UNIVERSITAT DE BARCELONA

Radiation in strongly coupled gauge theories

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A la memoria de mi abuelita

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Abstract

This thesis is devoted to the study of radiation in generic conformal field theories in four dimensions. In the first part of the thesis we explain at the classical level the implications of considering accelerated probes coupled to conformal scalar fields: the radiative energy density is not positive definite, the radiated power in not Lorentz invariant and the appearance of terms proportional to the derivative of the acceleration. Furthermore, we conjecture that the spacetime dependence of the expectation value of the energy-momentum tensor of a conformal field theory with extended supersymmetry is independent of the value of the coupling constant. In the second part of the thesis we focus on the determination of the coupling dependence of radiation for probes coupled to superconformal field theories. In order to do so we compute the vacuum expectation value of a circular Wilson loop for different theories preserving a certain amount of supersymmetry, and the way we compute it is using a novel technique coming from supersymmetric localization, which reduces the path integral computation to matrix models computations. The approach we take to compute both the expectation value of the Wilson loop and the partition function gives general results valid for different representations of different gauge groups. For $\mathcal{N} = 4$ super Yang-Mills theory we find an exact expression for the circular Wilson loop valid for arbitrary gauge groups and different representation, thus unifying known results. For $\mathcal{N} = 2$ superconformal quiver theories we find that the problem can be described as a multi-matrix model involving an infinite sum of single- and double-trace terms. We pay special attention to the case of a quiver theory with two nodes and we find an all-order expression for both the partition function and the expectation value of the circular Wilson loop in the limit where the number of colors tends to infinity. These expressions have a nice interpretation in terms of tree graphs and each of these graphs can be interpreted as a generalized Ising model; we conjecture that the contributions of each graph, as well as the sum of the contributions of the graphs with the same number of edges, satisfy the Lee-Yang property: the roots are unitary. Finally, we argue that every matrix model with double-trace terms in the potential can be described in the planar limit as a sum over tree graphs.

Resumen

Esta tesis está dedicada al estudio de la radiación en teorías de campos conformes en cuatro dimensiones. En la primera parte, explicamos a nivel clásico las implicaciones de considerar partículas de prueba aceleradas acopladas a campos conformes escalares: la densidad de energía radiativa no es positiva definida, la potencia radiada no es invariante de Lorentz y la aparición de términos proporcionales a la derivada de la aceleración. Conjeturamos que la dependencia espaciotemporal del valor de expectación del tensor de energía-momento de una teoría de campo conforme con supersimetría extendida es independiente del valor de la constante de acoplamiento. Posteriormente nos centramos en la determinación de la dependencia de la constante de acoplamiento de la radiación. Para lograrlo empleamos una novedosa técnica proveniente de localización supersimétrica que nos permite calcular, en ciertas clases de teorías supersimétricas, de manera general tanto el valor de expectación del Wilson loop como la función de partición. En el caso de la teoría de Yang-Mills con $\mathcal{N} = 4$ supersimetrías encontramos una expresión exacta para el Wilson loop circular válida para diferentes grupos de gauge en distintas representaciones. Para teorías quiver superconformes con $\mathcal{N} = 2$ supersimetrías encontramos que el problema puede ser descrito como un modelo de multi-matrices involucrando una suma infinita de términos de una y doble traza. En el caso especial de una teoría quiver con dos nodos, encontramos una expresión a todo orden para la función de partición y el valor de expectación del Wilson loop en el límite en que el número de colores tiende a infinito. Estas expresiones tienen una agradable interpretación en términos de grafos de árbol, donde cada uno de estos grafos puede ser interpretado como un modelo de Ising generalizado; conjeturamos que las contribuciones de cada grafo, así como las contribuciones de los grafos con el mismo número de aristas, satisfacen la propiedad de Lee-Yang: las raíces son unitarias. Finalmente, argumentamos que cada modelo de matrices con términos de doble traza en el potencial puede ser descrito en el límite planar como una suma sobre grafos de árbol.

Introduction

Understanding the natural phenomena has always led to progress, so when the first humans learned to manipulate the heat and light of a tree on fire, unknowingly they progressed as humankind (together with a lot more factors into play). Now we know that these two terms are two sides of the same coin, electromagnetic radiation. Nonetheless, the phenomenon of radiation is not exclusive of the electromagnetic theory, in the last years it was confirmed that massive objects can emit radiation in form of gravitational waves. The purpose of this thesis is to deepen the knowledge of radiation, specifically in gauge theories with massless quanta as photons, and help in the progress the first humans started thousands of years ago.

So far two fundamental forces have been mentioned: the electromagnetic and the gravitational one. The former along with the weak and strong force can be beautifully described in the so-called Standard Model. This model practically describes all the matter interactions, and it is the most successful quantum field theory (QFTs) considered so far. When dealing with QFTs the most common approach is perturbation theory, and even though this approach has predicted a lot of results, its range of validity is limited. It has been a constant desire in modern theoretical physics to unveil the non-perturbative nature of QFTs. A natural and rich arena for exploring the non-perturbative aspects of QFT is to include extra symmetries, like conformal symmetry, supersymmetry or even both.

Symmetries are often used as a tool to simplify problems, they always constrain the dynamics. As mentioned before when dealing with arbitrary QFTs, we are limited to perturbation theory around a parameter where the theory is solvable, but of course, the most interesting models are the ones that manage to have enough symmetry in order to have analytic control over them but also exhibit at least some of the phenomena present in more physical (less symmetrical) systems. A special class of QFTs with extra symmetry are conformal field theories (CFTs), and among all the theories described within this class is Maxwell theory. The benefits of these theories are twofold: first allow a nice explanation of the phenomenon of radiation and second reduce the level of difficulty of the problem. Even though the dynamics of these theories is far more constrained and they describe theories where there is no intrinsic scale, they are present in several areas of physics from string theory to condensed matter, it is within this last theory that they provide an accurate explanation for the existence and the features of critical phenomena, like phase transitions. Another good example are supersymmetric field theories not only because they are phenomenologically appealing for physics beyond the Standard Model, but they have yielded important physical insight on certain phenomena which are part of Nature, like confinement in quantum chromodynamics (QCD), yet they can be handled with much more control, serving as calculable models for four dimensional gauge theories. Also, conformal symmetry and supersymmetry contain the Poincaré group –which encompasses the most basic symmetries: translations, rotations and boosts– as a subgroup leading to non-trivial QFTs.

This thesis will be focused in the study of radiation in conformal field theories and superconformal field theories. In the following sections we will give a taste of how to deal with a particular set of these theories.

⊙ Classical radiation

To start describing the phenomenon of radiation, it is convenient to start with the canonical example of classical electromagnetic radiation [1, 2]. Lets consider an accelerated charged particle following a prescribed trajectory, L, like the one depicted in the figure below. The trajectory is described by z^μ, its derivative is $\dot{z}^\mu$ and x^μ is the point where the emitted radiation is observed, also we can define the null vector $\ell^\mu = x^\mu - z^\mu$.

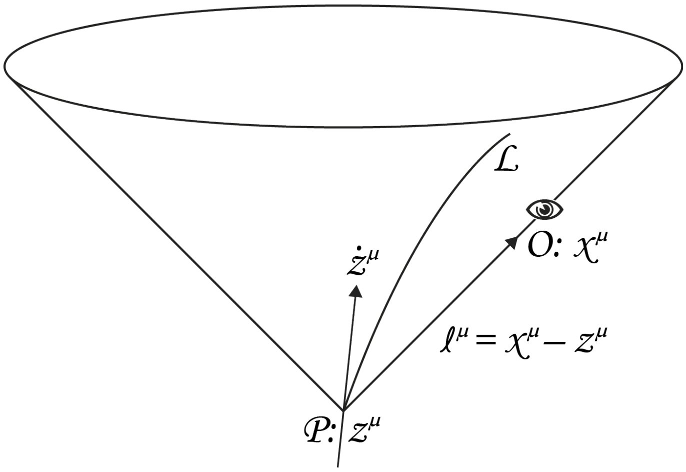


Diagram describing the radiation of a charged particle.

The equation of motion describing this situation is given by

$$\Box A^\mu = j^\mu, \tag{4.1}$$

where $\Box = \partial_\mu \partial^\mu$ is the d'Alembert operator, A^μ is the vector field and j^μ is the 4-current of charge q; to achieve the above equation we made use of the Lorenz gauge, $\partial_\mu A^\mu = 0$. We will be working in four spacetime dimensions, so the index runs $\mu = 0, 1, 2, 3$, also we will consider the speed of light $c = 1$ and there is an implicit sum whenever we see repeated indices. Equation (4.1) can be obtained from the corresponding Lagrangian, $\mathcal{L}$, in the following way

$$\frac{\partial \mathcal{L}}{\partial A^\mu} - \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu A^\mu)} \right) = 0. \tag{4.2}$$

The solution to the equation of motion (4.1) is

$$A^\mu = q \frac{\dot{z}^\mu}{\ell \cdot \dot{z}}, \tag{4.3}$$

which is known as the *Liénard-Wierchert potential* [1]. It is important to highlight that the vector field is evaluated at the *retarded time*, which is the time at which an observer at the point O detects the radiation emitted by the particle at the position P

$$t = t_{ret} + |\vec{x} - \vec{z}| \,. \tag{4.4}$$

Once we know the value of the vector field, A^μ, we proceed to the evaluation of its energy-momentum tensor, which is

$$T^{\mu\nu} = \frac{1}{4\pi} \Big(F^{\mu\alpha} F_\alpha^{\ \nu} + \frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} \Big), \tag{4.5}$$

with $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$ and $g^{\mu\nu}$ is the inverse of the metric $g_{\mu\nu}$. In the general case, we can obtain the energy-momentum tensor from the corresponding action, S, in the following way

$$T_{\mu\nu} = \frac{-2}{\sqrt{-g}}\frac{\delta S}{\delta g^{\mu\nu}}. \tag{4.6}$$

After evaluating (4.3) in (4.5) we obtain

$$\begin{aligned} T^{\mu\nu} = \frac{q^2}{4\pi}\Big(&\frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell\cdot u)^5}(1-\ell\cdot a) + \frac{\ell^\mu a^\nu + \ell^\nu a^\mu}{(\ell\cdot u)^4} - \frac{a^2}{(\ell\cdot u)^4}\ell^\mu\ell^\nu \\ &-\frac{(1-\ell\cdot a)^2}{(\ell\cdot u)^6}\ell^\mu\ell^\nu - \frac{1}{2}\frac{\eta^{\mu\nu}}{(\ell\cdot u)^4}\Big), \end{aligned} \tag{4.7}$$

here "$\cdot$" denotes contraction of 4-vectors, $A^\mu B_\mu = A\cdot B$, and $A^\mu A_\mu = A^2$, for simplicity we made the identification of $\dot{z}^\mu$ with the 4-velocity u^μ and $\ddot{z}^\mu$ with the 4-acceleration a^μ. Radiation is captured by the terms of the energy-momentum tensor decaying as $1/r^2$, where r is the distance form the source to the observation point. Also, radiation is energy and momentum that escapes to infinity and we can see this because after performing Gauss' law we obtain finite contributions arbitrarily away from the source and this part of the energy-momentum tensor is conserved by itself, $\partial_\mu T_r^{\mu\nu} = 0$, this is not the only definition of the radiative energy-momentum tensor, for a more restrictive definition please see [3, 4] In our case we have that the radiative part of the energy-momentum tensor is

$$T_r^{\mu\nu} = -\frac{q^2}{4\pi}\Big(a^2 + \frac{(\ell\cdot a)^2}{(\ell\cdot u)^2}\Big)\frac{\ell^\mu\ell^\nu}{(\ell\cdot u)^4}. \tag{4.8}$$

From the radiative energy-momentum tensor, $T_r^{\mu\nu}$, we can construct several quantities which will allow us to characterize the spacetime dependence of radiation. The first quantity is [2]

$$\frac{dP^\mu}{d\tau d\Omega} = r^2 T_r^{\mu\nu} u_\nu, \tag{4.9}$$

where τ is the proper time, Ω stands for the solid angle and u_ν is the 4-velocity. From the above equation derives a series of quantities relevant for our purposes, if we integrate it over the solid angle we obtain $dP^\mu/d\tau$, which tells us the rate at which the probe is losing energy and momentum due to radiation. Restricting to the zeroth component of this 4-vector we obtain the radiated power, $\mathcal{P}$,

$$\mathcal{P} = \frac{dP^0}{dt}, \tag{4.10}$$

but if instead of restricting to the first component of the 4-vector we contract it with the 4-velocity, we obtain the invariant radiation rate [2], $\mathcal{R}$,

$$\mathcal{R} = \frac{dP^\mu}{d\tau}u_\mu. \tag{4.11}$$

The difference between these two quantities, $\mathcal{P}$ and $\mathcal{R}$, is that while the former is not always Lorentz invariant, the latter is by construction. Another quantity of interest appears when we do not integrate over the solid angle and put our attention to the zeroth component of (4.9), this gives us the angular distribution of radiated power, $d\mathcal{P}/d\Omega$.

Having reviewed radiation in a classical field theory, the natural next step is to study it in quantum field theories. Indeed, there has been a lot of work on radiation in quantum electrodynamics [5]. For non-Abelian theories, like QCD, things are conceptually more complicated, since massless quanta do not appear in asymptotic states, due to confinement. In this thesis, we will focus on a very particular kind of quantum field theories, conformal field theories. As we discuss below, for these theories, the question is conceptually clean, and furthermore, due to their additional symmetry, we have more technical control over them.

⊙ Conformal field theories

The simplest definition of a conformal field theory is a quantum field theory invariant under the conformal group [6]. The conformal group is the symmetry group composed of translations, rotations and boosts (this forms the Poincaré group) plus transformations preserving angles: dilatations and special conformal transformations. A dilatation is just a rescaling by a factor, $x^\mu \to \lambda x^\mu$, and a special conformal transformation can be seen as an inversion followed by a translation and by an inversion again, where an inversion is $x^\mu \to \frac{x^\mu}{x^2}$. It is in this sense that the conformal group is an extension of the Poincaré group since the latter is enhanced, SO(1,3)→SO(2,4). Furthermore, in a conformal field theory the value of the beta function, which tells how the coupling of the theory varies when the energy scale changes, is zero, this means that CFTs are fixed points of RG flows [7].

In a conformal field theory, the energy-momentum tensor (4.6) has dimensions $[T^{\mu\nu}] = \mathrm{L}^{-4}$, where L stands for length dimension, and has the following properties

$$\begin{array}{ll} \text{Symmetric} & T^{\mu\nu} = T^{\nu\mu} \\ \text{Conserved} & \partial_\mu T^{\mu\nu} = 0 \end{array}$$

Also, for this kind of theories, the coupling constants, g, have dimension $[g] = 0$, these are known as a marginal couplings. Furthermore, the energy-momentum tensor shows an additional condition, the traceless condition, which translates into

$$\text{Conformality} \quad T^\mu{}_\mu = 0$$

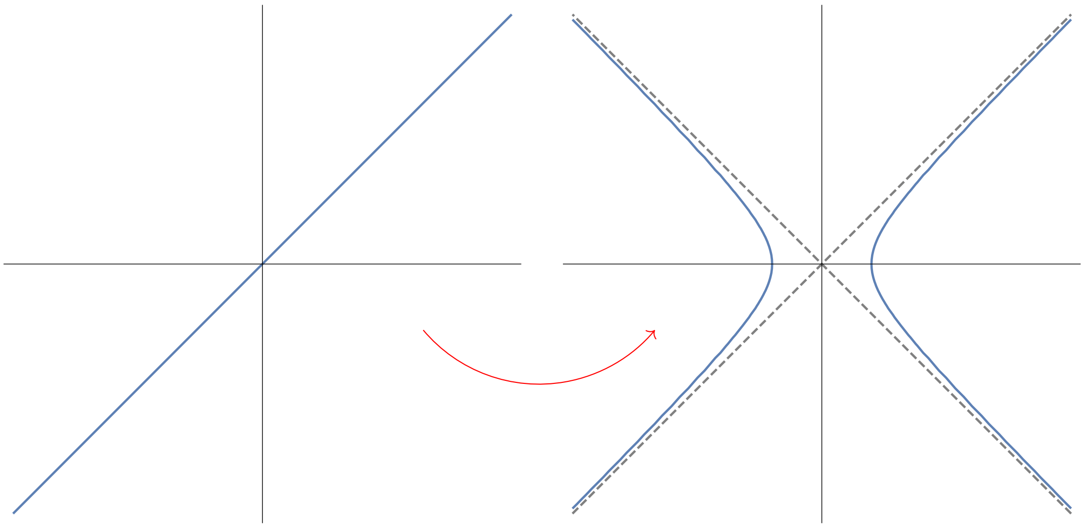
As a consequence of this property, the 1-point function of the energy-momentum tensor in the presence of a static line operator is totally fixed by symmetry [8]

$$\left\langle T^{00}(x)\right\rangle = \frac{h}{r^4}, \quad \left\langle T^{0i}(x)\right\rangle = 0, \quad \left\langle T^{ij}(x)\right\rangle = h\frac{\delta^{ij} - 2n^i n^j}{r^4}, \tag{4.12}$$

where h is a function of the marginal couplings of the CFT, which accounts for the energy measured at infinity [9], and n^i are unitary coordinates, $\frac{\vec{x}^i}{|x|}$.

One of the striking aspects of radiation in conformal field theories is that Coulombic fields and radiative fields are not independent: a very particular contour can be related to the static one. By a special conformal transformation, the straight line is mapped to a trajectory with constant proper acceleration (see figure below). In this case the 1-point function of the energy-momentum tensor is

$$\begin{aligned}\langle T^{\mu\nu}(x)\rangle = 2h\left[-\frac{1}{2}\frac{\eta^{\mu\nu}}{(\ell\cdot u)^4} + (1-\ell\cdot a)\frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell\cdot u)^5} - \frac{(1-\ell\cdot a)^2}{(\ell\cdot u)^6}\ell^\mu\ell^\nu\right.\\ \left. -a^2\frac{\ell^\mu\ell^\nu}{(\ell\cdot u)^4} + \frac{\ell^\mu a^\nu + \ell^\nu a^\mu}{(\ell\cdot u)^4}\right].\end{aligned} \tag{4.13}$$



A special conformal transformation relates a straight line to a hyperbola.

We can further constrain the energy-momentum tensor if we make a couple of assumptions. We are going to consider that $\langle T^{\mu\nu}\rangle$ depends only on $u(t_{ret})$, $a(t_{ret})$ and $\dot{a}(t_{ret})$, but not on the worldline for times $t < t_{ret}$, and the powers of u, a and $\dot{a}$ are integers, so $\langle T\rangle$ contains terms that decay as $1/r^4$, $1/r^3$, and $1/r^2$, but no other powers. Having this in mind it is possible to see that velocity terms (no acceleration) are fixed by conformal symmetry; they are universal for all probes in all CFTs. Now we pass to terms linear in the acceleration (they decay like $1/r^3$), there are six tensor structures one can consider, more than the three generated by a special conformal transformation that appear in (4.13). One might worry that a particular linear combination of these structures vanishes for motion with constant proper acceleration, but it is possible to prove that it is not the case. So terms linear in a are also universal for all CFTs, and are the linear terms that appear in (4.13). It is easy to check that these universal terms (velocity plus linear in the acceleration) are conserved, so it follows that the radiative terms are separately conserved, $\partial_\mu\langle T_r^{\mu\nu}\rangle = 0$, as mentioned before.

Regarding the radiative terms, we see that contain either $\dot{a}$ or two powers of a. For the case of constant proper acceleration $\dot{a}^\mu = -a^2 u^\mu$, therefore expressions written in terms of $\bar{a} = \dot{a} + a^2 u$ automatically vanish. The possible terms are

$$\begin{aligned}\langle T_r^{\mu\nu}(x)\rangle = &\left[A_1 a^2 + A_2\frac{(\ell\cdot a)^2}{(\ell\cdot u)^2} + A_3\frac{\ell\cdot\dot{a}}{\ell\cdot u}\right]\frac{\ell^\mu\ell^\nu}{(\ell\cdot u)^4}\\ &+ B_1\frac{u^\mu\bar{a}^\nu + u^\nu\bar{a}^\mu}{(\ell\cdot u)^2} + B_2\frac{\ell\cdot\bar{a}}{(\ell\cdot u)^3}\eta^{\mu\nu} + B_3\frac{\ell\cdot\bar{a}}{(\ell\cdot u)^3}u^\mu u^\nu + B_4\frac{\ell^\mu\bar{a}^\nu + \ell^\nu\bar{a}^\mu}{(\ell\cdot u)^3}\end{aligned} \tag{4.14}$$

where A_i and B_j are Lorentz scalars and functions of the marginal couplings of the CFT. Tracelessness imposes $4B_2 + B_3 + 2B_4 = 0$. To further constrain the coefficients, we require $\partial_\mu\langle T_r^{\mu\nu}(x)\rangle = 0$. The terms with A_i coefficients are individually conserved, so this constraint only applies to terms with B_i coefficients. This yields $B_1 = B_3 = 0$ and $B_2 + B_4 = 0$. Together with the previous relation, this implies that all B_i coefficients are zero. The radiative part of the energy-momentum tensor has thus the following form

$$\langle T_r^{\mu\nu}(x)\rangle = \left[A_1 a^2 + A_2\frac{(\ell\cdot a)^2}{(\ell\cdot u)^2} + A_3\frac{\ell\cdot\dot{a}}{\ell\cdot u}\right]\frac{\ell^\mu\ell^\nu}{(\ell\cdot u)^4}\,, \tag{4.15}$$

For a trajectory with constant proper acceleration, this must reduce to (4.13); this implies

$$A_1 - A_3 = -2h, \qquad A_2 = -2h. \tag{4.16}$$

Furthermore, from (4.9) if we integrate over the solid angle we obtain a 4-vector that gives the rate of energy and momentum emitted by the probe [10], explicitly we have

$$\frac{dP^\mu}{d\tau} = -\frac{4\pi}{3}(-3A_1 + A_2 + 4A_3)a^2 u^\mu - \frac{4\pi}{3}A_3 \dot{a}^\mu, \tag{4.17}$$

According to [11], we can identify from the term with a^2 the Bremsstrahlung function (which we will define properly later), B,

$$B = \frac{2}{3}(-3A_1 + A_2 + 4A_3)\,. \tag{4.18}$$

Thus, for CFTs such that $\langle T^{\mu\nu}\rangle$ depends only of kinematics of the particle at retarded time, the radiative part is

$$\langle T_r^{\mu\nu}(x)\rangle = \left[\left(\frac{3}{2}B - 6h\right)a^2 - 2h\frac{(\ell\cdot a)^2}{(\ell\cdot u)^2} + \left(\frac{3}{2}B - 4h\right)\frac{\ell\cdot\dot{a}}{\ell\cdot u}\right]\frac{\ell^\mu\ell^\nu}{(\ell\cdot u)^4}. \tag{4.19}$$

As we can see, the radiative part of the energy-momentum tensor is fixed up to two different functions, B and h. An intuitive argument for why we need two independent functions to characterize radiation in generic CFTs is that the probe can be coupled to scalar and vector fields, and the radiation pattern of these two types of fields are different. This result raises a couple of questions: first, to derive it we assumed that $\langle T\rangle$ depends only on the retarded time. Are there any interacting CFTs where this is true? As we will see later, the answer is positive for $\mathcal{N} = 4$ SYM theory, at least in the holographic regime. Second, how do we compute B and h efficiently? Since conformal symmetry does not help, we need other tools. As we will see, these tools exist for CFTs with extended supersymmetry. We thus conclude this section recalling the very basics of supersymmetric theories [12].

As we mentioned before, there exists another non-trivial extension of the Poincaré group, this is possible by including supersymmetry. The reason the Poincaré group allows an extension is because it is not simply connected, so it admits spinor representations. In other words, the Poincaré group can be extended if we include symmetry generators transforming in the spinor representations [13], the supersymmetry generators, Q. Furthermore, it is possible to introduce more than one copy of the Q, this corresponds to extended supersymmetry. We can introduce $\mathcal{N}$ copies of supersymmetry generators Q_α^A transforming in the fundamental representation of SL(2,$\mathbb{C}$), and $\bar{Q}_{\dot{\alpha}}^A$, transforming in the conjugate representation, with $\alpha = 1, 2$ and $A = 1, \ldots, \mathcal{N}$; this is because the universal cover of the Lorentz group is SL(2,$\mathbb{C}$). Requiring that the multiplets do not contain massless states with spin 2 imposes that $\mathcal{N} \leq 4$.

In this thesis, besides free conformal field theories, we are going to be particularly interested in conformal field theories with extended supersymmetry, $\mathcal{N} \geq 2$, that admit a Lagrangian description. Let's discuss a bit the casuistics: Lagrangian $\mathcal{N} = 2$ supersymmetric field theories have been classified [14, 15], but not all theories have conformal symmetry, if we want to preserve conformality we have to impose certain conditions on the matter the theory contains in order keep this symmetry. It is possible to prove that there are no weakly coupled $\mathcal{N} = 3$ SCFTs [16] and they don't have exactly marginal couplings, so they will not be considered in this thesis. Finally, all known $\mathcal{N}=4$ supersymmetric field theories are automatically conformal and have a Lagrangian description, but it is not known if there are non-Lagrangian $\mathcal{N}=4$ theories (see [17] for a discussion on this point).

The probes we will couple to these theories are of a particular kind, they will be all BPS particles. Let's define this property. In extended supersymmetry, the supersymmetry algebra contains the following anticommutator

$$\{Q^A_\alpha, Q^B_\beta\} = \epsilon_{\alpha\beta} Z^{AB}, \tag{4.20}$$

where Z^{AB} are called *central charges*. Since one of the two Casimirs of the Poincaré group, $P_\mu P^\mu$, is still a Casimir of the supersymmetry algebra, all the states of a supersymmetry multiplet share the same mass m. In this case the central charges are non-vanishing and one can prove the inequality

$$m \geq \frac{1}{2\mathcal{N}} \mathrm{Tr}\sqrt{Z^\dagger Z}, \tag{4.21}$$

where Z is the central charge matrix defined in (4.20). Multiplets that saturate this bound are called *BPS multiplets*. They are shorter than the generic massive multiplets, and are under better control.

As we will discuss later, for Lagrangian conformal field theories with extended supersimmetry, we have tools to compute the coupling dependence. Moreover, for arbitrary $\mathcal{N}=2$ SCFTs and 1/2 BPS line defects it can be proven that $B=3h$ [9] . Then (4.19) takes the form

$$\langle T^{\mu\nu}_r(x)\rangle_{\mathcal{N}=2} = h\left[-\frac{3}{2}a^2 - 2\frac{(\ell\cdot a)^2}{(\ell\cdot u)^2} + \frac{1}{2}\frac{\ell\cdot \dot{a}}{\ell\cdot u}\right]\frac{\ell^\mu \ell^\nu}{(\ell\cdot u)^4}. \tag{4.22}$$

An intuitive explanation as to why for $\mathcal{N}=2$ SCFTs we need only one function to describe radiation is that now the scalar and vector field content of the $\mathcal{N}=2$ vector multiplet is fixed by supersymmetry. This is the most general form the radiative part of the energy-momentum tensor can take assuming supersymmetry, conformal symmetry and that it depends only on the retarded time.

⊙ Wilson loops

Wilson loops are of great importance because they are observables, and in a broad sense an observable is a quantity containing physical information that can be measured. Before we proceed to the definition of a Wilson loop and how we are going to use it, it is worth to spend a few lines in its origin and evolution.

In QFT we have spacetime dependent fields, so at different points the same field will have different values, and if we want to make comparisons of a field evaluated at two different points we need a quantity relating both points, this quantity is the *Wilson line* [18]. This Wilson line plays the role of a comparator between two points, so if both points are infinitesimally together, we can define the difference of two field values as a derivative, this derivative is known as *covariant derivative*. Once we know what a Wilson line is, we can define it in the following way

$$W(x,y) = \exp\left(i\int_x^y A_\mu(z)dz^\mu\right), \tag{4.23}$$

where $A_\mu(z)$ is the gauge field; in more geometrical terms it plays the role of a connection. If we take the initial and final point to be the same we are performing a closed contour integral and it is known as *Wilson loop*, so we have

$$W = \exp\Big(i \oint A_\mu dx^\mu\Big). \tag{4.24}$$

The reason the Wilson loop is an observable is because it is invariant under gauge transformations. In an Abelian gauge theory, the Wilson line operator has the physical interpretation of representing the phase change experienced by a charged particle as it traverses a closed loop, the non-Abelian version is a generalization of this statement.

For the case of non-Abelian gauge theories, like Yang-Mills theory and its supersymmetric generalizations, if we want to preserve gauge invariance we need to add some extra ingredients to the previous definition. The first requirement is to include the *path-ordering operator*, $\mathcal{P}$, because now the gauge field is a matrix-valued quantity and the order of integration matters (matrices in general do not commute), and the second ingredient is to take the trace of the operator necessary to preserve the gauge invariance, so we arrive at

$$W = \mathrm{Tr}_R\, \mathcal{P}\Big(\exp\Big(i \oint \mathrm{A}_\mu dx^\mu\Big)\Big), \tag{4.25}$$

where Tr_R stands for the trace in the representation R of the gauge group we are considering and $\mathrm{A}_\mu(x) = A^a_\mu(x) T^a$ with T^a the generators of the gauge group.

So far we have kept the discussion about Wilson loops without considering supersymmetry. In $\mathcal{N} = 4$ SYM theory the Wilson loop in Euclidean signature we can include not only the gauge field but scalar fields too in the following way [19]

$$W = \mathrm{Tr}_R\, \mathcal{P}\Big(\exp\Big[\oint \Big(i\mathrm{A}_\mu \dot{x}^\mu + |\dot{x}|\Phi_I \Theta^I\Big) ds\Big]\Big), \tag{4.26}$$

where $I = 1, \ldots, 6$ are the six scalar fields of the theory and Θ^I are unitary coordinates of the 5-sphere. This Wilson loop for the case of a straight line and a circular trajectory preserves half of the supersymmetries, and it is called 1/2 BPS Wilson loop. The inclusion of the scalar field is a key component for our purposes, this is the Wilson loop which is solved using supersymmetric localization.

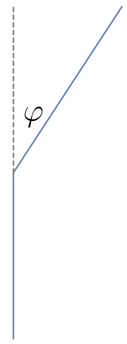
One crucial property of this last definition of Wilson loop is that it now, depending on the trajectory, preserves a fraction of the supersymmetry charges of the theory. This protects it from acquiring divergent contributions at quantum level. Nevertheless, their vacuum expectation value (vev) is in general a non-trivial function of the coupling constant of the theory.

As said before, Wilson loops are relevant because of their physical content, and in the context of SCFTs they are not the exception. Wilson loops with particular contours have connections with other physical quantities like the Bremsstrahlung function and the cusp anomalous dimension. The physical definition of the Bremsstrahlung function, B, can be seen as the generalization of the Larmor formula in electrodynamics and corresponds to the energy lost by an accelerated massive particle moving in the vacuum of a gauge theory [20]

$$E = 2\pi B \int a^2 dt, \tag{4.27}$$

where E is the energy and $a^2 = a_\mu a^\mu$ is the square of the 4-acceleration. In general B is a non-trivial function of the coupling constant of the theory.

The Bremsstrahlung function it is also related to the cusp anomalous dimension $\Gamma_{cusp}(\varphi)$. This is the quantity that controls the logarithmic divergence of a Wilson operator evaluated on a cusped contour, this contour is made by two semi-infinite straight lines that meet at a point forming an angle φ, as we can see in the figure below.



Cusped trajectory.

Close to the cusp short distance singularities appear, which exponentiate as

$$\langle W\rangle \sim e^{-\Gamma_{cusp}\log\frac{\epsilon}{\Lambda}}, \tag{4.28}$$

here ϵ is the IR regulator and Λ the UV regulator [21]. For small angles, $\varphi \ll 1$, the cusp anomalous dimension [11] behaves as

$$\Gamma_{cusp} = -B\varphi^2 + \mathcal{O}(\varphi^4), \tag{4.29}$$

where B is our Bremsstrahlung function.

Later it was shown [11] an explicit relation between the Bremsstrahlung function and the Wilson loop for the case of a circular trajectory in $\mathcal{N}=4$ super Yang-Mills theory. The relation goes like

$$B = \frac{1}{2\pi^2}\lambda\frac{\partial}{\partial\lambda}\log\langle W_\odot\rangle, \tag{4.30}$$

where λ is the 't Hooft coupling, $\lambda = g_{\text{YM}}^2 N$, and $\langle W_\odot\rangle$ stands for the vev of the circular Wilson loop.

The vev of the circular Wilson loop in the fundamental representation of SU(N) was first computed perturbatively for all values of the coupling constant in the planar limit [22]

$$\langle W_\odot\rangle = \frac{2}{\sqrt{\lambda}}I_1\left(\sqrt{\lambda}\right), \tag{4.31}$$

where $I_1(\lambda)$ is the modified Bessel function of the first kind. Later its exact value was found [23] fort the gauge group U(N)

$$\langle W_\odot\rangle = \frac{1}{N}L^1_{N-1}\left(\frac{-\lambda}{4N}\right)\exp\left(\frac{\lambda}{8N}\right), \tag{4.32}$$

which in the $1/N$ expansion reduces to (4.31); $L_n^m(x)$ is the Laguerre polynomial

$$L_n^m(x) = \frac{1}{n!}e^x\left(\frac{d}{dx}\right)^n\left(e^{-x}x^{m+n}\right). \tag{4.33}$$

This computation was possible because the contributions of planar diagrams are identical and the problem reduces to combinatorics. This result hints a remarkable property of the supersymmetric circular Wilson loop, its expectation value can be computed in terms of a matrix model. One way to see this is because the Wick contractions appearing in its computation are coordinate independent, so the path integral of a four-dimensional field theory turns into a matrix integral.

This conjecture was later proved and generalized to $\mathcal{N} = 2$ SYM theories [24], using localization techniques. The paper shows that the path integral over the four-dimensional fields reduces to an integral over a zero mode, which corresponds to the variable of integration in the matrix model integral.

These results suggested a possible generalization of (4.30) but now relating the h function for $\mathcal{N} = 2$ SCFTs and the vev of the circular Wilson loop. In fact it was first conjectured in [20] and later proved in [25] that these two quantities are related

$$h = \frac{1}{12\pi^2}\partial_b \ln \langle W_b \rangle|_{b=1}, \tag{4.34}$$

where $\langle W_b \rangle$ is the vev of the circular Wilson loop placed on an ellipsoid with parameter b instead of a sphere [26], and the limit $b = 1$ corresponds to the sphere. An intuitive idea of why the previous formula is valid lies in the fact that a deformation of the sphere is a change in the metric, which induces a perturbation in the energy-momentum tensor, this perturbation can be interpreted as radiation which is captured by the h function.

There is an extra relation coming from these computations valid first for $\mathcal{N} = 4$ SYM [27] and later proposed to hold for $\mathcal{N} = 2$ SCFTS [20] and later proved [9], $B = 3h$. This result, along the previous ones, will allow us to fully characterize the radiation of generic $\mathcal{N} = 2$ Lagrangian conformal field theories.

Now we will focus on the determination of the functions B and h within the framework of the AdS/CFT correspondence and supersymmetric localization.

⊙ The AdS/CFT correspondence and localization

One of the most promising approaches to study the strong coupling regime of certain QFTs is the holographic correspondence, also known as gauge/gravity duality, between QFTs in d spacetime dimensions and gravity/string theory in $d+1$ dimensions [28]. This new paradigm in physics relates strongly coupled quantum systems without gravity to purely (semi)classical gravitational phenomena, and has been applied to study from condensed matter problems at strong coupling to aspects of the strong nuclear interactions, to name a few examples. In all such situations the d-dimensional QFT under consideration is said to reside at the boundary of the dual $d+1$-dimensional gravitational system. The former is thus a hologram of the latter.

This correspondence is also a weak/strong duality. This means that while the gauge theory is weakly coupled, its holographic dual will be strongly coupled, and vice versa. In fact, the correspondence was first established for the four dimensional $\mathcal{N} = 4$ SYM theory in the strong coupling regime whose gravitational dual is a weakly coupled string theory living in ten dimensions, more concretely in a space formed by the product of anti-de Sitter space in five dimensions and a 5-sphere. There have been many applications of this correspondence, obtaining field theory observables in regimes where ordinary methods do not apply. One of these phenomena is radiation at strong coupling, which it was first studied for the case of a probe following circular motion [29] and later generalized for arbitrary motion [30].

Let us briefly describe how radiation is studied in this scenario. Consider a string with one of its ends free while the other is attached to the boundary of the spacetime under consideration, which is anti-de Sitter in five dimensions (AdS_5), the trajectory of the end that it is attached to the boundary will correspond to the trajectory of the probe in the dual theory. Once the string starts moving it will create an energy-momentum tensor in the bulk, but at the same time this tensor will cause perturbations in the bulk metric. The perturbations of the bulk metric will travel through the spacetime and once they reach the boundary, they will induce a new metric in the boundary of AdS_5, which is Minkowski spacetime in four dimension. Finally this induced metric will generate a energy-momentum tensor which will correspond to the one of the gauge theory. For our purposes this comes in handy because the dual theory in Minkowski spacetime is the $\lambda \to \infty$ and $N \to \infty$ limit of $\mathcal{N} = 4$ SYM theory, which cannot be easily studied by any other methods. The result they obtained was the following

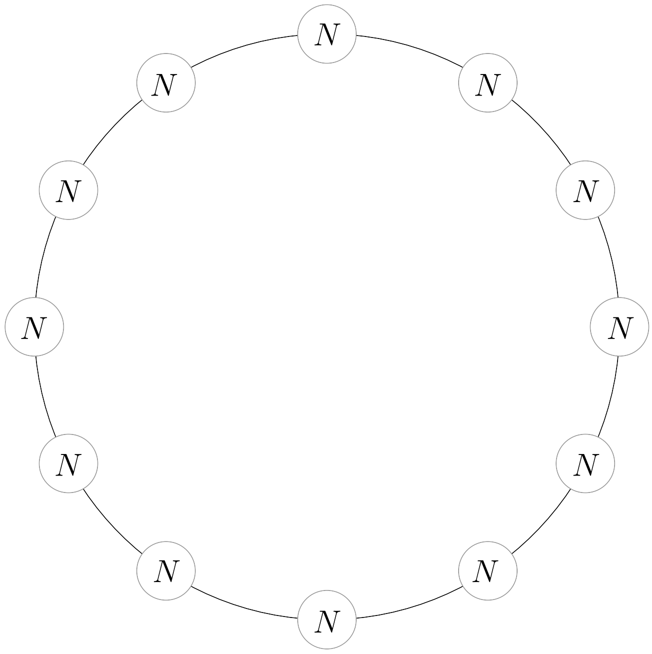
$$T^{00}_{\mathcal{N}=4} = \frac{\sqrt{\lambda}}{24\pi^2 r^2} \left(\frac{4\,|\vec{a}|^2 + 3\gamma^2(\vec{\beta}\cdot\vec{a})^2 + \vec{\beta}\cdot\dot{\vec{a}}}{(1-\vec{\beta}\cdot\vec{n})^4} + \frac{5(\vec{\beta}\cdot\vec{a})(\vec{n}\cdot\vec{a}) - \gamma^{-2}\vec{n}\cdot\dot{\vec{a}}}{(1-\vec{\beta}\cdot\vec{n})^5} - 4\frac{\gamma^{-2}(\vec{n}\cdot\vec{a})^2}{(1-\vec{\beta}\cdot\vec{n})^6} \right).$$

This result implies that in this regime $h = \frac{\sqrt{\lambda}}{12\pi^2}$. More strikingly, the answer depends only on the retarded time; this is currently the only example of an interacting CFT where this property holds. Finally, the radiative energy density exactly matches the general result (4.22).

Shortly after the AdS/CFT correspondence was first established with the canonical example of $\mathcal{N} = 4$ SYM theory, people started to look for new theories possessing a gravitational dual theory. Superconformal quiver theories with $\mathcal{N} = 2$ are one of the theories that in certain limits have a holographic description. The specific quiver theory we are going to consider is depicted in the figure below. Each node represents a vector multiplet of the SU(N) gauge group in the adjoint representation, while the lines connecting the nodes represent a chiral multiplet in the bi-fundamental representation. As we can see, all the gauge group have the same N, this is a necessary condition to preserve conformality [31].

In general, each node corresponds to a copy of $\mathcal{N} = 2$ superconformal quantum chromodynamics (SQCD), therefore each node has different coupling constant. Two relevant scenarios are the case where all the coupling constants are the same –it would correspond to an orbifold of $\mathcal{N} = 4$ SYM theory, thus it possess a dual description– and the case where all the coupling constants except one are zero –this corresponds to $\mathcal{N} = 2$ SQCD which lacks of a gravitational dual description [32]–.

In the previous section we discussed that Wilson loops are among the most interesting operators in a gauge theory and how they are related to the phenomenon of radiation. We also mentioned that supersymmetric localization applied to the calculation of exact physical observables is one of the most spectacular results in non-abelian gauge theories and it is intended to compute functional integrals exactly. Now we will describe how supersymmetric localization has reduced the calculation of the exact partition function and Wilson loop operators in supersymmetric gauge theories from a four dimensional to a zero dimensional problem, including all perturbative and non-perturbative contributions taking into account instanton effects [24]. In the next lines we will try to explain in a simple way the main idea behind supersymmetric localization and how we are going to exploit the result in order to determine the functions characterizing the radiation, B and h.



Quiver diagram containing 12 nodes; each node represents a SU(N) gauge group factor.

Lets consider an integral in one dimension in real space in the following way

$$I_\lambda = \int_a^b g(u) e^{-\lambda f(u)} du, \tag{4.35}$$

the variable of integration is u, λ is a parameter and $g(u)$ and $f(u)$ are real functions of u. This is an oversimplified version of a path integral if we make the identification of du with $\mathcal{D}x$, $f(u)$ with the action S and $g(u)$ with an observable. In general the integral (4.35) cannot be solved, but in the limit where λ is very large, an approximation can be made and the integral will be dominated by the minima of the function $f(u)$. To make the example even easier, we take $g(u) = 1$ and consider the minima of $f(u)$ such that $f'(u^*) = 0$ and $f''(u^*) > 0$. In this case and considering the expansion of $f(u)$ up to second order the integral will look like

$$I_\lambda \approx e^{-\lambda f(u^*)} \int_a^b d^{-\frac{\lambda}{2}(u-u^*)^2 f''(u)} du, \tag{4.36}$$

and we can solve this integral because it is Gaussian, leading to

$$I_\lambda = e^{-\lambda f(u^*)} \sqrt{\frac{2\pi}{\lambda f''(u^*)}}, \tag{4.37}$$

where the first term can be seen as the classical contribution and the integral the 1-loop determinant.

The idea behind localization is that in certain cases a similar approximation leads to exact results. Another simple example that might explain why the name localization is the following. Consider an integral over a 2-sphere

$$\begin{aligned} I_\lambda &= \int e^{i\lambda\cos(\theta)} ds^2 \\ &= \int_0^{2\pi} d\varphi \int_0^{\pi} e^{i\lambda\cos(\theta)} \sin(\theta) d\theta \\ &= \frac{4\pi}{\lambda}\sin(\lambda). \end{aligned}$$

The reason the above integral is exact is because the integrand does not depend on the variable φ, which means there is a symmetry, $\varphi \to \varphi + \alpha$, this is a rotation so there is an axis of symmetry and the two fixed points are 0 and π, in this sense the integral localizes around the fixed points, so the name localization.

The original calculation of [24] is more complicated than the examples above mentioned. Without going into details and following [33] what happens is the path integral is restricted to a space invariant under a fermionic symmetry, Q, so the Lagrangian, $\mathcal{L}$, is invariant under the action of the supercharge Q

$$\mathcal{L} \to \mathcal{L} + tQ \cdot V,$$

The restriction on the choice of V is such that if Q^2 generates a symmetry and a gauge transformation, then V must be gauge invariant and also invariant under the action of the symmetry. Another requirement is that the path integral must be convergent after the deformation. The supersymmetry generated by Q must be realized off-shell in order to localize the gauge fixed path integral, so a gauge fixing procedure must be implemented. This is done by introducing auxiliary fields which determine the measure of integration of the fluctuations.

Since the path integral is independent of t, we can consider the limit where t tends to infinity. In this limit the saddle points of the path integral are the saddle points of the deformed action $Q \cdot V$. It is in this limit where the integral becomes 1-loop exact and can be evaluated by summing over all saddle points. Then the path integral can be calculated by evaluating the original Lagrangian on the saddle points and by integrating out the quadratic fluctuations in the Lagrangian deformation $Q \cdot V$. Even though the path integral is 1-loop exact in t, it yields results to all orders in perturbation theory with respect to the original gauge coupling constant of the theory.

After a brief and simplified explanation of supersymmetric localization, without further ado the result of [24] for the partition function of $\mathcal{N} = 2$ super Yang-Mills theories (not necessarily conformal) is the following

$$Z_{S^4} = \int e^{-\frac{8\pi^2}{g^2_{\text{YM}}}\text{Tr}(a^2)} \mathcal{Z}_{\text{1-loop}} \left|\mathcal{Z}_{\text{inst}}\right|^2 da, \tag{4.38}$$

where $\mathcal{Z}_{\text{1-loop}}$ is a factor coming from a 1-loop computation, while $\mathcal{Z}_{\text{inst}}$ is the instanton contribution. The expression for the vev of a 1/2 BPS circular Wilson loop is

$$\langle W \rangle = \frac{1}{Z_{S^4}} \int \text{Tr} e^{-2\pi b a} e^{-\frac{8\pi^2}{g^2_{\text{YM}}}\text{Tr}(a^2)} \mathcal{Z}_{\text{1-loop}} |\mathcal{Z}_{\text{inst}}|^2 da. \tag{4.39}$$

We will consider the instanton contribution equals to 1 since in the large N limit are exponentially suppressed [31].

There are two cases to point out in these matrix integrals: first is that for the case of $\mathcal{N} = 4$ SYM theory the 1-loop determinant contributions are exactly 1, so we end up with a Gaussian matrix model; second is that the $\mathcal{Z}_{\text{1-loop}}$ will be treated as an effective action, e^{-S}, so the calculations will be perturbations around a Gaussian matrix model.

The way we are going to consider the vacuum expectation value of a quantity will be the following [34]: consider $f(a)$ which depends on the matrix a, so its vev it is given by

$$\begin{aligned}\langle f(a)\rangle &= \frac{1}{Z_{S^4}}\int f(a)e^{-\mathrm{Tr}(a^2)}\mathcal{Z}_{\text{1-loop}}\,da\\ &= \frac{\int f(a)e^{-\mathrm{Tr}(a^2)}e^{-S(a)}da}{\int e^{-\mathrm{Tr}(a^2)}e^{-S(a)}da},\end{aligned}$$

and as we said before, the effective action can be treated as an interaction in the Gaussian matrix model, so we can write the vev of $f(a)$ in the Gaussian matrix model as

$$\langle f(a)\rangle = \frac{\left\langle e^{-S(a)}f(a)\right\rangle_0}{\left\langle e^{-S(a)}\right\rangle_0}, \tag{4.40}$$

where subscript 0 denotes the vev is taken in the Gaussian matrix model. So the vev in the Gaussian matrix model is

$$\langle f(a)\rangle_0 = \frac{1}{Z_0}\int f(a)e^{-\mathrm{Tr}(a^2)}da, \tag{4.41}$$

with $Z_0 = \int e^{-\mathrm{Tr}(a^2)}da$. The computation of the vevs will be according to (4.40).

In the following chapters we will put to work the tools learned so far and finally we will determine the spacetime dependence of radiation of conformal field theories, as well as the coupling dependence of radiation in certain superconformal field theories in four dimensions.

Spacetime dependence of radiation

This chapter includes the publication:

- B. Fiol and J. Martínez-Montoya, *On scalar radiation*, JHEP **03**, 087 (2020), arXiv:1907.08161 [hep-th].

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On scalar radiation

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Abstract: We discuss radiation in theories with scalar fields. Our key observation is that even in flat spacetime, the radiative fields depend qualitatively on the coupling of the scalar field to the Ricci scalar: for non-minimally coupled scalars, the radiative energy density is not positive definite, the radiated power is not Lorentz invariant and it depends on the derivative of the acceleration. We explore implications of this observation for radiation in conformal field theories. First, we find a relation between two coefficients that characterize radiation, that holds in all the conformal field theories we consider. Furthermore, we find evidence that for a 1/2-BPS probe coupled to $\mathcal{N}=4$ super Yang-Mills, and following an arbitrary trajectory, the spacetime dependence of the one-point function of the energy-momentum tensor is independent of the Yang-Mills coupling.

Keywords: Conformal Field Theory, Supersymmetric Gauge Theory

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Contents

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1 Introduction

The study of the creation and propagation of field disturbances by sources is one of the basic questions in any field theory. In classical electrodynamics, emission of electromagnetic waves by charged particles is of paramount importance, both at the conceptual and practical level [1]. Similarly, the recent detection of gravitational waves [2] provides a striking confirmation of General Relativity, and opens a new way to explore the Universe.

Understandably, radiation of massless scalar fields due to accelerated probes coupled to them, has received much less attention [3]. An exception is the study of radiation in scalar-tensor theories of gravity, since the radiation pattern can differ from General Relativity [4].

The comments above refer to classical field theories. Recent formal developments, like holography and supersymmetric localization, have allowed to explore radiation in the strong coupling regime of conformal field theories (CFTs), which if they admit a Lagrangian formulation, very often include scalar fields. Some of the results of these explorations are, however, unexpected and even conflicting, as we now review.

In field theory, radiation is determined by the one-point function of the energy-momentum tensor of the field theory in the presence of an accelerated probe, which is described by a Wilson line W,

$$\langle T_{\mu\nu}\rangle_W = \frac{\langle W T_{\mu\nu}\rangle}{\langle W\rangle} \tag{1.1}$$

Instead of computing $\langle T_{\mu\nu}\rangle_W$ for arbitrary trajectories, one can consider particularly simple kinematical configurations. A first possibility is motion with constant proper acceleration. The reason for this choice is that in any CFT, a special conformal transformation maps a worldline with constant proper acceleration to a static one, for which $\langle T_{\mu\nu}\rangle_W$ is fixed up to a coefficient [5]

$$\left.\frac{\langle W T^{00}\rangle}{\langle W\rangle}\right|_{\vec{v}=0} = \frac{h}{|\vec{x}|^4} \tag{1.2}$$

where $|\vec{x}|$ is the distance between the static Wilson line, placed at the origin, and the point where the measure takes place. The coefficient h should thus capture the radiated power, at least for a probe with constant proper acceleration [6].

A second interesting kinematical situation is that of the probe receiving a sudden kick. The Wilson line associated to the probe exhibits a cusp, and its vacuum expectation value develops a divergence, characterized by the cusp anomalous dimension [7] $\Gamma(\varphi)$, that depends on the rapidity of the probe after the kick. The expansion of $\Gamma(\varphi)$ for small φ,

$$\Gamma(\varphi) = B\varphi^2 + \dots \tag{1.3}$$

defines the Bremsstrahlung function B [8]. It was argued in [8] that this function determines the energy radiated by a probe coupled to a CFT, since it appears in

$$E = 2\pi B \int dt (\vec{a})^2 \tag{1.4}$$

If one grants this relation and further assumes that for arbitrary CFTs the radiated power is Lorentz invariant, one arrives at a Larmor-type formula

$$\mathcal{P} = -2\pi B a^\lambda a_\lambda \tag{1.5}$$

where a^λ is the 4-acceleration. It was further argued in [8] that in any CFT, the Bremsstrahlung function is universally related to the coefficient C_D of the 2-point function of the displacement operator of any line defect [9], by $12B = C_D$. For Lagrangian CFTs with $\mathcal{N} = 2$ supersymmetry this function can be computed using supersymmetric localization [6, 10, 11]. For $\mathcal{N} = 2$ SCFTs it was argued [10, 12] and then proved [13] that $B = 3h$. This relation is not satisfied in Maxwell's theory [12], proving that no universal relation between B and h exists that is valid for all CFTs.

Turning to holography, radiation by accelerated charges in a CFT is studied by first introducing a holographic probe, a string or a D-brane. Computations can be done at the worldsheet/worldvolume level, or taking into account the linear response of the gravity solution due to the presence of the holographic probe. Intriguingly, these two methods do not fully agree. At the holographic probe level, the computation of [14], followed by [15, 16] indicated that for a 1/2-BPS probe coupled to $\mathcal{N} = 4$ super Yang-Mills, in the large N, large λ limit, the total radiated power is indeed of the form given by (1.5). The beautiful works [17, 18] dealt with the backreacted holographic computations, see also [19–21]. The work [17] considered only a probe in circular motion, and found agreement with (1.5). However, the work [18] dealt with arbitrary trajectories, and found

$$\mathcal{P} = -2\pi B \left(a^\lambda a_\lambda + \frac{1}{9} \frac{\dot{a}^0}{\gamma} \right) \tag{1.6}$$

where γ is the usual Lorentz factor. The additional term in (1.6) would imply that the radiated power in $\mathcal{N} = 4$ SYM is not Lorentz invariant. The work [17] was restricted to circular motion in a particular frame where $\dot{a}^0 = 0$, so by construction, it was not sensitive to the presence of the additional term in (1.6).

The angular distribution of radiated power is a more refined quantity than the total radiated power. At strong coupling it has been studied in [17, 18], where the angular distribution of radiation emitted by a 1/2-BPS probe coupled to $\mathcal{N} = 4$ super Yang-Mills was determined holographically. Some of the features of the angular distribution of radiation found in [17, 18] were unexpected, like regions with negative energy density, or its dependence on the derivative of the acceleration, eq. (1.6). This prompted [18] to consider them artifacts of the supergravity approximation.

In this work we revisit the issue of radiation in scalar field theory, bringing new insights to many of the issues reviewed above. Our key observation is rather elementary: scalar fields couple to the scalar curvature of spacetime via the term [22] $\xi R\phi^2$ so, even in flat spacetime, the energy-momentum tensor [23] and therefore the pattern of radiation, depend on ξ. In particular, radiation in conformal field theories requires considering conformally coupled scalars ($\xi = 1/6$) instead of minimally coupled ones, $\xi = 0$, as done in the field theory computations of [17, 18].

Once we take this observation into account, we find that already at the level of free theory, radiation for a free conformal scalar displays the features that were found holographically for $\mathcal{N} = 4$ super Yang-Mills: the radiated power is not Lorentz invariant, it depends on $\dot{a}$ and the radiated energy density is not everywhere positive. We conclude that these are generic features valid for all conformal field theories that include conformal scalars. In particular, eqs. (1.4) and (1.5) are not valid for arbitrary trajectories in CFTs with scalar fields.

Our observation also brings a new perspective to the lack of a universal relation between the coefficients B and h discussed above. In [1, 24] a manifestly Lorentz invariant quantity, the invariant radiation rate $\mathcal{R}$, was defined in the context of Maxwell theory. We extend the definition, and show that while in Maxwell theory $\mathcal{R} = \mathcal{P}$, this is not true in general CFTs. For the probes and CFTs considered in this work, $\mathcal{R}$ can be written as

$$\mathcal{R} = -2\pi B_{\mathcal{R}} a^\lambda a_\lambda \tag{1.7}$$

where $B_{\mathcal{R}}$ is a new coefficient that in general differs from the Bremsstrahlung function B. Furthermore we find that the relation

$$B_{\mathcal{R}} = \frac{8}{3} h \tag{1.8}$$

holds in all the cases considered. This relation has thus the potential to be universal for all probes and all CFTs.

We turn then our attention to Lagrangian $\mathcal{N} = 2$ SCFTs, and for $\mathcal{N} = 4$ super Yang-Mills, we do find a surprise. The full one-point function of the energy density in the presence of a probe following an arbitrary trajectory has exactly the same spacetime dependence at weak and at strong 't Hooft coupling. This leads us to conjecture that this quantity is protected by non-renormalization. This would be rather surprising, as for generic timelike trajectories, $\langle T^{\mu\nu}\rangle_W$ is not a BPS quantity.

The structure of the paper is the following. In section 2, we revisit radiation by probes coupled to free field theories. We show that once we take into account the improvement term of the energy-momentum tensor for non-minimally coupled scalars, the radiative energy density is not positive definite, which is just a manifestation of the more general fact that non-minimally coupled scalars can violate energy conditions even classically [25]. Furthermore, for non-minimally coupled scalars, the radiated power $\mathcal{P}$ is not Lorentz invariant. The new term that we find in the rate of 4-momentum loss is formally similar to the Schott term that appears in the Lorentz-Dirac equation in electrodynamics [1]. We will argue however that in theories with non-minimally coupled scalars its origin and meaning are different than the Schott term in classical electrodynamics.

In section 3, we discuss constraints imposed by conformal symmetry on the one-point function of the energy-momentum tensor of a conformal field theory, in the presence of an arbitrary timelike line defect.

In section 4 we discuss radiation by 1/2-BPS probes coupled to $\mathcal{N}=2$ SCFTs. Quite remarkably, for a 1/2-BPS probe coupled to $\mathcal{N}=4$ super Yang Mills following an arbitrary trajectory, the classical computation with conformally coupled scalars matches exactly the angular distribution found holographically [17, 18].

In section 5 we mention some open questions. Our conventions are as follows: we work with a mostly minus metric, so the 4-velocity u and the 4-acceleration a satisfy $u^2=1, a^2<0$. Dots have different meaning for vectors and 4-vectors: $\dot{a}=da/d\tau$, but $\dot{\vec{a}}=d\vec{a}/dt$. Our overall normalization of the energy-momentum tensor for scalars is not the usual one; it has been chosen for convenience when we add scalar and vector contributions in supersymmetric theories.

2 Radiation in free field theories

Consider a probe coupled to a field theory, following an arbitrary, prescribed, timelike trajectory $z^\mu(\tau)$. One first solves the equations of motion for the field theory, in the presence of this source, choosing the retarded solution. Let x^μ be the point where the field is being measured; define τ_{ret} by the intersection of the past light-cone of x^μ and the worldline of the probe, and the null vector $\ell = x - z(\tau_{\text{ret}})$.

One then evaluates the energy-momentum tensor with the retarded solution. Usually one defines the radiative part of the energy-momentum tensor $T_r^{\mu\nu}$ as the piece that decays as $1/r^2$ so it yields a nonzero flux arbitrarily far away from the source. A more restrictive definition of $T_r^{\mu\nu}$ was introduced in [26, 27], who required that

- $\partial_\mu T_r^{\mu\nu}=0$ away from the source.
- $\ell_\mu T_r^{\mu\nu}=0$ so flux through the light-cone emanating from the source is zero.
- $T_r^{\mu\nu}=\frac{A}{(\ell\cdot u)^4}\ell^\mu\ell^\nu$ with A a Lorentz scalar.
- $A\geq 0$ so the radiative energy density is nonnegative.

In this work we will consider theories that don't satisfy the weak energy condition classically; for these theories, the requirement that the radiative energy density is nonnegative is less well motivated. In this work we use the first definition of $T_r^{\mu\nu}$, but we will discuss the implications of considering the second one. From $T_r^{\mu\nu}$ we define [1]

$$\frac{dP^\mu}{d\tau d\Omega}=r^2 T_r^{\mu\nu}u_\nu \tag{2.1}$$

and integrating over the solid angle we obtain $dP^\mu/d\tau$. It is a 4-vector [28] that gives the rate of energy and momentum emitted by the probe. From it one can define two quantities. The first one is the radiated power $\mathcal{P}$,

$$\mathcal{P} = \frac{dP^0}{dt} \tag{2.2}$$

which is not manifestly Lorentz invariant. Following Rohrlich [1], we define a second quantity, the invariant radiation rate $\mathcal{R}$ as

$$\mathcal{R} = u_\mu \frac{dP^\mu}{d\tau} \tag{2.3}$$

which is manifestly Lorentz invariant. For free CFTs, this invariant radiation rate can be written as

$$\mathcal{R} = -2\pi B_{\mathcal{R}} a^\lambda a_\lambda \tag{2.4}$$

We don't have a proof that this is the most generic form that $\mathcal{R}$ can take in interacting CFTs, but let's mention some restrictions. In principle there could be also a term in (2.4) proportional $u\cdot\dot{a}$, but since $a^2 = -u\cdot\dot{a}$, it would be redundant. Furthermore, by dimensional analysis, terms with higher derivatives of a can't appear in (2.4). In conclusion, (2.4) is the most general form that $\mathcal{R}$ can take, if it depends only on Lorentz invariants evaluated at a single retarded time.

2.1 Maxwell field

The energy-momentum tensor is

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F^{\mu\lambda}F_\lambda^{\ \nu} + \frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\beta\alpha}\right) \tag{2.5}$$

It is traceless, without using the equations of motion. Consider a probe coupled to the Maxwell field, with charge q, following an arbitrary trajectory. The full energy-momentum tensor evaluated on the retarded solution is [28]

$$T^{\mu\nu} = \frac{q^2}{4\pi}\left(\frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell\cdot u)^5}(1-\ell\cdot a) + \frac{\ell^\mu a^\nu + \ell^\nu a^\mu}{(\ell\cdot u)^4} - \frac{a^2}{(\ell\cdot u)^4}\ell^\mu\ell^\nu - \frac{(1-\ell\cdot a)^2}{(\ell\cdot u)^6}\ell^\mu\ell^\nu - \frac{1}{2}\frac{\eta^{\mu\nu}}{(\ell\cdot u)^4}\right) \tag{2.6}$$

where all quantities are evaluated at retarded time. Evaluating (2.6) for a static probe we derive the h coefficient [5]

$$T^{00}\big|_{\vec{v}=\vec{0}} = \frac{q^2}{8\pi}\frac{1}{r^4} \qquad \Rightarrow \qquad h = \frac{q^2}{8\pi} \tag{2.7}$$

The part of (2.6) decaying as $1/r^2$ is

$$T_r^{\mu\nu} = -\frac{q^2}{4\pi}\left(\frac{a^2}{(\ell\cdot u)^4} + \frac{(\ell\cdot a)^2}{(\ell\cdot u)^6}\right)\ell^\mu\ell^\nu \tag{2.8}$$

It satisfies all the criteria of [26, 27], so it is the radiative part according to both definitions. Integration over angular variables yields

$$\frac{dP^\mu}{d\tau} = -\frac{2}{3}q^2 a^\lambda a_\lambda u^\mu \tag{2.9}$$

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It is a future-oriented timelike 4-vector, guaranteeing that all inertial observers agree that the particle is radiating away energy. The relativistic Larmor's formula follows

$$\mathcal{P} = \mathcal{R} = -\frac{2}{3} q^2 a^\lambda a_\lambda \tag{2.10}$$

recall that $a^2 < 0$ in our conventions. From (2.10) we derive the Bremsstrahlung coefficient for Maxwell's theory,

$$B = \frac{q^2}{3\pi} \tag{2.11}$$

It follows from (2.7) and (2.11) that [12]

$$B = \frac{8}{3} h. \tag{2.12}$$

2.2 Scalar fields

Consider a free massless scalar field, with arbitrary coupling ξ to the Ricci scalar. The energy-momentum tensor is [23]

$$4\pi T^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\alpha \phi \partial^\alpha \phi - \xi (\partial^\mu \partial^\nu - \eta^{\mu\nu} \Box) \phi^2 \tag{2.13}$$

In general, the trace of (2.13) does not vanish, even when applying the equations of motion. For the conformal value $\xi = \frac{1}{6}$ it vanishes away from the sources, if we apply the equations of motion. For $\xi \neq 0$, this energy-momentum tensor can violate the weak energy condition at the classical level [25], even in Minkowski space.

Now consider a probe coupled to the scalar field, following an arbitrary trajectory. The energy-momentum tensor (2.13) evaluated on the retarded solution of the equation of motion is

$$\begin{aligned} 4\pi T^{\mu\nu} = \frac{q^2}{(\ell\cdot u)^4} \bigg(& (1-6\xi) u^\mu u^\nu - (1-8\xi) \frac{1-\ell\cdot a}{\ell\cdot u} (\ell^\mu u^\nu + \ell^\nu u^\mu) + 2\xi (\ell^\mu a^\nu + \ell^\nu a^\mu) \\ & + (1-8\xi) \frac{(1-\ell\cdot a)^2}{(\ell\cdot u)^2} \ell^\mu \ell^\nu + 2\xi \frac{\ell\cdot\dot{a}}{\ell\cdot u} \ell^\mu \ell^\nu + \frac{1-8\xi}{2} \eta^{\mu\nu} - (1-6\xi)(\ell\cdot a) \eta^{\mu\nu} \bigg) \end{aligned} \tag{2.14}$$

evaluated at retarded time. It depends on $\dot{a} = da/d\tau$, because the improved energy-momentum tensor (2.13) involves second derivatives of the field, and the solution depends on the velocity of the probe.

In the conformal case $\xi = 1/6$ the terms independent or linear in the acceleration are the same as in (2.6), up to an overall factor. In the next section, we will argue that these terms are actually universal for all CFTs.

Evaluating (2.14) on a static probe for the conformal value $\xi = 1/6$, we derive [5]

$$T^{00}\big|_{\vec{v}=\vec{0}} = \frac{1-4\frac{1}{6}}{8\pi} \frac{q^2}{r^4} \;\Rightarrow\; h = \frac{1}{24\pi} q^2 \tag{2.15}$$

The part of (2.14) decaying as $1/r^2$ is

$$T_r^{\mu\nu} = \frac{q^2}{4\pi} \left((1-8\xi) \frac{(\ell\cdot a)^2}{(\ell\cdot u)^6} + 2\xi \frac{\ell\cdot\dot{a}}{(\ell\cdot u)^5} \right) \ell^\mu \ell^\nu \tag{2.16}$$

It satisfies the first three criteria of [27] to be the radiative part. It also satisfies $|T^{00}| = |T^{0i}|$. As a check, for $\xi = 0$, it reduces to the energy density found in [17], which is manifestly positive definite. However, for $\xi \neq 0$, T^{00} is not guaranteed to be positive. After integration over the angular variables, we find

$$\frac{dP^\mu}{d\tau} = -\frac{1}{3} q^2 a^\lambda a_\lambda u^\mu - \frac{2\xi}{3} q^2 \dot{a}^\mu \tag{2.17}$$

The improvement term in the energy-momentum tensor of the scalar field (2.13) induces a qualitatively new term in $dP^\mu/d\tau$, compared with the electrodynamics case. The additional term in (2.17) is a total derivative, and it is formally identical to the Schott term in classical electrodynamics [1]. However, the origin is different. In classical electrodynamics, the Schott term appears in the Lorentz-Dirac equation of motion of the probe, and it can be deduced from the fields created by the probe, in the zone near its worldline. It does not appear from evaluating the radiative part of the energy-momentum tensor (2.8). On the other hand, in (2.17) the new term appears directly from evaluating the energy-momentum tensor of the fields that decay like $1/r^2$, away from the probe.

This additional term that we have encountered in (2.17) in a free theory computation has the same form as the additional term found holographically by [18], eq. (1.6). In that context, the works [20, 21] have advocated using the more restrictive definition of $T_r^{\mu\nu}$, thus setting $\xi = 0$ in (2.16), (2.17). An argument in favor of doing so is that the new term in (2.17) is a total derivative so, for instance, its contribution vanishes for any periodic motion when integrated over a full period. This clashes with the intuition of radiated energy as something irretriavably lost by the particle. However, we think this intuition is built on the idea that the energy density is positive definite, which is not the case for non-minimally coupled fields.

For a minimally coupled scalar field, $\xi = 0$, $dP^\mu/d\tau$ is again a future-oriented, timelike 4-vector, and $\mathcal{P} = \mathcal{R}$, as in Maxwell's theory [3, 17]. On the other hand, for $\xi \neq 0$, this 4-vector is no longer guaranteed to be timelike. This is related with T^{00} no longer being positive definite. In the instantaneous rest frame,

$$\left.\frac{dP^\mu}{d\tau}\right|_{\vec{v}=0} = \left(\frac{1-2\xi}{3} q^2 \vec{a}^2, -\frac{2\xi}{3} q^2 \dot{\vec{a}}\right) \tag{2.18}$$

So for $\xi < 1/2$, in the instantaneous rest frame, there is energy loss. However, if $dP^\mu/d\tau$ is spacelike, the sign of its zeroth component is no longer the same in all inertial frames.

For a non-minimally coupled scalar, $\mathcal{P}$ and $\mathcal{R}$ no longer coincide, and $\mathcal{P}$ is not Lorentz invariant. Indeed,

$$\mathcal{P} = -\frac{1}{3} q^2 a^\lambda a_\lambda - \frac{2\xi}{3} q^2 \frac{\dot{a}^0}{\gamma} \tag{2.19}$$

and

$$\mathcal{R} = -\frac{1-2\xi}{3} q^2 a^\lambda a_\lambda \tag{2.20}$$

For non-minimally coupled scalars, we will still define $2\pi B$ as the coefficient in front of the $-a^\lambda a_\lambda$ term in (2.19). We furthermore introduce a new coefficient B_ξ, as the coefficient in

$\mathcal{R} = -2\pi B_\xi a^\lambda a_\lambda$. We obtain

$$B_\xi = \frac{1-2\xi}{6\pi} q^2 \tag{2.21}$$

Notice that $B_{\xi=0} = B$; we also define $B_{\mathcal{R}} = B_{\xi=1/6}$. In particular, for the conformally coupled scalar it follows that $B_{\mathcal{R}} = \frac{8}{3}h$. This ratio is the same as in Maxwell's theory, eq. (2.12).

3 One-point function of the energy-momentum tensor in CFTs

In this section we discuss the constraints that conformal invariance imposes on the one-point function of the energy-momentum tensor of a conformal field theory, in the presence of a timelike line defect. While in the rest of the paper we consider Lagrangian field theories and the line defects are Wilson lines, the arguments of this section apply to arbitrary line defects in general CFTs.

For classical conformal field theories, we have seen in the previous section that the full one-point function of the energy-momentum tensor at a point in spacetime depends on the value of the 4-velocity and the 4-acceleration evaluated at a single retarded time. It is far from obvious that this feature should hold for generic line defects in arbitrary CFTs. In fact, once one considers strongly coupled conformal non-Abelian gauge theories, there are compelling arguments [18] that virtual timelike quanta will decay into further quanta thus forming a cascade, so the radiation measured at a point in spacetime does not have its origin at just a single retarded time in the probe worldline. This picture suggests that at least in some theories, the full one-point function should include integrals over the worldline of the probe, up to the retarded time,

$$\langle T^{\mu\nu}\rangle_W = \int^{\tau_{\rm ret}} d\tau f(a) + \dots \tag{3.1}$$

to take into account radiation originated by the cascade of timelike virtual quanta. Intriguingly enough, the holographic computations of [17, 18] do not find such terms for $\mathcal{N}=4$ SYM in the planar limit. We will make a small comment about the presence or not of these terms for generic CFTs at the end of this section.

In the present discussion we will focus on the terms where the kinematic 4-vectors, like the 4-velocity and the 4-acceleration appear in the answer evaluated at a single time, without any integrals. Dimensional analysis, conformal symmetry and conservation of the energy-momentum tensor constraint the form of the answer.

The full energy-momentum tensor of a CFT in the presence of a static probe is fixed by conformal invariance [5], up to an overall coefficient,

$$\langle T^{00}\rangle_W = \frac{h}{|\vec{x}|^4} \qquad \langle T^{0i}\rangle_W = 0 \qquad \langle T^{ij}\rangle_W = \frac{h}{|\vec{x}|^4}\left(\delta^{ij} - 2\frac{x^i x^j}{|\vec{x}|^2}\right) \tag{3.2}$$

By applying a boost, it is then also fixed for a probe with constant velocity. This determines all the acceleration independent terms; since they are universal, they can be read off from (2.6) or (2.14). These terms decay as $1/r^4$ as dictated by dimensional analysis,

$$\langle T^{\mu\nu}\rangle_W|_{\vec{v}=constant} = h\left(-\frac{\eta^{\mu\nu}}{(\ell\cdot u)^4} + 2\frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell\cdot u)^5} - 2\frac{\ell^\mu \ell^\nu}{(\ell\cdot u)^6}\right) \tag{3.3}$$

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Furthermore, by applying a special conformal transformation to a static worldline, one obtains a worldline with constant proper acceleration. Therefore, for any CFT, the full energy-momentum tensor for a hyperbolic line defect is completely determined up to an overall constant. It is immediate to check that $\langle T_{\mu\nu}\rangle_W$ for Maxwell theory, eq. (2.6), and for a conformal scalar, eq. (2.14) with $\xi = 1/6$, have the same spacetime dependence for hyperbolic motion, since in this case $\dot{a} = -a^2 u$.

We will now argue that the previous property implies that the terms linear in the 4-acceleration a must also be universal. The argument goes as follows. Since a worldline with constant proper acceleration satisfies $\dot{a} = -a^2 u$, terms that are not universal in $T^{\mu\nu}$ and change from one CFT to another, must be such that they collapse to the same universal expression when $\dot{a} = -a^2 u$. But terms linear in a don't depend on $\dot{a}$ or a^2, so they must be universal for all CFTs. These terms decay as $1/r^3$ as dictated by dimensional analysis. All in all, the terms independent or linear in a are,

$$\langle T^{\mu\nu}\rangle_W|_{\mathcal{O}(a)} = 2h\left(-\frac{1}{2}\frac{\eta^{\mu\nu}}{(\ell\cdot u)^4} + (1-\ell\cdot a)\frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell\cdot u)^5} + \frac{\ell^\mu a^\nu + \ell^\nu a^\mu}{(\ell\cdot u)^4} - (1-2\ell\cdot a)\frac{\ell^\mu \ell^\nu}{(\ell\cdot u)^6}\right) \tag{3.4}$$

We then conclude that the terms in $\langle T_{\mu\nu}\rangle_W$ independent or linear in the 4-acceleration a^λ — which respectively decay as $1/r^4$ and $1/r^3$ — are universal for all CFTs. On the other hand, terms that involve a^2 or $\dot{a}$ and decay like $1/r^2$ are not uniquely fixed by conformal invariance. Indeed, the $1/r^2$ terms for Maxwell's theory (2.8) and a conformal scalar (2.16) are different.

The formula (3.4) refers only to terms that depend only on the probe worldline at the retarded time, and does not exclude potential additional terms of the schematic form (3.1). To conclude this section, let's comment on the restrictions that conservation of the energy-momentum tensor imposes on the presence of possible terms of the type (3.1), that depend on the worldline of the probe, and not just the retarded time. First of all, the full energy-momentum tensor is conserved. We can further require that the piece of the energy-momentum tensor that decays like $1/r^2$ is conserved by itself, since it corresponds to energy that is detached from the probe. It then follows that the piece of $\langle T^{\mu\nu}\rangle$ that doesn't decay like $1/r^2$ must also be conserved by itself. It is straightforward to check that the terms that appear explicitly in (3.4) are conserved. This implies that if there are additional terms of the type (3.1) that decay like faster than $1/r^2$ beyond the ones that appear in (3.4), they must be conserved on their own.

4 Radiation in $\mathcal{N} = 2$ superconformal theories

The discussion in the previous section was completely classical. In this section we consider $\mathcal{N} = 2$ Lagrangian SCFTs, for which powerful techniques to study the strong coupling regime are available.

Consider the energy-momentum tensor created by a 1/2-BPS probe coupled to a Lagrangian $\mathcal{N} = 2$ SCFT in the classical limit. The probe is coupled to a vector and a scalar in the adjoint representation of the gauge group. As argued in [17, 18], at very weak coupling this amounts to adding the contribution of the Maxwell (2.8) and free scalar (2.16) terms, with an effective charge. However [17, 18] considered a free minimally coupled scalar. In CFTs, the correct computation amounts to adding (2.6) and (2.14) with the conformal value, $\xi = 1/6$. We obtain

$$
\begin{aligned}
T^{\mu\nu}_{\mathcal{N}=2} = 2h^{\mathcal{N}=2}\left(-\frac{1}{2}\frac{\eta^{\mu\nu}}{(\ell\cdot u)^4} + (1-\ell\cdot a)\frac{\ell^\mu u^\nu + \ell^\nu u^\mu}{(\ell\cdot u)^5} + \frac{\ell^\mu a^\nu + \ell^\nu a^\mu}{(\ell\cdot u)^4} - (1-2\ell\cdot a)\frac{\ell^\mu\ell^\nu}{(\ell\cdot u)^6}\right) \\
+ \frac{h^{\mathcal{N}=2}}{2}\left(-\frac{3a^2}{(\ell\cdot u)^4} + \frac{\ell\cdot\dot{a}}{(\ell\cdot u)^5} - 4\frac{(\ell\cdot a)^2}{(\ell\cdot u)^6}\right)\ell^\mu\ell^\nu
\end{aligned}
\tag{4.1}
$$

In three-dimensional language, with $\vec{n} = \frac{\vec{r}-\vec{z}}{|\vec{r}-\vec{z}|}$, the radiative energy density is

$$
T^{00}_{\mathcal{N}=2} = \frac{h^{\mathcal{N}=2}}{2r^2}\left(\frac{4\,|\vec{a}|^2 + 3\gamma^2(\vec{\beta}\cdot\vec{a})^2 + \vec{\beta}\cdot\dot{\vec{a}}}{(1-\vec{\beta}\cdot\vec{n})^4} + \frac{5(\vec{\beta}\cdot\vec{a})(\vec{n}\cdot\vec{a}) - \gamma^{-2}\vec{n}\cdot\dot{\vec{a}}}{(1-\vec{\beta}\cdot\vec{n})^5} - 4\frac{\gamma^{-2}(\vec{n}\cdot\vec{a})^2}{(1-\vec{\beta}\cdot\vec{n})^6}\right)
\tag{4.2}
$$

Our free classical computation only guarantees (4.1), (4.2) at leading order in λ, for small λ. Strikingly, the 00 component of (4.1) is exactly the same result found by a rather elaborate holographic computation for a 1/2-BPS probe in the fundamental representation of $\mathcal{N} = 4$ SU(N) super Yang-Mills in [17, 18], in the planar limit and at strong 't Hooft coupling where [14] $3h = B = \sqrt{\lambda}/4\pi^2$! To elaborate, we have computed the $1/r^4, 1/r^3$ terms at strong coupling, using the results of the holographic computations of [17, 18] and have found exactly the first line of (4.1). The match of the spacetime dependence of these terms at weak and strong coupling is not surprising, as we have argued in section 3 that they are universal. Nevertheless, this match does provide a strong check of the holographic computations in [17, 18]. On the other hand, the $1/r^2$ term (4.2) was already computed at strong coupling in [17, 18], and again it displays the same spacetime dependence as the classical result. We stress that we find exact agreement at the level of energy density, before performing any time average. This agreement prompts us to conjecture that (4.1) is true for all values of λ, in the planar limit. It is tempting to conjecture that (4.1) is true even at finite N and finite λ, but we currently don't have evidence for this stronger claim. Conformal symmetry alone is not enough to explain this agreement: comparing (2.8), (2.16) and (4.1) it is clear that the radiative energy density of a probe in arbitrary motion is not the same for different conformal field theories. Furthemore, while the probe is 1/2-BPS, it is following an arbitrary trajectory, so the Wilson line does not preserve any supersymmetry globally.

Many of the unexpected features of (4.2) have simple classical explanations that arise from properties of conformally coupled scalars: the fact that (4.2) is not positive definite everywhere, was interpreted in [17] as an inherently quantum effect. In fact, it's a feature already present at the classical level, reflecting that conformally coupled scalar fields can violate energy conditions even classically. As first noticed in [18], (4.2) depends on the derivative of the acceleration; now we understand that this follows from the fact that the improved tensor (2.13) involves second derivatives of the field. Another puzzle raised in [18] is that in $\mathcal{N} = 4$ SYM, radiation was isotropic at weak coupling; as our classical derivation of (4.2) shows, this isotropy is just an artifact of considering minimally coupled scalars, instead of conformally coupled ones.

In [17] it was noticed that for circular motion, while the angular distribution of radiated power computed holographically did not match the classical computation of Maxwell plus minimally coupled scalar, the respective time averages over a period did match. The reason is now easy to understand: the details of the angular distribution depend on ξ, but after averaging over a period, the averaged angular distribution is independent of ξ.

Let's discuss now the total radiated power in $\mathcal{N}=2$ SCFTs. Integration of (4.2) over angular variables yields

$$\frac{dP^\mu}{d\tau}=-2\pi B^{\mathcal{N}=2}\left(a^\lambda a_\lambda u^\mu+\frac{1}{9}\dot{a}^\mu\right) \tag{4.3}$$

Our computation ensures that this formula is valid at the classical level. At strong coupling, the only evidence is the $\mathcal{N}=4$ SYM holographic computation of [18].

To conclude, let's comment on the relation $B^{\mathcal{N}=2}=3h^{\mathcal{N}=2}$ conjectured in [10, 12] and proved in [13] for generic, not necessarily Lagrangian, $\mathcal{N}=2$ SCFTs. This is a relation between the Bremsstrahlung coefficient as defined in (1.3) and the $h^{\mathcal{N}=2}$ coefficient, as defined in (1.2). The proof presented in [13] relies on $12B^{\mathcal{N}=2}=C_D$, but not on the argument [8] that identifies $B^{\mathcal{N}=2}$ defined in (1.3) with the $h^{\mathcal{N}=2}$ coefficient in (1.2). The values obtained in section 2 allow to test that this relation is satisfied by a free U(1) $\mathcal{N}=2$ SCFT, and in fact by any Lagrangian $\mathcal{N}=2$ SCFT at weak coupling,

$$B^{\mathcal{N}=2}=B^{EM}+B^{\text{scalar}}=3(h^{EM}+h^{\text{scalar}})=3h^{\mathcal{N}=2} \tag{4.4}$$

On the other hand, it also follows that the coefficients $B_{\mathcal{R}}^{\mathcal{N}=2}$ and $h^{\mathcal{N}=2}$ of any Lagrangian $\mathcal{N}=2$ SCFT satisfy, at weak coupling, the same relation as in Maxwell theory or for a conformal scalar,

$$B_{\mathcal{R}}^{\mathcal{N}=2}=B_{\mathcal{R}}^{EM}+B_{\mathcal{R}}^{\text{scalar}}=\frac{8}{3}h^{EM}+\frac{8}{3}h^{\text{scalar}}=\frac{8}{3}h^{\mathcal{N}=2} \tag{4.5}$$

At strong coupling, contracting (4.3) with u_μ and using $B^{\mathcal{N}=2}=3h^{\mathcal{N}=2}$, we again obtain

$$B_{\mathcal{R}}^{\mathcal{N}=2}=\frac{8}{9}B^{\mathcal{N}=2}=\frac{8}{3}h^{\mathcal{N}=2} \tag{4.6}$$

which is again the relation found for Maxwell's theory and for a free conformal scalar. So if (4.3) holds, (4.6) would be true for all the probes coupled to CFTs considered in this paper. Currently, the only evidence for (4.3) at strong coupling is the holographic computation of [18] for $\mathcal{N}=4$ SYM.

5 Discussion and outlook

In this work we have discussed radiation for theories with scalar fields. We have found that for non-minimally coupled scalars, the energy density is no longer positive definite, it depends on the derivative of the acceleration of the probe, and the radiated power is not Lorentz invariant. These three features were also encountered in the strongly coupled regime of $\mathcal{N}=4$ super Yang-Mills, by holographic computations [17, 18]. In the introduction we mentioned that these computations do not quite agree with the holographic

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computations at the probe string/brane level. The backreacted computations of [17, 18] are on a firmer theoretical ground, but the results they yielded were unexpected, casting doubts on their validity. Our work implies that these features are to be expected for any conformal field theory with conformal scalars, and confirm the validity of the holographic computations of [17, 18].

In this work we have not discussed radiation reaction on the probe coupled to the scalar field. It would be interesting to discuss it for the case of non-minimally coupled scalars.

We have shown that the relation (4.6) holds for probes of free CFTs, and we have presented evidence that it also holds for 1/2-BPS probes in $\mathcal{N} = 4$ SCFTs. At this point it is not clear whether it holds for arbitrary probes of generic CFTs. A possible case to further test it would be less supersymmetric probes of $\mathcal{N} = 4$ super Yang-Mills.

The fact that (4.2) holds both at weak and strong λ in the planar limit of $\mathcal{N} = 4$ super Yang-Mills is rather mysterious, as it is not a BPS quantity. It will be important to prove if (4.2) holds for any λ, in the planar limit, or even at finite N. An even stronger conjecture is that it holds for generic $\mathcal{N} = 2$ superconformal theories, but currently we lack techniques to study $\langle T^{\mu\nu}\rangle_W$ at strong coupling for generic $\mathcal{N} = 2$ SCFTs and arbitrary timelike worldlines.

Finally, this note has only considered radiation of scalar fields in Minkowski spacetime. It will be interesting to generalize our results to other spacetimes.

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References

[1] F. Rohrlich, *Classical charged particles*, Addison-Wesley, U.S.A. (1990).

[2] LIGO Scientific and Virgo collaborations, *Observation of gravitational waves from a binary black hole merger*, *Phys. Rev. Lett.* **116** (2016) 061102 [arXiv:1602.03837] [INSPIRE].

[3] R.G. Cawley, *Radiation of classical scalar fields from a point source — $m \to 0$ limit of massive theory vs. $m = 0$ theory*, *Annals Phys.* **54** (1969) 149 [INSPIRE].

[4] C.M. Will, *The confrontation between general relativity and experiment*, *Living Rev. Rel.* **17** (2014) 4 [arXiv:1403.7377] [INSPIRE].

[5] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev.* **D 74** (2006) 025005 [hep-th/0501015] [INSPIRE].

[6] B. Fiol, B. Garolera and A. Lewkowycz, *Exact results for static and radiative fields of a quark in $N = 4$ super Yang-Mills*, *JHEP* **05** (2012) 093 [arXiv:1202.5292] [INSPIRE].

[7] A.M. Polyakov, *Gauge fields as rings of glue*, *Nucl. Phys.* **B 164** (1980) 171 [INSPIRE].

[8] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2012) 048 [arXiv:1202.4455] [INSPIRE].

[9] M. Billó, M. Caselle, D. Gaiotto, F. Gliozzi, M. Meineri and R. Pellegrini, *Line defects in the 3d Ising model*, *JHEP* **07** (2013) 055 [arXiv:1304.4110] [INSPIRE].

[10] B. Fiol, E. Gerchkovitz and Z. Komargodski, *Exact Bremsstrahlung function in $N = 2$ superconformal field theories*, *Phys. Rev. Lett.* **116** (2016) 081601 [arXiv:1510.01332] [INSPIRE].

[11] B. Fiol, J. Martínez-Montoya and A. Rios Fukelman, *Wilson loops in terms of color invariants*, *JHEP* **05** (2019) 202 [arXiv:1812.06890] [INSPIRE].

[12] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, *JHEP* **05** (2014) 025 [arXiv:1312.5682] [INSPIRE].

[13] L. Bianchi, M. Lemos and M. Meineri, *Line defects and radiation in $N = 2$ conformal theories*, *Phys. Rev. Lett.* **121** (2018) 141601 [arXiv:1805.04111] [INSPIRE].

[14] A. Mikhailov, *Nonlinear waves in AdS/CFT correspondence*, hep-th/0305196 [INSPIRE].

[15] B. Fiol and B. Garolera, *Energy loss of an infinitely massive half-Bogomol'nyi-Prasad-Sommerfeld particle by radiation to all orders in $1/N$*, *Phys. Rev. Lett.* **107** (2011) 151601 [arXiv:1106.5418] [INSPIRE].

[16] B. Fiol, A. Güijosa and J.F. Pedraza, *Branes from light: embeddings and energetics for symmetric k-quarks in $N = 4$ SYM*, *JHEP* **01** (2015) 149 [arXiv:1410.0692] [INSPIRE].

[17] C. Athanasiou, P.M. Chesler, H. Liu, D. Nickel and K. Rajagopal, *Synchrotron radiation in strongly coupled conformal field theories*, *Phys. Rev.* **D 81** (2010) 126001 [*Erratum ibid.* **D 84** (2011) 069901] [arXiv:1001.3880] [INSPIRE].

[18] Y. Hatta, E. Iancu, A.H. Mueller and D.N. Triantafyllopoulos, *Radiation by a heavy quark in $N = 4$ SYM at strong coupling*, *Nucl. Phys.* **B 850** (2011) 31 [arXiv:1102.0232] [INSPIRE].

[19] V.E. Hubeny, *Holographic dual of collimated radiation*, *New J. Phys.* **13** (2011) 035006 [arXiv:1012.3561] [INSPIRE].

[20] R. Baier, *On radiation by a heavy quark in $N = 4$ SYM*, *Adv. High Energy Phys.* **2012** (2012) 592854 [arXiv:1107.4250] [INSPIRE].

[21] C.A. Agón, A. Guijosa and J.F. Pedraza, *Radiation and a dynamical UV/IR connection in AdS/CFT*, *JHEP* **06** (2014) 043 [arXiv:1402.5961] [INSPIRE].

[22] N.D. Birrell and P.C.W. Davies, *Quantum fields in curved space*, Cambridge University Press, Cambridge, U.K. (1982) [INSPIRE].

[23] C.G. Callan Jr., S.R. Coleman and R. Jackiw, *A new improved energy-momentum tensor*, *Annals Phys.* **59** (1970) 42 [INSPIRE].

[24] F. Rohrlich, *The definition of electromagnetic radiation*, *Nuovo Cim.* **21** (1961) 811.

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[25] J.D. Bekenstein, *Exact solutions of Einstein conformal scalar equations*, *Annals Phys.* **82** (1974) 535 [INSPIRE].

[26] C. Teitelboim, *Splitting of the Maxwell tensor-radiation reaction without advanced fields*, *Phys. Rev.* **D 1** (1970) 1572 [*Erratum ibid.* **D 2** (1970) 1763] [INSPIRE].

[27] C. Teitelboim, D. Villarroel and C. van Weert, *Classical electrodynamics of retarded fields and point particles*, *Riv. Nuovo Cim.* **3N9** (1980) 1 [INSPIRE].

[28] A. Schild, *On the radiation emitted by an accelerated point charge*, *J. Math. Anal. Appl.* **1** (1960) 127.

Coupling dependence of radiation: Part 1

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Wilson loops in terms of color invariants

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Abstract: We derive an expression for the vacuum expectation value (vev) of the 1/2 BPS circular Wilson loop of $\mathcal{N}=4$ super Yang Mills in terms of color invariants, valid for any representation R of any gauge group G. This expression allows us to discuss various exact relations among vevs in different representations. We also display the reduction of these color invariants to simpler ones, up to seventh order in perturbation theory, and verify that the resulting expression is considerably simpler for the logarithm of $\langle W\rangle_R$ than for $\langle W\rangle_R$ itself. We find that in the particular case of the symmetric and antisymmetric representations of SU(N), the logarithm of $\langle W\rangle_R$ satisfies a quadratic Casimir factorization up to seventh order, and argue that this property holds to all orders. Finally, we derive the large N expansion of $\langle W\rangle_R$ for an arbitrary, but fixed, representation of SU(N), up to order $1/N^2$.

Keywords: Wilson, 't Hooft and Polyakov loops, 1/N Expansion, Matrix Models, Supersymmetric Gauge Theory

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Contents

1 Introduction

Wilson loops are among the fundamental operators in gauge theories. Nevertheless, when it comes to extracting physically interesting quantities, many of them are determined in terms of the logarithm of the vacuum expectation value (vev) of certain Wilson loops, and not the vevs themselves. For instance, the quark anti-quark static potential is determined from the logarithm of the vev of a rectangular Wilson loop. Similarly, the cusp anomalous dimension [1] is the logarithm of the properly regularized vev of a Wilson loop with a cusp, dependent on the boost parameter φ

$$\langle W_\varphi \rangle \sim e^{\Gamma_{\mathrm{cusp}}(\varphi) \ln \frac{\Lambda_{\mathrm{UV}}}{\Lambda_{\mathrm{IR}}}} \tag{1.1}$$

The question then arises whether one can directly compute the logarithm of the vacuum expectation value of the Wilson loop, bypassing the computation of the vev of the Wilson loop itself. At the perturbative level, according to the non-Abelian exponentiation theorem [2, 3] (see [4] for a pedagogical review), for certain cases the answer is positive. One has to evaluate just a subset of the Feynman diagrams that would appear in the computation of the vev of the Wilson loop, with the proviso that each Feynman diagram carries now a modified color factor, and not the standard one assigned according to the ordinary Feynman rules. The application of the non-Abelian exponentiation theorem to the computation of the perturbative cusp anomalous dimension is discussed for QCD in [5] and for $\mathcal{N}=4$ super Yang-Mills in [6].

In order to understand the content of the non-Abelian exponentiation theorem, it is very clarifying to consider Wilson loops W_R in arbitrary representations R of the gauge group G. The perturbative expansion of their vevs can then be written in terms of color invariants. These color invariants involve contractions of the fully symmetrized traces [7, 8]

$$d_R^{a_1\dots a_n} = \frac{1}{n!}\,\mathrm{tr}\sum_{\sigma\in\mathcal{S}_n} T_R^{a_{\sigma(1)}}\dots T_R^{a_{\sigma(n)}} \tag{1.2}$$

where T_R^a are the generators of the Lie algebra of the group G, in the representation R.[1] Some examples of color invariants are d_R^{aabb} or $d_R^{abcd}d_A^{abcd}$. The non-Abelian exponentiation theorem implies that certain color invariants present in $\langle W\rangle_R$ are absent in $\ln\langle W\rangle_R$.

In the bulk of this paper we will consider the interplay of the non-Abelian exponentiation theorem and the evaluation of the vev of Wilson loops in $\mathcal{N}=4$ SYM, leaving the case of $\mathcal{N}=2$ SCFTs for future work. Nevertheless, before describing the results obtained that are specific of $\mathcal{N}=4$ SYM, we want to argue that this theorem — which is valid for non-Abelian gauge theories regardless of the amount of supersymmetry — also provides evidence for a conjecture formulated for generic $\mathcal{N}=2$ SCFTs [9]. To present the conjecture, and our argument, it is necessary to introduce a couple of quantities that will also appear in the main body of the paper.

First, the Bremsstrahlung function B associated to a heavy probe is defined [10] from the small boost limit of the cusp anomalous dimension (1.1),

$$\Gamma_{\text{cusp}}(\varphi) = B\varphi^2 + \mathcal{O}(\varphi^4) \tag{1.3}$$

This coefficient determines a number of interesting properties of a heavy probe coupled to a conformal field theory: its energy loss by radiation [10], its momentum diffusion coefficient [11] and the change in entanglement entropy it causes in a spherical region [12]. Since the cusp anomalous dimension satisfies the non-Abelian exponentiation theorem, so does the Bremsstrahlung function: only a subset of the most general color invariants will appear in its expansion. On the other hand, in any four-dimensional conformal field theory, the two-point function of the stress-energy tensor and a straight Wilson line is determined by conformal invariance, up to a coefficient h_W [13]

$$\frac{\langle T^{00}(x)W\rangle}{\langle W\rangle} = \frac{h_W}{|\vec{x}|^4} \tag{1.4}$$

This coefficient appears also in the two-point function of the stress-energy tensor and a circular Wilson loop. From this definition, there is no hint that h_W should involve only a subset of color invariants. Nevertheless, for $\mathcal{N}=2$ SCFTs, these two coefficients are related as

$$B = 3h_W \tag{1.5}$$

[1] We follow the convention that the previous definition with no indices means $d_R = \mathrm{tr}_R\,\mathbf{1} = \dim R$. The appendix contains our conventions for color invariants, a summary of techniques useful to evaluate them, and their evaluation for various representations and gauge groups.

This identity was first noticed to hold in $\mathcal{N}=4$ super Yang-Mills, by explicit computation [10, 14]; it was conjectured to hold for $\mathcal{N}=2$ SCFTs in [9, 12] and recently proven in [15]. However this identity is somewhat surprising in light of the previous comments. For arbitrary gauge group G and representation R, B can be expressed in terms of just a subset of color invariants. Why should that be the case also for h_W? In [9], it was further conjectured that for $\mathcal{N}=2$ SCFTs

$$h_W = \frac{1}{12\pi^2}\partial_b \ln \langle W_b\rangle|_{b=1} \tag{1.6}$$

where $\langle W_b\rangle$ is the vev of a circular Wilson loop in a squashed sphere of parameter b. This conjecture has been checked up to three loops [9, 16]; we want to show that the non-Abelian exponentiation theorem — which applies to non-Abelian gauge theories regardless of the amount of supersymmetry — provides evidence of this conjecture (1.6) by arguing that both sides of (1.6) involve at every order in perturbation theory the same subset of color invariants. On the one hand, given that (1.5) is now an established result [15], we know that h_W involves that same subset of color invariants as B. On the other hand, by virtue of the non-Abelian exponentiation theorem, the perturbative expansion of $\ln\langle W_b\rangle$ involves also just the reduced set of color invariants implied by this theorem. What this argument doesn't prove is that the coefficients that appear in front of the color invariants in the expansions of both sides of (1.5) also coincide; it doesn't address the non-perturbative validity of (1.5) either. The same comments apply to similar relations between various Bremsstrahlung functions and logarithms of Wilson loops in 3d ABJM theories [17–20].

After this detour, let's now describe the contents of the body of the paper. In this work we will focus on 1/2 BPS Wilson loops of $\mathcal{N}=4$ super Yang-Mills, and the quantities that can be obtained from these operators. Locally BPS Wilson loops of $\mathcal{N}=4$ super Yang-Mills depend on a representation R of the gauge group G, and a spacetime contour $\mathcal{C}$

$$W_R[\mathcal{C}] = \frac{1}{\text{dim R}}\text{tr}_R\mathcal{P}\exp\left(i\int_{\mathcal{C}}(A_\mu\dot{x}^\mu + |\dot{x}|\Phi_i\theta^i)ds\right) \tag{1.7}$$

When the contour is a circle (in Euclidean signature) the vev of this Wilson loop can be computed by supersymmetric localization [21] that reduces it to a Gaussian matrix model over the Lie algebra $\mathfrak{g}$

$$\langle W\rangle_R = \frac{1}{\text{dim } R}\frac{\int_{\mathfrak{g}} dM \text{tr}_R e^M\, e^{-\frac{1}{2g}\text{tr } M^2}}{\int_{\mathfrak{g}} dM e^{-\frac{1}{2g}\text{tr } M^2}} \tag{1.8}$$

The most common approach to tackle this type of matrix integrals is to first reduce the integral over the Lie algebra to an integral over a Cartan subalgebra $\mathfrak{h}$. This introduces a Jacobian, given by a Vandermonde determinant $\Delta(X)^2$,

$$\langle W\rangle_R = \frac{1}{\text{dim } R}\frac{\int_{\mathfrak{h}} dX\Delta(X)^2\ \text{tr}_R e^X\, e^{-\frac{1}{2g}\text{tr } X^2}}{\int_{\mathfrak{h}} dX\Delta(X)^2\, e^{-\frac{1}{2g}\text{tr } X^2}} \tag{1.9}$$

Then one applies either the method of orthogonal polynomials at finite N, or the saddle point approximation at large N (see [22] for a pedagogical review). This approach yields compact expressions for particular choices of G and R, but obscures the generic structure. In the current work, we are not going to follow this approach. Instead, following recent works [23, 24] we will not restrict the integrals to a Cartan subalgebra $\mathfrak{h}$ as in (1.9), but rather integrate over the full Lie algebra $\mathfrak{g}$, as in (1.8). At the technical level, the advantage is that the Vandermonde determinant is not generated, and the matrix integrals are truly trivial, since they are Gaussian. They can be carried out at once, for any R and G, just applying Wick's theorem. At the conceptual level, the benefit of this approach is that the results obtained are in terms of color invariants. Our first result is that the vev of W_R can be written in term of symmetrized traces (1.2), with pairwise contracted indices,

$$\langle W \rangle_R = \frac{1}{d_R} \sum_{k=0}^{\infty} d_R^{a_1 a_1 \ldots a_k a_k} \frac{1}{k!} \left(\frac{g_{\rm YM}^2}{4} \right)^k \tag{1.10}$$

This expression gives the vev of 1/2 BPS circular Wilson loop for any representation R of a gauge group G. It allows to discuss exact relations among vevs in different representations. For instance, if R^t is the transpose representation of R (in the sense of having Young diagrams transpose to each other), we will argue that

$$\langle W \rangle_{R^t}(\lambda, N) = \langle W \rangle_R(\lambda, -N) \tag{1.11}$$

thus relating, for instance, vevs in the symmetric and the antisymmetric representations of SU(N). It is possible to take the logarithm of (1.10), to obtain a closed expression for $\ln \langle W \rangle_R$, but this closed expression is of very little use; in particular, the non-Abelian exponentiation theorem is not manifest. On the other hand, the color invariants $d_R^{a_1 a_1 \ldots a_k a_k}$ in (1.10) can be reduced to lower order color invariants. As it will be illustrated in the main body of the paper, this expansion is simpler for $\ln \langle W \rangle_R$ than for $\langle W \rangle_R$ itself: the only color invariants that appear in the perturbative expansion of $\ln \langle W \rangle_R$ at a given order are those that can't be written as products of color invariants that appear at lower orders of the perturbative expansion, thus providing an illustration of the non-Abelian exponentiation theorem.

The structure of the papers is as follows. In section 2 we derive an exact expression for $\langle W \rangle_R$ in terms of color invariants, and present some exact relations among vevs of different representations. In section 3 we study the large N limit of $\langle W \rangle_R$ for arbitrary, but fixed, representations of SU(N), up to order $1/N^2$. In section 4, we present the expansion of $\ln \langle W \rangle_R$ in terms of color invariants; we provide a diagrammatic interpretation of the expansion, and discuss some patterns present in the perturbative expansion. The appendix contains our conventions for color invariants, a summary of the techniques we use to evaluate them, and tables of the evaluation of various color invariants.

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2 $\langle W \rangle_R$ in terms of color invariants

In this section we revisit the evaluation of $\langle W \rangle_R$ for $\mathcal{N} = 4$ super Yang Mills, for an arbitrary representation R of a generic Lie algebra G. Thanks to supersymmetric localization [21] this problem reduces to a Gaussian matrix model, and it has been solved exactly, for various choices of gauge group G and representation R [25–27]. As mentioned in the introduction, typically this is done by first reducing the matrix integral to an integral over the Cartan subalgebra, as in (1.9). While this procedure allows to obtain compact expressions for $\langle W \rangle_R$ for some choices of R, this has to be done in a case by case basis, and it obscures the dependence on the choice of G and R. Since in this work we are particularly interested in expressing $\langle W \rangle_R$ in terms of color invariants, we will follow a different route. We will instead carry out the integrals over the full Lie algebra. Specifically,

$$\langle W \rangle_R = \frac{1}{d_R} \langle \mathrm{tr}_R \, e^{2\pi M} \rangle = \frac{1}{\dim \, R} \frac{\int_{\mathfrak{g}} dM \, \mathrm{tr}_R e^{2\pi M} \, e^{-\frac{8\pi^2}{g_{\rm YM}^2} \mathrm{tr} \, M^2}}{\int_{\mathfrak{g}} dM \, e^{-\frac{8\pi^2}{g_{\rm YM}^2} \mathrm{tr} \, M^2}} \tag{2.1}$$

If we denote by m^a the coefficients of the matrix M in the Lie algebra, the two-point function in this Gaussian matrix model is

$$\left\langle m^a m^b \right\rangle = \frac{g_{\rm YM}^2}{8\pi^2} \delta^{ab} \qquad a, b = 1, \ldots, d_A \tag{2.2}$$

To compute the vev of the normalized Wilson loop, we expand the exponent insertion in (2.1), use the two-point function (2.2) and apply Wick's theorem,

$$\langle W \rangle_R = \frac{1}{d_R} \sum_{k=0}^{\infty} \frac{(2\pi)^{2k}}{(2k)!} \langle m^{a_1} \ldots m^{a_{2k}} \rangle \mathrm{tr} \, T_R^{a_1} \ldots T_R^{a_{2k}} = \frac{1}{d_R} \sum_{k=0}^{\infty} d_R^{a_1 a_1 \ldots a_k a_k} \frac{g^k}{k!} \tag{2.3}$$

where $g = g_{\rm YM}^2/4$ and $d_R^{a_1 \ldots a_k}$ are the symmetrized traces defined in (1.2). This expression for $\langle W \rangle_R^{\mathcal{N}=4}$ is exact — recall that there are no instanton corrections for $\langle W \rangle_R$ in $\mathcal{N} = 4$ [21] — and valid for any G and any R. It encompasses and unifies all the known results for particular choices of G and R [25–27].

Now it is a matter of evaluating $d_R^{a_1 a_1 \ldots a_k a_k}$, the fully symmetrized traces (1.2) with pairwise contracted indices. At every order, the outcome is a combination of lower order color invariants, involving the original representation R, and the adjoint representation A. At low orders, it's easy enough to evaluate them by hand, using the techniques detailed in [7]. For instance,

$$\begin{aligned} d_R^{aa} &= \mathrm{tr} \, T_R^a T_R^a = c_R d_R \\ d_R^{aabb} &= \frac{1}{3} \, \mathrm{tr} \left(2 T_R^a T_R^a T_R^b T_R^b + T_R^a T_R^b T_R^a T_R^b \right) = \left(c_R^2 - \frac{1}{6} c_A c_R \right) d_R \end{aligned}$$

To push the evaluation to higher orders, we use FormTracer [28]. Up to order g_{YM}^{14} we obtain

$$
\begin{aligned}
\langle W\rangle_R = & \; 1 + c_R g + \left(c_R^2 - \frac{1}{6}c_R c_A\right)\frac{g^2}{2!} + \left(c_R^3 - \frac{1}{2}c_R^2 c_A + \frac{1}{12}c_R c_A^2\right)\frac{g^3}{3!} \\
& + \left(c_R^4 - c_R^3 c_A + \frac{5}{12}c_R^2 c_A^2 - \frac{5}{72}c_R c_A^3 + \frac{1}{15}\frac{d_R^{abcd}d_A^{abcd}}{d_R}\right)\frac{g^4}{4!} \\
& + \left(c_R^5 - \frac{5}{3}c_R^4 c_A + \frac{5}{4}c_R^3 c_A^2 - \frac{35}{72}c_R^2 c_A^3 + \frac{35}{432}c_R c_A^4\right. \\
& \left. + \left(\frac{1}{3}c_R - \frac{2}{9}c_A\right)\frac{d_R^{abcd}d_A^{abcd}}{d_R} + \frac{1}{90}c_R\frac{d_A^{abcd}d_A^{abcd}}{d_A}\right)\frac{g^5}{5!} \\
& + \left(-\frac{35}{288}c_R c_A^5 + \frac{35}{48}c_A^4 c_R^2 - \frac{35}{18}c_A^3 c_R^3 + \frac{35}{12}c_A^2 c_R^4 - \frac{5}{2}c_A c_R^5 + c_R^6 + \frac{1}{10}\frac{d_R^{abcd}d_A^{cdef}d_A^{efab}}{d_R}\right. \\
& + \left(\frac{1}{15}c_R^2 - \frac{11}{180}c_R c_A\right)\frac{d_A^{abcd}d_A^{abcd}}{d_A} \\
& \left. + \left(c_R^2 - \frac{3}{2}c_A c_R + \frac{11}{18}c_A^2\right)\frac{d_R^{abcd}d_A^{abcd}}{d_R} - \frac{8}{63}\frac{d_R^{abcdef}d_A^{abcdef}}{d_R}\right)\frac{g^6}{6!} \\
& + \left(c_R^7 - \frac{7}{2}c_A c_R^6 + \frac{35}{6}(c_A^2 c_R^5 - c_A^3 c_R^4) + \frac{175}{48}c_A^4 c_R^3 - \frac{385}{288}c_A^5 c_R^2 + \frac{72757}{326592}c_R c_A^6\right. \\
& \left(\frac{7}{3}c_R^3 - \frac{35}{6}c_R^2 c_A + \frac{21}{4}c_R c_A^2 - \frac{91}{54}c_A^3\right)\frac{d_R^{abcd}d_A^{abcd}}{d_R} \\
& + \left(\frac{7}{30}c_R^3 - \frac{7}{15}c_R^2 c_A + \frac{817}{3240}c_R c_A^2\right)\frac{d_A^{abcd}d_A^{abcd}}{d_A} \\
& \frac{691}{18900}c_R\frac{d_A^{abcd}d_A^{cdef}d_A^{efab}}{d_A} - \frac{80}{27}\frac{d_A^{abcdef}d_R^{abcg}d_A^{defg}}{d_R} + \frac{14}{45}\frac{d_R^{abcdef}d_A^{abcg}d_A^{defg}}{d_R} \\
& \left. \frac{7}{10}(c_R - c_A)\frac{d_R^{abcd}d_A^{cdef}d_A^{efab}}{d_R} + \frac{8}{9}(c_A - c_R)\frac{d_R^{abcdef}d_A^{abcdef}}{d_R}\right)\frac{g^7}{7!} + \ldots
\end{aligned}
\tag{2.4}
$$

By construction, every color invariant in this expansion involves an even number of indices. Since $d_A^{a_1\ldots a_k} = 0$ for k odd, for every color invariant the adjoint representation contributes an even number of indices, and thus the representation R also contributes an even number. Up to the order computed, no color invariants involving $d_R^{a_1\ldots a_k}$ with an odd number of indices appear in the expansion. They would necessarily involve more than one $d_R^{a_1\ldots a_k}$, e.g. $d_R^{abc}d_R^{ade}d_A^{bcde}$. It is not clear to us whether such color invariants will appear at higher orders.

The reader that feels intimidated by the expansion (2.4) might find some comfort in the fact that, as we will show in section 4, the perturbative expansion of $\ln\langle W\rangle_R$ is considerably simpler.

Besides the possibility of evaluating $\langle W\rangle_R$ order by order in g_{YM} for all G and R at once, the result (2.3) allows to derive some general exact relations among vevs of 1/2 BPS Wilson loops in different representations. The first identity that we will point out is rather obvious. For a generic representation R, recall that the complex conjugate representation

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$\bar{R}$ of R has generators $T_{\bar{R}} = -T_R^t$. Then, since dim R = dim $\bar{R}$, it follows that

$$\langle W \rangle_{\bar{R}} = \langle W \rangle_R \tag{2.5}$$

As an illustration of this equality, we have that $\langle W \rangle_{A_k}^{\mathrm{SU}(N)} = \langle W \rangle_{A_{N-k}}^{\mathrm{SU}(N)}$, an identity that is readily seen to hold in the explicit results of [26].

A less trivial relation involves representations of classical Lie groups with transposed Young diagrams. For instance, irreducible representations of SU(N) are labelled by Young diagrams, and exchanging symmetrization and antisymmetrization of indices amounts to transposing the Young diagram. It is known [8, 29] that under this operation, color invariants change as $N \to -N$, up to an overall sign. For a representation R whose Young diagram has k boxes, the overall sign is

$$d_{R^t}^{a_1 a_1 \ldots a_m a_m}(N) = (-1)^{k+m} d_R^{a_1 a_1 \ldots a_m a_m}(-N) \tag{2.6}$$

In particular, dim $R^t(N) = (-1)^k$ dim $R(-N)$, as one can check for SU(N) using (A.3). Since we are considering normalized Wilson loops, divided by dim R, the $(-1)^k$ cancels in (2.3). The remaining $(-1)^m$ can be absorbed by expanding the vev in powers of the 't Hooft coupling $\lambda = g_{\mathrm{YM}}^2 N$ instead of in powers of g_{YM}^2. Overall, we arrive at the relation

$$\langle W \rangle_{R^t}(\lambda, N) = \langle W \rangle_R(\lambda, -N) \tag{2.7}$$

In the next section, we will provide an alternative derivation of this identity for SU(N). As a first illustration, a particular example of this identity is the relation

$$\langle W \rangle_F^{\mathrm{Sp}(N)}(N) = \langle W \rangle_F^{\mathrm{SO}(2N)}(-N) \tag{2.8}$$

found in [27]. Moreover, (2.7) implies that the vevs of Wilson loops in the symmetric and antisymmetric representations of SU(N) satisfy

$$\langle W \rangle_{S_k}(\lambda, N) = \langle W \rangle_{A_k}(\lambda, -N) \tag{2.9}$$

since the Young diagrams of the $k-$symmetric and $k-$antisymmetric representations are transpose of each other. Okuyama [30] recently found evidence for this particular consequence of the identity in eq. (2.7).[2] To illustrate the relation in eq. (2.9) we evaluate (2.3) for $G = \mathrm{SU}(N)$ and $R = S_k, A_k$ up to seventh order in λ, applying methods explained in the appendix. For compactness, we actually display the perturbative expansion of ln $\langle W \rangle_{S_k/A_k}$, with the upper signs corresponding to the symmetric representation, and the lower signs

[2] To compare our identity (2.9) with the one in [30], note that the Wilson loops in [30] are not normalized by the dimension of the representation. If the Young diagram associated with the representation R has k boxes, dim $R^t(N) = (-1)^k$ dim $R(-N)$, so normalizing the Wilson loop by dim R introduces an additional $(-1)^k$ factor in the relation, proving the equivalence of the result in [30] and ours.

to the antisymmetric one,

$$
\begin{aligned}
\ln \langle W\rangle_{S_k/A_k} = {} & \frac{k(N\pm k)(N\mp 1)}{2N^2}\left(\frac{\lambda}{4}-\frac{1}{12}\left(\frac{\lambda}{4}\right)^2+\frac{1}{72}\left(\frac{\lambda}{4}\right)^3+\frac{-4N^2\mp N+k(k\pm N)}{1440N^2}\left(\frac{\lambda}{4}\right)^4\right.\\
&+\frac{13N^2\pm 10N-10k(k\pm N)+3}{21600N^2}\left(\frac{\lambda}{4}\right)^5\\
&-\left(\frac{11}{112}+\frac{43(\pm N-k(k\pm N))}{280N^2}+\frac{109}{1008N^2}\right.\\
&\left.+\frac{\pm 73N-113Nk(N\pm k)+20k^2(N\pm k)^2}{2520N^4}\right)\frac{1}{6!}\left(\frac{\lambda}{4}\right)^6\\
&+\left(\frac{647}{20321280}\pm\frac{89}{1036800N}+\frac{10501}{101606400N^2}\pm\frac{17}{268800N^3}+\frac{197}{12700800N^4}\right.\\
&\mp\frac{5471k}{65318400N}-\frac{19}{268800N^2}-\frac{4499k^2}{65318400N^2}\mp\frac{k}{44800N^3}\mp\frac{19k^2}{268800N^3}\\
&\left.\left.\pm\frac{k^3}{33600N^3}-\frac{k^2}{44800N^4}+\frac{k^4}{67200N^4}\right)\left(\frac{\lambda}{4}\right)^7\right)+\dots
\end{aligned}
\tag{2.10}
$$

Notice that, at least up to order g_{YM}^{14}, all the coefficients factorize, and have a common factor that happens to be essentially the quadratic Casimir c_{S_k/A_k},

$$
c_{S_k/A_k}=\frac{k(N\pm k)(N\mp 1)}{2N}
\tag{2.11}
$$

This factorization is unexpected and, as the next example shows, it does not happen for generic representations. In the next section we will discuss this factorization in more detail, and argue that for $\ln\langle W\rangle_{S_k/A_k}$ it holds to all orders.

Another implication of the identity (2.7) is that if R is a $\mathrm{SU}(N)$ representation with a self-transpose Young diagram, $\langle W\rangle_R(\lambda,N)$ admits a $1/N^2$ rather than the more general $1/N$ expansion. A first illustration of this point is the fact that $\langle W\rangle_{\square}^{\mathrm{SU}(N)}$ has a $1/N^2$ expansion. As a second illustration of this point, we display the perturbative expansion of $\ln\langle W\rangle_{\tiny\yng(2,1)}^{\mathrm{SU}(N)}$ up to seventh order in λ, showing that every coefficient has a $1/N^2$ expansion,

$$
\begin{aligned}
\ln\langle W\rangle_{\tiny\yng(2,1)}^{\mathrm{SU}(N)} = {} & \frac{3(N^2-3)}{2N^2}\frac{\lambda}{4}-\frac{N^2-3}{8N^2}\left(\frac{\lambda}{4}\right)^2+\frac{N^2-3}{48N^2}\left(\frac{\lambda}{4}\right)^3+\frac{-4N^4+19N^2-27}{960N^4}\left(\frac{\lambda}{4}\right)^4\\
&+\frac{13N^4-106N^2+261}{14400N^4}\left(\frac{\lambda}{4}\right)^5+\frac{-495N^6+6796N^4-23269N^2+9720}{2419200N^6}\left(\frac{\lambda}{4}\right)^6\\
&+\frac{3235N^6-71360N^4+310273N^2-268572}{67737600N^6}\left(\frac{\lambda}{4}\right)^7+\dots
\end{aligned}
\tag{2.12}
$$

While we are discussing identities (2.5) and (2.7) for 1/2 BPS circular Wilson loops of $\mathcal{N}=4$ SYM, since they are mostly based on group theoretic properties of the color invariants, we expect that similar identities hold in more generic theories, for other observables defined in terms of a representation R of a classical Lie group G.

Equations (2.5) and (2.7) are exact relations, valid for finite λ and N. When the gauge group has a classical Lie algebra (and therefore a large N gravity dual), these exact

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relations have implications for the holographic dual. In particular, let's comment briefly on the implications of $\langle W\rangle_R^{\mathrm{SU}(N)}$ having a $1/N^2$ expansion when R is a representation with a self-transpose Young diagram.

In the probe limit, the holographic dual to a Wilson loop operator with an arbitrary Young diagram is a system of D3 and D5-branes in IIB [31–33]. Considering the transpose representation amounts to exchanging D3 and D5 branes. The identity (2.7) implies that in the particular case when the D-brane system is invariant under the exchange of D3 and D5 branes, corrections have a $1/N^2$ expansion. If we keep increasing the size of the Young diagram, the correct dual gravitational description eventually is in terms of bubbling geometries, half-BPS solutions of IIB supergravity, fully described in [34]. The representation R is geometrically encoded in a hyperelliptic curve, and a self-transpose Young diagram corresponds to hyperelliptic curves with an additional $\mathbb{Z}_2$ symmetry. Again, our results imply that corrections to the supergravity action evaluated on these backgrounds have $1/N^2$ as expansion parameter, instead of $1/N$. It would be interesting to check these predictions on the various regimes of the holographic dual.

3 Large N expansion of $\langle W\rangle_R^{\mathrm{SU}(N)}$

In this section we expand the vev of the unnormalized 1/2 BPS Wilson loop for a generic but fixed representation of $\mathrm{SU}(N)$ in the large N limit. We will obtain the leading term, the $1/N$ and the $1/N^2$ corrections. We do so for a fixed representation, i.e. we do not consider the interesting case where the number of boxes in the Young diagram of the representation scales with N. For recent work in that direction see [35–38].

The strategy to obtain this expansion will be the following. We will first recall some basic properties of the representation theory of the symmetric group, including Frobenius formula. By virtue of the Schur-Weyl duality, this formula yields an exact relation for vevs of Wilson loops of $\mathrm{U}(N)$: it gives the vev of the Wilson loop in an arbitrary representation as a linear combination of correlators of multiply-wound Wilson loops in the fundamental representation. The $1/N$ expansion of these correlators allows then to derive the large N expansion of $\langle W\rangle_R^{\mathrm{U}(N)}$ in arbitrary representations. Finally, since the rest of the paper deals with $\mathrm{SU}(N)$ rather than $\mathrm{U}(N)$, we will take into account the relation between $\langle W\rangle_R^{\mathrm{U}(N)}$ and $\langle W\rangle_R^{\mathrm{SU}(N)}$ in order to be able to make detailed comparisons with the results in the previous section.

Let R be an arbitrary irreducible representation of $\mathrm{U}(N)$, whose associated Young diagram has k boxes. Due to the Schur-Weyl duality, this Young diagram is also associated to an irreducible representation r of the symmetric group $\mathcal{S}_k$. For our purposes, it will be convenient to recall some basic facts about the symmetric group $\mathcal{S}_k$ [39]. A permutation $\pi \in \mathcal{S}_k$ is of cycle type $(1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$ if it has m_j cycles of length j. Two permutations of $\mathcal{S}_k$ are in the same conjugacy class if and only if have the same cycle type. Conjugacy classes of $\mathcal{S}_k$ are labelled by partitions of k, or equivalenty by Young diagrams with k boxes: the conjugacy class $\lambda = (1^{m_1}, 2^{m_2}, \ldots, k^{m_k})$ corresponds to a diagram with m_j rows of j boxes. Finally, if we define $z_\lambda = 1^{m_1} m_1! 2^{m_2} m_2! \ldots k^{m_k} k!$, the number of elements in the conjugacy class λ is $k!/z_\lambda$.

Frobenius formula allows to write the Schur polynomial of a irreducible representation as a linear combination of power sum symmetric polynomials [39],

$$s_\lambda = \sum_\pi \frac{\chi^\lambda(\pi)}{z_\lambda} p_\pi \tag{3.1}$$

where $\chi^\lambda(\pi)$ is the character of λ evaluated in the conjugacy class π, and the sum runs over conjugacy classes of $\mathcal{S}_k$, so there are $p(k)$ terms, the number of partitions of k.

Thanks to the Schur-Weyl duality, (3.1) can be uplifted to a statement relating representations of U(N). First, the Schur polynomial corresponds to the character of the representation R, so its integral precisely yields $\langle W\rangle_R$. On the other hand, the $n-$times wound Wilson loop in the fundamental representation, which we denote by $W(n)$, corresponds to tr $\left(e^{2\pi M}\right)^n$. Then, for the conjugacy class $(1^{m_1},\ldots,k^{m_k})$, the power sum symmetric polynomial corresponds to the insertion of $W(1)^{m_1}\ldots W(k)^{m_k}$. All in all, we arrive at

$$\langle W\rangle_R = \sum_\lambda \frac{\chi^r(\lambda)}{z_\lambda}\langle W(1)^{m_1}\ldots W(k)^{m_k}\rangle \tag{3.2}$$

where $\chi^r(\lambda)$ is the character of R evaluated in the conjugacy class λ. This relation is exact, valid for finite N. As already mentioned above, the sum in (3.2) involves $p(k)$ terms, the number of partitions of k. However in the large N limit, only a few of these terms contribute to the leading behavior and the first subleading corrections. In fact, we will argue that to compute the first three terms in the large N expansion of $\langle W\rangle_R$, one needs to consider only four terms in the sum (3.2).

Let's now recall a couple of properties of the $n-$point functions $\langle W(1)^{m_1}\ldots W(k)^{m_k}\rangle$. Large N factorization implies that the leading behavior is given by $N^{\sum_j m_j}$; notice that $\sum_j m_j$ is the number of rows of the corresponding Young diagram. Furthermore, all these correlators have a $1/N^2$ expansion. These two properties allow us to give a different derivation of (2.7) for SU(N), or more precisely, its formulation for unnormalized Wilson loops,

$$\langle W\rangle_{R^t}(\lambda,N) = (-1)^k\langle W\rangle_R(\lambda,-N) \tag{3.3}$$

The argument goes as follows. If R is an irreducible representation of $\mathcal{S}_k$, $r^t = r\otimes \text{sgn}$ is also an irreducible representation, and their Young diagrams are transpose of each other. We then have $\chi^{r^t}(\lambda) = \text{sgn}\,\lambda\,\chi^r(\lambda)$. The sign of a permutation can be easily read off from its Young diagram,

$$\text{sgn}\;\lambda = (-1)^{k-\sum_j m_j} \tag{3.4}$$

where k is the total number of boxes and $\sum_j m_j$ is the number of rows. In other words, the exponent is the total number of boxes not in the first column. On the other hand, according to the two properties explained above, $\langle W(1)^{m_1}\ldots W(k)^{m_k}\rangle$ picks a sign $(-1)^{\sum_j m_j}$ under $N\to -N$. Plugging these two results into (3.2) yields the relation (3.3).

Let's discuss now the correlators of multiply wound Wilson loops that contribute to the leading terms of the large N expansion of $\langle W\rangle_R$. There is just one λ whose Young diagram has k rows, the vertical column, see figure 1. This is the only $n-$point function contributing to the leading term, of order N^k, and because it has a $1/N^2$ expansion, it also contributes

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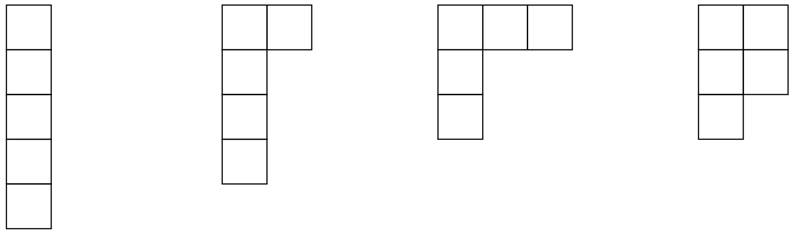

(a) 1^5 **(b)** $2^1 1^3$ **(c)** $3^1 1^2$ **(d)** $2^2 1^1$

Figure 1. The large N expansion of $\langle W\rangle_R^{\mathrm{SU}(N)}$ for any fixed R can be computed up to order $1/N^2$, in terms of four correlators of multiply-wound Wilson loops. Each correlator of multiply-wound Wilson loops has its own Young diagram, and the four relevant ones are displayed in this figure. They are shown for the particular example of an arbitrary irreducible representation R with $k=5$ boxes in its Young diagram. The first one contributes at leading order, and at $1/N^2$ order. The second one contributes at $1/N$ order. The last two ones contribute at $1/N^2$ order.

at order N^{k-2}, but not at order N^{k-1}. For its conjugacy class, $z_{1^k}=k!$. There is also just one $(k-1)-$point function contributing at order N^{k-1}, the one corresponding to the Young diagram where the k boxes are distributed in $k-1$ rows, $(2^1 1^{k-2})$, see figure 1. For its conjugacy class $z_{2^1 1^{k-2}}=2(k-2)!$. At order $1/N^2$, there are subleading contributions from $\left\langle W(1)^k\right\rangle$, and also leading contributions from two $(k-2)-$point functions, $\left\langle W(3)W(1)^{k-3}\right\rangle$ and $\left\langle W(2)^2 W(1)^{k-4}\right\rangle$, see figure 1. All in all,

$$\langle W\rangle_R = \frac{\chi^r(1^k)}{k!}\left\langle W(1)^k\right\rangle + \frac{\chi^r(2^1 1^{k-2})}{2(k-2)!}\left\langle W(2)W(1)^{k-2}\right\rangle + \frac{\chi^r(3^1 1^{k-3})}{3(k-3)!}\left\langle W(3)W(1)^{k-3}\right\rangle + \frac{\chi^r(2^2 1^{k-4})}{8(k-4)!}\left\langle W(2)^2 W(1)^{k-4}\right\rangle + \mathcal{O}(N^{k-3}) \tag{3.5}$$

To compute the leading contributions to the vevs, we use that in the large N limit, the $n-$point functions of the Gaussian matrix model factorize, and in the planar limit [40],

$$\left\langle \frac{1}{N} W(n)\right\rangle \to \frac{2}{n\sqrt{\lambda}} I_1(n\sqrt{\lambda}) \tag{3.6}$$

where $I_1(\sqrt{\lambda})$ is the modified Bessel function. Let's consider finally the subleading contributions of $\left\langle W(1)^k\right\rangle$, that contribute at order $1/N^2$. To do so, it is convenient to write $\left\langle W(1)^k\right\rangle$ in terms of connected correlators. At this order the relevant terms are

$$\left\langle W(1)^k\right\rangle = \langle W(1)\rangle^k + \binom{k}{2}\langle W(1)\rangle^{k-2}\langle W(1)W(1)\rangle_c + \dots \tag{3.7}$$

The dots correspond to more connected diagrams, which don't contribute at $1/N^2$ order. We see that $1/N^2$ contributions can come from two types of diagrams: first, from diagrams with k disconnected pieces, $k-1$ planar ones and a non-planar one; second, from planar diagrams with $k-1$ disconnected pieces ($k-2$ of them are $1-$point functions, the last one is a connected $2-$point function). The first contribution is obtained expanding the exact

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result of [25]

$$\left\langle \frac{1}{N} W(1) \right\rangle^k = \left(\frac{2I_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \frac{\lambda I_2(\sqrt{\lambda})}{48N^2} + \dots \right)^k = \frac{2^k I_1(\sqrt{\lambda})^k}{\lambda^{\frac{k}{2}}} + k \frac{2^k I_1(\sqrt{\lambda})^{k-1} I_2(\sqrt{\lambda})}{96 \lambda^{\frac{k-3}{2}}} \frac{1}{N^2} + \dots$$

For the second contribution we need the leading term of the connected two-point function of Wilson loops $\langle W(1)W(1)\rangle_c$ [41],

$$\langle W(1)W(1)\rangle_c = \frac{\sqrt{\lambda} I_0(\sqrt{\lambda}) I_1(\sqrt{\lambda})}{2} + \dots \tag{3.8}$$

All in all, the vev of the unnormalized Wilson loop has the following $1/N$ expansion,

$$\begin{aligned}
\langle W \rangle_R^{\mathrm{U}(N)} = & \frac{\chi^r(1^k)}{k!} \left(\frac{2}{\sqrt{\lambda}} I_1(\sqrt{\lambda}) \right)^k N^k + \frac{\chi^r(2^1 1^{k-2})}{4(k-2)!} \left(\frac{2}{\sqrt{\lambda}} \right)^{k-1} I_1(\sqrt{\lambda})^{k-2} I_1(\sqrt{4\lambda}) N^{k-1} \\
& + \left(\frac{\chi^r(1^k)}{(k-1)!} \frac{2^k I_1(\sqrt{\lambda})^{k-1}}{16 \lambda^{(k-3)/2}} \left(\frac{I_2(\sqrt{\lambda})}{6} + (k-1) I_0(\sqrt{\lambda}) \right) \right. \\
& + \frac{\chi^r(3^1 1^{k-3})}{3(k-3)!} \frac{2^k I_1(\sqrt{\lambda})^{k-3} I_1(\sqrt{9\lambda})}{12 \lambda^{(k-2)/2}} \\
& \left. + \frac{\chi^r(2^2 1^{k-4})}{8(k-4)!} \frac{2^k I_1(\sqrt{\lambda})^{k-4} I_1(\sqrt{4\lambda})^2}{16 \lambda^{(k-2)/2}} \right) N^{k-2} + \dots
\end{aligned} \tag{3.9}$$

We are now going to check that the general expansion (3.9) reproduces the explicit computations presented in the previous section. In order to make a detailed comparison, there are a couple of factors to take into account. The first one is that in the rest of the paper, the vevs are for $\mathrm{SU}(N)$ and not for $\mathrm{U}(N)$. This is not relevant for the leading term, but it affects the subleading terms. For the vev of 1/2 BPS Wilson loop in a representation R whose Young diagram has k boxes, they are related by

$$\langle W \rangle_R^{\mathrm{SU}(N)} = e^{-\frac{\lambda k^2}{8N^2}} \langle W \rangle_R^{\mathrm{U}(N)} \tag{3.10}$$

Since k is fixed (it does not scale with N), this introduces a correction at order $1/N^2$. The other issue is that in this section, unlike in the rest of the paper, we have been considering Wilson loops not normalized by the dimension. However, since we will compare the generic result with explicit computations of $\ln \langle W \rangle_R$, the dimension only contributes as a coupling-independent additive constant.

As a first check, let's consider the case of the S_k and A_k representations of $\mathrm{SU}(N)$. The corresponding representations of the symmetric group $\mathcal{S}_k$ are the trivial and the sign representations: k^1 and 1^k, respectively. Since these are one-dimensional representations of $\mathcal{S}_k$, their characters coincide with the representation elements: $\chi^k(\pi) = 1$, and $\chi^{1^k}(\pi) = \mathrm{sgn}\,\pi$. The signs of the four relevant permutations can be computed using (3.4) and consulting the figure 1. Applying then the formula (3.9) to S_k/A_k we obtain, up to a coupling-independent constant,

$$
\begin{aligned}
\ln \langle W\rangle^{\mathrm{SU}(N)}_{S_k/A_k} = k\ln\frac{I_1(\sqrt{\lambda})}{\sqrt{\lambda}} \pm \frac{k(k-1)}{8}\frac{\sqrt{\lambda}I_1(\sqrt{4\lambda})}{I_1(\sqrt{\lambda})^2}\frac{1}{N} + \Bigg(&-\frac{k^2}{8}\lambda + \frac{k\lambda^{3/2}I_2(\sqrt{\lambda})}{96I_1(\sqrt{\lambda})} \\
&+ \frac{k(k-1)\lambda^{3/2}I_0(\sqrt{\lambda})}{16I_1(\sqrt{\lambda})} + \frac{k(k-1)(k-2)\lambda I_1(\sqrt{9\lambda})}{36I_1(\sqrt{\lambda})^3} \\
&- \frac{k(k-1)(2k-3)\lambda I_1(\sqrt{4\lambda})^2}{64I_1(\sqrt{\lambda})^4}\Bigg)\frac{1}{N^2}
\end{aligned}
\tag{3.11}
$$

The $1/N$ correction vanishes for $k=1$, as it had to, since then the Wilson loop admits a $1/N^2$ expansion. This expression correctly reproduces the leading, $1/N$ and $1/N^2$ terms of the first orders computed in (2.10).

As a second check of the general result (3.9), consider the representation $\tiny\yng(2,1)$. Its character evaluated on the relevant conjugacy classes is $\chi^{\tiny\yng(2,1)}(\tiny\yng(1,1,1)) = 2$, $\chi^{\tiny\yng(2,1)}(\tiny\yng(2,1)) = 0$, $\chi^{\tiny\yng(2,1)}(\tiny\yng(3)) = -1$. The evaluation of (3.9) is then

$$
\ln \langle W\rangle^{\mathrm{SU}(N)}_{\tiny\yng(2,1)} = 3\ln\frac{I_1(\sqrt{\lambda})}{\sqrt{\lambda}} + \left(-\frac{9}{8}\lambda + \frac{3\lambda^{3/2}I_0(\sqrt{\lambda})}{8I_1(\sqrt{\lambda})} + \frac{\lambda^{3/2}I_2(\sqrt{\lambda})}{32I_1(\sqrt{\lambda})} - \frac{\lambda I_1(\sqrt{9\lambda})}{12I_1(\sqrt{\lambda})^3}\right)\frac{1}{N^2} + \ldots
\tag{3.12}
$$

which correctly reproduces the explicit computations displayed in (2.12).

4 Logarithm of $\langle W\rangle_R$

In section 2 we have obtained a formula (2.3) for the vev of the 1/2 BPS Wilson loop, for arbitrary G and R. For many applications, we are actually interested in $\ln\langle W\rangle_R$. In this section we will obtain a closed expression for $\ln\langle W\rangle_R$ and discuss its perturbative expansion.

From (2.3) it is possible to write the power series for the logarithm of $\langle W\rangle_R$, in terms of partial Bell polynomials $B_{n,k}$. Let's quickly recall the argument. A convenient way to define these polynomials is through its generating function,

$$
e^{\left(u\sum_{j=1}^{\infty} x_j \frac{t^j}{j!}\right)} = \sum_{n,k\geq 0} B_{n,k}(x_1,\ldots,x_{n-k+1})\frac{t^n}{n!}u^k
\tag{4.1}
$$

Then, for any power series of the form

$$
a(g) = 1 + \sum_{k=1}^{\infty}\frac{1}{k!}f_k g^k
\tag{4.2}
$$

making use of the Taylor series for $\ln(1+x)$ near the origin, we easily derive that

$$
\ln a(g) = \sum_{k=1}^{\infty}\frac{g^k}{k!}\sum_{j=1}^{k}(-1)^{j-1}(j-1)!B_{k,j}(f_1,f_2,\ldots,f_{k-j+1})
\tag{4.3}
$$

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Coming back to our specific problem, after defining $f_k = d_R^{a_1 a_1 \ldots a_k a_k}/N_R$, we have

$$\ln \langle W \rangle_R = \sum_{k=1}^{\infty} \frac{g^k}{k!} \sum_{j=1}^{k} (-1)^{j-1}(j-1)! B_{k,j}(f_1, f_2, \ldots, f_{k-j+1}) \tag{4.4}$$

As an application, using the result of [10], we obtain a closed formula for the Bremsstrahlung function (1.3) of any 1/2 BPS particle, for generic G and R

$$B_R(\lambda, N) = \frac{1}{2\pi^2}\lambda \frac{\partial \ln \langle W \rangle_R}{\partial \lambda} = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \frac{g^k}{(k-1)!} \sum_{j=1}^{k} (-1)^{j-1}(j-1)! B_{k,j}(f_1, f_2, \ldots, f_{k-j+1}) \tag{4.5}$$

Taking into account the $B = 3h_W$ relation (1.5), this also gives an expression for the coefficient h_w appearing in the two-point function of the 1/2 BPS Wilson loop and the stress-energy tensor, for arbitrary gauge group G and representation R.

While (4.4) is a closed expression for $\ln \langle W \rangle_R$, valid for any G and any R, it is extremely inefficient, and it obscures the fact that the perturbative expansion of $\ln \langle W \rangle_R$ is actually simpler than that of $\langle W \rangle_R$. To make this point manifest, let's compute $\ln \langle W \rangle_R$ from eq. (2.4), up to order $g_{\rm YM}^{14}$,

$$\begin{aligned}
\ln \langle W \rangle_R = {}& c_R g - \frac{1}{6} c_R c_A \frac{g^2}{2!} + \frac{1}{12} c_R c_A^2 \frac{g^3}{3!} + \left(-\frac{5}{72} c_R c_A^3 + \frac{1}{15} \frac{d_R^{abcd} d_A^{abcd}}{N_R} \right) \frac{g^4}{4!} \\
& + \left(\frac{35}{432} c_R c_A^4 - \frac{2}{9} c_A \frac{d_R^{abcd} d_A^{abcd}}{N_R} + \frac{1}{90} c_R \frac{d_A^{abcd} d_A^{abcd}}{d_A} \right) \frac{g^5}{5!} \\
& + \Bigg(-\frac{35}{288} c_R c_A^5 + \frac{11}{18} c_A^2 \frac{d_R^{abcd} d_A^{abcd}}{d_R} - \frac{11}{180} c_A c_R \frac{d_A^{abcd} d_A^{abcd}}{d_A} \\
& + \frac{1}{10} \frac{d_R^{abcd} d_A^{cdef} d_A^{efab}}{d_R} - \frac{8}{63} \frac{d_R^{abcdef} d_A^{abcdef}}{d_R} \Bigg) \frac{g^6}{6!} \\
& + \Bigg(\frac{72757}{326592} c_R c_A^6 - \frac{91}{54} c_A^3 \frac{d_R^{abcd} d_A^{abcd}}{d_R} + \frac{817}{3240} c_A^2 c_R \frac{d_A^{abcd} d_A^{abcd}}{d_A} \\
& - \frac{7}{10} c_A \frac{d_R^{abcd} d_A^{cdef} d_A^{efab}}{d_R} + \frac{691}{18900} c_R \frac{d_A^{abcd} d_A^{abef} d_A^{cdef}}{d_A} - \frac{8}{27} \frac{d_A^{abcdef} d_R^{abcg} d_A^{defg}}{d_R} \\
& + \frac{14}{45} \frac{d_R^{abcdef} d_A^{abcg} d_A^{defg}}{d_R} + \frac{8}{9} c_A \frac{d_R^{abcdef} d_A^{abcdef}}{d_R} \Bigg) \frac{g^7}{7!} + \ldots
\end{aligned} \tag{4.6}$$

Comparing with (2.4), we see that many color invariants present in the expansion of $\langle W \rangle_R$ are absent in the expansion of $\ln \langle W \rangle_R$. For instance, there are no color invariants in (4.6) involving c_R^k with $k \geq 2$. This simpler structure is a consequence of the non-Abelian exponentiation theorem [2, 3]: at every order in perturbation theory, the only color invariants that can appear in $\ln \langle W \rangle_R$ are the ones that can't be written as products of color invariants that appear at lower orders in the perturbative expansion of $\langle W \rangle_R$. So in practice, to obtain the expansion of $\ln \langle W \rangle_R$ in terms of color invariants, it is more efficient to expand $\langle W \rangle_R$ as in (2.4) and then discard by hand the terms that involve products of lower order color invariants.

4.1 Casimir factorization

In section 2, we noticed that the evaluation of the perturbative expansion (4.6) of $\ln \langle W \rangle_R$ for the case of the symmetric and antisymmetric representations of SU(N), eq. (2.10), showed an unexpected pattern up to the computed order, that we will refer to as Casimir factorization (not to be confused with the Casimir scaling hypothesis, as we discuss below).

On general grounds, the coefficients at every order in λ in $\ln \langle W \rangle_R$ are polynomials in $1/N$. Equation (2.10) shows that up to at least order λ^7, these coefficients factorize (as polynomials in $1/N$) with a universal factor, the quadratic Casimir divided by N, which is also quadratic polynomial in $1/N$. We refer to this feature as Casimir factorization. We will now argue that Casimir factorization of $\ln \langle W \rangle_{S_k/A_k}$ holds to all orders

$$\ln \langle W \rangle_{S_k/A_k} \stackrel{?}{=} \frac{c_{S_k/A_k}}{N} f_{S_k/A_k}(\lambda, N) \tag{4.7}$$

where f_{S_k/A_k} is such that at every order in λ the coefficient is a $k-$dependent *polynomial* in $1/N$. Recall that

$$c_{S_k/A_k} = \frac{k(N \pm k)(N \mp 1)}{2N} \tag{4.8}$$

so if we argue that at every order the coefficients of $\ln W_{S_k/A_k}$ are divisible by $(N \pm k)$ and $(N \mp 1)$, we are done. First, because of the relation $\langle W \rangle_{A_k} = \langle W \rangle_{A_{N-k}}$, that follows from the identity (2.5), $\ln \langle W \rangle_{A_k}$ must vanish when $k = N$, and together the identity (2.9), this implies that at every order the coefficients must have a $(\pm N + k)$ factor. Similarly, $\langle W \rangle^{\mathrm{SU}(N)}_{S_k}|_{N=1} = 1$, so $\ln \langle W \rangle^{\mathrm{SU}(N)}_{S_k}|_{N=1} = 0$. Again, together with the identity (2.9), this implies that at every order the coefficients must have a $(N \mp 1)$ factor, concluding the argument for (4.7).

The Casimir factorization (4.7) can't be true for generic representations, since the evaluation of $\ln \langle W \rangle_{\yng(2,1)}$, eq. (2.12), shows that it does not hold for that representation. Namely, as derived in the appendix, $c_{\yng(2,1)} = \frac{3(N^2-3)}{2N}$, but starting at order λ^4, the coefficients in the expansion of $\ln \langle W \rangle_{\yng(2,1)}$, eq. (2.12), are not divisible by $N^2 - 3$, so they don't satisfy the Casimir factorization. It would be interesting to determine if the Casimir factorization (4.7) of $\ln \langle W \rangle_R$ holds for other representations beyond the symmetric and the antisymmetric one.

Casimir factorization bears a superficial resemblance to the hypothesis of Casimir scaling, that states that various quantities derived from vevs of logarithms of Wilson loops in QCD — chiefly the quark-antiquark static potential [42] — depend on the choice of representation of the matter fields only through the quadratic Casimir c_R. Namely, for the logarithm of the vev of a Wilson loop,

$$\ln \langle W \rangle_R \stackrel{?}{=} c_R f(\lambda, N) \tag{4.9}$$

where $f(\lambda, N)$ is a universal function, independent of the representation R. In QCD, Casimir scaling of the quark-antiquark potential is known to be violated at three loops [43, 44]. For the cusp anomalous dimension, Casimir scaling holds up to three loops [45], but in QCD is violated starting at four loops [46]. For 1/2 BPS particles coupled to $\mathcal{N}=4$ SYM in arbitrary representations, it follows from the results of [10] — and our expression (4.6) makes it abundantly clear — that $\ln \langle W\rangle_R$ does not satisfy Casimir scaling, starting at four loops. Due to the relation (4.5), it follows that the Bremsstrahlung function, and therefore the full cusp anomalous dimension, also violates Casimir scaling starting at four loops. This violation at four-loops has also been observed by explicit computation in the light-like limit of the cusp anomalous dimension [47].

It is worth emphasizing that this Casimir factorization is a property of the color invariants themselves, and in this regard, doesn't provide any information about the dynamics of the theory. On the other hand, the original Casimir scaling is a statement about the vanishing of the coefficients in front of higher order color invariants.

Finally, let's remark that this discussion was at finite N. In the planar limit, it follows from (A.6) that for a representation R whose Young diagram has k boxes,

$$c_R \to \frac{k}{2N} \tag{4.10}$$

so it follows from eq. (3.9) that

$$\ln \langle W\rangle_R^{\text{planar}} = \frac{2c_R}{N} \ln \frac{2I_1(\sqrt{\lambda})}{\sqrt{\lambda}} \tag{4.11}$$

and we conclude that in the planar limit, the ordinary Casimir scaling actually holds for $\ln \langle W\rangle_R$ and the quantities derived from it, like the Bremsstrahlung function B_R.

4.2 Diagrammatic interpretation

We now want to provide a diagrammatic interpretation of the perturbative expansion (4.6) of $\ln \langle W\rangle_R$. It was argued in [25, 40] and proven in [21] that in the Feynman gauge, the only Feynman diagrams that contribute to $\langle W\rangle_R$ involve gluon propagators starting and ending on the Wilson line. In the Mathematics literature these diagrams have been studied thoroughly, and are called chord diagrams [48]. At order $2n$ there are $(2n-1)!!$ of them. On the other hand, by virtue of the non-Abelian exponentiation theorem [2, 3], to compute $\ln \langle W\rangle_R$ one only needs to take into account a subset of them, the so-called connected chord diagrams: diagrams where all gluon lines overlap with some other gluon line, see figure 2.

The number of connected chord diagrams with n chords satisfies the following recursion relation [49, 50]

$$a_1 = 1 \qquad a_n = (n-1)\sum_{k=1}^{n-1} a_k a_{n-k} \tag{4.12}$$

so the first values up to the seven loops considered in this work are

$$a_n = 1, 1, 4, 27, 248, 2830, 38232, \ldots \tag{4.13}$$

It can be proven [51] that asymptotically the ratio of the number of connected chord diagrams to the total number of chord diagrams with n gluons is given by

$$\lim_{n\to\infty} \frac{a_n}{(2n-1)!!} = \frac{1}{e} \tag{4.14}$$

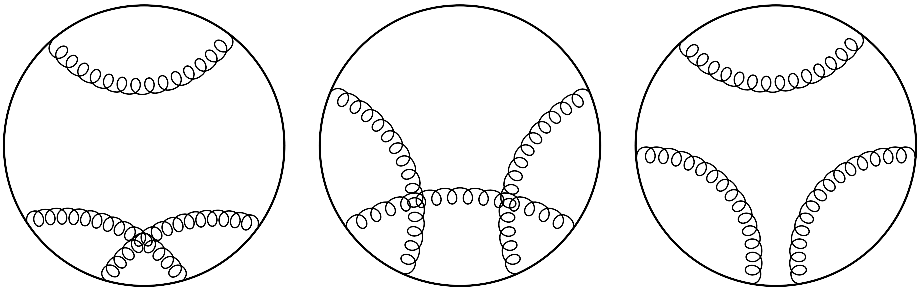
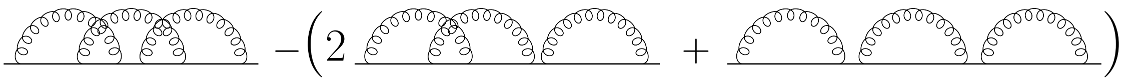


Figure 2. Various examples of chord diagrams: the first one is a generic gluon diagram, they contribute to $\langle W\rangle_R$ at finite N. For k gluons, by Wick's theorem there are $(2k-1)!!$ such diagrams. The second one is a connected chord diagram. They contribute to $\ln\langle W\rangle_R$. Their number is given by the recursion relation (4.12). The last diagram is a fully disconnected chord diagram. These are the diagrams that contribute to the planar limit of $\langle W\rangle_\square$ [40]. For k gluons, there are $\mathcal{C}_k = \frac{(2k)!}{(k+1)!k!}$ of them.



$$\bar{c} = c_R\left(c_R - \tfrac{1}{2}c_A\right)^2 - \left(2c_R\left(-\tfrac{1}{2}c_Rc_A\right) + c_R^3\right) = \tfrac{1}{4}c_Rc_A^3$$

Figure 3. Example of the determination of the modified color factor. The modified color factor of this connected diagrams with three gluons is obtained by considering the usual color factor and subtracting the color factor of all possible decompositions.

So, asymptotically, the number of connected Feynman diagrams is e times less than the total number of Feynman diagrams.

To compute $\ln\langle W\rangle_R$ by evaluating just the connected gluon diagrams, we have to take into account that according to the non-Abelian exponentiation theorem [2, 3], the color factor we have to assign to each diagram is not the ordinary one, but a modified color factor $\bar{c}_i$. To compute $\bar{c}_i$ of a given connected gluon diagram, we have to consider the original color factor, and subtract the color factor of all possible decompositions of the diagram, see figure 3 for an illustration of this procedure.

There is a further reduction on the number of gluon diagrams that one needs to consider, since many connected chord diagrams have the same reduced color factor. The relevant object that determines whether two chord diagrams have the same reduced color factor is the intersection graph associated to a given diagram. For every chord diagram one defines an intersection graph as follows [52]: for each chord introduce a point on the plane; if two chords cross, draw an edge between the two points, see figure 4 for an example.[3] If the crossing graphs are isomorphic, then the reduced color factors of the original chord diagrams are the same. Since only connected chord diagrams contribute to $\ln\langle W\rangle_R$, we can restrict our attention to connected intersection graphs. The number of non-isomorphic connected intersection graphs for chord diagrams has been discussed in [55]. Their num-

[3]Intersection graphs of chord diagrams have appeared recently in discussions of the SYK model [53, 54].

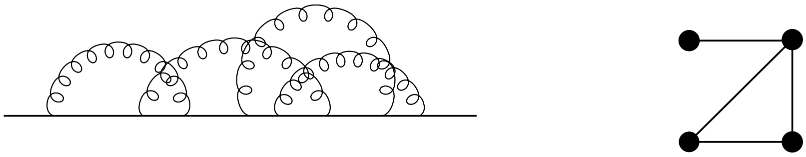


Figure 4. Example of intersection graph associated to a Feynman diagram with four gluons: for each gluon, draw a dot on the plane; each time two gluon lines intersect, draw a link between the corresponding two dots.

bers are

$$1, 1, 2, 6, 21, 110, 789, 8336, 117283, \dots \tag{4.15}$$

So for instance, at order g^4, there are $7!! = 105$ chord diagrams, 27 connected chord diagrams and only 6 connected intersection graphs.

We currently don't know how to read off the modified color factor directly from the intersection graph. Therefore, the procedure we propose is the following: first, group all connected chord diagrams, according to their intersection graphs. For each intersection graph, evaluate the modified color factor by computing it for any of the associated connected chord diagrams. Finally, add the contributions of all connected chord diagrams. We have carried out this procedure up to four loops. The results appear in figure 5. At first order there is a single diagram, with modified color factor $\bar{c} = c_R$. At second order there is a again a single diagram, with $\bar{c} = -\frac{1}{2}c_R c_A$. At third order, there are four connected chord diagrams; three of them share the first intersection graph with three dots, and have $\bar{c} = \frac{1}{4}c_R c_A^2$, while the fourth one has $\bar{c} = \frac{1}{2}c_R c_A^2$. At fourth order there are 27 connected chord diagrams, grouped according to the displayed six intersection graphs as follows $27 = 8 + 4 + 8 + 2 + 4 + 1$.

If n is the number of gluon propagators, we have

$$\ln \langle W \rangle_R = \sum_{n=1}^{\infty} \frac{1}{2n!} \left(\frac{{g_{\mathrm{YM}}}^2}{2} \right)^n \sum_{conn} \bar{c}_i \tag{4.16}$$

where the sum $\sum_{conn}$ runs over connected chord diagrams with n gluon propagators. Summing over all connected chord diagrams with up to four gluons, weighted by the modified color factors that appear in figure 5, we reproduce the expansion (4.6) up to fourth order.

4.3 Comments on the coefficients

In this concluding subsection, we indulge in a bit of numerology, and point out some patterns that we have spotted in the numerical coefficients that appear in the expansion (4.6) of $\ln \langle W \rangle_R$. Before we proceed, we must emphasize that starting at seventh order, color invariants are not all independent; the first identity they satisfy is [7]

$$d_A^{abcdef} d_A^{abcdef} - \frac{5}{8} d_A^{abcd} d_A^{cdef} d_A^{efab} + \frac{7}{240} c_A^2 d_A^{abcd} d_A^{abcd} + \frac{1}{864} c_A^6 d_A = 0 \tag{4.17}$$

For this reason, starting at seventh order, one must make a choice of color invariants to present any result, and any claim about the coefficients in front of the color invariants must take this ambiguity into account.

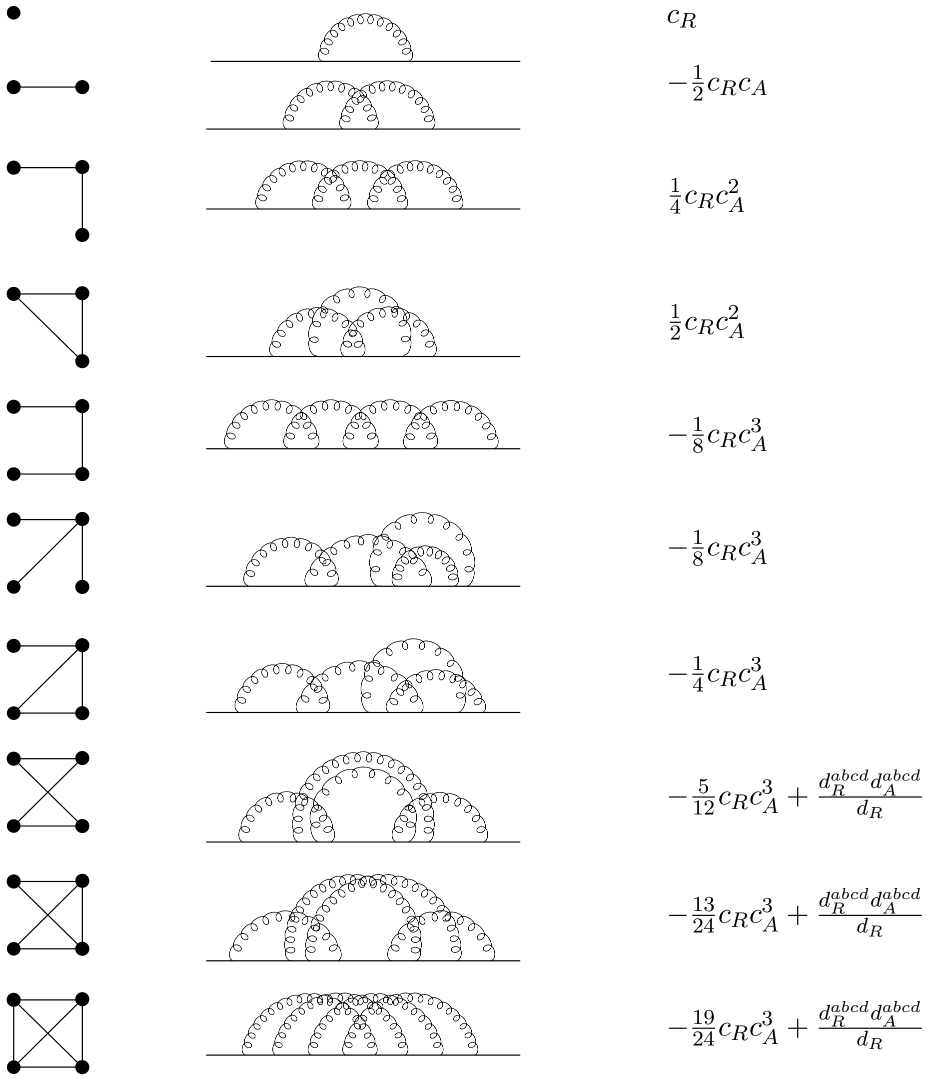


Figure 5. The first column displays all the non-isomorphic intersection graphs, up to four loops (graphs with four dots). The middle column shows a representative of the connected chord diagrams that share the given graph. The last column displays the modified color factor that must be assigned to all diagrams with the same intersection graph.

The first observation is that, up to sixth order, the coefficients of c_R in the perturbative expansion (4.6) are of the form

$$\mathcal{C}_k c_R \left(\frac{-c_A}{12}\right)^k g^{k+1} \tag{4.18}$$

where $\mathcal{C}_k$ are Catalan numbers

$$\mathcal{C}_k = \frac{1}{k+1}\binom{2k}{k} = 1, 1, 2, 5, 14, 42, 132, \ldots \tag{4.19}$$

At first sight, the appearance of Catalan numbers is hardly surprising, since they are ubiquitous in combinatorial problems, and in particular in graph enumeration. In fact, Catalan numbers appear in the planar approximation of the 1−point functions in the Hermitian Gaussian matrix model [56]

$$\left\langle tr\,\phi^{2k}\right\rangle = \mathcal{C}_k N^{k+1} + \mathcal{O}(N^{k-1}) \tag{4.20}$$

or equivalently, see figure 2, in the planar approximation to the vev of the Wilson loop in the fundamental representation, where $\mathcal{C}_k$ counts the number of diagrams with k non-crossing gluons [40]

$$\langle W\rangle_{\text{planar}} = \frac{1}{N}\sum_{k=0}^{\infty}\left\langle tr\,\phi^{2k}\right\rangle_{\text{planar}}\frac{g_{\text{YM}}^{2k}}{(2k)!} = \sum_{k=0}^{\infty}\frac{\mathcal{C}_k}{(2k)!}\left(\frac{\lambda}{4}\right)^k = \frac{2I_1\left(\sqrt{\lambda}\right)}{\sqrt{\lambda}} \tag{4.21}$$

However, we haven't been able to argue that the coefficients of c_R in (4.6) should follow the pattern (4.18). The difficulty in finding such an argument is that these coefficients arise from the interplay of combinatorics (diagram counting) and manipulations of Lie algebra generators, and we haven't managed to translate this interplay into a purely counting problem.

A second observation is that the coefficients of the $d_R^{abcd}d_A^{abcd}$ invariant, up to seventh order, follow a similar pattern, where now the numerators are given by Eulerian numbers, $A(k,1) = 2^k - k - 1$,

$$\frac{A(k,1)}{10}c_A^{k-2}\frac{d_R^{abcd}d_A^{abcd}}{d_R}\left(\frac{-g}{6}\right)^k g^2 \tag{4.22}$$

A third an final observation is that, again up to sixth order, when a color invariant appears for the first time in the expansion (4.6) of $\ln\langle W\rangle_R$, the coefficient in front of it is a unit fraction, a fraction with numerator equal to one.[4]

At seven loops, in the basis of color invariants chosen to present the result (4.6), the pattern (4.18) no longer holds. However, the numerical coefficient in front of $c_R c_A^6$ comes strikingly close to follow the pattern (4.18), if we recall that $\mathcal{C}_7 = 132$

$$\frac{72757}{1646023680}c_R c_A^6 = \frac{131.985\ldots}{12^6}c_R c_A^6 \tag{4.23}$$

Similarly, the third observation doesn't hold either: the color invariants that appear for the first time at seven loops in (4.6) have coefficients that are not unit fractions. At this order, the second observation is not affected by the ambiguity due to the relation (4.17), but presumably at higher orders it will be affected by similar identities involving $d_R^{a_1\ldots a_k}$.

As emphasized above, seven loops is precisely the first order where there are identities among color invariants, (4.17) being the first one. So it is natural to ask whether the breakdown of the patterns spotted up to six loops can be restored by the use of this relation. Since equation (4.17) is an identity among invariants, we can use it to impose by hand that the coefficient of $c_R c_A^6$ is indeed the one following the Catalan pattern, at the

[4]To avoid confussion, the coefficients that this observation refers to include the $\frac{1}{k!}$ factor in the g^k term in (4.6).

expense of introducing an overcomplete basis of color invariants. The terms that will be affected by the change are

$$\frac{72757}{1646023680}c_R c_A^6+\frac{817}{16329600}c_A^2 c_R\frac{d_A^{abcd}d_A^{abcd}}{d_A}+\frac{691}{95256000}c_R\frac{d_A^{abcd}d_A^{abef}d_A^{cdef}}{d_A} \tag{4.24}$$

and after the use of the identity (4.17), they turn into

$$\frac{132}{12^6}c_R c_A^6+\frac{13}{259200}c_A^2 c_R\frac{d_A^{abcd}d_A^{abcd}}{d_A}+\frac{1}{216000}c_R\frac{d_A^{abcd}d_A^{abef}d_A^{cdef}}{d_A}+\frac{1}{238140}c_R\frac{d_A^{abcdef}d_A^{abcdef}}{d_A} \tag{4.25}$$

Notice that if we impose by hand that the pattern (4.18) is preserved at seventh order, it turns out that the coefficients of the color invariants that appear for the first time at this order are now unit fractions, thus restoring also the validity of the third observation at seventh order. While the relevance of this fact is unclear to us, there was no a priori reason for it to happen.

So in closing, an open question is whether at higher orders in the expansion (4.6) it is always possible to use relations among color invariants to present the result in a way that the three observations presented above hold to all orders. If this turns out to be the case, a second question would be if these patterns hint at an alternative way of computing (4.6), in which they are easily explained.

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A Color invariants

In this appendix we collect our conventions for color invariants, which are largely those of [7]. We also present the explicit results we use in the main body of the paper; some of them are already listed in [7, 8, 59].

Let R be a representation of a Lie algebra: F and A denote the fundamental and the adjoint representations. The dimension of R is denoted by d_R. The generators T_R^a of the representation satisfy

$$[T_R^a, T_R^b] = i f^{abc} T_R^c \qquad a,b=1,\dots,d_A \tag{A.1}$$

$$\begin{array}{|c|c|c|}\hline N & N+1 & N+2 \\ \hline N-1 & N & \\ \cline{1-2}\end{array} \qquad d_{\yng(3,2)} = \frac{N(N+1)(N+2)(N-1)N}{4\times3\times1\times2\times1}$$

Figure 6. Example of the computation of the dimension of an irreducible representation of SU(N).

This does not fix the normalization of the generators T_R^a. We introduce two representation-dependent constants,

$$\begin{aligned}\mathrm{tr}\ T_R^a T_R^b &= I_2(R)\delta^{ab}\\ T_R^a T_R^a &= c_R \mathbb{1}_{d_R\times d_R}\end{aligned}$$

These two representation-dependent constants are related as follows

$$d_A I_2(R) = d_R c_R \tag{A.2}$$

In this work, we consider representations different from the fundamental only for the group SU(N). Irreducible representations of SU(N) are labelled by Young diagrams, given by $k \leq N-1$ rows of λ_i boxes, $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k$. We recall briefly how to compute the dimension and c_R of a representation from its Young diagram.

One can compute the dimension of the representation R of SU(N) from the Young diagram as follows [60]: given a box of the diagram, define its hook length h_i by the number of boxes in the hook formed by the boxes to its right (in the same row), the boxes below it (in the same column), and the box itself. Then, to compute the dimension of R, start writing N inside the box at the upper left corner of the Young diagram. Then fill the remaining boxes with numbers N_i, obtained by adding one every time one moves to the right, and subtracting one every time one moves down. The dimension of the representation is

$$d_R = \prod_i \frac{N_i}{h_i} \tag{A.3}$$

Figure 6 displays the computation of d_R for a particular example. It follows from this formula that if R is a representation whose Young diagram has k boxes, and R^t the representation with transpose Young diagram

$$d_{R^t}(N) = (-1)^k d_R(-N) \tag{A.4}$$

The quadratic Casimir c_R for the representation with Young diagram $(\lambda_1, \lambda_2, \ldots, \lambda_m)$ is given by [61]

$$c_R = I_2(F)\left(\sum_{i=1}^{m} \lambda_i(N + \lambda_i + 1 - 2i) - \frac{(\sum_i \lambda_i)^2}{N}\right) \tag{A.5}$$

This expression can be rewritten as follows [62],

$$c_R = I_2(F)\left(kN + \sum_i \lambda_i^2 - \sum_j (\lambda_j^T)^2 - \frac{k^2}{N}\right) \tag{A.6}$$

where k is the number of boxes of the Young diagram, $k = \sum_i \lambda_i$. This formula makes manifest that

$$c_{R^t}(-N) = -c_R(N) \tag{A.7}$$

Once one has d_R and c_R for a given representation R, $I_2(R)$ follows from eq. (A.2).

A.1 Higher order invariants

Define the fully symmetrized traces as a normalized sum over all the possible index permutations

$$d_R^{a_1 \dots a_k} = \frac{1}{k!} \sum_{\sigma \in \mathcal{S}_k} \mathrm{tr} \left(T_R^{a_{\sigma(1)}} \dots T_R^{a_{\sigma(k)}} \right) \tag{A.8}$$

It will be very useful to define the Chern character of a representation [7], as a function of dummy variables F^a. The symmetrized traces defined above appear in the expansion of the character,

$$ch_R(F) = tr\, e^{F^a T_R^a} = \sum_{k=0}^{\infty} \frac{1}{k!} d_R^{a_1 \dots a_k} F^{a_1} \dots F^{a_k} \tag{A.9}$$

In the main body of the paper we need the evaluation of color invariants for various representations of SU(N), and also for the fundamental representation of SO(N). The strategy we have used is to first derive results for the fundamental representation (most of them are already available in [7]). For higher dimensional representations, we will first relate their Chern character to that of the fundamental representation, and then evaluate their color invariants, making use of the results found for the fundamental representation.

A.2 Invariants for the fundamental representations of SU(N) and SO(N)

The following formulas have been computed using FORM [57], and in the SU(N) case, checked with FeynCalc [58].

A.2.1 Color invariants for SU(N)

For SU(N) we choose the usual normalization $I_2(F) = 1/2$. Then

$$d_F = N \qquad c_F = \frac{N^2-1}{2N} \qquad d_A = N^2 - 1 \qquad c_A = I_2(A) = N \tag{A.10}$$

The relevant color invariants are

$$\begin{aligned}
d_F^{abcd} d_A^{abcd} &= \frac{N(N^2-1)(N^2+6)}{48} \\
d_A^{abcd} d_A^{abcd} &= \frac{N^2(N^2-1)(N^2+36)}{24} \\
d_F^{abcd} d_A^{cdef} d_A^{efab} &= \frac{N^3(N^2-1)(N^2+51)}{432} \\
d_A^{abcd} d_A^{cdef} d_A^{efab} &= \frac{N^2(N^2-1)(N^4+135N^2+324)}{216} \\
d_F^{abcdef} d_A^{abcdef} &= \frac{(N^2-1)N(N^4+36N^2+120)}{3840}
\end{aligned}$$

$$d_F^{abcdef}d_A^{abcg}d_A^{defg} = \frac{N^2(N^2-1)(N^4+45N^2+84)}{3840}$$
$$d_A^{abcdef}d_F^{abcg}d_A^{defg} = \frac{N^2(N^2-1)(N^4+141N^2+540)}{3840}$$

A.2.2 Color invariants for SO(N)

For SO(N) we choose the usual normalization $I_2(F)=1$. Then

$$d_F = N \qquad c_F = \frac{N-1}{2} \qquad N_A = \frac{N(N-1)}{2} \qquad c_A = N-2 \tag{A.11}$$

The relevant color invariants are

$$d_F^{abcd}d_A^{abcd} = \frac{d_A c_A}{24}(N^2-7N+22)$$
$$d_A^{abcd}d_A^{abcd} = \frac{d_A c_A}{24}(N^3-15N^2+138N-296)$$
$$d_F^{abcd}d_A^{cdef}d_A^{efab} = \frac{d_A c_A}{432}(2N^4-31N^3+387N^2-1582N+2048)$$
$$d_A^{abcd}d_A^{cdef}d_A^{efab} = \frac{d_A c_A}{432}(2N^5-47N^4+971N^3-7018N^2+23272N-29440)$$
$$d_F^{abcdef}d_A^{abcdef} = \frac{d_A c_A}{960}\left(N^4-32N^3+273N^2-902N+1312\right)$$
$$d_F^{abcdef}d_A^{abcg}d_A^{defg} = \frac{d_A c_A}{960}\left(N^5-16N^4+193N^3-1214N^2+3656N-3920\right)$$
$$d_A^{abcdef}d_F^{abcg}d_A^{defg} = \frac{d_A c_A}{960}\left(N^5-40N^4+697N^3-4598N^2+14576N-17888\right)$$

A.3 Invariants for S_k/A_k representations of SU(N)

Applying the formulas (A.3) and (A.5) for the representations A_k/S_k of SU(N), we have

$$d_{\mathcal{A}_k} = \binom{N}{k} \qquad c_2(\mathcal{A}_k) = I_2(F)\frac{k(N+1)(N-k)}{N} \qquad I_2(\mathcal{A}_k) = I_2(F)\binom{N-2}{k-1}$$
$$d_{S_k} = \binom{N+k-1}{k} \qquad c_2(S_k) = I_2(F)\frac{k(N-1)(N+k)}{N} \qquad I_2(S_k) = I_2(F)\binom{N+k}{k-1}$$

We now turn to color invariants involving higher order symmetrized traces $d_R^{a_1\ldots a_m}$, for $R = S_k/A_k$. For invariants involving up to d_R^{abcd}, we first derive the formulas valid for arbitrary k, and then check them via an alternative computation, for $k=1,2,3,4$.

For invariants involving d_R^{abcdef}, we have explicitly computed the results for $k = 1,2,3,4$, and then we have guessed a formula for generic k, imposing that the formulas are invariant under $k \to N-k$ for A_k. So the formulas quoted have been only derived for $k=1,2,3,4$ but are probably true also for any k. According to [59]

$$d_{S_k}^{abcd} = \frac{N(N-1)+6k(N+k)}{(k-1)!(N+3)!}(N+k)!d_F^{abcd} + \binom{N+k+1}{k-2}I_2(F)^2\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right) \tag{A.12}$$

$$d_{A_k}^{abcd} = \frac{N(N+1)-6k(N-k)}{(k-1)!(N-k-1)!}(N-4)!d_F^{abcd} + \binom{N-4}{k-2}I_2(F)^2\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right) \tag{A.13}$$

To obtain the relevant color invariants, we contract these formulas with various symmetrized traces in the adjoint representation, and use

$$d_A^{aacd} = \frac{5}{6}c_A^2\delta^{cd} \tag{A.14}$$

$$d_A^{aacdef} = \frac{7}{10}c_A d_A^{cdef} \tag{A.15}$$

The results are as follows (the upper sign is for S_k, the lower one is for A_k),

$$\begin{aligned}
d_R^{abcd}d_A^{abcd} &= c_R d_R \frac{N}{24}\left(N^2 \mp 6N + 6k(k \pm N)\right) \\
d_R^{abcd}d_A^{abef}d_A^{cdef} &= c_R d_R \frac{N^2}{216}\left(N^3 \mp 6N^2 - 9N \mp 54 + 6Nk(k \pm N) + 54k(N \pm k)\right) \\
d_A^{abcdef}d_R^{abcg}d_A^{defg} &= c_R d_R \frac{N^2}{1920}\left(N^4 \mp 6N^3 + 81N^2 \mp 594N + (N^2 \pm 54N + 540)k(k \pm N)\right)
\end{aligned}$$

Note that they satisfy the $N \to -N$ symmetry when $S_k \to A_k$ (up to global sign) and for A_k the $k \to N-k$ symmetry. In order to repeat the same procedure to evaluate color invariants involving d_{S_k/A_k}^{abcdef}, we would need formulas similar to eqs. (A.12) and (A.13) for d_{S_k/A_k}^{abcdef}. From [59] one can derive the leading terms in such formulas

$$\begin{aligned}
d_{S_k}^{abcdef} &= \sum_{i=0}^{k-1}(k-i)^5\binom{N+i-1}{i}d_F^{abcdef} + \ldots \\
d_{A_k}^{abcdef} &= \sum_{i=0}^{k-1}(-1)^{k-1-i}(k-i)^5\binom{N}{i}d_F^{abcdef} + \ldots
\end{aligned}$$

but we are not aware of complete formulas for arbitrary k. Instead, we will compute them for small values of k, from the character formulas for the symmetric and antisymmetric representations [7]

$$\begin{aligned}
Ch_{S_k}(F) &= \sum_{\substack{n_i,m_i \\ k=n_im_i}}\prod_i \frac{1}{m_i!}\left(\frac{Ch(n_iF)}{n_i}\right)^{m_i} \\
Ch_{A_k}(F) &= (-1)^k \sum_{\substack{n_i,m_i \\ k=n_im_i}}\prod_i \frac{1}{m_i!}\left(-\frac{Ch(n_iF)}{n_i}\right)^{m_i}
\end{aligned}$$

where the sum is over all partitions of k into different integers n_i, each appearing with multiplicity m_i. From these formulas, we obtain the characters of S_k, A_k for $k = 2, 3, 4$,

$$\begin{aligned}
Ch_{S_2/A_2}(F) &= \frac{1}{2}(Ch\,F)^2 \pm \frac{1}{2}Ch(\,2F) \\
Ch_{S_3/A_3}(F) &= \frac{1}{3!}(Ch\,F)^3 \pm \frac{1}{2}Ch\,2F Ch\,F + \frac{1}{3}Ch\,3F \\
Ch_{S_4/A_4}(F) &= \frac{1}{4!}(Ch\,F)^4 \pm \frac{1}{4}Ch\,2F(Ch\,F)^2 + \frac{1}{8}(Ch\,2F)^2 + \frac{1}{3}Ch\,3F Ch\,F \pm \frac{1}{4}Ch\,4F
\end{aligned}$$

We expand in powers of F up to sixth order. At zeroth, second and fourth orders we recover the formulas for N_{S_k/A_k}, c_{S_k/A_k} and $d^{abcd}_{S_k/A_k}$ for $k=2,3,4$. At sixth order, we obtain the following formulas for $d^{abcdef}_{S_k/A_k}$,

$$d^{abcdef}_{S_2/A_2} = (N\pm 32)d^{abcdef}_F + I_2(F)\left(\delta^{ab}d^{cdef}_F+\dots\right)+\left(d^{abc}_F d^{def}_F+\dots\right) \tag{A.16}$$

$$\begin{aligned} d^{abcdef}_{S_3/A_3} &= \frac{N^2\pm 65N+486}{2}d^{abcdef}_F + (N\pm 10)I_2(F)(\delta^{ab}d^{cdef}_F+\dots)\\ &\quad + (N\pm 8)(d^{abc}_F d^{def}_F+\dots)+I_2(F)^3(\delta^{ab}\delta^{cd}\delta^{ef}+\dots) \end{aligned} \tag{A.17}$$

and

$$\begin{aligned} d^{abcdef}_{S_4/A_4} &= \frac{N^3\pm 99N^2+1556N\pm 6144}{6}d^{abcdef}_F + \frac{N^2\pm 21N+92}{2}I_2(F)(\delta^{ab}d^{cdef}+\dots)\\ &\quad + \frac{N^2\pm 17N+68}{2}(d^{abc}_F d^{def}_F+\dots)+(N\pm 6)I_2(F)^3(\delta^{ab}\delta^{cd}\delta^{ef}+\dots) \end{aligned} \tag{A.18}$$

Using these expressions, we evaluate the following color invariants,

$$\begin{aligned} d^{abcdef}_{S_k/A_k}d^{abcdef}_A &= \frac{c_R d_R N}{1920}\big(N^4\mp 30N^3+186N^2\\ &\quad \pm 60N+30N^2k(k\pm N)+120k^2(N\pm k)^2-300Nk(N\pm k)\big) \end{aligned} \tag{A.19}$$

$$\begin{aligned} d^{abcdef}_{S_k/A_k}d^{abcg}_A d^{defg}_A &= \frac{c_R d_R N^2}{1920}\big(N^4\mp 30N^3+105N^2\pm 150N+144\\ &\quad +30N^2k(k\pm N)+120k^2(N\pm k)^2-210Nk(N\pm k)-180k(k\pm N)\big) \end{aligned} \tag{A.20}$$

We emphasize that these last two formulas have been proven only for $k=1,2,3,4$, although we are confident that they are true for arbitrary k. We find that all the color invariants we have computed for S_k and A_k are related by $N\to -N$, as expected [29].

We can perform some checks for specific values of N. For SU(4) the invariants for A_2 coincide with those of SO(6) in the fundamental.

A.4 Results for the $\yng(2,1)$ representation of SU(N)

In the main body of the paper, we display various results for the $\yng(2,1)$ representation, since it is the simplest representation that is not fully symmetric or fully antisymmetric. Furthermore, its Young diagram is self-transpose, thus it allows to illustrate the $1/N^2$ expansion of $\langle W\rangle_R$ for these representations. Some of the results we need are already available in [59], but we have derived all the formulas below independently and checked them with [59] when possible.

To obtain the character for this representation, we recall

$$\yng(1)\times\yng(1)\times\yng(1) = \yng(3) + 2\,\yng(2,1) + \yng(1,1,1) \tag{A.21}$$

from which we deduce

$$Ch_{\yng(2,1)}F = \frac{1}{3}(Ch\,F)^3-\frac{1}{3}Ch\,3F \tag{A.22}$$

Expanding this result up to sixth order in F we obtain

$$d_{\yng(2,1)} = \frac{N(N^2-1)}{3} \qquad c_{\yng(2,1)} = \frac{3(N^2-3)}{2N} \qquad I_2(\yng(2,1)) = \frac{N^2-3}{2} \tag{A.23}$$

$$d_{\yng(2,1)}^{abcd} = (N^2-27)d_F^{abcd} + 2NI_2(F)^2\left(\delta^{ab}\delta^{cd}+\delta^{ac}\delta^{bd}+\delta^{ad}\delta^{bc}\right) \tag{A.24}$$

$$d_{\yng(2,1)}^{abcdef} = (N^2-3^5)d_F^{abcdef} + 2N(d^{ab}d^{cdef}+\dots) + 2N(d^{abc}d^{def}+\dots) + 2(d^{ab}d^{cd}d^{ef}+\dots) \tag{A.25}$$

The results for $d_{\yng(2,1)}$ and $c_{\yng(2,1)}$ can also be derived from the general formulas (A.3) and (A.5). With these formulas we derive the following color invariants

$$\begin{aligned}
d_{\yng(2,1)}^{abcd}d_A^{abcd} &= \frac{N(N^2-1)(N^4+39N^2-162)}{48} \\
d_{\yng(2,1)}^{abcd}d_A^{abef}d_A^{cdef} &= \frac{N^3(N^2-1)(N^4+192N^2-729)}{432} \\
d_{\yng(2,1)}^{abcdef}d_A^{abcdef} &= \frac{N(N^2-1)(N^6+213N^4+6492N^2-29160)}{3840} \\
d_{\yng(2,1)}^{abcdef}d_A^{abcg}d_A^{defg} &= \frac{N^2(N^2-1)(N^6+402N^4+1389N^2-11772)}{3840} \\
d_A^{abcdef}d_A^{abcg}d_{\yng(2,1)}^{defg} &= \frac{N^2(N^2-1)(N^6+282N^4+2781N^2-14580)}{3840}
\end{aligned}$$

We can provide two checks for these results. First, all the color invariants have a $1/N^2$ expansion, as expected since the Young diagram $\yng(2,1)$ is self-transpose. Also, for SU(2), the invariants evaluate to the same numbers if we replace $d_{\yng(2,1)}^{abcd}, d_{\yng(2,1)}^{abcdef}$ by d_F^{abcd}, d_F^{abcdef}.

References

[1] A.M. Polyakov, *Gauge Fields as Rings of Glue*, *Nucl. Phys.* **B 164** (1980) 171 [INSPIRE].

[2] J.G.M. Gatheral, *Exponentiation of Eikonal Cross-sections in Nonabelian Gauge Theories*, *Phys. Lett.* **133B** (1983) 90 [INSPIRE].

[3] J. Frenkel and J.C. Taylor, *Nonabelian eikonal exponentiation*, *Nucl. Phys.* **B 246** (1984) 231 [INSPIRE].

[4] C.D. White, *An Introduction to Webs*, *J. Phys.* **G 43** (2016) 033002 [arXiv:1507.02167] [INSPIRE].

[5] G.P. Korchemsky and A.V. Radyushkin, *Infrared factorization, Wilson lines and the heavy quark limit*, *Phys. Lett.* **B 279** (1992) 359 [hep-ph/9203222] [INSPIRE].

[6] J.M. Henn and T. Huber, *The four-loop cusp anomalous dimension in $\mathcal{N}=4$ super Yang-Mills and analytic integration techniques for Wilson line integrals*, *JHEP* **09** (2013) 147 [arXiv:1304.6418] [INSPIRE].

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[7] T. van Ritbergen, A.N. Schellekens and J.A.M. Vermaseren, *Group theory factors for Feynman diagrams*, *Int. J. Mod. Phys.* **A 14** (1999) 41 [hep-ph/9802376] [INSPIRE].

[8] P. Cvitanovic, *Group theory: Birdtracks, Lie's and exceptional groups*, Princeton University Press, Princeton U.S.A. (2008).

[9] B. Fiol, E. Gerchkovitz and Z. Komargodski, *Exact Bremsstrahlung Function in $N = 2$ Superconformal Field Theories*, *Phys. Rev. Lett.* **116** (2016) 081601 [arXiv:1510.01332] [INSPIRE].

[10] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2012) 048 [arXiv:1202.4455] [INSPIRE].

[11] B. Fiol, B. Garolera and G. Torrents, *Exact momentum fluctuations of an accelerated quark in $N = 4$ super Yang-Mills*, *JHEP* **06** (2013) 011 [arXiv:1302.6991] [INSPIRE].

[12] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, *JHEP* **05** (2014) 025 [arXiv:1312.5682] [INSPIRE].

[13] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, *Phys. Rev.* **D 74** (2006) 025005 [hep-th/0501015] [INSPIRE].

[14] B. Fiol, B. Garolera and A. Lewkowycz, *Exact results for static and radiative fields of a quark in $N = 4$ super Yang-Mills*, *JHEP* **05** (2012) 093 [arXiv:1202.5292] [INSPIRE].

[15] L. Bianchi, M. Lemos and M. Meineri, *Line Defects and Radiation in $\mathcal{N} = 2$ Conformal Theories*, *Phys. Rev. Lett.* **121** (2018) 141601 [arXiv:1805.04111] [INSPIRE].

[16] C. Gomez, A. Mauri and S. Penati, *The Bremsstrahlung function of $\mathcal{N} = 2$ SCQCD*, *JHEP* **03** (2019) 122 [arXiv:1811.08437] [INSPIRE].

[17] M.S. Bianchi, L. Griguolo, M. Leoni, S. Penati and D. Seminara, *BPS Wilson loops and Bremsstrahlung function in ABJ(M): a two loop analysis*, *JHEP* **06** (2014) 123 [arXiv:1402.4128] [INSPIRE].

[18] M.S. Bianchi, L. Griguolo, A. Mauri, S. Penati, M. Preti and D. Seminara, *Towards the exact Bremsstrahlung function of ABJM theory*, *JHEP* **08** (2017) 022 [arXiv:1705.10780] [INSPIRE].

[19] L. Bianchi, L. Griguolo, M. Preti and D. Seminara, *Wilson lines as superconformal defects in ABJM theory: a formula for the emitted radiation*, *JHEP* **10** (2017) 050 [arXiv:1706.06590] [INSPIRE].

[20] L. Bianchi, M. Preti and E. Vescovi, *Exact Bremsstrahlung functions in ABJM theory*, *JHEP* **07** (2018) 060 [arXiv:1802.07726] [INSPIRE].

[21] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, *Commun. Math. Phys.* **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[22] M. Mariño, *Les Houches lectures on matrix models and topological strings*, 2004, hep-th/0410165, http://weblib.cern.ch/abstract?CERN-PH-TH-2004-199 [INSPIRE].

[23] M. Billó, F. Fucito, A. Lerda, J.F. Morales, Ya. S. Stanev and C. Wen, *Two-point Correlators in $N = 2$ Gauge Theories*, *Nucl. Phys.* **B 926** (2018) 427 [arXiv:1705.02909] [INSPIRE].

[24] M. Billó, F. Galvagno, P. Gregori and A. Lerda, *Correlators between Wilson loop and chiral operators in $\mathcal{N} = 2$ conformal gauge theories*, *JHEP* **03** (2018) 193 [arXiv:1802.09813] [INSPIRE].

JHEP05(2019)202

[25] N. Drukker and D.J. Gross, *An Exact prediction of $N = 4$ SUSYM theory for string theory*, J. Math. Phys. **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[26] B. Fiol and G. Torrents, *Exact results for Wilson loops in arbitrary representations*, JHEP **01** (2014) 020 [arXiv:1311.2058] [INSPIRE].

[27] B. Fiol, B. Garolera and G. Torrents, *Exact probes of orientifolds*, JHEP **09** (2014) 169 [arXiv:1406.5129] [INSPIRE].

[28] A.K. Cyrol, M. Mitter and N. Strodthoff, *FormTracer — A Mathematica Tracing Package Using FORM*, Comput. Phys. Commun. **219** (2017) 346 [arXiv:1610.09331] [INSPIRE].

[29] P. Cvitanovic and A.D. Kennedy, *Spinors in Negative Dimensions*, Phys. Scripta **26** (1982) 5 [INSPIRE].

[30] K. Okuyama, *Connected correlator of 1/2 BPS Wilson loops in $\mathcal{N} = 4$ SYM*, JHEP **10** (2018) 037 [arXiv:1808.10161] [INSPIRE].

[31] J. Gomis and F. Passerini, *Holographic Wilson Loops*, JHEP **08** (2006) 074 [hep-th/0604007] [INSPIRE].

[32] N. Drukker and B. Fiol, *All-genus calculation of Wilson loops using D-branes*, JHEP **02** (2005) 010 [hep-th/0501109] [INSPIRE].

[33] S. Yamaguchi, *Wilson loops of anti-symmetric representation and D5-branes*, JHEP **05** (2006) 037 [hep-th/0603208] [INSPIRE].

[34] E. D'Hoker, J. Estes and M. Gutperle, *Gravity duals of half-BPS Wilson loops*, JHEP **06** (2007) 063 [arXiv:0705.1004] [INSPIRE].

[35] X. Chen-Lin, *Symmetric Wilson Loops beyond leading order*, SciPost Phys. **1** (2016) 013 [arXiv:1610.02914] [INSPIRE].

[36] J. Gordon, *Antisymmetric Wilson loops in $\mathcal{N} = 4$ SYM beyond the planar limit*, JHEP **01** (2018) 107 [arXiv:1708.05778] [INSPIRE].

[37] K. Okuyama, *Phase Transition of Anti-Symmetric Wilson Loops in $\mathcal{N} = 4$ SYM*, JHEP **12** (2017) 125 [arXiv:1709.04166] [INSPIRE].

[38] A.F. Canazas Garay, A. Faraggi and W. Mück, *Antisymmetric Wilson loops in $\mathcal{N} = 4$ SYM: from exact results to non-planar corrections*, JHEP **08** (2018) 149 [arXiv:1807.04052] [INSPIRE].

[39] B.E. Sagan, *The Symmetric Group*, Springer, Heidelberg Germany (2000).

[40] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory*, Nucl. Phys. **B 582** (2000) 155 [hep-th/0003055] [INSPIRE].

[41] G. Akemann and P.H. Damgaard, *Wilson loops in $N = 4$ supersymmetric Yang-Mills theory from random matrix theory*, Phys. Lett. **B 513** (2001) 179 [*Erratum ibid.* **B 524** (2002) 400] [hep-th/0101225] [INSPIRE].

[42] J. Ambjørn, P. Olesen and C. Peterson, *Stochastic Confinement and Dimensional Reduction. 2. Three-dimensional* SU(2) *Lattice Gauge Theory*, Nucl. Phys. **B 240** (1984) 533 [INSPIRE].

[43] C. Anzai, Y. Kiyo and Y. Sumino, *Violation of Casimir Scaling for Static QCD Potential at Three-loop Order*, Nucl. Phys. **B 838** (2010) 28 [*Erratum ibid.* **B 890** (2015) 569] [arXiv:1004.1562] [INSPIRE].

JHEP05(2019)202

[44] R.N. Lee, A.V. Smirnov, V.A. Smirnov and M. Steinhauser, *Analytic three-loop static potential*, *Phys. Rev.* **D 94** (2016) 054029 [arXiv:1608.02603] [INSPIRE].

[45] A. Grozin, J.M. Henn, G.P. Korchemsky and P. Marquard, *Three Loop Cusp Anomalous Dimension in QCD*, *Phys. Rev. Lett.* **114** (2015) 062006 [arXiv:1409.0023] [INSPIRE].

[46] A. Grozin, J. Henn and M. Stahlhofen, *On the Casimir scaling violation in the cusp anomalous dimension at small angle*, *JHEP* **10** (2017) 052 [arXiv:1708.01221] [INSPIRE].

[47] R.H. Boels, T. Huber and G. Yang, *Four-Loop Nonplanar Cusp Anomalous Dimension in $N = 4$ Supersymmetric Yang-Mills Theory*, *Phys. Rev. Lett.* **119** (2017) 201601 [arXiv:1705.03444] [INSPIRE].

[48] J. Touchard, *Sur un problème de configurations et sur les fractions continues*, *Canad. J. Math.* **4** (1952) 2.

[49] P.R. Stein, *On a class of linked diagrams, I. Enumeration*, *J. Comb. Theory* **A 24** (1978) 357.

[50] A. Nijenhuis and H.S. Wilf, *The enumeration of connected graphs and linked diagrams*, *J. Comb. Theory* **A 27** (1979) 356.

[51] P.R. Stein and C.J. Everett, *On a class of linked diagrams, II. Asymptotics*, *Discrete Math.* **21** (1978) 309.

[52] A. Bouchet, *Circle Graph Obstructions*, *J. Comb. Theory* **B 60** (1994) 107.

[53] A.M. García-García, Y. Jia and J.J.M. Verbaarschot, *Exact moments of the Sachdev-Ye-Kitaev model up to order $1/N^2$*, *JHEP* **04** (2018) 146 [arXiv:1801.02696] [INSPIRE].

[54] Y. Jia and J.J.M. Verbaarschot, *Large N expansion of the moments and free energy of Sachdev-Ye-Kitaev model and the enumeration of intersection graphs*, *JHEP* **11** (2018) 031 [arXiv:1806.03271] [INSPIRE].

[55] D.C. R. Arratia, B. Bollobás and G.B. Sorkin, *Euler circuits and DNA sequencing by hybridization*, *Discrete Appl. Math.* **104** (2000) 63.

[56] E. Brézin, C. Itzykson, G. Parisi and J.B. Zuber, *Planar Diagrams*, *Commun. Math. Phys.* **59** (1978) 35 [INSPIRE].

[57] J.A.M. Vermaseren, *New features of FORM*, math-ph/0010025 [INSPIRE].

[58] R. Mertig, M. Böhm and A. Denner, *FEYN CALC: Computer algebraic calculation of Feynman amplitudes*, *Comput. Phys. Commun.* **64** (1991) 345 [INSPIRE].

[59] S. Okubo and J. Patera, *Symmetrization of Product Representations and General Indices and Simple Lie Algebras*, *J. Math. Phys.* **24** (1983) 2722 [INSPIRE].

[60] G. de B. Robinson, *Representation Theory of the Symmetric Group*, University of Toronto Press, Toronto Canada (1961).

[61] A.O. Barut and R. Racza, *Theory of Group Representations and Applications*, World Scientific, New York U.S.A. (1986).

[62] D.J. Gross and W. Taylor, *Two-dimensional QCD is a string theory*, *Nucl. Phys.* **B 400** (1993) 181 [hep-th/9301068] [INSPIRE].

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Coupling dependence of radiation: Part 2

This chapter includes the publication:

- B. Fiol, J. Martínez-Montoya and A. Rios Fukelman, *The planar limit of $\mathcal{N} = 2$ superconformal quiver theories*, JHEP **08**, 161 (2020), `arXiv:2006.06379 [hep-th]`.

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The planar limit of $\mathcal{N} = 2$ superconformal quiver theories

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Abstract: We compute the planar limit of both the free energy and the expectation value of the 1/2 BPS Wilson loop for four dimensional $\mathcal{N} = 2$ superconformal quiver theories, with a product of SU(N)s as gauge group and bi-fundamental matter. Supersymmetric localization reduces the problem to a multi-matrix model, that we rewrite in the zero-instanton sector as an effective action involving an infinite number of double-trace terms, determined by the relevant extended Cartan matrix. We find that the results, as in the case of $\mathcal{N} = 2$ SCFTs with a simple gauge group, can be written as sums over tree graphs. For the $\widehat{A}_1$ case, we find that the contribution of each tree can be interpreted as the partition function of a generalized Ising model defined on the tree; we conjecture that the partition functions of these models defined on trees satisfy the Lee-Yang property, i.e. all their zeros lie on the unit circle.

Keywords: Matrix Models, Supersymmetric Gauge Theory, Wilson, 't Hooft and Polyakov loops

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Contents

1 Introduction

The emergence of quantum gravity from a gauge theory is one of the most fascinating issues that can be addressed with the AdS/CFT correspondence. Since the work of [1] it has been clear that not every conformal field theory (CFT) in the large N limit can be dual to a gravitational theory described by a two derivative Einstein-Hilbert action. For instance, for four dimensional CFTs a necessary condition is that the two central charges coincide in the large N limit, $a=c$ [1]. For instance, this property is satisfied by $\mathcal{N}=4$ super Yang-Mills, but it is not satisfied by $\mathcal{N}=2$ SU(N) with $n_F=2N$ hypermultiplets in the fundamental representation, thus ruling out that the large N limit of this CFT has a holographic dual well described by gravity.

Since the early days of the holographic correspondence, it has been important to find further examples of CFTs with holographic duals, beyond the original example of $\mathcal{N}=4$ SYM. Four dimensional quiver gauge theories with $\mathcal{N}=2$ superconformal symmetry satisfy an ADE classification [2], and for certain values of the marginal couplings, they are orbifolds of $\mathcal{N}=4$ SYM and have a gravity dual [3, 4]. These quiver gauge CFTs constitute thus an interesting laboratory, as variation of their marginal couplings allows to connect CFTs with and without gravity duals in the large N limit [5–12].

In this work we will consider $\mathcal{N}=2$ SCFTs with gauge group a product of SU(N)s, paying special attention to the simplest case, the $\widehat{A}_1$ theory, with gauge group SU(N)×SU(N). This theory has two marginal couplings (g_1,g_2) and varying them one can reach an orbifold of $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SU(N) SQCD. Our main technical tool will be supersymmetric localization [13]. Thanks to this tool, the planar free energy and expectation value of the 1/2 BPS circular Wilson loop are known to all orders in the 't Hooft coupling for the limiting theories mentioned above $\mathcal{N}=4$ SYM and $\mathcal{N}=2$ SU(N) SQCD [14–19].

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Four dimensional $\mathcal{N}=2$ quiver CFTs have already been studied using localization [7, 10–12, 20]. The novelty of this work is that we evaluate various quantities of these theories in the planar limit, to all orders in the 't Hooft couplings λ_i. We do so by applying the same strategy developed for CFTs with simple gauge groups in [19]. For these quiver CFTS, supersymmetric localization [13] reduces the evaluation of various quantities to matrix integrals. Compared to the case of $\mathcal{N}=2$ SCFTs with a simple gauge group, the main novelty is that the resulting matrix models are multi-matrix models. In the simplest case, the model to solve is a two-matrix model. As in our recent work [19], we rewrite the 1-loop factor as an effective action involving an infinite number of double-trace terms, in the fundamental representation of the respective gauge groups. We then show that this double-trace form of the potential implies that the perturbative series considered admit a combinatorial formulation, as sums over tree graphs.

While we will present results valid for all $\mathcal{N}=2$ quiver CFTs, we will pay special attention to the simplest theory, $\widehat{A_1}$. This theory has a $\mathbb{Z}_2$ symmetry exchanging the two nodes of the quiver. Since the ranks of the gauge groups are equal, this $\mathbb{Z}_2$ symmetry amounts to exchanging $g_1 \leftrightarrow g_2$. We will be particularly interested in observables that transform nicely under this symmetry: the free energy and particular linear combinations of the usual 1/2 BPS circular Wilson loop defined for each node [7].

In section 2, after introducing the theories we will consider, we derive the perturbative series of the planar free energy, to all orders in the 't Hooft couplings λ_i. Let's present here the answer for the $\widehat{A_1}$ theory. It is convenient to define $\mathcal{F}_0(\lambda_1,\lambda_2)=F_0(\lambda_1,\lambda_2)-F_0(\lambda_1)^{\mathcal{N}=4}-F_0(\lambda_2)^{\mathcal{N}=4}$. The perturbative series is given by a sum over tree graphs,

$$\begin{aligned}\mathcal{F}_0(\lambda_1,\lambda_2)=&\sum_{m=1}^{\infty}(-2)^m\sum_{n_1,\dots,n_m=2}^{\infty}\frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m}(-1)^{n_1+\dots+n_m}\\&\times\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\\text{with } m \text{ edges}}}\frac{1}{|\mathrm{Aut(T)}|}\prod_{i=1}^{m+1}\bar{\mathcal{V}}_i\,,\end{aligned}\tag{1.1}$$

where the product at the end of the last line runs over the vertices of a tree, and $\bar{\mathcal{V}}_i$ are factors to be defined below. This expression is formally identical to the one found for $\mathcal{N}=2$ SQCD in [19], except for the fact that now the factors $\bar{\mathcal{V}}_i$ depend on two 't Hooft couplings, λ_1 and λ_2. The terms in (1.1) with a single value of the ζ function have already appeared in [20]. In the perturbative expansion of $\mathcal{F}_0(\lambda_1,\lambda_2)$ above, each product of values of the ζ function is accompanied by a polynomial in λ_1 and λ_2, that can be rewritten as a palindromic polynomial in λ_2/λ_1. Intriguingly, up to the order we have checked explicitly, all such polynomials have all roots on the unit circle of the complex λ_2/λ_1 plane. This is of course reminiscent of the seminal work by Lee and Yang [21] for the zeros of the partition function of the ferromagnetic Ising model on a graph. We are able to prove this property for all the terms in (1.1) with a single value of ζ, and formulate two conjectures for general trees.

In section 3, we compute the planar limit of the expectation value of the 1/2 BPS circular Wilson loop defined for the gauge group in one of the two nodes of the $\widehat{A_1}$ theory, and in the fundamental representation. The answer is now given as a sum over rooted trees. This Wilson loop is defined for one of the two nodes of the quiver, so it does not transform nicely under the $\mathbb{Z}_2$ symmetry of the theory. For this reason we consider $\langle W\rangle_\pm = \langle W_1\rangle \pm \langle W_2\rangle$ (with the $\mathcal{N}=4$ results subtracted). For $\langle W\rangle_\pm$ we find again that, up to the orders we have checked explicitly, all the polynomials in λ_2/λ_1 that appear have all roots on the unit circle.

In the appendices, we write the first terms in the explicit expansion of the planar free energy and expectation value of various Wilson loop operators.

This work leaves open a number of interesting problems. First, there are general arguments that the perturbative series of the planar limit of quantum field theories have finite radius of convergence [22]. We have been able to determine the domain of convergence of just a small subset of the perturbative series found in this paper - see also [20] - but rigorously determining the full domain of convergence of the full perturbative series seems like a much harder problem. Second, in the main text we formulate two conjectures on the zeros of the polynomials that appear in the perturbative series of the planar free energy and expectation values of Wilson loops. It would be interesting to prove these conjectures, and further investigate if this property is related to the integrability of these theories, that has been encountered both in the planar limit [6, 9, 23, 24] and in the full theory [25, 26].

2 The partition function of $\mathcal{N}=2$ quiver CFT

In this section we introduce the theories we are going to study, and recall how supersymmetric localization reduces the evaluation of selected quantities to matrix integrals. In particular, we will study first the planar free energy of the theory. Following [27–29], the integrals are performed over the full Lie algebra instead of restricting to a Cartan subalgebra, and the 1-loop factor is rewritten as an effective action. We will focus on the planar limit and in this limit, as in [19], we will unravel the underlying graph structure of the perturbative expansion.

Let us start by briefly reviewing the classification and field content of $\mathcal{N}=2$ superconformal quiver gauge theories with $\mathrm{SU}(N)$ gauge groups. They are in one-to-one correspondence with simply-laced affine Lie algebras $\widehat{ADE}$, and thus follow an ADE classification [2]. The gauge sector and matter content are encoded in the extended Cartan matrix of the affine Lie algebra. The gauge group is

$$\prod_i \mathrm{SU}(n_i N)\,, \tag{2.1}$$

where n_i is the Dynkin index of the i-th node of the affine Dynkin diagram. The hypermultiplets transform in the representations

$$\oplus\, a_{ij}\left(n_i N, \overline{n_j N}\right)\,, \tag{2.2}$$

where a_{ij} is the adjacency matrix of the Dynkin diagram.

These theories have a marginal coupling for each gauge group and, in the particular case where the complexified couplings satisfy

$$\tau_i = n_i \tau\,, \tag{2.3}$$

the quiver theory can be obtained as an orbifold of $\mathcal{N}=4$ SU(N) super Yang-Mills by the discrete subgroup Γ of SU(2) [2], which also follow an ADE classification. These theories can be engineered in string theory via a suitable brane configuration and even more, in a suitable limit, they admit a weakly curved gravity dual in terms of the $AdS_5 \times S^5/\Gamma$ geometry [3, 4]. On the other hand, when all the couplings are set to zero except one, say g_1, the quiver theory reduces to $\mathcal{N}=2$ SQCD.

After having reviewed $\mathcal{N}=2$ superconformal quiver theories, let's discuss supersymmetric localization for them. Following [13] it is possible to localize the $\widehat{ADE}$ theories on S^4. It is also possible to localize the theory on a squashed sphere of parameter b for which in the limit $b \to 1$ we recover the sphere, in such configuration the exact partition function is given by

$$Z = \int \mathrm{d}a_I \, \mathcal{Z}_{\text{1-loop}}(a_I, b) |\, \mathcal{Z}_{\text{inst}}(a_I, b)|^2 e^{-\sum_{I=1}^{n} \frac{8\pi^2}{g_I^2} \mathrm{Tr} a_I^2}, \tag{2.4}$$

where a_I denotes the eigenvalues of the vector-multiplet scalars Φ_I restricted to the constant mode on S^4. In what follows we will be mostly interested in quantities that are relevant in the $b \simeq 1$ limit, such as the Wilson loop operator, or even more just observables defined on the sphere. As usual we will restrict our analysis to the zero-instanton sector, thus neglecting $|\mathcal{Z}_{\text{inst}}|^2$, and expanding (2.4) in b we obtain

$$Z = \int \mathrm{d}a_I \, \mathcal{Z}_{\text{1-loop}}(a_I) \, e^{-\sum_{I=1}^{n} \frac{8\pi^2}{g_I^2} \mathrm{Tr} a_I^2} + \mathcal{O}((b-1)^2) \,, \tag{2.5}$$

higher order terms in b were studied before in [10] and we refer the reader there for more details. The factor $\mathcal{Z}_{\text{1-loop}}$ is the 1-loop contribution determined by the matter content. For instance for the $\widehat{A}_{n-1}$ theory it is given by

$$\mathcal{Z}_{\text{1-loop}} = \prod_{I=1}^{n} \frac{\prod_{i<j} H^2(ia_i^I - ia_j^I)}{\prod_{i,j} H(ia_i^I - ia_j^{I+1})} \,, \tag{2.6}$$

where we identify the node $n+1$ with the first one and $H(x)$ is the Barnes function whose expansion is given by

$$\log H(x) = -(1+\gamma)x^2 - \sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} x^{2n} \,. \tag{2.7}$$

Following the previous works [27–29] the strategy will be once again to interpret the matter content as an effective action

$$S_{\text{int}} = -\log \mathcal{Z}_{\text{1-loop}} \,. \tag{2.8}$$

Given that the theory is conformal for arbitrary values of the couplings, the quadratic terms in (2.7) will exactly cancel and the effective action will start at order g_i^4.

Let us first illustrate the process with the $\widehat{A}_1$ quiver since the extension to the general case is straightforward. In this case the field content of the $\widehat{A}_1$ quiver consists of two vector multiplets in the adjoint: $(A_\mu^I, \Phi^I, \Phi'^I)$, $I = 1, 2$, and bi-fundamental matter: $(X, Y, X^\dagger, Y^\dagger)$: $D_\mu X = \partial_\mu X + A_\mu^1 X - X A_\mu^2$. The l-loop factor reduces to

$$\mathcal{Z}_{1\text{-loop}} = \frac{\prod_{i<j} H^2(ia_i^1 - ia_j^1) H^2(ia_i^2 - ia_j^2)}{\prod_{i,j} H^2(ia_i^1 - ia_j^2)} \,. \tag{2.9}$$

Following the procedure presented in [19] and using (2.7) it is possible to arrive to the effective action, obtaining

$$\begin{aligned} S_{\text{int}} = &\sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \\ &\times \Bigg[\sum_{k=1}^{n-1} \binom{2n}{2k} \left(\text{Tr}\, a_1^{2(n-k)} \text{Tr}\, a_1^{2k} + \text{Tr}\, a_2^{2(n-k)} \text{Tr}\, a_2^{2k} - 2\text{Tr}\, a_1^{2(n-k)} \text{Tr}\, a_2^{2k} \right) \\ &\quad - \sum_{k=1}^{n-2} \binom{2n}{2k+1} \Big(\text{Tr}\, a_1^{2(n-k)-1} \text{Tr}\, a_1^{2k+1} + \text{Tr}\, a_2^{2(n-k)-1} \text{Tr}\, a_2^{2k+1} \\ &\qquad\qquad - 2\text{Tr}\, a_1^{2(n-k)-1} \text{Tr}\, a_2^{2k+1} \Big) \Bigg] \,, \end{aligned} \tag{2.10}$$

where all traces are in the fundamental representation of the respective gauge group. Let's comment upon a couple of features of this result: first, as we already encountered in our previous work for theories with simple gauge groups [19], the effective action involves infinite sums of double-trace terms, that split into even and odd powers. By the same large N counting arguments as in [19], the odd powers will not contribute to the planar computations, so we discard such terms in what follows. Second, the pattern of double-trace terms in (2.10) is dictated by the Cartan matrix of $\widehat{A_1}$,

$$\frac{1}{2}\, C = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \,. \tag{2.11}$$

This shouldn't be a surprise, since for $\mathcal{N} = 2$ quiver superconformal field theories, the matter content is fixed by the 1-loop β functions, which are captured by the generalized Cartan matrix [2]. This last observation allows us to generalize (2.10) to arbitrary $\mathcal{N} = 2$ superconformal quiver theory. The effective action, keeping just the terms with even powers, is

$$S_{\text{int}} = \frac{1}{2} \sum_{I,J} C_{IJ} \sum_{n=2}^{\infty} \frac{\zeta(2n-1)(-1)^n}{n} \sum_{k=1}^{n-1} \binom{2n}{2k} \text{Tr} a_I^{2(n-k)} \text{Tr} a_J^{2k} \,, \tag{2.12}$$

where C_{IJ} is the Cartan matrix of the corresponding affine Lie algebra.

2.1 Planar free energy

We turn now to the large N limit of the free energy on S^4, $F(\lambda_i, N) = \log Z_{S^4}$. In fact, as usual, we will compute the difference of free energy with the Gaussian model,

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$\mathcal{F}(\lambda_i, N) \equiv F(\lambda_i, N) - \sum_i F(\lambda_i)^{\mathcal{N}=4}$. Our goal is to determine the leading term in the large N expansion, i.e. $F(\lambda_i, N) = F_0(\lambda_i)N^2 + \cdots$. In general we have

$$F(\lambda_i, N) = \log Z_{S^4} = \sum_{m=1}^{\infty} \frac{(-1)^{m+1}}{m} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \langle S_{\text{int}}^k \rangle \right)^m , \tag{2.13}$$

the free energy scales like N^2 in the planar limit, so there are many cancellations in (2.13) and we need to fully identify the N^2 terms from (2.13) that survive these cancellations. The argument to extract those terms is exactly the same as in our recent work [19]: for a disconnected $2m$-point function, the pieces that scale like N^2 are products of $m+1$ connected correlators. These connected correlators in the planar limit are given by [30] (see also [31] for a more recent derivation)

$$\langle \operatorname{Tr} a^{2k_1} \operatorname{Tr} a^{2k_2} \ldots \operatorname{Tr} a^{2k_n} \rangle_c = \mathcal{V}(k_1, \ldots, k_n) \tilde{\lambda}^d N^{2-n}, \quad \tilde{\lambda} = \frac{\lambda}{16\pi^2}, \tag{2.14}$$

with

$$\mathcal{V}(k_1, \ldots, k_n) = \frac{(d-1)!}{(d-n+2)!} \prod_{i=1}^{n} \frac{(2k_i)!}{(k_i-1)!k_i!}, \quad d = \sum_{i=1}^{n} k_i. \tag{2.15}$$

The products of $m+1$ connected correlators that contribute to the planar free energy are those where the $2m$ traces are distributed in a way that can be characterized by a tree graph [19]: for each correlator introduce a vertex, and join them by an edge if they have operators from the same double-trace. The contributions to $\mathcal{F}_0(\lambda)$ at fixed order in the number of values of ζ function are then obtained following a similar procedure as in our recent work [19], but with a couple of modifications. Terms with m values of the ζ function have m pairs of traces, coming from m double-trace terms, which are of the form $C_{IJ} \operatorname{Tr} a_I^{2(n-k)} \operatorname{Tr} a_J^{2k}$.

To find the contribution to the planar free energy at this order, first draw all the trees with m edges. For every tree, assign each of the m double-traces to one of the m edges; this labels the m edges of the tree, turning it into a edge-labeled tree. Next, add an arrow to each of the m edges, turning the tree into a directed edge-labeled tree. Assign $\operatorname{Tr} a_I^{2(n-k)}$ to the vertex at the start (i.e. origin of the arrow) of the i-th edge. Assign $\operatorname{Tr} a_J^{2k}$ to the vertex at the end (i.e. end of the arrow) of the i-th edge. This procedure assigns to each of the $m+1$ vertices a number of traces equal to its degree α_j, i.e. the number of edges connected to that vertex. For each vertex, consider now the connected correlator of all its trace operators and assign it its numerical factor $\mathcal{V}_j$, eq. (2.15), times $\tilde{\lambda}_j^{d_j}$, with $j = 1, \ldots, m+1$. For the connected correlator to be nonzero, all traces at a given vertex must be of the same matrix, and this enforces that they have the same index. Finally, multiply the contribution of this tree graph by a product of m components of the Cartan matrix, one per edge, with the indices fixed by those at the vertices of each edge. Summing

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over all the possible choices, we arrive at

$$\mathcal{F}_0(\tilde{\lambda}_1,\ldots,\tilde{\lambda}_n)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{n_1,\ldots,n_m=2}^{\infty}\frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m}(-1)^{n_1+\cdots+n_m}$$
$$\times\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\frac{1}{2^m}\sum_{\substack{\text{directed trees}\\ \text{with m labeled edges}}}\sum_{I,J}C_{I_1J_1}\ldots C_{I_mJ_m}\prod_{i=1}^{m+1}\tilde{\lambda}_{I_i}^{d_i}\mathcal{V}_i\,. \tag{2.16}$$

This expression is the perturbative series for the planar free energy of any $\mathcal{N}=2$ superconformal quiver theory, with quiver determined by the affine Lie algebra with Cartan matrix C. In what follows, we will discuss mostly the simplest quiver theory, $\widehat{A_1}$, that has gauge group SU(N)×SU(N), and Cartan matrix (2.11). This means that double-traces where both operators belong to the same gauge group, e.g. $\operatorname{Tr} a_1^{2(n-k)}\operatorname{Tr} a_1^{2k}$ are weighted with a $+1$, while mixed double-traces, e.g. $\operatorname{Tr} a_1^{2(n-k)}\operatorname{Tr} a_2^{2k}$ are weighted with a -1. The overall sign of a given product of correlators is then -1 raised to the number of mixed double-traces. These signs can be transferred from the edges to the vertices: just assign an extra factor $(-1)^{\alpha_j}$ to all vertices of the tree corresponding to correlators of, say, the second gauge group (this choice is arbitrary and the final result is independent of it). To convince oneself that these two rules are the same, write every sign on top of the edges of the tree: if it is a -1 assign it to the vertex with operators of the second gauge group. If it is a $+1$, and it is joining two vertices with operators of the second gauge group, just write $+1=(-1)(-1)$ and again assign one -1 to each vertex. Then each vertex contributes a factor

$$\bar{\mathcal{V}}(x_1,\ldots,x_\alpha)=\mathcal{V}(x_1,\ldots,x_\alpha)\left(\tilde{\lambda}_1^{\sum_i x_i}+(-1)^\alpha\tilde{\lambda}_2^{\sum_i x_i}\right), \tag{2.17}$$

and the generic expression (2.16) simplifies to

$$\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2)=\sum_{m=1}^{\infty}\frac{(-1)^m}{m!}\sum_{n_1,\ldots,n_m=2}^{\infty}\frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m}(-1)^{n_1+\cdots+n_m}$$
$$\times\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{directed trees}\\ \text{with } m \text{ labeled edges}}}\prod_{i=1}^{m+1}\bar{\mathcal{V}}_i\,. \tag{2.18}$$

Finally, by exactly the same arguments as in our previous paper [19], the last sum can be reduced to a sum over unlabeled trees

$$\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2)=\sum_{m=1}^{\infty}(-2)^m\sum_{n_1,\ldots,n_m=2}^{\infty}\frac{\zeta(2n_1-1)\ldots\zeta(2n_m-1)}{n_1\ldots n_m}(-1)^{n_1+\cdots+n_m}$$
$$\times\sum_{k_1=1}^{n_1-1}\binom{2n_1}{2k_1}\cdots\sum_{k_m=1}^{n_m-1}\binom{2n_m}{2k_m}\sum_{\substack{\text{unlabeled trees}\\ \text{with } m \text{ edges}}}\frac{1}{|\mathrm{Aut(T)}|}\prod_{i=1}^{m+1}\bar{\mathcal{V}}_i\,. \tag{2.19}$$

Let's mention a further property of $\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2)$. Since $\mathcal{F}_0(\tilde{\lambda}_2,\tilde{\lambda}_1)=\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2)$ and $\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_1)=0$, it follows that $\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2)$ has a double zero,

$$\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2)=(\tilde{\lambda}_1-\tilde{\lambda}_2)^2 f(\tilde{\lambda}_1,\tilde{\lambda}_2)\,, \tag{2.20}$$

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this implies that at the orbifold point $\lambda_1 = \lambda_2$ - see comment below (2.3) - not just the free energy, but also its first derivative with respect to λ coincides with the $\mathcal{N}=4$ result. To see that this property is implied by our result (2.19), we are going to prove that the contribution of every tree to (2.19) is of the form

$$(\tilde{\lambda}_1 - \tilde{\lambda}_2)^{v_{\text{odd}}}\, p(\tilde{\lambda}_1, \tilde{\lambda}_2)\,, \tag{2.21}$$

where v_{odd} is the number of vertices of the tree with odd degree, and $p(\tilde{\lambda}_1, \tilde{\lambda}_2)$ is a symmetric polynomial in $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ with positive coefficients. This follows from inspection of the factor attached to each vertex, (2.17). When the degree α of a vertex is odd, $\tilde{\lambda}_1 = \tilde{\lambda}_2$ is a simple root of that factor. After pulling out these factors, what is left is a polynomial with positive coefficients. As a check, notice that v_{odd} is always even: for a tree with $m+1$ vertices, $\sum_{i=1}^{m+1} \alpha_i = 2m$, and since $\sum_i \alpha_i^{\text{even}}$ is even, $\sum_i \alpha_i^{\text{odd}}$ must be even also, which implies that v_{odd} is even. This concludes the argument for (2.21). Now, since every tree has at least two vertices of degree one, $v_{\text{odd}} \geq 2$, and (2.20) follows.

To illustrate (2.19), let's work out the first terms. The $m=1$ terms in (2.19) are terms with a single value of ζ [20]. To write them, it is convenient to first recall the definition of the Narayana numbers

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}\,, \tag{2.22}$$

and the Narayana polynomials

$$\mathfrak{C}_n(t) = \sum_{k=0}^{n-1} N(n,k+1)t^k\,, \tag{2.23}$$

that satisfy $\mathfrak{C}_n(1) = \mathcal{C}_n$ with $\mathcal{C}_n$ the Catalan numbers. At this order, we have to consider trees with two vertices. There is just one such tree, and both vertices have degree one. Then,

$$\begin{aligned}
\mathcal{F}_0(\tilde{\lambda}_1, \tilde{\lambda}_2)|_\zeta &= -\sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n}(-1)^n \sum_{k=1}^{n-1} \binom{2n}{2k} \mathcal{C}_{n-k}\mathcal{C}_k \left(\tilde{\lambda}_1^{n-k} - \tilde{\lambda}_2^{n-k}\right)\left(\tilde{\lambda}_1^{k} - \tilde{\lambda}_2^{k}\right) \\
&= -\sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n}(-1)^n \mathcal{C}_n \tilde{\lambda}_1^n \left[\left(1 + \frac{\tilde{\lambda}_2^n}{\tilde{\lambda}_1^n}\right)\mathcal{C}_{n+1} - 2\mathfrak{C}_{n+1}\left(\frac{\tilde{\lambda}_2}{\tilde{\lambda}_1}\right)\right]\,,
\end{aligned} \tag{2.24}$$

where to avoid confusion, the first term in the parenthesis involves the Catalan number $\mathcal{C}_{n+1}$, and the second one the Narayana polynomial $\mathfrak{C}_{n+1}(\tilde{\lambda}_2/\tilde{\lambda}_1)$. A first question we can ask about this series is what is its domain of convergence in $\mathbb{C}^2$. As pointed out in [19, 20], when $\lambda_2 = 0$ it is straightforward to prove that the radius of convergence is $\lambda_1 = \pi^2$, and the same holds, *mutatis mutandi*, when $\lambda_1 = 0$. When both couplings are different from zero, since $\mathcal{F}_0(\lambda_1, \lambda_1) = 0$ the series trivially converges when both couplings are equal. When the two couplings are different, one of them is larger, say λ_1, applying the quotient criterion it follows that for any $|\lambda_2| < |\lambda_1| \leq \pi^2$, the series is convergent. All in all, this series is convergent in $|\lambda_1| \leq \pi^2$, $|\lambda_2| \leq \pi^2$ plus the $\lambda_1 = \lambda_2$ line.

For $\mathcal{N}=2$ superconformal field theories with a simple gauge group, terms with a fixed number of values of the ζ function form an infinite series. In [19] we sketched an argument that all these series have the same radius of convergence. It seems possible that this property extends to quiver theories.

Let's work out a couple more of terms in (2.19). Terms with two values of the ζ function are given by a sum over trees with two edges. There is just one tree with two edges, and its vertices have degrees $(1,2,1)$. As a last example, terms with three values of the ζ function are given by a sum over trees with three edges. There are two such unlabeled trees. The degrees are $(1,2,2,1)$ for the first tree, and $(3,1,1,1)$ for the second, all these trees are despicted in figure 1 and 2. Up to this order,

$$
\begin{aligned}
\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2) = & -\sum_{n=2}^{\infty}\frac{\zeta(2n-1)}{n}(-1)^n\sum_{k=1}^{n-1}\binom{2n}{2k}\mathcal{V}(n-k)\mathcal{V}(k)(\tilde{\lambda}_1^{n-k}-\tilde{\lambda}_2^{n-k})(\tilde{\lambda}_1^{k}-\tilde{\lambda}_2^{k}) \\
& +\frac{1}{2}\sum_{n_i=2}^{\infty}\frac{\zeta(2n_i-1)}{n_1n_2}(-1)^{n_1+n_2}\sum_{k_i=1}^{n_i-1}\binom{2n_i}{2k_i}4\mathcal{V}(k_1)\mathcal{V}(n_1-k_1,n_2-k_2)\mathcal{V}(k_2) \\
& \times(\tilde{\lambda}_1^{k_1}-\tilde{\lambda}_2^{k_1})(\tilde{\lambda}_1^{n_1-k_1+n_2-k_2}+\tilde{\lambda}_2^{n_1-k_1+n_2-k_2})(\tilde{\lambda}_1^{k_2}-\tilde{\lambda}_2^{k_2}) \\
& -\frac{1}{3!}\sum_{n_i=2}^{\infty}\frac{\zeta(2n_i-1)}{n_1n_2n_3}(-1)^{n_1+n_2+n_3}\sum_{k_i=1}^{n_i-1}\binom{2n_i}{2k_i}8\Big[3\mathcal{V}(n_1-k_1)\mathcal{V}(k_1,n_2-k_2) \\
& \times\mathcal{V}(k_2,n_3-k_3)\mathcal{V}(k_3)(\tilde{\lambda}_1^{n_1-k_1}-\tilde{\lambda}_2^{n_1-k_1})(\tilde{\lambda}_1^{k_1+n_2-k_2}+\tilde{\lambda}_2^{k_1+n_2-k_2}) \\
& \times(\tilde{\lambda}_1^{k_2+n_3-k_3}+\tilde{\lambda}_2^{k_2+n_3-k_3})(\tilde{\lambda}_1^{k_3}-\tilde{\lambda}_2^{k_3})+\mathcal{V}(n_1-k_1,n_2-k_2,n_3-k_3)\mathcal{V}(k_1)\mathcal{V}(k_2)\mathcal{V}(k_3) \\
& \times(\tilde{\lambda}_1^{n_1-k_1+n_2-k_2+n_3-k_3}-\tilde{\lambda}_2^{n_1-k_1+n_2-k_2+n_3-k_3})(\tilde{\lambda}_1^{k_1}-\tilde{\lambda}_2^{k_1})(\tilde{\lambda}_1^{k_2}-\tilde{\lambda}_2^{k_2})(\tilde{\lambda}_1^{k_3}-\tilde{\lambda}_2^{k_3})\Big] \\
& +\mathcal{O}(\zeta^4).
\end{aligned}
\tag{2.25}
$$

As a first check, when either of the two couplings vanishes, we recover the result of $\mathcal{N}=2$ SCQD presented in [19]. Also, in this expression we can see rather explicitly that at every order the contribution has at least a double zero $(\tilde{\lambda}_1-\tilde{\lambda}_2)^2$. In appendix A we have written the outcome of these sums, up to order $\tilde{\lambda}^6$.

2.2 The Lee-Yang property of the planar free energy expansion

We would like to discuss one further property of the perturbative expansion (2.19). Notice that the contribution of a given tree is obtained by summing over all the possible ways to assign one gauge group, 1 or 2, to each vertex in the tree, see figures 1 and 2. This is reminiscent of the Ising model defined on that tree, where on each vertex we can have a spin up or down. It is indeed possible to construct a generalized Ising-type model, with inhomogeneous external magnetic field, whose partition function yields each tree contribution in (2.19). This generalized Ising model is admittedly a bit contrived, but following the classical work by Lee and Yang [21], it motivates the study of the zeros of its partition function.

In more detail, every tree graph contributes to the planar energy in (2.19) a homogeneous polynomial in λ_1 and λ_2. Being homogeneous, these polynomials can be thought of as polynomials of a single variable λ_2/λ_1. Inspired by the classical work by Lee and Yang [21] on the ferromagnetic Ising model, we are going to put forward two conjectures regarding the zeros of these polynomials: first, that for a given tree, all the zeros of the

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corresponding polynomial are on the unit circle in the complex λ_2/λ_1 plane. Second, that when we sum the contributions from different trees with the same number of nodes, the same property holds.

To provide context, let's start by briefly recalling the definition of the Ising model on a graph and the Lee-Yang theorem. Let G be a finite graph, E its set of edges and V its set of vertices. The Ising model on G is defined by assigning to each vertex $i \in V$, a $\sigma_i = \pm 1$ (spin up/down). The Hamiltonian is

$$\mathcal{H} = -J \sum_{i-j \in E} \sigma_i \sigma_j - H \sum_{i \in V} \sigma_i \,, \tag{2.26}$$

with J the coupling among spins and H the external magnetic field. The partition function can be written as

$$Z(\beta J, \beta H) = \sum_{\text{all states}} e^{-\beta \mathcal{H}} = e^{\beta J|E| - \beta H|V|} \sum_{\text{all states}} e^{-2\beta J e_\pm} \, e^{2\beta H v_\uparrow} \,, \tag{2.27}$$

where $e_\pm$ is the number of edges connecting different spins, and $v_\uparrow$ the number of spins up in a given configuration. Define $\tau = e^{-2\beta J}$, $x = e^{2\beta H}$. The last sum defines a polynomial palindromic in x,

$$P(\tau, x) = \sum_{\text{all states}} \tau^{e_\pm} \, x^{v_\uparrow} \,. \tag{2.28}$$

In [21], Lee and Yang proved that for $\tau \in [-1, 1]$, the polynomials $P(\tau, x)$ have all their x roots on the unit circle. In fact, they proved it for arbitrary ferromagnetic couplings $J_{ij} \geq 0$, and different magnetic fields per site H_i.

To construct an Ising-type model whose partition function yields the polynomials that appear in (2.19), proceed as follows. Take the graph G to be a tree T,

1. Assign a positive integer n_i to each of the e edges of the tree graph.

2. For every edge, split n_i into two positive integers, $n_i = k_i + (n_i - k_i)$ and assign each of these two integers to one of the vertices at the ends of that edge.

3. Then, if a vertex has degree d_j this procedure assigns to that vertex d_j integers. Let m_j be the sum of these integers at a given vertex; the magnetic field at that vertex is then $m_j H$.

So far, for a fixed partition of all n_i, this is a peculiar way to assign external magnetic fields that are different at each vertex. This defines

$$P(\tau, x, k_i, n_i) = \sum_{\text{all states}} \tau^{e_\pm} \prod_{\substack{\text{vertices} \\ \text{with spin up}}} x^{m_j} \,, \tag{2.29}$$

Lee and Yang already proved (lemma in appendix II of [21]) that all the zeros of these polynomials are on the unit circle. Finally, consider the sum over all the partitions of each of the n_i into two

$$P(\tau, x, n_1, \ldots, n_e) = \sum_{k_1=1}^{n_1-1} \cdots \sum_{k_e=1}^{n_e-1} \rho(k_i, n_i) \sum_{\text{all states}} \tau^{e_\pm} \prod_{\substack{\text{vertices} \\ \text{with spin up}}} x^{m_j} \,, \tag{2.30}$$

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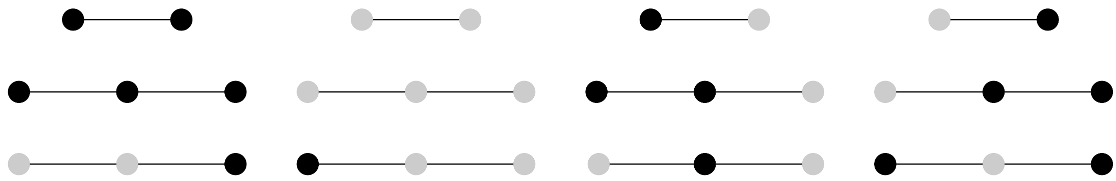

Figure 1. Trees contributing to the first and second order expansion of the free energy.

where $\rho(k_i, n_i)$ is a distribution that weights different configurations. The contribution of every tree to the planar free energy in (2.19) is obtained from the free energy of this Ising-type model, by setting $\tau = -1$, $x = \lambda_2/\lambda_1$, and the distribution

$$\rho(k_i, n_i) = \binom{2n_1}{2k_1} \cdots \binom{2n_m}{2k_m} \prod_{i=1}^{m} \mathcal{V}_i \,. \tag{2.31}$$

The main reason we have defined this family of Ising-type models is that there is numerical evidence that suggests that they share the Lee-Yang property with the original Ising model. This leads us to formulate the following two conjectures:

Conjecture 1. For any tree with e edges, any fixed positive integers $n_1, \ldots, n_e$ and arbitrary $\rho(k_i, n_i) > 0$ the polynomials $P(\tau, x, n_1, \ldots, n_e)$ have all their x roots on the unit circle.

Conjecture 2. If we sum the polynomials of all the trees with the same number of edges, the resulting polynomial still has the Lee-Yang property.

We can prove the first conjecture in the particular case of the simplest tree. In this case, (2.30) is simply

$$P(\tau, x, k, n) = x^n + \tau x^{n-k} + \tau x^k + 1 \,, \tag{2.32}$$

that for $|\tau| \leq 1$ has its roots on the unit circle. Then

$$P(\tau, x, k, n) = \sum_{k=1}^{n-1} \rho(n, k) \left(x^n + \tau x^{n-k} + \tau x^k + 1 \right) \,, \tag{2.33}$$

with arbitrary $\rho(n, k) > 0$. To prove that these polynomials have their roots on the unit circle, we make use of the following theorem [32]: if $P(x) = A_n x^n + A_{n-1} x^{n-1} + \cdots + A_1 x + A_0$ is a palindromic polynomial and $2|A_n| \geq \sum_{j=1}^{n-1} |A_j|$, then all its zeros are in the unit circle. In our case, the inequality in the theorem is satistifed as long as $|\tau| \leq 1$, so the result follows. Back to the free energy of the quiver theory, one can check indeed that the polynomials in the expansion (2.24) have the Lee-Yang property.

We haven't been able to prove these two conjectures for arbitrary tree graphs. After the seminal work [21], the proof of the Lee-Yang unit circle theorem has been extended to many other systems, see e.g. [33, 34]. It would be interesting to see if any of these arguments can be adapted to prove our conjectures.

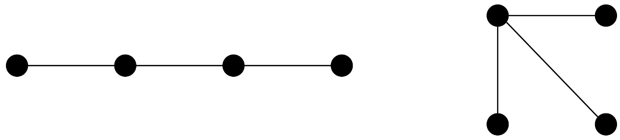

Figure 2. The two trees with three edges: (a) Tree with vertices of degrees (1,2,2,1). (b) Tree with vertices of degrees (3,1,1,1). There are 16 ways to color each of them.

3 Wilson loop in the large N limit

For each of the gauge groups of the quiver theory, we can define a 1/2 BPS Wilson loop, with circular contour in Euclidean signature. The evaluation of its expectation value reduces to a matrix integral thanks to supersymmetric localization. We will now evaluate the planar limit of this expectation value and show that the perturbative series involves a sum over rooted trees. While the Wilson loop can be defined for arbitrary representations of the gauge group, in order to take advantage of the results of [19, 31], we will restrict its study to the fundamental representation

$$\langle W^I \rangle = \langle \frac{1}{N} \mathrm{Tr}_F \mathcal{P} \exp \oint_{\mathcal{C}} ds \left(i A^I_\mu(x) \dot{x}^\mu + \Phi^I(x) |\dot{x}| \right) \rangle \,, \tag{3.1}$$

where $I = 1, \cdots, n$. The theory can be localized [13] on the sphere with squashing parameter b, where $b = 1$ corresponds to S^4, in such case the vev of the 1/2 BPS Wilson loop reduces to

$$\langle W_I^\pm \rangle = \frac{1}{Z} \int da_I \mathrm{Tr} \left(e^{-2\pi b^\pm a_I} \right) e^{-\sum_{I=1}^n \frac{8\pi^2}{g_I^2} \mathrm{Tr} a_I^2} \mathcal{Z}_{\text{1-loop}}(a_I, b) |\mathcal{Z}_{\text{inst}}(a_I, b)|^2 \,, \tag{3.2}$$

now $\pm$ represents the two different trajectories in which we can compute the Wilson loop on the squashed sphere [11]; from now on we will avoid the $\pm$ to make the notation less cumbersome, bearing in mind that in order to switch between trajectories we need to make the replacement $b \to b^{-1}$ in the following results. Once again we will consider the 1-loop contribution as an effective action, given by (2.10), and as discussed on the previous section we will compute the large N limit of this interacting theory while restricting ourselves to the zero-instanton sector. We are interested in observables that are only sensitive to the linear dependence of $\langle W_b \rangle$ in $(b-1)$, and since the dependence of $\mathcal{Z}_{\text{1-loop}}(a_I, b)$ is quadratic in $b-1$, for our purposes we can compute $\langle W_b \rangle$ directly on S^4 [35],

$$\langle W_I^\pm \rangle = \frac{1}{Z} \int da_I \mathrm{Tr} \left(e^{-2\pi b^\pm a_I} \right) e^{-\sum_{I=1}^n \frac{8\pi^2}{g_I^2} \mathrm{Tr} a_I^2} \mathcal{Z}_{\text{1-loop}}(a_I) + \mathcal{O}((b-1)^2) \,. \tag{3.3}$$

Let us expand the Wilson loop insertion

$$\langle W_I \rangle = \sum_{l=0}^{\infty} \frac{(4\pi^2 b^2)^l}{(2l)!} \frac{\langle N^{-1} \mathrm{Tr}\, a_I^{2l} e^{-S} \rangle}{\langle e^{-S} \rangle} \,. \tag{3.4}$$

As argued in our recent work [19], the large N expansion of this expectation value scales like N^0, so given the overall normalization factor $1/N$, the relevant terms to keep from

$\langle \mathrm{Tr}\, a_I^{2l} S^m \rangle$ are products of $m+1$ connected correlators. Now there are $2m+1$ traces to be distributed in $m+1$ correlators, but since $\langle \mathrm{Tr}\, a_I^{2l} \rangle$ can't be by itself, we effectively have to distribute $2m$ traces into the $m+1$ connected correlators, which is the by now familiar sign that the possibilities are given by tree graphs. As in [19], one of the vertices is singled out by the presence of $\langle \mathrm{Tr}\, a_I^{2l} \rangle$, so these are rooted trees. The correlator that contains $\langle \mathrm{Tr}\, a_I^{2l} \rangle$ is a correlator of a_I operators, so it involves the λ_I coupling; by convention, the root vertex corresponding to this correlator will be referred as the vertex 1. The remaining m correlators can be either products of a_I traces or a_J traces. As we found in the evaluation of the planar free energy in the previous section, this is accounted for by modifying the numerical factor of the correlator by a weighted sum over the coupling. eq. (2.17). All in all, for the case of $\widehat{A_1}$

$$\begin{aligned}\langle W_1\rangle - \langle W_1\rangle_0 = &\sum_{l=1} \frac{(2\pi b)^{2l}}{(2l)!} \sum_{m=1}^{\infty} (-2)^m \sum_{n_1,\dots,n_m=2}^{\infty} \frac{\zeta(2n_1-1)\dots\zeta(2n_m-1)}{n_1\dots n_m} (-1)^{n_1+\dots+n_m} \\ &\times \sum_{k_1=1}^{n_1-1} \binom{2n_1}{2k_1} \cdots \sum_{k_m=1}^{n_m-1} \binom{2n_m}{2k_m} \sum_{\substack{\text{unlabeled rooted trees}\\ \text{with } m \text{ edges}}} \frac{1}{|\mathrm{Aut(T)}|} \tilde{\lambda}_1^{d_1} \mathcal{V}_1 \prod_{i=2}^{m+1} \bar{\mathcal{V}}_i, \end{aligned} \tag{3.5}$$

In the language of Ising-type models on trees introduced in the previous section, we can think of the Wilson loop insertion as a spin that is pinned to be up, at the rooted vertex. To illustrate this result, let's expand it up to second order,

$$\begin{aligned}\langle W_1\rangle - \langle W_1\rangle_0 = &\sum_{l=1}^{\infty} \frac{(4\pi^2 b^2)^l}{(2l)!} \Bigg\{ -\sum_{n=2}^{\infty} \frac{\zeta(2n-1)}{n} (-1)^n \sum_{k=1}^{n-1} \binom{2n}{2k} 2\mathcal{V}(l,n-k)\mathcal{V}(k) \tilde{\lambda}_1^{l+n-k} \left(\tilde{\lambda}_1^k - \tilde{\lambda}_2^k\right) \\ &+ \frac{1}{2} \sum_{n_1,n_2=2}^{\infty} \frac{\zeta(2n_1-1)\zeta(2n_2-1)}{n_1 n_2} (-1)^{n_1+n_2} \sum_{k_i=1}^{n_i-1} \binom{2n_1}{2k_1}\binom{2n_2}{2k_2} \\ &\times \Big[8\mathcal{V}(l,n_1-k_1)\mathcal{V}(k_1,n_2-k_2)\mathcal{V}(k_2) \tilde{\lambda}_1^{l+n_1-k_1} \left(\tilde{\lambda}_1^{k_1+n_2-k_2} + \tilde{\lambda}_2^{k_1+n_2-k_2}\right) \left(\tilde{\lambda}_1^{k_2} - \tilde{\lambda}_2^{k_2}\right) \\ &+ 4\mathcal{V}(l,n_1-k_1,n_2-k_2)\mathcal{V}(k_1)\mathcal{V}(k_2) \tilde{\lambda}_1^{l+n_1-k_1+n_2-k_2} \left(\tilde{\lambda}_1^{k_1} - \tilde{\lambda}_2^{k_1}\right)\left(\tilde{\lambda}_1^{k_2} - \tilde{\lambda}_2^{k_2}\right) \Big] \Bigg\}, \end{aligned} \tag{3.6}$$

for which the corresponding rooted trees can be seen in figure 3.

In appendix B, we present the result of these sums up to order $\tilde{\lambda}^7$. We have checked that they reproduce the results of [10, 11]. Contrary to what happened for the free energy, the expectation value of this Wilson loop does not have nice properties under the exchange $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$. The reason is obvious, the Wilson loop is defined for one of the two gauge groups in the quiver, thus breaking the $\mathbb{Z}_2$ symmetry. For this reason, let's consider the linear combinations $\langle W_1\rangle \pm \langle W_2\rangle$, which were referred in [7] as twisted and untwisted. These are symmetric and antisymmetric under the $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$ exchange, so we can introduce

$$\langle W_1\rangle + \langle W_2\rangle - \langle W_1\rangle_0 - \langle W_2\rangle_0 = (\tilde{\lambda}_1 - \tilde{\lambda}_2)^2\, w_+(\tilde{\lambda}_1, \tilde{\lambda}_2)\,, \tag{3.7}$$
$$\langle W_1\rangle - \langle W_2\rangle - \langle W_1\rangle_0 + \langle W_2\rangle_0 = (\tilde{\lambda}_1 - \tilde{\lambda}_2)\, w_-(\tilde{\lambda}_1, \tilde{\lambda}_2)\,, \tag{3.8}$$

with $w_\pm$ symmetric under $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$. What is more, to the orders we have checked explicitly, again all the polynomials that appear in the expansion of $w_\pm$ have all their roots in the unit

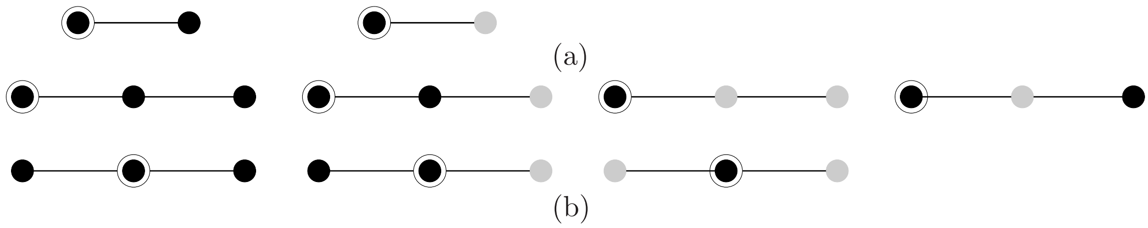

Figure 3. Rooted trees corresponding to the Wilson loop in the large N, we see that inserting the operator selects from figure 1 trees with the same color as the operator that we are inserting, trees containing two different colors arise from interaction terms in (2.10). (*a*) Terms corresponding to $\mathcal{V}(l, n_1 - k_1)\mathcal{V}(k_1)$. (*b*) Trees corresponding to $\mathcal{V}(l, n_1 - k_1)\mathcal{V}(k_1, n_2 - k_2)\mathcal{V}(k_2)$ and $\mathcal{V}(l, n_1 - k_1, n_2 - k_2)\mathcal{V}(k_1)\mathcal{V}(k_2)$.

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circle of the complex $\tilde{\lambda}_2/\tilde{\lambda}_1$ plane. We again conjecture that this is true for the polynomials generated by every tree.

For the polynomials that appear in $w_+(\tilde{\lambda}_1, \tilde{\lambda}_2)$, this would follow from our first conjecture if it is true. In particular, since in the previous section we proved the first conjecture for the simplest tree, it follows that it holds also for w_+, for the simplest tree. For w_- the argument does not apply immediately, since $\langle W_1\rangle - \langle W_2\rangle - \langle W_1\rangle_0 + \langle W_2\rangle_0$ produces antipalindromic polynomials.

To conclude, we can use these results to compute the one-point function of the energy-momentum tensor with these 1/2 BPS Wilson loops. This one-point function is fixed up to a coefficient h_W [36], which can be obtained from the expectation value of the deformed Wilson loop $\langle W_b\rangle$ by the formula [35, 37]

$$h_W = \frac{1}{12\pi^2}\,\partial_b \ln \langle W_b\rangle|_{b=1}\,. \tag{3.9}$$

finally, we can also compute the Bremsstrahlung function B [38] using the relation $B = 3h_W$ [35, 39, 40], valid for any $\mathcal{N} = 2$ superconformal field theory [41]. The results we obtain agree with those of [11].

Acknowledgments

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A Planar free energy up to 6th order

Here we present the explicit form of the planar free energy in terms of $\tilde{\lambda}_i = \frac{\lambda_i}{16\pi^2}$

$$
\begin{aligned}
\mathcal{F}_0(\tilde{\lambda}_1,\tilde{\lambda}_2) = (\tilde{\lambda}_1-\tilde{\lambda}_2)^2 \Big[& -3\zeta_3 + 20\zeta_5\left(\tilde{\lambda}_1+\tilde{\lambda}_2\right) - 70\zeta_7\left(2\tilde{\lambda}_1^2+3\tilde{\lambda}_1\tilde{\lambda}_2+2\tilde{\lambda}_2^2\right) \\
& +84\zeta_9\left(\tilde{\lambda}_1+\tilde{\lambda}_2\right)\left(13\tilde{\lambda}_1^2+10\tilde{\lambda}_1\tilde{\lambda}_2+13\tilde{\lambda}_2^2\right) \\
& -154\zeta_{11}\left(61\tilde{\lambda}_1^4+116\tilde{\lambda}_1^3\tilde{\lambda}_2+141\tilde{\lambda}_1^2\tilde{\lambda}_2^2+116\tilde{\lambda}_1\tilde{\lambda}_2^3+61\tilde{\lambda}_2^4\right) \\
& +36\zeta_3^2\left(\tilde{\lambda}_1^2+\tilde{\lambda}_2^2\right) - 240\zeta_3\zeta_5\left(\tilde{\lambda}_1+\tilde{\lambda}_2\right)\left(3\tilde{\lambda}_1^2-2\tilde{\lambda}_1\tilde{\lambda}_2+3\tilde{\lambda}_2^2\right) \\
& +840\zeta_3\zeta_7\left(8\tilde{\lambda}_1^4+5\tilde{\lambda}_1^3\tilde{\lambda}_2+2\tilde{\lambda}_1^2\tilde{\lambda}_2^2+5\tilde{\lambda}_1\tilde{\lambda}_2^3+8\tilde{\lambda}_2^4\right) \\
& +200\zeta_5^2\left(19\tilde{\lambda}_1^4+12\tilde{\lambda}_1^3\tilde{\lambda}_2+4\tilde{\lambda}_1^2\tilde{\lambda}_2^2+12\tilde{\lambda}_1\tilde{\lambda}_2^3+19\tilde{\lambda}_2^4\right) \\
& -144\zeta_3^3\left(5\tilde{\lambda}_1^4-2\tilde{\lambda}_1^3\tilde{\lambda}_2+6\tilde{\lambda}_1^2\tilde{\lambda}_2^2-2\tilde{\lambda}_1\tilde{\lambda}_2^3+5\tilde{\lambda}_2^4\right)\Big] + \mathcal{O}(\tilde{\lambda}^7).
\end{aligned} \tag{A.1}
$$

Up to the order we have explicitely checked, the polynomials have all unimodular roots.

B Wilson loop up to $\tilde{\lambda}^7$

Here we present the explicit expansion of the circular Wilson loop corresponding to an insertion in the first node of the quiver; it is possible to obtain the insertion in the second node by making the change $\tilde{\lambda}_1 \leftrightarrow \tilde{\lambda}_2$. For simplicity, in the expansion we have set $b=1$ and $\tilde{\lambda}_i = \frac{\lambda_i}{16\pi^2}$. If one wishes to restore the powers of b that appear in the perturbative expansion of $\langle W_b \rangle$ evaluated on S^4, one only needs to add in each term as many powers of b as powers of π there are.

$$
\begin{aligned}
\langle W_1 \rangle - \langle W_1 \rangle_0 = \left(\tilde{\lambda}_1 - \tilde{\lambda}_2\right)\Big[& -24\pi^2\zeta_3\tilde{\lambda}_1^2 - 32\pi^4\zeta_3\tilde{\lambda}_1^3 - 16\pi^6\zeta_3\tilde{\lambda}_1^4 - \frac{64}{15}\pi^8\zeta_3\tilde{\lambda}_1^5 - \frac{32}{45}\pi^{10}\zeta_3\tilde{\lambda}_1^6 \\
& + 80\pi^2\zeta_5\tilde{\lambda}_1^2\left(3\tilde{\lambda}_1+\tilde{\lambda}_2\right) + \frac{80}{3}\pi^4\zeta_5\tilde{\lambda}_1^3\left(13\tilde{\lambda}_1+4\tilde{\lambda}_2\right) \\
& + \frac{32}{3}\pi^6\zeta_5\tilde{\lambda}_1^4\left(17\tilde{\lambda}_1+5\tilde{\lambda}_2\right) + \frac{64}{9}\pi^8\zeta_5\tilde{\lambda}_1^5\left(7\tilde{\lambda}_1+2\tilde{\lambda}_2\right) \\
& - 280\pi^2\zeta_7\tilde{\lambda}_1^2\left(8\tilde{\lambda}_1^2+5\tilde{\lambda}_1\tilde{\lambda}_2+\tilde{\lambda}_2^2\right) \\
& - \frac{112}{3}\pi^4\zeta_7\tilde{\lambda}_1^3\left(91\tilde{\lambda}_1^2+55\tilde{\lambda}_1\tilde{\lambda}_2+10\tilde{\lambda}_2^2\right) \\
& - \frac{112}{3}\pi^6\zeta_7\tilde{\lambda}_1^4\left(49\tilde{\lambda}_1^2+29\tilde{\lambda}_1\tilde{\lambda}_2+5\tilde{\lambda}_2^2\right) \\
& + 336\pi^2\zeta_9\tilde{\lambda}_1^2\left(5\tilde{\lambda}_1+\tilde{\lambda}_2\right)\left(13\tilde{\lambda}_1^2+8\tilde{\lambda}_1\tilde{\lambda}_2+3\tilde{\lambda}_2^2\right) \\
& + 672\pi^4\zeta_9\tilde{\lambda}_1^3\left(51\tilde{\lambda}_1^3+41\tilde{\lambda}_1^2\tilde{\lambda}_2+17\tilde{\lambda}_1\tilde{\lambda}_2^2+2\tilde{\lambda}_2^3\right) \\
& - 3696\pi^2\zeta_{11}\tilde{\lambda}_1^2\left(61\tilde{\lambda}_1^4+56\tilde{\lambda}_1^3\tilde{\lambda}_2+36\tilde{\lambda}_1^2\tilde{\lambda}_2^2+11\tilde{\lambda}_1\tilde{\lambda}_2^3+\tilde{\lambda}_2^4\right) \\
& + 288\pi^2\zeta_3^2\tilde{\lambda}_1^2\left(2\tilde{\lambda}_1^2-\tilde{\lambda}_1\tilde{\lambda}_1+\tilde{\lambda}_2^2\right) \\
& + 192\pi^4\zeta_3^2\tilde{\lambda}_1^3\left(5\tilde{\lambda}_1^2-3\tilde{\lambda}_1\tilde{\lambda}_2+2\tilde{\lambda}_2^2\right)
\end{aligned} \tag{B.1}
$$

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$$
\begin{aligned}
&+192\pi^6\zeta_3^2\tilde\lambda_1^4\left(3\tilde\lambda_1^2-2\tilde\lambda_1\tilde\lambda_2+\tilde\lambda_2^2\right)\\
&-960\pi^2\zeta_3\zeta_5\tilde\lambda_1^2\left(15\tilde\lambda_1^3-5\tilde\lambda_1^2\tilde\lambda_2+\tilde\lambda_1\tilde\lambda_2^2+5\tilde\lambda_2^3\right)\\
&-320\pi^4\zeta_3\zeta_5\tilde\lambda_1^3\left(77\tilde\lambda_1^3-32\tilde\lambda_1^2\tilde\lambda_2+\tilde\lambda_1\tilde\lambda_2^2+20\tilde\lambda_2^3\right)\\
&+3360\pi^2\zeta_3\zeta_7\tilde\lambda_1^2\left(48\tilde\lambda_1^4-7\tilde\lambda_1^3\tilde\lambda_2-7\tilde\lambda_1^2\tilde\lambda_2^2+11\tilde\lambda_1\tilde\lambda_2^3+11\tilde\lambda_2^4\right)\\
&+1600\pi^2\zeta_5^2\tilde\lambda_1^2\left(57\tilde\lambda_1^4-8\tilde\lambda_1^3\tilde\lambda_2-10\tilde\lambda_1^2\tilde\lambda_2^2+14\tilde\lambda_1\tilde\lambda_2^3+13\tilde\lambda_2^4\right)\\
&-3456\pi^2\zeta_3^3\tilde\lambda_1^2\left(5\tilde\lambda_1^4-5\tilde\lambda_1^3\tilde\lambda_2+5\tilde\lambda_1^2\tilde\lambda_2^2-3\tilde\lambda_1\tilde\lambda_2^3+2\tilde\lambda_2^4\right)\Bigg].
\end{aligned}
$$

Note that we are inserting the operator in only one of the two nodes of the quiver thus breaking the $\mathbb{Z}_2$ invariance of the theory. This is the reason why the vev (B.1) does not exhibit the same properties as the free energy. It is possible to retain the $\mathbb{Z}_2$ invariance if we consider the sum and the difference, for the case of the sum we have

$$
\begin{aligned}
w_+(\tilde\lambda_1,\tilde\lambda_2)=\Bigg[&-24\pi^2\zeta_3\left(\tilde\lambda_1+\tilde\lambda_2\right)-32\pi^4\zeta_3\left(\tilde\lambda_1^2+\tilde\lambda_1\tilde\lambda_2+\tilde\lambda_2^2\right)\\
&-16\pi^6\zeta_3\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(\tilde\lambda_1^2+\tilde\lambda_2^2\right)-\frac{64}{15}\pi^8\zeta_3\left(\tilde\lambda_1^4+\tilde\lambda_1^3\tilde\lambda_2+\tilde\lambda_1^2\tilde\lambda_2^2+\tilde\lambda_1\tilde\lambda_2^3+\tilde\lambda_2^4\right)\\
&-\frac{32}{45}\pi^{10}\zeta_3\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(\tilde\lambda_1^4+\tilde\lambda_1^2\tilde\lambda_2^2+\tilde\lambda_2^4\right)+80\pi^2\zeta_5\left(3\tilde\lambda_1^2+4\tilde\lambda_1\tilde\lambda_2+3\tilde\lambda_2^2\right)\\
&+\frac{80}{3}\pi^4\zeta_5\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(13\tilde\lambda_1^2+4\tilde\lambda_1\tilde\lambda_2+13\tilde\lambda_2^2\right)\\
&+\frac{32}{3}\pi^6\zeta_5\left(17\tilde\lambda_1^4+22\tilde\lambda_1^3\tilde\lambda_2+22\tilde\lambda_1^2\tilde\lambda_2^2+22\tilde\lambda_1\tilde\lambda_2^3+17\tilde\lambda_2^4\right)\\
&+\frac{64}{9}\pi^8\zeta_5\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(7\tilde\lambda_1^4+2\tilde\lambda_1^3\tilde\lambda_2+7\tilde\lambda_1^2\tilde\lambda_2^2+2\tilde\lambda_1\tilde\lambda_2^3+7\tilde\lambda_2^4\right)\\
&-280\pi^2\zeta_7\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(8\tilde\lambda_1^2+5\tilde\lambda_1\tilde\lambda_2+8\tilde\lambda_2^2\right)\\
&-\frac{112}{3}\pi^4\zeta_7\left(91\tilde\lambda_1^4+146\tilde\lambda_1^3\tilde\lambda_2+156\tilde\lambda_1^2\tilde\lambda_2^2+146\tilde\lambda_1\tilde\lambda_2^3+91\tilde\lambda_2^4\right)\\
&-\frac{112}{3}\pi^6\zeta_7\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(49\tilde\lambda_1^4+29\tilde\lambda_1^3\tilde\lambda_2+54\tilde\lambda_1^2\tilde\lambda_2^2+29\tilde\lambda_1\tilde\lambda_2^3+49\tilde\lambda_2^4\right)\\
&+336\pi^2\zeta_9\left(65\tilde\lambda_1^4+118\tilde\lambda_1^3\tilde\lambda_2+138\tilde\lambda_1^2\tilde\lambda_2^2+118\tilde\lambda_1\tilde\lambda_2^3+65\tilde\lambda_2^4\right)\\
&+672\pi^4\zeta_9\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(51\tilde\lambda_1^4+41\tilde\lambda_1^3\tilde\lambda_2+68\tilde\lambda_1^2\tilde\lambda_2^2+41\tilde\lambda_1\tilde\lambda_2^3+51\tilde\lambda_2^4\right)\\
&-3696\pi^2\zeta_{11}\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(61\tilde\lambda_1^4+56\tilde\lambda_1^3\tilde\lambda_2+96\tilde\lambda_1^2\tilde\lambda_2^2+56\tilde\lambda_1\tilde\lambda_2^3+61\tilde\lambda_2^4\right)\\
&+288\pi^2\zeta_3^2\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(2\tilde\lambda_1^2-\tilde\lambda_1\tilde\lambda_2+2\tilde\lambda_2^2\right)\\
&+192\pi^4\zeta_3^2\left(5\tilde\lambda_1^4+2\tilde\lambda_1^3\tilde\lambda_2+4\tilde\lambda_1^2\tilde\lambda_2^2+2\tilde\lambda_1\tilde\lambda_2^3+5\tilde\lambda_2^4\right)\\
&+192\pi^6\zeta_3^2\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(3\tilde\lambda_1^4-2\tilde\lambda_1^3\tilde\lambda_2+4\tilde\lambda_1^2\tilde\lambda_2^2-2\tilde\lambda_1\tilde\lambda_2^3+3\tilde\lambda_2^4\right)\\
&-960\pi^2\zeta_3\zeta_5\left(15\tilde\lambda_1^4+10\tilde\lambda_1^3\tilde\lambda_2+6\tilde\lambda_1^2\tilde\lambda_2^2+10\tilde\lambda_1\tilde\lambda_2^3+15\tilde\lambda_2^4\right)\\
&-320\pi^4\zeta_3\zeta_5\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(77\tilde\lambda_1^4-32\tilde\lambda_1^3\tilde\lambda_2+78\tilde\lambda_1^2\tilde\lambda_2^2-32\tilde\lambda_1\tilde\lambda_2^3+77\tilde\lambda_2^4\right)
\end{aligned}
\tag{B.2}
$$

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$$
\begin{aligned}
&+3360\pi^2\zeta_3\zeta_7\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(48\tilde\lambda_1^4-7\tilde\lambda_1^3\tilde\lambda_2+30\tilde\lambda_1^2\tilde\lambda_2^2-7\tilde\lambda_1\tilde\lambda_2^3+48\tilde\lambda_2^4\right)\\
&+1600\pi^2\zeta_5^2\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(57\tilde\lambda_1^4-8\tilde\lambda_1^3\tilde\lambda_2+34\tilde\lambda_1^2\tilde\lambda_2^2-8\tilde\lambda_1\tilde\lambda_2^3+57\tilde\lambda_2^4\right)\\
&-3456\pi^2\zeta_3^3\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(5\tilde\lambda_1^4-5\tilde\lambda_1^3\tilde\lambda_2+8\tilde\lambda_1^2\tilde\lambda_2^2-5\tilde\lambda_1\tilde\lambda_2^3+5\tilde\lambda_2^4\right)\Bigg].
\end{aligned}
$$

For the case of the difference we have

$$
\begin{aligned}
w_-(\tilde\lambda_1,\tilde\lambda_2)=\Bigg[&-24\pi^2\zeta_3\left(\tilde\lambda_1^2+\tilde\lambda_2^2\right)-32\pi^4\zeta_3\left(\tilde\lambda_1^3+\tilde\lambda_2^3\right)-16\pi^6\zeta_3\left(\tilde\lambda_1^4+\tilde\lambda_2^4\right)\\
&-\frac{64}{15}\pi^8\zeta_3\left(\tilde\lambda_1^5+\tilde\lambda_2^5\right)-\frac{32}{45}\pi^{10}\zeta_3\left(\tilde\lambda_1^6+\tilde\lambda_2^6\right)\\
&+80\pi^2\zeta_5\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(3\tilde\lambda_1^2-2\tilde\lambda_1\tilde\lambda_2+3\tilde\lambda_2^2\right)\\
&+\frac{80}{3}\pi^4\zeta_5\left(13\tilde\lambda_1^4+4\tilde\lambda_1^3\tilde\lambda_2+4\tilde\lambda_1\tilde\lambda_2^3+13\tilde\lambda_2^4\right)\\
&+\frac{32}{3}\pi^6\zeta_5\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(17\tilde\lambda_1^4-12\tilde\lambda_1^3\tilde\lambda_2+12\tilde\lambda_1^2\tilde\lambda_2^2-12\tilde\lambda_1\tilde\lambda_2^3+17\tilde\lambda_2^4\right)\\
&+\frac{64}{9}\pi^8\zeta_5\left(7\tilde\lambda_1^6+2\tilde\lambda_1^5\tilde\lambda_2+2\tilde\lambda_1\tilde\lambda_2^5+7\tilde\lambda_2^6\right)\\
&-280\pi^2\zeta_7\left(8\tilde\lambda_1^4+5\tilde\lambda_1^3\tilde\lambda_2+2\tilde\lambda_1^2\tilde\lambda_2^2+5\tilde\lambda_1\tilde\lambda_2^3+8\tilde\lambda_2^4\right)\\
&-\frac{112}{3}\pi^4\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(91\tilde\lambda_1^4-36\tilde\lambda_1^3\tilde\lambda_2+46\tilde\lambda_1^2\tilde\lambda_2^2-36\tilde\lambda_1\tilde\lambda_2^3+91\tilde\lambda_2^4\right)\\
&-\frac{112}{3}\pi^6\zeta_7\left(49\tilde\lambda_1^6+29\tilde\lambda_1^5\tilde\lambda_2+5\tilde\lambda_1^4\tilde\lambda_2^2+5\tilde\lambda_1^2\tilde\lambda_2^4+29\tilde\lambda_1\tilde\lambda_2^5+49\tilde\lambda_2^6\right)\\
&+336\pi^2\zeta_9\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(65\tilde\lambda_1^4-12\tilde\lambda_1^3\tilde\lambda_2+38\tilde\lambda_1^2\tilde\lambda_2^2-12\tilde\lambda_1\tilde\lambda_2^3+65\tilde\lambda_2^4\right)\qquad\text{(B.3)}\\
&+672\pi^4\zeta_9\left(51\tilde\lambda_1^6+41\tilde\lambda_1^5\tilde\lambda_2+17\tilde\lambda_1^4\tilde\lambda_2^2+4\tilde\lambda_1^3\tilde\lambda_2^3+17\tilde\lambda_1^2\tilde\lambda_2^4+41\tilde\lambda_1\tilde\lambda_2^5+51\tilde\lambda_2^6\right)\\
&-3696\pi^2\zeta_{11}\left(61\tilde\lambda_1^6+56\tilde\lambda_1^5\tilde\lambda_2+37\tilde\lambda_1^4\tilde\lambda_2^2+22\tilde\lambda_1^3\tilde\lambda_2^3+37\tilde\lambda_1^2\tilde\lambda_2^4+56\tilde\lambda_1\tilde\lambda_2^5+61\tilde\lambda_2^6\right)\\
&+288\pi^2\zeta_3^2\left(2\tilde\lambda_1^4-\tilde\lambda_1^3\tilde\lambda_2+2\tilde\lambda_1^2\tilde\lambda_2^2-\tilde\lambda_1\tilde\lambda_2^3+2\tilde\lambda_2^4\right)\\
&+192\pi^4\zeta_3^2\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(\tilde\lambda_1^2+\tilde\lambda_2^2\right)\left(5\tilde\lambda_1^2-8\tilde\lambda_1\tilde\lambda_2+5\tilde\lambda_2^2\right)\\
&+192\pi^6\zeta_3^2\left(3\tilde\lambda_1^6-2\tilde\lambda_1^5\tilde\lambda_2+\tilde\lambda_1^4\tilde\lambda_2^2+\tilde\lambda_1^2\tilde\lambda_2^4-2\tilde\lambda_1\tilde\lambda_2^5+3\tilde\lambda_2^6\right)\\
&-960\pi^2\zeta_3\zeta_5\left(\tilde\lambda_1+\tilde\lambda_2\right)\left(15\tilde\lambda_1^4-20\tilde\lambda_1^3\tilde\lambda_2+26\tilde\lambda_1^2\tilde\lambda_2^2-20\tilde\lambda_1\tilde\lambda_2^3+15\tilde\lambda_2^4\right)\\
&-320\pi^4\zeta_3\zeta_5\left(77\tilde\lambda_1^6-32\tilde\lambda_1^5\tilde\lambda_2+\tilde\lambda_1^4\tilde\lambda_2^2+40\tilde\lambda_1^3\tilde\lambda_2^3+\tilde\lambda_1^2\tilde\lambda_2^4-32\tilde\lambda_1\tilde\lambda_2^5+77\tilde\lambda_2^6\right)\\
&+3360\pi^2\zeta_3\zeta_7\left(48\tilde\lambda_1^6-7\tilde\lambda_1^5\tilde\lambda_2+4\tilde\lambda_1^4\tilde\lambda_2^2+22\tilde\lambda_1^3\tilde\lambda_2^3+4\tilde\lambda_1^2\tilde\lambda_2^4-7\tilde\lambda_1\tilde\lambda_2^5+48\tilde\lambda_2^6\right)\\
&+1600\pi^2\zeta_5^2\left(57\tilde\lambda_1^6-8\tilde\lambda_1^5\tilde\lambda_2+3\tilde\lambda_1^4\tilde\lambda_2^2+28\tilde\lambda_1^3\tilde\lambda_2^3+3\tilde\lambda_1^2\tilde\lambda_2^4-8\tilde\lambda_1\tilde\lambda_2^5+57\tilde\lambda_2^6\right)\\
&-3456\pi^2\zeta_3^3\left(5\tilde\lambda_1^6-5\tilde\lambda_1^5\tilde\lambda_2+7\tilde\lambda_1^4\tilde\lambda_2^2-6\tilde\lambda_1^3\tilde\lambda_2^3+7\tilde\lambda_1^2\tilde\lambda_2^4-5\tilde\lambda_1\tilde\lambda_2^5+5\tilde\lambda_2^6\right)\Bigg].
\end{aligned}
$$

The series $w_\pm(\tilde\lambda_1,\tilde\lambda_2)$ are symmetric. At the considered orders, the polynomials that appear also have all unimodular roots.

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References

[1] M. Henningson and K. Skenderis, *The holographic Weyl anomaly*, JHEP **07** (1998) 023 [hep-th/9806087] [INSPIRE].

[2] S. Katz, P. Mayr and C. Vafa, *Mirror symmetry and exact solution of* 4*D* $N = 2$ *gauge theories:* 1, Adv. Theor. Math. Phys. **1** (1998) 53 [hep-th/9706110] [INSPIRE].

[3] S. Kachru and E. Silverstein, 4*D conformal theories and strings on orbifolds*, Phys. Rev. Lett. **80** (1998) 4855 [hep-th/9802183] [INSPIRE].

[4] A.E. Lawrence, N. Nekrasov and C. Vafa, *On conformal field theories in four-dimensions*, Nucl. Phys. B **533** (1998) 199 [hep-th/9803015] [INSPIRE].

[5] A. Gadde, E. Pomoni and L. Rastelli, *The Veneziano limit of* $N = 2$ *superconformal QCD: towards the string dual of* $N = 2$ $\mathrm{SU}(N_c)$ *SYM with* $N_f = 2N_c$, arXiv:0912.4918 [INSPIRE].

[6] A. Gadde, E. Pomoni and L. Rastelli, *Spin chains in* $N = 2$ *superconformal theories: from the* Z_2 *quiver to superconformal QCD*, JHEP **06** (2012) 107 [arXiv:1006.0015] [INSPIRE].

[7] S.-J. Rey and T. Suyama, *Exact results and holography of Wilson loops in* $N = 2$ *superconformal (quiver) gauge theories*, JHEP **01** (2011) 136 [arXiv:1001.0016] [INSPIRE].

[8] E. Pomoni and C. Sieg, *From* $N = 4$ *gauge theory to* $N = 2$ *conformal QCD: three-loop mixing of scalar composite operators*, arXiv:1105.3487 [INSPIRE].

[9] A. Gadde, P. Liendo, L. Rastelli and W. Yan, *On the integrability of planar* $N = 2$ *superconformal gauge theories*, JHEP **08** (2013) 015 [arXiv:1211.0271] [INSPIRE].

[10] V. Mitev and E. Pomoni, *Exact effective couplings of four dimensional gauge theories with* $N = 2$ *supersymmetry*, Phys. Rev. D **92** (2015) 125034 [arXiv:1406.3629] [INSPIRE].

[11] V. Mitev and E. Pomoni, *Exact Bremsstrahlung and effective couplings*, JHEP **06** (2016) 078 [arXiv:1511.02217] [INSPIRE].

[12] K. Zarembo, *Quiver CFT at strong coupling*, JHEP **06** (2020) 055 [arXiv:2003.00993] [INSPIRE].

[13] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313** (2012) 71 [arXiv:0712.2824] [INSPIRE].

[14] J.K. Erickson, G.W. Semenoff and K. Zarembo, *Wilson loops in* $N = 4$ *supersymmetric Yang-Mills theory*, Nucl. Phys. B **582** (2000) 155 [hep-th/0003055] [INSPIRE].

[15] N. Drukker and D.J. Gross, *An exact prediction of* $N = 4$ *SUSYM theory for string theory*, J. Math. Phys. **42** (2001) 2896 [hep-th/0010274] [INSPIRE].

[16] F. Passerini and K. Zarembo, *Wilson loops in* $N = 2$ *super-Yang-Mills from matrix model*, JHEP **09** (2011) 102 [*Erratum ibid.* **10** (2011) 065] [arXiv:1106.5763] [INSPIRE].

[17] J.G. Russo and K. Zarembo, *Large* N *limit of* $N = 2$ $\mathrm{SU}(N)$ *gauge theories from localization*, JHEP **10** (2012) 082 [arXiv:1207.3806] [INSPIRE].

[18] B. Fiol, J. Martínez-Montoya and A. Rios Fukelman, *Wilson loops in terms of color invariants*, JHEP **05** (2019) 202 [arXiv:1812.06890] [INSPIRE].

[19] B. Fiol, J. Martínez-Montoya and A. Rios Fukelman, *The planar limit of* $N = 2$ *superconformal field theories*, JHEP **05** (2020) 136 [arXiv:2003.02879] [INSPIRE].

JHEP08(2020)161

- [20] A. Pini, D. Rodriguez-Gomez and J.G. Russo, *Large N correlation functions $\mathcal{N}=2$ superconformal quivers*, JHEP **08** (2017) 066 [arXiv:1701.02315] [INSPIRE].
- [21] T.D. Lee and C.-N. Yang, *Statistical theory of equations of state and phase transitions. 2. Lattice gas and Ising model*, Phys. Rev. **87** (1952) 410 [INSPIRE].
- [22] J. Koplik, A. Neveu and S. Nussinov, *Some aspects of the planar perturbation series*, Nucl. Phys. B **123** (1977) 109 [INSPIRE].
- [23] E. Pomoni, *Integrability in $\mathcal{N}=2$ superconformal gauge theories*, Nucl. Phys. B **893** (2015) 21 [arXiv:1310.5709] [INSPIRE].
- [24] E. Pomoni, *4D $\mathcal{N}=2$ SCFTs and spin chains*, J. Phys. A **53** (2020) 283005 [arXiv:1912.00870] [INSPIRE].
- [25] K. Papadodimas, *Topological anti-topological fusion in four-dimensional superconformal field theories*, JHEP **08** (2010) 118 [arXiv:0910.4963] [INSPIRE].
- [26] M. Baggio, V. Niarchos and K. Papadodimas, *Exact correlation functions in* SU(2) $\mathcal{N}=2$ *superconformal QCD*, Phys. Rev. Lett. **113** (2014) 251601 [arXiv:1409.4217] [INSPIRE].
- [27] M. Billò, F. Fucito, A. Lerda, J.F. Morales, Y.S. Stanev and C. Wen, *Two-point correlators in $\mathcal{N}=2$ gauge theories*, Nucl. Phys. B **926** (2018) 427 [arXiv:1705.02909] [INSPIRE].
- [28] M. Billò, F. Galvagno, P. Gregori and A. Lerda, *Correlators between Wilson loop and chiral operators in $\mathcal{N}=2$ conformal gauge theories*, JHEP **03** (2018) 193 [arXiv:1802.09813] [INSPIRE].
- [29] M. Billò, F. Galvagno and A. Lerda, *BPS Wilson loops in generic conformal $\mathcal{N}=2$* SU(N) *SYM theories*, JHEP **08** (2019) 108 [arXiv:1906.07085] [INSPIRE].
- [30] W.T. Tutte, *A census of slicings*, Canad. J. Math. **14** (1962) 708.
- [31] R. Gopakumar and R. Pius, *Correlators in the simplest gauge-string duality*, JHEP **03** (2013) 175 [arXiv:1212.1236] [INSPIRE].
- [32] P. Lakatos and L. Losonczi, *Polynomials with all zeros on the unit circle*, Acta Math. Hungarica **125** (2009) 341.
- [33] T. Asano, *Theorems on the partition functions of the Heisenberg ferromagnets*, J. Phys. Soc. Jpn. **29** (1970) 350.
- [34] D. Ruelle, *Zeros of graph-counting polynomials*, Commun. Math. Phys. **200** (1999) 43.
- [35] B. Fiol, E. Gerchkovitz and Z. Komargodski, *Exact Bremsstrahlung function in $\mathcal{N}=2$ superconformal field theories*, Phys. Rev. Lett. **116** (2016) 081601 [arXiv:1510.01332] [INSPIRE].
- [36] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, Phys. Rev. D **74** (2006) 025005 [hep-th/0501015] [INSPIRE].
- [37] L. Bianchi, M. Billò, F. Galvagno and A. Lerda, *Emitted radiation and geometry*, JHEP **01** (2020) 075 [arXiv:1910.06332] [INSPIRE].
- [38] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in $\mathcal{N}=4$ super Yang-Mills*, JHEP **06** (2012) 048 [arXiv:1202.4455] [INSPIRE].
- [39] B. Fiol, B. Garolera and A. Lewkowycz, *Exact results for static and radiative fields of a quark in $\mathcal{N}=4$ super Yang-Mills*, JHEP **05** (2012) 093 [arXiv:1202.5292] [INSPIRE].
- [40] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, JHEP **05** (2014) 025 [arXiv:1312.5682] [INSPIRE].
- [41] L. Bianchi, M. Lemos and M. Meineri, *Line defects and radiation in $\mathcal{N}=2$ conformal theories*, Phys. Rev. Lett. **121** (2018) 141601 [arXiv:1805.04111] [INSPIRE].

Conclusions

This thesis has been mostly devoted to the study of radiation in generic conformal field theories in the regimes of parameters where ordinary perturbative methods are not applicable and its essence can be captured in three main achievements: an improved comprehension of the angular distribution of radiation for generic conformal field theories, the implementation of a new method to compute the partition function and the expectation value of the Wilson loop operator using supersymmetric localization and the determination of the coupling dependence of radiation for generic Lagrangian $\mathcal{N}=2$ superconformal quiver theories.

The first result stems from the realization of the peculiar features of radiation in theories with conformally coupled scalars, even at the classical level: the radiative energy density is not positive definite, the radiated power is not Lorentz invariant and as a matter of fact, it depends on the derivative of the acceleration. The determination of the angular distribution of radiation for theories with holographic dual reveals two surprising results: first, it depends on the probe worldline only through the retarded time, and second, it matches exactly the angular distribution obtained by a free theory computation.

Additionally we conjectured that the energy-momentum tensor for a conformal field theory with $\mathcal{N}=2$ supersymmetry has the same spacetime dependence independently of the value of the coupling. Following this line of research, we are currently trying to determine for which theories and scenarios this unexpected factorization of the coupling dependence for the angular distribution of radiation is valid. So far we have been able to show what is the most general form of the expectation value of the energy-momentum tensor by arguments of conformal symmetry; if we further impose it depends only on the retarded time and it has $\mathcal{N}=2$ supersymmetry, its radiative part will be exactly the same as the one found holographically. This raises a question: for what interacting CFTs does the angular distribution of radiation depend only on the retarded time? One approach to tackle this interrogation is to observe the response of an arbitrary $4d$ CFT in the presence of a line defect within the formalism of defect conformal field theory.

Another issue that could be addressed in the future is to find the proper analogue of the Abraham-Lorentz-Dirac equation for probes of CFTs with scalar fields [35]. In classical electrodynamics the so-called "Schott term" amounts to the self-force of a charged particle, an identical term appeared in the radiated power of conformally coupled scalar fields, but the origin of both terms is totally different. The former comes from the fields created by the probe near its worldline, the latter comes from evaluating the energy-momentum tensor away from the probe. The possible generalization of the Abraham-Lorentz-Dirac equation, which describes this problem at the classical level, for the conformally coupled scalar case could help gain some insight about this similarity.

The second result comes from the implementation of a new method to compute exactly the vacuum expectation value of the circular Wilson loop in $\mathcal{N}=4$ SYM theory. This new approach was obtained via supersymmetric localization techniques in which the problem reduces to a matrix model computation. The beauty of the formula obtained encompasses and unifies many known partial results, already present in the literature. Furthermore, it allows to derive various exact relations among different cases. Some of these relations had been noticed but not explained, and some of these relations appear to be new.

The third result is based on the generalization of the method explained above in the derivation of an all-order expression for the vacuum expectation value of the circular Wilson loop for $\mathcal{N} = 2$ superconformal quiver theories possessing a Lagrangian description, in the limit where the number of colors tends to infinity. Likewise we found a compact expression for the planar free energy for these theories. All the expressions found are given by a purely combinatorial expression and have a simple diagrammatic representation in terms of tree graphs.

Various observables in these theories are effectively described by a multi-matrix model containing single- and double-trace terms, but the relevant contributions come only from the double-trace terms. We proved that any matrix model with double-trace terms in the potential, the planar free energy will be described by a sum over tree graphs. For the specific case of the quiver theory containing two nodes the planar free energy is still given by a sum of tree graphs, but now each tree can be interpreted as the partition function of a generalized Ising model defined on the tree structure. We claim the zeros of each partition function lie on the unit circle, as well as the sum of the contributions of each tree at a given order still has its zeros on the unit circle. It would be interesting to further investigate this relation, as well as to prove the conjectures above mentioned.

References

[1] J. D. Jackson, *Classical Electrodynamics*, New York: Wiley, (1999).

[2] F. Rohrlich, *Classical Charged Particles*, World Scientific, (2007).

[3] C. Teitelboim, *Splitting of the maxwell tensor - radiation reaction without advanced fields*, Phys. Rev. D **1**, 1572–1582 (1970).

[4] C. Teitelboim, D. Villarroel and C. van Weert, *Classical Electrodynamics of Retarded Fields and Point Particles*, Riv. Nuovo Cim. **3N9**, 1–64 (1980).

[5] M. E. Peskin and D. V. Schroeder, *An introduction to quantum field theory*, Westview Press, (1995).

[6] J. D. Qualls, *Lectures on Conformal Field Theory*, [arXiv:1511.04074 [hep-th]].

[7] K. G. Wilson, *The Renormalization Group: Critical Phenomena and the Kondo Problem*, Rev. Mod. Phys. **47**, 773 (1975).

[8] A. Kapustin, *Wilson-'t Hooft operators in four-dimensional gauge theories and S-duality*, Phys. Rev. D **74**, 025005 (2006).

[9] L. Bianchi, M. Lemos and M. Meineri, *Line Defects and Radiation in $\mathcal{N} = 2$ Conformal Theories*, Phys. Rev. Lett. **121**, no.14, 141601 (2018).

[10] A. Schild, *On the radiation emitted by an accelerated point charge*, Journal of Mathematical Analysis and Applications **1**, 127–131 (1960).

[11] D. Correa, J. Henn, J. Maldacena and A. Sever, *An exact formula for the radiation of a moving quark in N=4 super Yang Mills*, JHEP **06**, 048 (2012).

[12] F. Quevedo, S. Krippendorf and O. Schlotterer, *Cambridge Lectures on Supersymmetry and Extra Dimensions*, [arXiv:1011.1491 [hep-th]].

[13] R. Haag, J. T. Łopuszański and M. Sohnius, *All possible generators of supersymmetries of the S-matrix*, Nucl. Phys. B **88**, 257–274 (1975).

[14] I. G. Koh and S. Rajpoot, *Finite N=2 extended supersymmetric field theories*, Phys. Lett. B **135**, 397–401 (1984).

[15] L. Bhardwaj and Y. Tachikawa, *Classification of 4d N=2 gauge theories*, JHEP **12**, 100 (2013).

[16] O. Aharony and M. Evtikhiev, *On four dimensional $N = 3$ superconformal theories*, JHEP **04**, 040 (2016).

[17] K. Papadodimas, *Topological Anti-Topological Fusion in Four-Dimensional Superconformal Field Theories*, JHEP **08**, 118 (2010).

[18] M. D. Schwartz, *Quantum Field Theory and the Standard Model*, Cambridge University Press, (2013).

[19] J. M. Maldacena, *Wilson loops in large N field theories*, Phys. Rev. Lett. **80**, 4859–4862 (1998).

[20] B. Fiol, E. Gerchkovitz and Z. Komargodski, *Exact Bremsstrahlung Function in $N = 2$ Superconformal Field Theories*, Phys. Rev. Lett. **116**, no.8, 081601 (2016).

[21] A. M. Polyakov, *Gauge Fields as Rings of Glue*, Nucl. Phys. B **164**, 171–188 (1980).

[22] J. K. Erickson, G. W. Semenoff and K. Zarembo, *Wilson loops in N=4 supersymmetric Yang-Mills theory*, Nucl. Phys. B **582**, 155–175 (2000).

[23] N. Drukker and D. J. Gross, *An Exact prediction of N=4 SUSYM theory for string theory*, J. Math. Phys. **42** ,2896–2914 (2001).

[24] V. Pestun, *Localization of gauge theory on a four-sphere and supersymmetric Wilson loops*, Commun. Math. Phys. **313**, 71–129 (2012).

[25] L. Bianchi, M. Billò, F. Galvagno and A. Lerda, *Emitted Radiation and Geometry*, JHEP **01**, 075 (2020).

[26] N. Hama and K. Hosomichi, *Seiberg-Witten Theories on Ellipsoids*, JHEP **09**, 033 (2012).

[27] A. Lewkowycz and J. Maldacena, *Exact results for the entanglement entropy and the energy radiated by a quark*, JHEP **05** 025 (2014).

[28] J. M. Maldacena, *The Large N limit of superconformal field theories and supergravity*, Adv. Theor. Math. Phys. **2**, 231–252 (1998).

[29] C. Athanasiou, P. M. Chesler, H. Liu, D. Nickel and K. Rajagopal, *Synchrotron radiation in strongly coupled conformal field theories*, Phys. Rev. D **81**, 126001 (2010).

[30] Y. Hatta, E. Iancu, A. H. Mueller and D. N. Triantafyllopoulos, *Radiation by a heavy quark in N=4 SYM at strong coupling*, Nucl. Phys. B **850**, 31–52 (2011).

[31] A. Pini, D. Rodríguez-Gómez and J. G. Russo, *Large N correlation functions $\mathcal{N} = 2$ superconformal quivers*, JHEP **08**, 066 (2017).

[32] M. Henningson and K. Skenderis, *The Holographic Weyl anomaly*, JHEP **07**, 023 (1998).

[33] J. Gomis, T. Okuda and V. Pestun, *Exact Results for 't Hooft Loops in Gauge Theories on S^4*, JHEP **05**, 141 (2012).

[34] M. Billo, F. Fucito, A. Lerda, J. F. Morales, Y. S. Stanev and C. Wen, *Two-point correlators in $N = 2$ gauge theories*, Nucl. Phys. B **926** , 427–466 (2018).

[35] E. Poisson, A. Pound and I. Vega, *The Motion of point particles in curved spacetime*, Living Rev. Rel. **14**, 7 (2011).