)_L^{\mathrm{hom}}$ over K with coefficients in L is defined by *i*) The objects are triples (X, e, m) where X is a smooth projective scheme over K, $e = e^2 \in \mathrm{Corr}^0_{\mathrm{hom}}(X, X)_L$ is a projector and m is an integer;

ii) For $M_i := (X_i, e_i, m_i) \in \mathcal{M}(K)_L^{\mathrm{hom}}$, with X_1 of pure dimension,

$$\mathrm{Hom}(M_1, M_2) := e_1 \circ \mathrm{Corr}^{m_2 - m_1}_{\mathrm{hom}}(X_1, X_2)_L \circ e_2.$$

Since rational equivalence is finer than homological equivalence, there is a natural functor

$$\mathcal{M}(K)_L \longrightarrow \mathcal{M}(K)_L^{\mathrm{hom}}.$$

Chow and Grothendieck motives $M = (X, e, m) \in \mathcal{M}(K)_L$ come equipped with their realisations, we give now a description of some of them.

i) Fixing an embedding $K \subseteq \mathbb{C}$, we get the *Betti realisation* of M

$$M_{\mathrm{B}} := e \cdot (\mathrm{H}^*(X(\mathbb{C}), \mathbb{Q})(m) \otimes L),$$

which is an L-vector space equipped with Hodge decomposition described in (1.1).

ii) The *de Rham realisation* of M is

$$M_{\mathrm{dR}} := e \cdot (\mathrm{H}^*_{\mathrm{dR}}(X/K)(m) \otimes_{\mathbb{Q}} L),$$

where $\mathrm{H}^*_{\mathrm{dR}}(X/K)$ denotes the algebraic de Rham cohomology of X over K. Then M_{dR} is a free $K \otimes L$-module with an action of G_K with the Hodge filtration (1.1.2), whose graded pieces are related to the Hodge decomposition of M_{B} by the comparison isomorphism (1.1.4).

iii) For every prime ℓ, the *ℓ-adic étale realisation* of M is

$$M_{\mathrm{et},\ell} := e \cdot (\mathrm{H}^*_{\mathrm{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell(m)) \otimes L),$$

which is a free $L \otimes \mathbb{Q}_\ell$-module with an action of G_K. It then decomposes as a direct sum of λ-adic representations $M_{\mathrm{et},\lambda}$ as λ runs over the ideals of L lying above ℓ.

We say that a motive is *pure of weight w* if its étale realisations are pure of this weight (see Definition 1.28). In particular, by Deligne's proof of the Weil conjectures, the étale realisations of the form

$$\{\mathrm{H}^w_{\mathrm{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell)\}_\ell$$

give a compatible system of ℓ-adic representations pure of weight w and whose set of ramification is $S := \{\text{primes of bad reduction for } X\}$. It is not known in general that $\{M_{\mathrm{et},\lambda}\}_\lambda$ is compatible. Nevertheless, in this thesis will only appear motives M attached to modular forms and Hecke characters, on which we will focus in §1.10.6 and §1.10.4). The realisations of these motives arise in a fixed degree of the corresponding cohomologies, and moreover their étale realisations are the representations appearing in §1.6.4 and §1.6.4.5, so

$$M_{\mathrm{et}} := \{M_{\mathrm{et},\lambda}\}_\lambda \tag{1.10.1}$$

form a compatible system of λ-adic representations.

Conjecture 1.71 (Tate's conjecture)**.** Let K be a number field and denote by $\mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)$ the category of ℓ-adic representations of G_K. The functor

$$(\cdot)_{\mathrm{et},\ell} : \mathcal{M}(K)_{\mathbb{Q}} \longrightarrow \mathrm{Rep}_{\mathbb{Q}_\ell}(G_K)$$

that sends a motive over K to its étale ℓ-adic realisation is fully faithful.

Some basic examples of motives are the following:

1. $\mathbb{Q} := (\mathrm{Spec}(\mathbb{Q}), \mathrm{id}, 0) \in \mathcal{M}(\mathbb{Q})_{\mathbb{Q}}$ is the trivial character.
2. $\mathbb{L} := (\mathrm{Spec}(K), \mathrm{id}, -1) \in \mathcal{M}(K)_{\mathbb{Q}}$ is called the *Lefschetz motive*;
3. $\mathbb{Q}(-1) := (\mathbb{P}^1_{\mathbb{Q}}, e, 0) \in \mathcal{M}(\mathbb{Q})_{\mathbb{Q}}$ where e is the idempotent that annihilates the 0th cohomology of $\mathbb{P}^1_{\mathbb{Q}}$ and which acts trivially on $\mathrm{H}^2(\mathbb{P}^1_{\mathbb{Q}})$. So $\mathbb{Q}(-1)$ is pure of weight 2.
4. $\mathbb{Q}(1) := \mathbb{Q}(-1)^\vee \in \mathcal{M}(\mathbb{Q})_{\mathbb{Q}}$ and, in general, $\mathbb{Q}(n) := \mathbb{Q}(1)^{\otimes n}$ for all $n \in \mathbb{Z}$ (see §1.10.2 for the definition of tensor products and duals of motives). $\mathbb{Q}(1)$ is called the *Tate motive*, is pure of weight -2, and its étale ℓ-adic realisation is the one dimensional representation $\mathbb{Q}_\ell(1)$ attached to the cyclotomic character (i.e. the ℓ-adic Tate module of $\mathbb{G}_m$).
5. Let X be a smooth projective variety defined over a number field K, and let $j, n \in \mathbb{Z}$, with $j \geq 0$. This triple defines a motive

$$h^j(X)(n) := (X, e_j, n),$$

which is pure of weight $j - 2n$, where e_j is a suitable projector such that the realisations of $h^j(X)(n)$ are of the form $H^j(X)(n)$. In particular,

$$h^j(X)(n)_{\mathrm{dR}} = \mathrm{H}^j_{\mathrm{dR}}(X/K)(n), \qquad h^j(X)(n)_{\ell,\mathrm{et}} = \mathrm{H}^j_{\mathrm{et}}(X_{\bar{K}}, \mathbb{Q}_\ell)(n).$$

1.10.1 Chow groups of motives

The *Chow group* of a motive $M \in \mathcal{M}(K)_L$ is defined as

$$\mathrm{CH}^d(M) := \mathrm{Hom}(\mathbb{L}^d, M),$$

where $\mathbb{L}$ is the Lefschetz motive. The Chow group of a motive can also be interpreted as a group of cycles, since

$$\mathrm{CH}^d((X, e, m)) \cong e \cdot \mathrm{CH}^{d+m}(X/K)_L.$$

We denote $\mathrm{CH}^d(M)_0$ the subgroup of the null-homologous cycles of $\mathrm{CH}^d(M)$. We will occasionally use the notation $\mathrm{CH}^d(M)_{0,L}$ if we need to emphasize the field of coefficients of the Chow group.

1.10.2 Basic operations with motives

Let $M := (X, e, m)$, and $M_i = (X_i, e_i, m_i) \in \mathcal{M}(K)_L$ for $i = 1, 2$, be motives over K with coefficients in L.

Definition 1.72. The *tensor product* of M_1 and M_2 is

$$M_1 \otimes M_2 := (X_1 \times X_2, e_1 \otimes e_2, m_1 + m_2) \in \mathcal{M}(K)_L.$$

The realisations of $M_1 \otimes M_2$ are the tensor product of the realisations of M_1 and M_2. In particular, if each M_i is pure of weight w_i, then $M_1 \otimes M_2$ is pure of weight $w_1 + w_2$.

Given an integer n, the *Tate twist* $M(n)$ is defined as

$$M(n) := M \otimes \mathbb{Q}(n) \in \mathcal{M}(K)_L.$$

Notice that, if M is a pure motive of weight w, then $M(n)$ is pure of weight $w - 2n$.

We can also define the *dual* $M^\vee$ of M, which, if X is of pure dimension d, is defined as

$$M^\vee = (X, e^t, d - m) \in \mathcal{M}(K)_L,$$

where e^t denotes the transpose of the correspondence e. If M is pure of weight w, then $M^\vee$ is pure of weight $-w$. The relation between these two operations is given by the isomorphism

$$\mathrm{Hom}(M \otimes N, P) \cong \mathrm{Hom}(M, N^\vee \otimes P), \tag{1.10.2}$$

for $M, N, P \in \mathcal{M}(K)_L$.

There is another dual that we will need in the following, which is the *Kummer dual* of M, which is defined as

$$M^* := \mathrm{Hom}(M, \mathbb{Q}(1)) = M^\vee(1).$$

A motive is *selfdual* if $M^* \cong M$. If M is pure of weight w, then M^* is pure of weight $-2-w$. In particular, a selfdual motive is necessarily pure of weight -1.

Definition 1.73. *i)* If $m_1 = m_2$, the *direct sum* of M_1 and M_2 is

$$M_1 \oplus M_2 := (X_1 \coprod X_2, e_1 \oplus e_2, m).$$

ii) If $m_1 < m_2$, notice that

$$M_1 = (X_1, e_1, m_2) \otimes \mathbb{L}^{m_2-m_1} = (X_1, e_1, m_2) \otimes h^2(\mathbb{P}^1)^{m_2-m_1} = (X_1 \times (\mathbb{P}^1)^{m_2-m_1}, e_1', m_2)$$

and the *direct sum* of M_1 and M_2 is

$$M_1 \oplus M_2 := (X_1 \times (\mathbb{P}^1)^{m_2-m_1}, e_1', m_2) \oplus (X_2, e_2, m_2) = (X_1 \times (\mathbb{P}^1)^{m_2-m_1} \coprod X_2, e_1' \oplus e_2, m_2).$$

The realisations of the motive $M_1 \oplus M_2$ coincide with the direct sum of the realisations of the two motives.

1.10.3 Restriction of scalars of motives

For an extension K/F of number fields, there is a restriction of scalar functor

$$\mathrm{Res}_{K/F} : \mathcal{M}(K) \longrightarrow \mathcal{M}(F),$$

which extends the restriction of scalars on algebraic varieties to the category of motives [Kar00]. Suppose that $M \in \mathcal{M}(K)$, and put $R = \mathrm{Res}_{K/F}(M)$. Also, for $Y \in \mathcal{M}(F)$, denote by $Y_K := Y \otimes_F K$ the extension of scalars of Y from F to K. Then there is a canonical morphism

$$w : M \longrightarrow R_K$$

satisfying the following universal property: if $Y \in \mathcal{M}(F)$ and f is a morphism $f : Y_K \longrightarrow M$, then there exists a unique morphism $s : Y \longrightarrow R$ such that $w \circ f = s$. In other words, there is a canonical identification

$$\mathrm{Hom}(Y_K, M) \cong \mathrm{Hom}(Y, \mathrm{Res}_{K/F}(M)).$$

For any integer d, we denote $\mathbb{L}^d := \mathbb{L}^{\otimes d}$ the tensor product of the Lefschetz motive with itself d times. In particular, taking $Y = \mathbb{L}^c \in \mathcal{M}(\mathbb{Q})$ with c some integer, we have

$$\mathrm{CH}^c(M) \cong \mathrm{Hom}(\mathbb{L}^c_K, M) \cong \mathrm{Hom}(\mathbb{L}^c, \mathrm{Res}_{K/\mathbb{Q}} M) = \mathrm{CH}^c(\mathrm{Res}_{K/\mathbb{Q}} M). \tag{1.10.3}$$

Here we have used the fact that the Lefschetz motive over K is the base extension $\mathbb{L}_K$. We will need the following generalisation of (1.10.3).

Lemma 1.74. *Suppose that M is a motive over K and that N is a motive over $\mathbb{Q}$. There is a canonical isomorphism of Chow groups*

$$\mathrm{CH}^c(N \otimes \mathrm{Res}_{K/\mathbb{Q}} M) \cong \mathrm{CH}^c(N_K \otimes M).$$

Proof. By definition of Chow group, and using the formula (1.10.2) relating tensor product and duals we have:

$$\begin{aligned}\mathrm{CH}^c(N \otimes \mathrm{Res}_{K/\mathbb{Q}}(M)) &= \mathrm{Hom}(\mathbb{L}^c, N \otimes \mathrm{Res}_{K/\mathbb{Q}}(M)) = \mathrm{Hom}(\mathbb{L}^c \otimes N^\vee, \mathrm{Res}_{K/\mathbb{Q}}(M)) \\ &= \mathrm{Hom}(\mathbb{L}^c_K \otimes N^\vee_K, M) = \mathrm{Hom}(\mathbb{L}^c_K, N^\vee_K \otimes M) = \mathrm{CH}^c(N_K \otimes M).\end{aligned}$$

□

There is a natural isomorphism of $\mathbb{Q}$-vector spaces, preserving the Hodge filtration (cf. [Jan90, p. 16])

$$\mathrm{H}_{\mathrm{dR}}(M) \cong \mathrm{H}_{\mathrm{dR}}(\mathrm{Res}_{K/\mathbb{Q}}(M)). \tag{1.10.4}$$

1.10.4 Motives attached to modular forms of weight $k \geq 2$

Let $f \in S_k(N, \chi)$ be a p-ordinary nomalised cuspform of weight $k \geq 2$ and let K_f be the number field generated by the Fourier coefficients of f. In [Sch90], Scholl attached to f a motive M_f over K and coefficients in K_f. This is constructed in loc. cit. as a Grothendieck motive, although it is expected to be also a Chow motive (see [Sch90, Remark 1.2.6], [DS91, p. 16]). Using the notation of §1.6.4.3, the motive M_f is given by the triple $(W_{k-2}, e_f, 0)$, where e_f is a certain projector in the ring of correspondences of W_{k-2}, which is constructed from Hecke correspondences. By functoriality, e_f acts on the different cohomological realisations of M_f, acting as the projection projecting onto the f-isotypical component of the cohomology of W_{k-2}. For example

$$e_f \cdot \mathrm{H}^{k-1}_{\mathrm{et}}((W_{k-2})_{\bar{\mathbb{Q}}}, \mathbb{Q}_p),$$

which in fact is the 2-dimensional p-adic representation V_f attached to f. Similarly, we denote, for any number field L containing K_f, $e_f \cdot S_k(N)_L$ to be the projection onto the eigenspace of f relative to the action of the Hecke operators T_ℓ with $(\ell, N) = 1$. This f-isotypical component is isomorphic to a piece of the de Rham realisation of M_f:

$$e_f \cdot S_k(N)_L \cong \mathrm{Fil}^{k-1}\left(e_f \cdot \mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/L)\right).$$

Remark 1.75. Recall that if $f \in S_2(N)$ has rational Fourier coefficients, then Shimura's construction attaches to f an elliptic curve $E/\mathbb{Q}$ so that $V_f = T_pE \otimes \mathbb{Q}_p$. On the other hand, the motive attached to f is $M_f = (X_1(N), e_f, 0)$. In this case, we have

$$M_f = (X_1(N), e_f, 0) = (E, e_1, 0) = h^1(E).$$

1.10.5 Motives attached to elliptic curves with complex multiplication

Let E be an elliptic curve with complex multiplication by the ring of integers $\mathcal{O}_K$ of an imaginary quadratic field K, i.e. such that $\mathrm{End}(E) \cong \mathcal{O}_K$. More precisely, fix the identification

$$\mathrm{End}(E) \otimes \mathbb{Q} \cong K \tag{1.10.5}$$

and for each $\alpha \in K$ let α^* denote the pullback in differentials of the endomorphism of E corresponding to α. Recall that such an elliptic curve is defined over the Hilbert class field H of K, and let $\psi_E : \mathbb{A}_H^\times \longrightarrow \mathbb{C}^\times$ be the Hecke character attached to E as in [Sil94, Theorem 9.2]. The theta series $\theta(\psi_E)$ of ψ_E is the weight 2 cuspform attached to E by modularity. The motive attached to ψ_E belongs to $\mathcal{M}(H)_{\mathbb{Q}}$ and is of the form

$$M(\psi_E) = (E, e_E, 0),$$

for an appropriate projector e_E as described in [BDP14, §2.2], which only depends on K. We can attach to $M(\psi_E)$ a motive over H with coefficients in K

$$M(\psi_E)_K \in \mathcal{M}(H)_K,$$

by letting K act on E via (1.10.5). The ℓ-adic étale realisation of $M(\psi_E)_K$ is

$$M(\psi_E)_{K,\ell} = e_E \cdot \mathrm{H}^1_{\mathrm{et}}(E_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = (K \otimes \mathbb{Q}_\ell)(\psi_E),$$

which is free of rank one over $K \otimes \mathbb{Q}_\ell$. The de Rham realisation of this motive is the K-vector space

$$M(\psi_E)_{K,\mathrm{dR}} = e_{_E} \cdot \mathrm{H}^1_{\mathrm{dR}}(E/H).$$

It is endowed with an action $[\cdot]$ of K given as follows: if ω is a differential form on E and $\alpha \in K$, then $[\alpha]\omega := \alpha^*\omega$. Let ω_E be a generator of $\Omega(E/H) \subseteq \mathrm{H}^1_{\mathrm{dR}}(E/H)$ and considering the pairing induced by the short exact sequence (1.11.8), $\eta_E \in \mathrm{H}^1_{\mathrm{dR}}(E/H)$ be the element such that $\langle \omega_E, \eta_E \rangle = 1$.

1.10.6 Motives attached to Hecke characters

Generalising the construction of the previous section, for $r \in \mathbb{Z}_{\geq 1}$, we can consider the motives attached to ψ_E^r

$$M(\psi_E^r) = (E^r, e_{E^r}, 0) \in \mathcal{M}(H)_{\mathbb{Q}}, \qquad M(\psi_E^r)_K \in M(H)_K \tag{1.10.6}$$

for a certain projector e_{E^r} depending on r and K (see §2.2[BDP14] for the precise definition). The étale ℓ-adic realisation of $M(\psi_E^r)_K$ is

$$M(\psi_E^r)_{K,\ell} = e_{E^r} \cdot \mathrm{H}^r_{\mathrm{et}}(E^r_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) = (K \otimes \mathbb{Q}_\ell)(\psi_E^r).$$

and its deRham realisation is

$$M(\psi_E^r)_{K,\mathrm{dR}} = e_{E^r}\, \mathrm{H}^r_{\mathrm{dR}}(E^r/H) = \mathrm{Sym}^r\, \mathrm{H}^1_{\mathrm{dR}}(E/H).$$

For each $j \in \{0, \ldots, r\}$, we define

$$\omega_E^j \eta_E^{r-j} := e^*_{E^r}(p_1^*\omega_E \wedge \cdots \wedge p_j^*\omega_E \wedge p_{j+1}^*\eta_E \wedge \cdots \wedge p_r^*\eta_E), \tag{1.10.7}$$

where $p_1, \ldots, p_r : E^r \longrightarrow E$ are the projections. The set

$$\{\omega_E^j \eta_E^{r-j} \mid j = 0, \ldots, r\}$$

for a basis for $\mathrm{Sym}^r\, \mathrm{H}^1_{\mathrm{dR}}(E/H)$.

Although the examples of §1.10.5 and §1.10.6 may seem specific, the motive attached to a Hecke character of an imaginary quadratic field is composed of the motives described in the previous sections. More precisely, retaining the notation of the previous paragraphs, let

$$\psi : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

be any Hecke character of infinity type $(0, r)$ with $r \geq 0$. Fix an elliptic curve E/H with complex multiplication by $\mathcal{O}_K$, where H is the Hilbert class field of K. Define

$$\psi_H := \psi \circ \mathrm{N}_{H/K}.$$

Then

$$\psi_H = \chi \cdot \psi_E^r$$

for some finite order character $\chi : \mathrm{Gal}(F/H) \longrightarrow L^\times$, where F and L are number fields. Then the motive attached to ψ over L is

$$M(\psi)_L = M(\psi_E^r)_L \otimes M(\chi) \in \mathcal{M}(H \cdot F)_L,$$

whose ℓ-adic étale realisation is

$$e_{E^r} \cdot \mathrm{H}^\ell_{\mathrm{et}}(E^r_{\mathbb{Q}}, \mathbb{Q}_\ell) \otimes e_\chi \cdot (F \otimes \mathbb{Q}_\ell) = e_\chi e_{E^r} \cdot \left(\mathrm{H}^r_{\mathrm{et}}(E^r_{\bar{\mathbb{Q}}}, \mathbb{Q}_\ell) \otimes F\right).$$

1.11 Cycle class maps and Abel–Jacobi maps

Let X be a smooth projective variety defined over $\mathbb{Q}$. Recall the cycle class map attached to the Chow group of X in §1.9. There is a more abstract expression for the cycle class map that permits to generalise these construction to motives. We now explain roughly this construction, and we refer to [Nek94, (4.2)]. Secondly, we give the definition of the so-called *intermediate Jacobian* of X, generalising the definition given in §1.6.4.1, and we introduce the *Abel–Jacobian maps*, which also have a motivic formulation. The de Rham p-adic realisation of this map can be regarded as a generalisation of the p-adic logarithm, as explained in Remark 2.11.

Recall that there is an isomorphism $\mathrm{H}^{2c}_{\mathrm{B}}(X(\mathbb{C}),\mathbb{Z})\otimes\mathbb{C}\cong\mathrm{H}^{2c}_{\mathrm{dR}}(X/\mathbb{C})$, so we can regard the cycle class map as

$$\mathrm{cl}:\mathrm{CH}^c(X)\longrightarrow\mathrm{H}^{2c}_{\mathrm{dR}}(X/\mathbb{C})(c). \tag{1.11.1}$$

Let $Z\in X^c$ be an irreducible subvariety of X of codimension c. Attached to the triple $(Z,X,U:=X\smallsetminus Z)$ there is the *Gysin long exact sequence*

$$0\longrightarrow\mathrm{H}^{2c-1}_{\mathrm{dR}}(X/K)\longrightarrow\mathrm{H}^{2c-1}_{\mathrm{dR}}(U/K,\mathbb{Z})\longrightarrow\mathrm{H}^{2c}_{Z}(X/K)\longrightarrow\mathrm{H}^{2c}_{\mathrm{dR}}(X/K)\longrightarrow\cdots \tag{1.11.2}$$

for every field K over which the triple (Z,X,U) is defined, and where $\mathrm{H}^{*}_{Z}(X/K,\mathbb{Z})$ denotes the de Rham cohomology with support on Z. Moreover, there is an isomorphism

$$\mathrm{H}^{0}_{\mathrm{dR}}(Z/K)\cong\mathrm{H}^{2c}_{Z}(X/K)(c), \tag{1.11.3}$$

Combining (1.11.2) and (1.11.3), we get a map

$$\mathrm{H}^{0}_{\mathrm{dR}}(Z/K)\longrightarrow\mathrm{H}^{2c}_{\mathrm{dR}}(X/K)(c), \tag{1.11.4}$$

and the image via (1.11.1) of the class of Z in $\mathrm{CH}^c(X)$ equals the image of 1 via (1.11.4) with $K=\mathbb{C}$. Note that, via this interpretation, it is naturally defined

$$\mathrm{cl}_K:\mathrm{CH}^c(X/K)\longrightarrow\mathrm{H}^{2c}_{\mathrm{dR}}(X/K)(c)$$

for all fields K.

Let $Z\in X^c$ and let $U:=X\smallsetminus Z$ as above. The Gysin long exact sequence is actually defined at the level of motives:

$$0\longrightarrow h^{2c-1}(X)(c)\longrightarrow h^{2c-1}(U)(c)\longrightarrow h^{2c}_{Z}(X)(c)\xrightarrow{\ \beta\ }h^{2c}(X)(c)\longrightarrow\cdots, \tag{1.11.5}$$

(see [Har77, Ex. 2.3] and [Mil13, Chapter 9] for the realisations of $h^{2c}_{Z}(X)(c)$). And we can look at cl as the Betti realisation of the map

$$\mathrm{cl}_{\mathrm{mot}}:\mathrm{CH}^c(X)\longrightarrow h^{2c}(X)(c),\qquad \mathrm{cl}_K([Z])=\gamma_{\mathrm{mot}}(1), \tag{1.11.6}$$

where

$$\gamma_{\mathrm{mot}}:\mathbb{Q}\longrightarrow h^{2c}_{Z}(X)(c)\xrightarrow{\ \beta\ }h^{2c}(X)(c).$$

Analogously, $\mathrm{cl}_{\mathrm{mot}}$ is the de Rham realisation (over K) of $\mathrm{cl}_{\mathrm{mot}}$.

This description of (1.11.1), allows to define *étale p-adic class maps*

$$\mathrm{cl}^{\mathrm{et}}_p:\mathrm{CH}^c(X)\longrightarrow\mathrm{H}^{2c}_{\mathrm{et}}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p(c))$$

as follows. For $Z\in X^c$, there is an isomorphism $\mathrm{H}^{0}_{\mathrm{et}}(Z_{\bar{\mathbb{Q}}},\mathbb{Q}_p)\cong\mathrm{H}^{2c}_{Z}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p(c))$, which, combined with the étale realisation of (1.11.5) gives a map

$$\gamma_p:\mathrm{H}^{0}_{\mathrm{et}}(Z_{\bar{\mathbb{Q}}},\mathbb{Q}_p)\longrightarrow\mathrm{H}^{2c}_{\mathrm{et}}(X_{\bar{\mathbb{Q}}},\mathbb{Q}_p(c)), \tag{1.11.7}$$

which is the analogous of (1.11.4), and

$$\mathrm{cl}_p^{\mathrm{et}}([Z]) := \gamma_p(1).$$

By definition, these cycle class maps are compatible with the comparison isomorphisms between étale and de Rham cohomology of X.

Let X be a nonsingular curve defined over a field K of characteristic zero together with the embedding $K \subseteq \mathbb{C}$. As explained in Remark (1.66), $\mathrm{CH}^1(X)_0$ is the group of degree 0 divisors on X. Recall the Jacobian of X over $\mathbb{C}$

$$J(X) := \mathrm{H}^0(X, \Omega_X)^\vee / \mathrm{H}_1(X, \mathbb{Z})$$

introduced in §1.6.4.1. There is an isomorphism

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^1(X/\mathbb{C})_0 \xrightarrow{\cong} J(X)$$

called *complex Abel–Jacobi map*, which sends a class $[P - Q]$ to the class of the map $\omega \mapsto \int_P^Q \omega$.

These constructions can be generalised to a nonsingular irreducible variety of dimension $d \geq 1$ as follows. Let $c \in \{1, \dots, d\}$ and let F any field containing the field of definition K of X.

Definition 1.76. The *c-th intermediate Jacobian* of X is

$$J^c(X) := \mathrm{Fil}^{d-c+1} \mathrm{H}_{\mathrm{dR}}^{2d-2c+1}(X/\mathbb{C})^\vee / \mathrm{H}_{2d-2c+1}(X(\mathbb{C}), \mathbb{Z}).$$

The complex Abel–Jacobi generalises as

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^c(X/\mathbb{C})_0 \longrightarrow J^{2c-1}(X),$$

sending the class of a subvariety $W \in X^c$ to $\omega \mapsto \int_\gamma \omega$, where γ is a smooth $2d - 2c + 1$-(real) dimensional chain in $X(\mathbb{C})$ with boundary $\partial\gamma = W$. One can show that this assignment is well-defined, i.e. it does not depend on the choice of W and of γ.

Remark 1.77. If $d = c = 1$, then $J^1(X) := \mathrm{Fil}^1 \mathrm{H}_{\mathrm{dR}}^1(X/\mathbb{C})^\vee / \mathrm{H}_1(X(\mathbb{C}), \mathbb{Z})$. Moreover, the fact that the spectral sequence (1.1.1) degenerates at the first page implies that there is a short exact sequence

$$0 \longrightarrow \mathrm{H}^0(X, \Omega_X) \longrightarrow \mathrm{H}_{\mathrm{dR}}^1(X/K) \longrightarrow \mathrm{H}^1(X, \mathcal{O}_X) \longrightarrow 0, \tag{1.11.8}$$

and via this sequence there is an identification

$$\mathrm{Fil}^1 \mathrm{H}_{\mathrm{dR}}^1(X/K) = \mathrm{H}^0(X, \Omega_X),$$

so that $J^1(X) = J(X)$. One can show that $J(X)$ can be defined over K. Moreover,

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^1(X/\mathbb{C})_0 = \mathrm{Pic}^0(X) \xrightarrow{\cong} J(X) = \mathrm{Fil}^1 \mathrm{H}_{\mathrm{dR}}^1(X/\mathbb{C}) / \mathrm{H}_1(X(\mathbb{C}), \mathbb{Z})$$

is an isomorphism by Abel's theorem and Jacobi's injection. In particular, if $X = E$ is an elliptic curve, via the isomorphism (1.6.10), we have

$$J^1(E) = \mathrm{H}^1(E, \Omega_E) / \mathrm{H}_1(E(\mathbb{C}), \mathbb{Z}) \cong \mathbb{C}/\Lambda_E,$$

where Λ_E is the lattice determined by the Néron differential ω_E of E, i. e. $\Lambda_E := \left\{ \int_\gamma \omega_E \mid \gamma \in \mathrm{H}_1(E(\mathbb{C}), \mathbb{Z}) \right\}$. In this setting, the complex Abel–Jacobi map gives the well known identification of the elliptic curve with its Picard group

$$\mathrm{AJ}_{\mathbb{C}} : \mathrm{CH}^1(E/\mathbb{C})_0 = \mathrm{Pic}^0(E) \longrightarrow \mathbb{C}/\Lambda_E = E(\mathbb{C}).$$

As for the cycle class map, also the complex Abel–Jacobi has a motivic formulation

$$\mathrm{AJ}_{\mathrm{mot}} : \mathrm{CH}^c(X)_0 \longrightarrow \mathrm{Ext}^1_{\mathcal{M}(\mathbb{Q})_{\mathbb{Q}}}(\mathbb{Q}, h^{2c-1}(X)(c))$$

whose Betti realisation coincides with $\mathrm{AJ}_{\mathbb{C}}$ and that permits to define also de Rham and étale p-adic versions of $\mathrm{AJ}_{\mathbb{C}}$. Explicitly, for an irreducible subvariety $Z \in X^c$ such that $\mathrm{cl}([Z]) = 0$, $\mathrm{AJ}_{\mathrm{mot}}([Z])$ is the extension determined by the following diagram with exact rows

$$\begin{array}{ccccccccc} 0 & \longrightarrow & h^{2c-1}(X)(c) & \longrightarrow & \mathrm{AJ}_{\mathrm{mot}}([Z]) & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\ & & \downarrow \mathrm{id} & & \downarrow & & \downarrow & \searrow^{\gamma_{\mathrm{mot}}} & \\ 0 & \longrightarrow & h^{2c-1}(X)(c) & \longrightarrow & h^{2c-1}(X \smallsetminus Z)(c) & \longrightarrow & h^{2c}_Z(X)(c) & \xrightarrow{\beta} & h^{2c}(X)(c) \end{array}$$

where the bottom row is Gysin exact sequence and $\gamma_{\mathrm{mot}} = 0$ since $[Z] \in \mathrm{CH}^c(X)_0$.

For a prime p and an integer $c \geq 0$, let

$$V := \mathrm{H}^{2c}_{\mathrm{et}}(X_{\bar{\mathbb{Q}}}, \mathbb{Q}_p(c)).$$

The *étale p-adic Abel–Jacobi map*

$$\mathrm{AJ}^{\mathrm{et}}_p : \mathrm{CH}^c(X/\mathbb{Q})_0 \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V)$$

is the map obtained by taking the étale realisation of the above construction, and using the fact that $\mathrm{H}^1(\mathbb{Q}_p, V) = \mathrm{Ext}^1(\mathbb{Q}_p, V)$ (see Remark (1.10)). Analogously we get a *(de Rham) p-adic Abel–Jacobi map*

$$\mathrm{AJ}_p : \mathrm{CH}(X/\mathbb{Q}_p)_0 \longrightarrow \mathrm{Fil}^{d-c+1}\, \mathrm{H}^{2d-2c+1}_{\mathrm{dR}}(X/\mathbb{Q}_p)^{\vee}. \tag{1.11.9}$$

Here we are using that $\mathrm{Fil}^{d-c+1}\, \mathrm{H}^{2d-2c+1}_{\mathrm{dR}}(X/\mathbb{C})^{\vee} \cong \mathrm{Ext}^1_{\mathrm{ffm}}(\mathbb{Q}_p, \mathrm{H}^{2c-1}_{\mathrm{dR}}(X/\mathbb{Q}_p))$, where $\mathrm{Ext}^1_{\mathrm{ffm}}$ denotes extensions of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules endowed with filtration and Frobenius (see, e.g. [BDP13, Proposition 3.5]). Notice that we can extend the above definition to

$$\mathrm{AJ}_p : \mathrm{CH}(X/K)_0 \longrightarrow \mathrm{Fil}^{d-c+1}\, \mathrm{H}^{2d-2c+1}_{\mathrm{dR}}(X/K)^{\vee}.$$

for all p-adic fields K.

1.12 Complex L-functions

We recall here the definition and the main properties and conjectures of the complex L-functions attached to Galois representations and to motives. We will then focus on the complex L-functions that will be used in Chapters 2, 3, and 4, and which are involved in the construction of the p-adic L-functions appearing in §1.15.

Let K, L be two number fields and let

$$\{\rho_\lambda : G_K \longrightarrow \mathrm{Aut}(V_\lambda)\}_\lambda$$

be a compatible system of λ-adic representations, where V_λ are L_λ vector spaces, and let S be the ramification set for $V := \{V_\lambda\}_\lambda$ as in Definition 1.26. For each prime $\mathfrak{p}$ of K, denote

$$L_{\mathfrak{p}}(V, x) := L_{\mathfrak{p}}(V_\lambda, s) := \det(1 - x \cdot \rho_\lambda(\mathrm{Frob}_{\mathfrak{p}}^{-1}) | V_\lambda^{I_{\mathfrak{p}}}) \tag{1.12.1}$$

the characteristic polynomial of $\mathrm{Frob}_{\mathfrak{p}}^{-1}$ acting on $V_\lambda^{I_{\mathfrak{p}}}$, where λ is any prime of L such that $\mathfrak{p} \notin S_\lambda$. Recall that, for $\mathfrak{p} \notin S$, the polynomial $L_{\mathfrak{p}}(V_\lambda, s) \in L[x]$ does not depend on λ. We will futher assume that $L_{\mathfrak{p}}(V_\lambda, s)$ is independent of λ *for all* $\mathfrak{p}$.

Definition 1.78. For a prime $\mathfrak{p}$ of K, let $q_\mathfrak{p} := \mathrm{N}_K(\mathfrak{p})$. The complex L-function attached to the compatible system of λ-adic representations $V = \{V_\lambda\}$ is

$$L(V, s) := \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)} L_\mathfrak{p}(V, q_\mathfrak{p}^{-s}).$$

In the definition above, s is a complex variable, and the product defining $L(\{V_\lambda\}_\lambda, s)$ converges for $\mathrm{Re}(s) >> 0$.

Let $M = (X, e, m)$ be a Chow (or a Grothendieck) motive over K with coefficients in L which is pure of weight w. Assume that

$$M_{\mathrm{et}} = \{M_{\mathrm{et},\lambda} \subseteq \mathrm{H}^i_{\mathrm{et}}(X_{\overline{K}}, \mathbb{Q}_\ell(n)) \otimes L\}_{\lambda \in \mathrm{Spec}(\mathcal{O}_L)}$$

form a compatible system of λ-adic representations (then $w = i - 2n$). Let S be the ramification set for M_{et}. Using the definition above, we can attach to M a complex L-function.

Definition 1.79. The complex L-function attached to the motive $M \in \mathcal{M}(K)_L$ is

$$L(M, s) := \prod_{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_K)} L_\mathfrak{p}(M, q_\mathfrak{p}^{-s}),$$

where, for each prime $\mathfrak{p}$ of K,

$$L_\mathfrak{p}(M, q_\mathfrak{p}^{-s}) := L_\mathfrak{p}(M_{\mathrm{et}}, q_\mathfrak{p}^{-s}).$$

The following properties follow directly by computing the local factors (1.12.1). Let M, M_1, M_2 be motives as above. Then for all prime $\mathfrak{p}$ of K we have

1. $L_\mathfrak{p}(M_1 \oplus M_2, s) = L(M_1, s) \cdot L(M_2, s)$;
2. $L_\mathfrak{p}(M(n), s) = L_\mathfrak{p}(M, s + n)$ for all $n \in \mathbb{Z}$;
3. $L_p(\mathrm{Res}_{K/\mathbb{Q}}(M), s) = \prod_{\mathfrak{p}|p} L_\mathfrak{p}(M, s)$

There is an explicit receipt using Gamma functions for the definition of an *Euler factor at* ∞ in terms of the Hodge decomposition of the realisation M_B.

Let $\nu : K \hookrightarrow \mathbb{C}$ be an embedding and let $X_\nu := X \times_{\nu, K} \mathbb{C}$. Consider the corresponding Betti realisation and Hodge decomposition

$$M_{\mathrm{B},\nu} = \mathrm{H}^i_{\mathrm{B}}(X_\nu(\mathbb{C}), \mathbb{Q}), \qquad M_{\mathrm{B},\nu} \otimes \mathbb{C} = \bigoplus_{p+q=i} \mathrm{H}^{p,q}.$$

Let F_∞ be the involution acting on $M_{\mathrm{B},\nu} \otimes \mathbb{C}$ given by complex conjugation, and recall that $F_\infty(\mathrm{H}^{p,q}) = \mathrm{H}^{q,p}$, so there is a decomposition $\mathrm{H}^{p,p} = \mathrm{H}^{p,p,+} \oplus \mathrm{H}^{p,p,-}$ in F_∞-eigenspaces. Denote $h^{p,q} := \dim \mathrm{H}^{p,q}$ and $h^{p,\pm} := \dim \mathrm{H}^{p,p,\pm}$.

Definition 1.80. The *Euler factor at* ∞ attached to M is

$$L_\infty(M, s) := \prod_{\nu : K \hookrightarrow \mathbb{C}} \begin{cases} \prod_{p+q=i, p<q} \Gamma_\mathbb{C}(s-p)^{h^{p,q}} & i \text{ odd} \\ (\prod_{p+q=i, p<q} \Gamma_\mathbb{C}(s-p)^{h^{p,q}}) \Gamma_\mathbb{R}(s - i/2)^{h^{i/2,+}} \Gamma_\mathbb{R}(s + 1 - i/2)^{h^{i/2,-}} & i \text{ even,} \end{cases}$$

where

$$\Gamma_\mathbb{R}(s) := \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_\mathbb{C}(s) := 2(2\pi)^{-s}\Gamma(s).$$

The *completed L-function* attached to M is

$$\Lambda(M, s) := L(M, s) \cdot L_\infty(M, s).$$

The function $L_\infty(M,s)$ is meromorphic, and can only have poles at points $s = m \in \mathbb{Z}$ such that $m \leq w/2$.

Conjecture 1.81. *i)* The complex L-function $L(M,s)$ admits a meromorphic continuation to $\mathbb{C}$ and it has no zeroes on $\mathrm{Re}(s) \geq w/2+1$. Moreover, if the étale realisation M_{et} are irreducible, then $L(M,s)$ is entire, except if $M = \mathbb{Q}(n)$, in which case $L(M,s)$ has a unique pole at $s = n+1$.

ii) The completed L-function satisfies the following functional equation

$$\Lambda(M,s) = \varepsilon(M,s) \cdot \Lambda(M^\vee(1), -s) \ (= \varepsilon(M,s) \cdot \Lambda(M, w+1-s)), \tag{1.12.2}$$

where $\varepsilon(M,s) = a \cdot b^s$ with $a \in \mathbb{C}, b \in \mathbb{R}_{>0}$.

We will see in the next sections that this conjecture in known to hold for many of the motives we will be considering in this thesis. If M satisfies Conjecture 1.81, then the *center of the functional equation* (1.12.2) is $c := (w+1)/2$, $L(M,s)$ has no pole at $s = c$ and in this case $\varepsilon(M,c) \in \{\pm 1\}$ is called the *sign* of M. The motive $M = h^i(X)(n)$ is *critical* if $L_\infty(M,s)$ has no zeroes nor poles at $s = c$.

1.12.1 The complex L-functions attached to a modular form

Let

$$f = \sum_{n \geq 1} a_n(f) q^n \in S_k(N, \chi)$$

be a newform and let M_f be the motive attached to f as in §1.10.4. The L-function attached to f can be expressed as a power series

$$L(f,s) := L(M_f, s) = \prod_p (1 - a_p(f)p^{-s} + \chi_f(p)p^{k-1}p^{-2s})^{-1} = \sum_{n \geq 1} \frac{a_n(f)}{n^s}.$$

The Euler product (and the power series) above converges uniformly to a holomorphic function on $\mathrm{Re}(s) > 1 + k/2$, and extends to an entire function on $\mathbb{C}$. Moreover, the completed L-function

$$\Lambda(f,s) = L_\infty(f,s) \cdot L(f,s) = N^{s/2}(2\pi)^{-s}\Gamma(s) \cdot L(f,s)$$

is entire and satisfies the functional equation

$$\Lambda(f,s) = \varepsilon(f) \cdot \Lambda(f, k-s),$$

with $\varepsilon(f) \in \{\pm 1\}$. In other words, the motive attached to f satisfies Conjecture 1.81. In particular, if f is the newform attached to an elliptic curve E defined over $\mathbb{Q}$, then $L(f,s) = L(E/\mathbb{Q}, s)$ and the center of the functional equation is $s = 1$. The order of vanishing of $L(E,s)$ at this point is the main object of the Birch and Swinnerton-Dyer Conjecture (see §1.14). Recall that, by Theorem 1.40, the complex L-function attached to an elliptic curve defined over $\mathbb{Q}$ is always of this form.

1.12.2 The complex L-function attached to a modular form and a Hecke character

Let $f \in S_k(N, \chi)$ be a normalised newform and let ψ be a Hecke character of an imaginary quadratic field K of infinity type (ℓ_1, ℓ_2), conductor $\mathfrak{c}_\psi$ and central character ε_ψ. We can attach to the pair (f, ψ) the complex L-function

$$L(f,\psi,s) := L(M_f \otimes M(\psi), s) = L(V_f \otimes \mathrm{Ind}_K^{\mathbb{Q}}(\psi), s).$$

It is defined as an Euler product as in Definition 1.79, and it can be completed to a meromorphic function

$$\Lambda(f, \psi, s) = L_\infty(f, \psi, s)L(f, \psi, s),$$

that is an entire function if $\chi \cdot \chi_K \cdot \varepsilon_\psi \neq 1$, where χ_K is the quadratic Dirichlet character attached to K. Moreover, it satisfies a functional equation of the form

$$\Lambda(f, \psi, s) = \epsilon(f, \psi)\Lambda(\bar{f}, \bar{\psi}, k + \ell_1 + \ell_2 - s).$$

Following the terminology of [BDP13], the character ψ is said to be *central critical for* f if $\Lambda(f, \psi^{-1}, s)$ is selfdual, $s = 0$ is the center of simmetry in the functional equation, and the factor $L_\infty(f, \psi^{-1}, s)$ has no poles at $s = 0$. It will be useful to recall that, if ψ is central critical for f, then it satisfies

$$k = \ell_1 + \ell_2, \quad \text{and} \quad \varepsilon_\psi = \chi. \tag{1.12.3}$$

1.12.3 The Garrett–Rankin triple product complex L-function

Let

$$f \in S_k(N_f, \chi_f), \; g \in S_\ell(N_g, \chi_g), \; h \in S_m(N_h, \chi_h)$$

be three normalised newforms, cuspidal if they have weight ≥ 2, and assume that

$$\chi_f \cdot \chi_g \cdot \chi_h = 1. \tag{1.12.4}$$

We denote by V_f, V_g and V_h the corresponding 2-dimensional p-adic Galois representations.

Definition 1.82. The *Garrett–Rankin triple product L-function* is the complex L-function

$$L(f \otimes g \otimes h, s) := L(M_f \otimes M_g \otimes M_h, s) = L(V_f \otimes V_g \otimes V_h, s)$$

attached to the tensor product of the motives attached to f, g and h.

It is defined by an Euler product which is absolutely convergent in the half plane $\mathrm{Re}(s) > \frac{k+\ell+m-1}{2}$. With the appropriate Euler factors at infinity, the completed function

$$\Lambda(f \otimes g \otimes h, s) = L_\infty(f \otimes g \otimes h, s)L(f \otimes g \otimes h, s)$$

extends to the whole complex plane and satisfies a functional equation of the form

$$\Lambda(f \otimes g \otimes h, s) = \varepsilon(f, g, h)\Lambda(f \otimes g \otimes h, k + \ell + m - 2 - s), \tag{1.12.5}$$

where $\varepsilon(f, g, h) \in \{\pm 1\}$ is the *sign* of the functional equation. The center of symmetry with respect to (1.12.5) is then

$$c := \frac{k + \ell + m - 2}{2},$$

at which $L(f \otimes g \otimes h, s)$ has no poles. Note that the condition (1.12.4) implies that $k + \ell + m$ is even, so that $c \in \mathbb{Z}$, and moreover, c is a *critical point* for the L-function, meaning that $L_\infty(f \otimes g \otimes h, s)$ has no poles at $s = c$.

Definition 1.83. A triple of weights $(k, \ell, m) \in \mathbb{Z}^3_{\geq 1} \subseteq \mathcal{W}^3$ is called *unbalanced* if one of the weights is greater or equal to the sum of the other two (in which case the greater weight is called *dominant weight*). Otherwise, the triple is called *balanced.*

The sign of the functional equation can be expressed as a product of *local signs* over the places of $\mathbb{Q}$. More precisely, if $N := \mathrm{lcm}(N_f, N_g, N_h)$, then

$$\varepsilon(f, g, h) = \prod_{v | N \cdot \infty} \varepsilon_v(f, g, h), \tag{1.12.6}$$

and the local sign at infinity depends on whether the weights are balaced or unbalanced:

$$\varepsilon_\infty(f, g, h) = \begin{cases} +1 & \iff (k, \ell, m) \text{ unbalanced} \\ -1 & \iff (k, \ell, m) \text{ balanced.} \end{cases} \tag{1.12.7}$$

For more details in the study of the complex L-function, see [PSR87].

1.13 Selmer groups

Let K be a number field and let V be a $\mathbb{Q}_p[G_K]$-module. For each prime ideal λ of K, denote

$$\mathrm{res}_\lambda : \mathrm{H}^1(K, V) \longrightarrow \mathrm{H}^1(K_\lambda, V)$$

the restriction map in Galois cohomology, induced by the inclusion $G_{K_\lambda} \subseteq G_K$, and recall the Bloch–Kato submodule $\mathrm{H}^1_f(K_\lambda, V) \subseteq \mathrm{H}^1(K_\lambda, V)$ introduced in §1.3.

Definition 1.84. The *Bloch–Kato Selmer group* of V is

$$\mathrm{Sel}_p(K, V) := \{x \in \mathrm{H}^1(K, V) \mid \mathrm{res}_\lambda(x) \in \mathrm{H}^1_f(K_\lambda, V)\ \forall\lambda\}.$$

For each prime λ, we denote by ∂_λ the composition

$$\partial_\lambda : \mathrm{H}^1(K, V) \xrightarrow{\mathrm{res}_\lambda} \mathrm{H}^1(K_\lambda, V) \longrightarrow \mathrm{H}^1_s(K_\lambda, V),$$

where the second morphism is the natural quotient map.

Definition 1.85. The *relaxed Selmer group* of V is defined as

$$\mathrm{Sel}_{(p)}(K, V) := \{x \in \mathrm{H}^1(K, V) \mid \mathrm{res}_\lambda(x) \in \mathrm{H}^1_f(K_\lambda, V)\ \forall\lambda \nmid p\}.$$

Notice that

$$\mathrm{Sel}_p(K, V) \subseteq \mathrm{Sel}_{(p)}(K, V).$$

Let $V^* := \mathrm{Hom}_{\mathbb{Q}_p}(V, \mathbb{Q}_p(1))$ be the Kummer dual of V. The Selmer group $\mathrm{Sel}_{p,*}(K, V^*)$ is defined by the local conditions which are dual to $\{\mathrm{H}^1_f(K_\lambda, V)\}_\lambda$ with respect to the local Tate pairings

$$\langle\ ,\ \rangle_\lambda : \mathrm{H}^1(K_\lambda, V) \times \mathrm{H}^1(K_\lambda, V^*) \longrightarrow \mathbb{Q}_p \tag{1.13.1}$$

(cf. (1.4.7)). More precisely, for each λ, define $\mathrm{H}^1_{f,*}(K_\lambda, V^*)$ to be the orthogonal complement of $\mathrm{H}^1_f(K_\lambda, V)$ with respect to (1.13.1); the Selmer group attached to V^* is then

$$\mathrm{Sel}_{p,*}(K, V^*) := \{x \in \mathrm{H}^1(K, V^*) \mid \mathrm{res}_\lambda \in \mathrm{H}^1_{f,*}(K_\lambda, V^*) \ \text{ for all prime } \lambda\}.$$

Finally, the *strict Selmer group* of V^* is the subspace of $\mathrm{Sel}_{p,*}(K, V^*)$ defined as

$$\mathrm{Sel}_{[p],*}(K, V^*) := \{x \in \mathrm{H}^1(K, V^*) \mid \mathrm{res}_\lambda(x) \in \mathrm{H}^1_{f,*}(K_\lambda, V^*)\ \forall\lambda \text{ and } \mathrm{res}_{\mathfrak{p}}(x) = 0 \quad \text{for all } \ \mathfrak{p} \mid p\}.$$

By Poitou–Tate duality (see, for example, [MR04, Theorem 2.3.4]) there is an exact sequence

$$0 \longrightarrow \mathrm{Sel}_p(K, V) \longrightarrow \mathrm{Sel}_{(p)}(K, V) \longrightarrow \oplus_{\mathfrak{p}|p} \mathrm{H}^1_s(K_{\mathfrak{p}}, V) \longrightarrow \mathrm{Sel}_{p,*}(K, V^*)^\vee \longrightarrow \mathrm{Sel}_{[p],*}(K, V^*)^\vee, \tag{1.13.2}$$

where $^\vee$ stands for the $\mathbb{Q}_p$-dual.

We will be interested in the case in which $V = T_pE \otimes \mathbb{Q}_p$ for an elliptic curve E or $V = V_f \otimes V_g \otimes V_h$ in the setting of §1.12.3, so we describe now more explicitly the corresponding Selmer groups.

1.13.1 The Selmer group of an elliptic curve

Let E/K be an elliptic curve defined over a number field K and let $V_E := T_pE \otimes \mathbb{Q}_p$. The Bloch–Kato Selmer group of V_E has to be interpreted, at least conjecturally, as the set of *geometric* cohomology classes in $\mathrm{H}^1(\mathbb{Q}_p, V_E)$, in the sense that we now explain. For all prime λ of K, the local Kummer map

$$\delta_\lambda : E(K_\lambda) \otimes \mathbb{Q}_p \longrightarrow \mathrm{H}^1(K_\lambda, V_E),$$

is injective. The Selmer group of E/K

$$\mathrm{Sel}_p(E/K) := \mathrm{Sel}_p(K, V_E) = \{x \in \mathrm{H}^1(K, V_E) \ \mid \mathrm{res}_\lambda(x) \in \mathrm{image}(\delta_\lambda)\}$$

is then the set of global classes $x \in \mathrm{H}^1(K, V_E)$ which arise from local points $P_\lambda \in E(K_\lambda) \otimes \mathbb{Q}_p$ for all primes λ. Conjecturally, this should be equivalent to the fact that x arises from a global point. More precisely, recall the *Shafarevich–Tate* group

$$Ш(E/K) := \ker\left(\mathrm{H}^1(K, V_E) \longrightarrow \prod_\lambda \mathrm{H}^1(K_\lambda, V_E)\right).$$

The global Kummer map gives an injection

$$\delta : E(K) \otimes \mathbb{Q}_p \longrightarrow \mathrm{Sel}_p(E/K),$$

which is an isomorphism if and only if the p-primary part of $Ш(E/K)$ is finite, which is conjectured to be true by the Shafarevich–Tate conjecture.

1.13.2 The Selmer group of an elliptic curve twisted by a certain Artin representation

Let g, h be two weight one normalised newforms with Nebentype characters χ_g, χ_h respectively such that $\chi_g\chi_h = 1$. Let V_g°, V_h° be the Artin representations attached to g and h and let V_g, V_h be the p-adic representations. Let L be a number field containing the Fourier coefficients of g and h and fix a completion L_p of L at a prime above p. Recall that there are isomorphisms $j_g : V_g^\circ \otimes_L L_p \xrightarrow{\cong} V_g$ and $j_h : V_h^\circ \otimes_L L_p \xrightarrow{\cong} V_h$. We denote $V_{gh}^\circ := V_g^\circ \otimes V_h^\circ$,

$$\rho_{gh} : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_{gh}^\circ) \cong \mathrm{GL}_4(L) \tag{1.13.3}$$

the corresponding Artin representation, where H is the number field cut out by the action of $G_\mathbb{Q}$ on V_{gh}° and

$$j_{gh} : V_{gh}^\circ \otimes_L L_p \xrightarrow{\cong} V_{gh} := V_g \otimes V_h.$$

Let E be an elliptic curve defined over $\mathbb{Q}$ and let f be the corresponding weight 2 normalised newform and let

$$V := V_f \otimes V_g \otimes V_h = V_E \otimes V_{gh}.$$

Since the product of the Nebentype characters of f, g and h is trivial, using the notation of §1.13 we have that $V^* \cong V$. Denoting $E(H)_L := E(H) \otimes L$, the Kummer map and j_{gh} induce an injection

$$\mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}^\circ, E(H)_L) \cong (E(H)_L \otimes V_{gh}^\circ)^{\mathrm{Gal}(H/\mathbb{Q})} \hookrightarrow \mathrm{Sel}_p(\mathbb{Q}, V).$$

1.14 Birch and Swinnerton–Dyer and Beilinson's conjectures

Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$. By the Mordell–Weil Theorem, the abelian group of rational points $E(\mathbb{Q})$ is finitely generated. Let $r(E/\mathbb{Q})$ be the *algebraic rank* of $E/\mathbb{Q}$, i.e. the integer such that

$$E(\mathbb{Q}) \cong \mathbb{Z}^{r(E/\mathbb{Q})} \oplus E(\mathbb{Q})_{\mathrm{tors}},$$

as abelian groups, where $E(\mathbb{Q})_{\mathrm{tors}}$ is the torsion of $E(\mathbb{Q})$, and is a finite group. As explained in §1.12.1, by modularity, the complex L-function $L(E, s)$ admits a holomorphic continuation to the whole complex plane and satisfies a functional equation whose center of symmetry is $s = 1$. The conjecture of Birch and Swinnerton–Dyer (BSD) relates the algebraic rank of

E with the order of vanishing at $s = 1$ of the L-function $L(E/\mathbb{Q}, s)$ attached to the motive $h^1(E) \in \mathcal{M}(\mathbb{Q})_{\mathbb{Q}}$, or, equivalently, to the (compatible system of) representation given by the Tate module of E. Moreover, BSD predicts the exact value of the first nonzero term of the Taylor expansion of $L(E/\mathbb{Q})$, which can be conjecturally expressed in terms of a certain number of arithmetic invariants attached to E. In particular, there should appear the *(real) regulator* attached to $E/\mathbb{Q}$, which we now describe. Let $\hat{h} : E(\bar{\mathbb{Q}}) \longrightarrow \mathbb{R}$ be the Néron–Tate height on E, as described, e. g. in [Sil86, VIII §9]. It induces a bilinear and positive definite pairing

$$\langle\ ,\ \rangle_{\mathrm{NT}} : E(\mathbb{Q}) \otimes \mathbb{R} \times E(\mathbb{Q}) \otimes \mathbb{R} \longrightarrow \mathbb{R}$$

by $\langle P, Q\rangle_{\mathrm{NT}} := \frac{1}{2}(\hat{h}(P + Q) - \hat{h}(P) - \hat{h}(Q))$.

Definition 1.86. Choose a $\mathbb{Z}$-basis $\{P_1, \ldots, P_{r(E)}\}$ of $E(\mathbb{Q})/E(\mathbb{Q})_{\mathrm{tors}}$. The *real regulator* of $E/\mathbb{Q}$ is

$$\mathrm{Reg}_{\infty}(E/\mathbb{Q}) := \det((\langle P_i, P_j\rangle_{\mathrm{NT}})_{i,j}) \in \mathbb{R}_{>0}.$$

For each prime number p choose a minimal Weierstrass equation for E, so that we can consider the curve $E_{\mathbb{F}_p} := E \times_{\mathbb{Z}_p} \mathbb{F}_p$ over $\mathbb{F}_p$, which is defined looking at the Weierstrass equation modulo p, and let $\mathrm{red}_p : E(\mathbb{Q}_p) \longrightarrow E_{\mathbb{F}_p}(\mathbb{F}_p)$ be the reduction map.

Definition 1.87. Let $E_0(\mathbb{Q}_p)$ be the set of points $P \in E(\mathbb{Q}_p)$ such that $\mathrm{red}_p(P)$ is a nonsingular point of the curve $E_{\mathbb{F}_p}$. The integer

$$c_p := [E(\mathbb{Q}_p) : E_0(\mathbb{Q}_p)]$$

is called the *Tamagawa number* at p.

Notice that $c_p = 1$ for almost all p. More precisely, it is > 1 precisely when $E_{\mathbb{F}_p}$ is a singular curve, i.e. when p divides the conductor of E.

Conjecture 1.88 (BSD). *i)* $\mathrm{ord}_{s=1}L(E/\mathbb{Q}, s) = r(E/\mathbb{Q})$.

ii)

$$\lim_{s\to 1} \frac{L(E/\mathbb{Q}, s)}{(s-1)^{r(E/\mathbb{Q})}} = \frac{2^{r(E/\mathbb{Q})}\Omega_E \#\text{Ш}(E/\mathbb{Q})\mathrm{Reg}_{\infty}(E/\mathbb{Q}) \prod_p c_p}{\#E(\mathbb{Q})^2_{\mathrm{tors}}}, \tag{1.14.1}$$

where, if ω_E denotes the Néron differential of E, then $\Omega_E := \int_{E(\mathbb{R})} |\omega_E|$.

Remark 1.89.
1. In Chapter 2, we will state the elliptic Stark conjecture, which predicts the expression of a special value of a *p-adic* L-function. This value conjecturally involves a *p-adic* regulator attached to the elliptic curve twisted by an Artin representation (see Definition 2.5). The generalisation to higher weights will involve a p-adic regulator attached to a triple of modular forms (cf. Definition 2.12 and Definition 2.15).

2. [1]

3. If E is an elliptic curve defined over a number field, then the abelian group of K-rational points of E is still finite generated, and we can naturally consider the algebraic rank $r(E/K)$ satifying $E(K) \cong \mathbb{Z}^{r(E/K)} \oplus E(K)_{\mathrm{tors}}$. In this generality, it is not known if the complex L-function attached to E/K has analytic continuation or satisfies a functional equation, although it conjecturally does. There is a more general version of Conjecture 1.88 which predicts, in particular, that

$$\mathrm{ord}_{s=1}L(E/K, s) = r(E/K).$$

[1]**FIXME:** express c_p in term of the group of connected component: notation of Chapter 4

4. For an elliptic curve E with complex multiplication by an imaginary quadratic field K which is defined over $F \in \{K, \mathbb{Q}\}$, Coates and Wiles in [CW77] proved that if $\mathrm{ord}_{s=1}L(E/F,s) = 0$, then $r(E/F) = 0$.

5. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N defined over $\mathbb{Q}$ and let K be an imaginary quadratic field K of discriminant $-D_K$. Assume that the pair (N, K) satisfies the following *Heegner Hypothesis*

   Assumption 1.90. There exists an ideal $\mathcal{N}$ of $\mathcal{O}_K$ coprime to D_K such that $\mathcal{O}_K/\mathcal{N} \cong \mathbb{Z}/N\mathbb{Z}$.

   The theory of complex multiplication allows to construct a so-called *Heegner point* $y_K \in E(K)$, which is related to the derivative of the complex L-function of E over K at $s = 1$ via the *Gross–Zagier formula* of [GZ86]

   $$L'(E/K,1) = \frac{\int_{E(\mathbb{C})} \omega_E \wedge i\overline{\omega_E}}{\#(\mathcal{O}_K^\times)^2\sqrt{D_K}}\hat{h}(y_K). \tag{1.14.2}$$

   If $\mathrm{ord}_{s=1}L(E/K,s) = 1$, then y_K is a point of $E(K)$ of infinite order, so $r(E/K) \geq 1$. Conversely, using the *Euler system of Heegner points*, Kolyvagin proved in [Kol90] that $r(E/K) \leq 1$.

6. Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$. If $\mathrm{ord}_{s=1}L(E/\mathbb{Q},s) \leq 1$, then $\mathrm{ord}_{s=1}L(E/\mathbb{Q},s) = r(E/\mathbb{Q})$. The proof of this fact, due to Kolyvagin, uses the results described in the previous point, together with Heegner points' theory for bounding Selmer groups.

Let $\rho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_\rho)$ be an Artin representation, where V_ρ stands for a finite dimensional vector space over a number field $L \subseteq \mathbb{C}$. If we denote $V_E := T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, then $V_E \otimes_{\mathbb{Q}} V_\rho$ is a system of compatible p-adic representations and one can attach to it an L-series $L(E \otimes \rho, s)$ as explained in §1.12. It is defined by an Euler product that converges for $\mathrm{Re}(s) > 3/2$, but it is expected to have analytic continuation to the whole complex plane. The Galois equivariant refinement of the BSD conjecture then predicts that the analytic order of this complex L-function should control the multiplicity of ρ in the representation $E(H) \otimes L$. More precisely, the role of the Mordell–Weil group in this setting is played by

$$\mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_\rho, E(H)_L),$$

and the generalisation of the BSD conjecture to this setting is the following.

Conjecture 1.91 (Galois equivariant BSD)**.** Let $r(E,\rho) := \dim_L \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_\rho, E(H)_L)$. Then

$$\mathrm{ord}_{s=1}L(E \otimes \rho, s) = r(E,\rho).$$

Remark 1.92. 1. The group $\mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_\rho, E(H)_L)$ has to be regarded as a generalisation of the Mordell–Weil group $E(\mathbb{Q})$ in the setting of Conjecture 1.88. Indeed, if $\rho = \rho_{gh}$ is the Artin representation (1.13.3) attached to two weight one modular forms of Nebentype χ_g, χ_h such that $\chi_g\chi_h = 1$, then V_{gh}° is selfdual and so

$$\mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_\rho, E(H)_L) = \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}^\circ, E(H)_L) \cong (E(H)_L \otimes V_{gh}^\circ)^{\mathrm{Gal}(H/\mathbb{Q})}$$

is the group considered in §1.13.2, which embed into $\mathrm{Sel}_p(\mathbb{Q}, V_E \otimes V_{gh})$.

2. If $\rho : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow L^\times$ is a character, then Kato in [Kat04] proved that if $\mathrm{ord}_{s=1}L(E \otimes \rho, s) = 0$, then $r(E,\rho) = 0$.

3. Let K be an imaginary quadratic field such that (E, K) satisfy the Heegner Hypothesis and let $\psi : \mathrm{Gal}(H/K) \longrightarrow L^\times$ be an *anticyclotomic* character, i.e. such that H/K is abelian and $H/\mathbb{Q}$ is dihedral. Let $\rho_\psi := \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$ and consider the complex L-function

$$L(E \otimes \rho_\psi, s) = L(E \otimes \psi, s) = L(f_E, \psi, s)$$

of §1.12.2, where f is the newform attached to $E/\mathbb{Q}$ by modularity. Then Zhang proved in [Zha01a] a generalisation of the Gross–Zagier formula in this setting. On the other hand, Kolyvagin's method has been generalised in [BD90], [YZZ13]. Combining these results we have that, if $\mathrm{ord}_{s=1} L(E \otimes \rho_\psi, s) \leq 1$, then $\mathrm{ord}_{s=1} L(E \otimes \rho_\psi, s) = r(E, \rho_\psi)$.

4. If ρ is an odd irreducible two-dimensional Artin representation satisfying some mild restrictions, by [BDR15], if $\mathrm{ord}_{s=1} L(E \otimes \rho, s) = 0$, then $r(E, \rho) = 0$.

5. If $\rho = \rho_{gh}$ as in (1.13.3), then by [DR17], if $\mathrm{ord}_{s=1} L(E \otimes \rho_{gh}, s) = 0$, then $r(E, \rho_{gh}) = 0$.

Yet very little progress has been made for the general case and for ranks ≥ 2, essentially due to the lack of a systematic construction of points in that case (as Heegner points are nontorsion essentially only in the rank 1 setting). Generalisations of these conjectures are known for general motives and are due to Beilinson, Bloch and Kato. We describe now some specific instances of these conjectures which are related to the p-adic conjectures described in Chapter 2 and to the p-adic formulas that we will prove in Chapters 3 and 4.

Let $f \in S_k(N, \chi)$ be a normalised newform of even weight $k \geq 2$ and recall the motive $M_f = (W_{k-2}, e_f, 0)$ over $\mathbb{Q}$ attached to f as in §1.10.4. Let $\rho : G_{\mathbb{Q}} \longrightarrow \mathrm{Aut}(V)$ be an Artin representation and let H be the number field cut out by ρ. Let L be a number field containing the Fourier coefficients of f and such that V is an L-vector space. A conjecture due to Beilinson predicts that

$$\mathrm{ord}_{s=k/2} L(M_f \otimes \rho, s) \stackrel{?}{=} \dim_L \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V, e_f \cdot \mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L}).$$

Let now

$$f \in S_k(N_f, \chi_f), \qquad g \in S_\ell(N_g, \chi_g), \qquad h \in S_m(N_h, \chi_h)$$

be normalised newforms such that $\chi_f \chi_g \chi_h = 1$ and $k, \ell, m \geq 2$, and L be a number field containing the all Fourier coefficients of f, g and h. Recall the complex L-function

$$L(f \otimes g \otimes h, s) = L(M_f \otimes M_g \otimes M_h, s)$$

described in §1.12.3, and let $c := (k+\ell+m-2)/2$ be the centre of symmetry for the corresponding functional equation. Then Beilinson's conjecture predicts that

$$\mathrm{ord}_{s=c} L(f \otimes g \otimes h, s) \stackrel{?}{=} \dim_L e_{fgh} \, \mathrm{CH}^c(W_{k-2} \times W_{\ell-2} \times W_{m-2}/\mathbb{Q})_{0,L}, \qquad (1.14.3)$$

where $e_{fgh} := e_f \otimes e_g \otimes e_h$. Notice that, with the notation of §1.10.1,

$$e_{fgh} \, \mathrm{CH}^c(W_{k-2} \times W_{\ell-2} \times W_{m-2}/\mathbb{Q})_0 = \mathrm{CH}(M_f \otimes M_g \otimes M_h).$$

1.15 p-adic L-functions

A common characteristic of p-adic L-functions is that they can be defined by interpolating special values of complex L-functions. Since complex L-functions are attached (at least conjecturally) to motives, this process can be seen in many instances as associating a p-adic L-function to a p-adic family of motives. In these cases, if the domain of the p-adic L-function is a p-adic space $\mathcal{W}$ there is a family of motives $\mathbf{M} = \{M_x\}_{x \in \Sigma^{\mathrm{cl}}}$ parametrised by a subset $\Sigma^{\mathrm{cl}} \subseteq \mathcal{W}$ of *classical* points which is dense with respect to the Zariski topology. The region of interpolation is a subset $\Sigma^{\mathrm{int}} \subseteq \Sigma^{\mathrm{cl}}$, again dense within $\mathcal{W}$, with the property that for each $x \in \Sigma^{\mathrm{int}}$ there is a canonical period $\Omega_x \in \mathbb{C}$ in the sense of Deligne such that the value of the complex L-function $L(M_x, s)/\Omega_x$ at its critical point c_x is algebraic. When multiplied by an appropriate Euler p-factor $\mathcal{E}(M_x) \in \bar{\mathbb{Q}}$, these values can be p-adically interpolated to a rigid-analytic function on $\mathcal{W}$. The p-adic L-function attached to $\mathbf{M}$ is then a function $\mathcal{L}_p(\mathbf{M}, s) : \mathcal{W} \longrightarrow \mathbb{C}_p$ such that

$$\mathcal{L}_p(\mathbf{M}, x) = \frac{\mathcal{E}(M_x)}{\Omega_x} L(M_x, c_x), \qquad \forall\, x \in \Sigma^{\mathrm{int}}.$$

A prototypical and classical example of this situation is the Kubota–Leopoldt p-adic L-function associated to an even Dirichlet character χ. The interpolation formula for this p-adic L-function $\mathcal{L}_p(\chi, s)$ reads

$$\mathcal{L}_p(\chi, k) = (1 - \chi(p)\omega(p)^{-k}p^{-k})L(\chi\omega^{-k}, k) \qquad \forall\, k \in \mathbb{Z}_{\leq 0},$$

where $\omega : \mathbb{Z}/p\mathbb{Z} \longrightarrow \mathbb{C}^\times$ is the Teichmüller character. Therefore, $\mathcal{L}_p(\chi, s)$ for $s \in \mathbb{Z}_p$ can be interpreted as the p-adic L-function associated to $\{\mathbb{Q}(\chi)(k)\}_{k\in\mathbb{Z}}$, the family if Tate twists of the Dirichlet motive $\mathbb{Q}(\chi)$ attached to χ, with region of interpolation $\Sigma^{\mathrm{int}} = \mathbb{Z}_{\leq 0}$.

1.15.1 Katz's p-adic L-function

Let K be an imaginary quadratic field of discriminant $-D_K < 0$ and class number h_K. Fix an ideal $\mathfrak{c}$ of the ring of integers $\mathcal{O}_K$ of K, and let $\Sigma(\mathfrak{c})$ be the set of Hecke characters of K with conductor dividing $\mathfrak{c}$. Let p be a prime number such that splits in K as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. Define

$$\begin{aligned}
\Sigma_K^{(1)}(\mathfrak{c}) &:= \{\psi \in \Sigma(\mathfrak{c}) \text{ of infinity type } (\ell_1, \ell_2) \mid \ell_1 \leq 0, \ell_2 \geq 1\};\\
\Sigma_K^{(2)}(\mathfrak{c}) &:= \{\psi \in \Sigma(\mathfrak{c}) \text{ of infinity type } (\ell_1, \ell_2) \mid \ell_1 \geq 1, \ell_2 \leq 0\};\\
\Sigma_K(\mathfrak{c}) &:= \Sigma_K^{(1)}(\mathfrak{c}) \sqcup \Sigma_K^{(2)}(\mathfrak{c}).
\end{aligned}$$

For each $\psi \in \Sigma_K(\mathfrak{c})$, the point $s = 0$ is central critical for the complex L-function $L(\psi^{-1}, s)$. In [Kat76], Katz attached to K a p-adic L-function $\mathcal{L}_p(K)$, defined on the completion of $\Sigma_K^{(2)}(\mathfrak{c})$ with respect to an adequate p-adic topology and characterised by the following interpolation property.

Theorem 1.93. *For each $\psi \in \Sigma_K^{(2)}(\mathfrak{c})$ with infinity type (ℓ_1, ℓ_2), we have*

$$\mathcal{L}_p(K)(\psi) = \mathfrak{a}(\psi)\mathfrak{e}(\psi)\mathfrak{f}(\psi)\left(\frac{\Omega_p}{\Omega}\right)^{\ell_1-\ell_2} L_{\mathfrak{c}}(\psi^{-1}, 0),$$

where

i) $L_{\mathfrak{c}}(\psi^{-1}, s)$ is the product of all the Euler factors defining $L(\psi^{-1}, s)$ except the ones corresponding to the primes dividing $\mathfrak{c}$,

ii) $\mathfrak{a}(\psi) = (\ell_1 - 1)!\pi^{-\ell_2}$,

iii) $\mathfrak{e}(\psi) = (1 - \psi(\mathfrak{p})p^{-1})(1 - \psi^{-1}(\bar{\mathfrak{p}}))$,

iv) $\mathfrak{f}(\psi) = (D_K)^{\ell_2/2}2^{-\ell_2}$,

v) If c denotes the smallest positive integer in $\mathfrak{c}$ and $\mathcal{O}_c := \mathbb{Z} + c \cdot \mathcal{O}_K$ is the order of K of conductor c, then Ω and Ω_p are complex and p-adic periods attached to an elliptic curve with complex multiplication by $\mathcal{O}_c$, as defined in [BDP13, (5.1.15) and (5.2.2)].

From the functional equation satisfied by the complex Hecke L-function $L(\psi^{-1}, s)$, it follows a functional equation for Katz's p-adic L-function. In particular, let ψ' be the Hecke character defined by

$$\psi'(\sigma) = \psi(\sigma_0 \sigma \sigma_0^{-1}) \tag{1.15.1}$$

for $\sigma \in G_{\mathbb{Q}}$, where σ_0 is any lift of the nontrivial involution of $K/\mathbb{Q}$. If ψ is a finite order character such that $(\psi')^{-1} = \psi$, then

$$\mathcal{L}_p(K)(\psi) = \mathcal{L}_p(K)(\psi \, \mathrm{N}_K) \tag{1.15.2}$$

where $\mathrm{N}_K = \mathrm{N} \circ \mathrm{N}_{K/\mathbb{Q}}$ is the norm character of K.

1.15.2 The Bertolini–Darmon–Prasanna p-adic L-function

Let $f \in S_k(N, \chi)$ be a normalised newform, fix an imaginary quadratic field K of determinant $-D_K$ and let χ_K be the quadratic character attached to K. Let Σ be the set Hecke characters of K which are central critical for f. Since each $\psi \in \Sigma$ satisfies (1.12.3), the set Σ decomposes as

$$\Sigma = \Sigma^{(1)} \sqcup \Sigma^{(2)} \sqcup \Sigma^{(2')},$$

where

1. $\Sigma^{(1)} := \{\psi \in \Sigma \mid 1 \leq \ell_1 \leq k-1, \ 1 \leq \ell_2 \leq k-1\}$,
2. $\Sigma^{(2)} := \{\psi \in \Sigma \mid \ell_1 \geq k, \ \ell_2 \leq 0\}$,
3. $\Sigma^{(2')} := \{\psi \in \Sigma \mid \ell_1 \leq 0, \ \ell_2 \geq k\}$.

Assume that the pair (N, K) satisfies the Heegner Hypothesis (see 1.90) and let $\mathcal{N}$ be a cyclic ideal of $\mathcal{O}_K$ of norm N. We now recall the definition of the p-adic L-function attached to f and K constructed in [BDP13] that interpolates central critical values $L(f, \psi^{-1}, 0)$. We will denote $\Sigma(c, \mathcal{N}, \chi)^{(j)}$ the elements of $\Sigma^{(j)}$ of finite type $(c, \mathcal{N}, \chi)$ (see Definition 1.103). In [BDP13] the p-adic L-function

$$\mathcal{L}_p(f, K) : \hat{\Sigma}(c, \mathcal{N}, \chi)^{(2)} \longrightarrow \mathbb{C}_p$$

is defined on the completion of $\Sigma(c, \mathcal{N}, \chi)^{(2)}$ with respect to an adequate p-adic topology, and it is characterised by the following interpolation property.

Proposition 1.94. *For each $\psi \in \Sigma(c, \mathcal{N}, \chi)^{(2)}$ of infinity type $(k+j, -j)$ with $j \geq 0$,*

$$\mathcal{L}_p(f, K)(\psi) = \left(\frac{\Omega_p}{\Omega}\right)^{2(k+j)} \mathfrak{e}(f, \psi)^2 \mathfrak{a}(f, \psi) \mathfrak{f}(f, \psi) L(f, \psi^{-1}, 0)$$

where

i) Ω, Ω_p *are the periods appearing in Theorem 1.93,*

ii) $\mathfrak{e}(f, \psi) = 1 - \psi^{-1}(\bar{\mathfrak{p}}) a_p(f) + \psi^{-2}(\bar{\mathfrak{p}}) \chi(p) p^{k-1}$,

iii) $\mathfrak{a}(f, \psi) = \pi^{k+2j-1} j! (k+j-1)!$,

iv) $\mathfrak{f}(f, \psi) = \dfrac{2^{k+2j-2}}{(c\sqrt{D_K})^{k+2k-1}} \prod_{q|c} \dfrac{q - \chi_K(q)}{q-1} \omega(f, \psi)^{-1} \#(\mathcal{O}_K^{\times})$, *where $\omega(f, \psi)$ is the scalar of complex norm 1 defined in [BDP13, (5.1.11)].*

Castella constructed in [Cas19] a two variable p-adic L-function that interpolates the square roots of $\mathcal{L}_p(f, K)(\psi)$ for forms f varying in a Hida family. More precisely, if $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ is a Hida family, Castella's contruction provides a two variable function $\mathcal{L}_p(\mathbf{f}, K)(k, \psi)$ (defined on an appropriate weight space $\mathcal{W}$) such that for $(k, \psi) \in \mathcal{W}^{\mathrm{cl}}$ one has

$$\mathcal{L}_p(\mathbf{f}, K)(k, \psi)^2 = \mathcal{L}_p(f_k, K)(\psi).$$

1.15.3 Triple product p-adic L-functions

In this section we introduce the p-adic L-functions attached to a triple of Hida families. These are the main objects of study of this thesis, whose results appear in Chapters 2, 3 and 4.

Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be three Hida families of tame levels $N_{\mathbf{f}}, N_{\mathbf{g}}, N_{\mathbf{h}}$ and tame Nebentype characters $\chi_{\mathbf{f}}, \chi_{\mathbf{g}}, \chi_{\mathbf{h}}$ such that

$$\chi_{\mathbf{f}} \cdot \chi_{\mathbf{g}} \cdot \chi_{\mathbf{h}} = 1.$$

As in [Hsi19, Hypothesis (sf) and (CR)], we assume the following hypothesis.

Assumption 1.95.
1. $\gcd(N_{\mathbf{f}}, N_{\mathbf{g}}, N_{\mathbf{h}})$ is squarefree;
2. the residual representation $\bar{\rho}_{\mathbf{g}} : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}_2(\bar{\mathbb{F}}_p)$ is absolutely irreducible and p-distinguished (i.e. its semisimplification does not act as multiplication by scalard when restricted to a decomposition group at p).

Define the set $\mathcal{W}^{\circ}_{\mathbf{fgh}} := \mathcal{W}^{\circ}_{\mathbf{f}} \times \mathcal{W}^{\circ}_{\mathbf{g}} \times \mathcal{W}^{\circ}_{\mathbf{h}}$ of triples of classical cristalline points for $\mathbf{f}, \mathbf{g}, \mathbf{h}$. It can be decomposed as

$$\mathcal{W}^{\circ}_{\mathbf{fgh}} = \mathcal{W}^{f}_{\mathbf{fgh}} \sqcup \mathcal{W}^{g}_{\mathbf{fgh}} \sqcup \mathcal{W}^{h}_{\mathbf{fgh}} \sqcup \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}},$$

where $\mathcal{W}^{f}_{\mathbf{fgh}}$ is the set of triples $(\nu_1, \nu_2, \nu_3) \in \mathcal{W}^{\circ}_{\mathbf{fgh}}$ of unbalanced weights with ν_i dominant, i.e. such that, if ν_i have weight k_i for $i \in \{1, 2, 3\}$, then $k_1 \geq k_2 + k_3$. The sets $\mathcal{W}^{g}_{\mathbf{fgh}}$ and $\mathcal{W}^{h}_{\mathbf{fgh}}$ are defined similarly, with the weight ν_2 and ν_3 dominant respectively, and $\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}} := \{(\nu_1, \nu_2, \nu_3) \in \mathcal{W}^{\circ}_{\mathbf{fgh}}$ of balanced weights$\}$.

Let $N := \mathrm{lcm}(N_{\mathbf{f}}, N_{\mathbf{g}}, N_{\mathbf{h}})$ and define

$$S^{\mathrm{ord}}_{\Lambda_{\mathbf{f}}}(N, \chi_f)[\mathbf{f}] := \{\breve{\mathbf{f}} \in S^{\mathrm{ord}}_{\Lambda_{\mathbf{f}}}(N, \chi_f) \mid T_\ell \breve{\mathbf{f}} = a_\ell(\mathbf{f})\breve{\mathbf{f}} \;\text{ for } \ell \nmid Np;\; U_p\breve{\mathbf{f}} = a_p(\mathbf{f})\breve{\mathbf{f}}\} \tag{1.15.3}$$

the set of Λ-*adic test vectors for* $\mathbf{f}$ *of tame level* N. Analogously we define $S^{\mathrm{ord}}_{\Lambda_{\mathbf{g}}}(N, \chi_g)[\mathbf{g}], S^{\mathrm{ord}}_{\Lambda_{\mathbf{h}}}(N, \chi_h)[\mathbf{h}]$.

For each choice of a triple of test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$, in [DR14] the authors constructed the *triple product L-function*

$$\mathcal{L}^{g}_{p}(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) \in \Lambda_{\mathbf{f}} \hat{\otimes} \mathrm{Frac}(\Lambda_{\mathbf{g}}) \hat{\otimes} \Lambda_{\mathbf{h}}.$$

It interpolates the square-root of the central critical values $L(f_k \otimes g_\ell \otimes h_m, \frac{k+\ell+m-2}{2})$ as the triple of weights (k, ℓ, m) varies in $\mathcal{W}^{g}_{\mathbf{fgh}}$. Recall that f_k denotes the newform whose p-stabilisation is the weight k specialisation of $\mathbf{f}$, and analogously for g_ℓ and h_m and that $L(f_k \otimes g_\ell \otimes h_m, s)$ is the complex L-function introduced in 1.12.3. In [Hsi19, Chapter 3], Hsieh constructed an explicit choice of test vectors $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ for which

$$\mathcal{L}^{g}_{p}(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathcal{L}^{g}_{p}(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$$

belongs to $\Lambda_{\mathbf{fgh}} := \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$ and it satisfied a simpler interpolation formula. We fix this choice of test vector once and for all. Moreover, while for the construction of [DR14] the specialisation at each classical point of $\mathbf{f}, \mathbf{g}, \mathbf{h}$ has to be assumed to be old at p, in [Hsi19] the specialisations of the three Hida families are allowed to be either old or new at p. We next summarize the interpolation properties of the triple product p-adic L-function attached to Hsieh's test vectors.

We recall that for an eigenform $\varphi \in S_w(N_\varphi, \chi_\varphi)$, we denote by α_φ and β_φ the two toots of the characteristic polynomial $x^2 - a_p(\varphi)x + p^{w-1}\chi_\varphi(p)$, ordered in such a way that $\mathrm{ord}_p(\alpha_\varphi) \leq \mathrm{ord}_p(\beta_\varphi)$. We will use the convention that, if p divides the level of φ, then $\beta_\varphi = 0$.

Theorem 1.96 (Hsieh)**.** *Let* $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star) \in S^{\mathrm{ord}}_{\Lambda_{\mathbf{f}}}(N, \chi_f)[\mathbf{f}] \times S^{\mathrm{ord}}_{\Lambda_{\mathbf{g}}}(N, \chi_g)[\mathbf{g}] \times S^{\mathrm{ord}}_{\Lambda_{\mathbf{h}}}(N, \chi_h)[\mathbf{h}]$ *be the triple of* Λ-*adic test vectors for* $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ *defined in [Hsi19, Chapter 3] and let* $\mathcal{W}_{\mathbf{fgh}} := \mathrm{Spf}(\Lambda_{\mathbf{fgh}})$. *Then the p-adic L-function*

$$\mathcal{L}^{g}_{p}(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathcal{L}^{g}_{p}(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star) : \mathcal{W}_{\mathbf{fgh}} \longrightarrow \mathbb{C}_p$$

is uniquely characterised by the following interpolation property: for each $(k,\ell,m)\in\mathcal{W}^g_{\mathbf{fgh}}$*,*

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})^2(k,\ell,m)=L(f_k\otimes g_\ell\otimes h_m,c)\frac{\mathcal{E}(f_k,g_\ell,h_m)^2}{(-4)^\ell\langle g_\ell,g_\ell\rangle^2\mathcal{E}_0(g_\ell)^2\mathcal{E}_1(g_\ell)^2}\mathfrak{a}(k,\ell,m)\prod_{g\in\Sigma_{\mathrm{exc}}}(1+q^{-1})$$

where:

i) $\langle\cdot,\cdot\rangle$ *is the Petersson product;*

ii) $c=\frac{k+\ell+m-2}{2}$*;*

iii) Σ_{exc} *is the set of exceptional primes defined in [Hsi19, §1.5];*

iv) $\mathfrak{a}(k,\ell,m)=\Gamma_{\mathbb{C}}(\frac{k+\ell+m-2}{2})\Gamma_{\mathbb{C}}(\frac{-k+\ell-m+2}{2})\Gamma_{\mathbb{C}}(\frac{k+\ell-m}{2})\Gamma_{\mathbb{C}}(\frac{-k+\ell+m}{2})$ *and* $\Gamma_{\mathbb{C}}(s)=\frac{\Gamma(s)}{(2\pi)^s}$*.*

v) $\mathcal{E}(f_k,g_\ell,h_m)=(1-\beta_{g_\ell}\alpha_{f_k}\alpha_{h_m}p^{-c})(1-\beta_{g_\ell}\alpha_{f_k}\beta_{h_m}p^{-c})(1-\beta_{g_\ell}\beta_{f_k}\alpha_{h_m}p^{-c})(1-\beta_{g_\ell}\beta_{f_k}\beta_{h_m}p^{-c})$*;*

vi) $\mathcal{E}_0(g_\ell)=1-\beta^2_{g_\ell}\chi_g^{-1}(p)p^{1-\ell}$*;*

vii) $\mathcal{E}_1(g_\ell)=1-\beta^2_{g_\ell}\chi_g^{-1}(p)p^{-\ell}$*.*

Remark 1.97. As explained in [DR17] and [DLR15], different choices of Λ-adic test vectors for $(\mathbf{f},\mathbf{g},\mathbf{h})$ yields differents p-adic L-functions, but the values of these functions agree modulo $L^\times$.

The same construction of [DR14] yields two others triple product L-functions, namely

$$\mathcal{L}_p^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})\ \text{ and }\ \mathcal{L}_p^h(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$$

which interpolates the square roots of the central values of the classical L-function $L(f_k\otimes g_\ell,h_m,s)$ in the regions $\mathcal{W}^f_{\mathbf{fgh}}$ and $\mathcal{W}^h_{\mathbf{fgh}}$ respectively, and an analogous statement of Theorem 1.96 hold for these functions. Finally, there is also a four-variables *balanced* p-adic L-function

$$\mathcal{L}_p^{\mathrm{bal}}(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}):\mathcal{W}_{\mathbf{fgh}}\times\mathcal{W}\longrightarrow\mathbb{C}_p$$

interpolating the special values $L(f_k,g_\ell,h_m,s)$ for $(k,\ell,m)\in\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ and $1\le s\le k+\ell+m-3$. In this thesis we will not make use of this function, and for further details on its contruction and properties, we refer the reader to [BP06] and [Hsi19].

In this thesis we will focus on the values of $\mathcal{L}_p^g(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ at cristallline weights $(k,\ell,m)\in\mathcal{W}^\circ_{\mathbf{fgh}}$. By Theorem 1.96, we know the values in the region $\mathcal{W}^g_{\mathbf{fgh}}$ of classical interpolation. As we will see in §1.16.4, the main result of [DR14] gives a formula for the values of $\mathcal{L}_p^g(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ at balanced triples in $\mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$. Finally, the study of the values of the p-adic L-function at points in $\mathcal{W}^f_{\mathbf{fgh}}$ has been initiated in [DLR15] with the *elliptic Stark Conjecture* that we will describe in §2.1 and which has been proved in loc. cit. in some special cases. In this thesis we will continue this study in Chapters 2, 3 and 4.

1.16 Algebraic cycles, cohomology classes and p-adic L-functions

In the spirit of the conjectures of §1.14, special values of complex L-functions (or their derivatives) attached to motives are, at least conjecturally related to arithmetic objects. The formulas (1.14.1) and (1.14.2) and Heegner points are instances of this philosophy in the case of elliptic curves. As we will see in this section and in the conjectures of Chapter 2, also p-adic L-functions are related to arithmetic objects in various ways. With the notation of the beginning of §1.15, let $\mathcal{L}_p(\mathbf{M},s):\mathcal{W}\longrightarrow\mathbb{C}_p$ be the p-adic L-function attached to a p-adic family of motives

$\mathbf{M} = \{M_x\}_{x\in\Sigma^{\rm cl}}$ which interpolates the central values of the complex L-function $L(M_x, c_x)$ for $x \in \Sigma^{\rm int} \subseteq \Sigma^{\rm cl} \subseteq \mathcal{W}$. In this framework, it is usually of great interest to study the values $\mathcal{L}_p(\mathbf{M}, x)$ at classical points $x \in \Sigma^{\rm cl} \smallsetminus \Sigma^{\rm int}$ lying outside the region of interpolation, as it is believed they should encode a p-adic invariant associated with the relevant motivic cohomology group of the motive M_x. An example of this situation is Leopoldt's p-adic formula for the Kubota–Leopoldt p-adic L-function. As described at the beginning of §1.15, the point $k = 1$ is outside the region of classical interpolation for $\mathcal{L}_p(\chi, s)$, and Leopoldt's formula delates $\mathcal{L}_p(\chi, 1)$ to the p-adic logarithm of a *circular unit* in the cyclotomic field $\mathbb{Q}(\zeta_N)$:

$$\mathcal{L}_p(\chi, 1) = -\frac{(1-\chi(p)p^{-1})}{\mathfrak{g}(\chi)} \sum_{j=1}^{N-1} \chi^{-1}(j) \log_p(1-\zeta_N^j),$$

where N is the conductor of χ, ζ_N is a N-th root of unity, and $\mathfrak{g}$ denotes the Gauss sum.

There are many other illustrative examples of this philosophy. Some classical and relatively recent formulas exhibiting this phenomenon are summarized in the survey [BCD+14], but very recently there have been exciting developments in this direction, including [BSV], [DRb], [LSZ19], [LSZ17]. In spite of that, all these scattered formulas in the literature do not provide a systematic and thorough study of the collection of special values of a p-adic L-function as a whole and is not always easy to have a good understanding of the complete picture. The aim of this section is to describe these objects in the specific settings that we will encounter in the next chapters of this thesis, namely elliptic units, generalised Heegner cycles and generalised Kato classes.

1.16.1 Elliptic units and Katz's p-adic L-functions

The arithmetic objects attached to Katz's L-function are the so-called *elliptic units*. We recall briefly the defintion of these units and their relation with p-adic L-functions in the specific case that we will need in Chapter 2. Let K be an imaginary quadratic field of class number h_K, let $\mathfrak{c}$ be an ideal of K and fix a prime p which splits in K as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. Let $c > 0$ be the smallest positive integer of belonging to $\mathfrak{c}$ and let K_c be the ring class field of K of conductor c. Fix a primitive c-th root of unit ζ_c, and consider the so-callad *Siegel unit* $g \in \mathcal{O}_{Y_1(c)}^\times$ with q-expansion

$$g(q) = q^{1/12}(1-\zeta_c)\prod_{n>0}(1-q^n\zeta_c)(1-q^n\zeta_c^{-1}).$$

Using the notation of Theorem 1.93, consider the elliptic curve with complex multiplication whose complex points are $A_c := \mathbb{C}/\mathcal{O}_c$ and let $t_c \in A_c$ be the point corresponding to $1/c$. Via the moduli interpretation of the modular curve described in §1.5.1, it makes sense to evaluate g at the point (A_c, t_c), and

$$u := g(A_c, t_c) \in \mathcal{O}_{K_c}^\times$$

is an *elliptic unit*. In [Kat76], Katz related the values of $\mathcal{L}_p(K)$ at finite order characters to elliptic units.

Theorem 1.98. *Let ψ be a finite order character of K of conductor $\mathfrak{c}$. Then*

$$\mathcal{L}_p(K)(\psi) = \begin{cases} \frac{1}{2}(\frac{1}{p}-1)\log_p(u_{\mathfrak{p}}) & \text{if } \psi = 1, \\ -\frac{1}{24c}\mathfrak{e}(\psi)\sum_{\sigma\in{\rm Gal}(K_c/K)}\psi^{-1}(\sigma)\log_p(\sigma(u)) & \text{if } \psi \neq 1, \end{cases}$$

where $u_{\mathfrak{p}} \in K^\times$ is a generator of the principal ideal $\mathfrak{p}^{h_K}$ and $u \in \mathcal{O}_{K_c}^\times$ is an elliptic unit.

1.16.2 Kolyvagin classes

In this section we recall briefly the construction and the main properties of the Kolyvagin classes defined in [BD97, §6]. These objects will appear in the main result of Chapter 4: in the setting of loc. cit. we will obtain a formula for a special value of the triple product L-function described in §1.15.3 in terms of the logarithm of a cristalline projection of $\mathbf{K}$. As in the previous section, K will denote an imaginary quadratic field, and we will assume for simplicity that $\mathcal{O}_K^\times = \{\pm 1\}$.

Fix an odd prime p which is *inert* in K, and let $H := K_c$ be a ring class field of K of conductor c prime to p. For every integer $m \geq 1$, let $H(p^m)$ be the ring class field of K of conductor $c \cdot p^m$. The Galois group $\mathrm{Gal}(H(p^m)/H)$ is cyclic of order $e_m := (p+1)p^{m-1}$. Let F_m be the intermediate field $H \subseteq F_m \subseteq H(p^{m+1})$ such that $\mathrm{Gal}(F_m/H)$ is cyclic of order p^m. Since p is inert in K, the prime ideal $p\mathcal{O}_K$ splits completely in H. Fix once and for all a prime $\mathfrak{p}$ of H above p; it ramifies totally in $H(p^m)$ as $\mathfrak{p}\mathcal{O}_{F_m} = \mathfrak{p}_m^{p^m}$. Let us denote $F_{m,\mathfrak{p}}$ the completion of F_m at $\mathfrak{p}_m$, $\mathcal{O}_{m,\mathfrak{p}}$ its ring of integers and $\mathbb{F}_{m,\mathfrak{p}}$ its residue field. For each m, fix a generator σ_m of $G_m := \mathrm{Gal}(F_m/H)$ such that the image of σ_m via the projection

$$G_m \longrightarrow G_m/\,\mathrm{Gal}(F_m/F_{m-1}) \cong G_{m-1} \tag{1.16.1}$$

equals σ_{m-1}. Moreover, the elements $\{\sigma_m\}_m$ can be chosen so that $\sigma_{\mathrm{ant}} = (\sigma_{m|G_{F_{m,\mathfrak{p}}}})_m$, where σ_{ant} is the generator of Γ_{ant} we fixed in the first example of §1.3.1. In [BD96], Bertolini and Darmon constructed a collection of points $\{\tilde{\alpha}_m \in E(H(p^m))\}_m$ (in the notation of [BD96], $\tilde{\alpha}_m = \alpha_m(1)$) such that

$$\mathrm{N}_{H(p^m)/H}\,\tilde{\alpha}_m = 0. \tag{1.16.2}$$

For each $m \geq 1$, let

$$\alpha_m := \mathrm{N}_{H(p^{m+1})/F_n}\,\tilde{\alpha}_{m+1} \in E(F_m).$$

To each point α_m, in [BD97] the authors attached a Kolyvagin class $\mathbf{K}_m \in \mathrm{H}^1(H, E[p^m])$ in the following way. The *Kolyvagin derivative operator* is the element $D_m := \sum_{i=1}^{p^m-1} i\sigma_m^i \in \mathbb{Z}[G_m]$.

Lemma 1.99. *The Kolyvagin derivative satisfies the following equality in* $\mathbb{Z}[G_m]$

$$(\sigma_m - 1)D_m = p^m - \mathrm{N}_{F_m/H}\,. \tag{1.16.3}$$

Proof. Note that $D_n = -\sum_{i=1}^{p^n} \dfrac{\sigma_n^i - 1}{\sigma_n - 1}$. Then $(\sigma_n - 1)D_n = -\sum_{i=1}^{p^n}(\sigma_n - 1) = -\mathrm{N}_{F_n/H} + p^n$. ☐

Define $P_m := D_m\alpha_m \in E(F_m)$.

Lemma 1.100. *1. The point $P_m \in E(F_m)$ only depends on the choice of the generator σ_m of G_m up to the multiplication by an element $a_m \in (\mathbb{Z}/p^m\mathbb{Z})^\times$. Moreover, if we choose the elements $\{\sigma_m\}_m$ to be compatible in the sense of* (1.16.1)*, then $a_{m-1} \equiv A_m \mod p^m$.*

2. The class $[P_m]$ of P_m in $E(F_m)/p^mE(F_m)$ is fixed by $G_m = \mathrm{Gal}(F_m/H)$.

Proof. 1. Let σ_m' any generator of $G_m \cong \mathbb{Z}/p^m\mathbb{Z}$ and define $D_m' := \sum i(\sigma_m')^i$. Then there exists an element $a_m \in (\mathbb{Z}/p^m\mathbb{Z})^\times$ such that $\sigma_m' = \sigma_m^{a_m}$. Then $D_m' = \sum i\sigma_n^{a_m i} = a_m^{-1}D_m$.

2. It suffices to prove that the class of $D_m\alpha_m$ is fixed by the generator σ_m of G_m, i.e. that $\sigma_nD_n\alpha_n - D_n\alpha_n \in p^nE(F_n)$. Combining (1.16.3) and (1.16.2) we obtain $(\sigma_m - 1)D_m\alpha_m = p^m\alpha_m - \mathrm{N}_{F_m/H}\,\alpha_m = p^m\alpha_m$.

☐

The inflation-restriction and Kummer exact sequences yield the following commutative diagram

$$\begin{array}{ccccccccc}
 & & & & \mathrm{H}^1(F_m/H, E(F_m)[p^m]) & & & & \\
 & & & & \downarrow{\scriptstyle \mathrm{Inf}} & & & & \\
0 & \longrightarrow & \Big(E(H)/p^m E(H)\Big) & \xrightarrow{\delta_p} & \mathrm{H}^1(H, E[p^m]) & \longrightarrow & \mathrm{H}^1(H, E)[p^m] & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \downarrow{\scriptstyle \mathrm{Res}} & & \\
0 & \longrightarrow & \Big(E(F_m)/p^m E(F_m)\Big)^{G_m} & \xrightarrow{\delta_p} & \mathrm{H}^1(F_m, E[p^m])^{G_m} & \longrightarrow & \mathrm{H}^1(F_m, E)[p^m]^{G_m} & & \\
 & & & & \downarrow & & & & \\
 & & & & \mathrm{H}^2(F_m/H, E(F_m)[p^m]). & & & &
\end{array} \tag{1.16.4}$$

whose rows and columns are exact. Let C be a constant annihilating $E(F_m)[p^m]$, which can be chosen to be independent of m by [BD97, Lemma 6.3]. Define

$$\mathbf{K}_m := C\mathbf{K}_m^\circ \in \mathrm{H}^1(H, E[p^m]),$$

where $\mathbf{K}_m^\circ$ is a preimage via Res of $-\delta_p[P_m]$. For each m, denote

$$\mathcal{G}_m := \mathrm{Gal}(F_m/K) \supseteq G_m := \mathrm{Gal}(F_m/H)$$

and fix a complete set $S = S_m = \{\sigma_1^{(m)}, \ldots, \sigma_h^{(m)}\}$ of representatives for the G_m-cosets in $\mathcal{G}_m$. We require these sets to satisfy the following compatibility condition. Let $\pi_m : \mathcal{G}_m \longrightarrow \mathcal{G}_{m-1}$ denote the projection induced by the inclusion $F_{m-1} \subseteq F_m$. Then

$$\pi_m(\sigma_i^{(m)}) = \sigma_i^{(m-1)} \tag{1.16.5}$$

for all i and all m. In other words, under this compatibility condition we can consider a complete set $\tilde{S} = \{\tilde{\sigma}_1, \ldots, \tilde{\sigma}_m\}$ of representatives of G_H-cosets in G_K such that, under the projection $\tilde{\pi}_m : G_K \longrightarrow \mathrm{Gal}(F_m/K)$ we have

$$\tilde{\pi}_m(\tilde{\sigma}_i) = \sigma_i^{(m)}. \tag{1.16.6}$$

Lemma 1.101. *The classes $\mathbf{K}_m$ form a compatible system*

$$\mathbf{K} := (\mathbf{K}_m)_m \in \varprojlim_m \mathrm{H}^1(H, E[p^m]) = \mathrm{H}^1(H, T_pE) \subset \mathrm{H}^1(H, V_f).$$

Proof. The norm map $\mathrm{N}_{F_{m+1}/F_m} : F_{m+1} \longrightarrow F_m$ induces a map $\mathrm{N}_{F_{m+1}/F_m} : E(F_{m+1}) \longrightarrow E(F_m)$. By [BD96, §2.45], we have

$$\mathrm{N}_{F_{m+1}/F_m}(\alpha_{m+1}) = \alpha_m. \tag{1.16.7}$$

Consider the composition

$$f_{m+1} : E(F_{m+1})/p^{m+1}E(F_{m+1}) \overset{\mathrm{N}_{F_{m+1}/F_m}}{\longrightarrow} E(F_m)/p^{m+1}E(F_m) \longrightarrow E(F_m)/p^m E(F_m)$$

where the second morphism is the projection given by the inclusion $p^{m+1}E(F_m) \subseteq p^m E(F_m)$. Hence (1.16.7) implies that

$$f_{m+1}([\alpha_{m+1}]) = [\alpha_m]. \tag{1.16.8}$$

Consider the following commutative diagram

$$\begin{array}{ccc} E(F_{m+1})/p^{m+1}E(F_{m+1}) & \xrightarrow{\delta_p} & \mathrm{H}^1(F_{m+1}, E[p^{m+1}]) \\ \Big\downarrow f_{m+1} & & \Big\downarrow f_{m+1,*} \\ E(F_m)/p^m E(F_m) & \xrightarrow{\delta_p} & \mathrm{H}^1(F_m, E[p^m]). \end{array}$$

Here $f_{m+1,*}$ is the map induced in cohomology by f_{m+1}, i.e. it is the composition

$$\mathrm{H}^1(F_{m+1}, E[p^{m+1}]) \overset{\mathrm{N}_{F_{m+1}/F_m}}{\longrightarrow} \mathrm{H}^1(F_m, E[p^{m+1}]) \xrightarrow{p_*} \mathrm{H}^1(F_m, E[p^m]),$$

where the last map is the composition with the multiplication by p map $E[p^{m+1}] \xrightarrow{p} E[p^m]$. Then (1.16.8) implies that

$$f_{m+1,*}\delta_p[D_{m+1}\alpha_{m+1}] = \delta_p[D_m\alpha_m]. \tag{1.16.9}$$

□

1.16.3 Generalised Heegner cycles and BDP p-adic L-function

Let K be an imaginary quadratic field of discriminant $-D_K$ and let N be a squarefree positive integer coprime to D_K. The aim of this section is to describe the so-called *generalised Heegner cycles* introduced by Bertolini, Darmon and Prasanna. These objects are defined under the *Heegner Hypothesis* of Assumption 1.90 for the pair (K, N). It is worth to mention that the construction of generalised Heegner cycles under relaxed hypotheses has been carried out in [Bro15], but we will not need them in this thesis. Notice that Assumption 1.90 is equivalent to the condition that all the primes dividing N split in K. Fix an elliptic curve E defined over the Hilbert class field of K and with complex multiplication by $\mathcal{O}_K$, and a generator t of $E[\mathcal{N}]$ so that the pair (E, t) corresponds to a point P on the modular curve $X_1(N)$ as explained in §1.5.1. In [BDP13] and [BDP17], Bertolini, Darmon and Prasanna constructed a family of so-called generalised Heegner cycles in the product of a Kuga–Sato variety with a power of E, extending Nekovář's construction of *Heegner cycles* of [Nek92]. As we will recall in §1.15.2, these cycles are related to special values of a p-adic L-function, and we will use this relation in Chapter 2 to prove a special case of Conjecture 2.13 and Conjecture 2.17. We now briefly recall the definition of the cycles.

Let c be a positive integer coprime to ND_K, and let $E_c := \mathbb{C}/\mathcal{O}_c$ be an elliptic curve with complex multiplication by $\mathcal{O}_c$, which we can assume is defined over the ring class field K_c of K of conductor c. Let $\phi_c : E \longrightarrow E_c$ be an isogeny of degree c. Given an ideal $\mathfrak{a}$ of $\mathcal{O}_c$ prime to $\mathcal{N}_c := \mathcal{N} \cap \mathcal{O}_c$, denote by $E_\mathfrak{a}$ the elliptic curve $\mathbb{C}/\mathfrak{a}^{-1}$ and by $\phi_\mathfrak{a}$ the isogeny

$$\phi_\mathfrak{a} : E_c \longrightarrow E_\mathfrak{a}.$$

The isogeny $\phi_\mathfrak{a} \circ \phi_c$ defines a $\Gamma_1(N)$-level on $E_\mathfrak{a}$, i.e. a point $t_\mathfrak{a} := \phi_\mathfrak{a} \circ \phi_c(t)$ of exact order $\mathcal{N}_c$.

Let $r_0 \geq r_1$ be two non-negative integers with the same parity, set $s := \dfrac{r_0 + r_1}{2}, u := \dfrac{r_0 - r_1}{2}$ and let

$$X_{r_0,r_1} := W_{r_0} \times E^{r_1},$$

where W_{r_0} is the Kuga–Sato variety introduced in Definition 1.41. The variety X_{r_0,r_1} is defined over the ring class field of K and has dimension $r_0 + r_1 + 1 = 2s + 1$. Let

$$\pi : X_{r_0,r_1} \xrightarrow{p_1} W_{r_0} \longrightarrow X_1(N)$$

be the composition of the projection on the first component of X_{r_0,r_1} with the canonical map of the Kuga–Sato variety onto $X_1(N)$. For each ideal $\mathfrak{a}$ of $\mathcal{O}_c$ prime to $\mathcal{N}_c$, the fibre of the point $P_{\mathfrak{a}} = (E_{\mathfrak{a}}, t_{\mathfrak{a}})$ is

$$\pi^{-1}(P_{\mathfrak{a}}) \cong E_{\mathfrak{a}}^{r_0} \times E^{r_1} \cong (E_{\mathfrak{a}} \times E)^{r_1} \times (E_{\mathfrak{a}} \times E_{\mathfrak{a}})^{u}.$$

Write $\mathrm{End}(E_{\mathfrak{a}})$ as

$$\mathrm{End}(E_{\mathfrak{a}}) = \mathbb{Z}\left[\frac{d_c + \sqrt{d_c}}{2}\right], \tag{1.16.10}$$

where we regard $\sqrt{d_c}$ as an endomorphism of the curve, and define

$$\Gamma_{\mathfrak{a}} := (\mathrm{Graph}(\sqrt{d_c}))^{\mathrm{tr}} \subseteq E_{\mathfrak{a}} \times E_{\mathfrak{a}}; \qquad \Gamma_{c,\mathfrak{a}} := (\mathrm{Graph}(\phi_{\mathfrak{a}} \circ \phi_c))^{\mathrm{tr}} \subseteq E_{\mathfrak{a}} \times E;$$

$$\Gamma_{r_0,r_1,c,\mathfrak{a}} := \Gamma_{c,\mathfrak{a}}^{r_1} \times \Gamma_{\mathfrak{a}}^{u} \subseteq X_{r_0,r_1},$$

where tr denotes the transpose. Notice that $\Gamma_{r_0,r_1,c,\mathfrak{a}}$ is a cycle of codimension $s+1$ in X_{r_0,r_1} supported on the fibre $\pi^{-1}(P_{\mathfrak{a}})$.

Definition 1.102. The *generalised Heegner cycle* attached to the data $r_0, r_1, \mathfrak{a}, c$ is defined as

$$\Delta_{r_0,r_1,c,\mathfrak{a}} := \varepsilon_{X_{r_0,r_1}}(\Gamma_{r_0,r_1,c,\mathfrak{a}}) \in \mathrm{CH}^{s+1}(X_{r_0,r_1})_{0,\mathbb{Q}},$$

where $\varepsilon_{X_{r_0,r_1}}$ is the projector defined in [BDP17, §4.1].

Let $f \in S_{r_0+2}(N,\chi)$ be a modular form. We want to consider the projection onto the "(f,ψ)-component" of these cycles, for certain Hecke characters ψ of K. Let ϵ be a Dirichlet character of conductor $N_\epsilon \mid N$ and let $\mathcal{N}_\epsilon$ be the ideal of $\mathcal{O}_K$ such that $\mathcal{N}_\epsilon \mid \mathcal{N}$ and $\mathcal{O}_K/\mathcal{N}_\epsilon \cong \mathbb{Z}/N_\epsilon\mathbb{Z}$, whose existence follows from Assumption 1.90. Let c be a positive integer such that $\gcd(c, ND_K) = 1$, and let $U_c := \hat{\mathcal{O}}_c^\times = \prod_p (\mathcal{O}_c \otimes \mathbb{Z}_p)^\times$.

Definition 1.103. A Hecke character ψ of K is of *finite type* $(c, \mathcal{N}, \epsilon)$ if the conductor of ψ is $c\mathcal{N}$ and if, denoting $\mathcal{N}_{c,\epsilon} := \mathcal{N}_\epsilon \cap \mathcal{O}_c$, the restriction of ψ to U_c coincides with the composition

$$U_c = \hat{\mathcal{O}}_c^\times \longrightarrow \left(\hat{\mathcal{O}}_c/\mathcal{N}_{c,\epsilon}\hat{\mathcal{O}}_c\right)^\times \cong (\mathcal{O}_K/\mathcal{N}_\epsilon\mathcal{O}_K)^\times \cong (\mathbb{Z}/N_\epsilon\mathbb{Z})^\times \xrightarrow{\epsilon^{-1}} \mathbb{C}^\times.$$

Let ψ be a Hecke character of K of conductor $\mathfrak{c}_\psi$, central character ε_ψ, infinity type $(r_1 - j, j)$ for some $0 \le j \le r_1$, and such that $\varepsilon_\psi = \chi$. The condition on the central character implies that the complex L-function $L(f, \psi^{-1}, s)$ is selfdual. Let $\mathcal{N}_\chi$ be the ideal of K dividing $\mathcal{N}$ whose norm equals the conductor of χ, and suppose that the conductor of $\mathfrak{c}_\psi$ is of the form $\mathfrak{c}_\psi = \mathcal{N}_\chi$, for some $c \in \mathbb{Z}$ coprime to N. By [BDP17, §4.2] the cycle $\Delta_{r_0,r_1,c,\mathfrak{a}}$ is then defined over the number field F that corresponds by class field theory to the subgroup $K^\times W \subseteq \mathbb{A}_K^\times$, where

$$W = \{x \in \mathbb{A}_K^\times \mid x\mathcal{O}_c = \mathcal{O}_c, xt = t\}.$$

Arguing as in [BDP17, (4.2.1)] we find that $\mathrm{Gal}(F/K_c) \cong (\mathbb{Z}/N\mathbb{Z})^\times/\{\pm 1\}$. Define $H_{c,f}$ to be the subextension of F/K that corresponds to $\ker\chi$ under the isomorphism and let

$$\tilde{\Delta}_{r_0,r_1,c,\mathfrak{a}} := \frac{1}{[F : H_{c,f}]} \mathrm{Tr}_{F/H_{c,f}}(\Delta_{r_0,r_1,\mathfrak{a},c}). \tag{1.16.11}$$

Finally, following [BDP17, Definition 4.2.3], we define the following cycle.

Definition 1.104.

$$\tilde{\Delta}_{r_0,r_1,c}^{\psi} := e_f\left(\sum_{\mathfrak{a} \in S} \psi(\mathfrak{a})^{-1} \cdot \tilde{\Delta}_{r_0,r_1,c,\mathfrak{a}}\right),$$

where S is a set of representatives for $\mathrm{Pic}(\mathcal{O}_c)$ that are prime to $c \cdot \mathcal{N}$.

This definition might depend on the choice of S, but its image under the p-adic Abel–Jacobi map (1.11.9) does not by [BDP17, Remark 4.2.4]. Observe that

$$\tilde{\Delta}^{\psi}_{r_0,r_1,c} \in \mathrm{CH}^{s+1}(X_{r_0,r_1}/H_{c,f})_{0,\mathbb{Q}(\psi)},$$

where $\mathbb{Q}(\psi)$ denotes the number field generated by the values of ψ.

Recall the p-adic L-function $\mathcal{L}_p(f,K)$ described in §1.15.2, and we resume the notation used there. The set $\Sigma(c,\mathcal{N},\chi)^{(1)}$ is contained in the completion $\hat{\Sigma}(c,\mathcal{N},\chi)^{(2)}$, and the main theorem of [BDP13] (and its extension of [BDP17]) relates the values of the p-adic L-function at characters in $\Sigma(c,\mathcal{N},\chi)^{(1)}$ to the generalised Heegner cycles. As in §1.10.5, we fix an elliptic curve E with complex multiplication by $\mathcal{O}_K$, defined over the ring class field of K. Assume that E has good reduction at p and let $r \leq k-2$ be an integer such that $k \equiv r \mod 2$. Let H be a number field over which all the structures above are defined. The following theorem is due to Bertolini–Darmon–Prasanna and Castella.

Theorem 1.105. *For each Hecke character $\psi \in \Sigma(c,\mathcal{N},\chi)^{(1)}$ of infinity type $(r-j,j)$ with $0 \leq j \leq r$,*

$$\mathcal{L}_p(f,K)(\psi\,\mathrm{N}_K^{\frac{k-r}{2}}) = \mathfrak{e}(f,\psi\,\mathrm{N}_K^{\frac{k-2-r}{2}})^2 \frac{\Omega_p^{r-2j}}{(j+1)!\cdot c^{2j}\cdot(4d_c)^{\frac{k-2-r}{2}}}\left(\mathrm{AJ}_p(\Delta^{\psi}_{k-2,r,c})(\omega_f\wedge\omega_E\eta_E^{r-j}\right)^2,$$

where $\omega_f \in \mathrm{Fil}^{k-1}\,\mathrm{H}^{k-1}_{\mathrm{dR}}(W_{k-2}/H) \cong \mathrm{D}_{\mathrm{dR}}(V_f^-)$ is the differential attached to f defined in §1.8 and d_c is the integer defined by (1.16.10).

The result above is [BDP17, Theorem 4.2.5] when W_{k-2} has good reduction at p (in loc. cit. only the case $c=1$ is treated, but the proofs therein generalise to $c>1$ with $(c,ND_K)=1$). The formula was extended by Castella to the case of semistable reduction [Cas18, Theorem 2.11] (again, Castella works with trivial character but the proofs extend to the case of nontrivial character).

1.16.4 Diagonal cycles, generalised Kato classes and the triple product p-adic L-function

2

Let f,g,h be normalised newforms of *balanced* weights $k,\ell,m \geq 2$, levels N_f,N_g,N_h and nebentype characters χ_f,χ_g,χ_h such that

$$\chi_f\cdot\chi_g\cdot\chi_h = 1.$$

Assume that the local signs of (1.12.6) are $\varepsilon_v(f,g,h)=+1$ for all $v \mid \mathrm{lcm}(N_f,N_g,N_h)$ and let $\mathbf{f},\mathbf{g},\mathbf{h}$ be the Hida families passing through them. In this section we will introduce the cycles and the Λ-adic classes which are related to the triple product p-adic L-functions

$$\mathcal{L}_p^f(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}),\ \mathcal{L}_p^g(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}}),\ \mathcal{L}_p^h(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$$

for a fixed triple of Λ-adic test vectors $(\breve{\mathbf{f}},\breve{\mathbf{g}},\breve{\mathbf{h}})$ for $(\mathbf{f},\mathbf{g},\mathbf{h})$. Recall from (1.12.7) that, since the weights (k,ℓ,m) are balanced, the sign of the functional equation satisfied by the complex L-function $L(f\otimes g\otimes h,s)$ is -1, thus its central value is

$$L(f\otimes g\otimes h,(k+\ell+m-2)/2)=0.$$

Recall from §1.14 that Beilinson's conjecture predicts in this case that

$$e_{fgh}\,\mathrm{CH}^{\frac{k+\ell+m-2}{2}}(W_{k-2}\times W_{\ell-2}\times W_{m-2})_0 \overset{?}{\neq} \{0\}.$$

[2]**FIXME:** non finita: finire dopo kolyvagin

The geometric construction of the diagonal cycles, as defined in [DR14] depends on *balanced* the weights (k, ℓ, m). Let $N := \mathrm{lcm}(N_f, N_g, N_h)$ and recall the Kuga–Sato variety. For a subset $\emptyset \neq I \subseteq \{1, 2, 3\}$, we define

$$X_I := \{(P_1, P_2, P_3) \in X_1(N)^3 \mid P_i = P_j \text{ for all } \{i, j\} \subseteq I, P_j = 0 \text{ for all} j \notin I\}.$$

The *diagonal cycle* attached to the weights $(k, \ell, m) = (2, 2, 2)$ is

$$\Delta_{2,2,2} := X_{123} - X_{12} - X_{13} - X_{23} + X_1 + X_2 + X_3 \in \mathrm{CH}^2(X_1(N)^3).$$

Here we regard the cycle $\Delta_{2,2,2}$ inside $X_1(N)^3$ via the identification $X_1 \times X_2 \times X_3 = X_1(N) \times X_1(N) \times X_1(N)$. In order to define the cycles for the other weights, let $A := a_1, \dots, a_{k-2}, B := \{b_1, \dots, b_{\ell-2}\}, C := \{c_1, \dots, c_{m-2}\}$ be subsets of $\{1, \dots, \frac{k+\ell+m-6}{2}\}$ satisfying $A \cap B \cap C = \emptyset$. Recall from §1.6.4.3 the form of the generic point of a Kuga–Sato vairety. Let $r := \frac{k+\ell+m-6}{2}$Consider the closed embeddings

$$\varphi_{ABC} : W_r \longrightarrow W_{k-2} \times W_{\ell-2} \times W_{m-2} \text{ and } \varphi_{BC} : W_r \longrightarrow W_{\ell-2} \times W_{m-2},$$

given by

$$(x; P_1, \dots, P_r) \mapsto ((x; P_1, \dots, P_{k-2}), (x; P_1, \dots, P_{\ell-2}), (x; P_1, \dots, P_{m-2}))$$

and

$$(x; P_1, \dots, P_r) \mapsto ((x; P_1, \dots, P_{\ell-2}), (x; P_1, \dots, P_{m-2}))$$

respectively. Then we define, for $\ell > 2$,

$$\Delta_{2,\ell,\ell} := (\mathrm{Id}, e_{\ell-2}, e_{\ell-2})(\varphi_{ABC}(W_r) \smallsetminus \{0\} \times \varphi_{BC}(W_r)) \in \mathrm{CH}^{r+2}(X_1(N) \times W_{\ell-2} \times W_{\ell-2})$$

and, for $k, \ell, m > 2$,

$$\Delta_{k,\ell,m} := (e_{k-2}, e_{\ell-2}, e_{m-2})\varphi_{ABC}(W_r) \in \mathrm{CH}^{r+2}(W_{k-2} \times W_{\ell-2} \times W_{m-2}).$$

Here e_j is the projector defined in [DR14, Display (62)]. In loc. cit. the autors proved that in all cases, $\Delta_{k,\ell,m} \in \mathrm{CH}^{r+2}(W_{k-2} \times W_{\ell-2} \times W_{m-2})_0$.

Recall from §1.11 the p-adic Abel–Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^{r+2}(W_{k-2} \times W_{\ell-2} \times W_{m-2})_0 \longrightarrow \mathrm{Fil}^{r+2} \mathrm{H}^{2r+3}_{\mathrm{dR}}(W_{k-2} \times W_{\ell-2} \times W_{m-2})^\vee.$$

The main result of [DR14], relates the Abel–Jacobi image of these diagonal cycles to the triple product p-adic L-function at *balanced* weights.

Theorem 1.106 (Darmon–Rotger). *For all* $(x, y, z) \in \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$,

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(x, y, z) = (-1)^{(y-x-z+2)/2} \frac{\mathcal{E}(f_x, g_y, h_z)}{\mathcal{E}(g_y)\mathcal{E}_1(g_y)} \mathrm{AJ}_p(\Delta_{x,y,z})(\omega_{f_x} \otimes \eta_{g_y} \otimes \omega_{h_z})$$

where $\omega_{f_x} \otimes \eta_{g_y} \otimes \omega_{h_z} \in \mathrm{H}^{x+y+z-3}_{\mathrm{dR}}(W/\mathbb{Q}_p)$ *are the differentials defined in §1.8 and* $\mathcal{E}(f_x, g_y, h_z), \mathcal{E}_0(g_y), \mathcal{E}(g_y)$ *are the Euler factors defined in §1.15.3.*

Proof. [DR14, Theorem 5.1] ∎

1.16.4.1 The Λ-adic generalised Kato class

Fix three cuspidal Hida families

$$\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]], \qquad \mathbf{g} \in \Lambda_{\mathbf{g}}[[q]], \qquad \mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$$

that we fixed at the beginning of the section.

Write $\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{f}} \hat{\otimes} \Lambda_{\mathbf{g}} \hat{\otimes} \Lambda_{\mathbf{h}}$ and $\mathcal{O}_{\mathbf{fgh}} := \Lambda_{\mathbf{f}}[1/p] \hat{\otimes} \Lambda_{\mathbf{g}}[1/p] \hat{\otimes} \Lambda_{\mathbf{h}}[1/p]$. Set

$$\mathbb{V} := V_{\mathbf{f}} \otimes V_{\mathbf{g}} \otimes V_{\mathbf{h}} \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi,$$

where $\Xi : G_{\mathbb{Q}_p} \longrightarrow \mathcal{O}_{\mathbf{fgh}}^{\times}$ is the character defined in [BSV, §4.6.2] and is characterized by the property that, if $\underline{x} = (k, \ell, m) \in \mathcal{W}_{\mathbf{fgh}}^{\circ}$ is a triple of cristalline points and $\rho_{\underline{x}} := \nu_k \otimes \nu_\ell \otimes \nu_m : \mathcal{O}_{\mathbf{fgh}}^{\times} \longrightarrow L_p$ is the corresponding specialisation map, then $\rho_{\underline{x}} \circ \Xi = \chi_{\mathrm{cyc}}^{(4-k-\ell-m)/2}$. We define a filtration on $\mathbb{V}$ by

$$\mathrm{Fil}^q(\mathbb{V}) := [\oplus_{i+j+t=q} \mathrm{Fil}^i(V_{\mathbf{f}}) \hat{\otimes} \mathrm{Fil}^j(V_{\mathbf{g}}) \hat{\otimes} \mathrm{Fil}^t(V_{\mathbf{h}})] \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi,$$

Since $\mathbb{V}/\mathrm{Fil}^2(\mathbb{V})$ has no $G_{\mathbb{Q}_p}$-invariants, the natural inclusion

$$\mathrm{Fil}^2(\mathbb{V}) = [V_{\mathbf{f}} \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi] \oplus [V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}} \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi] \oplus [V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}} \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi] \hookrightarrow \mathbb{V}$$

induces an injective morphism in cohomology

$$\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2(\mathbb{V})) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{V}),$$

and henceforth we shall identify $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2(\mathbb{V}))$ with its image in $\mathrm{H}^1(\mathbb{Q}_p, \mathbb{V})$. Defining

$$\mathbb{V}^f := V_{\mathbf{f}}^- \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi, \quad \mathbb{V}^g := V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}}^- \otimes V_{\mathbf{h}}^+ \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi, \quad \mathbb{V}^h := V_{\mathbf{f}}^+ \otimes V_{\mathbf{g}}^+ \otimes V_{\mathbf{h}}^- \otimes_{\mathcal{O}_{\mathbf{fgh}}} \Xi, \tag{1.16.12}$$

it follows that $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2(\mathbb{V})/\mathrm{Fil}^3(\mathbb{V}))$ decomposes as

$$\mathrm{H}^1(\mathbb{Q}_p, \mathbb{V}^f) \oplus \mathrm{H}^1(\mathbb{Q}_p, \mathbb{V}^g) \oplus \mathrm{H}^1(\mathbb{Q}_p, \mathbb{V}^h).$$

For $\varphi \in \{f, g, h\}$, we denote

$$\pi_\varphi : \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2(\mathbb{V})) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{V}^\varphi)$$

the natural projection map.

In [BSV] and [DRa] the authors constructed a global cohomology class

$$\boldsymbol{\kappa} := \boldsymbol{\kappa}(\mathbf{f}, \mathbf{g}, \mathbf{h}) \in \mathrm{H}^1(\mathbb{Q}, \mathbb{V})$$

related to the triple product p-adic L-functions of §1.15.3 as follows. By [BSV, Corollary 4.7.1] and [DRa, Proposition 3.5.7], the local class

$$\boldsymbol{\kappa}_p := \mathrm{res}_p(\boldsymbol{\kappa}) \in \mathrm{H}^1(\mathbb{Q}_p, \mathbb{V})$$

belongs to $\mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2(\mathbb{V}))$. Moreover, for $\varphi \in \{f, g, h\}$, the p-adic L-function $\mathcal{L}_p^\varphi(\mathbf{f}, \mathbf{g}, \mathbf{h})$ can be recast as the image of $\boldsymbol{\kappa}_p$ under the idoneous Perrin-Riou's Λ-adic logarithm: more precisely, [BSV, Theorem A], [DRa, Theorem 2.29] assert that

$$\mathcal{L}_p^\varphi(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathcal{L}_\varphi(\mathrm{res}_p(\boldsymbol{\kappa})), \tag{1.16.13}$$

where $\mathcal{L}_\varphi : \mathrm{H}^1(\mathbb{Q}_p, \mathrm{Fil}^2(\mathbb{V})) \longrightarrow \mathcal{O}_{\mathbf{fgh}}$ is the homomorphism described in [BSV, Proposition 4.6.2], [DRa, Proposition 3.5.6].

1.16.4.2 The generalised Kato class at weights $(2, 1, 1)$

Let f, g, h be normalised newforms of weights $2, 1, 1$, levels N_f, N_g, N_h and nebentype characters $1, \chi, \chi^{-1}$ respectively. Fix a prime number p such that

$$p \nmid N_f N_g N_h.$$

Let α_g, β_g be the eigenvalues for the action of the Frobenius element Frob_p on the representation V_g and let $V_g^{\alpha}, V_g^{\beta}$ be the corresponding eigenspaces, and use the analogous notation for α_h, β_h. Assume that

$$\alpha_g \neq \beta_g, \quad \text{and} \quad \alpha_h \neq \beta_h.$$

Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be the Hida families passing through $f, g_\alpha, h_\alpha)$ and let

$$\kappa := \boldsymbol{\kappa}(2, 1, 1) \in \mathrm{Sel}_{(p)}(\mathbb{Q}, V)$$

be the specialisation of $\boldsymbol{\kappa}$ at weight $(2, 1, 1)$. Next, we briefly recall the explicit relation between κ and the p-adic L-values $\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ and $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$. Let $V := V_f \otimes V_g \otimes V_h$, and denote $V^{\alpha\alpha} := V_f \otimes V_g^{\alpha} \otimes V_h^{\alpha}, \ldots$. There is a decomposition of the $G_{\mathbb{Q}_p}$-representation

$$V = V^{\alpha\alpha} \oplus V^{\alpha\beta} \oplus V^{\beta\alpha} \oplus V^{\beta\beta}.$$

Denote

$$\pi_{\alpha\beta} : \mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V^{\alpha\beta})$$

the projection map induced by the natural map $V \longrightarrow V^{\alpha\beta}$. And recall from §1.13 the maps

$$\mathrm{res}_p : \mathrm{H}^1(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}, V), \qquad \partial_p : \mathrm{H}^1(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_p, V).$$

Proposition 1.107 (Darmon–Rotger)**.** *Let* $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ *be the triple of test vectors of Theorem 1.96.*

i) The element $\partial_p \kappa$ *lies in the image of the natural map*

$$\mathrm{H}^1_s(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_p, V).$$

In particular, $\partial_p \kappa$ *can be viewed as an element of* $\mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta})$*. Moreover,*

$$\exp^*_{\beta\beta}(\partial_p \kappa) = \frac{2(1 - p\alpha_f \alpha_g^{-1} \alpha_h^{-1})}{\alpha_g \alpha_h (1 - \alpha_f \alpha_g \alpha_h)(1 - \chi^{-1}(p) \alpha_f^{-1} \alpha_g \alpha_h^{-1})} \times \mathcal{L}_p^f(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)(2, 1, 1). \tag{1.16.14}$$

ii) The element $\pi_{\alpha\beta} \,\mathrm{res}_p \,\kappa \in \mathrm{H}^1(\mathbb{Q}_p, V^{\alpha\beta})$ *belongs to* $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$*, and*

$$\log_{\alpha\beta}(\pi_{\alpha\beta} \,\mathrm{res}_p \,\kappa) = 2(1 - \chi(p) p^{-1} \alpha_f a_p(g)^{-1} a_p(h))^{-1} \times \mathcal{L}_p^g(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)(2, 1, 1). \tag{1.16.15}$$

1.16.4.3 Exceptional cases and improved Euler systems

Resuming the notation of the last section, let f, g, h of weights $2, 1, 1$ respectively. Let K be an imaginary quadratic field of discriminant $-D_K$ relatively prime to N_f. Let

$$\psi_g, \psi_h : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

be two finite order Hecke characters of K of conductors $\mathfrak{c}_g, \mathfrak{c}_h$ respectively and let $g := \theta(\psi_g)$ and $h := \theta(\psi_h)$ denote the theta series associated to them as in §1.6.4.5. Assume that the nebentype characters of g and h are mutually inverse. Fix a prime number p such that

$$p \,||\, N_f \ \text{ and } p \text{ is inert in } K. \tag{1.16.16}$$

Let $E/\mathbb{Q}$ be the elliptic curve attached to f. Condition (1.16.16) implies that

$$\alpha_f = \begin{cases} +1 & \text{if } E \text{ has split multiplicative reduction at } p \\ -1 & \text{if } E \text{ has nonsplit multiplicative reduction at } p, \end{cases} \tag{1.16.17}$$

$$\beta_g = -\alpha_g, \qquad (\alpha_h, \beta_h) = \begin{cases} (-1/\alpha_g, 1/\alpha_g) & \text{if } \alpha_f = +1; \\ (1/\alpha_g, -1/\alpha_g) & \text{if } \alpha_f = -1. \end{cases} \tag{1.16.18}$$

Let $\mathcal{H}$ denote the surface cut out by the equation $k = 2 + \ell - m$ in $\mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}}$ and let $\mathcal{F}_{\mathcal{H}}$ be the fraction field of the ring of Iwasawa functions on $\mathcal{H}$.

Define the two-variable meromorphic Iwasawa function

$$\boldsymbol{\mathcal{E}}_g := \mathcal{E}_g(\mathbf{f}, \mathbf{g}, \mathbf{h}) := 1 - \frac{a_p(\mathbf{g}_\ell)}{\chi(p) a_p(\mathbf{f}_{\ell-m+2}) a_p(\mathbf{h}_m)} \in \mathcal{F}_{\mathcal{H}}.$$

Conditions (1.16.17), (1.16.18) imply that

$$\boldsymbol{\mathcal{E}}_g(2,1,1) = 1 - \frac{\alpha_g \beta_h}{\alpha_f} = 0. \tag{1.16.19}$$

This fact forces the vanishing of the class $\boldsymbol{\kappa}(2,1,1)$, as explained in [BSV] (cf. also Prop. 1.108 below).

Let $\rho_{\mathcal{H}} : \mathcal{F}_{\mathbf{fgh}} \longrightarrow \mathcal{F}_{\mathcal{H}}$ be the map taking a function $F(k, \ell, m)$ to its restriction $F(2 + \ell - m, \ell, 1)$ to the plane $\mathcal{H}$. Denote $\mathbb{V}_{|\mathcal{H}} := \mathbb{V} \otimes_{\mathcal{F}_{\mathbf{fgh}}, \rho_{\mathcal{H}}} \mathcal{F}_{\mathcal{H}}$ and let

$$\boldsymbol{\kappa}_{|\mathcal{H}} := \rho_{\mathcal{H},\star}(\boldsymbol{\kappa}) \in \mathrm{H}^1(\mathbb{Q}, \mathbb{V}_{|\mathcal{H}})$$

denote the restriction of $\boldsymbol{\kappa}$ to $\mathcal{H}$. Recall the triple of test vectors $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ of Theorem 1.96.

Proposition 1.108. *There exists a global cohomology class*

$$\boldsymbol{\kappa}_g^* \in \mathrm{H}^1(\mathbb{Q}, \mathbb{V}_{|\mathcal{H}})$$

satisfying, for each $\ell, m \in \mathbb{Z}_{\geq 1}$*:*

$$\boldsymbol{\kappa}(2 + \ell - m, \ell, m) = \boldsymbol{\mathcal{E}}_g(2 + \ell - m, \ell, m) \cdot \boldsymbol{\kappa}_g^*(2 + \ell - m, \ell, m). \tag{1.16.20}$$

Proof. This is shown in [BSV, §8.3] □

Let $\mathcal{C} \subset \mathcal{H}$ denote the curve given on which the set of points $(\ell+1, \ell, 1)$ for $\ell \in \mathbb{Z}_{\geq 1}$ is dense.

Proposition 1.109. *There exists an analytic function* $\mathcal{L}_p^{f,*}$ *on* $\mathcal{C}$ *satisfying, for each* $\ell \in \mathbb{Z}_{\geq 1}$*:*

$$\mathcal{L}_p^{f,*}(\ell) = \frac{\langle w_N(f_{\ell+1})^{(p)}, h\mathbf{g}_\ell \rangle}{\langle w_N(f_{\ell+1})^{(p)}, w_N(f_{\ell+1})^{(p)} \rangle}$$

and

$$\mathcal{L}_p^f(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)(\ell+1, \ell, 1) = \left(1 - \frac{a_p(\mathbf{g}_\ell)\alpha_h}{a_p(\mathbf{f}_{\ell+1})}\right) \boldsymbol{\mathcal{E}}_g(\ell+1, \ell, 1) \mathcal{L}_p^{f,*}(\ell). \tag{1.16.21}$$

Moreover,

$$\mathcal{L}_p^{f,*}(1) = \frac{1}{2(1-1/p)} \exp^*_{\beta\beta}(\pi_{\beta\beta} \partial_p \boldsymbol{\kappa}_g^*(2,1,1)) = \frac{1}{2(1-1/p)} \exp^*_{\beta\beta}(\partial_p \boldsymbol{\kappa}_g^*(2,1,1)). \tag{1.16.22}$$

Proof. This is proved in [BSV, Lemma 8.6 + equation (170)]. In particular, combining Theorem A, part 3 of Proposition 8.2 and Lemma 8.6 of loc. cit. it follows that the factor λ_{w_0} in equation (170) is $1/2(1-1/p)$. □

Remark 1.110. By (1.16.17) and (1.16.18), the factor $1 - \dfrac{a_p(\mathbf{g}_\ell)\alpha_h}{a_p(\mathbf{f}_{\ell+1})}$ does not vanish in a neighborhood of $(2,1,1)$ in $\mathcal{C}$.

Chapter 2

Special values of the triple product p-adic L-function and the elliptic Stark conjecture in higher weight

Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be Λ-adic cuspidal Hida families of tame levels N_f, N_g, N_h and tame Nebentype characters χ_f, χ_g, χ_h satisfying that $\gcd(N_f, N_g, N_h)$ is squarefree and $\chi_f\chi_g\chi_h = 1$. Put $N = \mathrm{lcm}(N_f, N_g, N_h)$ and suppose that $p \nmid N$. If $\breve{\mathbf{f}}, \breve{\mathbf{g}}$ and $\breve{\mathbf{h}}$ are test vectors of tame level N associated to $\mathbf{f}, \mathbf{g}$ and $\mathbf{h}$, we can consider the three variable p-adic L-function $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ introduced in §1.15.3. Recall that the set of cristalline triples in the domain of $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ can be divided into four regions

$$\mathcal{W}^\circ_{\mathbf{fgh}} = \mathcal{W}^f_{\mathbf{fgh}} \sqcup \mathcal{W}^g_{\mathbf{fgh}} \sqcup \mathcal{W}^h_{\mathbf{fgh}} \sqcup \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}.$$

The type of arithmetic information encoded by

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k, \ell, m)$$

depends on the region where (k, ℓ, m) lies. For example, $\mathcal{W}^g_{\mathbf{fgh}}$ is the region of classical interpolation and therefore $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k, \ell, m)$ can be expressed in terms of the algebraic part of the central value of the classical L-function $L(\breve{f}_k, \breve{g}_\ell, \breve{h}_m, s)$ by the interpolation formula of Theorem 1.96. On the other hand, by Theorem 1.106 if $(k, \ell, m) \in \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$, then $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k, \ell, m)$ can be expressed in terms of the Abel–Jacobi image of a generalised diagonal cycle $\Delta_{k,\ell,m}$ in the product of Kuga–Sato varieties $W = W_{k-2} \times W_{\ell-2} \times W_{m-2}$ (cf. §1.6.4.3 for the definition).

The cases $\mathcal{W}^f_{\mathbf{fgh}}$ and $\mathcal{W}^h_{\mathbf{fgh}}$ turn out to be symmetric, so it remains to consider $\mathcal{W}^f_{\mathbf{fgh}}$. In this case, the article [DLR15] can be viewed as the first step towards understanding the values of $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ at classical weights in $\mathcal{W}^f_{\mathbf{fgh}}$ by means of the so-called *elliptic Stark conjecture (ESC)*, which we recall in §2.1. In the setting in which the complex L-function $L(f_2, g_1, h_1, s)$ vanishes to even order ≥ 2 at its central point $s = 1$, the ESC gives a conjectural formula for $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2, 1, 1)$, under an additional classicality assumption on the weight one specialisation of $\mathbf{g}$. The main theoretical evidence supporting the elliptic Stark conjecture stems from [DLR15, Theorem 3.1], which proves it in the particular case where g and h are theta series of the same imaginary quadratic field in which p splits.

The aim of this chapter, which contains the results of [GG20], is to study natural generalisations of the elliptic Stark conjecture regarding the special values of the triple product L-function $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ introduced at general unbalanced weights $(k, \ell, m) \in \mathcal{W}^g_{\mathbf{fgh}}$, in a setting in which the complex L-function attached to the triple (f_k, g_ℓ, h_m) vanishes at its central point. In order to describe this more accurately, we devote the following section to describe the setting and the structure of the elliptic Stark Conjecture of Darmon–Lauder–Rotger. In §2.2 we describe a generalisation of ESC to triples of newforms (f, g, h) of weight $(k, 1, 1)$ and finally in §2.3 we allow the weights (k, ℓ, m) of (f, g, h) to be general, provided $k, \ell, m \geq 2$ and $k \geq \ell + m$.

2.1 The elliptic Stark conjecture

Let $E/\mathbb{Q}$ be an elliptic curve defined over $\mathbb{Q}$ and conductor N_f, let

$$g \in M_1(N_g, \chi_g), \qquad h \in M_1(N_h, \chi_h)$$

be eigenforms of weight one, Fourier coefficients in an number field L and whose Nebentype characters satisfy that

$$\chi_g \cdot \chi_h = 1.$$

As explained in §1.12.3, if f is the newform attached to $E/\mathbb{Q}$ and ρ_{gh} is the Artin representation of (1.13.3), the complex L-function $L(E\otimes\rho_{gh}, s) = L(f\otimes g\otimes h, s)$ can be completed to an entire function satisfying a functional equation with center of symmetry $s = 1$, and it is expected from the Galois equivariant version of BSD that

$$\mathrm{ord}_{s=1} L(E\otimes\rho_{gh}, s) \stackrel{?}{=} \dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}^{\circ}, E(H)_L). \tag{2.1.1}$$

As reported in §1.14, this conjecture is known in some cases of analytic rank 0, where the seminal ideas of Kolyvagin to bound Selmer groups have been applied to several instances of Euler systems, and rank 1, where Heegner point constructions are used to provide the supply of points in $E(H)_L$ predicted by the conjecture when the rank is positive. For ranks ≥ 2, BSD conjecture is widely open, as no construction of rational points in that case is known. In the work [DLR15], Darmon, Lauder, and Rotger came up with a p-adic construction which conjecturally produces (a certain combination of) points in certain rank 2 settings. This so-called elliptic Stark Conjecture (ESC), for which they provide a great deal of empirical and theoretical evidence, is promising to shed light on the equivariant BSD conjecture in higher rank.

In this section, we recall the statement of the elliptic Stark conjecture of [DLR15], in order to explain the original conjecture that will be generalised in the next sections and to introduce objects that will appear also in Conjectures 2.13 and 2.17. We put ourselves in the setting of §1.13.2, and we retain the notation used there and above. In particular

$$f \in S_2(N_f), \quad g \in S_1(N_g, \chi_g), \quad h \in S_1(N_h, \chi_h)$$

are newforms with $\chi_g\chi_h = 1$ and f is attached to an elliptic curve $E/\mathbb{Q}$ of conductor N_f. Recall that in this setting

$$L(E\otimes\rho_{gh}, s) = L(f\otimes g\otimes h, s)$$

is the triple product complex L-function described in §1.12.3. In order to state the conjecture, we need to fix the exact setting and to introduce the objects attached to the triple (f, g, h) of weight $(2, 1, 1)$ such as the *regulator* $\mathrm{Reg}_{g_\alpha}(E\otimes\rho_{gh})$, the *iterated integral* and the *Stark unit* u_{g_α}. These constructions will be used or generalised in the next sections in order to state Conjecture 2.13 and Conjecture 2.17.

Recall that the completed L-function $\Lambda(E\otimes\rho_{gh}, s)$ satisfies a functional equation of the form

$$\Lambda(f\otimes g\otimes h, s) = \varepsilon(f, g, h)\Lambda(f\otimes g\otimes h, 2-s), \tag{2.1.2}$$

where $\varepsilon(f, g, h) \in \{\pm 1\}$ decomposes in a product of local signs

$$\varepsilon(f, g, h) = \prod_{v \mid N\cdot\infty} \varepsilon_v(f, g, h),$$

where $N := \mathrm{lcm}(N_f, N_g, N_h)$. The weights of the triple (f, g, h) are $(2, 1, 1)$, so by (1.12.7) we have

$$\varepsilon_\infty(f, g, h) = +1.$$

The elliptic Stark conjecture is stated under the following setting.

Assumption 2.1. 1. $\varepsilon_v(f, g, h) = +1$ for all $v \mid \widetilde{N}$;

2. $L(f \otimes g \otimes h, 1) = 0$.

By (2.1.2) and by the fact that $s = 1$ is a central critical point for $L(E \otimes \rho_{gh}, s)$, the first part of the assumption implies that the order of vanishing of this L-function at $s = 1$ is even. So Assumption 2.1 implies that

$$\mathrm{ord}_{s=1} L(f \otimes g \otimes h, s) \geq 2.$$

In particular, we are interested in the study of the rank 2 setting i.e. under the following hypothesis.

Assumption 2.2.

$$\mathrm{ord}_{s=1} L(f \otimes g \otimes h, s) = 2.$$

Fix a prime number $p \geq 3$ such that

$$\mathrm{ord}_p(N_f) \leq 1, \text{ and } p \nmid N_g N_h, \tag{2.1.3}$$

and assume that f, g and h are ordinary at p. Denote by N the prime-to-p part of $\widetilde{N} = \mathrm{lcm}(N_f, N_g, N_h)$. Recall that we denoted by α_g, β_g the two roots of the Hecke polynomial of g at p and

$$g_\alpha(q) = g(q) - \beta_g \cdot g(q^p) \tag{2.1.4}$$

the p-stabilisation of g such that $U_p \cdot g_\alpha = \alpha_g \cdot g_\alpha$. Let

$$(\breve{f}, \breve{\gamma}, \breve{h}) \in e_f S_2(Np) \times e_{g_\alpha} M_1(Np, \chi_g)^\vee \times e_h M_1(Np, \chi_h) \tag{2.1.5}$$

be a triple on which the good Hecke operators and U_p act with the same eigenvalues as f, g_α and h. A key notion introduced in [DLR15], and the first ingredient of the elliptic Stark conjecture, is the *p-adic iterated integral*

$$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} \in \mathbb{C}_p$$

attached to such a test vector, under a certain technical (but essential) classicality hypothesis on g, labelled as Hypothesis C in [DLR15]. In fact, we will assume the following more explicit condition, labelled as Hypothesis C' in loc. cit.

Assumption 2.3. The modular form g satisfies one of the following conditions:

1. it is a cuspform *regular* at p (i.e. $\alpha_g \neq \beta_g$), and it is not the theta series of a character of a real quadratic field in which p splits;

2. it is an Eisenstein form which is *irregular* (i.e. $\alpha_g = \beta_g$).

Under Assumption 2.3 one can define a 1-dimensional L-subspace $V_g^{\circ\alpha}$ of V_g°:

1. if g satisfies the first condition, then the attached Artin representation V_g° decomposes as the direct sum of the eigenspaces $V_g^{\circ\alpha}, V_g^{\circ\beta}$ with respect to the action of Frob_p, with eigenvalues α_g, β_g respectively.

2. If g is an irregular Eisenstein form, we take $V_g^{\circ\alpha}$ to be any 1-dimensional subspace of V_g° which is not stable under the action of $G_\mathbb{Q}$.

Let $g_\alpha^* := g_\alpha \otimes \chi_g^{-1} \in M_1(Np, \chi_g^{-1})$ be the twist of g_α by the inverse of its Nebentype character and denote $S_1^{\mathrm{oc,ord}}(N, \chi_g)[[g_\alpha^*]]$ the generalised eigenspace of cuspidal ordinary overconvergent modular forms attached to g_α^* and relative to the action of the Hecke operators T_ℓ with $\ell \nmid Np$ and U_p. Assumption 2.3 ensures that $S_1^{\mathrm{oc,ord}}(N, \chi_g)[[g_\alpha^*]]$ is nontrivial and consists entirely of classical modular forms, i.e. that there is an equality

$$S_1^{\mathrm{oc,ord}}(N, \chi_g)[[g_\alpha^*]] = M_1(Np, \chi_g)_{\mathbb{C}_p}[[g_\alpha^*]] \tag{2.1.6}$$

(see §1.7.4). In case 1. of Assumption 2.3, the equality (2.1.6) is a consequence of a result of Bellaïche–Dimitrov in [BD16] (cf. [DLR15, §1]), and in case 2. it follows from the recent work of Betina–Dimitrov–Pozzi [BDP18].

Denote

$$e_{g_\alpha^*} : S_1^{\mathrm{oc,ord}}(N, \chi_g) \longrightarrow S_1^{\mathrm{oc,ord}}(N, \chi_g)[[g_\alpha^*]] = M_1(Np, \chi_g)_{\mathbb{C}_p}[[g_\alpha^*]]$$

the projection onto the generalised eigenspace attached to g_α^*.

Definition 2.4. For a triple $(\breve{f}, \breve{\gamma}, \breve{h})$ as in (2.1.5), the *p-adic iterated integral* is defined as

$$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} := \breve{\gamma}(e_{g_\alpha^*} e_{\mathrm{ord}}(\breve{F} \cdot \breve{h})) \in \mathbb{C}_p,$$

where $\breve{F} \in S_0^{\mathrm{oc}}(N)$ is the overconvergent primitive of $\breve{f}$, as defined in 1.7.4.1.

Note that $e_{g_\alpha^*} e_{\mathrm{ord}}(\breve{F} \cdot \breve{h})$ belongs to $S_1(N, \chi_g)[[g_\alpha^*]]$, so Assumption 2.3 is necessary for the definition above. We put ourselves in the case in which

$$\dim_L \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}^\circ, E(H) \otimes L) = 2. \tag{2.1.7}$$

Recall that the Galois equivariant BSD conjecture in this case predicts the equality (2.1.1), then Assumption 2.2 should imply (2.1.7). Fix L-bases

$$\{\Phi_1, \Phi_2\} \text{ for } \mathrm{Hom}_{G_{\mathbb{Q}}}(V_{gh}^\circ, E(H) \otimes L), \qquad \{v_1, v_2\} \text{ for } V_g^{\circ\,\alpha} \otimes V_h^\circ.$$

We attach to these bases a p-adic regulator as follows. Fix an embedding of H into $\mathbb{Q}_p^{\mathrm{ur}}$ and let H_p be the completion of H at the corresponding prime above p. For $i, j \in \{1, 2\}$, we can consider the points

$$\Phi_i(v_j) \in E(H) \otimes L \hookrightarrow E(H_p) \otimes L.$$

Definition 2.5. The *regulator* attached to the pair (E, ρ_{gh}) and to the choice of the eigenvalue α_g of V_g is

$$\mathrm{Reg}_{g_\alpha}(E, \rho_{gh}) := \det \begin{pmatrix} \log_{E,p}(\Phi_1(v_1)) & \log_{E,p}(\Phi_1(v_2)) \\ \log_{E,p}(\Phi_2(v_1)) & \log_{E,p}(\Phi_2(v_2)) \end{pmatrix},$$

where $\log_{E,p} : E(H_p) \otimes L \longrightarrow H_p \otimes L$ is the p-adic logarithm of the elliptic curve E.

Notice that $\mathrm{Reg}_{g_\alpha}(E, \rho_{gh})$ depends on the choice of the bases above only up to multiplication by $L^\times$.

Remark 2.6. Recall the isomorphism

$$\mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}^\circ, E(H)_L) \cong (E(H)_L \otimes V_{gh}^\circ)^{\mathrm{Gal}(H/\mathbb{Q})} \tag{2.1.8}$$

of Remark 1.92 and choose Frob_p-eigenvectors

$$v_{gh}^{\alpha\alpha},\ v_{gh}^{\alpha\beta},\ v_{gh}^{\beta\alpha},\ v_{gh}^{\beta\beta}\ \in V_{gh}^\circ$$

with eigenvalues $\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h$ respectively. Under (2.1.7), fix a basis $\{P, Q\}$ of $(E(H)_L \otimes V_{gh}^\circ)^{\mathrm{Gal}(H/\mathbb{Q})}$ and write

$$P = P^{1/\alpha\alpha} \otimes v_{gh}^{\alpha\alpha} + P^{1/\alpha\beta} \otimes v_{gh}^{\alpha\beta} + P^{1/\beta\alpha} \otimes v_{gh}^{\beta\alpha} + P^{1/\beta\beta} \otimes v_{gh}^{\beta\beta},$$
$$Q = Q^{1/\alpha\alpha} \otimes v_{gh}^{\alpha\alpha} + Q^{1/\alpha\beta} \otimes v_{gh}^{\alpha\beta} + Q^{1/\beta\alpha} \otimes v_{gh}^{\beta\alpha} + Q^{1/\beta\beta} \otimes v_{gh}^{\beta\beta},$$

with $P^{1/\alpha\alpha}, Q^{1/\alpha\alpha} \in E(H_p)^{1/\alpha\alpha}$ and analogously for the other local points. With this notation,

$$\mathrm{Reg}_{g_\alpha}(E, \rho_{gh}) := \det \begin{pmatrix} \log_{E,p}(P^{1/\alpha\alpha}) & \log_{E,p}(P^{1/\alpha\beta}) \\ \log_{E,p}(Q^{1/\alpha\alpha}) & \log_{E,p}(Q^{1/\alpha\beta}) \end{pmatrix}$$

See [DR16] for further details. This formulation also shows that Frob_p acts on $\mathrm{Reg}_{g_\alpha}(E, \rho_{gh})$ as multiplication by $(\alpha_g^2 \alpha_h \beta_h)^{-1}$.

The last ingredient we need in order to state the elliptic Stark conjecture is the unit u_{g_α} attached to the *adjoint representation* Ad_g of V_g°. As a vector space,

$$\mathrm{Ad}_g := \left\{ M \in \mathrm{End}(V_g^\circ) \mid \mathrm{tr}(M) = 0 \right\} \cong \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in L \right\},$$

so it has dimension 3 over L. The Galois group $G_\mathbb{Q}$ acts on this space by conjugation by ρ_g, i.e. the representation $\rho_{\mathrm{Ad}_g} : G_\mathbb{Q} \longrightarrow \mathrm{Aut}(\mathrm{Ad}_g)$ is given by

$$\rho_{\mathrm{Ad}_g}(\sigma)(M) := \rho_g(\sigma) \cdot M \cdot \rho_g(\sigma)^{-1} \qquad \text{for } \sigma \in G_\mathbb{Q},\ M \in \mathrm{Ad}_g .$$

If Frob_p acts on V_g with eigenvalues α_g, β_g, then the eigenvalues for the Frobenius on Ad_g are $1, \alpha_g/\beta_g, \beta_g/\alpha_g$. Let $H_g \subseteq H$ be the field cut out by ρ_{Ad_g} and let

$$(\mathcal{O}_{H_g}^\times)_L^{\mathrm{Ad}_g} := \sum_\varphi \varphi(\mathrm{Ad}_g),$$

where the sum runs over a basis $\{\varphi\}$ of $\mathrm{Hom}_{G_\mathbb{Q}}(\mathrm{Ad}_g, \mathcal{O}_{H_g}^\times \otimes L)$. In [DLR15, §1.5] it is defined a *Stark unit*

$$u_{g_\alpha} \in \begin{cases} \left(\mathcal{O}_{H_g}[1/p]^\times\right)_L^{\mathrm{Ad}_g^\circ} & \text{if } g \text{ is Eisenstein} \\ (\mathcal{O}_{H_g}^\times)_L^{\mathrm{Ad}_g^\circ} & \text{if } g \text{ is cuspidal} \end{cases} \tag{2.1.9}$$

on which Frob_p acts with eigenvalue β_g/α_g, and which is well-defined up to $L^\times$. More precisely, if g is a cuspform, then $\mathrm{Hom}_{G_\mathbb{Q}}(\mathrm{Ad}_g, \mathcal{O}_{H_g}^\times \otimes L)$ is one-dimensional by [DLR15, Proposition 1.5]. If φ is a generator for this space, then and u_{g_α} is the image via φ of a Frob_p-eigenvector of Ad_g with eigenvalue β_g/α_g. In the case in which the eigenspace $\mathrm{Ad}_g^{\beta_g/\alpha_g}$ is one dimensional, then u_{g_α} is uniquely determined by this condition up to $L^\times$, and this is always under Assumption (2.3), provided $\alpha_g \neq -\beta_g$. For further details, and for the definition of u_{g_α} in the Eisenstein case, we refer to [DLR15, §1.2].

Remark 2.7. The unit u_{g_α} will appear in various forms throughout this thesis. Let $\log_p : \mathcal{O}_{H_g}^\times \otimes L \longrightarrow \mathcal{O}_{H_p} \otimes L \longrightarrow H_p \otimes L$ be the p-adic logarithm of $\mathcal{O}_{H_p}$.

1. The element $\log_p(u_{g_\alpha})$ appears at the denominator of the conjectural formulas (2.1.12) and (2.2.3) for special values of the triple product p-adic L-function $\mathcal{L}_p^g$ at points $(2,1,1)$ and $(k,1,1)$ with $k \geq 2$ respectively.

2. In the case in which g and h are theta series of finite order Hecke characters of the same imaginary quadratic field in which p splits, u_{g_α} [1] an elliptic unit, and its p-adic logarithm is related to a special value of Katz's p-adic L-function as in §1.16.1. This fact is used in [DLR15] to prove a special case of the elliptic Stark conjecture, and, using the same strategy of loc. cit. we will use it in this chapter to prove some instances of Conjecture 2.13 and Conjecture 2.17.

[1] **FIXME:** is? is related to?

3. Recall the elements $\Omega_{g_\alpha} \in H_p^{1/\alpha_g}, \Theta_{g_\alpha} \in H_p^{1/\beta_g}$ of Definition 1.63, and define

$$\mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}} \in H_p^{\beta_g/\alpha_g}. \tag{2.1.10}$$

This element is often expected to be related to the Gross–Stark unit u_{g_α}. More precisely, under the first case of Assumption 2.3, [DR16, Conjecture 2.1] gives the following prediction.

Conjecture 2.8.

$$\mathcal{L}_{g_\alpha} = \log_p(u_{g_\alpha}).$$

In Chapters 3 and 4 we will consider the case in which the complex L-function $L(E\otimes\rho_{gh}, s)$ does not vanish at $s = 1$. In this setting, we obtain the formulas (3.2.2) and [2] for the special value of the triple product p-adic L-function $\mathcal{L}_p^g$ at the point $(2, 1, 1)$, and the element $\mathcal{L}_{g_\alpha}$ appears in the denominator of these formulas. This fact has to be related to the conjectural formulas of this Chapter via Conjecture 2.8.

We are finally able to state the elliptic Stark conjecture.

Conjecture 2.9. Let $r(E, \rho_{gh}) := \dim_L \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, E(H) \otimes L)$. Under Assumptions 2.3, 2.1,

i) if $r(E, \rho_{gh}) > 2$, then $\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = 0$ for any choice of $(\breve{f}, \breve{\gamma}, \breve{h})$ in $e_f S_2(Np) \times e_{g_\alpha} M_1(Np, \chi_g)^\vee \times e_h M_1(Np, \chi_h)$;

ii) if $r(E, \rho_{gh}) = 2$, then there exist a triple

$$(\breve{f}, \breve{\gamma}, \breve{h}) \in e_f \cdot S_2(Np) \times e_{g_\alpha} \cdot M_1(Np, \chi_g)^\vee \times e_h \cdot M_1(Np, \chi_h)$$

such that

$$\int_{\breve{\gamma}} \breve{f} \cdot \breve{h} = \frac{\mathrm{Reg}_{g_\alpha}(E, \rho_{gh})}{\log_p(u_{g_\alpha})} \mod L^\times. \tag{2.1.11}$$

The statement above is [DLR15, Conjecture ES]. We give now an equivalent formulation of this conjecture in terms of the special value at weight $(2, 1, 1)$ of a triple product p-adic L-function, which is suitable for the generalisations to higher weight of Conjecture 2.9 that will be stated in the following sections. The equivalence between the two formulations follows from [DLR15, Proposition 2.6].

Conjecture 2.10. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be the Hida families passing through the p-stabilisations $f_\alpha, g_\alpha, h_\alpha$ of f, g and h, and let $r(E, \rho_{gh}) := \dim_L \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, E(H) \otimes L)$. Under Assumptions 2.3, 2.1,

i) sf $r(E, \rho_{gh}) > 2$, then $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2, 1, 1) = 0$ for any choice of test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$;

ii) if $r(E, \rho_{gh}) = 2$, then there exist a triple

$$(\breve{f}, \breve{\gamma}, \breve{h}) \in e_f S_2(Np) \times e_{g_\alpha} M_1(Np, \chi_g)^\vee \times e_h M_1(Np, \chi_h)$$

and Hida families $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$ with $\breve{f}_2 = f,\ \breve{g}_1 = g,\ \breve{h}_1 = h$, such that

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2, 1, 1) = \frac{\mathrm{Reg}_{g_\alpha}(E, \rho_{gh})}{\log_p(u_{g_\alpha})} \mod L^\times. \tag{2.1.12}$$

[2]**FIXME:** mettere ref del capitolo kolyvagin

2.2 The conjecture for $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(k,1,1)$

Let

$$f \in S_k(N_f,\chi_f), \quad g \in M_1(N_g,\chi_g), \quad h \in M_1(N_h,\chi_h)$$

be three normalised eigenforms, with $k \geq 2$ and

$$\chi_f \cdot \chi_g \cdot \chi_h = 1. \tag{2.2.1}$$

Notice that this condition forces k to be even, so that $k/2$, which is the central point for the complex L-function attached to $f \otimes g \otimes h$, is an integer. As in the previous section, fix a prime number p satisfying (2.1.3), assume that f, g and h are p-ordinary, set N to be the prime-to-p-part of $\mathrm{lcm}(N_f,N_g,N_h)$ and fix the p-stabilisation g_α of g. We begin by defining, under certain additional conditions, a regulator $\mathrm{Reg}(f,g_\alpha,h)$ which generalises to weight $k \geq 2$ the one defined in [DLR15] for $k=2$, that we recalled in §2.1. We still require g to satisfy Assumption 2.3 and the complex L-function $L(f\otimes g\otimes h,s)$ to satisfy Assumption 2.1(substituing 1 with $k/2$). Since the weights $(k,1,1)$ of the triple (f,g,h) are unbalanced, then also in this case $\varepsilon_\infty(f,g,h)=+1$ and arguing as in the previous section, we conclude that Assumption 2.1 implies

$$\mathrm{ord}_{s=k/2} L(f\otimes g\otimes h,s) \geq 2.$$

In order to simplify a bit the notation for the f-isotypical component, we put

$$\mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L}^{[f]} := e_f \cdot \mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L}.$$

As we mentioned in §1.14, a conjecture due to Beilinson predicts that

$$\mathrm{ord}_{s=k/2} L(f\otimes g\otimes h,s) \stackrel{?}{=} \dim_L \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, \mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L}^{[f]}).$$

If $\dim_L \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, \mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L}^{[f]}) = 2$, then we can define a regulator by fixing an L-basis $\{\Phi_1,\Phi_2\}$ of this space and an L-basis $\{v_1,v_2\}$ of $V_{gh}^{\circ\alpha} = V_g^{\circ\alpha} \otimes V_h^\circ$. Here $V_g^{\circ\alpha}$ is defined exactly as in the previous section, under Assumption 2.3.

Remark 2.11. In this setting the p-adic logarithm appearing in Definition 2.5 is replaced by the p-adic Abel–Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^{k/2}(W_{k-2}/H_p)_{0,L} \longrightarrow \mathrm{Fil}^{k-2}\,\mathrm{H}_{\mathrm{dR}}^{k-1}(W_{k-2}/H_p)^\vee \tag{2.2.2}$$

introduced in §1.11, where H_p is the completion of H with respect to the prime relative to our fixed embedding $H \subseteq \mathbb{Q}_p^{\mathrm{ur}}$. This is justified by the fact that (2.2.2) can be seen as a generalisation of the formal group logarithm attached to a differential form on $X_1(N)$. Indeed, if $\log_{E,p}$ denotes the formal group logarithm of E attached to the differential ω_f, one has that

$$\log_{E,p}(P) = \mathrm{AJ}_p(P)(\omega_f)$$

for any $P \in X_1(N)(\mathbb{Q}_p)$.

Definition 2.12. The *regulator* attached to the triple (f,g_α,h) is

$$\mathrm{Reg}(f,g_\alpha,h) := \det\begin{pmatrix} \mathrm{AJ}_p(\Phi_1(v_1))(\omega_f) & \mathrm{AJ}_p(\Phi_1(v_2))(\omega_f) \\ \mathrm{AJ}_p(\Phi_2(v_1))(\omega_f) & \mathrm{AJ}_p(\Phi_2(v_2))(\omega_f) \end{pmatrix},$$

where $\omega_f \in e_f \cdot \mathrm{Fil}^{k-2}\,\mathrm{H}_{\mathrm{dR}}^{k-1}(W_{k-2}/H_p)$ is defined in §1.8.

In particular, when $k=2$ we recover from Definition 2.12 the regulator of §2.1. The following conjecture, which is [GG20, Conjecture 2.9], is a generalisation of Conjecture 2.9 in the setting of this section.

Conjecture 2.13. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be the Hida families passing through the p-stabilisations $f_\alpha, g_\alpha, h_\alpha$ of f, g and h. Set $r := \dim_L \mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, \mathrm{CH}^{k/2}(W_{k-2}/H)_{0,L}^{[f]})$.

i) If $r > 2$, then $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,1,1) = 0$ for any choice of test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$;

ii) if $\mathrm{ord}_{s=k/2} L(f \otimes g \otimes h, s) = 2$, then there exists a triple

$$(\breve{f}, \breve{g}_\alpha, \breve{h}) \in e_f S_k(Np, \chi_f)_L \times e_{g_\alpha} M_1(Np, \chi_g) \times e_h M_1(Np, \chi_h)_L$$

and Hida families $\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}$ with $\breve{f}_k = \breve{f}, \breve{g}_1 = \breve{g}, \breve{h}_1 = \breve{h}$, such that

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,1,1) = \frac{\mathrm{Reg}(f, g_\alpha, h)}{\mathfrak{g}(\chi_f) \log_p(u_{g_\alpha})} \qquad \mathrm{mod}\ L^\times, \tag{2.2.3}$$

where $\mathfrak{g}(\chi_f)$ denotes the Gauss sum of the character χ_f and u_{g_α} is the Stark unit (2.1.9).

Remark 2.14. Since $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k,1,1)$ belongs to $L_p \subseteq H_p \otimes L_p$, the Frobenius at p acts trivially on the left-hand part of (2.2.3). On the other hand, $\mathrm{Frob}_p(\log_p(u_{g_\alpha})) = \beta_g/\alpha_g$ and arguing as in Remark 2.6 one sees that the element Frob_p acts on the regulator as multiplication by $(\alpha_g^2 \alpha_h \beta_h)^{-1}$; finally, Frob_p acts on the Gauss sum with eigenvalue $\chi_f(p)$. So Frob_p also acts trivially on the right hand side of (2.2.3). Indeed, $\chi_f(p) = (\chi_g(p)\chi_h(p))^{-1} = (\alpha_g \beta_g \alpha_h \beta_h)^{-1}$, so

$$\frac{(\alpha_g^2 \alpha_h \beta_h)^{-1}}{\chi_f(p)\beta_g/\alpha_g} = \frac{\alpha_g \beta_g \alpha_h \beta_h \alpha_g}{\alpha_g^2 \alpha_h \beta_h \beta_g} = 1.$$

In §2.4.2 we will provide theoretical evidence in support of Conjecture 2.13 in the particular case where g and h are theta series of the same imaginary quadratic field where p splits. The main result in this direction is Theorem 2.19, which relates the triple product p-adic L-function in this setting with the p-adic Abel–Jacobi image of certain generalised Heegner cycles described in §1.16.3 which were introduced in the works of Bertolini–Darmon–Prasanna. The structure of the proof of Theorem 2.19 follows the strategy devised in [DLR15, §3.2] and it is a consequence of a factorisation formula for the triple product p-adic L-function that we prove in this special case.

2.3 The conjecture for general unbalanced weights

In this section we let the triple of normalised newforms (f, g, h) of levels (N_f, N_g, N_h) to have weights (k, ℓ, m) with

$$k \geq \ell + m, \text{ and } k, \ell, m \geq 2.$$

As in the previous sections, we fix the prime p satisfying 2.1.3 at which f, g and h are ordinary. The aim of this section is to formulate a version of the elliptic Stark conjecture for (f, g, h) in this setting. Let L be a number field that contains the Fourier coefficients of f, g and h. As explained in §1.10.4, the motive attached to $f \otimes g \otimes h$ is the object of $\mathcal{M}(\mathbb{Q})_L$ obtained as the tensor product of motives attached to f, g and h:

$$M(f \otimes g \otimes h) := M_f \otimes M_g \otimes M_h,$$

whose underlying variety is

$$X := W_{k-2} \times W_{\ell-2} \times W_{m-2}.$$

Put $c := (k + \ell + m - 2)/2$, which is the centre of symmetry for the functional equation satsfied by the complex L-function $L(f \otimes g \otimes h, s)$. We still work under Assumption 2.1(where we substitute 1 with c), and we are interested in the setting in which

$$\mathrm{ord}_{s=c} L(f \otimes g \otimes h, s) = 2.$$

Suppose that

$$\dim_L \mathrm{CH}^c(M(f \otimes g \otimes h))_{0,L} = 2$$

(which is the case if we believe in Beilinson's conjecture (1.14.3)).

Definition 2.15. Let $\{\Delta_1, \Delta_2\}$ be a basis of $\mathrm{CH}^c(M(f \otimes g \otimes h))_{0,L}$. The *regulator* attached to (f, g, h) is

$$\mathrm{Reg}(f,g,h) := \det \begin{pmatrix} \mathrm{AJ}_p(\Delta_1)(\omega_f \wedge \eta_g \wedge \omega_h) & \mathrm{AJ}_p(\Delta_1)(\omega_f \wedge \eta_g \wedge \eta_h) \\ \mathrm{AJ}_p(\Delta_2)(\omega_f \wedge \eta_g \wedge \omega_h) & \mathrm{AJ}_p(\Delta_2)(\omega_f \wedge \eta_g \wedge \eta_h) \end{pmatrix},$$

where $\omega_f, \eta_g, \eta_h, \omega_h$ are the de Rham classes defined in §1.8.

Remark 2.16. Since the definition of the regulator involves the choice of an L-basis of $\mathrm{CH}^c(M(f \otimes g \otimes h))_{0,L}$, it is only defined up to multiplication by an element of $L^\times$.

Denote $\mathbf{f}, \mathbf{g}, \mathbf{h}$ the Hida families passing through the ordinary p-stabilisations of f, g and h. Notice that in this setting we do not have to choose the stabilisation: since the weight of the modular forms is strictly greater then 2, only one of the roots of the Hecke polynomial at p is a p-adic unit. The following is the analog of the elliptic Stark conjecture in this setting.

Conjecture 2.17. Set $r := \dim_L \mathrm{CH}^c(M(f \otimes g \otimes h))_{0,L}$.

i) If $r > 2$, then $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k, \ell, m) = 0$ for any choice of test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$;

ii) if $\mathrm{ord}_{s=c} L(f \otimes g \otimes h, s) = 2$, then there exists a finite extension L_0 of L, a triple of test vectors

$$(\breve{f}, \breve{g}, \breve{h}) \in e_f S_k(Np, \chi_f)_L \times e_g M_\ell(Np, \chi_g)_L \times e_h M_m(Np, \chi_h)_L$$

and Hida families $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ such that $\breve{f}_k = \breve{f}, \breve{g}_\ell = \breve{g}, \breve{h}_m = \breve{h}$ such that

$$\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(k, \ell, m) = \mathrm{Reg}(f, g, h) \mod L_0^\times.$$

2.4 Proof of special cases and theoretical evidences

In this section we will give some theoretical evidence for Conjecture 2.13 and Conjecture 2.17 in some special cases. Retain the assumptions and the notation of the previous sections, and we fix an imaginary quadratic field K of discriminant $-D_K < 0$ in which p splits as $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$. We will focus on the case in which g and h are theta series of K. Throughout all this section, $f \in S_k(N_f, \chi_f)$ will be a normalised newform of weight $k \geq 2$ and squarefree level N_f. Suppose that the level N_f is coprime to D_K and that the pair (K, N_f) satisfies the Heegner Hypothesis 1.90. Fix from now on an ideal $\mathcal{N}_f$ of $\mathcal{O}_K$ of norm N_f and let $\mathcal{N}_{\chi_f} \mid \mathcal{N}_f$ be the ideal whose norm is the conductor of χ_f.

The argument of the proofs of the main results of this section follow essentially the strategy introduced in [DLR15, §3], which we now summarise. As a first step, we prove in §2.4.1 a factorisation formula for $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ (using the test vector of [Hsi19]), which involves Katz's and BDP p-adic L-functions introduced in §1.15.1 and §1.15.2. Then in §2.4.2 and in §2.4.3 we use this factorisation formula to prove Theorem 2.19 and Proposition 2.25, in which we relate special values of $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ to the p-adic Abel–Jacobi image of certain generalised Heegner cycles, using Theorem 1.105. In the case of weights $(k, 1, 1)$, Theorem 2.19 gives a proof of Conjecture 2.13 under some natural assumptions on Heegner cycles (see Assumption 2.22). Finally, in §2.4.3.1 we are interested in the special value of $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at a point (k, ℓ, m) with $k, \ell, m \geq 2$ (and $k \geq \ell + m$). In this case we need to particularise the special value formula of Proposition 2.25 to a triple of forms (f, g, h) satisfying some additional hypotheses in order to obtain a proof of the conjecture for such triple, conditional on the validity of Tate's conjecture for motives and of certain standard conjectures on the p-adic Abel–Jacobi map.

2.4.1 A factorisation formula for the triple product p-adic L-function

We resume the notations described in §1.15.3, and we consider the special case in which **g** and **h** are Hida families of theta series as described in §1.7.6. Recall that, using the notation of §1.7.6, the specialisation of **g** at a point of weight ℓ is the p-stabilisation of $g_\ell := \theta(\psi_{g,\ell-1})$, where $\psi_{g,\ell-1}$ is a Hecke character of K of infinity type $(0, \ell-1)$ and similarly $h_m := \theta(\psi_{h,m-1})$.

Define for each $k, \ell, m \in \mathbb{Z}_{\geq 1}$ the following Hecke characters:

1. $\Psi_g(\ell) := \psi_{g,\ell-1}^{-2}\chi_g \mathrm{N}_K^{\ell}$;
2. $\Psi_{gh}(k, \ell, m) := (\psi_{g,\ell-1}\psi_{h,m-1})^{-1}\,\mathrm{N}_K^{\frac{k+\ell+m-2}{2}}$;
3. $\Psi_{gh'}(k, \ell, m)' := (\psi_{g,\ell-1}\psi_{h,m-1})^{-1}\,\mathrm{N}_K^{\frac{k+\ell+m-2}{2}}$.

For each $k, \ell, m \in \mathbb{Z}_{\geq 1}$, we have the following decomposition of the triple tensor product of representations

$$V_{f_k}\otimes V_{g_\ell}\otimes V_{h_m} = V_{f_k}\otimes V_{\psi_{g,\ell-1}}\otimes V_{\psi_{h,m-1}} = \Big(V_{f_k}\otimes V_{\psi_{g,\ell-1}\psi_{h,m-1}}\Big)\oplus\Big(V_{f_k}\otimes V_{\psi_{g,\ell-1}\psi'_{h,m-1}}\Big). \quad (2.4.1)$$

This induces a factorisation of complex L-functions, up to a finite number of factors at the bad reduction primes. Evaluating at the central critical point $c_0 := \frac{k+\ell+m-2}{2}$ we obtain a factorisation of the form

$$L(f_k\otimes g_\ell\otimes h_m, c_0) = \mathfrak{f}_1(f_k, g_\ell, h_m)L(f_k, \psi_{g,\ell-1}\psi_{h,m-1}, c)L(f_k, \psi_{g,\ell-1}\psi_{h,m-1}, c_0) \quad (2.4.2)$$

$$= \mathfrak{f}_1(f_k, g_\ell, h_m)L(f_k, \Psi_{gh}(k,\ell,m)^{-1}, c_0)L(f_k, \Psi_{gh'}(k,\ell,m)^{-1}, c_0),$$

where $\mathfrak{f}_1(f_k, g_\ell, h_m)$ accounts for the evaluation of the Euler factors at the bad reduction primes. From this decomposition it follows a factorisation of the triple product p-adic L-function in terms of Katz's and Castella's p-adic L-functions. Recall that we denoted

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h}) := \mathcal{L}_p^g(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star),$$

where $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ is the triple of Λ-adic test vector of [Hsi19, Chapter 3].

Theorem 2.18. *For each* $(k, \ell, m) \in \mathcal{W}^\circ_{\mathbf{fgh}}$ *we have*

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})^2(k,\ell,m)\mathcal{L}_p(K)(\Psi_g(\ell))^2 = \mathcal{L}_p(\mathbf{f}, K)(k, \Psi_{gh}(k,\ell,m))^2\mathcal{L}_p(\mathbf{f}, K)(k, \Psi_{gh'}(k,\ell,m))^2\mathfrak{f}(k,\ell,m),$$

where

i) $\mathfrak{f}(k,\ell,m) := \dfrac{\prod_{q|\Sigma_{\mathrm{exc}}}(1+q^{-1})}{(-4)^{\ell-2}}\dfrac{\mathfrak{f}(\Psi_g(\ell))^2\mathfrak{f}_1(f_k, g_\ell, h_m)}{\mathfrak{f}_2(g_\ell)^2\mathfrak{f}_3(g_\ell)^2\mathfrak{f}(k, \Psi_{gh}(k,\ell,m))\mathfrak{f}(k, \Psi_{gh'}(k,\ell,m))}$;

ii) $\mathfrak{f}(\Psi_g(\ell))$ *is the factor appearing in Theorem 1.93;*

iii) $\mathfrak{f}_1(f_k, g_\ell, h_m)$ *is the factor appearing in* (2.4.2)*;*

iv) $\mathfrak{f}_2(g_\ell)$ *is the factor defined by the equality* $L(\Psi_g(\ell), 0) = \mathfrak{f}_2(g_\ell)L_{\mathfrak{c}}(\Psi_g(\ell), 0)$*, where* $\mathfrak{c} := lcm(\mathfrak{c}_g, \mathfrak{c}_h)$ *and* $L_{\mathfrak{c}}(\Psi_g(\ell), s)$ *is the product of all the Euler factors defining* $L(\Psi_g(\ell), s)$ *except the ones corresponding to the primes dividing* $\mathfrak{c}$*;*

v) $\mathfrak{f}_3(g_\ell)$ *is the factor defined by the equality* $\langle g_\ell^*, g_\ell^*\rangle = (\ell-1)!\pi^{-\ell}\mathfrak{f}_3(g_\ell)L(\Psi_g(\ell), 0)$ *of [DLR15, Lemma 3.7], where* $g_\ell^* := g_\ell\otimes\chi_g^{-1}$.

Proof. By Theorem 1.96, for each $(k,\ell,m) \in \mathcal{W}^g_{\mathbf{fgh}}$ we have

$$L(f_k \otimes g_\ell \otimes h_m, c_0) = \mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(k,\ell,m)^2 \frac{(-4)^\ell \langle g_\ell, g_\ell\rangle^2 \mathcal{E}_0(g_\ell)\mathcal{E}_1(g_\ell)}{\mathcal{E}(f_k,g_\ell,h_m)^2} \times \\ \times \frac{1}{\mathfrak{a}(k,\ell,m)} \frac{1}{\prod_{q\in\Sigma_{\mathrm{exc}}}(1+q^{-1})}.$$

Let c be the smallest positive integer in $\mathfrak{c}$, then if $(x,y,z) \in \mathcal{W}^g_{\mathbf{fgh}}$, the characters $\Psi_{gh}(k,\ell,m)$ and $\Psi_{gh'}(k,\ell,m)$ belong to the region of interpolation $\Sigma(c,\mathcal{N}_f,\chi_f)^{(2)}$ defined in §1.15.2. Indeed, $L(f_k,\Psi_{gh}(k,\ell,m)^{-1},0) = L(f_k,\psi_{g,\ell-1}\psi_{h,m-1},c_0)$, and c_0 is central critical for this complex L-function. Moreover, $\Psi_{gh}(k,\ell,m)$ has infinity type

$$\Big(\frac{k+\ell+m-2}{2}, \frac{k-\ell-m+2}{2}\Big) = (k+j,-j) \tag{2.4.3}$$

with $j = \frac{-k+\ell+m-2}{2} \geq 0$. Similarly, $\Psi_{gh'}(k,\ell,m)$ has infinity type

$$\Big(\frac{k+\ell-m}{2}, \frac{k-\ell+m}{2}\Big) = (k+j,-j) \tag{2.4.4}$$

with $j = \frac{\ell-k-m}{2} \geq 0$. Then using (2.4.2) and Proposition 1.94 we obtain

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})^2(x,y,z)\Big(\frac{\Omega}{\Omega_p}\Big)^{4-4\ell}\langle g_\ell,g_\ell\rangle^2 \\ = \mathcal{L}_p(K,\mathbf{f})(k,\Psi_{gh}(k,\ell,m))^2\mathcal{L}_p(K,\mathbf{f})(k,\Psi_{gh'}(k,\ell,m))^2 \times \frac{\prod_{q\in\Sigma_{\mathrm{exc}}}(1+q^{-1})\mathfrak{f}_1(f_k,g_\ell,h_m)}{(-4)^\ell\mathfrak{f}(k,\Psi_{gh}(k,\ell,m))\mathfrak{f}(k,\Psi_{gh'}(k,\ell,m))} \\ \times \frac{\mathfrak{a}(k,\ell,m)}{\mathfrak{a}(\Psi_{gh}(k,\ell,m))\mathfrak{a}(\Psi_{gh'}(k,\ell,m))} \times \frac{\mathcal{E}(f_k,g_\ell,h_m)^2}{\mathcal{E}_0(g_\ell)^2\mathcal{E}_1(g_\ell)^2\mathfrak{e}(k,\Psi_{gh}(k,\ell,m))^2\mathfrak{e}(k,\Psi_{gh'}(k,\ell,m))^2}. \tag{2.4.5}$$

On the other hand, the character $\Psi_g(\ell)$ has infinite type $(\ell, 2-\ell)$ and conductor dividing $\mathfrak{c}$, so for $\ell \geq 2$ it belongs to the set of classical interpolation $\Sigma(\mathfrak{c})^{(2)}$ for Katz's p-adic L-function (see §1.15.1). Substituting [DLR15, (53) and Lemma 3.7] in the interpolation formula of Theorem 1.93, we obtain, for each $\ell \geq 2$,

$$L_p(K)(\Psi_g(\ell)) = \mathfrak{a}(\Psi_g(\ell))\mathfrak{e}(\Psi_g(\ell))\frac{\mathfrak{f}(\Psi_g(\ell))}{\mathfrak{f}_2(\ell)\mathfrak{f}_3(\ell)}\Big(\frac{\Omega_p}{\Omega}\Big)^{2\ell-2}\langle g_\ell,g_\ell\rangle\frac{\pi^\ell}{(\ell-1)!}. \tag{2.4.6}$$

Plugging (2.4.6) into (2.4.5) it follows that:

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(x,y,z)^2\mathcal{L}_p(K)(\Psi_g(\ell))^2 = \mathcal{L}_p(K,\mathbf{f})(k,\Psi_{gh}(k,\ell,m))^2\mathcal{L}_p(K,\mathbf{f})(k,\Psi_{gh'}(k,\ell,m))^2 \\ \times \frac{\mathfrak{f}(k,\ell,m)\pi^{2\ell}\mathfrak{a}(\Psi_g(\ell))^2\mathfrak{a}(f_k,g_\ell,h_m)}{2^4[(\ell-1)!]^2\mathfrak{a}(\Psi_{gh}(k,\ell,m))\mathfrak{a}(\Psi_{gh'}(k,\ell,m))} \times \frac{\mathfrak{e}(\Psi_g(\ell))^2\mathcal{E}(f_k,g_\ell,h_m)^2}{\mathcal{E}_0(g_\ell)^2\mathcal{E}_1(g_\ell)^2\mathfrak{e}(k,\Psi_{gh}(k,\ell,m))^2\mathfrak{e}(k,\Psi_{gh'}(k,\ell,m))^2}.$$

Then the statement of the theorem follows from the identities

1. $\dfrac{\pi^{2\ell}\mathfrak{a}(\Psi_g(\ell))^2\mathfrak{a}(f_k,g_\ell,h_m)}{[(\ell-1)!]^2\mathfrak{a}(\Psi_{gh}(k,\ell,m))\mathfrak{a}(\Psi_{gh'}(k,\ell,m))} = 2^4,$

2. $\mathcal{E}(f_k,g_\ell,h_m) = \mathfrak{e}(k,\Psi_{gh}(k,\ell,m))\mathfrak{e}(k,\Psi_{gh'}(k,\ell,m)),$

3. $\mathfrak{e}(\Psi_g(\ell)) = \mathcal{E}_0(g_\ell)\mathcal{E}_1(g_\ell),$

and by continuity. □

2.4.2 A special case of Conjecture 2.13

We retain the notations introduced in the beginning of §2.4 so in particular $f \in S_k(N_f, \chi_f)$ is a normalised newform of weight $k \geq 2$. Let

$$\psi_g, \psi_h : \mathbb{A}_K^\times \longrightarrow \mathbb{C}^\times$$

be finite order Hecke characters of conductors $\mathfrak{c}_g, \mathfrak{c}_h$ and central characters $\varepsilon_g, \varepsilon_h$, and let g and h be the theta series attached to ψ_g and ψ_h, respectively, as described in §1.5.4. They are weight one modular forms with Nebentype; more precisely, if χ_K dentes the quadratic character attached to K, then

$$g \in M_1(N_g, \chi_g) \quad \text{and} \quad h \in M_1(N_h, \chi_h),$$

where

$$N_g = D_K \cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c}_g), \quad N_h = D_K \cdot \mathrm{N}_{K/\mathbb{Q}}(\mathfrak{c}_h), \quad \chi_g = \chi_K \cdot \varepsilon_g, \quad \chi_h = \chi_K \cdot \varepsilon_h.$$

Assume that

$$\chi_f \varepsilon_g \varepsilon_h = 1,$$

so that the Nebentype characters of f, g and h satisfy (2.2.1). Assume that the prime p (which splits in K) satisfies (2.1.3) and that f, g and h are p-ordinary.

In this section we are interested in Conjecture 2.13 for the modular forms f, g and h we just defined. Recall that the field of coefficients L can be taken as the field generated by the Fourier coefficients of f, g and h. With the notation introduced in §1.15.1 , let ψ_h' be the Hecke character defined as in (1.15.1). Define also the characters

$$\psi_1 := \psi_g \psi_h, \qquad \psi_2 := \psi_g \psi_h'.$$

Condition (2.2.1) on the Nebentype characters implies that

$$\psi_{1|\mathbb{A}_\mathbb{Q}^\times} = \psi_{2|\mathbb{A}_\mathbb{Q}^\times} = \chi_f^{-1}$$

and that the conductor of ψ_i is of the form $c_i \mathcal{N}_i$, where $\mathcal{N}_i \mid \mathcal{N}_{\chi_f}$ and c_i is an integer coprime to $\mathcal{N}_{\chi_f}$. We will assume from now on that:

1. $\mathcal{N}_i = \mathcal{N}_{\chi_f}$ for $i = 1, 2$;
2. ψ_i has finite type $(c_i, \mathcal{N}_f, \chi_f)$ (cf. Definition 1.103).

In this setting, by looking at the Euler factors one checks that there is a decomposition of Artin representations

$$V_{gh}^\circ = V_{\psi_1} \oplus V_{\psi_2}, \tag{2.4.7}$$

where we denote $V_{\psi_i} := \mathrm{Ind}_K^\mathbb{Q}(\psi_i)$ for $i = 1, 2$. This decomposition induces a factorisation of L-functions

$$L(f \otimes g \otimes h, s) = L(f, \psi_1, s) \cdot L(f, \psi_2, s)$$

As explained in [BDP13, pages 1036, 1093], the conditions imposed so far imply that all the finite local signs of $L(f, \psi_i, s)$ are $+1$ and the global sign of $L(f, \psi_i, s)$ is -1 , so

$$L(f, \psi_1, k/2) = L(f, \psi_2, k/2) = 0.$$

Then Assumption 2.1 is satisfied and

$$\mathrm{ord}_{s=k/2} L(f \otimes g \otimes h, s) \geq 2.$$

Thanks to our assumptions on the characters ψ_1 and ψ_2, we can speak of the cycles $\tilde{\Delta}^{\overline{\psi}_i}_{k-2,0,c_i}$ as defined in §1.16.3. Observe that in this particular case in which $r_1 = 0$, these are in fact classical Heegner cycles, as defined in [Nek95]. If we let H_i be the field denoted as H_{c_i,χ_f} in §1.16.3, then we have that

$$\tilde{\Delta}^{\overline{\psi}_i}_{k-2,0,c_i} \in \mathrm{CH}^{k/2}(W_{k-2}/H_i)^{[f]}_{0,L}.$$

Since we want to view the two cycles as being defined over the same field, we set $H := H_1 \cdot H_2$, the composition of H_1 and H_2, and $c := \mathrm{lcm}(c_1, c_2)$. In this way, the cycle $\tilde{\Delta}^{\overline{\psi}_i}_{k-2,0,c_i}$ belongs to $\mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}$. The following is the main result of this section. It is the generalisation to weights $k \geq 2$ of [DLR15, Theorem 3.3]. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be Hida families passing through f_α, g_α and h_α. Recall that we denoted $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h}) = \mathcal{L}_p^g(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ for the test vector $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ of [Hsi19, Chapter 3].

Theorem 2.19. *There exists a quadratic extension L_0 of L and $\lambda \in L_0^\times$ such that*

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, 1, 1) = \lambda \frac{\mathrm{AJ}_p(\tilde{\Delta}^{\overline{\psi}_1}_{k-2,0,c_1})(\omega_f) \cdot \mathrm{AJ}_p(\tilde{\Delta}^{\overline{\psi}_2}_{k-2,0,c_2})(\omega_f)}{\mathfrak{g}(\chi_f) \log_p(u_{g_\alpha})}.$$

Remark 2.20. We stress that the nonzero scalar λ lies in a quadratic extension of the field of coefficients of f, g and h. In this sense, this also represents a slight strengthening of [DLR15, Theorem 3.3], in which one had a less precise control of the degree of such extension.

Theorem 2.19 can be seen as giving evidence towards Conjecture 2.13, as we now explain. The decomposition of representations (2.4.7) induces a decomposition

$$\begin{aligned}&\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, \mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}) \cong \\ &\mathrm{Hom}_{G_\mathbb{Q}}(V_{\psi_1}, \mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}) \oplus \mathrm{Hom}_{G_\mathbb{Q}}(V_{\psi_2}, \mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}),\end{aligned}$$

Consider now the Heegner cycle $\tilde{\Delta}_c := \tilde{\Delta}_{k-2,0,\mathcal{O}_c,c}$ defined in (1.16.11) with $r_0 = k-2, r_1 = 0$ and associated to the trivial ideal $\mathcal{O}_c$. It belongs to $\mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}$, and we consider the projection to the $\overline{\psi}_i$ component

$$\Delta_c^{\overline{\psi}_i} := \sum_{\sigma \in \mathrm{Gal}(H/K)} \psi_i(\sigma)(\tilde{\Delta}_c)^\sigma. \tag{2.4.8}$$

Observe that $\Delta_c^{\overline{\psi}_i}$ gives an element in $\mathrm{Hom}_{G_\mathbb{Q}}(V_{\psi_i}, \mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L})$. Indeed, since ψ_i is anticyclotomic (see Remark 1.92 part 3. for the definition) we have that $V_{\psi_i} = V_{\overline{\psi}_i}$, and by Frobenius reciprocity (see (1.6.2)), giving an element in

$$\mathrm{Hom}_{G_\mathbb{Q}}(V_{\overline{\psi}_i}, \mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L})$$

is equivalent to giving a G_K-homomorphism form L (viewed as a G_K-module via the action of $\overline{\psi}_i$) to $\mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}$; the map that sends 1 to $\Delta_c^{\overline{\psi}_i}$ is one such homomorphism.

As we mentioned before, in our setting the sign of the functional equation of $L(f, \psi_i, s)$ for $i = 1, 2$ is -1 and therefore these L-functions vanish at the central point $s = k/2$. Suppose that the following assumption (which implies Assumption 2.2) holds true.

Assumption 2.21.

$$\mathrm{ord}_{s=k/2} L(f, \psi_i, s) = 1 \quad \text{for } i = 1, 2.$$

This assumption is actually expected to hold for "generic" Hecke characters. Then, by the general philosophy of Heegner points and Heegner cycles it is expected that also the following assumption holds.

Assumption 2.22.
$$\mathrm{Hom}_{G_{\mathbb{Q}}}(V_{\psi_i}, \mathrm{CH}^{k/2}(W_{k-2}/H)^{[f]}_{0,L}) = \langle \Delta_c^{\overline{\psi}_i} \rangle. \tag{2.4.9}$$

That is to say, the above space is generated by the homomorphism given by the Heegner cycle. This has been proven for $k = 2$ by the results of Gross–Zagier [GZ86], Kolyvagin [Kol90], Zhang [Zha01b] and Bertolini–Darmon [BD90]. For $k > 2$, in the particular case where $c = 1$ and $\chi_f = 1$, it follows from results of Zhang [Zha97] and Nekovář [Nek95] on Heegner cycles if one assume Gillet–Soule' conjecture on the non-degeneracy of the height pairing.

Arguing as in [DLR15, Lemma 3.2] we see that, under Assumption 2.22, we have

$$\mathrm{Reg}(f, g_\alpha, h) = \mathrm{AJ}_p(\Delta_c^{\overline{\psi}_1})(\omega_f) \cdot \mathrm{AJ}_p(\Delta_c^{\overline{\psi}_2})(\omega_f).$$

Finally, observe that $\Delta_c^{\overline{\psi}_i}$ and $\tilde{\Delta}_{k-2,0,c_i}^{\overline{\psi}_i}$ are defined differently (see (2.4.8) and Definition 1.104), but we have the following result

Lemma 2.23.

$$\mathrm{AJ}_p(\Delta_c^{\overline{\psi}_i})(\omega_f) = [H : K_c] \cdot \mathrm{AJ}_p(\tilde{\Delta}_{k-2,0,c_i}^{\overline{\psi}_i})(\omega_f) \mod L^\times.$$

Proof. By Shimura's reciprocity law

$$\left(\tilde{\Delta}_{r_0,r_1,\mathcal{O}_c,c}\right)^\sigma = \tilde{\Delta}_{r_0,r_1,\mathfrak{a}^{-1},c} \tag{2.4.10}$$

when σ corresponds to the class of the ideal $\mathfrak{a}$ under the reciprocity map of class field theory. Let S be a complete set of representatives for $\mathrm{Pic}(\mathcal{O}_c) \cong \mathrm{Gal}(K_c/K)$ and write $\mathrm{Gal}(H/K) = \sqcup\, \sigma\, \mathrm{Gal}(H/K_c)$ where $\{\sigma\}$ is then a complete set of representatives for $\mathrm{Gal}(K_c/K)$. Then

$$\begin{aligned}
\mathrm{AJ}_p(\Delta_c^{\overline{\psi}_i}) &= \mathrm{AJ}_p\Big(\sum_{\tau \in \mathrm{Gal}(H/K)} \psi_i(\sigma)(\tilde{\Delta}^\tau_{k-2,0,\mathcal{O}_c,c})\Big) \\
&= \mathrm{AJ}_p\Big(\sum_{\tau \in \mathrm{Gal}(H/K)} \chi_f^{-1}(\tau)(\tilde{\Delta}^\tau_{k-2,0,\mathcal{O}_c,c})\Big) && \psi_i \text{ is of finite type } (c_i, \mathcal{N}_f, \chi_f) \\
&= \sum_\sigma \sum_{\tau \in \mathrm{Gal}(H/K_c)} \chi_f^{-1}(\sigma\tau)\, \mathrm{AJ}_p((\tilde{\Delta}^{\sigma\tau}_{k-2,0,\mathcal{O}_c,c})) \\
&= \sum_\sigma \chi_f^{-1}(\sigma) \sum_{\tau \in \mathrm{Gal}(H/K_c)} \chi_f^{-1}(\tau)\, \mathrm{AJ}_p((\tilde{\Delta}^{\sigma\tau}_{k-2,0,\mathcal{O}_c,c})) \\
&= [H : K_c] \sum_\sigma \chi_f^{-1}(\sigma)\, \mathrm{AJ}_p((\tilde{\Delta}^\sigma_{k-2,0,\mathcal{O}_c,c})) && \text{[BDP17, Prop 4.2.1]} \\
&= [H : K_c] \sum_{\mathfrak{a} \in S} \chi_f^{-1}(\sigma)\, \mathrm{AJ}_p((\tilde{\Delta}_{k-2,0,\mathfrak{a},c})) && (2.4.10)
\end{aligned}$$

□

Using Lemma 2.23 we see that

$$\mathrm{Reg}(f, g_\alpha, h) = \mathrm{AJ}_p(\Delta_c^{\overline{\psi}_1})(\omega_f) \cdot \mathrm{AJ}_p(\Delta_c^{\overline{\psi}_2})(\omega_f) \mod L^\times,$$

and therefore Theorem 2.19 proves Conjecture 2.13 in this case (up to the fact that in Theorem 2.19 λ lies in a quadratic extension of L rather than in L itself).

2.4.2.1 Proof of Theorem 2.19

As we will see, the proof of Theorem 2.19 follows from evaluating the formula of Theorem 2.18 at weights $(k, 1, 1)$. Since it will be needed, we record the following result on the field of definition of the factor $\mathfrak{f}(k, 1, 1)$.

Proposition 2.24. *If k is even, then $\mathfrak{f}(k, 1, 1)\mathfrak{g}(\chi_f)^2$ belongs to L.*

Proof. It follows readily from the definitions that the several factors that enter into the definition of $\mathfrak{f}(k, 1, 1)$ belong to L, except the factors $\mathfrak{f}(k, \Psi_{gh}(k, 1, 1))$ and $\mathfrak{f}(k, \Psi_{gh'}(k, 1, 1))$. Indeed, these factors are defined in terms of certain scalars $\omega(f_k, \Psi_{gh}(k, 1, 1))$ and $\omega(f_k, \Psi_{gh'}(k, 1, 1))$. By [BDP13, (5.1.11)], we have that

$$\omega(f_k, \Psi_{gh}(k, 1, 1))\omega(f_k, \Psi_{gh'}(k, 1, 1)) = \frac{w_f^2(-N_f)^{\ell-1}\,\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b})^{\ell-1}}{\chi_f(\mathrm{N}_{K/\mathbb{Q}}(\mathfrak{b}))^2\psi_g(\mathfrak{b})\psi'_h(\mathfrak{b})b^{2\ell+2}}.$$

Here $\mathfrak{b}$ is a choice of an ideal of $\mathcal{O}_c$ prime to pN_fc such that $\mathfrak{b} \cdot \mathcal{N}_f = (b)$ and w_f is the scalar such that $W_{N_f} f_k^* = w_f \cdot f_k$ (here W_{N_f} is the Atkin–Lehner involution). The statement then follows from [AL78, Theorem 2.1], which implies, when k is even, that $w_f \cdot \mathfrak{g}(\chi_f)$ belongs to L. □

Proof of Theorem 2.19. Evaluating the factorisation formula of Theorem 2.18 at $(k, 1, 1)$ and taking square roots we obtain:

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)\mathcal{L}_p(K)(\Psi_g(1)) = \mathcal{L}_p(K, \mathbf{f})(k, \Psi_{gh}(k, 1, 1))\mathcal{L}_p(K, \mathbf{f})(k, \Psi_{gh'}(k, 1, 1))\mathfrak{f}'(k, 1, 1),$$

where $\mathfrak{f}'(k, 1, 1) := \sqrt{\mathfrak{f}(k, 1, 1)}$. Then the statement of Theorem 2.19 follows applying Theorem 1.98 and Theorem 1.105, and Proposition 2.24, after observing that:

1. the fact that $\chi_f\chi_g\chi_h = 1$ implies that k is even;
2. $\Psi_g(1) = \psi\,\mathrm{N}_K$, where $\psi := \psi'_g/\psi_g$ has finite order and it is selfdual, so that, by (1.15.2), $\mathcal{L}_p(K)(\Psi_g(1)) = \mathcal{L}_p(K)(\psi)$.
3. $\Psi_{gh}(k, 1, 1) = (\psi_g\psi_h)^{-1}\,\mathrm{N}_K^{k/2} = \psi_1^{-1}\,\mathrm{N}_K^{\frac{k-r}{2}}$ and $\psi_1^{-1} = (\psi_g\psi_h)^{-1}$ has infinity type $(r - j, j)$ with $r := 0$, $j := 0$, and analogously for $\Psi_{gh'}(k, 1, 1)$.

□

2.4.3 A special case of Conjecture 2.17

In this section we consider the case in which g and h are theta series of two Hecke characters ψ_g, ψ_h of K. We will use the same notations and assume the same hypotheses of §2.4.2, with only two differences. The first one is that in §2.4.2 the characters ψ_g, ψ_h were assumed to be of infinity type $(0, 0)$, whereas we now suppose that they are of infinity type $(0, \ell - 1), (0, m - 1)$ for some $\ell, m \geq 2$. The second difference is that now we will define the characters ψ_1 and ψ_2 to be

$$\psi_1 := \psi_g\psi_h\,\mathrm{N}_K^{2-\ell-m},\ \psi_2 := \psi_g\psi'_h\,\mathrm{N}_K^{2-\ell-m}.$$

As in §2.4.2, we assume that for $i = 1, 2$ the conductor of ψ_i is of the form $c_i\mathcal{N}_i$ with $c_i \in \mathbb{Z}$ coprime to $D_K N_f$ and $\mathcal{N}_i \mid \mathcal{N}_{\chi_f}$. Notice that, in particular, the factorisations (2.4.1) and (2.4.2) and Theorem 2.18 still hold in this setting. Using the notation introduced in §1.16.3, we consider the generalised Heegner cycles

$$\tilde{\Delta}^{\psi_i^{-1}} := \tilde{\Delta}^{\psi_i^{-1}}_{k-2,\ell+m-2,c_i} \in \mathrm{CH}^c(W_{k-2} \times E^{\ell+m-2}/H_{c_i,f})_{0,L},$$

where $c := \frac{k+\ell+m-2}{2}$ and E is an elliptic curve defined over the Hilbert class field K_1 of K with complex multiplication by $\mathcal{O}_K$ that we fix once and for all. Recall that we denoted

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h}) := \mathcal{L}_p^g(\mathbf{f}^\star,\mathbf{g}^\star,\mathbf{h}^\star),$$

where $(\mathbf{f}^\star,\mathbf{g}^\star,\mathbf{h}^\star)$ is the triple of Λ-adic test vector of [Hsi19, Chapter 3].

Proposition 2.25. *There exist a quadratic extension L_0/L and $\lambda \in L_0$ such that*

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(k,\ell,m) = \frac{\lambda}{\mu}\cdot \mathrm{AJ}_p(\tilde{\Delta}^{\psi_1^{-1}})(\omega_f\wedge\eta_E^{\ell+m-2})\cdot \mathrm{AJ}_p(\tilde{\Delta}^{\psi_2^{-1}})(\omega_f\wedge\eta_E^{\ell-1}\omega_E^{m-1}), \quad (2.4.11)$$

where $\mu := \Omega^{2-2\ell}\pi^{\ell-2}L(\Psi_g(\ell)^{-1},0)\in\overline{\mathbb{Q}}$ and $\eta_E^i\omega_E^j$ are the differentials defined in (1.10.7).

Proof. Let $r := \ell+m-2$. Applying Theorem 1.105 to the characters

1. $\Psi_{gh}(k,\ell,m) = \psi_1^{-1}\,\mathrm{N}_K^{\frac{k-r}{2}}$ where ψ_1^{-1} has infinity type $(r-j,j)$ with $j=0$;
2. $\Psi_{gh'}(k,\ell,m) = \psi_2^{-1}\,\mathrm{N}_K^{\frac{k-r}{2}}$ where ψ_2^{-1} has infinity type $(r-j,j)$ with $j=m-1$

and substituting into equation (2.4.5), we obtain

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})^2(k,\ell,m)\times\Omega^{4-4\ell}\langle g,g\rangle^2\frac{\mathfrak{a}(k,\Psi_{gh}(k,\ell,m))\mathfrak{a}(k,\Psi_{gh}(k,\ell,m))}{\mathfrak{a}(k,\ell,m)} =$$

$$= \frac{(-1)^\ell\prod_{q\in\Sigma_{\mathrm{exc}}}(1+q^{-1})}{m!c_2^{2m-2}4^{k-\ell-m}(d_{c_1}d_{c_2})^{\frac{k-\ell-m}{2}}}\frac{\mathfrak{f}_1(f,g,h)}{\mathfrak{f}(k,\Psi_{gh}(k,\ell,m))\mathfrak{f}(k,\Psi_{gh'}(k,\ell,m))}\frac{\mathcal{E}(f,g,h)^2}{\mathcal{E}_0(g)^2\mathcal{E}_1(g)^2}$$
$$\times\,\mathrm{AJ}_p(\tilde{\Delta}^{\psi_1^{-1}})^2(\omega_f\wedge\eta_E^{\ell+m-2})\cdot\mathrm{AJ}_p(\tilde{\Delta}^{\psi_2^{-1}})^2(\omega_f\wedge\eta_E^{\ell-1}\omega_E^{m-1}).$$

Using the definition of the factors involved, the left hand side of the previous equality is

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})^2(k,\ell,m)(\Omega^{2-2\ell}\langle g,g\rangle\pi^{2\ell-2})^2.$$

As $\ell\geq 2$, the character $\Psi_g(\ell)$ belongs to the region of classical interpolation for Katz's p-adic L-function, and following the computations of the proof of [DLR15, Lemma 3.7], we obtain

$$\langle g,g\rangle = \frac{L(\Psi_g(\ell)^{-1},0)}{\pi^\ell\sqrt{D_K}} \mod \mathbb{Q}^\times.$$

Then we see that

$$\mu = (\Omega^{2-2\ell}\langle g,g\rangle\pi^{2\ell-2})^{-1} = (\Omega^{2-2\ell}\pi^{\ell-2}L(\Psi_g(\ell)^{-1},0))^{-1},$$

is algebraic by [BDP12, Proposition 2.11 (1) and Theorem 2.12], and the factors $\mathfrak{f}_1(k,\ell,m)$ and $\frac{\mathcal{E}(f,g,h)}{\mathcal{E}_0(g)\mathcal{E}_1(g)}$ belong to L by the definition of these factors. □

2.4.3.1 Statement and proof of the special case

The rest of the section will be devoted to analysing the connection between Proposition 2.25 and Conjecture 2.17 in a particular case where ψ_g and ψ_h are powers of the Hecke character of an elliptic curve with complex multiplication by $\mathcal{O}_K$. More precisely, in this subsection we continue to denote by f a normalised newform of weight $k\geq 2$, level N_f and Nebentype character χ_f, and we make the following additional assumptions regarding K,ψ_g and ψ_h:

1. K is an imaginary quadratic field of class number 1.

2. We fix an elliptic curve $E_0/\mathbb{Q}$ with complex multiplication by $\mathcal{O}_K$. We denote $E := E_0 \otimes K$ its extension of scalars to K and by ψ_E the Hecke character attached to E as in §1.10.5. Then we assume that $\psi_g = \psi_E^{\ell-1}$ and $\psi_h = \psi_E^{m-1}$, with $\ell > m \geq 2$ and $k \geq \ell + m$.

As usual, we denote

$$g := \theta(\psi_g) \in S_\ell(N_g, \chi_g) \text{ and } h := \theta(\psi_h) \in S_m(N_h, \chi_h).$$

We simplify further the setting making the following assumption on the discriminant of K.

Assumption 2.26. The discriminant $-D_K$ of K satisfies one of the following conditions:

1. D_K is odd;
2. $8 \mid D_K$;
3. there exists a prime $\ell \mid D_K$ such that $\ell \equiv 3 \mod 4$.

Under this assumption, the elliptic curve $E_0/\mathbb{Q}$ can be constructed as in [Gro80, §11], so that the conductor of ψ_E is generated by $\sqrt{-D_K}$, a condition that we will assume from now on. From the conditions imposed in this section, and using the fact that $\theta(\psi_E)$ is the cuspform attached to the elliptic curve E that descends to $\mathbb{Q}$, it follows that

$$N_g = N_h = D_K^2 \tag{2.4.12}$$

and

$$\chi_g = \chi_K^\ell,\ \chi_h = \chi_K^m,\ \chi_f = \chi_K^{\ell+m} = \begin{cases} 1 & \text{if } \ell + m \text{ is even} \\ \chi_K & \text{if } \ell + m \text{ is odd.} \end{cases} \tag{2.4.13}$$

In this setting, the involved Hecke characters are

$$\psi_1 = \psi_g \psi_h \, \mathrm{N}_K^{2-\ell-m} = \psi_E^{\ell+m-2} \, \mathrm{N}_K^{2-\ell-m};$$
$$\psi_2 = \psi_g \psi_h' \, \mathrm{N}_K^{2-\ell-m} = \psi_E^{\ell-1} \overline{\psi}_E^{m-1} \, \mathrm{N}_K^{2-\ell-m} = \psi_E^{\ell-m} \, \mathrm{N}_K^{1-\ell},$$

where we have used that $\psi_E' = \overline{\psi}_E$ and $\psi_E \cdot \overline{\psi}_E = \mathrm{N}_K$.

Let us assume that, as in §2.3, that we are in a rank 2 setting. That is to say,

$$\dim_L \mathrm{CH}^c(M(f \otimes g \otimes h))_{0,L} = 2,$$

say with basis $\{\Delta_1, \Delta_2\}$. The main result of this section is Theorem 2.34 below. It states that assuming Tate's conjecture for motives (cf. Conjecture 1.71) and a natural property of the p-adic Abel–Jacobi map (cf. Assumption 2.29), if $\mathrm{ord}_{s=c} L(f \otimes g \otimes h, s) = 2$ and $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m) \neq 0$ then the regulator $\mathrm{Reg}(f, g, h)$ of Definition 2.15 is a nonzero algebraic multiple of $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)$. It can thus be viewed as the (conditional) proof of a particular case of Conjecture 2.17.

The strategy of the proof is roughly as follows. Under Tate's Conjecture, the motive $M(f \otimes g \otimes h)$ decomposes as a sum of motives whose underlying varieties are $W_{k-2} \times E^{\ell+m-2}$ and $W_{k-2} \times E^{\ell-m}$. Using this decomposition and Assumption 2.29 we are able to write the regulator $\mathrm{Reg}(f, g, h)$ in terms of cycles in these varieties (Proposition 2.30). Then in Proposition 2.33 we relate the p-adic Abel–Jacobi image of these cycles to that of the generalised Heegner cycles

$$\tilde{\Delta}^{\psi_i^{-1}} := \tilde{\Delta}_{k-2,\ell+m-2,c_i}^{\psi_i^{-1}} \in \mathrm{CH}^c(W_{k-2} \times E^{\ell+m-2}/H_{c_i,f})_0,$$

and then Proposition 2.25 provides the relation with the special value of the p-adic L-function.

Recall that the motive over $\mathbb{Q}$ associated to f is $M_f = (W_{k-2}, e_f, 0)$, and let $M_{f/K}$ be its base change to K. As explained in §1.10.6, since K has class number 1, we have the following motives in $\mathcal{M}(K)_{\mathbb{Q}}$

$$M(\psi_E^{\ell+m-2}) = (E^{\ell+m-2}, e_1, 0), \quad M(\psi_E^{\ell-m}, e_2, 0)$$

where we denoted $e_1 := e_{E^{\ell+m-2}}, e_2 := e_{E^{\ell-m}}$ to simplify the notation. Define the Hecke characters

$$\tilde{\psi}_1 := \psi_g \psi_h = \psi_E^{\ell+m-2}, \quad \tilde{\psi}_2 := \psi_g \psi_h' = \psi_E^{\ell-m} \mathrm{N}_K^{m-1}.$$

For $i = 1, 2$, denote by M_i the motive associated with $\tilde{\psi}_i$. Observe that (cf. [Sch88, p. 98])

$$M_1 = M(\psi_E^{\ell+m-2}), \qquad M_2(\psi_E^{\ell-m})(1-m) \quad \text{(the Tate twist)}.$$

Proposition 2.27. *Assuming Conjecture 1.71, there are natural isomorphisms*

$$\beta_{\mathrm{CH}} : \mathrm{CH}^c(M(f \otimes g \otimes h))_0 \cong \mathrm{CH}^c(M_{f/K} \otimes M(\psi_E^{\ell+m-2}))_0 \oplus \mathrm{CH}^c(M_{f/K} \otimes M(\psi_E^{\ell-m}))_0$$

and

$$\beta_{\mathrm{dR}} : (M_f)_{\mathrm{dR}} \otimes M(g \otimes h)_{\mathrm{dR}} \cong (M_f)_{\mathrm{dR}} \otimes \left[M(\psi_E^{\ell+m-2})_{\mathrm{dR}} \oplus (M(\psi_E^{\ell-m})(1-m))_{\mathrm{dR}} \right].$$

Proof. By the decomposition (2.4.1) and by Artin formalism (see §1.12) we have that

$$\begin{aligned} L(f \otimes g \otimes h, s) &= L(V_f \otimes (V_{\tilde{\psi}_1} \oplus V_{\tilde{\psi}_2}), s) = L(V_f \otimes V_{\tilde{\psi}_1}, s) \cdot L(V_f \otimes V_{\tilde{\psi}_2}, s) \\ &= L(f/K \otimes \tilde{\psi}_1, s) \cdot L(f/K \otimes \tilde{\psi}_2, s) = L(M_{f/K} \otimes M(\tilde{\psi}_1), s) \cdot L(M_{f/K} \otimes M(\tilde{\psi}_2), s) \\ &= L(M_f \otimes \mathrm{Res}_{K/\mathbb{Q}}(M_1), s) \cdot L(M_f \otimes \mathrm{Res}_{K/\mathbb{Q}}(M_1), s). \end{aligned}$$

Tate's conjecture implies then the existence of an isomorphism of motives

$$M(f \otimes g \otimes h, s) \cong M_f \otimes \left(\mathrm{Res}_{K/\mathbb{Q}}(M_1) \oplus \mathrm{Res}_{K/\mathbb{Q}}(M_2) \right),$$

which induces isomorphisms at the level of Chow groups and de Rham realisations:

$$\begin{aligned} \mathrm{CH}^c(M(f \otimes g \otimes h))_0 &\cong \mathrm{CH}^c\left(M_f \otimes \left(\mathrm{Res}_{K/\mathbb{Q}}(M_1) \oplus \mathrm{Res}_{K/\mathbb{Q}}(M_2) \right) \right)_0; \\ M(f \otimes g \otimes h, s)_{\mathrm{dR}} &\cong M_f \otimes \left(\mathrm{Res}_{K/\mathbb{Q}}(M_1) \oplus \mathrm{Res}_{K/\mathbb{Q}}(M_2) \right)_{\mathrm{dR}}. \end{aligned}$$

By Lemma 1.74 and using the fact that the cycle class map of §1.11 commutes with the restriction of scalars (see [Jan90, p. 75]), we see that there is a natural isomorphism

$$\mathrm{CH}^c(M(f \otimes g \otimes h))_0 \cong \mathrm{CH}^c(M_{f/K} \otimes M_1)_0 \oplus \mathrm{CH}^c(M_{f/K} \otimes M_2)_0.$$

Observe also that there is a canonical isomorphism

$$\mathrm{CH}^c(M_{f/K} \otimes M_2) \cong \mathrm{CH}^{c-m+1}(M_{f/K} \otimes M(\psi_E^{\ell-m})).$$

Indeed, this follows from the very definition of the Chow group of a motive and the fact that

$$M_2(1-m) = M_2 \otimes \mathbb{L}^{m-1}.$$

Therefore we obtain the canonical isomorphism β_{CH}. Also, the isomorphism (1.10.4) gives the natural isomorphism β_{dR}.

□

Recall that we are in a situation of algebraic rank 2.

Assumption 2.28. We will furher assume that we are in a *rank* $(1,1)$*-setting*, meaning that the rank of $\mathrm{CH}^c(M_{f/K} \otimes M(\psi_E^{\ell+m-2}))_0$ and $\mathrm{CH}^{c-m+1}(M_{f/K} \otimes M(\psi_E^{\ell-m}))_0$ is one.

This hypothesis is not too restrictive for the aim of this section. Indeed, the proof of Theorem 2.34 shows that it is satisfied whenever $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(k,\ell,m) \neq 0$, and we will prove the main result under this nonvanishing hypothesis.

Since in the definition of $\mathrm{Reg}(f,g,h)$ we are free to choose the basis of $\mathrm{CH}^c(M(f\otimes g\otimes h))_0$ we can, and do, assume that Δ_1, Δ_2 are chosen to be adapted to the decomposition of Chow groups given by the isomorphism β_{CH}. That is to say,

$$\beta_{\mathrm{CH}}(\Delta_1) = (\Delta_1^1, 0), \quad \beta_{\mathrm{CH}}(\Delta_2) = (0, \Delta_2^2) \tag{2.4.14}$$

for some cycles

$$\Delta_1^1 \in \mathrm{CH}^c(M_{f/K}\otimes M(\psi_E^{\ell+m-2}))_0, \quad \Delta_2^2 \in \mathrm{CH}^{c-m+1}(M_{f/K}\otimes M(\psi_E^{\ell-m}))_0.$$

In view of the naturalness of the isomorphism of Proposition 2.27, it is also natural to assume that they behave well with respect to the Abel–Jacobi map.

Assumption 2.29. For any cycle Δ and de Rham class ω, we have that

$$\mathrm{AJ}_p(\Delta)(\omega) = \mathrm{AJ}_p(\beta_{\mathrm{CH}}(\Delta))(\beta_{\mathrm{dR}}(\omega)).$$

Proposition 2.30. *Under the assumptions of this section, the matrix defining* $\mathrm{Reg}(f,g,h)$ *can be chosen to be diagonal. More precisely,*

$$\mathrm{Reg}(f,g,h) = \mathrm{AJ}_p(\Delta_1^1)(\eta_E^{\ell+m-2})\cdot \mathrm{AJ}_p(\Delta_2^2)(\eta_E^{\ell-m}) \mod K^\times.$$

In order to prove this proposition, we need a lemma on the behaviour of the de Rham classes via the isomorphism β_{dR}.

Lemma 2.31. *If we regard the target of* β_{dR} *as the direct sum*

$$\Big((M_f)_{\mathrm{dR}}\otimes M(\psi_E^{\ell+m-2})_{\mathrm{dR}}\Big) \oplus \Big((M_f)_{\mathrm{dR}}\otimes M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}}\Big),$$

then we have

$$\begin{aligned}\beta_{\mathrm{dR}}(\omega_f\wedge\eta_g\wedge\eta_h) &= (\omega_f\wedge\eta_E^{\ell+m-2}, 0) \mod K^\times,\\ \beta_{\mathrm{dR}}(\omega_f\wedge\eta_g\wedge\omega_h) &= (0, \omega_f\wedge\eta_E^{\ell-m}) \mod K^\times.\end{aligned}$$

Proof. The isomorphism β_{dR} is induced by a isomorphism

$$M(g\otimes h)_{\mathrm{dR}} \cong M(\psi_E^{\ell+m-2})_{\mathrm{dR}} \oplus M(\psi_E^{\ell-m}(1-m))_{\mathrm{dR}},$$

that respects the Hodge filtration. Recall the filtration described in §1.8. In particular by (1.8.2) we have

$$\begin{aligned}\mathrm{Fil}^0 M(g\otimes h)_{\mathrm{dR}}/\mathrm{Fil}^{m-1} M(g\otimes h)_{\mathrm{dR}} &= \langle \eta_g\wedge\eta_h\rangle;\\ \mathrm{Fil}^{m-1} M(g\otimes h)_{\mathrm{dR}}/\mathrm{Fil}^{\ell-1} M(g\otimes h)_{\mathrm{dR}} &= \langle \eta_g\wedge\omega_h\rangle.\end{aligned}$$

On the other hand, the Hodge filtration of $M(\psi_E)_{\mathrm{dR}}$ is:

$$\begin{aligned}\mathrm{Fil}^0(M(\psi_E)_{\mathrm{dR}}) &= K\cdot\omega_E + K\cdot\eta_E;\\ \mathrm{Fil}^1(M(\psi_E)_{\mathrm{dR}}) &= K\cdot\omega_E;\\ \mathrm{Fil}^j(M(\psi_E)_{\mathrm{dR}}) &= 0 \text{ for } j\geq 2.\end{aligned}$$

More in general, if $r \in \mathbb{Z}_{\geq 0}$, the Hodge filtration of $M(\psi_E^r)_{\mathrm{dR}}$ is:

$$\begin{aligned}\mathrm{Fil}^0(M(\psi_E^r)_{\mathrm{dR}}) &= K \cdot \omega_E^r + K \cdot \eta_E^r;\\ \mathrm{Fil}^j(M(\psi_E^r)_{\mathrm{dR}}) &= K \cdot \omega_E^r \quad \text{for } j = 1, \ldots, r;\\ \mathrm{Fil}^j(M(\psi_E^r)_{\mathrm{dR}}) &= 0 \quad \text{for } j > r.\end{aligned}$$

So we have

$$\begin{aligned}\mathrm{Fil}^0(M(\psi_E^{\ell+m-1})_{\mathrm{dR}}) / \mathrm{Fil}^{m-1}(M(\psi_E^{\ell+m-1}))_{\mathrm{dR}} &= \langle \eta_E^{\ell+m-2} \rangle;\\ \mathrm{Fil}^{m-1}(M(\psi_E^{\ell+m-1})_{\mathrm{dR}}) / \mathrm{Fil}^{\ell-1}(M(\psi_E^{\ell+m-1}))_{\mathrm{dR}} &= 0.\end{aligned}$$

As for $M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}}$, recall that it is isomorphic to $M(\psi_E^{\ell-m})_{\mathrm{dR}}$ with the Hodge filtration shifted $(m-1)$-positions. That is:

$$\begin{aligned}\mathrm{Fil}^0 M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^{m-1} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} &= \langle \omega_E^{\ell-m}, \eta_E^{\ell-m} \rangle;\\ \mathrm{Fil}^{m} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} = \cdots = \mathrm{Fil}^{\ell-1} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} &= \langle \omega_E^{\ell-m} \rangle;\\ \mathrm{Fil}^{\ell} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} &= 0.\end{aligned}$$

Therefore

$$\begin{aligned}\mathrm{Fil}^0 M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} / \mathrm{Fil}^{m-1} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} &= 0;\\ \mathrm{Fil}^{m-1} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} / \mathrm{Fil}^{\ell-1} M(\psi_E^{\ell-m})(1-m)_{\mathrm{dR}} &= \langle \eta_E^{\ell-m} \rangle.\end{aligned}$$

□

Proof of Proposition 2.30. By Assumption 2.29, the regulator of f, g and h can be computed as

$$\mathrm{Reg}(f,g,h) = \det \begin{pmatrix} \mathrm{AJ}_p(\beta_{\mathrm{CH}}(\Delta_1))(\beta_{\mathrm{dR}}(\omega_f \wedge \eta_g \wedge \omega_h)) & \mathrm{AJ}_p(\beta_{\mathrm{CH}}(\Delta_1))(\beta_{\mathrm{dR}}(\omega_f \wedge \eta_g \wedge \eta_h)) \\ \mathrm{AJ}_p(\beta_{\mathrm{CH}}(\Delta_2))(\beta_{\mathrm{dR}}(\omega_f \wedge \eta_g \wedge \omega_h)) & \mathrm{AJ}_p(\beta_{\mathrm{CH}}(\Delta_2))(\beta_{\mathrm{dR}}(\omega_f \wedge \eta_g \wedge \eta_h)) \end{pmatrix}.$$

By choosing a basis of the Chow group satisfying (2.4.14), we find that

$$\begin{aligned}\mathrm{Reg}(f,g,h) &= \det \begin{pmatrix} \mathrm{AJ}_p(\Delta_1^1, 0)(\eta_E^{\ell+m-2}, 0) & \mathrm{AJ}_p(\Delta_1^1, 0)(0, \eta_E^{\ell-m}) \\ \mathrm{AJ}_p(0, \Delta_2^2)(\eta_E^{\ell+m-2}, 0) & \mathrm{AJ}_p(0, \Delta_2^2)(0, \eta_E^{\ell-m}) \end{pmatrix} \mod K^\times\\ &= \mathrm{AJ}_p(\Delta_1^1)(\eta_E^{\ell+m-2}) \cdot \mathrm{AJ}_p(\Delta_2^2)(\eta_E^{\ell-m}) \mod K^\times.\end{aligned}$$

□

In order to compare the regulator expressed as in Proposition 2.30 with the right hand side of the formula in Proposition 2.25, we focus on the generalised Heegner cycles appearing in this setting.

Lemma 2.32. *In the setting of this subsection we have that $H_{c_i,f} = K$. That is to say, the Heegner cycles $\tilde{\Delta}^{\psi_i^{-1}}$ are defined over K.*

Proof. Recall that

$$\psi_1 = \psi_E^{\ell+m-2} \mathrm{N}_K^{2-\ell-m}, \quad \psi_2 = \psi_E^{\ell-m} \mathrm{N}_K^{1-\ell}.$$

For $i \in \{1,2\}$, the conductor of ψ_i is of the form $c_i \cdot \mathcal{N}_{\chi_f}$, where the norm of $\mathcal{N}_{\chi_f}$ equals the conductor N_{χ_f} of χ_f and $(c_i, N_{\chi_f}) = 1$.

By (2.4.12) and (2.4.13), the conductor of χ_f is $N_{\chi_f} = (D_K)^\varepsilon$ where $\varepsilon \in \{0,1\}$. On the other hand, the conductor of ψ_E is only divisible by the primes above D_K, so $c_i = 1$. Recall the extension F/K defined in [BDP17, §4.2] such that $\mathrm{Gal}(F/K) \cong (\mathbb{Z}/N_{\chi_f}\mathbb{Z})^\times/\{\pm 1\}$. The field $H_{c_1,f} = H_{c_2,f} = H_{1,f}$ is the subextension of F/K corresponding to $\ker(\chi_f)(\{\pm 1\}) \subseteq (\mathbb{Z}/N_f\mathbb{Z})^\times/\{\pm 1\}$, which is K by (2.4.13). □

Denote
$$\tilde{\Delta}^{\psi_2^{-1}\,\mathrm{N}_K^{1-m}} := \tilde{\Delta}_{k-1,\ell-m,1}^{\psi_2^{-1}\,\mathrm{N}_K^{1-m}} \in \mathrm{CH}^{c-m+1}(W_{k-1}\times E^{\ell-m}/K)_0.$$

Proposition 2.33.

$$\mathrm{AJ}_p(\tilde{\Delta}^{\psi_2^{-1}})(\omega_f\wedge\omega_E^{m-1}\eta_E^{\ell-1}) = (2\sqrt{-D_K})^{1-m}\,\mathrm{AJ}_p(\tilde{\Delta}^{\psi_2^{-1}\,\mathrm{N}_K^{1-m}})(\omega_f\wedge\eta_E^{\ell-m}).$$

Proof. We refer to §1.9.1 for the notation and the properties of algebraic correspondences. By [BDP17, Proposition 4.1.1], there is a correspondence

$$Z : W_{k-2}\times E^{\ell+m-2} \rightsquigarrow W_{k-2}\times E^{\ell-m}$$

induced by the cycle

$$Z := W_{k-2}\times E^{\ell-m}\times E^{m-1} \in \mathrm{CH}^{k+\ell-2}(W_{k-2}\times E^{\ell+m-2}\times W_{k-2}\times E^{\ell-m})$$

embedded into

$$W_{k-2}\times E^{\ell+m-2}\times W_{k-2}\times E^{\ell-m} = W_{k-2}\times E^{\ell-m}\times (E\times E)^{m-1}\times W_{k-2}\times E^{\ell-m}$$

via
$$\mathrm{Id}\times(\sqrt{-D_K}\times\mathrm{Id})^{m-1}\times\mathrm{Id}.$$

It induces a homomorphism of Chow groups

$$Z_* : \mathrm{CH}^{\frac{k+\ell+m-2}{2}}(W_{k-2}\times E^{\ell+m-2}) \longrightarrow \mathrm{CH}^{\frac{k+\ell-m}{2}}(W_{k-2}\times E^{\ell-m}).$$

For each $\mathfrak{a}$ ideal of $\mathcal{O}_K$ prime to $\mathcal{N}$, let

$$\Delta_{k-2,\mathfrak{a}} \in \mathrm{CH}^{k-1}(W_{k-2}\times E^{k-2}),$$

be the generalised Heegner cycle defined in [BDP13]. As we recalled in §1.16.3, for each $b\le k-2$ such that $b\equiv k \mod 2$ there is a cycle

$$\Delta_{k-1,b,\mathfrak{a}} \in \mathrm{CH}^{\frac{k+b}{2}}(W_{k-2}\times E^{b}),$$

as defined in [BDP17].

The correspondence above gives the relations between these cycles:

$$Z_*(\Delta_{k-1,\ell+m-2,\mathfrak{a}}) = (\mathrm{N}_K\,\mathfrak{a})^{m-1}\Delta_{k-1,\ell-m,\mathfrak{a}}.$$

Using this relation, as in [BDP17, Proposition 4.1.2], we obtain

$$(2\sqrt{-D_K})^{m-1}\,\mathrm{AJ}_p(\Delta_{k-1,\ell+m-2,\mathfrak{a}})(\omega_f\wedge\omega_E^{m-1}\eta_E^{\ell-1}) = (\mathrm{N}_K\,\mathfrak{a})^{m-1}\,\mathrm{AJ}_p(\Delta_{k-1,\ell-m,\mathfrak{a}})(\omega_f\wedge\eta_E^{\ell-m}).$$

Then, finally,

$$\mathrm{AJ}_p(\tilde{\Delta}^{\psi_2^{-1}})(\omega_f\wedge\omega_E^{m-1}\eta_E^{\ell-1}) = (2\sqrt{-D_K})^{1-m}\,\mathrm{AJ}_p(\tilde{\Delta}^{\psi_2^{-1}\,\mathrm{N}_K^{1-m}})(\omega_f\wedge\eta_E^{\ell-m}).$$

□

Finally, we state and prove the main result of this section. Recall that f, g and h are modular forms of weights k,ℓ and m respectively with $\ell > m \ge 2$ and $k \ge \ell+m$. In addition, g and h are theta series of an imaginary quadratic field K with class number 1 that satisfies Assumption and in which p splits. More precisely, $g=\theta(\psi_E^{\ell-1})$ and $h=\theta(\psi_E^{m-1})$, where E/K is an elliptic curve of conductor $\sqrt{-D_K}$ which has complex multiplication by $\mathcal{O}_K$.

Theorem 2.34. *Assume Conjecture 1.71 and Assumption 2.29. If*

$$\dim_L \mathrm{CH}^c(M(f \otimes g \otimes h))_{0,L} = 2, \quad \textit{and} \quad \mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m) \neq 0,$$

then there exists a quadratic extension L_0 of L and a $\lambda \in L_0$ such that

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m) = \mathrm{Reg}(f, g, h) \mod (K \cdot L_0)^\times.$$

Proof. Assume that $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m) \neq 0$. Combining Proposition 2.25 and Proposition 2.33 with the fact that the kernel of the p-adic Abel–Jacobi map contains all torsion cycles, we obtain that the generalised Heegner cycles $\tilde{\Delta}^{\psi_1^{-1}}$ and $\tilde{\Delta}^{\psi_2^{-1}\mathrm{N}_K^{1-m}}$ are nontorsion. Since we are in a situation of algebraic rank 2, this implies that the preimages via β_{CH} of

$$(\tilde{\Delta}^{\psi_1^{-1}}, 0) \quad (0, \tilde{\Delta}^{\psi_2^{-1}\mathrm{N}_K^{1-m}})$$

generate $\mathrm{CH}^c(M(f \otimes g \otimes h))_0$. In other words, we can choose Δ_1, Δ_2 in such a way that

$$\Delta_1^1 = \tilde{\Delta}^{\psi_1^{-1}}, \quad \Delta_2^2 = \tilde{\Delta}^{\psi_2^{-1}\mathrm{N}_K^{1-m}}.$$

On the other hand, the period Ω attached to the elliptic curve E/K coincides with the period $\Omega(\psi_E)$ attached to the Hecke character ψ_E asi in [BDP17, §2.3]. It follows from [BDP17, Proposition 2.11(2)] that $\Omega(\psi_E^r) = \Omega^r \mod K^\times$ for $r \geq 0$. Using [BDP17, Proposition 2.11(2)], we conclude that the factor μ appearing in the formula of Proposition 2.25 lies in $K^\times$.

The result then follows by combining Proposition 2.30, Proposition 2.33 and Proposition 2.25.

□

Chapter 3

Special values of the triple product p-adic L-function and non-cristalline diagonal classes

In this chapter we put ourselves in the setting of the elliptic Stark conjecture of §2.1 with the only difference that now we assume that the relevant complex L-function does not vanish at its central point $s = 1$. We recall briefly the setting and the assumptions under which we will work, and we resume the main results of this chapter, which corresponds to [GGMR19]. Let E be an elliptic curve defined over $\mathbb{Q}$ and let $f \in S_2(N_f)$ be the newform attached to E. Moreover, we consider

$$g \in S_1(N_g, \chi) \quad \text{and} \quad h \in S_1(N_h, \bar{\chi}),$$

which are cuspforms of weight one, inverse Nebentype characters and with Fourier coefficients contained in a number field L. Let ρ_g and ρ_h be the Artin representations attached to g and h. The tensor product $\rho_{gh} := \rho_g \otimes \rho_h$ is a self-dual Artin representation of dimension 4 of the form

$$\rho_{gh} : \mathrm{Gal}(H/\mathbb{Q}) \longrightarrow \mathrm{Aut}(V_g^\circ \otimes V_h^\circ) \cong \mathrm{GL}_4(L),$$

where $H/\mathbb{Q}$ is a finite extension. As explained in §1.12, the complex L-function $L(E \otimes \rho_{gh}, s)$ attached to the (Tate module V_E of the) elliptic curve E twisted by the Artin representation ρ_{gh} coincides with the Garrett–Rankin L-function $L(f \otimes g \otimes h, s)$ attached to the triple (f, g, h) of modular forms. Recall that, by multiplying this L-function by an appropriate archimedean factor $L_\infty(f, g, h)$, one obtains the entire function $\Lambda(f \otimes g \otimes h, s)$ which satisfies the functional equation

$$\Lambda(f \otimes g \otimes h, s) = \varepsilon(f, g, h) \cdot \Lambda(f \otimes g \otimes h, 2 - s). \tag{3.0.1}$$

Here $\varepsilon(f, g, h) = \prod_v \varepsilon_v(f, g, h) \in \{\pm 1\}$, where v runs over the places of $\mathbb{Q}$, and $\varepsilon_v(f, g, h) = +1$ if v is a finite prime which does not divide $\mathrm{lcm}(N_f, N_g, N_h)$ or if $v = \infty$. Moreover, $L_\infty(f \otimes g \otimes h, s)$ does not have zeroes nor poles at $s = 1$. We will work under the following assumption, which coincides with the first part of Assumption 2.1.

Assumption 3.1. $\varepsilon_v(f, g, h) = +1$ for all v.

Assumption 3.1 holds most of the time: this is the case for instance if the greatest common divisor of the levels of f, g and h is 1.

Fix an odd prime number p such that

$$p \nmid N_f N_g N_h,$$

and denote by α_g, β_g the eigenvalues for the action of the Frobenius element at p acting on V_g. We use the analogous notation for h and we assume

$$\alpha_g \neq \beta_g, \quad \text{and} \quad \alpha_h \neq \beta_h.$$

Fix once and for all completions H_p, L_p of the number fields H, L at primes above p and choose ordinary p-stabilisations g_α, h_α of g and h. Let $\mathbf{f}, \mathbf{g}, \mathbf{h}$ be Hida families passing through the unique ordinary p-stabilisation of f and g_α and h_α reprectively, and consider the Garret–Hida p-adic L-function

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$$

of [DR14] described in §1.15.3, associated to the specific choice of test vector $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ of [Hsi19, Chapter 3]. Recall that this p-adic L-function interpolates the square roots of the central values of the classical L-function $L(f_k \otimes g_\ell \otimes h_m, s)$ attached to the specialisations of the Hida families at classical points of weights $k, \ell, m \geq 2$ with $\ell \geq k + m$. Notice that the point $(2, 1, 1)$, which corresponds to our triple of modular forms (f, g, h), lies outside the region of classical interpolation for $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$. We are interested in studying the value

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$$

under the following assumption:

Assumption 3.2. $L(E \otimes \rho_{gh}, 1) \neq 0$ and $\mathrm{Sel}_p(E \otimes \rho_{gh}) = 0$.

Here $\mathrm{Sel}_p(E \otimes \rho_{gh})$ denotes the Bloch–Kato Selmer group attached to the representation

$$V := V_E \otimes V_g \otimes V_h$$

described in §1.13.2. Under Assumption 3.1, the sign $\varepsilon(f, g, h)$ of the functional equation (3.0.1) is $+1$, and thus the order of vanishing of $L(E \otimes \rho_{gh}, s)$ at $s = 1$ is even. One hence expects that $L(E \otimes \rho_{gh}, 1)$ is generically nonzero. If this L-value is nonzero, by [DR17] we know that the ρ_{gh}-isotypical component $\mathrm{Hom}_{G_\mathbb{Q}}(V_{gh}^\circ, E(H)_L)$ of the Mordell–Weil group $E(H)$ is trivial. As explained in §1.13.1, by the Shafarevich–Tate conjecture one also expects the Selmer group $\mathrm{Sel}_p(E \otimes \rho_{gh})$ to be trivial, although this conjecture is widely open.

Recall from §2.1 that the value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ in the case in which the classical L-value $L(E \otimes \rho_{gh}, 1)$ vanishes has been analysed in [DLR15]. While in loc. cit. the authors give a conjectural formula for the p-adic L-value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ as a 2×2 regulator of p-adic logarithms of global points, under our running Assumption 3.2 one can not expect a similar formula for the above p-adic L-value, as no global points are naturally present in this scenario. The main result of this chapter consists in an explicit formula for the value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ which involves the algebraic part of the classical L-value $L(E \otimes \rho_{gh}, 1)$ and the *logarithm* of a canonical non-cristalline class along a certain *cristalline direction*.

In §3.0.1 we give a precise description of the relaxed p-Selmer group $\mathrm{Sel}_{(p)}(E \otimes \rho_{gh})$ under Assumption 3.2. More precisely, the projection to the singular quotient gives an isomorphism

$$\partial_p : \mathrm{Sel}_{(p)}(E \otimes \rho_{gh}) \xrightarrow{\cong} \mathrm{H}^1_s(\mathbb{Q}_p, V). \tag{3.0.2}$$

Let V_g^α, V_g^β, with basis v_g^α, v_g^β respectively, be the eigenspaces of V_g for the action of Frob_p with eigenvalues α_g, β_g, and use the analogous notation for V_h. The $G_{\mathbb{Q}_p}$-representation V decomposes as a direct sum as

$$V = V^{\alpha\alpha} \oplus V^{\alpha\beta} \oplus V^{\beta\alpha} \oplus V^{\beta\beta},$$

where $V^{\alpha\alpha} := V_E \otimes V_g^\alpha \otimes V_h^\alpha$ and similarly for the other pieces. It induces the decomposition

$$\mathrm{H}^1_s(\mathbb{Q}_p, V) = \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\beta}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}), \tag{3.0.3}$$

and the Bloch–Kato dual exponential described in §1.4 gives isomorphisms

$$\exp^*_{\alpha\alpha} : \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \xrightarrow{\cong} L_p$$

and similarly for the other pieces of the decomposition (3.0.3). Combining it with (3.0.2), we get a basis

$$\{\xi^{\alpha\alpha},\ \xi^{\alpha\beta},\ \xi^{\beta\alpha},\ \xi^{\beta\beta}\}$$

for $\mathrm{Sel}_{(p)}(E \otimes \rho_{gh})$ characterised by the fact that

$$\partial_p \xi^{\alpha\alpha} \in \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \quad \text{and} \quad \exp^*_{\alpha\alpha} \partial_p \xi^{\alpha\alpha} = 1,$$

and similarly for $\xi^{\alpha\beta},\ \xi^{\beta\alpha},\ \xi^{\beta\beta}$.

The $G_{\mathbb{Q}_p}$-cohomology of V and its submodule of cristalline classes $\mathrm{H}^1_f(\mathbb{Q}_p, V) \subseteq \mathrm{H}^1(\mathbb{Q}_p, V)$ also have decompositions analogous to (3.0.3). Moreover, if

$$\pi_{\alpha\beta} : \mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V^{\alpha\beta})$$

denotes the projection, then $\pi_{\alpha\beta}\xi^{\beta\beta}$ lies in $\mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$. Finally, we can write

$$\pi_{\alpha\beta}\xi^{\beta\beta} = R_{\beta\alpha} \otimes v_g^{\alpha} \otimes v_h^{\beta} \in (E(H_p) \otimes V_g^{\alpha} \otimes V_h^{\beta})^{G_{\mathbb{Q}_p}} \cong \mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta})$$

where $R_{\beta\alpha} \in E(H_p)$ is a local point on which Frob_p acts as multiplication by $\beta_g \alpha_h$.

We can finally state the main result of this chapter.

Theorem (cf Theorem 3.10)**.** Under Assumptions 3.1 and 3.2,

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1) = \frac{A \cdot \mathcal{E}}{\pi \langle f, f \rangle} \times \frac{\log_p(R_{\beta\alpha})}{\mathcal{L}_{g_\alpha}} \times \sqrt{L(E \otimes \rho_{gh}, 1)}, \tag{3.0.4}$$

where $A \in \mathbb{Q}^\times$ is an explicit number, $\mathcal{E} \in L_p$ is a product of Euler factors, $\langle f, f \rangle$ denotes the Petersson norm of f, $\mathcal{L}_{g_\alpha} \in H_p$ is the element defined in Remark 2.7, on which Frob_p acts as multiplication by β_g/α_g and which only depends on g_α, and $\log_p : E(H_p) \longrightarrow H_p$ denotes the p-adic logarithm.

We refer to Theorem 3.10 for a more precise statement of the result and of the objects appearing in (3.0.4). Recall that the element $\mathcal{L}_{g_\alpha}$ is expected to be related to the so-called Gross–Stark unit attached to g_α appearing in Conjectures 2.9 and 2.13, as conjectured in [DR16, Conjecture 2.1].

Under the additional assumption that g is the theta series of a Hecke character of a real quadratic field in which p splits (cf Assumption 2.3), the value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ can be recast in a more explicit way in terms of *p-adic iterated integrals*, as explained in §2.1. The numerical computations we offer in 3.4 have been carried out by Masdeu in [GGMR19] and they are obtained by calculating such integrals, where a key input are Lauder's algorithms [Lau14] for the computation of overconvergent projections.

As an application of the main result, in §3.3 we explore the situation where g and h are theta series of the same imaginary quadratic field in which p splits, in analogy to §2.4. The following theorem is stated as Theorem 3.12 in the text.

Theorem (cf Theorem 3.12)**.** Let K be an imaginary quadratic field in which p is split, and let ψ_g (resp. ψ_h) be a finite order Hecke character of K of conductor $\mathfrak{c}_g$ (resp. of conductor $\mathfrak{c}_h$). Denote g and h the theta series attached to ψ_g and ψ_h respectively. Suppose that $\gcd(N_f, \mathfrak{c}_g\mathfrak{c}_h) = 1$ and that the Nebentype characters of g and h are inverses to each other. If $L(E \otimes \rho_{gh}, 1) \neq 0$ then $\mathcal{L}_p^g(\mathbf{g}, \mathbf{h}, \mathbf{h})(2, 1, 1) = 0$.

3.0.1 The Selmer groups of $E \otimes \rho_{gh}$

The aim of this section is to describe explicitly the structure of the Selmer group relaxed at p of $E \otimes \rho_{gh}$. We retain the notation and all assumptions introduced at the beginning of the chapter.

Lemma 3.3. *There are isomorphisms*

$$\mathrm{H}^1(\mathbb{Q}, V) \cong (\mathrm{H}^1(H, V_f) \otimes V_{gh})^{\mathrm{Gal}(H/\mathbb{Q})} \cong \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, \mathrm{H}^1(H, V_f));$$

$$\mathrm{H}^1(\mathbb{Q}_p, V) \cong (\mathrm{H}^1(H_p, V_f) \otimes V_{gh})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \cong \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}, \mathrm{H}^1(H_p, V_f)). \tag{3.0.5}$$

Proof. We prove only the isomorphisms in the second row of the statement, the first row is given analogously. By Inflation–Restriction we have the exact sequence

$$0 \longrightarrow \mathrm{H}^1(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(H_p, V)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \longrightarrow \mathrm{H}^2(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}).$$

Since $\mathrm{H}^1(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}) = \mathrm{H}^2(\mathrm{Gal}(H_p/\mathbb{Q}_p), V^{G_{H_p}}) = 0$, the restriction to G_{H_p} gives an isomorphism

$$\mathrm{H}^1(\mathbb{Q}_p, V) \longrightarrow \mathrm{H}^1(H_p, V)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}.$$

Composing it with the identifications

$$\mathrm{H}^1(H_p, V)^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} = \mathrm{H}^1(H_p, V_f \otimes V_{gh})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} = (\mathrm{H}^1(H_p, V_f) \otimes V_{gh})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)},$$

we get the first isomorphism (3.0.5). Finally, the second isomorphism follows from the relation between Hom and tensor and from the selfduality $V_{gh}^{\vee} \cong V_{gh}$. □

Remark 3.4. It will be useful for similar computations carried out in Chapter 4 to note that the previous lemma holds independently of our running assumption 3.2 and of the request that $p \nmid N_f$.

Denote $E(H)_L := E(H) \otimes L$. The Kummer homomorphism $\delta : E(H)_L \longrightarrow \mathrm{H}^1(H, V_f)$ induces a homomorphism

$$\delta : \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, E(H)_L) \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, \mathrm{H}^1(H, V_f)),$$

which using the first row of Lemma 3.3 can be seen as an isomorphism

$$\delta : \mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}, E(H)_L) \longrightarrow \mathrm{H}^1(\mathbb{Q}, V).$$

Let α_g, β_g be the eigenvalues for the action of Frob_p on V_g and denote $V_g^{\alpha}, V_g^{\beta}$ the corresponding eigenspaces. As described in §1.8, we can consider the L-structure V_g^L of V_g and fix an L-basis of eigenvectors $v_g^{\alpha}, v_g^{\beta}$ for $V_g^L \cap V_g^{\alpha}$ and $V_g^L \cap V_g^{\beta}$ respectively. Using the analogous notation for h, denote

$$V_{gh}^{\triangle\heartsuit} := V_g^{\triangle} \otimes V_h^{\heartsuit} \quad \text{and} \quad V^{\triangle\heartsuit} := V_f \otimes V_g^{\triangle} \otimes V_h^{\heartsuit}.$$

for $\triangle, \heartsuit \in \{\alpha, \beta\}$. Recall by §1.8 that there is a short exact sequence of $G_{\mathbb{Q}_p}$-representations

$$0 \longrightarrow V_g^{\beta} \longrightarrow V_g \longrightarrow V_g^{\alpha} \longrightarrow 0$$

obtained by specialising (1.8.10) via y_g using the identifications (1.8.12). For $\triangle \neq \heartsuit$ the pairing (1.8.11) and its analogue for h induce perfect pairings

$$\langle\ ,\ \rangle : V_{gh}^{\triangle\triangle} \times V_{gh}^{\heartsuit\heartsuit} \longrightarrow L_p, \qquad \langle\ ,\ \rangle : V_{gh}^{\triangle\heartsuit} \times V_{gh}^{\heartsuit\triangle} \longrightarrow L_p.$$

The identifications

$$V_{gh}^{\triangle\triangle} \cong \mathrm{Hom}_{L_p[G_{\mathbb{Q}_p}]}(V_{gh}^{\heartsuit\heartsuit}, L_p) \quad \text{and} \quad V_{gh}^{\triangle\heartsuit} \cong \mathrm{Hom}_{L_p[G_{\mathbb{Q}_p}]}(V_{gh}^{\heartsuit\triangle}, L_p), \tag{3.0.6}$$

together with (3.0.5) give the following isomorphisms:

$$\mathrm{H}^1(\mathbb{Q}_p, V^{\triangle\triangle}) \cong (\mathrm{H}^1(H_p, V_f) \otimes V_{gh}^{\triangle\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \cong \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\heartsuit\heartsuit}, \mathrm{H}^1(H_p, V_f));$$
$$\mathrm{H}^1(\mathbb{Q}_p, V^{\triangle\heartsuit}) \cong (\mathrm{H}^1(H_p, V_f) \otimes V_{gh}^{\triangle\heartsuit})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \cong \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\heartsuit\triangle}, \mathrm{H}^1(H_p, V_f)).$$

Recall from §1.3 the Bloch–Kato module $\mathrm{H}^1_f(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit})$ and the singular quotient $\mathrm{H}^1_s(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit})$. It follows from [DRb, Lemma 2.4.1] that they can be written in terms of the filtration (1.8.1) of V_f as follows:

$$\mathrm{H}^1_s(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) = \mathrm{H}^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\triangle\heartsuit}) \cong (V_{gh}^{\heartsuit\triangle} \otimes \mathrm{H}^1_s(H_p, V_f))^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}; \tag{3.0.7}$$

$$\mathrm{H}^1_f(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) = \ker(\mathrm{H}^1(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) \longrightarrow \mathrm{H}^1(I_p, V_f^- \otimes V_{gh}^{\triangle\heartsuit})) = \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}), \tag{3.0.8}$$

where I_p is the inertia group of $G_{\mathbb{Q}_p}$.

Lemma 3.5. *For $\triangle, \heartsuit \in \{\alpha, \beta\}, \triangle \neq \heartsuit$, there are isomorphisms*

$$\delta_p : \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}, E(H_p)_{L_p}) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, V);$$
$$\delta_p^{\triangle\heartsuit} : \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\heartsuit}, E(H_p)_{L_p}) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, V^{\heartsuit\triangle}),$$
$$\delta_p^{\triangle\triangle} : \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\triangle}, E(H_p)_{L_p}) \longrightarrow \mathrm{H}^1_f(\mathbb{Q}_p, V^{\heartsuit\heartsuit}).$$

Proof. We prove the existence of the isomorphism $\delta_p^{\triangle\heartsuit}$, the others are similar. By Kummer theory, there is an injective morphism

$$\delta_p : E(H_p)_{L_p} \longrightarrow \mathrm{H}^1(H_p, V_f), \tag{3.0.9}$$

which is an isomorphism on its image $\mathrm{H}^1_f(H_p, V_f) \cong \mathrm{H}^1(H_p, V_f^+)$. It induces an isomorphism

$$\delta_p^{\triangle\heartsuit} : \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\heartsuit}, E(H_p)_{L_p}) \longrightarrow \mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\heartsuit}, \mathrm{H}^1(H_p, V_f^+)).$$

Using the isomorphisms (3.0.6) we obtain

$$\mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\triangle\heartsuit}, \mathrm{H}^1(H_p, V_f^+)) \xrightarrow{\cong} (\mathrm{H}^1(H_p, V_f^+) \otimes V_{gh}^{\heartsuit\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)}.$$

Arguing as in the proof of Lemma 3.3, we get the isomorphisms

$$\begin{aligned}(\mathrm{H}^1(H_p, V_f^+) \otimes V_{gh}^{\heartsuit\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} &\cong \mathrm{H}^1(H_p, V_f^+ \otimes V_{gh}^{\heartsuit\triangle})^{\mathrm{Gal}(H_p/\mathbb{Q}_p)} \\ &\cong \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\heartsuit\triangle}) \\ &\cong \mathrm{H}^1_f(\mathbb{Q}_p, V^{\heartsuit\triangle}).\end{aligned}$$

□

From now on, the fact that we are in a *rank* 0 setting (i.e. under Assumption 3.2), becomes crucial. Indeed, under this assumption one can identify the relaxed Selmer group with the singular quotient. Recall that we are also assuming that $p \nmid N_f N_g N_h$.

Lemma 3.6. *The natural map*

$$\partial_p : \mathrm{Sel}_p(\mathbb{Q}, V) \longrightarrow \mathrm{H}^1_s(\mathbb{Q}_p, V)$$

is an isomorphism. In particular, there is an isomorphism

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V) \cong \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\beta}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}). \tag{3.0.10}$$

Proof. Using the notation of §1.13, since the representation V is selfdual, there is an isomorphism

$$\mathrm{Sel}_{p,*}(\mathbb{Q}, V^*) \cong \mathrm{Sel}_p(\mathbb{Q}, V)$$

(see, for example,[BK90] or [Bel, Theorem 2.1]). Then the lemma follows immediately from the Poitou–Tate exact sequence (1.13.2). □

3.1 Block–Kato logarithms and dual exponentials

For $\triangle, \heartsuit \in \{\alpha, \beta\}$, the $G_{\mathbb{Q}_p}$-representation $V_f^+ \otimes V_{gh}^{\triangle\heartsuit}$ is one dimensional and, therefore, given by characters. More precisely, recall the unramified character

$$\psi_f : G_{\mathbb{Q}_p} \longrightarrow L_p^\times$$

such that $\psi_f(\mathrm{Frob}_p) = \alpha_f$ introduced in §1.6.4.3, and define analogously ψ_g, ψ_h. Then $G_{\mathbb{Q}_p}$ acts on V_f^+ as $\chi_{\mathrm{cyc}}\psi_f^{-1}$, and it acts as ψ_g (resp. ψ_g^{-1}) on V_g^α (resp. V_g^β) and as ψ_h (resp. ψ_h^{-1}) on V_h^α (resp. V_h^β). Therefore we have that

$$V_f^+ \otimes V_{gh}^{\alpha\alpha} = L_p(\chi_{\mathrm{cyc}}\psi_f^{-1}\psi_g\psi_h), \qquad V_f^+ \otimes V_{gh}^{\alpha\beta} = L_p(\chi_{\mathrm{cyc}}\psi_f^{-1}\psi_g\psi_h^{-1}),$$
$$V_f^+ \otimes V_{gh}^{\beta\alpha} = L_p(\chi_{\mathrm{cyc}}\psi_f^{-1}\psi_g^{-1}\psi_h), \qquad V_f^+ \otimes V_{gh}^{\beta\beta} = L_p(\chi_{\mathrm{cyc}}\psi_f^{-1}\psi_g^{-1}\psi_h^{-1}).$$

In particular, $V_f^+ \otimes V_{gh}^{\triangle\heartsuit}$ is of the form $L_p(\psi\chi_{\mathrm{cyc}})$ for some nontrivial unramified character. By (3.0.8), we have that

$$\mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) \cong \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}),$$

and the Bloch–Kato logarithm gives an isomorphism (cf. part 4. of §1.3.1 and §1.4.1)

$$\log_{\triangle\heartsuit} : \mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) \longrightarrow \mathrm{D_{dR}}(V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) = \mathrm{D_{dR}}(L_p(\psi\chi_{\mathrm{cyc}})). \tag{3.1.1}$$

For $(\triangle, \heartsuit) = (\alpha, \alpha)$, the pairings (1.8.13) and (1.8.8) give rise to pairings

$$\langle\ ,\ \rangle : V_f^+ \otimes V_g^\alpha \otimes V_h^\alpha \times V_{f^*}^- \otimes V_g^\beta \otimes V_h^\beta \longrightarrow L_p$$

and

$$\langle\ ,\ \rangle : \mathrm{D_{dR}}(V_f^+ \otimes V_g^\alpha \otimes V_h^\alpha) \times \mathrm{D_{dR}}(V_{f^*}^- \otimes V_g^\beta \otimes V_h^\beta) \longrightarrow \mathrm{D_{dR}}(L_p) = L_p. \tag{3.1.2}$$

Recall also the differentials attached to modular forms defined in (1.8). In (3.1.2), pairing with the class $\omega_{f^*} \otimes \eta_g \otimes \eta_h$ gives then an isomorphism

$$\langle \cdot, \omega_{f^*} \otimes \eta_g \otimes \eta_h \rangle : \mathrm{D_{dR}}(L_p(\psi\chi_{\mathrm{cyc}})) = \mathrm{D_{dR}}(V_f^+ \otimes V_g^\alpha \otimes V_h^\alpha) \longrightarrow L_p. \tag{3.1.3}$$

There are similar pairings and isomorphisms for the remaining pairs $(\triangle, \heartsuit)$. We still denote

$$\log_{\triangle\heartsuit} : \mathrm{H}^1_f(\mathbb{Q}_p, V_f \otimes V_{gh}^{\triangle\heartsuit}) \longrightarrow L_p \tag{3.1.4}$$

the map obtained by composing (3.1.1) with (3.1.3).

Remark 3.7. The logarithm maps of (3.1.4) are related to the usual p-adic logarithm on E as follows. The differential ω_f gives rise to an invariant differential on E, and we denote by

$$\log_{E,p} : E(H_p) \longrightarrow H_p$$

the corresponding formal group logarithm on E. The map $\log_{\alpha\beta}$ coincides with the inverse of the isomorphism of Lemma 3.3

$$\mathrm{Hom}_{\mathrm{Gal}(H_p/\mathbb{Q}_p)}(V_{gh}^{\beta\alpha}, E(H_p)_{L_p}) \cong E(H_p)^{\beta_g\alpha_g} \otimes V_{gh}^{\alpha\beta}$$

composed with the maps

$$\begin{array}{ccccc} E(H_p)^{\beta_g\alpha_h}\otimes V_{gh}^{\alpha\beta} & \longrightarrow & H_p^{\beta_g\alpha_h}\otimes V_{gh}^{\alpha\beta}=D(V_{gh}^{\alpha\beta}) & \longrightarrow & L_p \\ x\otimes v_g^{\alpha}v_h^{\beta} & \longmapsto & \log_{E,p}(x)\otimes v_g^{\alpha}v_h^{\beta} & & \\ & & y & \longmapsto & \langle y,\eta_g\omega_h\rangle. \end{array}$$

Above and in what follows, for a $G_{\mathbb{Q}_p}$-representation M, we denote M^{γ} the subspace of M on which Frob_p acts as multiplication by γ. Notice that we have used the equality

$$\alpha_g\beta_h\cdot\beta_g\alpha_h=\chi(p)\bar{\chi}(p)=1.$$

Analogous equalities hold for the other maps $\log_{\triangle\heartsuit}$.

A similar discussion can be applied to the representations of the form $V_f^-\otimes V_{gh}^{\triangle\heartsuit}$. In this case we have the following isomorphisms of 1-dimensional representations

$$\begin{array}{cc} V_f^-\otimes V_{gh}^{\alpha\alpha}=L_p(\psi_f\psi_g\psi_h), & V_f^-\otimes V_{gh}^{\alpha\beta}=L_p(\psi_f\psi_g\psi_h^{-1}\bar{\chi}), \\ V_f^-\otimes V_{gh}^{\beta\alpha}=L_p(\psi_f\psi_g^{-1}\psi_h\chi), & V_f^-\otimes V_{gh}^{\beta\beta}=L_p(\psi_f\psi_g^{-1}\psi_h^{-1}). \end{array}$$

Therefore, $V_f^-\otimes V_{gh}^{\triangle\heartsuit}$ is isomorphic to a $G_{\mathbb{Q}_p}$ representationof the form $L_p(\psi)$ for some unramified nontrivial character ψ. By (3.0.7) there is an identification

$$\mathrm{H}^1_s(\mathbb{Q}_p,V^{\triangle\heartsuit})=\mathrm{H}^1(\mathbb{Q}_p,L_p(\psi)),$$

and by part 3 of §1.3.1 and §1.4.1, the dual exponential gives isomorphisms

$$\exp^*_{\triangle\heartsuit}:\mathrm{H}^1_s(\mathbb{Q}_p,V^{\triangle\heartsuit})\longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p(\psi))\cong L_p,$$

where the last isomorphism is induced by pairing with the appropriate class of

$$\mathrm{D}_{\mathrm{dR}}(L_p(\psi^{-1}))=\mathrm{D}_{\mathrm{dR}}(V_{f^*}^+\otimes V_g^{\heartsuit}\otimes V_h^{\triangle})$$

similarly as in (3.1.3). Arguing as in Remark 3.7, recall the dual exponential

$$\exp^*_{E,p}:\mathrm{H}^1_s(H_p,V_f)\longrightarrow H_p$$

attached to E/H_p. Then $\exp^*_{\beta\beta}$ can be identified with the composition

$$\begin{array}{ccccc} \mathrm{H}^1_s(H_p,V_f)^{\alpha_g\alpha_h}\otimes V_{gh}^{\beta\beta} & \longrightarrow & H_p^{\alpha_g\alpha_h}\otimes V_{gh}^{\beta\beta}=D(V_{gh}^{\beta\beta}) & \longrightarrow & L_p \\ x\otimes v_g^{\beta}v_h^{\beta} & \longmapsto & \exp^*_{E,p}(x)\otimes v_g^{\beta}v_h^{\beta} & & \\ & & y & \longmapsto & \langle y,\omega_g\omega_h\rangle, \end{array}$$

after taking into account the identification

$$\mathrm{H}^1_s(\mathbb{Q}_p,V^{\beta\beta})\cong\mathrm{H}^1_s(H_p,V_f)^{\alpha_g\alpha_h}\otimes V_{gh}^{\beta\beta}.$$

Analogous formulas hold for the dual exponential $\exp^*_{\triangle\heartsuit}$ on the remaining components.

To sum up the discussion of this subsection, we conclude that the relaxed Selmer group of V admits a basis adapted to the decomposition (3.0.10) with respect to the dual exponential maps.

Proposition 3.8. *Under Assumption 3.2,* $\mathrm{Sel}_{(p)}(\mathbb{Q}, V)$ *has a basis*

$$\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$$

characterised by the fact that there exist elements

$$X_{\beta\beta} \in \mathrm{H}^1_s(H_p, V_f)^{\beta_g\beta_h},\ X_{\beta\alpha} \in \mathrm{H}^1_s(H_p, V_f)^{\beta_g\alpha_h},\ X_{\alpha\beta} \in \mathrm{H}^1_s(H_p, V_f)^{\alpha_g\beta_h},\ X_{\alpha\alpha} \in \mathrm{H}^1_s(H_p, V_f)^{\alpha_g\alpha_h}$$

such that

$$\partial_p\xi^{\alpha\alpha} = (X_{\beta\beta} \otimes v_g^\alpha v_h^\alpha, 0, 0, 0), \qquad \partial_p\xi^{\alpha\beta} = (0, X_{\beta\alpha} \otimes v_g^\alpha v_h^\beta, 0, 0)$$
$$\partial_p\xi^{\beta\alpha} = (0, 0, X_{\alpha\beta} \otimes v_g^\beta v_h^\alpha, 0), \qquad \partial_p\xi^{\beta\beta} = (0, 0, 0, X_{\beta\beta} \otimes v_g^\beta v_h^\beta)$$

and

$$\exp^*_{E,p}(X_{\beta\beta}) = \exp^*_{E,p}(X_{\alpha\beta}) = \exp^*_{E,p}(X_{\beta\alpha}) = \exp^*_{E,p}(X_{\alpha\alpha}) = 1.$$

Remark 3.9. Note that the basis of Proposition 3.8 depends on the choice of the L-basis $\{v_g^\alpha, v_g^\beta\}$ of V_g and the L-basis $\{v_h^\alpha, v_h^\beta\}$ of V_h. Then each element of the basis $\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$ depends on this choice up to multiplication by an element of $L^\times$.

3.2 The special value formula for the triple product p-adic L-function

We continue with the notation and assumptions of the previous sections.

Recall that p stands for a prime that does not divide $N := \mathrm{lcm}(N_f, N_g, N_h)$, and that $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$ (resp. $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$) is a Hida family passing through the p-stabilisation g_α (resp. h_α) such that $U_p g_\alpha = \alpha_g g_\alpha$ (resp. $U_p h_\alpha = \alpha_h h_\alpha$). Similarly, denote by $\mathbf{f} \in \Lambda_{\mathbf{f}}[[q]]$ the Hida family passing through the ordinary p-stabilisation f_α of f.

Denote by $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ the triple product p-adic L-function described in §1.15.3, attached to the choice of Λ-adic test vector $(\mathbf{f}^\star, \mathbf{g}^\star, \mathbf{h}^\star)$ of [Hsi19, Ch. 3]. Recall that the values

$$\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(k, \ell, m)$$

of this p-adic L-function at triples of classical weights (k, ℓ, m) with $\ell \geq k + m$ interpolate the square root of the algebraic part of

$$L(f_k \otimes g_\ell \otimes h_m, \frac{k + \ell + m - 2}{2}), \tag{3.2.1}$$

where f_k, g_ℓ, h_m are the newforms attached to the specialisations of $\mathbf{f}, \mathbf{g}$ and $\mathbf{h}$ at the weights k, ℓ and m. The aim of this chapter is to compute the value of $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})$ at the point $(2, 1, 1)$, which lies *outside* the region of classical interpolation. The formula for this special value will involve another triple product p-adic L-function, namely $\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})$, that interpolates (3.2.1) but for the values (k, ℓ, m) with $k \geq \ell + m$. In particular, the point $(2, 1, 1)$ belongs to the region of interpolation of this p-adic L-function, so $\mathcal{L}_p^f(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ is directly related to $L(E \otimes \rho, 1)$. Recall that one of our running assumptions is that

$$\mathrm{Sel}_p(\mathbb{Q}, V) = 0.$$

This implies that $\mathrm{Hom}_{\mathrm{Gal}(H/\mathbb{Q})}(V_{gh}^\circ, E(H)_L) = 0$ and, conjecturally, it also implies that

$$L(E \otimes \rho, 1) \neq 0$$

as explained in §1.14. This is the reason why we directly work under Assumption 3.2. The main result of this section and of all this chapter is an explicit formula for $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ in this case, and this can be seen as completing the study of $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ initiated in [DLR15]. The main tool that we will use are the generalised Kato classes

$$\kappa := \kappa(f, g_\alpha, h_\alpha) \in \mathrm{Sel}_{(p)}(\mathbb{Q}, V)$$

introduced in [DR17, §3] and described in §1.16.4. Using the relation between this class and the value at $(2,1,1)$ of the triple product p-adic L-functions given by Proposition 1.107, and using the basis of $\mathrm{Sel}_{(p)}(\mathbb{Q}, V)$ defined in Proposition 3.8, we can give a precise formula for $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ in the rank 0 setting.

Recall the Kummer map (3.0.9) and define $P_{\alpha\beta} \in E(H_p)^{\beta\alpha}$ by the equality

$$\pi_{\alpha\beta}\,\mathrm{res}_p\,\xi^{\beta\beta} = \delta_p P_{\alpha\beta} \otimes v_g^\alpha v_h^\beta \in \mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta}) = (\mathrm{H}^1_f(H_p, V_f)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta})^{G_{\mathbb{Q}_p}},$$

and recall the elements $\Theta_{g_\alpha}, \Theta_{h_\alpha}, \Omega_{g_\alpha}, \Omega_{h_\alpha}, \mathcal{L}_{g_\alpha}$ attached to g and h as in Definition 1.63.

Theorem 3.10. *The class κ is a multiple of $\xi^{\beta\beta}$. More precisely,*

$$\kappa = \frac{\Theta_{g_\alpha}\Theta_{h_\alpha} 2(1 - p\alpha_f\alpha_g^{-1}\alpha_h^{-1})\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_h)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h)^{-1}} \times \xi^{\beta\beta}.$$

Moreover, if we define the quantity

$$\mathcal{E} := \frac{(1-\chi(p)p^{-1}\alpha_g^{-1}\alpha_h)(1-p\alpha_f\alpha_g^{-1}\alpha_h^{-1})}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_h)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})},$$

then we have that

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1) = \mathcal{E} \times \frac{\log_{E,p}(P_{\alpha\beta})}{\mathcal{L}_{g_\alpha}} \times \mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1) \mod L^\times. \tag{3.2.2}$$

Proof. By Proposition 1.107, κ is an element of $\mathrm{Sel}_{(p)}(\mathbb{Q}, V)$ such that

$$\exp^*(\partial_p\kappa) = \left(0,0,0,\frac{2(1-p\alpha_f\alpha_g^{-1}\alpha_h^{-1})\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_g)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})}\right). \tag{3.2.3}$$

Then κ is a multiple of the element $\xi^{\beta\beta}$; indeed

$$\kappa = \frac{\exp^*_{\beta\beta}(\partial_p\kappa)}{\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta})} \times \xi^{\beta\beta}.$$

Observe that (3.2.3) gives us the expression for the numerator. We now compute the denominator.

$$\begin{aligned}\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta}) &= \langle \exp^*_{E,p}(X_{\alpha\alpha}) \otimes v_g^\beta v_h^\beta, \omega_g\omega_h\rangle \\ &= \frac{\exp^*_{E,p}(X_{\alpha\alpha})}{\Theta_{g_\alpha}\Theta_{h_\alpha}} \\ &= \frac{1}{\Theta_{g_\alpha}\Theta_{h_\alpha}}.\end{aligned}$$

Here we used the fact that $\eta_g\eta_h = \Theta_{g_\alpha}\Theta_{h_\alpha} v_g^\beta v_g^\beta$. So we get

$$\begin{aligned}\kappa &= \frac{\exp^*_{\beta\beta}(\partial_p\kappa)}{\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta})} \times \xi^{\beta\beta} \\ &= \frac{2\Theta_{g_\alpha}\Theta_{h_\alpha}(1-p\alpha_f\alpha_g^{-1}\alpha_h^{-1})\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_g)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})} \times \xi^{\beta\beta}.\end{aligned}$$

By (1.16.15),

$$\begin{aligned}
\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1) &= \frac{1}{2}(1-\chi(p)p^{-1}\alpha_g^{-1}\alpha_h)\log_{\alpha\beta}(\pi_{\alpha\beta}\operatorname{res}_p\kappa)\\
&= \frac{\Theta_{g_\alpha}\Theta_{h_\alpha}(1-\chi(p)p^{-1}\alpha_g^{-1}\alpha_h)(1-p\alpha_f\alpha_g^{-1}\alpha_h^{-1})\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)}{\alpha_g\alpha_h(1-\alpha_f^{-1}\alpha_g\alpha_g)(1-\chi^{-1}(p)\alpha_f^{-1}\alpha_g\alpha_h^{-1})}\log_{\alpha\beta}(\pi_{\alpha\beta}\operatorname{res}_p\xi^{\beta\beta})\\
&= \mathcal{E}\Theta_{g_\alpha}\Theta_{h_\alpha}\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)\log_{\alpha\beta}(\pi_{\alpha\beta}\operatorname{res}_p\xi^{\beta\beta})\\
&= \mathcal{E}\Theta_{g_\alpha}\Theta_{h_\alpha}\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)\langle\log_{E,p}(P_{\beta\alpha})\otimes v_g^\alpha v_h^\beta,\eta_g\omega_h\rangle\\
&= \mathcal{E}\frac{\Theta_{g_\alpha}\Theta_{h_\alpha}}{\Omega_{g_\alpha}\Theta_{h_\alpha}}\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)\log_{E,p}(P_{\beta\alpha})\\
&= \mathcal{E}\frac{\Theta_{g_\alpha}}{\Omega_{g_\alpha}}\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)\log_{E,p}(P_{\beta\alpha})\\
&= \frac{\mathcal{E}\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)}{\mathcal{L}_{g_\alpha}}\log_{E,p}(P_{\beta\alpha})
\end{aligned}$$

since $\omega_g\eta_h = \Omega_{g_\alpha}\Theta_{h_\alpha}\otimes v_g^\alpha v_h^\beta$. □

We end this section recalling that Conjecture 2.8 relates $\mathcal{L}_{g_\alpha}$ to the p-adic logarithm of the Gross–Stark unit u_{g_α} appearing in Chapter 2, under the additional assumption that g is not the theta series of a Hecke character of a real quadratic field in which p splits.

Corollary 3.11. *Assuming Conjecture 2.8, if* $\mathrm{Sel}_p(\mathbb{Q},V)=0$ *then*

$$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1) = \mathcal{E}\times\frac{\log_{E,p}(P_{\beta\alpha})}{\log_p(u_{g_\alpha})}\times\mathcal{L}_p^f(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1).$$

3.3 The case of theta series of an imaginary quadratic field K where p splits

In this section we will consider a particular case where g and h are theta series of the same imaginary quadratic field in which p slits. We will see that in this setting the representation V decomposes in a way that forces $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ to vanish when the complex L-function does not vanish at the central critical point; that is, the special value of the p-adic L-function vanishes in analytic rank 0.

The setting is analogous to the one of §2.4.2. Let K be an imaginary quadratic field of discriminant $-D_K$ and let $\psi_g,\psi_h:\mathbb{A}_K^\times\longrightarrow\mathbb{C}^\times$ be two finite order Hecke characters of K of conductor $\mathfrak{c}_g,\mathfrak{c}_h$ and central characters $\epsilon,\bar\epsilon$ respectively. Here $\epsilon:\mathbb{A}_\mathbb{Q}^\times\longrightarrow\mathbb{C}^\times$ is a finite order character and $\bar\epsilon$ denotes its complex conjugate. Let g and h be the theta series attached to ψ_g and ψ_h as in §1.5.4. They are modular forms of weight one, and their levels and Nebentype characters are given by

$$N_g := D_K\cdot\mathrm{N}_K(\mathfrak{c}_g),\ N_h := D_K\cdot\mathrm{N}_K(\mathfrak{c}_h),\ \chi:=\chi_K\cdot\epsilon,\ \bar\chi:=\chi_K\cdot\bar\epsilon,$$

where χ_K is the quadratic Dirichlet character attached to the field K and we regard $\epsilon,\bar\epsilon$ as Dirichlet characters via class field theory. That is to say,

$$g\in M_1(N_g,\chi),\quad\text{and}\quad h\in M_1(N_h,\bar\chi).$$

Let $f\in S_2(N_f)$ be a newform with rational coefficients an let E be the associated elliptic curve over $\mathbb{Q}$ as in §1.6.4.2. We will particularise some of the results of the previous sections to this choice of forms f,g and h, so we will use the same notations as before. In particular, ρ_{gh} stands for the Artin representation afforded by $V_g^\circ\otimes V_h^\circ$ and p is a prime that does not divide $N_f\cdot N_g\cdot N_h$. In this section, we will make the following additional assumptions:

1. $\gcd(N_f, \mathfrak{c}_g\mathfrak{c}_h) = 1$;
2. p splits in K.

A finite order Hecke character ψ of K can be regarded, via class field theory, as Galois character $\psi : G_K \longrightarrow \mathbb{C}^\times$. We denote by ψ' the character defined as in (1.15.1). Also, ψ gives rise to a 1-dimensional representation of G_K, and we let $V_\psi := \mathrm{Ind}_K^{\mathbb{Q}}(\psi)$ denote the induced representation; it is a 2-dimensional representation of $G_{\mathbb{Q}}$. As explained in §1.6.4.5, with this notation, $V_g = V_{\psi_g}$ and $V_h = V_{\psi_h}$.

As explained in §2.4.2, there is a decomposition of $V_g \otimes V_h$ as the direct sum of two representations:

$$V_g \otimes V_h = V_{\psi_1} \oplus V_{\psi_2},$$

where the characters ψ_1 and ψ_2 are

$$\psi_1 := \psi_g\psi_h, \quad \text{and} \quad \psi_2 := \psi_g\psi_h'.$$

This induces a decomposition of the representation $V := V_f \otimes V_g \otimes V_h$ as a direct sum of two representations:

$$V = V_1 \oplus V_2,$$

where $V_1 := V_f \otimes V_{\psi_1}$, and $V_2 := V_f \otimes V_{\psi_2}$. This induces a factorisation of complex L-functions

$$L(E \otimes \rho_{gh}, s) = L(f, \psi_1, s) \cdot L(f, \psi_2, s). \tag{3.3.1}$$

As explained in §2.4.2, under our assumption that $\gcd(N_f, \mathfrak{c}_g\mathfrak{c}_h) = 1$ the local signs of $L(f, \psi_1, s)$ and $L(f, \psi_2, s)$ are $+1$. An important difference with Chapter 2, is that now we are not assuming Heegner hypothesis to hold for (N_f, K), since we are interested in a *rank* 0 situation. Indeed, Assumption 1.90 would imply that $L(E \otimes \rho_{gh}, 1) = 0$, so we are actually requiring Heegner hypothesis *not* to hold. In this case, the sign $\varepsilon_\infty(f, \psi_1) = \varepsilon_\infty(f, \psi_2) = +1$, therefore the local signs of $L(f \otimes \rho, s)$ are all equal to $+1$ and Assumption 3.1 on local signs is satisfied. Moreover, this setting is compatible with the request that $L(E \otimes \rho_{gh}, 1) \neq 0$.

Theorem 3.12. *In the setting of this section, if $L(E \otimes \rho, 1) \neq 0$ then $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1) = 0$.*

Proof. By (3.3.1), if $L(E \otimes \rho_{gh}, 1) \neq 0$ then $L(E, \psi_i) \neq 0$ for $i = 1, 2$. Note that ψ_1 and ψ_2 are ring class characters of the imaginary quadratic field K. Then, by results of Gross-Zagier and Kolyvagin, and its generalisation of Bertolini and Darmon in [BD97],

$$\mathrm{Sel}_p(\mathbb{Q}, V_1) = \mathrm{Sel}_p(\mathbb{Q}, V_2) = 0. \tag{3.3.2}$$

The decomposition of V induces a decomposition of the Selmer groups

$$\mathrm{Sel}_p(\mathbb{Q}, V) = \mathrm{Sel}_p(\mathbb{Q}, V_1) \oplus \mathrm{Sel}_p(\mathbb{Q}, V_2),$$

and analogously for the relaxed and the strict Selmer groups of V. In particular $\mathrm{Sel}_p(\mathbb{Q}, V) = 0$.

Since p splits in K we can write $p\mathcal{O}_K = \mathfrak{p}\bar{\mathfrak{p}}$, and from our assumption that $p \nmid N_f N_g N_h$ we see that $p \nmid \mathfrak{c}_g\mathfrak{c}_h$. Without loss of generality we can suppose that

$$\psi_g(\mathfrak{p}) = \alpha_g, \ \psi_g(\bar{\mathfrak{p}}), \ \psi_h(\mathfrak{p}) = \alpha_h, \ \psi_h(\bar{\psi}) = \beta_h,$$

so that

$$V_1 = V^{\alpha\alpha} \oplus V^{\beta\beta} \quad \text{and} \quad V_2 = V^{\alpha\beta} \oplus V^{\beta\alpha}.$$

By (3.3.2), the same computation as in §3.0.1 show that there are isomorphisms

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V_1) \xrightarrow{\partial_p} \mathrm{H}^1_s(\mathbb{Q}_p, V_1) \xrightarrow{(\pi^s_{\alpha\alpha}, \pi^s_{\beta\beta})} \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}),$$

where $\pi^s_{\alpha\alpha}$ denotes the natural map in the singular quotient induced by the projection $V \longrightarrow V^{\alpha\alpha}$, and analogously for $\pi^s_{\beta\beta}$. Similarly, there are dual exponential maps

$$\exp^*_{\alpha\alpha} : \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) = \mathrm{H}^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\alpha\alpha}) \longrightarrow L_p$$

and

$$\exp^*_{\beta\beta} \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}) = \mathrm{H}^1(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\beta\beta}) \longrightarrow L_p$$

which are in fact isomorphisms. Then $\mathrm{Sel}_{(p)}(\mathbb{Q}, V_1)$ has dimension 2 over $\mathbb{Q}_p$ with the canonical basis

$$\{\zeta^{\alpha\alpha}, \zeta^{\beta\beta}\},$$

where $\zeta^{\alpha\alpha}$ is characterised (up to scalars in $L^\times$) by the fact that

$$\exp^*_{\alpha\alpha}(\pi_{\alpha\alpha}\partial_p\zeta^{\alpha\alpha}) = 1, \quad \text{and} \quad \exp^*_{\beta\beta}(\pi_{\beta\beta}\partial_p\zeta^{\alpha\alpha}) = 0.$$

Similarly,

$$\exp^*_{\alpha\alpha}(\pi_{\alpha\alpha}\partial_p\zeta^{\beta\beta}) = 0, \quad \text{and} \quad \exp^*_{\beta\beta}(\pi_{\beta\beta}\partial_p\zeta^{\beta\beta}) = 1.$$

Analogously, $\mathrm{Sel}_{(p)}(\mathbb{Q}, V_2)$ has dimension 2 with basis $\{\zeta^{\alpha\beta}, \zeta^{\beta\alpha}\}$.

By Theorem 3.10, the value $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1)$ is a multiple of $\log_{\alpha\beta} \mathrm{res}_p\, \xi^{\beta\beta}$. On the other hand, using the decomposition

$$\mathrm{Sel}_{(p)}(\mathbb{Q}, V) = \mathrm{Sel}_{(p)}(\mathbb{Q}, V_1) \oplus \mathrm{Sel}_{(p)}(\mathbb{Q}, V_2),$$

the element $\xi^{\beta\beta} \in \mathrm{Sel}_{(p)}(\mathbb{Q}, V)$ corresponds to a multiple of $(0, \zeta^{\beta\beta})$, and this implies that

$$\pi_{\alpha\beta} \mathrm{res}_p\, \xi^{\beta\beta} = 0.$$

□

3.4 Numerical computation

In this section we present some numerical examples. They have been computed by Marc Masdeu with a Sage ([S+19]) implementation of Lauder's algorithms ([Lau14]), adapted to work in the current setting. The code and the following computations have been performed by Marc Masdeu, to whom I am grateful for letting me reproduce them here. The code is available at `github.com/mmasdeu/ellipticstarkconjecture` and the data for the weight-one modular forms can be found in Alan Lauder's website.[1]

3.4.1 Dihedral case

(*a*) We computed $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{g})(2, 1, 1)$ with $\mathbf{f}$ the Hida family passing through the modular form f of weight 2 attached to an elliptic curve $E_f/\mathbb{Q}$ of conductor N_f and $\mathbf{g}$ attached to the weight-one modular form $g = \theta(1_K)$ for some imaginary quadratic field K. The modular form g belongs then to $M_1(N_g, \chi_K)_{\mathbb{Q}}$. For each of the entries in the table we give the Cremona label for the elliptic curve E_f, its conductor N_f, the field K, the level N_g of g, the level N such that $p^\alpha N = \mathrm{lcm}(N_f, N_g)$ with $\alpha \geq 0$ and $p \nmid N$. In all of these cases, we obtained $\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{g}) = 0$ up to the working precision of p^{10}. Due to computational restrictions, only in the ramified case we have been able to compute examples where p divides the conductor of the elliptic curve.

Note that all the elliptic curves arising in Table 1 below have rank 0 over K, and thus the zeros obtained in this table are accounted for by Theorem 3.12.

E_f	K	N_g	p	N	$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$
11a	$\mathbb{Q}(\sqrt{-5})$	20	7	220	0
11a	$\mathbb{Q}(\sqrt{-11})$	11	5	11	0
19a	$\mathbb{Q}(\sqrt{-19})$	19	5	19	0
19a	$\mathbb{Q}(\sqrt{-19})$	19	7	19	0
39a	$\mathbb{Q}(\sqrt{-39})$	39	5	39	0
51a	$\mathbb{Q}(\sqrt{-51})$	51	5	51	0
55a	$\mathbb{Q}(\sqrt{-55})$	55	7	55	0
187a	$\mathbb{Q}(\sqrt{-187})$	187	7	187	0

Table 3.1: Cases with p split in K.

E_f	K	N_g	p	N	$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$
11a	$\mathbb{Q}(\sqrt{-3})$	3	5	33	0
11a	$\mathbb{Q}(\sqrt{-11})$	11	7	11	0
15a	$\mathbb{Q}(\sqrt{-15})$	15	7	15	0
39a	$\mathbb{Q}(\sqrt{-39})$	39	7	39	0
51a	$\mathbb{Q}(\sqrt{-51})$	51	7	51	0
67a	$\mathbb{Q}(\sqrt{-67})$	67	5	67	0
67a	$\mathbb{Q}(\sqrt{-67})$	67	7	67	0
187a	$\mathbb{Q}(\sqrt{-187})$	187	5	187	0

Table 3.2: Cases with p inert in K.

E_f	K	N_g	p	N	$\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$
15a	$\mathbb{Q}(\sqrt{-15})$	15	5	3	0
35a	$\mathbb{Q}(\sqrt{-35})$	35	5	7	0
35a	$\mathbb{Q}(\sqrt{-35})$	35	7	5	0
55a	$\mathbb{Q}(\sqrt{-55})$	55	5	11	0

Table 3.3: Cases with p ramified in K.

In the next two tables we see instances of zeros which we expect are explained by the sign of the action of the level N Atkin-Lehner operator although we have not verified this in detail.

In what follows we illustrate with examples the fact that the quantity $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{g})$ is not always zero.

(*b*) In this example we fix f to be attached to the elliptic curve $E_f\colon y^2 = x^3 + x^2 - 15x + 18$, of conductor $N_f = 120$. The weight-one form g we consider has level $N_g = 120$ also, and has q-expansion

$$g(q) = q + iq^2 + iq^3 - q^4 - iq^5 - q^6 - iq^8 - q^9 + q^{10} - iq^{12} + q^{15} + q^{16} - iq^{18} + iq^{20} + q^{24} - q^{25} - iq^{27} + iq^{30} - 2q^{31} + iq^{32} + O(q^{34}),$$

where $i^2 = -1$. It is the theta series attached to the Dirichlet character ϵ modulo 120

[1]See `http://people.maths.ox.ac.uk/lauder/weight1/`.

defined by

$$\epsilon(97) = -1, \quad \epsilon(31) = 1, \quad \epsilon(41) = -1, \quad \epsilon(61) = -1.$$

The field cut out by ϵ is $K = \mathbb{Q}(\sqrt{-6})$, and we take $p = 5$ which is split in both $L = \mathbb{Q}(\sqrt{-1})$ and K. Note that p divides N_f and N_g. We compute to precision 10 the quantity

$$\mathcal{L}_5^g(\mathbf{f}, \mathbf{g}, \mathbf{g})(2, 1, 1) = 4 \cdot 5 + 3 \cdot 5^2 + 4 \cdot 5^3 + 3 \cdot 5^5 + 4 \cdot 5^6 + 3 \cdot 5^7 + 5^8 + 2 \cdot 5^9 + O(5^{10}).$$

With the same setting, we take $p = 13$ (now p is split in L but inert in K). We obtain

$$\mathcal{L}_{13}^g(\mathbf{f}, \mathbf{g}, \mathbf{g})(2, 1, 1) = 7+3\cdot 13+10\cdot 13^2+13^4+11\cdot 13^5+13^6+6\cdot 13^7+4\cdot 13^8+5\cdot 13^9+O(13^{10})$$

(*c*) Let E_f be the elliptic curve $y^2+y = x^3+x^2+42x-131$ with label `175c1`. It has conductor $N_f = 175$ and rank 0. Let $g = h$ be the theta series of the character ϵ_1 of $K = \mathbb{Q}(\alpha)$ with α satisfying $\alpha^2 - \alpha + 2 = 0$, of discriminant $D_K = -7$ and conductor $5\mathcal{O}_K$ (which is inert, of norm 25), satisfying

$$\epsilon_1(127) = -1, \quad \epsilon_1(101) = -1.$$

The modular form g has q-expansion

$$g(q) = q + iq^2 - iq^7 + iq^8 - q^9 - q^{11} + q^{14} - q^{16} - iq^{18} - iq^{22} - iq^{23} + q^{29} + O(q^{30}),$$

where again $i^2 = 1$. For $p = 13$ (which is inert in K and split in L), we obtain

$$\mathcal{L}_{13}^g(\mathbf{f}, \mathbf{g}, \mathbf{g})(2, 1, 1) = 1 + 3 \cdot 13 + 2 \cdot 13^2 + 13^3 + 12 \cdot 13^4 + 9 \cdot 13^5 + 3 \cdot 13^8 + 5 \cdot 13^9 + O(13^{10}).$$

(*d*) Finally, consider the elliptic curve E_f of conductor 175 from the previous example, and for $g = h$ consider the theta series of another character ϵ_2 of $K = \mathbb{Q}(\alpha)$, $\alpha^2 - \alpha + 2 = 0$, of discriminant $D_K = -7$ and conductor $5\mathcal{O}_K$ (inert, of norm 25), now taking the values

$$\epsilon_2(127) = 1, \quad \epsilon_2(101) = -1.$$

This yields a modular form g with q-expansion

$$g(q) = q + q^2 - q^7 - q^8 + q^9 - q^{11} - q^{14} - q^{16} + q^{18} - q^{22} + q^{23} - q^{29} + O(q^{30}).$$

We numerically obtain for $p = 13$ that

$$\mathcal{L}_{13}^g(\mathbf{f}, \mathbf{g}, \mathbf{g})(2, 1, 1) = 0.$$

Again, we do not have a way to prove that $\mathcal{L}_{13}^g(\mathbf{f}, \mathbf{g}, \mathbf{g})(2, 1, 1)$ is actually zero.

3.4.2 Exotic image case

In the non-CM setting, we have been able to compute the following example. Consider $E_f\colon y^2 = x^3 - 17x - 27$, which has conductor $N_f = 124$. Let g be the modular form of level $N_g = 124$ and projective image A_4, defined as the theta series of the character ϵ of conductor 124 having values

$$\epsilon(65) = \alpha^2 - 1, \quad \epsilon(63) = -1,$$

where α satisfies $\alpha^4 - \alpha^2 + 1 = 0$. The modular form g has q-expansion

$$\begin{aligned} g(q) = {} & q - \alpha^3 q^2 + \left(-\alpha^3 + \alpha\right) q^3 - q^4 + \left(\alpha^2 - 1\right) q^5 - \alpha^2 q^6 + \left(\alpha^3 - \alpha\right) q^7 + \alpha^3 q^8 + \alpha q^{10} - \alpha q^{11} \\ & + \left(\alpha^3 - \alpha\right) q^{12} + \left(-\alpha^2 + 1\right) q^{13} + \alpha^2 q^{14} + \alpha^3 q^{15} + q^{16} - \alpha^2 q^{17} + \left(-\alpha^3 + \alpha\right) q^{19} + \left(-\alpha^2 + 1\right) q^{20} \\ & + \left(\alpha^2 - 1\right) q^{21} + \left(\alpha^2 - 1\right) q^{22} + \alpha^2 q^{24} - \alpha q^{26} + \alpha^3 q^{27} + O(q^{28}). \end{aligned}$$

We let $h = g^*$ be its complex conjugate, and compute with $p = 13$, obtaining

$$\mathcal{L}_{13}^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2, 1, 1) = 1+5\cdot 13+5\cdot 13^2+4\cdot 13^3+6\cdot 13^4+6\cdot 13^5+6\cdot 13^6+13^7+3\cdot 13^8+9\cdot 13^9+9\cdot 13^{10}+O(13^{11}).$$

Chapter 4

Special values of the triple product p-adic L-function and Kolyvagin classes versus non-cristalline diagonal classes

As in the previous chapter, we are now interested in the case of a triple of newforms (f, g, h) of weights $(2, 1, 1)$ whose corresponding complex L-function does not vanish at the central point $s = 1$, and we study the special value $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})(2, 1, 1)$. The main feature of the setting of this chapter, which contains the results of [GR20] is that the elliptic curve attached to f will have *multiplicative reduction* at p, and we will take g and h to be theta series of the same imaginary quadratic field where p is *inert*.

More precisely, let $E/\mathbb{Q}$ be an elliptic curve and let $f \in S_2(N_f)$ be the normalized newform attached to it by modularity. Let K be an imaginary quadratic field of discriminant $-D_K$ relatively prime to N_f. Let

$$\psi_g, \psi_h : \mathbb{A}_K^\times / K^\times \longrightarrow \mathbb{C}^\times$$

be two finite order Hecke characters of K of conductors $\mathfrak{c}_g, \mathfrak{c}_h$ respectively and let $g := \theta(\psi_g)$ and $h := \theta(\psi_h)$ denote the theta series associated to them. Assume that the central characters of g and h are mutually inverse. Then

$$g \in M_1(N_g, \chi), \qquad h \in M_1(N_h, \chi^{-1})$$

are weight one modular forms of level $N_g = D_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}_g), N_h = D_K \cdot N_{K/\mathbb{Q}}(\mathfrak{c}_h)$ respectively. For simplicity we assume throughout that

$$\gcd(N_f, N_g N_h) = 1 \tag{4.0.1}$$

and we set $N := \operatorname{lcm}(N_f, N_g, N_h)$. Let p be an odd prime number such that

$$p \,||\, N_f \;\text{ and }\; p \text{ is inert in } K. \tag{4.0.2}$$

Hence E has multiplicative reduction at p and the completion K_p of K at p is the unramified quadratic extension of $\mathbb{Q}_p$. Recall from §1.6.3.1 Tate's uniformization

$$\varphi_{\mathrm{Tate}} : \bar{\mathbb{Q}}_p^\times / q_E^{\mathbb{Z}} \xrightarrow{\cong} E(\bar{\mathbb{Q}}_p), \tag{4.0.3}$$

with $q_E \in p\mathbb{Z}_p$, and assume throughout that

$$p \nmid \operatorname{ord}_p(q_E). \tag{4.0.4}$$

Set

$$a := a_p(E) = a_p(f) \in \{\pm 1\}.$$

In other words, $a = 1$ (resp. $a = -1$) according to whether E has split (non-split) multiplicative reduction at p.

Let α_g, β_g (resp. α_h, β_h) denote the roots of the p-th Hecke polynomial of g (resp. of h). Note that (4.0.2) implies that

$$\alpha_g = -\beta_g \quad \text{and} \quad \alpha_h = -\beta_h.$$

Since the nebentype character of h is the inverse of that of g, one either has

$$(\alpha_h, \beta_h) = (1/\alpha_g, -1/\alpha_g) \quad \text{or} \quad (\alpha_h, \beta_h) = (-1/\alpha_g, 1/\alpha_g). \tag{4.0.5}$$

Throughout this chapter we fix the ordering of the pair (α_h, β_h) in such a way that

$$\alpha_g \cdot \alpha_h = -a. \tag{4.0.6}$$

As in the previous chapters, $L \subset \bar{\mathbb{Q}}$ denotes the number field generated by the traces of ψ_g and ψ_h, together with the roots $\alpha_g, \beta_g, \alpha_h, \beta_h$. and L_p denotes the completion of L within $\bar{\mathbb{Q}}_p$. Recall the p-stabilisations g_α, h_α of (2.1.4) and the Hida families $\mathbf{f}, \mathbf{g}_\alpha, \mathbf{h}_\alpha$ passing through f, g_α, h_α respectively. As explained in §1.16.4, associated to any choice of Λ-adic test vectors $(\breve{\mathbf{f}}, \breve{\mathbf{g}}_\alpha, \breve{\mathbf{h}}_\alpha)$ of tame level N there is a three-variable p-adic L-function $\mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}_\alpha, \breve{\mathbf{h}}_\alpha)$. As in [DLR15] and in Chapter 2, we are interested in studying the p-adic L-value

$$I_p(f, g_\alpha, h_\alpha) := \mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}_\alpha, \breve{\mathbf{h}}_\alpha)(2, 1, 1), \tag{4.0.7}$$

regarded as an element of $\bar{\mathbb{Q}}_p^\times / L^\times$. As explained in §1.16.4, different choices of Λ-adic test vectors yield the same p-adic value up to algebraic scalars in $L^\times$. This entitles us to denote it simply $I_p(f, g_\alpha, h_\alpha)$, dropping from the notation this choice. Recall that the triple-product L-function $L(f \otimes g \otimes h, s) = L(E \otimes \rho, s)$ may be recast as the Hasse-Weil L-series of the twist of E by the tensor product $\rho_{gh} = \rho_g \otimes \rho_h$ of the two Artin representations associated to g and h. In light of (4.0.1), the order of vanishing of $L(f \otimes g \otimes h, s)$ at $s = 1$ is always even and one thus expects the central critical value $L(E \otimes \rho_{gh}, 1)$ to be generically nonzero. We assume throughout that we fall indeed in this generic case, that is to say, we have

Assumption 4.1. $L(E \otimes \rho_{gh}, 1) \neq 0$.

The main purpose of this chapter is proving a formula relating the p-adic iterated integral $I_p(f, g_\alpha, h_\alpha)$ to the Kolyvagin class associated by Bertolini and Darmon in [BD97]. As we recalled in §1.16.2, given a ring class field H of K of conductor prime to p, the *Kolyvagin class* $\mathbf{K} \in \mathrm{H}^1(H, V_f)$ associated to E/H is attached to a canonical collection of *Heegner points* $\{\alpha_m \in E(F_m)\}_{m \geq 1}$, where F_m/H is the layer of degree p^m within the anticyclotomic $\mathbb{Z}_p$-extension of H.

Fix a prime $\mathfrak{p}$ of H above p and denote $\Phi_{m,\mathfrak{p}}$ the p-primary part of the group of connected components of the Néron model of E over the completion of F_m at the unique prime ideal above $\mathfrak{p}$. Denote $\Phi_{\infty,\mathfrak{p}} := \varprojlim_m \Phi_{m,\mathfrak{p}}$ the projective limit with respect to the natural projection maps $\Phi_{m,\mathfrak{p}} \to \Phi_{m-1,\mathfrak{p}}$.

As explained in Lemma 4.3 there is a canonical isomorphism $\varphi : \Phi_{\infty,\mathfrak{p}} \xrightarrow{\cong} \mathbb{Z}_p$ induced by Tate's uniformization. Denote by $\bar{\alpha}_m$ the image of α_m in $\Phi_{m,\mathfrak{p}}$ and set $\bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty,\mathfrak{p}}$. Define the period

$$\Pi_p := \varphi(\bar{\alpha}) \in \mathbb{Z}_p.$$

Note that $H^1(K_p, V_f)$ is naturally a $\mathrm{Gal}(K_p/\mathbb{Q}_p)$-module and we let $H^1(K_p, V_f)^\pm$ denote the $\mathbb{Q}_p$-subspace on which Frob_p acts with eigenvalue ± 1.

The Kolyvagin class $\mathbf{K}$ is not in general cristalline at p, that is to say, the local class $\mathrm{res}_p(\mathbf{K}) \in H^1(K_p, V_f)$ is not expected to lie in Bloch-Kato's finite subspace $H^1_f(K_p, V_f)$. Nevertheless, one can show (cf. Proposition 4.27) that $\mathrm{res}_p(\mathbf{K})^a := \mathrm{res}_p(\mathbf{K}) + a\,\mathrm{res}_p(\mathbf{K}^{\mathrm{Frob}_p})$ does lie in $H^1_f(K_p, V_f)^a$ and therefore there exists a local point

$$Q_p^a \in (E(K_p) \otimes \mathbb{Q}_p)^a \quad \text{such that} \quad \delta_p(Q_p^a) = \mathrm{res}_p(\mathbf{K})^a, \tag{4.0.8}$$

where $\delta_p : E(K_p) \otimes \mathbb{Q}_p \xrightarrow{\sim} H^1_f(K_p, V_f)$ stands for Kummer's isomorphism. In spite of the notation we have chosen, beware that Q_p^a is *not* expected to be the a-component of any local point $Q_p \in E(K_p) \otimes \mathbb{Q}_p$, precisely because $\mathrm{res}_p(\mathbf{K})$ does not lie in $H^1_f(K_p, V_f)$. In other words, while $\mathrm{res}_p(\mathbf{K})^a$ is cristalline, the local class $\mathrm{res}_p(\mathbf{K})^{-a}$ is not.

Let $c = \prod_{v \nmid N\infty} c_v(\breve{f}, \breve{g}_\alpha, \breve{h}_\alpha) \in L$ denote the product of local automorphic factors appearing in [DLR15, Prop. 2.1 (a) (iii)] associated to the choice of test vectors at $(2, 1, 1)$, and define the algebraic L-value

$$L^{\mathrm{alg}}(E \otimes \rho, 1) := \frac{L(E \otimes \rho, 1)}{\pi^4 \langle f, f \rangle^2}.$$

It follows from the work of Harris and Kudla [HK91] that the above ratio lies in L and is in fact non-zero by (4.1). The following is the main theorem of this note.

Theorem 4.2. *For any choice of test vectors, we have*

$$I_p(f, g_\alpha, h_\alpha) = \frac{\sqrt{c} \cdot \sqrt{L^{\mathrm{alg}}(E \otimes \rho, 1)}}{\Pi_p \cdot \mathcal{L}_{g_\alpha}} \times \log_p(Q_p^a) \qquad (\mod L^\times)$$

where

- $\mathcal{L}_{g_\alpha} \in K_p$ *is a period defined in* (2.1.10)*.*
- $\log_p : E(K_p) \longrightarrow K_p$ *denotes the* p*-adic logarithm.*

The body of this chapter is devoted to the proof of this result. In §4.1 we relate Bloch-Kato's dual exponential map on E to the group of connected components. This is an exercise in p-adic Hodge theory which is surely well-known to experts, but we included because we did not find precise references in the literature. We are most grateful to A. Iovita for his assistance in this section. In §4.2 we review the theory of Hida families and Selmer groups, and culminates with the description of an explicit basis of the relaxed Selmer group associated to the triple (f, g, h). In §4.3 we exploit the fine work [BSV] of Bertolini, Seveso and Venerucci on the exceptional zero phenomena appearing in our scenario in order to prove a formula relating the iterated integral $I_p(\breve{f}, \breve{g}_\alpha, \breve{h}_\alpha)$ in terms of the basis described in the previous section. Finally, in the last section we introduce Kolyvagin classes and prove our main result, which is found in Corollary 4.29.

It would be of great interest to investigate whether the recent p-adic L-functions of Andreatta and Iovita [AI19] could be exploited in order to provide an alternative proof of our formula, although this does not appear to be a straight-forward project.

4.1 p-adic Hodge theory for elliptic curves with multiplicative reduction

Recall by §1.6.3.1 the filtration

$$0 \longrightarrow T_E^+ \xrightarrow{\iota} T_E \xrightarrow{\pi} T_E^- \longrightarrow 0$$

of the Tate module of E, and the corresponding filtration for $V_f = T_E \otimes_{\mathbb{Q}_p} \mathbb{Q}_p$.

4.1.1 The group of connected components

Fix a ring class field H of K of conductor c prime to p and we resume the notation of §1.16.2: for every integer $m \geq 1$, we denote $H(p^m)$ be the ring class field of K of conductor $c \cdot p^m$ and F_m the intermediate field $H \subseteq F_m \subseteq H(p^{m+1})$ such that $\mathrm{Gal}(F_m/H)$ is cyclic of order p^m. Recall that the prime ideal $p\mathcal{O}_K$ splits completely in H. Fix once and for all a prime $\mathfrak{p}$ of H above p, which ramifies totally in $H(p^m)$ as $\mathfrak{p}\mathcal{O}_{F_m} = \mathfrak{p}_m^{p^m}$. Finally, we denote $F_{m,\mathfrak{p}}$ the completion of F_m at $\mathfrak{p}_m$, $\mathcal{O}_{m,\mathfrak{p}}$ its ring of integers and $\mathbb{F}_{m,\mathfrak{p}}$ its residue field.

Let $\mathcal{E}$ be the Néron model of E over $\mathbb{Z}_p$, and let $\tilde{\mathcal{E}} := \mathcal{E} \times_{\mathbb{Z}_p} \mathbb{F}_p$ denote its special fiber. For all $m \geq 1$, let $\Phi_{m,\mathfrak{p}}$ denote the p-Sylow subgroup of the group

$$\tilde{\mathcal{E}}(\mathbb{F}_{m,\mathfrak{p}})/\tilde{\mathcal{E}}_0(\mathbb{F}_{m,\mathfrak{p}}) \cong E(F_{m,\mathfrak{p}})/E_0(F_{m,\mathfrak{p}}) \tag{4.1.1}$$

of connected components of the base change of $\mathcal{E}$ to $\mathcal{O}_{m,\mathfrak{p}}$.

Lemma 4.3. *There are isomorphisms of $G_{\mathbb{Q}_p}$-modules*

$$T_E^- \;\cong\; \mathbb{Z}_p(\chi_E) \;\cong\; \Phi_{\infty,\mathfrak{p}}. \tag{4.1.2}$$

Proof. The first isomorphism is (1.6.6). Tate's uniformisation provides a description of the group (4.1.1) as

$$\overline{\varphi}_{\mathrm{Tate}} : F_{m,\mathfrak{p}}^\times/q_E^{\mathbb{Z}}\mathcal{O}_{m,\mathfrak{p}}^\times(\chi_E) \cong (\mathbb{Z}/\mathrm{ord}_{m,\mathfrak{p}}(q_E)\mathbb{Z})\,(\chi_E) \xrightarrow{\cong} E(F_{m,\mathfrak{p}})/E_0(F_{m,\mathfrak{p}}), \tag{4.1.3}$$

where $\mathrm{ord}_{m,\mathfrak{p}}$ is the discrete valuation on $\mathcal{O}_{m,\mathfrak{p}}$. We have $p\mathcal{O}_{F_m} = \prod_{\mathfrak{p}_m|p} \mathfrak{p}_m^{p^m}$, where $\mathfrak{p}_m = (\pi_m)$ is the maximal ideal of $\mathcal{O}_{m,\mathfrak{p}}$. Hence $q_E = p^n\alpha = \pi_m^{np^m}\alpha'$ where $\alpha, \alpha' \in \mathcal{O}_{F_{\mathfrak{p},m}}^\times$, i.e.

$$\mathrm{ord}_{m,\mathfrak{p}}(q_E) = n \cdot p^m.$$

Under condition (4.0.4), then (4.1.3) gives the isomorphism

$$\overline{\varphi}_{\mathrm{Tate}} : (\mathbb{Z}/np^m\mathbb{Z})\,(\chi_E) \cong (\mathbb{Z}/n\mathbb{Z})\,(\chi_E) \oplus (\mathbb{Z}/p^m\mathbb{Z})\,(\chi_E) \xrightarrow{\cong} E(F_{m,\mathfrak{p}})/E_0(F_{m,\mathfrak{p}}). \tag{4.1.4}$$

The lemma follows after taking p-primary parts and passing to the inverse limit.

□

4.1.2 The G_{K_p}-cohomology of E

Recall from the first example of §1.3.1 the decomposition

$$G_{K_p}^{\mathrm{ab}} = \Gamma_{\mathrm{nr}} \times \Gamma_{\mathrm{cyc}} \times \Gamma_{\mathrm{ant}} \times \mathrm{Gal}(K'/K)$$

of the Galois group of the maximal abelian extension of K_p over K_p. Recall the Artin map

$$\mathrm{Art} : K_p^\times \longrightarrow G_{K_p}^{\mathrm{ab}} \tag{4.1.5}$$

denote the Artin map which gives an isomorphism once extended to the profinite completion $\widehat{K}_p^\times$ of $K_p^\times$. As explained in §1.3.1, the decomposition

$$K_p^\times = p^{\mathbb{Z}} \oplus (1 + p\mathcal{O}_{K_p}) \oplus \left(\mathcal{O}_{K_p}/p\mathcal{O}_{K_p}\right)^\times,$$

corresponds via (4.1.5) to

$$G_{K_p}^{\mathrm{ab}} \cong \mathrm{Gal}(K_p^{\mathrm{nr}}/K_p) \times \mathrm{Gal}(K_\infty/K_p) \times \mathrm{Gal}(K_p'/K_p)$$

where K_p^{nr} is the maximal unramified extension of K_p, K_∞/K_p is a $\mathbb{Z}_p^2$-extension and K_p'/K_p is finite. More precisely, via the Artin map, $p^{\widehat{\mathbb{Z}}} \cong \mathrm{Gal}(K_p^{\mathrm{nr}}/K_p)$. Fix a prime $\mathfrak{p}$ of H above p, and denote

$$K_p(\mu_{p^\infty}) := \varinjlim K_p(\mu_{p^m}); \qquad H(p^\infty)_{\mathfrak{p}} := \varinjlim H(p^m)_{\mathfrak{p}_m},$$

where $\mathfrak{p}_m$ is the unique prime of $H(p^m)$ lying above $\mathfrak{p}$. Recall also the $\mathbb{Q}_p$-basis

$$\{\xi_{\mathrm{nr}}, \xi_{\mathrm{cyc}}, \xi_{\mathrm{ant}}\}$$

of $\mathrm{Hom}_{\mathrm{cont}}(G_{K_p}^{\mathrm{ab}}, \mathbb{Q}_p)$. After composing with the isomorphism $\mathbb{Q}_p \overset{\overline{\varphi}_{\mathrm{Tate}}}{\longrightarrow} V^-_{f|G_{K_p}}$ provided by Tate's uniformization this further yields a $\mathbb{Q}_p$-basis of $\mathrm{H}^1(K_p, V_f^-)$, which in turn may be regarded as a basis of $\mathrm{H}^1(K_p, \Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p)$ by means of Lemma 4.3.

For a class $\xi \in \mathrm{H}^1(K_p, V_f^-)$, we denote $\bar{\xi}$ its image in the singular quotient $\mathrm{H}^1_s(K_p, V_f^-)$.

Lemma 4.4. *$\{\bar{\xi}_{\mathrm{cyc}}, \bar{\xi}_{\mathrm{ant}}\}$ is a $\mathbb{Q}_p$-basis for $\mathrm{H}^1_s(K_p, V_f^-)$.*

Proof. By definition, the submodule $\mathrm{H}^1_g(K_p, \mathbb{Q}_p) = \mathrm{H}^1_f(K_p, \mathbb{Q}_p)$ of $\mathrm{H}^1(K_p, \mathbb{Q}_p) = \mathrm{Hom}_{\mathrm{cont}}(G_{K_p}, \mathbb{Q}_p)$ is given by $\mathrm{H}^1_f(K_p, \mathbb{Q}_p(\chi_E)) \cong \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{nr}}, \mathbb{Q}_p)$ and is generated by ξ_{nr}. The lemma follows because $\mathrm{H}^1_s(K_p, \mathbb{Q}_p) = \mathrm{H}^1_s(K_p, V_f^-) = \mathrm{H}^1(K_p, V_f^-)/\,\mathrm{H}^1_f(K_p, V_f^-)$. □

Write the uniformizer of the elliptic curve $E/\mathbb{Q}_p$ as

$$q_E = p^n u^s x \in p\mathbb{Z}_p, \qquad n \geq 1, s \in \mathbb{Z}_p, x \in \mu_{p-1}, \tag{4.1.6}$$

so that $n = \mathrm{ord}_p(q_E)$ and $ps = \log_p(q_E)$. Let

$$\pi_* : \mathrm{H}^1(K_p, V_f) \longrightarrow \mathrm{H}^1(K_p, V_f^-) \tag{4.1.7}$$

be the morphism induced in G_{K_p}-cohomology by the projection $\pi : V_f \longrightarrow V_f^-$.

For a $\mathrm{Gal}(K_p/\mathbb{Q}_p)$-module M, let us write $M^\pm$ for the eigenspace on which Frob_p acts as ± 1. Set $a := a_p(E)$ as in the introduction.

Proposition 4.5. *Let $x_{\mathrm{cyc}}, x_{\mathrm{ant}} \in \mathrm{H}^1(K_p, V_f)$ be elements such that*

$$\pi_*(x_{\mathrm{cyc}}) = n\xi_{\mathrm{cyc}} - s\xi_{\mathrm{nr}}, \qquad \pi_*(x_{\mathrm{ant}}) = \xi_{\mathrm{ant}} \ \text{ in } \ \mathrm{H}^1(K_p, V_f^-). \tag{4.1.8}$$

Then $\{\bar{x}_{\mathrm{cyc}}, \bar{x}_{\mathrm{ant}}\}$ is a $\mathbb{Q}_p$-basis for $\mathrm{H}^1_s(K_p, V_f)$ and π_* descends to an isomorphism

$$\bar{\pi}_* : \mathrm{H}^1_s(K_p, V_f) \overset{\cong}{\longrightarrow} \mathrm{H}^1_s(K_p, V_f^-), \qquad \bar{x}_{\mathrm{cyc}} \mapsto n \cdot \bar{\xi}_{\mathrm{cyc}}, \qquad \bar{x}_{\mathrm{ant}} \mapsto \bar{\xi}_{\mathrm{ant}}.$$

Moreover, the Frobenius eigenspaces $\mathrm{H}^1_s(K_p, V_f)^a$ and $\mathrm{H}^1_s(K_p, V_f)^{-a}$ are generated respectively by $\bar{x}_{\mathrm{cyc}}$ and $\bar{x}_{\mathrm{ant}}$.

Proof. Consider the long exact sequence

$$0 \to V_f^- \overset{\partial^0}{\longrightarrow} \mathrm{H}^1(K_p, V_f^+) \overset{\iota_*}{\longrightarrow} \mathrm{H}^1(K_p, V_f) \overset{\pi_*}{\longrightarrow} \mathrm{H}^1(K_p, V_f^-) \overset{\partial^1}{\longrightarrow} \mathrm{H}^2(K_p, V_f^+) \to 0 \tag{4.1.9}$$

induced in G_{K_p}-cohomology by (1.6.7). The connecting homomophisms ∂^0, ∂^1 can be expressed in terms of q_E as follows. Cup product and the trace map give a pairing

$$\langle\ ,\ \rangle : \mathrm{H}^1(K_p, V_f^+) \times \mathrm{H}^1(K_p, V_f^-) \longrightarrow \mathrm{H}^2(K_p, V_f^+) \cong \mathbb{Q}_p$$

satisfying

$$\langle \delta_p(q_E), \xi \rangle = \xi(\mathrm{Art}(q_E))$$

for all $\xi \in \mathrm{H}^1(K_p, V_f^-)$. If we still call ∂^1 its composition with $\mathrm{H}^2(K_p, V_f^+) \cong \mathbb{Q}_p$, then it coincides with the map $\langle \delta_p(q_E), \cdot \rangle$. Using the notation of (4.1.6), we have $\mathrm{Art}(q_E) = \mathrm{Frob}_p^n\, \sigma_{\mathrm{cyc}}^s$ and thus

1. $\partial^1(\xi_{\mathrm{cyc}}) = \xi_{\mathrm{cyc}}(\mathrm{Art}(q_E)) = \xi_{\mathrm{cyc}}(\sigma_{\mathrm{cyc}}^s) = \log_p(\chi_{\mathrm{cyc}}(\sigma_{\mathrm{cyc}}^s) = s\log_p(u) = sp$;
2. $\partial^1(\xi_{\mathrm{ant}}) = \xi_{\mathrm{ant}}(\mathrm{Art}(q_E)) = \log_p(\chi_{\mathrm{ant}}(\sigma_{\mathrm{ant}}^0)) = 0$;
3. $\partial^1(\xi_{\mathrm{nr}}) = \xi_{\mathrm{nr}}(\mathrm{Art}(q_E)) = \xi_{\mathrm{nr}}(\mathrm{Frob}_p^n) = \log_p(\chi_{\mathrm{nr}}(\sigma_{\mathrm{cyc}})^n) = n\log_p(u) = np$.

Combining these computations with the exactness of (4.1.9) we deduce that the image of π_* is generated by $\{n\cdot\xi_{\mathrm{cyc}} - s\cdot\xi_{\mathrm{nr}}, \xi_{\mathrm{ant}}\}$, and this gives the existence of $x_{\mathrm{cyc}}, x_{\mathrm{ant}}$ as in the statement.

From (4.1.8) and Lemma 4.4 we deduce that $\bar{\pi}_*(\bar{x}_{\mathrm{cyc}}) = n\cdot\bar{\xi}_{\mathrm{cyc}}, \bar{\pi}_*(\bar{x}_{\mathrm{ant}}) = \bar{\xi}_{\mathrm{ant}}$ and that $\bar{\pi}_*$ is surjective. The map ι_* of (4.1.9) restricts to an isomorphism $\iota_* : \mathrm{H}^1_f(K_p, V_f^+) \cong \mathrm{H}^1_f(K_p, V_f)$, hence $\bar{\pi}_*$ is also injective.

Finally, in order to understand the action of Frobenius, note that $\mathrm{H}^1_s(K_p, V_f) \simeq \mathrm{H}^1_s(K_p, V_f^-)$ is naturally a $\mathrm{Gal}(K_p/\mathbb{Q}_p)$-module. As explained in the proof of Lemma 4.4, $\{\xi_{\mathrm{cyc}}, \xi_{\mathrm{ant}}\}$ is a basis of

$$\mathrm{H}^1_s(K_p, V_f^-) \cong \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p(\chi_E)) \oplus \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p(\chi_E)). \tag{4.1.10}$$

We have

$$\xi_{\mathrm{ant}}^{\mathrm{Frob}_p}(\sigma_{\mathrm{ant}}) = \chi_E(\mathrm{Frob}_p)\cdot\xi_{\mathrm{ant}}(\mathrm{Frob}_p\,\sigma_{\mathrm{ant}}\,\mathrm{Frob}_p^{-1}) = a\cdot\xi_{\mathrm{ant}}(\sigma_{\mathrm{ant}}^{-1}) = -a\cdot\xi_{\mathrm{ant}}(\sigma_{\mathrm{ant}})$$

and via (4.1.10) it follows that

$$\mathrm{H}^1_s(K_p, V_f^-)^{-a} \cong \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p(\chi_E))$$

is generated by $\bar{\xi}_{\mathrm{ant}}$. Analogously, $\bar{\xi}_{\mathrm{cyc}}$ generates $\mathrm{H}^1_s(K_p, V_f^-)^a \cong \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p(\chi_E))$.

□

4.1.3 Bloch–Kato logarithm and dual exponential maps

We wish to describe explicitly the Bloch–Kato logarithm and the dual exponential maps relative to the representation V_f regarded as a G_{K_p}-module. First we need to study $\mathrm{D_{dR}}(V_{K_p})$, $\mathrm{D_{dR}}(V_{K_p}^{\pm})$ and their filtration, where $V_{K_p} := V_{f|G_{K_p}}$. Let $\mathbb{C}_p$ be the completion of a fixed algebraic closure of $\mathbb{Q}_p$, and let

$$\mathcal{R} := \varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}.$$

Let $W(\mathcal{R})$ be the ring of Witt vectors of $\mathcal{R}$ and denote $[\cdot] : \mathcal{R} \longrightarrow W(\mathcal{R})$ the Teichmuller lift. There is an isomorphism

$$\varprojlim_{x\mapsto x^p} \mathcal{O}_{\mathbb{C}_p} \xrightarrow{\cong} \mathcal{R}, \qquad (x^{(n)})_n \mapsto (x_n := x^{(n)} \mod p)_n.$$

Recall the basis $\tilde{\varepsilon} := (\tilde{\varepsilon}^{(m)})_m$ of $\mathbb{Z}_p(1)$ we introduced in §4.1.1. It can be regarded as an element of $\lim_{x\mapsto x^p}\mathcal{O}_{\mathbb{C}_p} \cong \mathcal{R}$, and if $\log_{q_E}$ denotes the branch of the p-adic logarithm such that $\log_{q_E}(q_E) = 0$, then

$$t := \log_{q_E}([\tilde{\varepsilon}])$$

generates the maximal ideal $\mathrm{Fil}^1\,\mathrm{B_{dR}}$ of $\mathrm{B}^+_{\mathrm{dR}} = \mathrm{Fil}^0\,\mathrm{B_{dR}}$. Analogously, we can regard $\tilde{q}$ as an element of $\mathcal{R}$ and we define

$$q_E^{\#} := \log_{q_E}([\tilde{q}]/q_E) \in \mathrm{Fil}^1\,\mathrm{B_{dR}}$$

Recall that the images ε, q of $\tilde{\varepsilon}, \tilde{q}$ under Tate's uniformization form a basis of V_f. By elementary calculations one obtains the following lemma.

Lemma 4.6. *Let*

$$e_1 := q \otimes 1 - \varepsilon \otimes \frac{q_E^{\#}}{t} \in V_f \otimes \mathrm{Fil}^0 \mathrm{B}_{\mathrm{dR}}, \quad e_2 := \varepsilon \otimes \frac{1}{t} \quad \in V_f \otimes \mathrm{Fil}^{-1} \mathrm{B}_{\mathrm{dR}}.$$

These elements are invariant with respect to the action of G_{K_p}, they form a K_p-basis of $\mathrm{D}_{\mathrm{dR}}(V_{K_p})$ and e_1 is a K_p-basis of $\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p})$. Moreover, there are isomorphisms

$$\iota_* : \mathrm{D}_{\mathrm{dR}}(V_{K_p}^+)/\,\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}^+) \xrightarrow{\cong} \mathrm{D}_{\mathrm{dR}}(V_{K_p})/\,\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}), \qquad \varepsilon \otimes t^{-1} \mapsto e_2; \tag{4.1.11}$$

$$\pi_* : \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}) \xrightarrow{\cong} \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}^-), \qquad e_1 \mapsto \bar{q}, \tag{4.1.12}$$

where, with a slight abuse of notation, $\bar{q} := \bar{q} \otimes 1$ in $(V_f^- \otimes \mathrm{B}_{\mathrm{dR}}^+)^{G_{K_p}} = \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_f^-)$.

Kummer theory provides isomorphisms

$$\delta_p : \mathcal{O}_{K_p}^\times \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{H}^1_f(K_p, \mathbb{Q}_p(1)), \quad E(K_p) \otimes \mathbb{Q}_p \xrightarrow{\cong} \mathrm{H}^1_f(K_p, V_f), \tag{4.1.13}$$

which combined with (1.6.6) and Tate's uniformization yield canonical identifications of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$\mathcal{O}_{K_p}^\times(\chi_E) \otimes \mathbb{Q}_p \cong \mathrm{H}^1_f(K_p, V_f^+) \overset{\iota_*}{\cong} \mathrm{H}^1_f(K_p, V_f) = E(K_p) \otimes \mathbb{Q}_p. \tag{4.1.14}$$

Besides, the p-adic logarithm induces a morphism of $\mathbb{Q}_p[G_{\mathbb{Q}_p}]$-modules

$$\log_p : \mathcal{O}_{K_p}^\times(\chi_E) \otimes \mathbb{Q}_p \longrightarrow K_p(\chi_E). \tag{4.1.15}$$

Via the identifications (4.1.14), it gives rise to the map

$$\log_E \colon E(K_p) \longrightarrow K_p(\chi_E), \tag{4.1.16}$$

which is the formal group logarithm on E associated with the invariant differential ω_f on E introduced in §1.8. Finally, (4.1.14) and the isomorphism

$$\mathrm{D}_{\mathrm{dR}}(V_{K_p})/\,\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}) \xrightarrow{\cong} K_p(\chi_E), \qquad e_2 \mapsto 1,$$

allow us to identify $\log_E$ with the composition of the Bloch–Kato logarithm

$$\log : \mathrm{H}^1_f(K_p, V_f) \longrightarrow \mathrm{D}_{\mathrm{dR}}(V_{K_p})/\,\mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(V_{K_p}) \tag{4.1.17}$$

with the pairing with ω_f.

For a p-adic de Rham representation W of G_{K_p}, let

$$\gamma : \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(W) = (W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}^+)^{G_{K_p}} \longrightarrow \mathrm{H}^1(K_p, W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}^+)$$

be the isomorphism defined by $x \mapsto \gamma(x)$, where $\gamma(x)$ is the cohomology class of the map

$$\sigma \mapsto x \cdot \log_p(\chi_{\mathrm{cyc}}(\sigma)). \tag{4.1.18}$$

Definition 4.7. The *Bloch–Kato dual exponential* of W is the map

$$\exp^* : \mathrm{H}^1(K_p, W) \xrightarrow{\alpha_*} \mathrm{H}^1(K_p, W \otimes_{\mathbb{Q}_p} \mathrm{B}_{\mathrm{dR}}^+) \xrightarrow{\gamma^{-1}} \mathrm{Fil}^0 \mathrm{D}_{\mathrm{dR}}(W).$$

The map $\mathrm{H}^1(K_p, W\otimes\mathrm{B}_{\mathrm{dR}}^+) \longrightarrow \mathrm{H}^1(K_p, W\otimes\mathrm{B}_{\mathrm{dR}})$ is injective and thus $\ker(\exp^*) = \mathrm{H}^1_g(K_p, W)$. We shall regard the dual exponential associated to W as the map

$$\exp^* : \mathrm{H}^1_s(K_p, W) = \mathrm{H}^1(K_p, W)/\,\mathrm{H}^1_g(K_p, W) \longrightarrow \mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(W). \tag{4.1.19}$$

Similarly as in the case of the Block–Kato logarithm, the dual exponential for $W = K_p$ together with the identification $\mathrm{H}^1_s(K_p, \mathbb{Q}_p)(\chi_E) = \mathrm{H}^1_s(K_p, \mathbb{Q}_p(\chi_E))$ gives rise to an isomorphism of $K_p[G_{\mathbb{Q}_p}]$-modules

$$\exp^* : \mathrm{H}^1_s(K_p, \mathbb{Q}_p(\chi_E)) \longrightarrow K_p(\chi_E). \tag{4.1.20}$$

It coincides with the dual exponential

$$\exp^* : \mathrm{H}^1_s(K_p, V) \longrightarrow K_p(\chi_E) \tag{4.1.21}$$

via the identifications

$$\mathrm{H}^1_s(K_p, \mathbb{Q}_p(\chi_E)) \cong \mathrm{H}^1_s(K_p, V^-) \cong \mathrm{H}^1_s(K_p, V)$$

explained in the previous section, where

$$\mathrm{Fil}^0\,\mathrm{D}_{\mathrm{dR}}(V_{K_p}) \xrightarrow{\cong} K_p(\chi_E), \qquad e_1 \mapsto 1.$$

By Proposition 1.15 we have

$$\exp^*(\xi_{\mathrm{cyc}}) = 1 \quad \text{and} \quad z := \exp^*(\xi_{\mathrm{ant}}) \in K_p^-. \tag{4.1.22}$$

The map (4.1.21) and its $\pm$-components can be explicitly described as follows.

Proposition 4.8. *The dual exponential* $\exp^* : \mathrm{H}^1_s(K_p, V_f) \longrightarrow K_p(\chi_E)$ *is characterized by*

$$\exp^*(\bar{x}_{\mathrm{cyc}}) = n, \quad \exp^*(\bar{x}_{\mathrm{ant}}) = z \in K_p^-.$$

Proof. By (the proof of) Proposition 4.5,

$$\mathrm{H}^1_s(K_p, \mathbb{Q}_p(\chi_E))^a \oplus \mathrm{H}^1_s(K_p, \mathbb{Q}_p(\chi_E))^{-a} \cong \mathrm{Hom}(\Gamma_{\mathrm{cyc}}, \mathbb{Q}_p(\chi_E)) \oplus \mathrm{Hom}(\Gamma_{\mathrm{ant}}, \mathbb{Q}_p(\chi_E)),$$

whose basis $\{\xi_{\mathrm{cyc}}, \xi_{\mathrm{ant}}\}$ is compatible with the above decomposition.

The claim follows by using Lemma 4.6, Proposition 4.5 and the description of (4.1.21) discussed before this proposition. ☐

Corollary 4.9. *There is a commutative diagram whose arrows are isomorphisms of $G_{\mathbb{Q}_p}$-modules:*

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{cyc}}, \Phi_{\infty,\mathfrak{p}}) \oplus \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,\mathfrak{p}}) & \xrightarrow{\varphi} & \mathrm{H}^1_s(K_p, V_f) \\
\Big\downarrow {\scriptstyle \frac{1}{p}\mathrm{ev}_{\mathrm{cyc}} \oplus \frac{1}{p}\mathrm{ev}_{\mathrm{ant}}} & & \Big\downarrow {\scriptstyle \exp^*} \\
(\Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p) \oplus (\Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p) & \xrightarrow{\varrho} & K_p(\chi_E)
\end{array}$$

where

1. $\varphi := (\bar{\varphi}_{\mathrm{Tate}} \circ \bar{\pi}_*)^{-1}$*;*
2. $\varrho(\bar{q}, 0) = 1$*,* $\varrho(0, \bar{q}) = z$*, where* $z = \exp^*(\xi_{\mathrm{ant}}) \in K_p^-$ *is as in Proposition 4.8;*
3. $\mathrm{ev}_{\mathrm{cyc}}$ *denotes the valuation at* $\sigma_{\mathrm{cyc}} \in \Gamma_{\mathrm{cyc}}$*, and analogously for* $\mathrm{ev}_{\mathrm{ant}}$*.*

Moreover, φ *decomposes with respect to the action of* Frob_p *as*

$$\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{cyc}}, \Phi_{\infty,\mathfrak{p}}) \longrightarrow \mathrm{H}^1_s(K_p, V_f)^a, \qquad \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,\mathfrak{p}}) \longrightarrow \mathrm{H}^1_s(K_p, V_f)^{-a}. \tag{4.1.23}$$

Proof. By definition we have

$$\xi_{\mathrm{cyc}}(\sigma_{\mathrm{cyc}}) = \overline{\varphi}_{\mathrm{Tate}}(\log_p(\chi_{\mathrm{cyc}}(\sigma_{\mathrm{cyc}}))) = \overline{\varphi}_{\mathrm{Tate}}(\log_p(u)) = \overline{\varphi}_{\mathrm{Tate}}(p) = p \cdot \bar{q}$$

and similarly, $\xi_{\mathrm{ant}}(\sigma_{\mathrm{ant}}) = p \cdot \bar{q}$. The statement follows by combining (the proof of) Proposition 4.8, Proposition 4.5 and Lemma 4.3. ☐

4.2 The Selmer groups associated to (f, g, h)

Let now (f, g, h) be the triple of eigenforms of weights $(2, 1, 1)$ introduced at the beginning of the section. Recall Assumption 3.2 imposed on $V := V_f \otimes V_g \otimes V_h$ in the introduction, which implies

$$\mathrm{Sel}_p(V) = 0.$$

Indeed, as discussed in detail in §4.4.1, the representation V decomposes as a direct sum

$$V = [V_f \otimes V_{\psi_1}] \oplus [V_f \otimes V_{\psi_2}],$$

where, for $i = 1, 2$, the $G_{\mathbb{Q}}$-representation V_{ψ_i} is induced by a ring class character ψ_i of the imaginary quadratic field K. This decomposition induces factorisations

$$L(E, \rho, s) = L(E/K, \psi_1, s) \cdot L(E/K, \psi_2, s), \qquad \mathrm{Sel}_p(V) = \mathrm{Sel}_p(E \otimes \psi_1) \oplus \mathrm{Sel}_p(E \otimes \psi_2).$$

Combining the factorisation of the complex L-function with Assumption 3.2 we obtain that $L(E/K, \psi_i, 1) \neq 0$ for $i = 1, 2$, which, by [BD97, Theorem B], implies that $\mathrm{Sel}_p(E \otimes \psi_1) = \mathrm{Sel}_p(E \otimes \psi_2) = 0$.

Lemma 4.10. *There is an isomorphism*

$$\partial_p : \mathrm{Sel}_{(p)}(V) \xrightarrow{\cong} \mathrm{H}^1_s(\mathbb{Q}_p, V)$$

Proof. This follows from Poitou–Tate duality as in [MR04, Theorem 2.3.4], where a similar statement is proved, under the (irrelevant) assumption that $p \nmid N_f$. (See also the analogous statements proved in the previous chapter, and Remark 3.3). □

The characteristic polynomial for the action of Frob_p on V_g is $x^2 - a_p(g)x + \chi(p) = x^2 - \varepsilon(p)$ and we may thus write the eigenvalues of Frob_p as $\alpha_g = \lambda$, $\beta_g = -\lambda$ for some root of unity λ. The same holds for h and since its nebentype is the inverse of that of g we have

$$(\alpha_h, \beta_h) = (1/\lambda, -1/\lambda) \quad \text{or} \quad (\alpha_h, \beta_h) = (-1/\lambda, 1/\lambda).$$

Using the notation of §1.8.2, denote $\{v_g^\alpha, v_g^\beta\}$ and $\{v_h^\alpha, v_h^\beta\}$ a pair of L-bases for V_g and V_h consisting of eigenvectors with eigenvalue $\alpha_g, \beta_g, \alpha_h, \beta_h$ respectively. Denote V_g^α, V_g^β, etc. the corresponding eigenspaces. Set $V_{gh}^{\alpha\alpha} := V_g^\alpha \otimes V_h^\alpha$ and $V^{\alpha\alpha} := V_f \otimes V_{gh}^{\alpha\alpha}$, and likewise for the remaining pairs of eigenvalues. There is a decomposition of $L_p[G_{\mathbb{Q}_p}]$-modules

$$V := V_{fgh} = V^{\alpha\alpha} \oplus V^{\alpha\beta} \oplus V^{\beta\alpha} \oplus V^{\beta\beta}. \tag{4.2.1}$$

Lemma 4.11. *Let $\triangle, \heartsuit \in \{\alpha, \beta\}$. Then*

i) the Bloch–Kato logarithm gives an isomorphism

$$\log_{\triangle\heartsuit} : \mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) = \mathrm{H}^1_f(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) \xrightarrow{\cong} L_p;$$

ii) the Bloch–Kato dual exponential gives an isomorphism

$$\exp^*_{\triangle\heartsuit} : \mathrm{H}^1_s(\mathbb{Q}_p, V^{\triangle\heartsuit}) = \mathrm{H}^1_s(\mathbb{Q}_p, V_f^- \otimes V_{gh}^{\triangle\heartsuit}) \xrightarrow{\cong} L_p; .$$

Proof. Let $W := V_{gh}^{\triangle\heartsuit}$. By [DRb, Lemma 2.4.1]

$$\mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) = \begin{cases} \mathrm{H}^1_f(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) & \text{if } \triangle \cdot \heartsuit \cdot a = +1 \\ \mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) & \text{if } \triangle \cdot \heartsuit \cdot a = -1. \end{cases} \tag{4.2.2}$$

Moreover, in the case in which $\triangle \cdot \heartsuit \cdot a = -1$, we have

$$\mathrm{H}^1(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) = \mathrm{H}^1_f(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit})$$

(see §1.3.1). Since $V_f^+ \cong \mathbb{Q}_p(1)(\chi_E)$, $i)$ follows from the fact that

$$V_f^+ \otimes V_{gh}^{\triangle\heartsuit} = L_p(\zeta\chi_{\mathrm{cyc}})$$

where $\zeta : G_{\mathbb{Q}_p} \longrightarrow \mathbb{Q}_p^\times$ is the unramified character given by $\zeta(\mathrm{Frob}_p) = \triangle\cdot\heartsuit\cdot a$ (for more details, see see §1.3.1). The statement on the singular quotients is proven similarly using (4.2.2) and the fact that

$$V_f^- \otimes V_{gh}^{\triangle\heartsuit} = L_p(\zeta).$$

□

Set $L_p^4 = L_p \oplus L_p \oplus L_p \oplus L_p$. We obtain from the previous lemma isomorphisms

$$\log : \mathrm{H}^1_f(\mathbb{Q}_p, V) \longrightarrow L_p^4, \qquad \exp^* : \mathrm{H}^1_s(\mathbb{Q}_p, V) \longrightarrow L_p^4.$$

where $\log = \log_{\alpha\alpha} \oplus \log_{\alpha\beta} \oplus \log_{\beta\alpha} \oplus \log_{\beta\beta}$ and $\exp^* = \exp^*_{\alpha\alpha} \oplus \exp^*_{\alpha\beta} \oplus \exp^*_{\beta\alpha} \oplus \exp^*_{\beta\beta}$.

Combining Lemma 4.10 with Lemma 4.11 we obtain the following result on the L_p-structure of the relaxed Selmer group attached to V.

Corollary 4.12. *There are isomorphisms*

$$\mathrm{Sel}_{(p)}(V) \xrightarrow{\partial_p} \mathrm{H}^1_s(\mathbb{Q}_p, V) \xrightarrow{\exp^*} L_p^4.$$

We now turn to pin down a particular basis of $\mathrm{Sel}_{(p)}(V)$ which is canonical up to multiplication by elements in $L^\times$. In order to do this, we need a more precise description of the finite and singular parts of $\mathrm{H}^1(\mathbb{Q}_p, V^{\heartsuit\triangle})$ in terms of $\mathrm{H}^1(K_p, V_f)$. Note that the latter space is equipped with a natural action of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$.

In what follows, for any $\mathrm{Gal}(K_p/\mathbb{Q}_p)$-module M we denote $M^\pm$ the subspace of M on which Frob_p acts as multiplication by ± 1. Note that $\alpha_g\alpha_h, \alpha_g\beta_h, \beta_g\alpha_h, \beta_g\beta_h \in \{\pm 1\}$ and thus it makes sense to consider $M^{\alpha_g\alpha_h}$, etc.

Lemma 4.13. *Let $\heartsuit, \triangle \in \{\alpha, \beta\}$. There are canonical isomorphisms of L_p-vector spaces*

$$\mathrm{H}^1_f(\mathbb{Q}_p, V_f \otimes V_{gh}^{\heartsuit\triangle}) \cong \mathrm{H}^1_f(K_p, V_f)^{\triangle\heartsuit} \otimes V_{gh}^{\heartsuit\triangle} \tag{4.2.3}$$

and

$$\mathrm{H}^1_s(\mathbb{Q}_p, V_f \otimes V_{gh}^{\heartsuit\triangle}) \cong \mathrm{H}^1_s(K_p, V_f)^{\triangle\heartsuit} \otimes V_{gh}^{\heartsuit\triangle}. \tag{4.2.4}$$

Proof. Let χ be the quadratic character of $G_{\mathbb{Q}_p}$ with $\chi(\mathrm{Frob}_p) = -1$. We describe the isomorphisms in the case in which E has split multiplicative reduction at p and $\triangle \cdot \heartsuit = -1$. In this setting,

$$\begin{aligned}
\mathrm{H}^1_f(\mathbb{Q}_p, V^{\triangle\heartsuit}) &= \mathrm{H}^1_f(\mathbb{Q}_p, V_f^+ \otimes V_{gh}^{\triangle\heartsuit}) && \text{by Lemma 4.11}\\
&= \mathrm{H}^1_f(\mathbb{Q}_p, \mathbb{Q}_p(1) \otimes L_p(\chi)) && \text{by §1.6.3.1}\\
&= \mathrm{H}^1_f(K_p, \mathbb{Q}_p(1) \otimes L_p(\chi))^{G_{\mathbb{Q}_p}} && \text{by the inflation-restriction exact sequence}\\
&= (\mathrm{H}^1_f(K_p, \mathbb{Q}_p(1)) \otimes L_p(\chi))^{G_{\mathbb{Q}_p}} && \text{because } \chi_{|G_{K_p}} = 1\\
&= (\mathrm{H}^1_f(K_p, V_f) \otimes L_p(\chi))^{G_{\mathbb{Q}_p}} && \text{by (4.1.14).}
\end{aligned}$$

The claim follows, because $V_{gh}^{\triangle\heartsuit} = L_p(\chi)$ as $G_{\mathbb{Q}_p}$-modules and the subspace of $G_{\mathbb{Q}_p}$-invariants of $\mathrm{H}^1_f(K_p, V_f) \otimes V_{gh}^{\triangle\heartsuit}$ is $\mathrm{H}^1_f(K_p, V_f)^{\triangle\heartsuit} \otimes V_{gh}^{\heartsuit\triangle}$. Using similar computations as above, one also verifies that

$$\mathrm{H}^1_s(\mathbb{Q}_p, V^{\triangle\heartsuit}) \cong (\mathrm{H}^1(K_p, V_f) \otimes V_{gh}^{\triangle\heartsuit})^{G_{\mathbb{Q}_p}} / (\mathrm{H}^1_f(K_p, V_f) \otimes V_{gh}^{\triangle\heartsuit})^{G_{\mathbb{Q}_p}} \cong (\mathrm{H}^1_s(K_p, V_f) \otimes V_{gh}^{\triangle\heartsuit})^{G_{\mathbb{Q}_p}}.$$

The remaining cases are proven similarly. □

Remark 4.14. The logarithm maps of Lemma 4.11 are twisted versions of (4.1.16). More precisely, $\log_{\alpha\beta}$ may be accordingly recast as

$$\begin{array}{rcl} H^1_f(\mathbb{Q}_p, V_f \otimes V_{gh}^{\alpha\beta}) = & E(K_p)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta} & \longrightarrow & K_p(\chi_E)^{\beta\alpha} \otimes K_p^{\alpha\beta} \otimes L_p \cong L_p \\ & x \otimes v_g^\alpha v_h^\beta & \longmapsto & \log_E(x) \cdot \langle v_g^\alpha v_h^\beta, \eta_{g_\alpha}\omega_{h_\alpha}\rangle. \end{array} \tag{4.2.5}$$

Using Proposition 4.8 we can define the following generators k^+ and k^- of $K_p(\chi_E)^+$ and $K_p(\chi_E)^-$ respectively:

$$k^+ := \begin{cases} 1 & \text{if } a = +1; \\ z & \text{if } a = -1 \end{cases} \quad \text{and} \quad k^- := \begin{cases} z & \text{if } a = +1; \\ 1 & \text{if } a = -1. \end{cases}$$

Using (1.8.14) and (1.63), the latter pairing is

$$\langle v_g^\alpha v_h^\beta, \eta_{g_\alpha}\omega_{h_\alpha}\rangle = \frac{1}{\Omega_{g_\alpha}\Theta_{h_\alpha}}\langle \omega_{g_\alpha}\eta_{h_\alpha}, \eta_{g_\alpha}\omega_{h_\alpha}\rangle = \frac{1}{\Omega_{g_\alpha}\Theta_{h_\alpha}} \otimes 1 \in K_p^{\alpha\beta} \otimes L_p,$$

and the right-most isomorphism of (4.2.5) is given by

$$\begin{array}{ccccc} K_p(\chi_E)^{\beta\alpha} \otimes K_p^{\alpha\beta} & \xrightarrow{\cong} & K_p(\chi_E)^+ & \xrightarrow{\cong} & \mathbb{Q}_p \\ x \otimes y & \longmapsto & xy & & \\ & & k^+ & \longmapsto & 1. \end{array}$$

Analogously, the dual exponential maps of Lemma 4.11 are twisted versions of (4.1.21), and $\exp^*_{\alpha\beta}$ may be recast under the identification provided by Lemma 4.13 as

$$\begin{array}{rcl} \exp^*_{\alpha\beta} : \mathrm{H}^1_s(K_p, V_f)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta} & \longrightarrow & L_p \\ x \otimes v_g^\alpha v_h^\beta & \longmapsto & \exp^*(x) \cdot \langle v_g^\alpha v_h^\beta, \eta_{g_\alpha}\omega_{h_\alpha}\rangle. \end{array}$$

We will still denote by

$$\exp^*_\pm : \mathrm{H}^1_s(K_p, V_f)^\pm \longrightarrow \mathbb{Q}_p \tag{4.2.6}$$

the composition of (4.1.21) with the isomorphism $K_p(\chi_E)^\pm \cong \mathbb{Q}_p$ sending $k^\pm$ to 1.

Corollary 4.15. *Define $X_\pm \in \mathrm{H}^1_s(K_p, V_f)^\pm$ such that $\exp^*_\pm(X_\pm) = 1$. The p-relaxed Selmer group $\mathrm{Sel}_{(p)}(V)$ admits a basis*

$$\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$$

characterized as

$$\begin{array}{ll} \partial_p\xi^{\alpha\alpha} = X_{\alpha\alpha} \otimes v_g^\alpha \otimes v_h^\alpha, & \partial_p\xi^{\alpha\beta} = X_{\alpha\beta} \otimes v_g^\alpha \otimes v_h^\beta, \\ \partial_p\xi^{\beta\alpha} = X_{\beta\alpha} \otimes v_g^\beta \otimes v_h^\alpha, & \partial_p\xi^{\beta\beta} = X_{\beta\beta} \otimes v_g^\beta \otimes v_h^\beta. \end{array}$$

Proof. By Corollary 4.12 and using the decomposition (4.2.1), there are isomorphisms

$$\mathrm{Sel}_{(p)}(V) \xrightarrow{\partial_p} \mathrm{H}^1_s(\mathbb{Q}_p, V) \cong \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\alpha\beta}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\alpha}) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V^{\beta\beta}). \tag{4.2.7}$$

By Lemma 4.13, each of the four components in (4.2.7) is isomorphic to

$$\mathrm{H}^1_s(K_p, V_f)^{\beta\beta} \otimes V_{gh}^{\alpha\alpha},\ \mathrm{H}^1_s(K_p, V_f)^{\beta\alpha} \otimes V_{gh}^{\alpha\beta},\ \mathrm{H}^1_s(K_p, V_f)^{\alpha\beta} \otimes V_{gh}^{\beta\alpha},\ \mathrm{H}^1_s(K_p, V_f)^{\alpha\alpha} \otimes V_{gh}^{\beta\beta},$$

respectively. These are in turn isomorphic to $V_{gh}^{\alpha\alpha} \oplus V_{gh}^{\alpha\beta} \oplus V_{gh}^{\beta\alpha} \oplus V_{gh}^{\beta\beta}$ via $\exp^*_{\alpha\alpha} \oplus \exp^*_{\alpha\beta} \oplus \exp^*_{\beta\alpha} \oplus \exp^*_{\beta\beta}$. □

4.3 A special value formula for the triple product p-adic L-function in rank 0

The aim of this section is to describe the p-adic L-value $I_p(\breve{f}, \breve{g}_\alpha, \breve{h}_\alpha)$ in terms of the basis $\{\xi^{\alpha\alpha}, \xi^{\alpha\beta}, \xi^{\beta\alpha}, \xi^{\beta\beta}\}$ of $\mathrm{Sel}_{(p)}(V)$ appearing in Corollary 4.15. This section lies within the framework of the *exceptional setting* of [BSV] and we recall here the notation and the main results from loc. cit. that we shall use.

4.3.1 The triple product p-adic L-function

If $\varphi = \sum a_n(\varphi)q^n$ is a modular form of weight w, level M and nebentype character χ_φ and $p \nmid M$, the Hecke polynomial at p of φ is

$$x^2 - a_p(\varphi)x + \chi_\varphi(p)p^{w-1} = (x - \alpha_\varphi)(x - \beta_\varphi),$$

where we label the eigenvalues so that $\mathrm{ord}_p(\alpha_\varphi) \le \mathrm{ord}_p(\beta_\varphi)$. Recall that if φ is ordinary at p, then α_φ is a p-adic unit. If $p \mid M$ we have $\alpha_\varphi = a_p(\varphi)$.

Since g has weight $w = 1$, both α_g and β_g are p-adic units. Recall that we are assuming that $\alpha_g \ne \beta_g$, thus g has two different ordinary p-stabilisations. We denote g_α and g_β the stabilisation satisfying $U_p g_\alpha = \alpha_g g_\alpha$ and $U_p g_\beta = \beta_g g_\beta$ respectively. The same holds for h, and we denote similarly its p-stabilisations. Since E has multiplicative reduction at p, the level of the newform is divisible by p and it is p-stabilised, meaning that $f_\alpha = f$.

Let $\mathbf{f} = \sum_{n\ge 1} a_n(\mathbf{f}) \in \Lambda_{\mathbf{f}}[[q]]$, $\mathbf{g} \in \Lambda_{\mathbf{g}}[[q]]$, $\mathbf{h} \in \Lambda_{\mathbf{h}}[[q]]$ be the Hida families passing through f, g_α, h_α respectively, where $\Lambda_{\mathbf{f}}, \Lambda_{\mathbf{g}}$ and $\Lambda_{\mathbf{h}}$ are finite flat extensions of the Iwasawa algebra Λ. These Hida families have tame level $N_f/p, N_g, N_h$ and tame character $1, \chi, \bar{\chi}$ respectively.

Let $N :=\mathrm{lcm}(N_f/p, N_g, N_h)$. A *test vector* for $(\mathbf{f}, \mathbf{g}, \mathbf{h})$ is a triple of Hida families $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$ of tame level N such that $\breve{\mathbf{f}}$ is of the form $\sum \lambda_d \mathbf{f}(q^d) \in \Lambda_{\mathbf{f}}[[q]]$ with $\lambda_d \in \Lambda_{\mathbf{f}}$, where d runs over the divisors of N/N_f, and similarly for $\breve{\mathbf{g}}$ and $\breve{\mathbf{h}}$.

Recall from §1.15.3 the set $\mathcal{W}^\circ_{\mathbf{fgh}}$ of cristalline points, the subsets $\mathcal{W}^f_{\mathbf{fgh}}, \mathcal{W}^g_{\mathbf{fgh}}, \mathcal{W}^h_{\mathbf{fgh}}, \mathcal{W}^{\mathrm{bal}}_{\mathbf{fgh}}$ and the triple product p-adic L-functions attached to $(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}})$

$$\mathcal{L}_p^f(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}), \mathcal{L}_p^g(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}), \mathcal{L}_p^h(\breve{\mathbf{f}}, \breve{\mathbf{g}}, \breve{\mathbf{h}}) : \mathcal{W}_{\mathbf{f}} \times \mathcal{W}_{\mathbf{g}} \times \mathcal{W}_{\mathbf{h}} \longrightarrow \mathbb{C}_p$$

constructed in [DR14].

4.3.2 $I_p(f, g_\alpha, h_\alpha)$ in terms of the basis

In this section we finally obtain, in Theorem 4.19, the formula for $I_p(f, g_\alpha, h_\alpha)$. Recall from the introduction the Galois representation

$$V = V_f \otimes V_g \otimes V_h.$$

Note that the character Ξ introduced in §1.16.4.1 specializes at $(2, 1, 1)$ to the trivial character, and thus the specialization of the improved Λ-adic cohomology class of Prop. 1.108 yields a global cohomology class

$$\boldsymbol{\kappa}_g^*(2, 1, 1) \in \mathrm{Sel}_{(p)}(V).$$

Recall the periods

$$\Omega_{g_\alpha} \in K_p^{1/\alpha_g}, \quad \Theta_{g_\alpha} \in K_p^{1/\beta_g}, \quad \mathcal{L}_{g_\alpha} := \frac{\Omega_{g_\alpha}}{\Theta_{g_\alpha}} \in K_p^{\beta_g/\alpha_g} \tag{4.3.1}$$

introduced in Definition 1.63.

Proposition 4.16. *We have*

$$\boldsymbol{\kappa}_g^*(2,1,1) = \Theta_{g_\alpha}\Theta_{h_\alpha}\frac{2(1-1/p)\sqrt{c}\sqrt{L(E\otimes\rho,1)}}{\pi^2\langle f,f\rangle}\cdot\xi^{\beta\beta},$$

where $c\in L^\times$ is the product of the local terms appearing in [DLR15, Proposition 2.1 (iii)].

Proof. According to [BSV, §8.3] one has $\mathrm{res}_p(\boldsymbol{\kappa}_g^*(2,1,1))\in \mathrm{H}^1(\mathbb{Q}_p,\mathrm{Fil}^2(V))$. Recall that

$$\mathrm{H}^1(\mathbb{Q}_p,\mathrm{Fil}^2(V)/\mathrm{Fil}^3(V)) = \mathrm{H}^1(\mathbb{Q}_p,V_f^-\otimes V_{gh}^{\beta\beta})\oplus \mathrm{H}^1(\mathbb{Q}_p,V_f^+\otimes V_{gh}^{\alpha\beta})\oplus \mathrm{H}^1(\mathbb{Q}_p,V_f^+\otimes V_{gh}^{\beta\alpha})$$

and hence $\partial_p\boldsymbol{\kappa}_g^*(2,1,1)\in \mathrm{H}^1_s(\mathbb{Q}_p,V) = \mathrm{H}^1_s(\mathbb{Q}_p,V_f^-\otimes V_{gh})$ lies in $\mathrm{H}^1_s(\mathbb{Q}_p,V^{\beta\beta})$. Then, by definition of the basis of $\mathrm{Sel}_{(p)}(V)$ described in Corollary 4.15,

$$\boldsymbol{\kappa}_g^*(2,1,1) = \frac{\exp^*_{\beta\beta}(\partial_p\boldsymbol{\kappa}_g^*(2,1,1))}{\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta})}\cdot\xi^{\beta\beta}.$$

We need to compute the numerator and denominator of the previous formula. By (1.16.22),

$$\begin{aligned}\exp^*_{\beta\beta}(\partial_p\boldsymbol{\kappa}_g^*(2,1,1)) &= 2(1-1/p)\mathcal{L}_p^{f*}(1)\\ &= \frac{2(1-1/p)\langle w_N(f),hg\rangle}{\langle w_N(f),w_N(f)\rangle}\\ &= \frac{2(1-1/p)\sqrt{c}\sqrt{L(E\otimes\rho,1)}}{\pi^2\langle f,f\rangle},\end{aligned}$$

where the last equality is given by [Ich08]. Besides,

$$\begin{aligned}\exp^*_{\beta\beta}(\partial_p\xi^{\beta\beta}) &= \langle \exp^*_{\beta\beta}\partial_p\xi^{\beta\beta},\eta_f\omega_{g_\alpha}\omega_{h_\alpha}\rangle\\ &= \exp^*(X_{\beta\beta})\langle v_g^\beta\otimes v_h^\beta,\omega_{g_\alpha}\otimes\omega_{h_\alpha}\rangle\\ &= \frac{1}{\Theta_{g_\alpha}\Theta_{h_\alpha}}\langle \eta_{g_\alpha}\otimes\eta_{h_\alpha},\omega_{g_\alpha}\otimes\omega_{h_\alpha}\rangle\\ &= \frac{1}{\Theta_{g_\alpha}\Theta_{h_\alpha}}.\end{aligned}$$

The proposition follows. □

Let $\Lambda_{\mathrm{cyc}} = \mathbb{Z}_p[[j+1]]$ denote the usual Iwasawa algebra regarded as the ring of bounded analytic functions on an open disc centered at $j=-1$, and let $\boldsymbol{\chi}_{\mathrm{cyc}}: G_\mathbb{Q}\longrightarrow\Lambda_{\mathrm{cyc}}^\times$ denote the Λ-adic cyclotomic character characterized by the property that $\nu_j(\boldsymbol{\chi}_{\mathrm{cyc}}) = \chi_{\mathrm{cyc}}^{-j}$ for all $j\in\mathbb{Z}$.

Recall the three-variable Iwasawa algebra $\Lambda_{\mathbf{fgh}}$ and set

$$\bar\Lambda_{\mathbf{fgh}} = \Lambda_{\mathbf{fgh}}\hat\otimes\Lambda_{\mathrm{cyc}},\quad \overline{\mathcal{W}}_{\mathbf{fgh}} := \mathrm{Spf}(\bar\Lambda_{\mathbf{fgh}}) = \mathcal{W}_{\mathbf{fgh}}\times\mathcal{W},\quad \overline{\mathcal{W}}^\circ_{\mathbf{fgh}} := \mathcal{W}^\circ_{\mathbf{fgh}}\times\mathcal{W}^\circ.$$

Let $\Psi: G_{\mathbb{Q}_p}\longrightarrow\Lambda_{\mathbf{fgh}}^\times$ denote the unramified character taking Frob_p to $\dfrac{a_p(\mathbf{g})}{a_p(\mathbf{f})a_p(\mathbf{h})\chi(p)}$. Define

$$\mathbb{M}^g := \Lambda_{\mathbf{fgh}}(\Psi),\qquad \overline{\mathbb{M}}^g := \mathbb{M}^g\otimes\Lambda_{\mathrm{cyc}}(\boldsymbol{\chi}_{\mathrm{cyc}})$$

and note that $\mathbb{M}^g$ is the unramified twist of the local Galois representation $\mathbb{V}^g$ introduced in (1.16.12).

Let $\theta:\bar\Lambda_{\mathbf{fgh}}\longrightarrow\Lambda_{\mathbf{fgh}}$ denote the homomorphism taking a function $F(k,\ell,m,j)$ to its restriction $F(k,\ell,m,(\ell-k-m)/2)$. As shown in the proof of [BSV, Prop. 4.6.2], there is an isomorphism

$$\theta:\overline{\mathbb{M}}^g\otimes_\theta\mathcal{O}_{\mathbf{fgh}}\cong\mathbb{V}^g. \tag{4.3.2}$$

Define also the Λ-adic Dieudonné modules

$$\mathbb{D}^g := (\mathbb{M}^g \hat{\otimes} \hat{\mathbb{Z}}_p^{\mathrm{ur}})^{G_{\mathbb{Q}_p}}, \quad \overline{\mathbb{D}}^g := \mathbb{D}^g \otimes \Lambda_{\mathrm{cyc}},$$

where $\hat{\mathbb{Z}}_p^{\mathrm{ur}}$ is the ring of integers of the p-adic completion $\hat{\mathbb{Q}}_p^{\mathrm{ur}}$ of the maximal unramified extension of $\mathbb{Q}_p$.

As it directly follows from the above definitions, the specialization of $\overline{\mathbb{M}}^g$ at a point $\underline{x} = (k,\ell,m,j) \in \overline{\mathcal{W}}^{\circ}_{\mathbf{fgh}}$ is

$$\overline{\mathbb{M}}^g_{\underline{x}} := \overline{\mathbb{M}}^g \otimes_{\underline{x}} L_p = L_p(\Psi_{(k,\ell,m)})(-j) = V^+_{\mathbf{f}_k} \otimes V^-_{\mathbf{g}_\ell} \otimes V^+_{\mathbf{h}_m}(2-k-m-j). \tag{4.3.3}$$

Bloch–Kato's logarithm and dual exponential maps give rise to an isomorphism

$$\log_{\underline{x}} : \mathrm{H}^1(\mathbb{Q}_p, L_p(\Psi_{(k,\ell,m)})(-j)) \longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p(\Psi_{(k,\ell,m)})(-j)) \qquad \text{if } j<0 \tag{4.3.4}$$

$$\exp^*_{\underline{x}} : \mathrm{H}^1(\mathbb{Q}_p, L_p(\Psi_{(k,\ell,m)})(-j)) \longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p(\Psi_{(k,\ell,m)})(-j)) \qquad \text{if } j\geq 0. \tag{4.3.5}$$

In particular, at $\underline{x} = (2,1,1,-1)$ this isomorphism becomes the map

$$\log_{\alpha\beta} : \mathrm{H}^1_f(\mathbb{Q}_p, V^{\alpha\beta}) \longrightarrow L_p,$$

after taking the pairing with the differential $\eta_{g_\alpha}\omega_{h_\alpha}$ (cf. Remark 4.14).

Proposition 4.17. *There exists a single homomorphism of $\bar{\Lambda}_{\mathbf{fgh}}$-modules*

$$\bar{\mathcal{L}}_g : \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g) \longrightarrow \overline{\mathbb{D}}^g,$$

such that for all $\mathbf{Z} \in \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g)$ and $\underline{x} = (k,\ell,m,j) = (w,j) \in \overline{\mathcal{W}}^{\circ}_{\mathbf{fgh}}$ with

$$1 - \Psi_w(\mathrm{Frob}_p)p^{-j-1} \neq 0,$$

we have

$$\bar{\mathcal{L}}_g(\mathbf{Z})(\underline{x}) = \left(1 - \frac{p^j}{\Psi_w(\mathrm{Frob}_p)}\right)(1-\Psi_w(\mathrm{Frob}_p)p^{-j-1})^{-1} \cdot \begin{cases} \dfrac{(-1)^{j+1}}{(-j-1)!}\log_{\underline{x}}(\mathbf{Z}(\underline{x})) & j<0; \\ j!\ \exp^*_{\underline{x}}(\mathbf{Z}(\underline{x})) & j \geq 0. \end{cases} \tag{4.3.6}$$

Proof. The proposition as stated here is [BSV, Proposition 4.6.1] for $\mathbf{g}$, and it is a consequence of [LZ14]. □

By the proof of [BSV, Proposition 4.6.2], the relation between $\bar{\mathcal{L}}_g$ and the logarithm $\mathcal{L}_g$ that appears in (1.16.13) is given, for all $\mathbf{Z} \in \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g)$ by

$$\mathcal{L}_g(\theta_*(\mathbf{Z}))(k,\ell,m) = \langle \bar{\mathcal{L}}_g(\mathbf{Z})(k,\ell,m,(\ell-k-m)/2), \omega_{\mathbf{f}_k}\eta_{\mathbf{g}_\ell}\omega_{\mathbf{h}_m}\rangle. \tag{4.3.7}$$

Here, if $\underline{x} = (k,\ell,m,(\ell-k-m)/2)$, then

$$\omega_{\mathbf{f}_k}\eta_{\mathbf{g}_\ell}\omega_{\mathbf{h}_m} \in \overline{\mathbb{D}}^g \otimes_{\underline{x}} L_p \cong \mathrm{D}_{\mathrm{dR}}(V^+_{\mathbf{f}_k} \otimes V^-_{\mathbf{g}_\ell} \otimes V^+_{\mathbf{h}_m}((\ell-k-m+4)/2)) \tag{4.3.8}$$

are the differentials defined as in [BSV, (228)] and are natural generalizations of the ones introduced in §**??**.

When j is a fixed integer, note that the Euler-like factors appearing above vary analytically. As it will suffice or our purposes, set $j=-1$ and recall the plane $\mathcal{H}$ parametrized by points of weights $(2+\ell-m,\ell,m)$ in $\mathcal{W}_{\mathbf{fgh}}$. By a slight abuse of notation we also regard $\mathcal{H}$ as the plane in $\bar{\mathcal{W}}_{\mathbf{fgh}}$ parametrizing points of weights $(2+\ell-m,\ell,m,-1)$.

Define the homomorphism

$$\bar{\mathcal{L}}_g^* : \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g_{|\mathcal{H}}) \longrightarrow \overline{\mathbb{D}}^g_{|\mathcal{H}}$$

of $\mathcal{O}_{\mathcal{H}}$-modules given by

$$\bar{\mathcal{L}}_g^* = \boldsymbol{\mathcal{E}}_g \times \bar{\mathcal{L}}_{g|\mathcal{H}}. \tag{4.3.9}$$

In light of the above proposition, for any $\boldsymbol{Z} \in \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g_{|\mathcal{H}})$ and $\ell \geq 1$ one then has

$$\bar{\mathcal{L}}_g^*(\boldsymbol{Z})(\ell) = (1 - \frac{a_p(\mathbf{g}_\ell)\beta_h}{a_p(\mathbf{f}_{\ell+1})p}) \cdot \langle \log_{(\ell+1,\ell,1,-1)}(\mathbf{Z}(\ell+1,\ell,1,-1)), \omega_{\mathbf{f}_{\ell+1}}\eta_{\mathbf{g}_\ell}\omega_{h_\alpha}\rangle. \tag{4.3.10}$$

Proposition 4.18. *We have*

$$I_p(f, g_\alpha, h_\alpha) = \left(1 - \frac{1}{p}\right) \log_{\alpha\beta}(\pi_{\alpha\beta}\boldsymbol{\kappa}_g^*(2,1,1)).$$

Proof. To avoid notational clutter, we continue to denote $\mathcal{H}$ the plane in $\overline{\mathcal{W}}_{\mathbf{fgh}}$ containing the points $(2+\ell-m, \ell, m, -1)$ as a dense subset. Let $\widetilde{\boldsymbol{\kappa}}_{g,p}^* \in \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g_{|\mathcal{H}})$ be any lift of $\mathrm{res}_p(\boldsymbol{\kappa}_g^*)$, whose existence is assured by the fact that the map

$$\theta_* : \mathrm{H}^1(\mathbb{Q}_p, \overline{\mathbb{M}}^g) \otimes_\theta \mathcal{O}_{\mathbf{fgh}} \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, \mathbb{V}^g)$$

induced by (4.3.2) is an isomorphism (cf. the proof of [BSV, Proposition 4.6.2]).

Combining Proposition 4.17 with (1.16.13), (1.16.20), (4.3.7) we deduce

$$\begin{aligned}\mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})_{|\mathcal{H}} &= \mathcal{L}_{g|\mathcal{H}}(\mathrm{res}_p(\boldsymbol{\kappa}_{|\mathcal{H}})) = \boldsymbol{\mathcal{E}}_{g|\mathcal{H}} \cdot \mathcal{L}_{g|\mathcal{H}}(\mathrm{res}_p(\boldsymbol{\kappa}_g^*)) \\ &= \boldsymbol{\mathcal{E}}_{g|\mathcal{H}} \cdot \langle \bar{\mathcal{L}}_{g|\mathcal{H}}(\widetilde{\boldsymbol{\kappa}}_{g,p}^*), \omega_{\mathbf{f}_{\ell+1}}\eta_{\mathbf{g}_\ell}\omega_{h_\alpha}\rangle.\end{aligned}$$

By (4.3.9), (4.3.10) and (4.3.4), the latter quantity is equal to

$$\langle \bar{\mathcal{L}}_g^*(\widetilde{\boldsymbol{\kappa}}_{g,p}^*), \omega_{\mathbf{f}_{\ell+1}}\eta_{\mathbf{g}_\ell}\omega_{h_\alpha}\rangle = (1 - \frac{a_p(\mathbf{g}_\ell)\beta_h}{a_p(\mathbf{f}_{\ell+1})p}) \cdot \log_{\alpha\beta}(\pi_{\alpha\beta}\boldsymbol{\kappa}_g^*(\ell+1,\ell,1)).$$

Since $I_p(f, g_\alpha, h_\alpha) = \mathcal{L}_p^g(\mathbf{f}, \mathbf{g}, \mathbf{h})(2,1,1)$ by definition, the proposition follows by using (**??**). ☐

Define the local point

$$P_{\alpha\beta} \in E(K_p)^{\alpha\beta}_{L_p}$$

as the one satisfying

$$\pi_{\alpha\beta}\,\mathrm{res}_p(\xi^{\beta\beta}) = \delta_p(P_{\alpha\beta}) \otimes v_g^\alpha \otimes v_h^\beta \in \left(\mathrm{H}^1_f(K_p, V_f) \otimes V_{gh}^{\alpha\beta}\right)^{G_{\mathbb{Q}_p}}, \tag{4.3.11}$$

where $\delta_p : E(K_p) \longrightarrow \mathrm{H}^1_f(K_p, V_f)$ is the Kummer map and $E(K_p)^{\alpha\beta}_{L_p} := E(K_p)^{\alpha\beta} \otimes L_p$.

Theorem 4.19. *We have*

$$I_p(f, g_\alpha, h_\alpha) = \frac{2(1-1/p)^2\sqrt{c}}{\pi^2\langle f, f\rangle} \times \sqrt{L(E \otimes \rho, 1)} \times \frac{\log_p(P_{\alpha\beta})}{\mathcal{L}_{g_\alpha}} \qquad L^\times.$$

Proof. By Proposition 4.16, $\boldsymbol{\kappa}_g^*(2,1,1) = \dfrac{\Theta_{g_\alpha}\Theta_{h_\alpha}2(1-1/p)\sqrt{c}\sqrt{L(E\otimes\rho,1)}}{\pi^2\langle f,f\rangle}\xi^{\beta\beta}$. It thus follows from Proposition 4.18 that

$$\begin{aligned} I_p(f,g_\alpha,h_\alpha) &= \left(1-\frac{1}{p}\right)\log_{\alpha\beta}(\pi_{\alpha\beta}\boldsymbol{\kappa}_g^*(2,1,1)) \\ &= \frac{\Theta_{g_\alpha}\Theta_{h_\alpha}\sqrt{c}2(1-1/p)^2\sqrt{L(E\otimes\rho,1)}}{\pi^2\langle f,f\rangle}\log_{\alpha\beta}(\pi_{\alpha\beta}\xi^{\beta\beta}) \\ &= \frac{\Theta_{g_\alpha}\Theta_{h_\alpha}\sqrt{c}2(1-1/p)^2\sqrt{L(E\otimes\rho,1)}}{\pi^2\langle f,f\rangle}\log_p(P_{\alpha\beta})\langle v_g^\alpha\otimes v_h^\beta,\eta_{g_\alpha}\omega_{h_\alpha}\rangle \\ &= \frac{\Theta_{g_\alpha}\Theta_{h_\alpha}\sqrt{c}2(1-1/p)^2\sqrt{L(E\otimes\rho,1)}}{\Omega_{g_\alpha}\Theta_{h_\alpha}\pi^2\langle f,f\rangle}\log_p(P_{\alpha\beta}). \end{aligned}$$

The theorem follows in light of (4.3.1). □

4.4 A special value formula for $I_p(f,g_\alpha,h_\alpha)$ in terms of Kolyvagin classes

The aim of this section is to relate Theorem 4.19 to Bertolini-Darmon's Kolyvagin classes constructed in [BD97, §6].

4.4.1 Decomposition of the representation V and consequences

The underlying reason why the p-adic L-value $\mathcal{L}_p^g(\mathbf{f},\mathbf{g},\mathbf{h})(2,1,1)$ may be related to Kolyvagin classes relies on the decomposition of the Galois representation $V_g\otimes V_h$ as the direct sum of a pair of 2-dimensional representations induced from characters of an imaginary quadratic field.

More precisely, recall that g and h are theta series of two finite order Hecke characters $\psi_g,\psi_h : G_K \longrightarrow L^\times$. As explained in the introduction, there is a decomposition

$$V_g\otimes V_h \cong V_{\psi_1}\oplus V_{\psi_2} \tag{4.4.1}$$

where $\psi_1 := \psi_g\psi_h,\ \psi_2 := \psi_g\psi_h'$ and $V_{\psi_i} := \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\psi_i)$.

For $i=1,2$, ψ_i is a ring class character, i.e. it factors through the absolute Galois group of a number field H_i which is a ring class field of K of some conductor c_i. Recall we have assumed that p does not divide the conductor of ψ_g and ψ_h, hence that the characters ψ_1 and ψ_2 are unramified at p, and this implies that $p\nmid c_1c_2$. The Artin representation

$$\rho = \rho_g\otimes\rho_h : G_\mathbb{Q} \to \mathrm{GL}(V_g\otimes_{L_p}V_h)$$

thus factors through $\mathrm{Gal}(H/K)$, where H is the ring class field of K of conductor $c := \mathrm{lcm}(c_1,c_2)$. Moreover, since the prime p is inert in K and $p\nmid c$, the principal ideal $p\mathcal{O}_K$ splits completely in H. Fix once and for all a prime $\mathfrak{p}$ of the field H dividing p and denote $H_\mathfrak{p}$ the completion of H at $\mathfrak{p}$. Then $H_\mathfrak{p} = K_p$ and, in particular, the restriction of ψ_i to the decomposition group $G_{K_p} = G_{H_\mathfrak{p}}$ is trivial for $i=1,2$.

Recall that Frob_p acts on V_g with eigenvalues $\alpha_g=\lambda,\ \beta_g=-\lambda$ and on V_h with eigenvalues

$$(\alpha_h,\beta_h) = \begin{cases} (-1/\lambda, 1/\lambda) & \text{if } a=+1; \\ (1/\lambda,-1/\lambda) & \text{if } a=-1, \end{cases}$$

so we always have $\alpha_g\alpha_h = -a$. As in the previous sections, choose an L-basis of eigenvectors v_g^α, v_g^β of V_g and likewise a basis v_h^α, v_h^β of V_h.

Note also that Frob_p acts on V_{ψ_1} and V_{ψ_2} with eigenvalues ±1.

Lemma 4.20. *There is a $G_{\mathbb{Q}}$-equivariant isomorphism*

$$\Psi : V_g \otimes V_h \cong V_{\psi_1} \oplus V_{\psi_2}.$$

Moreover, V_{ψ_i} admits an L-basis $\{v_i^+, v_i^-\}$ of eigenvectors with relative eigenvalues $\{+1, -1\}$ satisfying:

- *if $a = +1$, then*

$$\Psi(v_g^\alpha \otimes v_h^\alpha) = v_1^- - v_2^-, \quad \Psi(v_g^\beta \otimes v_h^\beta) = v_1^- + v_2^-, \quad \Psi(v_g^\alpha \otimes v_h^\beta) = v_1^+ + v_2^+, \quad \Psi(v_g^\beta \otimes v_h^\alpha) = v_1^+ - v_2^+.$$

- *If $a = -1$, then*

$$\Psi(v_g^\alpha \otimes v_h^\alpha) = v_1^+ + v_2^+, \quad \Psi(v_g^\beta \otimes v_h^\beta) = v_1^+ - v_2^+, \quad \Psi(v_g^\alpha \otimes v_h^\beta) = v_1^- - v_2^-, \quad \Psi(v_g^\beta \otimes v_h^\alpha) = v_1^- + v_2^-.$$

Proof. Let ψ be a finite order Hecke character of K, let $\varphi := \theta(\psi)$ and denote $\mathrm{Frob}_p \in G_{\mathbb{Q}} \smallsetminus G_K$ a Frobenius element at p. As explained in [?, §2.2], we can choose a basis $\{u_\varphi, v_\varphi\}$ of V_φ such that, if

$$\rho_\varphi : G_{\mathbb{Q}} \longrightarrow \mathrm{GL}(V_\varphi)$$

is the ℓ-adic representation attached to φ, then $v_\varphi = \rho_\varphi(\mathrm{Frob}_p)u_\varphi$ and with respect to this basis,

$$\rho_\varphi(\sigma) = \begin{pmatrix} \psi(\sigma) & 0 \\ 0 & \psi'(\sigma) \end{pmatrix} \text{ for } \sigma \in G_K; \qquad \rho_\varphi(\tau) = \begin{pmatrix} 0 & \eta(\tau) \\ \eta'(\tau) & 0 \end{pmatrix} \text{ for } \tau \in G_{\mathbb{Q}} \smallsetminus G_K.$$

Here ψ' denotes the character defined by $\psi'(\sigma) := \psi(\mathrm{Frob}_p \, \sigma \, \mathrm{Frob}_p^{-1})$, and η is a function on $G_{\mathbb{Q}} \smallsetminus G_K$. Recall that the characteristic polynomial for the action of Frob_p on V_φ is $x^2 - a_p(\varphi)x + \chi_\varphi(p) = x^2 - \epsilon_\psi(p)$, where $\chi_\varphi = \chi_K \epsilon_\psi$ and ϵ_ψ is the central character of ψ. We can always choose $\{u_\varphi, u_\varphi\}$ such that $\rho_\varphi(\mathrm{Frob}_p) = \begin{pmatrix} 0 & \zeta \\ \zeta & 0 \end{pmatrix}$, with $\zeta^2 = \epsilon_\psi(p)$. Then a basis of eigenvectors for the action of Frob_p on V_φ is

$$\{v_\varphi^\zeta = u_\varphi + v_\varphi, \qquad v_\varphi^{-\zeta} = u_\varphi - v_\varphi\}$$

with eigenvalue ζ, $-\zeta$ respectively. Using this notation we obtain the basis

$$(u_g, v_g), \quad (u_h, v_h), \quad (u_1, v_1), \quad (u_2, v_2)$$

of V_g, V_h, V_{ψ_1}, V_{ψ_2} where we denoted $u_i := u_{\theta(\psi_i)}$ and $v_i := v_{\theta(\psi_i)}$. The corresponding matrix $\rho_\varphi(\mathrm{Frob}_p)$ is

$$\begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1/\lambda \\ 1/\lambda & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{4.4.2}$$

Consider the basis

$$(u_g \otimes u_h, u_g \otimes v_h, v_g \otimes u_h, v_g \otimes v_h)$$

of $V_g \otimes V_h$, and the basis

$$(u_1, v_1, u_2, v_2)$$

of $V_{\psi_1} \oplus V_{\psi_2}$. Comparing the matrices above, we conclude that the isomorphism Ψ is given by the rule

$$u_g \otimes u_h \mapsto u_1, \quad u_g \otimes v_h \mapsto u_2, \quad v_g \otimes u_h \mapsto v_2, \quad v_g \otimes v_h \mapsto v_1.$$

Moreover, since the eigenvalues of Frob_p acting on V_g and V_h are $\lambda, -\lambda, 1/\lambda, 1/\lambda$, a basis of eigenvectors for $V_g \otimes V_h$ is

$$v_g^\lambda \otimes v_h^{1/\lambda} = (u_g + v_g) \otimes (u_h + v_h), \qquad v_g^\lambda \otimes v_h^{-1/\lambda} = (u_g + v_g) \otimes (u_h - v_h),$$
$$v_g^{-\lambda} \otimes v_h^{1/\lambda} = (u_g - v_g) \otimes (u_h + v_h), \qquad v_g^{-\lambda} \otimes v_h^{-1/\lambda} = (u_g - v_g) \otimes (u_h - v_h).$$

with eigenvalues $+1, -1, -1, +1$ respectively. On the other hand, a basis of eigenvectors for $V_{\psi_1} \oplus V_{\psi_2}$ is

$$(v_1^+ := u_1 + v_1, \quad v_1^- := u_1 - v_1, \quad v_2^+ := u_2 + v_2, \quad v_2^- := u_2 - v_2)$$

In conclusion,

$$v_g^\lambda \otimes v_h^{1/\lambda} \mapsto v_1^+ + v_2^+, \;\; v_g^\lambda \otimes v_h^{-1/\lambda} \mapsto v_1^- - v_2^-, \;\; v_g^{-\lambda} \otimes v_h^{1/\lambda} \mapsto v_1^- + v_2^-, \;\; v_g^{-\lambda} \otimes v_h^{-1/\lambda} \mapsto v_1^+ - v_2^+.$$

□

Recall that

$$V_{\psi_i} = \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\psi_i) = \{v : G_\mathbb{Q} \longrightarrow L(\psi) \mid v(\sigma\tau) = \psi_i(\sigma)v(\tau) \;\forall \sigma \in G_K, \tau \in G_\mathbb{Q}\} = L \oplus L \cdot \mathrm{Frob}_p\,.$$

The proof of the previous lemma shows that we may choose the vectors $v_i^\pm$ such that

$$v_i^+(1) = v_i^-(1) = 1. \tag{4.4.3}$$

Any other choice of basis would be of the form $(a_i v_i^+, b_i v_i^-)$ for some scalars $a_i, b_i \in L^\times$, and would yield the same formula for $I_p(f, g_\alpha, h_\alpha)$ up to a scalar in $L^\times$. This is fine because that is precisely the ambiguity we are working with in our framework, as explained in the introduction.

As in previous sections, denote

$$V := V_f \otimes V_g \otimes V_h, \quad V_1 := V_f \otimes V_{\psi_1}, \quad V_2 := V_f \otimes V_{\psi_2}$$

The map Ψ of Lemma 4.20 induces the isomorphism

$$\Psi : V \xrightarrow{\cong} V_1 \oplus V_2. \tag{4.4.4}$$

There is then a decomposition of the Selmer group

$$\Psi_* : \mathrm{Sel}_p(V) \xrightarrow{\cong} \mathrm{Sel}_p(V_1) \oplus \mathrm{Sel}_p(V_2), \tag{4.4.5}$$

and the analogous decomposition holds for the relaxed and the strict Selmer groups associated to V. By Artin's formalism, decomposition (4.4.4) yields the factorization

$$L(E, \rho, s) = L(E, \psi_1, s)L(E, \psi_2, s) \tag{4.4.6}$$

of classical L-series.

As explained in the introduction, Assumption 3.2 on the analytic rank of $E \otimes \rho$ implies that

$$\mathrm{Sel}_p(V_1) = \mathrm{Sel}_p(V_2) = 0.$$

Denote $V_{\psi_i}^\pm$ the eigenspace of V_{ψ_i} on which Frob_p acts as ± 1, and let

$$V_i^\pm := V_f \otimes V_{\psi_i}^\pm.$$

Lemma 4.21. *There are isomorphisms*

$$\mathrm{H}^1_f(\mathbb{Q}_p, V_i^\pm) \;\cong\; \mathrm{H}^1_f(K_p, V_f)^\pm \otimes V_{\psi_i}^\pm \quad \textit{and} \quad \mathrm{H}^1_s(\mathbb{Q}_p, V_i^\pm) \;\cong\; \mathrm{H}^1_s(K_p, V_f)^\pm \otimes V_{\psi_i}^\pm. \tag{4.4.7}$$

The Bloch–Kato logarithm and dual exponential yield isomorphisms

$$\mathrm{H}^1_f(\mathbb{Q}_p, V_i^\pm) \xrightarrow{\log_\pm} L_p, \qquad \mathrm{H}^1_s(\mathbb{Q}_p, V_i^\pm) \xrightarrow{\exp^*_\pm} L_p. \tag{4.4.8}$$

Moreover, for $i \in \{1, 2\}$, there are isomorphisms

$$\mathrm{Sel}_{(p)}(V_i) \xrightarrow{\partial_p} \mathrm{H}^1_s(\mathbb{Q}_p, V_i) \xrightarrow{\exp^*} L_p^2. \tag{4.4.9}$$

Proof. The isomorphisms are obtained as in the proofs of Lemma 4.13, Lemma 4.11 and Corollary 4.12. □

Similarly as in Remark 4.14, the Bloch–Kato maps (4.4.8) are related to the logarithm and the dual exponential of the representation V_f as follows. Since V_{ψ_i} is the Galois representation attached to the theta series $\theta(\psi_i)$, (1.8.13) yields a pairing

$$\langle\ ,\ \rangle : \mathrm{D}_{\mathrm{dR}}(V_{\psi_i}^+) \times \mathrm{D}_{\mathrm{dR}}(V_{\psi_i}^-) \longrightarrow \mathrm{D}_{\mathrm{dR}}(L_p(\chi_i)) \tag{4.4.10}$$

where χ_i is the Nebentype character of $\theta(\psi_i)$. Since V_{ψ_i} is unramified, we have

$$\mathrm{D}_{\mathrm{dR}}(V_{\psi_i}^\pm) = (V_{\psi_i}^\pm \otimes K_p)^{G_{\mathbb{Q}_p}} = V_{\psi_i}^\pm \otimes K_p^\pm.$$

One can introduce differentials $\omega_i^\pm \in \mathrm{D}_{\mathrm{dR}}(V_{\psi_i}^\pm)$ as in Definition 1.62, and elements $\Omega_i^\pm \in K_p^\pm$ characterised by the following relation

$$v_i^\pm \otimes \Omega_i^\pm = \omega_i^\pm.$$

Thus (4.4.10) in turn induces a pairing

$$\langle\ ,\ \rangle : V_{\psi_i}^+ \times \mathrm{D}_{\mathrm{dR}}(V_{\psi_i}^-) \longrightarrow L_p$$

by setting

$$\langle v_i^+, \omega\rangle := \frac{1}{\Omega_i^+}\langle \omega_i^+, \omega\rangle$$

for all $\omega \in \mathrm{D}_{\mathrm{dR}}(V_{\psi_i}^-)$. Hence we have

$$\exp_\pm^*(x \otimes v) = \exp_{V_f}^*(x) \cdot \langle v, \omega_i^\pm\rangle \tag{4.4.11}$$

for any $x \in \mathrm{H}_s^1(K_p, V_f)^\pm$, $v \in V_{\psi_i}^\pm$.

Recall from Lemma 4.20 the basis $\{v_i^+, v_i^-\}$ for V_{ψ_i}, and from Corollary 4.15 the local classes $X_\pm \in \mathrm{H}_s(K_p, V_f)_{L_p}^\pm$ such that $\exp_\pm^*(X_\pm) = 1$.

Corollary 4.22. *For $i \in \{1, 2\}$, $\mathrm{Sel}_{(p)}(V_i)$ admits a basis $\{\xi_i^+, \xi_i^-\}$ characterized as*

$$\partial_p \xi_i^\pm = X_\pm \otimes v_i^\pm.$$

under the identifications given in (4.4.7).

Proof. The statement follows after applying Lemma 4.21 and using the same argument as in the proof of Corollary 4.15. □

Proposition 4.23. *Let*

$$\Psi_* : \mathrm{Sel}_{(p)}(V) \xrightarrow{\cong} \mathrm{Sel}_{(p)}(V_1) \oplus \mathrm{Sel}_{(p)}(V_2)$$

denote the isomorphism induced by (4.4.5).

- *If $a = +1$ then*

$$\Psi_*\xi^{\alpha\alpha} = \xi_1^- - \xi_2^-, \qquad \Psi_*\xi^{\alpha\beta} = \xi_1^+ + \xi_2^+, \qquad \Psi_*\xi^{\beta\alpha} = \xi_1^+ - \xi_2^+, \qquad \Psi_*\xi^{\beta\beta} = \xi_1^- + \xi_2^-.$$

- *If $a = -1$ then*

$$\Psi_*\xi^{\alpha\alpha} = \xi_1^+ + \xi_2^+, \qquad \Psi_*\xi^{\alpha\beta} = \xi_1^- - \xi_2^-, \qquad \Psi_*\xi^{\beta\alpha} = \xi_1^- + \xi_2^-, \qquad \Psi_*\xi^{\beta\beta} = \xi_1^+ - \xi_2^+.$$

Proof. Combining (4.4.9) and Lemma 4.10, we obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{Sel}_{(p)}(V) & \xrightarrow{\Psi_*} & \mathrm{Sel}_{(p)}(V_1) \oplus \mathrm{Sel}_{(p)}(V_2) \\ \downarrow{\scriptstyle \partial_p} & & \downarrow{\scriptstyle \partial_p \oplus \partial_p} \\ \mathrm{H}^1_s(\mathbb{Q}_p, V) & \xrightarrow{\Psi_*} & \mathrm{H}^1_s(\mathbb{Q}_p, V_1) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V_2) \end{array}$$

where each arrow is an isomorphism. Moreover, by Lemma 4.13 and Lemma 4.21, we also have the commutative diagram

$$\begin{array}{ccc} \mathrm{H}^1_s(\mathbb{Q}_p, V) & \xrightarrow{\Psi_*} & \mathrm{H}^1_s(\mathbb{Q}_p, V_1) \oplus \mathrm{H}^1_s(\mathbb{Q}_p, V_2) \\ \downarrow & & \downarrow \\ (\mathrm{H}^1_s(K_p, V_f) \otimes V_{gh})^{G_{\mathbb{Q}_p}} & \longrightarrow & (\mathrm{H}^1_s(K_p, V_f) \otimes V_1)^{G_{\mathbb{Q}_p}} \oplus (\mathrm{H}^1_s(K_p, V_f) \otimes V_2)^{G_{\mathbb{Q}_p}} \end{array}$$

where each arrow is an isomorphism and the bottom horizontal arrow is $1 \otimes \Psi$. By definition,

$$\partial_p \xi^{\alpha\alpha} = X_{\alpha\alpha} \otimes v_g^{\alpha} \otimes v_h^{\alpha} \in (\mathrm{H}^1_s(K_p, V_f) \otimes V_{gh}^{\alpha\alpha})^{G_{\mathbb{Q}_p}} \subseteq (\mathrm{H}^1_s(K_p, V_f) \otimes V_{gh})^{G_{\mathbb{Q}_p}}.$$

Since

$$X_{\alpha\alpha} = \begin{cases} X_+ & \text{if } \alpha_g \cdot \alpha_h = +1; \\ X_- & \text{if } \alpha_g \cdot \alpha_h = -1, \end{cases}$$

we deduce

$$(1 \otimes \Psi)(\partial_p \xi^{\alpha\alpha}) = \begin{cases} X_{\alpha\alpha} \otimes (v_1^+ + v_2^+) = \partial_p \xi_1^+ + \partial_p \xi_2^+ & \text{if } \alpha_g \cdot \alpha_h = +1 \\ X_{\alpha\alpha} \otimes (v_1^- - v_2^-) = \partial_p \xi_1^- - \partial_p \xi_2^- & \text{if } \alpha_g \cdot \alpha_h = -1. \end{cases}$$

Similar computations prove the remaining cases. □

Lemma 4.24. *Let $\delta_p : E(K_p) \otimes \mathbb{Q}_p \longrightarrow \mathrm{H}^1_f(K_p, V_f)$ denote Kummer's map.*

- *If $a = +1$,there is a local point $P^+ \in E(K_p)^+ \otimes \mathbb{Q}_p$ such that*

$$\pi_1^+ \operatorname{res}_p(\xi_1^-) = \delta_p P^+ \otimes v_1^+, \qquad \pi_2^+ \operatorname{res}_p(\xi_2^-), = \delta_p P^+ \otimes v_2^+ \qquad \pi_{\alpha\beta} \operatorname{res}_p(\xi^{\beta\beta}) = \delta_p P^+ \otimes v_g^{\alpha} \otimes v_h^{\beta}, \tag{4.4.12}$$

- *If $a = -1$, there is a local point $P^- \in E(K_p)^- \otimes \mathbb{Q}_p$ such that*

$$\pi_1^- \operatorname{res}_p(\xi_1^+) = \delta_p P^- \otimes v_1^-, \qquad \pi_2^- \operatorname{res}_p(\xi_2^+) = \delta_p P^- \otimes v_2^-, \qquad \pi_{\alpha\beta} \operatorname{res}_p(\xi^{\beta\beta}) = \delta_p P^- \otimes v_g^{\alpha} \otimes v_h^{\beta}; \tag{4.4.13}$$

Proof. We prove the claim when $a = -1$, as the other case works similarly. By Proposition 4.23,

$$\Psi_* \operatorname{res}_p(\xi^{\beta\beta}) = \operatorname{res}_p(\xi_1^+) - \operatorname{res}_p(\xi_2^+) \in \mathrm{H}^1(\mathbb{Q}_p, V_1) \oplus \mathrm{H}^1(\mathbb{Q}_p, V_2)$$

Write

$$\pi_{\alpha\beta} \operatorname{res}_p(\xi^{\beta\beta}) = Q_- \otimes v_g^{\alpha} \otimes v_h^{\beta} \in \mathrm{H}^1_f(K_p, V_f)^- \otimes V_{gh}^{\alpha\beta}.$$

Then, by Proposition 4.23 and (4.4.7)

$$\Psi_* \pi_{\alpha\beta} \operatorname{res}_p(\xi^{\beta\beta}) = Q_- \otimes (v_1^- - v_2^-) = Q_- \otimes v_1^- - Q_- \otimes v_2^- \tag{4.4.14}$$

in

$$(\mathrm{H}^1_f(K_p, V_f) \otimes (V_{\psi_1}^- \oplus V_{\psi_2}^-))^{G_{\mathbb{Q}_p}} \cong \mathrm{H}^1_f(K_p, V_1^-) \oplus \mathrm{H}^1_f(K_p, V_2^-).$$

On the other hand, by definition of the basis $\xi_i^\pm$ and using (4.4.7),

$$\pi_1^- \operatorname{res}_p(\xi_1^+) - \pi_2^- \operatorname{res}_p(\xi_2^+) \in \mathrm{H}^1_f(\mathbb{Q}_p, V_1^-) \oplus \mathrm{H}^1_f(\mathbb{Q}_p, V_2^-) \cong (\mathrm{H}^1_f(K_p, V_f) \otimes (V_{\psi_1}^- \oplus V_{\psi_2}^-))^{G_{\mathbb{Q}_p}}.$$

More precisely, we can write

$$\pi_1^- \operatorname{res}_p(\xi_1^+) - \pi_2^- \operatorname{res}_p(\xi_2^+) = P_1^- \otimes v_1^- - P_2^- \otimes v_2^- \tag{4.4.15}$$

for points $P_i^- \in E(K_p)^-$. Hence, comparing (4.4.15) and (4.4.14) we deduce that $Q_- = P_1^- = P_2^-$. □

Theorem 4.25. *Let $P^\pm \in E(K_p)^\pm$ be the local points of Lemma 4.24. Then*

$$I_p(f, g_\alpha, h_\alpha) = \frac{\sqrt{c} \cdot 2(1-1/p)^2 \sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle} \times \frac{1}{\mathcal{L}_{g_\alpha}} \times \log_p(P^a) \mod L^\times.$$

Proof. Combine Theorem 4.19 with (4.3.11) and Lemma 4.24 and use the relation between the Bloch–Kato logarithms given in (4.2.5). □

4.4.2 $I_p(f, g_\alpha, h_\alpha)$ and Kolyvagin classes

Recall the ring class characters ψ_1, ψ_2 appearing in the decomposition (4.4.1). Since they are ring class characters unramified at p, they factor through the Galois group $\mathrm{Gal}(H/K)$, where H is a ring class field of conductor prime to p.

Using the Kolyvagin classes described in §1.16.2, in this section we define elements $\mathbf{K}^{\psi_i}$ in the relaxed Selmer groups $\mathrm{Sel}_{(p)}(K, V_f \otimes \psi_i)$ for $i = 1, 2$. The aim of this section is to compare the classes $\mathbf{K}^{\psi_1}, \mathbf{K}^{\psi_2}$ with the local points in $E(K_p)^\pm$ appearing in Theorem 4.25, in order to obtain a formula for the value $I_p(f, g_\alpha, h_\alpha)$ in terms of these Kolyvagin classes. Using the notation of the previous sections, consider the dual exponential

$$\exp_\pm^* : \mathrm{H}^1_s(K_p, V_f)^\pm \longrightarrow \mathbb{Q}_p$$

of (4.2.6).

Recall the groups $\Phi_{m,\mathfrak{p}}$ and $\Phi_{\infty,\mathfrak{p}}$ defined in §4.1.1. As explained in §4.1, Tate's uniformisation induces the isomorphism $\varphi : \Phi_{\infty,\mathfrak{p}} \xrightarrow{\cong} \mathbb{Z}_p$ (in the notation of Lemma 4.3, $\varphi := \overline{\varphi}_{\mathrm{Tate}}$ maps $\bar{q}$ to 1). By Lemma 4.4 and Proposition 4.5, there is an injection

$$\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p) \subseteq \mathrm{H}^1_s(K_p, V_f).$$

Recall the choice σ_m of generator of G_m; if we still denote σ_m its restriction to $\mathrm{Gal}(F_{m,\mathfrak{p}}/H_\mathfrak{p}) = \mathrm{Gal}(F_{m,\mathfrak{p}}/K_p)$, then Γ_{ant} is generated by $\sigma_{\mathrm{ant}} := (\sigma_m)_m$. Recall the map $\partial_p : \mathrm{H}^1(H, V_f) \to \mathrm{H}^1_s(K_p, V_f)$ onto the singular quotient.

Proposition 4.26. *The class $\mathbf{K}$ lies in $\mathrm{Sel}_{(p)}(H, V_f)$ and $\partial_p \mathbf{K} \in \mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p)$. Moreover, if $\bar{\alpha}_m$ denotes the image of α_m in $\Phi_{m,\mathfrak{p}}$ and $\bar{\alpha} := (\bar{\alpha}_m)_m \in \Phi_{\infty,\mathfrak{p}}$, then*

$$\partial_p \mathbf{K}(\sigma_{\mathrm{ant}}) = \bar{\alpha}.$$

Proof. [BD97, Proposition 6.9, 1,2]. □

Recall the sign

$$a := a_p(E) \in \{\pm 1\}$$

and the period

$$\Pi_p := \varphi(\bar{\alpha}) \in \mathbb{Z}_p.$$

Proposition 4.27. *We have*

$$\partial_p \mathbf{K} \in \mathrm{H}^1_s(K_p, V_f)^{-a} \quad \text{and} \quad \exp^*_{-a} \partial_p \mathbf{K} = \frac{\mathrm{ord}_p(q_E)}{p} \Pi_p.$$

In particular

$$\mathrm{res}_p(\mathbf{K})^a := \mathrm{res}_p(\mathbf{K}) + a\, \mathrm{res}_p(\mathbf{K}^{\mathrm{Frob}_p})$$

is cristalline, i.e. lies in $\mathrm{H}^1_f(\mathbb{Q}_p, V_f)^a$.

Proof. Recall we can regard $\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p)$ as a subspace of $\mathrm{H}^1_s(K_p, V_f)$. By (4.1.23) Frob_p acts on $\mathrm{Hom}_{\mathrm{cont}}(\Gamma_{\mathrm{ant}}, \Phi_{\infty,\mathfrak{p}} \otimes \mathbb{Q}_p)$ as multiplication by $-a$. Hence $\mathrm{res}_p(\mathbf{K})$ belongs to $\mathrm{H}^1_s(K_p, V_f)^{-a}$ by Proposition 4.26, and the formula for $\exp^*_{-a} \partial_p \mathbf{K}$ follows from Corollary 4.9. ☐

For $i = 1, 2$ let

$$\mathrm{Tr}_i = \sum_{\sigma \in \tilde{S}} \psi_i(\sigma)\sigma : \mathrm{H}^1(H, V_f) \longrightarrow \mathrm{H}^1(K, V_f \otimes \psi_i)$$

denote the trace map onto the ψ_i-isotypic component and consider the isomorphism given by Shapiro's Lemma

$$\mathrm{Sh} : \mathrm{H}^1(K, V_f \otimes \psi_i) \xrightarrow{\cong} \mathrm{H}^1(\mathbb{Q}, V_f \otimes \mathrm{Ind}_{G_K}^{G_\mathbb{Q}}(\psi_i)) =: \mathrm{H}^1(\mathbb{Q}, V_i). \tag{4.4.16}$$

Define

$$\mathbf{K}^{\psi_i} := \mathrm{Sh}(\mathrm{Tr}_i(\mathbf{K})) \in \mathrm{H}^1(\mathbb{Q}, V_i).$$

It follows from Proposition 4.26 that $\mathbf{K}^{\psi_i}$ lies in $\mathrm{Sel}_{(p)}(V_i)$.

Frobenius element $\mathrm{Frob}_p \in G_{\mathbb{Q}_p}$ acts on $\mathrm{H}^1(K_p, V_f)$ as an involution and we may consider the decomposition in $\pm$-eigenspaces

$$\mathrm{H}^1(K_p, V_f \otimes L_p) = \mathrm{H}^1(K_p, V_f \otimes L_p)^+ \oplus \mathrm{H}^1(K_p, V_f \otimes L_p)^-.$$

As one readily verifies, Shapiro's isomorphism restricts to

$$\mathrm{Sh}_p : \mathrm{H}^1(K_p, V_f \otimes L_p)^{\pm} \xrightarrow{\cong} \mathrm{H}^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i}^{\pm}). \tag{4.4.17}$$

Corollary 4.28. *The Kolyvagin class* $\mathbf{K}^{\psi_i}$ *satisfies*

$$\mathbf{K}^{\psi_i} = \frac{h \mathrm{ord}_p(q_E)\Pi_p}{p} \cdot \xi_i^{-a}.$$

Proof. Since $\psi_{i|G_{K_p}} = 1$, the restriction of Tr_i to $\mathrm{H}^1(H_\mathfrak{p}, V_f \otimes L_p) = \mathrm{H}^1(K_p, V_f \otimes L_p)$ is multiplication by $h = [H : K]$. Hence $\mathrm{res}_p(\mathbf{K}^{\psi_i}) = h \cdot \mathrm{Sh}_p(\mathrm{res}_p(\mathbf{K}))$.

By Proposition 4.27 we have $\partial_p \mathbf{K} \in \mathrm{H}^1_s(K_p, V_f)^{-a}$, and by (4.4.17), it follows that $\partial_p \mathbf{K}^{\psi_i} \in \mathrm{H}^1_s(\mathbb{Q}_p, V_{\psi_i}^{-a})$. Recall the choice of basis $\xi_i^{\pm}$ made in Corollary 4.22. We may thus write

$$\mathbf{K}^{\psi_i} = \frac{\exp^*_{-a}(\partial_p \mathbf{K}^{\psi_i})}{\exp^*_{-a}(\partial_p \xi_i^{-a})} \cdot \xi_i^{-a}. \tag{4.4.18}$$

By definition of the basis $\{\xi_i^+, \xi_i^-\}$ and by (4.4.11), the denominator in the above expression is $\exp^*(X_{-a})\langle v_i^{-a}, \omega_i^{-a}\rangle = \langle v_i^{-a}, \omega_i^{-a}\rangle$. Let

$$R \otimes v_i^{-a} \in \mathrm{H}^1_s(K_p, V_f) \otimes V_{\psi_i}^{-a}$$

denote the image of $\partial_p \mathbf{K}^{\psi_i}$ via the isomorphism (4.4.7). Then

$$\mathbf{K}^{\psi_i} = \frac{\exp^*(R)\langle v_i^{-a}, \omega_i^{-a}\rangle}{\langle v_i^{-a}, \omega_i^{-a}\rangle} \cdot \xi_i^{-a} = \exp^*(R) \cdot \xi_i^{-a}.$$

In order to compute the dual exponential of R, we need the following explicit expression for (4.4.21) for Sh_p. Recall that, as a $L_p[G_{\mathbb{Q}_p}]$-module,

$$V_{\psi_i} = \mathrm{Ind}_{G_{K_p}}^{G_{\mathbb{Q}_p}}(1) = \{v : G_{\mathbb{Q}_p} \longrightarrow L_p \mid v(\sigma\tau) = v(\tau)\ \forall \sigma \in G_{K_p}, \tau \in G_{\mathbb{Q}_p}\}.$$

Consider the map

$$ev : V_{\psi_i} \longrightarrow L_p, \qquad v \mapsto v(1). \tag{4.4.19}$$

It is an equivariant G_{K_p}-morphism which is compatible with the inclusion $G_{K_p} \hookrightarrow G_{\mathbb{Q}_p}$, so it induces a morphism

$$\mathrm{Sh}_p^{-1} : \mathrm{H}^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i}) \longrightarrow \mathrm{H}^1(K_p, V_f \otimes L_p) = \mathrm{H}^1(K_p, V_f \otimes \psi_i), \tag{4.4.20}$$

which is the inverse of Shapiro's isomorphism (4.4.16) restricted to G_{K_p}. More explicitly, if $\xi : G_{\mathbb{Q}_p} \longrightarrow V_f \otimes V_{\psi_i}$ represents a class in $\mathrm{H}^1(\mathbb{Q}_p, V_f \otimes V_{\psi_i})$, then

$$\mathrm{Sh}_p^{-1}(\xi) := (\mathrm{id} \otimes ev) \circ \xi_{|G_{K_p}} \tag{4.4.21}$$

represents its image via (4.4.20). Recall that $R \otimes v_i^{-a}$ is the image of $h \cdot \partial_p \mathbf{K}$ via the composition

$$\mathrm{H}^1_s(K_p, V_f \otimes L_p)^{-a} \xrightarrow{\mathrm{Sh}_p} \mathrm{H}^1_s(\mathbb{Q}_p, V_f \otimes V_{\psi_i}^{-a}) \xrightarrow{\mathrm{Res}} \mathrm{H}^1_s(K_p, V_f \otimes V_{\psi_i}^{-a})^{G_{\mathbb{Q}_p}} \cong \mathrm{H}^1_s(K_p, V_f)^{-a} \otimes V_{\psi_i}^{-a}.$$

We conclude that the element $R \otimes v_i^{-a}$ is represented by the cocycle

$$h \cdot \mathrm{Res}(\mathrm{Sh}_p(\partial_p \mathbf{K})) : G_{K_p} \longrightarrow V_f \otimes V_{\psi_i}^{-a}$$

satisfying

$$R \cdot (v_i^{-a}(1)) = h \cdot \partial_p \mathbf{K}.$$

In other words, by (4.4.3), we have

$$\exp^*(R) = h \cdot \exp^*(\partial_p \mathbf{K}).$$

The statement follows by applying Proposition 4.27. □

Corollary 4.29. *We have*

$$I_p(f, g_\alpha, h_\alpha) = \frac{\sqrt{c} \cdot 2p(1 - 1/p)^2}{\mathrm{ord}_p(q_E)} \cdot \frac{\sqrt{L(E \otimes \rho, 1)}}{\pi^2 \langle f, f \rangle \Pi_p} \cdot \frac{1}{\mathcal{L}_{g_\alpha}} \times \log_p(Q_p^a) \quad \mod L^\times,$$

where $Q_p^a \in E(K_p)^a$ *is characterized by* $\delta_p(Q_p^a) = \mathrm{res}_p(\mathbf{K})^a \in \mathrm{H}^1_f(K_p, V_f)^a$.

Proof. By (4.4.17), if $\pi_a : \mathrm{H}^1(\mathbb{Q}_p, V_i) \longrightarrow \mathrm{H}^1(\mathbb{Q}_p, V_i^a)$ denotes the natural projection, then

$$\pi_a \,\mathrm{res}_p(\mathbf{K}^{\psi_i}) = h \cdot \mathrm{Sh}_p(\mathrm{res}_p(\mathbf{K})^a) \in \mathrm{H}^1_f(\mathbb{Q}_p, V_i^a). \tag{4.4.22}$$

Arguing as in the proof of Corollary 4.28, write

$$A \otimes v_i^a \in \mathrm{H}^1_f(K_p, V_f)^a \otimes V_{\psi_i}^a$$

for the image of (4.4.22) via the isomorphism (4.4.7). Then

$$A = \frac{h \cdot \mathrm{res}_p(\mathbf{K})^a}{v_i^a(1)} = h \cdot \mathrm{res}_p(\mathbf{K})^a.$$

Combining this with Corollary 4.28 and Lemma 4.24, we obtain

$$h \cdot \mathrm{res}_p(\mathbf{K})^a \otimes v_i^a = \pi_a \,\mathrm{res}_p(\mathbf{K}^{\psi_i}) = \frac{h \mathrm{ord}_p(q_E) \Pi_p}{p} \delta_p(P^a) \otimes v_i^a.$$

Then

$$\delta_p(P^a) = \frac{p}{\mathrm{ord}_p(q_E) \Pi_p} \cdot \mathrm{res}_p(\mathbf{K})^a,$$

and the thesis follows by applying Theorem 4.25.

□

Bibliography

[AI19] F. Andreatta and A. Iovita. Katz type p-adic L-functions for primes p non-split in the CM field. *Preprint*, 2019.

[AL78] A.O.L. Atkin and W.C.W. Li. Twists of newforms and pseudo-eigenvalues of W-operators. *Invent. Math.*, 48(3):221–243, 1978.

[BCD$^+$14] M. Bertolini, F. Castella, H. Darmon, S. Dasgupta, K. Prasanna, and V. Rotger. p-adic L-functions and Euler systems: a tale in two trilogies. In *Automorphic forms and Galois representations. Vol. 1*, volume 414 of *London Math. Soc. Lecture Note Ser.*, pages 52–101. Cambridge Univ. Press, Cambridge, 2014.

[BD90] M. Bertolini and H. Darmon. Kolyvagin's descent and Mordell-Weil groups over ring class fields. *J. Reine Angew. Math.*, 412:63–74, 1990.

[BD96] M. Bertolini and H. Darmon. Heegner points on Mumford-Tate curves. *Invent. Math.*, 126(3):413–456, 1996.

[BD97] M. Bertolini and H. Darmon. A rigid analytic gross-zagier formula and arithmetic applications. *Annals of Mathematics*, 146(1):111–147, 1997.

[BD16] J. Bellaïche and M. Dimitrov. On the eigencurve at classical weight 1 points. *Duke Math. J.*, 165(2):245–266, 2016.

[BDP12] M. Bertolini, H. Darmon, and K. Prasanna. p-adic Rankin L-series and rational points on CM elliptic curves. *Pacific J. Math.*, 260(2):261–303, 2012.

[BDP13] M. Bertolini, H. Darmon, and K. Prasanna. Generalized Heegner cycles and p-adic Rankin L-series. *Duke Math. J.*, 162(6):1033–1148, 2013. With an appendix by Brian Conrad.

[BDP14] M. Bertolini, H. Darmon, and K. Prasanna. Chow-Heegner points on CM elliptic curves and values of p-adic L-functions. *Int. Math. Res. Not. IMRN*, (3):745–793, 2014.

[BDP17] M. Bertolini, H. Darmon, and K. Prasanna. p-adic L-functions and the coniveau filtration on Chow groups. *J. Reine Angew. Math.*, 731:21–86, 2017. With an appendix by Brian Conrad.

[BDP18] A. Betina, M. Dimitrov, and A. Pozzi. On the failure of Gorensteinness at weight 1 Eisenstein points of the eigencurve. *Preprint. arXiv:1804.00648*, 2018.

[BDR15] M. Bertolini, H. Darmon, and V. Rotger. Beilinson-Flach elements and Euler systems II: the Birch-Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-series. *J. Algebraic Geom.*, 24(3):569–604, 2015.

[Bel] J. Belaïche. An introduction to the conjecture of Bloch and Kato. *Two lectures at the Clay Mathematical Institute Summer School, Honolulu, Hawaii, 2009.* http://people.brandeis.edu/ jbellaic/bkhawaii5.pdf.

[BK90] S. Bloch and K. Kato. L-functions and Tamagawa numbers of motives. In *The Grothendieck Festschrift, Vol. I*, volume 86 of *Progr. Math.*, pages 333–400. Birkhäuser Boston, Boston, MA, 1990.

[BP06] Siegfried Böcherer and A. A. Panchishkin. Admissible p-adic measures attached to triple products of elliptic cusp forms. *Doc. Math.*, (Extra Vol.):77–132, 2006.

[Bro15] E. H. Brooks. Shimura curves and special values of p-adic l-functions. *International Mathematics Research Notices*, (12):65. 4177–4241, 2015. National Licences.

[BSV] M. Bertolini, M. A. Seveso, and R. Venerucci. Reciprocity laws for balanced diagonal classes. In *Heegner points, Stark–Heegner points and diagonal classes,*.

[Cas18] F. Castella. On the exceptional specialisations of big Heegner points. *J. Inst. Math. Jussieu*, 17(1):207–240, 2018.

[Cas19] F. Castella. On the p-adic variation of Heegner points. *J. Inst. Math. Jussieu*, 2019.

[CW77] J. Coates and A. Wiles. On the conjecture of Birch and Swinnerton-Dyer. *Invent. Math.*, 39(3):223–251, 1977.

[DI95] F. Diamond and J. Im. *Modular forms and modular curves*, volume 17 of *CMS Conf. Proc.* Amer. Math. Soc., Providence, RI, 1995.

[DLR15] H. Darmon, A. Lauder, and V. Rotger. Stark points and p-adic iterated integrals attached to modular forms of weight one. *Forum Math. Pi*, 3:e8, 95, 2015.

[DRa] H. Darmon and V. Rotger. p-adic families of diagonal cycles. In *Heegner points, Stark–Heegner points and diagonal classes,*.

[DRb] H. Darmon and V. Rotger. Stark–Heegner points and diagonal classes. In *Heegner points, Stark–Heegner points and diagonal classes,*.

[DR73] P. Deligne and M. Rapoport. Les schémas de modules de courbes elliptiques. In *Modular functions of one variable, II (Proc. Internat. Summer School, Univ. Antwerp, Antwerp, 1972)*, pages 143–316. Lecture Notes in Math., Vol. 349, 1973.

[DR14] H. Darmon and V. Rotger. Diagonal cycles and Euler systems I: A p-adic Gross-Zagier formula. *Ann. Sci. Éc. Norm. Supér. (4)*, 47(4):779–832, 2014.

[DR16] H. Darmon and V. Rotger. Elliptic curves of rank two and generalised kato classes. *Research in the Mathematical Sciences*, 3(1):27, Aug 2016.

[DR17] H. Darmon and V. Rotger. Diagonal cycles and Euler systems II: The Birch and Swinnerton-Dyer conjecture for Hasse-Weil-Artin L-functions. *J. Amer. Math. Soc.*, 30(3):601–672, 2017.

[DS74] P. Deligne and J.-P. Serre. Formes modulaires de poids 1. *Ann. Sci. École Norm. Sup. (4)*, 7:507–530 (1975), 1974.

[DS91] C. Deninger and A. J. Scholl. The Beĭlinson conjectures. In *L-functions and arithmetic (Durham, 1989)*, volume 153 of *London Math. Soc. Lecture Note Ser.*, pages 173–209. Cambridge Univ. Press, Cambridge, 1991.

[GG20] F. Gatti and X. Guitart. On the elliptic Stark conjecture in higher weight. *Publ. Mat.*, 64(2):577–619, 2020.

[GGMR19] F. Gatti, X. Guitart, M. Masdeu, and V. Rotger. Special values of triple-product p-adic L-functions and non-crystalline diagonal classes. *Preprint. arXiv:1912.07892*, 2019.

[Gha05] E. Ghate. Ordinary forms and their local galois representations. In Rajat Tandon, editor, *Algebra and Number Theory: Proceedings of the Silver Jubilee Conference University of Hyderabad*, pages 226–242. Hindustan Book Agency, Gurgaon, 2005.

[GR20] F. Gatti and V. Rotger. Kolyvagin classes versus non-cristalline diagonal classes. *Preprint*, 2020.

[Gro80] B. H. Gross. Arithmetic on elliptic curves with complex multiplication. volume 776 of *Lecture Notes in Mathematics*, pages iii+95. Springer, Berlin, 1980. With an appendix by B. Mazur.

[GZ86] B. H. Gross and D. B. Zagier. Heegner points and derivatives of L-series. *Invent. Math.*, 84(2):225–320, 1986.

[Har77] R. Hartshorne. *Algebraic geometry.* Springer-Verlag, New York-Heidelberg, 1977. Graduate Texts in Mathematics, No. 52.

[Hid86] H. Hida. Galois representations into $\mathrm{GL}_2(\mathbf{Z}_p[[X]])$ attached to ordinary cusp forms. *Invent. Math.*, 85(3):545–613, 1986.

[Hid93] H. Hida. p-ordinary cohomology groups for SL(2) over number fields. *Duke Math. J.*, 69(2):259–314, 1993.

[HK91] Michael Harris and Stephen S. Kudla. The central critical value of a triple product L-function. *Ann. of Math. (2)*, 133(3):605–672, 1991.

[How07] B. Howard. Variation of heegner points in hida families. *Invent. Math.*, 167(1):91–128, 2007.

[Hsi19] M.-L. Hsieh. Hida families and p-adic triple product L-functions. *American Journal of Mathematics*, 2019.

[Ich08] Atsushi Ichino. Trilinear forms and the central values of triple product L-functions. *Duke Math. J.*, 145(2):281–307, 2008.

[Jan90] U. Jannsen. *Mixed motives and algebraic K-theory*, volume 1400 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1990. With appendices by S. Bloch and C. Schoen.

[Kar00] N. A. Karpenko. Weil transfer of algebraic cycles. *Indag. Math. (N.S.)*, 11(1):73–86, 2000.

[Kat76] N. M. Katz. p-adic interpolation of real analytic Eisenstein series. *Ann. of Math. (2)*, 104(3):459–571, 1976.

[Kat93] K. Kato. Lectures on the approach to Iwasawa theory for Hasse-Weil L-functions via B_{dR}. I. 1553:50–163, 1993.

[Kat04] K. Kato. p-adic Hodge theory and values of zeta functions of modular forms. Number 295, pages ix, 117–290. 2004. Cohomologies p-adiques et applications arithmétiques. III.

[Kna92] A. W. Knapp. *Elliptic curves*, volume 40 of *Mathematical Notes*. Princeton University Press, Princeton, NJ, 1992.

[Kol90] V. A. Kolyvagin. Euler systems. In *The Grothendieck Festschrift, Vol. II*, volume 87 of *Progr. Math.*, pages 435–483. Birkhäuser Boston, Boston, MA, 1990.

[Lau14] A.G.B. Lauder. Efficient computation of Rankin p-adic L-functions. In *Computations with modular forms*, volume 6 of *Contrib. Math. Comput. Sci.*, pages 181–200. Springer, Cham, 2014.

[LSZ17] D. Loeffler, C. Skinner, and S. L. Zerbes. Euler systems for $GSp(4)$. *To appear in J. Eur. Math. Soc.*, 2017.

[LSZ19] D. Loeffler, C. Skinner, and S. L. Zerbes. Syntomic regulators of Asai–Flach classes. In *Iwasawa 2017, Tokyo, Japan, 19 -28 Jul 2017.* Advanced Studies in Pure Mathematics, 2019.

[LZ14] D. Loeffler and S. L. Zerbes. Iwasawa theory and p-adic L-functions over $\mathbb{Z}_p^2$-extensions. *Int. J. Number Theory*, 10(8):2045–2095, 2014.

[Mil13] J. S. Milne. Lectures on etale cohomology (v2.21), 2013. Available at www.jmilne.org/math/.

[MR04] B. Mazur and K. Rubin. Kolyvagin systems. *Mem. Amer. Math. Soc.*, 168(799):viii+96, 2004.

[Nek92] J. Nekovář. Kolyvagin's method for Chow groups of Kuga-Sato varieties. *Invent. Math.*, 107(1):99–125, 1992.

[Nek94] J. Nekovář. Beĭlinson's conjectures. In *Motives (Seattle, WA, 1991)*, volume 55 of *Proc. Sympos. Pure Math.*, pages 537–570. Amer. Math. Soc., Providence, RI, 1994.

[Nek95] J. Nekovář. On the p-adic height of Heegner cycles. *Math. Ann.*, 302(4):609–686, 1995.

[Neu99] J. Neukirch. *Algebraic number theory*, volume 322 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999. Translated from the 1992 German original and with a note by Norbert Schappacher, With a foreword by G. Harder.

[Oht95] M. Ohta. On the p-adic Eichler-Shimura isomorphism for Λ-adic cusp forms. *J. Reine Angew. Math.*, 463:49–98, 1995.

[PSR87] I. Piatetski-Shapiro and S. Rallis. Rankin triple L-functions. *Compositio Mathematica*, 64(1):31–115, 1987.

[S$^+$19] W. A. Stein et al. *Sage Mathematics Software (Version 8.9).* The Sage Development Team, 2019. `http://www.sagemath.org`.

[Sch88] N. Schappacher. *Periods of Hecke characters*, volume 1301 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1988.

[Sch90] A. J. Scholl. Motives for modular forms. *Invent. Math.*, 100(2):419–430, 1990.

[Sil86] J. H. Silverman. *The arithmetic of elliptic curves*, volume 106 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1986.

[Sil94] J. H. Silverman. *Advanced topics in the arithmetic of elliptic curves*, volume 151 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1994.

[TW95] R. Taylor and A. Wiles. Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)*, 141(3):553–572, 1995.

[Wil88] A. Wiles. On ordinary λ-adic representations associated to modular forms. *Invent. Math.*, 94(3):529–573, 1988.

[Wil95] A. Wiles. Modular elliptic curves and Fermat's Last Theorem. *Ann. of Math. (2)*, 141(3):443–551, 1995.

[YZZ13] X. Yuan, S. Zhang, and W. Zhang. *The Gross-Zagier formula on Shimura curves*, volume 184 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2013.

[Zha97] S. Zhang. Heights of Heegner cycles and derivatives of L-series. *Invent. Math.*, 130(1):99–152, 1997.

[Zha01a] S. Zhang. Gross-Zagier formula for GL_2. *Asian J. Math.*, 5(2):183–290, 2001.

[Zha01b] S. Zhang. Heights of Heegner points on Shimura curves. *Ann. of Math. (2)*, 153(1):27–147, 2001.