Universitat Autònoma de Barcelona

Essays on Expectations and Option Prices

Gaurav Mehta

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Essays on Expectations and Option Prices

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To my parents, Harish and Veena

To my wife, Ankita and my son, Atharv

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Introduction

In my dissertation, I explain different empirical features of stock option prices through a framework which considers departure from rational expectations. Rational expectations assumes that agents in a model hold beliefs which are consistent with the *"true"* data generating process. However, as has been discussed extensively in the literature, rational expectations imposes unrealistic requirement on the information set of the agents in a model. However, I consider a framework where agents in the model are not aware of the pricing function that maps the equilibrium stock price with its fundamentals. In particular, they form beliefs about future stock price based on its historical realizations and their beliefs may be temporarily disconnected from movement in fundamentals. I show that introduction of such beliefs in a standard asset pricing model can help reconcile theory with the data.

The dissertation is composed of three chapters. In the first chapter, I derive a closed form option price formula for a one period European call option. I show that given this belief structure, investor's subjective expectations about next period price growth are priced in an option, creating a wedge between option implied-variance and conditional variance. Time variation in the agent's subjective expectations link this wedge to future stock returns, making it a strong predictor of stock returns. Furthermore, the model also generates different shapes of option implied volatility curve in line with the data. The findings suggest that option-implied variance such as VIX are not capturing the true uncertainty expected by agents but are biased in the direction of the investors expectations of future capital gains on the underlying asset. Finally, I propose a trading strategy that exploits the inefficiency of the VIX market and generates abnormal returns after adjustment for risk.

In the second chapter, I address the empirical behavior of the term structure of option-implied volatility. Implied volatility (BS IV) calculated from market option prices using Black and Scholes (1973) model appears to be related to price-strike ratio and time to expiry of an option. In particular, the relationship between BS IV and price-strike ratio or the BS IV curve is considerably flat for long duration options compared to short duration options. Furthermore, the BS IV surface which characterize the joint relationship of BS IV along direction of moneyness and maturity itself is stochastic with time. It is a challenge for most option pricing models to replicate these empirical features of the BS IV surface. In this chapter, I extend the framework developed in the first chapter to multi-period options. It is assumed that investors use the same learning rule as in Chapter 1 to form expectations of future capital gains for different horizons. I derive a multi-period option pricing formula in this setup. BS IV surfaces generated by the model replicates the features observed in the data.

Finally, in the third chapter of this dissertation, I formally estimate the model developed in Chapter 2 on index options data. The model fit in terms of root mean square pricing error (RMSE) is significantly better than BS model. The estimation exercise delivers a new time series of option implied volatility which is adjusted for subjective capital gains expectations. VIX premium calculated using this option implied volatility strongly predicts future index returns in the data. Furthermore, the estimation also delivers a new series of investors subjective expectations of risk adjusted returns which I call the "***option implied expectations***". I find that the option implied expectations are strongly correlated with survey based expectations and price to dividends ratio. The option implied expectations appears to be a more reliable indicator of investor sentiments that is free from measurement error and can be obtained on a real time basis.

Chapter 1

Expectations and the Option Implied-Variance

1.1 Introduction

Investors beliefs about future stock market returns has been at the center of debate in asset pricing research recently. It has been shown in many studies that investors beliefs as documented in survey data reject rational expectations hypothesis. Introducing these beliefs in option pricing models can be useful in explaining puzzles observed in the options price data. In this paper, I show that a simple equilibrium asset pricing model in an economic environment with homogeneous agents having time-separable preferences can explain many salient features of index options prices if one allows for slight deviations from rational expectations (henceforth, RE).[1]

There is a vast literature that builds on the seminal work of Black and Scholes (1973) (henceforth, BS) to explain the behavior of options prices in the data. This literature has contributed immensely to our understanding of options. However, still some facts stand out which have proved to be a challenge for models with RE. The first such stylized fact is that the expected volatility implied by Black and Scholes (1973) model (henceforth, BS IV) in the data is inconsistent with the model used to derive it. It has been documented since at least Rubinstein (1994) that BS IV of an option appears to be related to it's price-

[1]Index in this thesis refers to S&P 500 index, unless specified otherwise.

strike ratio which is inconsistent with the model's assumption of a constant volatility. For example, for the index options, BS IV is on an average increasing with price-strike ratio and for the individual stock options, BS IV curve can either be increasing or decreasing.[2] Predominant part of the literature has focused on explaining this feature in the data by appealing to stochastic volatility and non-gaussian nature of the underlying asset's returns distribution (see for example, Heston (1993), Bakshi and Kapadia (2003), Du (2011), Shaliastovich (2015)). However, Bollen and Whaley (2004) show that the properties of BS IV curve in the data is inconsistent with most models of option pricing. [3]

The next set of stylized facts relate to the stochastic properties of a variable, which I call the VIX premium. The VIX premium is defined as the difference between (square of) VIX and the conditional variance of the stock returns (henceforth, CV).[4] [5]VIX premium is interesting because the equilibrium asset pricing models with RE that employ Gaussian shocks to state variables predict VIX premium equals zero. Bekaert, Engstrom, and Ermolov (2020) point out that a non-zero VIX premium refutes the standard habit model (Campbell and Cochrane (1999)) and the long run risk model (Bansal and Yaron (2004)) which do well in explaining asset pricing facts such as excess volatility of stock returns, and high equity premium. In this paper, I discuss the following stylized facts about VIX premium that have been documented in the literature: 2(a) VIX premium is substantially positive on average;[6]2(b) VIX premium is moderately correlated with the VIX index or the conditional variance; 2(c) VIX premium can be negative, especially during market turmoil;[7]2(d)VIX premium is a reliable predictor of S&P 500 index returns for a short horizon over and above the well-known predictors such as the price-dividend ratio.[8] The

[2]For brevity, I call the relationship between BS IV and price strike ratio as "BS IV curve".

[3]Bollen and Whaley (2004) show that the net buying pressure affects the shape of BS IV curve.

[4]The CBOE volatility index (VIX) is the value of a portfolio of 1 month S&P 500 index options (see, Section 2 for details). It is published by the Chicago Board Option Exchange (CBOE).

[5]In some studies for example, Bollerslev, Tauchen, and Zhou (2009), Drechsler and Yaron (2011), VIX premium is called the variance risk premium. In this paper, I avoid such interpretation of VIX premium and treat it purely as a data artifact.

[6]See, for example, Dew-Becker, Giglio, Le, and Rodriguez (2017) who notes that on an average (squared) VIX has been trading significantly higher than realized variance.

[7]Cheng (2018) provides evidence for negative VIX futures premium and a fall in VIX futures premium when ex ante measures of risk rise. Bekaert and Hoerova (2014) show that the VIX premium is negative in the data even after taking into account the possibility for a measurement error in forecasting realized variance.

[8]Bollerslev, Tauchen, and Zhou (2009) show that the difference between squared VIX index and histrorical monthly realized variance can explain monthly S&P 500 index returns upto 1 year in future.

empirical behavior of VIX premium is also puzzling because it is tough to find variables that predict stock index returns for a short horizon.

Nevertheless, there is a growing literature that explores the properties of VIX premium in equilibrium asset pricing models with non-Gaussian shocks while maintaining RE. For example, Drechsler and Yaron (2011) (henceforth, DY) is a version of the long run risk model that attempt to explain the behavior of VIX premium by relating it to the variance risk premium (henceforth, VRP) which is the difference between risk neutralized conditional variance and the physical conditional variance. These papers argue that VRP captures equity risk premium and thereby predict stock market future returns. In this paper I argue that VRP is not equivalent to VIX premium.[9] I simulate the VIX index in these models and show that these papers fall well short of replicating the empirical facts 2(a)-(d) related to VIX premium.

This paper focuses on a different departure from BS model: I assume agents form their beliefs about stock prices according to their perceived model. Investors are rational; prices are fully flexible and determined in equilibrium. Since investors' beliefs are "close" to RE we think of this as a slight departure from RE. One advantage of this paradigm is that it explains stock returns volatility very well (see Adam, Marcet, and Nicolini (2016) and Adam, Marcet, and Beutel (2017)). Under the Lucas Jr (1978) framework, we relax the standard assumption that agents have perfect knowledge about the pricing function that maps each sequence of fundamental shocks to a market outcome of the stock price.[10] In particular, agents optimally update their subjective expectations about next period stock price growth using realized market outcomes. Adam, Marcet, and Beutel (2017) show that such a model is consistent with the behavior of investors' capital gains expectations, as measured from survey data.

While the investors are only learning about the expected future capital gains on the underlying stock, the model produces interesting implications for its option price. Importantly, the model delivers a closed-form formula for a call option price that nests the

Further, Bollerslev, Marrone, Xu, and Zhou (2014) show that this predictability is robust and holds for benchmark indices in many developed economies.

[9] It has been known that VIX is subject to approximation error when returns distribution is not normal (See for example Carr and Wu (2008)). However, to my knowledge, the effect of relaxation of the assumption of RE on the VIX has not been studied in the literature which is the focus of this paper.

[10] In other words, equilibrium stock price is not the present discounted value of all future dividends.

BS formula. In contrast to models under RE, the beliefs about the first moment of the underlying's returns distribution matter for the option price. In particular, for a given conditional variance, optimistic (pessimistic) capital gains expectations leads to a positive (negative) difference between the model call option price and the BS price. The the positive (negative) price difference between the model price and BS price is weakly increasing (decreasing) with price-strike ratio leading to upward (downward) sloping BS IV curve.

The model's ability to replicate time-series properties of VIX premium can be understood by noting how the VIX index is calculated. Strictly speaking, VIX is a weighted average of option prices. Rather than following the standard approach of deriving the risk-neutral conditional variance, I study how VIX may deviate from conditional variance when options are priced according to the framework developed in the paper. For this purpose, I calculate VIX from the model generated option prices. It turns out that, in the model under RE, (squared) VIX equals the conditional variance of stock returns. As a slight deviation from RE is allowed, a wedge emerges between VIX and conditional variance i.e. VIX premium is non-zero. In contrast to extant literature that interprets VIX premium as a compensation for variance risk, in this model, the dynamics of VIX premium are primarily driven by investor's beliefs. VIX premium is positive (negative) with optimistic (pessimistic) capital gains expectations. Given that in the data and in the model, the stock prices follow an upward trend with rising cash flows, investors on an average hold optimistic capital gains expectations which leads to average positive VIX premium, this explains fact 2(a). In the model, while the level of the wedge (VIX premium) depends upon the level of conditional variance and subjective expectations, the sign depends exclusively on subjective capital gains expectations, which explains moderate correlation between VIX premium and the VIX index i.e. fact 2(b). Furthermore, since the subjective expectations of investors are revised based on realized capital gains, during market turmoil, investors are more likely to revise subjective expectations to pessimistic which leads to a negative VIX premium in the model explaining fact 2(c). Fact 2(d) can be rationalized in my model by the self-referential aspect of subjective expectations that enables VIX premium to predict returns for a short horizon. In the model, subjective expectations, thereby VIX premium, influence realized returns positively.

In this article, all the stylized facts mentioned above are explained by a single variable, which is investors subjective capital gains expectations. One does not need to rely on complicated preferences and complicated stochastic processes in a model to explain behavior of option prices in the data. Importantly, quantitative simulations show that the model can explain empirically consistent levels of above-mentioned facts with coefficient of risk aversion close to 1.

One of the model's key implications is that VIX premium is largely unrelated to the conditional variance. In other words, VIX index is a systematically biased estimate of conditional variance and VIX premium offers a profitable opportunity for traders to exploit. The trading strategy that involves selling the VIX index to receive realized variance after one month when VIX premium is positive and not trading when VIX premium is negative generated an annualized Sharpe ratio of 2.228 for the sample period of February 1990 to December 2019. The same strategy, which involves selling, over the counter traded, variance swap rate instead of VIX index generated an annualized Sharpe ratio of 2.145 from January 1996 to September 2013. The Sharpe ratios of these strategies are striking when compared to investing S&P 500 total return index which generated annualized Sharpe ratio of 0.59 for the sample period of February 1990 to December 2019. Overall, this paper's findings suggest that features of index options discussed above are natural if option traders are not aware of the pricing function that maps the fundamentals to underlying asset price.

In terms of contribution to literature, this chapter contributes to a sizable literature on option pricing with learning. For example: Guidolin and Timmermann (2003), David and Veronesi (2000), Shaliastovich (2015), Benzoni, Collin-Dufresne, and Goldstein (2011) study Bayesian RE learning about parameters of the stochastic process of fundamentals to explain option implied volatility curve. In these models agents are aware of the pricing function that maps fundamentals to equilibrium price. In contrast, the learning modeled here is different in the sense that agents are directly learning about future capital gains. Furthermore, this chapter also studies VIX premium.

The remainder of the paper is structured as follows. Section 2 briefly discusses data sources and documents key facts related to the S&P 500 index option prices and VIX premium. In Section 3, I confront a few models, that explore properties of variance risk premium, with facts related to VIX premium. Section 4 sets up the basic asset pricing model, specifies the agents' beliefs, and contains analytical results from the model. Section 5 contains the results of the quantitative analysis. Section 6 contains potential trading strategies that are implied by the model. Finally, Section 7 concludes.

1.2 Definitions and Data

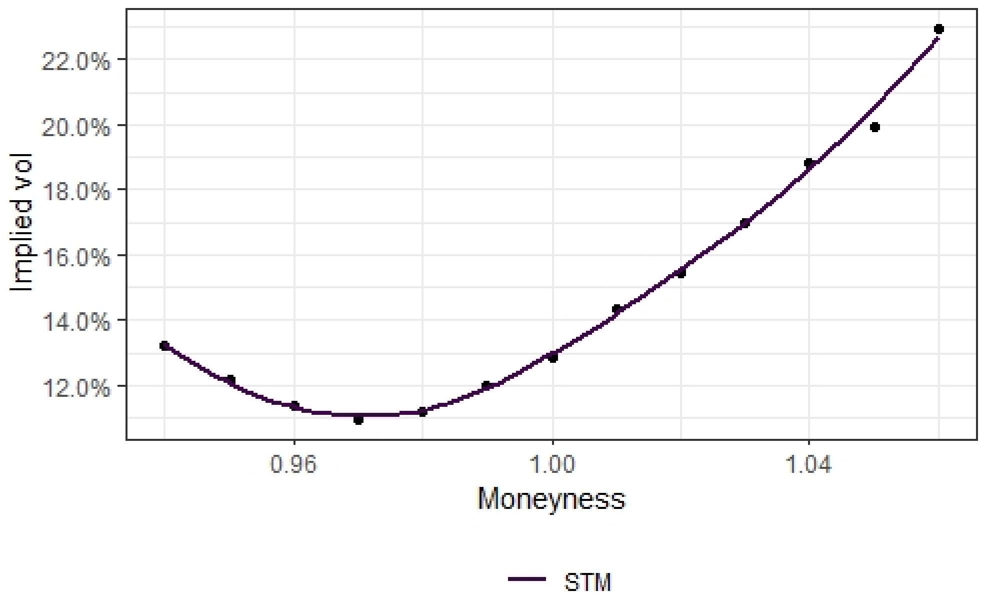
This section describes the stylized facts related to US options market data.

It has been well documented that BS IV is increasing with respect to price-strike ratio for index call (put) options. Figure 1.1 plots the average BS IV curve for the cross section of call options with time to expiry between 25 and 35 days during the period from Jan-2004 to Dec-2017. As can be seen in the figure that average BS IV from call options is increasing with price-strike ratio. From the put-call parity relationship, this also implies that out of the money ("OTM") put options have higher BS IV compared to at the money ("ATM") put options. The average annualized BS IV for in the money ("ITM") call options with price strike ratio of 1.06 is 24% compared to 13% for ATM call options. Under the framework of BS, the curve should be a flat line i.e. same BS IV for all price-strike ratios.

VIX premium is defined in as the difference between the (square) VIX Index and conditional variance i.e. $VP_{t,t+1} = VIX_{t,t+1} - CV_{t,t+1}$. VIX index is a weighted average of OTM options' prices on S&P 500 with time to expiry of one month. VIX index is calculated using the following formula:

$$VIX^2_{t,t+1} \equiv \left[2 \sum_i \frac{\Delta X_i}{X_i^2} (1+r) P^j_{t,t+1}(X_i) \right] \tag{1.1}$$

where X_0 is ATM strike price, $P^j_{t,t+1}(X_i)$ is the price of an out of the money option at date t with strike price X_i and a calendar month to expiry ; $j = C$ (call option) when $X_i \geq X_0$ and $j = P$ (put option) when $X_i \leq X_0$. VIX is calculated in a "model-free" way using market information. VIX is interesting because it has been shown that if the underlying asset's returns follows a log normal distribution, then under assumption of no-



Notes: The figure present the average implied volatility with respect to price-strike ratio of European call options on S&P 500 futures between Jan-2004 to Dec-2017. STM is for short term maturity options with moneyness between 25 and 35 calendar days.

Figure 1.1: Implied Volatility Curve

arbitrage, VIX equals conditional variance of returns under the risk neutral measure (see for example Carr and Wu (2008)). [11]It is important to note that while VIX is calculated using only market information, any asset pricing statement about VIX relies on certain assumptions. In this paper, I use monthly closing values of VIX obtained from CBOE starting from January 1990 to December 2019.

Many techniques are available in the literature to estimate CV. However, famous among practitioners and researchers is the realized variance which is calculated by summing the high frequency squared log returns on the underlying asset i.e.,

$$RV_{t,t+1} \equiv \sum_{j=1}^{n} r^2_{t+\frac{j}{n}},$$

for n is arbitrary number of time intervals chosen between any two dates t and $t+1$ where $r_{t+\frac{j}{n}}$ is the log return on the index for the time interval $[t+\frac{j-1}{n},\ t+\frac{j}{n}]$.[12] Below I report statistics based on realized variance (RV-BTZ) as used in Bollerslev et al. (henceforth, BTZ) which is calculated by summing squared five-minute returns on

[11]Carr2008 show that $VIX^2_{t,T} = E^Q_t[\int_t^T \sigma^2_z dz]$ if price of the underlying P_t follows an Ito process $dP_t = rP_t dt + \sigma_t P_t dZ^Q_t$ under the risk neutral measure. E^Q stands for expectations under the risk neutral measure.

[12]As $n \to \infty$, $RV_{t,t+1} = \int_t^{t+1} \sigma^2_u du$.

the index for the whole month with returns overnight or over a weekend treated as one "five minute interval". [13] Bekaert and Hoerova (2014) document that $VP_{t,t+1}$ using BTZ approach outperforms all other models considered by them in future excess returns predictability.

Additionally, I report statistics for the difference between VIX index and realized volatility which is called VIX-V premium ("VVP"). Units of estimated $VVP_t = VIX_{t,t+1} - \sqrt{RV_{t,t-1}}$) are easier to interpret than VIX premium. Figure 1.2 plots the VIX premium and VIX-V premium from January 1990 to December 2019. In Panel A, the solid blue line is the estimated VIX premium and similarly in Panel B the solid blue line is the estimated VIX-V premium. The figure shows that both series are highly volatile and frequently negative. The fact that VIX premium is frequently negative has been subject of a debate in the literature. The papers that argue that VIX premium is compensation for variance risk, claim that negative VIX premium is a result a of measurement errors or due to errors made by investors in forecasting conditional variance. However, Bekaert and Hoerova (2014) find that many leading variance forecasting models produce a negative VIX premium.

Table 1.2 below report the summary statistics for VIX premium, realized VIX premium, VIX^2 and realized variance. Average estimated VIX premium is 15.45 with standard deviation of 19.94. The realized VIX premium, VP_t realized $= VIX^2_{t-1,t} - RV_{t-1,t}$ has higher standard deviation than the estimated variance premium. Note that VIX premium is only weakly correlated with VIX^2 or RV which is our fact 2(c). Similarly, Table 1.2 provides statistics for VIX-V premium, VIX-V premium realized, VIX and $\sqrt{\text{RV}}$.

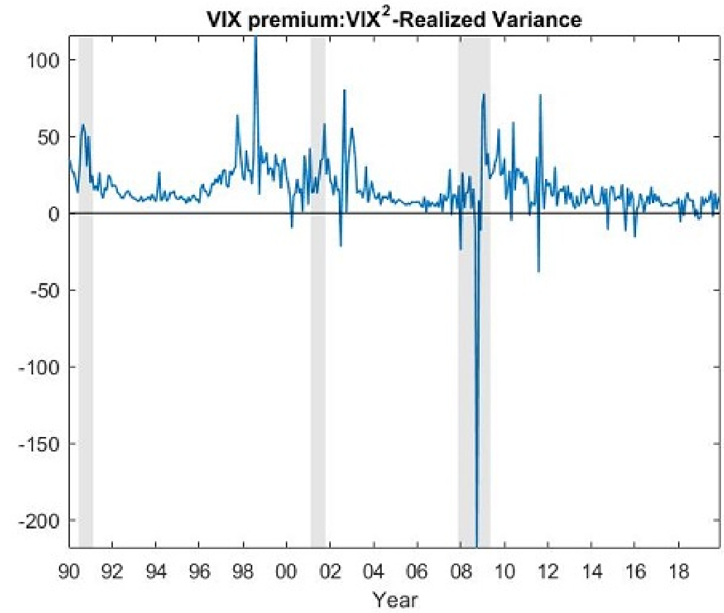
Table 1.3 presents the evidence for the fact 2(d) which is a significant ability of the VIX premium to predict returns for a short horizon. I estimate the following regression equations:

$$\sum_h er_{t+h} = \alpha + \beta VP_t + \epsilon_{t+h} \tag{1.2}$$

where h takes values 1, 3 and 6 and $er_{t+h} = \frac{S\&P500_{t+h}}{S\&P500_t} - R_{f,t+h}$; $R_{f,t+h}$ is the h months treasury bill rate. The results are in line with the finding of BTZ with the maximum R^2 of 6.503% for the 3 month holding returns.

[13] The data for RV-BTZ is obtained from Hao Zhou's website

Panel A



Panel B

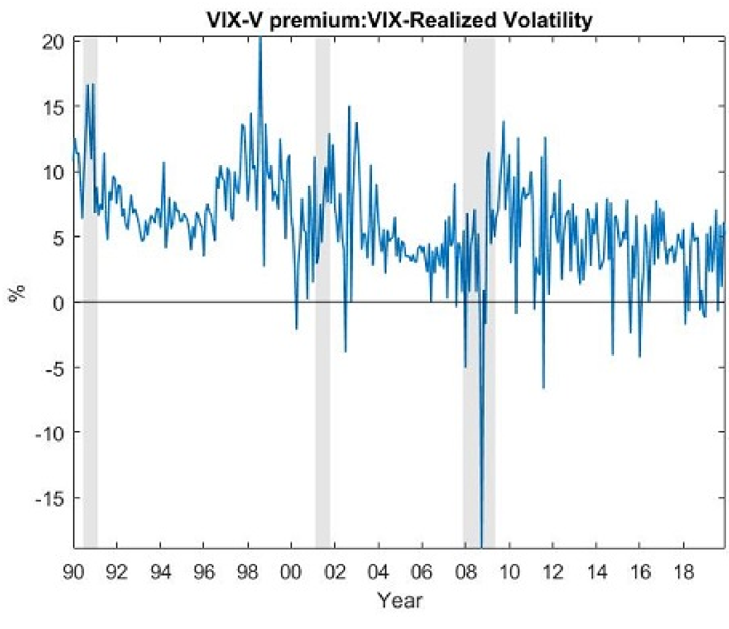


Figure 1.2: Estimated VIX premium and VIX-V premium

Notes: This figure plots estimated monthly VIX premium and VIX-V premium from Jan 1990 - Dec 2009. Data is obtained from Hao Zhou's website.

	VP	VP realized	VIX^2	RV
Mean	15.45	15.56	35.09	19.64
Median	12.64	12.90	24.48	10.44
Std.Dev	19.94	29.70	32.68	35.78
Min	-218.56	-388.17	7.54	1.73
Skewness	-3.62	-6.97	3.65	8.82
Kurtosis	57.10	98.11	22.59	111.11
AC(1)	0.28	0.28	0.81	0.64
Correlation Matrix				
VP	1.00	0.73	0.14	-0.43
VP realized		1.00	-0.12	-0.51
VIX^2			1.00	0.83
RV				1.00

Notes: The table present summary statistics for estimated VIX premium (VP), realized VIX premium (VP realized), VIX^2 and monthly realized variance (RV).VIX^2 is calculated as squaring the monthly closing values of the VIX index and dividing by 12. VIX premium is calculated as monthly closing value ofVIX^2-RV for the contemporaneous month. Realized VIX premium is calculated as the difference between monthly closing value ofVIX^2 and RV for the following month. The sample used is from Jan-1990 to Dec- 2019. AC(1) is first order auto-correlation. Data is obtained from Hao Zhou's website.

Table 1.1: Summary Statistics

Data for S&P500 index and nominal yields on 1 month U.S. government bill is obtained from St. Louis FED's FRED database. Data for monthly consumption is obtained from U.S. Bureau of Economic Analysis and dividends data is constructed using S&P dividend point index and and data from Robert Shiller's website.[14]

1.3 VIX premium versus Variance risk premium

Strictly speaking, VIX premium is a feature of index options data. The information set and rationality of an option trader is not known to a researcher. Therefore, making a definitive statement about the economic forces driving VIX premium is a challenging exercise. However, there is an increasing literature that studies the properties of a related concept called variance risk premium as a proxy for VIX premium in consumption based asset pricing models without explicitly studying the models' implications for option prices. Variance risk premium is defined as the difference between conditional variance of stock returns under the risk neutral probability measure and the objective measure. In other words, variance risk premium is a premium paid to hedge against fluctuations in future

[14]http://www.econ.yale.edu/~shiller/

	VVP	VVP realized	VIX	$\sqrt{RV}$
Mean	5.88	5.91	19.1588	13.2761
Median	5.96	6.21	17.14	11.1905
Std. Dev	3.88	5.41	7.3637	7.7205
Min	-18.91	-39.41	9.51	4.5547
Skewness	-0.63	-2.47	1.7724	3.0818
Kurtosis	7.97	19.17	7.8372	19.954
AC(1)	0.35	0.32	0.842	0.7582
Correlation Matrix				
VVP	1	0.6471	0.169	-0.341
VVP realized	0.6471	1	-0.0938	-0.4144
VIX	0.169	-0.0938	1	0.8689
$\sqrt{RV}$	-0.341	-0.4144	0.8689	1

Notes: The table present summary statistics for estimated VIX-V premium (VVP), realized VIX-V premium (VVP realized), the monthly closing value of VIX index and annualized realized volatility ($\sqrt{12\text{RV}}$). VIX-V premium is calculated as monthly closing value ofVIX-$\sqrt{12\text{RV}}$ for the contemporaneous month. Realized VIX-V premium is calculated as the difference between monthly closing value ofVIX and $\sqrt{12\text{RV}}$ for the following month. The sample used is from Jan-1990 to Dec- 2019. AC(1) is first order auto-correlation. Data is obtained from Hao Zhou's website.

Table 1.2: Summary Statistics

Excess Stock Return Regressions.

Monthly Return Horizon	1	3	6	9	12
Constant	0.003	0.008	0.027	0.052	0.076
	[1.213]	[1.719]	[3.967]	[6.02]	[7.185]
VP_t	0.027	0.088	0.103	0.090	0.091
	[2.987]	[5.07]	[5.069]	[2.637]	[2.214]
$Adj.R^2(\%)$	2.171	6.503	3.821	1.678	1.115

Notes: The regression result of S&P500 cumulative excess returns for 1, 3, 6, 9 and 12 months on VIX premium for the sample of Jan-1990 to Dec-2019. T-stats are reported in square brackets. Data is obtained from Hao Zhou's website.

Table 1.3: S&P 500 return predictability

variance of stock returns. While the concept of variance risk premium is intuitive, however what is not clear is why an investor would need to hedge against fluctuations in variance when she presumably only cares about his lifetime consumption path. Moreover, one needs to make strong assumptions in a model to make variance of variance relevant for agents utility.

On the other hand, it can be argued that treating VIX premium and variance risk premium as equivalent is misleading. This is because models with Gaussian dynamics and RE predict a zero variance risk premium as has been shown in DY. A separate possibility is that VIX premium could be a premium to hedge against higher than 2nd moments of *variance* of returns. One might again question why would an investor care about higher than second moment of the variance (second moment) of returns. Furthermore, it has been shown in Carr and Wu (2008), that empirical estimates of higher moments of the variance of stock index returns do not justify the magnitude of VIX premium in the data. To further demonstrate this point, I reconsider the models in BTZ and DY which study variance risk premium in consumption based equilibrium model. I confront both BTZ and DY with facts 1-4 by calculating VIX premium in these models.

Bollerslev, Tauchen and Zhou (2009)

The model is an equilibrium model where a representative agent has Epstein-Zin-Weil recursive preferences. The model considers a stochastic process of log consumption growth, $g_{t+1} = log(C_{t+1}/C_t)$ as the following:

$$g_{t+1} = \mu_g + \sigma_{g,t} z_{g,t+1}, \tag{1.3}$$

$$\sigma^2_{g,t+1} = a_\sigma + \rho_\sigma \sigma^2_{g,t} + \sqrt{q_t} z_{\sigma,t+1}, \tag{1.4}$$

$$q_{t+1} = a_q + \rho_q q_t + \psi_q \sqrt{q_t} z_{q,t+1}, \tag{1.5}$$

where the parameters satisfy $a_\sigma > 0$, $a_q > 0$, $|\rho_\sigma| < 1$, $|\rho_q| < 1$, $\psi_q > 0$, and $\{z_{\sigma,t}\}$ and $\{z_{q,t}\}$ are independent i.i.d. $N(0,1)$ processes jointly independently of $\{z_{g,t}\}$ which is also i.i.d $N(0,1)$. The stochastic variance process $\sigma^2_{g,t+1}$ represents time-varying economic uncertainty in consumption growth, with the volatility-of-volatility process q_t in effect inducing an additional source of temporal variation in that same process. Variance risk premium in BTZ is calculated as $VRP^{BTZ}_{t,t+1} = E^{\mathcal{Q}}_t(\sigma^2_{g,t+2}) - E_t(\sigma^2_{g,t+2})$ where $E^{\mathcal{Q}}_t(x_{t+1}) = E_t[M_{t+1}x_{t+1}](E_t[M_{t+1}])^{-1}$ with M_{t+1} being the stochastic discount factor.

Drechsler and Yaron (2010)

This model is a version of the long run risk asset pricing model popularized by Bansal and Yaron (2004). Drechsler and Yaron (2011), mainly, adds two features to the original Bansal and Yaron (2004) model: slow moving long run component to the volatility and non Gaussian shocks to long run risk variable and conditional variance. The utility function is:

$$V_t = [(1-\delta)C_t^{\frac{1-\gamma}{\theta}} + \delta(E_t[V_{t+1}^{1-\gamma}])^{\frac{1}{\theta}}]^{\frac{\theta}{1-\gamma}} \tag{1.6}$$

The state vector of the economy is given by $Y_t \in \mathcal{R}^5$ which follows a VAR structure given by:

$$Y_{t+1} = \mu + FY_t + G_t z_{t+1} + J_{t+1} \tag{1.7}$$

where

$$Y_{t+1} = \begin{bmatrix} \Delta c_{t+1} \\ x_{t+1} \\ \tilde{\sigma}^2_{t+1} \\ \sigma^2_{t+1} \\ \Delta d_{t+1} \end{bmatrix}, F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & \rho_x & 0 & 0 & 0 \\ 0 & 0 & \rho_{\tilde{\sigma}} & 0 & 0 \\ 0 & 0 & (1-\rho_{\tilde{\sigma}}) & \rho_\sigma & 0 \\ 0 & \phi & 0 & 0 & 0 \end{bmatrix} \tag{1.8}$$

, μ is unconditional expectation of Y_t, $z_{t+1} \sim N(0, I_5)$, $G_t G_t'$ is variance-covariance matrix of the Gaussian shocks z_{t+1} and J_{t+1} is the vector of compound-Poisson jump shocks. Δc_{t+1} is log consumption growth, x_{t+1} is the persistent component of consumption and dividend growth, σ^2_{t+1} is the conditional variance, $\tilde{\sigma}^2_{t+1}$ is a variable that drives the long run mean of variance and Δd_{t+1} is log dividend growth. The ith component of J_{t+1} is given by $J_{t+1,i} = \sum_{J=1}^{\{N^i_{t+1}\}} \xi^j_i$, where N^i_{t+1} is the Poisson counting process upon the jth increment of N^i_{t+1} and ξ^j_i are i.i.d. The intensity process of N^i_{t+1} is given by the ith component of vector λ_t. Further,

$$G_t G_t' = h + \sum_k H_k Y_{t,k}$$

$$\lambda_t = l_0 + l_1 Y_t,$$

where $h \in \mathcal{R}^{n*n}$, $H_k \in \mathcal{R}^{n*n}$, $l_0 \in \mathcal{R}^n$and $l_1 \in \mathcal{R}^{n*n}$.

Variance risk premium in DY, $VRP^{DY}_{t,t+1}$is defined as the sum of the level difference i.e. $\text{var}^Q_t(r_{m,t+1}) - \text{var}_t(r_{m,t+1})$ and the drift difference $\text{var}^Q_t(r_{m,t+2}) - \text{var}^Q_t(r_{m,t+2})$. Here,

$\text{var}_t^Q(r_{m,t+1})$ represent the conditional expectation of variance in period $t+1$ of market returns under the risk neutral measure $\mathcal{Q}$ and $\text{var}_t(r_{m,t+1})$ is the conditional variance under the physical probability measure.

Note that BTZ and DY define variance risk premium differently, both of which are inconsistent with definition of VIX premium. To calculate VIX premium in these models, firstly, I calibrate the model to the parameter values as mentioned in the paper. Subsequently, I generate simulated series of 10,000 observations for stock prices, stochastic discount factors, conditional variance of stock returns and risk free rate. At each point of observation, I calculate a cross-section of option prices at various strike prices by calculating the expectations $E_t[M_{t+1}(Max\{P_{t+1}^S - X_i, 0\})]$ for each strike price X_i, here, M_{t+1} and P_{t+1}^S signifies the stochastic discount factor and stock price, respectively, generated by the model at $t+1$.[15] Finally, VIX series in these models is calculated using formula in (1.1) and the VIX premium, $VP_{t,t+1}$is calculated as $VIX_{t,t+1}^2 - \text{var}_t(r_{m,t+1})$.

Upper panel of Table 1.4 reports the statistics related VVP, VIX and the conditional variance. The model moments are calculated from a sample of 10,000 observations generated by the model. The data moments are reported in the first column of the table. The second column reports the moments that are implied by BS model. Since the BS model assumes a constant variance, the model predicts zero VVP and has no chance of matching any moments related to VVP. On the other hand, while the BTZ model allows for stochastic volatility and stochastic 'volatility of volatility', it is still conditionally log-normal. Hence, as shown in the third column of Table 1.4, in BTZ model the VVP is close to zero. The small standard deviation of VVP in BTZ is most likely due to computational errors. In general, BTZ model fails to match any of the facts 2(a)-2(d) mentioned above. The fourth column of the table reports the statistics for DY model. DY being non-Gaussian is able to generate a positive VVP of around 2.0% which falls well short of 5.88% in the data. Additionally, DY is not able to generate a negative VVP while in the data it is frequently negative. In DY, the VVP is extremely persistent with one period auto-correlation of 0.85 while in the data, it has a low auto-correlation of 0.35. DY struggles to match the persistence of VVP because the state variables that

[15] I use numerical integration to calculate the expectations of option's payoffs.

	Data	Black-Scholes	BTZ	DY
	1990:01 to 2019:12		Mean	Mean
$E[VVP]$	5.88	0.00	0.53	2.02
$\sigma(VVP)$	3.88	0.00	1.01	2.22
$\min(VVP)$	-18.91	0.00	-1.88	-0.09
$AC1(VVP)$	0.35	0.00	0.86	0.85
$AC1(VIX)$	0.84	0.00	0.93	0.88
$AC1(\sqrt{RV})$	0.76	0.00	0.93	0.90
$\hat{E}(VIX)$	19.16	13.28	13.33	18.59
$\hat{E}(\sqrt{RV})$	13.28	13.28	12.80	16.57
$\hat{\sigma}(VIX)$	7.36	0.00	5.12	7.00
$\hat{\sigma}(\sqrt{RV})$	7.72	0.00	4.35	4.80
correl(VIX,VVP)	0.17	0.00	0.88	0.97
correl($\sqrt{\text{RV}}$,VVP)	-0.34	0.00	0.81	0.94
correl($\sqrt{\text{RV}}$,VIX)	0.87	1.00	0.99	0.99
Predictability				
$\beta(1)$	0.027	0.00	0.08	0.03
$Adj.R^2(1)$	2.171	0.00	0.20	0.68
$\beta(2)$	0.088	0.00	0.20	0.02
$Adj.R^2(3)$	6.503	0.00	0.50	0.44
$\beta(3)$	0.103	0.00	0.32	0.01
$Adj.R^2(6)$	3.821	0.00	0.78	0.09

Notes: In upper panel, report the moments of annualized VIX, VVP. Data moments are based on a monthly sample from Jan-1990 to Dec-2019. Moments for BTZ and DY are calculated from model generated series of 10,000 observations. VVP for BTZ and DY is calculated as VIX- conditional volatility. Lower panel reports predictive regression of excess market returns on VIX premium for horizon of 1,3 and 6 months.

Table 1.4: VIX premium and VIX : BTZ and DY

drives it are highly persistent in the model. As far as the correlation between VVP and VIX is concerned, DY model produces a correlation of 0.97 compared to 0.17 in the data. The lower panel of Table 1.4 reports the results for OLS regression of ex-dividend stock returns on VIX premium. In DY, the ability of VIX premium to predict returns is quite low compared to data. The 3 month horizon R^2 is only 0.44% versus 6.503% in the data. Taken together, one can conclude that both BTZ and DY fails to match any of the above mentioned stylized facts about VIX premium.

1.4 Model

In this section, I first describe the basic asset pricing model with subjective beliefs. The model is similar to Lucas Jr (1978) asset pricing model albeit agents hold subjective beliefs about future stock price.

Environment: The environment assumed in this model is standard, of the type considered in Adam, Marcet, and Nicolini (2016) (henceforth, AMN). The economy has $t = 0, 1, 2, \ldots$ periods and is populated by unit mass of infinitely-lived risk averse investor types. Aggregate consumption and aggregate dividends follow a standard log normal process with same expected growth rate a : $\frac{C_{t+1}}{C_t} = a\varepsilon^c_{t+1}$ where $ln\ \varepsilon^c_{t+1} \sim iiN(\frac{-s_c^2}{2}, s_c)$ and $\frac{D_{t+1}}{D_t} = a\varepsilon^d_{t+1}$ where $ln\ \varepsilon^d_{t+1} \sim iiN(\frac{-s_d 2}{2}, s_d^2)$ with $(ln\ \varepsilon^c_t, ln\ \varepsilon^d_t)$ jointly normal with co-variance $\rho_{c,d}$. It has been widely documented that consumption is less volatile than dividends. To allow for this, it is assumed that agent also receives endowment of Y_t amount of perishable goods every period. In particular, $C_t = Y_t + D_t$ is the total supply of the consumption goods in the economy.

Objective Function and Probability Space: The problem of an agent $i \in [0, 1]$ is to maximize his lifetime utility given by standard time separable CRRA utility function

$$E_0^{\mathcal{P}} \sum_{t\ =0}^{\infty} (\delta)^t \frac{(C_t^i)^{1-\gamma}}{1-\gamma},$$

subject to:

$$C_t^i + P_t S_t^i + B_t^i + P^C_{t,t+1} O_t^i = (P_t + D_t)\, S^i_{t-1}\ + (1 + r_{t-1})\, B^i_{t-1} + Y_t + max\{P_t - X, 0\} O^i_{t-1} \tag{1.9}$$

$$\underline{S} \leq S_t^i \leq \bar{S}$$

$$\underline{B} \leq B_t^i \leq \bar{B}$$

$$\underline{O} \leq O_t^i \leq \bar{O}$$

for all $t \geq 0$ where r_{t-1} denotes the real interest rate on a risk-less bond issued in period *t-1* and maturing in period *t*, $\gamma > 0$ is the risk aversion coefficient of the agent, C_t^i is consumption demand in period t for individual i, S_t^i is the holdings of stocks in period t for individual i, B_t^i is the holdings of bonds in period t and O_t^i is the positions in a one-period European call option contract for individual i. Importantly, agent's consumption *demand* is denoted by C_t^i, and *the aggregate supply* of consumption goods is denoted by C_t. The initial endowments given by $S_{-1}^i = 1$, $B_{-1}^i = 0$ and $O_{-1}^i = 0$ are such that stock are in fixed supply of 1 unit, risk-free bonds and options contracts are in zero-net supply. Furthermore, there are bounds to holdings of stocks, bonds and option contracts i.e. $\underline{S} \leq 1 \leq \bar{S}$, $\underline{B} \leq 0 \leq \bar{B}$ and $\underline{O} \leq 0 \leq \bar{O}$ such that ponzi schemes are ruled out. I only consider three assets namely; bonds, stock and stock options, which are not assumed to span the consumption state space i.e. markets are incomplete.

The non-standard part in this setup is the underlying probability space $\mathcal{P}$ that assigns probabilities to all external variables. Agents in the model are internally rational as defined in Adam and Marcet (2011). Agents behave rationally given their perceived model i.e. they are utility maximisers subject to a budget constraint. Markets are competitive; dividend and income processes are exogenous. The underlying probability space is given by $(\Omega, \mathcal{B}, \mathcal{P})$ with $\mathcal{B}$ denoting the corresponding σ-algebra of Borel subsets of Ω and $\mathcal{P}$ the agent's subjective probability measure on $(\Omega, \mathcal{B})$. Important thing to note is that the stock price histories are also part of the probability space which is a deviation from standard practice in RE models where stock prices carry redundant information and can be fully mapped to fundamentals. The internal rational agents are not aware of the pricing function $P_t(.)$ that link fundamentals to the stock price. Consequently, agents consider the process for stock prices $\{P_t\}$ along with the income and dividend process $\{Y_t, D_t\}$ as exogenous to their decision problem. In other words, the underlying sample state space Ω consists of realization for prices, dividends and income. Adam and Marcet (2011) show that assuming the knowledge of such pricing function is very restrictive. AMN show that slightly relaxing this assumption enables a standard model to explain key asset pricing facts, especially high volatility of stock prices relative to fundamentals. On the other hand, the price of options are excluded from the probability space, without loss of generality, because options expire after one period and the eventual pay-off from an option contract only depends on the strike price and the price of the underlying stock.

Although, $\mathcal{P}$ contains beliefs of the agents for all dates, the conditional distribution of the variables will be revised as agents record new information. The algorithm according to which agents revise their beliefs is also part of $\mathcal{P}$.

Agent of type i choose consumption, bond holdings, stock holdings and option positions in period t, denoted by $(C_t^i, B_t^i, S_t^i, O_t^i)$ contingent on the observed history $\omega^t = \{P^t, Y^t, D^t\}$, i.e. investors choose a function $(C_t^i, B_t^i, S_t^i, O_t^i) : \Omega^t \to R^4$ for all $t \geq 0$ to maximize expected utility subject to his budget constraint and the asset limits given by (1.9) . Since the objective function is concave with a convex feasible set, the following first order conditions characterizes the agent's optimal plan

$$(C_t^i)^{-\gamma} P_t = \delta E_t^{\mathcal{P}} \left[(C_{t+1}^i)^{-\gamma} P_{t+1} \right] + \delta E_t^{\mathcal{P}} \left[(C_{t+1}^i)^{-\gamma} D_{t+1} \right] \tag{1.10}$$

$$(C_t^i)^{-\gamma} = \delta(1 + r_t) E_t^{\mathcal{P}} \left[(C_{t+1}^i)^{-\gamma} \right] \tag{1.11}$$

$$P_{t,t+1}^C = \delta E_t^{\mathcal{P}} \left[\left(\frac{C_{t+1}^i}{C_t^i} \right)^{-\gamma} max\{P_{t+1} - X_t, 0\} \right]. \tag{1.12}$$

These conditions are standard except the conditional expectations are taken with respect to the subjective probability measure $\mathcal{P}$.

1.4.1 Rational Expectations

This sub-section reviews standard results under rational expectations. The stock price under rational expectations P_t^{RE}, is derived by solving equation (1.10)

$$P_t^{RE} = \frac{\delta a^{1-\gamma} \rho_\varepsilon}{1 - \delta a^{1-\gamma} \rho_\varepsilon} D_t \tag{1.13}$$

where

$$\begin{aligned} \rho_\varepsilon &= E_t \left[(\varepsilon_{t+1}^c)^{-\gamma} \varepsilon_{t+1}^d \right] \\ &= e^{\gamma(1+\gamma)\frac{s_c^2}{2}} e^{-\gamma \rho_{c,d} s_c s_d} \end{aligned}$$

Equation (1.13) show that price to dividends (henceforth, PD) ratio is constant. Consequently, the returns volatility should be approximately equal to the volatility of dividend growth which, as documented in many papers, is at odds with the data. [16]

[16] The proof of (1.13) is fairly straightforward, interested readers may refer to AMN for the proof.

Call option price under RE $P^{C,RE}_{t,t+1}$ is given by the following:[17]

$$P^{C,RE}_{t,t+1} = P_t \Delta N\left(d_{1,t}\right) - \frac{X}{1+r_t} N\left(d_{2,t}\right) \tag{1.14}$$

where $\Delta = 1 - E_t \left[\left(\frac{C^i_{t+1}}{C^i_t} \right)^{-\gamma} \frac{D_{t+1}}{P_t} \right] = \delta a^{1-\gamma} \rho_\varepsilon$, $d_{1,t} = \frac{ln\ P_t - ln\ X_t + r_t + ln\ \Delta + \frac{sd^2}{2}}{sd}$, $d_{2,t} = d_{1,t} - sd$ and $N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{z^2}{2}} dz$.

The formula in (1.14) is identical to the one period B-S formula. The price of a call option $P^{C,RE}_{t,t+1}$ is increasing in P_t, sd and r_t and non-increasing in X_t. Under RE, option implied volatility equals the conditional volatility under the objective measure sd which is a constant. The model implies $VP_t = 0$ for all t, hence, this model under RE cannot explain any of the stylized facts 2(a)-2(d). It is important to note that options price $P^{C,RE}_{t,t+1}$ is independent of agent's expected return which is interesting because the option price is derived by directly solving the first order condition under the objective probability measure. The standard practice in the reduced form literature is to a-priori transform the probability measure to a risk neutral measure for deriving the option price which makes the option price independent of expected returns on the underlying and preferences of the investors. The above result in equation (1.14) shows that the two approaches are equivalent under RE.[18]

1.4.2 Beliefs

In this sub-section, I specify the exact structure of beliefs that agent of type i hold and characterize the agent's optimal plan under learning. I assume that agents of all types share the same beliefs and preferences which is not common knowledge i.e. the agent i is not aware that she is a representative agent. Furthermore, I assume that agents are aware of the true stochastic processes of D_t and Y_t. The latter assumption is made to highlight how subjective beliefs exclusively about stock prices can influence asset prices in equilibrium. As discussed above and comprehensively argued in Adam and Marcet (2011), in this setting it is highly unlikely that a representative agent can determine the unique relationship between fundamentals and price.

[17] A proof in a similar setup is available in Rubinstein (1976)

[18] This is true because under RE the options can be perfectly hedged. Black and Scholes (1973) is the first paper to note this famous result.

Under the belief structure $\mathcal{P}$ the agent also takes into account the realized price outcome P_t while making choices for consumption, hence, it may not be strictly straightforward to assume that his pricing kernel is determined by the aggregate consumption as in the Lucas Jr (1978) tree model. Despite the fact that $S_t^i = 1$, $B_t^i = 0$, $O_t^i = 0$ and $C_t^i = C_t$ in equilibrium, we may have $E_t^{\mathcal{P}}\left[\left(C_{t+1}^i\right)^{-\gamma}\right] \neq E_t^{\mathcal{P}}\left[\left(C_{t+1}\right)^{-\gamma}\right]$. To rule out the possibility where agent's expected capital gains on his stock portfolio effect his discount factor, I assume the following as in AMN.

Assumption 1: I assume Y_t for all $t \geq 0$ is large enough such that given bounds to asset holding, and that $E_t^{\mathcal{P}}\left[\left(\frac{P_{t+1}}{D_t}\right)\right] < \bar{M}$ for some $M < \infty$ the condition (1.15) holds even ex ante in an identical agent environment.

$$E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}\right] \approxeq E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right] \tag{1.15}$$

(1.15) will be true with very high $\mathcal{P}$-probability when agent in the model is assumed to know the true stochastic process of Y_t and D_t. Stochastic discount factor for an agent $\left(\frac{C_{t+1}^i}{C_t^i}\right)^{-\gamma}$ will be little influenced by his trading decisions when Y_t is sufficiently high. If condition (1.15) holds, one can price assets using $\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}$ as the discount factor also under the framework of learning. Firstly, the risk free rate r_t is derived by standard Euler equation

$$1 = \delta(1 + r_t)E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]. \tag{1.16}$$

Additionally, agent subjective expectation of risk-adjusted stock price growth is defined as

$$\beta_t \equiv E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_{t+1}}{P_t}\right] \tag{1.17}$$

and the subjective expectation of risk-adjusted dividend growth

$$\beta_t^D \equiv E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{D_{t+1}}{D_t}\right]. \tag{1.18}$$

after substituting (1.17) and (1.18) in the first order condition for stock price 1.10, the expression for stock price P_t under subjective beliefs can be written as follows:

$$P_t = \frac{\delta\beta_t^D}{1 - \delta\beta_t}D_t \tag{1.19}$$

provided that β_t is bounded above by less than δ. Under the assumptions that agent know the true stochastic process of D_t, β^D_{t+1} is constant. However, when agent are learning about β_{t+1}, which is risk adjusted stock price growth, realized stock prices can fluctuate much more than dividends. As can be seen in (1.19), a slight change in expectations about next period price growth can lead to a large fluctuation in prices, especially, when β_t fluctuates around δ.[19]

Add derivation of variance?

Constant-Gain Learning: Now, I specify the exact structure of $\mathcal{P}$. An internally rational agent in this model follow a constant gain learning algorithm to update his expectation of log of risk adjusted capital gains. Agent believes that there is a persistent component in risk adjusted capital gains, which is independent and additional to what above specified dividend process would imply. This type of learning is motivated by the fact that PD ratio in the data is highly persistent while innovations to dividend growth is largely unpredictable. In particular, agent beliefs are specified as following:

$$ln\left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{P_t}{P_{t-1}}\right) = ln\ b_t + ln\ \epsilon_t \tag{1.20}$$

$$ln\ b_t = ln\ b_{t-1} + ln\ \xi_t$$

where $ln\ b_t|\omega^t \sim N(ln\ m_t, \sigma_0^2)$, $ln\ \varepsilon_t \sim iiN(\frac{-\sigma_\varepsilon^2}{2}, \sigma_\varepsilon^2)$ and $ln\ \xi_t \sim iiN(\frac{-\sigma_\xi^2}{2}, \sigma_\xi^2)$ and $E_{t-1}[(\varepsilon_t, \xi_t)] = 0$.

Further, I assume that agent's prior beliefs are centered at the RE value i.e $ln\ b_0 \sim N(a^{1-\gamma}\rho_\varepsilon, \sigma_0^2)$. Since prior is specified normally distributed, agents can optimally filter the persistent component of risk adjusted price growth using a Kalman filter. Agents' posterior beliefs at any time t are given by $ln\ b_t \sim N(ln\ m_t, \sigma_0^2)$ where $ln\ m_t$ evolves recursively according to the following equation:

$$ln\ m_t = ln\ m_{t-1} - \frac{\sigma_\xi^2}{2} + g\left(ln\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{P_t}{P_{t-1}}\right) + \frac{\sigma_\varepsilon^2 + \sigma_\xi^2}{2} - ln\ m_{t-1}\right), \tag{1.21}$$

where the constant Kalman gain parameter is given by $g = (\sigma_0^2)/(\sigma_0^2 + \sigma_\varepsilon^2)$ with steady state variance σ_0^2, given by

$$\sigma_0^2 = \frac{-\sigma_\xi^0 + \sqrt{(\sigma_\xi^2)^2 + 4\sigma_\xi^2\sigma_\varepsilon^2}}{2}$$

[19] β_{t+1} is bounded above in the quantitative simulations to ensure the price is positive for all possibilities in $\mathcal{B}$. I use the projection facility as used in AMN to bound the beliefs.

These beliefs constitute only a small deviation from RE beliefs whenever the gain parameter g is sufficiently small. Using equation (1.20) and given the distribution of $\left(\frac{C_t}{C_{t-1}}\right)$ as specified above, the posterior distribution of the log price ratio

$$ln\ \left(\frac{P_{t+1}}{P_t}\right) \sim N\left(\mu_{p,t}, \sigma^2_{p,t}\right) \tag{1.22}$$

where $\mu_{p,t} = ln\ m_t + r_t - \frac{\left(\sigma^2_\varepsilon + \sigma^2_\xi - \gamma^2 s^2_c\right)}{2}$ and $\sigma^2_{p,t} = \sigma^2_\varepsilon + \sigma^2_0 + \sigma^2_\xi - \gamma^2 s^2_c - 2\sigma_{r_t,c_t}$
$\sigma_{r_t,c_t} = cov_t\left[ln\left(\frac{P_{t+1}}{P_t}\right), ln\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right]$

Notice that the conditional variance $\sigma^2_{p,t}$ is stochastic and is influenced by agent expectations.

1.4.3 Valuation of Options

In this section, I provide analytical results that show how the learning mechanism assumed above can help rationalize the stylized facts related to option markets as documented in Section 2. The model is a discrete-time model and all the analysis here is based upon one-period options. Firstly, I provide a closed form formula for a call option price, $P^{C,CG}_{t,t+1}$ in the proposition 1 below which is derived by solving equation (1.12)

Proposition 1. *Under constant gains learning, price of 1 period European Call option is given by the following:*

$$P^{C,CG}_{t,t+1} = P_t e^{\left(ln\ m_t + \frac{\sigma^2_0}{2}\right)} N\left(\tilde{d_{1,t}}\right) - \frac{X_t}{1+r} N\left(\tilde{d_{2,t}}\right) \tag{1.23}$$

where $\tilde{d_{1,t}} = \frac{ln\ P_t - ln\ X_t + ln\ m_t + r + \frac{\sigma^2_{p,t}}{2}}{\sigma_{p,t}}$ and $\tilde{d_{2,t}} = \tilde{d_1} - \sigma_{p,t}$

Proof. See Appendix A.2 □

Notice that the formula for a call option in equation (1.23) (henceforth called the CG formula) is similar to formula in equation (1.14) (henceforth called the RE formula). This is due to the fact that underlying state variables in both cases are log normally distributed. The option price is a function of risk-adjusted price growth which is $\exp\{ln\ m_t + \frac{\sigma^2_0}{2}\}$ under learning and Δ in RE. In RE equilibrium, Δ can be expressed as a funcation of only fundamentals i.e. D_t and C_t and given i.i.d. shocks to fundamentals Δ is a constant. Whereas under learning $ln\ m_t + \frac{\sigma_0^2}{2}$ fluctuates as investors revise their beliefs after observing realized returns. Additionally, the conditional volatility paramter in the CG formula $\sigma_{p,t}$ in general is much higher than sd in the RE formula. However, as variance of idiosyncratic shocks to b_t, $\sigma^2_{\xi_t}$ approaches 0, the price of a call under constant gains learning converges to price under RE.

In the ensuing analysis, I explain the various option pricing stylized facts through studying the difference between the CG formula and the RE formula for a given conditional variance of stock returns. [20] [21]Proposition 2 below shows that higher expectations will have a first-order effect on a call-option price i.e. higher (lower) expectations *ceteris paribus* will lead to higher (lower) call option price.

Proposition 2. *Expectation of returns, $ln\ m_t$ has a positive effect on price of call options, which is given by the following:*

$$\begin{aligned}\frac{\partial P^{C,CG}_{t,t+1}}{\partial lnm_t} &= P_t N(\tilde{d_{1,t}}) e^{\left(\frac{\sigma_0^2}{2} + ln\ m_t\right)} \\ &> 0\end{aligned} \tag{1.24}$$

Proof. See Appendix A.3 □

This result is in contrast to RE models where expected returns on the underlying do not influence the price of an option (See equation (1.14)). Under RE, options can be perfectly hedged which makes the cost of hedging i.e. the risk free rate r_t rather than the expected returns, important for pricing of options.

Proposition 2 also offers insight into the shape of BS IV curve implied by the model. This can be seen through the following example: suppose learning model is the true model determining market option prices and conditional variance of returns is given by some value $\hat{\sigma_{p,t}}$. Furthermore, investors are optimistic i.e. $exp\{ln\ m_t + \sigma_0^2/2\}$ is greater than Δ. Given the above, if we use $\hat{\sigma_{p,t}}$ as unknown volatility parameter in the BS formula, then the BS formula price $P^{C,RE}(\hat{\sigma_{p,t}}, X_i, P_t, ...)$ will be less than the market

[20]In this setup, RE formula is equivalent to BS formula.

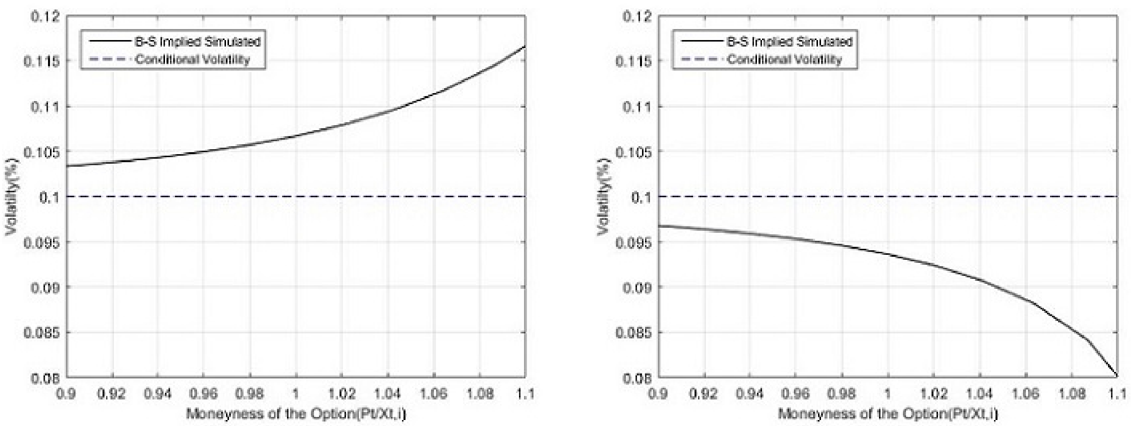
[21]We ignore here that conditional variance of stock returns in RE equilibrium and in CG equilibrium is different. The reason being that the stylized facts documented above are with respect to reduced form models which assume a stock price process that describes empirical data rather than deriving that process from a general equilibrium model. It can be argued that reduced form option pricing models are not truly rational if the stock price stochastic process assumed in the model reject the RE hypothesis.

price of option $P^{C,CG}(\hat{\sigma_{p,t}}, X_i, P_t, ...)$.Therefore, BS IV, which is the value σ_t^{BS-IV} that equates $P^{C,RE}(\sigma_t^{BS-IV}, ...)$ to the market option price, has to be greater than $\hat{\sigma_{p,t}}$ irrespective of moneyness. Now, the price difference $\Gamma(\hat{\sigma_{p,t}}, X_i, P_t, ...) \equiv P^{C,CG}(\hat{\sigma_{p,t}}, X_i, P_t, ...) - P^{C,RE}(\hat{\sigma_{p,t}}, X_i, P_t, ...)$ increases weakly as moneyness increases. In the limit as $X_i \to 0$, the probability $P(P_{t+1} > X_i) \to 1$ and $\Gamma(\hat{\sigma_{p,t}}, X_i, P_t, ...) \to P_t \left(e^{\left(\frac{\sigma_0^2}{2} + ln\ m_t\right)} - \Delta \right)$. To compensate for the increasing price difference $\Gamma(\hat{\sigma_{p,t}}, X_i, P_t, ...)$, BS IV increases with moneyness. The convex shape of BS IV curve has to do with the concave shape of the cdf of a standard normal variable. In the limit as $X_i \to 0$, BS IV $\to \infty$. Finally, it is important to note that BS IV is unrelated to conditional variance sd^2 in the RE model when learning model is the true model.

Figure 1.3 plots the BS IV curve when $exp\{\frac{\sigma_0^2}{2} + ln\ m_t\} > (<)\Delta$. As can be seen in the left (right) panel of the figure that when investors are optimistic (pessimistic), BS IV curve is upward (downward) sloping and convex (concave). The model being able to produce different shapes of BS IV curve is an advantage over the extant literature which has focused primarily on explaining the upward sloping smile. While BS IV curve calculated from the index call options is typically upward sloping, BS IV curves calculated from options on individual stocks have varied shapes which are not explainable by the extant models as documented by Bollen and Whaley (2004). Notice that in this model, the curvature is endogenously generated as result of agents holding subjective beliefs rather than by introducing exogenous non-gaussian dynamics to the stochastic process of underlying asset's returns. [22]

Stylized facts 1 to 4 described in Section 1 pertains to the properties of VIX premium. In the model, VIX premium is defined as the difference between VIX and conditional volatility i.e. $VP = VIX^2_{t,t+1} - \sigma^2_{p,t}$. In contrast to the extant literature which interpret (squared) VIX as a model-free risk neutral expectation of future variance of underlying's returns i.e $VIX^2_{t,t+1} = E_t^Q[\sigma^2_{p,t+1}]$, under internal rationality, VIX as such does not have any asset pricing interpretation if we deviate from rational expectations. Hence, I study the model predictions about VIX and VIX premium more thoroughly in the quantitative

[22]For example, Du (2011) in asset pricing models showed that the B-S IV smile can be generated if agents in the model expect a big crash in cash flows or consumption in the future.



Notes: The figure shows the implied volatility (B-S IV) calculated by inverting Black and Scholes (1973) option pricing formula, from option prices generated for moneyness [0.9,1.1] using the constant gains formula in (1.23)with conditional volatility =0.1, price =100 and r =0. The solid line in the figure is the B-S IV and the dashed line is the conditional volatility. The figure on the left (right) panel assumes $\left(\frac{\sigma_0^2}{2} + ln\ m_t\right) - \log\Delta =$ 0.005(-0,005).

Figure 1.3: Expectations and B-S implied volatility

sections. Nevertheless, in the CG model VIX premium can be understood by observing the properties of BS IV under the model. This is illustrated in Figure 1.3 where a optimism (pessimism) leads to a positive (negative) wedge between BS IV and conditional volatility at all moneyness. In the model under RE, VIX = BS-IV = sd, hence, the capital gains expectations that drive a wedge between BS IV and the conditional volatility is the same channel driving the VIX premium. VIX premium is on an average positive because investors for a majority of time are optimistic in a growing economy. The positive returns expectations have been noted in many surveys of investor expectations (See for example Greenwood and Shleifer (2014)). The following proposition shows that subjective expectations create a wedge between conditional volatility and BS IV.

Proposition 3. *Under the environment of Constant Gain Learning , the effect of higher expectations, $ln\ m_t$ will lead to higher implied volatility in B-S Model which is given by the following*

$$\frac{d\sigma_t^{BS}}{dln\ m_t} = \frac{N(\tilde{d_{1,t}})e^{\left(\frac{\sigma_0^2}{2}+ln\ m_t\right)}}{N'(\tilde{d_{1,t}})} > 0 \qquad (1.25)$$

Proof. See Appendix A.4 □

It has been documented in some studies (for example Cheng (2018)) that VIX premium is negative when ex ante measure of risk rises. This can be rationalized through the difference equation (1.21) that characterizes the evolution of $ln\ m_t$. Investors over predicting returns or when surprised by fall in prices may turn pessimistic leading to a negative VIX premium. This is an advantage over existing theories (see for example BTZ and DY) which predict only a positive VIX premium at all times. Since, the factor driving VIX premium is subjective capital gains expectations, it should be a effective predictor of stock returns because investor expectations, as documented in many places, are to some extent self-fulfilling (see for example Greenwood and Shleifer (2014)). The fact that VIX premium can predict returns for an extended horizon upto few quarters can also be rationalized in our model, since, capital gains expectations $ln\ m_t$ adjust very slowly to errors with a very small kalman gain parameter g. Importantly, the CG model implies that the directional behavior of VIX premium is largely independent of conditional volatility, however, the levels of VIX premium is largely determined by conditional variance. As a result, VIX premium exhibit considerably lower persistence than the conditional variance.

1.5 Quantitative Analysis

1.5.1 Cash flows and Equity Returns

In this section, I first report the estimation of the structural asset pricing model outlined in Section 4. The estimation is done using Methods of Simulated Moments (MSM). The approach is very similar to AMN albeit with monthly data. The goal of this estimation, firstly, is to demonstrate that the basic asset pricing model with subjective beliefs outlined above is a good approximation of the reality. Secondly, the output of the model using the estimated parameters will then be used to match key moments related to index option prices. It is important to note that matching index returns is not the primary objective of this paper since other studies (see for example AMN and Adam, Marcet, and Beutel (2017)) have shown that model with similar set up can replicated wide array of asset pricing moments. The objective of this paper is to match moments and patterns related to index option prices, hence, I select only 8 key moments related to index returns.

The parameters are estimated to match moments of the simulated data with the actual data moments. The data sample is from January 1990 to December 2019 with 360 monthly observations. The model is simulated for 10,000 times, with each simulation generating output of the same sample length as the actual data sample.

The parameters are summarized in the vector

$$\theta \equiv (\gamma, sd, g, \delta)$$

where γ is degree of risk aversion, sd refers to the standard deviation of dividend growth, g is the constant gain parameter and δ is the discount factor. The dividend growth rate, a is calibrated at the rate estimated in the data.

θ is estimated to match 8 sample moments as mentioned in Table 1.5.

$$\hat{\mathcal{S}_N} \equiv (\hat{E}[PD], \hat{E}(r^s), \hat{E}[dg], \hat{Std}[r^s], \hat{Std}[d], \hat{E}[r^b], \hat{Std}[PD], \hat{E}[PD_t, PD_{t-1}])$$

where $\hat{\mathcal{S}_N} \in \mathcal{R}^8$ denotes the vector of the sample moments for data sample of length N. Additionally, $\tilde{\mathcal{S}}(\theta)$ denotes the moments implied by the model for some parameter value θ. The MSM parameter estimate $\hat{\theta}_N$ is defined as

$$\hat{\theta}_N \equiv arg \min_{\theta} [\hat{\mathcal{S}_N} - \tilde{\mathcal{S}}(\theta)]' \sum [\hat{\mathcal{S}_N} - \tilde{\mathcal{S}}(\theta)] \tag{1.26}$$

where $\sum$ is an identity matrix. $\hat{\theta}_N$ is given by the parameters that provide the best possible fit of the moments estimated using model simulated data $\tilde{\mathcal{S}}(\theta)$ with sample moments of data $\hat{\mathcal{S}_N}$.

Table 1.5 below report the results of the method of simulated moments. The first column in the upper panel of the table reports the moments calculated using the data sample and the second column in the upper panel of the table reports the moments estimated using model simulations. As can be seen in the table 1.5 the model does a reasonable job of matching the selected moments of asset prices. In particular, the model does a good job in replicating the volatility of stock returns $\hat{Std}[r^s]$, volatility of dividends $\hat{Std}[d]$and high persistence of the PD ratio $\hat{E}[PD_t, PD_{t-1}]$. The model is also able to reasonably match average PD ratio $\hat{E}[PD]$ and average stock returns $\hat{E}(r^s)$. However, as has been noted in other papers using similar models (see for example AMN), the model fails to match the risk free rate $\hat{E}[r^b]$. Nevertheless, the model still generates average equity risk premium of 0.14% per month compared to the average data estimate

	U.S. Data 1990:1-2014:12	Subj. Belief Model
	Moment	Moment
E[PD]	628.29	543.96
E(r^s)	0.46	0.38
E[dg]	0.2	0.2
Std[r^s]	3.61	2.55
Std[d]	0.78	0.8
E[r^b]	0	0.24
Std[PD]	181.68	155.44
E[PD_t, PD_{t-1}]	1	0.97
$\hat{g}$		0.00125
$\hat{a}$		1.002
$\hat{sd}$		0.008
$\hat{\gamma}$		1
$\hat{\delta}$		0.9996

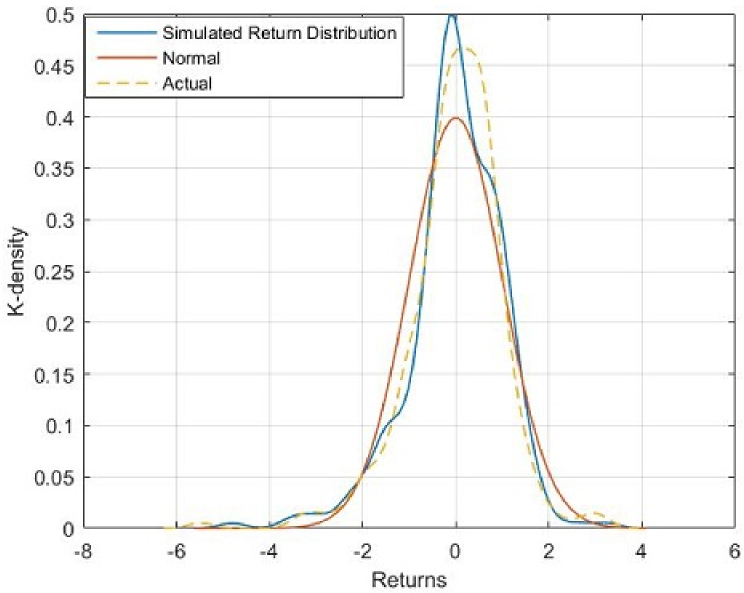
Notes: The table reports U.S. asset pricing moments (second column) using the data sources described in Section 2, the moments of the estimated models (columns three). The reported moments are as follows: E[PD], Std(PD) and Corr[PD_t, PD_{t-1}] denote the mean, standard deviation and auto-correlation of the quarterly price dividend ratio, respectively; E[r^s] and Std[r^s] denote the mean and standard deviation of the real quarterly stock return, expressed in percentage points, respectively; E[r^b] is the mean risk free interest rate, expressed in percentage points, dg is the dividend growth and Std[d] is the standard deviation of dividends. The estimated parameters are the updating gain g, from (1.21), the time discount factor δ, the coefficient of relative risk aversion γ and the standard deviation of the distribution of dividends sd.

Table 1.5: Estimation Outcome

0.38% per month.[23] As far as the parameters are concerned, the estimate of coefficient of risk aversion $\hat{\gamma}$ at only 1 is well within the range of risk aversion parameter used in the literature to match asset pricing moments. The constant gain parameter estimate $\hat{g}$ is 0.00125 which means that agents adjust their expectations by only 0.125% in the direction of the expectations error. This estimate is in line with estimate of AMN who estimate a gain of 0.7% using quarterly data in a similar model. It is important to note that at $g = 0$, the model is back to rational expectations. Therefore, from value of $\hat{g}$, one can conclude that a very small deviation from rational expectations can bring this model quite close to data.

Further, using the parameter estimates in Table 1.5, I simulate the S&P 500 series through the model using the actual dividend data of S&P 500. In Figure 1.4 the solid

[23]McGrattan and Prescott (2005) show that changes in dividend taxes can explain a large proportion of equity premium which is outside the scope of this paper.



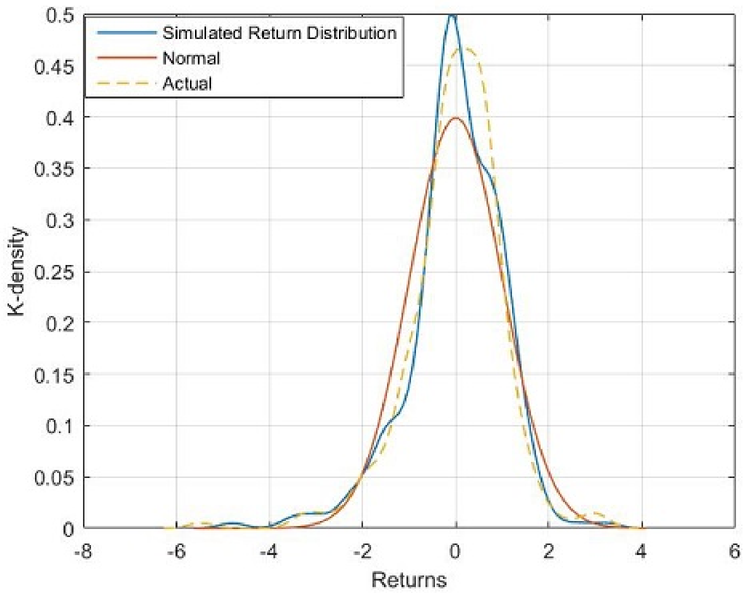
Notes: The figure show the simulation of the S&P500 produced by the learning model and corresponding actual value of S&P500 index adjusted for inflation. Monthly observations from Jan 1990 to December 2019.

Figure 1.4: Simulated S&P 500

green line is a simulated path of S&P 500 from the model while the dashed orange line shows the actual series of S&P500 index adjusted for inflation from Jan-1990 to Dec-2019. The model is able to fit the S&P 500 series fairly well for the sample in consideration. In Figure 1.5, I plot the sample distribution of simulated returns and compare it to actual sample distribution of S&P 500 real returns. Notice, that the model distribution is much closer to the actual distribution. The model is able to produce higher kurtosis of 6.29 and negative skewness of -0.7021 compared to kurtosis of 6.8 and skewness of -0.9877 in the data. In all, taking into account the results of MSM and the fit of the model to the actual data, the simple model can be considered a good approximation of the actual data. In the next subsection, I report results for the index option prices using the model simulations.

1.5.2 Volatility Dynamics and VIX premium

Table 1.6 provides the empirical moments and the corresponding statistics related to VIX premium for the estimated model. In particular, I provide the model-based 5%, 50% and 95% percentile for the statistics of interest generated from 10,000 simulations, each based on sample length corresponding to its data counterpart. For the purpose of generating



Notes: The figure shows the realized /actual distribution (shaded line) of S&P500 real returns, the simulated distribution (light blue solid line) of S&P500 real returns and the normal distribution (dark orange solid line)

Figure 1.5: Distribution of real returns.

option prices using (1.23), the model-based expected volatility input is estimated by running the Glosten, Jagannathan, and Runkle (1993) GARCH (1,1) procedure on model simulations of S&P 500 returns.[24]

Table 1.6 upper panel reports moments of VIX Index, realized volatility and VIX premium.[25] The first column in the upper panel reports moments that are calculated from the data sample of January 1990 to December 2019. The second column in the upper panel reports moments that are implied by B-S formula. The third and fourth column reports moments estimated using output from BTZ and DY. The last three columns report the percentiles of moments using the model simulations. Firstly, Table 1.6 shows that the model successfully generates a large average VVP. The model generates an mean VVP of 3.38% which is lower than empirical average of 5.88%. The model also respects the moments related to VIX. The average annualized VIX in the model is 18.08% compared to 19.16% in the data. Moderate persistence of VIX premium is very hard to replicate

[24]The real risk free rate is kept constant at 0.0005 % per month to isolate the impact of expectations on option prices. I convert the model simulated samples of next period real capital gains expectations to nominal terms by adding inflation expectations generated from a AR(1) process with mean 0.3% and standard deviation of 0.0015. This process is close to empirical distribution of non seasonally adjusted monthly consumer price inflation in the US.

[25]The model based model-based VIX is calculated according to formula in 1.1. Put prices are calculated using the put-call parity relationship.

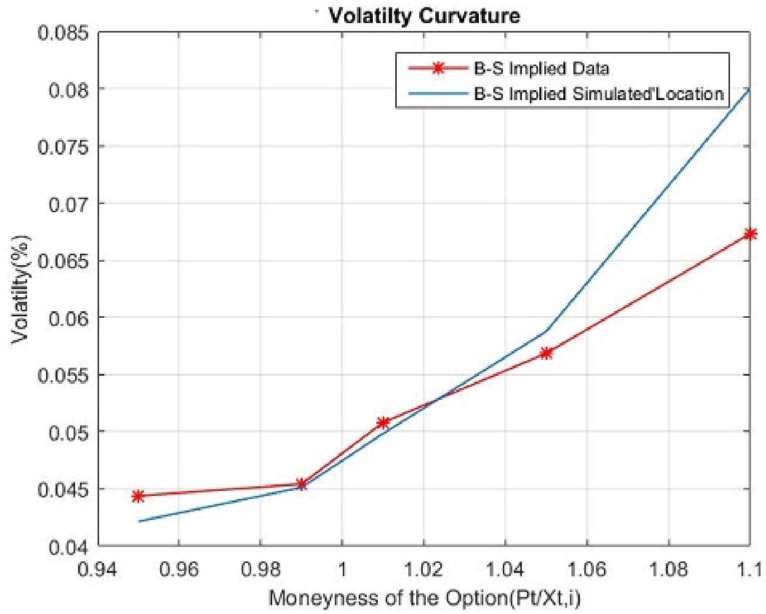
	Data	Black-Scholes	BTZ	DY	CG Model		
	1990:01 to 2014:06				5%	Mean	95%
$E[VVP]$	5.88	0.00	0.53	2.02	2.92	3.38	3.85
$\sigma(VVP)$	3.88	0.00	1.01	2.22	3.01	3.35	3.69
$\min(VVP)$	-18.91	0.00	-1.88	-0.09	-7.18	-4.82	-2.45
$AC1(VVP)$	0.35	0.00	0.86	0.85	0.41	0.50	0.59
$AC1(VIX)$	0.84	0.00	0.93	0.88	0.77	0.83	0.87
$AC1(\sqrt{RV})$	0.76	0.00	0.93	0.90	0.83	0.88	0.91
$\hat{E}(VIX)$	19.16	13.28	13.33	18.59	17.14	18.08	19.04
$\hat{E}(\sqrt{RV})$	13.28	13.28	12.80	16.57	13.83	14.7	15.58
$\hat{\sigma}(VIX)$	7.36	0.00	5.12	7.00	6.61	9.11	11.14
$\hat{\sigma}(\sqrt{RV})$	7.72	0.00	4.35	4.80	6.85	9.92	12.4
correl(VIX,VVP)	0.17	0.00	0.88	0.97	-0.23	-0.06	0.16
correl($\sqrt{\text{RV}}$,VVP)	-0.34	0.00	0.81	0.94	-0.50	-0.40	-0.29
correl($\sqrt{\text{RV}}$,VIX)	0.87	1.00	0.99	0.99	0.89	0.94	0.96
Predictability							
$\beta(1)$	0.027	0.00	0.08	0.03	0.02	0.07	0.12
$Adj.R^2(1)$	2.171	0.00	0.20	0.68	0.46	5.05	10.79
$\beta(2)$	0.088	0.00	0.20	0.02	0.01	0.14	0.25
$Adj.R^2(3)$	6.503	0.00	0.50	0.44	0.09	2.81	6.88
$\beta(3)$	0.103	0.00	0.32	0.01	-0.05	0.17	0.36
$Adj.R^2(6)$	3.821	0.00	0.78	0.09	0.02	1.55	4.56

Notes: Upper panel report the moments of annualized VIX and VVP. VVP in the CG model is VIX-conditional volatility estimated using Glosten, Jagannathan, and Runkle (1993) GARCH (1,1) procedure. Data moments are based on a monthly sample from Jan-1990 to Dec-2019. CG Model moments are mean, 5th and 95th percentiles based on 10,000 simulations of the model. Moments for BTZ and DY are calculated from model generated series of 10,000 observations. Lower panel reports predictive regression of excess market returns on VIX premium for horizon of 1, 3 and 6 months.

Table 1.6: VIX premium and VIX

in existing RE models, the one-period auto-correlation of VIX premium in the CG model 0.50 which is a bit higher than 0.35 in the data but still the persistence in the CG model is quite low compared to DY and BTZ.

Using the VIX premium generated in the model, I run predictive regressions of cumulative stock returns up to 6 months in future on VIX premium as given in (1.2). In Table 1.6 lower panel, I report the results for the predictability regression for 1, 3, and 6 months horizon. The model average R^2 of 2.81% for three month ahead predictability is lower than the data estimate of 6.5%. However, the data estimates lie with in the confidence bands of the model.



Notes: The figure shows the Black-Scholes curvature (line with asterisks) using implied volatility calculated from option prices data for 30 days European Call Option data and the Black-Scholes implied volatility backed out option prices generated through CG (blue line) model using a constant monthly volatility of 3.8% and model simulated m_t. Monthly date used from Jan-2004 to Dec-2013.

Figure 1.6: Implied Volatility: Data vs. CG Model

1.5.3 Implied volatility curvature

The Figure 1.6 plots the average BS IV curve. The blue line is the average BS IV curve generated using model simulations and orange line with asterisks represents the average BS IV curve calculated from the options data. For calculating the data BS IV curve, I calculate implied volatility using B-S formula from closing prices of selected 30-day call options and calculate the average BS IV curve based on monthly data from Jan 2004 to Dec 2013. To calculate the model BS IV curve, I generate call option prices from CG formula using the same price of underlying, strike price, and risk-free nominal rate as for the options selected in the data. The monthly volatility is assumed constant at 3.5% which is the standard deviation of monthly returns for the period of Jan 1990-Dec 2019 and capital gains expectations are estimated by simulating the model. Subsequently, I calculate BS IV implied by option prices generated using CG formula. The model captures well near the money BS IV but it generates higher BS IV for deep in the money call options.

1.6 Trading Strategies

To further test the predictions of the model, in this subsection, I consider trading strategies that are implied by the CG model. In the above sections, I show that, in the model, VIX premium is essentially a function of investors capital gains expectations i.e. optimism (pessimism) results in positive (negative) VIX premium. In the model, agents expected variance as given in 1.22 is also derived from his optimal learning algorithm. The model implies that investors follow a martingale model for forecasting conditional (realized) variance and are as such not paying any variance premium. In other words, (squared) VIX index is not a fair price for future variance. Taking cue from the model, one can exploit this inefficiency in the VIX market by following a long-short (L/S) trading strategy which entails trading in 1 month VIX index, selling (buying) squared VIX index when VIX premium>0 (VIX premium≤0).[26] [27] The payoff of the LS strategy is the following:

$$Payoff^{L/S}_{t+22} = \begin{cases} RV_{t,t+22} - VIX^2_t & \text{if } VP_t \leq 0 \\ VIX^2_{t,} - RV_{t,t+22} & \text{otherwise.} \end{cases}, \tag{1.27}$$

where VIX^2_t is the monthly value of squared VIX index at date t and $RV_{t,t+22}$ is the sum of squared log returns on the following 22 trading days. Another strategy that is implied by the model is where a trader opts for a cash position when $VP_t \leq 0$ and short VIX^2_t when $VP_t > 0$. This strategy is called the cash-short (C/S) strategy has the following payoff

$$Payoff^{C/S}_{t+22} = \begin{cases} 0 & \text{if } VP_t \leq 0 \\ VIX^2_t - RV_{t,t+22} & \text{otherwise.} \end{cases}. \tag{1.28}$$

Lastly, I also consider the strategy where a trader assumes that estimated VP_t is always positive and hence shorts the VIX index every month which has the following payoff

[26] VIX premium for this strategy is calculated as $VP_t = VIX^2_t - RV_{t-22,t}$where VIX^2_t is square of VIX index divided by 12 to get the monthly value and $RV_{t-22,t} = \sum_{h=t-22}^{t} r^2_h$ is the sum of squared daily log returns on the index for the last 22 trading days.

[27] It is well know that VIX index itself, as such, cannot be traded, however, 1 month variance swaps are traded and the one months variance swap rate tracks VIX quite closely. Since VIX index is interpreted as the synthetic one-month variance swap rate. These strategies can be implemented in the variance swap market at very low costs.

	1990:02-2019:12			
	L/S VIX	C/S VIX	S/S VIX	S&P500TR
Mean return	0.3013	0.3009	0.3006	0.007
Standard deviation	0.6557	0.4678	0.656	0.0411
Annualized Sharpe Ratio	1.592	2.228	1.587	0.590
	2004:01-2019:12			
	L/S VIX	C/S VIX	S/S VIX	S&P500TR
Mean return	0.2718	0.2514	0.231	0.0063
Standard deviation	0.8049	0.5277	0.8176	0.0388
Annualized Sharpe Ratio	1.170	1.650	0.979	0.562
	2008:04-2019:12			
	L/S VIX	C/S VIX	S/S VIX	S&P500TR
Mean return	0.2819	0.2447	0.2075	0.0077
Standard deviation	0.9042	0.5715	0.9242	0.0427
Annualized Sharpe Ratio	1.080	1.483	0.778	0.625

Notes: The Table report summary statistics for monthly returns on S&P500 total return index (S&P500TR), as well as returns on trading strategies that involve long (L), short (S) on the VIX index or in cash(C). L/S VIX strategy involves going long (short) on VIX index for a month when VIX premium (VP) $\leq 0 (> 0)$ at the start of the month and receiving (paying) monthly realized variance, $RV - VIX^2/12$ at the end of the month. C/S VIX strategy involves going cash (short) on VIX index for a month when VIX premium (VP) $\leq 0 (> 0)$ at the start of the month. C/S VIX strategy receives $VIX^2/12 - RV$ at the end of the month when trader is Short VIX and receives 0 when in cash. Short VIX strategy involves going short the VIX index at start of every month and receiving payoff of $VIX^2/12 - RV$ at the end of each month. The upper panel report summary statistics for returns from Feb-1990 to Dec-2019, the middle panel report summary statistics for returns from Jan-2004 to Dec-2019 and the lower panel report summary statistics for returns from Apr-2008 to Dec-2019.

Table 1.7: Trading strategies

$$Payoff^{S/S}_{t+22} = VIX^2_t - RV_{t,t+22} \tag{1.29}$$

.

The table 1.7 presents the mean, variance and annualized Sharpe ratio of the strategies for various sample periods. As can be seen in the table C/S strategy outperforms all other trading strategies by a wide margin with a Sharpe ratio of 2.228 for the full sample of February 1990 to December 2019. Furthermore, L/S trading strategy outperforms the pure short S/S strategy for all sample periods. Overall, all the trading strategies involving trading in VIX index outperforms the returns on S&P500 total return index in all sample periods. Additional, the table 1.8 presents the mean, variance and annualized Sharpe ratio for the same strategies when trading is done in variance swap rate instead of VIX index. Variance swaps are traded over the counter and the variance swap rate pays $RV_{t,t+22}$ after one month. See for example, Dew-Becker, Giglio, Le, and Rodriguez

1996:01-2013:09				
	L/S VSWP	C/S VSWP	Short VSWP	S&P500TR
Mean return	0.2827	0.2696	0.2564	0.0053
Standard deviation	0.6797	0.4354	0.6901	0.0458
AnnualizedSharpe Ratio	1.441	2.145	1.287	0.401
2004:01-2013:09				
	L/S VSWP	C/S VSWP	Short VSWP	S&P500TR
Mean return	0.3199	0.2761	0.2323	0.0041
Standard deviation	0.814	0.446	0.8434	0.0423
AnnualizedSharpe Ratio	1.361	2.144	0.954	0.336
2008:04-2013:09				
	L/S VSWP	C/S VSWP	Short VSWP	S&P500TR
Mean return	0.3977	0.2955	0.1934	0.0055
Standard deviation	1.0162	0.4962	1.0748	0.0521
AnnualizedSharpe Ratio	1.356	2.063	0.623	0.366

Notes: The Table report summary statistics for monthly returns on S&P500 total return index (S&P500TR), as well as returns on trading strategies that involve long (L), short (S) on the Variance swap rate (VSWP) or in cash(C). L/S VSWP strategy involves going long (short) on VSWP for a month when VIX premium (VP) $\leq 0 (> 0)$ at the start of the month and receiving (paying) monthly realized variance, $RV - VSWP$ at the end of the month. C/S VIX strategy involves going cash (short) on VSWP for a month when VIX premium (VP) $\leq 0 (> 0)$ at the start of the month. C/S VSWP receives $VSWP - RV$ at the end of the month when trader is Short VSWP and receives 0 when in cash. Short VSWP strategy involves going short the VSWP index at start of every month and receiving payoff of $VSWP - RV$ at the end of each month. The upper panel report summary statistics for returns from Jan-1996 to Sep-2013, the middle panel report summary statistics for returns from Jan-2004 to Sep-2013 and the lower panel report summary statistics for returns from Apr-2008 to Sep-2013. The data for VSWP is obtained from Stefano Giglio website.

Table 1.8: Trading strategies

(2017) for a discussion on the depth of the over the counter variance swap market. The end of month variance swap rate and the squared VIX index has correlation of 99.3% in the sample period of January 1996 to September 2013. [28] The results in the table 1.8 are largely in line with results presented in the table 1.7.

1.7 Discussion

Naturally, question arises why the above mentioned trading strategies are this profitable? It appears unlikely that traders would leave these opportunities unexploited. If variance swap market is indeed efficient then this may imply that the risk hedged by VIX is some unknown type of risk for which investors are willing to pay extremely high premium.

[28]The data for end of month variance swap rate is obtained from Stefan Giglio website.

Alternatively, risk-return trade off is perhaps not a linear relationship as has been studied in the CAPM based approach.

On the other hand, the analysis in this paper suggest that the variance swap market and the stock options market are inefficient. It is the investors subjective beliefs, which are not necessarily consistent with fundamentals, that drives a large part of VIX premium. Therefore, such an opportunity exists because trading VIX is costly or highly complex.

1.8 Conclusions

I show that by making only a slight deviation from rational expectations, a very simple asset pricing model can explain salient features of index option price data. In this model, investors subjective capital gains expectations on the underlying stock are relevant for the pricing of options. A closed-form expression for European call-option price is derived under this setup. The model is able to generate curvature in "Black and Scholes" option implied volatility. Optimistic (pessimistic) subjective capital gains expectations lead to upward (downward) sloping curve of Black and Scholes option implied volatility and price-strike ratio.

The model suggests that the difference between the squared VIX index and conditional variance or the VIX premium contains information on investors capital gains expectations; hence, it can predict index returns for a short horizon. Further, the model also suggests that option implied volatility measures such as VIX are biased estimates of conditional volatility in the direction of investors subjective expectations. Overall, the model shows that relaxing the strict assumption of rational expectations can be very informative in explaining many features of financial derivatives data. For future research, it will be useful to test whether one can uncover return expectations for different assets by analyzing the options data.

Appendix A

A.1

Proof. [Proposition 1]Using 1.12 the price, $P^C_{t,t+1}$, of a one period option contracted in period t and set to expire in period *t+1* is given by the following:

$$\begin{aligned}P^C_{t,t+1} &= \delta E_t^{Pi}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma} max(P_{t+1}-X,\ 0)\right] \\ &= \delta E_t^{Pi}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}(P_{t+1}-X)|P_{t+1}>X\right].\end{aligned}$$

To use the setup build in section 1.4, we can split the above equation and write it as following,

$$\begin{aligned}P^C_{t,t+1} = \delta P_t E_t^{Pi}&\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+1}}{P_t}\right)|ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_{t+1}}{P_t}\right] > ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{X}{P_t}\right]\right] \\ &- \delta P_t E_t^{Pi}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{X}{P_t}\right)|ln\left[\frac{P_{t+1}}{P_t}\right] > ln\left[\frac{X}{P_t}\right]\right]. \quad (1.30)\end{aligned}$$

Defining $K_{t+1} = ln\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{X}{P_t}\right]$, and using (1.20) implies,

$$\begin{aligned}&\delta P_t E_t^{Pi}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+1}}{P_t}\right)|ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_{t+1}}{P_t}\right] > ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{X}{P_t}\right]\right] \\ &\quad = P_t\int_{-\infty}^{\infty}\int_{K_{t+1}}^{\infty}\left(e^{ln\ b_{t+1}+ln\ \epsilon_{t+1}}\right)dF_t\left(ln\ b_{t+1}+ln\ \epsilon_{t+1}\right)dF_t(K_{t+1}) \quad (1.31)\end{aligned}$$

To solve the above expression, we can use the property of the normal distribution, where if $x \sim N(\mu_x, \sigma_x^2)$ then

$$\int_a^{\infty} e^x f(x)dx = (e^{\mu_x+\frac{1}{2}\sigma_x^2})N(\frac{-a+\mu_x}{\sigma_x}+\sigma_x)$$

for $a \in (-\infty, \infty)$. As a result (1.31) can be simplified to

$$\delta P_t E_t^{\mathcal{P}i}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+1}}{P_t}\right)|ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_{t+1}}{P_t}\right] > ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{X}{P_t}\right]\right]$$
$$= P_t e^{\left(ln\ m_t+\frac{\sigma_0^2}{2}\right)}\int_{-\infty}^{\infty} N\left(\frac{-K_{t+1}+ln\ m_t+\frac{\sigma_0^2}{2}+\frac{\left(\sigma_0^2+\sigma_\epsilon^2+\sigma_\xi^2\right)}{2}}{\left(\sigma_0^2+\sigma_\epsilon^2+\sigma_\xi^2\right)/2}\right) dF_t(K_{t+1})$$

.Since, K_{t+1} is a random variable, to get an analytical expression we need to integrate the above expression for values of K_{t+1}. However, if in the above expression we replace for $y_{t+1} = ln\ b_{t+1} + ln\ \epsilon_{t+1} - K_{t+1}$, then after simple algebra , it can be shown that

$$\delta P_t E_t^{Pi}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+1}}{P_t}\right)|ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_{t+1}}{P_t}\right] > ln\left[\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{X}{P_t}\right]\right]$$
$$= P_t e^{\left(ln\ m_t+\frac{\sigma_0^2}{2}\right)} N\left(\frac{ln\ P_t - ln\ X_t + ln\ m_t + r + \frac{\sigma_{p,t}^2}{2}}{\sigma_{p,t}^2}\right) \quad (1.32)$$

.For the second term on the right hand side in the equation (1.30), we can use the property of the conditional normal distribution, since $ln\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)$ and $ln\left(\frac{P_{t+1}}{P_t}\right)$ have joint normal distribution as assumed in (1.20). For any joint normally distributed random variables x and y,

$$\int_{-\infty}^{\infty}\int_{a}^{\infty} e^y f(y,x)dxdy = \int_{a}^{\infty}\int_{-\infty}^{\infty} e^y f(y|x)f(x)dydx$$
$$= (e^{\mu_y+\frac{1}{2}\sigma_y^2})N(\frac{-a+\mu_x}{\sigma_x}+\kappa\sigma_y)$$

for $a \in (-\infty, \infty)$. Using this result and after straight forward algebra it follows that Expectations and the term structure of Option implied volatility

$$\delta P_t E_t^{Pi}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\left(\frac{X}{P_t}\right)|ln\left[\frac{P_{t+1}}{P_t}\right] > ln\left[\frac{X}{P_t}\right]\right] = \frac{X_t}{1+r}N\left(\frac{ln\ P_t - ln\ X_t + ln\ m_t + r - \frac{\sigma_{p,t}^2}{2}}{\sigma_p}\right) \quad (1.33)$$

Finally, substituting (1.32) and (1.33) in (1.30), the analytical expression in Proposition 1 is as follows

$$P_{t,t+1}^{C,CG} = P_t e^{\left(ln\ m_t+\frac{\sigma_0^2}{2}\right)} N\left(\frac{ln\ P_t - ln\ X_t + ln\ m_t + r + \frac{\sigma_{p,t}^2}{2}}{\sigma_{p,t}}\right)$$
$$- \frac{X_t}{1+r}N\left(\frac{ln\ P_t - ln\ X_t + ln\ m_t + r - \frac{\sigma_{p,t}^2}{2}}{\sigma_{p,t}}\right)$$

□

A.2

Proof. [Proposition 2]Taking partial derivative of the option price, $P^{C,CG}_{t,t+1}$ with respect to $ln\ m_t$, we have the following:

$$\frac{\partial P^{C,CG}_{t,t+1}}{\partial ln m_t} = P_t N(\tilde{d_{1,t}}) e^{\left(\frac{\sigma_0^2}{2}+ln\ m_t\right)} + P_t e^{\left(\frac{\sigma_0^2}{2}+ln\ m_t\right)} N'(\tilde{d_{1,t}}) \frac{1}{\sigma_{p,t}} - \frac{X_t}{1+r} N'\left(\tilde{d_{2,t}}\right) \frac{1}{\sigma_{p,t}} \quad (1.34)$$

where $\tilde{d_{1,t}}=\frac{ln\ P_t - ln\ X_t + ln\ m_t + r + \frac{\sigma_{p,t}^2}{2}}{\sigma_p}$ and $\tilde{d_{2,t}}=\tilde{d_{1,t}} - \sigma_{p,t}$. Given the relation between $\tilde{d_{1,t}}$and $\tilde{d_{2,t}}$, we substitute the value of $N'\left(\tilde{d_{2,t}}\right)$ using the following:

$$\begin{aligned} N'\left(\tilde{d_{2,t}}\right) &= \frac{\partial N(\tilde{d_{2,t}})}{\partial \tilde{d_{2,t}}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tilde{d_{2,t}})^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tilde{d_{1,t}}-\sigma_{p,t})^2}{2}} \\ &= \frac{1}{\sqrt{2\pi}} e^{-\frac{(\tilde{d_{1,t}})^2}{2}} e^{\frac{2d_1 \tilde{\sigma}_{p,t} - \sigma_{p,t}^2}{2}} \end{aligned}$$

This can be further solved as the following:

$$N'\left(\tilde{d_{2,t}}\right) = N'\left(\tilde{d_{1,t}}\right) . \frac{(1+r) P_t e^{\left(ln\ m_t + \frac{\sigma_0^2}{2}\right)}}{X_t} \quad (1.35)$$

Replacing $N'\left(\tilde{d_{2,t}}\right)$ in (1.35), we get Proposition 2,

$$\begin{aligned} \frac{\partial P^{C,CG}_{t,t+1}}{\partial ln m_t} &= P_t N(\tilde{d_{1,t}}) e^{\left(\frac{\sigma_0^2}{2}+ln\ m_t\right)} \\ &> 0 \end{aligned}$$

□

A.3

Proof. [Proposition 3]Under the assumption that CG model is the true model given some conditional volatility $\sigma_{p,t}$, price of the underlying stock P_t, the strike price X_t, subjective expectations m_t and risk free rate r_t. We can find a B-S implied volatility σ_t^{BS} that solves the equation

$$0 = P^{C,BS}_{t,t+1} - P^{C,CG}_{t,t+1} \quad (1.36)$$

.

Subsequently, using implicit function theorem we can derive the relationship between σ_t^{BS} and lnm_t as the following:

$$
\begin{aligned}
\frac{d\sigma_t^{BS}}{dlnm_t} &= \frac{\frac{\partial P_{t,t+1}^C}{\partial lnm_t}}{\frac{\partial P_{t,t+1}^C}{\partial \sigma_t^{BS}}} \\
&= \frac{P_t N(\tilde{d_{1,t}}) e^{\left(\frac{\sigma_0^2}{2} + ln\ m_t\right)}}{N'(d_{1,t})} > 0
\end{aligned}
$$

□

Chapter 2

Expectations and the term structure of Option implied volatility

2.1 Introduction

A typical test of an option pricing model is whether the model can replicate the empirical facts regarding the option implied volatility. One of the well known type of option implied volatility (IV, henceforth) is the unknown parameter of volatility that equates the BS model price with the market option price. Essentially, the idea is to determine whether the option prices generated by a model produce the same pattern of BS IV as observed in the data. There are a few important stylized facts regarding BS IV of index options that have been extensively studied in the literature. Firstly, at each point in time and for options with the same underlying asset, BS IV varies with respect to the ratio of price of the underlying and the strike price (henceforth, moneyness), sometimes called the BS IV curve. Secondly, the slope of the BS IV curve changes with maturity and is typically more flat as maturity of an option increases. Thirdly, BS IV surface which characterize the joint relationship of BS IV along direction of moneyness and maturity itself is stochastic with time. These empirical features of BS IV are in violation of the BS model which assumes a constant instantaneous volatility with respect to moneyness and time to maturity of an option.

In the literature, most reduced form models of option pricing only partially explain the stochastic behavior of volatility surface. For example, single factor stochastic volatility model such as Heston (1993) or simple jump process model can explain any fixed slope of BS IV curve but cannot replicate a fluctuating BS IV curve with a fixed set of estimated parameters.Bakshi, Cao, and Chen (1997) considers a general class of option pricing models and show that all models are inconsistent in the sense that the estimated parameters differ by maturity of options as well as are highly unstable with time. Furthermore, the stochastic process of underlying index implied by these models cannot replicate the index data. Two factor stochastic volatility model as proposed by Christoffersen, Heston, and Jacobs (2009) offers some flexibility in matching the stochastic BS IV surface. But the estimated structural parameters as provided in the paper are very unstable and fluctuate significantly when estimated to different year samples. Aït-Sahalia, Li, and Li (2021a) find that various models can explain different features of the BS IV curve and its term structure but they do not propose a single model that can explain all the stylized facts mentioned above.

Another approach is to explicitly model BS IV rather than looking for an option pricing model that produces the empirical features of BS IV, for example, see Cont and Da Fonseca (2002). In this paper, IV is strictly considered as a derivative of option prices. In other words, the objective is to study how an option can be correctly valued rather than model the IV itself. This is because the information that IV captures depends on the true model that generates the market option price data. While it may appear that IV is a function of time, moneyness and the underlying volatility but the underlying state variable that actually determines IV could be totally unrelated to underlying volatility expected by market participant.

I extend the discrete time general equilibrium model studied in Chapter 1 by including multi-period options. An internally rational representative agent is learning about log of future risk adjusted price growth from realized price growth. In particular, the agent revises his next period subjective capital gains expectations based on his last period forecast error. The persistent component of the learning rule follows a random walk model implying that expectations in the further future periods are closely related to expectations for the next period. In this setup, I derive a closed form expression for multi-period option price. I simulate option prices from the model using reasonable values of parameters and show that model can replicate many features of BS IV surface. The result further demonstrates that a small departure from rational expectations should be a norm rather than exception in asset pricing models.

The remainder of the paper is structured as follows. Section 2 describes the data and stylized facts. Section 3 describes the asset pricing model and the beliefs system of the agents. Section 4 contains analytical results from the model and simulations of the model. Finally, Section 5 concludes.

2.2 Data

In this section, I documents some key facts about the term structure of the option implied volatility. The daily data on call options on S&P 500 index futures for the period from Jan 2004 through Dec 2017 is obtained from Chicago Board Options Exchange. The data includes bid-price, ask-price quotes at 1545 for all outstanding call options and its underlying on each trading day during the sample period. It also includes strike price, date of expiry, BS implied volatility for all respective call options. Options contracts with fewer than 15 days or greater than 365 days to expiry or with open interest less than 50 or trading volume less than 10 or closing mid price less 0.1 are omitted from the sample. After cleaning, the data contains total of 346,867 contracts.

Table 2.1 below summarizes the data set. S/X also called moneyness is the price of underlying divided by the strike price of the option contract and DTM stands for days to maturity. In panel A, we can see that there are more out of the money $\left(\frac{S}{X} < 1\right)$ contracts outstanding than in the money $\left(\frac{S}{X} > 1\right)$ contracts. Columns in Panel C highlights presence of option implied volatility curve at all maturities with steepest curvature present in the short term options. Notice that the curve becomes flatter as the maturity increase.

Figure 1 plots the sample average of BS IV curve by maturity where the yellow dashed line with plus marker is for call option contracts with short term maturity (STM) between 15 and 45 days, the green line with solid square is for contracts with short to medium term maturity (SMTM) between 45 and 90 days, the blue line with solid triangles is for

Table 1	S&P 500 Index Call Option Data, 2004-2017			
	15<DTM<45	45<=DTM<90	90<=DTM<180	180<=DTM¡366
	Panel A. Number of call option contracts			
0.935<S/X<0.975	76,379	45,874	13,824	8,382
0.975<=S/X<0.995	51,197	21,130	6,863	4,719
0.995<=S/X<1.005	22,296	11,027	3,993	3,284
1.005<=S/X¡1.025	24,329	11,389	4,668	3,852
1.025<=S/X	17,798	8,446	4,151	3,266
	Panel B. Average call price			
0.935<S/X<0.975	3.16	7.90	21.27	48.59
0.975<=S/X<0.995	11.02	23.72	42.93	74.16
0.995<=S/X<1.005	23.83	38.19	56.37	88.27
1.005<=S/X<1.025	40.16	52.40	69.62	99.47
1.025<=S/X	80.31	91.71	110.49	133.85
	Panel C. Average implied volatility from call options			
0.935<S/X<0.975	11.84	11.93	13.95	16.20
0.975<=S/X<0.995	11.45	13.43	15.59	17.63
0.995<=S/X<1.005	12.78	14.85	16.41	17.71
1.005<=S/X<1.025	14.59	16.30	17.48	18.70
1.025<=S/X	19.55	19.28	19.38	20.06

Table 2.1: Summary Statistics: S&P 500 Index Call Option Data, 2004-2017

contracts with medium term maturity (MTM) between 90 to 180 days and the purple line with solid circles is for contracts with long term maturity (LTM). The flatness of the curve with maturity is clearly evident in the figure. Black and Scholes (1973) model will imply that there should be no curve at any maturity. In other words the implied volatility should be equal to a unique expected volatility for all moneyness and maturity.

Figure 2.2 plots the BS IV curve for STM options and LTM options under three types of volatility regimes. High (Low) volatility regime include trading days with in the sample where closing value of VIX index is >30(<15), and moderate volatility regime include remaining trading days in the sample. It can be seen that the during high volatility regime the BS IV curve for the LTM options lies below the STM options whereas in the other two regimes BI curve for LTM options mostly lie above the curve for STM options except deep in the money options. It appears that when VIX is extremely high, STM options are expensive relative to LTM options.

Table 2.2 contains the correlation between the VIX index and the slope of BS IV curve. The slope is calculated as ratio of ITM/ATM implied volatility. It can be seen in the table VIX index is negative correlated with the slope of BS IV curve for LTM options, MTM and SMTM options. Further, Figure 2.3 shows the daily evolution of slope ratio

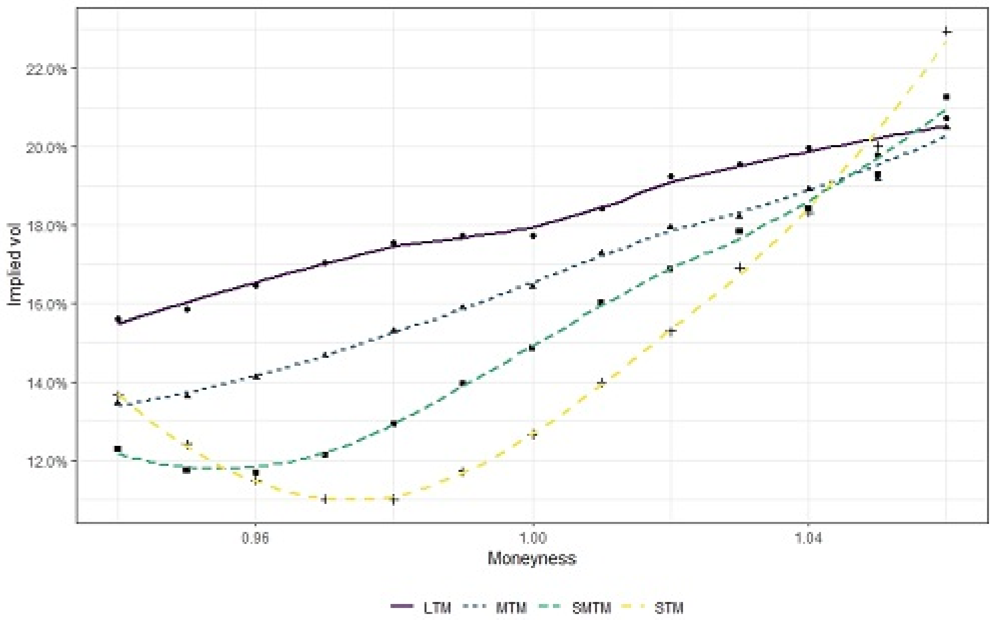


Figure 2.1: BS IV by maturity of option,
Notes: Sample average of BS IV curve by maturity where yellow dashed line with plus marker plots call option contracts with short term maturity (STM) between 15 and 45 days, green line with solid square plots contracts with short to medium term maturity (SMTM) between 45 and 90 days, blue line with solid triangles plots contracts with medium term maturity (MTM) between 90 to 180 days and purple line with solid circles plots contracts with long term maturity (LTM). Sample includes all trading days between Jan-2004 through Dec2017.

by maturity of options along and closing value of VIX index/100. The blue dashed dot, green double dashed, mustard triple dashed and orange solid are smoothed series of slope ratio for STM, SMTM, MTM and LTM options, respectively. The purple dotted line is the smoothed series of VIX index/100. It appears from the graph that high VIX index is associated with convergence of slope ratio of all maturities.[1]

To summarize, the two main facts about BS IV that I study in this paper are as follows:

Stylized fact1: The slope of BS IV curve varies with maturity and becomes flatter as maturity increases.

Stylized fact 2: The slope of BS IV curve for all maturities changes with time and during market turmoil, BS IV curves for all maturities flatten.

[1]Smoothing is done using LOESS procedure.

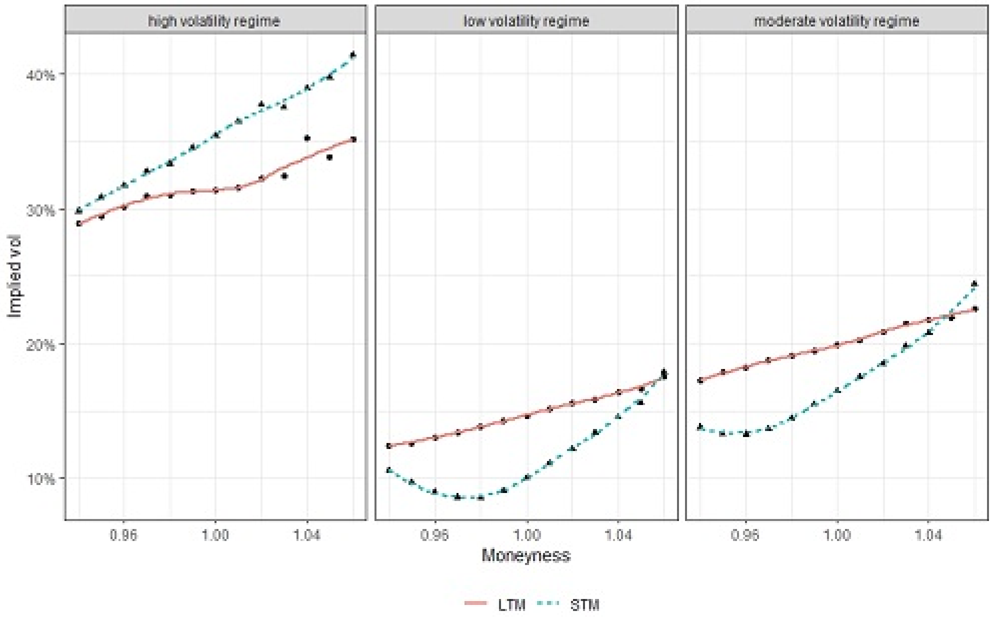


Figure 2.2: BS IV curve by volatility regimes
Notes: Average BS IV curve for STM options and LTM options under three types of volatility regimes. High (Low) volatility regime include trading days with in the sample where closing value of VIX index is >30(<15), and moderate volatility regime include remaining working days of the sample. Sample includes all trading days between Jan-2004- Dec2017.

	VIX.Close	LTM	MTM	SMTM	STM
VIX.Close		-0.62	-0.38	-0.64	-0.06
LTM			0.38	0.72	-0.001
MTM				0.42	-0.03
SMTM					0.03
STM					

Table 2.2: Correlation
Notes: The correlation between the VIX index and the slope measure of BS IV curve. The slope is calculated as ratio of ITM/ATM implied volatility.

2.3 Model

The framework considered in this chapter is the same as the one studied in Chapter 1. Section 3 Chapter 1 describes the environment in detail. The stochastic processes of the fundamentals are given by the following

$$logC_{t+1} - logC_t = loga + log\varepsilon^c_{t+1} \tag{2.1}$$

$$logD_{t+1} - logD_t = loga + log\varepsilon^d_{t+1} \tag{2.2}$$

for all $t > 0$ and where C_t denotes aggregate consumption, D_t denotes aggregate dividends, $log\varepsilon^c_{t+1} \sim i.i.N(-\frac{s_c^2}{2}, s_c^2)$ and $log\varepsilon^d_{t+1} \sim i.i.N(-\frac{s_d^2}{2}, s_d^2)$.

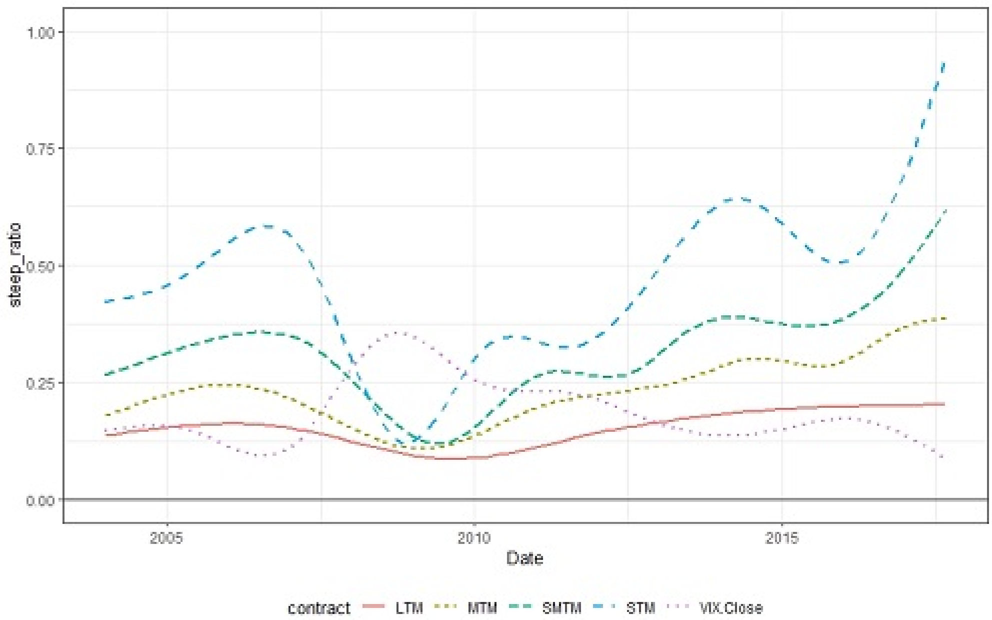


Figure 2.3: Slope and VIX
Notes: Daily chart of slope ratio by maturity of options along and closing value of VIX index/100. The blue, green, mustard and orange lines are smoothed series of slope ratio for STM, SMTM, MTM and LTM options, respectively. The dotted purple line plots the slope ratio of VIX index/100. Sample includes each trading day between Jan-2004 through Dec2017.

Objective Function and Probability Space: A representative agent[2] maximize lifetime utility which is given by standard time separable CRRA utility function by choosing $\{C_t \geq 0, B_t, S_t, O_t^1, ..., O_t^N\}_{t=0}^{\infty}$:

$$E_0^{\mathcal{P}} \sum_{t=0}^{\infty} (\delta)^t \frac{(C_t)^{1-\gamma}}{1-\gamma} \tag{2.3}$$

subject to:

$$\begin{aligned} C_t + P_t S_t + B_t + \sum_{n=1}^{N} P_{t,n}^{call} O_{t,n} = (P_t + D_t) S_{t-1} \ &+ (1 + r_{t-1,1}) B_{t-1} \\ &+ Y_t + \sum_{n=1}^{N} max\{P_t - X_n, 0\} O_{t-n,n} \end{aligned} \tag{2.4}$$

$$\underline{S} \leq S_t \leq \bar{S}$$

$$\underline{B} \leq B_t \leq \bar{B}$$

[2]As assumed in Chapter 1, the representative agent holds subjective beliefs and is not aware that she is a representative agent.

$$\begin{aligned} \underline{O} &\leq O_{t,n} \\ &\leq \bar{O} \end{aligned}$$

for all $t \geq 0$ where $r_{t-1,1}$ denotes the real interest rate on a one-period risk-less bonds issued in period *t-1* and maturing in period t, P_t is the price of a stock in period t, $P^{call}_{t,n}$ is the price of a European call option in period t which matures in period $t+n$, $\gamma > 0$ is the risk aversion coefficient of the agent, δ is the time discount factor, C_t is the consumption demand in period t , S_t is the holdings of stocks in period t, B_t is the holdings of bonds in period t and $O_{t,n}$ is positions in European call options in period which expires in period $t+n$ where $n \in \{1, 2, ..., N\}$ for some positive integer N. The initial endowments given by $S_{-1} = 1$, $B_{-1} = 0$ and $O_{-1,n} = 0$ for all n are such that stock are in fixed supply of 1 unit, risk-free bonds and options contracts are in zero-net supply. Furthermore, there are bounds to holdings of stocks, bonds and option contracts i.e. $\underline{S} \leq 1 \leq \bar{S}$, $\underline{B} \leq 0 \leq \bar{B}$ and $\underline{O} \leq 0 \leq \bar{O}$ such that ponzi schemes are ruled out. The only difference from the setup in Chapter 1 is that investors can also trade in multi-period options. As is typical in the literature, I abstain from discussion about secondary market trading in options. Since these are European call options, they can only be exercised at the date of expiry.[3]

The probability measure $\mathcal{P}$ is the subjective probability measure of the representative investor. It contains beliefs about all state variables that are exogenous to his decision making which includes consumption, dividends and stock prices. This is the key departure from the RE paradigm, that is, investor also hold subjective beliefs about the stochastic process of stock price which are potentially independent of beliefs about fundamentals. Nevertheless, beliefs about future option prices are redundant analogous to treatment in RE models.

In particular, the representative investor in the model is internally rational as defined in Adam and Marcet (2011). Agent is not aware of the pricing function $P_t(.)$ that maps fundamentals to equilibrium stock price. Agent behave rationally given their perceived model i.e she is a utility maximiser subject to a budget constraint. Markets are competitive; dividend and income processes are exogenous. The underlying sample state space Ω

[3]Secondary market trading could be potentially important but since new option contracts can be created at any point in time, there is no clear distinction between primary and secondary markets for derivatives unlike for stocks.

consists of realization for prices, dividends and income.

Agent chooses consumption, bond holdings, stock holdings and option positions in period t, denoted by $(C_t, B_t, S_t, O_{t,1}, ..., O_{t,n})$ contingent on the observed history $\omega^t = \{P^t, Y^t, D^t\}$, to maximize (2.3) subject to his budget constraint and the asset limits given by (2.4). Since the objective function is concave with a convex feasible set, the following first order conditions characterizes the agent's optimal plan

$$(C_t^i)^{-\gamma} P_t = \delta E_t^{\mathcal{P}} \left[(C_{t+1}^i)^{-\gamma} P_{t+1} \right] + \delta E_t^{\mathcal{P}} \left[(C_{t+1}^i)^{-\gamma} D_{t+1} \right] \tag{2.5}$$

$$(C_t^i)^{-\gamma} = \delta (1 + r_{t,1}) E_t^{\mathcal{P}} \left[(C_{t+1}^i)^{-\gamma} \right] \tag{2.6}$$

$$P_{t,n}^{call} = \delta E_t^{\mathcal{P}} \left[\left(\frac{C_{t+n}^i}{C_t^i} \right)^{-\gamma} max\{P_{t+n} - X_n, 0\} \right]. \tag{2.7}$$

These conditions are standard except the conditional expectations are taken with respect to the subjective probability measure $\mathcal{P}$.

2.3.1 Rational Expectations

This subsection reviews the results of this model under rational expectations. The stock price under rational expectations P_t^{RE}, is derived by solving the forward difference equation (2.5) and is given by

$$P_t^{RE} = \frac{\delta a^{1-\gamma} \rho_\varepsilon}{1 - \delta a^{1-\gamma} \rho_\varepsilon} D_t \tag{2.8}$$

where

$$\begin{aligned} \rho_\varepsilon &= E \left[(\varepsilon_{t+1}^c)^{-\gamma} \varepsilon_{t+1}^d \right] \\ &= e^{-\gamma(1-\gamma)\frac{s_c^2}{2}} e^{-\gamma \rho_{c,d} s_c s_d} \end{aligned}$$

and E denotes the statistical expectations under the objective probability measure. Equation (2.8) show that price to dividends (henceforth, PD) ratio is constant. Consequently, the ex-dividend stock returns volatility $std\left(\frac{P_{t+1}^{RE} - P_t^{RE}}{P_t^{RE}}\right)$ is approximately equal to the volatility of dividend growth which, as documented in many papers, is at odds with the real word data. [4]

The one-period risk free rate $r_{t,1}$is standard i.e. $1-$inverse of the expectations of 1 period stochastic discount factor $\left(\delta\frac{E_t\left[(C_{t+1})^{-\gamma}\right]}{C_t^{-\gamma}}\right)^{-1}$. Note that in this environment where shocks to consumption growth are distributed $i.i.N$, the yield curve is flat i.e. $1+r_{t,n} = \left(\delta^n\frac{E_t\left[(C_{t+n})^{-\gamma}\right]}{C_t^{-\gamma}}\right)^{-1/n} = \left(\delta\frac{E_t\left[(C_{t+1})^{-\gamma}\right]}{C_t^{-\gamma}}\right)^{-1} = \delta^{-1}a^{\gamma}e^{\frac{s_C^2\gamma(1-\gamma)}{2}}$.

Proposition 4. *Call option price under RE $P_{t,n}^{Call-RE}$ is given by the following*

$$P_{t,n}^{Call-RE} = P_t\Delta^n N\left(d_{1,t}\right) - \frac{X}{(1+r_{t,n})^n}N\left(d_{2,t}\right) \tag{2.9}$$

where $\Delta^n = E\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{D_{t+n}}{D_t}\right) = \left(\delta a^{(1-\gamma)}\rho_{\varepsilon}\right)^n$, $d_{1,t}$=$\frac{ln\ P_t - ln\ X_t + nlog\Delta + nr_t + n\frac{sd^2}{2}}{\sqrt{n}s_d}$ and $d_{2,t}$=$d_{1,t}-\sqrt{n}sd$ and $N(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{z^2}{2}}dz$.

Proof. Appendix A 1. □

The formula in (2.9) is identical to the BS formula. The price of a call option $P_{t,n}^{Call-,RE}$ is increasing in P_t, Δ sd, n and r_t and non-increasing in X_t. Under RE, option implied volatility equals the one period volatility of the dividends sd scaled by $\sqrt{n}$.

2.3.2 Beliefs

In this section, I discuss the exact structure of beliefs that the representative agent holds and its implication on equilibrium asset prices in the model. As in Chapter 1, the true stochastic processes governing $\{D_t\}_{t=0}^{\infty}$ and $\{C_t\}_{t=0}^{\infty}$ is part of the information set of the representative agent. Consequently, these beliefs imply same equilibrium risk free rate and the yield curve as in RE. The risk free rate is given by standard Euler equation in (2.10).

$$1 = \delta(1+r_t)E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right]. \tag{2.10}$$

Additionally, agent's subjective expectation of risk-adjusted stock price growth is defined as

[4]The proof of (2.8) is fairly straightforward, interested readers may refer to Adam, Marcet, and Nicolini (2016)for the proof.

$$\beta_t \equiv E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{P_{t+1}}{P_t}\right] \tag{2.11}$$

and the subjective expectation of risk-adjusted dividend growth

$$\beta_t^D \equiv E_t^{\mathcal{P}}\left[\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{D_{t+1}}{D_t}\right]. \tag{2.12}$$

after substituting (2.11) and (2.12) in the first order condition for stock price (2.5), the expression for stock price P_t under subjective beliefs can be written as follows:

$$P_t = \frac{\delta\beta_t^D}{1-\delta\beta_t}D_t \tag{2.13}$$

provided that β_t is bounded above by less than δ. Given the dividend growth is distributed $i.i.N$, β_t^D is constant. However, when agent are learning β_t, i.e. risk adjusted stock price growth, equilibrium stock prices can fluctuate much more than dividends. As can be seen in (2.13), a small change in β_t can lead to a large fluctuation in prices, especially, when β_t fluctuates around δ.

Constant-Gain Learning: Now, I specify the exact structure of $\mathcal{P}$. Internally rational representative agent follow a constant gain learning algorithm to update his expectation of risk adjusted capital gains. she believes that the process for log of risk ajdusted capital gains is composed of a persistent component and noise. In particular, agent beliefs are specified as following:

$$log\left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{P_t}{P_{t-1}}\right) = log\ \ b_t + log\ \ \varepsilon_t \tag{2.14}$$

$$log\ \ b_t = log\ \ b_{t-1} + log\ \ \xi_t$$

where $log\ b_t|\omega^t \sim N(ln\ m_t,\sigma_0^2)$, $log\ \varepsilon_t \sim iiN(\frac{-\sigma_\varepsilon^2}{2},\sigma_\varepsilon^2)$ and $log\ \xi_t \sim iiN(\frac{-\sigma_\xi^2}{2},\sigma_\xi^2)$ and $E_{t-1}[(log\ \varepsilon_t, ln\ \xi_t)] = 0$.

Further, I assume that agent's prior beliefs are centered at the RE value i.e $log\ b_0 \sim N(log\ \delta + (1-\gamma)log\ a + \gamma(1-\gamma)\frac{s_c^2}{2} - 2\gamma\rho_{c,d}s_d s_c, \sigma_0^2)$. Since prior is specified normally distributed, agents can optimally filter the persistent component of risk adjusted price growth using a Kalman filter. Agent's posterior beliefs at any time t are given by $log\ b_t \sim N(log\ m_t, \sigma_0^2)$ where $log\ m_t$ evolves recursively according to the following equation:

$$log\ m_t = log\ m_{t-1} - \frac{\sigma_\xi^2}{2} + g\left(log\ \left(\delta\left(\frac{C_t}{C_{t-1}}\right)^{-\gamma}\frac{P_t}{P_{t-1}}\right) + \frac{\sigma_\varepsilon^2+\sigma_\xi^2}{2} - log\ m_{t-1}\right), \tag{2.15}$$

where the constant Kalman gain parameter is given by $g = (\sigma_0^2)/(\sigma_0^2 + \sigma_\varepsilon^2)$ with steady state variance σ_0^2, given by

$$\sigma_0^2 = \frac{-\sigma_\xi^0 + \sqrt{(\sigma_\xi^2)^2 + 4\sigma_\xi^2\sigma_\varepsilon^2}}{2}$$

These beliefs constitute only a small deviation from RE beliefs whenever the gain parameter g is sufficiently small. Using (2.17) and given the distribution of $\left(\frac{C_{t+1}}{C_t}\right)$ as specified above, the posterior distribution of the log price ratio is given by the following

$$log \ \left(\frac{P_{t+1}}{P_t}\right)|t \sim N\left(\mu_{p,t,1}, \sigma_{p,t,1}\right) \tag{2.16}$$

where $\mu_{p,t,1}^{\mathcal{P}} = ln \ m_t + log(1+r_t) - \frac{\left(\sigma_\varepsilon^2+\sigma_\xi^2-\gamma^2 s_c^2\right)}{2}$ and $(\sigma_{p,t,1}^{\mathcal{P}})^2 = \sigma_\varepsilon^2+\sigma_0^2+\sigma_\xi^2+\gamma^2 s_c^2 - 2\sigma_{b_t,c_t}^1$
$\sigma 1_{b_t,c_t} = cov_t \left[logb_{t+1} + log\varepsilon_{t+1}, log\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right]$

Notice that the conditional variance $(\sigma_{p,t,1}^{\mathcal{P}})^2$ is stochastic and is influenced by agent expectations . Since, we are interested in multi-period options in this paper, I assume that agents extrapolate the one period learning rule to form beliefs about log risk adjusted price growth in the future periods. In particular,

$$log\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right) = \sum_{i=1}^{n}\left[log \ \ b_{t+i} + log \ \ \varepsilon_{t+i}\right] \tag{2.17}$$

$$log \ \ b_{t+n} = log \ \ b_t + \sum_{i=1}^{n} log \ \ \xi_{t+i}$$

. One could argue that this assumption is strong, that is, the long term beliefs should be anchored by long term average returns. However, given that the capital gains expectations are quite persistent as documented in survey data, these beliefs could be a reasonable description of near term beliefs for example up to 1 year. [5]Further, since the variance of beliefs increase at a rate faster than n, investors will have a very low confidence in their expectations for the distant future, it perhaps could be useful to model distant beliefs differently. However, this is outside the scope of this paper. Nevertheless for the facts that I analyze in this paper, the beliefs modeled in equation (2.17) are sufficient.[6] From (2.17) we get that

[5]Adam, Marcet and Beutel (2018) show that a simple auto regressive model can accurately describe the investors expectations as documented in survey data.

[6]In a world of geometric Brownian motion and rational expectations, the expected variance of returns increases at the rate of time.

$$E_t^{\mathcal{P}}\left[log\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right)\right]=nlog\ m_t-n\frac{\sigma_\varepsilon^2}{2}-n(n+1)\frac{\sigma_\xi^2}{4}$$

,

$$Var_t^{\mathcal{P}}\left[log\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right)\right]=n^2\sigma_0^2+n\sigma_\varepsilon^2+n(n+1)(2n+1)\frac{\sigma_\xi^2}{6}$$

and

$$ln\ \left(\frac{P_{t+n}}{P_t}\right)|t\sim N\left(\mu_{p,t,n},\sigma_{p,t,n}\right) \tag{2.18}$$

where $\mu_{p,t,n}^{\mathcal{P}}=n\mu_{p,t,1}^{\mathcal{P}}-\frac{n^2}{2}\sigma_\xi^2+$ and $(\sigma_{p,t,n}^{\mathcal{P}})^2=n(\sigma_{p,t,1}^{\mathcal{P}})^2+n(n-1)\sigma_0^2+n(n-1)(2n+5)\frac{\sigma_\xi^2}{6}$.[7]

Note that the stochastic variance of log price growth increases at a rate faster than n because the uncertainty around posterior beliefs parameterized σ_0^2 and σ_ξ^2 increases at the rate faster than n.

Valuation of Options: In this section, I provide analytical results that show how the learning mechanism assumed above can help rationalize the stylized facts related to option markets as documented in section 2. Firstly, I provide a closed form formula for a European call option price, $P_{t,n}^{Call}$ in the proposition 2 below which is derived by solving equation (2.7)

Proposition 5. *Under constant gains learning, price of n period European Call option is given by the following:*

$$P_{t,n}^{Call}=P_t\Delta_{CG}^nN\left(\tilde{d_{1,t}^n}\right)-\frac{X_t}{(1+r_{t,n})^n}N\left(\tilde{d_{2,t}^n}\right) \tag{2.19}$$

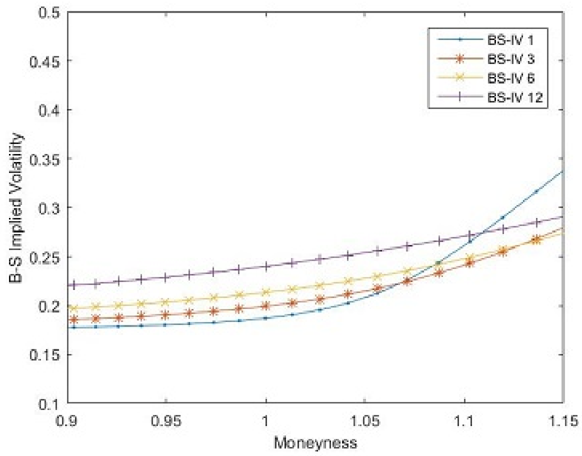
where $\Delta_{CG}^n=e^{n(logm_t+\frac{n}{2}\sigma_0^2+\frac{(n+1)(2n-5)}{12}\sigma_\xi^2)}$,$\tilde{d_{1,t}^n}=\frac{ln\ P_t-ln\ X_t+nlog(\Delta_{CG})+nlog(1+r_{t,n})+\frac{(\sigma_{p,t,n}^{\mathcal{P}})^2}{2}}{\sigma_{p,t,n}^{\mathcal{P}}}$ and $\tilde{d_{2,t}^n}=\tilde{d_1^n}-\sigma_{p,t,n}^{\mathcal{P}}$

Proof. See Appendix A.2 □

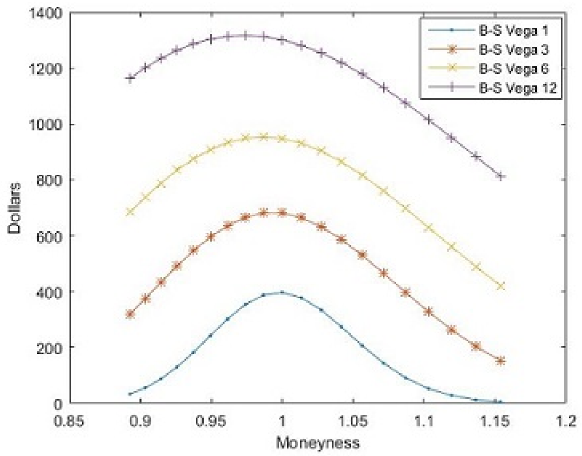
Note that the formula for price of n period European call option $P_{t,n}^{Call}$ under learning nests the RE formula in (2.9). $P_{t,n}^{Call}$ differs from $P_{t,t+n}^{Call-RE}$ due to two important factors. Firstly, the volatility parameter $\sigma_{p,t,n}^{\mathcal{P}}$ in the CG formula is a function of agent beliefs and

[7] Here I make an assumption that $cov_t\left[logb_{t+1}+log\varepsilon_{t+1},ln\left(\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\right)\right]=cov_t\left[logb_{t+n}+log\varepsilon_{t+n},log\left(\left(\frac{C_{t+n}}{C_{t+n-1}}\right)^{-\gamma}\right)\right]$.

Panel A



Panel B



Notes: Panel A shows the implied volatility (BS IV) curve for European call options with maturity 1 ,3, 6 and 12. BS IV is calculated using Black and Scholes (1973) option pricing formula, with option prices generated at moneyness [0.9,1.15] using the constant gains formula in (2.19) using the following parametrization:
$P = 1000, r_{t,n} = 0.005, m_t = -0.001, d = 4, \sigma^{\mathcal{P}}_{p,t,1} = 0.05, \sigma^2_{\xi} = 0.0002$ and $\sigma^2_0 = 0.0002$.
Panel B show BS vega for European call options with maturity 1 ,3, 6 and 12. at moneyness [0.9,1.15] using the following parametrization $P = 1000, r_{t,n} = 0.005, d = 4, \sigma_{p,t,1} = 0.05$.

Figure 2.4: Model BS IV curve and BS Vega

uncertainty around those beliefs while in the RE formula it is just n times the periodic volatility of dividends s_d. Secondly, due to risk adjusted expectations of capital gains which is $\Delta^n_{CG} \equiv E^{\mathcal{P}}_t \left[\delta^n \left(\frac{C_{t+n}}{C_t} \right)^{-\gamma} \frac{P_{t+n}}{P_t} \right]$ under subjective probability measure $\mathcal{P}$ and Δ^n under rational expectations. Importantly, higher uncertainty around belief captured by parameters σ^2_0, $\sigma^2_{\varepsilon}$and σ^2_{ξ} leads to higher subjective expected future variance and secondly, it also increases subjective expectation of future risk adjusted price growth due to adjustment for convexity.

In the literature, mainly two channels have been studied as potential explanations for the appearance of BS IV curve in the data. One being the leverage effect which is the apparent negative correlation between stock returns and its variance and secondly, positive probability of jumps or discontinuity in stock returns. These two factors contribute to fatter tails and negative skewness in distribution of returns compared to log normal assumption of BS model. Consequently, put and call options that are written on strike prices on the left tail of the price distribution could be expensive relative to BS model price which in turn would imply an upward sloping BS IV curve as observed in the data. [8] However, it is unclear in these models why the curve should become flatter as the maturity increases which is our Stylized Fact 1. These models do not imply that the distribution of returns approaches normal as horizon increases. Bakshi et al. (1997) find that implied parameters of the underlying returns distribution estimated from market options prices for most classes of reduced form option pricing model differ significantly by maturity which makes these models inconsistent. In other words, these models imply that investor assume different stochastic dynamics for the same underlying instrument at same point in time when valuing options with different maturity. This inconsistency is over and above the fact that the estimated parameters cannot explain observed underlying's return dynamics. To sum up, matching both the cross-section and term structure of option prices with unique parametrization is extremely challenging for most RE option pricing models.

Here we have a subtle but important difference from the above mentioned approaches. Firstly, the expected variance under the subjective probability measure $\mathcal{P}$ (henceforth, $\mathcal{P}$ expected future variance) given in (2.18), in general, is not equal to statistical expectations of future variance (henceforth, $\mathcal{S}$ expected future variance) . $\mathcal{S}$ expected future variance can potentially be estimated from the time series of equilibrium stock returns in the model. Secondly, the BS IV is derived from equilibrium option prices which are also influenced by the difference between Δ_{CG} and Δ. This is another source of wedge between BS IV and $\mathcal{S}$ expected future variance. The emphasis here is on the subjective beliefs rather than the tails of the statistical distribution of returns.

Analysis of how the $\mathcal{P}$ expected future variance and $\mathcal{S}$ expected future variance differ

[8]For put options this implies OTM options are expensive and for call options ITM options are expensive relative to BS price.

is left for future research. Here, I take a simplified approach by treating $\mathcal{P}$ expected future variance $\approx \mathcal{S}$ expected future variance and focus only on the influence of $\Delta_{CG} - \Delta$ on the option price and consequently the BS IV curve. To calculate BS IV, I first calculate the CG model price $\hat{P^{C}_{CG,t,n}}$ for a European call option using equation (2.19) with a maturity n with particular value of parameters say, $\hat{P}$,$\hat{X}$, $\hat{r_{t,n}}$,$\hat{d}$, $\hat{m_t}$,$\hat{\sigma_\xi}$,$\hat{\sigma_\varepsilon}$ and $\hat{\sigma^{\mathcal{P}}_{p,t,n}}$. Then the BS IV $\hat{IV_{BS,t,n}}$ implied by that particular option price is the volatility parameter that makes BS model price calculated using equation (2.9) equal CG price i.e. $\Phi^{BS}(\hat{P},\hat{X}, \hat{r_{t,n}},\hat{d}, n, \hat{IV_{BS,t,n}}) = \hat{P^{C}_{CG,t,n}}$ where $\Phi^{BS}()$ is the expression on the right side in equation (2.9).

It is common practice in the literature to study the BS IV curve with reference to *at the money* implied volatility, disregarding, how *at the money* implied volatility differ from the S expected future volatility. I follow the same approach but it is also important to note that *at the money* BS IV in the learning model, in general, is different from $\mathcal{P}$ expected future variance because $\Delta_{CG} - \Delta \neq 0$ and not due to so called "change of probability measure".

Panel A in Figure 2.4 presents the BS IV curves for maturities of 1, 3, 6 and 12 periods implied by this model using parameter values of $P = 1000, r_{t,n} = 0.005, m_t = -0.001, d = 4, \sigma^{\mathcal{P}}_{p,t,1} = 0.05, \sigma^2_\xi = 0.0002$ and $\sigma^2_0 = 0.0002$. It can be seen that the BS IV curve becomes flatter as maturity increases replicating our **stylized fact 1**.

The reason for the appearance of BS IV curve in this model is more mathematical than intuitive. To understand the shape of curve, we need to consider prices of a European call option calculated using the BS formula and CG formula with same volatility $\sigma^{\tilde{\mathcal{P}}}_{p,t,n}$. If, $\Delta^n_{CG} - \Delta^n = 0$, then BS model price and CG model price at $\sigma^{\tilde{\mathcal{P}}}_{p,t,n}$ are identical therefore BS IV calculated from CG option price $= \sigma^{\tilde{\mathcal{P}}}_{p,t,n}$ and when $\Delta^n_{CG} - \Delta^n > (<)0 \implies$ BS-IV $> (<)\sigma^{\tilde{\mathcal{P}}}_{p,t,n}$. The magnitude by which BS IV differ from $\sigma^{\tilde{\mathcal{P}}}_{p,t,n}$ depends on the magnitude of price difference between BS model price and CG model price at $\sigma^{\tilde{\mathcal{P}}}_{p,t,n}$ and the BS vega. The BS vega which is the sensitivity of option price to a unit change in volatility according to BS formula follows Gaussian type shape i.e. it approaches 0 as moneyness increases. [9] This is because BS vega is a function of the probability of price

[9]The BS Vega, $V_t = \frac{\partial P^{Call}}{\partial \sigma} = P_t e^{-\Delta}\sqrt{n}N'(d_{1,t})$ where σ is the volatility of the underlying asset returns and $N'(d_{1,t}) = \frac{1}{\sqrt{2\pi}}e^{-(d_{1,t})^2/2}$.

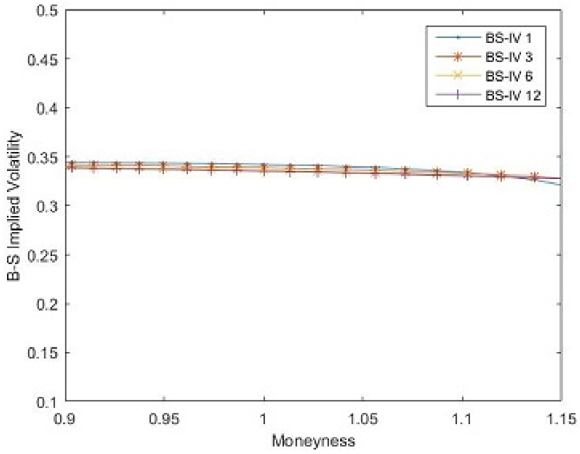


Figure 2.5: Model BS IV curve

Notes: The implied volatility (BS IV) curve for European call options with maturity 1, 3, 6 and 12. BS IV is calculated using Black and Scholes (1973) option pricing formula, with option prices generated at moneyness [0.9,1.15] using the constant gains formula in (2.19) using the following parametrization: $P = 1000, r_{t,n} = 0.005, m_t = -0.005, d = 4, \sigma^{\mathcal{P}}_{p,t,1} = 0.1, \sigma^2_{\xi} = 0.0002$ and $\sigma^2_0 = 0.0002$.

of underlying at maturity P_{t+n} is greater than X and as option is deeper in the money this probability approaches 1 independent of the level of volatility. When vega is low then for a given $\Delta^n_{CG} - \Delta^n$, increase in BS IV needed to bring BS model price equal to CG price is much higher compared to when vega is high. But the difference between CG price and BS price is weakly increasing as moneyness increase. Therefore, BS IV that brings the BS price equal to CG price increases as moneyness increases. The curve becomes flatter as the maturity increases because vega increases at the rate faster than $\sqrt{n}$ but the difference between CG price and BS price increases at a rate slower than $\sqrt{n}$.

In this model Δ^n_{CG} fluctuates with time with change in equilibrium price , thereby allowing the model to reproduce different BS IV surface which is the **stylized fact 2**. Finally, the vertical difference between BS IV curves is also a function of difference between $\Delta^n_{CG}$and Δ^n, if investor are less optimistic and the uncertainty around beliefs is not large than all the BS IV curves will be flatter, close to each other as it happened during the great financial crisis in 2008 which is shown in Figure 2.3. Figure 2.5 plots the BS IV curve with model parameters $P = 1000, r_{t,n} = 0.005, m_t = -0.005, d = 4, \sigma^{\mathcal{P}}_{p,t,1} = 0.10, \sigma^2_{\xi} = 0.0002$ and $\sigma^2_0 = 0.0002$. It can be seen that all IV curves are flat lines and are stacked close to each other because m_t= -0.005 . If higher volatility is associated with lower expectations of future capital gains then the negative relationship between the slope of smile and implied volatility can also be explained by this model.

2.4 Conclusion

The discrete time general equilibrium model studied in Chapter 1 is extended by including multi-period options. An internally rational representative agent is learning about log of future risk adjusted price growth from realized price growth. The persistent component of the learning rule follows a random walk model implying that expectations about capital gains at distant horizons are closely related to expectations for the next period. In this setup, a closed form expression for multi-period option price is derived. I simulate option prices from the model using reasonable values of parameters and show that model can replicate many feature of BS IV surface. The model correctly predicts that BS IV curve gets flatter as maturity increases. The model is also able to generate BS IV surface that fluctuates with changing subjective capital gains expectations. The result further demonstrates that a small departure from rational expectations should be a norm rather than exception in asset pricing models.

Appendix A

A 1

Rational Expectations

Proof. [Proposition 1] The price $P^{Call}_{t,n}$ of a European call option in period t set to expire in period t+n satisfy the following equation:

$$\begin{aligned} P^{Call-RE}_{t,n} &= \delta^n E_t\left[\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma} max(P_{t+n} - X,\ 0)\right] \\ &= \delta^n E_t\left[\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma} (P_{t+n} - X)|P_{t+n} > X\right] \end{aligned}$$

To simplify notation, I define a variable $g_{t+n} \equiv log\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\right)$. Since $\left(\frac{C_{t+n}}{C_t}\right)$ follows a lognormal distribution, $E_t(g_{t+n}) = -nlog\ \delta - n\gamma log\ a - n\gamma\frac{s_c^2}{2}$ and $var_t(g_{t+n}) = n\gamma^2 s_c^2$.

Additionaly, I define $p_{t+n} \equiv log\left(\frac{P_{t+n}}{P_t}\right)$. Given expression for P_t^{RE} in equation (2.8), we get that $log\left(\frac{P_{t+n}}{P_t}\right) = log\left(\frac{D_{t+n}}{D_t}\right)$. Therefore,$E_t(p_{t+n}) = nlog\ a - n\frac{s_d^2}{2}$and $var_t(p_{t+n}) = ns_d^2$. Furthermore $cov_t(g_{t+n}, p_{t+n}) = -n\gamma\rho_{c,d}s_c s_d$. Consequently,

$$P_{t,n}^{Call-RE} = P_t E_t\left[e^{g_{t+n}}e^{p_{t+n}}|p_{t+n} > ln\left[\frac{X}{P_t}\right]\right] - XE_t\left[e^{g_{t+n}}|p_{t+n} > ln\left[\frac{X}{P_t}\right]\right]. \quad (2.20)$$

The expression,

$$P_t E_t\left[e^{g_{t+n}}e^{p_{t+n}}|p_{t+n} > ln\left[\frac{X}{P_t}\right]\right] = P_t\int_{-\infty}^{\infty}\int_{k_t}^{\infty} e^{g_{t+n}}e^{p_{t+n}}f(g_{t+n}, p_{t+n})dg_{t+n}dp_{t+n}$$

. To solve the above expression, we can make use of the result i.e. if $x \sim N(\mu_x, \sigma_x^2)$ and $y \sim N(\mu_y, \sigma_y^2)$ are joint normally distributed with correlation coeffient κ, then

$$\int_{-\infty}^{\infty}\int_{a}^{\infty} e^x e^y f(x,y)dxdy = (e^{\mu_x+\mu_y+\frac{1}{2}(\sigma_x^2+\sigma_y^2+2\kappa\sigma_x\sigma_y)})N(\frac{-a+\mu_x}{\sigma_x} + \kappa\sigma_y + \sigma_x)$$

for $a \in (-\infty, \infty)$. Therefore,

$$P_t E_t\left[e^{g_{t+n}}e^{p_{t+n}}|p_{t+n} > ln\left[\frac{X}{P_t}\right]\right] = P_t E_t\left(\delta\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right)N\left(\frac{-lnX+lnP_t+E_t[p_{t+n}]+cov_t(g_{t+n},p_{t+n})+var_t(p_{t-}}{\sqrt{nvar_t(p_{t+n})}}\right.$$

$$P_t E_t\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right)$$

$$N\left(\frac{-lnX+lnP_t+E_t[p_{t+n}]+E_t[g_{t+n}]+\frac{var_t(g_{t+n})}{2}+\frac{var_t(p_{t+n})}{2}+cov_t(g_{t+n},p_{t+n})-E_t[g_{t+n}]-\frac{var_t(g_{t+n})}{2}+\frac{var_t(p_{t+n})}{2}}{\sqrt{nvar_t(p_{t+n})}}\right).$$

Notice that $log(1+r_{t+n}) = -E_t[g_{t+n}] - \frac{var_t(g_{t+n})}{2}$and $log\ E_t\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right) = E_t[p_{t+n}] + E_t[g_{t+n}] + \frac{var_t(g_{t+n})}{2} + \frac{var_t(p_{t+n})}{2} + cov_t(g_{t+n}, p_{t+n})$. Furthermore, since the price-dividend ratio is constant under RE therefore $E_t\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{P_{t+n}}{P_t}\right) = E_t\left(\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{D_{t+n}}{D_t}\right) = E_t\left(\prod_{i=1}^{n}\delta\left(\frac{C_{t+i}}{C_{t+i-1}}\right)^{-\gamma}\frac{D_{t+i}}{D_{t+i-1}}\right)$. This can be compactly written as Δ^n where $\Delta = E_t\left(\delta\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\frac{D_{t+1}}{D_t}\right) =$ $\delta a^{(1-\gamma)}\rho_\varepsilon$ because shocks to consumption growth and dividend growth are *i.i.d.* By subsitution for the above expressions we get that,

$$P_t E_t\left[e^{g_{t+n}}e^{p_{t+n}}|p_{t+n} > ln\left[\frac{X}{P_t}\right]\right] = P_t\Delta^n N\left(\frac{-lnX + lnP_t + nln\ \Delta + nln(1+r_{t+n})}{\sqrt{n}sd} + \frac{\sqrt{n}sd}{2}\right)$$

. To solve the other expression we can make use of another result i.e if $x \sim N(\mu_x, \sigma_x^2)$ and $y \sim N(\mu_y, \sigma_y^2)$ are joint normally distributed with correlation coeffient κ,

$$\int_{-\infty}^{\infty}\int_{a}^{\infty} e^y f(x,y)dxdy = (e^{\mu_y+\frac{1}{2}\sigma_y^2})N(\frac{-a+\mu_x}{\sigma_x}+\kappa\sigma_y)$$

.Consequently,

$$XE_t\left[e^{g_{t+n}}|p_{t+n} > ln\left[\frac{X}{P_t}\right]\right] = X(1 + r_{t+n})^{-n}N\left(\frac{-lnX + lnP_t + nln\ \Delta + nln(1+r_{t+n})}{\sqrt{n}sd} - \frac{\sqrt{n}sd}{2}\right)$$

. Therefore under rational expectations we get exactly the Black-Scholes formula i.e.

$$P_{t,n}^{Call-RE} = P_t\Delta^n N(d_{t+n}) + X(1+r_{t+n})^{-n}N(d_{t+n} - \sqrt{n}sd) \tag{2.21}$$

where $d_{t+n} = \frac{-lnX+lnP_t+nln\Delta+nln(1+r_{t+n})}{\sqrt{n}sd} + \frac{\sqrt{n}sd}{2}$. □

A.2

Proof. The expression for price $P_{t,n}^{Call}$ of European call at date t with maturity at $t+n$ under constant gains learning is derived from the following first order condition:

$$P_{t,n}^{Call} = \delta^n E_t^{\mathcal{P}}\left[\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}(P_{t+n} - X)|P_{t+n} > X\right]$$

, where $\mathcal{P}$is the subjective probability measure. Further, we can write

$$P_{t,n}^{Call} = P_t E_t^{\mathcal{P}}\left[\delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\left(\frac{P_{t+n}}{P_t}\right) - \delta^n\left(\frac{C_{t+n}}{C_t}\right)^{-\gamma}\frac{X}{P_t}|ln\left[\frac{P_{t+n}}{P_t}\right] > ln\left[\frac{X}{P_t}\right]\right] \tag{2.22}$$

In the above expression I replace $z_{t+n} \equiv g_{t+n} + p_{t+n}$.

$$P_{t,n}^{Call} = P_t\int_{-\infty}^{\infty}\int_{ln\frac{X}{P_t}}^{\infty} e^{z_{t+n}}f(z_{t+n}, p_{t+n})dp_{t+n}dz_{t+n} - X\int_{-\infty}^{\infty}\int_{ln\frac{X}{P_t}}^{\infty} e^{ln\ g_{t+n}}f(g_{t+n}, p_{t+n})dp_{t+n}dg_{t+n}$$

Given the learning algorithm described in 2.17 z_{t+n} is equal to $n\ ln\ b_{t+1} + \sum_{j=1}^{n} ln\ \varepsilon_{t+j} + \sum_{k=1}^{n}\sum_{j=1}^{k} ln\ \xi_{t+j}$. Further the $log\ E_t^{\mathcal{P}}[e^{z_{t+n}}] = n(ln\ m_t + \frac{n}{2}\sigma_0^2 + \frac{(n+1)(2n-5)}{12}\sigma_\xi^2)$.Therefore

$$P_{t,,n}^{Call} = P_t e^{E_t^{\mathcal{P}}[z_{t+n}]+\frac{1}{2}var_t^{\mathcal{P}}[z_{t+n}]} N\left(\frac{-log\ X + log\ P_t + \mu_{p,t,n} + Cov_t(p_{t+n}, z_{t+n})}{\sigma_{p,t,n}}\right)$$
$$- X_t e^{E_t^{\mathcal{P}}[g_{t+n}]+\frac{1}{2}var_t^{\mathcal{P}}[g_{t+n}]} N\left(\frac{-ln\ X + log\ P_t + \mu_{p,t,n} + Cov_t(p_{t+n}, g_{t+n})}{\sigma_{p,t,n}}\right) \quad (2.23)$$

.Notice that $\mu_{p,t,n}+Cov_t(p_{t+n}, z_{t+n}) = \mu_{p,t,n}+\frac{\sigma_{p,t,n}^2}{2}+E_t^{\mathcal{P}}[g_{t+n}]+\frac{1}{2}var_t^{\mathcal{P}}[g_{t+n}]+Cov_t(p_{t+n}, p_{t+n}+g_{t+n})-\frac{\sigma_{p,t,n}^2}{2}-E_t^{\mathcal{P}}[g_{t+n}]-\frac{1}{2}var_t^{\mathcal{P}}[g_{t+n}] = log\ E_t^{\mathcal{P}}[e^{z_{t+n}}]+\frac{\sigma_{p,t,n}^2}{2}+nlog(1+r_{t+n})$. Consequently,

$$P_{t,n}^{Call} = P_t e^{n(logm_t+\frac{n}{2}\sigma_0^2+\frac{(n+1)(2n-5)}{12}\sigma_\xi^2)} N\left(\frac{-ln\ X+ln\ P_t+n(logm_t+\frac{n}{2}\sigma_0^2+\frac{(n+1)(2n-5)}{12}\sigma_\xi^2)+nlog(1+r_{t,n})}{\sigma_{p_{t+n}}} + \frac{\sigma_{p,t,n}}{2}\right)$$
$$-X_t(1+r_{t,n})^{-n} N\left(\frac{-ln\ X+ln\ P_t+n(logm_t+\frac{n}{2}\sigma_0^2+\frac{(n+1)(2n-5)}{12}\sigma_\xi^2)+nlog(1+r_{t,n})}{\sigma_{p,t,n}} - \frac{\sigma_{p,t,n}}{2}\right).$$

□

Chapter 3

Option Implied Expectations: An alternative to survey expectations of stock return

3.1 Introduction

Expectations of future stock returns as documented in various surveys of investors and investment professionals has attracted great attention from researchers recently. It has been shown in many studies (see for example: Adam, Marcet, and Beutel (2017), Adam, Matveev, and Nagel (2021), Greenwood and Shleifer (2014)) that survey data on expectations of stock returns strongly rejects the rational expectations (RE) hypothesis which underpins much of the asset pricing literature. In particular, it has been documented that investors expectations of stock returns are pro-cyclical as reported in surveys as against a counter-cyclical temporal pattern of realized returns. On the other hand, proponents of rational expectations argue that survey data is extremely noisy and respondents of surveys perhaps report risk adjusted expectations (see for example, (Cochrane, 2011, 2017)) rather than only stock price growth. That said, the quality of survey data remains to be a contentious issue preventing the researchers from settling the debate on either side.

In this chapter, building on the analysis of the previous two chapters, I estimate risk adjusted return expectations from market option prices which I call the *option implied ex-*

pectations (henceforth, IE). IE supports the conclusion that expectations of stock returns of the market participants reject the RE hypothesis. Another way to look at IE is that it represents the magnitude of departure from RE. This is because one of the core results in option pricing under RE is that option prices are independent of expected returns on the underlying. Therefore, any deviation of IE from zero is a departure from RE. In fact, IE different from zero rejects the core equation in most asset pricing models i.e conditional expectations of risk adjusted returns equals zero.

Additionally, estimated IE has some useful properties. Firstly, IE is substantially positive most of the times only turning negative during crisis periods i.e. it is pro-cyclical in line with survey expectations. Secondly, IE is strongly negatively correlated with implied volatility (IV) estimated using the learning model and other types of conditional volatility such as the VIX index. Thereby, it confirms the leverage effect. Furthermore, IE is strongly positive correlated with survey based expectations. Additionally, in a regression of IE on monthly price-dividend (PD) ratio, the coefficient on PD ratio is significantly positive. Taken together, IE offers an alternative to survey measures of expectations with an advantage that the data is available on a real-time basis as well as free from measurement errors.

The estimation procedure involves estimating the structural parameters of the multi-period call option pricing model derived in Chapter 2. In particular, IE, IV and two parameters related to uncertainty of beliefs are selected that minimize the daily sum of squared pricing error of the model. Importantly, the option pricing model with learning has an in-sample average RMSE lower by almost 60 percent compared to Black and Scholes (1973) model when estimated on data of all traded call options with maturity from 14 days to 365 days. The IV estimated is considerably lower on average than BS IV and the VIX index which is a departure from estimates from alternative models where IV estimates are relatively close to each other. Furthermore, VIX premium calculated as monthly average of VIX(squared)-IV(squared) positively predict stock market excess returns upto a horizon of 12 months. The VIX premium estimated here outperform the variance risk premium that is proposed by Bollerslev, Tauchen, and Zhou (2009) in predictability regressions. I also find that IV negatively predict future stock excess returns in a regression when the VIX premium is also used as a regressor suggesting that standard risk-return regressions such as GARCH (for example, Glosten, Jagannathan, and Runkle (1993)) suffer from the omitted variable bias.

To my knowledge, this is the first study estimating expected returns from market data on option prices. Other studies in the literature try to estimate the risk neutral distribution of the expected stock returns but in those studies IE is equal to zero by construction (see for example, Jackwerth and Rubinstein (1996)).

The remainder of the paper is structured as follows. Section 2 describes the data and stylized facts. Section 3 describes the estimation procedure and model estimation. Section 4 contains discussion on VIX premium and evidence of the predictability of stock market excess returns on VIX premium. Section 5, discussed IE and the survey data. Finally, Section 6 concludes.

3.2 Data

3.2.1 Call options

To estimate the model parameters I use daily data on call options on the S&P 500 index futures for the period from Jan 2004 through Dec 2017 which is obtained from the Chicago Board Options Exchange. The data set includes bid-price, ask-price quotes at 1545 for all outstanding call options and its underlying on each trading day during the sample period. The data set also includes strike price and date of expiry. Options contracts with fewer than 14 days or greater than 365 days to expiry or with open interest less than 50 or trading volume less than 10 or closing mid price less 0.5 are omitted from the sample. To get the present value of the S&P 500 index futures price, I adjust the futures price by corresponding constant maturity zero coupon treasury yield obtained from Liu and Wu (2021).Table 3.1 contains the summary statistics of this data set. Call options with days to expiry >180 days and <366 days are classified as Long term maturity (LTM), options with days to expiry >90 days and <180 days are classified as medium term maturity (MTM), options with days to expiry >45 days and <90 days are classified as medium term maturity (MTM) and options with days to expiry >14 and <45 are classified as

	LTM	MTM	SMTM	STM
Average Price	82.40	48.45	27.26	19.39
Average BS IV	0.18	0.16	0.14	0.13
n	21,616	32,650	96,300	189,969

Notes: STM is short term maturity, SMTM is short to medium term maturity, MTM is medium term maturity and LTM is long term maturity. Sample period is Jan 2004 to Dec 2017

Table 3.1: Summary Statistics: S&P 500 Index Call Option Data

short term maturity (STM). As expected most number of options are STM with total of 189, 969 options. The implied volatility (henceforth, BS IV) calculated using Black and Scholes (1973) formula is highest for the LTM options on average and decreases with maturity.

3.2.2 Survey Expectations

I use data from various surveys which ask for point estimates of expected returns on the stock market. The first data set is from the DUKE CFO Global Business Outlook, the survey is conducted quarterly by Duke University's Fuqua School of Business and CFO Magazine. The survey include responses for various economic and business conditions including expected return on the S&P500 index for 1 year ahead and 10 year ahead. Each survey participant is asked to provide his/her minimum estimate, maximum estimate and expectations of the S&P 500 returns. I use the average of minimum estimate, expectations and maximum estimate of 1 year ahead returns from this survey. This range of estimates provided by participants are potentially useful in understanding the expected returns distribution of each participant. In this paper, I use the three responses to study which responses better reflect option implied returns expectations.

The second data-set is from the Livingston survey, conducted by the Federal Reserve Bank of Philadelphia. The respondents include economists from industry, government, banking and academia. The survey is sent to the participants at the end of April and October every year and is published in June and December. Respondents are asked to forecast level of the S&P 500 index for the zero month (June or December), 6 months ahead, 1 year and 10 year ahead. Expectations of returns are calculated by dividing the expected S&P 500 index reported in the survey by the value of the index at the end of month in which survey was sent. For this paper, I only consider responses upto 1 year

	Period	Frequency	Forecast Horizon	Mean	SD	Max	Min
CFO -Low	03/03 to 10/17	Quarterly	1y	-2.2	1.9	0.8	-8.6
CFO- Expected				5.3	1.3	7.7	2.2
CFO - High				10.4	1.5	13.7	7.1
Livingston	03/03 to 06/17	half-yearly	2m	1.1	3.6	6.8	-10.3
			8m	6.2	9.9	52.8	-0.7
			1y 2m	8.5	4.6	23.2	3.2
UBS (NX)	03/03 to 10/17	Quarterly	1y	0.1	0.0	0.1	0.0

Notes: The table shows summary statistics for the survey data used in this paper. The last four colums on the right contains mean, standard deviation (SD), mimimum (Min) and maximum (Max) of the time series of mean responses in the survey

Table 3.2: Survey Data

from this survey.

Finally, I also use a data-set constructed by Nagel and Xu (2022) that uses additional surveys data to extend the UBS/Gallup survey forward and backward in time. Table 3.2 contains mean, standard deviation (SD), minimum (Min) and maximum (Max) of the time series of the mean responses of various surveys considered here.

3.3 Structural Parameters Estimation

In this section, I discuss the estimation procedure and results from the estimation of the model. The formula for a call option $P_{t,n}^{Call}$ with maturity of n periods at current level of underlying price P_t, n period risk free interest rate $r_{t,n}$, with subjective expectations of risk adjusted one period price growth m_t, two parameters reflecting uncertainty of beliefs σ_0^2, σ_ξ^2 and subjective expectations of future variance $\sigma_{p,t,n}^{2\mathcal{P}}$ is the following:

$$\begin{aligned} P_{t,n}^{Call} &\equiv \Phi(P_t, X, r_{t,n}, m_t, \sigma_{p,t,n}^{\mathcal{P}}, \sigma_0, \sigma_\xi, n) \\ &= P_t \Delta_{CG}^n N\left(d_{1,t}^{\tilde{n}}\right) - \frac{X_t}{(1+r_{t,n})^n} N\left(d_{2,t}^{\tilde{n}}\right) \end{aligned} \tag{3.1}$$

where $\Delta_{CG}^n = e^{n(logm_t + \frac{n}{2}\sigma_0^2 + \frac{(n+1)(2n-5)}{12}\sigma_\xi^2)}$,$d_{1,t}^{\tilde{n}}$=$\frac{ln\ P_t - ln\ X_t + nlog(\Delta_{CG}) + nlog(1+r_{t,n}) + \frac{(\sigma_{p,t,n}^{\mathcal{P}})^2}{2}}{\sigma_{p,t,n}^{\mathcal{P}}}$ and $d_{2,t}^{\tilde{n}}$=$d_1^{\tilde{n}} - \sigma_{p,t,n}^{\mathcal{P}}$. [1]

For a pricing a call option from (3.1) we have $X$and n which are specified in the option contract while P_t, $r_{t,n}$ can be taken from published market data. However, there

[1]This proof of this formula is available in Appendix A.2 Chapter 2.

are four unobservable structural parameters namely $\{m_t, \sigma^{\mathcal{P}}_{p,t,n}, \sigma_0, \sigma_\xi\}$ that need to be estimated. I use the standard approach in the literature that minimizes the sum of squared pricing errors. In particular, I take all traded call options for each day. Suppose the number of qualifying option contracts available on day t is N_t where $N_t > 4$. For each $i = 1, 2, ...N_t$, we have its maturity n_i, strike price X_i and observed call option market price $P^{Call-market}_{t,n_i}(n_i, X_i)$. Let $P^{Call}_{t,n_i}(P_t, X_i, r_{t,n_i}, n_i, \theta)$ be the model price for the same option i calculated using (3.1) with P_t and r_{t,n_i} at date t taken from market data and for some parameter vector $\theta_t \equiv \{m_t, \sigma^{\mathcal{P}}_{p,t,n}, \sigma_0, \sigma_\xi\}$. Lets define a pricing error $\epsilon_{i,t}$ as difference between $P^{Call-market}_{t,n_i}(n_i, X_i)$ and $P^{Call}_{t,n_i}(P_t, X_i, r_{t,n_i}, n_i, \theta)$ for some θ i.e.

$$\epsilon_{i,t}(\theta) \equiv P^{Call-market}_{t,n_i}(n_i, X_i) - P^{Call}_{t,n_i}(P_t, X_i, r_{t,n_i}, n_i, \theta) \tag{3.2}$$

The estimated structural parameters at date t,

$$\hat{\theta}_t = \underset{\theta}{argmin} \sum_{i=1}^{N_t} \epsilon^2_{i,t}(\theta) \tag{3.3}$$

. The objective function is the sum of squared pricing errors (SSE). The estimation procedure minimizes this objective function and may put more weight to matching highly priced options which are ITM options and long maturity options and less weight to matching OTM options and short maturity options. This approach is not free from its shortcomings but has been the standard procedure followed in many studies (see for example, Bakshi, Cao, and Chen (1997)).

Table 3.3 contain summary statistics for the estimated parameters and average root mean square error of the models (RMSE). First thing to note in the table is that m_t has a positive mean. These estimates of m_t rejects the standard optimality condition in almost all asset pricing models.[2]

An implication of this could be that financially constrained investors undertake desirable exposure to the stock index through its options at a fraction of investment than buying index itself. It is well known that options positions are akin to highly leveraged bet on the underlying, hence, it appears that speculative investors with subjective beliefs

[2]This core asset pricing equation for an asset that does not pay dividends is $E^{\mathcal{Z}}(\frac{M_{t+1}P_{t+1}}{P_t}) = 1$where $\mathcal{Z}$ is any subjective or objective probability measure, $M_{t+1}$is the stochastic discount factor in period $t+1$ and P_{t+s} is the price of an asset in period $t+s$. Adam and Marcet (2011) show that $E^{\mathcal{Z}}(\frac{M_{t+1}P_{t+1}}{P_t}) \neq 1$ is possible when investors are financial constrained and internally rational.

parameters	mean	max	min	sd	skew	kurt	n
mt	0.23	0.74	-5.83	0.32	-7.43	91.72	3326
RMSE	1.79	21.83	0.28	1.39	3.98	33.19	3326
RMSE$_{BS}$	4.26	36.97	1.21	1.77	3.20	42.30	3326
σ_0	0.00	0.04	0.00	0.00	3.02	34.94	3326
σ_ξ	0.00	0.01	0.00	0.00	10.53	158.94	3326
IV-L	14.40	83.70	6.98	8.59	3.21	16.85	3326
IV$_{BS}$	16.82	59.13	0.00	6.76	2.33	10.52	3326

Table 3.3: Implied Parameters

influence pricing in the index options market. Importantly, m_t is the key parameter driving improvement in model's pricing ability over the BS model. Root mean squared error (RMSE) is substantially reduced from 4.26 for BS to 1.79 for the learning model. The maximum monthly risk adjusted expected return is 0.74% and minimum is -5.4% which is estimated near the depth of the Great Financial Crisis. The other parameters σ_0 and σ_ξ are estimated to be extremely small and have insignificant pricing implications on the whole. The average annualized implied volatility $\sigma^{\bar{\hat{\mathcal{P}}}}_{p,t,1}$ (henceforth, IV-L) is 14.40 percent which is much lower than average BS IV of 16.85 percent. This is interesting because, typically, implied volatility estimated from most reduced form models is much closer to each other (see for example Bakshi, Cao, and Chen (1997, Pg 2018)) [3]

Figure 3.1 plots the evolution of monthly average of option implied expectations (IE) and option implied volatility (IV-L) as estimated from this model. In Panel A, we see that average IE is mostly positive during the sample period only becoming significantly negative during the depths of the Great Financial Crisis and Euro-Zone Debt crisis. Furthermore, it appears that IE and IV are negatively correlated i.e when investors are optimistic the estimated IV is low. These estimates point to another type of leverage effect which is negative correlation between risk adjusted expected returns and expected volatility. It is worth noting that nothing in the estimation procedure target this negative relationship between IE and IV. [4]This can be seen in Table 3.4 where monthly average of expectations m_t is highly negative correlated with monthly average of all three types of implied volatility the VIX, IV BS and IV-L. Notice that the estimated IV-L is lower but

[3]Daily BS implied volatility is estimated using the standard procedure of minimizing the SSE for all call options on each day.

[4]The so called leverage effect is a negative contemporaneous correlation between returns and volatility. This is an assumption in many asset pricing models and option pricing models, see for example, Bansal and Yaron (2004) and Heston (1992).

	m_t	vol	vol$_{BS}$	VIX
m_t	1.00	-0.79	-0.66	-0.76
vol	-0.79	1.00	0.98	0.99
vol$_{BS}$	-0.66	0.98	1.00	0.97
VIX	-0.76	0.99	0.97	1.00

Table 3.4: Correlations

highly positively correlated with IV BS and the VIX.

3.4 VIX Premium and Forecasting Stock Market Returns

In the last section we saw that average IV-L is lower than average VIX index. Recall that IV-L is an estimate of subjective conditional volatility according to the learning model. Therefore, the difference between VIX and IV-L is an estimate of VIX premium. As discussed in Chapter 1, VIX premium estimated using different methods has an interesting property that it predict index returns for a short horizon (see for example, Bekaert and Hoerova (2014)). Therefore, one way to show that learning model is indeed a consistent option pricing model is by testing whether VIX premium estimated using learning model (henceforth, VP-L) has comparable predictability of stock returns with respect to alternative estimates of VIX premium. To test this, I run the standard monthly predictability regressions of future excess returns on S&P 500 on VP-L which is calculated as the monthly average of difference between squared VIX and squared IV-L i.e.

$$\text{VP-L}_t = \frac{1}{22} \sum_{i=t-22}^{t} \text{VIX}_i^2 - \text{IV-L}_i^2 \tag{3.4}$$

where 22 stands for the number of working days in a calendar month. Correlation of VP-L is 0.72 with VIX^2 and 0.579 with IV-L^2. An alternative estimate of VIX premium considered is one proposed by Bollersev et al. (2009) called VP-BTZ calculated as the difference between VIX^2 and the historical realized volatility.[5]

Table 3.5 reports results of the predictability regressions where the left hand side variable is the S&P 500 excess returns at monthly, quarterly, semi-annually and annual

[5]The data for VP-BTZ is downloaded from Hao Zhou website : https://sites.google.com/site/haozhouspersonalhomepage/.

horizon. All returns are annualized and excess returns are calculated by deducting corresponding zero-coupon treasury yield from gross returns. Panel A reports the result of uni-variate regression on excess returns on VP-L and excess returns on VP-BTZ for the respective horizons. Standard errors are estimates according to Newey-West (1986) method with 3 lags to account for overlap in monthly data. In a uni-variate regression, both VP-L and VP-BTZ does not have much predictability power at the monthly horizon where as for the quarterly horizon VP-BTZ has highly significant coefficient and $\mathrm{R}^2$of 4.7% while VP-L has very little predict predictability at the quarterly horizon. On the other hand, at longer horizons VP-L outperforms VP-BTZ where both the coefficient estimate and R^2 is higher. However, note that R^2 in these regressions are smaller than found in other studies which may be due to different samples. Separately, in Table 3.6 we can see that VIX and IV does appear to have some predictability in uni-variate regressions albeit only at the very near term horizon of 1 month with R^2 close to 5% but the coefficient which is negative is not significant. This is in line with findings in the literature where it has been hard to find conclusive evidence of relationship between conditional variance and future returns contrary to a clear prediction in the theory of a positive relation between conditional variance and future returns.

It is important to note that the in the learning model, the ability of VP-L to predict future returns is primarily driven by subjective beliefs related parameters in the model. In rational expectations, the model converges to BS and VIX premium in that case is the difference between VIX and BS IV which does not much predictability of returns (results not reported here).

Furthermore, remember that VP-L is correlated with IV-L and both appear to have predictability power at different horizons. This implies that uni-variate regressions are suspect and suffer from omitted variable bias. Panel B reports results of multivariate regressions which includes IV-L and VP-L as predictors and future excess returns as the left hand side variable. This is a better specification to test for risk-return relationship and effect of subjective beliefs on returns. We can see here that the R^2 at all horizons is substantially higher with highest being 11% at the 6 month horizon. Importantly, coefficient of VP-L is positive and IV-L is negative at all horizons. These estimates provide evidence that controlling for subjective beliefs the risk-return relationship is negative in the short term contrary to prediction of RE asset pricing models where it is the conditional variance that determines the equity premium even in the short term. Notice that however, including VP-BTZ with IV-L does not change the results much. In results not shown here, replacing VIX with IV-L produces similar results. Finally, In Panel C of the table, I consider a specification where VP-BTZ is also included with VP-L and IV-L, notice that the predictability of VP-BTZ goes away and we get similar results as in the specification with only IV-L and VP-L. This brings into question the robustness of VP-BTZ as a predictor of future stock returns. Overall these regression results provide compelling evidence that VP-L is a robust predictor of future index returns.

3.5 Option Implied Expectations and Survey Data

In the last section, we saw that the option pricing model proposed here appears consistent to the extent that it is able to explain challenging facts related to VIX premium. Furthermore, in Chapter 1 and Chapter 2, we saw that model is also consistent with both the cross-sectional and term structure properties of BS -IV curve. Therefore, IE can be considered a realistic estimate of subjective expectations of market participants.

Having a market based measure of investors expectations can be very useful as it circumvents the issues around measurement error which is a common concern with survey based estimates. At the same time, it would be interesting to see, how IE is related with survey based estimates of returns expectations. Although these estimates are not strictly comparable, still, a priori, one would expect a positive relationship between the survey based estimates and IE. [6]

Table 3.7 reports correlation coefficients of IE with various available survey estimates. All surveys report expectations of returns on S&P 500 index. However, it is important to note that different surveys are available at different frequencies as mentioned in Table 3.2. To calculate sample correlations, I take the IE for the month survey expectations

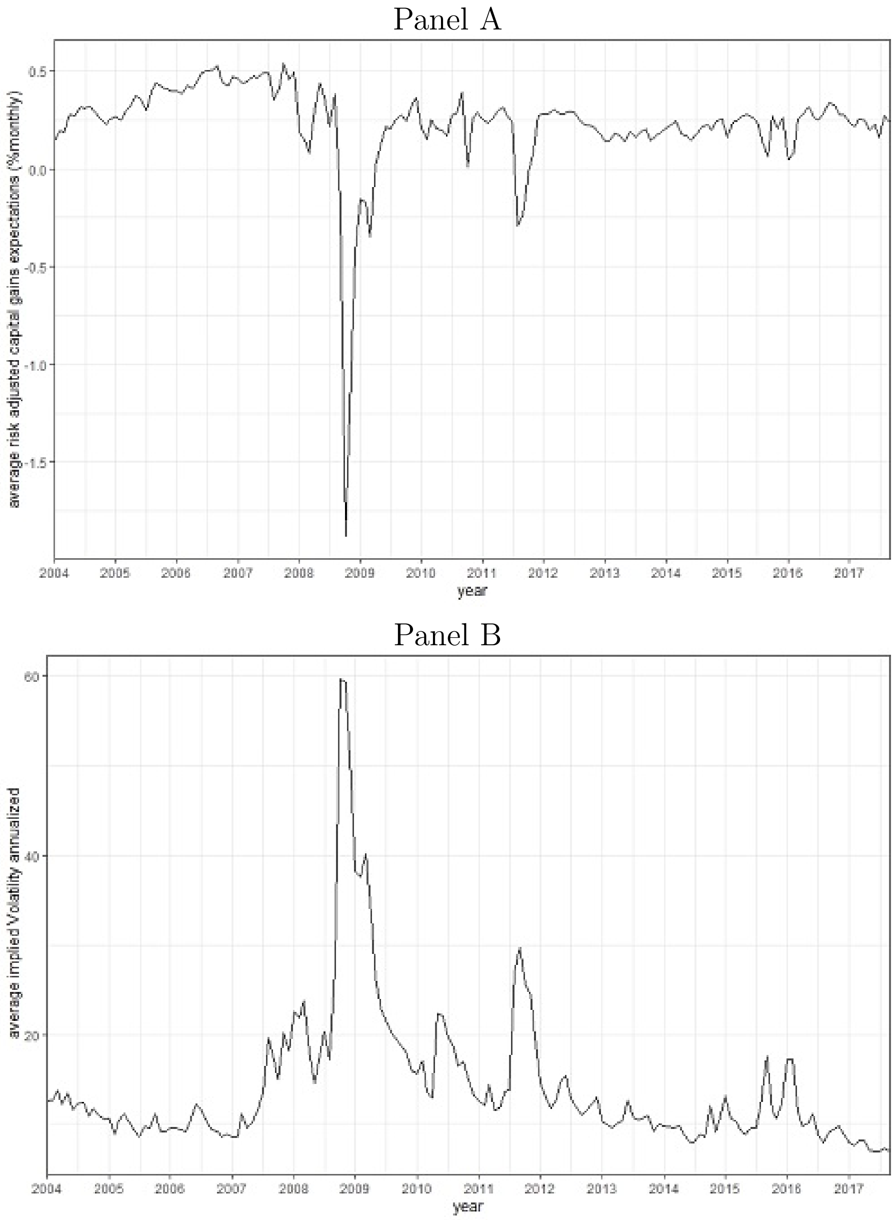
[6] IE are risk adjusted expectations while survey based expectations are probably not as discussed in Adam, Matveev, and Nagel (2021) and surveys ask for expectations at different horizon while IE is an estimate of monthly expectations,

are reported. For example: for Livingston survey, the base survey months are April and October of each year and the correlation coefficient is calculated using IE for April and October of each year. Secondly, the horizon of expectations is different for different surveys and IE is an estimate of one month ahead expectations. Therefore one would expect IE to be highly correlated with survey responses for horizon which are nearer rather than further. With the Livingston survey, as expected, IE is strongly correlated with the near horizon expectations with correlation coefficient of 0.58 than expectations for returns 8 months or 12 months in future confirming our conjecture. Interestingly, with the CFO survey which reports 1 year ahead returns expectations, IE is highly correlated with low estimate of expected returns with correlation of 0.70. This is, perhaps, due to the fact that IE reflects risk adjusted returns rather than nominal returns. However, even with CFO survey expected returns IE is strongly positively correlated with correlation coefficient 0.46. While the CFO survey high estimate of expected returns and IE have low sample correlation. With the UBS-Gallup extended survey provided by Nagel and Xu (2022) which report expectations at 1 year horizon, IE is again strongly positively correlated with the correlation coefficient of 0.54. Additionally, from this evidence it appears that when survey participants are not asked for separate responses for different horizon, there responses reflect short term beliefs. Finally, IE is strongly positively correlated with PD ratio with the correlation of 0.618 at the monthly frequency. This is consistent with findings noted in many studies that investors expectations are pro-cyclical i.e. high PD ratio is associated with high returns expectations.

Table 3.8 reports regression result of IE on lagged PD ratio. It can be seen that the coefficient of PD ratio is strongly significant in a univariate regression, however the point estimate drops from 0.882 to 0.704 when 1 month zero coupon treasury yield (1 m risk free rate) is included in the regression. The coefficient of 1 m risk free rate is also positive and statistically significant indicating that higher interest rates are associated with higher IE also pointing to procyclical nature of IE. The standard error are corrected according to Newey-West method with 3 lags to account for persistence in the variables.

3.6 Conclusion

In this paper, I estimate a multi-period option price model derived in a setup where investors are internally rational and learning about log of risk adjusted price growth. The model performance in matching option price data is considerably superior to Black and Scholes (1973) model. The estimation exercise delivers a new time series of option implied volatility which is adjusted for subjective capital gains expectations. VIX premium calculated using this option implied volatility strongly predicts future index returns in the data. The estimation exercise also delivers a new series of investors subjective expectations of risk adjusted returns which I call the "***option implied expectations***". Magnitude of option implied expectations (IE) suggest that the core equation in asset pricing with respect to stock prices is not satisfied in the option markets. IE is consistent with the evidence presented in various investors surveys i.e. expectations of returns of market participant is pro-cyclical rejecting the rational expectation hypothesis. IE can be very useful in future research as it provides real time assessment of investor sentiment.



Notes: Panel A plots the monthly average of m_t which is option implied expectations of log of risk adjusted returns. Panel B plots the monthly average of option implied volatility estimated using the learning option pricing formula. Data sample is from Jan 2004- Oct 2017.

Figure 3.1: Option Implied Expectations and Option Implied Volatility

Horizon	1	3	6	12	1	3	6	12
Panel A: Monthly, quarterly, semi-annual and annual regression with VIX premium								
VP-L	−0.002	0.005	0.006**	0.005***				
	(0.004)	(0.004)	(0.003)	(0.001)				
VP-BTZ					0.003	0.003***	0.002***	0.001
					(0.003)	(0.001)	(0.001)	(0.001)
Constant	0.066	−0.012	−0.035	−0.019	0.015	0.014	0.021	0.036
	(0.057)	(0.065)	(0.085)	(0.116)	(0.056)	(0.052)	(0.064)	(0.095)
N	162	160	157	151	162	160	157	151
Adj. R^2	−0.005	0.013	0.047	0.052	0.018	0.047	0.040	0.001
Panel B: Monthly, quarterly, semi-annual and annual regression with VIX premium and IV-L								
VP-L	0.015	0.015***	0.013***	0.008				
	(0.010)	(0.006)	(0.005)	(0.005)				
VP-BTZ					0.003*	0.003**	0.002**	0.001
					(0.001)	(0.001)	(0.001)	(0.0005)
IV-L	−0.026*	−0.016*	−0.011	−0.005	−0.013*	−0.004	−0.00002	0.001
	(0.014)	(0.009)	(0.008)	(0.009)	(0.007)	(0.007)	(0.006)	(0.006)
Constant	0.221***	0.086	0.035	0.015	0.206**	0.066	0.022	0.017
	(0.072)	(0.059)	(0.049)	(0.061)	(0.086)	(0.082)	(0.060)	(0.050)
N	162	160	157	151	162	160	157	151
Adj. R^2	0.096	0.099	0.110	0.077	0.069	0.050	0.033	−0.002
Panel B: Monthly, quarterly, semi-annual and annual regression with VIX premium and IV-L								
VP-L	−0.003	0.003	0.005**	0.005***	0.013	0.013*	0.012**	0.008
	(0.005)	(0.004)	(0.002)	(0.001)	(0.011)	(0.007)	(0.006)	(0.005)
VP-BTZ	0.003	0.003***	0.002***	0.0003	0.001	0.001	0.001	−0.0002
	(0.002)	(0.0005)	(0.0004)	(0.0004)	(0.002)	(0.002)	(0.001)	(0.001)
IV-L					−0.024*	−0.014	−0.010	−0.006
					(0.014)	(0.010)	(0.009)	(0.009)
Constant	0.055	−0.022	−0.041	−0.020	0.207***	0.068	0.023	0.018
	(0.069)	(0.063)	(0.071)	(0.107)	(0.079)	(0.069)	(0.049)	(0.057)
N	162	160	157	151	162	160	157	151
Adj. R^2	0.016	0.049	0.070	0.047	0.094	0.105	0.111	0.071

Notes: Sample period January 2003-October-2017. All regressions are based on monthly observations. IV-L is the implied volatility estimated using the CGL option pricing formula and N is the sample size. The standard errors reported in brackets are computed using Newey-West method using 2 lags.$^{*}p < 0.1$;$^{**}p < 0.05$;$^{***}p < 0.01$

Table 3.5: S&P 500 returns regressions.

Horizon	1	3	6	12	1	3	6	12
IV-L	-0.014	-0.004	-0.001	0.001				
	(0.010)	(0.011)	(0.009)	(0.008)				
VIX					-0.012	-0.003	0.001	0.002
					(0.008)	(0.010)	(0.008)	(0.007)
Constant	0.241^{**}	0.105	0.053	0.027	0.276^{**}	0.099	0.035	0.005
	(0.112)	(0.109)	(0.080)	(0.051)	(0.124)	(0.140)	(0.100)	(0.063)
Observations	162	160	157	151	162	160	157	151
Adjusted R^2	0.056	0.006	-0.006	-0.004	0.050	0.0002	-0.006	0.002

Notes:Sample period January 2003-October-2017. All regressions are based on monthly observations. The standard errors reported in brackets are computed using 2 Newey-West lags.
$^{*}p < 0.1$;$^{**}p < 0.05$;$^{***}p < 0.01$

Table 3.6: Risk-returns regression

	m_t	Livingston 2m	Livingston 6m	Livingston 12m	CFO low	CFO exp	CFO high	UBS(NX)
m_t	1.00	0.58	0.08	-0.21	0.71	0.46	0.06	0.54
Livingston 2m	0.58	1.00	0.45	0.52	N.A	N.A	N.A	N.A
Livingston 6m	0.08	0.45	1.00	0.58	N.A	N.A	N.A	N.A
Livingston 12m	-0.21	0.52	0.58	1.00	N.A	N.A	N.A	N.A
CFO low	0.71	N.A	N.A	N.A	1.00	0.76	0.29	0.74
CFO exp	0.46	N.A	N.A	N.A	0.76	1.00	0.82	0.58
CFO high	0.06	N.A	N.A	N.A	0.29	0.82	1.00	0.16
UBS (NX)	0.54	N.A	N.A	N.A	0.74	0.58	0.16	1.00

Notes: Table reports correlation of with survey based expectations. N.A. is reported where the two time series have no overlapping values. CFO is CFO survey, Livingston is for Livingston survey, UBS (NX) is for extended UBS-Gallup survey provided in Nagel and Xu (2022) and m_t is option implied expectations.

Table 3.7: Correlation: Option Implied Expectations and Survey data

	Implied Expectations	
PD ratio	0.882^{**}	0.704^{*}
	(0.357)	(0.395)
1m risk free rate		0.031^{***}
		(0.011)
Constant	-3.207^{**}	-2.551^{*}
	(1.413)	(1.550)
Observations	162	162
Adjusted R^2	0.262	0.286

Notes:$^{*}p < 0.1$;$^{**}p < 0.05$;$^{***}p < 0.01$

Table 3.8: Implied Expectations and PD ratio

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