Anexo 3.1

Deducción de las ecuaciónes del movimiento de un sólido rígido giratorio accionado mediante un actuador lineal

En el apartado cinemático se había obtenido:

$$\vartheta_4 = \frac{1}{\overline{O_4 B}} \sin(\vartheta_2 \quad \vartheta_4)\ v_{A3/2} \tag{3.6}$$

$$\vartheta_2 = \frac{1}{\overline{O_2 B}} \cos(\vartheta_2 \quad \vartheta_4) \sin(\vartheta_2 \quad \vartheta_4)\ v_{A3/2} \tag{3.7}$$

Estas expresiones serán muy útiles al derivar el Lagrangiano.

La ecuación del movimiento, aplicando Lagrange, es :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial x_{A3/2}}\right) \quad \frac{\partial L}{\partial x_{A3/2}} = F_{cil} \tag{A3.1.1}$$

en la que L es:

$$L = L2 \quad L3 \quad L4 = Ec2 \quad Ep2 \quad Ec3 \quad Ep3 \quad Ec4 \quad Ep4 \tag{A3.1.2}$$

y las expresiones de las energías cinética y potencial de cada sólido son:

$$Ec2 = \frac{1}{2}\ m_2\ \overline{O_2 G_2}^2\ \vartheta_2^2 \quad \frac{1}{2}\ I_{G2}\ \vartheta_2^2 \tag{A3.1.3}$$

$$Ep2 = m_2\ g\ \overline{O_2 G_2} \sin\vartheta_2 \tag{A3.1.4}$$

$$Ec3 = \frac{1}{2}\ m_3\ (\overline{O_2 A} \quad \overline{A G_3})^2\ \vartheta_2^2 \quad \frac{1}{2}\ I_{G3}\ \vartheta_2^2 \quad \frac{1}{2}\ m_3\ v_{A3/2}^2 \tag{A3.1.5}$$

$$Ep3 = m_3\ g\ (\overline{O_2 A} \quad \overline{A G_3}) \sin\vartheta_2 \tag{A3.1.6}$$

$$Ec4 = \frac{1}{2}\ m_4\ \overline{O_4 G_4}^2\ \vartheta_4^2 \quad \frac{1}{2}\ I_{G4}\ \vartheta_4^2 \tag{A3.1.7}$$

$$Ep4 = m_4\ g\ \overline{O_4 G_4} \sin\vartheta_4 \quad . \tag{A3.1.8}$$

Para facilitar el cálculo se aplicará varias veces la regla de la cadena:

$$\frac{\partial L2}{\partial x_{A3/2}}=\frac{\partial Ec2}{\partial x_{A3/2}}=\frac{\partial Ec2}{\partial \vartheta_2}\ \frac{\partial \vartheta_2}{\partial x_{A3/2}} \qquad \text{(A3.1.9)}$$

$$\frac{\partial L2}{\partial x_{A3/2}}=\frac{\partial Ec2}{\partial x_{A3/2}} \quad \frac{\partial Ep2}{\partial \vartheta_2}\ \frac{\partial \vartheta_2}{\partial x_{A3/2}} \qquad \text{(A3.1.10)}$$

pues:

$$\frac{d\vartheta_2}{\partial x_{A3/2}}=\frac{d\vartheta_2}{\partial x_{A3/2}} \quad . \qquad \text{(A3.1.11)}$$

Realizando los cálculos se obtiene:

$$\frac{\partial L2}{\partial x_{A3/2}}=\frac{\partial Ec2}{\partial \vartheta_2}\ \frac{d\vartheta_2}{\partial x_{A3/2}}=(m_2\ \overline{O_2G_2}^2\ \vartheta_2 \quad I_{G2}\ \vartheta_2)\ \frac{1}{\overline{O_2B}}\ \cos(\vartheta_2 \quad \vartheta_4)\ \sin(\vartheta_2 \quad \vartheta_4) \qquad \text{(A3.1.12)}$$

y de aquí:

$$\frac{\partial L2}{\partial x_{A3/2}}=(m_2\ \overline{O_2G_2}^2 \quad I_{G2})\ [\frac{1}{x_{A3/2} \quad \overline{AB}}\ \frac{1}{2}\sin 2(\vartheta_2 \quad \vartheta_4)]^2\ v_{A3/2} \qquad \text{(A3.1.13)}$$

derivando respecto del tiempo:

$$\frac{d}{dt}(\frac{\partial L2}{\partial x_{A3/2}})=(m_2\ \overline{O_2G_2}^2 \quad I_{G2})\ [\frac{1}{x_{A3/2} \quad \overline{AB}}\ \frac{1}{2}\sin 2(\vartheta_2 \quad \vartheta_4)]^2\ a_{A3/2} \quad ...$$

$$2\ [\frac{(m_2\ \overline{O_2G_2}^2 \quad I_{G2})}{x_{A3/2} \quad \overline{AB}}\ \frac{1}{2}\sin 2(\vartheta_2 \quad \vartheta_4)]\ v_{A3/2}\ [\frac{(\vartheta_2 \quad \vartheta_4)\cos 2(\vartheta_2 \quad \vartheta_4)}{(x_{A3/2} \quad \overline{AB})}] \quad ... \qquad \text{(A3.1.14)}$$

$$2\ [\frac{(m_2\ \overline{O_2G_2}^2 \quad I_{G2})}{x_{A3/2} \quad \overline{AB}}\ \frac{1}{2}\sin 2(\vartheta_2 \quad \vartheta_4)]\ v_{A3/2}\ [\frac{\frac{1}{2}\sin 2(\vartheta_2 \quad \vartheta_4)\ v_{A3/2}}{(x_{A3/2} \quad \overline{AB})^2}]$$

Así mismo:

$$\frac{\partial Ec2}{\partial x_{A3/2}}= \quad \frac{\partial Ec2}{\partial \vartheta_2}\ \frac{\partial \vartheta_2}{\partial x_{A3/2}}=(m_2\ \overline{O_2G_2}^2\ \vartheta_2 \quad I_{G2}\ \vartheta_2)\ \frac{\partial \vartheta_2}{\partial x_{A3/2}}=...$$

$$(m_2\ \overline{O_2G_2}^2\ \vartheta_2 \quad I_{G2}\ \vartheta_2)\ (\frac{\vartheta_2}{x_{A3/2} \quad \overline{AB}} \quad \frac{1}{x_{A3/2} \quad \overline{AB}}\ (\vartheta_2 \quad \vartheta_4)\cos 2(\vartheta_2 \quad \vartheta_4)) \qquad \text{(A3.1.15)}$$

finalmente:

$$\frac{\partial Ep2}{\partial x_{A3/2}}=\frac{\partial Ep2}{\partial \vartheta_2}\ \frac{\partial \vartheta_2}{\partial x_{A3/2}}=m_2\ g\ \overline{O_2G_2}\cos(\vartheta_2)\ \frac{1}{\overline{O_2B}}\ \cos(\vartheta_2 \quad \vartheta_4)\ \sin(\vartheta_2 \quad \vartheta_4) \qquad \text{(A3.1.16)}$$

De manera similar se hace el cálculo del sólido 3:

$$\frac{\partial Ec3}{\partial x_{A3/2}}=\frac{\partial(\frac{1}{2}\ m_3\ (\overline{O_2A} \quad \overline{AG_3})^2\ \vartheta_2^2 \quad \frac{1}{2}\ I_{G3}\ \vartheta_2^2)}{\partial \vartheta_2}\ \frac{d\vartheta_2}{\partial x_{A3/2}} \quad \frac{\partial(\frac{1}{2}\ m_3\ v_{A3/2}^2)}{\partial x_{A3/2}}=... \qquad \text{(A3.1.17)}$$

$$[m_3\ (x_{A3/2} \quad \overline{AG_3})^2 \quad I_{G3}]\ [\frac{1}{x_{A3/2} \quad \overline{AB}}\ \frac{1}{2}\sin 2(\vartheta_2 \quad \vartheta_4)]^2\ v_{A3/2} \quad m_3\ v_{A3/2}$$

$$\frac{d}{dt}(\frac{\partial L3}{\partial x_{A3/2}})=[m_3\,(x_{A3/2}\quad \overline{AG_3})^2\quad I_{G3}]\,[\frac{1}{x_{A3/2}\quad \overline{AB}}\,\frac{1}{2}\sin 2(\vartheta_2\quad \vartheta_4)]^2\,a_{A3/2}\quad \ldots$$

$$2\,[\frac{[m_3\,(x_{A3/2}\quad \overline{AG_3})^2\quad I_{G3}]}{x_{A3/2}\quad \overline{AB}}\,\frac{1}{2}\sin 2(\vartheta_2\quad \vartheta_4)]\,v_{A3/2}\,[\frac{(\vartheta_2\quad \vartheta_4)\cos 2(\vartheta_2\quad \vartheta_4)}{(x_{A3/2}\quad \overline{AB})}]\quad \ldots$$

$$2\,[\frac{[m_3\,(x_{A3/2}\quad \overline{AG_3})^2\quad I_{G3}]}{x_{A3/2}\quad \overline{AB}}\,\frac{1}{2}\sin 2(\vartheta_2\quad \vartheta_4)]\,v_{A3/2}\,[\frac{\frac{1}{2}\sin 2(\vartheta_2\quad \vartheta_4)\,v_{A3/2}}{(x_{A3/2}\quad \overline{AB})^2}]\quad \ldots$$

$$2\;m_3\,(x_{A3/2}\quad \overline{AG_3})\,[\frac{1}{x_{A3/2}\quad \overline{AB}}\,\frac{1}{2}\sin 2(\vartheta_2\quad \vartheta_4)]^2\,v_{A3/2}^2\quad m3\;a_{A3/2}$$

(A3.1.18)

$$\frac{\partial Ec3}{\partial x_{A3/2}}=\quad\frac{\partial(\frac{1}{2}\,m_3\,(\overline{O_2A}\quad \overline{AG_3})^2\,\vartheta_2^2\quad \frac{1}{2}\,I_{G3}\,\vartheta_2^2)}{\partial\vartheta_2}\,\frac{\partial\vartheta_2}{\partial x_{A3/2}}=\ldots$$

$$[m_3\,(x_{A3/2}\quad \overline{AG_3})^2\,\vartheta_2\quad I_{G3}\,\vartheta_2]\,\frac{\partial\vartheta_2}{\partial x_{A3/2}}=\ldots$$

$$[m_3\,(x_{A3/2}\quad \overline{AG_3})^2\,\vartheta_2\quad I_{G3}\,\vartheta_2]\,(\frac{\vartheta_2}{x_{A3/2}\quad \overline{AB}}\quad \frac{1}{x_{A3/2}\quad \overline{AB}}\,(\vartheta_2\quad \vartheta_4)\cos 2(\vartheta_2\quad \vartheta_4))$$

(A3.1.19)

$$\frac{\partial Ep3}{\partial x_{A3/2}}=m_3\,g\,\sin\vartheta_2\quad m_3\,g\,(x_{A3/2}\quad \overline{AG_3})\cos(\vartheta_2)\,\frac{1}{\overline{O_2B}}\cos(\vartheta_2\quad \vartheta_4)\sin(\vartheta_2\quad \vartheta_4)$$

(A3.1.20)

y finalmente, para el sólido 4, se obtienen estas ecuaciones :

$$\frac{\partial L4}{\partial x_{A3/2}}=\frac{\partial Ec4}{\partial\vartheta_4}\,\frac{d\,\vartheta_4}{\partial x_{A3/2}}=(m_4\,\overline{O_4G_4}^2\,\vartheta_4\quad I_{G4}\,\vartheta_4)\,\frac{1}{\overline{O_4B}}\sin(\vartheta_2\quad \vartheta_4)=\ldots$$

$$(m_4\,\overline{O_4G_4}^2\quad I_{G4})\,[\frac{1}{\overline{O_4B}}\sin(\vartheta_2\quad \vartheta_4)]^2\,v_{A3/2}$$

(A3.1.21)

$$\frac{d}{dt}(\frac{\partial L4}{\partial x_{A3/2}})=(m_4\,\overline{O_4G_4}^2\quad I_{G4})\,[\frac{1}{\overline{O_4B}}\sin(\vartheta_2\quad \vartheta_4)]^2\,a_{A3/2}\quad \ldots$$

$$2\,(m_4\,\overline{O_4G_4}^2\quad I_{G4})\,[\frac{1}{\overline{O_4B}}\sin(\vartheta_2\quad \vartheta_4)]\,v_{A3/2}\,[\frac{(\vartheta_2\quad \vartheta_4)\cos(\vartheta_2\quad \vartheta_4)}{\overline{O_4B}}]$$

(A3.1.22)

$$\frac{\partial Ec4}{\partial x_{A3/2}}=\quad\frac{\partial Ec4}{\partial\vartheta_4}\,\frac{\partial\vartheta_4}{\partial x_{A3/2}}=\quad(m_4\,\overline{O_4G_4}^2\,\vartheta_4\quad I_{G4}\,\vartheta_4)\,\frac{\partial\vartheta_4}{\partial x_{A3/2}}=\ldots$$

$$(m_4\,\overline{O_4G_4}^2\,\vartheta_4\quad I_{G4}\,\vartheta_4)\,\frac{1}{\overline{O_4B}}\,(\vartheta_2\quad \vartheta_4)\cos(\vartheta_2\quad \vartheta_4)$$

(A3.1.23)

$$\frac{\partial Ep4}{\partial x_{A3/2}} = \frac{\partial Ep4}{\partial \vartheta_4} \frac{\partial \vartheta_4}{\partial x_{A3/2}} = m_4\, g\, \overline{O_4 G_4} \cos(\vartheta_4) \frac{1}{\overline{O_2 B}} \sin(\vartheta_2 \quad \vartheta_4) \qquad \text{(A3.1.24)}$$

Agrupando todos estos términos se obtiene finalmente:

$$F_{cil} = (m_2\, \overline{O_2 G_2}^2 \quad I_{G2}) \left[\frac{1}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right]^2 a_{A3/2} \quad ...$$

$$2 \left[\frac{(m_2\, \overline{O_2 G_2}^2 \quad I_{G2})}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right] v_{A3/2} \left[\frac{(\vartheta_2 \quad \vartheta_4) \cos 2(\vartheta_2 \quad \vartheta_4)}{(x_{A3/2} \quad \overline{AB})}\right] \quad ...$$

$$2 \left[\frac{(m_2\, \overline{O_2 G_2}^2 \quad I_{G2})}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right] v_{A3/2} \left[\frac{\frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\, v_{A3/2}}{(x_{A3/2} \quad \overline{AB})^2}\right] \quad ...$$

$$(m_2\, \overline{O_2 G_2}^2\, \vartheta_2 \quad I_{G2}\, \vartheta_2) \left(\frac{\vartheta_2}{x_{A3/2} \quad \overline{AB}} \quad \frac{1}{x_{A3/2} \quad \overline{AB}} (\vartheta_2 \quad \vartheta_4) \cos 2(\vartheta_2 \quad \vartheta_4)\right) \quad ...$$

$$m_2\, g\, \overline{O_2 G_2} \cos(\vartheta_2) \frac{1}{\overline{O_2 B}} \cos(\vartheta_2 \quad \vartheta_4) \sin(\vartheta_2 \quad \vartheta_4) \quad ...$$

$$[m_3 (x_{A3/2} \quad \overline{AG_3})^2 \quad I_{G3}] \left[\frac{1}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right]^2 a_{A3/2} \quad ...$$

$$2 \left[\frac{[m_3 (x_{A3/2} \quad \overline{AG_3})^2 \quad I_{G3}]}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right] v_{A3/2} \left[\frac{(\vartheta_2 \quad \vartheta_4) \cos 2(\vartheta_2 \quad \vartheta_4)}{(x_{A3/2} \quad \overline{AB})}\right] \quad ...$$

$$2 \left[\frac{[m_3 (x_{A3/2} \quad \overline{AG_3})^2 \quad I_{G3}]}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right] v_{A3/2} \left[\frac{\frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\, v_{A3/2}}{(x_{A3/2} \quad \overline{AB})^2}\right] \quad ...$$

$$2\, m_3 (x_{A3/2} \quad \overline{AG_3}) \left[\frac{1}{x_{A3/2} \quad \overline{AB}} \frac{1}{2} \sin 2(\vartheta_2 \quad \vartheta_4)\right]^2 v_{A3/2}^2 \quad m3\, a_{A3/2} \quad ...$$

$$[m_3 (x_{A3/2} \quad \overline{AG_3})^2\, \vartheta_2 \quad I_{G3}\, \vartheta_2] \left(\frac{\vartheta_2}{x_{A3/2} \quad \overline{AB}} \quad \frac{1}{x_{A3/2} \quad \overline{AB}} (\vartheta_2 \quad \vartheta_4) \cos 2(\vartheta_2 \quad \vartheta_4)\right) \quad ...$$

$$m_3\, g \sin\vartheta_2 \quad m_3\, g\, (x_{A3/2} \quad \overline{AG_3}) \cos(\vartheta_2) \frac{1}{\overline{O_2 B}} \cos(\vartheta_2 \quad \vartheta_4) \sin(\vartheta_2 \quad \vartheta_4) \quad ...$$

$$(m_4\, \overline{O_4 G_4}^2 \quad I_{G4}) \left[\frac{1}{\overline{O_4 B}} \sin(\vartheta_2 \quad \vartheta_4)\right]^2 a_{A3/2} \quad ...$$

$$2\, (m_4\, \overline{O_4 G_4}^2 \quad I_{G4}) \left[\frac{1}{\overline{O_4 B}} \sin(\vartheta_2 \quad \vartheta_4)\right] v_{A3/2} \left[\frac{(\vartheta_2 \quad \vartheta_4) \cos(\vartheta_2 \quad \vartheta_4)}{\overline{O_4 B}}\right] \quad ...$$

$$(m_4\, \overline{O_4 G_4}^2\, \vartheta_4 \quad I_{G4}\, \vartheta_4) \frac{1}{\overline{O_4 B}} (\vartheta_2 \quad \vartheta_4) \cos(\vartheta_2 \quad \vartheta_4) \quad ...$$

$$m_4\, g\, \overline{O_4 G_4} \cos(\vartheta_4) \frac{1}{\overline{O_2 B}} \sin(\vartheta_2 \quad \vartheta_4)$$

(A3.1.25)