Universitat Autònoma de Barcelona

# Transport equations via smooth kernels

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UNIVERSITAT AUTÒNOMA DE BARCELONA

DOCTORAL THESIS

# Transport equations via smooth kernels

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*in*

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# Declaration of Authorship

I, Juan Carlos CANTERO GUARDEÑO, declare that this thesis titled, "Transport equations via smooth kernels" and the work presented in it are my own. I confirm that:

- This work was done wholly or mainly while in candidature for a research degree at this University.
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Signed:

Date: June 3, 2021

*"¿Quién iba a decir que sin borrón no hay trato?*
*(...)*
*¿Quién iba a decir que sin carbón no hay Reyes Magos?"*

Guille Galván, *Los Días Raros*, Vetusta Morla

UNIVERSITAT AUTÒNOMA DE BARCELONA

# *Abstract*

Facultat de Ciències
Departament de Matemàtiques

Doctor of Philosophy

**Transport equations
via
smooth kernels**

by Juan Carlos CANTERO GUARDEÑO

Throughout this thesis we consider the non-linear non-local homogeneous transport equation

$$\begin{cases} \rho_t(x,t) + v(x,t) \cdot \nabla \rho(x,t) = 0, \\ v(x,t) = [K * \rho(\cdot,t)](x), \\ \rho(x,0) = \rho_0(x). \end{cases}$$

By non-linear we mean that the velocity field is not given: it is also an unknown of the partial differential equation and it is related to the scalar $\rho$. This makes that if we have $\rho_1$ and $\rho_2$ solutions of the PDE then their sum $\rho_1 + \rho_2$ does not have to be a solution for sure. By non-local we mean that the relation between the velocity field at a time $v(\cdot,t)$ and the scalar $\rho(\cdot,t)$ is by the convolution with a kernel $K$. So, the value of the velocity field at a point depends not only on the scalar at that point but on its values in the whole space. This class of equations are also known as *active scalar equations*.

The two master examples of this type of equation are:

(a) the vorticity formulation of the Euler equation in dimension 2 (2D Euler), where the scalar is usually written as $\omega$ (and we call it the vorticity) instead of $\rho$ and the kernel is

$$K_{BS}(x_1,x_2) = \frac{1}{2\pi(x_1^2+x_2^2)}(-x_2,x_1),$$

for $(x_1,x_2) \in \mathbb{R}^2$;

(b) and the aggregation equation when the initial data $\rho_0$ is the characteristic function of a set. In this case the kernel is

$$K_{\mathrm{Ag}}(x) = -\nabla N(x)$$

for $x \in \mathbb{R}^n$, where $N$ is the fundamental solution of the laplacian.

Other examples of active scalar equations can be found. For instance, the surface quasi-geostrophic equation it is one.

The thesis is structured in two large blocks and a smaller one. In a few words, the first one corresponds to the Hölder well-posedness of the PDE, the second one to the persistence of the regularity of the boundary of a *density patch* and the third one to the limit structure as time goes to infinity of a certain type of solutions. We want to mention that each chapter of this dissertation has its own introduction.

- In Chapters 1 and 2 we prove the $C^\gamma$ well-posedness of the transport equation for some families of kernels in $\mathbb{R}^n$ and in the complex plane respectively. Special cases of the theorems presented here recover the $C^\gamma$ well-posedness for 2D Euler (see [MB, Chapter 4]) and the transport equation with the aggregation kernel (see [CGK, Theorem 5.3]).

- In Chapters 3 and 4 we prove the persistence of the $C^{1,\gamma}$ regularity of the boundary of a density patch, that is, we prove that if the initial data $\rho_0$ is the characteristic function of a $C^{1,\gamma}$ domain, then the solution $\rho(\cdot, t)$ of the transport equation is for every time the characteristic function of such a regular domain. This is done for the same families of kernels considered in the first block and it also covers the equivalent result for 2D Euler (done first by Chemin in [Ch] and later on by Bertozzi and Constantin in [BC]) and for the aggregation equation (see [BGLV]).

- Finally, in Chapter 5 we study the *skeleton* for a one-dimensional aggregation equation. We call *skeleton* to the limit domain at the blow-up time when we start with a density patch. This equation is equivalent to a transport equation for the case of the evolution of the characteristic function of a domain. This was one of the first questions when I started my doctoral studies, but we finally did not follow this line of work.

The main technical arduousness appearing is related with the divergence of the velocity field, which depends obviously on the choice of the kernel. In 2D Euler the velocity field is incompressible, which means that it has zero divergence. For a velocity field obtained by the convolution of the aggregation kernel $K_{\mathrm{Ag}}$ with a scalar, the divergence is exactly minus the scalar function, meaning that the regularity of the scalar is inherited by the divergence. As we will se later, this is not the situation for the kernels that we consider in the two first blocks of the thesis.

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# Contents

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# 1 $C_c^\gamma$ well-posedness in $\mathbb{R}^n$

## 1.1 Introduction

Let $\rho(x,t)$ a scalar quantity usually known as the *density* and let $v(x,t)$ a vector field called *velocity* both depending on the position $x \in \mathbb{R}^n$ and on the time $t \in \mathbb{R}$ . The *(homogeneous) transport equation* for the pair ($\rho$,$v$) is the partial differential equation defined by

$$\begin{cases} \rho_t + v \cdot \nabla\rho = 0, \\ \rho(\cdot,0) = \rho_0. \end{cases} \tag{1.1}$$

We shall now give an explanation of such a name for (1.1). Given a velocity field $v$ and a point $\alpha \in \mathbb{R}^n$ we set, whenever it is well defined, the *flow map*

$$\begin{aligned} X(\alpha,\cdot) : \mathbb{R} &\to \mathbb{R}^n, \\ t &\to X(\alpha,t) \end{aligned}$$

as the solution of the ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X(\alpha,t) = v(X(\alpha,t),t), \\ X(\alpha,0) = \alpha. \end{cases} \tag{1.2}$$

This map indicates the position at time $t$ of the particle that was initially at $\alpha$ and that has moved following the velocity field at every moment. It is also called the *trajectory* of the particle $\alpha$.

Then let $\rho$ the density and set $g(\alpha,t) = \rho(X(\alpha,t),t)$. If we compute the derivative of $g$ with respect to the time variable, we get, by an application of the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\alpha,t) = \rho_t(X(\alpha,t),t) + \nabla\rho(X(\alpha,t),t) \cdot \frac{\mathrm{d}}{\mathrm{d}t}X(\alpha,t),$$

and by (1.2),

$$\frac{\mathrm{d}}{\mathrm{d}t}g(\alpha,t) = \rho_t(X(\alpha,t),t) + v(X(\alpha,t),t) \cdot \nabla\rho(X(\alpha,t),t). \tag{1.3}$$

For a pair $(\rho,v)$ satisfying the transport equation, the right hand side of (1.3) vanishes, meaning that $g(x,t)$ does not depend on time and hence

$$\rho(X(\alpha,t),t) = \rho(X(\alpha,0),0) = \rho(\alpha,0) = \rho_0(\alpha).$$

That means that the density at time 0 and at position $\alpha$ takes the same value as the density evaluated at $t$ and at the future position of $\alpha$ at time $t$. So, it can be said that $\rho$ is transported with the flow defined by the velocity field $v$. This is a good reason for calling *transport equation* to (1.1).

For a fixed time $t$, the functions $\rho$ and $v$ in (1.1) can be related by some functional $T$ so that $v(\cdot,t) = T(\rho(\cdot,t))$ and often $T$ can be expressed as a convolution with a given kernel. The most important example of a transport equation of this kind for $n = 2$ is the Euler equation. Let $N$ be the fundamental solution of the Laplacian ($N(x) = \frac{1}{2\pi}\ln(|x|)$) and let

$$K_{BS}(x_1,x_2) = \frac{1}{2\pi\,|x|^2}(-x_2,x_1) = \nabla^{\perp}N(x)$$

(we call $K_{BS}$ the Biot-Savart kernel). If we set $\omega(\cdot,t) = \nabla \times v(\cdot,t)$, then the vorticity formulation of the Euler equation is

$$\begin{cases} \omega_t + v\cdot\nabla\omega = 0, \\ v(\cdot,t) = K_{BS} * \omega(\cdot,t), \\ \omega(\cdot,0) = \omega_0, \end{cases} \tag{1.4}$$

which is a non-linear transport equation for $(\omega, v)$.

Another example in $\mathbb{R}^n$ (see [BLL, Section 4.2] for more details of its derivation) is the aggregation equation when the initial condition $\rho_0$ is the characteristic function of some domain $D_0$, $\chi_{D_0}$. We will call solutions for this type of initial data *density patches* and they will be discused in depth in Chapters 3 and 4 of this dissertation. Let $w_n$ the volume of the $n$-dimensional unit ball and set

$$N(x) = -\frac{1}{n(n-2)w_n}\frac{1}{|x|^{n-2}}, \quad n \geq 3 \tag{1.5}$$

the fundamental solution of the Laplacian in $\mathbb{R}^n$. In this case, given $K_{Ag} = -\nabla N$ we get

$$\begin{cases} \rho_t + v\cdot\nabla\rho = 0, \\ v(\cdot,t) = K_{Ag} * \rho(\cdot,t), \\ \rho(\cdot,0) = \rho_0 = \chi_{D_0}. \end{cases}$$

In the spirit of generalizing these example equations we will consider throughout this paper a matrix $L \in M_{n\times n}(\mathbb{R})$ and the corresponding kernel

$$K(x) = L\cdot\nabla N(x) \tag{1.6}$$

where $\cdot$ stands for the usual product of matrices and $\nabla N(x)$ is seen as a column vector. Then (1.6) produces a relation $v(\cdot,t) = K * \rho(\cdot,t)$ as in the previous cases.

Our goal is to prove a well-posedness result for the transport equation and for the kernel (1.6) in some space of functions that will be defined in a moment.

We would like to anticipate that the divergence of $v$ is an important quantity appearing in the computations and in the proofs that we are going to develop. For the Euler equation the divergence vanishes everywhere for any time and for the aggregation equation the divergence at a given time $t$ is equal to $-\rho(\cdot, t)$. In both cases, the proofs are easier due to the simplicity of the divergence.

**Definition 1.1.** *Given* $0 < \gamma < 1$ *and* $f : \mathbb{R}^n \to \mathbb{R}$ *let*

$$||f||_{L^\infty} = \sup_{x \in \mathbb{R}^n} |f(x)| \quad \text{and} \quad |f|_\gamma = \sup_{\substack{x,y \in \mathbb{R}^n \\ x \neq y}} \frac{|f(x) - f(y)|}{|x-y|^\gamma}.$$

*We define the norm*

$$||f||_\gamma := ||f||_{L^\infty} + |f|_\gamma \, .$$

*For* $F : \mathbb{R}^n \to \mathbb{R}^d$, $x \to F(x) = (f_1(x), \dots, f_d(x))$, *we define*

$$||F||_\gamma := \sup_{i=1,\dots,d} ||f_i||_\gamma$$

*and then the space*

$$C^\gamma(\mathbb{R}^n; \mathbb{R}^d) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^d : ||f||_\gamma < \infty \right\}.$$

*Finally, given an integer* $m$ *and a multiindex* $\alpha = (\alpha_1, \dots, \alpha_n)$ *consider* $|\alpha| = \sum_{i=1}^n \alpha_i$ *and* $\frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} = \frac{\mathrm{d}^{|\alpha|}}{\mathrm{d}x_1^{\alpha_1} \dots \mathrm{d}x_n^{\alpha_n}}$. *Then define*

$$|F|_{m,\gamma} = |F(0)| + \sup_{1 \le |\alpha| \le m} \left|\left| \frac{\mathrm{d}^\alpha}{\mathrm{d}x^\alpha} F \right|\right|_\gamma .$$

*For* $m = 1$ *we simply write*

$$||\nabla F||_{L^\infty} = \sup_{i=1,\dots,d} \left( \sup_{j=1,\dots,n} \left|\left| \frac{\partial}{\partial x_j} f_i \right|\right|_{L^\infty} \right),$$

$$|\nabla F|_\gamma = \sup_{i=1,\dots,d} \left( \sup_{j=1,\dots,n} \left| \frac{\partial}{\partial x_j} f_i \right|_\gamma \right),$$

*and then*

$$|F|_{1,\gamma} = |F(0)| + \|\nabla F\|_{L^\infty} + |\nabla F|_\gamma \, .$$

*We define the Hölder space $C^{m,\gamma}(\mathbb{R}^n;\mathbb{R}^d)$ as*

$$C^{m,\gamma}(\mathbb{R}^n;\mathbb{R}^d) = \left\{ f : \mathbb{R}^n \to \mathbb{R}^d : ||f||_{m,\gamma} < \infty \right\}.$$

*When it is clear enough, we will just write $C^{m,\gamma}$.*

*Additionally, we set $C_c^{m,\gamma}$ as the space of functions in $C^{m,\gamma}$ which are also compactly supported.*

We are ready to anticipate the main theorem of this chapter.

**Theorem 1.2.** *Let $N$ the fundamental solution of the Laplacian in $\mathbb{R}^n$. Consider $L \in M_{n\times n}(\mathbb{R})$. For $0 < \gamma < 1$, if $\rho_0 \in C_c^{m,\gamma}(\mathbb{R}^n, \mathbb{R})$, then the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot, t) = L \cdot \nabla N * \rho(\cdot, t), \\ \rho(\cdot, 0) = \rho_0, \end{cases}$$

*has a unique weak solution $\rho(\cdot, t) \in C_c^{m,\gamma}(\mathbb{R}^n, \mathbb{R})$ for any time $t \in \mathbb{R}$.*

The definition of weak solution will be given in Chapter 4 (see Definition 4.7). Nevertheless, in the smooth framework of the present chapter it is possible to check that a (transported by trajectories) solution is a weak solution of the PDE.

The reason we have chosen this space is double. Firstly, the result was proved for the Euler equation (see [MB, Chapter 4]) and for $L$ the identity matrix, i.e., for the aggregation kernel (see [CGK, Theorem 5.3]). Secondly, we wanted to be sure about having well-posedness of the smooth case before moving to other situations (for instance, density patches). The proof presented here does not change much from the one for the Euler equation, but at some steps we will need more involved arguments due to the fact that the velocity field is not divergence free in general. We will stress this fact in the next sections whenever those differences appear.

### 1.1.1 Outline of the chapter

The present chapter is structured as follows. In Section 1.2 we give some preliminary results on Hölder spaces and on Calderón-Zygmund Operators (CZO) acting on them. In Section 1.3 we prove a local-in-time version of Theorem 1.2. In Section 1.4 we prove that this local solution is actually global. Finally, in Section 1.5 we explain how to proof higher regularity provided the initial data is of class $C_c^{m,\gamma}$.

## 1.2 Preliminaries

First of all, we have the following elementary properties for elements of the Hölder spaces.

**Lemma 1.3.** *Let $f, g$ be $C^\gamma$ functions, $0 < \gamma < 1$. Then*

$$|fg|_\gamma \le ||f||_{L^\infty} |g|_\gamma + |f|_\gamma ||g||_{L^\infty}, \tag{1.7}$$

$$||fg||_\gamma \le ||f||_\gamma ||g||_\gamma. \tag{1.8}$$

*If moreover $X$ is a smooth invertible transformation in $\mathbb{R}^n$ satisfying*

$$|\det \nabla X(\alpha)| \ge c_1 > 0,$$

*then there exists $c > 0$ such that*

$$\left|\left|(\nabla X)^{-1}\right|\right|_\gamma \le c\, ||\nabla X||_\gamma^{2n-1}, \tag{1.9}$$

$$\left|X^{-1}\right|_{1,\gamma} \le c\, |X|_{1,\gamma}^{2n-1}, \tag{1.10}$$

$$|f \circ X|_\gamma \le |f|_\gamma ||\nabla X||_{L^\infty}^\gamma, \tag{1.11}$$

$$||f \circ X||_\gamma \le ||f||_\gamma (1 + |X|_{1,\gamma}^\gamma), \tag{1.12}$$

$$\left|\left|f \circ X^{-1}\right|\right|_\gamma \le ||f||_\gamma (1 + |X|_{1,\gamma}^{\gamma(2n-1)}). \tag{1.13}$$

The proof of Lemma 1.3 can be found in [MB, p. 159] (see Lemmas 4.1, 4.2 and 4.3). Note that (1.8) implies that $C^\gamma$ is an algebra.

We also have the following bounds for Calderón-Zygmund operators acting on Hölder spaces. They will be used repeatedly in the proofs developed in the upcoming sections.

**Lemma 1.4.** *Let $K : \mathbb{R}^n \to \mathbb{R}$, $K \in C^2(\mathbb{R}^n \setminus \{0\})$ a kernel homogeneous of degree $1-n$. That is,*

$$K(\lambda x) = \frac{1}{\lambda^{n-1}} K(x), \quad \forall \lambda > 0. \ \forall x \neq 0, \tag{1.14}$$

*Let $P = \partial_i K$, $i = 1, \dots, n$. Set*

$$Tf(x) = \int_{\mathbb{R}^n} K(x-x') f(x')\, dx'; \qquad Sf(x) = p.v. \int_{\mathbb{R}^n} P(x-x') f(x')\, dx'.$$

*For $0 < \gamma < 1$ let $f \in C_c^\gamma(\mathbb{R}^n; \mathbb{R})$. Set $R^n := m(supp(f)) < \infty$, that is, the measure of the support of $f$. Then, there exists a constant $c$, independent of $f$ and $R$, such that*

$$||Tf||_{L^\infty} \le cR\, ||f||_{L^\infty}, \tag{1.15}$$

$$||Sf||_{L^\infty} \le c \left\{ |f|_\gamma \varepsilon^\gamma + \max\left(1, \ln \frac{R}{\varepsilon}\right) ||f||_{L^\infty} \right\} \quad \forall \varepsilon > 0, \tag{1.16}$$

$$|Sf|_\gamma \le c\, |f|_\gamma. \tag{1.17}$$

**Remark 1.5.** *The proof of Lemma 1.4 can be found in [MB, pp. 159-163] (see Lemmas 4.5 and 4.6). There more hypothesis on the kernels K and P are required but we remark here that they are not needed. In particular, for K as in the previous lemma, by differentiating with respect to $x_i$ the equation (1.14) it is clear that this derivative is homogeneous of degree $-n$. Also, we can see that $\partial_i K$ has zero mean integral on the sphere. Let $0 < a < b$. By Stokes' theorem we can write*

$$\begin{aligned}\int_{a\leq|x|\leq b} \partial_i K(x)\,\mathrm{d}x &= \\ &= \int_{\partial B(0,b)} K(x)n_i(x)\,\mathrm{d}\sigma(x) - \int_{\partial B(0,a)} K(x)n_i(x)\,\mathrm{d}\sigma(x),\end{aligned} \tag{1.18}$$

*where $n_i(x)$ is the i-th component of the unitary normal vector to each surface at the point x. By homogeneity of the kernel K it is clear that the two integrals in the second line of (1.18) are equal and then the difference is 0. By doing a hyperspherical coordinates change of variables and again by homogeneity of the kernel, the first line of (1.18) can be written as*

$$\int_{a\leq|x|\leq b} \partial_i K(x)\,\mathrm{d}x = (\log(b) - \log(a)) \int_{\partial B(0,1)} \partial_i K(w)\,\mathrm{d}\sigma(w),$$

*and so we can conclude that*

$$\int_{\partial B(0,1)} \partial_i K(w)\,\mathrm{d}\sigma(w) = 0$$

*as it is required in the mentioned proof done in [MB].*

**Remark 1.6.** *For N the fundamental solution of the Laplacian, each component of the kernel $K = \nabla N$ satisfies (1.14) and, consequently, Lemma 1.4 holds.*

## 1.3 Local Theorem

As in the case of Euler equation, a good way to prove an existence and uniqueness result is by dealing with an, in some sense, equivalent equation rather than the one presented in Theorem 1.2. Recall that $\rho$ is transported with the flow, i.e., $\rho(x,t) = \rho_0(X^{-1}(x,t))$ for $X^{-1}(\cdot,t)$ the inverse of the flow $X(\cdot,t)$. Therefore by (1.13) we have

$$||\rho(\cdot,t)||_\gamma \leq ||\rho_0||_\gamma \left(1 + |X(\cdot,t)|_{1,\gamma}^{\gamma(2n-1)}\right).$$

Thus, $\rho(\cdot,t) \in C^\gamma$ provided $X(\cdot,t) \in C^{1,\gamma}$.

Furthermore, eventually we will need to control the measure of the support of $\rho(\cdot,t)$. In order to do it, we have the next lemma, which will be also needed in the rest of the chapters of this dissertation. Note that in the zero divergence case there is no need to control the support of $\rho(\cdot,t)$ since its measure is conserved with the time and therefore it is equal to the measure of the support of $\rho_0$.

**Lemma 1.7.** *Let $(\rho, v)$ be a solution of (1.1) and let $X$ be the flow map associated to $v(\cdot,t)$ as in (1.2). Then*

$$m(supp(\rho(\cdot,t))) \leq c(n)m(supp(\rho_0))\,||\nabla X(\cdot,t)||^n_{L^\infty}.$$

*Proof.* Given $A \subseteq \mathbb{R}^n$, let $\chi_A$ be the function taking value 1 in $A$ and 0 otherwise. Then

$$m(\text{supp}(\rho(\cdot,t))) = \int_{\mathbb{R}^n} \chi_{\text{supp}(\rho(\cdot,t))}(x)\,\mathrm{d}x.$$

Taking the change of variables $x = X(\alpha,t)$ we get

$$m(\text{supp}(\rho(\cdot,t))) = \int_{\mathbb{R}^n} \chi_{\text{supp}(\rho(\cdot,t))}(X(\alpha,t))\,\det DX(\alpha,t)\,\mathrm{d}\alpha.$$

Since $\rho$ is transported with the flow, it is clear that $X(\alpha,t) \in \text{supp}(\rho(\cdot,t))$ if and only if $\alpha \in \text{supp}(\rho_0)$. Thus,

$$m(\text{supp}(\rho(\cdot,t))) = \int_{\mathbb{R}^n} \chi_{\text{supp}(\rho_0)}(\alpha)\,\det DX(\alpha,t)\,\mathrm{d}\alpha.$$

Taking absolute value on the previous equation and having into account that $||\det DX(\cdot,t)||_{L^\infty} \leq c(n)\,||\nabla X(\cdot,t)||^n_{L^\infty}$ we get

$$m(\text{supp}(\rho(\cdot,t))) \leq c(n)m(\text{supp}(\rho_0))\,||\nabla X(\cdot,t)||^n_{L^\infty}.$$

□

We can focus then on proving existence, uniqueness and regularity for $X$. We know $X$ satisfies (1.2) and then, as $v(\cdot,t) = K * \rho(\cdot,t)$, we obtain

$$\frac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = v(X(\alpha,t),t) = \int_{\mathbb{R}^n} K(X(\alpha,t)-x')\rho(x',t)\,\mathrm{d}x'.$$

Applying a change of variables $x' = X(\alpha',t)$

$$\begin{aligned}\frac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) &= \int_{\mathbb{R}^n} K(X(\alpha,t)-X(\alpha',t))\rho(X(\alpha',t),t)\ \det[DX(\alpha',t)]\,\mathrm{d}\alpha' = \\ &= \int_{\mathbb{R}^n} K(X(\alpha,t)-X(\alpha',t))\rho_0(\alpha')\ \det[DX(\alpha',t)]\,\mathrm{d}\alpha',\end{aligned}$$

where, in the last equality, we have used that $\rho$ is conserved along the flow.

Consequently, we have an ordinary differential equation (ODE) for $X$. A standard way to prove existence and uniqueness for an ODE is to apply Picard-Lindelöf's theorem. It can be stated as follows.

**Theorem 1.8** (Picard-Lindelöf). *Let $O \subseteq B$ be an open subset of a Banach space $B$ and let $F : O \to B$ be a locally Lipschitz continuous mapping.*

*Then given $X_0 \in O$, there exists a time $T > 0$ such that the ordinary differential equation*

$$\frac{\mathrm{d}X}{\mathrm{d}t} = F(X), \qquad X(\cdot, t = 0) = X_0 \in O,$$

*has a unique (local) solution $X \in C^1\left[(-T, T); O\right]$.*

So, in order to apply Theorem 1.8, we first need an equation of type $\frac{\mathrm{d}X}{\mathrm{d}t} = F(X)$. As we have seen, we have it for

$$F(X(\alpha, t)) := \int_{\mathbb{R}^n} K(X(\alpha, t) - X(\alpha', t))\rho_0(\alpha') \det[DX(\alpha', t)] \, \mathrm{d}\alpha'. \tag{1.19}$$

Then we need a Banach space $B$ and an open subspace of $B$ such that the flow maps $X(\cdot, t)$ belong to $O_M$. We also need a functional $F$ mapping $O_M$ to $B$ being this map locally Lipschitz continuous and satisfying that $F(X(\alpha, t))$ is equal to (1.19). Let $B = C^{1,\gamma}(\mathbb{R}^n; \mathbb{R}^n)$ and

$$O_M = B \cap \left\{ X : \mathbb{R}^n \to \mathbb{R}^n : \frac{1}{M} < \sup_{\alpha \neq \beta} \frac{|X(\alpha) - X(\beta)|}{|\alpha - \beta|} < M \right\}. \tag{1.20}$$

**Remark 1.9.** *Then we have:*

- *$O_M$ is non-empty: $Id \in O_M$ $\forall M > 1$.*
- *It is an open set since it is the preimage of the open set $(\frac{1}{M}, M)$ for some norm function (which is continuous).*
- *If $X \in O_M$, then the image of $X$ is open because $X$ is locally a diffeomorphism and it is also closed because it is complete ($X$ is a bilipschitz function) . Then the image of $X$ is the whole space and so $X$ is a homeomorphism.*

After this, we have to check that the hypothesis in Picard-Lindelöf's theorem are satisfied. Since computations of derivatives of $F(X)$ will be needed, we first look how distributional derivatives of our kernels are.

**Lemma 1.10.** *Given $L = (l_{ij})_{i,j=1}^n \in M_{n\times n}(\mathbb{R})$, then for $K = L \cdot \nabla N$ we have, distributionally,*

$$\partial_i K_j = \frac{l_{ji}}{n}\delta_0 + p.v. \sum_{r=1}^n l_{jr}\partial_i\partial_r N,$$

*where $\delta_0$ is the Dirac delta at 0.*

*Proof.* It is well known that for $N$, fundamental solution of the Laplacian, we have $\partial_i^2 N = \frac{1}{n}\delta_0 + p.v.\partial_i^2 N$ and for $i \neq j$ we have $\partial_i\partial_j N = p.v.\partial_i\partial_j N$. Then, since $K_j$ can be expressed as

$$K_j = \sum_{r=1}^n l_{jr}\partial_r N,$$

we apply $\partial_i$ and we get the result. $\square$

We are now in position to show that $F : O_M \to B$.

**Proposition 1.11.** *Let $O_M$ as defined in (1.20). Then, the functional $F$ defined by*

$$F(X)(\alpha) = \int_{\mathbb{R}^n} K(X(\alpha) - X(\alpha'))\rho_0(\alpha')\, \det[DX(\alpha')]\, \mathrm{d}\alpha' \tag{1.21}$$

*maps $O_M$ to $C^{1,\gamma}(\mathbb{R}^n;\mathbb{R}^n)$.*

*Proof.* Let $X \in O_M$. In order to prove the proposition, we need to verify

$$||F(X)||_{L^\infty} + \sup_{i\in\{1,\ldots,n\}} \left( \left|\left| \frac{\mathrm{d}}{\mathrm{d}\alpha_i} F(X) \right|\right|_\gamma \right) < \infty. \tag{1.22}$$

If we consider the change of variables $x' = X(\alpha')$ in (1.21) we get, for the $j$-th component

$$\begin{aligned} F_j(X)(\alpha) &= \int_{\mathbb{R}^n} K_j(X(\alpha) - x')\rho_0(X^{-1}(x'))\, \mathrm{d}x' = \\ &= (K_j * (\rho_0 \circ X^{-1}))(X(\alpha)). \end{aligned} \tag{1.23}$$

Let $R = \mathrm{m}(\mathrm{supp}\ (\rho_0 \circ X^{-1}))^{1/n}$. Then, as in Lemma 1.7, we have, for $R_0 = \mathrm{m}(\mathrm{supp}\ (\rho_0))^{1/n}$ that

$$R \le c_n R_0 \, ||\nabla X||_{L^\infty} .$$

Since $K = L \cdot \nabla N$ satisfies (1.14) (recall that $\nabla N$ satisfies it and every component of our kernel is just a linear combination of components of it), by (1.15) in Lemma 1.4 we have

$$||F(X)||_{L^\infty} \le cR \left|\left| \rho_0 \circ X^{-1} \right|\right|_{L^\infty} = cR\, ||\rho_0||_{L^\infty} \le c_n R_0 \, ||\nabla X||_{L^\infty} \, ||\rho_0||_{L^\infty} ,$$

which is bounded for $X \in C^{1,\gamma}$ and $\rho_0 \in C_c^\gamma$.

We focus then on the norms of derivatives of $F(X)$. We write $\partial_i = \frac{\partial}{\partial \alpha_i}$. We have, by definition of the norm, $||\partial_i F(X)||_\gamma = \sup_{j\in\{1,\ldots,n\}} \left|\left|\partial_i F_j(X)\right|\right|_\gamma$. We work then with $\partial_i F_j(X)$ for $i, j \in \{1, \ldots, n\}$. An application of the chain rule, combined with Lemma 1.10, yields

$$\begin{aligned} \partial_i F_j(X)(\alpha) &= \nabla(K_j * (\rho_0 \circ X^{-1}))(X(\alpha)) \cdot \partial_i X(\alpha) = \\ &= \sum_{r=1}^n \partial_r (K_j * (\rho_0 \circ X^{-1}))(X(\alpha)) \partial_i X_r(\alpha) = \\ &= \sum_{r=1}^n \left( \frac{l_{jr}}{n} \rho_0(\alpha) + \mathrm{p.v.}\ (\partial_r K_j * (\rho_0 \circ X^{-1}))(X(\alpha)) \right) \partial_i X_r(\alpha) = \\ &= \sum_{r=1}^n \left( \frac{l_{jr}}{n} \rho_0(\alpha) + S_{rj}(\alpha) \right) \partial_i X_r(\alpha), \end{aligned} \tag{1.24}$$

where $S_{rj}(\alpha) := \text{p.v.}\ (\partial_r K_j * (\rho_0 \circ X^{-1}))(X(\alpha))$ and $\cdot$ stands for the usual scalar product.

Since $\rho_0, \partial_i X_r \in C^\gamma$ and $C^\gamma$ is an algebra, then it suffices to control the $C^\gamma$ norm of $S_{rj}$. Clearly, hypothesis in Lemma 1.4 are satisfied if $P = \partial_r K_j$. Then we set $\varepsilon = \left|\rho_0 \circ X^{-1}\right|_\gamma^{1/\gamma}$ and apply bound (1.16) in Lemma 1.4 to get

$$||S_{rj}||_{L^\infty} \leq c\left\{1 + \max\left(1, \frac{1}{\gamma}\ln(R\left|\rho_0 \circ X^{-1}\right|_\gamma)\right)\left|\left|\rho_0 \circ X^{-1}\right|\right|_{L^\infty}\right\} \tag{1.25}$$

where $R = \text{m}(\text{supp}\ (\rho_0 \circ X^{-1}))^{1/n}$. As previously, $R$ is bounded by $c_n R_0 \,||\nabla X||_{L^\infty}$. Since both $||\rho_0 \circ X^{-1}||_{L^\infty}$ and $\left|\rho_0 \circ X^{-1}\right|_\gamma$ are bounded above by $||\rho_0 \circ X^{-1}||_\gamma$ and taking also into account that

$$\left|\left|\rho_0 \circ X^{-1}\right|\right|_\gamma \leq ||\rho_0||_\gamma \,(1 + |X|_{1,\gamma}^{\gamma(2n-1)}) < \infty,$$

then we have that the right hand side of (1.25) is finite. On the other hand, by (1.17) we have

$$\left|S_{rj}\right|_\gamma \leq c\left|\rho_0 \circ X^{-1}\right|_\gamma ||\nabla X||_{L^\infty}^\gamma \leq c\,||\rho_0||_\gamma \,(1 + |X|_{1,\gamma}^{\gamma(2n-1)})\,|X|_{1,\gamma}^\gamma .$$

Then, as we argued before, the $C^\gamma$ norm of $\partial_i F_j(X)$ is finite for any $i, j$ so the supremum in (1.22) is finite as well, completing the proof of the proposition. □

Hence, we have that $F$ satisfies the first hypothesis in Picard-Lindelöf's theorem. It remains to check that $F$ is locally Lipschitz. We claim (and prove later) that if the directional derivative $F'(X)$ is bounded as a linear operator between $O_M$ and $B$ then $F$ is locally Lipschitz. So, first of all we have to compute this directional derivative. An auxiliary lemma is useful for this computation and we need to give a previous definition to write it.

**Definition 1.12.** *Given $A \in M_{n\times n}(\mathbb{R}^n)$ we define $A^c_{i,j} \in M_{(n-1)\times(n-1)}(\mathbb{R}^n)$ as the submatrix of $A$ obtained by erasing the i-th row and the j-th column .*

The following lemma is not needed whenever the velocity field is divergence free, as in the Euler equation. In that case, $\det(DX(\cdot, t)) \equiv 1$ and consequently the functional in (1.21) does not contain the determinant inside the integral. In the general case, as we will see later, an expression for the sum of determinants will be necessary.

**Lemma 1.13.** *Given $X, Y : \mathbb{R}^n \to \mathbb{R}^n$ differentiable homeomorphisms.*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\det(DX + \varepsilon DY)\big|_{\varepsilon=0} = \sum_{i,j=1}^n (-1)^{i+j}\partial_j Y_i \det(DX^c_{i,j}).$$

*Proof.* First we use a formula for the determinant of a sum of square matrices. The proof can be found in [Ma, pp. 162-163]. Let $A, B \in M_{n\times n}(\mathbb{R})$ and let $\alpha, \beta$ strictly increasing integer sequences chosen from $\{1, \dots, n\}$. Let $|\alpha|$ (resp. $|\beta|$) the number of elements of $\alpha$ (resp. $\beta$). If $|\alpha| = |\beta|$ then let $A[\alpha|\beta] \in M_{|\alpha|\times|\alpha|}(\mathbb{R})$ the submatrix of $A$ lying in rows $\alpha$ and columns $\beta$ and $B[\alpha|\beta] \in M_{(n-|\alpha|)\times(n-|\alpha|)}(\mathbb{R})$ the submatrix of $B$ lying in rows complementary to $\alpha$ and columns complementary to $\beta$. Let $s(\alpha)$ (resp. $s(\beta)$) the sum of the integers in $\alpha$ (resp. $\beta$). Then

$$\det(A+B) = \sum_{r=0}^{n} \sum_{\substack{\alpha,\beta \\ |\alpha|=|\beta|=r}} (-1)^{s(\alpha)+s(\beta)} \det(A[\alpha|\beta]) \det(B[\alpha|\beta]). \tag{1.26}$$

Note that for a matrix $M \in M_{s\times s}(\mathbb{R})$ and a constant $c \in \mathbb{R}$ we have $det(c\cdot M) = c^s \det(M)$. Then setting $A = DX$ and $B = \varepsilon DY$ in (1.26) we get

$$\det(DX + \varepsilon DY) = \sum_{r=0}^{n} \varepsilon^{n-r} \sum_{|\alpha|=|\beta|=r} (-1)^{s(\alpha)+s(\beta)} \det(DX[\alpha|\beta]) \det(DY[\alpha|\beta]).$$

Differentiating with respect to $\varepsilon$ the previous equation and setting $\varepsilon = 0$ make some terms vanish and, in consequence,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \det(DX + \varepsilon DY)\big|_{\varepsilon=0} = \sum_{|\alpha|=|\beta|=n-1} (-1)^{s(\alpha)+s(\beta)} \det(DX[\alpha|\beta]) \det(DY[\alpha|\beta]).$$

Note that a strictly increasing sequence taking $n-1$ elements of $\{1, \dots, n\}$ is a sequence avoiding just one of them. So

$$\alpha = (1, \dots, i-1, i+1, \dots, n), \quad \beta = (1, \dots, j-1, j+1, \dots, n),$$

for $i, j = 1, \dots, n$. Then $s(\alpha) + s(\beta) = n(n+1) - (i+j)$ and hence $(-1)^{s(\alpha)+s(\beta)} = (-1)^{i+j}$. For these special sequences we can simplify and write $DY[\alpha|\beta] = \partial_j Y_i$ and $DX[\alpha|\beta] = DX^c_{i,j}$ as in Definition 1.12. Summing up,

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon} \det(DX + \varepsilon DY)\big|_{\varepsilon=0} = \sum_{i,j=1}^{n} (-1)^{i+j} \partial_j Y_i \det(DX^c_{i,j})$$

and the lemma is proved.

□

We already have the tools to compute the directional derivative of $F$.

**Proposition 1.14.** *Let $X \in O_M, Y \in B$ . For $F = (F_j)_{j=1}^n$ defined in (1.21) we have $F_j'(X)Y$ =I+II where*

$$I := \int_{\mathbb{R}^n} \nabla K_j(X(\alpha) - X(\alpha')) \cdot (Y(\alpha) - Y(\alpha'))\rho_0(\alpha') \det(DX)(\alpha')\, d\alpha',$$
$$II := \sum_{r,s=1}^{n} (-1)^{r+s} \int_{\mathbb{R}^n} K_j(X(\alpha) - X(\alpha'))\rho_0(\alpha')\partial_s Y_r(\alpha') \det(DX^c_{r,s})(\alpha')\, d\alpha'.$$

*Proof.* Let $j = 1, \ldots, n$. Consider $X \in O_M$ and $Y \in B$. Firstly, we apply the chain rule to see

$$\begin{aligned}\frac{d}{d\varepsilon}&\left(K_j(X(\alpha) - X(\alpha') + \varepsilon(Y(\alpha) - Y(\alpha')))\right)_{\varepsilon=0} =\\ &= \sum_{i=1}^{n} \partial_i K_j(X(\alpha) - X(\alpha'))(Y_i(\alpha) - Y_i(\alpha')) =\\ &= \nabla K_j(X(\alpha) - X(\alpha')) \cdot (Y(\alpha) - Y(\alpha')),\end{aligned}$$

where $\cdot$ is the usual scalar product. Thus having into account the above computation and applying Lemma 1.13 we get

$$\begin{aligned}&(F_j'(X)Y)(\alpha) = \frac{d}{d\varepsilon}(F_j(X + \varepsilon Y)(\alpha))_{\varepsilon=0} =\\ &= \int_{\mathbb{R}^n} \nabla K_j(X(\alpha) - X(\alpha')) \cdot (Y(\alpha) - Y(\alpha'))\rho_0(\alpha') \det(DX)(\alpha')\, d\alpha' +\\ &+ \sum_{r,s=1}^{n} (-1)^{r+s} \int_{\mathbb{R}^n} K_j(X(\alpha) - X(\alpha'))\rho_0(\alpha')\partial_s Y_r(\alpha') \det(DX^c_{r,s})(\alpha')\, d\alpha',\end{aligned}$$

as we wanted to prove. □

**Remark 1.15.** *Note that there is no need to write principal value in the term I of the proposition because the singularity of $\nabla K_j$ when $\alpha = \alpha'$ is compensated with the term $Y(\alpha) - Y(\alpha')$.*

The directional derivative computed in Proposition 1.14 has been decomposed as the sum of two terms. The second one is very similar to the one treated in Proposition 1.11, but the first one looks different. It can be written as an integral with respect to a kernel which is not of convolution type and so its derivatives may be tricky to handle. In the following lemma, which is somehow technical, we compute exactly those derivatives.

**Lemma 1.16.** *Let I defined in Proposition 1.14. Then $I = \sum_{i=1}^n I_i(\alpha)$ where*

$$I_i(\alpha) := \int_{\mathbb{R}^n} \partial_i K_j(X(\alpha) - X(\alpha'))(Y_i(\alpha) - Y_i(\alpha'))\rho_0(\alpha') \det(DX)(\alpha')\, d\alpha',$$

*and its distributional derivatives are* $\partial_l I_i(\alpha) = \nabla \tilde{I}_i(X(\alpha)) \cdot \partial_l X(\alpha)$ *where*

$$\begin{aligned}
&\partial_k \tilde{I}_i(x) = \\
&= p.v. \int_{\mathbb{R}^n} \partial_k \partial_i K_j(x-x')(Y_i(X^{-1}(x)) - Y_i(X^{-1}(x')))(\rho_0 \circ X^{-1})(x')\,\mathrm{d}x' + \\
&+ \partial_k [Y_i \circ X^{-1}](x) p.v. \int_{\mathbb{R}^n} \partial_i K_j(x-x')(\rho_0 \circ X^{-1})(x')\,\mathrm{d}x' + \\
&+ c_k (\nabla [Y_i \circ X^{-1}](x) \cdot \xi_k)(\rho_0 \circ X^{-1})(x),
\end{aligned}$$

*where* $\xi_k$ *is a vector in* $\mathbb{R}^n$ *depending on* $k$.

*Proof.* Consider $\alpha = X^{-1}(x)$, then after the change of variables $\alpha' = X^{-1}(x')$, we have

$$\begin{aligned}
\tilde{I}_i(x) &= I_i(X^{-1}(x)) = \\
&= \int_{\mathbb{R}^n} \partial_i K_j(x-x')(Y_i(X^{-1}(x)) - Y_i(X^{-1}(x')))\rho_0(X^{-1}(x'))\,\mathrm{d}x'.
\end{aligned}$$

Following the scheme in [MB, p. 165] for the Euler equation, let $R(x,x') = \partial_i K_j(x-x')(Y_i(X^{-1}(x)) - Y_i(X^{-1}(x')))$. Firstly, we compute the partial distributional derivative with respect to $x_k$ of $R(x+x',x')$. In order to do that, note previously that, given $h > 0$ and $a \in \mathbb{R}^n$, $|a| = 1$, we have, by Taylor expansion of $Y \circ X^{-1}$,

$$\begin{aligned}
&\lim_{h\to 0} R(x+ah,x)h^{n-1} = \\
&\qquad = \lim_{h\to 0} \partial_i K_j(ah)(Y_i(X^{-1}(x+ah)) - Y_i(X^{-1}(x)))h^{n-1} = \\
&\qquad = \lim_{h\to 0} \partial_i K_j(a) h^{-n} h^{n-1} (\nabla [Y_i \circ X^{-1}](x) \cdot ah + o(h)) = \\
&\qquad = \partial_i K_j(a) \nabla [Y_i \circ X^{-1}](x) \cdot a.
\end{aligned} \tag{1.27}$$

We then compute the distributional derivative of $R(x+x',x')$. Let $\varphi \in C_c^\infty(\mathbb{R}^n;\mathbb{R})$ a test function. Then

$$\begin{aligned}
\langle \partial_{x_k} R(\cdot + x', x'), \varphi \rangle &= -\int_{\mathbb{R}^n} \partial_{x_k}\varphi(x) R(x+x',x')\,\mathrm{d}x = \\
= \lim_{\varepsilon\to 0} &\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \varphi(x)\partial_{x_k} R(x+x',x')\,\mathrm{d}x - \\
- \lim_{\varepsilon\to 0} &\int_{\mathbb{R}^n \setminus B(0,\varepsilon)} \partial_{x_k}\left[\varphi(x) R(x+x',x')\right]\,\mathrm{d}x.
\end{aligned}$$

Applying Stokes' theorem, and since $\varphi$ has compact support we obtain

$$\begin{aligned}
\langle \partial_{x_k} R(\cdot + x', x'), \varphi \rangle &= p.v. \langle \partial_{x_k} R(\cdot + x', x'), \varphi \rangle + \\
&+ \lim_{\varepsilon\to 0} \int_{\partial B(0,\varepsilon)} \varphi(x) R(x+x',x') n_k(x)\,\mathrm{d}\sigma(x),
\end{aligned}$$

where $n_k(x)$ is the $k$-th component of the unitary normal vector to $\partial B(0,\varepsilon)$ at the point $x$. We apply the observation made in (1.27) to conclude that

$$\lim_{\varepsilon\to 0}\int_{\partial B(0,\varepsilon)} \varphi(x)R(x+x',x')n_k(x)\,\mathrm{d}\sigma(x) = \varphi(0)\nabla[Y_i\circ X^{-1}](x')\cdot\xi_k,$$

where the $l$-th component of $\xi_k$ is

$$(\xi_k)_l = \int_{\partial B(0,1)} \partial_i K_j(a)n_k(a)\,a_l\,\mathrm{d}\sigma(a).$$

Therefore, distributionally

$$\partial_{x_k}R(\cdot+x',x') = \text{p.v. }\partial_{x_k}R(\cdot+x',x') + (c_k\nabla[Y_i\circ X^{-1}](x')\cdot\xi_k)\delta_0.$$

Then, since $\partial_{x_k}R(x,x') = \partial_{x_k}[R(\cdot+x',x')](x-x')$ we finally get for $H(x)=\int_{\mathbb{R}^n}R(x,x')f(x')\,\mathrm{d}x'$,

$$\begin{aligned}\partial_k H(x) = \text{p.v.}&\int_{\mathbb{R}^n}\partial_k R(x,x')f(x')\,\mathrm{d}x'+\\ &+\int_{\mathbb{R}^n}\delta_0(x-x')c_k\nabla[Y_i\circ X^{-1}](x')\cdot\xi_k f(x')\,\mathrm{d}x' =\\ &=\text{p.v.}\int_{\mathbb{R}^n}\partial_k\partial_i K_j(x-x')(Y_i(X^{-1}(x))-Y_i(X^{-1}(x')))f(x')\,\mathrm{d}x'+\\ &+\partial_k[Y_i\circ X^{-1}](x)\text{p.v.}\int_{\mathbb{R}^n}\partial_i K_j(x-x')f(x')\,\mathrm{d}x'+\\ &+c_k\nabla[Y_i\circ X^{-1}](x)\cdot\xi_k f(x).\end{aligned}$$

The proof is completed setting $f=\rho_0\circ X^{-1}$ in the previous expression and applying the chain rule to $I_i(\alpha)=\tilde{I}_i(X(\alpha))$.

□

Remember that our goal is to bound the $C^{1,\gamma}$ norm of the map $\alpha\to F'(X)Y(\alpha)$ in such a way that we get

$$|F'(X)Y|_{1,\gamma}\le c\,|Y|_{1,\gamma}$$

for $c$ depending maybe on $n,\rho_0$ and $X$ in order to have that $F'(X)$ is bounded as a linear operator. Note that the first term in $\partial_k\tilde{I}_i$ in Lemma 1.16 is an integral containing $\partial_k\partial_i K_j$, which is a hypersingular kernel. Nevertheless, the term $Y(\alpha)-Y(\alpha')$ will, in some sense, kill this excess of singularity. We quantify this effect in the following Lemma.

**Lemma 1.17.** *Let $H:\mathbb{R}^n\to\mathbb{R}$, $H\in C^1(\mathbb{R}^n\setminus\{0\})$, be a kernel homogeneous of degree $-n-1$ such that $H^i_1(x)=x_iH(x)$, $i=1,\dots,n$, define a CZO of convolution type. Let $g\in C^{1,\gamma}(\mathbb{R}^n;\mathbb{R})$ and $f\in C^\gamma_c(\mathbb{R}^n;\mathbb{R})$. Then for*

$$Tf(x) = p.v.\int_{\mathbb{R}^n}H(x-x')(g(x)-g(x'))f(x')\,\mathrm{d}x'$$

*we have*

$$||Tf||_\gamma \leq c\,|g|_{1,\gamma}\,||f||_\gamma\,,$$

*for c depending on m(supp(f)).*

*Proof.* Since $g \in C^{1,\gamma}$ we can write its Taylor series centered at $x'$ as

$$g(x) = g(x') + \sum_{i=1}^{n} \partial_i g(x')(x_i - x_i') + R(x,x')$$

with $|R(x,x')| \leq c\,|g|_{1,\gamma}\,|x-x'|^{1+\gamma}$. Now, if we add and subtract some term we obtain

$$\begin{aligned}Tf(x) = &\int_{\mathbb{R}^n} H(x-x')(g(x)-g(x')-\nabla g(x')\cdot(x-x'))f(x')\,\mathrm{d}x' + \\ &+ \sum_{i=1}^{n} \text{p.v.} \int_{\mathbb{R}^n} (x_i - x_i')H(x-x')\partial_i g(x')f(x')\,\mathrm{d}x' =: T_1 f(x) + T_2 f(x),\end{aligned}$$

The kernel $H_g(x,x') := H(x-x')(g(x)-g(x')-\nabla g(x')\cdot(x-x'))$ satisfies the bound $\left|H_g(x,x')\right| \leq \frac{c|g|_{1,\gamma}}{|x-x'|^{n-\gamma}}$ and its gradient

$$\begin{aligned}\nabla_x H_g(x,x') = &\nabla H(x-x')(g(x)-g(x')-\nabla g(x')(x-x')) + \\ &+ H(x-x')(\nabla g(x) - \nabla g(x'))\end{aligned}$$

satisfies $\left|\nabla_x H_g(x,x')\right| \leq \frac{c|g|_{1,\gamma}}{|x-x'|^{n+1-\gamma}}$. To check that $T_1 f$ belongs to $C^\gamma$ we use an usual argument. We can see $|T_1 f(x)| \leq c\,||f||_{L^\infty}\,|g|_{1,\gamma}$ for every $x \in \mathbb{R}^n$. Then $||T_1 f||_{L^\infty} \leq c\,||f||_{L^\infty}\,|g|_{1,\gamma}$. Now, let $x_1, x_2 \in \mathbb{R}^n$ and $B := B(x_1, 3\,|x_1 - x_2|)$. We decompose

$$\begin{aligned}T_1 f(x_1) - T_1 f(x_2) = &\int_{\mathbb{R}^n \setminus B} (H_g(x_1,x') - H_g(x_2,x'))f(x')\,\mathrm{d}x' + \\ &+ \int_B H_g(x_1,x')f(x')\,\mathrm{d}x' - \int_B H_g(x_2,x')f(x')\,\mathrm{d}x'.\end{aligned}$$

Then, by the Mean Value Theorem and the bounds for $H_g$ and $\nabla_x H_g$, we have

$$\begin{aligned}&|T_1 f(x_1) - T_1 f(x_2)| \leq c\,|g|_{1,\gamma} \left\{ |x_1 - x_2| \int_{\mathbb{R}^n \setminus B} \frac{|f(x')|}{|x_1 - x'|^{n+1-\gamma}}\,\mathrm{d}x' + \right. \\ &\left. + \int_B \frac{|f(x')|}{|x_1 - x'|^{n-\gamma}}\,\mathrm{d}x' + \int_B \frac{|f(x')|}{|x_2 - x'|^{n-\gamma}}\,\mathrm{d}x' \right\} \leq \\ &\leq c\,|g|_{1,\gamma} \left\{ |x_1 - x_2|\,||f||_{L^\infty}\,|x_1 - x_2|^{\gamma - 1} + ||f||_{L^\infty}\,|x_1 - x_2|^\gamma \right\} \leq \\ &\leq c\,|g|_{1,\gamma}\,||f||_{L^\infty}\,|x_1 - x_2|^\gamma\,.\end{aligned}$$

So $|T_1 f|_\gamma \le c\,||f||_{L^\infty}\,|g|_{1,\gamma}$. To finish we need to bound $T_2 f$. If we set $H_1^i(x) = x_i H(x)$, which is a CZO of convolution type by hypothesis, then since $T_2 f = \sum_{i=1}^n H_1^i * (f\partial_i g)$, by Lemma 1.4 we have

$$||T_2 f||_\gamma \le c\,||f\nabla g||_\gamma \le c\,|g|_{1,\gamma}\,||f||_\gamma,$$

finishing the proof of the lemma.

□

The constant $c$ in Lemma 1.17 is finite whenever $f$ is compactly supported, but we know this is true by Lemma 1.7. In a few words, Lemma 1.7 states that $\rho(\cdot,t)$ is compactly supported when it is transported by a flow with bounded gradient as it happens, in particular, if $X \in O_M$. We are ready to check that $F'(X)$ is bounded. As promised, taking into account that boundedness, we can verify that the second hypothesis in Picard-Lindelöf holds for $F$.

**Proposition 1.18.** *Let $O_M$ as defined in (1.20). Then, the functional $F : O_M \to C^{1,\gamma}(\mathbb{R}^n;\mathbb{R}^n)$ defined in (1.21) is locally Lipschitz.*

*Proof.* First of all, by the Fundamental Theorem of Calculus, given $X_1, X_2 \in O_M$,

$$\begin{aligned}|F(X_1) - F(X_2)|_{1,\gamma} &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}\varepsilon} F(X_1 + \varepsilon(X_2 - X_1))\,\mathrm{d}\varepsilon\right|_{1,\gamma} = \\ &= \left|\left(\int_0^1 F'(X_1 + \varepsilon(X_2 - X_1)) \cdot (X_2 - X_1)\,\mathrm{d}\varepsilon\right)\right|_{1,\gamma} \le \\ &\le \left\{\int_0^1 ||F'(X_1 + \varepsilon(X_2 - X_1))||_{B\to B}\,\mathrm{d}\varepsilon\right\} |X_2 - X_1|_{1,\gamma},\end{aligned}$$

where $F'(X)$ is the operator defined by $Y \to F'(X)Y$. So, provided $||F'(X)||_{B\to B} < \infty$ for every $X \in B$ then the integral in the previous expression is finite and therefore $F$ is Lipschitz. Thus, it suffices to prove this boundedness in order to prove the Proposition.

By Proposition 1.14 we have seen that every component of $F'(X)Y$ can be written as the sum of two terms $I$ and $II$. The arguments in Proposition 1.11 can be repeated for each element appearing in the sum in which $II$ is decomposed, just by changing the role of $\rho_0$ to $\rho_0\partial_s Y_r$. Then we can conclude, similarly, that

$$|II|_{1,\gamma} \le c(n,\rho_0)\,|Y|_{1,\gamma}\,|X|_{1,\gamma}^n.$$

Hence, we just have to work with the first term, $I$, and bound its $C^{1,\gamma}$ norm. Before considering derivatives, note that $I$ can be compared to any derivative of $II$, so also in similar fashion to Proposition 1.11, we get

$$||I||_{L^\infty} \le c(n,\rho_0)\,|Y|_{1,\gamma}\,|X|_{1,\gamma}^n.$$

We then need to consider derivatives of $I$. By Lemma 1.16 we write $I = \sum_{i=1}^n I_i$ and also $\partial_l I_i(\alpha) = \sum_{k=1}^n \partial_k \tilde{I}_i(X(\alpha))\partial_l X_k(\alpha)$. Since $C^\gamma$ is an algebra and also, by (1.12), we have

$$||\partial_i I_i||_\gamma \leq \sum_{k=1}^{n} ||\partial_k \tilde{I}_i \circ X||_\gamma \, ||\partial_l X_k||_\gamma \leq$$
$$\leq \sum_{k=1}^{n} ||\partial_k \tilde{I}_i||_\gamma \, (1 + |X|_{1,\gamma}^{\gamma}) \, |X|_{1,\gamma} \, .$$

Now we focus on $||\partial_k \tilde{I}_i||_\gamma$. We consider the expression given by Lemma 1.16

$$\begin{aligned}
&\partial_k \tilde{I}_i(x) = \\
&= \text{p.v.} \int_{\mathbb{R}^n} \partial_k \partial_i K_j(x - x')(Y_i(X^{-1}(x)) - Y_i(X^{-1}(x')))(\rho_0 \circ X^{-1})(x') \, \mathrm{d}x' + \\
&+ \partial_k [Y_i \circ X^{-1}](x) \text{p.v.} \int_{\mathbb{R}^n} \partial_i K_j(x - x')(\rho_0 \circ X^{-1})(x') \, \mathrm{d}x' + \\
&+ c_k(\nabla [Y_i \circ X^{-1}](x) \cdot \xi_k)(\rho_0 \circ X^{-1})(x) = A(x) + B(x) + C(x).
\end{aligned}$$

A straightforward repetition of the arguments done before let us verify that

$$||C||_\gamma \leq c(n, \rho_0) \, |Y|_{1,\gamma} \, |X|_{1,\gamma}^m \, ,$$

where $m$ is a finite constant depending on $\gamma$ and $n$. Also, as in Proposition 1.11, we get

$$||B||_\gamma \leq c(n, \rho_0) \, |Y|_{1,\gamma} \, |X|_{1,\gamma}^m \, .$$

The term $A$ is more involved than the rest in the decomposition of $\partial_k \tilde{I}_i$. Nevertheless, we claim that we can apply Lemma 1.17 for $H = \partial_k \partial_i K_j$, $g = Y_i \circ X^{-1}$ and $f = \rho_0 \circ X^{-1}$ to obtain also

$$||A||_\gamma \leq c(n, \rho_0, R) \, |Y|_{1,\gamma} \, |X|_{1,\gamma}^m \, ,$$

where $R = \mathrm{m}(\mathrm{supp}(\rho_0 \circ X^{-1})) = \mathrm{m}(\mathrm{supp}(\rho))$. We conclude that

$$|I|_{1,\gamma} \leq c(n, \rho_0, R) \, |Y|_{1,\gamma} \, |X|_{1,\gamma}^n \, ,$$

where $c(n, \rho, R)$ is finite by Lemma 1.7.

Summing up, for any $X \in B$ and any $Y \in O_M$ we have seen

$$\left| F'(X)Y \right|_{1,\gamma} \leq c(n, \rho_0, R) \, |X|_{1,\gamma}^m \, |Y|_{1,\gamma} \, ,$$

so $F'(X)$ is bounded as a linear operator from $O_M$ to $B$ and then the Proposition is proved. □

**Remark 1.19.** *We have been able to apply Lemma 1.17 since the kernels* $x_i \partial_j \partial_k \partial_l N$ *are CZO. In general, if* $K$ *satisfies the hypothesis in Lemma 1.4 then its second derivatives* $\partial_j \partial_k K$ *satisfies Lemma 1.17. It is clear that* $\partial_j \partial_k K$ *is homogeneous* *of degree $-n-1$. Also, if $i \neq j$ (and similarly if $i \neq k$),*

$$\int_{|w|=1} w_i \partial_j \partial_k K(w)\, \mathrm{d}\sigma(w) = \int_{|w|=1} \partial_j [w_i \partial_k K(w)]\, \mathrm{d}\sigma(w) = 0.$$

*The last integral vanishes for similar reasons as explained in Remark 1.5. Otherwise, if $i = j = k$,*

$$\int_{|w|=1} w_i \partial_i^2 K(w)\, \mathrm{d}\sigma(w) =$$
$$= \int_{|w|=1} \partial_i [w_i \partial_i K(w)]\, \mathrm{d}\sigma(w) - \int_{|w|=1} \partial_i K(w)\, \mathrm{d}\sigma(w) = 0$$

*since both integrals in the right vanishes, similarly as before. If $K = \partial_l N$ then we have seen that $x_i \partial_j \partial_k \partial_l N$ are CZO.*

Finally, since all the hypothesis in Theorem 1.8 are verified, we prove the existence result for the trajectory maps.

**Theorem 1.20.** *Let $\rho_0 \in C_c^\gamma(\mathbb{R}^n; \mathbb{R})$. Then there exists $T^* > 0$ such that the ordinary differential equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(\alpha, t) = F(X(\alpha, t)), \\ X(\alpha, 0) = \alpha, \end{cases}$$

*for*

$$F(X(\alpha, t)) = \int_{\mathbb{R}^n} K(X(\alpha, t) - X(\alpha', t)) \rho_0(\alpha')\, \det[DX(\alpha', t)]\, \mathrm{d}\alpha',$$

*has a unique solution $X(\cdot, t) \in C^{1,\gamma}(\mathbb{R}^n; \mathbb{R}^n)$ for $t \in (-T^*, T^*)$.*

*Proof.* Let $B = C^{1,\gamma}(\mathbb{R}^n; \mathbb{R}^n)$ and let $O_M$ defined in (1.20). Then, by Propositions 1.11 and 1.18 the functional $F$ satisfies the hypothesis of Picard-Lindelöf's theorem 1.8 and therefore we conclude that the statement holds. □

Given the flow map $X(\cdot, t)$ we can define the solution to (1.1) in an unique way: since the velocity field is smooth enough, any solution of the transport equation (1.1) can be described through the trajectories. So we have the main result of this section: well-posedness in the Hölder class for the transport equation and the family of kernels $L \cdot \nabla N$ described in (1.6).

**Theorem 1.21.** *Let $\rho_0 \in C_c^\gamma(\mathbb{R}^n; \mathbb{R})$. Let $L \in M_{n \times n}(\mathbb{R})$ and consider $K = L \cdot \nabla N$, where $N$ is the fundamental solution of the Laplacian. Then there exists $T^* > 0$ such that the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot, t) = K * \rho(\cdot, t), \\ \rho(\cdot, 0) = \rho_0. \end{cases} \tag{1.28}$$

*has a unique solution* $\rho(\cdot,t) \in C_c^\gamma(\mathbb{R}^n;\mathbb{R})$, $v(\cdot,t) \in C^{1+\gamma}(\mathbb{R}^n;\mathbb{R}^n)$ *for* $t \in (-T^*, T^*)$.

*Proof.* By Theorem 1.20 up to time $T^*$ there exists a unique solution $X(\cdot,t) \in C^{1+\gamma}$. We define $\rho$ and $v$ via the flow $X$ as $\rho(\cdot,t) = \rho_0(X^{-1}(\cdot,t))$ and $v(\cdot,t) = L \cdot \nabla N * \rho(\cdot,t)$. Assume there exist $\tilde{\rho} \in C_c^\gamma$ and $\tilde{v} \in C^{1+\gamma}$ satisfying (1.28). Then we can find a trajectory $\tilde{X}(\cdot,t)$ associated to $\tilde{v}(\cdot,t)$ such that $\tilde{\rho}$ is transported by $\tilde{X}$. Since we have uniqueness of trajectory by Theorem 1.20 then $\tilde{X} = X$. Therefore

$$\tilde{\rho}(\cdot,t) = \rho_0(\tilde{X}^{-1}(\cdot,t)) = \rho_0(X^{-1}(\cdot,t)) = \rho(\cdot,t)$$

and hence, by convoluting the density with the kernel $K$ we can see that $\tilde{v} = v$. □

## 1.4 Global Theorem

We want to show that the solution defined in Theorem 1.20 does exist for any time, that is, we want to show that $T^* = \infty$. In order to do that, we need to invoke a Continuation Theorem which gives us a necessary condition for that to happen. The theorem is stated as in [MB, p. 148] and a proof for a general version of it can be found in [LL, p. 161]. We would like to remark that it is valid since we have been able to state the problem with a functional $F$ which does not depend explicitly on time.

**Theorem 1.22.** *In the situation of Theorem 1.8 the unique solution* $X \in C^1((-T,T);O)$ *either exists globally in time or* $T$ *is finite and* $X(t)$ *leaves the open set* $O$ *as* $|t|$ *approaches* $T$.

In a nutshell, we need to check that at time $T^*$ the flow $X(\cdot,T^*)$ still belongs to $O_M$. As we will see later, it is enough to verify that the $C^{1,\gamma}$ norm of $X(\cdot,T^*)$ is a priori bounded. The following lemma is an auxiliary result needed in order to achieve bounds that allow us to prove that boundedness.

**Lemma 1.23.** *Let* $X(\cdot,t)$ *defined in* (1.2). *Then for the inverse flow at time* $t$*, we have* $X^{-1}(\cdot,t) = \tilde{X}(0,t,\cdot)$*, where* $\tilde{X}(s,t,x)$ *is the solution of the integro-differential equation*

$$\tilde{X}(s,t,x) = x - \int_s^t v(\tilde{X}(r,t,x),r)\,dr.$$

*Proof.* We define a generalized flow map $\hat{X} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfying

$$\hat{X}(s,t,x) = x + \int_s^t v(\hat{X}(s,r,x),r)\,dr. \tag{1.29}$$

Then $\hat{X}(s,t,x)$ is the position at time $t$ of the particle that was at the position $x$ at time $s$. It is clear that $X(x,t) = \hat{X}(0,t,x)$.

First, we can check that $\hat{X}$ has the semigroup structure

$$\hat{X}(s,t,x) = \hat{X}(\tau,t,\hat{X}(s,\tau,x)). \tag{1.30}$$

By definition of $\hat{X}$ in (1.29), the right hand side of (1.30) can be expressed as

$$\hat{X}(\tau,t,\hat{X}(s,\tau,x)) = \hat{X}(s,\tau,x) + \int_\tau^t v(\hat{X}(\tau,u,\hat{X}(s,\tau,x)),u)\,\mathrm{d}u. \tag{1.31}$$

We differentiate (1.31) with respect to $\tau$ to get

$$\begin{aligned}\partial_\tau(\hat{X}(\tau,t,\hat{X}(s,\tau,x))) &= \partial_\tau \hat{X}(s,\tau,x) - v(\hat{X}(\tau,\tau,\hat{X}(s,\tau,x)),\tau) = \\ &= v(\hat{X}(s,\tau,x),\tau) - v(\hat{X}(s,\tau,x),\tau) = 0.\end{aligned}$$

So $\hat{X}(\tau,t,\hat{X}(s,\tau,x))$ does not depend on $\tau$ and hence

$$\hat{X}(\tau,t,\hat{X}(s,\tau,x)) = \hat{X}(\tau,t,\hat{X}(s,\tau,x))|_{\tau=t} = \hat{X}(t,t,\hat{X}(s,t,x)) = \hat{X}(s,t,x),$$

which proves equation (1.30).

Secondly, we want to see that $\hat{X}$ satisfies a transport equation. Differentiating (1.30) with respect to $s$ we obtain

$$\begin{aligned}\partial_s \hat{X}(s,t,x) &= \nabla \hat{X}(\tau,t,\hat{X}(s,\tau,x))\partial_s \hat{X}(s,\tau,x) = \\ &= -\nabla \hat{X}(\tau,t,\hat{X}(s,\tau,x))v(\hat{X}(s,s,x),s) = \\ &= -\nabla \hat{X}(\tau,t,\hat{X}(s,\tau,x))v(x,s),\end{aligned} \tag{1.32}$$

and putting $\tau = s$ in (1.32)

$$\partial_s \hat{X}(s,t,x) + v(x,s)\nabla \hat{X}(s,t,x) = 0. \tag{1.33}$$

Thus, $\hat{X}(\cdot,t,\cdot) : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies a transport equation.

Then, we define a map $\tilde{X} : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n$ via

$$\tilde{X}(s,t,x) = x - \int_s^t v(\tilde{X}(r,t,x),r)\,\mathrm{d}r. \tag{1.34}$$

We want to check that for any $t,s \in \mathbb{R}$ and any $x \in \mathbb{R}^n$ the maps $\hat{X}$ and $\tilde{X}$ are inverse in the following sense

$$\hat{X}(s,t,\tilde{X}(s,t,x)) = x. \tag{1.35}$$

In order to prove (1.35) we differentiate its left hand side with respect to $s$

$$\partial_s \hat{X}(s,t,\tilde{X}(s,t,x)) + \nabla \hat{X}(s,t,\tilde{X}(s,t,x))\partial_s \tilde{X}(s,t,x).$$

By (1.34) we can see that $\partial_s \tilde{X}(s,t,x) = v(\tilde{X}(s,t,x),s)$ and then the above expression can be written as

$$\partial_s \hat{X}(s,t,\tilde{X}(s,t,x)) + v(\tilde{X}(s,t,x),s)\nabla \hat{X}(s,t,\tilde{X}(s,t,x)) = \\ = \left[\partial_s \hat{X}(\cdot,t,\cdot) + v(\cdot,\cdot)\nabla \hat{X}(\cdot,t,\cdot)\right](x,\tilde{X}(s,t,x)) = 0$$

since $\hat{X}(\cdot,t,\cdot)$ satisfies the transport equation (1.33).

Thus, the left hand side of (1.35) does not depend on $s$ and hence

$$\hat{X}(s,t,\tilde{X}(s,t,x)) = \hat{X}(t,t,\tilde{X}(t,t,x)) = x,$$

as we wanted to prove.

Taking $s = 0$ we get $\hat{X}(0,t,\tilde{X}(0,t,x)) = X(\tilde{X}(0,t,x),t)$, so $X(x,t)^{-1} = \tilde{X}(s,t,x)|_{s=0}$ and the Lemma is proved. □

As a consequence of the previous Lemma, we see that the flow map $X(\cdot,t)$ and its inverse $X^{-1}(\cdot,t)$ share the same regularity properties since its integral expressions are similar. We will use this fact later on.

The next lemma is a well known and very classical fact in the theory of ordinary differential differential equations. For its proof see, for instance, [Pa, p. 13].

**Lemma 1.24** (Gronwall Lemma)**.** *Let $u$ and $f$ be continuous and nonnegative functions defined on $I = [a,b]$, and let $n$ be a continuous, positive and non-decreasing function defined on $I$. If*

$$u(t) \le n(t) + \int_a^t f(s)u(s)\,\mathrm{d}s$$

*for $t \in I$, then*

$$u(t) \le n(t)\exp\left(\int_a^t f(s)\,\mathrm{d}s\right).$$

We are ready to give a first condition ensuring the a priori $C^{1,\gamma}$ boundedness of the flow map at any time. The following bounds are very similar to those obtained for the Euler equation. The only difference that appears in the proof is the fact that the measure of the support of $\rho(\cdot,t)$ is not constant over time, because the divergence is not zero in general.

**Proposition 1.25.** *Let $X(\cdot,t)$ be the solution given in Theorem 1.20 and $c(n)$ a constant depending on the dimension $n$. Then, for a certain function $G : \mathbb{R} \to \mathbb{R}^+$ with $G(t) < \infty$ whenever $\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s < \infty$, we have* *the following inequalities*

$$|X(0,t)| \leq c(n)m(supp(\rho_0))^{1/n} \, ||\rho_0||_{L^\infty} \int_0^t ||\nabla X(\cdot,s)||_{L^\infty} \, \mathrm{d}s,$$
$$||\nabla X(\cdot,t)||_{L^\infty} \leq \exp\left(\int_0^t ||\nabla v(\cdot,s)||_{L^\infty} \, \mathrm{d}s\right),$$
$$|\nabla X(\cdot,t)|_\gamma \leq G(t) \exp\left(c\int_0^t ||\nabla v(\cdot,s)||_{L^\infty} \, \mathrm{d}s\right).$$

*In particular,* $|X(\cdot,t)|_{1,\gamma}$ *is bounded provided* $\int_0^t ||\nabla v(\cdot,s)||_{L^\infty} \, \mathrm{d}s$ *also is.*

*Proof.* By definition (1.2) we have, for any $\alpha \in \mathbb{R}^n$,

$$X(\alpha,t) = \alpha + \int_0^t v(X(\alpha,s),s) \, \mathrm{d}s.$$

Setting $\alpha = 0$ and taking absolute value we get

$$|X(0,t)| = \left|\int_0^t v(X(0,s),s) \, \mathrm{d}s\right| \leq \int_0^t ||v(X(\cdot,s),s)||_{L^\infty} \, \mathrm{d}s.$$

Since $X(\cdot,s)$ is an homeomorphism then $||v(X(\cdot,s),s)||_{L^\infty} = ||v(\cdot,s)||_{L^\infty}$ and therefore we simply have

$$|X(0,t)| \leq \int_0^t ||v(\cdot,s)||_{L^\infty} \, \mathrm{d}s.$$

Now let $R(s) = m(\mathrm{supp}(\rho(\cdot,s)))^{1/n}$. Then as $v(\cdot,s) = K * \rho(\cdot,s)$ by (1.15) we get the bound

$$||v(\cdot,s)||_{L^\infty} \leq cR(s) \, ||\rho(\cdot,s)||_{L^\infty} = cR(s) \, ||\rho_0||_{L^\infty}, \tag{1.36}$$

where the last equality stands provided $\rho$ is transported with the flow and so, the $L^\infty$ norm is conserved in time.

By Lemma 1.7 we have a control for $R(s)$ and then

$$|X(0,t)| \leq c(n)m(\mathrm{supp}(\rho_0))^{1/n} \, ||\rho_0||_{L^\infty} \int_0^t ||\nabla X(\cdot,s)||_{L^\infty} \, \mathrm{d}s.$$

In order to achieve bounds on derivatives of the flow map, we compute the partial derivative with respect to $\alpha_i$ (denoted as $\partial_i$ from now on) of the $j$-th component of (1.2). By the chain rule, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\partial_i X_j(\alpha,t)) &= \sum_{k=1}^n \frac{\partial v_j(X(\alpha,t),t)}{\partial X_k(\alpha,t)} \partial_i X_k(\alpha,t) = \\ &= \nabla v_j(X(\alpha,t)) \cdot \partial_i X(\alpha,t),\end{aligned} \tag{1.37}$$

where $\cdot$ stands for the scalar product between vectors in $\mathbb{R}^n$.

Taking $L^\infty$ norm on (1.37) and considering supremum over $i, j$ we get

$$\frac{\mathrm{d}}{\mathrm{d}t} ||\nabla X(\cdot,t)||_{L^\infty} \leq ||\nabla v(X(\cdot,t),t)||_{L^\infty} ||\nabla X(\cdot,t)||_{L^\infty} = \\ = ||\nabla v(\cdot,t)||_{L^\infty} ||\nabla X(\cdot,t)||_{L^\infty} . \tag{1.38}$$

Therefore, by direct integration on (1.38) we have the desired bound

$$||\nabla X(\cdot,t)||_{L^\infty} \leq \exp\left(\int_0^t ||\nabla v(\cdot,s)||_{L^\infty} \,\mathrm{d}s\right). \tag{1.39}$$

Finally, taking the $|\cdot|_\gamma$ seminorm and considering supremum over $i, j$ on (1.37) we have

$$\frac{\mathrm{d}}{\mathrm{d}t} |\nabla X(\cdot,t)|_\gamma \leq \sup_{i,j} \left|\nabla v_j(X(\cdot,t)) \cdot \partial_i X(\cdot,t)\right|_\gamma \leq \\ \leq c\left(||\nabla v(\cdot,t)||_{L^\infty} |\nabla X(\cdot,t)|_\gamma + |\nabla v(X(\cdot,t),t)|_\gamma ||\nabla X(\cdot,t)||_{L^\infty}\right), \tag{1.40}$$

where we have used inequality (1.7) to bound the $|\cdot|_\gamma$ seminorm of a product.

By (1.11) we have

$$|\nabla v(X(\cdot,t),t)|_\gamma \leq |\nabla v(\cdot,t)|_\gamma ||\nabla X(\cdot,t)||^\gamma_{L^\infty} . \tag{1.41}$$

Also, by (1.16),

$$|\nabla v(\cdot,t)|_\gamma \leq c\,|\rho(\cdot,t)|_\gamma \leq c\,|\rho_0|_\gamma \left|\left|\nabla X^{-1}(\cdot,t)\right|\right|^\gamma_{L^\infty}, \tag{1.42}$$

where we have used that $\rho(\cdot,t) = \rho_0(X^{-1}(\cdot,t))$. Using the equation for $X^{-1}(\cdot,t)$ described in Lemma 1.23 and similarly as done in (1.38) we have

$$\left|\left|\nabla X^{-1}(\cdot,t)\right|\right|_{L^\infty} \leq \exp\left(\int_0^t ||\nabla v(\cdot,s)||_{L^\infty} \,\mathrm{d}s\right). \tag{1.43}$$

Combining inequalities (1.38), (1.41), (1.42), (1.43), we get a bound for (1.40)

$$\frac{\mathrm{d}}{\mathrm{d}t} |\nabla X(\cdot,t)|_\gamma \leq c\left(||\nabla v(\cdot,t)||_{L^\infty} |\nabla X(\cdot,t)|_\gamma + \right. \\ \left. + |\rho_0|_\gamma \exp\left((1+2\gamma)\int_0^t ||\nabla v(\cdot,s)||_{L^\infty} \,\mathrm{d}s\right)\right). \tag{1.44}$$

Setting

$$g(t) := c\,|\rho_0|_\gamma \exp\left((1+2\gamma)\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s\right)$$

and $G(t) := \int_0^t g(s)\,\mathrm{d}s$, and then applying Lemma 1.24 to (1.44) we get

$$|\nabla X(\cdot,t)|_\gamma \le G(t)\exp\left(c\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s\right),$$

which completes the proof of the proposition. □

Having proved the inequalities in Proposition 1.25 we can see that, in fact, the $C^{1,\gamma}$ norm of the flow is finite for any time which was our first goal.

**Proposition 1.26.** *Let $X(\cdot,t)$ be the solution given in Theorem 1.20. Then $|X(\cdot,t)|_{1,\gamma}$ is finite for any time.*

*Proof.* By Proposition 1.25 $|X(\cdot,t)|_{1,\gamma}$ is finite provided $\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s$ is. Then it suffices to check that this integral is bounded for any time. Let $i,j\in\{1,\dots,n\}$. By Lemma 1.10 we have

$$\partial_i v_j(\cdot,t) = \frac{l_{ji}}{n}\rho(\cdot,t) + p.v.\partial_i K_j * \rho(\cdot,t). \tag{1.45}$$

Let $\varepsilon = |\rho(\cdot,t)|_\gamma^{-1/\gamma}$ and let $R(t) = m(\mathrm{supp}(\rho(\cdot,t)))^{1/n}$ and apply inequality (1.16) to the equation (1.45) to obtain

$$||\partial_i v_j(\cdot,t)||_{L^\infty} \le c\{1+\ln[R(t)\,|\rho(\cdot,t)|_\gamma^{1/\gamma}]\,||\rho_0||_{L^\infty}\}. \tag{1.46}$$

Since $\rho(\cdot,t) = \rho_0(X^{-1}(\cdot,t))$ then $|\rho(\cdot,t)|_\gamma \le |\rho_0|_\gamma\,||\nabla X(\cdot,t)||_{L^\infty}^\gamma$. Also, taking into account Lemma 1.7 we get

$$R(t)\,|\rho(\cdot,t)|_\gamma^{1/\gamma} \le c(n)m(\mathrm{supp}(\rho_0))^{1/n}\,|\rho_0|_\gamma^{1/\gamma}\,||\nabla X(\cdot,t)||_{L^\infty}^2.$$

By Proposition 1.25 we can bound $||\nabla X(\cdot,t)||_{L^\infty}$ and then

$$R(t)\,|\rho(\cdot,t)|_\gamma^{1/\gamma} \le c(n)m(\mathrm{supp}(\rho_0))^{1/n}\,|\rho_0|_\gamma^{1/\gamma}\exp\left(2\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s\right).$$

Therefore inequality (1.46) can be written as

$$||\partial_i v_j(\cdot,t)||_{L^\infty} \le c(n,\rho_0) + c\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s.$$

Taking supremum over $i,j\in\{1,\dots,n\}$

$$||\nabla v(\cdot,t)||_{L^\infty} \le c(n,\rho_0) + c\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,\mathrm{d}s.$$

and applying Gronwall's Lemma (1.24) we finally get

$$||\nabla v(\cdot,t)||_{L^\infty} \leq c(n,\rho_0)\exp(ct),$$

which makes $\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s$ finite for any time. □

Finally, as we anticipated at the beginning of the section, using the a priori bound for the flow and by the Continuation Theorem 1.22 we can prove that the solution $X(\cdot,t)$ is global in time.

**Theorem 1.27.** *Let $\rho_0 \in C_c^\gamma(\mathbb{R}^n;\mathbb{R})$. Then the ordinary differential equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X(\alpha,t) = F(X(\alpha,t)), \\ X(\alpha,0) = \alpha, \end{cases}$$

*for*

$$F(X(\alpha,t)) = \int_{\mathbb{R}^n} K(X(\alpha,t) - X(\alpha',t))\rho_0(\alpha')\ \det[DX(\alpha',t)]\,\mathrm{d}\alpha'$$

*has a unique solution $X(\cdot,t) \in C^{1,\gamma}(\mathbb{R}^n;\mathbb{R}^n)$ for any time $t \in \mathbb{R}$.*

*Proof.* We want to apply Theorem 1.22 in order to ensure the globalness of the solution $X(\cdot,t)$. So, we need to check that, for any given $T$, the map $X(\cdot,T)$ belongs to $O_M$ where

$$O_M = B \cap \left\{ X : \mathbb{R}^n \to \mathbb{R}^n : \frac{1}{M} < \sup_{\alpha\neq\beta} \frac{|X(\alpha) - X(\beta)|}{|\alpha-\beta|} < M \right\}.$$

Let us first prove that we can avoid to check that the condition for $M$ is satisfied at time $T$. By the Mean Value Theorem,

$$|X(\alpha,t) - X(\beta,t)| \leq ||\nabla X(\cdot,t)||_{L^\infty} |\alpha-\beta| \leq |X(\cdot,t)|_{1,\gamma} |\alpha-\beta|,$$

and also since we can express

$$\alpha = X^{-1}((X(\alpha,t),t)) \quad \text{and} \quad \beta = X^{-1}((X(\beta,t),t)),$$

we then get

$$|\alpha-\beta| \leq \left|\left|\nabla X^{-1}(\cdot,t)\right|\right|_{L^\infty} |X(\alpha,t) - X(\beta,t)| \leq |X(\cdot,t)|_{1,\gamma}^{2n-1} |X(\alpha) - X(\beta)|.$$

Consider now $M'$ such that

$$\sup_{t\in[-T,T]} \max\{|X(\cdot,t)|_{1,\gamma}, |X(\cdot,t)|_{1,\gamma}^{2n-1}\} \leq M' < \infty.$$

Such an $M'$ exists since $|X(\cdot,t)|_{1,\gamma}$ is finite for every time by Proposition 1.26. For this choice of $M'$ it is sure that $X(\cdot,t) \in O_{M'}$ for every time $t \in [-T,T]$.

Throughout the proofs of Proposition 1.11 and 1.18 we can check that those statements are independent of $M$. Due to this independence we can modify $O_M$ to $O_{M'}$ without changing neither the solution nor the maximal time of existence given by Picard-Lindelöf's theorem.

Then as soon as $|X(\cdot,t)|_{1,\gamma}$ is finite at time $T$, $X(\cdot,T) \in O_{M'}$. But we know that this is true by Proposition 1.26. So $X(\cdot,T) \in O_{M'}$ and this does not depend on the choice of $T$ so we have existence and uniqueness of $X(\cdot,t) \in C^{1,\gamma}$ for any time $t$ by Theorem 1.22. □

As a direct consequence of Theorems 1.21 and 1.27 we finally have the next result, which corresponds to Theorem 1.2 for $m = 0$.

**Theorem 1.28.** *Let $N$ the fundamental solution of the Laplacian in $\mathbb{R}^n$. Consider $L \in M_{n\times n}(\mathbb{R})$. For $0 < \gamma < 1$, if $\rho_0 \in C_c^\gamma(\mathbb{R}^n, \mathbb{R})$, then the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot,t) = L \cdot \nabla N * \rho(\cdot,t), \\ \rho(\cdot,0) = \rho_0, \end{cases}$$

*has a unique solution $\rho(\cdot,t) \in C_c^\gamma(\mathbb{R}^n,\mathbb{R})$ for any time $t \in \mathbb{R}$.*

## 1.5 Higher regularity

Up to now we have proved a special case of Theorem 1.2. In this Section, we explain, skipping most of the computations (which are much more tedious in the general situation), how to move from $m = 0$ to the general version. We will sometimes focus on $m = 1$ since some detail for it can be given explicitly and also the reader can deduce the behavior of $m \geq 2$ from this case.

For $\rho_0 \in C_c^{m,\gamma}$ we expect $\rho(\cdot,t) \in C_c^{m,\gamma}$, $v(\cdot,t) \in C^{m+1,\gamma}$ and hence $X(\cdot,t) \in C^{m+1,\gamma}$ as well, so this will be a natural working space in order to apply Picard-Lindelöf's theorem to the ODE satisfied by $X(\cdot,t)$. Then, for $B = C^{m+1,\gamma}$, we take

$$O_M^{m+1} = C^{m+1,\gamma} \cap \left\{ X : \mathbb{R}^n \to \mathbb{R}^n : |X|_{m+1,\gamma} < M \right\}. \tag{1.47}$$

As in Section 1.3, we need to prove first that $F : O_M^{m+1} \to C^{m+1,\gamma}$. Roughly speaking, we have to deal with the $m+1$-th derivatives of the functional

$$F(X(\alpha,t)) := \int_{\mathbb{R}^n} K(X(\alpha,t) - X(\alpha',t))\rho_0(\alpha') \det[DX(\alpha',t)]\, d\alpha'.$$

The first derivative can be applied to the kernel $K$ as it was done in Proposition 1.11, obtaining again Equation (1.24). Then the difficult term will be

$$S_{ij}(\alpha) = \text{p.v.} \int_{\mathbb{R}^n} \partial_i K_j(X(\alpha) - X(\alpha'))\rho_0(\alpha') \det DX(\alpha')\, d\alpha'.$$

We want to use the fact that now $\rho_0$ is more regular than in Section 1.3. So we want to put the rest of derivatives up to the $m+1$-th on it. We can undo the change of variables performed and get

$$S_{ij}(\alpha) = \text{p.v.} \int_{\mathbb{R}^n} \partial_i K_j(X(\alpha) - x')\rho_0(X^{-1}(x'))\,\mathrm{d}x' = \\ = \text{p.v.}\ (\partial_i K_j * (\rho_0 \circ X^{-1}))(X(\alpha)).$$

Since $\rho_0 \in C_c^{m,\gamma}$ and $X \in O_M^{m+1}$ the term $\rho_0 \circ X^{-1}$ can receive up to the $m+1$-th derivative. For instance, if $m=1$

$$\partial_k S_{ij}(\alpha) = \text{p.v.}\ (\partial_i K_j * (\nabla[\rho_0 \circ X^{-1}]))(X(\alpha)) \cdot \partial_k X(\alpha).$$

Therefore, repeating the arguments on Proposition 1.11 we can get

$$|S_{ij}|_{1,\gamma} \simeq ||\partial_k S_{ij}||_\gamma \le c(|\rho_0|_{1,\gamma}, |X|_{1,\gamma})\,|X|_{2,\gamma}.$$

In general, and dealing with the rest of the terms (which are easier) we have the following statement, analogous (and more precise) to Proposition 1.11.

**Proposition 1.29.** *Let $O_M^{m+1}$ as defined in (1.47). Then, the functional $F$ defined in (1.21) maps $O_M^{m+1}$ to $C^{m+1,\gamma}(\mathbb{R}^n;\mathbb{R}^n)$. In particular,*

$$|F(X(\cdot,t))|_{m+1,\gamma} \le c(|\rho_0|_{m,\gamma}, |X(\cdot,t)|_{m,\gamma})\,|X(\cdot,t)|_{m+1,\gamma}.$$

We need to verify that $F : O_M^{m+1} \to C^{m+1,\gamma}$ is locally Lipschitz continuous. Again, it suffices to bound $F'(X)$ as a linear operator. The expression for $F'(X)Y$ in Proposition 1.14 and the derivatives of the tricky term that were computed in Lemma 1.16 remain both valid. We need to adapt the Lemma 1.17 to the higher regularity of $\rho_0$. For example, for $m=1$, we have the following Lemma.

**Lemma 1.30.** *Let $H \in C^2(\mathbb{R}^n \setminus \{0\})$ be a real-valued kernel, homogeneous of degree $-n-1$ and such that*

- *for any $i \in \{1,\dots,n\}$ the kernel $H_1^i(x) = x_i H(x)$,*
- *and for any $i,j,k \in \{1,\dots,n\}$ the kernel $H_2^{i,j,k}(x) = x_i x_j \partial_k H(x)$*

*both define a SIO of convolution type. Let $g \in C^{2,\gamma}(\mathbb{R}^n;\mathbb{R})$ and $f \in C_c^{1,\gamma}(\mathbb{R}^n;\mathbb{R})$. Then for*

$$Tf(x) = p.v. \int_{\mathbb{R}^n} H(x-x')(g(x)-g(x'))f(x')\,\mathrm{d}x'$$

*we have*

$$||Tf||_{1,\gamma} \le c\,|g|_{2,\gamma}\,||f||_{1,\gamma},$$

*for $c$ depending on $m(supp(f))$.*

*Proof.* Since $g \in C^{2,\gamma}$ we can write its Taylor series centered at $x'$ as

$$g(x) = g(x') + \sum_{i=1}^{n} \partial_i g(x')(x_i - x'_i) + \\ + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \partial_i\partial_j g(x')(x_i - x'_i)(x_j - x'_j) + R(x, x')$$

with $|R(x,x')| \leq c\,|g|_{2,\gamma}\,|x - x'|^{2+\gamma}$. Now, if we add and subtract the terms in the Taylor series written above, we obtain, by defining $\hat{H}_2^{i,j}(y) := y_i y_j H(y)$,

(1.48)
$$\begin{aligned} Tf(x) = &\int_{\mathbb{R}^n} H(x-x')R(x,x')f(x')\,\mathrm{d}x' + \\ &+ \sum_{i=1}^{n} \text{p.v.} \int_{\mathbb{R}^n} H_1^i(x-x')\partial_i g(x')f(x')\,\mathrm{d}x' + \\ &+ \sum_{i=1}^{n}\sum_{j=1}^{n} \int_{\mathbb{R}^n} \hat{H}_2^{i,j}(x-x')\partial_i\partial_j g(x')f(x')\,\mathrm{d}x' =: \\ &=: T_1 f(x) + \sum_{i=1}^{n}(H_1^i * (f\partial_i g))(x) + \sum_{i=1}^{n}\sum_{j=1}^{n}(\hat{H}_2^{i,j} * (f\partial_i\partial_j g))(x), \end{aligned}$$

The operator $T_1$ can be treated similarly to the respective term in the proof of Lemma 1.17 and we can benefit from the fact that first derivatives of $H(x - x')R(x, x')$ are bounded by a constant times $|x - x'|^{-n+\gamma}$ (as it happen to the kernel in $T_1$ from the mentioned Lemma) and second derivatives are bounded by a constan times $|x - x'|^{-n-1+\gamma}$ (as happened with derivatives of the kernel in $T_1$ from the Lemma). Arguing in a similar way then we can verify that

(1.49)
$$|T_1 f|_{1,\gamma} \leq c\,|f|_{1,\gamma}\,|g|_{2,\gamma}\,.$$

Since $H_1^i$ is a SIO of convolution type,

$$\left|\partial_k[H_1^i * (f(\partial_i g))]\right|_{\gamma} = \left|H_1^i * (\partial_k(f(\partial_i g)))\right|_{\gamma} \leq c\,|f|_{1,\gamma}\,|g|_{2,\gamma}\,,$$

and then taking supremum over $k$ we can see

(1.50)
$$\left|H_1^i * (f(\partial_i g))\right|_{1,\gamma} \leq c\,|f|_{1,\gamma}\,|g|_{2,\gamma}\,.$$

To finish we need to bound derivatives of $\hat{H}_2^{i,j} * (f(\partial_i\partial_j g))$. Since second derivatives of $g$ are involved in this term and $g \in C^{2,\gamma}$ if we want to differentiate the convolution these derivatives must go to $\hat{H}_2^{i,j}$. Recall that $\hat{H}_2^{i,j}(y) = y_i y_j H(y)$. Then if we consider, for instance, the partial derivative with respect to $k$ of this term, it may contain terms of type $H_1^i$ and terms of type $H_2^{i,j,k}$ (as in the statement of the Lemma) depending on the values of $i, j, k \in \{1, \ldots, n\}$. In any case since all of them are SIO of convolution type we also have

$$\left| \hat{H}_2^{i,j} * (f(\partial_i \partial_j g)) \right|_{1,\gamma} \leq c\, |f|_{1,\gamma}\, |g|_{2,\gamma}\,. \tag{1.51}$$

Combining (1.49), (1.50) and (1.51) into (1.48) the Lemma is proved.

□

**Remark 1.31.** *Note that we can apply the previous lemma to the hypersingular kernels $\partial_j \partial_l \partial_t N$ because of Remark 1.19 and because of the fact that also*

$$x_r x_s \partial_r \partial_j \partial_l \partial_t N$$

*define CZOs. In fact, if $K$ satisfies the hypothesis in Lemma 1.4 then its second derivatives $\partial_j \partial_l K$ satisfy the second item in the hypothesis in Lemma 1.30. Since the homogeneity of $x_r x_s \partial_i \partial_j \partial_l K$ is clearly of degree $-n$ then we just have to verify that we have the zero mean integration over spheres. So, we have to check that*

$$\int_{|w|=1} w_r w_s \partial_i \partial_j \partial_l K(w)\, \mathrm{d}\sigma(w) = 0.$$

*If $i$ (similarly for $j, l$) is such that $i \neq r$ and $i \neq s$ then*

$$\int_{|w|=1} w_r w_s \partial_i \partial_j \partial_l K(w)\, \mathrm{d}\sigma(w) = \int_{|w|=1} \partial_i \left[ w_r w_s \partial_j \partial_l K(w) \right]\, \mathrm{d}\sigma(w) = 0,$$

*the last inequality being true for similar reasons as argued in Remark 1.5. Otherwise, if at least one of the elements in $\{r, s\}$ (namely $r$) is equal to one in $\{i, j, l\}$ (namely $i$) then*

$$\int_{|w|=1} w_r w_s \partial_r \partial_j \partial_l K(w)\, \mathrm{d}\sigma(w) =$$
$$= \int_{|w|=1} \partial_r [w_r w_s \partial_j \partial_l K(w)]\, \mathrm{d}\sigma(w) - \int_{|w|=1} w_s \partial_r \partial_j \partial_l K(w)\, \mathrm{d}\sigma(w) = 0.$$

*The two integrals in the last expression vanishes similarly as seen in Remark 1.19.*

We can apply this bound to the first term in Lemma 1.16 (the rest of the terms involved are easier to handle) and in similar fashion to what was done in Section 1.3 get

$$\left| F'(X) Y \right|_{2,\gamma} \leq c(n, \rho_0, R)\, |X|_{2,\gamma}^n\, |Y|_{2,\gamma}\,.$$

In general we have an equivalent result to Proposition 1.18.

**Proposition 1.32.** *Let $O_M^{m+1}$ as defined in (1.47). Then, the functional $F : O_M^{m+1} \to C^{m+1,\gamma}(\mathbb{R}^n; \mathbb{R}^n)$ defined in (1.21) is Lipschitz.*

Then by Propositions 1.11 and 1.18 we can apply Picard-Lindelöf's theorem again and obtain local-in-time existence and uniqueness of solution for the trajectory map $X(\cdot,t)$ and thus for $(\rho(\cdot,t), v(\cdot,t))$.

For the global-in-time version of the theorem, notice that the precise bound given in Proposition 1.29 is very important. By making use of it, we can see

$$\frac{\mathrm{d}}{\mathrm{d}t}\left|X(\cdot,t)\right|_{m+1,\gamma} \leq \left|F(X(\cdot,t))\right|_{m+1,\gamma} \leq c(|\rho_0|_{m,\gamma}, |X(\cdot,t)|_{m,\gamma})\left|X(\cdot,t)\right|_{m+1,\gamma},$$

which implies

$$|X(\cdot,t)|_{m+1,\gamma} \leq |X(\cdot,0)|_{m+1,\gamma} \exp\left(\int_0^t c(|\rho_0|_{m,\gamma}, |X(\cdot,s)|_{m,\gamma})\,\mathrm{d}s\right).$$

That is, $|X(\cdot,t)|_{m+1,\gamma}$ is bounded a priori if $|X(\cdot,t)|_{m,\gamma}$ is and so on. Hence we can reduce to bound $|X(\cdot,t)|_{1,\gamma}$ and we know by Proposition 1.26 that this is finite for any time. This is sufficient to ensure the globalness of the solution.

# 2 $C_c^\gamma$ well-posedness in $\mathbb{C}$

## 2.1 Introduction

We want to translate Equation (1.4) to the two-dimensional euclidean space seen as the complex plane. So, we consider $x = (x_1, x_2) \equiv x_1 + ix_2 = z$. We note that $x^\perp = (-x_2, x_1) \equiv -x_2 + ix_1 = iz$ and $|x|^2 = z\bar{z}$. Hence,

$$K_{BS}(x) = \frac{1}{2\pi}\frac{x^\perp}{|x|^2} \equiv \frac{iz}{2\pi z\bar{z}} = \frac{i}{2\pi\bar{z}} = K_{\mathrm{BS}}(z) = \nabla^\perp N(z),$$

that is, we can reformulate the original problem using complex variable and the corresponding kernel still has a simple form. Our goal is to study the mentioned transport equation (1.4) but changing the kernel $K_{BS}$ for a similar one (with the same regularity properties). In the case of the aggregation equation the kernel can be written in complex variable as

$$K_{\mathrm{Ag}}(z) = \frac{-1}{2\pi\bar{z}} = -\nabla N(z).$$

A natural way to choose a kernel that resembles to $K_{\mathrm{BS}}$ and $K_{\mathrm{Ag}}$ is to take

$$K_C = \frac{1}{2\pi z},$$

where $C$ stands for Cauchy.

Despite there is no physical model described by the transport equation with the Cauchy kernel (at least as far as we are aware), the problem makes sense from the mathematical point of view and by comparison with $K_{\mathrm{BS}}$ and $K_{\mathrm{Ag}}$ it seems very reasonable to consider it.

Since for a velocity field in the complex plane $v(z) \equiv v_x + iv_y$ the derivative with respect to $z$ –which we denote by $\partial$– can be computed as

$$\partial v(z) \equiv \frac{1}{2}(\partial_x - i\partial_y)(v_x + iv_y) = \frac{1}{2}([\partial_x v_x + \partial_y v_y] + i[\partial_x v_y - \partial_y v_x]).$$

That means that in complex notation the divergence of $v$ can be expressed simply as $2\Re(\partial v)$. So, if $v(z) = K_C * \rho(\cdot, t)$ then

$$\mathrm{div} v(z) = 2\Re[\text{p.v. } \partial K_C * \rho(\cdot, t)] = 2\Re[B(\rho(\cdot, t))],$$

where $B$ is the Beurling Transform. This Cauchy kernel produces then a velocity field with a more singular divergence than in the previous cases.

Nevertheless, in Chapter 1 we dealt with kernels of type $K = L \cdot \nabla N$ (as in equation (1.6)). We can easily check that

$$\frac{1}{2\pi z} \equiv L \cdot \nabla N(z) \quad \text{for} \quad L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Thus, the $C_c^\gamma$ well-posedness for the Cauchy transport equation has been already proved as a special case of Theorem 1.2. In fact, when we consider the kernels

$$L \cdot \nabla N(z), \quad L = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

in the case of the complex plane we see that they correspond to

$$K(z) = \frac{1}{2\pi} \frac{w_1 z + w_2 \bar{z}}{|z|^2} = \frac{w_1}{2\pi \bar{z}} + \frac{w_2}{2\pi z},$$

for the complex numbers

$$w_1 = \frac{a+d}{2} + \frac{c-b}{2} i, \quad \text{and} \quad w_2 = \frac{a-d}{2} + \frac{b+c}{2} i.$$

Since the hypothesis in Picard-Lindelöf's theorem are satisfied for linear combinations of functionals satisfying them, we can say that we dealt with kernels of type $\frac{\alpha}{z}$ and $\frac{\beta}{\bar{z}}$ for $\alpha, \beta \in \mathbb{C}$. In the spirit of generalizing the kernels for which the transport equation is well-posed we consider a wider class containing those two simpler cases. We will consider kernels of the type

$$K_1(z) = \frac{1}{\pi} \frac{(z + \varepsilon \bar{z})^k}{(\bar{z} + \varepsilon z)^{k+1}} \quad \text{or} \quad K_2(z) = \overline{K_1(z)} = \frac{1}{\pi} \frac{(\bar{z} + \varepsilon z)^k}{(z + \varepsilon \bar{z})^{k+1}}, \tag{2.1}$$

for $\varepsilon \in \mathbb{R}$, $|\varepsilon| < 1$, $k \in \mathbb{N} \cup \{0\}$, so setting $k = 0$ and $\varepsilon = 0$ we recover the kernels in the previous chapter.

The reason to this choice (and not a more general class) will become clear in Chapter 4. Nevertheless, we will explain at the end of the chapter how –for the $C_c^\gamma$ well-posedness– we can consider other kernels as soon as they satisfy certain conditions.

### 2.1.1 Outline of the chapter

The chapter is structured as follows: in Section 2.2 the distributional derivatives of the kernels (2.1) are computed and so we have an expression for the derivatives of the velocity field. Then, in Section 2.3 we prove local-in-time well-posedness for the transport equation and these kernels. This can be done –since derivatives of the velocity field behave likely to the ones there–, at almost every moment, as a repetition of the arguments in Chapter 1, so we will write the partial results without a proof and just mention the little differences. In Section 2.4 we check that the local solutions are indeed global also by a straightforward repetition of the results in the previous chapter. Finally, Section 2.5 is a remark about how one can consider a bigger family of kernels for which we have $C_c^\gamma$ well-posedness.

## 2.2 Distributional derivatives of the kernel.

Since the velocity field is computed by $v(\cdot,t) = K * \rho(\cdot,t)$ in order to control its partial derivatives we need to compute the distributional derivatives of the kernels.

**Lemma 2.1.** *Let*

$$K_1(z) = \frac{1}{\pi}\frac{(z+\varepsilon\bar{z})^k}{(\bar{z}+\varepsilon z)^{k+1}}$$

*Then, distributionally we have*

$$\begin{cases} \partial K_1 & = p.v.\ \partial K_1 - \varepsilon^k \delta_0, \\ \bar{\partial} K_1 & = p.v.\ \bar{\partial} K_1. \end{cases}$$

*Proof.* Let $\varphi \in C^\infty$ function with compact support. Then,

$$\begin{aligned} \langle (\partial\,(K_1 * f)\,, \varphi\rangle &= -\langle K_1 * f, \partial\varphi\rangle = \\ &= -\int_{\mathbb{C}} \left\{ \int_{\mathbb{C}} K_1(z-w) f(w)\,\mathrm{d}A(w) \right\} \partial\varphi(z)\,\mathrm{d}A(z) = \\ &= -\int_{\mathbb{C}} \left\{ \int_{\mathbb{C}} K_1(z-w)\partial\varphi(z)\,\mathrm{d}A(z) \right\} f(w)\,\mathrm{d}A(w), \end{aligned} \tag{2.2}$$

where we have applied Fubini's theorem to change the integrals.

Having into account that for $\partial = \partial_z$ and at points $z \neq w$ we have

$$\partial[K_1(z-w)\varphi(z)] = \partial K_1(z-w)\varphi(z) + K_1(z-w)\partial\varphi(z),$$

we can write the last integral in (2.2) as

$$\begin{aligned} &-\int_{\mathbb{C}} \lim_{\delta\to 0} \left\{ \int_{\mathbb{C}\setminus B(w,\delta)} \partial(K_1(z-w)\varphi(z))\,\mathrm{d}A(z) \right\} f(w)\,\mathrm{d}A(w) + \\ &+ \int_{\mathbb{C}} \lim_{\delta\to 0} \left\{ \int_{\mathbb{C}\setminus B(w,\delta)} \varphi(z)\partial\left(K_1(z-w)\right)\mathrm{d}A(z) \right\} f(w)\,\mathrm{d}A(w) = I + II, \end{aligned} \tag{2.3}$$

and applying again Fubini's theorem, we can see that the term $II$ is equal to

$$\langle \text{p.v. } \partial K_1 * f, \varphi\rangle.$$

On the other hand, by Stokes' theorem, the term $I$ in (2.3) can be written as

$$-\frac{i}{2}\int_{\mathbb{C}}\lim_{\delta\to 0}\left\{\int_{\partial B(w,\delta)} K_1(z-w)\varphi(z)\,\mathrm{d}\bar{z}\right\} f(w)\,\mathrm{d}A(w) = \\ = -\int_{\mathbb{C}} g(w)f(w)\,\mathrm{dA}(w),$$

where we have used that, for $z = x + iy$, we have

$$\mathrm{dA}(z) = \mathrm{d}x \wedge \mathrm{d}y = \frac{i}{2}\mathrm{d}z \wedge \mathrm{d}\bar{z}$$

and that $\varphi$ has compact support. Furthermore, we have defined

$$g(w) := \frac{i}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} K_1(z-w)\varphi(z)\,\mathrm{d}\bar{z}.$$

Now, for $g$ we have

$$g(w) = \frac{i}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} K_1(z-w)(\varphi(z)-\varphi(w)+\varphi(w))\,\mathrm{d}\bar{z} = \\ = \frac{i}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} K_1(z-w)(\varphi(z)-\varphi(w))\,\mathrm{d}\bar{z} + \\ + \frac{i}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} K_1(z-w)\varphi(w)\,\mathrm{d}\bar{z} = \mathrm{A} + \mathrm{B}.$$

We check that the integral in A vanishes as we let $\delta \to 0$. Let $C_\varphi$ the modulus of continuity of $\varphi$. Thus,

$$\left|\frac{i}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} K_1(z-w)(\varphi(z)-\varphi(w))\,\mathrm{d}\bar{z}\right| \leq \\ \leq \frac{1}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} |K_1(z-w)|\,|\varphi(z)-\varphi(w)|\,\mathrm{d}\bar{z} \leq \\ \leq \frac{1}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} C_\varphi(z,w)\,|z-w|\,|K_1(z-w)|\,\mathrm{d}\bar{z} = 0,$$

because $C_\varphi(z,w) \to 0$ as $\delta \to 0$ and $|K_1(z-w)|\,|z-w|$ is bounded. Therefore,

$$g(w) = \frac{i\varphi(w)}{2}\lim_{\delta\to 0}\int_{\partial B(w,\delta)} K_1(z-w)\,\mathrm{d}\bar{z}.$$

By a change of variables $z = w + \delta e^{i\theta}$ we can simplify the computation of the complex line integral around $\partial B(w,\delta)$ appearing in $g(w)$.

$$
\begin{aligned}
g(w) &= -\frac{i\varphi(w)}{2} \lim_{\delta\to 0} \int_0^{2\pi} K_1(\delta e^{i\theta})\, i\delta e^{-i\theta}\, d\theta = \\
&= -\frac{i\varphi(w)}{2\pi} \lim_{\delta\to 0} \int_0^{2\pi} \frac{(\delta e^{i\theta} + \varepsilon\delta e^{-i\theta})^k}{(\delta e^{-i\theta} + \varepsilon\delta e^{i\theta})^{k+1}} i\delta e^{-i\theta}\, d\theta = \\
&= -\frac{i\varphi(w)}{2\pi} \int_0^{2\pi} \frac{(e^{i\theta} + \varepsilon e^{-i\theta})^k}{(e^{-i\theta} + \varepsilon e^{i\theta})^{k+1}} ie^{-i\theta}\, d\theta = \\
&= -\frac{i\varphi(w)}{2\pi} \int_0^{2\pi} \frac{(e^{i\theta} + \varepsilon e^{-i\theta})^k}{(e^{-i\theta} + \varepsilon e^{i\theta})^{k+1}} \frac{ie^{i\theta}}{e^{i2\theta}}\, d\theta = \\
&= -\frac{i\varphi(w)}{2\pi} \int_{\partial B(0,1)} \frac{(z + \varepsilon\frac{1}{z})^k}{z^2(\frac{1}{z} + \varepsilon z)^{k+1}}\, dz = \\
&= -\frac{i\varphi(w)}{2\pi} \int_{\partial B(0,1)} \frac{1}{z(1+\varepsilon z^2)} \frac{(z + \varepsilon\frac{1}{z})^k}{(\frac{1}{z} + \varepsilon z)^k}\, dz.
\end{aligned}
\tag{2.4}
$$

In order to compute the integral in the right hand side of (2.4) we will use the Residue Theorem (see [Ru, p. 215] for instance). Since $|\varepsilon| < 1$, the only pole of the function

$$z \to \frac{1}{z(1+\varepsilon z^2)} \frac{(z + \varepsilon\frac{1}{z})^k}{(\frac{1}{z} + \varepsilon z)^k}$$

inside the unit ball is $z = 0$ and for it

$$\operatorname{Res}\left(\frac{1}{z(1+\varepsilon z^2)} \frac{(z + \varepsilon\frac{1}{z})^k}{(\frac{1}{z} + \varepsilon z)^k}, 0\right) = \lim_{z\to 0} z \frac{1}{z(1+\varepsilon z^2)} \frac{(z + \varepsilon\frac{1}{z})^k}{(\frac{1}{z} + \varepsilon z)^k} = \varepsilon^k.$$

Thus, by Residue Theorem

$$g(w) = -\frac{i\varphi(w)}{2\pi} 2\pi i \varepsilon^k = \varphi(w)\varepsilon^k$$

and hence

$$I = -\varepsilon^k \int_{\mathbb{C}} \varphi(w) f(w)\, dA(w).$$

Summing up,

$$\langle \partial(K_1 * f), \varphi\rangle = \langle \text{p.v. } \partial K_1 * f + \varepsilon^k f, \varphi\rangle$$

so, distributionally,

$$\partial K_1 = \text{p.v. } \partial K_1 - \varepsilon^k \delta_0,$$

where $\delta_0$ is the Diract delta at the origin.

Repeating the arguments above one can see that

$$\langle \overline{\partial}(K_1 * f), \varphi\rangle = I' + II'$$

for

$$\begin{aligned} I' &= -\int_{\mathbb{C}} \lim_{\delta\to 0}\left\{\int_{\mathbb{C}\setminus B(w,\delta)} \bar\partial(K_1(z-w)\varphi(z))\,\mathrm{dA}(z)\right\} f(w)\,\mathrm{dA}(w) \\ &= \int_{\mathbb{C}} g'(w)f(w)\,\mathrm{dA}(w) \end{aligned}$$

and

$$\begin{aligned} II' &= \int_{\mathbb{C}} \lim_{\delta\to 0}\left\{\int_{\mathbb{C}\setminus B(w,\delta)} \varphi(z)\bar\partial\left(K_1(z-w)\right)\mathrm{dA}(z)\right\} f(w)\,\mathrm{dA}(w) = \\ &= \langle \text{p.v. } \bar\partial K_1 * f, \varphi\rangle. \end{aligned}$$

Similarly as done before, by the continuity of $\varphi$ and by a straightforward application of Stokes' theorem we get

$$g(w) = \frac{i\varphi(w)}{2\pi}\int_{\partial B(0,1)} \frac{z}{1+\varepsilon z^2}\frac{(z+\varepsilon\frac{1}{z})^k}{(\frac{1}{z}+\varepsilon z)^k}\,\mathrm{d}z.$$

Since the function inside the integral has no poles in $B(0,1)$ then it is equal to 0. Then $g(w) = 0$ and we simply have that distributionally

$$\bar\partial K_1 = \text{p.v. } \bar\partial K_1.$$

□

For the conjugate kernels we have a similar expression for their distributional derivatives.

**Lemma 2.2.** *Let*

$$K_2(z) = \frac{1}{\pi}\frac{(\bar z+\varepsilon z)^k}{(z+\varepsilon\bar z)^{k+1}}$$

*Then, distributionally we have*

$$\begin{cases} \partial K_2 &= p.v.\ \partial K_2, \\ \bar\partial K_2 &= p.v.\ \bar\partial K_2 - \varepsilon^k\delta_0. \end{cases}$$

*Proof.* Since $K_2(z) = \overline{K_1(z)}$ we then have, by Lemma (2.1),

$$\partial K_2 = \partial\overline{K_1} = \overline{\bar\partial K_1} = \text{p.v. } \overline{\bar\partial K_1} = \text{p.v. } \partial\overline{K_1} = \text{p.v. } \partial K_2$$

and similarly

$$\bar\partial K_2 = \bar\partial\overline{K_1} = \overline{\partial K_1} = \text{p.v. } \overline{\partial K_1} - \overline{\varepsilon^k\delta_0} = \text{p.v. } \bar\partial K_2 - \varepsilon^k\delta_0.$$

□

## 2.3 Local Theorem

In this section we will explain the reasons why we have $C_c^\gamma$ well-posedness for the kernel $K_1$ as in (2.1). Since $K_2$ is the complex conjugate of $K_1$ the result holds immediately for it too.

We want to apply Picard-Lindelöf's theorem for an ODE satisfied by the flow map. We need, as in Section 1.3 in Chapter 1 a space of functions $B$, a subspace $O$ of $B$ and a functional between $O$ and $B$. Since the flow satisfies (1.2), we have

$$\frac{\mathrm{d}X}{\mathrm{d}t}(\alpha,t) = v(X(\alpha,t),t) = \\ = \int_{\mathbb{C}} K(X(\alpha,t)-X(\alpha',t))\rho_0(\alpha')\ \det[DX(\alpha',t)]\,\mathrm{d}A(\alpha').$$

Thus, we can consider

$$F(X)(\alpha) = \int_{\mathbb{C}} K(X(\alpha)-X(\alpha'))\rho_0(\alpha')\ \det[DX(\alpha')]\,\mathrm{d}A(\alpha') \tag{2.5}$$

and so $\frac{\mathrm{d}}{\mathrm{d}t}X(\cdot,t) = F(X(\cdot,t))$. We now set $B = C^{1,\gamma}(\mathbb{C};\mathbb{C})$ and

$$O_M = B \cap \left\{ X : \mathbb{C} \to \mathbb{C} : \frac{1}{M} < \sup_{\alpha\neq\beta} \frac{|X(\alpha)-X(\beta)|}{|\alpha-\beta|} < M \right\}. \tag{2.6}$$

We then have, as in Remark 1.9 that $O_M$ is an open, non-empty subspace containing homeomorphisms of the complex plane.

Then, first of all, we need to check that $F$ maps $O_M$ to $B$. That is, we need to proof Proposition 1.11 for our functional. If we take a look to that proof we conclude that it will remain valid for our complex kernel as soon as the partial derivatives of $F$ are comparable. Applying the Complex Chain Rule to (2.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F(X(\alpha)) = -\varepsilon^k \partial X(\alpha)\rho_0(\alpha) + \\ + \mathrm{p.v.}\int_{\mathbb{C}} \nabla K_1(X(\alpha)-X(\alpha'))\mathcal{D}X(\alpha)\rho_0(\alpha')\det DX(\alpha')\,\mathrm{d}A(\alpha') \tag{2.7}$$

for

$$\nabla K_1 := \begin{pmatrix}\partial K_1, & \bar{\partial} K_1\end{pmatrix} \text{ (row vector)}, \\ \mathcal{D}X := \begin{pmatrix}\partial X \\ \partial \overline{X}\end{pmatrix} \text{ (column vector)}.$$

Also,

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}\overline{\alpha}}F(X(\alpha)) = -\varepsilon^k \partial X(\alpha)\rho_0(\alpha) + \\ + \text{p.v.} \int_{\mathbb{C}} \nabla K_1(X(\alpha) - X(\alpha'))\overline{\mathcal{D}}X(\alpha)\rho_0(\alpha') \det DX(\alpha')\, \mathrm{d}A(\alpha') \end{aligned} \tag{2.8}$$

for

$$\overline{\mathcal{D}}X := \begin{pmatrix} \overline{\partial} X \\ \overline{\partial X} \end{pmatrix} \text{ (column vector).}$$

Comparing the expression (2.5) with (1.21) on one hand and the expressions (2.7)-(2.8) with (1.24) on the other hand, one can see that they are likely the same. The derivatives of the functional are the sum of a constant times the scalar $\rho_0$ plus a SIO acting on $\rho_0 \det DX$. So, a repetition of the arguments in Proposition 1.11 in Section 1.3 yields the following.

**Proposition 2.3.** *Let $O_M$ as defined in* (2.6)*. Then, the functional $F$ defined by* (2.5) *maps $O_M$ to $C^{1,\gamma}(\mathbb{C};\mathbb{C})$.*

Secondly, we need to prove that the functional $F$ is locally Lipschitz between $O_M$ and $B$. Again, it will be sufficient to have a bound for the directional derivative $F'(X)Y$. The computation giving an expression for $F'(X)Y$ is easier in the two-dimensional space. For instance, in dimension 2, the formula for the determinant of a sum of two square matrices $A$ and $B$ can be stated easily as

$$\det(A+B) = \det A + \det B + \mathrm{tr}A\mathrm{tr}B - \mathrm{tr}AB.$$

Then, we simply have

$$\det(DX + \varepsilon DY) = \det DX + \varepsilon^2 \det DY + \varepsilon(\mathrm{tr}DX\mathrm{tr}DY - \mathrm{tr}DXDY)$$

which yields

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\det(DX+\varepsilon DY)|_{\varepsilon=0} = \mathrm{tr}DX\mathrm{tr}DY - \mathrm{tr}DXDY$$

and one can compute the directional derivative as in Lemma 1.14 to get

$$\begin{aligned} F'(X)Y(\alpha) &= \frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(X+\varepsilon Y)(\alpha) = \\ &= \int_{\mathbb{C}} \nabla K_1(X(\alpha)-X(\alpha'))(Y(\alpha)-Y(\alpha'))^* \rho_0(\alpha') \det DX(\alpha')\, \mathrm{d}A(\alpha') + \\ &+ \int_{\mathbb{C}} K_1(X(\alpha)-X(\alpha'))\rho_0(\alpha') \times \\ &\qquad \times [\mathrm{tr}DX(\alpha')\mathrm{tr}DY(\alpha') - \mathrm{tr}DX(\alpha')DY(\alpha')]\, \mathrm{d}A(\alpha'), \end{aligned} \tag{2.9}$$

where

$$(Y(\alpha)-Y(\alpha))^* = \begin{pmatrix} Y(\alpha)-Y(\alpha') \\ \overline{Y(\alpha)-Y(\alpha')} \end{pmatrix}, \text{ (column vector).}$$

The expression (2.9) is equivalent to the one obtained in Proposition 1.14. Following up the procedure in Chapter 1, we can see that the each of the distributional derivatives of the first term in (2.9) can be written as some expression similar to the one in Lemma 1.16, so again the proof in Proposition 1.18 can be adapted and so we have the second hypothesis in Picard-Lindelöf's theorem.

**Proposition 2.4.** *Let $O_M$ as defined in (2.6). Then, the functional $F : O_M \to C^{1,\gamma}(\mathbb{C};\mathbb{C})$ defined in (2.5) is locally Lipschitz.*

Then we can apply Picard-Lindelöf and so we have assured existence and uniqueness of a flow map $X(\cdot,t)$ for $t \in [-T^*, T^*]$ and this time $T^*$ does not depend on the constant $M$ in (2.6). With this flow map, we can define in an unique way the density $\rho(\cdot,t) = \rho_0$ and the velocity field as $v(\cdot,t) = K_1 * \rho(\cdot,t)$ and then we finally have the local theorem.

**Theorem 2.5.** *Let $\rho_0 \in C_c^\gamma(\mathbb{C};\mathbb{R})$ and $K_1$ as in (2.1). Then there exists $T^* > 0$ such that the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot,t) = K_1 * \rho(\cdot,t), \\ \rho(\cdot,0) = \rho_0. \end{cases} \tag{2.10}$$

*has a unique solution $\rho(\cdot,t) \in C_c^\gamma(\mathbb{C};\mathbb{R})$, $v(\cdot,t) \in C^{1+\gamma}(\mathbb{C};\mathbb{C})$ for $t \in (-T^*, T^*)$.*

## 2.4 Global Theorem

The validity of the a priori bound for the planar case is even easier to check. The proof of Proposition 1.25 does not depend on the choice of the kernel as soon as it was locally integrable. This integrability of $K$ was the one that produces the bound

$$\|v(\cdot,t)\|_{L^\infty} = \|K * \rho(\cdot,t)\|_{L^\infty} \le cR(t)\|\rho_0\|_{L^\infty}$$

as in (1.36). The rest of the proof of Proposition 1.25 concerns bounds depending implicitly on $\nabla v$ and thus they hold for any given kernel $K$ such that $v(\cdot,t) = K * \rho(\cdot,t)$. We can state it in a compact way as follows.

**Proposition 2.6.** *Let $X(\cdot,t)$ be the solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t}X(\cdot,t) = F(X(\cdot,t)), \quad X(\cdot,0) = Id,$$

*for $F$ as in (2.5). Then $|X(\cdot,t)|_{1,\gamma}$ is finite provided*

$$\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s < \infty. \tag{2.11}$$

Now it all reduces to check again that the integral in (2.11) is finite for any time. The proof for this does depend on the choice of the kernel. However it relies on the derivatives of the velocity field. Since for $K_1$ those derivatives are equal to a SIO applied to the density $\rho$ plus a constant times $\rho$ the boundedness proved for the kernels in Chapter 1 can be proved also for $K_1$. Indeed, we can substitute the expression for $\partial v(\cdot,t)$ and $\overline{\partial} v(\cdot,t)$ in (1.45) and then finish the proof in the same way to check that the integral in (2.11) is finite for any time and so we have the following:

**Proposition 2.7.** *Let $X(\cdot,t)$ be the solution of*

$$\frac{\mathrm{d}}{\mathrm{d}t}X(\cdot,t) = F(X(\cdot,t)), \quad X(\cdot,0) = Id,$$

*for $F$ as in (2.5). Then $|X(\cdot,t)|_{1,\gamma}$ is finite for any time $t$.*

We state then the global result which is a consequence of the local theorem 2.5 and Proposition 2.7. Also, since $K_2 = \overline{K_1}$, by Lemma 2.1 we see that the derivatives of the velocity field obtained by $v(\cdot,t) = K_2 * \rho(\cdot,t)$ are also a combination of the density $\rho$ and a SIO acting also on $\rho$. So all the procedure can be restated for $K_2$ instead $K_1$. To sum up we have the main theorem of the chapter.

**Theorem 2.8.** *Let $\rho_0 \in C_c^\gamma(\mathbb{C};\mathbb{R})$ and $K_i$, $i = 1,2$ as in (2.1). Then, the transport equation*

$$\begin{cases} \rho_t + v\cdot\nabla\rho = 0, \\ v(\cdot,t) = K_i * \rho(\cdot,t), \;\; i = 1,2 \\ \rho(\cdot,0) = \rho_0. \end{cases} \tag{2.12}$$

*has a unique solution $\rho(\cdot,t) \in C_c^\gamma(\mathbb{C};\mathbb{R})$, $v(\cdot,t) \in C^{1+\gamma}(\mathbb{C};\mathbb{C})$ for $t \in \mathbb{R}$.*

## 2.5 Further comments about $C_c^\gamma$ well-posedness

We want to stress that for the relationship $v(\cdot,t) = K * \rho(\cdot,t)$ we have focused on kernels of the type

- $L\cdot\nabla N$ for $N$ the fundamental solution of the Laplacian and $L \in M_{n\times n}(\mathbb{R})$ for the transport equation in $\mathbb{R}^n$, as done in Chapter 1,
- $K_i$, $i = 1,2$ as in (2.1) for the transport equation in the complex plane.

The reason to choose these kernels is the following: we have a well-posedness result for them in the case of a *density patch*. Although the $C_c^\gamma$ result is not necessary for the study of patches, we strongly thought it would be good to prove it too for completeness.

Notwithstanding, if we take a closer look to the proof in Chapter 1 and the remarks for it to be adapted in Sections 2.3 and 2.4 we see that a wider class of kernels could have been considered instead of $L \cdot \nabla N$ or $K_i$, $i = 1, 2$. In fact, note that by Remark 1.5 we reduced the hypothesis in [MB, Lemmas 4.5 and 4.6] to produce Lemma 1.4. Then, in conclusion, for any kernel $K \in C^2(\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $1-n$ we have bounds (1.15), (1.16) and (1.17) and so we can repeat the proof of the $C_c^\gamma$ well-posedness for $K$ provided distributional derivatives of $K$ are a combination of Dirac deltas and SIOs. Observing the proof of Lemma 2.1 we see that the Dirac delta appears since the terms

$$\lim_{\delta \to 0} \int_{\partial B(w,\delta)} K_1(z-w)\,\mathrm{d}z \quad \text{or} \quad \lim_{\delta \to 0} \int_{\partial B(w,\delta)} K_1(z-w)\,\mathrm{d}\bar{z}$$

are equal to a well defined quantity. In general, in dimension $n$ and for a kernel $K \in C^2(\mathbb{R}^n \setminus \{0\})$ homogeneous of degree $1-n$, when repeating the proof of Lemma 2.1 one has to take care of a term

$$\lim_{\delta \to 0} \int_{\partial B(w,\delta)} K(z-w) n_k(z)\,\mathrm{d}\sigma(z),$$

where $n_k(z)$ is the $k$-th component of the unitary normal vector $n(z)$ at the point $z \in \partial B(w,\delta)$. By a change of variable $z = w + \delta s$, $s \in \partial B(0,1)$ and by the homogeneity of the kernel $K$, we have

$$\lim_{\delta \to 0} \int_{\partial B(w,\delta)} K(z-w) n_k(z)\,\mathrm{d}\sigma(z) = c \int_{\partial B(0,1)} K(s) n_k(s) \mathrm{d}\sigma(s),$$

which is equal to a quantity independent of $w$. If this quantity is equal to 0 then no Dirac delta appear but in any case since it is well defined for the kernel $K$ we would have for the partial derivative in the direction $i$

$$\partial_i K = \text{p.v. } \partial_i K + c_i \delta_0$$

for $c_i \in \mathbb{R}$ and then the $C_c^\gamma$ well-posedness works also for $K$.

On the other hand, the arguments in Chapter 1, Section 1.5 can be adapted for $K$ this way and finally we can conclude that the following theorem holds.

**Theorem 2.9.** *Let $\rho_0 \in C_c^{m,\gamma}(\mathbb{C}; \mathbb{R})$ and let $K \in C^2(\mathbb{R}^n \setminus \{0\})$ be a kernel homogeneous of degree $1-n$. Then, the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot, t) = K * \rho(\cdot, t), \quad i = 1, 2 \\ \rho(\cdot, 0) = \rho_0. \end{cases}$$

*has a unique solution $\rho(\cdot, t) \in C_c^{m,\gamma}(\mathbb{R}^n; \mathbb{R})$, $v(\cdot, t) \in C^{m+1,\gamma}(\mathbb{R}^n; \mathbb{R}^n)$ for $t \in \mathbb{R}$.*

# 3 $C^{1,\gamma}$ regularity for patches and for the kernel $L \cdot \nabla N$ in $\mathbb{R}^n$

## 3.1 Introduction

The vorticity form of the Euler equation in the plane is

$$\begin{aligned} &\partial_t \omega + v \cdot \nabla \omega = 0, \\ &v(\cdot,t) = \nabla^{\perp} N * \omega(\cdot,t), \\ &\omega(\cdot,0) = \omega_0, \end{aligned} \tag{3.1}$$

where $t \in \mathbb{R}$, $N = \frac{1}{2\pi} \log |x|$ is the fundamental solution of the laplacian in the plane and $\omega_0$ is the initial vorticity.

A very well-known result of Yudovich [Y] states that the vorticity equation is well-posed in $L^1 \cap L^\infty$ the measurable bounded and integrable functions. In particular the result holds for vorticities in $L^\infty_c$, that is, vorticities measurable bounded and with compact support.

We call a *vortex patch* to a special weak solution of (3.1) for an initial condition which is the characteristic function of a bounded domain $D_0$. Since the vorticity equation is a transport equation, vorticity is conserved along trajectories and thus $\omega(x,t) = \chi_{D_t}(x)$ for some domain $D_t$. A challenging problem, posed in the eighties, was to show that boundary smoothness persists for all times. Specifically, if $D_0$ has boundary of class $C^{1,\gamma}$, $0 < \gamma < 1$, then one would like $D_t$ to have boundary of the same class for all times. This was viewed as a 2 dimensional problem which featured some of the main difficulties of the regularity problem for the Euler equation in $\mathbb{R}^3$. On one hand, and based on numerical simulations, it was conjectured that the boundary of $D_t$ could become of infinite length in finite time [M]. Nevertheless, Chemin proved that boundary regularity persists for all times [Ch] using paradifferential calculus, and Bertozzi and Constantin found shortly after a minimal beautiful proof in [BC] based on methods of classical analysis with a geometric flavor.

Later on, the *density patch* problem (when the initial density is equal to $\chi_{D_0}$) was considered for the aggregation equation with newtonian kernel in higher dimensions in [BGLV]. The equation is

$$\begin{aligned} &\partial_t \rho + \operatorname{div}(\rho v) = 0, \\ &v(\cdot,t) = -(\nabla N * \rho(\cdot,t)), \\ &\rho(\cdot,0) = \rho_0, \end{aligned} \tag{3.2}$$

$x \in \mathbb{R}^n$ and $t \in \mathbb{R}$. In [BLL] a well-posedness theory in $L^\infty_c$ was developed, following the path of [Y] and [MB, Theorem 8.1]. When the initial condition is the characteristic function of a domain one calls the unique weak solution a *density patch*, analogously to the vorticity equation. One proves in [BGLV] that if the boundary of $D_0$ is of class $C^{1,\gamma}$, $0 < \gamma < 1$, then the solution of (3.2) with initial condition $\rho_0 = \chi_{D_0}$ is of the form $\rho(x,t) = \frac{1}{1-t}\chi_{D_t}(x)$, $x \in \mathbb{R}^n$, $0 \le t < 1$ where $D_t$ is a $C^{1,\gamma}$ domain for all $t < 1$. The restriction to times less than 1 obeys a blow up phenomenon studied in [BLL]. Hence the preceding result is the analog of Chemin's theorem for the aggregation equation.

Equation (3.2) is not a transport equation, but for density patches and after a change in the time scale it becomes the non-linear transport equation

$$\begin{aligned} &\partial_t \rho + v \cdot \nabla \rho = 0, \\ &v(\cdot,t) = -(\nabla N * \rho(\cdot,t)), \\ &\rho(\cdot,0) = \chi_{D_0}, \end{aligned} \tag{3.3}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, where $N$ is the fundamental solution of the laplacian in $\mathbb{R}^n$ and $D_0$ is a bounded domain. Due to this equivalence, the result in [BGLV] proves that if $D_0$ is of class $C^{1,\gamma}$, then there is a solution of (3.3) of the form $\chi_{D_t}(x)$ with $D_t$ a domain of class $C^{1+\gamma}$.

To the best of our knowledge there is no well-posedness theory in $L^1 \cap L^\infty_c$ (or even in $L^\infty_c$) for (3.3), so that there could be other weak solutions in $L^\infty_c$ with initial condition $\chi_{D_0}$. Nevertheless, one has uniqueness in the class of characteristic functions of domains with $C^{1,\gamma}$ boundary.

In this chapter we consider, as in Chapter 1, that the velocity field can be recovered from the density by $v(\cdot,t) = L \cdot \nabla N * \rho(\cdot,t)$, for $L$ an $n$-dimensional square matrix and $N$ the fundamental solution of the laplacian defined by

$$N(x) = \begin{cases} \frac{1}{2\pi} \log|y|, & n = 2, \\ -\frac{1}{n(n-2)w_n} \frac{1}{|x|^{n-2}}, & n \ge 3, \end{cases}$$

where $w_n$ is the volume of the $n$-dimensional unit ball.

**Theorem 3.1.** *Let $L \in M_{n\times n}(\mathbb{R})$ and $D_0$ a domain with boundary of class $C^{1,\gamma}$, $0 < \gamma < 1$. Then the non-linear transport equation*

$$\begin{aligned} &\partial_t \rho + v \cdot \nabla \rho = 0, \\ &v(\cdot,t) = (L \cdot \nabla N) * \rho(\cdot,t), \\ &\rho(\cdot,0) = \chi_{D_0}(x) \end{aligned} \tag{3.4}$$

$x \in \mathbb{R}^n$, $t \in \mathbb{R}$, *has a weak solution of the form*

$$\rho(x,t) = \chi_{D_t}(x), \quad x \in \mathbb{R}^n, \quad t \in \mathbb{R},$$

*with $D_t$ a domain with boundary of class $C^{1,\gamma}$.*

*This solution is unique in the class of characteristic functions of domains with boundary of class $C^{1,\gamma}$.*

We recall what we mean by weak solution of the transport equation. We say $(\rho, v)$ is a weak solution to equation (3.4) if, for any $0 \le t \le T$, we have $\rho(\cdot,t) \in L^1_{\text{loc}}(\mathbb{R}^n)$, the relationship $v(\cdot,t) = L \cdot \nabla N * \rho(\cdot,t)$ holds and for each $\varphi \in C^\infty([0,T] \times \mathbb{R}^n)$ with compact support we have

$$-\int_0^T \int_{\mathbb{R}^n} \rho\varphi_t \,\mathrm{d}x\,\mathrm{d}t - \int_{\mathbb{R}^n} \rho_0 \varphi(\cdot,0)\,\mathrm{d}x - \int_0^T \int_{\mathbb{R}^n} \rho \mathrm{div}\,(\varphi v)\,\mathrm{d}x\,\mathrm{d}t = 0.$$

The proof of Theorem 3.1 follows the scheme of [BC] and overcomes difficulties related to the fact that the velocity field has a non-zero divergence and to the higher dimensional context.

The results in this chapter belong to a joint work with Joan Mateu, Joan Orobitg and Joan Verdera (see [CMOV]). We present them here with a different and more detailed structure in their proofs.

In the next chapter we will prove a result analogous to Theorem 3.1 for a wider class of kernels replacing those of the form $L \cdot \nabla N$ but reducing to dimension 2.

### 3.1.1 Outline of the chapter

The chapter is structured as follows. In Section 3.2 we give an important instrument, the –non unique– defining function, used to measure the smoothness of a domain in $\mathbb{R}^n$. Also, we define some quantities strictly related to the boundary of a domain that will be used throughout the chapter. Then a version of the classical logarithmic inequality for the gradient of the velocity field is established. Again, this new inequality relies just on the behavior of the domain at its boundary. Later on, it will become clear why these adaptations have to be made. In Section 3.3 we set a Contour Dynamics Equation (CDE) for the evolution of the domain describing the patch and we state a local-in-time existence and uniqueness result for this equation. We take profit of the work done in [BGLV] for $L$ equal to the $n$ dimensional identity matrix and adapt it to the general case.

Once we have proved the existence and uniqueness of local solutions, the goal of the chapter is to check that these solutions are in fact global. In Section 3.4, we choose a good defining function for the transported domain and compute a partial differential equation for its material derivative. In Section 3.5 we get rid of a disturbing term appearing when computing the material derivative of the gradient of the defining function. Once this term, that we will call *solitary*, is controlled we will verify that in fact those material derivatives are equal to a difference of commutators when looked at the boundary of the domain. Since the equality to commutators is just valid at the boundary, in Section 3.6 we extend, via Whitney's Extension theorem, the defining function at the boundary to the whole space. Finally, in Section 3.7 we combine all the results coming from previous sections and we prove the global-in-time version of the existence and uniqueness results and so we have Theorem 3.1.

## 3.2 Smoothness of domains and the (refined) logarithmic inequality

We start this section by giving some basic information about how to quantify the smoothness of a domain. Classically, we define a $C^{1,\gamma}$ domain $D$ as a bounded domain whose boundary is locally, and possibly after a rotation, the graph of a $C^{1,\gamma}$ function. An important tool when dealing with these kind of domains is the following.

**Definition 3.2.** *Let $D$ a bounded domain in $\mathbb{R}^n$. We say $\varphi : \mathbb{R}^n \to \mathbb{R}$ is a $C^{1,\gamma}$-defining function for $D$ if*

*(a)* $\varphi \in C^{1,\gamma}(\mathbb{R}^n; \mathbb{R})$,

*(b)* $D = \{x \in \mathbb{R}^n : \varphi(x) < 0\}$,

*(c)* $\partial D = \{x \in \mathbb{R}^n : \varphi(x) = 0\}$,

*(d)* $\nabla \varphi(x) \neq 0$, *for* $x \in \partial D$.

A standard argument based on a partition of unity shows that if $D$ is a $C^{1,\gamma}$ function then there exists a $C^{1,\gamma}$-defining function $\Phi$ (which is not unique) and on the other hand, by the Implicit Function Theorem, if there exists $\Phi$ satisfying Definition 3.2 then the domain $D$ is of class $C^{1,\gamma}$ in the classical sense.

For these domains we can define some important quantities that will be used in the next sections. Set

$$|f|_{\gamma,\partial D} = \sup \left\{ \frac{|f(x) - f(y)|}{|x - y|^{\gamma}}, \ x, y \in \partial D \right\}$$

and

$$|f|_{\inf} = \inf \{|f(x)|, \ x \in \partial D\}.$$

Then we define $\sigma(D)$ as the $n-1$ dimensional surface measure of $\partial D$ and also

$$q(D) := \inf \left\{ \frac{|\nabla \Phi|_{\gamma,\partial D}}{|\nabla \Phi|_{\inf}} : \Phi \text{ is a } C^{1,\gamma}\text{-defining function for } D \right\}.$$

Furthermore, if in the numerator inside the infimum we consider the Hölder norm in the whole space, we can define the bigger quantity

$$Q(D) := \inf \left\{ \frac{|\nabla \Phi|_{\gamma}}{|\nabla \Phi|_{\inf}} : \Phi \text{ is a } C^{1,\gamma}\text{-defining function for } D \right\}. \tag{3.5}$$

The classical definition -given at the beginning of the section- of being $D$ a $C^{1,\gamma}$ domain means that given any $x \in \partial D$, there exists a radius $r_x$ such that the points of the boundary of $D$ closer than $r_x$ to $x$ are a graph. The following Lemma quantifies the size of a radius $r$ which does not depend on the point $x \in \partial D$ and such that the property holds for every point. In [BGLV, see Lemma 6.4], this universal radius is related to the quotient appearing in $Q(D)$ as in (3.5). The following lemma is a new version of that result where the dependence is on the quotient in $q(D)$ as defined in (3.2). The proof is very similar to the one presented in [BGLV], but for convenience of the reader we repeat it here adding the mentioned improvement.

**Lemma 3.3.** *Let $D$ a $C^{1,\gamma}$ domain and let $\Phi$ a $C^{1,\gamma}$-defining function for $D$. If we consider $\delta > 0$ such that*

$$\delta^\gamma \frac{|\nabla\Phi|_{\gamma,\partial D}}{|\nabla\Phi|_{\inf}} \leq \frac{1}{2},$$

*then for each $x \in \partial D$ the set $\partial D \cap B(x,\delta)$ is, after a rotation around $x$, the graph of a $C^{1,\gamma}$ function $\varphi$ and $D \cap B(x,\delta)$ is the part of $B(x,\delta)$ lying below the graph of $\varphi$. Also we have the estimate*

$$|\varphi(x')| \leq \sqrt{2^\gamma}\frac{|\nabla\Phi|_{\gamma,\partial D}\, r^\gamma}{|\nabla\Phi|_{\inf}} \quad \text{for } |x'| \leq r < \frac{\delta}{\sqrt{2}}. \tag{3.6}$$

*Proof.* Without loss of generality we can assume $x = 0$ and $\nabla\Phi(0) = (0,\dots,0,\partial_n\Phi(0))$ with $\partial_n\Phi(0) > 0$. Consider $p,q \in \partial D \cap B(0,\delta)$ and set $p = (p',p_n)$ with $p' = (p_1,\dots,p_{n-1})$ and equivalently $q = (q',q_n)$. Then, by doing a Taylor expasion of $\Phi$ around 0 we get

$$0 = \Phi(p) = \Phi(0) + \nabla\Phi(0)\cdot p + E(p) = |\nabla\Phi(0)|\, p_n + E(p),$$

and similarly

$$0 = |\nabla\Phi(0)|\, q_n + E(q).$$

Subtracting and taking absolute value we get, since $\Phi \in C^{1,\gamma}$,

$$\begin{aligned} |\nabla\Phi(0)|\,|p_n - q_n| &= |E(p) - E(q)| \leq \sup_{x\in[p,q]} |\nabla E(x)|\,|p-q| = \\ &= \sup_{x\in[p,q]} |\nabla\Phi(x) - \nabla\Phi(0)|\,|p-q| \leq \\ &\leq \sup_{x\in[p,q]} |x|^\gamma\, |\nabla\Phi|_\gamma\, |p-q| \leq \\ &\leq |\nabla\Phi|_\gamma\, \delta^\gamma (|p'-q'| + |p_n - q_n|). \end{aligned} \tag{3.7}$$

Thus

$$\frac{|p_n - q_n|}{|p'-q'|} \leq \frac{|\nabla\Phi|_\gamma}{|\nabla\Phi|_{\inf}}\delta^\gamma \left(1 + \frac{|p_n-q_n|}{|p'-q'|}\right),$$

which implies, by setting $\delta$ satisfying $\delta^\gamma \frac{|\nabla\Phi|_\gamma}{|\nabla\Phi|_{\inf}} \leq \frac{1}{2}$, that

$$\frac{|p_n - q_n|}{|p' - q'|} \leq 1.$$

This inequality states that $\partial D \cap B(x, \delta)$ is the graph of a Lipschitz function $\varphi$ with $p_n = \varphi(p')$ with domain an open subset $U$ of $\{x' \in \mathbb{R}^{n-1} : |x'| < \delta\}$ and it satisfies $|\nabla\varphi(x')| \leq 1$ for $x' \in U$. By the Implicit Function Theorem $\varphi$ is of class $C^{1,\gamma}$ on its domain.

We can get a better bound in Equation (3.7) now that we know that the surface $\partial D$ is smooth. In particular, given $p, q \in \partial D$ consider the (differentiable) curve $c \subset \partial D$ parametrized by arc length (i.e. $|c'| \equiv 1$) such that $c(0) = p$, $c(l) = q$, and minimizing $l$ the length of $c$ between $p$ and $q$. Then, by the Mean Value Theorem, there exists $s \in [0, l]$ such that

$$E(q) - E(p) = E(c(l)) - E(c(0)) = l\nabla E(c(s)) \cdot c'(s),$$

and thus

$$\begin{aligned} |\nabla\Phi(0)|\,|p_n - q_n| &= |E(p) - E(q)| = |E(c(l)) - E(c(0)))| \leq \\ &\leq l\,|\nabla E(c(s))|\,|c'(s)| = l\,|\nabla E(c(s))| \leq l \sup_{s\in[0,l]} |\nabla E(c(s))| \leq \\ &\leq l \sup_{s\in[0,l]} |\nabla\Phi(c(s)) - \nabla\Phi(0)| \leq |\nabla\Phi|_{\gamma,\partial D}\, \delta^\gamma l. \end{aligned}$$

Since $c$ is chord-arc then $l \leq c\,|p - q|$ and hence

$$|\nabla\Phi(0)|\,|p_n - q_n| \leq c\,|\nabla\Phi|_{\gamma,\partial D}\, \delta^\gamma\,|p - q|\,,$$

which allow us to repeat the argument done before but setting $\delta^\gamma \frac{|\nabla\Phi|_{\gamma,\partial D}}{|\nabla\Phi|_{\inf}} \leq \frac{1}{2}$ instead of $\delta^\gamma \frac{|\nabla\Phi|_\gamma}{|\nabla\Phi|_{\inf}} \leq \frac{1}{2}$.

By implicit differentiation we have for $i = 1, \ldots, n-1$,

$$0 = \partial_i[\Phi(x', \varphi(x'))] = \partial_i\Phi(x', \varphi(x')) + \partial_n\Phi(x', \varphi(x'))\partial_i\varphi(x'),$$

and then

$$\partial_i\Phi(x', \varphi(x')) = -\frac{\partial_i\Phi(x', \varphi(x'))}{\partial_n\Phi(x', \varphi(x'))}.$$

Since $|\nabla\varphi(x')| \leq 1$ we have, taking supremum over $i$

$$\left|\nabla\Phi(x', \varphi(x'))\right| \leq \left|\partial_n\Phi(x', \varphi(x'))\right|.$$

Note that the ball $B(x', \frac{\delta}{\sqrt{2}})$ is contained in $U$. Then, since $\partial_j \Phi(0) = 0$ for $j = 1, \ldots, n-1$, if we consider $r < \frac{\delta}{\sqrt{2}}$ we have

$$|\nabla \varphi(x')| \leq \frac{|\nabla \Phi|_{\gamma,\partial D} (\sqrt{2}r)^{\gamma}}{|\nabla \Phi|_{\inf}}, \quad \text{for } |x'| \leq r,$$

completing the proof of the Lemma. □

We need a logarithmic inequality for the $L^\infty$ norm of the gradient of the velocity field. Such a bound has been established both for Euler (e.g. [MB, Proposition 8.12]) and aggregation (e.g. [BGLV, Corollary 6.3]) equations. Since the kernel $L \cdot \nabla N$ has the same regularity properties than the kernel $-\nabla N$ (each one of them can be recovered as a linear combination of the other) the results in [BGLV] can be adapted to the present case. Since second derivatives of the fundamental solution of the laplacian has zero mean integral on the sphere and recalling Lemma 1.10, then the distributional derivatives of the velocity field can be computed as in the following lemma.

**Lemma 3.4.** *Given $v = L \cdot \nabla N * \chi_D$, let $x \notin \partial D$, and set $\varepsilon = \varepsilon(x) = dist(x, \partial D)$. Then for $j,k \in \{1, \ldots, n\}$, distributionally we have*

$$\partial_k v_j = \frac{l_{jk}}{n} \chi_D + \sum_{i=1}^{n} l_{ji} \partial_k \partial_i N * \chi_{D \setminus B(x,\varepsilon)}.$$

The following estimate can be recovered by repeating the proof of Theorem 6.2 in [BGLV] step by step but changing the value of $\delta$ from

$$\delta^{\gamma} \frac{|\nabla \Phi|_{\gamma}}{|\nabla \Phi|_{\inf}} = \frac{1}{2}$$

to the refined choice that is emerged from Lemma 3.3,

$$\delta^{\gamma} \frac{|\nabla \Phi|_{\gamma,\partial D}}{|\nabla \Phi|_{\inf}} = \frac{1}{2}.$$

At some point in the proof we need to split integrals in two parts: one for the region of integration of points closer to a distance related to $\delta$ and the other one for the complementary. Furthermore, estimate (3.6) (depending on $|\nabla \Phi|_{\gamma,\partial D}$ instead of $|\nabla \Phi|_{\gamma}$ as in [BGLV]) is used to give the following theorem.

**Theorem 3.5.** *Let $D$ a $C^{1,\gamma}$ domain and let $\Phi$ be a $C^{1,\gamma}$-defining function for $D$. Then, for every $i, j \in \{1, \ldots, n\}$, $x \in \mathbb{R}^n$ and $\varepsilon > 0$ there exists a constant $c_n$ depending on $n$ such that*

$$\left| \int_{|y-x|>\varepsilon} \partial_i \partial_j N(x-y) \chi_D(y) \, dy \right| \leq \frac{c_n}{\gamma} \left( 1 + \log^+ \left( |D|^{1/d} \frac{|\nabla \Phi|_{\gamma,\partial D}}{|\nabla \Phi|_{inf}} \right) \right),$$

*where* $\log^+ x = \max\{\log x, 0\}$.

Finally, as a combination of Lemma 3.4 with Theorem 3.5 we get the desired logarithmic inequality that will be used later on.

**Theorem 3.6.** *Let $D$ a domain in $\mathbb{R}^n$ with a $C^{1,\gamma}$-defining function $\Phi$. If $v = L \cdot \nabla N * \chi_D$, then for $R := m(D)^{1/n}$ we have*

$$||\nabla v||_{L^\infty} \leq \frac{c_n}{\gamma}\left(1 + \log^+\left(R\frac{|\nabla\Phi|_{\gamma,\partial D}}{|\nabla\Phi|_{\inf}}\right)\right).$$

## 3.3 Local Theorem

As we have said, in [BGLV] the proofs were developed for the kernel $-\nabla N$. In the case of local well-posedness of regular patches we do not have to repeat a similar argument since we can reduce to the situation for that kernel and take advantage of the work done in that paper.

Let $DX$ be the differential of the flow $X$ as a differentiable mapping from $\partial D_0$ into $\mathbb{R}^n$. Take $\beta \in \partial D_0$ and set $n(\beta)$ the normal vector to $\partial D_0$ at $\beta$ and $T_1(\beta), \ldots, T_{n-1}(\beta)$ an orthonormal basis of the tangent space to $\partial D_0$ at $\beta$. We choose all of them in such a way that $n(\beta), T_1(\beta), \ldots, T_{n-1}(\beta)$ gives the standard orientation in $\mathbb{R}^n$. Then, consider, as in [BGLV, Eq. (2.3)],

$$F^{(\mathrm{Ag})}(X)(\alpha) = \int_{\partial D_0} N(X(\alpha) - X(\beta)) \bigwedge_{j=1}^{n-1} DX(\beta)(T_j(\beta))\, \mathrm{d}\sigma(\beta)$$

(where $Ag$ stands for aggregation). Having into account the way the functional has been computed we have

$$F^{(\mathrm{Ag})}(X(\cdot,t))(\alpha) = (-\nabla N * \chi_{D_t})(X(\alpha,t)),$$

meaning that component-wise $F_i^{(\mathrm{Ag})} = (-\partial_i N * \chi_{D_t})(X(\alpha,t))$. As usual, the functional $F$ defining the Contour Dynamics Equation satisfies

$$\frac{\mathrm{d}X(\cdot,t)}{\mathrm{d}t} = F(X(\cdot,t)).$$

Since in our working case $v(\cdot,t) = (L \cdot \nabla N) * \chi_{D_t}$ and so $v_i(\cdot,t) = \sum_{j=1}^n l_{ij}\partial_j N * \chi_{D_t}$, then we write the $i$-th component of $F$ as

$$\begin{aligned} F_i(X(\cdot,t))(\alpha) &= v_i(X(\alpha,t),t) = \sum_{j=1}^n l_{ij}(\partial_j N * \chi_{D_t})(X(\alpha,t)) = \\ &= -\sum_{j=1}^n l_{ij} F_j^{(\mathrm{Ag})}(X(\cdot,t))(\alpha). \end{aligned} \tag{3.8}$$

That is, each component of $F$ is a linear combination of components of $F^{(\mathrm{Ag})}$.

In the paper by Bertozzi, Garnett, Laurent and Verdera, they proved (see [BGLV, Theorem 2.2]) that $F^{(\mathrm{Ag})}$ satisfies the hypothesis of Picard-Lindelöf's theorem. More precisely, for

$$\Omega = \Big\{ X \in C^{1,\gamma}(\partial D_0, \mathbb{R}^n) \text{ for which } \exists\, \mu \geq 1 \text{ such that} \\ |X(\alpha) - X(\beta)| \geq \frac{1}{\mu}|\alpha - \beta| \;\; \alpha, \beta \in \partial D_0 \Big\},$$

then $F^{(\mathrm{Ag})} : \Omega \to C^{1,\gamma}(\partial D_0, \mathbb{R}^n)$ and also $F^{(\mathrm{Ag})}$ is Lipschitz with constants depending on some parameters referring to $D_0$ (one of them being $Q(D)$ as in (3.5)). Also, as the functional $F$ whose components are defined in (3.8) is a linear combination (with the coeficients of the matrix $L$) of $F^{(\mathrm{Ag})}$ we then conclude with the same result but depending also on $L$.

**Theorem 3.7.** *Let $F$ defined component-wise in (3.8). If $X \in \Omega$, then*

$$|F(X)|_{1,\gamma} \leq c\mu(X)^{3n+2}(1 + |X|_{1,\gamma}^{2n+4}), \tag{3.9}$$

*and*

$$||F'(X)||_{C^{1,\gamma} \to C^{1,\gamma}} \leq c\mu(X)^{3n+8}(1 + |X|_{1,\gamma}^{3n+7}), \tag{3.10}$$

*where $c$ is a constant depending on $L$, $n$, $q(D_0)$, $\sigma(D_0)$ and $\mathrm{diam}(D_0)$ and*

$$\mu(X) := \inf\left\{ \mu > 0 : \textit{ for every } \alpha, \beta \in \partial D_0, |X(\alpha) - X(\beta)| \geq \frac{1}{\mu}|\alpha - \beta| \right\}.$$

The main difference between Theorem 3.7 and its equivalent version in [BGLV, Theorem 2.2] is that the universal radius of Lemma 3.3 (with norms referring to the boundary of the domain) is used in the proof instead of the one given in [BGLV, Lemma 6.4]. The proof of Theorem 3.7 can be done following the one in [BGLV] with this new choice of $\delta$ when needed. The new dependence on the matrix $L$ relies on equation (3.8). In particular, Theorem 3.7 implies that hypothesis in Picard-Lindelöf's theorem 1.8 are satisfied and so local-in-time existence and uniqueness for the CDE holds. Thus, we have the following theorem.

**Theorem 3.8.** *Let $D_0$ a $C^{1,\gamma}$ domain. Let $F(X(\cdot,t)) = (F_i(X(\cdot,t)))_{i=1}^n$ defined by*

$$F_i(X(\cdot,t))(\alpha) = -\sum_{j=1}^{n} l_{ij} F_j^{(Ag)}(X(\cdot,t))(\alpha).$$

*Then there exists $T^* > 0$ depending on $L, n, q(D_0), \sigma(D_0)$ and $\mathrm{diam}(D_0)$, such that the ordinary differential equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} X(\alpha, t) = F(X(\alpha, t)), \\ X(\alpha, 0) = \alpha, \end{cases}$$

*has a unique solution $X(\cdot, t) \in C^{1,\gamma}(\partial D_0; \mathbb{R}^n)$ for $t \in (-T^*, T^*)$.*

The Contour Dynamics Equation in Theorem 3.8 can be thought of as an ODE in the open set $\Omega$. We want to show that a solution $X(\cdot, t)$ to the CDE in an interval $(-T, T)$ provides a weak solution of the non-linear transport equation (3.4).

**Theorem 3.9.** *Let $L \in M_{n\times n}(\mathbb{R}^n)$ and $D_0$ a domain with boundary of class $C^{1,\gamma}$, $0 < \gamma < 1$. Then, there exists $T^* > 0$ depending on $L, n, q(D_0), \sigma(D_0)$ and $\mathrm{diam}(D_0)$ such that the non-linear transport equation (3.4) has a weak solution of the form $\rho(\cdot, t) = \chi_{D_t}$ for $t \in (-T^*, T^*)$, with $D_t$ a domain with boundary of class $C^{1,\gamma}$. This solution is unique in the class of characteristic functions of domains with boundary of class $C^{1,\gamma}$.*

*Proof.* Clearly $X(\cdot, t)$ maps $\partial D_0$ onto a $n-1$ dimensional hypersurface $S_t$. The goal now is to identify an open set $D_t$ with boundary $S_t$. First assume that $\partial D_0$ is connected, and hence a connected $n-1$ dimensional hypersurface of class $C^{1,\gamma}$, then the analog of the Jordan curve theorem holds [GP, p. 89]. Then the complement of $\partial D_0$ in $\mathbb{R}^n$ has only one bounded connected component which is $D_0$. In the same vein, the complement of $S_t$ has only one bounded connected component, which we denote by $D_t$, so that the boundary of $D_t$ is $S_t$.

Secondly, if we drop the assumption that $\partial D_0$ is connected then we proceed as follows. Let $S_t^j, 1 \le j \le m$, be the connected components of $S_t$. Denote by $U_t^j$ be the bounded connected component of the complement of $S_t^j$ in $\mathbb{R}^n$. Among the $U_t^j$ there is one, say $U_t^1$, that contains all the others. This is so at time $t = 0$ because $D_0$ is connected and this property is preserved by the flow $X(\cdot, t)$. We set $D_t = U_t^1 \setminus (\cup_{j=2}^m \overline{U_t^j})$, so that the boundary of $D_t$ is $S_t$.

Thus, we define a velocity field by

$$v(\cdot, t) = L \cdot \nabla N * \chi_{D_t}, \quad t \in (-T, T). \tag{3.11}$$

Since its gradient is a SIO acting on a bounded function, a first look to (3.11) tells us that $v(\cdot, t)$ belongs to the continuous Zygmund class. Nevertheless, since $D_t$ has boundary of class $C^{1,\gamma}$, the field $v(\cdot, t)$ is Lipschitz for each $t \in (-T, T)$ and the equation of the flow (1.2) has a unique solution which is a bilipschitz mapping of $\mathbb{R}^n$ onto itself whose restriction to $\partial D_0$ is the solution of the CDE we were given. Thus $X(D_0, t) = D_t$ and $\chi_{D_t}$ is a weak solution of the non-linear transport equation (3.4).

□

## 3.4 The choice of the defining function

In Theorem 3.9, we have seen that the unique solution to (3.4) at time $t$ is the characteristic function of a $C^{1,\gamma}$ domain $D_t$ up to a certain time $T^*$. From now on, we want to proof that this weak solution is indeed global, that is, that we have $T^* = \infty$. We make use of the $C^{1,\gamma}$-defining functions introduced in Section 3.2 to measure the smoothness of $D_t$. By Definition 3.2 it is clear that given a $C^{1,\gamma}$ domain the defining function associated to that domain is not unique (for instance multiplying by a positive constant the function does not change the validity of the requirements in the definition). A natural ansatz in order to get a $C^{1,\gamma}$-defining function for a domain evolving with a flow is the following. We consider $\varphi_0$ a $C^{1,\gamma}$ defining function for the initial domain $D_0$. For the domain $D_t = X(D_0, t)$ we consider

$$\varphi(\cdot, t) = \varphi_0(X^{-1}(\cdot, t)),$$

that is, we also let the defining function evolve with the flow. In the case of Euler equation the function $\varphi(\cdot, t)$ defined this way is a $C^{1,\gamma}$-defining function for the domain $D_t$ determining the density patch and we can develop an argument using $\varphi(\cdot, t)$ to prove that the domain keeps the $C^{1,\gamma}$ smoothness for any time. For the aggregation equation one can see in [BGLV] that this function $\varphi$ does not behave well and a correction (depending, as usual, on the divergence of the velocity field) has to be made in order to addapt the proof of the Euler case. In this section we get the good correction to $\varphi$ in the case of the kernel $L \cdot \nabla N$.

We start with a technical lemma that will be used when proving that our choice is a nice defining function. This is a result that was not needed neither for the Euler nor the aggregation equations.

**Lemma 3.10.** *Let $\Omega$ a subset of $\mathbb{R}^n$ and let*

$$u \in Har(\Omega) \cap C^{\gamma}(\Omega).$$

*Then there exists a constant $c$ such that*

$$|\nabla u(x)| \leq c\, |u|_{\gamma,\Omega}\, d(x, \partial\Omega)^{\gamma-1}.$$

*Proof.* Let $\varphi$ be a $C^\infty$ function, radial and supported in the unit ball and such that $\int_{\mathbb{R}^n} \varphi(x)\, dx = 1$. Let $x \in \Omega$ and consider $r > 0$ such that $r < d(x, \partial\Omega)$. Define $\varphi_r = \frac{1}{r^n}\varphi(y/r)$. Since $u$ is harmonic on $B(x,r)$ then by the mean-value property

$$u(x) = \int_{|x-y|<r} u(y)\, \varphi_r(x-y)\, dy.$$

We compute the partial derivative $\partial_i$ as

$$\partial_i u(x) = \int_{|x-y|<r} u(y) \partial_i \varphi_r(x-y) \, \mathrm{d}y.$$

Since by Stokes' theorem we have

$$\int_{|x-y|<r} \partial_i \varphi_r(x-y) \, \mathrm{d}y = 0,$$

then

$$\partial_i u(x) = \int_{|x-y|<r} (u(y) - u(x)) \partial_i \varphi_r(x-y) \, \mathrm{d}y$$

and so

$$\begin{aligned} |\partial_i u(x)| &= \int_{|x-y|<r} |u(y) - u(x)| \, |\partial_i \varphi_r(x-y)| \, \mathrm{d}y \leq \\ &\leq \frac{|u|_{\gamma,\Omega}}{r^n} \int_{|x-y|<r} |y-x|^\gamma \left| \partial_i \left[ \varphi\left( \frac{x-y}{r} \right) \right] \right| \, \mathrm{d}y \leq \\ &\leq c \frac{|u|_{\gamma,\Omega}}{r^n} \int_0^r s^\gamma \frac{1}{s} s^{n-1} \, \mathrm{d}s = c \frac{|u|_{\gamma,\Omega}}{r^n} r^{\gamma+n-1} = c \, |u|_{\gamma,\Omega} \, r^{\gamma-1}. \end{aligned}$$

Letting $r \to d(x, \partial\Omega)$ we obtain the result. □

With this lemma we will set in a moment a right defining function for $D_t$ in terms of a partial differential equation. The lemma will be used to verify that the partial derivatives of the divergence of the velocity field blow up at the boundary but in a controlled way since they satisfy the hypothesis. Note that for the Euler equation, since the velocity field was incompressible these partial derivatives are equal to 0 so the previous lemma is not needed. In the aggregation case although the divergence is not equal to 0, its partial (spatial) derivatives vanishes almost everywhere and we can also avoid this general argument.

**Proposition 3.11.** *Given $\{D_t\}_{0 \leq t \leq T^*}$ a family of $C^{1,\gamma}$ domains in $\mathbb{R}^n$. If we consider a velocity field $v(\cdot, t) = (L \cdot \nabla N) * \chi_{D_t}$, then the solution $\Phi : \mathbb{R}^n \to \mathbb{R}$ of the linear non-homogeneous partial differential equation*

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \operatorname{div}(v) \, \Phi \tag{3.12}$$

*is a $C^{1,\gamma}$-defining function for $D_t$ whose gradient is continuous.*

*Proof.* Let $\varphi_0$ a $C^{1,\gamma}$ Firstly we see that the solution of (3.12) is

$$\Phi(x,t) = \begin{cases} 0, & x \in \partial D_t, \\ \det \nabla X(X^{-1}(x,t), t) \; \varphi(x,t), & x \notin \partial D_t, \end{cases} \tag{3.13}$$

where $X(\cdot,t)$ is the flow map defined in (1.2) and $\varphi(x,t) = \varphi_0(X^{-1}(x,t))$ for $\varphi_0$ a $C^{1,\gamma}$-defining function for $D_0$. We check it by computing the material derivative of $\Phi$.

$$\begin{aligned} \frac{D\Phi}{Dt}(x,t) &= \frac{D\varphi}{Dt}(x,t) \det \nabla X(X^{-1}(x,t),t) + \\ &+ \frac{D}{Dt}(\det \nabla X(X^{-1}(x,t),t)\varphi(x,t) = \\ &= \frac{D}{Dt}(\det \nabla X(X^{-1}(x,t),t)\varphi(x,t), \end{aligned} \tag{3.14}$$

since $\frac{D\varphi}{Dt} \equiv 0$ because it is a function transported by the flow map. The evolution of the jacobian of the flow map

$$J(\alpha,t) = \det \nabla X(\alpha,t) \tag{3.15}$$

is determined by the equation (see [MB, Proposition 1.2])

$$\frac{dJ}{dt}(\alpha,t) = \operatorname{div}(v(X(\alpha,t),t))J(\alpha,t). \tag{3.16}$$

If we apply this to (3.14) we simply get the partial differential equation (3.12).

Now, we have to verify that the gradient of $\Phi$ defined in (3.13) is continuous. The Main Lemma in [MOV] states that if $T$ is an even smooth convolution homogeneous Calderón-Zygmund operator and $D$ a domain with boundary of class $C^{1+\gamma}$, $0 < \gamma < 1$, then $T(\chi_D)$ satisfies a Hölder condition of order $\gamma$ in $D$ and in $\mathbb{R}^n \setminus \overline{D}$. Then

$$||\nabla v(\cdot,s)||_{\gamma,D_s} + ||\nabla v(\cdot,s)||_{\gamma,\mathbb{R}^n\setminus\overline{D_s}} \leq C(t), \quad 0 \leq s \leq t, \tag{3.17}$$

where $C(t)$ denotes here and in the sequel a positive constant depending on $t$ but not on $s \in [0,t]$. Equation (3.17) implies a similar bound for the gradient of $\Phi$, that is,

$$||\nabla \Phi(\cdot,s)||_{\gamma,D_s} + ||\nabla \Phi(\cdot,s)||_{\gamma,\mathbb{R}^n\setminus\overline{D_s}} \leq C(t), \quad 0 \leq s \leq t,$$

and this way $\Phi(\cdot,s)$ is of class $C^{1,\gamma}$ both in the interior of $D_s$ and in the complement of the closure of $D_s$. Then we just have to check the continuity of the gradient of $\Phi$ in the boundary of the domain. It will ensure that $\Phi(\cdot,s) \in C^{1,\gamma}$ in the whole euclidean space.

As it was pointed out in [BGLV, Section 8] if one transports a defining function $\varphi_0$ of $D_0$ by $\varphi(\cdot,t) = \varphi_0 \circ X^{-1}(\cdot,t)$, then $\nabla\varphi(\cdot,t)$ may have jumps at the boundary of $D_t$ for $t \neq 0$ and so $\varphi(\cdot,t)$ is not necessarily differentiable. In [BGLV] one shows that, for $x \in \partial D_t$,

$$\lim_{D_t \ni y \to x} \nabla\varphi(y,t) = \lim_{D_t \ni y \to x} \det \nabla X^{-1}(y,t) \frac{|\nabla\varphi_0(X^{-1}(x,t))|}{\det D(x)} \vec{n}(x) \tag{3.18}$$

and
(3.19)

$$\lim_{\mathbb{R}^n \setminus \overline{D_t} \ni y \to x} \nabla \varphi(y,t) = \lim_{\mathbb{R}^n \setminus \overline{D_t} \ni y \to x} \det \nabla X^{-1}(y,t) \, \frac{|\nabla \varphi_0(X^{-1}(x,t))|}{\det D(x)} \, \vec{n}(x),$$

where $X^{-1}(\cdot,t)$ is the inverse mapping of $X(\cdot,t)$, $\vec{n}(x)$ is the unitary exterior normal vector to $\partial D_t$ at $x$ and $D(x)$ is the differential at $x$ of the restriction of $X^{-1}(\cdot,t)$ to $\partial D_t$, as a differentiable mapping from $\partial D_t$ onto $\partial D_0$. Equations (3.18) and (3.19) can be interpreted as follows. The gradient of $\varphi(\cdot,t)$ at a point $x \in \partial D_t$ may have a jump and this jump is related with the jump of the jacobian at $x$. By equation (3.16) we see that the possible discontinuity at the boundary has to be related with the divergence. We are going to check that this jump appearing when the divergence is not continuous is in fact compensated with the determinant in equation (3.13).

In particular, if we take gradient in (3.13) we get, for $x \notin \partial D_t$,

$$\begin{aligned} \nabla \Phi(x,t) &= \det \nabla X(X^{-1}(x,t),t)) \, \nabla \varphi(x,t) + \\ &\quad + \nabla \left( \det \nabla X(X^{-1}(x,t),t) ) \right) \varphi(x,t) = I(x) + II(x). \end{aligned} \tag{3.20}$$

Since second order derivatives of the velocity field are harmonic on the complement of $\partial D_s$ an application of Lemma 3.10 yields that for $x \notin \partial D_s$ and for any $0 \le s \le t$ and any $j,k \in \{1,\ldots,n\}$

$$|\partial_j \partial_k v(x,s)| \le C(t) \, \operatorname{dist}(x, \partial D_s)^{\gamma-1}. \tag{3.21}$$

As we have seen in Chapter 1 (see Lemma 1.23 and Proposition 1.25) both the flow map and its inverse satisfy

$$\|\nabla X(\cdot,t)\|_{L^\infty}, \; \|\nabla X^{-1}(\cdot,t)\| \le \exp \int_0^t \|\nabla v(\cdot,s)\|_{L^\infty} \, ds \tag{3.22}$$

and then

$$C(t)^{-1} \le \|\nabla X(\cdot,s)\|_{L^\infty} \le C(t), \quad 0 \le s \le t. \tag{3.23}$$

Consequently, for all $\alpha \in R^n$, and for $0 \le s \le t$,

$$C(t)^{-1} \operatorname{dist}(\alpha, \partial D_0) \le \operatorname{dist}(X(\alpha,s), \partial D_s) \le C(t) \operatorname{dist}(\alpha, \partial D_0). \tag{3.24}$$

Now let us turn to the second term in the right hand side of (3.20). We write

$$II(x) = \varphi_0(\alpha) \, \nabla_x J(\alpha,t), \tag{3.25}$$

where we have set $x = X(\alpha, t)$ and $J(\alpha, t)$ as defined in (3.15). The jacobian satisfies (3.16) and so

$$J(\alpha, t) = \exp \int_0^t \operatorname{div}(v(X(\alpha, s), s))\, ds.$$

Hence $\nabla_x J(\alpha, t)$ is

$$J(\alpha, t) \int_0^t \operatorname{div}\left((\nabla v)^t (X(\alpha, s), s)\right)\, \nabla X(\alpha, s)\, ds\, \nabla X^{-1}(x, t), \tag{3.26}$$

where the divergence of a matrix is the vector with components the divergence of rows. Combining (3.21), (3.22), (3.23), (3.24), (3.25) and (3.26) we get

$$\begin{aligned} |II(x)| &\leq C(t)\, |\varphi_0(\alpha)| \int_0^t \operatorname{dist}(X(\alpha, s), \partial D_s)^{\gamma - 1}\, ds \\ &\leq C(t)\, |\varphi_0(\alpha)|\, \operatorname{dist}(\alpha, \partial D_0)^{\gamma - 1} \\ &\leq C(t)\, \operatorname{dist}(\alpha, \partial D_0)^{\gamma}. \end{aligned}$$

If $\operatorname{dist}(x, \partial D_t) \to 0$ then $\operatorname{dist}(\alpha, \partial D_0) \to 0$ and thus $II(x) \to 0$.

Therefore, if we let $x \to \partial D_t$ equation (3.20) becomes

$$\nabla \Phi(x, t) = \det \nabla X(X^{-1}(x, t), t))\, \nabla \varphi(x, t) \tag{3.27}$$

and then a straightforward application of equations (3.18) and (3.19) shows us that $\nabla \Phi(\cdot, t)$ can be extended continuously to any point $x \in \partial D_t$ and, in particular, we have

$$\lim_{\mathbb{R}^n \setminus \partial D_t \ni y \to x} \nabla \Phi(x, t) = \frac{|\nabla \varphi_0(X^{-1}(x, t))|}{\det D(x)}\, \vec{n}(x).$$

□

As we have pointed out in the proof of the previous lemma, the function $\Phi(\cdot, t)$ is a $C^{1,\gamma}$-defining function for $D_t$ for $t$ up to the time $T^*$ given by Picard-Lindelöf's theorem. The problem is that we need an a priori control of the smoothness of the domain for bigger times. For this reason, in the next section we describe the evolution of the derivatives of this defining function in terms of commutators (as in the Euler equation or the aggregation one).

## 3.5 Commutators for the material derivative of $\nabla\Phi$

At the moment we have set a defining function $\Phi(\cdot,t)$ for the domain $D_t$ for $0 \le t \le T^*$ and we have seen that this $\Phi$ satisfies

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \operatorname{div}(v)\Phi.$$

This partial differential equation for $\Phi$ is general and does not depend on the choice of the kernel $K$ such that $v(\cdot,t) = K * \chi_{D_t}$ (as soon as we can invoke Lemma 3.10 or a similar result) but the divergence of the velocity field does. For instance, if $K$ is such that the velocity field is divergence free then the material derivative of $\Phi$ vanishes and we simply have that $\Phi(\cdot,t) = \varphi_0(X^{-1}(\cdot,t))$ for $\varphi_0$ a defining function for $D_0$. Also if we consider the aggregation kernel then $\operatorname{div}(v(\cdot,t)) = -\chi_{D_t}$ (note that the spatial derivative of this divergence is 0 except maybe at $\partial D_t$) and then we recover the defining function $\Phi$ computed in [BGLV]. Nevertheless, in general, the divergence of the velocity field for $K = L \cdot \nabla N$ will be a SIO acting on the characteristic of a domain.

Both in Euler and aggregation equations one can see that the material derivative of the gradient of a good defining function is equal to a commutator with a singular kernel, that is, it is equal to an expression of the form

$$\text{p.v.} \int_{\mathbb{R}^n} K(x-y)[f(x)-f(y)]\omega(y)\,\mathrm{d}y.$$

The importance of having a commutator is that it allows us to have a proper bound for the Hölder norm of it, and this will be needed later on in our arguments. More specifically, we have the next control of the Hölder norm.

**Lemma 3.12.** *Let $K$ be a Calderon-Zygmund kernel, homogeneous of degree $-n$, with mean zero on spheres, satisfying $|\nabla K(x)| \le C\,|x|^{-n-1}$. For $f \in C^\gamma$ and $\omega \in L^\infty$ set*

$$G(x) = p.v. \int_{\mathbb{R}^n} K(x-y)[f(x)-f(y)]\omega(y)\,\mathrm{d}y.$$

*Then, there exists a constant $C_0$ depending on $\gamma$ and $n$ such that*

$$|G|_\gamma \le C_0\,|f|_\gamma\,(|K*\omega|_{L^\infty} + |\omega|_{L^\infty}).$$

*Proof.* See [BGLV, p. 355]. □

The goal of this section is to show that the material derivative of the gradient of $\Phi$ is equal to a commutator also when the kernel is $L \cdot \nabla N$. Notice that taking the gradient of $\operatorname{div}(v)\Phi$ a term involving second derivatives of $v$ appears. This is

$$\nabla(\operatorname{div}(v))\Phi$$

and it is annoying for our purposes. It does not combine with the rest of the terms to yield a commutator. Also, apparently this is the most singular term appearing. If our velocity field is divergence free this *solitary term* vanishes, and for the aggregation equation, although the is not equal to 0, it also disappears because $\operatorname{div}(v)$ is constant on each component of $\mathbb{R}^n \setminus \partial D_t$ and $\Phi$ vanishes on $\partial D_t$. In general, the solitary term is present but, as we will see in a moment, at least it vanishes when we look the PDE for $\nabla\Phi$ just at points of the boundary of the domain.

To simplify the computations instead of a general matrix $L$ we consider $n$-square matrices having an entry equal to 1 and the rest of them null. It is clear that these matrices form a base of the space $M_{n\times n}(\mathbb{R})$ and then, as we discuss later, it will be sufficient to get the results in this section for them. We define the basis in a precise way.

**Definition 3.13.** *Let $i, j = 1, \ldots, n$. We define the matrix $M^{ij} \in M_{n\times n}(\mathbb{R})$ as $M^{ij} = (M^{ij})_{kl}$ for*

$$(M^{ij})_{kl} = \begin{cases} 1 & \text{if } k = i \text{ and } l = j, \\ 0 & \text{otherwise.} \end{cases}$$

First of all we get the PDE involving the commutator for matrices of the basis and for points of the boundary of the domain. The restriction to the boundary has to be done, as already mentioned, due to the presence of the solitary term.

**Lemma 3.14.** *Let $M^{ij}$ the matrix defined in (3.13) and let $v = M^{ij} \cdot \nabla N * \chi_D$. Then, for $\Phi$ a defining function of $D$ satisfying*

$$\frac{D\Phi}{Dt} = \operatorname{div}(v)\Phi$$

*we have that for every $k = 1, \ldots, n$ the material derivative of $\partial_k\Phi$ restricted to $\partial D$ is either equal to 0 or equal to the difference of two commutators, each one of them of the form*

$$S_{ijk}[\Phi] := p.v.\ \partial_i\partial_j N * (\chi_D \partial_k\Phi) - (p.v.\ \partial_i\partial_j N * \chi_D)\partial_k\Phi.$$

*Proof.* In the conditions of the lemma, the field $v$ is 0 en each component except $v_j$. Also, we have $v_j = \partial_i N * \chi_D$. Then, the equation for $\Phi$ can be written as

$$\partial_t\Phi + v_j\partial_j\Phi = \partial_j v_j \Phi.$$

Computing the partial derivative $\partial_k$ of the previous equation and rearranging terms we obtain

$$\frac{\mathrm{D}(\partial_k\Phi)}{\mathrm{D}t} = \partial_k\partial_j v_j\Phi + \partial_j v_j \partial_k\Phi - \partial_k v_j \partial_j\Phi. \tag{3.28}$$

We claim that the term $\partial_k\partial_j v_j\Phi$ vanishes when we restrict to $\partial D$. On one hand, since $\Phi$ is equally 0 and its gradient is not null at $\partial D$ one has, by Taylor expansion, that near the boundary $\Phi(x) \simeq d(x,\partial D)$. On the other hand, By Main Lemma in [MOV], derivatives of the velocity field belong to $C^\gamma(D\cap\bar{D}^c)$. Also, these derivatives are harmonic in this region because of the choice of the kernel $L\cdot\nabla N$ defining the velocity. Thus, having into account these facts and Lemma 3.10 we have

$$\partial_k\partial_j v_j\Phi \le Cd(x,\partial D)^{\gamma-1}d(x,\partial D) = Cd(x,\partial D)^\gamma \to 0 \quad \text{as } x\to\partial D,$$

which proves the claim. Then equation (3.28) can be written at the boundary as

$$\frac{\mathrm{D}(\partial_k\Phi)}{\mathrm{D}t} = \partial_j v_j\partial_k\Phi - \partial_k v_j\partial_j\Phi = (\partial_i\partial_j N*\chi_D)\partial_k\Phi - (\partial_k\partial_i N*\chi_D)\partial_j\Phi. \tag{3.29}$$

As $i,j,k\in\{1,\dots,n\}$ then different combinations and repetitions of the indexes can appear and we have to distinguish between some cases.

First of all, if $i=j=k$ then the two terms in the right hand side of (3.29) are equal and $\frac{\mathrm{D}(\partial_k\Phi)}{\mathrm{D}t}=0$.

If $i=j\neq k$, since $\partial_i^2 N = \frac{1}{n}\delta_0 + \text{p.v. }\partial_i^2 N$ for any $i=1,\dots,n$ then equation (3.29) is equal to

$$\begin{aligned}\frac{\mathrm{D}(\partial_k\Phi)}{\mathrm{D}t} &= (\partial_i^2 N*\chi_D)\partial_k\Phi - (\partial_k\partial_i N*\chi_D)\partial_i\Phi = \\ &= \frac{1}{n}\chi_D\partial_k\Phi + (\text{p.v. }\partial_i^2 N*\chi_D)\partial_k\Phi - (\text{p.v. }\partial_k\partial_i N*\chi_D)\partial_i\Phi.\end{aligned} \tag{3.30}$$

The first term of the right hand side can be expressed as

$$\frac{1}{n}\chi_D\partial_k\Phi = \frac{1}{n}\delta_0*(\chi_D\partial_k\Phi) = \partial_i^2 N*(\chi_D\partial_k\Phi) - \text{p.v. }\partial_i^2 N*(\chi_D\partial_k\Phi). \tag{3.31}$$

Since $\Phi$ vanishes in $\partial D$ then $\chi_D\partial_k\Phi = \partial_k(\chi_D\Phi)$. Thus, switching derivatives in the convolution we have

$$\begin{aligned}\partial_i^2 N*(\chi_D\partial_k\Phi) &= \partial_i(\partial_i N)*(\partial_k[\chi_D\Phi]) = \partial_k(\partial_i N)*(\partial_i[\chi_D\Phi])) = \\ &= \partial_k\partial_i*(\chi_D\partial_i\Phi) = \text{p.v. }\partial_i\partial_k N*(\chi_D\partial_i\Phi),\end{aligned}$$

and we can write (3.31) as

$$\frac{1}{n}\chi_D\partial_k\Phi = \text{p.v. }\partial_k\partial_i N*(\chi_D\partial_i\Phi) - \text{p.v. }\partial_i^2 N*(\chi_D\partial_k\Phi). \tag{3.32}$$

Putting (3.32) inside (3.30) we get

$$\begin{aligned}\frac{\mathrm{D}(\partial_k\Phi)}{\mathrm{D}t} &= \text{p.v. }\partial_k\partial_i N*(\chi_D\partial_i\Phi) - (\text{p.v. }\partial_k\partial_i N*\chi_D)\partial_i\Phi \\ &- [\text{p.v. }\partial_i^2 N*(\chi_D\partial_k\Phi) - (\text{p.v. }\partial_i^2 N*\chi_D)\partial_k\Phi] = S_{kii}[\Phi] - S_{iik}[\Phi].\end{aligned}$$

Up to now we have discussed the cases where $i = j$, that is, when the matrix $M^{ij}$ has the non-zero value at the diagonal. Consider now $i \neq j, k = i$. Then

$$\frac{\mathrm{D}(\partial_i \Phi)}{\mathrm{D}t} = (\partial_i \partial_j N * \chi_D)\partial_i \Phi - (\partial_i^2 N * \chi_D)\partial_j \Phi.$$

Note that the above expression is very similar to (3.30). Then, a straightforward repetition of the argument in that case shows

$$\frac{\mathrm{D}(\partial_i \Phi)}{\mathrm{D}t} = S_{iij}[\Phi] - S_{iji}[\Phi].$$

If $i \neq j$ but $k = j$ then (3.29) is

$$\frac{\mathrm{D}(\partial_j \Phi)}{\mathrm{D}t} = (\text{p.v. } \partial_i \partial_j N * \chi_D)\partial_j \Phi) - (\text{p.v. } \partial_i \partial_j N * \chi_D)\partial_j \Phi) = 0$$

and there is nothing to prove.

Finally, if the three indexes $i, j, k$ are pairwise different the expression in (3.29) can be written with principal values for sure. That is, in this case

$$\frac{\mathrm{D}(\partial_k \Phi)}{\mathrm{D}t} = \text{p.v. } (\partial_i \partial_j N * \chi_D)\partial_k \Phi - \text{p.v. } (\partial_k \partial_i N * \chi_D)\partial_j \Phi, \tag{3.33}$$

since no Dirac deltas appear.

Note that since $i, j, k$ are different from each other (and therefore the principal values can be avoided), then we can switch derivatives in the following convolutions and get

$$\begin{aligned} \text{p.v. } \partial_i \partial_j N * (\chi_D \partial_k \Phi) &= \partial_i \partial_j N * (\chi_D \partial_k \Phi) = \\ &= \partial_k \partial_i N * (\chi_D \partial_j \Phi) = \text{p.v. } \partial_k \partial_i N * (\chi_D \partial_j \Phi). \end{aligned}$$

Then we can add and subtract some term in (3.33) to have

$$\frac{\mathrm{D}(\partial_k \Phi)}{\mathrm{D}t} = S_{kij}[\Phi] - S_{ijk}[\Phi].$$

□

If we apply Lemma 3.12 we see that the Hölder semi-norm of order $\gamma$ of each of the commutators appearing in the previous lemma can be estimated by

$$C_n \|\nabla v(\cdot, t)\|_\infty \|\nabla \Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}.$$

This is not enough in our situation. Since the commutator is just valid at the boundary of the domain then we need that the presence of the factor $\|\nabla \Phi(\cdot, t)\|_{\gamma, \mathbb{R}^n}$ (which appears since the region of integration of the commutator is the whole space) is replaced by a boundary quantity like $\|\nabla \Phi(\cdot, t)\|_{\gamma, \partial D_t}$. This attempt could be tried since we can transform the differences of commutators obtained in Lemma 3.14 in what we call *boundary* *commutators*, that is, commutators described by an integral just taking values of points in the boundary of the domain.

In order to simplify the proof of the equivalence between differences of *solid commutators* and differences of *boundary* ones we have the following general relationship between them.

**Lemma 3.15.** *Let $f \in C^2(\mathbb{R}^n, \mathbb{R})$ and $D$ a smooth domain in $\mathbb{R}^n$. If we define*

$$B_{ijk}[f](x) := (-1)^{k-1} \int_{\partial D} \partial_i N(x-y)[\partial_j f(y) - \partial_j f(x)]\, dy_{\bar{k}},$$

*for $dy_{\bar{k}} = dy_1 \wedge \ldots \wedge dy_{k-1} \wedge dy_{k+1} \wedge \ldots \wedge dy_n$, then we have*

$$S_{ijk}[f] = -B_{jki}[f] + \partial_j N * (\chi_D \partial_i \partial_k f).$$

*Proof.* Let $A_\varepsilon = D \cap B(x, \varepsilon)$. Thus,

$$S_{ijk}[f](x) = \lim_{\varepsilon \to 0} \int_{D \setminus A_\varepsilon} \partial_i \partial_j N(x-y)[\partial_k f(y) - \partial_k f(x)]\, dy. \tag{3.34}$$

For $y \in D \setminus A_\varepsilon$ we have

$$\begin{aligned} &\frac{\partial}{\partial y_i}(\partial_j N(x-y)[\partial_k f(y) - \partial_k f(x)]) = \\ &\quad = -\partial_i \partial_j N(x-y)[\partial_k f(y) - \partial_k f(x)] + \partial_j N(x-y) \partial_i \partial_k f(y) \end{aligned} \tag{3.35}$$

By (3.35) we can write (3.34) as

$$\begin{aligned} S_{ijk}[f](x) &= \lim_{\varepsilon \to 0} \int_{D \setminus A_\varepsilon} \partial_j N(x-y) \partial_i \partial_k f(y)\, dy - \\ &\quad - \lim_{\varepsilon \to 0} \int_{D \setminus A_\varepsilon} \frac{\partial}{\partial y_i}(\partial_j N(x-y)[\partial_k f(y) - \partial_k f(x)])\, dy = \\ &= \lim_{\varepsilon \to 0} \int_{D \setminus A_\varepsilon} \partial_j N(x-y) \partial_i \partial_k f(y)\, dy - \\ &\quad -(-1)^{i-1} \lim_{\varepsilon \to 0} \int_{D \setminus A_\varepsilon} d\left(\partial_j N(x-y)[\partial_k f(y) - \partial_k f(x)]\, dy_{\bar{i}}\right), \end{aligned} \tag{3.36}$$

where

$$dy_{\bar{i}} = dy_1 \wedge \ldots \wedge dy_{i-1} \wedge dy_{i+1} \wedge \ldots \wedge dy_n.$$

If we apply Stokes' Theorem to the last integral in (3.36) we get

$$\begin{aligned} S_{ijk}[f](x) &= \lim_{\varepsilon \to 0} \int_{D \setminus A_\varepsilon} \partial_j N(x-y) \partial_i \partial_k f(y)\, dy - \\ &\quad -(-1)^{i-1} \text{p.v.} \int_{\partial D} \partial_j N(x-y)[\partial_k f(y) - \partial_k f(x)]\, dy_{\bar{k}} = \\ &= \partial_j N(x-y) * (\chi_D \partial_i \partial_k f) - B_{jki}[f](x), \end{aligned} \tag{3.37}$$

because the term

$$\lim_{\varepsilon\to 0}\int_{\partial B(x,\varepsilon)\cap D}\partial_j N(x-y)[\partial_k f(y)-\partial_k f(x)]\,\mathrm{d}y_{\bar{k}}$$

vanishes since $\partial_k f$ is continuous at $x$. □

The above result needs the function $f$ to have two derivatives but we want $\Phi$ to take the role of $f$ and in general we just can assure that $\Phi\in C^{1,\gamma}$. We need the following technicalities to justify that we can use the previous lemma. We start by setting a new space of functions related to the Hölder space.

**Definition 3.16.** *Let $0<\gamma<1$. We define the little-Hölder space of functions $c^\gamma(\mathbb{R}^n;\mathbb{R}^d)$ as*

$$c^\gamma(\mathbb{R}^n;\mathbb{R}^d)=\left\{f\in C^\gamma(\mathbb{R}^n;\mathbb{R}^d):\lim_{\delta\to 0}\sup_{\substack{x,y\in\mathbb{R}^n\\|x-y|<\delta}}\frac{|f(x)-f(y)|}{|x-y|^\gamma}=0\right\}.$$

**Remark 3.17.** *One can check that, given $0<\gamma<1$ then for every $\varepsilon>0$,*

$$C^{\gamma+\varepsilon}(\mathbb{R}^n;\mathbb{R}^d)\subset c^\gamma(\mathbb{R}^n;\mathbb{R}^d)\subset C^\gamma(\mathbb{R}^n;\mathbb{R}^d).$$

We have that this space we just defined is the closure of the classical Hölder space.

**Proposition 3.18.** *Let $0<\gamma'<1$ and let $\gamma>\gamma'$. Then $c^{\gamma'}(\mathbb{R}^n;\mathbb{R}^d)$ is the closure of $C^\gamma(\mathbb{R}^n;\mathbb{R}^d)$ in $C^{\gamma'}(\mathbb{R}^n;\mathbb{R}^d)$.*

*Proof.* See e.g. [Lu, Proposition 0.2.1]. □

We have all the tools to check that the PDE for the gradient of $\Phi$ can be expressed (at least at the boundary of the domain) as a difference of *boundary commutators*.

**Lemma 3.19.** *Let $M^{ij}$ the matrix defined in (3.13) and let $v=M^{ij}\cdot\nabla N*\chi_D$. Then, for $\Phi$ a defining function of $D$ satisfying*

$$\frac{D\Phi}{Dt}=\operatorname{div}(v)\Phi$$

*we have that for every $k=1,\dots,n$ the material derivative of $\partial_k\Phi$ restricted to $\partial D$ is either equal to 0 or equal to the difference of two commutators on the boundary of $D$, each one of them of the form*

$$B_{ijk}[f](x):=(-1)^{k-1}\int_{\partial D}\partial_i N(x-y)[\partial_j f(y)-\partial_j f(x)]\,\mathrm{d}y_{\bar{k}}.$$

*Proof.* There are two type differences of commutators that appear in the proof of Lemma 3.14. When $i = j \neq k$ (similarly for $k = i \neq j$) the material derivative of $\partial_k \Phi$ at the boundary can be written as $S_{kii}[\Phi] - S_{iik}[\Phi]$. Assume $\Phi \in C^2$. Then, by Lemma 3.15 we have

$$S_{kii}[\Phi] - S_{iik}[\Phi] = -B_{iik}[\Phi] + \partial_i N * (\chi_D \partial_i \partial_k \Phi) + \\ + B_{iki}[\Phi] - \partial_i N * (\chi_D \partial_i \partial_k \Phi) = B_{iki}[\Phi] - B_{iik}[\Phi].$$

Also, for $i, j, k$ pairwise different we saw that the material derivative of $\partial_k \Phi$ was equal to $S_{kij}[\Phi] - S_{ijk}[\Phi]$. Again assuming $\Phi \in C^2$ we can apply Lemma 3.15 to get

$$\begin{aligned} S_{kij}[\Phi] - S_{ijk}[\Phi] &= S_{kij}[\Phi] - S_{jik}[\Phi] = \\ &= -B_{ijk}[\Phi] + \partial_i N * (\chi_D \partial_j \partial_k \Phi) + B_{ikj}[\Phi] - \partial_i N * (\chi_D \partial_j \partial_k \Phi) = \\ &= B_{ikj}[\Phi] - B_{ijk}[\Phi], \end{aligned}$$

where we have used that $S_{ijk}[\Phi] = S_{jik}[\Phi]$ which is clear by definition.

If $\Phi \notin C^2$, consider $\varphi$ a smooth mollifier and $\varphi_\varepsilon = \varepsilon^{-n}\varphi(x/\varepsilon)$. Let $\Phi_\varepsilon = \varphi_\varepsilon * \Phi$. In particular, $\Phi_\varepsilon \in C^2$. Let $T_1$ one of the differences of commutators appearing in Lemma 3.14 and let $T_2$ the corresponding difference of commutators on the boundary, as computed above. Thus, since $\Phi_\varepsilon \in C^2$, we have

$$T_1 \Phi_\varepsilon = T_2 \Phi_\varepsilon.$$

Since the operators $T_i$, $i \in \{1, 2\}$ are linear we have

$$\lim_{\varepsilon \to 0} T_i(\Phi_\varepsilon) - T_i(\Phi) = \lim_{\varepsilon \to 0} T_i(\Phi_\varepsilon - \Phi) = 0$$

where the limit has been taken in the $C^{\gamma'}$ norm for $\gamma' < \gamma$. The second limit is equal to 0 by virtue of Lemma 3.12 and Proposition 3.18 since

$$\lim_{\varepsilon \to 0} |T_i(\Phi_\varepsilon - \Phi)|_{\gamma'} \leq \lim_{\varepsilon \to 0} C\, |\Phi_\varepsilon - \Phi|_{1,\gamma'} = 0.$$

Consequently, in the $C^{\gamma'}$ norm we get

$$T_1(\Phi) = \lim_{\varepsilon \to 0} T_1(\Phi_\varepsilon) = \lim_{\varepsilon \to 0} T_2(\Phi_\varepsilon) = T_2(\Phi)$$

even though $\Phi \notin C^2$.

A straightforward argument shows that $T_1(\Phi) = T_2(\Phi)$ pointwise and hence the proposition holds.

□

At the moment, we have the commutators satisfied by the material derivative of the components of $\nabla \Phi$. Unfortunately, we cannot apply Lemma 3.12 directly as some difficulties appear. We explain the nature of this problem and how to solve it in the next section.

## 3.6 The (controlled) extension of $\Phi$

As we explained before, our objective is to bound the $C^\gamma$ norm of the gradient of the defining function $\Phi$ at the boundary of the domain. In order to do that, we bound the material derivative of this gradient in such a way that we can apply Gronwall's Lemma and get what we desire. As stated before, we cannot do it for the material derivative written as the difference of two solid commutators since that would lead us to a bound depending on the $C^\gamma$ of the gradient of $\Phi$ in the whole space. On the other hand and with respect to the boundary commutators, if we try to adapt the Lemma 3.12 to the underlying measure $d\sigma_t$ on $\partial D_t$, we would get a constant of the type

$$C_t = \sup_{x\in\partial D_t} \sup_{r>0} \frac{\sigma_t(B(x,r))}{r^{n-1}}.$$

The constant $C_t$ can be estimated by the Lipschitz constant of $X(\cdot,t)$, namely, $\exp \int_0^t \|\nabla v(\cdot,s)\|_\infty \, ds$, but this exponential constant is far too large. The goal of the present section will be to solve these problems and achieve the right estimate for the material derivative of the gradient of $\Phi$ (at the boundary) in terms of the quantity $\|\nabla\Phi(\cdot,t)\|_{\gamma,\partial D_t}$. We will make use of Whitney's Extension theorem (see e.g. [Ste, Chapter VI, p.177]) and also we will take profit again of the equivalence between solid and boundary commutators, in this case turning back to the solids once the extension is done.

We start by giving a technical lemma that will be used to control the future extension by just its behavior at the boundary.

**Lemma 3.20.** *Let $D$ a domain with $C^{1,\gamma}$-defining function $\Phi$. Then*

$$\sup\left\{\frac{|\nabla\Phi(x)\cdot(y-x)|}{|y-x|^{1+\gamma}} : y\neq x,\ y,x\in\partial D\right\} \leq 2^{1+\gamma/2}\,||\nabla\Phi||_{\gamma,\partial D}\,.$$

*Proof.* Let $x \in \partial D$. Without loss of generality we can consider $x = 0$ and $\nabla\Phi(0) = (0,\ldots,0,\partial_n\Phi(0))$ with $\partial_n\Phi(0) > 0$. We define $\delta = \delta_x$ by

$$\delta^{-\gamma} = 2\frac{||\nabla\Phi||_{\gamma,\partial D}}{|\nabla\Phi(0)|}.$$

Let $y \in B(0,\delta)\cap\partial D$. Since

$$|\nabla\Phi(y) - \nabla\Phi(0)| \leq ||\nabla\Phi||_{\gamma,\partial D}\,\delta^\gamma = \frac{|\nabla\Phi(0)|}{2},$$

then we have that $\nabla\Phi(y) \in B(\nabla\Phi(0), \frac{|\nabla\Phi(0)|}{2})$. That is, the tangent hyperplane to $\partial D$ at $y$ forms an angle less than 30 degrees with the horizontal plane and thus $\partial D\cap B(0,\delta)$ is the graph of a function $y_n = \varphi(y_n')$ which satisfies a Lipschitz condition with constant less than 1. The function $\varphi$ is defined in an open set $U$ which is the projection of $B(0,\delta) \cap \partial D$ into $\mathbb{R}^{n-1}$ defined by $y = (y', y_n) \to y'$. By the Implicit Function Theorem $\varphi$ is of class $C^{1,\gamma}$ in its domain.

Note that the segment $[0, y'] = \{ty' : 0 \le t \le 1\}$ is contained in $U$ since for any $z' \in [0, y']$ we have

$$\left|\varphi(z') - \varphi(0)\right| = \left|\varphi(z')\right| \le \left|z'\right| \le \left|y'\right|.$$

The Mean Value Theorem on the segment $[0, y']$ for the function $t \to \varphi(ty')$ yields

$$\frac{|\nabla\Phi(0) \cdot y|}{|y|^{1+\gamma}} = \frac{|\nabla\Phi(0)|\,|\varphi(y')|}{|y|^{1+\gamma}} \le \le \frac{|\nabla\Phi(0)|}{|y|^{1+\gamma}} \sup\left\{\left|\nabla\varphi(z')\right| : z' \in U, |z'| \le |y'|\right\} \left|y'\right|. \tag{3.38}$$

By implicit differentiation

$$\partial_j\varphi(z') = -\frac{\partial_j\Phi(z', \varphi(z'))}{\partial_n\Phi(z', \varphi(z'))}, \quad 1 \le j \le n-1$$

and recalling that $\partial_j\Phi(0) = 0, 1 \le j \le n-1$ and that $z = (z', \varphi(z'))$ we get

$$\left|\nabla\varphi(z')\right| \le \frac{||\nabla\Phi||_{\gamma,\partial D}}{|\partial_n\Phi(z)|} |z|^\gamma, \quad \left|z'\right| \le \left|y'\right|. \tag{3.39}$$

We also have

$$|\partial_n\Phi(z)| \ge |\partial_n\Phi(0)| - |\partial_n\Phi(z) - \partial_n\Phi(0)| \ge \ge |\nabla\Phi(0)| - ||\nabla\Phi||_{\gamma,\partial D}\,\delta^\gamma = \frac{|\nabla(0)|}{2} \tag{3.40}$$

and

$$|z| = (\left|z'\right|^2 + \varphi(z')^2)^{1/2} \le \sqrt{2}\left|z'\right|. \tag{3.41}$$

Putting (3.40) and (3.41) into (3.39) we get

$$\left|\nabla\varphi(z')\right| \le \frac{2}{|\nabla\Phi(0)|} ||\nabla\Phi||_{\gamma,\partial D}\, 2^{\gamma/2} \left|z'\right|^\gamma, \quad \left|z'\right| \le \left|y'\right|.$$

Thus, by the inequality above we can bound (3.38) as

$$\frac{|\nabla\Phi(0) \cdot y|}{|y|^{1+\gamma}} \le 2^{1+\gamma/2} ||\nabla\Phi||_{\gamma,\partial D}.$$

Otherwise, if $y \in \partial D \setminus B(0,\delta)$

$$\frac{|\nabla\Phi(0)\cdot y|}{|y|^{1+\gamma}} \leq \frac{|\nabla\Phi(0)|}{|y|^{\gamma}} \leq \frac{|\nabla\Phi(0)|}{\delta^{\gamma}} = 2\,||\nabla\Phi||_{\gamma,\partial D}$$

completing the proof of the lemma. □

Then we can state the result that we anticipated in the beginning of the section.

**Proposition 3.21.** *In the situation of Lemma 3.19, the $|\cdot|_{\gamma,\partial D}$ norm of each of the differences of two commutators on the boundary of $D$ appearing for the material derivative of $\partial_k\Phi$ is bounded by*

$$C_n\,||\nabla v||_{L^\infty}\,|\nabla\Phi|_{\gamma,\partial D}\,,$$

*where $C_n$ is a constant depending on the dimension $n$.*

*Proof.* We start by considering the jet

$$(0,\partial_1\Phi,\ldots,\partial_n\Phi)$$

on $\partial D$. By Whitney's Extension theorem there exists $\Psi$ of class $C^{1+\gamma}(\mathbb{R}^n)$ such that $\Psi = 0$ and $\nabla\Psi = \nabla\Phi$ on $\partial D$, satisfying

$$\|\nabla\Psi\|_{\gamma,\mathbb{R}^n} \leq C_n\left(\|\nabla\Phi\|_{\gamma,\partial D} + \sup\{\frac{|\nabla\Phi(x)\cdot(y-x)|}{|y-x|^{1+\gamma}} : y \neq x,\ y,x\in\partial D\}\right).$$

This estimate is not stated explicitly in the theorem in [Ste, p. 177] but it follows from the proof. In Lemma 3.20 the supremum in the expression above has been bounded and then we have the simpler bound

$$\|\nabla\Psi\|_{\gamma,\mathbb{R}^n} \leq C_n\|\nabla\Phi\|_{\gamma,\partial D}.$$

We set $DS$ as a difference of solid commutators described in Lemma 3.14 and $DB$ the associate difference of boundary commutators as in Lemma 3.19. Then, since $\nabla\Psi = \nabla\Phi$ on $\partial D$ the differences of solid commutators $DS(\Phi)$ and $DS(\Psi)$ are equal. Thus

$$\begin{aligned}\|DB\|_{\gamma,\partial D} &= \|DS(\Psi)\|_{\gamma,\partial D} \leq \|DS(\Psi)\|_{\gamma,\mathbb{R}^n}\\ &\leq C_n\,||\nabla v||_{L^\infty}\,\|\nabla\Psi\|_{\gamma,\mathbb{R}^n} \leq C_n\,||\nabla v||_{L^\infty}\,\|\nabla\Phi\|_{\gamma,\partial D}\end{aligned}$$

and so the proposition is proved. □

Finally we have achieved a good control of the commutators. This would lead us to an a priori control of the smoothness of the domain.

## 3.7 Proof of the Main Theorem

Before proving Theorem 3.1, we present the a priori estimates for the defining function $\Phi$. Getting these estimates is the core of the proof and are a direct consequence of the bound obtained thanks to have a commutator for the material derivative of $\nabla\Phi$. The same estimates were obtained both for the Euler and for the aggregation equations.

**Lemma 3.22.** *Let $v(\cdot,t) = L \cdot \nabla N * \chi_{D_t}$ and let $\Phi(\cdot,t)$ the defining function for $D_t$ determined by*

$$\frac{D\Phi}{Dt} = \operatorname{div}(v)\,\Phi. \tag{3.42}$$

*Then, for $\|\nabla\Phi(\cdot,t)\|_{L^\infty,\partial D_t} := \|\nabla\Phi(\cdot,t)\chi_{\partial D_t}\|_{L^\infty}$ we have*

$$\|\nabla\Phi(\cdot,t)\|_{L^\infty,\partial D_t} \leq \|\nabla\Phi(\cdot,0)\|_{L^\infty,\partial D_t} \exp\left(2n\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,ds\right) \tag{3.43}$$

$$|\nabla\Phi(\cdot,t)|_{inf} \geq |\nabla\Phi(\cdot,0)|_{inf} \exp\left(-2\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,ds\right), \tag{3.44}$$

*and*

$$|\nabla\Phi(\cdot,t)|_{\gamma,\partial D_t} \leq |\nabla\Phi(\cdot,0)|_{\gamma,\partial D_0} \exp\left(C_n\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,ds\right). \tag{3.45}$$

*Proof.* Getting rid of the solitary term $\partial_k(\operatorname{div}(v))\Phi$ –we know it vanishes at the boundary by an application of Lemma 3.10 as done in the previous section–, we get by taking a partial derivative to equation (3.42) and after a rearrangement of terms,

$$\frac{D(\partial_k\Phi)}{Dt} = \operatorname{div}(v)\partial_k\Phi - \sum_{j=1}^n \partial_k v_j \partial_j \Phi = \sum_{j=1}^n (\partial_j v_j\, \partial_k\Phi - \partial_k v_j \partial_j\Phi). \tag{3.46}$$

If we consider the $\|\cdot\|_{L^\infty,\partial D_t}$ norm in the previous equation we obtain

$$\begin{aligned}\frac{D}{Dt}\|\partial_k\Phi(\cdot,t)\|_{L^\infty,\partial D_t} \leq \sum_{i=1}^n \big(&\|\partial_j v_j(\cdot,t)\|_{L^\infty,\partial D_t}\|\partial_k\Phi(\cdot,t)\|_{L^\infty,\partial D_t} + \\ &+ \|\partial_k v_j(\cdot,t)\|_{L^\infty,\partial D_t}\|\partial_j\Phi(\cdot,t)\|_{L^\infty,\partial D_t}\big) \leq \\ &\leq 2n\|\nabla v(\cdot,t)\|_{L^\infty,\partial D_t}\|\nabla\Phi(\cdot,t)\|_{L^\infty,\partial D_t}.\end{aligned}$$

Taking supremum over $k$ then

$$\frac{D}{Dt}\|\nabla\Phi(\cdot,t)\|_{L^\infty,\partial D_t} \leq 2n\|\nabla v(\cdot,t)\|_{L^\infty,\partial D_t}\|\nabla\Phi(\cdot,t)\|_{L^\infty,\partial D_t}$$

which yields

$$\|\nabla\Phi(\cdot,t)\|_{L^\infty,\partial D_t} \le \|\nabla\Phi(\cdot,0)\|_{L^\infty,\partial D_t} \exp\left(2n\int_0^t \|\nabla v(\cdot,s)\|\,\mathrm{d}s\right).$$

Secondly, for $x \in \partial D_t$ choose $k \in \{1,\dots,n\}$ such that $\partial_k\Phi(\cdot,t)$ does not vanishes at $x$ –such a $k$ exists since $\nabla\Phi(\cdot,t)$ is not zero for every point in the boundary since $\Phi$ is a $C^{1,\gamma}$-defining function–

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}[\log(|\partial_k\Phi(x,t)|)] &= \frac{1}{|\partial_k\Phi(x,t)|}\frac{\mathrm{D}}{\mathrm{D}t}\,|\partial_k\Phi(x,t)| \ge \\ &\ge \frac{1}{\|\nabla\Phi(\cdot,t)\|_{L^\infty}}\frac{\mathrm{D}}{\mathrm{D}t}\,|\partial_k\Phi(x,t)|\,.\end{aligned}$$

By equation (3.46) we can bound $\frac{\mathrm{D}}{\mathrm{D}t}\,|\partial_k\Phi(x,t)|$ from below by

$$-2n\|\nabla v(\cdot,t)\|_{L^\infty}\|\nabla\Phi(\cdot,t)\|_{L^\infty}$$

and obtain

$$\frac{\mathrm{D}}{\mathrm{D}t}[\log(|\partial_k\Phi(x,t)|)] \ge -2n\|\nabla v(\cdot,t)\|_{L^\infty}. \tag{3.47}$$

Taking supremum over $k$ and by direct integration we get, for $x \in \partial D_t$

$$|\nabla\Phi(x,t)| \ge |\nabla\Phi(x,0)|\exp\left(-2n\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right)$$

which implies inequality (3.44).

We finally prove inequality (3.45). Since the right hand side of (3.46) depends linearly on $v$ then the material derivative of $\partial_k$ is a linear combination of differences of commutators on the boundary of $D_t$. Therefore, by Proposition 3.21

$$\frac{\mathrm{D}}{\mathrm{D}t}\,|\partial_k\Phi(\cdot,t)|_{\gamma,\partial D_t} \le C_n\|\nabla v(\cdot,t)\|_{L^\infty}\,|\nabla\Phi(\cdot,t)|_{\gamma,\partial_{D_t}}\,.$$

Taking supremum over $k \in \{1,\dots,n\}$ and integrating we get inequality (3.45). □

We already have all the ingredients to complete the proof of the main theorem of the chapter.

*Proof of Theorem 3.1.* By Theorem 3.8 there exists a solution $X(\cdot,t)$ with maximal time $T^*$. By this we mean that $X(\cdot,t)$ is defined for $t \in (-T^*,T^*)$ but cannot be extended to a larger interval. We want to prove that $T^* = \infty$. For that it suffices to prove that for some constant $C = C(T^*)$ one has

$$\operatorname{diam}(D_t) + \sigma_t(\partial D_t) + q(D_t) \le C, \quad t \in (-T^*,T^*). \tag{3.48}$$

If the preceding inequality holds, then we take $t_0 < T^*$ close enough to $T^*$ so that after the application of the existence and uniqueness theorem for the CDE to the domain $D_{t_0}$ at time $t_0$ we get an interval of existence for the solution which goes beyond $T^*$ (the same argument applies to the lower extreme $-T^*$).

Given $\chi_{D_t}$ the weak solution to the transport equation given by Theorem 3.9 and $v(\cdot,t) = L \cdot \nabla N * \chi_{D_t}$, we consider the function $\Phi(x,t)$ defined by the partial differential equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\Phi(x,t) = \operatorname{div}(v(x,t))\Phi(x,t).$$

By Proposition 3.11 we know that $\Phi(\cdot,t)$ is a $C^{1,\gamma}$-defining function for $D_t$. Also, by inequalities (3.44) and (3.45) in Lemma 3.22 we get

$$\frac{|\nabla\Phi(\cdot,t)|_{\gamma,\partial D_t}}{|\nabla\Phi(\cdot,t)|_{\inf}} \leq \frac{|\nabla\Phi(\cdot,0)|_{\gamma,\partial D_0}}{|\nabla\Phi(\cdot,0)|_{\inf}} \exp\left(C_n \int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right). \tag{3.49}$$

By an application of Lemma 1.7 and the classic bound

$$\|\nabla X(\cdot,t)\|_{L^\infty} \leq c \exp\left(\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right) \tag{3.50}$$

we also have for $R(t) = m(D_t)^{1/n}$

$$R(t) \leq R(0) \exp\left(\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right). \tag{3.51}$$

Combining now the bounds (3.49) and (3.51) and the logarithmic inequality in Theorem 3.6 one gets, for a dimensional constant $C$,

$$||\nabla v(\cdot,t)||_{L^\infty} \leq C + C\int_0^t ||\nabla v(\cdot,s)||_{L^\infty}\,ds,$$

which yields, by Gronwall,

$$||\nabla v(\cdot,t)||_{L^\infty} \leq C\,e^{Ct}, \quad -T^* < t < T^*. \tag{3.52}$$

Inequality (3.52) allow us to control the left hand side of (3.48). Indeed, as seen in [BGLV, Section 7], one has

$$\sigma_t(\partial D_t) \leq (n-1)^{1/2}\sigma_0(\partial D_0)\exp\left((n-1)\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right)$$

and then by (3.52), for $-T^* < t < T^*$,

$$\sigma_t(\partial D_t) \leq (n-1)^{1/2}\sigma_0(\partial D_0)\exp\left(C\exp(Ct)\right).$$

An argument similar to the one developed in the proof of Lemma 1.7 allow us to give an expression for the diameter of $D_t$ in terms of the gradient of the flow map and therefore by (3.50) and (3.52), for $-T^* < t < T^*$ we have

$$\mathrm{diam}(D_t) \leq \mathrm{diam}(D_0) \exp(C \exp(Ct)).$$

Finally by inequalities (3.49) and (3.52) it is clear that

$$q(D_t) \leq q(D_0) \exp(C \exp(Ct))$$

which completes the proof of the theorem. □

# 4 Patches in $\mathbb{C}$

## 4.1 Introduction

Throughout this chapter we will study the *density patch* problem as done in Chapter 3 but reducing to dimension 2. This reduction allow us to consider more kernels than the ones of the form $L \cdot \nabla N$. As in Chapter 2 we will work in the complex plane (details describing the change of language and the notation used can be found in Section 2.1) and we will deal with the same general family of kernels described in (2.1), namely

$$K_1(z) = \frac{1}{\pi}\frac{(z+\varepsilon\bar{z})^k}{(\bar{z}+\varepsilon z)^{k+1}} \quad \text{or} \quad K_2(z) = \overline{K_1(z)} = \frac{1}{\pi}\frac{(\bar{z}+\varepsilon z)^k}{(z+\varepsilon\bar{z})^{k+1}}.$$

For these kernels we will recover the $C^{1,\gamma}$ regularity result for the boundary of a domain, exactly as in Chapter 3. That is, the main result of the present chapter is the following.

**Theorem 4.1.** *Let $K_i$, $i = 1,2$ as defined in (2.1) and let $D_0$ be a simply connected $C^{1,\gamma}$ domain in the complex plane. Then the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla\rho = 0, \\ v(\cdot,t) = K_i * \rho(\cdot,t), \\ \rho(\cdot,0) = \chi_{D_0}, \end{cases} \tag{4.1}$$

*has a unique weak solution in the sense of (4.8) such that $\rho(\cdot,t) = \chi_{D_t}$ for $D_t$ a simply connected $C^{1,\gamma}$ domain in the complex plane, for every time $t \in \mathbb{R}$.*

We will write the proof just for the kernel $K_2$ but since $K_1$ is the conjugate of it, a straightforward repetition of the argument will work for $K_1$ too. We want to stress that at some steps –specially in the last section– we will refer to the previous chapter to see details that are not presented here. However, we will highlight the differences and difficulties appearing when dealing with the kernels in (2.1) when these come up. In particular, we prove in detail a local version of Theorem 4.1, that is, for short times. The proof of the local Theorem was avoided in Chapter 3 because it could be reduced to the proof for the case of patches for the aggregation kernel, and this was done in [BGLV]. Nevertheless, for the kernels in (2.1) we cannot reduce to that case and so the proof is needed.

Recall that the kernels $K_1$ and $K_2$ (or its sum) are a generalization of the kernels $L \cdot \nabla N$ considered in Chapters 1 and 3, so Theorem 4.1 encompasses the equivalent result for $L \cdot \nabla N$ when reducing to dimension 2.

### 4.1.1 Outline of the chapter

The chapter has the following structure. In Section 4.2 we present some Hölder estimates around integral operators acting on a curve in $\mathbb{C}$. In Section 4.3 we have a complete proof of the local-in-time version of Theorem 4.1 by using again the Picard-Lindelöf's theorem. Finally, in Section 4.4 we adapt the proof in Chapter 3 in order to check that the local patch solution is in fact global. We explain the details concerning the differences appearing and refer to Chapter 3 whenever we need results that were already done there.

## 4.2 Hölder estimates

In order to show Theorem 4.1, we need some auxiliary results concerning the kernels $K_1$ and $K_2$ that should be used on the proof. The techniques used in this section might be already known, but we prefer to write the proofs in detail for the sake of the reader.

**Definition 4.2.** *Let $D$ be a simply connected domain with boundary $\Gamma = \partial D$, a Jordan-Ahlfors regular curve. Let*

$$N_{k,\varepsilon}(z) = \frac{(\bar{z} + \varepsilon z)^k}{(z + \varepsilon \bar{z})^{k+1}}, \qquad k \in \mathbb{N} \cup \{0\},\ 0 \leq |\varepsilon| < 1.$$

*We define the operator $\mathcal{C}_{k,\varepsilon}$ as*

$$[\mathcal{C}_{k,\varepsilon} f](z) = p.v. \int_\Gamma N_{k,\varepsilon}(z - w) f(w)\, \mathrm{d}w, \qquad z \in \Gamma$$

*for sufficiently good functions $f$.*

**Definition 4.3.** *For $\Gamma$ as in Definition 4.2 we define the norm*

$$||f||_{Lip(\gamma,\Gamma)} = \sup_{z \in \Gamma} |f(z)| + \sup_{\substack{z,w \in \Gamma \\ z \neq w}} \frac{|f(z) - f(w)|}{|z - w|^\gamma} := ||f||_{L^\infty} + |f|_\gamma .$$

*Also, we define the space $Lip(\gamma, \Gamma)$ of functions having this norm bounded.*

This norm presented here is not new and we write it using this notation when working with curves in the plane.

The result of this section shows that the operator $\mathcal{C}_{k,\varepsilon}$ sends the space $Lip(\gamma, \Gamma)$ into itself. First of all, we verify that the result holds for the function taking constant value 1, as this would be used in the proof later on.

**Lemma 4.4.** *Let $\mathcal{C}_{k,\varepsilon}$ as in Definition 4.2. Then, for each $k \in \mathbb{N} \cup \{0\}$ and each $\varepsilon \in \mathbb{R}$ such that $0 \leq |\varepsilon| < 1$, we have*

$$[\mathcal{C}_{k,\varepsilon} 1](z) \in Lip(\gamma, \Gamma).$$

*Proof.* First of all, recall that by Plemelj's Formula (e.g. [To, Thm. 1.1]),

$$[\mathcal{C}_{k,\varepsilon} 1](z) = \frac{1}{2}\left([\mathcal{C}^{+}_{k,\varepsilon} 1](z) + [\mathcal{C}^{-}_{k,\varepsilon} 1](z)\right), \qquad a.e.\ z \in \Gamma;$$

where

$$[\mathcal{C}^{+}_{k,\varepsilon} 1](z) = \lim_{\substack{D \ni y \to z \\ \text{non-tangencially}}} \int_\Gamma N_{k,\varepsilon}(y-w)\, f(w)\, \mathrm{d}w,$$

$$[\mathcal{C}^{-}_{k,\varepsilon} 1](z) = \lim_{\substack{\overline{D}^c \ni y \to z \\ \text{non-tangencially}}} \int_\Gamma N_{k,\varepsilon}(y-w)\, f(w)\, \mathrm{d}w.$$

We observe that for $\partial B(y,\delta) := \{|w-y| = \delta\}$, by a change of variables $w = y + \delta e^{i\theta}$ we have

$$\int_{\partial B(y,\delta)} N_{k,\varepsilon}(y-w)\ \mathrm{d}w = \int_0^{2\pi} N_{k,\varepsilon}(\delta \mathrm{e}^{i\theta}) i\delta \mathrm{e}^{i\theta}\, \mathrm{d}\theta = \int_0^{2\pi} N_{k,\varepsilon}(\mathrm{e}^{i\theta}) i\mathrm{e}^{i\theta}\, \mathrm{d}\theta,$$

where the last equality stands due to the homogeneity of the kernel $N_{k,\varepsilon}$. If we do now a change of variables $z = \mathrm{e}^{i\theta}$ we then have

$$\int_{\partial B(y,\delta)} N_{k,\varepsilon}(y-w)\ \mathrm{d}w = \int_{\partial B(0,1)} N_{k,\varepsilon}(z)\, \mathrm{d}z =: c_{k,\varepsilon}.$$

Let $y \in D$. We observe then that $c_{k,\varepsilon}$ is a well-defined quantity which is independent of $\delta$. Thus, we can subtract the integral around $\partial B(y,\delta)$ by paying the quantity $c_{k,\varepsilon}$.

$$\begin{aligned} \int_\Gamma N_{k,\varepsilon}(y-w)\, \mathrm{d}w &= \int_\Gamma N_{k,\varepsilon}(y-w)\, \mathrm{d}w - \\ &\quad - \int_{\partial B(y,\delta)} N_{k,\varepsilon}(y-w)\, \mathrm{d}w + c_{k,\varepsilon} = \\ &= \int_{\Gamma \setminus \partial B(y,\delta)} N_{k,\varepsilon}(y-w)\, \mathrm{d}w + c_{k,\varepsilon}. \end{aligned} \tag{4.2}$$

We just have to focus on the integral in the right hand side of (4.2). Applying Stokes' theorem

$$\int_{\Gamma \setminus \partial B(y,\delta)} N_{k,\varepsilon}(y-w)\, \mathrm{d}w = 2i \int_{D \setminus B(y,\delta)} \bar{\partial} N_{k,\varepsilon}(y-w)\, \mathrm{d}A(w) := \bar{T}_{k,\varepsilon}(\chi_D)(y),$$

where $\bar{T}_{k,\varepsilon}$ denotes the Calderón-Zygmund operator with kernel

$$2i\left[\varepsilon\frac{(\bar{z}+\varepsilon z)^{k-1}}{(z+\varepsilon\bar{z})^{k+1}}-(k+1)\frac{(\bar{z}+\varepsilon z)^{k}}{(z+\varepsilon\bar{z})^{k+2}}\right].$$

These operators are even, smooth homogeneous Calderón-Zygmund operators and by the Main Lemma in [MOV, p. 407] we have

$$\bar{T}_{k,\varepsilon}(\chi_D)\in \operatorname{Lip}(\gamma,D)$$

because $\Gamma=\partial D$ is a curve of class $C^{1,\gamma}$. Proceeding in a similar way, we get when $y\in\overline{D}^c$

$$\int_\Gamma N_{k,\varepsilon}(y-w)\,\mathrm{d}w=-\bar{T}_{k,\varepsilon}(\mathbb{C}\setminus\chi_D)(y)=\bar{T}_{k,\varepsilon}(\chi_D)(y)$$

and by the mentioned Main Lemma in [MOV] $\bar{T}_{k,\varepsilon}(\chi_\Omega)\in\operatorname{Lip}(\gamma,\overline{D}^c)$.

Therefore $[\mathcal{C}^+_{k,\varepsilon}1]$ and $[\mathcal{C}^-_{k,\varepsilon}1]$ belong to $\operatorname{Lip}(\gamma,\Gamma)$ and finally

$$[\mathcal{C}_{k,\varepsilon}1]=\frac{1}{2}([\mathcal{C}^+_{k,\varepsilon}1]+[\mathcal{C}^-_{k,\varepsilon}1])\in\operatorname{Lip}(\gamma,\Gamma),$$

proving the lemma. □

We also need the following lemma in order to prove our goal.

**Lemma 4.5.** *Let $N_{k,\varepsilon}$ as in Definition 4.2. Given $\eta>0$, we set for $z\in\Gamma$ and $k\in\mathbb{N}\cup\{0\}$,*

$$[\mathcal{C}^\eta_{k,\varepsilon}f](z):=\text{p.v.}\int_{\Gamma\setminus D(z,\eta)}N_{k,\varepsilon}(z-w)f(w)\,\mathrm{d}w.$$

*Then, $\left|[\mathcal{C}^\eta_{k,\varepsilon}1](z)\right|\le C$, where $C$ is a constant independent of $\eta$.*

*Proof.* Recall the definition of the maximal singular operator

$$\mathcal{C}^*_{k,\varepsilon}f(z):=\sup_{\eta>0}\left|[\mathcal{C}^\eta_{k,\varepsilon}f](z)\right|.$$

Now, since $|\mathrm{d}w|$ on $\Gamma$ is a doubling measure we can apply Cotlar's inequality (e.g. [Ar, Thm. 3.7]) to have the estimate

$$\mathcal{C}^*_{k,\varepsilon}f(z)\le C\left(M([\mathcal{C}_{k,\varepsilon}f])(z)+M(f)(z)\right),$$

where $M$ denotes the Hardy-Littlewood maximal operator. In particular,

$$\mathcal{C}^*_{k,\varepsilon}1(z)\le C\left(M([\mathcal{C}_{k,\varepsilon}1])(z)+M(1)(z)\right),\qquad z\in\Gamma.$$

By Lemma 4.4 we have that $\mathcal{C}_{k,\varepsilon}1$ is bounded. Obviously, 1 is bounded on $\Gamma$ and Hardy-Littlewood Maximal Operator preserves bounded functions. Then,

$$\left|[\mathcal{C}^{\eta}_{k,\varepsilon}1](z)\right| \le \mathcal{C}^{*}_{k,\varepsilon}1(z) \le C$$

as we claimed. □

Now, we are in position to prove the boundedness of $\mathcal{C}_{k,\varepsilon}$ in $\mathrm{Lip}(\gamma,\Gamma)$.

**Proposition 4.6.** *Let $\mathcal{C}_{k,\varepsilon}$ and $\Gamma$ as in Definition 4.2. If $\Gamma$ is a $C^{1,\gamma}$ curve, then the operators $\mathcal{C}_{k,\varepsilon}$, $k \in \mathbb{N}\cup\{0\}$, $0 \le |\varepsilon| < 1$, are bounded on Lip($\gamma,\Gamma$) for $0<\gamma<1$.*

*Proof.* Let $f \in \mathrm{Lip}(\gamma,\Gamma)$. We write

$$\begin{aligned}[\mathcal{C}_{k,\varepsilon}f](z) &= [\mathcal{C}_{k,\varepsilon}f](z) - f(z)[\mathcal{C}_{k,\varepsilon}1](z) + f(z)[\mathcal{C}_{k,\varepsilon}1](z) = \\ &= \int_\Gamma N_{k,\varepsilon}(z-w)(f(w)-f(z))\,\mathrm{d}w + f(z)[\mathcal{C}_{k,\varepsilon}1](z) =: \\ &=: T_{k,\varepsilon}f(z) + f(z)[\mathcal{C}_{k,\varepsilon}1](z).\end{aligned} \tag{4.3}$$

Note there is no principal value in $T_{k,\varepsilon}$ because the term $f(w)-f(z)$ makes it integrable.

By Lemma 4.4 the second term in the right hand side of (4.3) belongs to $\mathrm{Lip}(\gamma,\Gamma)$ and it is enough to check that $T_{k,\varepsilon}f \in \mathrm{Lip}(\gamma,\Gamma)$. Clearly,

$$|T_{k,\varepsilon}f(z)| \le \int_\Gamma \frac{|f(w)-f(z)|}{|w-z|}\,|\mathrm{d}w| \le |f|_\gamma \int_\Gamma \frac{1}{|w-z|^{1-\gamma}}\,|\mathrm{d}w| \le C\,|f|_\gamma\,.$$

Now, let $z_1, z_2 \in \Gamma$ and let $d := |z_1-z_2|$. By definition,

$$\begin{aligned}&T_{k,\varepsilon}f(z_1) - T_{k,\varepsilon}f(z_2) = \\ &= \int_\Gamma [(f(w)-f(z_1))N_{k,\varepsilon}(z_1-w) - (f(w)-f(z_2))N_{k,\varepsilon}(z_2-w)]\,\mathrm{d}w.\end{aligned}$$

Hence, by taking absolute value, we get

$$\begin{aligned}&|T_{k,\varepsilon}f(z_1) - T_{k,\varepsilon}f(z_2)| = \\ &= \left|\int_{\Gamma\setminus D_1} [(f(w)-f(z_1))N_{k,\varepsilon}(z_1-w) - (f(w)-f(z_2))N_{k,\varepsilon}(z_2-w)]\,\mathrm{d}w\right| + \\ &+ \int_{\Gamma\setminus D_1} \frac{|f(w)-f(z_1)|}{|w-z_1|}\,|\mathrm{d}w| + \int_{\Gamma\setminus D_2} \frac{|f(w)-f(z_2)|}{|w-z_2|}\,|\mathrm{d}w| =: \mathrm{I}+\mathrm{II}+\mathrm{III},\end{aligned}$$

where $D_1 := D(z_1, 2d)$ and $D_2 := D(z_2, 3d)$.

The term II can be estimated by

$$\mathrm{II} \le |f|_\gamma \int_{\Gamma\cap D(z_1,2d)} \frac{|\mathrm{d}w|}{|w-z_1|^{1-\gamma}}. \tag{4.4}$$

Let $\sigma(\rho) = \text{length}(\Gamma \cap D(z,\rho))$. Recall that $\Gamma$ is Ahlfors, so $\sigma(\rho) \simeq \rho$. Hence

$$\mathrm{II} \leq C \int_0^{2d} \frac{\mathrm{d}\sigma(\rho)}{\rho^{1-\gamma}} = C \left|\frac{\sigma(\rho)}{\rho^{1-\gamma}}\right|_0^{2d} + C(1-\gamma)\int_0^{2d} \frac{\sigma(\rho)}{\rho^{2-\gamma}}\,\mathrm{d}\rho =$$
$$= Cd^\gamma + C\int_0^{2d} \rho^{\gamma-1}\,\mathrm{d}\rho = Cd^\gamma.$$

An estimate for III follows in similar fashion and so we have

$$\mathrm{III} \leq |f|_\gamma \int_{\Gamma\cap D(z_2,3d)} \frac{|\mathrm{d}w|}{|w-z_2|^{1-\gamma}} \leq Cd^\gamma.$$

Now, we focus on I. Adding and subtracting

$$(f(w)-f(z_2))N_{k,\varepsilon}(z_1-w)$$

we get

$$\begin{aligned} \mathrm{I} \leq & \left|\int_{|w-z_1|>2d} (f(z_2)-f(z_1))N_{k,\varepsilon}(z_1-w)\,\mathrm{d}w\right| + \\ & + \left|\int_{|w-z_1|>2d} (f(w)-f(z_2))\left[N_{k,\varepsilon}(z_1-w)-N_{k,\varepsilon}(z_2-w)\right]\mathrm{d}w\right|. \end{aligned} \tag{4.5}$$

By Lemma 4.5 the first term of (4.5) is bounded by $C\,|f(z_2)-f(z_1)|$. The second one is bounded by

$$C\,||f||_\gamma\,|z_1-z_2| \int_{|w-z_1|>2d} \frac{|\mathrm{d}w|}{|w-z|^{2-\gamma}} \leq Cd\int_{2d}^{\infty} \frac{\mathrm{d}\sigma(\rho)}{\rho^{2-\gamma}} =$$
$$= Cd\left|\frac{\sigma(\rho)}{\rho^{2-\gamma}}\right|_{2d}^{\infty} + Cd\int_{2d}^{\infty} \frac{\sigma(\rho)}{\rho^{3-\gamma}}\,\mathrm{d}\rho = Cd^\gamma.$$

Therefore, I, II and III are bounded by $Cd^\gamma = C\,|z_1-z_2|^\gamma$, which proves the proposition.

□

## 4.3 Local Theorem

### 4.3.1 Contour Dynamics Equation

We derive an equation for the boundary of the domain of the patch at any time. Such a derivation will be done formally and the equation will make completely sense whenever the existence theorem is proved. Our goal is to prove that the $C^{1,\gamma}$ regularity of the boundary is preserved.

First of all, we compute the distributional derivative with respect to $\bar{z}$ for the indicator function of a domain $\Omega$, $\chi_\Omega$. Let $\varphi \in C_0^\infty$ a test function. Then,

$$\langle\bar{\partial}\chi_\Omega,\varphi\rangle = -\langle\chi_\Omega,\bar{\partial}\varphi\rangle = -\int_{\mathbb{C}}\chi_\Omega(w)\bar{\partial}\varphi(w)\,\mathrm{d}A(w) = -\int_\Omega \bar{\partial}\varphi(w)\,\mathrm{d}A(w)$$
$$= -\frac{i}{2}\int_\Omega \bar{\partial}\varphi(w)\,\mathrm{d}w\wedge\mathrm{d}\bar{w} = \frac{i}{2}\int_{\partial\Omega}\varphi(w)\,\mathrm{d}w = \langle\frac{i}{2}\,\mathrm{d}w|_{\partial\Omega},\varphi\rangle,$$

where we have applied Stokes' theorem.

That is, $\bar{\partial}\chi_\Omega = \frac{i}{2}\,\mathrm{d}w|_{\partial\Omega}$. Note that we have used

$$\mathrm{d}A(w) = \mathrm{d}x\wedge\mathrm{d}y = \frac{1}{2}(\mathrm{d}w+\mathrm{d}\bar{w})\wedge\frac{1}{2i}(\mathrm{d}w-\mathrm{d}\bar{w}) =$$
$$= \frac{1}{4i}(-2\mathrm{d}z\wedge\mathrm{d}\bar{z}) = \frac{i}{2}\mathrm{d}z\wedge\mathrm{d}\bar{z}.$$

Similarly, we can compute $\partial\chi_\Omega$.

$$\langle\partial\chi_\Omega,\varphi\rangle = -\langle\chi_\Omega,\partial\varphi\rangle = -\int_{\mathbb{C}}\chi_\Omega(w)\partial\varphi(w)\,\mathrm{d}A(w) = -\int_\Omega \partial\varphi(w)\,\mathrm{d}A(w) =$$
$$= -\frac{i}{2}\int_\Omega \partial\varphi(w)\,\mathrm{d}w\wedge\mathrm{d}\bar{w} = -\frac{i}{2}\int_{\partial\Omega}\varphi(w)\,\mathrm{d}\overline{w} = -\langle\frac{i}{2}\,\mathrm{d}\overline{w}|_{\partial\Omega},\varphi\rangle.$$

Thus, $\partial\chi_\Omega = -\frac{i}{2}\,\mathrm{d}\bar{w}|_{\partial\Omega}$.

Consider the kernel $K_2$ as defined in (2.1)

$$K_2(z) = \frac{1}{\pi}\frac{(\bar{z}+\varepsilon z)^k}{(z+\varepsilon\bar{z})^{k+1}}.$$

Then, given $z\neq 0$ we have $K_2(z) = (\overline{\partial}-\varepsilon\partial)H(z)$ for

$$H(z) := \frac{1}{(1-\varepsilon^2)(k+1)}\frac{1}{\pi}\frac{(\bar{z}+\varepsilon z)^{k+1}}{(z+\varepsilon\bar{z})^{k+1}}.$$

We compute the velocity field by

$$v(z,t) = (K_2 * \rho(\cdot,t))(z) = \Big((\overline{\partial}-\varepsilon\partial)H * \rho(\cdot,t)\Big)(z) =$$
$$= \Big(H * (\overline{\partial}-\varepsilon\partial)\rho(\cdot,t)\Big)(z).$$

Now, since solutions are transported by trajectories, we know that for $\rho_0 = \chi_{\Omega_0}$, we have $\rho(\cdot,t) = \chi_{\Omega_t}$ for some domain $\Omega_t$. Let

$$z : [0,2\pi]\times\mathbb{R}^+ \longrightarrow \mathbb{C},$$
$$(\alpha,t) \longrightarrow z(\alpha,t),$$

such that $z(\cdot,t):[0,2\pi]\to\mathbb{C}$ is a parametrization of $\partial\Omega_t$, the boundary of $\Omega_t$. Then, we compute the velocity at time $t$ and at the point $z(\alpha,t)$ of $\partial\Omega_t$.

$$\begin{aligned}
v(z(\alpha,t),t) &= \left(H*(\overline{\partial}-\varepsilon\partial)\rho(\cdot,t)\right)(z(\alpha,t)) = \\
&= \int_{\mathbb{C}} H(z(\alpha,t)-z)(\overline{\partial}-\varepsilon\partial)\chi_{\Omega_t}(z)\,\mathrm{d}A(w) = \\
&= \frac{i}{2}\int_{\partial\Omega_t} H(z(\alpha,t)-z)\,\mathrm{d}z + \varepsilon\frac{i}{2}\int_{\partial\Omega_t} H(z(\alpha,t)-z)\,\mathrm{d}\bar{z} = \\
&= \frac{i}{2}\int_0^{2\pi} H(z(\alpha,t)-z(\alpha',t))z_\alpha(\alpha',t)\,\mathrm{d}\alpha' + \\
&+ \varepsilon\frac{i}{2}\int_0^{2\pi} H(z(\alpha,t)-z(\alpha',t))\overline{z_\alpha}(\alpha',t)\,\mathrm{d}\alpha' =: F(z(\cdot,t))(\alpha),
\end{aligned}$$

where $z_\alpha(\alpha',t)=(\frac{\mathrm{d}}{\mathrm{d}\alpha}z(\cdot,t))(\alpha')$. We have assumed that the parametrization is differentiable.

Considering $z(\alpha,\cdot)$ as the trajectory starting at $\alpha$, we have the ordinary differential equation

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}z(\alpha,t) &= v(z(\alpha,t),t) = F(z(\cdot,t))(\alpha), \\ z(\alpha,0) &= z_0(\alpha), \end{cases} \tag{4.6}$$

for the functional

$$\begin{aligned}
F(z)(\alpha) &= \frac{i}{2}\int_0^{2\pi} H(z(\alpha)-z(\alpha'))z_\alpha(\alpha')\,\mathrm{d}\alpha' + \\
&+ \varepsilon\frac{i}{2}\int_0^{2\pi} H(z(\alpha)-z(\alpha'))\overline{z_\alpha(\alpha')}\,\mathrm{d}\alpha' = \\
&= \frac{i}{2}[F_1(z)(\alpha)+\varepsilon F_2(z)(\alpha)].
\end{aligned} \tag{4.7}$$

We will check that a solution of (4.6) is a solution of the transport equation in the weak sense, i.e., satisfying the following.

**Definition 4.7.** *Let $\rho\in L^1\cap L^\infty$. We say $(\rho,v)$ is a weak solution of (4.1) if $v(\cdot,t)=K*\rho(\cdot,t)$ and for any $\varphi\in C^\infty(\mathbb{C}\times[0,T])$ with compact support,*

$$\begin{aligned}
&\int_{\mathbb{C}}\varphi(z,T)\rho(z,T)\,\mathrm{d}A(z) - \int_{\mathbb{C}}\varphi(z,0)\rho_0(z)\,\mathrm{d}A(z) = \\
&= \int_0^T\int_{\mathbb{C}}[\varphi_t(z,t)+\operatorname{div}(v(z,t)\varphi(z,t))]\rho(z,t)\,\mathrm{d}A(z)\,\mathrm{d}t
\end{aligned} \tag{4.8}$$

*holds.*

Before proving that a solution of the Contour Dynamics Equation defines a weak solution, we have to be sure of the existence and uniqueness of the trajectory maps $X(\cdot,t)$ defined by

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t}X(\alpha,t) = v(X(\alpha,t),t), \\ X(\alpha,0) = \alpha. \end{cases} \tag{4.9}$$

Firstly, we assume $\rho(\cdot,t) \in L^\infty(\mathbb{C})$ with compact support. Then since

$$\begin{cases} \partial v(\cdot,t) = \partial K * \rho(\cdot,t) \\ \bar{\partial} v(\cdot,t) = \bar{\partial} K * \rho(\cdot,t) \end{cases}$$

and both $\partial K$ and $\bar{\partial} K$ are even kernels defining a Calderón-Zygmund operator of convolution type, we clearly have $\partial v(\cdot,t), \bar{\partial} v(\cdot,t) \in \mathrm{BMO}(\mathbb{C})$. That is, $v(\cdot,t) \in$I(BMO). By [Str], $v(\cdot,t)$ belongs to the Zygmund class and hence, it satisfies a log-Lipschitz condition, which assures existence and uniqueness of solution of equation (4.9) (see [AL, Thm. 1.5.1], for instance).

Therefore, provided $\rho(\cdot,t) \in L^\infty(\mathbb{C})$ with compact support, the trajectory maps $X(\cdot,t)$ are well defined.

**Proposition 4.8.** *Assume that for* $0 < t < T^*$, *we have* $z(\cdot,t)$ *solution of* (4.6). *Then, the pair* $(v,\rho)$ *defined by*

$$\begin{aligned} v(z,t) &= \frac{i}{2}\int_0^{2\pi} H(z - z(\alpha',t)) z_\alpha(\alpha',t)\,\mathrm{d}\alpha' - \\ &\quad + \varepsilon\frac{i}{2}\int_0^{2\pi} H(z-z(\alpha',t))\overline{z_\alpha(\alpha',t)}\,\mathrm{d}\alpha', \\ \rho(z,t) &= \rho_0(X^{-1}(z,t)), \end{aligned} \tag{4.10}$$

*is a weak solution of* (4.1) *in the sense of* (4.8).

*Proof.* For simplicity we will consider $k = 0$ and $\varepsilon = 0$. We have

$$\begin{aligned} v(z,t) &= \frac{i}{2\pi}\int_0^{2\pi} \frac{\overline{z - z(\alpha',t)}}{z - z(\alpha',t)} z_\alpha(\alpha',t)\,\mathrm{d}\alpha' = \\ &= \frac{i}{2\pi}\int_{\Gamma_t} \frac{\overline{z-z'}}{z-z'}\,\mathrm{d}z', \end{aligned}$$

where $\Gamma_t$ is the curve defined by $z(\cdot,t)$. Thus,

$$\bar{\partial} v(z,t) = \frac{i}{2\pi}\int_{\Gamma_t}\frac{1}{z-z'}\,\mathrm{d}z' = \frac{i}{2\pi}2\pi i \chi_{D_t}(z) = -\chi_{D_t}(z),$$

by Cauchy Integral Formula, where $D_t$ is the interior of the curve $\Gamma_t$. Since $v$ is one half of the Cauchy operator applied to the density $\chi_\Omega$ we verify this computation is correct. □

### 4.3.2 Checking the hypothesis in Picard-Lindelöf

This section requires detailed, accurate and sometimes tedious computations that resemble to [MB, Section 8.3.2].

Once again we want to apply the Picard-Lindelöf's theorem 1.8. We need a functional (the one given by the CDE), a suitable function space and a subspace of it. Our choice is similar to the one in the previous chapters. We take $B = C^{1,\gamma}([0,2\pi];\mathbb{C})$, the space of functions whose derivatives are bounded and belong to the Hölder class with exponent $\gamma$. For the open subset, we consider the ones that are also bilipschitz, that is,

$$O_M = B \cap \left\{ f : [0,2\pi] \to \mathbb{C} : \ \exists M > 0 \text{ such that } \frac{1}{M} < \frac{|f(x) - f(y)|}{|x-y|} < M \right\}.$$

We need to check that $F : O_M \to B$ and it is locally Lipschitz continuous. First of all, we see that $F$ is well defined between $O_M$ and $B$. As we will explain later, due to the likelihood between the functionals $F_1$ and $F_2$ in (4.7) we can just check the hypothesis for one of them, in this case for $F_1$.

**Proposition 4.9.** *Let $F_1$ defined in (4.7). For $O_M$ defined in (4.7) we have that $F_1 : O_M \to C^{1,\gamma}([0,2\pi];\mathbb{C})$.*

*Proof.* First of all, we estimate $||F_1(z)||_{L^\infty}$.

$$\begin{aligned} ||F_1(z)||_{L^\infty} &= \sup_{\alpha\in[0,2\pi]} \left| \int_0^{2\pi} H(z(\alpha) - z(\alpha'))z_\alpha(\alpha')\, \mathrm{d}\alpha' \right| \leq \\ &\leq \sup_{\alpha\in[0,2\pi]} \int_0^{2\pi} \left|H(z(\alpha) - z(\alpha'))\right| \left|z_\alpha(\alpha')\right| \, \mathrm{d}\alpha' \leq \\ &\leq 2\pi c(k,\varepsilon)\, ||z_\alpha||_{L^\infty} < \infty, \end{aligned}$$

where we have used

$$|H(z)| = \frac{1}{\pi(1-\varepsilon^2)(k+1)} =: c(k,\varepsilon).$$

Secondly, we want to control the $L^\infty$ norm of the derivative. We begin with the case $k = 0$ and $\varepsilon = 0$ in order to understand better the procedure, but we will explain later on the differences when $k > 0$, $\varepsilon \neq 0$.

We have to differentiate with respect to $\alpha$ the functional $F_1$.

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\alpha} F_1(z)(\alpha) = \\ &= \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{\overline{z_\alpha(\alpha)}[z(\alpha) - z(\alpha')] - z_\alpha(\alpha)[\overline{z(\alpha)} - \overline{z(\alpha')}]}{(z(\alpha) - z(\alpha'))^2} z_\alpha(\alpha')\, \mathrm{d}\alpha' = \\ &=: \frac{1}{\pi}\text{p.v.} \int_0^{2\pi} L(z(\alpha), z(\alpha'))z_\alpha(\alpha')\, \mathrm{d}\alpha', \end{aligned} \tag{4.11}$$

where the identity defines $L(z(\alpha), z(\alpha'))$.

The integral can be expressed as the sum of four integrals which are not singular. In fact, we can write

$$\frac{\mathrm{d}}{\mathrm{d}\alpha} F_1(z)(\alpha) = \frac{1}{\pi} \sum_{i=1}^{4} \int_0^{2\pi} L_i(z(\alpha), z(\alpha')) z_\alpha(\alpha')\, \mathrm{d}\alpha'$$

where we have defined

$$\begin{aligned}
L_1(z(\alpha), z(\alpha')) &= \frac{\overline{z_\alpha(\alpha)}[z(\alpha) - z(\alpha') + z_\alpha(\alpha)(\alpha' - \alpha)]}{(z(\alpha) - z(\alpha'))^2}, \\
L_2(z(\alpha), z(\alpha')) &= \frac{[z_\alpha(\alpha) - z_\alpha(\alpha')][\overline{z(\alpha)} - \overline{z(\alpha')}]}{(z(\alpha) - z(\alpha'))^2}, \\
L_3(z(\alpha) z(\alpha')) &= \frac{z_\alpha(\alpha')[\overline{z(\alpha)} - \overline{z(\alpha')} - \overline{z_\alpha(\alpha')}(\alpha - \alpha')]}{(z(\alpha) - z(\alpha'))^2}, \\
L_4(z(\alpha), z(\alpha')) &= \frac{(|z_\alpha(\alpha')| + |z_\alpha(\alpha)|)(|z_\alpha(\alpha')| - |z_\alpha(\alpha)|)(\alpha' - \alpha)}{(z(\alpha) - z(\alpha'))^2}.
\end{aligned} \tag{4.12}$$

Note that, since $z \in O_M$ we have $|L_i(z(\alpha), z(\alpha'))| \leq \frac{C_i}{|\alpha - \alpha'|^{1-\gamma}}$. Therefore, the kernels are summable and we easily have

$$\left|\left| \frac{\mathrm{d}}{\mathrm{d}\alpha} F_1(z) \right|\right|_{L^\infty} \leq c \, ||z_\alpha||_{L^\infty} < \infty. \tag{4.13}$$

Finally, in order to see that the norm $\left| \frac{\mathrm{d}}{\mathrm{d}\alpha} F(z) \right|_\gamma$ is bounded we return to the original expression for $\frac{\mathrm{d}}{\mathrm{d}\alpha} F$.

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\alpha} F_1(z)(\alpha) = \frac{\overline{z_\alpha(\alpha)}}{\pi} \text{p.v.} \int_0^{2\pi} \frac{z_\alpha(\alpha')}{z(\alpha) - z(\alpha')} \, \mathrm{d}\alpha' - \\
- \frac{z_\alpha(\alpha)}{\pi} \text{p.v.} \int_0^{2\pi} \frac{\overline{z(\alpha)} - \overline{z(\alpha')}}{(z(\alpha) - z(\alpha'))^2} z_\alpha(\alpha') \, \mathrm{d}\alpha' =: F_{\alpha,1}(\alpha) - F_{\alpha,2}(\alpha),
\end{aligned}$$

and treat each integral separately.

$$\begin{aligned}
F_{\alpha,1}(\alpha) &= \frac{\overline{z_\alpha(\alpha)}}{\pi} \int_0^{2\pi} \frac{z_\alpha(\alpha')}{z(\alpha) - z(\alpha')} \, \mathrm{d}\alpha' = \\
&= \frac{\overline{z_\alpha(\alpha)}}{\pi} \int_\Gamma \frac{\mathrm{d}z}{z(\alpha) - z} = \frac{\overline{z_\alpha(\alpha)}}{\pi} [\mathcal{C}_{0,0} 1](z(\alpha)),
\end{aligned} \tag{4.14}$$

where we have performed a change of variables $z = z(\alpha')$ and where $\mathcal{C}_{0,0}$ is as in Definition 4.2. Similarly,

$$F_{\alpha,2}(\alpha) = \frac{z_\alpha(\alpha)}{\pi}\int_0^{2\pi}\frac{\overline{z(\alpha)}-\overline{z(\alpha')}}{(z(\alpha)-z(\alpha'))^2}z_\alpha(\alpha')\,\mathrm{d}\alpha' = = \frac{z_\alpha(\alpha)}{\pi}\int_\Gamma\frac{\bar{z}-\overline{z(\alpha)}}{(z(\alpha)-z)^2}\,\mathrm{d}z = \frac{z_\alpha(\alpha)}{\pi}[\mathcal{C}_{1,0}1](z(\alpha)). \tag{4.15}$$

Then, applying Theorem 4.6 we have $[\mathcal{C}_{0,0}1](z(\alpha))$ and $[\mathcal{C}_{1,0}1](z(\alpha))$ belong to $\mathrm{Lip}(\gamma,\Gamma)$. Also, $z_\alpha, \overline{z_\alpha} \in \mathrm{Lip}(\gamma,\Gamma)$ and since this space is an algebra, we can conclude that both $F_{\alpha,1}$ and $F_{\alpha,2}$ belong to it. Hence, $F$ maps $O$ to $B$ whenever $k=0$ and $\varepsilon = 0$.

Consider now $\varepsilon \neq 0$ but $k=0$. Then, for $L$ as in (4.11), if we define

$$L^\varepsilon(z(\alpha),z(\alpha')) := L\left(z(\alpha)+\varepsilon\bar{z}(\alpha), z(\alpha')+\varepsilon\bar{z}(\alpha')\right),$$

it is easy to check that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1(z)(\alpha) = \frac{1}{\pi(1-\varepsilon^2)}\mathrm{p.v.}\int_0^{2\pi} L^\varepsilon(z(\alpha),z(\alpha'))z_\alpha(\alpha')\,\mathrm{d}\alpha'$$

and then we can repeat the arguments above for $\varepsilon = 0$. In fact we can write

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1(z)(\alpha) = \frac{1}{\pi(1-\varepsilon^2)}\sum_{i=1}^4\int_0^{2\pi} L_i^\varepsilon(z(\alpha),z(\alpha'))z_\alpha(\alpha')\,\mathrm{d}\alpha'$$

for

$$L_i^\varepsilon(z(\alpha),z(\alpha')) := L_i\left(z(\alpha)+\varepsilon\bar{z}(\alpha), z(\alpha')+\varepsilon\bar{z}(\alpha')\right),$$

and $L_i$ as in (4.12).

Also, one can see that

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1(z)(\alpha) = F^\varepsilon_{\alpha,1}(\alpha) + F^\varepsilon_{\alpha,2}(\alpha)$$

for

$$\begin{cases} F^\varepsilon_{\alpha,1}(\alpha) &= \frac{\overline{z_\alpha}(\alpha)+\varepsilon z_\alpha(\alpha)}{\pi(1-\varepsilon^2)}[\mathcal{C}_{0,\varepsilon}1](z(\alpha)), \\ F^\varepsilon_{\alpha,2}(\alpha) &= \frac{z_\alpha(\alpha)+\varepsilon\overline{z_\alpha}(\alpha)}{\pi(1-\varepsilon^2)}[\mathcal{C}_{1,\varepsilon}1](z(\alpha)). \end{cases}$$

Thus, similar arguments allow us to verify that $F: O_M \to B$ when $k=0$ and $\varepsilon \neq 0$.

Finally, if $k>0$ and $\varepsilon \neq 0$ we can write

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1(z)(\alpha) = \frac{1}{\pi(1-\varepsilon^2)}\mathrm{p.v.}\int_0^{2\pi}[P_0(z(\alpha),z(\alpha'))]^k L^\varepsilon(z(\alpha),z(\alpha'))z_\alpha(\alpha')\,\mathrm{d}\alpha'$$

for

$$P_0(z(\alpha),z(\alpha')) := \frac{\overline{z(\alpha)}+\varepsilon z(\alpha)-\overline{z(\alpha')}-\varepsilon z(\alpha')}{z(\alpha)+\varepsilon\overline{z(\alpha)}-z(\alpha')-\varepsilon\overline{z(\alpha')}}.$$

Also,

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1(z)(\alpha) = \frac{1}{\pi(1-\varepsilon^2)}\sum_{i=1}^{4}\int_0^{2\pi}[P_0(z(\alpha),z(\alpha'))]^k L_i^{\varepsilon}(z(\alpha),z(\alpha'))z_\alpha(\alpha')\,\mathrm{d}\alpha'$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1(z)(\alpha) = F^{k,\varepsilon}_{\alpha,1}(\alpha)+F^{k,\varepsilon}_{\alpha,2}(\alpha)$$

for

$$\begin{cases} F^{k,\varepsilon}_{\alpha,1}(\alpha) & = \frac{\overline{z_\alpha}(\alpha)+\varepsilon z_\alpha(\alpha)}{\pi(1-\varepsilon^2)}[\mathcal{C}_{k,\varepsilon}1](z(\alpha)),\\ F^{k,\varepsilon}_{\alpha,2}(\alpha) & = \frac{z_\alpha(\alpha)+\varepsilon\overline{z_\alpha}(\alpha)}{\pi(1-\varepsilon^2)}[\mathcal{C}_{k+1,\varepsilon}1](z(\alpha)).\end{cases}$$

So, for the reasons explained above for the case $k=0$ and $\varepsilon=0$ (and since the operator $\mathcal{C}_{k,\varepsilon}$ sends $\mathrm{Lip}(\gamma,\Gamma)$ into itself in general) we have that $F: O_M \to B$ generally. □

Before proving the second hypothesis in Picard-Lindelöf's theorem we need to compute the directional derivative of $F_1$.

**Lemma 4.10.** *Let $F_1 : O_M \to C^{1,\gamma}([0,2\pi];\mathbb{C})$ as defined in (4.7). Let $z \in O_M$ and $Y \in C^{1,\gamma}([0,2\pi];\mathbb{C})$. Let $Q:\mathbb{C}^2\to\mathbb{C}$ defined by*

$$Q(w_1,w_2) := \frac{\overline{w_2}w_1-\overline{w_1}w_2}{w_1^2}$$

*and*

$$c(k,\varepsilon) := \frac{1}{\pi(1-\varepsilon^2)(k+1)}.$$

*Then, we have $F_1'(z)w = I(\alpha)+II(\alpha)$ for*

$$\begin{aligned} I(\alpha) :&= c(k,\varepsilon)\int_0^{2\pi} Q(z_\varepsilon(\alpha)-z_\varepsilon(\alpha'),w_\varepsilon(\alpha)-w_\varepsilon(\alpha'))\,\mathrm{d}\alpha' \\ II(\alpha) :&= c(k,\varepsilon)\int_0^{2\pi} H(z(\alpha)-z(\alpha'))\,w_\alpha(\alpha')\,\mathrm{d}\alpha' \end{aligned} \tag{4.16}$$

*where*

$$w_\varepsilon(\alpha) = w(\alpha)+\varepsilon\overline{w(\alpha)} \text{ and } z_\varepsilon(\alpha) = z(\alpha)+\varepsilon\overline{z(\alpha)}.$$

*Proof.* Since

$$F_1(z)(\alpha) = \int_0^{2\pi} H(z(\alpha)-z(\alpha'))\,z_\alpha(\alpha')\,\mathrm{d}\alpha'$$

then

$$F_1'(z)(\alpha) = \frac{\mathrm{d}}{\mathrm{d}\eta}[F_1(z+\eta w)]|_{\eta=0}(\alpha) =$$
$$= \int_0^{2\pi} \frac{\mathrm{d}}{\mathrm{d}\eta}[H(z(\alpha)+\eta w(\alpha) - z(\alpha') - \eta w(\alpha'))]|_{\eta=0}\, z_\alpha(\alpha')\,\mathrm{d}\alpha' +$$
$$+ \int_0^{2\pi} H(z(\alpha)-z(\alpha'))\frac{\mathrm{d}}{\mathrm{d}\eta}(z_\alpha(\alpha') + \eta w_\alpha(\alpha'))|_{\eta=0}\,\mathrm{d}\alpha' = I(\alpha) + II(\alpha).$$

The second integral in the expression above corresponds clearly with

$$II(\alpha) = \int_0^{2\pi} H(z(\alpha)-z(\alpha'))w_\alpha(\alpha')\,\mathrm{d}\alpha'.$$

To compute $I(\alpha)$ recall the expression for the kernel $H$,

$$H(z) = \frac{1}{\pi(1-\varepsilon^2)(k+1)}\left(\frac{\overline{z}+\varepsilon z}{z+\varepsilon\overline{z}}\right)^{k+1},$$

and after some computations we can see that $I(\alpha)$ can be written as in (4.16). □

Then, we need to check that $F_1$ is locally Lipschitz continuous.

**Proposition 4.11.** *Given $k = 0$ and $\varepsilon = 0$, let $F_1 : O_M \to C^{1,\gamma}([0,2\pi];\mathbb{C})$ defined in (4.7). Then $F_1$ is locally Lipschitz continuous.*

*Proof.* It can be seen that it is enough to prove that the directional derivative of $F_1$ is bounded as a linear operator. Details for this simplification can be found in Proposition 1.18.

For $k = 0$, $\varepsilon = 0$ the directional derivative computed in Lemma 4.16 can be simply written as $F_1'(z)w = I(\alpha) + II(\alpha)$ for

$$I(\alpha) = \frac{1}{\pi}\int_0^{2\pi} S(\alpha,\alpha')z_\alpha(\alpha')\,\mathrm{d}\alpha',\; II(\alpha) = \frac{1}{\pi}\int_0^{2\pi} \frac{\overline{z(\alpha)}-\overline{z(\alpha')}}{z(\alpha)-z(\alpha')}w_\alpha(\alpha')\,\mathrm{d}\alpha',$$

where

$$S(\alpha,\alpha') = \frac{(\overline{w(\alpha)}-\overline{w(\alpha')})(z(\alpha)-z(\alpha')) - (\overline{z(\alpha)}-\overline{z(\alpha')})(w(\alpha)-w(\alpha'))}{(z(\alpha)-z(\alpha'))^2}.$$

Note that we can write

$$\mathrm{I}(\alpha) = \frac{1}{\pi}\int_0^{2\pi} K_{\mathrm{I}}(\alpha,\alpha')z_\alpha(\alpha')\,\mathrm{d}\alpha', \qquad \mathrm{II}(\alpha) = \frac{1}{\pi}\int_0^{2\pi} K_{\mathrm{II}}(\alpha,\alpha')w_\alpha(\alpha')\,\mathrm{d}\alpha',$$

with $|K_j(\alpha,\alpha')| \leq M^4$, $j \in \{\mathrm{I},\mathrm{II}\}$ (because $z$ and $w$ are bilipschitz, and we let $M$ the bigger of the constants). Recall that $z_\alpha, w_\alpha \in L^\infty$ (because they belong to $C^{1,\gamma}$). Hence,

$$||\mathrm{I}||_{L^\infty} \leq \pi M^4 \, ||z_\alpha||_{L^\infty} < \infty;$$
$$||\mathrm{II}||_{L^\infty} \leq \pi M^4 \, ||w_\alpha||_{L^\infty} < \infty.$$

We compute now the derivative with respect to $\alpha$ of $F_1'(z)w$. With the same notation as before, $\frac{\mathrm{d}}{\mathrm{d}\alpha}F_1'(z)w = \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{I} + \frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{II}$. Explicitly for the first one,

$$(4.17)\qquad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{I} = \mathrm{I}_1 - \mathrm{I}_2 - \mathrm{I}_3 - \mathrm{I}_4 - \mathrm{I}_5; \quad \text{where} \\ &\mathrm{I}_1 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{\overline{w_\alpha(\alpha)}}{z(\alpha)-z(\alpha')} z_\alpha(\alpha')\,\mathrm{d}\alpha', \\ &\mathrm{I}_2 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{z_\alpha(\alpha)\big(\overline{w(\alpha)}-\overline{w(\alpha')}\big)}{(z(\alpha)-z(\alpha'))^2} z_\alpha(\alpha')\,\mathrm{d}\alpha', \\ &\mathrm{I}_3 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{\overline{z_\alpha(\alpha)}\big(w(\alpha)-w(\alpha')\big)}{(z(\alpha)-z(\alpha'))^2} z_\alpha(\alpha')\,\mathrm{d}\alpha', \\ &\mathrm{I}_4 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{w_\alpha(\alpha)\big(\overline{z(\alpha)}-\overline{z(\alpha')}\big)}{(z(\alpha)-z(\alpha'))^2} z_\alpha(\alpha')\,\mathrm{d}\alpha', \\ &\mathrm{I}_5 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{2z_\alpha(\alpha)\big(\overline{z(\alpha)}-\overline{z(\alpha')}\big)\big(w(\alpha)-w(\alpha')\big)}{(z(\alpha)-z(\alpha'))^3} z_\alpha(\alpha')\,\mathrm{d}\alpha'. \end{aligned}$$

and for the second term

$$(4.18)\qquad \begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{II} = \mathrm{II}_1 + \mathrm{II}_2; \\ &\mathrm{II}_1 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{\overline{z_\alpha(\alpha)}}{z(\alpha)-z(\alpha')} w_\alpha(\alpha')\,\mathrm{d}\alpha', \\ &\mathrm{II}_2 = \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{z_\alpha(\alpha)\big(\overline{z(\alpha)}-\overline{z(\alpha')}\big)}{(z(\alpha)-z(\alpha'))^2} w_\alpha(\alpha')\,\mathrm{d}\alpha'. \end{aligned}$$

In order to see that the seven integrals in equations (4.17) and (4.18) are bounded in $\mathrm{Lip}(\gamma,\Gamma)$ we want to use Theorem 4.6, so we need to write them in a convenient way. For instance,

$$\begin{aligned} \mathrm{I}_1 &= \frac{1}{\pi}\text{p.v.}\int_0^{2\pi} \frac{\overline{w_\alpha(\alpha)}}{z(\alpha)-z(\alpha')} z_\alpha(\alpha')\,\mathrm{d}\alpha' = \\ &= \frac{\overline{w_\alpha(\alpha)}}{\pi}\text{p.v.}\int_\Gamma \frac{\mathrm{d}z}{z(\alpha)-z} = \overline{w_\alpha(\alpha)}[\mathcal{C}_{0,0}1](z(\alpha)). \end{aligned}$$

Thus, $\mathrm{I}_1 \in \mathrm{Lip}(\gamma,\Gamma)$ since $w \in C^{1,\gamma}$ and $\mathcal{C}_{0,0}1 \in C^{1,\gamma}$ (by Theorem 4.6). Also we have used that $\mathrm{Lip}(\gamma,\Gamma)$ is an algebra.

To verify that $\mathrm{I}_2$ belongs to $\mathrm{Lip}(\gamma,\Gamma)$ we have to proceed more carefully. If we undo the change of variables corresponding to the parametrization in $\mathrm{I}_2$, that is, for $z(\alpha')=z'$, we have

$$\begin{aligned}\mathrm{I}_2 &= \frac{1}{\pi}\mathrm{p.v.}\int_0^{2\pi}\frac{z_\alpha(\alpha)(\overline{w(\alpha)}-\overline{w(\alpha')})}{(z(\alpha)-z(\alpha'))^2}z_\alpha(\alpha')\,\mathrm{d}\alpha' = \\ &= \frac{z_\alpha(\alpha)}{\pi}\mathrm{p.v.}\int_\Gamma \frac{\overline{w\circ z^{-1}(z(\alpha))}-\overline{w\circ z^{-1}(z')}}{(z(\alpha)-z')^2}\,\mathrm{d}z'.\end{aligned}$$

Since $w\in B$ and $z\in O$, the composition $g:=w\circ z^{-1}$ is also in $B$. By Whitney's Extension theorem, there exists an extension of $g$, called G, with $G\in C^{1,\gamma}(\Omega^*,\mathbb{C})$, where $\Omega^*$ is an open neighborhood of $\Gamma$. Hence, the complex Taylor expansion for $G(z(\alpha))$ centered at $z'$ is

$$\begin{aligned}&G(z(\alpha)) = \\ &= G(z')+\partial G(z')(z(\alpha)-z')+\bar\partial G(z')(\overline{z(\alpha)-z'})+R(z(\alpha),z'),\end{aligned} \tag{4.19}$$

where $R(z(\alpha),z')$ is the remainder. Since $G\in C^{1,\gamma}$, it satisfies

$$\left|R(z(\alpha),z')\right|\le C\left|z(\alpha)-z'\right|^{1+\gamma}.$$

Hence, we have,

$$\begin{aligned}\mathrm{I}_2 &= \frac{z_\alpha(\alpha)}{\pi}\times \\ &\times \mathrm{p.v.}\int_\Gamma \frac{\overline{\partial G(z')(z(\alpha)-z')}+\overline{\bar\partial G(z')(\overline{z(\alpha)-z'})}+\overline{R(z(\alpha),z')}}{(z(\alpha)-z')^2}\,\mathrm{d}z' = \\ &= \frac{z_\alpha(\alpha)}{\pi}\left([\mathcal{C}_{1,0}\bar\partial G](z(\alpha))+[\mathcal{C}_{0,0}\partial G](z(\alpha))+J(z(\alpha))\right),\end{aligned} \tag{4.20}$$

where $J(z(\alpha))=\int_\Gamma \hat R(z(\alpha),z')\,\mathrm{d}z'$ and

$$\begin{aligned}\hat R(z_1,z_2) &= \frac{\overline{R(z_1,z_2)}}{(z_1-z_2)^2} = \\ &= \frac{\overline{G(z_1)}-\overline{G(z_2)}-\overline{\partial G}(z_2)(\overline{z_1}-\overline{z_2})-\partial\overline{G}(z_2)(z_1-z_2)}{(z_1-z_2)^2}.\end{aligned}$$

The derivatives with respect to the first variable of the kernel $\hat{R}$ are

$$\partial_{z_1}\hat{R}(z_1,z_2) = \frac{\partial\overline{G}(z_1)-\partial\overline{G}(z_2)}{(z_1-z_2)^2} - \\ -2\frac{\overline{G(z_1)}-\overline{G(z_2)}-\overline{\partial G}(z_2)(\overline{z_1}-\overline{z_2})-\partial\overline{G}(z_2)(z_1-z_2)}{(z_1-z_2)^3},$$

$$\bar{\partial}_{z_1}\hat{R}(z_1,z_2) = \frac{\overline{\partial G}(z_1)-\overline{\partial G}(z_2)}{(z_1-z_2)^2}.$$

Thus, we have the easy bounds,

$$\left|\hat{R}(z_1,z_2)\right| \le \frac{C\,|G|_{1+\gamma}}{|z_1-z_2|^{1-\gamma}}, \qquad \left|\partial_{z_1}\hat{R}(z_1,z_2)\right|,\left|\bar{\partial}_{z_1}\hat{R}(z_1,z_2)\right| \le \frac{C\,|G|_{1+\gamma}}{|z_1-z_2|^{2-\gamma}}.$$

Now, given $w_1, w_2 \in \Gamma$, let $d = |w_1 - w_2|$ and let $D_1 = \Gamma \cap D(w_1, 3d)$. Then

$$J(w_1) - J(w_2) = \int_\Gamma \hat{R}(w_1,z') - \hat{R}(w_2,z')\,\mathrm{d}z' = \\ = \int_{D_1} \hat{R}(w_1,z')\,\mathrm{d}z' - \int_{D_1} \hat{R}(w_2,z')\,\mathrm{d}z' + \int_{\Gamma\setminus D_1} \left(\hat{R}(w_1,z') - \hat{R}(w_2,z')\right)\,\mathrm{d}z'.$$

By taking absolute value of the previous equation, and by triangular inequality, we have

$$|J(w_1)-J(w_2)| \le C_1 \left[\int_{D_1} \frac{\mathrm{d}z'}{|w_1-z'|^{1-\gamma}} + \int_{D_1} \frac{\mathrm{d}z'}{|w_2-z'|^{1-\gamma}} + \right. \\ \left. + \int_{\Gamma\setminus D_1} \left|\hat{R}(w_1,z')-\hat{R}(w_2,z')\right|\,\mathrm{d}z'\right]. \tag{4.21}$$

The first two integrals in the right hand side of (4.21) can be estimated by $C_2\,|w_1-w_2|^\gamma$ (note that our domain of integration is close to the points $w_1, w_2$ along the curve). For the third integral, we can apply the mean-value theorem and obtain

$$\int_{\Gamma\setminus D_1} \left|\hat{R}(w_1,z')-\hat{R}(w_2,z')\right|\,\mathrm{d}z' \le \int_{\Gamma\setminus D_1} \left|\nabla\hat{R}(\xi,z')\right|\,\mathrm{d}z'\,|w_1-w_2| \le \\ \le C\,|w_1-w_2| \int_{\Gamma\setminus D_1} \frac{\mathrm{d}z'}{|w_1-z'|^{2-\gamma}} \le C\,|w_1-w_2|\,d^{\gamma-1} = \\ = C\,|w_1-w_2|^\gamma.$$

Hence, $J \in C^{1,\gamma}$ and the other two terms in (4.20) also are bounded by Theorem 4.6.

In similar fashion with $\mathrm{I}_2$ we can write

$$\mathrm{I}_3 = \frac{\overline{z_\alpha(\alpha)}}{\pi}\left([\mathcal{C}_{0,0}\partial G](z(\alpha)) + [\mathcal{C}_{1,0}\bar{\partial}G](z(\alpha)) + \bar{J}(z(\alpha))\right),$$

where
$$\bar{J}(z(\alpha)) = \int_\Gamma \frac{R(z(\alpha),z')}{(z(\alpha)-z')^2}\,\mathrm{d}z'$$
and $R$ as in (4.19). For the same arguments above, $\mathrm{I}_3 \in \mathrm{Lip}(\gamma,\Gamma)$.

$$\begin{aligned}\mathrm{I}_4 &= \frac{1}{\pi}\int_0^{2\pi} \frac{w_\alpha(\alpha)(\overline{z(\alpha)}-\overline{z(\alpha')})}{(z(\alpha)-z(\alpha'))^2} z_\alpha(\alpha')\,\mathrm{d}\alpha' = \\ &= \frac{w_\alpha(\alpha)}{\pi}\int_\Gamma \frac{\overline{z(\alpha)}-\overline{z'}}{(z(\alpha)-z')^2}\,\mathrm{d}z' = \frac{w_\alpha(\alpha)}{\pi}[\mathcal{C}_{1,0}1](z(\alpha)),\end{aligned}$$

so, $\mathrm{I}_4 \in \mathrm{Lip}(\gamma,\Gamma)$ since $w \in C^{1,\gamma}$. Similarly to $\mathrm{I}_2$, we can write

$$\begin{aligned}\mathrm{I}_5 &= \frac{2z_\alpha(\alpha)}{\pi}\mathrm{p.v.}\int_\Gamma \frac{(\overline{z(\alpha)}-\overline{z'})(G(z(\alpha))-G(z'))}{(z(\alpha)-z')^3}\,\mathrm{d}z' = \\ &= \frac{2z_\alpha(\alpha)}{\pi}\left([\mathcal{C}_{1,0}\partial G](z(\alpha)) + [\mathcal{C}_2\bar{\partial} G](z(\alpha)) + \tilde{J}(z(\alpha))\right),\end{aligned}$$

where
$$\tilde{J}(z(\alpha)) = \int_\Gamma \frac{R(z(\alpha),z')(\overline{z(\alpha)}-\overline{z'})}{(z(\alpha)-z')^3}\,\mathrm{d}z',$$
and $R$ is defined as in (4.19). As it is done for $J$ and $\bar{J}$, we can see $\tilde{J} \in \mathrm{Lip}(\gamma,\Gamma)$ as before. Therefore, once again by Theorem 4.6, we can conclude that $\mathrm{I}_5 \in \mathrm{Lip}(\gamma,\Gamma)$.

For the terms related with $\frac{\mathrm{d}}{\mathrm{d}\alpha}\mathrm{II}$, we get

$$\begin{aligned}\mathrm{II}_1 &= \frac{1}{\pi}\,\mathrm{p.v.}\int_0^{2\pi} \frac{\overline{z_\alpha(\alpha)}}{z(\alpha)-z(\alpha')} w_\alpha(\alpha')\,\mathrm{d}\alpha' = \\ &= \frac{\overline{z_\alpha(\alpha)}}{\pi}\,\mathrm{p.v.}\int_0^{2\pi} \frac{w_\alpha(\alpha')}{z_\alpha(\alpha')}\frac{z_\alpha(\alpha')\,\mathrm{d}\alpha'}{z(\alpha)-z(\alpha')} = \\ &= \frac{\overline{z_\alpha(\alpha)}}{\pi}\,\mathrm{p.v.}\int_\Gamma \frac{w_\alpha(z^{-1}(z'))}{z_\alpha(z^{-1}(z'))}\frac{\mathrm{d}z}{z(\alpha)-z'} = \frac{\overline{z_\alpha(\alpha)}}{\pi}\left[\mathcal{C}_{0,0}\frac{w_\alpha\circ z^{-1}}{z_\alpha\circ z^{-1}}\right](z(\alpha)).\end{aligned}$$

Since $z \in \mathcal{O}$ and $w \in \mathcal{B}$ we have that the function $\frac{w_\alpha\circ z^{-1}}{z_\alpha\circ z^{-1}}$ belongs to $\mathrm{Lip}(\gamma,\Gamma)$ and hence, by Theorem 4.6, $\mathrm{II}_1 \in \mathrm{Lip}(\gamma,\Gamma)$. Analogously

$$\begin{aligned}
\mathrm{II}_2 &= \frac{z_\alpha(\alpha)}{\pi}\,\mathrm{p.v.}\int_0^{2\pi} \frac{(\overline{z(\alpha)}-\overline{z(\alpha')})}{(z(\alpha)-z(\alpha'))^2} w_\alpha(\alpha')\,\mathrm{d}\alpha' = \\
&= \frac{z_\alpha(\alpha)}{\pi}\,\mathrm{p.v.}\int_0^{2\pi} \frac{(\overline{z(\alpha)}-\overline{z(\alpha')})}{(z(\alpha)-z(\alpha'))^2} \frac{w_\alpha(\alpha')}{z_\alpha(\alpha')} z_\alpha(\alpha')\,\mathrm{d}\alpha' = \\
&= \frac{z_\alpha(\alpha)}{\pi}\,\mathrm{p.v.}\int_\Gamma \frac{(\overline{z(\alpha)}-\overline{z'})}{(z(\alpha)-z')^2} \frac{w_\alpha(z^{-1}(z'))}{z_\alpha(z^{-1}(z'))}\mathrm{d}z' = \\
&= \frac{z_\alpha(\alpha)}{\pi}\left[\mathcal{C}_{1,0}\frac{w_\alpha \circ z^{-1}}{z_\alpha \circ z^{-1}}\right](z(\alpha)),
\end{aligned}$$

which belongs to $\mathrm{Lip}(\gamma, \Gamma)$ for the same reasons explained before.

Summing up, $F_1'(z)w$ is a bounded linear operator from $O$ to $B$, and hence $F$ is locally Lipschitz from $O$ to $B$.

□

**Remark 4.12.** *We have just written the proof of Proposition 4.11 for $k = 0$ and $\varepsilon = 0$ to keep the expressions simple. Nevertheless, as it can be seen in the proof of Proposition 4.9 this particular case contain all the information of the proof of the general case. In fact, the split of the integrals appearing in the proof remain valid and one just should work with the operators $\mathcal{C}_{k+i,\varepsilon}$ instead of $\mathcal{C}_{i,0}$ when one of them appears in the computations. But by Lemma 4.6 we know that all of them share the same regularity properties so the proofs will be valid in general, meaning that we have the next proposition.*

**Proposition 4.13.** *Let $F_1 : O_M \to C^{1,\gamma}([0, 2\pi]; \mathbb{C})$ defined in (4.7). Then $F_1$ is locally Lipschitz continuous.*

### 4.3.3 The local theorem

As in the previous chapters, once we have verified that the hypothesis are satisfied, we can apply Picard-Lindelöf's theorem to prove existence and uniqueness of solution

**Theorem 4.14.** *Let*

$$H(z) := \frac{1}{(1-\varepsilon^2)(k+1)}\frac{1}{\pi}\frac{(\bar{z}+\varepsilon z)^{k+1}}{(z+\varepsilon\bar{z})^{k+1}}.$$

*Let $z_0 \in C^{1,\gamma}([0, 2\pi]; \mathbb{C})$ a parametrization of $\partial D_0$, where $D_0$ is a simply connected domain in the complex plane. Then there exists $T^* > 0$ such that the ordinary differential equation*

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} z(\alpha, t) = F(z(\cdot, t))(\alpha), \\ z(\cdot, 0) = z_0, \end{cases}$$

*for*

$$\begin{aligned}F(z)(\alpha) = &\frac{i}{2}\int_0^{2\pi} H(z(\alpha)-z(\alpha'))z_\alpha(\alpha')\,\mathrm{d}\alpha' + \\ &+\varepsilon\frac{i}{2}\int_0^{2\pi} H(z(\alpha)-z(\alpha'))\overline{z_\alpha(\alpha')}\,\mathrm{d}\alpha' = \\ &=\frac{i}{2}[F_1(z)(\alpha)+\varepsilon F_2(z)(\alpha)],\end{aligned}$$

*has a unique solution* $z(\cdot,t)\in C^{1,\gamma}([0,2\pi];\mathbb{C})$ *for* $t\in(-T^*,T^*)$.

*Proof.* It is clear that it suffices to verify the hypothesis in Picard-Lindelöf's theorem for $F_1$. By linearity and by the likelihood between $F_1$ and $F_2$ they will also hold for $F$. By Propositions 4.9 and 4.13 the functional $F_1$ satisfies those hypothesis (Proposition 4.13 is proved for $k=0$, $\varepsilon=0$ in Proposition 4.11), but by Remark 4.12 it is enough to treat this particular case) and hence, by Picard-Lindelöf's theorem the statement holds. □

We can conclude by establishing the corresponding weak solution to (4.1).

**Theorem 4.15.** *Let* $K_2$ *as defined in* (2.1) *and let* $D_0$ *be a simply connected* $C^{1,\gamma}$ *domain in the complex plane. Then, there exists* $T^*>0$ *such that the transport equation*

$$\begin{cases}\rho_t + v\cdot\nabla\rho = 0,\\ v(\cdot,t) = K_2 * \rho(\cdot,t),\\ \rho(\cdot,0)=\chi_{D_0},\end{cases}$$

*has a unique weak solution in the sense of* (4.8) *such that* $\rho(\cdot,t)=\chi_{D_t}$ *for* $D_t$ *a simply connected* $C^{1,\gamma}$ *domain in the complex plane, for* $t\in[-T^*,T^*]$.

*Proof.* The result holds by setting $\rho(\cdot,t)=\chi_{D_t}$, where $D_t$ is the domain corresponding to the interior of the curve defined by $z(\cdot,t)$. □

**Remark 4.16.** *As we said in the introduction, since* $K_2$ *and* $K_1$ *are complex conjugate kernels it is obvious that all the intermediate results for* $K_2$ *can be also proved for* $K_1$. *Thus, Theorem 4.15 is valid for* $K_1$.

## 4.4 Globalness

In this section we will follow the scheme done in Chapter 3 (Sections 3.4 to 3.7), to prove that the local-in-time solution for the density patch given in Theorem 4.15 is in fact global. We will skip the arguments that are a repetition of the ones there but we will stress the new difficulties. For instance, in Chapter 3 we took advantage of the harmonicity of the the velocity field since it was obtained by the convolution of the density and a kernel involving partial derivatives of the fundamental solution of the laplacian. Thanks to this regularity we saw that the annoying *solitary term* vanishes at the boundary of the domain defining the patch. For the kernels $K_1$ or $K_2$ we lose the harmonicity but a recent result from A. V. Vasin allow us to deal with the *solitary term* in a more general situation, as we will explain.

### 4.4.1 The defining function $\Phi$

As we recently said, we need to control the disturbing term involving the blow up of second (spatial) derivatives of the velocity field when approaching the boundary of the domain. The next proposition will help us in this task.

**Proposition 4.17.** *Let $D$ be a $C^{1,\gamma}$ domain in $\mathbb{C}$. Let $T$ be a smooth homogeneous Calderón-Zygmund operator defined by the convolution with an even kernel. Then, there exists $C_0$ such that for each $z \in D$ we have*

$$|\partial T(\chi_D)(z)|, \left|\bar{\partial} T(\chi_D)(z)\right| \leq C_0[dist(x, \partial D)]^{\gamma-1}.$$

*Proof.* See [Va, Proposition 5.1] for the proof of a general version of the result. □

In the simpler case $k = 0$, that is, when we have the kernel

$$K(z) = \frac{1}{\pi}\frac{1}{z + \varepsilon\bar{z}} \tag{4.22}$$

we can proof the estimate in the previous proposition directly by making use of the next lemma.

**Lemma 4.18.** *Let $f \in C^\gamma(\Omega)$ such that*

$$(\bar{\partial} - \varepsilon\partial)f = 0 \quad in\ \Omega.$$

*Then, we have the following bounds*

$$|\partial f(z)|, \left|\bar{\partial} f(z)\right| \leq \frac{C}{d(z, \partial\Omega)^{1-\gamma}}.$$

*Proof.* If $\varepsilon = 0$, then $f$ is holomorphic in $\Omega$ and by Cauchy integral formula, we have, for $a \in \Omega$ and $0 < r < d(a, \partial\Omega)$,

$$\partial f(a) = \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z)}{(z-a)^2}\, dz.$$

Since $\int\limits_{\partial B(a,r)} \frac{1}{(z-a)^2}\, dz = 0$, we can write

$$\partial f(a) = \frac{1}{2\pi i}\int_{\partial B(a,r)} \frac{f(z) - f(a)}{(z-a)^2}\, dz.$$

Now taking absolute value and having into account $f \in C^\gamma(\Omega)$, we have

$$|\partial f(a)| = \frac{1}{2\pi}\int_{\partial B(a,r)} \frac{|f(z)-f(a)|}{|z-a|^2}\,|dz| \le \frac{||f||_\gamma}{2\pi}\int_{\partial B(a,r)} \frac{1}{|z-a|^{2-\gamma}}\,|dz| =$$
$$= \frac{||f||_\gamma}{2\pi}\int_0^{2\pi}\frac{1}{r^{2-\gamma}}\left|ire^{i\theta}\right| d\theta = \frac{||f||_\gamma}{2\pi}\int_0^{2\pi}\frac{1}{r^{1-\gamma}}\,d\theta = \frac{||f||_\gamma}{r^{1-\gamma}},$$

where we have set $z = a + re^{i\theta}$ in the change of variables. To finish just consider $r \to d(a, \partial\Omega)$.

If $\varepsilon \neq 0$, then for

$$s = \frac{1}{1-\varepsilon^2}z + \frac{\varepsilon}{1-\varepsilon^2}\bar{z}$$

we have $\partial_{\bar{s}} := \frac{\partial}{\partial \bar{s}} = 0$.

Then, as in the previous case

$$|\partial_s f(a)| := \left|\frac{\partial}{\partial s} f(a)\right| \le \frac{||f||_\gamma}{d(a,\partial\Omega)^{1-\gamma}}.$$

So, since derivatives with respect to $z$ i $\bar{z}$ are linear combination of derivatives with respect to $s$ and $\bar{s}$, we can conclude the proof and get

$$|\partial_z f(a)|, |\partial_{\bar{z}} f(a)| \le \frac{c\,||f||_\gamma}{d(a,\partial\Omega)^{1-\gamma}}.$$

□

If $k = 0$ then $v(\cdot, t) = K * \chi_{D_t}$ with $K$ as in (4.22), then the partial derivatives of $v$ can take the role of $f$ in Lemma 4.18.

Once this detail is solved, we can obtain the same PDE for a $C^{1,\gamma}$ defining function (see Definition 3.2 in Chapter 3) associated to $D_t$.

**Proposition 4.19.** *Given $\{D_t\}_{0\le t\le T^*}$ a family of $C^{1,\gamma}$ domains in $\mathbb{C}$. If we consider a velocity field $v(\cdot,t) = K_i * \chi_{D_t}$, $i = 1,2$ as defined in (2.1), then the solution $\Phi : \mathbb{C} \to \mathbb{R}$ of the linear non-homogeneous partial differential equation*

$$\frac{D\Phi}{Dt} = \operatorname{div}(v)\,\Phi \tag{4.23}$$

*is a $C^{1,\gamma}$-defining function for $D_t$ whose gradient is continuous.*

*Proof.* We can repeat the proof of Proposition 3.11 but with the next change. When trying to get a bound of type (3.21), we use Proposition 4.17 (or Lemma 4.18 if $k = 0$) since the kernels corresponding of derivatives of $K_i$ are even and hence derivatives of the velocity field are operators as $T$ in Proposition 4.17. Thus, second derivatives of $v$ should satisfy (3.21) even $v$ is not harmonic anymore. The rest of the steps in the proof of Proposition 3.11 will follow then. □

### 4.4.2 Commutators for $\partial\Phi$

Now, in order to control the spatial derivative of the defining function $\Phi$ we need to check that its material derivative is equal to: first, a commutator integrating on $D_t$ and second, a commutator on the boundary of $D_t$. For the first of them we have the following. Although $K_1$ is not included in the family of kernels in Chapter 3, the procedure in order to achieve this commutator expression is very similar to the one there. We write it in full detail.

**Proposition 4.20.** *For $K_1$ defined in (2.1), let $v = K_1 * \chi_D$. Then, for $\Phi$ a defining function of $D$ satisfying (4.23) we have that*

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi)(z) = p.v. \int_D \partial K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\,\mathrm{d}A(w) - \\ - \left[p.v. \int_D \bar\partial K_2(z-w)(\partial\Phi(w) - \partial\Phi(z))\,\mathrm{d}A(w)\right],\end{aligned} \tag{4.24}$$

*that is, the material derivative of $\partial\Phi$ is equal to the difference of two commutators.*

*Proof.* Equation (4.23) can be written in complex variable as

$$\Phi_t + v\partial\Phi + \bar v\bar\partial\Phi = \mathrm{div}(v)\Phi.$$

Applying the derivative $\partial$ to the previous expression and rearranging terms we get

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi) &= (\partial\Phi)_t + v\partial(\partial\Phi) + \bar v\bar\partial(\partial\Phi) = \\ &= \partial(\mathrm{div}(v))\Phi + \mathrm{div}(v)\partial\Phi - \partial v\partial\Phi - \partial\bar v\bar\partial\Phi = \\ &= \partial(\mathrm{div}(v))\Phi + \overline{\partial v}\partial\Phi - \partial\bar v\bar\partial\Phi,\end{aligned}$$

where we have used that $\mathrm{div}(v) = 2\Re(\partial v)$. By Proposition 4.17 and since $|\Phi(z)| \leq C\mathrm{dist}(z, \partial D)$ (because $\Phi$ is a defining function) we have

$$|\partial(\mathrm{div}(v))\Phi| \leq C\mathrm{dist}(z,\partial D)^{\gamma-1}\mathrm{dist}(z,\partial D) = C\mathrm{dist}(z,\partial D)^{\gamma} \to 0 \quad \text{if } z \to \partial D.$$

Then we can simply write

$$\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi) = \overline{\partial v}\partial\Phi - \partial\bar v\bar\partial\Phi, \tag{4.25}$$

By Lemma 2.1 we have

$$\overline{\partial v} = \overline{\mathrm{p.v.}\ \partial K_1 * \chi_D - \varepsilon^k\chi_D} = \mathrm{p.v.}\ \bar\partial K_2 * \chi_D - \varepsilon^k\chi_D, \tag{4.26}$$

$$\partial\bar v = \overline{\bar\partial v} = \overline{\mathrm{p.v.}\ \bar\partial K_1 * \chi_D} = \mathrm{p.v.}\ \partial K_2 * \chi_D, \tag{4.27}$$

because $\overline{K_2} = K_1$. Putting (4.26) and (4.27) into (4.25) we get

$$\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi) = (\text{p.v. } \bar{\partial}K_2 * \chi_D)\partial\Phi - \varepsilon^k\chi_D\partial\Phi - \text{p.v. } \partial K_2 * \chi_D\bar{\partial}\Phi. \tag{4.28}$$

Note that, also by Lemma 2.1 we can write

$$\begin{aligned}
-\varepsilon^k\chi_D\partial\Phi &= -\varepsilon^k\delta_0 * (\chi_D\partial\Phi) = \\
&= \bar{\partial}(K_2 * (\chi_D\partial\Phi)) - \text{p.v. } \bar{\partial}K_2 * (\chi_D\partial\Phi) = \\
&= \partial K_2 * (\chi_D\bar{\partial}\Phi) - \text{p.v. } \bar{\partial}K_2 * (\chi_D\partial\Phi) = \\
&= \text{p.v. } \partial K_2 * (\chi_D\bar{\partial}\Phi) - \text{p.v. } \bar{\partial}K_2 * (\chi_D\partial\Phi).
\end{aligned} \tag{4.29}$$

Inserting (4.29) into (4.28) we obtain

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi) = \text{p.v. } \partial K_2 * (\chi_D\bar{\partial}\Phi) - \text{p.v. } \partial K_2 * \chi_D\bar{\partial}\Phi - \\
- \left[\text{p.v. } \bar{\partial}K_2 * (\chi_D\partial\Phi) - (\text{p.v. } \bar{\partial}K_2 * \chi_D)\partial\Phi\right]
\end{aligned}$$

which is equal to (4.30). □

Furthermore, we can express the difference of commutators in (4.30) as a difference of commutators acting on the boundary of the domain.

**Proposition 4.21.** *For $K_1$ defined in (2.1), let $v = K_1 * \chi_D$. Then, for $\Phi$ a defining function of $D$ satisfying (4.23) we have that*

$$\begin{aligned}
\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi)(z) = &-\frac{i}{2}p.v. \int_{\partial D} K_2(z-w)(\partial\Phi(w) - \partial\Phi(z))\,\mathrm{d}w - \\
&-\frac{i}{2}p.v. \int_{\partial D} K_2(z-w)(\bar{\partial}\Phi(w) - \bar{\partial}\Phi(z))\,\mathrm{d}\bar{w},
\end{aligned} \tag{4.30}$$

*that is, the material derivative of $\partial\Phi$ is equal to the difference of two commutators on the boundary of $D$.*

*Proof.* First of all, assume $\Phi \in C^2$. If not check in the proof of Lemma 3.19 to see how to proceed. Then, since ($\partial$ is the differentiation with respect to $w$)

$$\begin{aligned}
\partial[K_2(z-w)(\bar{\partial}\Phi(w) - \bar{\partial}\Phi(z))] = &- \partial K_2(z-w)(\bar{\partial}\Phi(w) - \bar{\partial}\Phi(z)) + \\
&+ K_2(z-w)\partial\bar{\partial}\Phi(w)
\end{aligned}$$

we can write the first integral in the right hand side of (4.30) as

$$\begin{aligned}\text{p.v.} \int_D \partial K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}A(w) &= \\ &= \lim_{\delta\to 0}\int_{D\setminus D(z,\delta)} \partial K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}A(w) = \\ &= -\lim_{\delta\to 0}\int_{D\setminus D(z,\delta)} \partial[K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))]\, \mathrm{d}A(w) + \\ &+ \int_D K_2(z-w)\partial\bar\partial\Phi(w)\, \mathrm{d}A(w).\end{aligned}$$

If we apply Stokes' theorem we can write

$$\begin{aligned}\text{p.v.} \int_D \partial K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}A(w) &= \\ &= -\frac{i}{2}\int_{\partial D} K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}\bar w + \\ &+ \frac{i}{2}\lim_{\delta\to 0}\int_{\partial D(z,\delta)\cap D} K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}\bar w + \\ &+ \int_D K_2(z-w)\partial\bar\partial\Phi(w)\, \mathrm{d}A(w).\end{aligned}$$

By continuity of $\bar\partial\Phi$ we have

$$\int_{\partial D(z,\delta)\cap D} K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}\bar w \to 0 \quad \text{as} \quad \delta \to 0$$

and hence

$$\begin{aligned}\text{p.v.} \int_D \partial K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}A(w) &= \\ &= -\frac{i}{2}\int_{\partial D} K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}\bar w + \\ &+ K_2 * (\chi_D \partial\bar\partial\Phi)(z).\end{aligned} \tag{4.31}$$

Analogously for the second integral in the right hand side of (4.30)

$$\begin{aligned}\text{p.v.} \int_D \bar\partial K_2(z-w)(\partial\Phi(w) - \partial\Phi(z))\, \mathrm{d}A(w) &= \\ &= \frac{i}{2}\int_{\partial D} K_2(z-w)(\partial\Phi(w) - \partial\Phi(z))\, \mathrm{d}w + K_2 * (\chi_D \partial\bar\partial\Phi)(z).\end{aligned} \tag{4.32}$$

Finally, putting (4.31) and (4.32) into (4.30) we get

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}(\partial\Phi)(z) = &-\frac{i}{2}\int_{\partial D} K_2(z-w)(\partial\Phi(w) - \partial\Phi(z))\, \mathrm{d}w - \\ &-\frac{i}{2}\int_{\partial D} K_2(z-w)(\bar\partial\Phi(w) - \bar\partial\Phi(z))\, \mathrm{d}\bar w,\end{aligned}$$

which proves the Proposition. □

### 4.4.3 Global Theorem

As a consequence of being the material derivative of $\partial\Phi$ a commutator and making use of Whitney's Extension theorem (see Section 3.6 in Chapter 3 for more details) we get the same a priori estimates than in the previous chapter, but for $\partial\Phi$.

**Lemma 4.22.** *Let $v(\cdot,t) = K_1 * \chi_{D_t}$ and let $\Phi(\cdot,t)$ the defining function for $D_t$ determined by*

$$\frac{\mathrm{D}\Phi}{\mathrm{D}t} = \operatorname{div}(v)\,\Phi. \tag{4.33}$$

*Then, for $\|\partial\Phi(\cdot,t)\|_{L^\infty,\partial D_t} := \|\partial\Phi(\cdot,t)\chi_{\partial D_t}\|_{L^\infty}$ we have*

$$\|\partial\Phi(\cdot,t)\|_{L^\infty,\partial D_t} \leq \|\partial\Phi(\cdot,0)\|_{L^\infty,\partial D_t} \exp\left(4\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right) \tag{4.34}$$

$$|\partial\Phi(\cdot,t)|_{inf} \geq |\partial\Phi(\cdot,0)|_{inf} \exp\left(-2\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right), \tag{4.35}$$

*and*

$$|\partial\Phi(\cdot,t)|_{\gamma,\partial D_t} \leq |\partial\Phi(\cdot,0)|_{\gamma,\partial D_0} \exp\left(C\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right). \tag{4.36}$$

Besides, the well-known logarithmic inequality for the velocity fields of the form $v = K * \chi_D$ can be derived provided the derivatives of the kernel are even, smooth and produce a Calderón-Zygmund operator. This is the case for $K_1$ and $K_2$ in (2.1) and for this reason we have the following.

**Theorem 4.23.** *Let $D$ a domain in $\mathbb{C}$ with a $C^{1,\gamma}$-defining function $\Phi$. If $v = K_1 * \chi_D$, then for $R := m(D)^{1/2}$ we have*

$$||\nabla v||_{L^\infty} \leq \frac{c}{\gamma}\left(1 + \log^+\left(R\frac{|\partial\Phi|_{\gamma,\partial D}}{|\partial\Phi|_{\inf}}\right)\right).$$

*Proof.* See [BGLV, Theorem 6.2] and also check Section 3.2 for the adaptation done in order to get the precise quotient inside the logarithm in the estimate in Theorem 4.23. □

We prove now the global version of the main theorem of the chapter.

**Theorem 4.24.** *Let $K_1$ as defined in (2.1) and let $D_0$ be a simply connected $C^{1,\gamma}$ domain in the complex plane. Then the transport equation*

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot,t) = K_1 * \rho(\cdot,t), \\ \rho(\cdot,0) = \chi_{D_0}, \end{cases}$$

*has a unique weak solution in the sense of (4.8) such that $\rho(\cdot,t) = \chi_{D_t}$ for $D_t$ a simply connected $C^{1,\gamma}$ domain in the complex plane, for every time $t \in \mathbb{R}$.*

*Proof.* First of all, we check that the $L^\infty$ norm of the gradient of the velocity field is finite for every time.

Combining inequality (4.35), (4.36) and the logarithmic inequality in Theorem 4.23 we get (see the proof of Theorem 3.1 in Section 3.7 for details)

$$\|\nabla v(\cdot,t)\|_{L^\infty} \le C\exp(Ct). \tag{4.37}$$

Then, by (4.36) again, we can bound also the Hölder norm of the derivative of the defining function as

$$\|\partial\Phi(\cdot,t)\|_{\gamma,\partial D_t} \le C\exp(C\exp(Ct)).$$

We know that, for $z \in \partial D_t$, the complex number $\nabla\Phi(z,t)$ points in the direction of the normal vector to $\partial D_t$ at $z$ (see (3.18) and (3.19)). Then, for $n^\perp(z,t) = \nabla^\perp\Phi(z,t)$, a tangent vector to $\partial D_t$ at $z$ we have

$$\|n^\perp(\cdot,t)\|_{\gamma,\partial D_t} = \|\nabla^\perp\Phi(\cdot,t)\|_{\gamma,\partial D_t} = \frac{1}{2}\|\partial\Phi(\cdot,t)\|_{\gamma,\partial D_t} \le \frac{1}{2}C\exp(C\exp(Ct)).$$

So, the Hölder norm of the tangent vector $n^\perp(\cdot,t)$ is finite for every time $t \in \mathbb{R}$. Since $z_\alpha$ is comparable to $n^\perp$ we then have that $\|z_\alpha(\cdot,t)\|_{\gamma,\partial D_t}$ is also bounded by the double exponential term. On the other hand, since

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\alpha-\alpha'}{z(\alpha,t)-z(\alpha',t)}\right) = -\frac{(\alpha-\alpha')(v(z(\alpha,t),t)-v(z(\alpha',t),t))}{(z(\alpha,t)-z(\alpha',t))^2},$$

taking absolute value we get

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\left|\left(\frac{\alpha-\alpha'}{z(\alpha,t)-z(\alpha',t)}\right)\right| &= \left|\frac{(\alpha-\alpha')(v(z(\alpha,t),t)-v(z(\alpha',t),t))}{(z(\alpha,t)-z(\alpha',t))^2}\right| = \\ &= \left|\frac{v(z(\alpha,t),t)-v(z(\alpha',t),t)}{z(\alpha,t)-z(\alpha',t)}\right|\left|\frac{\alpha-\alpha'}{z(\alpha,t)-z(\alpha',t)}\right| \le \\ &\le \|\nabla v(\cdot,t)\|_{L^\infty}\left|\frac{\alpha-\alpha'}{z(\alpha,t)-z(\alpha',t)}\right|, \end{aligned} \tag{4.38}$$

by the Mean Value Theorem. By direct integration on (4.38) and using the exponential bound for $\|\nabla v(\cdot,t)\|_{L^\infty}$ in (4.37) once more, we have

$$\left|\frac{\alpha-\alpha'}{z(\alpha,t)-z(\alpha',t)}\right| \leq \exp\left(\int_0^t \|\nabla v(\cdot,s)\|_{L^\infty}\,\mathrm{d}s\right) \leq \exp(C\exp(Ct)),$$

and therefore the bilipschitz norm of $z(\cdot,t)$ is bounded for any time $t$. Hence by the Continuation Theorem 1.22 the solution $z(\alpha,t)$ in Theorem 4.14 is global in time, which proves the theorem. $\square$

# 5 Skeleton of a one-dimensional aggregation patch

## 5.1 Introduction

In the present chapter, we want to study the limit behavior of aggregation patches, that is, we want to take the limit as $t \to \infty$ of the solution $\rho(\cdot, t) = \chi_{D_t}$ of the transport equation

$$\begin{cases} \rho_t + v \cdot \nabla \rho = 0, \\ v(\cdot, t) = -\nabla N * \rho(\cdot, t), \\ \rho(\cdot, 0) = \rho_0 = \chi_{D_0}, \end{cases} \tag{5.1}$$

where $N$ is the fundamental solution of the laplacian. In [BLL, Section 4.2] –after a change of variables in the time coordinate and by a renormalization of the unknowns $\rho$ and $v$– it was shown that the aggregation equation in its divergence form was equivalent to the transport equation (5.1). In order to show this equivalence it was completely necessary that the kernel was exactly $-\nabla N$ (so the divergence of the velocity field was equal to $-\rho$) and that we were in the patch setting (so $\rho^2 = \rho$). Alternatively, the transport equation (5.1) and the continuity equation below are equivalent also if the velocity field is incompressible.

To achieve the equivalent partial differential equation in the aggregation case that we are concerned in, consider explicitly

$$s = 1 - \mathrm{e}^{-t}, \qquad \rho = (1-s)\tilde{\rho}, \qquad v = (1-s)\tilde{v}.$$

Then, equation (5.1) can be written as

$$\begin{cases} \tilde{\rho}_s + \operatorname{div}(\tilde{\rho}\tilde{v}) = 0, \\ \tilde{v}(\cdot, s) = -\nabla N * \tilde{\rho}(\cdot, s), \\ \rho(\cdot, 0) = \rho_0 = \chi_{\Omega_0}. \end{cases} \tag{5.2}$$

Since we have changed the time scale $t \to s(t) = 1 - \mathrm{e}^{-t}$ it is clear that the limit time $t \to \infty$ corresponds now to $s \to 1$, which is, of course, the time of the blow-up for the aggregation equation in its divergence form (see [BLL] for details).

We can compute an explicit solution to (5.2) along the flow map

$$\rho(X(\alpha,s),s) = \left(\frac{1}{\rho_0(\alpha)} - s\right)^{-1} = \frac{\rho_0(\alpha)}{1 - s\rho_0(\alpha)},$$

where $X(\cdot,s)$ defined, as usual, by the ODE

$$\begin{cases} \frac{d}{ds}X(\alpha,s) = v(X(\alpha,s),s), \\ X(\alpha,0) = \alpha. \end{cases}$$

As we said, we are interested in the case of the evolution of an aggregation patch, that is, when $\rho_0 = \chi_{\Omega_0}$ for some bounded domain $\Omega_0$. In this case we simply have

$$\rho(\cdot,s) = \frac{1}{1-s}\chi_{\Omega_s}$$

where $\Omega_s = X(\Omega_0, s)$. With this expression, it is clear that the blow-up occurs at time $s = 1$. It is well known that equation (5.1) preserves the $L^1$ norm of the scalar $\rho$, then

$$\|\rho(\cdot,s)\|_{L^1} = \frac{1}{1-s}|\Omega_s| = \|\rho_0\|_{L^1} = |\Omega_0|,$$

and therefore

$$|\Omega_s| = (1-s)|\Omega_0| \to 0 \quad \text{as} \quad s \to 1.$$

We want to study in detail the structure of the *skeleton*, that is, the blow-up domain

$$\Omega_1 = \lim_{t \to 1^-} X(\Omega_s, s).$$

We consider a toy model which corresponds to equation (5.2) but just for dimension 1. In this case, the fundamental solution of the laplacian is simply $N(x) = \frac{1}{2}|x|$, and so the kernel is

$$K(x) = -N'(x) = -\frac{1}{2}\text{sign}(x) = \begin{cases} 1/2 & \text{if } x < 0, \\ -1/2 & \text{if } x \geq 0. \end{cases}$$

The reason to reduce to this case is the following: as we will see later, the velocity $v(X(\alpha,t),t)$ is independent of $t$ and hence it can be computed at the original time as $v_0(\alpha) = (K * \chi_{D_0})(\alpha)$. So the particle trajectories are "straight lines" (as a function of $t$) in the sense that it can be written as

$$X(\alpha,t) = \alpha + v_0(\alpha)t.$$

In dimension 2 or bigger we lose this nice property which is fundamental to develop explicit computations around the behavior of the skeleton.

### 5.1.1 Outline of the chapter

This chapter has a simple structure: in Section 5.2 we prove the evolution of an open set towards a numerable collection of Dirac deltas and in Section 5.3 we consider the evolution of a compact set.

## 5.2 Open domain

As we said in the introduction, we are just considering the one-dimensional aggregation equation. Explicitly, we have

$$\begin{cases} \rho_t + (\rho v)_x = 0, \\ v(\cdot,t) = -N' * \rho(\cdot,t), \\ \rho(\cdot,0) = \rho_0, \end{cases} \tag{5.3}$$

where $\rho : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ and $v : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$, and $N$ is the fundamental solution of the Laplace operator, i.e. $N(x) = \frac{1}{2}|x|$, and therefore $N'(x) = \frac{1}{2}\text{sign}(x)$.

Then we have the following theorem, stating the evolution of a general open domain in $\mathbb{R}$ under equation (5.3).

**Theorem 5.1.** *Let $\Omega_0 = \cup_{i=1}^{\infty} I_i^0 \subset \mathbb{R}$ be an open domain satisfying the following:*

*1) the intervals $\{I_i^0\}_{i\in\mathbb{N}}$ are open and pairwise disjoint,*

*2) $|\Omega_0| = \sum_{i=1}^{\infty} |I_i^0| = 1$,*

*3) $\Omega_0 \subset [0,2]$ and $|\Omega_0 \cap [\delta, 2-\delta]| < 1$ $\forall \delta > 0$.*

*4) Given $x \in I_j$, $y \in I_k$ amb $j \neq k$, we have*

$$|I(x,y) \cap \Omega_0| < |x-y|,$$

*where $I(x,y)$ indicates the minimum interval containing $x$ and $y$.*

*Therefore,*

*a) $X(\Omega_0, 1) \subset [1/2, 3/2]$.*

*b) $X(\Omega_0, 1) = \cup_{i=1}^{\infty}\{x_i\}$.*

*c) If $\mathrm{d}\mu_t = \rho(\cdot,t)\,\mathrm{d}x$, then $\mu_t \to \mu_1 = \sum_{i=1}^{\infty} |I_i^0|\, \delta_{x_i}$.*

*Proof.* First of all, let us see that under these assumptions, there exists spatial derivative for the trajectory map at least for $\alpha \in \Omega_0$. Recall that, for any $0 \le t < 1$ the trajectory map $X(\cdot,t)$ is the unique homeomorphism solution to the ODE

$$\begin{cases} \frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}t} = v(X(\alpha,t),t), \\ X(\alpha,0) = \alpha. \end{cases}$$

Hence, differentiating with respect to $\alpha$ the previous equation, we obtain

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(\frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}t}\right) &= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left(v(X(\alpha,t),t)\right) = \\
&= \frac{\mathrm{d}}{\mathrm{d}\alpha}\left[\frac{1}{2}\int_{X(\alpha,t)}^{+\infty}\rho(y,t)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^{X(\alpha,t)}\rho(y,t)\,\mathrm{d}y\right] \\
&= -\rho(X(\alpha,t),t)\frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}\alpha} = -\frac{1}{1-t}\rho_0(\alpha)\frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}\alpha}.
\end{aligned} \tag{5.4}$$

Since Schwarz lemma holds for the trajectory map, we can state

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}\alpha}\right) = -\frac{1}{1-t}\rho_0(\alpha)\left(\frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}\alpha}\right).$$

Integrating and using that the homeomorphism $X(\cdot,0)$ is the identity map, we obtain the spatial derivative (for any $\alpha \in \Omega_0$).

$$\frac{\mathrm{d}X(\alpha,t)}{\mathrm{d}\alpha} = 1-t.$$

Secondly, we can prove that the velocity of a particle is constant along the trajectory, this is,

$$v(X(\alpha,t),t) = v(\alpha,0) =: v_0(\alpha). \tag{5.5}$$

This just requires a simple computation, involving a change of variable $y = X(\alpha',t)$.

$$\begin{aligned}
v(X(\alpha,t),t) &= \frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(X(\alpha,t)-y)\rho(y,t)\,\mathrm{d}y = \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(X(\alpha,t)-X(\alpha',t))\rho(X(\alpha',t),t)\frac{\mathrm{d}X(\alpha',t)}{\mathrm{d}\alpha}\,\mathrm{d}\alpha' = \\
&= \frac{1}{2}\int_{-\infty}^{+\infty}\operatorname{sign}(X(\alpha,t)-X(\alpha',t))\frac{1}{1-t}\rho_0(\alpha)\frac{\mathrm{d}X(\alpha',t)}{\mathrm{d}\alpha}\,\mathrm{d}\alpha' = \\
&= \frac{1}{2}\int_{\alpha'\in\Omega_0}\operatorname{sign}(X(\alpha,t)-X(\alpha',t))\frac{1}{1-t}\frac{\mathrm{d}X(\alpha',t)}{\mathrm{d}\alpha}\,\mathrm{d}\alpha' = \\
&= \frac{1}{2}\int_{\alpha'\in\Omega_0}\operatorname{sign}(X(\alpha,t)-X(\alpha',t))\frac{1}{1-t}(1-t)\,\mathrm{d}\alpha' = \\
&= \frac{1}{2}\int_{\alpha'\in\Omega_0}\operatorname{sign}(X(\alpha,t)-X(\alpha',t))\,\mathrm{d}\alpha' = \\
&= \frac{1}{2}\int_{\alpha'\in\Omega_0}\operatorname{sign}(\alpha-\alpha')\,\mathrm{d}\alpha' = v_0(\alpha);
\end{aligned}$$

where we have used that $X(\alpha,t) - X(\alpha',t)$ and $\alpha - \alpha'$ have the same sign since $X(\cdot,t)$ is a non-decreasing homeomorphism. By (5.5) it is clear that all particle trajectory maps are straight lines. Indeed, for $0 \le t < 1$, we have

$$X(\alpha,t) = \alpha + \int_0^t v(X(\alpha,s),s)\,\mathrm{d}s = \alpha + \int_0^t v_0(\alpha)\,\mathrm{d}s = \alpha + v_0(\alpha)t.$$

Now we can see that any $x \in [\alpha_i, \beta_i]$ has the same limit point

$$\lim_{t\to 1^-} X(x,t).$$

In fact,

$$\begin{aligned} v_0(x) &= \frac{1}{2}\int_x^\infty \rho_0(y)\,\mathrm{d}y - \frac{1}{2}\int_{-\infty}^x \rho_0(y)\,\mathrm{d}y = \\ &= \frac{1}{2}\left[\int_{\alpha_j}^\infty \rho_0(y)\,\mathrm{d}y - \int_{\alpha_j}^x \rho_0(y)\,\mathrm{d}y - \int_{-\infty}^{\alpha_j} \rho_0(y)\,\mathrm{d}y - \int_{\alpha_j}^x \rho_0(y)\,\mathrm{d}y\right] = \\ &= v_0(\alpha_j) - \int_{\alpha_j}^x \rho_0(y)\,\mathrm{d}y = v_0(\alpha_j) - (x-\alpha_j), \end{aligned} \tag{5.6}$$

where we have used the fact that $\rho_0 \equiv 1$ in $(\alpha_j, x)$. Hence, for any $x \in [\alpha_i, \beta_i]$, we have

$$X(x,t) = x + \left(v_0(\alpha_j) - (x-\alpha_j)\right) t \overset{t\to 1^-}{\longrightarrow} \alpha_j + v_0(\alpha_j), \tag{5.7}$$

which does not depend on the choice of $x$. From now on, we denote the limit point for each interval as $x_j := \alpha_j + v_0(\alpha_j)$. Finally, we have to see the convergence of the measure $\mu_t$ defined as $\mathrm{d}\mu_t = \rho(x,t)\,\mathrm{d}x$ towards $\mu_1 = \sum_{i=1}^\infty \left|I_i^0\right| \delta_{x_i}$.

In order to prove this, let $f$ be a continuous function on $\mathbb{R}$. Then, recall

$$\rho(x,t) = \sum_{i=1}^\infty \frac{1}{1-t}\chi_{(\alpha_{i,t},\beta_{i,t})},$$

where $\alpha_{i,t} = X(\alpha_i, t)$ and $\beta_{i,t} = X(\beta_i, t)$. Summing up, we have

$$\langle f, \mu_t\rangle = \frac{1}{1-t}\sum_{i=1}^\infty \int_{\alpha_{i,t}}^{\beta_{i,t}} f(x)\,\mathrm{d}x.$$

Let $m_{i,t} := \min_{x\in(\alpha_{i,t},\beta_{i,t})} f(x)$ and $M_{i,t} := \max_{x\in(\alpha_{i,t},\beta_{i,t})} f(x)$. Then, it is clear that for any $i$,

$$\frac{1}{1-t} m_{i,t}(\beta_{i,t}-\alpha_{i,t}) \le \frac{1}{1-t}\int_{\alpha_{i,t}}^{\beta_{i,t}} f(x)\,\mathrm{d}x \le \frac{1}{1-t} M_{i,t}(\beta_{i,t}-\alpha_{i,t}).$$

On the other hand, from (5.7) we can see that

$$\beta_{i,t} - \alpha_{i,t} = (1-t)(\beta_i - \alpha_i)$$

and hence

$$m_{i,t}(\beta_i - \alpha_i) \leq \frac{1}{1-t} \int_{\alpha_{i,t}}^{\beta_{i,t}} f(x)\,\mathrm{d}x \leq M_{i,t}(\beta_i - \alpha_i).$$

Both the left and the right hand sides of the previous inequality clearly tend to the same value $f(x_i)(\beta_i - \alpha_i)$, by definition of $m_{i,t}$ and $M_{i,t}$. Therefore, by Sandwich rule we have

$$\frac{1}{1-t} \int_{\alpha_{i,t}}^{\beta_{i,t}} f(x)\,\mathrm{d}x \longrightarrow f(s_i)(\beta_i - \alpha_i),$$

as $t \to 1^-$. Then

$$\langle f, \mu_t \rangle \longrightarrow \langle f, \sum_{i=1}^{\infty} |I_i|\, \delta_{x_i} \rangle,$$

which proves the result.

□

We can also formulate the converse theorem.

**Theorem 5.2.** *Let $\{x_j\}_{j=1}^{\infty}$ a numerable collection of separate points such that*

$$\cup_{j=1}^{\infty} \{x_j\} \subsetneq \left[\frac{1}{2} + \varepsilon, \frac{3}{2} - \varepsilon\right].$$

*Let $\mu_1 = \sum_{j=1}^{\infty} c_j \delta_{x_j}$, for $c_j > 0$ and such that $\sum_{j=1}^{\infty} c_j = 1$, where $\delta_{x_j}$ is the Dirac delta centered at $x_j$. Then there exists an open and bounded set $\Omega_0 \subseteq [0, 2]$ such that*

*i)* $X(\Omega_0, 1) = \cup_{i=1}^{\infty} \{x_i\}$.

*ii) for $\Omega_t = X(\Omega_0, t)$, and for the measure*

$$\mathrm{d}\mu_t = \rho(x,t)\,\mathrm{d}x = \frac{1}{1-t} \chi_{\Omega_t}(x)\,\mathrm{d}x$$

*we have that $\mu_t \to \mu_1$ as $t \to 1$.*

*Proof.* For any $i \in \mathbb{N}$, let $l_i := \sum_{x_j < x_l} c_j$ and define

$$\begin{cases} a_i := x_i + l_i - \frac{1}{2}, \\ b_i := a_i + c_i. \end{cases}$$

Then, if we set $\Omega_0 := \cup_{i=1}^{\infty} (a_i, b_i)$, repeating the construction in the proof of Theorem 5.1 the theorem is proved. □

**Remark 5.3.** *It is trivial to check that in Theorems 5.1 and 5.2, the original set $\Omega_0$ is not unique and the result also holds for any $\tilde{\Omega}_0$ such that*

$$\Omega_0 \subseteq \tilde{\Omega}_0 \subseteq \cup_{i=1}^{\infty} \overline{I_i}^0 = \cup_{i=1}^{\infty} [a_i, b_i].$$

## 5.3 Compact domain case

On the previous section we have seen that, when $\Omega_0$ is an open domain, the limit measure is a numerable combination of Dirac deltas. Therefore, the Hausdorff dimension of the skeleton $\Omega_1$ is 0. Now, we shall prove that if we do not require the domain to be open, we can obtain a skeleton of any Hausdorff dimension. Specifically, we shall prove that given $\mu_1$ supported on $\Omega_1$ with zero length (but no necessarily having Hausdorff dimension equal to 0) we can construct $\Omega_0$ such that, if $\rho_0 = \chi_{\Omega_0}$, then the solution $\rho$ of (5.3) evolves towards $\mu_1$ as a measure.

**Theorem 5.4.** *Given $\mu_1$ supported on $K_1 \subseteq [1/2, 3/2]$ such that*

$$|K_1| = 0 \ \text{ and } \mu_1(\mathbb{R}) = \mu_1(K_1) = 1.$$

*Then, there exists $K_0$ with $|K_0| = 1$ and such that the solution $\rho(\cdot, t)$ to the transport equation (5.3) with initial data*

$$\rho_0 = \chi_{K_0}$$

*satisfies*

$$\lim_{t\to 1} \rho(x,t)\mathrm{d}x \to \mathrm{d}\mu_1.$$

*Proof.* Since $K_1$ is compact, then the set $U_1 = [1/2, 3/2] \setminus K_1$ is open. Then it can be written as a numerable union of open intervals as

$$U_1 = \cup_{j=1}^{\infty} (a_{j,1}, b_{j,1}).$$

Then, for a point $x \in (a_{i,1}, b_{i,1})$ we associate the following velocity (recall that in Theorem 5.1 we saw that velocity is constant along trajectories)

$$v_i = \frac{1}{2}\left\{\mu_1\left(K_1 \cap \left[\frac{a_{i,1}+b_{i,1}}{2}, \frac{3}{2}\right]\right) - \mu_1\left(K_1 \cap \left[\frac{1}{2}, \frac{a_{i,1}+b_{i,1}}{2}\right]\right)\right\}.$$

Now we define

$$\begin{cases} a_{i,0} = a_{i,1} - v_i, \\ b_{i,1} = a_{i,0} + (b_{i,1} - a_{i,1}) = b_{i,1} - v_i. \end{cases}$$

and also we let $U_0 = \cup_{i=1}^{\infty} (a_{i,0}, b_{i,0})$ and set $K_0 = [0,2] \setminus U_0$.

The spirit of this procedure is the following: we have observed in the proof of Theorem 5.1 that the intervals in the complementary set of $K_0$ just move by keeping its length, because the velocity is constant in these intervals. What we have done here is keeping the length of the intervals in the complementary of $K_1$ and move them with the expected velocity (constant for each interval) for them.

Consider now the uniform measure defined on $K_0$, that is, $d\mu_0 = \chi_{K_0}\, dx$. It remains to check that $\mu_0 \to \mu_1$. Let

$$v(x) = (-\text{sign} * \chi_{K_0})(x) = \frac{1}{2}\left\{|K_0 \cap (x,2)| - |\Omega_0 \cap (0,x)|\right\}.$$

Then, the flow can be written as

$$X(\alpha, t) = \begin{cases} x + \frac{t}{2} & \text{if } x \leq 0, \\ x + v(x)t & \text{if } 0 < x < 2, \\ x - \frac{t}{2} & \text{if } 2 \leq x. \end{cases}$$

It is clear by construction that $U_1 = X(U_0, 1)$ and hence $K_1 = X(K_0, 1)$ too. We define $\mu_t = (X(\cdot, t))_{\#}\mu_0$, that is, for a measurable set $A$ we have

$$\mu_t(A) = (X(\cdot, t))_{\#}\mu_0(A) = \mu_0(X^{-1}(A, t)).$$

We have seen in Theorem 5.2 that the length of $K_t = X(K_0, t)$ is uniformly shrinking with a ratio $1 - t$, meaning that for any set $\tilde{K} \subseteq K_0$ we have

$$\left|X(\tilde{K}, t)\right| = (1 - t)\left|\tilde{K}\right|.$$

Then we clearly have

$$\mu_t(A) = \int_{X^{-1}(A,t)\cap K_0} dx = \frac{1}{1-t}\int_{A\cap K_t} dx = \frac{1}{1-t}\left|A \cap K_t\right|.$$

Hence, given $(a, b) \subseteq (0, 2)$ we have

$$\begin{aligned} \mu_t(a + v(a)t, b + v(b)t) &= \frac{1}{1-t}\left|(a + v(a)t, b + v(b)t) \cap K_t\right| = \\ &= \left|(a, b) \cap K_0\right|, \end{aligned} \tag{5.8}$$

which is indepenent of $t$. We want to check that whenever $t \to 1^-$ then $\mu_t(A) \to \mu_1(A)$ for a set $A$. Without loss of generality we can reduce to the case when $A$ is an interval. An interval $(y, z)$ can be written as $(a + v(a), b + v(b))$ for some $a, b$ since $X(\cdot, t)$ is a homeomorphism. Then,

$$\begin{aligned} \mu_t(a + v(a), &b + v(b)) - \mu_1(a + v(a), b + v(b)) = \\ &= \mu_t(a + v(a), b + v(b)) - \mu_t(a + v(a)t, b + v(b)t) + \\ &+ \mu_t(a + v(a)t, b + v(b)t) - \mu_1(a + v(a), b + v(b)) = \\ &= \mu_t(a + v(a), b + v(b)) - \mu_t(a + v(a)t, b + v(b)t) \to 0 \end{aligned}$$

where we have used equation (5.8). So $\mu_t(A) \to \mu_1(A)$. Since $\{\mu_t\}_{0\leq t\leq 1}$ are probability measures we can conclude that $\mu_t \to \mu_1$, which proves the result. □

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