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Models for dealing with uncertainty in decision-making

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1. Introduction

1.1. Introduction

We all agree that, over time, the world has become more complex, volatile, and uncertain. For instance, the pandemic of coronavirus disease 2019 (COVID-19) represents this uncertainty very well (Rutter et al., 2020). In the early years, there were many questions about what will happen and the absolute magnitude of this crisis, especially because there was no reference case. Thus, in this context, making correct decisions became more complicated and challenging. Moreover, decision-making in times of COVID-19 had a significant impact. To give an insight, governments had to constantly decide on various issues, such as which population group should be first vaccinated, what vaccines to use and with how many doses and frequency, order a lockdown on a national level or on a regional one, which countries need to be considered as high coronavirus risk zones in order to require foreign tourists a negative COVID-19 test or proof of vaccination to enter, and how to distribute the subsidies. Also, the firms needed to make important decisions such as closing the company, selling online, repurposing the production line (to give an example, many businesses took advantage of the soaring demand for face masks), implementing homeworking, laying off part of the staff, reduce wages, decide to sell stock, and much more. Individuals also had to make critical decisions, among others, to move out of the city to save money, use private transport to prevent getting COVID-19, and avoid meeting friends and family.

Therefore, it can be said that decision-making under uncertainty has taken an increasingly important role in today's society.

Decision-making can be defined as the process of identifying and choosing alternatives based on the values and preferences of the decision-maker. The origin of the decision-making theory in uncertain environments can be found in the publication of the seminal paper "Fuzzy sets. *Information and Control*" made by professor Lotfi Asker Zadeh (1965). This paper introduced the concept of fuzzy sets, which may be seen as a generalization of classical/crisp sets to deal with vagueness. Since its introduction in back 1965, fuzzy set theory has been rapidly developed and successfully applied in various fields (Merigó et al., 2015). Twenty-three years later, Ronald Robert Yager (1988) published the scientific article "On ordered weighted averaging aggregation operators in multicriteria decisionmaking. *IEEE Transactions on Systems, Man, and Cybernetics*", in which he presented one of the most popular aggregation operators, called the ordered weighted averaging (OWA) operator. The OWA operator provides a parametrized family of aggregation operators between the maximum and the minimum. This operator has proven to be a potent tool for decision-making problems under uncertainty, as they allow to consider the attitudinal character of the decision-maker. Since its introduction, many contributions have been made (Csiszar, 2021).

The overall goal and motivation of this doctoral thesis is, to contribute to the decision-making under uncertainty literature by developing new aggregation functions as well as applications in the field of pensions.

Through an extensive review of the literature, it has been found that there exist several extensions of the ordered weighted averaging adequacy coefficient (OWAAC) operator (Merigó & Gil-Lafuente, 2008, 2010), but none of these uses linguistic variables. However, in real-life, we can find some situations where the available information can only be assessed with linguistic variables. Nor is there an OWAAC operator which incorporates interval numbers. Therefore, it is necessary to provide new OWAAC operator formulations that use both linguistic and interval information. This doctoral thesis aims to fill these two gaps.

One of the current topics that the whole world worries most about is the pension crisis caused mainly by demographic changes. For example, according to Eurostat (2023), the old-age dependency ratio of Spain has increased from 25.2% in 2011 to 30.5% in 2022. Pensions are subject to different uncertainties, including economic uncertainties (such as interest rates and labor participation) and pension policy uncertainties (like increasing the statutory retirement age and changing the contribution rates). This is why new algorithms for improving decision-making concerning pensions will be investigated in this doctoral thesis, thus providing practical solutions to the actual pension crisis. To do so, different mathematical methods for dealing with uncertainty will be used, including the OWA operator, the Hamming distance (Hamming, 1950), and the linguistic variables.

This doctoral thesis will offer governments, companies, and citizens alternative tools to improve their decision-making in real-life problems when a high degree of uncertainty and ambiguity is involved. This is why it is believed that the results of this thesis work will be very valuable to the development of science as well as society.

1.2. Objectives

The main objectives of this doctoral thesis are the following five:

1. Develop the state of the art of OWA aggregation operators through a bibliometric analysis.
2. Review of the mathematical theories used for decision-making in uncertain environments.
3. Analyze new extensions of the OWA operator.
4. Analyze new applications of the OWA operator and its extensions in the field of pensions.
5. Make scientific contributions through international publications.

First objective. *Develop the state of the art of OWA aggregation operators through a bibliometric analysis.*

In order to demonstrate the novelty of the research results of this doctoral thesis, a stat of the art will be carried out in the form of a paper. To do so, a bibliometric analysis of the OWA operator will be conducted using the Web of Science (WoS), one of the most powerful databases of research publications and citations in the world. This method allows to study and evaluate scientific publications quantitatively. This bibliometric analysis aims to show the 50 most prevailing papers, the 50 most productive authors, the 50 most productive institutions (e.g., colleges and universities), the 50 most frequent countries, the 50 most common journals, the 50 most common applications, and some other interesting bibliometric indicators.

Second objective. *Review of the mathematical theories used for decision-making in uncertain environments.*

The second objective of this thesis work is to conduct an extensive analysis of the different existing decision-making methods in uncertain environments. This includes the concepts of decision-making, interval numbers (Moore, 1966), fuzzy sets, fuzzy numbers (Chang & Zadeh, 1972), fuzzy arithmetic, linguistic variables, intuitionistic fuzzy sets (IFS) (Atanassov, 1986), basic uncertain information (BUI) (Jin et al., 2018; Mesiar et al., 2018), similarity measures, and aggregation operators. Some of the similarity measures that will be discussed are the Hamming distance and the adequacy coefficient from Kaufmann and Gil-Aluja (1986, 1987). Concerning the aggregation operators, a further review of the OWA operator presented by Yager and its most important extensions will be carried out. These extensions are the induced OWA (IOWA) operator (Yager & Filev, 1999), the heavy OWA (HOWA) operator (Yager, 2002), the generalized OWA (GOWA) operator (Yager, 2004), the Quasi-OWA operator (Fodor et al., 1995), the probabilistic OWA (POWA) operator (Merigó, 2012), the uncertain OWA (UOWA) operator (Xu & Da, 2002), the fuzzy OWA (FOWA) operator, the linguistic OWA (LOWA) operator (Bordogna & Pasi, 1995; Herrera et al., 1995; Herrera & Martínez, 2000; Xu, 2004), the OWA distance (OWAD) operator (Merigó & Gil-Lafuente, 2007, 2010), the OWAAC operator, and the OWA index of maximum and minimum level (OWAIMAM) operator (Merigó & Gil-Lafuente, 2012).

Third objective. *Analyze new extensions of the OWA operator.*

Another objective of this thesis work is to investigate and create new extensions of the OWA operator. As it is well known, the OWA operator has many extensions. For example, the IOWA operator, which uses order-inducing variables; the GOWA operator, which uses generalized means (Dyckhoff & Pedrycz, 1984); and many others. A very recent extension of the OWA operator is the OWAAC operator, which is based on the use of the adequacy coefficient. However, the OWAAC operator considers only precise numerical information, which is not always possible. That is why this doctoral thesis aims to develop new aggregations operators based on the use of linguistic variables and the adequacy coefficient in a single formulation. Similarly, this thesis work aspires to discuss the utilization of interval numbers in the OWAAC operator.

Fourth objective. *Analyze new applications of the OWA operator and its extensions in the field of pensions.*

The OWA operator has been successfully applied in a wide range of fields (Kacprzyk et al., 2019), mainly due to the flexibility that this operator offers. E.g., sales forecasting (Merigó et al., 2015), social choice and voting (Kacprzyk & Zadrożny, 2009), insurance (Casanovas et al., 2015, 2016), and inflation calculations (Espinoza-Audelo et al., 2020; León-Castro et al., 2020). Nonetheless, the OWA operator has not yet been applied in the field of pensions. Thus, the fourth objective of this doctoral thesis is to study new applications of OWA operators in the field of pensions and thereby provide solutions to the global pension crisis. For example, the use of the OWA operator in a decision-making problem about the selection of supplementary pension products in Spain will be studied.

Fifth objective. *Make scientific contributions through international publications.*

The fifth and last objective consists in sharing the research results and knowledge obtained during the elaboration of this doctoral thesis and thereby contribute to the betterment of society. In order to meet this goal, this research aims to write and publish papers in international journals and participate in several scientific conferences. Some key journals in the field of decision-making under uncertainty are the *International Journal of Intelligent Systems*, the *Fuzzy Set and Systems*, the *Information Sciences*, the *IEEE Transactions on Fuzzy Systems*, the *Expert Systems with Applications*, the *Knowledge-Based Systems*, and the *Computers & Industrial Engineering*.

1.3. Methodology

The methodological approach used in this doctoral thesis is mainly based on the idea of the fuzzy set theory introduced by Zadeh more than 50 years ago, the interval numbers, the linguistic variables, the Hamming distance, the adequacy coefficient presented by Kaufmann and Gil-Aluja, and the OWA operator developed by Yager. Below, a more detailed explanation of these methods is given.

Fuzzy set theory

The theory of fuzzy sets (also known as the theory of fuzzy subsets) is very useful for dealing with uncertainty, subjectivity, ambiguity, and vagueness. Fuzzy sets are seen as an extension of the classical notion of sets (known as crisp sets). A crisp set (also called classical set or ordinary set) is a collection of well-defined elements (or objects) out of some universal set. To know if an element belongs or not to the crisp set, it is used what is named "membership value". Thus, for a crisp set, if an element is present in the crisp set, its membership value is equal to 1. Otherwise, if an element does not belong to the crisp set, its membership value equals 0. The fuzzy sets are also defined by their membership functions, but with the difference that their values can take any number between zero and one, thus enabling partial memberships. Therefore, crisp sets are employed when there is no uncertainty involved; meanwhile fuzzy sets are used when the environment is uncertain and a certain degree of flexibility is needed.

Interval numbers

The interval numbers (sometimes referred as confidence intervals), suggested by Moore almost six decades ago, is a practical technique for representing uncertain information. An interval number can be simply defined as a set of real numbers lying between two specific real numbers. Moreover, interval numbers can take various forms.

Linguistic variables

Zadeh (1975a, 1975b, 1975c) defined a linguistic variable as a variable whose values are not numbers but words or phrases in a natural or synthetic language. For example, the linguistic variable "speed" can assume different linguistic values, such as "very slow", "slow", "normal", "fast", and "very fast". Sometimes, there are situations where the information cannot be assessed precisely in a quantitative way due the presence of uncertainty and ambiguity. In such situations, using an approach based on linguistic variables for assessing the information may be more convenient.

The Hamming distance

The Hamming distance is a well-known method for calculating the difference between two elements, two sets, or two fuzzy sets. It is very useful for those situations where the decision-maker wants to calculate the distance between the available alternatives and the ideal result. Furthermore, 24 years later, the professors and researchers Merigó and Gil-Lafuente analyzed the use of the OWA operator in the Hamming distance, thus obtaining the OWAD operator. The main characteristic of this operator is that it allows to aggregate individual distances according to the attitudinal character of the decision-maker. Besides the Hamming distance, in the literature, we can find other practical distance measures, such as the Euclidean distance, the Hausdorff distance (Huttenlocher et al., 1993), the Minkowski distance, and many others.

The adequacy coefficient

Similar to the Hamming distance, the adequacy coefficient is an index used for calculating the differences between two elements, two sets, or two fuzzy sets. The main characteristic of this coefficient is that it allows to neutralize the result when the comparison shows that the real element is higher than the ideal one. Furthermore, Merigó and Gil-Lafuente developed the OWAAC operator, which uses the adequacy coefficient and the OWA operator in the same formulation.

The OWA operator

The OWA operator is an increasingly popular method (Blanco-Mesa et al., 2019; Emrouznejad & Marra, 2014; He et al., 2017) used for information aggregation with the aim of obtaining a representative value. It provides a parameterized class of mean type aggregation operators that lies between the minimum and the maximum. The main advantage of this operator is the possibility to aggregate the information according to the attitudinal character of the decision-maker. Thus, it provides excellent flexibility to the decision-maker, which is not possible when using other traditional methods. An interesting characteristic of this type of operator is that it includes the classical methods for decision-making into a single formulation. Some of these classical methods are the optimistic criterion, the pessimistic criterion, the Laplace criterion, and the Hurwicz criterion. Since its introduction, many authors have developed a wide range of extensions of this operator. Some of them are the IOWA operator, the Heavy OWA operator, the GOWA operator, the Quasi-OWA operator, the POWA operator, the UOWA operator, the FOWA operator, the LOWA operator, the OWAD operator, the OWAAC operator, and the OWAIMAM operator.

1.4. Structure and content

This doctoral thesis is divided into five chapters that treat in a complementary way the subject of study.

The **first chapter** of this doctoral thesis starts with a general background introduction. Then, the objectives of the research project are exposed. Also, the methodology followed is briefly explained. This chapter ends with a detailed description of the structure of the doctoral thesis.

The **second chapter** corresponds to the literature review and is divided into four parts. The first part contains the state of the art of the OWA operator, based on a bibliometric analysis which will be conducted using the WoS database. The second part studies some of the most important notions and mathematical tools regarding decision-making under uncertainty. This includes the concepts of decision-making, interval numbers, fuzzy sets, BUI, distance measures, and aggregation operators. The third part describes the OWA operator from Yager and its most important extensions and applications in detail. The fourth and last part of this chapter offers a general survey of pensions.

The **third chapter** is the most important of this doctoral thesis as it provides the main contributions made to the field of decision-making under uncertainty. Specifically, it comprises all the published papers as well as the publishable ones at the time of presentation of this doctoral thesis.

The **fourth chapter** summarizes the main conclusions of this doctoral thesis and discloses future research paths.

The **fifth chapter** is the last of this doctoral thesis and contains the annexes. This chapter provides additional research.

Fig. 1.1 provides an overview of the structure of this doctoral thesis.

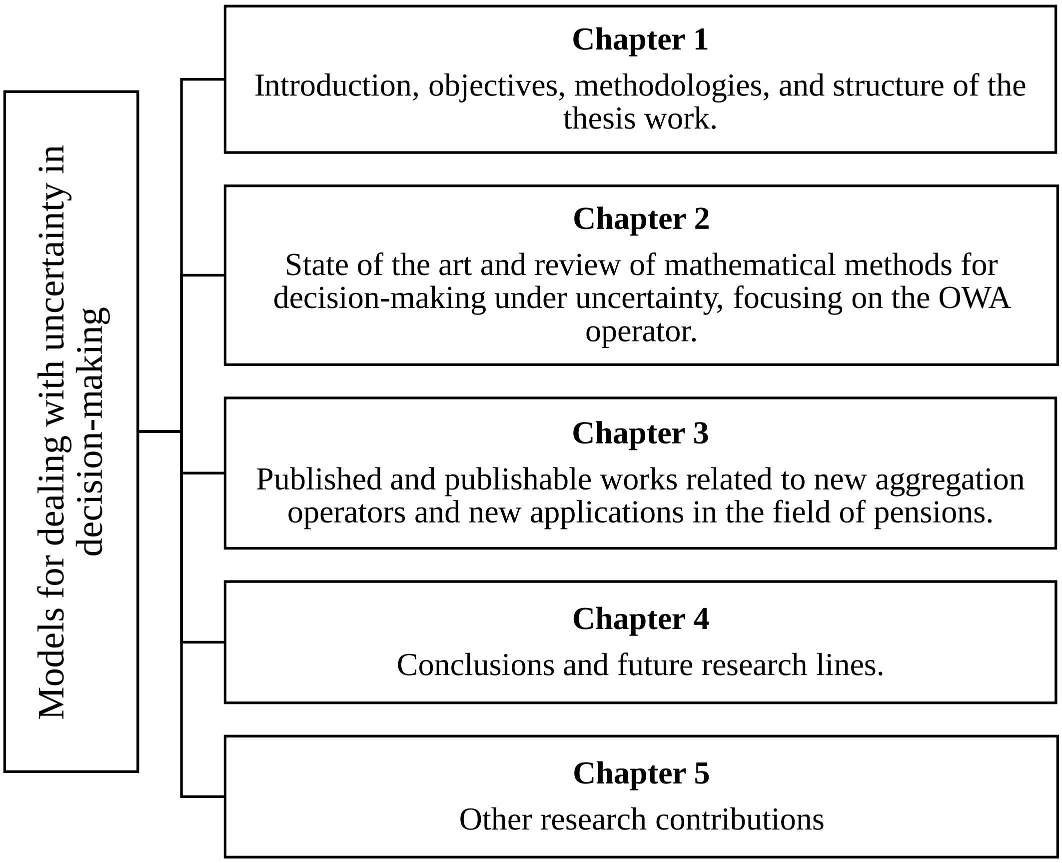

Fig. 1.1. Thesis structure
Source: Own elaboration

1.5. References

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2. Literature review

2.1. A bibliometric analysis of the OWA operator from 1988 to 2021

The following research paper was submitted to the Q1 journal *Fuzzy Sets and Systems*. For the year 2021, the Impact Factor is 4.462 and the CiteScore 7.1.

The authors of this paper are Anton Figuerola Wischke (University of Barcelona), José María Merigó Lindahl (University of Technology Sydney), Anna Maria Gil Lafuente (University of Barcelona), and Sefa Boria Reverter (University of Barcelona).

Abstract

The ordered weighted averaging (OWA) operator was proposed by Yager back in 1988 and it constitutes a parameterized family of aggregation functions between the minimum and the maximum. The purpose of this paper is to perform a bibliometric analysis of this aggregation operator during the last 34 years through the Web of Science (WoS) Core Collection database and the Visualization of Similarities (VOS) viewer software. The results allow the assertion that the OWA operator is an increasingly popular aggregation operator. The results also show that Yager, as expected, is still the most influential and productive author. Other interesting findings are presented in order to provide a comprehensive and up-to-date analysis of the OWA operator literature.

Keywords: Aggregation operator, bibliometric analysis, OWA operator, VOS viewer, Web of Science.

1. Introduction

Aggregation can be described as the process of combining multiple values into a single representative one, and an aggregation operator conducts this operation (Grabisch et al., 2009). The ordered weighted averaging (OWA) operator was presented by Yager (1988) and provides a parametrized class of aggregation operators, ranging from the minimum to the maximum. Since its appearance, this operator has been applied to various problems (Kacprzyk et al., 2019).

Likewise, the OWA has also been widely extended. Some well-known extensions are the induced OWA (IOWA) operator (Yager & Filev, 1999), the generalized OWA (GOWA) operator (Yager, 2004a), the quasi OWA (QOWA) operator (Fodor et al., 1995), the probabilistic OWA (POWA) operator (Merigó, 2012), the uncertain OWA (UOWA) operator (Xu & Da, 2002b), the linguistic OWA (LOWA) operator (Herrera et al., 1995; Xu, 2004a), and the OWA distance (OWAD) operator (Merigó & Gil-Lafuente, 2010).

The main objective of this paper is to provide an up-to-date state of the art of the OWA operator knowledge domain and to identify research trends. In order to achieve this, a bibliometric analysis of the OWA operator between the years 1988 and 2021 is developed, using the Web of Science (WoS) Core Collection database in conjunction with the Visualization of Similarities (VOS) viewer software.

This paper is structured as follows. Section 2 reviews the followed methodology and data collection. Section 3 presents the obtained results. Primarily, the publication and citation structure, the major authors/institutions/countries/journals/research areas (both from a static and dynamic perspective), and the co-citation and co-occurrence networks. Finally, Section 4 summarizes the main conclusions and limitations.

2. Methodology and data

Bibliometric analysis is becoming more commonplace as it allows to analyze quantitatively large amounts of bibliographic information (Donthu et al., 2021). Accordingly, bibliometric studies have been carried out in a large variety of fields, including economics (Bonilla et al., 2015; Wang et al., 2022), blockchain (Guo et al., 2021), marketing (Kim et al., 2021), and tourism (Khanra et al., 2021). Also, in (He et al., 2017) the authors conducted an interesting bibliometric analysis of the OWA operator for the period of 1988-2015, and in (Emrouznejad & Marra, 2014) during the years 1988-

2014. Similarly, in (Blanco-Mesa et al., 2019) the researchers prepared a survey of aggregation operators as a whole.

When conducting a bibliometric analysis, it is critical to choose the right bibliometric indicators (Joshi, 2014). This study considers different types of indicators, which are the number of documents published, the number of citations, and the *h* index, among others. The number of published documents and citations are used to evaluate the productivity and influence, respectively, while the *h* index unifies these two. The *h* index was proposed by Hirsch (2005) and can be interpreted as the number of documents that have *h* or more citations.

Currently, there are several databases for conducting a bibliometric analysis, such as Scopus, PubMed, Web of Science (WoS), and Google Scholar. This study uses the WoS Core Collection to collect all the scientific data. As of the date of this study, the WoS is owned by the company Clarivate Analytics.

The retrieval strategy was carried out as follows. The search topics were “OWA” and “ordered weighted averag*”. The asterisk (*) is used in order to represent any group of characters, including no character. For example, searching for “ordered weighted averag*” will find “ordered weighted averaging”, “ordered weighted average”, and more. The time range applied was 1988-2021. This search was conducted in October 2022 and a total of 3,060 publications were found. However, this number reduces to 2,307 publications, as only articles (2,289), review articles (14), letters (2), and notes (2) were considered.

Additionally, the software VOS viewer was employed to provide a more comprehensive view of the bibliographic networks. Specifically, maps were drawn up in terms of co-citation and co-occurrence. Co-citation can be described as the frequency with which two documents are cited in conjunction (Small, 1973). With regard to the co-occurrence, the number of co-occurrences of two keywords is the number of documents in which both keywords appear jointly (Van Eck & Waltman, 2014). Furthermore, by using the VOS viewer, it is possible to detect the most cited references/authors/journals and the most frequent keywords within OWA publications. Lastly, indicate that in some cases, the VOS viewer thesaurus file was operated to perform data cleaning.

3. Results

Publication and citation structure

The annual evolution of the number of documents published in OWA is exhibited in Fig. 2.1 The graph line shows a clear growing trend. Additionally, it can be seen that most of the documents have been published during the last decade. Also, a total of 221 documents published in OWA were reached during the peak year of 2019. While in the last year analyzed, i.e., 2021, 206 documents were recorded.

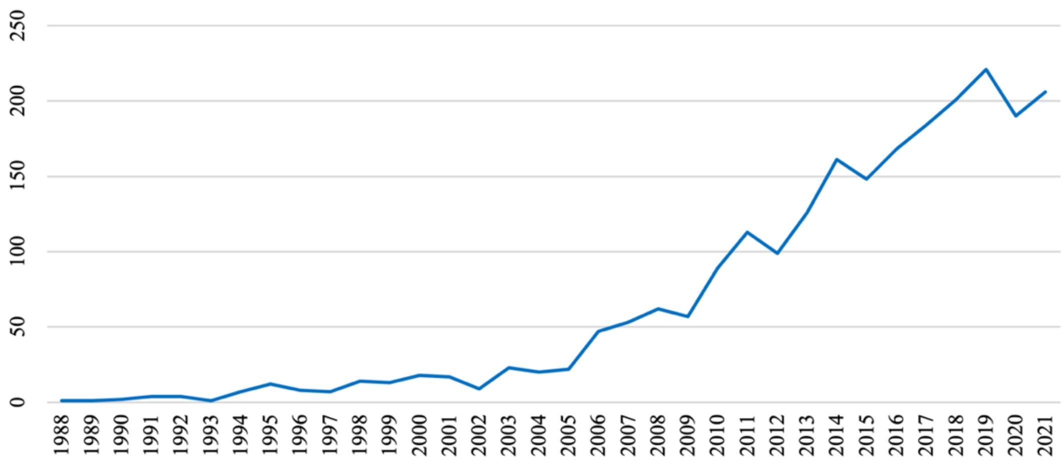

Fig. 2.1. Evolution of the annual number of documents published in OWA
Source: Own elaboration

Another interesting issue is the citation structure in OWA within the WoS Core Collection, which is shown in Table 2.1. There is only one document that exceeds the 4,000 citations. Specifically, it is the letter "*On ordered weighted averaging aggregation operators in multicriteria decisionmaking*", written by Yager in 1988 (Yager, 1988). Likewise, there are three documents with between 1,000 and 4,000 citations. Although most of the documents have between 0 and 25 citations, equivalent to almost 67% of the total.

Table 2.1. Citation structure in OWA

TC	TP	% TP	TC	TP	% TP
[4,000, +∞)	1	0.04%	≥ 4,000	1	0.04%
[1,000, 4,000)	3	0.13%	≥ 1,000	4	0.17%
[500, 1,000)	11	0.48%	≥ 500	15	0.65%
[400, 500)	11	0.48%	≥ 400	26	1.13%
[300, 400)	10	0.43%	≥ 300	36	1.56%
[200, 300)	37	1.65%	≥ 200	73	3.21%
[100, 200)	117	5.07%	≥ 100	190	8.28%
[50, 100)	242	10.49%	≥ 50	432	18.76%
[25, 50)	336	14.56%	≥ 25	768	33.32%
[0, 25)	1,539	66.68%	≥ 0	2,307	100%

Source: Own elaboration through WoS. Abbreviations: TC = Total citations; TP = Total publications; % TP = Percentage of total publications.

The fifty most cited documents ranged from 242 to 4,902 citations, which can be seen in Table 2.2. This equates to an average of 555 cites per document and a median of 405. The most cited document is the already mentioned "*On ordered weighted averaging aggregation operators in multicriteria decisionmaking*" from Yager (1988) and published in the *IEEE Transactions on Systems, Man, and Cybernetics* journal in 1988. Specifically, it has been cited 4,902 times until October 2022, which is 3,178 times more than the second most cited document. Considering that this publication introduces the OWA operator for the first time, it is not surprising that it is the most cited document.

The second most influential publication comprising the OWA topic was written by Xu (2007) and which is entitled "*Intuitionistic fuzzy aggregation operators*". In this document, the author developed different types of aggregation operators for aggregating intuitionistic fuzzy information. One of them is the intuitionistic fuzzy OWA (IFOWA) operator, which extends the OWA operator by using intuitionistic fuzzy values.

In the third position appears the document "*Linguistic decision analysis: Steps for solving decision problems under linguistic information*", prepared by the authors Herrera and Herrera-Viedma (2000b). This document describes the steps to follow for addressing a multi-criteria decision-making

(MCDM) problem with linguistic information, including the analysis of the LOWA operator.

Table 2.2. Top 50 most cited documents in OWA

R	Article	Author	Journal	TC	PY
1	On ordered weighted averaging aggregation operators in multicriteria decisionmaking (Yager, 1988)	Yager, RR	IEEE Transactions on Systems, Man, and Cybernetics	4,902	1988
2	Intuitionistic fuzzy aggregation operators (Xu, 2007)	Xu, ZS	IEEE Transactions on Fuzzy Systems	1,724	2007
3	Linguistic decision analysis: Steps for solving decision problems under linguistic information (Herrera & Herrera-Viedma, 2000b)	Herrera, F; Herrera-Viedma, E	Fuzzy Sets and Systems	1,175	2000
4	Hesitant fuzzy information aggregation in decision making (Xia & Xu, 2011)	Xia, MM; Xu, ZS	International Journal of Approximate Reasoning	1,118	2011
5	Families of OWA operators (Yager, 1993)	Yager, RR	Fuzzy Sets and Systems	905	1993
6	Quantifier guided aggregation using OWA operators (Yager, 1998)	Yager, RR	International Journal of Intelligent Systems	836	1996
7	Induced ordered weighted averaging operators (Yager & Filev, 1999)	Yager, RR; Filev, DP	IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)	767	1999
8	A model based on linguistic 2-tuples for dealing with multigranular hierarchical linguistic contexts in multi-expert decision-making (Herrera & Martinez, 2001)	Herrera, F; Martínez, L	IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)	694	2001
9	An overview of operators for aggregating information (Xu & Da, 2003)	Xu, ZS; Da, QL	International Journal of Intelligent Systems	635	2003
10	Uncertain linguistic aggregation operators based approach to multiple attribute group decision making under uncertain linguistic environment (Xu, 2004b)	Xu, ZS	Information Sciences	629	2004
11	A fusion approach for managing multi-granularity linguistic term sets in decision making (Herrera et al., 2000)	Herrera, F; Herrera-Viedma, E; Martínez, L	Fuzzy Sets and Systems	613	2000
12	Integrating three representation models in fuzzy multipurpose decision making based on fuzzy preference relations (Chiclana et al., 1998)	Chiclana, F; Herrera, F; Herrera-Viedma, E	Fuzzy Sets and Systems	596	1998
13	An overview of methods for determining OWA weights (Xu, 2005)	Xu, ZS	International Journal of Intelligent Systems	566	2005

R	Article	Author	Journal	TC	PY
14	A consensus model for multiperson decision making with different preference structures (Herrera-Viedma et al., 2002)	Herrera-Viedma, E; Herrera, F; Chiclana, F	IEEE Transactions on Systems, Man, and Cybernetics, Part A (Systems and Humans)	533	2002
15	Some induced geometric aggregation operators with intuitionistic fuzzy information and their application to group decision making (Wei, 2010)	Wei, GW	Applied Soft Computing	503	2010
16	A consensus model for group decision making with incomplete fuzzy preference relations (Herrera-Viedma, Alonso, et al., 2007)	Herrera-Viedma, E; Alonso, S; Chiclana, F; Herrera, F	IEEE Transactions on Fuzzy Systems	497	2007
17	The weighted OWA operator (Torra, 1997)	Torra, V	International Journal of Intelligent Systems	470	1997
18	A sequential selection process in group decision making with a linguistic assessment approach (Herrera et al., 1995)	Herrera, F; Herrera-Viedma, E; Verdegay, JL	Information Sciences	468	1995
19	The power average operator (Yager, 2001)	Yager, RR	IEEE Transactions on Systems, Man, and Cybernetics, Part A (Systems and Humans)	464	2001
20	Dynamic intuitionistic fuzzy multi-attribute decision making (Xu & Yager, 2008)	Xu, ZS; Yager, RR	International Journal of Approximate Reasoning	455	2008
21	Group decision-making model with incomplete fuzzy preference relations based on additive consistency (Herrera-Viedma, Chiclana, et al., 2007)	Herrera-Viedma, E; Chiclana, F; Herrera, F; Alonso, S	IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)	454	2007
22	Application of fuzzy measures in multi-criteria evaluation in GIS (Jiang & Eastman, 2000)	Jiang, H; Eastman, JR	International Journal of Geographical Information Science	433	2000
23	The uncertain OWA operator (Xu & Da, 2002b)	Xu, ZS; Da, QL	International Journal of Intelligent Systems	424	2002
24	A new generalized Pythagorean fuzzy information aggregation using Einstein operations and its application to decision making (Garg, 2016)	Garg, H	International Journal of Intelligent Systems	419	2016
25	On the issue of obtaining OWA operator weights (Filev & Yager, 1998)	Filev, DP; Yager, RR	Fuzzy Sets and Systems	409	1998
26	A linguistic modeling of consensus in group decision making based on OWA operators (Bordogna et al., 1997)	Bordogna, G; Fedrizzi, M; Pasi, G	IEEE Transactions on Systems, Man, and Cybernetics, Part A (Systems and Humans)	405	1997
27	Generalized aggregation operators for intuitionistic fuzzy sets (Zhao et al., 2010)	Zhao, H; Xu, ZS; Ni, MF; Liu, SS	International Journal of Intelligent Systems	389	2010
28	Induced uncertain linguistic OWA operators applied to group decision making (Xu, 2006)	Xu, ZS	Information Fusion	368	2006

R	Article	Author	Journal	TC	PY
29	The induced generalized OWA operator (Merigó & Gil-Lafuente, 2009)	Merigó, JM; Gil-Lafuente, AM	Information Sciences	361	2009
30	Consistency and consensus measures for linguistic preference relations based on distribution assessments (Zhang et al., 2014)	Zhang, GQ; Dong, YC; Xu, YF	Information Fusion	334	2014
31	An approach for combining linguistic and numerical information based on the 2-tuple fuzzy linguistic representation model in decision-making (Herrera & Martínez, 2000)	Herrera, F; Martínez, L	International Journal of Uncertainty Fuzziness and Knowledge-Based Systems	329	2000
32	Induced aggregation operators (Yager, 2003)	Yager, RR	Fuzzy Sets and Systems	307	2003
32	Direct approach processes in group decision making using linguistic OWA operators (Herrera et al., 1996)	Herrera, F; Herrera-Viedma, E; Verdegay, JL	Fuzzy Sets and Systems	307	1996
34	OWA aggregation over a continuous interval argument with applications to decision making (Yager, 2004b)	Yager, RR	IEEE Transactions on Systems, Man, and Cybernetics, Part B (Cybernetics)	306	2004
35	Ordered weighted averaging with fuzzy quantifiers: GIS-based multicriteria evaluation for land-use suitability analysis (Malczewski, 2006)	Malczewski, J	International Journal of Applied Earth Observation and Geoinformation	302	2006
36	Intuitionistic fuzzy Choquet integral operator for multi-criteria decision making (Tan & Chen, 2010)	Tan, CQ; Chen, XH	Expert Systems with Applications	300	2010
37	Some induced ordered weighted averaging operators and their use for solving group decision-making problems based on fuzzy preference relations (Chiclana et al., 2007)	Chiclana, F; Herrera-Viedma, E; Herrera, F; Alonso, S	European Journal of Operational Research	284	2007
38	A fuzzy envelope for hesitant fuzzy linguistic term set and its application to multicriteria decision making (H. B. Liu & Rodríguez, 2014)	Liu, HB; Rodríguez, RM	Information Sciences	279	2014
38	Some Hamacher aggregation operators based on the interval-valued intuitionistic fuzzy numbers and their application to group decision making (P. D. Liu, 2014)	Liu, PD	IEEE Transactions on Fuzzy Systems	279	2014
40	An adaptive consensus support model for group decision-making problems in a multigranular fuzzy linguistic context (Mata et al., 2009)	Mata, F; Martínez, L; Herrera-Viedma, E	IEEE Transactions on Fuzzy Systems	278	2009
41	An analytic approach for obtaining maximal entropy OWA operator weights (Fuller & Majlender, 2001)	Fuller, R; Majlender, P	Fuzzy Sets and Systems	274	2001

R	Article	Author	Journal	TC	PY
42	Integrating multi-criteria evaluation techniques with geographic information systems for landfill site selection: A case study using ordered weighted average (Gorsevski et al., 2012)	Gorsevski, PV; Donevska, KR; Mitrovski, CD; Frizado, JP	Waste Management	268	2012
43	Choice functions and mechanisms for linguistic preference relations (Herrera & Herrera-Viedma, 2000a)	Herrera, F; Herrera-Viedma, E	European Journal of Operational Research	261	2000
44	Power-geometric operators and their use in group decision making (Xu & Yager, 2010)	Xu, ZS; Yager, RR	IEEE Transactions on Fuzzy Systems	256	2010
45	An overview of distance and similarity measures of intuitionistic fuzzy sets (Xu & Chen, 2008)	Xu, ZS; Chen, J	International Journal of Uncertainty Fuzziness and Knowledge-Based Systems	255	2008
45	The ordered weighted geometric averaging operators (Xu & Da, 2002a)	Xu, ZS; Da, QL	International Journal of Intelligent Systems	255	2002
47	EOWA and EOWG operators for aggregating linguistic labels based on linguistic preference relations (Xu, 2004a)	Xu, ZS	International Journal of Uncertainty Fuzziness and Knowledge-Based Systems	247	2004
48	On generalized Bonferroni mean operators for multi-criteria aggregation (Yager, 2009)	Yager, RR	International Journal of Approximate Reasoning	243	2009
49	Pythagorean fuzzy power aggregation operators in multiple attribute decision making (Wei & Lu, 2018)	Wei, GW; Lu, M	International Journal of Intelligent Systems	242	2018
49	Analyzing consensus approaches in fuzzy group decision making: Advantages and drawbacks (Cabrerizo et al., 2010)	Cabrerizo, FJ; Moreno, JM; Pérez, IJ; Herrera-Viedma, E	Soft Computing	242	2010
49	An approach to ordinal decision making (Yager, 1995)	Yager, RR	International Journal of Approximate Reasoning	242	1995

Source: Own elaboration through WoS. Abbreviations are available in Table 2.1 except for: R = Ranking; PY = Publication year.

Leading authors in OWA

Since Yager introduced the OWA operator, many authors and himself have made several significant contributions. Table 2.3 lists the top 50 authors with the most publications in OWA for the last 34 years. We can see that Yager followed by Merigó are by large the authors with the highest amount of published documents. Specifically, they contributed with 133 and 122 publications, respectively. Additionally, they have the highest *h* index in the ranking too. It is also noteworthy the average cites per publication achieved by Herrera, with a value of 316.35.

Table 2.3. Top 50 most productive authors in OWA

R	Author	TP	% TP	TC	Avg	*h*	≥ 500	≥ 100	≥ 50
1	Yager, RR	133	5.77%	14,002	105.28	45	4	22	40
2	Merigó, JM	122	5.29%	5,111	41.89	39	0	14	33
3	Xu, ZS	58	2.51%	10,080	173.79	37	5	25	31
3	Mesiar, R	58	2.51%	986	17.00	18	0	1	5
5	Zeng, SZ	45	1.95%	1,480	32.89	20	0	2	8
6	Chen, HY	44	1.91%	1,541	35.02	24	0	3	9
7	Liu, XW	40	1.73%	1,457	36.43	20	0	3	11
8	Jin, LS	39	1.69%	400	10.26	11	0	0	0
9	Herrera-Viedma, E	36	1.56%	7,877	218.81	30	4	22	29
9	Wei, GW	36	1.56%	3,714	103.17	29	1	16	27
11	Abdullah, S	34	1.47%	586	17.24	12	0	0	4
11	Bustince, H	34	1.47%	990	29.12	14	0	2	8
13	Liu, PD	32	1.39%	1,663	51.97	19	0	5	11
14	Gil-Lafuente, AM	30	1.30%	1,499	49.97	17	0	4	8
15	Chiclana, F	29	1.26%	4,122	142.14	22	2	12	20
16	León-Castro, E	25	1.08%	296	11.84	9	0	0	0
17	Herrera, F	23	1.00%	7,276	316.35	20	5	17	20
18	Garg, H	22	0.95%	1,821	82.77	18	0	5	14
19	Ibrahim, RW	20	0.87%	207	10.35	8	0	0	0
19	Wang, JQ	20	0.87%	911	45.55	15	0	2	7
21	Beliakov, G	19	0.82%	1,019	53.63	12	0	3	10
21	Casanovas, M	19	0.82%	1,406	74.00	14	0	7	10
21	Chen, XH	19	0.82%	1,264	66.53	14	0	4	9
24	Akram, M	18	0.78%	473	26.28	13	0	0	2
24	Dong, YC	18	0.78%	2,280	126.67	16	0	12	12
24	Liu, JP	18	0.78%	491	27.28	13	0	1	2
24	Ahn, BS	18	0.78%	436	24.22	13	0	0	2
24	Martínez, L	18	0.78%	3,018	167.67	15	2	8	11
29	Paternain, D	17	0.74%	181	10.65	6	0	0	2
29	Calvo, T	17	0.74%	551	32.41	10	0	2	5
31	Xu, YJ	16	0.69%	610	38.13	12	0	2	3
31	Llamazares, B	16	0.69%	252	15.75	9	0	0	1
31	Amin, GR	16	0.69%	361	22.56	11	0	0	3
34	Zarghami, M	14	0.61%	296	21.14	10	0	0	2
34	Blanco-Mesa, F	14	0.61%	223	15.93	8	0	0	1
34	Alajlan, N	14	0.61%	258	18.43	10	0	0	1
34	Aouf, MK	14	0.61%	82	5.86	6	0	0	0
34	Torra, V	14	0.61%	846	60.43	10	0	2	3
39	Su, WH	13	0.56%	475	36.54	10	0	1	2
39	Chang, KH	13	0.56%	522	40.15	9	0	2	5
39	Wan, SP	13	0.56%	569	43.77	11	0	2	4
39	Cheng, CH	13	0.56%	390	30.00	9	0	1	3
43	Xian, SD	12	0.52%	227	18.92	9	0	0	0
43	Rahman, K	12	0.52%	168	14.00	8	0	0	0
43	Wu, J	12	0.52%	718	59.83	11	0	3	6
43	Bordogna, G	12	0.52%	564	47.00	6	0	1	1
47	Li, WW	11	0.48%	46	4.18	4	0	0	0

R	Author	TP	% TP	TC	Avg	*h*	≥ 500	≥ 100	≥ 50
47	Moshiri, B	11	0.48%	178	16.18	5	0	0	0
47	Kacprzyk, J	11	0.48%	407	37.00	8	0	1	3
47	Sadiq, R	11	0.48%	416	37.82	9	0	0	4
47	Yi, PT	11	0.48%	53	4.82	4	0	0	0
47	Aggarwal, M	11	0.48%	159	14.45	9	0	0	0
47	Pei, Z	11	0.48%	317	28.82	8	0	1	2
47	James, S	11	0.48%	385	35.00	7	0	1	4
47	Hong, DH	11	0.48%	46	4.18	5	0	0	0
47	Zhang, HY	11	0.48%	732	66.55	10	0	2	7
47	Emrouznejad, A	11	0.48%	366	33.27	8	0	1	3

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1 and 2.2 except for: Avg = Average cites per publication; *h* = *h* index; ≥ 500, ≥ 100, ≥ 50 = Number of publications with equal or more than 500, 100, and 50 citations.

Leading institutions in OWA

Next, Table 2.4 lists the most productive institutions in OWA. Note that the institutions represent the affiliation of the author at the time of publication. Among the top 50 most productive institutions, 19 of them are from China and 7 from Spain. Despite this, Iona College from United States of America, occupies the first position in the ranking with 137 publications. This is explained by the fact that Yager was, and still is, professor at the Iona College.

Table 2.4. Top 50 most productive institutions in OWA

R	Institution	TP	% TP	TC	Avg	*h*	≥ 500	≥ 100	≥ 50
1	Iona College	137	5.94%	14,328	104.58	46	4	23	41
2	University of Barcelona	79	3.42%	4,146	52.48	33	0	13	28
3	Southeast University China	73	3.16%	7,257	99.41	34	4	16	30
4	Slovak University of Technology Bratislava	65	2.82%	1,300	20.00	20	0	3	7
5	University of Granada	63	2.73%	9,839	156.17	38	5	28	37
6	University of Chile	58	2.51%	1,147	19.78	20	0	1	6
7	University of Tehran	46	1.99%	938	20.39	20	0	0	4
8	Anhui University	45	1.95%	1,556	34.58	24	0	3	9
9	Nanjing Normal University	43	1.86%	469	10.91	12	0	0	0
10	Public University of Navarre	42	1.82%	1,054	25.10	15	0	2	8
11	Central South University	37	1.60%	1,719	46.46	21	0	4	11
11	Abdul Wali Khan University Mardan	37	1.60%	599	16.19	12	0	0	4
11	Shandong University of Finance Economics	37	1.60%	1,758	47.51	20	0	5	11
11	Sichuan University	37	1.60%	2,181	58.95	20	0	9	12
15	Deakin University	30	1.30%	1,325	44.17	14	0	5	12

R	Institution	TP	% TP	TC	Avg	*h*	≥ 500	≥ 100	≥ 50
15	University of Manchester	30	1.30%	1,100	36.67	17	0	2	7
15	Zhejiang Wanli University	30	1.30%	791	26.37	16	0	1	3
18	De Montfort University	28	1.21%	2,928	104.57	21	0	10	17
18	University of Technology Sydney	28	1.21%	276	9.86	9	0	0	0
20	Islamic Azad University	27	1.17%	621	23.00	13	0	1	5
21	Palacky University Olomouc	26	1.13%	155	5.96	6	0	0	0
21	Sichuan Normal University	26	1.13%	1,802	69.31	18	0	7	12
21	Zhejiang University of Finance Economics	26	1.13%	645	24.81	16	0	0	4
24	Northeastern University China	25	1.08%	358	14.32	9	0	0	2
24	University of Jaen	25	1.08%	3,539	141.56	19	2	9	15
24	University of Valladolid	25	1.08%	411	16.44	12	0	0	1
27	Thapar Institute of Engineering Technology	24	1.04%	1,902	79.25	19	0	5	15
27	University of Tabriz	24	1.04%	833	34.71	13	0	2	5
29	King Abdulaziz University	23	1.00%	1,429	62.13	14	0	8	10
29	King Saud University	23	1.00%	447	19.43	12	0	1	2
29	Ningbo University	23	1.00%	791	34.39	14	0	1	5
32	Ghent University	22	0.95%	632	28.73	10	0	1	5
32	Hohai University	22	0.95%	923	41.95	13	0	4	5
32	Polish Academy of Sciences	22	0.95%	556	25.27	10	0	1	4
35	National Centre for Scientific Research	21	0.91%	267	12.71	9	0	0	0
35	Chinese Academy of Sciences	21	0.91%	356	16.95	11	0	0	2
35	Chongqing University of Arts and Sciences	21	0.91%	2,075	98.81	18	1	9	15
35	Egyptian Knowledge Bank	21	0.91%	151	7.19	7	0	0	0
35	University of Ostrava	21	0.91%	449	21.38	12	0	0	2
35	University of Trento	21	0.91%	771	36.71	12	0	1	3
41	Army Engineering University of Pla	20	0.87%	1,190	59.50	13	0	4	4
41	Hazara University	20	0.87%	281	14.05	9	0	0	2
41	Udice French Research Universities	20	0.87%	404	20.20	9	0	0	3
41	University of Punjab	20	0.87%	494	24.70	14	0	0	2
45	Beijing Institute of Technology	19	0.82%	341	17.95	11	0	0	1
45	North China Electric Power University	19	0.82%	261	13.74	8	0	0	0
45	University of Malaya	19	0.82%	199	10.47	8	0	0	0
45	Zhejiang Gongshang University	19	0.82%	542	28.53	11	0	1	2
49	Fuzhou University	18	0.78%	1,011	56.17	15	0	3	8
49	University of Alcala	18	0.78%	481	26.72	8	0	2	5
49	Polytechnic University of Valencia	18	0.78%	384	21.33	12	0	0	3

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1, 2.2, and 2.3.

Leading countries in OWA

In Table 2.5, the most productive countries in OWA are highlighted. Nowadays, China is the leading contributor to the development of OWA research. In concrete terms, China has the largest number of publications, citations, and *h* index. However, the average cites per publication is lower compared to other countries, occupying the sixth place. The second country with the widest number of publications as well as citations is Spain, with a record of 341 and 18,621, respectively. The United States of America, which has a total of 268 publications, ranks third.

Table 2.5. Top 50 most productive countries in OWA

R	Country	TP	% TP	TC	Avg	*h*	≥ 500	≥ 100	≥ 50
1	China	884	38.32%	37,101	41.97	97	6	91	193
2	Spain	341	14.78%	18,621	54.61	67	5	48	88
3	United States of America	268	11.62%	18,089	67.50	55	4	30	62
4	Iran	149	6.46%	3,058	20.52	31	0	3	18
5	India	111	4.81%	3,095	27.88	26	0	6	18
5	Pakistan	111	4.81%	2,249	20.26	27	0	4	12
7	England	93	4.03%	5,178	55.68	37	0	16	34
8	Italy	91	3.94%	2,403	26.41	23	0	4	12
9	Australia	76	3.29%	2,237	29.43	24	0	7	16
10	Slovakia	73	3.16%	1,394	19.10	21	0	3	7
11	Canada	72	3.12%	2,274	31.58	24	0	3	16
12	Saudi Arabia	71	3.08%	2,105	29.65	21	0	9	12
13	Chile	70	3.03%	1,204	17.20	20	0	1	6
14	Taiwan	69	2.99%	2,330	33.77	28	0	6	15
15	Poland	61	2.64%	1,469	24.08	21	0	3	8
16	Czech Republic	57	2.47%	1,124	19.72	18	0	2	7
17	Turkey	50	2.17%	1,198	23.96	15	0	4	8
18	France	48	2.08%	1,156	24.08	19	0	1	8
19	South Korea	45	1.95%	778	17.29	16	0	0	5
20	Malaysia	43	1.86%	449	10.44	11	0	0	1
21	Japan	38	1.65%	1,141	30.03	17	0	1	9
22	Mexico	32	1.39%	304	9.50	9	0	0	0
23	Belgium	31	1.34%	956	30.84	13	0	2	6
24	Brazil	24	1.04%	522	21.75	10	0	1	4
24	Colombia	24	1.04%	255	10.63	8	0	0	1
26	Germany	23	1.00%	673	29.26	12	0	3	4
27	Egypt	21	0.91%	151	7.19	7	0	0	0
28	Finland	19	0.82%	532	28.00	10	0	1	3
29	Greece	17	0.74%	305	17.94	9	0	1	2
29	Oman	17	0.74%	569	33.47	11	0	2	3
31	Netherlands	15	0.65%	471	31.40	10	0	2	4
32	Austria	14	0.61%	737	52.64	11	0	3	4

R	Country	TP	% TP	TC	Avg	*h*	≥ 500	≥ 100	≥ 50
33	Romania	12	0.52%	205	17.08	5	0	1	1
33	Wales	12	0.52%	418	34.83	10	0	0	5
35	Cuba	11	0.48%	89	8.09	4	0	0	1
35	Hungary	11	0.48%	822	74.73	7	0	3	4
37	Israel	10	0.43%	242	24.20	8	0	0	1
37	Lithuania	10	0.43%	271	27.10	8	0	0	2
39	Nigeria	9	0.39%	71	7.89	5	0	0	0
39	Serbia	9	0.39%	69	7.67	5	0	0	0
41	Algeria	8	0.35%	50	6.25	4	0	0	0
41	Argentina	8	0.35%	69	8.63	7	0	0	0
41	Portugal	8	0.35%	312	39.00	5	0	1	3
41	Thailand	8	0.35%	110	13.75	6	0	0	0
45	Ireland	7	0.30%	50	7.14	5	0	0	0
46	Denmark	6	0.26%	79	13.17	5	0	0	0
46	North Ireland	6	0.26%	120	20.00	5	0	0	0
46	Singapore	6	0.26%	76	12.67	3	0	0	0
49	Russia	5	0.22%	41	8.20	3	0	0	0
49	South Africa	5	0.22%	56	11.20	4	0	0	0
49	United Arab Emirates	5	0.22%	70	14.00	4	0	0	0

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1, 2.2, and 2.3.

Leading journals in OWA

Journals play a particularly important role in the dissemination and advance of science. Table 2.6 presents the top 50 journals with the most publications in OWA. The *International Journal of Intelligent Systems* is the one with most publications, with a record of 202 publications, which equals to an 8.76% over the total. Currently, this prestigious journal is part of a partnership between two publishers, which are Wiley and Hindawi. The second most productive is the *Journal of Intelligent & Fuzzy Systems*, with a total of 131 publications and a 5.68% share. The publisher of this well-known journal is IOS Press. Nevertheless, the number of citations is well below the third most productive journal, which is *Fuzzy Sets and Systems*. This respected journal is published by Elsevier.

It should be also emphasized that the *Information Fusion* journal from Elsevier, is the one with the highest impact factor (IF), also referred to as journal impact factor (JIF). Recall that the IF is a scientometric index calculated by Clarivate Analytics, and it reflects the number of times an average paper in a journal has been cited during a specific year or period.

Table 2.6. Top 50 most productive journals in OWA

R	Journal	TP	% TP	TC	Avg	*h*	IF 2021	IF 5Y
1	International Journal of Intelligent Systems	202	8.76%	8,970	44.41	43	8.993	8.926
2	Journal of Intelligent & Fuzzy Systems	131	5.68%	2,938	22.43	28	1.737	1.664
3	Fuzzy Sets and Systems	93	4.03%	7,791	83.77	35	4.462	3.581
4	Information Sciences	83	3.60%	6,217	74.90	38	8.233	7.299
5	IEEE Transactions on Fuzzy Systems	71	3.08%	6,004	84.56	36	12.253	11.637
6	International Journal of Uncertainty Fuzziness and Knowledge-Based Systems	65	2.82%	2,065	31.77	22	1.027	1.234
7	Expert Systems with Applications	61	2.64%	3,749	61.46	34	8.665	8.093
8	Soft Computing	51	2.21%	1,448	28.39	18	3.732	3.524
9	Knowledge-Based Systems	47	2.04%	2,171	46.19	25	8.139	8.153
10	Computers & Industrial Engineering	45	1.95%	2,178	48.40	26	7.18	6.876
11	Applied Soft Computing	41	1.78%	2,523	61.54	26	8.263	7.595
12	International Journal of Fuzzy Systems	33	1.43%	1,110	33.64	16	4.085	3.718
13	European Journal of Operational Research	29	1.26%	1,998	68.90	18	6.363	6.598
13	International Journal of Approximate Reasoning	29	1.26%	3,082	106.28	19	4.452	3.544
15	Group Decision and Negotiation	25	1.08%	1,174	46.96	17	2.928	2.527
16	International Journal of General Systems	22	0.95%	816	37.09	12	2.435	2.088
17	Symmetry-Basel	21	0.91%	177	8.43	9	2.94	2.834
18	International Journal of Computational Intelligence Systems	20	0.87%	812	40.60	12	2.259	2.244
18	Mathematical Problems in Engineering	20	0.87%	202	10.10	7	1.43	1.393
20	Information Fusion	18	0.78%	1,958	108.78	16	17.564	16.58
20	Mathematics	18	0.78%	166	9.22	6	2.592	2.542
22	Technological and Economic Development of Economy	17	0.74%	433	25.47	11	5.656	4.502
23	IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics	16	0.69%	2,788	174.25	14	6.22[a]	6.184[a]
24	Applied Mathematical Modelling	15	0.65%	837	55.80	14	5.336	4.522
24	International Journal of Information Technology & Decision Making	15	0.65%	451	30.07	9	3.508	2.956
26	Economic Computation and Economic Cybernetics Studies and Research	14	0.61%	169	12.07	8	0.899	0.997
26	Granular Computing	14	0.61%	276	19.71	9	-	-
26	IEEE Access	14	0.61%	198	14.14	8	3.476	3.758
26	Sustainability	14	0.61%	158	11.29	8	3.889	4.089
30	Iranian Journal of Fuzzy Systems	13	0.56%	307	23.62	6	2.006	1.866

R	Journal	TP	% TP	TC	Avg	*h*	IF 2021	IF 5Y
31	Annals of Operations Research	12	0.52%	147	12.25	8	4.82	4.46
31	Cybernetics and Systems	12	0.52%	294	24.50	9	1.859	1.832
31	Fuzzy Optimization and Decision Making	12	0.52%	385	32.08	8	5.274	4.614
34	International Journal of Knowledge-Based and Intelligent Engineering Systems	11	0.48%	226	20.55	6	-	-
34	International Journal of Machine Learning and Cybernetics	11	0.48%	130	11.82	8	4.377	3.764
34	Kybernetes	11	0.48%	143	13.00	7	2.352	2.158
37	Water Resources Management	10	0.43%	129	12.90	7	4.426	4.415
38	Applied Intelligence	9	0.39%	152	16.89	6	5.019	4.76
38	Ecological Indicators	9	0.39%	208	23.11	6	6.263	6.643
38	Journal of Applied Mathematics	9	0.39%	85	9.44	4	0.72	0.735
38	Journal of Systems Engineering and Electronics	9	0.39%	183	20.33	7	1.363	1.369
42	Arabian Journal for Science and Engineering	8	0.35%	316	39.50	5	2.807	2.621
42	Engineering Applications of Artificial Intelligence	8	0.35%	134	16.75	6	7.802	6.694
42	Informatica	8	0.35%	63	7.88	5	3.429	2.553
42	Land Use Policy	8	0.35%	338	42.25	7	6.189	6.158
46	IEEE Transactions on Systems, Man, and Cybernetics, Part A: Systems and Humans	7	0.30%	1,813	259.00	7	2.183[b]	2.44[b]
46	International Journal of Advanced Manufacturing Technology	7	0.30%	77	11.00	6	3.563	3.471
46	Scientia Iranica	7	0.30%	164	23.43	6	1.416	1.387
49	Computational & Applied Mathematics	6	0.26%	123	20.50	4	2.998	2.408
49	Journal of Ambient Intelligence and Humanized Computing	6	0.26%	184	30.67	4	3.662	3.718
49	Journal of Cleaner Production	6	0.26%	186	31.00	6	11.072	11.016
49	Journal of Environmental Management	6	0.26%	264	44.00	5	8.91	8.549
49	Journal of Intelligent Systems	6	0.26%	56	9.33	4	-	-
49	Natural Hazards	6	0.26%	288	48.00	5	3.158	3.685
49	Neural Computing & Applications	6	0.26%	105	17.50	5	5.102	5.13

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1, 2.2, and 2.3 except for: IF 2021 = 2021 impact factor; IF 5Y = 5-year impact factor. Footnotes: a = Latest available year 2014; b = Latest available year 2012.

Leading research areas in OWA

In order to get an enhanced understanding of the OWA research areas, Table 2.7 lists the top 50. It can clearly be seen that Computer Science is leading the ranking of the most productive research areas. Similarly, the OWA operator plays a key role in other fields such as Engineering and Mathematics.

Table 2.7. Top 50 most productive research areas in OWA

R	Research area	TP	TC	Avg	*h*
1	Computer Science	1,450	71,881	49.57	123
2	Engineering	463	23,241	50.20	70
3	Mathematics	354	11,880	33.56	52
4	Operations Research Management Science	201	8,584	42.71	52
5	Business Economics	141	4,947	35.09	39
6	Environmental Sciences Ecology	124	3,053	24.62	31
7	Science Technology Other Topics	101	1,961	19.42	24
8	Automation Control Systems	98	5,272	53.80	31
9	Water Resources	47	1,172	24.94	19
10	Geology	36	1,315	36.53	17
11	Telecommunications	35	597	17.06	13
12	Social Sciences Other Topics	30	1,219	40.63	18
13	Energy Fuels	27	1,313	48.63	19
14	Remote Sensing	21	677	32.24	12
15	Agriculture	20	398	19.90	10
16	Physics	18	218	12.11	6
17	Geography	17	932	54.82	10
17	Materials Science	17	140	8.24	5
17	Mechanics	17	1,009	59.35	15
20	Physical Geography	16	751	46.94	10
21	Meteorology Atmospheric Sciences	15	560	37.33	10
22	Imaging Science Photographic Technology	13	156	12.00	7
23	Chemistry	12	59	4.92	5
23	Forestry	12	219	18.25	7
23	Information Science Library Science	12	867	72.25	8
23	Mathematical Computational Biology	12	93	7.75	5
27	Biodiversity Conservation	10	219	21.90	7
27	Instruments Instrumentation	10	124	12.40	6
29	Mathematical Methods in Social Sciences	9	223	24.78	5
29	Neurosciences Neurology	9	139	15.44	5
29	Public Environmental Occupational Health	9	276	30.67	7
32	Construction Building Technology	8	70	8.75	4
33	Robotics	7	10	1.43	2
33	Thermodynamics	7	420	60.00	5
35	Geochemistry Geophysics	6	87	14.50	4
35	Marine Freshwater Biology	6	71	11.83	4
35	Transportation	6	210	35.00	3
38	Biochemistry Molecular Biology	5	26	5.20	4
38	Radiology Nuclear Medicine Medical Imaging	5	60	12.00	3
40	Education Educational Research	4	31	7.75	2
40	Life Sciences Biomedicine Other Topics	4	57	14.25	3
42	Biotechnology Applied Microbiology	3	17	5.67	2
42	Health Care Sciences Services	3	25	8.33	3
42	Medical Informatics	3	40	13.33	3
42	Oceanography	3	41	13.67	3

R	Research area	TP	TC	Avg	*h*
42	Psychology	3	155	51.67	2
47	Dentistry Oral Surgery Medicine	2	4	2.00	1
47	Evolutionary Biology	2	33	16.50	1
47	International Relations	2	33	16.50	2
47	Linguistics	2	2	1.00	1
47	Nuclear Science Technology	2	74	37.00	2
47	Otorhinolaryngology	2	5	2.50	2
47	Pharmacology Pharmacy	2	13	6.50	2
47	Plant Sciences	2	6	3.00	1
47	Public Administration	2	46	23.00	2
47	Urban Studies	2	25	12.50	2

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1, 2.2, and 2.3.

Temporal evolution of the most productive authors, institutions, countries, journals, and research areas in OWA

Next, Tables 2.8, 2.9, 2.10, 2.11, and 2.12 display the evolution of the most productive authors, institutions, countries, journals, and research areas in OWA through the last three decades. Starting with the results of the authors, during the periods of 1992-2001 and 2002-2011, Yager was the most productive with 30 and 38 publications, respectively. Nevertheless, during the period of 2012-2021, it was Merigó with 93 publications.

If we analyze the most productive institutions through time, as is apparent, the Iona College, represented primarily by Yager, was the leading institution during the past three decades. Additionally, during the period of 1992-2001, the University of Granada was the second institution, basically explained by the professors Herrera, Herrera-Viedma, and Verdegay. Nonetheless, during the period of 2002-2011, the Southeast University China has managed to establish itself as the second most productive institution, mainly driven by the researchers X. W. Liu and Z. S. Xu. However, during the period of 2012-2021, the University of Chile took the second place, which came from the contributions made by Merigó.

Likewise, during the past decades, China has experienced a significant growth in academic research productivity in OWA. Spain and the United States of America, on the other hand, have remained almost constant over the past 30 years.

Table 2.8. Productivity evolution of the authors over the last three decades

R	1992-2001 Author	TP	TC	R	2002-2011 Author	TP	TC
1	Yager, RR	30	4,933	1	Yager, RR	38	3,269
2	Herrera, F	10	4,647	2	Xu, ZS	32	9,263
3	Herrera-viedma, E	9	3,713	3	Merigó, JM	27	2,310
4	Filev, DP	7	1,626	4	Liu, XW	20	777
5	Mitchell, HB	6	194	5	Herrera-Viedma, E	18	3,115
5	Torra, V	6	634				

R	2012-2021 Author	TP	TC
1	Merigó, JM	93	2,428
2	Yager, RR	61	826
3	Mesiar, R	45	605
4	Zeng, SZ	44	1,296
5	Chen, HY	40	1,208

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1 and 2.2.

Table 2.9. Productivity evolution of the institutions over the last three decades

R	1992-2001 Institution	TP	TC	R	2002-2011 Institution	TP	TC
1	Iona College	30	4,933	1	Iona College	39	3,575
2	University of Granada	12	4,856	2	Southeast University China	36	5,897
3	Elta Elect Ind Ltd	5	123	3	University of Barcelona	28	2,671
3	University of the Balearic Islands	5	96	4	University of Granada	22	3,373
3	Rovira i Virgili University	5	546	5	Slovak University of Technology Bratislava	13	526

R	2012-2021 Institution	TP	TC
1	Iona College	66	884
2	University of Chile	57	1,147
3	Slovak University of Technology Bratislava	51	759
3	University of Barcelona	51	1,475
5	Nanjing Normal University	43	469

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1 and 2.2.

Moreover, the *Fuzzy Sets and Systems* journal has been progressively losing weight. In contrast, the *International Journal of Intelligent Systems* has managed to consolidate its position. Both journals are placed in the Q1 quartile (2021). Also outstanding is the number of documents successfully published by the *Journal of Intelligent & Fuzzy Systems* during the period of 2012-2021.

Table 2.10. Productivity evolution of the countries over the last three decades

R	1992-2001 Country	TP	TC	R	2002-2011 Country	TP	TC
1	United States of America	39	5,511	1	China	126	14,981
2	Spain	31	5,728	2	Spain	86	7,358
3	Belgium	7	359	3	United States of America	81	4,927
3	Israel	7	203	4	Iran	33	972
5	Italy	4	479	5	Taiwan	29	1,214
5	Japan	4	97				

R	2012-2021 Country	TP	TC
1	China	755	22,048
2	Spain	224	5,535
3	United States of America	144	2,713
4	Iran	116	2,086
5	Pakistan	110	2,190

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1 and 2.2.

Table 2.11. Productivity evolution of the journals over the last three decades

R	1992-2001 Journal	TP	TC	R	2002-2011 Journal	TP	TC
1	Fuzzy Sets and Systems	19	4,783	1	International Journal of Intelligent Systems	43	3,832
2	International Journal of Intelligent Systems	15	1,615	2	Fuzzy Sets and Systems	37	2,437
3	International Journal of Uncertainty Fuzziness and Knowledge-Based Systems	14	557	3	Expert Systems with Applications	30	2,152
4	International Journal of Approximate Reasoning	8	570	4	Information Sciences	26	3,203
5	European Journal of Operational Research	4	361	5	IEEE Transactions on Fuzzy Systems	19	3,570
5	IEEE Transactions on Fuzzy Systems	4	412				
5	Information Sciences	4	820				
5	International Journal of General Systems	4	235				

R	2012-2021 Journal	TP	TC
1	International Journal of Intelligent Systems	144	3,523
2	Journal of Intelligent & Fuzzy Systems	129	2,877
3	Information Sciences	53	2,194
4	IEEE Transactions on Fuzzy Systems	48	2,022
5	Knowledge-Based Systems	40	1,761

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1 and 2.2.

With regard to the research fields, Computer Science, Engineering, Mathematics, and Operations Research Management Science have always been the most popular. Although the research area of Environmental Sciences Ecology has become more relevant in the last decade of the study.

Table 2.12. Productivity evolution of the research areas over the last three decades

R	1992-2001			R	2002-2011		
	Research area	TP	TC		Research area	TP	TC
1	Computer Science	82	12,205	1	Computer Science	362	27,811
2	Mathematics	25	4,858	2	Engineering	119	8,630
3	Engineering	10	822	3	Mathematics	78	3,274
4	Business Economics	7	518	4	Operations Research Management Science	76	4,673
5	Agriculture	4	119	5	Automation Control Systems	32	1,948
5	Operations Research Management Science	4	361				

R	2012-2021		
	Research area	TP	TC
1	Computer Science	1,002	26,917
2	Engineering	332	8,847
3	Mathematics	251	3,746
4	Operations Research Management Science	121	3,548
5	Environmental Sciences Ecology	115	2,694

Source: Own elaboration through WoS. Abbreviations are available in Tables 2.1 and 2.2.

Analysis with VOS viewer

With the VOS viewer software, it is possible to obtain the citation and co-citation of cited references, author, and journals, as well as the occurrence and co-occurrence of keywords. Table 2.13 presents the most cited references among OWA publications. First, we have the document "*On ordered weighted averaging aggregation operators in multicriteria decisionmaking*", written by Yager (1988). Second, we find the document "*Fuzzy sets*", authoring Zadeh (1965). Third, we get the document "*Families of OWA operators*", from Yager (1993).

Table 2.13. Top 20 citing references of OWA

R	Cited reference (only first author)	Citations	TLS	PY
1	Yager RR, IEEE T Syst Man Cyb, V18, P183	1587	1535	1988
2	Zadeh LA, Inform Control, V8, P338	547	544	1965
3	Yager RR, Fuzzy Set Syst, V59, P125	473	470	1993
4	Atanassov KT, Fuzzy Set Syst, V20, P87	410	410	1986
5	Yager RR, IEEE T Syst Man Cy B, V29, P141	404	403	1999
6	Xu ZS, Int J Intell Syst, V20, P843	294	292	2005
7	Yager RR, The Ordered Weighted Averaging Operators	283	282	1997
8	Xu ZS, Int J Intell Syst, V18, P953	267	267	2003
9	Zadeh LA, Inform Sciences, V8, P199	256	256	1975
10	Xu ZS, IEEE T Fuzzy Syst, V15, P1179	242	242	2007
11	Beliakov G, Aggregation Functions	232	232	2007
12	Filev DP, Fuzzy Set Syst, V94, P157	229	228	1998
13	Yagger RR, Fuzzy Optim Decis Ma, V3, P93	224	224	2004
14	Merigó JM, Inform Sciences, V179, P729	221	221	2009
15	Xu ZS, Int J Gen Syst, V35, P417	219	219	2006
16	Zadeh LA, Comput Math Appl, V9, P149	211	211	1983
17	Herrera F, IEEE T Fuzzy Syst, V8, P746	189	188	2000
18	Fuller R, Fuzzy Set Syst, V124, P53	185	185	2001
19	Herrera F, Fuzzy Set Syst, V115, P67	175	175	2000
20	Yager RR, Fuzzy Set Syst, V137, P59	161	160	2003

Source: Own elaboration through VOS viewer. Abbreviations are available in Tables 2.1 and 2.2 except for: TLS = Total link strength.

The originality of the OWA operator has drawn the attention of many researcher from all over the world. Fig. 2.2 displays the co-citation network of cited authors among OWA publications. To do so, a minimum of 60 citations of an author are contemplated. Note that only the first author of a cited document is considered in the co-citation analysis of cited authors. Each node or circle constitutes an author, and the size of the node is proportional to the number of citations. Likewise, the lines represent the strongest co-citation relations between authors. Also, clusters are differentiated by colors. As can be seen, the biggest nodes correspond to the researchers Yager, Z. S. Xu, Merigó, Wei, Herrera, and Zadeh, respectively.

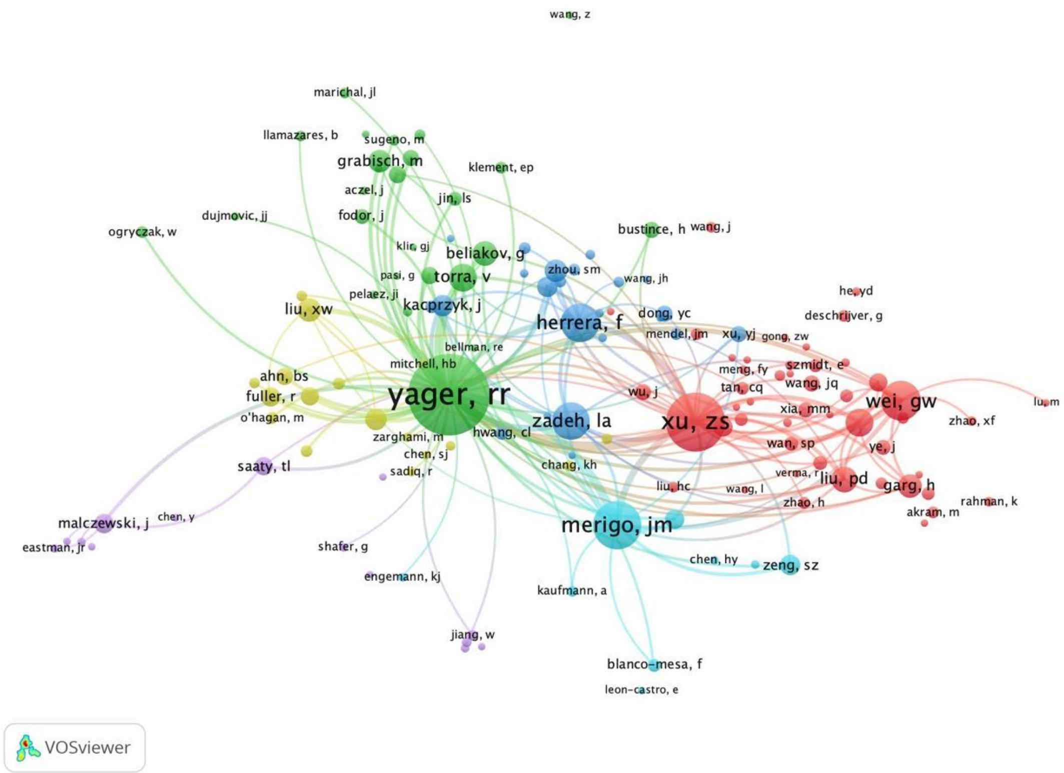

Fig. 2.2. Co-citation network of cited authors
Source: Own elaboration through VOS viewer

Similarly, Fig. 2.3 visualizes the co-citation network of cited journals among OWA publications, taking into account a minimum of 130 citations of a journal. In this case, each node represents a journal. The bigger the node, the higher the number of citations received by the journal. The major co-citation links between journals are illustrated with lines. The color of the node indicates the cluster. It can be seen that the largest nodes are those from *Fuzzy Sets and Systems*, *International Journal of Intelligent Systems*, and *Information Sciences*. Further, these last two journals are likely to be strongly related, as they are placed very close to each other.

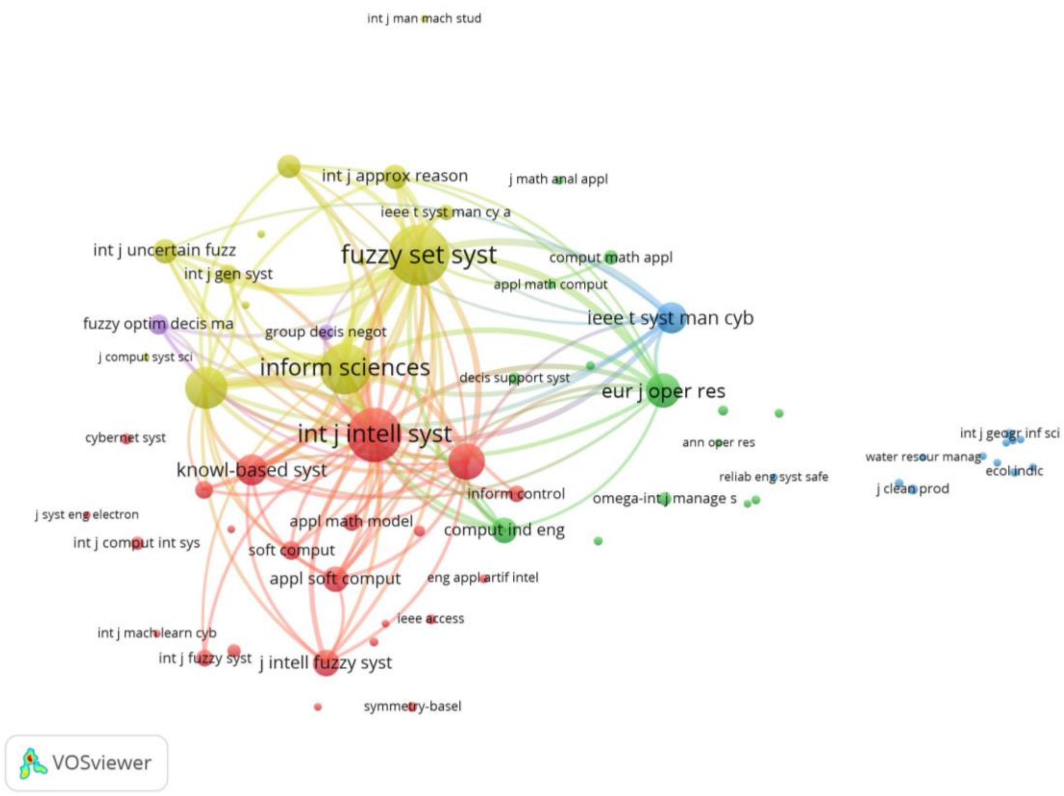

Fig. 2.3. Co-citation network of cited journals
Source: Own elaboration through VOS viewer

Next, Fig. 2.4 presents the co-occurrence network of keywords, while considering a threshold of 20 occurrences of a keyword. Each node represents a keyword. The node size reflects the keyword frequency (the higher the frequency, the larger the node). The node color indicates the cluster to which keywords belong. The lines denote the strongest co-occurrence links. We can observe five different clusters and that the most frequent keywords are “OWA operators”, “aggregation operators”, “model”, and “group decision-making”, respectively.

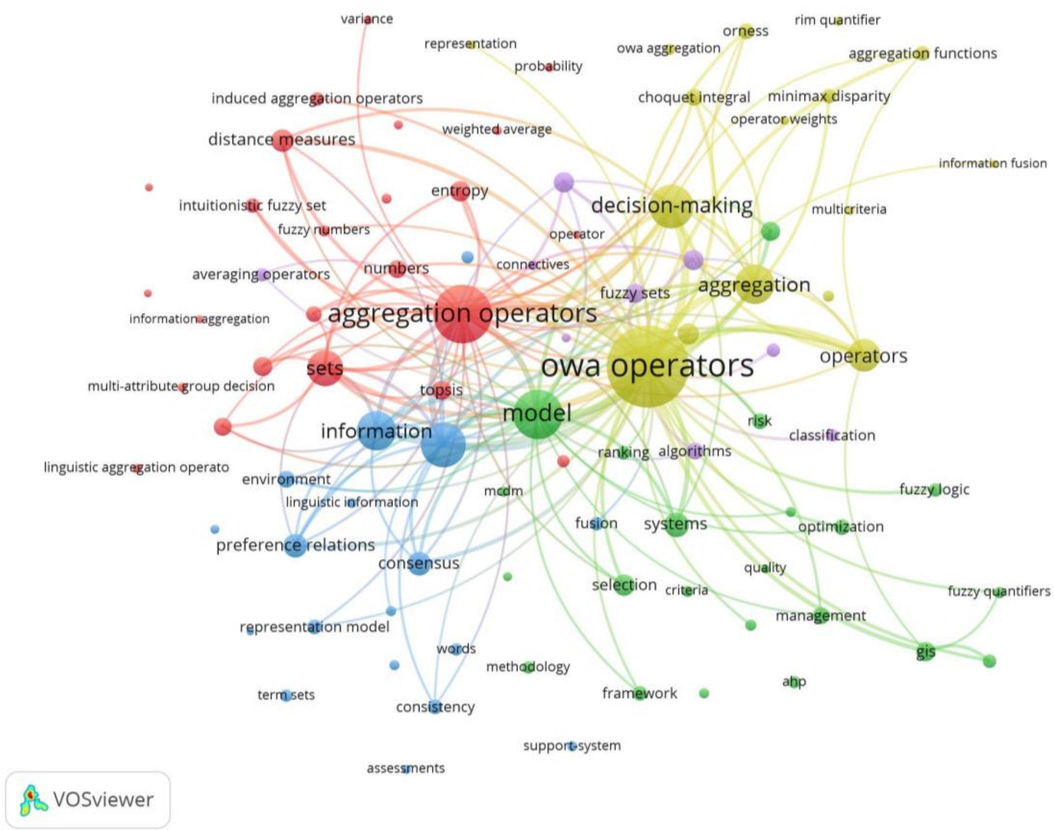

Fig. 2.4. Co-occurrence network of keywords
Source: Own elaboration through VOS viewer

4. Conclusions

This paper presented a very comprehensive bibliometric analysis of the OWA operator from 1988 to 2021 based on the WoS Core Collection database and also the VOS viewer software. One of the main conclusions is that there is a growing interest in the OWA operator. This is reflected in the fact that the number of publications has increased significantly since 1988.

Another finding is that Yager is the most prolific and influential author regarding the OWA operator. He also has written the most cited document, which is the "*On ordered weighted averaging aggregation operators in multicriteria decisionmaking*", where the OWA operator is introduced for the very first time. Additionally, he represents the Iona College, which is the leading institution in OWA. Moreover, in the analysis of the journals with most publications as well as citations, the distinguished *International Journal* *of Intelligent Systems* heads the ranking. Likewise, based on the obtained results, we can confirm that China has the largest number of publications and citations. Besides, it is noteworthy that Computer Science is by far the preferred research area.

Furthermore, some conclusions can be drawn from the citation and co-citation analysis of cited references, authors, and journals, as well as occurrence and co-occurrence of keywords. For example, that among OWA publications, the most cited reference is the "*On ordered weighted averaging aggregation operators in multicriteria decisionmaking*", the most cited author is Yager, the most cited journal is the *Fuzzy Sets and Systems*, and the most frequent keyword is "OWA operators".

This research has also some limitations. One of these limitations is the use of only WoS Core Collection database. Thus, future research should include additional databases like Elsevier's Scopus. Another restriction is the selection of solely articles, review articles, letters, and notes, disregarding other types of documents, such as proceeding papers. A limitation is also the fact that through time some authors may change the institution to which they belong.

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2.2. Mathematical methods under uncertainty

2.2.1. Decision-making

Decision-making can be defined as the process of identifying and evaluating a set of alternatives and choosing the best one.

Decision-making can be done under three different condition (Riabacke, 2006): certainty, risk, and uncertainty. Below, a generic explanation is provided for each of these states.

- Decision-making under certainty: When the decision-maker is fully informed and thus knows for sure the outcome of each alternative.
- Decision-making under risk: When the decision-maker does not know what the exact outcomes of each alternative will be. However, he/she knows the probabilities.
- Decision-making under uncertainty: A decision under uncertainty occurs when the probabilities associated with the outcomes of each alternative are completely unknown by the decision-maker.

2.2.2. Interval numbers

2.2.2.1. Defining interval numbers

The origin of the interval numbers, also known as confidence intervals, is found in the work "*Interval analysis*" (Moore, 1966). Suppose a magnitude whose exact value x is unknown; however, it is known that it is greater than or equal to a_1, and lower than or equal to a_2. So, it is assumed that the value of the magnitude belongs to the segment $[a_1, a_2]$, called "interval number". The formal definition of an interval number is as follows:

$$A = [a_1, a_2] = \{x \in R: a_1 \leq x \leq a_2\}. \tag{2.1}$$

Although the closed interval is the most common one, it is possible to consider other types of intervals (Kaufmann & Gil-Aluja, 1990):

- The right half-open interval and defined as $[a_1, a_2) = [a_1, a_2[= \{x \in R: a_1 \leq x < a_2\}$.
- The left half-open interval and defined as $(a_1, a_2] =]a_1, a_2] = \{x \in R: a_1 < x \leq a_2\}$.
- The open interval and defined as $(a_1, a_2) =]a_1, a_2[= \{x \in R: a_1 < x < a_2\}$.

Also, it is possible to represent an uncertain value of a magnitude through confidence triplets and confidence quadruples:

- A confidence triplet $[a_1, a_2, a_3]$ is formed by three values, where a_1 is the lower limit, a_3 is the upper limit, and a_2 corresponds to the maximum presumption, i.e., maximum probability.
- A confidence quadruple $[a_1, a_2, a_3, a_4]$ is formed by four values, where a_1 is the lower limit, a_4 is the upper limit, and the subinterval $[a_2, a_3]$ corresponds to the maximum presumption.

2.2.2.2. Operations on interval numbers

Consider the interval numbers $A = [a_1, a_2] \subset R$, $B = [b_1, b_2] \subset R$, and $C = [c_1, c_2] \subset R$ for the following basic operations.

Addition of two interval numbers

The addition of the interval numbers A and B results in:

$$A(+)B = [a_1, a_2](+)[b_1, b_2] = [a_1 + b_1, a_2 + b_2]. \quad (2.2)$$

The properties are the following three:

- Commutative. The sum is always the same regardless of the order in which the interval numbers are added., i.e., $A(+)B = B(+)A$.
- Associative. The sum is always the same regardless of the way in which the interval numbers are grouped., i.e., $(A(+)B)(+)C = A(+)(B(+)C)$.

- Identity. The sum of zero and any number is always the number, i.e., $A(+)0 = 0(+)A = A$.

Subtraction of two interval numbers

The subtraction of the interval numbers A and B results in:

$$A(-)B = [a_1, a_2](-)[b_1, b_2] = [a_1 - b_2, a_2 - b_1]. \tag{2.3}$$

As is evident, the properties of commutative and associative are not applicable to the subtraction.

Multiplication of two interval numbers

The multiplication of the interval numbers A and B results in:

$$\begin{aligned} A(\cdot)B &= [a_1, a_2](\cdot)[b_1, b_2] \\ &= [Min(a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2), \\ &Max(a_1 \cdot b_1, a_1 \cdot b_2, a_2 \cdot b_1, a_2 \cdot b_2)]. \end{aligned} \tag{2.4}$$

The properties are the following three:

- Commutative. The product is always the same regardless of the order in which the interval numbers are multiplied., i.e., $A(\cdot)B = B(\cdot)A$.
- Associative. The product is always the same regardless of the way in which the interval numbers are multiplied., i.e., $(A(\cdot)B)(\cdot)C = A(\cdot)(B(\cdot)C)$.
- Identity. The product of one and any number is always the number, i.e., $A(\cdot)1 = 1(\cdot)A = A$.

Division of two interval numbers

The division of the interval numbers A and B results in:

$$\begin{aligned} A(:)B &= [a_1, a_2](:)[b_1, b_2] \\ &= \left[Min\left(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right), Max\left(\frac{a_1}{b_1}, \frac{a_1}{b_2}, \frac{a_2}{b_1}, \frac{a_2}{b_2}\right)\right]. \end{aligned} \tag{2.5}$$

Particular case of interval numbers in R^+

For the multiplication, we have:

$$A(\cdot)B = [a_1, a_2](\cdot)[b_1, b_2] = [a_1 \cdot b_1, a_2 \cdot b_2]. \tag{2.6}$$

For the division, we have:

$$A(:)B = [a_1, a_2](:)[b_1, b_2] = \left[\frac{a_1}{b_2}, \frac{a_2}{b_1}\right]. \tag{2.7}$$

For the addition and subtraction there is no change.

Other operations on interval numbers in R

Minimization:

$$\begin{aligned} A(\wedge)B = [a_1, a_2](\wedge)[b_1, b_2] &= [Min(a_1, b_1), Min(a_2, b_2)] \\ &= [a_1 \wedge b_1, a_2 \wedge b_2]. \end{aligned} \tag{2.8}$$

Maximization:

$$\begin{aligned} A(\vee)B = [a_1, a_2](\vee)[b_1, b_2] &= [Max(a_1, b_1), Max(a_2, b_2)] \\ &= [a_1 \vee b_1, a_2 \vee b_2]. \end{aligned} \tag{2.9}$$

There are five mathematical properties, which are:

- Commutative $\begin{cases} A(\wedge)B = B(\wedge)A \\ A(\vee)B = B(\vee)A \end{cases}.$ (2.10)
- Associative $\begin{cases} (A(\wedge)B)(\wedge)C = A(\wedge)(B(\wedge)C) \\ (A(\vee)B)(\vee)C = A(\vee)(B(\vee)C) \end{cases}.$ (2.11)
- Idempotent $\begin{cases} A(\wedge)A = A \\ A(\vee)A = A \end{cases}.$ (2.12)
- Absorption $\begin{cases} A(\wedge)(A(\vee)B) = A \\ A(\vee)(A(\wedge)B) = A \end{cases}.$ (2.13)

- Distributive $\begin{cases} A(\vee)(B(\wedge)C) = (A(\vee)B)(\wedge)(A(\vee)C) \\ A(\wedge)(B(\vee)C) = (A(\wedge)B)(\vee)(A(\wedge)C) \end{cases}$. (2.14)

2.2.3. Fuzzy set theory

2.2.3.1. Classical sets

Before entering into the description of the fuzzy set theory, it is worth to review some basic but necessary terminologies with regard to classical set theory.

Universal set

A universal set X, or universe of discourse, is a nonempty set that contains all the possible elements x of concern in a particular context. The membership function (or characteristic function) of the universal set X, written as $\mu_X(x)$, is defined as:

$$\mu_X(x) = 1 \quad \forall x \in X. \tag{2.15}$$

Empty set

An empty set $\emptyset$ is described as a set that has no elements. The membership function of the empty set $\emptyset$, written as $\mu_\emptyset(x)$, is defined as:

$$\mu_\emptyset(x) = 0 \quad \forall x \in X. \tag{2.16}$$

Classical set

A classical set A, also referred as crisp set or ordinary set, can be described as a collection of well-defined elements x from a universal set X. So, it has precise boundaries. In this case, the membership function can take two different values which are 1 or 0. Thus, the membership is considered to be binary. If an element x of the universal set X belongs to the set A, then the membership value (or degree of membership) is 1. Otherwise, if an element x of the universal set X does not belong to the set A, then the membership value is 0. Hence, the membership function of the set $A \subseteq X$ is $\mu_A: X \to \{0,1\}$ defined as:

$$\mu_A(x) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases}. \tag{2.17}$$

For example, consider the universal set $X = \{a, b, c, d, e, f\}$ and the classical set $A = \{a, d, f\}$. It can be written as:

	a	b	c	d	e	f
$X =$	1	1	1	1	1	1

	a	b	c	d	e	f
$A =$	1	0	0	1	0	1

2.2.3.2. Fuzzy sets

The fuzzy set theory (also known as fuzzy subset theory) was introduced by Zadeh (1965) as an extension of the classical notion of sets. Later, Bellman and Zadeh (1970) surveyed for the first time fuzzy set theory within multi-criteria decision-making (MCDM). The fuzzy set theory has proven to be a powerful mathematical tool for dealing with uncertainty, imprecision, vagueness, and ambiguity in several fields (Dubois & Prade, 1980; Zimmermann, 1987). A fuzzy set can be defined as follows.

Definition. If X is a universal set with a collection of elements represented by x, then a fuzzy set $\tilde{A}$ of X is a set of ordered pairs defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x)) | x \in X\}, \tag{2.18}$$

where $\mu_{\tilde{A}}(x)$ is the membership function of x in $\tilde{A}$ that takes values in the closed interval $[0,1]$.

Thus, the main difference between a fuzzy set and a classical set is that the fuzzy set allows to consider elements with partial degree of membership. In fact, classical sets are special cases of fuzzy sets where $\mu_{\tilde{A}}(x)$ is only 1 or 0.

For example, consider the same universal set as the previous example, i.e., $X = \{a, b, c, d, e, f\}$, and a fuzzy set $\tilde{A}$ whose elements have the following membership values:

	a	b	c	d	e	f
$\tilde{A} =$	1	0.4	0.6	1	0	1

Fig. 2.5 compares with Venn diagrams the classical set A and the fuzzy set $\tilde{A}$ from the previous examples. For the fuzzy set $\tilde{A}$ it can be seen that a, d, and f take full membership, b and c take partial membership, and e take non-membership.

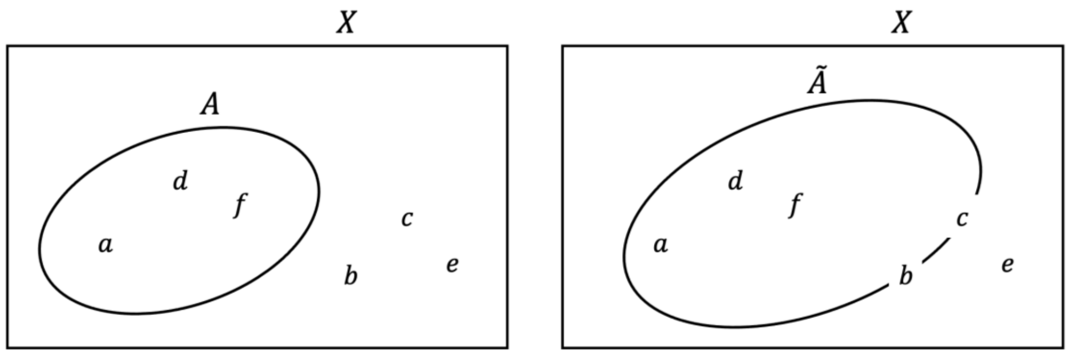

Fig. 2.5. Boundary region of the classical set and the fuzzy set
Source: Own elaboration

2.2.3.3. Fuzzy numbers

The concept of fuzzy numbers was first introduced by Chang and Zadeh (1972) in order to deal with imprecise numerical quantities in a practical way. Thus, fuzzy numbers are used for uncertainty modelling. A fuzzy number is defined as follows.

Definition. A fuzzy number $\tilde{A}$ is a fuzzy set of a universe of discourse that is both normal and convex.

Normality implies that $\exists x \in R$, $\mu_{\tilde{A}}(x) = 1$. Convexity means that for all $x_1, x_2 \in R$ and $\lambda \in [0,1]$, $\mu_{\tilde{A}}(\lambda x_1 + (1 - \lambda)x_2) \geq min\big(\mu_{\tilde{A}}(x_1), \mu_{\tilde{A}}(x_2)\big)$.

A comparison between a normalized fuzzy number and a non-normalized fuzzy number (or subnormalized fuzzy number) is shown in Fig. 2.6 and Fig. 2.7.

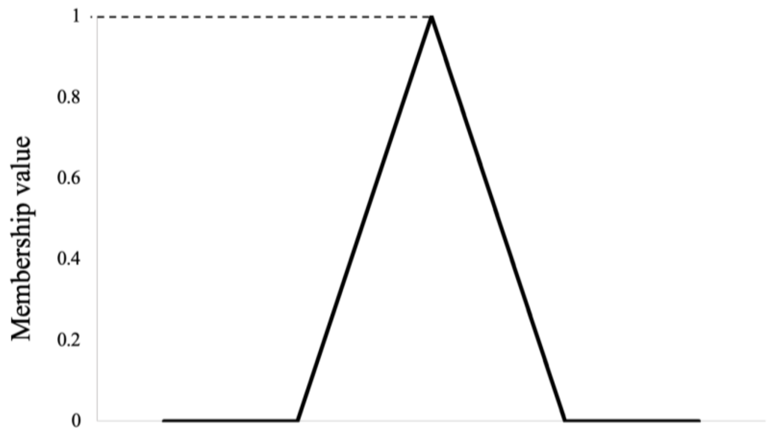

Fig. 2.6. Normalized fuzzy number
Source: Own elaboration

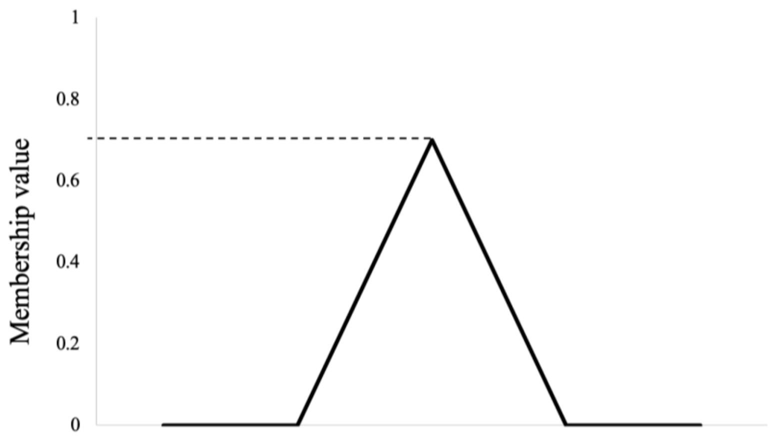

Fig. 2.7. Non-normalized fuzzy number
Source: Own elaboration

There exist different types of fuzzy numbers, such as triangular fuzzy numbers, trapezoidal fuzzy numbers, interval-valued fuzzy numbers, intuitionistic fuzzy numbers, generalized fuzzy numbers, and more.

The membership function of a triangular fuzzy number $\tilde{A} = (a, b, c)$ can be written as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ \dfrac{c-x}{c-b}, & b \leq x \leq c \\ 0, & x \geq c \end{cases}. \tag{2.19}$$

A triangular fuzzy number can be graphically represented as in Fig. 2.8.

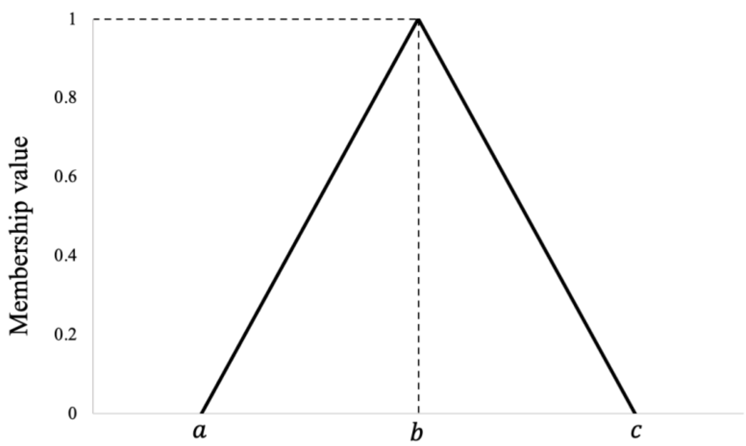

Fig. 2.8. Graphical representation of a triangular fuzzy number
Source: Own elaboration

The membership function of a trapezoidal fuzzy number $\tilde{A} = (a, b, c, d)$ can be written as follows:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0, & x \leq a \\ \dfrac{x-a}{b-a}, & a \leq x \leq b \\ 1, & b \leq x \leq c \\ \dfrac{d-x}{d-c}, & c \leq x \leq d \\ 0, & x \geq d \end{cases}. \tag{2.20}$$

A trapezoidal fuzzy number can be graphically represented as in Fig. 2.9.

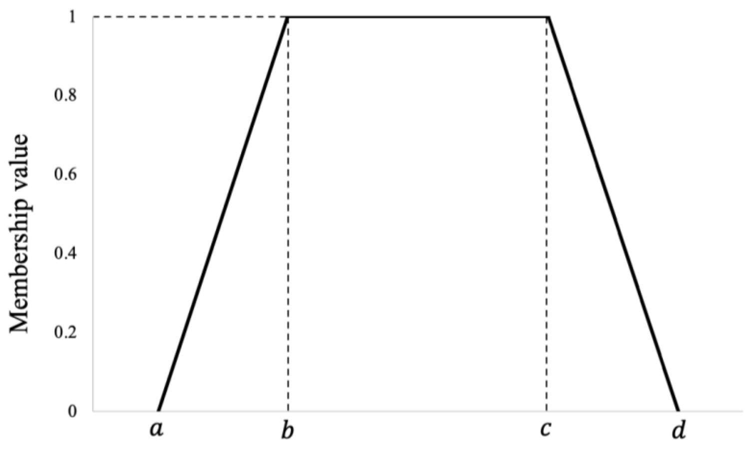

Fig. 2.9. Graphical representation of a trapezoidal fuzzy number
Source: Own elaboration

2.2.3.4. Fuzzy arithmetic

The four basic arithmetic operations are addition, subtraction, multiplication, and division. There are predominantly two approaches for implementing fuzzy arithmetic operations on fuzzy numbers (Fayek & Lourenzutti, 2018), which are: the α-cut (or α-level) approach and the Zadeh's extension principle approach (Zadeh, 1965).

Fuzzy arithmetic based on α-cuts

Given two fuzzy numbers $\tilde{A}$ and $\tilde{B}$, the following operations in R can be defined in terms of α-cuts.

Addition of two fuzzy numbers:

$$\tilde{A}(+)\tilde{B} \Leftrightarrow \forall\alpha \in [0,1], \quad (2.21)$$
$$A^{\alpha}(+)B^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}](+)[b_1^{\alpha}, b_2^{\alpha}] = [a_1^{\alpha} + b_1^{\alpha}, a_2^{\alpha} + b_2^{\alpha}].$$

Subtraction of two fuzzy numbers:

$$\tilde{A}(-)\tilde{B} \Leftrightarrow \forall\alpha \in [0,1], \tag{2.22}$$
$$A^{\alpha}(-)B^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}](-)[b_1^{\alpha}, b_2^{\alpha}] = [a_1^{\alpha} - b_1^{\alpha}, a_2^{\alpha} - b_2^{\alpha}].$$

Multiplication of two fuzzy numbers:

$$\tilde{A}(\cdot)\tilde{B} \Leftrightarrow \forall\alpha \in [0,1], \tag{2.23}$$
$$A^{\alpha}(\cdot)B^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}](\cdot)[b_1^{\alpha}, b_2^{\alpha}] = \begin{bmatrix} Min(a_1^{\alpha} \cdot b_1^{\alpha}, a_1^{\alpha} \cdot b_2^{\alpha}, a_2^{\alpha} \cdot b_1^{\alpha}, a_2^{\alpha} \cdot b_2^{\alpha}), \\ Max(a_1^{\alpha} \cdot b_1^{\alpha}, a_1^{\alpha} \cdot b_2^{\alpha}, a_2^{\alpha} \cdot b_1^{\alpha}, a_2^{\alpha} \cdot b_2^{\alpha}) \end{bmatrix}.$$

Division of two fuzzy numbers:

$$\tilde{A}(:)\tilde{B} \Leftrightarrow \forall\alpha \in [0,1], \tag{2.24}$$
$$A^{\alpha}(:)B^{\alpha} = [a_1^{\alpha}, a_2^{\alpha}](:)[b_1^{\alpha}, b_2^{\alpha}] = A^{\alpha}(\cdot)[B^{\alpha}]^{-1} = [a_1^{\alpha}, a_2^{\alpha}](\cdot)[b_1^{\alpha}, b_2^{\alpha}]^{-1}.$$

Fuzzy arithmetic based on the extension principle

Using the extension principle, the same operations, i.e., $\{+, -, \times, \div\}$, can be accomplished as shown below.

Addition of two fuzzy numbers:

$$\mu_{\tilde{A}(+)\tilde{B}}(z) = \bigvee_{z=x+y} \left(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)\right). \tag{2.25}$$

Subtraction of two fuzzy numbers:

$$\mu_{\tilde{A}(-)\tilde{B}}(z) = \bigvee_{z=x-y} \left(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)\right). \tag{2.26}$$

Multiplication of two fuzzy numbers:

$$\mu_{\tilde{A}(\cdot)\tilde{B}}(z) = \bigvee_{z=x\cdot y} \left(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y)\right). \tag{2.27}$$

Division of two fuzzy numbers:

$$\mu_{\tilde{A}(:)\tilde{B}}(z) = \bigvee_{z=x:y} \left(\mu_{\tilde{A}}(x) \wedge \mu_{\tilde{B}}(y) \right). \tag{2.28}$$

2.2.3.5. Linguistic variables

A linguistic variable is a variable whose values are words or sentences in a natural or artificial language (Zadeh, 1975a, 1975b, 1975c).

Formally, a linguistic variable can be defined as a quintuple $(N, T(N), U, G, M)$, where N is the name of the variable, $T(N)$ is the set of linguistic terms (or labels) which can be a value of the variable, U is the universe of discourse, G is a syntactic rule (a grammar) that produces the linguistic terms in $T(N)$, and M is a semantic rule that maps to each linguistic term its meaning, which is a fuzzy set on U.

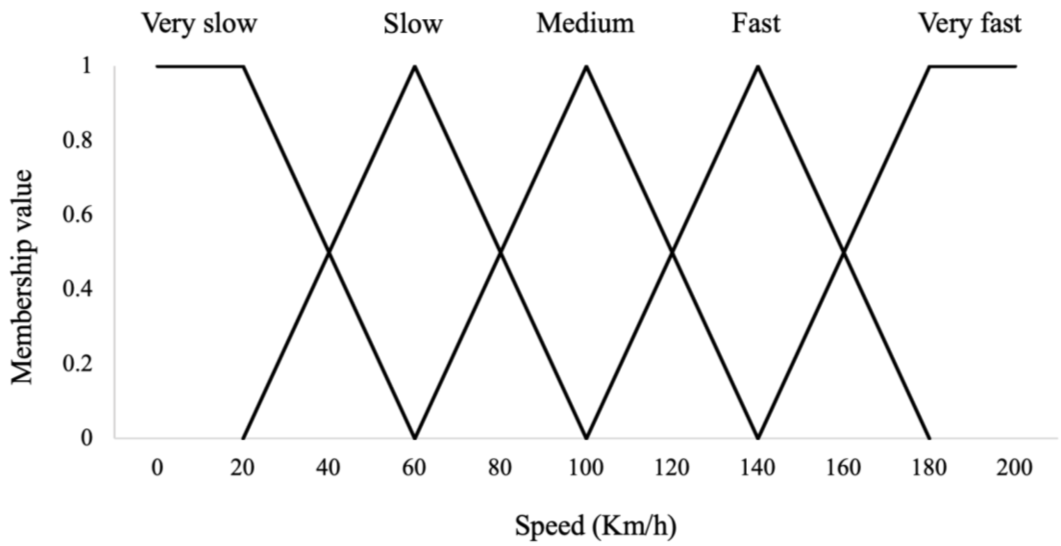

Fig. 2.10. Triangular and partially trapezoidal membership functions representing the terms of the linguistic variable "speed"
Source: Own elaboration

An example of a linguistic variable is given in Fig. 2.10. In this example, "speed" is the linguistic variable with a universe of discourse $U = [0{,}180]$, i.e., a range between 0 and 180 km/h. The linguistic terms of the linguistic variable are “very slow”, “slow”, “medium”, “fast”, and “very fast”. Each of these linguistic terms is assigned one of the fuzzy numbers (whose membership functions have triangular and partially trapezoidal shapes) through a semantic rule.

The membership function of each term can be defined as follows:

$$\mu_{very\ slow}(x) = \begin{cases} 1, & 0 \le x \le 20 \\ \dfrac{60 - x}{40}, & 20 \le x \le 60 \end{cases},$$

$$\mu_{slow}(x) = \begin{cases} \dfrac{x - 20}{40}, & 20 \le x \le 60 \\ \dfrac{100 - x}{40}, & 60 \le x \le 100 \end{cases},$$

$$\mu_{medium}(x) = \begin{cases} \dfrac{x - 60}{40}, & 60 \le x \le 100 \\ \dfrac{140 - x}{40}, & 100 \le x \le 140 \end{cases},$$

$$\mu_{fast}(x) = \begin{cases} \dfrac{x - 100}{40}, & 100 \le x \le 140 \\ \dfrac{180 - x}{40}, & 140 \le x \le 180 \end{cases},$$

$$\mu_{very\ fast}(x) = \begin{cases} \dfrac{x - 140}{40}, & 140 \le x \le 180 \\ 1, & 180 \le x \le 200 \end{cases}.$$

For example, a car whose speed is 110 km/h is considered to be more “medium” and less “fast”, as the degree of membership is 0.75 for the first term and 0.25 for the second one. This relation can be seen in Fig. 2.11.

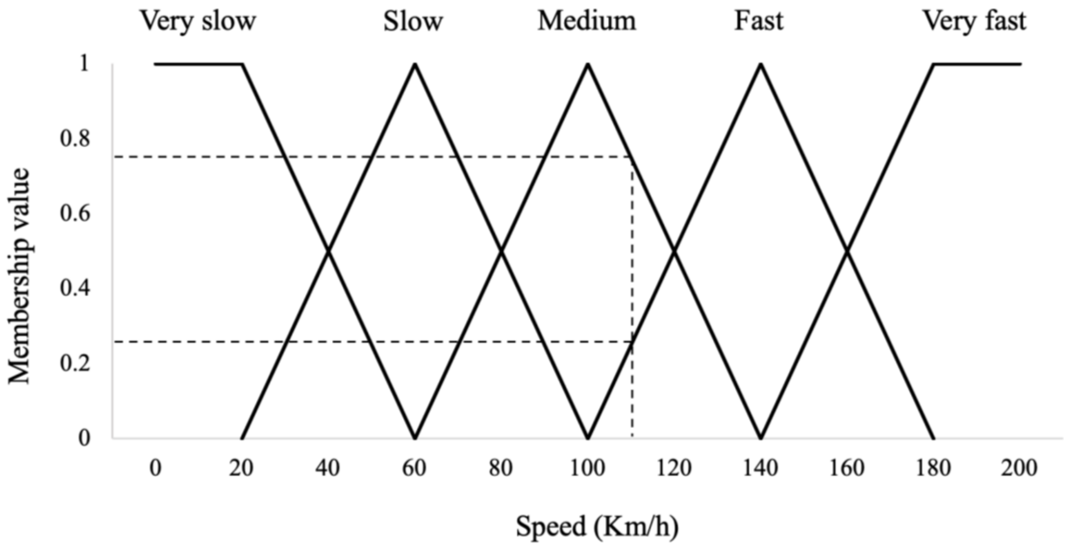

Fig. 2.11. Membership values for the terms "medium" and "fast"
Source: Own elaboration

2.2.3.6. Intuitionistic fuzzy sets

Atanassov (1986) introduced the concept of intuitionistic fuzzy set (IFS), which can be seen as a generalization of Zadeh's fuzzy set. Fuzzy sets only consider the membership of an element to a certain set. By contrast, IFSs contemplate the membership and the non-membership of an element to a certain set.

Definition. An IFS $\tilde{A}$ in X is defined as:

$$\tilde{A} = \{(x, \mu_{\tilde{A}}(x), v_{\tilde{A}}(x)) | x \in X\}, \quad (2.29)$$

where $\mu_{\tilde{A}}(x): X \to [0,1]$ and $v_{\tilde{A}}(x): X \to [0,1]$, with the condition $0 \leq \mu_{\tilde{A}}(x) + v_{\tilde{A}}(x) \leq 1$ for every $x \in X$. $\mu_{\tilde{A}}(x)$ and $v_{\tilde{A}}(x)$ denote, respectively, the degree of membership and non-membership of x to $\tilde{A}$.

2.2.4. Basic uncertain information

The basic uncertain information (BUI) (Jin et al., 2018; Mesiar et al., 2018) is a concept recently introduced that allows to generalize a wide range of uncertain information.

Definition. A BUI is as real pair $\tilde{x} =< x; c >$, where $x(x \in [0,1])$ is the input value and $c(c \in [0,1])$ the certainty degree of x.

Each BUI can be transformed into a closed interval $[a, b]$, where $a = cx$ and $b = cx + 1 - c$.

2.2.5. The concept of distance

2.2.5.1. Distance measures

In the literature we can find different distance or similarity measures: The Hamming distance (Hamming, 1950), the Euclidean distance, the Minkowski distance, the Hausdorff distance (Huttenlocher et al., 1993), the Chebyshev distance, and more. Among them, some of the most popular are the Hamming distance and the Euclidean distance, which can be defined as follows.

For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, we have.

The Hamming distance (also called Manhattan, taxicab, or city-block distance):

$$d_H(X,Y) = \sum_{i=1}^{n} |x_i - y_i| . \tag{2.30}$$

The normalized Hamming distance:

$$d_{NH}(X,Y) = \frac{1}{n} \sum_{i=1}^{n} |x_i - y_i| . \tag{2.31}$$

The Euclidean distance:

$$d_E(X,Y) = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2} . \tag{2.32}$$

The normalized Euclidean distance:

$$d_{NE}(X,Y) = \sqrt{\frac{1}{n}\sum_{i=1}^{n}(x_i - y_i)^2}\,. \tag{2.33}$$

The Minkowski distance is a distance measure that generalizes a wide range of other distance measures such as the Hamming distance and Euclidean distance. For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, and an order parameter λ such that $\lambda \in (-\infty, \infty)$, the Minkowski distance is defined as follows.

The Minkowski distance:

$$d_{MK} = \left(\sum_{i=1}^{n}|x_i - y_i|^{\lambda}\right)^{1/\lambda}. \tag{2.34}$$

Note that if $\lambda = 1$ we obtain the standard Hamming distance, and if $\lambda = 2$ we obtain the standard Euclidean distance.

The normalized Minkowski distance:

$$d_{NMK} = \frac{1}{n}\left(\sum_{i=1}^{n}|x_i - y_i|^{\lambda}\right)^{1/\lambda}. \tag{2.35}$$

Similarly, when $\lambda = 1$ the normalized Hamming distance is found, and when $\lambda = 2$, the normalized Minkowski distance is equivalent to the normalized Euclidean distance.

The Hausdorff distance is a measure commonly used for image matching (Sim et al., 1999; Zhao et al., 2005). This distance can be defined as

$$H(X,Y) = max\big(h(X,Y), h(Y,X)\big), \tag{2.36}$$

$$h(X,Y) = \max_{x \in X} \left(\min_{y \in Y} \left(d_E(x,y) \right) \right), \tag{2.37}$$

$$h(Y,X) = \max_{y \in Y} \left(\min_{x \in X} \left(d_E(y,x) \right) \right), \tag{2.38}$$

where $h(X,Y)$ is referred as the direct Hausdorff distance from X to Y, and $d_E(x,y)$ is the Euclidean distance between x and y.

2.2.5.2. The adequacy coefficient

The adequacy coefficient (Kaufmann & Gil-Aluja, 1986, 1987) is an index used for calculating the differences between two elements, two sets, two fuzzy subsets, etc. The main advantage of the adequacy coefficient is that it neutralizes the result when the real element is higher than the ideal one. This characteristic differentiates it from the Hamming distance.

The adequacy coefficient can be defined in the following way.

Definition. Let x and y be two real numbers such that $x, y \in [0,1]$. Then, the adequacy coefficient between x and y is obtained by applying the following formula:

$$\text{AC}(x,y) = [1 \wedge (1 - x + y)]. \tag{2.39}$$

Note that the symbol $\wedge$ indicates the lesser of the $(1 - x + y)$ value and 1.

Furthermore, for two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, the weighted adequacy coefficient (WAC) can be defined as follows.

Definition. A WAC of dimension n is a mapping WAC: $[0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, such that:

$$\text{WAC}(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) = \sum_{i=1}^{n} w_i \, [1 \wedge (1 - x_i + y_i)], \tag{2.40}$$

where x_i and y_i are the ith arguments of the sets X and Y.

Observe that the normalized adequacy coefficient is obtained when $w_i = 1/n$.

2.2.6. Aggregation operators

Aggregation can be formally defined as the process of combining several values into a single representative value. Aggregation operators (also called aggregation functions) perform this operation (Grabisch et al., 2009). Two very common aggregation operator are the arithmetic mean (also referred as average) and the weighted mean.

Another important and increasingly popular aggregation operator is the OWA operator from Yager (1988). Since its introduction, several authors developed a range of interesting extensions and applications, which will be reviewed in the following sections.

2.2.7. References

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2.3. The OWA operator

2.3.1. Yager's OWA operator

The ordered weighted averaging (OWA) operator was introduced by Yager (1988) and it is an aggregation function that provides a parameterized family of aggregation operators between the minimum and the maximum.

The OWA operator can be defined as follows.

Definition. An OWA operator of dimension n is a mapping $\mathrm{OWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{OWA}(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j b_j , \qquad (2.41)$$

where b_j is the jth largest element of the arguments $a_1, \dots, a_n$, that is $(b_1, \dots, b_n)$ is $(a_1, \dots, a_n)$ reordered from largest to smallest.

Note that it is possible to distinguish between the descending OWA (DOWA) operator and the ascending OWA (AOWA) operator (Yager, 1992). The weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DOWA (or OWA) operator and w^*_{n-j+1} is the jth weight of the AOWA operator.

An interesting property of the OWA operator is that it includes the classical methods for decision-making as particular cases. This can be achieved through choosing different manifestations of the weighting vector W. The optimistic criterion or maximax criterion selects the most favorable result of each alternative, that is $w_1 = 1$ and $w_j = 0$ for $\forall\ j \neq 1$. The pessimistic criterion or maximin criterion selects the most unfavorable result of each alternative, that is $w_j = 0$ and $w_n = 1$ for $\forall\ j \neq n$. The Laplace criterion is obtained when $w_j = 1/n$ for $\forall\ j$, so it is assumed that all alternatives have equal probability to occur. The Hurwicz criterion is found when $w_1 = \alpha$,

$w_n = 1 - \alpha$ and $w_j = 0$ for $\forall\, j \neq 1, n$, so it takes into account both the best and the worst alternative.

The OWA operator is a mean operator that satisfies the properties of monotonicity, commutativity (sometimes referred to as symmetry), boundedness, and idempotency. These properties are expressed with the following theorems.

Theorem. Monotonicity. Let F be the OWA operator. If $a_i \geq \hat{a}_i$ for all i, then, $F(a_1, \dots, a_n) \geq F(\hat{a}_1, \dots, \hat{a}_n)$.

Theorem. Commutativity. In the sense that the initial indexing of the arguments does not meter. So, if F is the OWA operator, then, $F(a_1, \dots, a_n) = F(\hat{a}_1, \dots, \hat{a}_n)$, where $(\hat{a}_1, \dots, \hat{a}_n)$ is any permutation of $(a_1, \dots, a_n)$.

Theorem. Boundedness. Since the aggregation is delimited. Let F be the OWA operator. Then, $Min\{a_i\} \leq F(a_1, \dots, a_n) \leq Max\{a_i\}$.

Theorem. Idempotency. Let F be the OWA operator. If $a_i = a$ for all i, then, $F(a_1, \dots, a_n) = a$.

Another noteworthy aspect is the measures for characterizing the weighting vector W and the type of aggregation it performs. The four characterizing measures introduced by Yager are: The attitudinal character measure (Yager, 1988), the dispersion measure (Yager, 1988), the balance operator (Yager, 1996), and the divergence of W (Yager, 2002).

The first measure refers to the attitudinal character (degree of or-ness) associated with a weighting vector and is denoted as $\alpha(W)$ or also as $\mathrm{AC}(W)$. It can be defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \qquad (2.42)$$

As we can see, $\alpha(W) \in [0,1]$. The closer $\alpha(W)$ is to 1, the higher the level of preference for larger values in the aggregation.

The second measure is the measure of dispersion or entropy and is denoted as $\mathrm{H}(W)$ or also as $\mathrm{Disp}(W)$. Its definition is as follows:

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln\left(w_j\right). \tag{2.43}$$

It can be shown that $\mathrm{H}(W)$ has a value between 0 and the natural logarithm of n. That is $\mathrm{H}(W) \in [0, \ln(n)]$.

The third, is the balance operator $\mathrm{Bal}(W)$, which measures the degree of favoritism towards higher values (optimistic values) or lower values (pessimistic values). Its formula is as follows:

$$\mathrm{Bal}(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1}\right). \tag{2.44}$$

The balance operator can range from –1 to 1, that is $\mathrm{Bal}(W) \in [-1,1]$. For values of $\mathrm{Bal}(W)$ close to –1 the aggregation emphasizes the lower values. For values of $\mathrm{Bal}(W)$ close to 1 the aggregation emphasizes the higher values.

The fourth, is the measure of divergence $\mathrm{Div}(W)$. It measures the divergence of the weights against the degree of or-ness measure. It can be defined by using the following expression:

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \tag{2.45}$$

In the Table 2.14, we can see when the special cases of the pessimistic, Laplace and optimistic criterion are met according to the measure outcome.

Table 2.14. Particular cases of measures for characterizing a weighting vector

Measure	Criterion		
	Pessimistic	Laplace	Optimistic
α(W)	0	0.5	1
H(W)	0	$\ln(n)$	0
Bal(W)	–1	0	1
Div(W)	0	$\frac{n+1}{12(n-1)}$	0

Furthermore, the OWA operator has been used in a vast range of areas (Kacprzyk et al., 2019). This operator is commonly used in decision-making situations. Moreover, the OWA operator has been applied in the fields of computer science, engineering, business, and economics, among others.

Example. In the following, an illustrative example is provided of the OWA operator in a decision-making problem regarding the selection of a car. Suppose that a woman called Kathrin is considering buying a new car and she is hesitating between five options, which are the following ones:

- A_1: Audi Q5 Sportback Advanced 45 TFSI quattro S tronic.
- A_2: BMW X4 xDrive 20i.
- A_3: Range Rover Evoque P160.
- A_4: Mercedes-Benz GLC 200 4MATIC Coupé.
- A_5: Porsche Macan.

She considers five different characteristics in order to assess individually each alternative. These characteristics are the following ones:

- C_1: Price.
- C_2: Safety.
- C_3: Power.
- C_4: Consumption.
- C_5: Emissions.

After collecting some data, she is able to conduct the assessments for each car. Table 2.15 shows numerically these valuations.

Table 2.15. Individual assessments

	C_1	C_2	C_3	C_4	C_5
A_1	2	3	5	3	3
A_2	4	2	2	4	4
A_3	5	4	1	5	5
A_4	3	5	3	2	2
A_5	1	1	4	1	1

Next, Kathrin aggregates the information by using the OWA operator and the AOWA operator. To do so, she considers the following weighting vector $W = (0.3,0.3,0.2,0.1,0.1)$. The results are shown in Table 2.16 and the order of preferences in Table 2.17. Note that AM means arithmetic mean and $\succ$ "preferred to".

Table 2.16. Aggregated results

	Min	Max	AM	OWA	AOWA
A_1	2	5	3.2	3.5	2.9
A_2	2	4	3.2	3.6	2.8
A_3	1	5	4	4.5	3.5
A_4	2	5	3	3.4	2.6
A_5	1	4	1.6	1.9	1.3

Table 2.17. Ordering of the results

Operator	Ordering
Min	$A_1 = A_2 = A_4 \succ A_3 = A_5$
Max	$A_1 = A_3 = A_4 \succ A_2 = A_5$
AM	$A_3 \succ A_1 = A_2 \succ A_4 \succ A_5$
OWA	$A_3 \succ A_2 \succ A_1 \succ A_4 \succ A_5$
AOWA	$A_3 \succ A_1 \succ A_2 \succ A_4 \succ A_5$

In Table 2.17 we can see that depending on the aggregation operator used the ordering can vary substantially.

2.3.2. Extensions

2.3.2.1. The IOWA operator

One appealing extension of the OWA operator is the induced OWA (IOWA) operator developed by Yager and Filev (1999). In this operator, the step of reordering the input arguments is carried out using order-inducing variables. This allows to consider other factors in the reordering step and not only to the degree of optimism and pessimism. The IOWA operator can be defined as follows.

Definition. An IOWA operator of dimension n is a function IOWA: $R^n \times R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{IOWA}(\langle u_1, a_1 \rangle, \dots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j \,, \qquad (2.46)$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and a_i as the argument variable.

The IOWA operator satisfies the same properties than the OWA operator, i.e., monotonicity, commutativity, boundedness, and idempotency.

2.3.2.2. The HOWA operator

Another common extension of the OWA operator is the heavy OWA (HOWA) operator (Yager, 2002). The key feature of this operator is that the sum of the weights is allowed to be between 1 and n instead of being restricted to sum up to 1. The HOWA operator can be defined as follows.

Definition. A HOWA operator of dimension n is a mapping HOWA: $R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $1 \leq \sum_{j=1}^{n} w_j \leq n$, such that:

$$\text{HOWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \,, \tag{2.47}$$

where b_j is the jth largest of the a_i.

Note that the HOWA operator is a monotonic and commutative function, however it is not bounded by the minimum and the maximum. In this case, it is bounded by the minimum and the total operator, i.e., the sum of all the arguments.

2.3.2.3. The GOWA operator

The generalized OWA (GOWA) operator was introduced by Yager (2004) and it combines the OWA operator with generalized means (Dyckhoff & Pedrycz, 1984). The GOWA operator can be defined as follows.

Definition. A GOWA operator of dimension n is a mapping GOWA: $R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{GOWA}(a_1, \ldots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda} , \tag{2.48}$$

where λ is a parameter such that $\lambda \in (-\infty, \infty)$ and b_j is the jth largest of the argument variable a_i.

The GOWA operator is monotonic, commutative, idempotent, and limited by the minimum and the maximum.

Depending on the value that we give to the parameter λ, it is possible to obtain a wide range of particular cases of the GOWA operator. If $\lambda = -1$, we obtain the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004); if $\lambda = 0$, the ordered weighted geometric (OWG) operator (Chiclana et al., 2000, 2002); if $\lambda = 1$, the OWA operator; and if $\lambda = 2$, the ordered weighted quadratic averaging (OWQA) operator (Yager, 2004).

2.3.2.4. The Quasi-OWA operator

Another interesting generalization of the OWA operator is the Quasi-OWA operator, presented by Fodor et al. (1995). By using quasi-arithmetic means (Kolmogorov, 1930; Nagumo, 1930), the Quasi-OWA operator provides a more general formulation, including a wide range of particular cases that are not considered in the GOWA operator. The Quasi-OWA operator can be defined as follows.

Definition. A Quasi-OWA operator of dimension n is a mapping $\text{Quasi} - \text{OWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{Quasi} - \text{OWA}(a_1, \ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right), \tag{2.49}$$

where b_j is the jth largest of the a_i and $g(b)$ is a strictly continuous monotonic function.

Note that if $g(b) = b$ we get the OWA operator and if $g(b) = b^{\lambda}$ the GOWA operator. Also, similar to the OWA operator, the Quasi-OWA operator is commutative, monotonic, bounded, and idempotent.

2.3.2.5. The POWA operator

The probabilistic aggregation (PA) operator is an aggregation function where the aggregation process is done according to the probability associated to each argument. A PA operator is defined as follows.

Definition. A PA operator of dimension n is a function $\text{PA}: R^n \rightarrow R$ such that:

$$\text{PA}(a_1, \dots, a_n) = \sum_{i=1}^{n} v_i a_i \,, \tag{2.50}$$

where a_i is the ith argument variable and each argument a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$.

The probabilistic OWA (POWA) operator introduced by Merigó (2012), is an aggregation function that unifies the probability and the OWA operator in the same formulation and according to the degree of importance of these two concepts in the aggregation process. Therefore, it provides a unified framework between decision-making problems under risk with the use of probabilities and under uncertainty with the use of OWA operators. The POWA operator can be defined as follows.

Definition. A POWA operator of dimension n is a function $\text{POWA}: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{POWA}(a_1, \dots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j = \beta \sum_{j=1}^{n} w_j b_j + (1 - \beta) \sum_{i=1}^{n} v_i a_i \,, \tag{2.51}$$

where b_j is the jth largest of the a_i, each argument a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to b_j, that is, according to the jth largest of a_i.

Note that if the parameter β is equal to 1, we obtain the normal OWA operator, and if β is equal to 0, we get the PA operator. Then, by taking into consideration the last two equations, the POWA operator can be formulated alternatively as $\text{POWA} = \beta(\text{OWA}) + (1 - \beta)\text{PA}$.

Also, the POWA operator shares the properties of the OWA operator: monotonicity, commutativity, boundedness, and idempotency.

2.3.2.6. The UOWA operator

The uncertain OWA (UOWA) operator was developed by Xu and Da (2002) as an extension of the OWA operator for uncertain environments where the available information can be assessed with confidence intervals. These confidence intervals can take different forms. For example, confidence triplets, confidence quadruples, and so on. The UOWA operator can be defined as follows.

Definition. Let Ω be the set of confidence intervals. An UOWA operator of dimension n is a function $\text{UOWA}: \Omega^n \rightarrow \Omega$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{UOWA}(\tilde{a}_1, \dots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j \,, \tag{2.52}$$

where b_j is the jth largest of the $\tilde{a}_i$, and $\tilde{a}_i$ is the argument variable represented in the form of confidence intervals.

Furthermore, the UOWA operator is commutative, monotonic, bounded, and idempotent.

2.3.2.7. The FOWA operator

The fuzzy OWA (FOWA) operator can be described as an extension of the OWA operator that uses uncertain information represented in the form of fuzzy numbers (Chang & Zadeh, 1972). Furthermore, the FOWA operator provides a parametrized family of aggregation operators, among others, the fuzzy minimum, the fuzzy maximum, and the fuzzy average criteria. The FOWA operator can be defined as follows.

Definition. Let Ψ be the set of fuzzy numbers. A FOWA operator of dimension n is a function FOWA: $\Psi^n \rightarrow \Psi$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{FOWA}(\tilde{a}_1, \dots, \tilde{a}_n) = \sum_{j=1}^{n} w_j b_j\,, \qquad (2.53)$$

where b_j is the jth largest of the $\tilde{a}_i$, and $\tilde{a}_i$ is the argument variable represented in the form of fuzzy numbers.

Note that the purpose of the FOWA operator is the same that the UOWA operator, i.e., analyze vague or imprecise information when it is not possible to do it with exact numbers. However, for dealing with this type of information, the FOWA operator uses fuzzy numbers instead of confidence intervals.

Moreover, the FOWA operator is monotonic, commutative, idempotent, and bounded by the minimum and the maximum.

2.3.2.8. The LOWA operator

Sometimes, it is not possible to analyze the available information with numerical values. When this occurs, a more suitable approach may be the use of linguistic values by means of linguistic variables (Zadeh, 1975a, 1975b, 1975c).

In the literature we can find different linguistic models for decision-making that use the OWA operator. The first ones were presented in (Bordogna & Pasi, 1995; Herrera et al., 1995). However, the main problem of these classical linguistic models is the loss of information when operations are performed and therefore produce biased results. To address this problem, additional linguistic models were proposed. Notable among these are the ones presented by Herrera and Martínez (2000) and Xu (2004a, 2004b).

On the one hand, the approach from Herrera and Martínez (2000) uses means of 2-tuples, (s, α), to represent the linguistic information. The 2-tuples are formed by a linguistic term, s, and a number, α, representing the value of the symbolic translation.

Definition. (Herrera & Martínez, 2000) "Let β be the result of an aggregation of the indexes of a set of labels assessed in the linguistic label set $S = \{s_0, s_1, \ldots, s_g\}$, i.e., the result of a symbolic aggregation operation. $\beta \in [0, g]$, being $g + 1$ the cardinality of S. Let $i = round(\beta)$ and $\alpha = \beta - i$ be two values such that $i \in [0, g]$ and $\alpha \in [-0.5, 0.5)$, then α is called a symbolic translation."

Definition. The 2-tuple that expresses the equivalent information to β is found with the following function:

$$\Delta: [0, g] \to S \times [-0.5, 0.5),$$
$$\Delta(\beta) = (s_i, \beta), with \begin{cases} s_i & i = round(\beta) \\ \alpha = \beta - i & \alpha \in [-0.5, 0.5), \end{cases} \tag{2.54}$$

where round is the usual round operation, s_i has the closest index label to β, and α is the symbolic translation value.

Let $X = \{(s_1, \alpha_1), \ldots, (s_n, \alpha_n)\}$ be a set of 2-tuples. Then, taking into account the previous concepts, the 2-tuple OWA (2-TOWA) operator can be defined as follows.

Definition. A 2-TOWA operator of dimension n is a mapping $2-$TOWA$: S^n \times R^n \to S$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$2 - \text{TOWA}\big((s_1, \alpha_1), \ldots, (s_n, \alpha_n)\big) = \Delta\left(\sum_{j=1}^{n} w_j \beta_j^*\right), \tag{2.55}$$

where β_j^* is the jth largest of the β_j.

On the other hand, the method from Xu (2004a, 2004b) basically extends a discrete term set S to a continuous linguistic term set $\bar{S} = \{s_\alpha | s_1 < s_\alpha \leq s_t, \alpha \in [1, t]\}$. In the case that $s_\alpha \in S$, then s_α is called original linguistic term. But if $s_\alpha \notin S$, then s_α is called virtual linguistic term. An example of a discrete linguistic term set can be seen in Fig. 2.12, where $\{s_1, s_2, s_3, s_4, s_5\}$ are the original linguistic terms. Conversely, e.g., $s_{4.2}$ is a virtual linguistic term set. Usually, the original linguistic terms are used to conduct the initial assessments, while the virtual linguistic terms appear in operations.

$$S = \{s_1 = very\ slow;\ s_2 = slow;\ s_3 = medium;\ s_4 = fast;\ s_5 = very\ fast\}$$

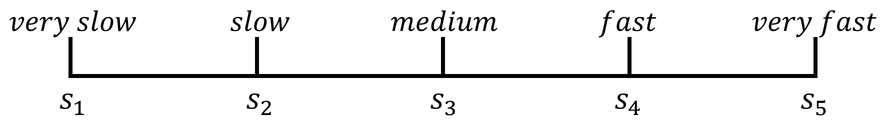

Fig. 2.12. Discrete linguistic term set
Source: Own elaboration

Note that this method considers that the linguistic term set is uniformly and symmetrically distributed.

In keeping with the method from Xu (2004a, 2004b), a formal definition of the linguistic OWA (LOWA) operator, also called extended OWA (EOWA) operator, would be the following one.

Definition. A LOWA operator of dimension n is a mapping LOWA: $S^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{LOWA}(s_{\alpha_1}, \ldots, s_{\alpha_n}) = \sum_{j=1}^{n} w_j s_{\beta_j}, \qquad (2.56)$$

where s_{β_j} is the jth largest of the s_{α_i}.

2.3.2.9. The OWAD operator

The ordered weighted averaging distance (OWAD) operator (Merigó & Gil-Lafuente, 2007, 2010) is an aggregation operator that integrates the OWA operator with the Hamming distance measure (Hamming, 1950).

For two sets $X = \{x_1, \ldots, x_n\}$ and $Y = \{y_1, \ldots, y_n\}$, the OWAD operator can be defined as follows.

Definition. An OWAD operator of dimension n is a function $\text{OWAD}: [0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has associated a weighting vector W of dimension n $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{OWAD}(X, Y) = \sum_{j=1}^{n} w_j D_j \,, \tag{2.57}$$

where D_j is the jth largest individual distance of the $|x_i - y_i|$ and $x_i, y_i \in [0,1]$ are the ith arguments of the sets X and Y.

The OWAD operator satisfies the same properties as the OWA operator, i.e., monotonicity, commutativity, boundedness, and idempotency. But, in addition, the OWAD operator also accomplishes the conditions of nonnegativity and reflexivity. These last two properties can be demonstrated with the following theorems.

Theorem. Nonnegativity. Let F be the OWAD operator. Then, $F(|a_1 - b_1|, \ldots, |a_n - b_n|) \geq 0$.

Theorem. Reflexivity. Let F be the OWAD operator. If $A = B$, then, $F(|a_1 - b_1|, \ldots, |a_n - b_n|) = 0$.

2.3.2.10. The OWAAC operator

The researchers Merigó and Gil-Lafuente (2008, 2010) presented the ordered weighted averaging adequacy coefficient (OWAAC) operator, which uses the adequacy coefficient (Kaufmann & Gil-Aluja, 1986, 1987) and the OWA operator in the same formulation. This operator is used for comparing two sets in a more complete way, as it neutralizes the result when one or more elements of a set are greater than the elements of the other set.

For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, the OWAAC operator is defined as follows.

Definition. An OWAAC operator of dimension n is a mapping $\text{OWAAC}: [0,1]^n \times [0,1]^n \to [0,1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWAAC}(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j K_j, \tag{2.58}$$

where K_j is the jth largest of the $[1 \wedge (1 - x_i + y_i)]$, and x_i and y_i are the ith arguments of the sets X and Y.

Like the OWAD operator, the OWAAC operator is commutative, monotonic, bounded, idempotent, nonnegative, and reflexive.

2.3.2.11. The OWAIMAM operator

Merigó and Gil-Lafuente (2012) first introduced the ordered weighted averaging index of maximum and minimum level (OWAIMAM) operator. The particularity of this operator is that it utilizes in the same formulation the Hamming distance, the adequacy coefficient, and the OWA operator.

For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, the OWAIMAM operator is defined as follows.

Definition. An OWAIMAM operator of dimension n is a mapping $\text{OWAIMAM}: [0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWAIMAM}(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j K_j, \qquad (2.59)$$

where K_j is the jth largest of all the $|x_i - y_i|$ and the $[0 \vee (x_i - y_i)]$, and x_i and y_i are the ith arguments of the sets X and Y. The symbol $\vee$ indicates the greater of the $(x_i - y_i)$ value and 0.

Note also that the OWAIMAM is a commutative, monotonic, bounded, and idempotent aggregation function.

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2.4. Overview of pensions

2.4.1. Theoretical foundation

2.4.1.1. Current situation

The main goal of pension systems (also referred as retirement income systems) is to ensure an adequate retirement income for all elderly people, while maintaining sound financial prospects. However, the acceleration of population aging and the adverse economic conditions imply a serious challenge for the pension systems (Mercer, 2022; Organization for Economic Cooperation and Development [OECD], 2021). To address this issue, certain reforms have been carried out over the past years, such as raising the retirement age. However, these measures are not enough.

2.4.1.2. Types of pension systems

Over the years, several approaches have been presented to classify pension systems. First, the World Bank (1994) recommended a three-pillar classification. This three-pillar classification can be summarized as follows:

- Pillar 1: Mandatory and publicly managed.
- Pillar 2: Mandatory and privately managed.
- Pillar 3: Voluntary.

However, a few years later, the same international institution introduced a new approach to classify pension systems (World Bank, 2008) based on the following five pillars:

- Pillar 0: Non-contributory and financed by the state.
- Pillar 1: Mandatory with contributions linked to earnings.
- Pillar 2: Mandatory defined contribution plan.
- Pillar 3: Voluntary.
- Pillar 4: Non-financial.

The OECD (2021) used a different approach based on a three-tier taxonomy, outlined in the following bullet points:

- Tier 1: Mandatory, non-earnings-related, and provided by the public sector.
- Tier 2: Mandatory, earnings-related, and provided by either the public or private sector.
- Tier 3: Voluntary, earnings-related, and provided by the private sector.

The Mercer CFA Institute Global Pension Index compares the pension systems of various countries worldwide. This comparison is based on three sub-indices: adequacy, sustainability, and integrity. According to the 2022 results of this index (Mercer, 2022), Iceland, Netherlands, and Denmark are considered the countries with the best pension systems. By contrast, the pension systems of Indonesia, Turkey, India, Argentina, Philippines, and Thailand are the ones with major weaknesses and gaps.

2.4.1.3. Public plan and private plan

Public pension plans refer to the Social Security and similar systems, where pension benefit payments are administered by the general government (i.e., central, state, or local government as well as other public-sector bodies) (Kumar, 2014; Yermo, 2002). In private pension plans, pension benefit payments are administered by an institution other than the general government. Specifically, these plans are administered by employers who function as the plan sponsor, pension entities, or private sector providers (Kumar, 2014; Yermo, 2002).

2.4.1.4. Defined benefit plan and defined contribution plan

On the one hand, defined benefit (DB) pension plans provide a predetermined retirement income based on the number of years of service and the salary history. On the other hand, in defined contribution (DC) pension plans, the retirement income depends on the amount of contributions as well as investment returns. Also, in DC pension plans, the financial and longevity risk is borne by the member, which does not occur with DB pension plans (Yermo, 2002).

2.4.2. Bibliometric analysis

2.4.2.1. Introduction

In order to quantitatively analyze the scientific literature on pensions and retirement, a bibliometric analysis is conducted. Specifically, the annual evolution of the number of published documents is examined. Also, the most influential publications are assessed. Lastly, authors, institutions, countries, journals, and research areas are investigated regarding productivity.

2.4.2.2. Methodology

In this study, the database Web of Science (WoS) Core Collection is used; however, other sources of information exist. The variant words selected were ‘retirement*’ or ‘pension*’. The period of the search was from 1990 to 2022. Only articles and review articles were contemplated, obtaining a total of 37,504 publications. This investigation was performed in January 2023.

2.4.2.3. Results

Publication and citation structure

Fig. 2.13 presents the number of annual publications on retirement/pension. A high productivity growth can be observed over the years of the period studied. This reflects the increasing concern about pensions in society. The maximum is reached in the year 2021, with a volume of 3,447 annual publications.

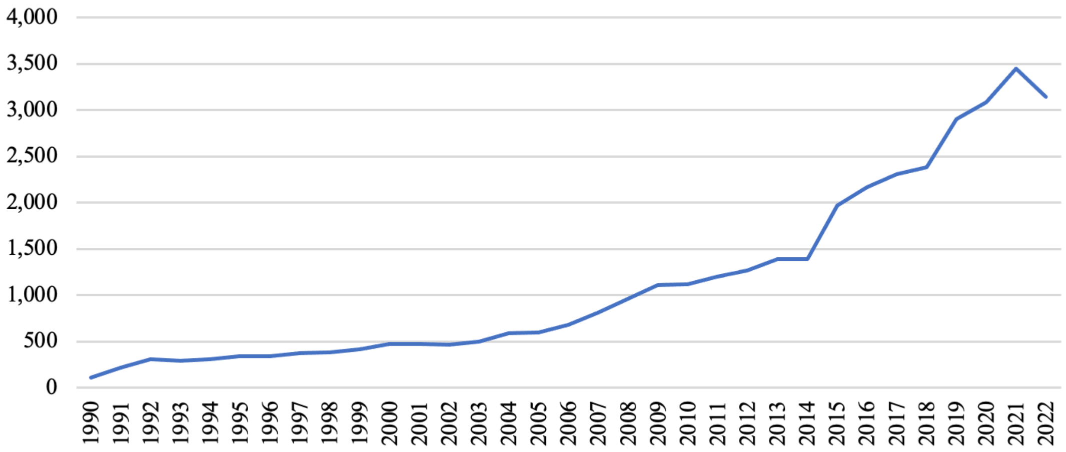

Fig. 2.13. Annual trend of publications related to retirement/pension
Source: Own elaboration

Table 2.18 shows the ten publications on retirement/pension with a higher number of citations. A general remark is that all these publications have reached more than 1,000 citations.

Table 2.18. Top 10 most influential publications with retirement/pension

R	Article	Author/s	PY	TC
1	The size and burden of mental disorders and other disorders of the brain in Europe 2010 (Wittchen et al., 2011)	Wittchen, HU et al.	2011	2,245
2	Econometric methods for fractional response variables with an application to 401(k) plan participation rates (Papke & Wooldridge, 1996)	Papke, LE; Wooldridge, JM	1996	1,985
3	Long-term cognitive impairment and functional disability among survivors of severe sepsis (Iwashyna et al., 2010)	Iwashyna, TJ et al.	2010	1,411
4	Social jetlag: Misalignment of biological and social time (Wittmann et al., 2006)	Wittmann, M et al.	2006	1,331
5	Long-term impact of overweight and obesity in childhood and adolescence on morbidity and premature mortality in adulthood: Systematic review (Reilly & Kelly, 2011)	Reilly, JJ; Kelly, J	2011	1,329
6	Prevalence of dementia in the United States: The aging, demographics, and memory study (Plassman et al., 2007)	Plassman, BL et al.	2007	1,194

R	Article	Author/s	PY	TC
7	The China Syndrome: Local labor market effects of import competition in the United States (Autor et al., 2013)	Autor, DH et al.	2013	1,184
8	The economic importance of financial literacy: Theory and evidence (Lusardi & Mitchell, 2014)	Lusardi, A; Mitchell, OS	2014	1,175
9	Global, regional, and national disability-adjusted life-years (DALYs) for 359 diseases and injuries and healthy life expectancy (HALE) for 195 countries and territories, 1990-2017: A systematic analysis for the Global Burden of Disease Study 2017 (Kyu et al., 2018)	Kyu, HH et al.	2018	1,129
10	Effect of exposure to natural environment on health inequalities: An observational population study (Mitchell & Popham, 2008)	Mitchell, R; Popham, F	2008	1,092

Source: Own elaboration through WoS. Abbreviations: R = Ranking; PY = Publication year; TC = Total citations.

Leading authors

Table 2.19 lists the ten authors with a higher volume of publications related to retirement/pension. Alexanderson ranks first with a total of 173 publications. She is affiliated with the Karolinska Institute in Sweden. Langa and Vahtera rank second and third, respectively.

Table 2.19. Top 10 most productive authors on retirement/pension

R	Author	TP
1	Alexanderson K	173
2	Langa KM	148
3	Vahtera J	117
4	Kivimaki M	114
5	Henkens K	101
6	Glymour MM	94
7	Mittendorfer-Rutz E	90
7	Pentti J	90
9	Stephan Y	85
10	Sutin AR	81

Source: Own elaboration through WoS. Abbreviations: R = Ranking; TP = Total publications.

Leading institutions

The leading institutions in terms of scientific productivity are presented in Table 2.20. As can be seen, the University of London leads the ranking with 1,090 publications. The University of California and the University of Michigan occupy second and third places, respectively. Also, it is remarkable that 6 of the ten most productive institutions are American.

Table 2.20. Top 10 most productive institutions on retirement/pension

R	Institution	TP
1	University of London	1,090
2	University of California	963
3	University of Michigan	843
4	Harvard University	686
5	Karolinska Institute	668
6	State University System of Florida	564
7	University College London	500
8	University of North Carolina	474
9	National Bureau of Economic Research	465
10	University of Helsinki	436

Source: Own elaboration through WoS. Abbreviations: R = Ranking; TP = Total publications.

Leading countries

Fig. 2.14 illustrates the foremost productive countries in retirement/pension. The United States of America (USA) is clearly dominant, with more than 13,000 articles and review articles published. It is followed in second and third position by England and Germany, respectively.

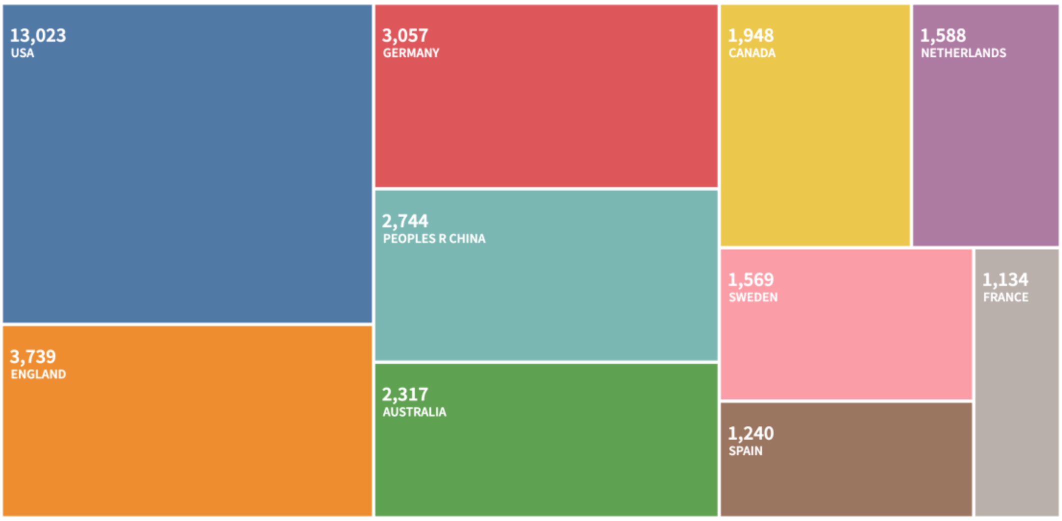

Fig. 2.14. Top 10 most productive countries on retirement/pension
Source: Own elaboration through WoS

Leading journals

The ten journals with the most publications on retirement/pension can be found in Table 2.21. The leading journal is *Journals of Gerontology, Series B: Psychological Sciences and Social Sciences*, with 517 publications. The *International Journal of Environmental Research and Public Health* ranked the second greatest producer, followed by the *Ageing & Society*.

Table 2.21. Top 10 most productive journals on retirement/pension

R	Journal	TP
1	Journals of Gerontology, Series B: Psychological Sciences and Social Sciences	517
2	International Journal of Environmental Research and Public Health	381
3	Ageing & Society	377
4	Journal of Pension Economics and Finance	352
5	BMC Public Health	332
6	PLOS ONE	307
7	The Gerontologist	285
8	Insurance: Mathematics and Economics	253
9	Social Science and Medicine	239
10	Research on Aging	228

Source: Own elaboration through WoS. Abbreviations: R = Ranking; TP = Total publications.

Leading fields

As shown in Fig. 2.15, “Business Economics” is the predominant research area, with a total of 10,632 publications that included the word retirement/pension and its different forms. “Geriatrics Gerontology” occupies the second position with 4,979 publications, followed by “Public Environmental Occupational Health” with 4,193.

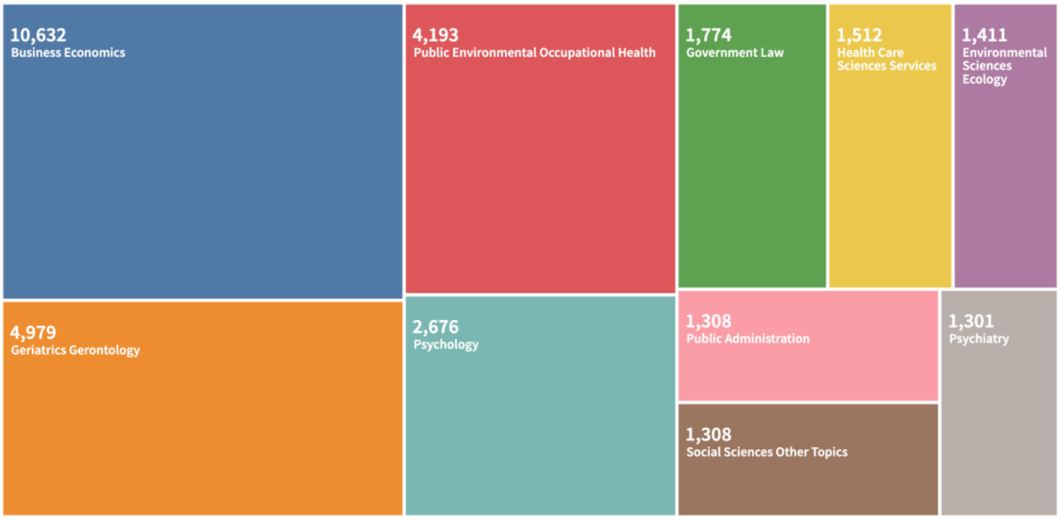

Fig. 2.15. Top 10 research areas with most publications on retirement/pension
Source: Own elaboration through WoS

2.4.3. References

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3. Research contributions

3.1. The linguistic OWA adequacy coefficient operator and its application to decision-making

The following research paper will be submitted to the *Expert Systems with Applications* Q1 journal. According to Journal Citation Reports (JCR), published by Clarivate Analytics, the Impact Factor in 2021 is 8.665. Also, based on the Scopus database, the 2021 CiteScore metric of the journal is 12.2.

The authors of this paper are Anton Figuerola Wischke (University of Barcelona), Anna Maria Gil Lafuente (University of Barcelona), Finn Erling Kydland (University of California – Santa Barbara), and José María Merigó Lindahl (University of Technology Sydney).

Abstract

This article presents the linguistic adequacy coefficient (LAC), a new index for calculating the difference between two linguistic variables. It is similar to the linguistic Hamming distance with the distinction that it establishes a threshold from which the result is always the same. Furthermore, the linguistic ordered weighted averaging adequacy coefficient (LOWAAC) operator is introduced, which is an enhanced aggregation function that utilizes linguistic information and the LAC in the OWA operator. Moreover, the LOWAAC operator is extended by using order-inducing variables in the reordering step of the linguistic arguments, thus obtaining a new operator called induced LOWAAC (ILOWAAC). Additionally, the LOWAAC and ILOWAAC operators are both generalized by employing generalized means, thus attaining the generalized LOWAAC (GLOWAAC) operator and the generalized ILOWAAC (GILOWAAC) operator, respectively. Likewise, the LOWAAC and ILOWAAC operators can also be generalized with the use of quasi-arithmetic means, getting, in this case, the Quasi-LOWAAC operator and the Quasi-ILOWAAC operator, respectively. These new aggregation operators are helpful when the decision-maker needs to compare a range of alternatives with an ideal, but without giving any reward or penalty if the ideal levels are surpassed. This comparison especially applies to situations of great uncertainty, where the information available cannot be assessed by means of exact numbers. However, it is possible to employ linguistic assessments. Finally, the article develops an application of the proposed approach in a multi-expert decision-making (MEDM) problem regarding the selection of human resources in a football club.

Keywords: Adequacy coefficient, decision-making, human resource management, linguistic aggregation operator, OWA operator.

1. Introduction

Aggregation is the process of combining several numerical or linguistic values into a single representative value, and an aggregation operator (also called aggregation function) performs this operation (Beliakov et al., 2016; Grabisch et al., 2009). Yager first introduced the ordered weighted averaging (OWA) operator in (Yager, 1988). It is an increasingly popular aggregation operator that has been applied successfully to various fields (Csiszar, 2021; Shu, 2022; Yager et al., 2011), primarily due to its effectiveness in modeling the bipolar (pessimism/optimism) preference. Observing some of the most recent publications, the OWA operator has been utilized for customer classification (Pons-Vives et al., 2022) and risk assessment (Cables et al., 2022; Garg et al., 2022), among others.

In the existing literature, we can find a lot of interesting extensions of the OWA operator. One of them is the induced OWA (IOWA) operator from Yager and Filev (1999), where the reordering process of the input arguments uses order-inducing variables, rather than based upon the values of the arguments. Another extension is the generalized OWA (GOWA) operator presented by Yager (2004), and which uses generalized means (Dyckhoff & Pedrycz, 1984). Moreover, the OWA operator has been further generalized by incorporating quasi-arithmetic means (Kolmogorov, 1930; Nagumo, 1930). This form of aggregation is called Quasi-OWA operator (Fodor et al., 1995).

Furthermore, the OWA operator has been studied by deploying the adequacy coefficient similarity measure developed by Kaufmann and Gil-Aluja (1986, 1987), obtaining the OWA adequacy coefficient (OWAAC) operator (Merigó & Gil-Lafuente, 2008, 2010). The OWAAC operator, developed by

Merigó and Gil-Lafuente, compares an ideal set with a real one. But, unlike the OWA distance (OWAD) operator (Merigó & Gil-Lafuente, 2010), it does not penalize the result when the ideal levels are surpassed. Since its introduction, further contributions have been made to this operator. See, e.g., Ref. (Figuerola-Wischke et al., 2022).

Nonetheless, we can find some situations where the available information cannot be assessed with precise numerical values. When this occurs, it is necessary to apply a different approach, such as the utilization of linguistic assessments (Zadeh, 1975a, 1975b, 1975c). In (Herrera et al., 1995), Herrera et al. proposed one of the first linguistic version of the OWA operator, called linguistic OWA (LOWA) operator. From then on, different authors have suggested interesting developments (Herrera et al., 2009; Xu, 2008), highlighting the contributions made by Herrera and Martínez (2000) and Xu (2004a). However, presently there is not a linguistic aggregation function based on the idea of the OWAAC operator.

Therefore, the aim of this article is to create a linguistic version of the adequacy coefficient, called the linguistic adequacy coefficient (LAC). The LAC is a new index used to calculate the distance between two linguistic variables with certain distinctive features. Through the LAC, it is possible to develop two novel operators: the linguistic weighted adequacy coefficient (LWAC) operator, and the linguistic OWAAC (LOWAAC) operator.

These new operators are for situations where a high level of uncertainty is present, and the available information is only assessable with words. In addition, these operators are practical when comparing a real set with an ideal one. This comparison especially applies when the ideal set does not take the form of a Boolean set and the decision-maker does not want to reward or penalize the result when one or more elements of the real set exceed the ideal ones. It only penalizes the result when the element in question, belonging to the real set, is lower than the ideal one. This feature differentiates it from other well-known methods, such as the LOWA distance (LOWAD) operator (Merigó & Casanovas, 2010), which uses the Hamming distance (Hamming, 1950), the linguistic variables, and the OWA operator, all together in a single formulation.

Also, this article extends the LOWAAC operator by adding order-inducing variables, thereby attaining the induced LOWAAC (ILOWAAC) operator. Moreover, the LOWAAC operator, and the ILOWAAC operator, are generalized by using generalized means. Therefore, generating the generalized LOWAAC (GLOWAAC) operator and the generalized ILOWAAC (GILOWAAC) operator. Further generalizations of the LOWAAC and ILOWAAC operators are also introduced, specifically by integrating quasi-arithmetic means. They are named Quasi-LOWAAC operator and Quasi-ILOWAAC operator, respectively.

Furthermore, an application of the new approach in a human resources multi-expert decision-making (MEDM) problem is conducted. The exercise focuses on the selection of the most suitable football player. The complex undertaking of recruiting the right players is essential to the success of any football club (Taylor et al., 2008). Additionally, by including a comparative analysis utilizing classical techniques, the effectiveness of the LOWAAC operator and its variations has been demonstrated.

This article is organized as follows. Section 2 briefly reviews some basic but necessary concepts. Section 3 explains the linguistic approach used in this work. Section 4 presents the LAC index and the LWAC operator. Section 5 defines the LOWAAC operator. Section 6 formulates some extensions of the LOWAAC operator. Section 7 develops an illustrative example applying the new method in the context of human resources. Section 8 performs a comparative analysis. Finally, Section 9 summarizes the article’s main conclusions and identifies directions for future research.

2. Preliminaries

The following section comprises a concise review of the OWA operator, the IOWA operator, the GOWA operator, the Quasi-OWA operator, the LOWA operator, the adequacy coefficient, and the OWAAC operator.

The OWA operator

The OWA operator from Yager (1988) provides a parameterized class of mean type aggregation operators that lies between the minimum and the maximum. The OWA operator can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping $\text{OWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWA}(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j b_j \,, \tag{1}$$

where b_j is the jth largest element of the arguments $a_1, \dots, a_n$, i.e., $(b_1, \dots, b_n)$ is $(a_1, \dots, a_n)$ reordered from largest to smallest.

The IOWA operator

One of the most important extensions of the OWA operator is the IOWA operator (Yager & Filev, 1999). The main difference is that the reordering process uses order-inducing variables. Because of this particularity, a substantial advantage of this extension is that it can contemplate the complex attitudes of the decision-maker. The definition of an IOWA operator is as follows.

Definition 2. An IOWA operator of dimension n is a mapping $\text{IOWA}: R^n \times R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{IOWA}(\langle u_1, a_1 \rangle, \dots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j \,, \tag{2}$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value, u_i is referred as the order-inducing variable, and a_i as the argument variable.

The GOWA operator

The GOWA operator, introduced by Yager (2004), combines the OWA operator with generalized means (Dyckhoff & Pedrycz, 1984). The GOWA operator can be defined as follows.

Definition 3. A GOWA operator of dimension n is a mapping GOWA: $R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{GOWA}(a_1, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \qquad (3)$$

where λ is a parameter such that $\lambda \in (-\infty, \infty)$ and b_j is the jth largest of the argument variable a_i.

The Quasi-OWA operator

Another interesting generalization of the OWA operator is the Quasi-OWA operator, presented in (Fodor et al., 1995). Through the use of quasi-arithmetic means (Kolmogorov, 1930; Nagumo, 1930), the Quasi-OWA operator provides a more general formulation, encompassing a wide range of particular cases that are not considered in the GOWA operator. The Quasi-OWA operator can be defined as follows.

Definition 4. A Quasi-OWA operator of dimension n is a mapping $\mathrm{Quasi} - \mathrm{OWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{Quasi} - \mathrm{OWA}(a_1, \dots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j) \right), \qquad (4)$$

where b_j is the jth largest of the argument variable a_i and $g(b)$ is a strictly continuous monotonic function.

Note that if $g(b) = b$, we get the OWA operator and if $g(b) = b^{\lambda}$, the GOWA operator.

The LOWA operator

Sometimes, it is not feasible to analyze the available information with exact numerical values. When this occurs, a suitable solution may be employing linguistic values by means of linguistic variables (Zadeh, 1975a, 1975b, 1975c). Among the literature, various types of OWA operators that allow to combine linguistic information can be found (Herrera et al., 2009; Xu, 2008). One of them is the LOWA operator from Xu (2004a), which can be defined as follows.

Definition 5. Let S be a discrete linguistic term set, and $\bar{S}$ a continuous linguistic term set. Then, a LOWA operator of dimension n is a mapping LOWA: $S^n \rightarrow \bar{S}$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{LOWA}\left(s_{\alpha_1}, \dots, s_{\alpha_n}\right) = \sum_{j=1}^{n} w_j s_{\beta_j}, \tag{5}$$

where s_{β_j} is the jth greatest of the linguistic argument variable s_{α_i}.

The adequacy coefficient

Distance measures, such as the adequacy coefficient (Kaufmann & Gil-Aluja, 1986, 1987), are widely used in decision-making. The adequacy coefficient is mathematically defined as follows.

Definition 6. The adequacy coefficient between two real numbers $x \in [0,1]$ and $y \in [0,1]$ is obtained with the following formula:

$$\mathrm{AC}(x, y) = [1 \wedge (1 - x + y)]. \tag{6}$$

Regarding the mathematical symbol $\wedge$, it is used to indicate the lesser of the $(1 - x + y)$ value and 1. Also, note that the adequacy coefficient is similar to the Hamming distance (Hamming, 1950) and the Euclidean distance, however, with the advantage that it neutralizes the result when $x < y$.

For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, the weighted adequacy coefficient (WAC) is defined as follows.

Definition 7. A WAC of dimension n is a mapping WAC: $[0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, such that:

$$\mathrm{WAC}(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) = \sum_{i=1}^{n} w_i \, \mathrm{AC}(x_i, y_i), \quad (7)$$

where $\mathrm{AC}(x_i, y_i)$ is the adequacy coefficient between x_i and y_i, with x_i and y_i as the ith arguments of the sets X and Y.

The OWAAC operator

Often, when using similarity measures with the OWA operator, an ideal set is established. It is then compared with different alternatives to rank them according to their closeness. Such is the case of the OWAD operator. Nevertheless, in some situations, utilizing this operator can be inappropriate.

Usually, this is the case when one or more characteristics of an alternative have higher values than the ones of the ideal. In this case, the OWAD operator would penalize the result, which is inconsistent (on the assumption that greater values are preferred). E.g., within a personnel selection process, if a candidate presents higher skills than the required ones, he/she should not be penalized.

Some reader may have noticed that the problem mentioned in the previous paragraph would not occur if the ideal set is fixed at the maximum. However, following such criteria could also lead to inconsistent results. One explanation is that, within a given alternative, a high value in a characteristic may compensate for a low value in another attribute. For more detailed information in this regard, see Ref. (Figuerola-Wischke et al., 2022).

The OWAAC operator (Merigó & Gil-Lafuente, 2008, 2010) provides a practical solution to this limitation by using the adequacy coefficient. For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$ (in practice, X being the ideal set and Y the real or alternative set), the OWAAC operator can be defined as follows.

Definition 8. An OWAAC operator of dimension n is a mapping $\text{OWAAC}: [0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWAAC}(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j \, \text{AC}(x_j, y_j), \tag{8}$$

where $\text{AC}(x_j, y_j)$ is the jth largest $\text{AC}(x_i, y_i)$ value of the OWAAC pair (x_i, y_i) and $\text{AC}(x_i, y_i)$ is the adequacy coefficient between x_i and y_i.

3. The linguistic approach

To correctly use the LOWAAC operator and its different manifestations, we first need to define the linguistic term set (also called linguistic label set). The linguistic approach adopts symmetrically and uniformly distributed linguistic term sets.

Definition 9. Let $S = \{s_0, s_\delta, s_{2\delta}, s_{3\delta}, \dots, s_1\}$ be a finite and totally ordered discrete linguistic term set with at least two elements, that is $|S| > 1$, and a parameter δ calculated as $\delta = \frac{1}{|S|-1}$. $|S|$ refers to the cardinality value of the linguistic term set S. Alternatively, it can be written as $n(S)$.

Furthermore, the following characteristics are stated:

- The set is ordered: $s_i < s_j \Leftrightarrow i < j$.
- The negation operator: $Neg(s_i) = s_j$ such that $j = 1 - i$.
- The maximum operator is s_1.
- The minimum operator is s_0.

E.g., a linguistic set of five terms could be presented as follows (Fig. 3.1):

$$S = \{s_0 = extremely\ poor;\ s_{0.25} = poor;\ s_{0.5} = fair;\ s_{0.75} = good;\ s_1 = extremely\ good\}$$

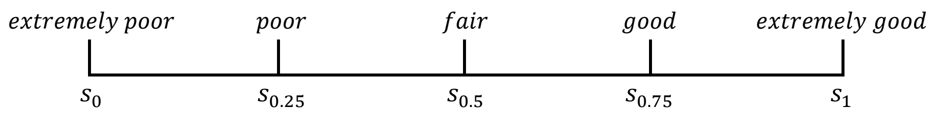

Fig. 3.1. Five term linguistic set
Source: Own elaboration

Moreover, the cardinal value of the linguistic term set S should be small enough so as not to impose useless precision on the decision-maker and rich enough to allow discrimination of the assessments in a limited number of grades (Bordogna et al., 1997).

Also, when the operations are carried out, the idea of (Xu, 2004b) is followed because it allows the preservation of all the information. Specifically, the discrete linguistic term set S is extended to a continuous linguistic term set $\bar{S} = \{s_i | s_0 < s_i \leq s_1, i \in [0,1]\}$. Thus, if $s_i \in S$, then s_i is called the original linguistic term. Otherwise, s_i is called the virtual linguistic term. As can be inferred, the original linguistic terms are used to assess the alternatives, and the virtual linguistic terms appear in the operations (Xu, 2005).

4. The linguistic adequacy coefficient

Within real-life decision-making, there are some situations where the Hamming distance combined with linguistic variables may not be helpful, rather the contrary. That is because it penalizes the result when it either does not reach the required level of a characteristic or if it exceeds it. As mentioned previously, in human resources, e.g., if a candidate for a job has a higher level for an attribute than the required one, it makes no sense that the individual gets penalized for it. It only makes sense if the candidate performs to a lower standard than required.

In such situations, a very suitable alternative may be the use of the LAC presented in this article, as it only penalizes the result when the real level of a characteristic is lower than the required one. With the LAC, when the result exceeds the required level, there is no penalty nor reward. The LAC is formally defined as follows.

Definition 10. Let s_x and s_y be two linguistic variables, with $x, y \in [0,1]$. Then, the LAC between s_x and s_y is obtained with the following formula:

$$\mathrm{LAC}(s_x, s_y) = [s_1 \wedge (s_1 - s_x + s_y)]. \quad (9)$$

Recall that the mathematical symbol $\wedge$ is used to denote the lower of the s_1 value and the $(s_1 - s_x + s_y)$ resulting value. Alternatively, the notation $\mathrm{Min}\{s_1, (s_1 - s_x + s_y)\}$ could be used.

For two linguistic sets $X = \{s_{x_1}, \dots, s_{x_n}\}$ and $Y = \{s_{y_1}, \dots, s_{y_n}\}$, the LWAC is defined as follows.

Definition 11. A LWAC of dimension n is a mapping LWAC: $[s_0, s_1]^n \times [s_0, s_1]^n \rightarrow [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, such that:

$$\mathrm{LWAC}(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle) = \sum_{i=1}^{n} w_i \, \mathrm{LAC}(s_{x_i}, s_{y_i}), \quad (10)$$

where $\mathrm{LAC}(s_{x_i}, s_{y_i})$ is the LAC between s_{x_i} and s_{y_i}, with s_{x_i} and s_{y_i} as the ith linguistic arguments of the linguistic sets X and Y.

It can be deduced that, when $w_i = 1/n$, $\forall i$, the linguistic normalized adequacy coefficient (LNAC) is obtained.

5. The LOWAAC operator

The LOWAAC operator is a linguistic aggregation operator that uses the LAC and the OWA operator in the same formulation. This new operator is helpful when decision-making under uncertainty for ranking alternatives and then selecting the optimal one. It is mainly useful when the decision-maker needs to compare the linguistic evaluations of the characteristics of each alternative with the desired ones, and he/she does not want to penalize the result when the desired levels are surpassed, but he/she wants to apply a penalization when not meet. This mechanism differentiates the LOWAAC operator from other more common, such as the LOWAD. For two linguistic sets $X = \{s_{x_1}, \dots, s_{x_n}\}$ and $Y = \{s_{y_1}, \dots, s_{y_n}\}$, the LOWAAC operator is defined as follows.

Definition 12. A LOWAAC operator of dimension n is a mapping LOWAAC: $[s_0, s_1]^n \times [s_0, s_1]^n \to [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{LOWAAC}(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle) = \sum_{j=1}^{n} w_j \,\text{LAC}\left(s_{x_j}, s_{y_j}\right), \tag{11}$$

where $\text{LAC}\left(s_{x_j}, s_{y_j}\right)$ is the jth largest $\text{LAC}(s_{x_i}, s_{y_i})$ value of the LOWAAC pair (s_{x_i}, s_{y_i}).

Example 1. Assume two linguistic sets $X = \{s_{0.6}, s_{0.6}, s_{0.3}, s_{0.8}\}$ and $Y = \{s_{0.8}, s_{0.4}, s_{0.6}, s_{0.7}\}$. If the weighting vector is $W = (0.4,0.3,0.2,0.1)$, then the LOWAAC operator can be calculated as follows:

$$\begin{aligned}\text{LOWAAC} = {} & 0.4 \times [s_1 \wedge (s_1 - s_{0.6} + s_{0.8})] + 0.3 \times [s_1 \wedge (s_1 - s_{0.3} + s_{0.6})] \\ & + 0.2 \times [s_1 \wedge (s_1 - s_{0.8} + s_{0.7})] \\ & + 0.1 \times [s_1 \wedge (s_1 - s_{0.6} + s_{0.4})] = s_{0.96}.\end{aligned}$$

Furthermore, a distinction can be drawn between the descending LOWAAC (DLOWAAC) operator and the ascending LOWAAC (ALOWAAC) operator. The weights of these two operators are connected by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the DLOWAAC (or just LOWAAC) operator and w^*_{n-j+1} is the jth weight of the ALOWAAC operator.

Additionally, the LOWAAC operator is commutative, monotonic, bounded, idempotent, and reflexive. The following theorems and proofs explain these mathematical properties attributed to the LOWAAC operator:

Theorem 1. Commutativity. Suppose that f is the LOWAAC operator, then: $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = f\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right)$, where $\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right)$ is any permutation of the arguments $\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right)$.

Proof. Let

$$f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{x_j}, s_{y_j}\right),$$
$$f\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{z_j}, s_{g_j}\right).$$

Since $\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right)$ is any permutation of the arguments $\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right)$, we have $\mathrm{LAC}\left(s_{x_j}, s_{y_j}\right) = \mathrm{LAC}\left(s_{z_j}, s_{g_j}\right)$, $\forall j$, and hence $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = f\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right)$.

Theorem 2. Monotonicity. Suppose that f is the LOWAAC operator. If $\mathrm{LAC}\left(s_{x_i}, s_{y_i}\right) \geq \mathrm{LAC}\left(s_{z_i}, s_{g_i}\right)$, $\forall i$, then: $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) \geq f\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right)$.

Proof. Let

$$f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{x_j}, s_{y_j}\right),$$
$$f\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{z_j}, s_{g_j}\right).$$

Since $\mathrm{LAC}(s_{x_i}, s_{y_i}) \geq \mathrm{LAC}(s_{z_i}, s_{g_i})$, $\forall i$, it follows that $\mathrm{LAC}\left(s_{x_j}, s_{y_j}\right) \geq \mathrm{LAC}\left(s_{z_j}, s_{g_j}\right)$, and then $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) \geq f\left(\langle s_{z_1}, s_{g_1}\rangle, \dots, \langle s_{z_n}, s_{g_n}\rangle\right)$.

Theorem 3. Boundedness. Suppose that f is the LOWAAC operator, then: $\mathrm{Max}\{\mathrm{LAC}(s_{x_i}, s_{y_i})\} \geq f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) \geq \mathrm{Min}\{\mathrm{LAC}(s_{x_i}, s_{y_i})\}$.

Proof. Let $\mathrm{Max}\{\mathrm{LAC}(s_{x_i}, s_{y_i})\} = s_z$ and $\mathrm{Min}\{\mathrm{LAC}(s_{x_i}, s_{y_i})\} = s_g$, then

$$f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{x_j}, s_{y_j}\right) \leq \sum_{j=1}^{n} w_j s_z = s_z \sum_{j=1}^{n} w_j,$$
$$f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{x_j}, s_{y_j}\right) \geq \sum_{j=1}^{n} w_j s_g = s_g \sum_{j=1}^{n} w_j.$$

Since $\sum_{j=1}^{n} w_j = 1$, we get $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) \leq s_z$ and $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) \geq s_g$, and therefore $\mathrm{Max}\{\mathrm{LAC}(s_{x_i}, s_{y_i})\} \geq f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) \geq \mathrm{Min}\{\mathrm{LAC}(s_{x_i}, s_{y_i})\}$.

Theorem 4. Idempotency. Suppose that f is the LOWAAC operator. If $\mathrm{LAC}(s_{x_i}, s_{y_i}) = \mathrm{LAC}(s_x, s_y)$, $\forall i$, then: $f\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = \mathrm{LAC}(s_x, s_y)$.

Proof. As $[s_1 \wedge (s_1 - s_{x_i} + s_{y_i})] = [s_1 \wedge (s_1 - s_x + s_y)]$, $\forall i$, we have

$$f(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{x_j}, s_{y_j}\right)$$
$$= \sum_{j=1}^{n} w_j \left[s_1 \wedge (s_1 - s_x + s_y)\right]$$
$$= \left[s_1 \wedge (s_1 - s_x + s_y)\right] \sum_{j=1}^{n} w_j \,.$$

Since $\sum_{j=1}^{n} w_j = 1$, we get $f(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle) = \mathrm{LAC}(s_x, s_y)$.

Theorem 5. Reflexivity. Suppose that f is the LOWAAC operator, then: $f(\langle s_{x_1}, s_{x_1}\rangle, \dots, \langle s_{x_n}, s_{x_n}\rangle) = s_1$.

Proof. Let

$$f(\langle s_{x_1}, s_{x_1}\rangle, \dots, \langle s_{x_n}, s_{x_n}\rangle) = \sum_{j=1}^{n} w_j \mathrm{LAC}\left(s_{x_j}, s_{y_j}\right).$$

Since $[s_1 \wedge (s_1 - s_{x_i} + s_{x_i})] = s_1$, $\forall i$, we get $f(\langle s_{x_1}, s_{x_1}\rangle, \dots, \langle s_{x_n}, s_{x_n}\rangle) = s_1$.

Another noteworthy aspect is the characterizing measures related to the weighting vector W of the LOWAAC operator and the type of aggregation performed. Below are briefly stated the measures of attitudinal character (Yager, 1988), entropy of dispersion (Yager, 1988), balance operator (Yager, 1996), and divergence of W (Yager, 2002).

The attitudinal character measure, also referred as the degree of orness, is defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \qquad (12)$$

The measure of entropy of dispersion is defined as follows:

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{13}$$

The balance operator characterizing measure is defined as follows:

$$\mathrm{Bal}(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1}\right). \tag{14}$$

And the divergence of W is defined as follows:

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \tag{15}$$

6. Families and extensions of the LOWAAC operator

The following section first formulates a few particular families of the LOWAAC operator. Then, some new extensions of the LOWAAC operator are defined: the ILOWAAC operator, the GLOWAAC operator, the GILOWAAC operator, the Quasi-LOWAAC operator, and the Quasi-ILOWAAC operator.

Families of the LOWAAC operator

Locating different families of the LOWAAC operator is done by choosing different manifestations of the weighting vector W. Some families of the LOWAAC operator are:

- The linguistic maximum adequacy coefficient (Max-LAC), which is found when $w_1 = 1$ and $w_j = 0, \forall j \neq 1$.
- The linguistic minimum adequacy coefficient (Min-LAC), which is found when $w_n = 1$ and $w_j = 0, \forall j \neq n$.

- The LNAC, which is found when $w_j = 1/n, \forall j$.
- The Median-LOWAAC, which is found when $w_{(n+1)/2} = 1$ and $w_j = 0, \forall j \neq (n+1)/2$, provided the dimension of W is odd. In the case where n is even, the Median-LOWAAC is found when $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_j = 0, \forall j \neq n/2, (n/2)+1$.
- The Hurwicz-LOWAAC, which is found when $w_1 = \alpha$, $w_n = 1 - \alpha$, and $w_j = 0, \forall j \neq 1, n$.
- The Olympic-LOWAAC, which is found when $w_1 = w_n = 0$ and $w_j = 1/(n-2), \forall j \neq 1, n$.
- Lastly, the Step-LOWAAC, which is found when $w_k = 1$ and $w_j = 0, \forall j \neq k$.

The ILOWAAC operator

Defined as an extension of the LOWAAC operator, the ILOWAAC operator uses order-inducing variables to dictate the aggregation order of the linguistic arguments. This results in a broader formulation of the reordering process that allows the representation of more complex situations. For two linguistic sets $X = \{s_{x_1}, \ldots, s_{x_n}\}$ and $Y = \{s_{y_1}, \ldots, s_{y_n}\}$, the ILOWAAC operator is mathematically defined as follows.

Definition 13. An ILOWAAC operator of dimension n is a mapping ILOWAAC: $R^n \times [s_0, s_1]^n \times [s_0, s_1]^n \to [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{ILOWAAC}(\langle u_1, s_{x_1}, s_{y_1}\rangle, \ldots, \langle u_n, s_{x_n}, s_{y_n}\rangle) = \sum_{j=1}^{n} w_j \, \text{LAC}\left(s_{x_j}, s_{y_j}\right), \quad (16)$$

where $\text{LAC}\left(s_{x_j}, s_{y_j}\right)$ is the $\text{LAC}(s_{x_i}, s_{y_i})$ value of the ILOWAAC triplet $\langle u_i, s_{x_i}, s_{y_i}\rangle$ having the jth largest u_i and u_i is the order-inducing variable.

Example 2. Assume two linguistic sets $X = \{s_{0.6}, s_{0.6}, s_{0.3}, s_{0.8}\}$ and $Y = \{s_{0.8}, s_{0.4}, s_{0.6}, s_{0.7}\}$. If the weighting vector is $W = (0.4,0.3,0.2,0.1)$ and the order-inducing variables are $U = (9,7,3,6)$, then the ILOWAAC operator can be calculated as follows:

$$\begin{aligned}\text{ILOWAAC} = {} & 0.4 \times [s_1 \wedge (s_1 - s_{0.6} + s_{0.8})] \\ & + 0.3 \times [s_1 \wedge (s_1 - s_{0.6} + s_{0.4})] \\ & + 0.2 \times [s_1 \wedge (s_1 - s_{0.8} + s_{0.7})] \\ & + 0.1 \times [s_1 \wedge (s_1 - s_{0.3} + s_{0.6})] = s_{0.92}.\end{aligned}$$

The GLOWAAC operator

The GLOWAAC operator is a new mathematical function that generalizes the LOWAAC operator and the generalized mean. For two linguistic sets $X = \{s_{x_1}, \dots, s_{x_n}\}$ and $Y = \{s_{y_1}, \dots, s_{y_n}\}$, the GLOWAAC operator is defined as follows.

Definition 14. A GLOWAAC operator of dimension n is a mapping GLOWAAC: $[s_0, s_1]^n \times [s_0, s_1]^n \rightarrow [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{GLOWAAC}(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle) = \left(\sum_{j=1}^{n} w_j \left(\text{LAC}\left(s_{x_j}, s_{y_j}\right)\right)^{\lambda}\right)^{1/\lambda}, \quad (17)$$

where $\text{LAC}\left(s_{x_j}, s_{y_j}\right)$ is the jth largest $\text{LAC}(s_{x_i}, s_{y_i})$ value of the GLOWAAC pair (s_{x_i}, s_{y_i}) and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Example 3. Assume two linguistic sets $X = \{s_{0.6}, s_{0.6}, s_{0.3}, s_{0.8}\}$ and $Y = \{s_{0.8}, s_{0.4}, s_{0.6}, s_{0.7}\}$. If the weighting vector is $W = (0.4,0.3,0.2,0.1)$ and the parameter λ is equal to 4, then the GLOWAAC operator can be calculated as follows:

$$\begin{aligned}\text{GLOWAAC} = {} & (0.4 \times [s_1 \wedge (s_1 - s_{0.6} + s_{0.8})]^4 \\ & + 0.3 \times [s_1 \wedge (s_1 - s_{0.3} + s_{0.6})]^4 \\ & + 0.2 \times [s_1 \wedge (s_1 - s_{0.8} + s_{0.7})]^4 \\ & + 0.1 \times [s_1 \wedge (s_1 - s_{0.6} + s_{0.4})]^4)^{1/4} = s_{0.97}.\end{aligned}$$

One can observe that, based on the value assigned to the parameter λ, a wide range of particular cases of the GLOWAAC operator are possible. Some particular cases of the GLOWAAC operator are:

- The linguistic ordered weighted harmonic averaging adequacy coefficient (LOWHAAC) operator, which is obtained when $\lambda = -1$.
- The linguistic ordered weighted geometric adequacy coefficient (LOWGAC) operator, which is obtained when $\lambda = 0$.
- The original LOWAAC operator, which is obtained when $\lambda = 1$.
- Lastly, the linguistic ordered weighted quadratic averaging adequacy coefficient (LOWQAAC) operator, which is obtained when $\lambda = 2$.

The GILOWAAC operator

The GILOWAAC operator is basically an extension of the GLOWAAC operator in which the process of reordering depends on the order-inducing variables. For two linguistic sets $X = \{s_{x_1}, \dots, s_{x_n}\}$ and $Y = \{s_{y_1}, \dots, s_{y_n}\}$, the GILOWAAC operator is defined as follows.

Definition 15. A GILOWAAC operator of dimension n is a mapping GILOWAAC: $R^n \times [s_0, s_1]^n \times [s_0, s_1]^n \to [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{GILOWAAC}(\langle u_1, s_{x_1}, s_{y_1}\rangle, \dots, \langle u_n, s_{x_n}, s_{y_n}\rangle) = \left(\sum_{j=1}^{n} w_j \left(\text{LAC}\left(s_{x_j}, s_{y_j}\right)\right)^{\lambda}\right)^{1/\lambda}, \tag{18}$$

where $\text{LAC}\left(s_{x_j}, s_{y_j}\right)$ is the $\text{LAC}(s_{x_i}, s_{y_i})$ value of the GILOWAAC triplet $\langle u_i, s_{x_i}, s_{y_i}\rangle$ having the jth largest u_i, u_i is the order-inducing variable, and λ is a parameter such that $\lambda \in (-\infty, \infty)$.

Note that when λ approaches 1, the ILOWAAC operator is obtained.

The Quasi-LOWAAC operator

The Quasi-LOWAAC operator is a greater generalization of the LOWAAC operator that uses quasi-arithmetic means. For two linguistic sets $X = \left\{s_{x_1}, \dots, s_{x_n}\right\}$ and $Y = \left\{s_{y_1}, \dots, s_{y_n}\right\}$, the Quasi-LOWAAC operator is defined as follows.

Definition 16. A Quasi-LOWAAC operator of dimension n is a mapping $\text{Quasi} - \text{LOWAAC}: [s_0, s_1]^n \times [s_0, s_1]^n \rightarrow [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{Quasi} - \text{LOWAAC}\left(\langle s_{x_1}, s_{y_1}\rangle, \dots, \langle s_{x_n}, s_{y_n}\rangle\right) = g^{-1}\left(\sum_{j=1}^{n} w_j\, g\left(\text{LAC}\left(s_{x_j}, s_{y_j}\right)\right)\right), \tag{19}$$

where $\text{LAC}\left(s_{x_j}, s_{y_j}\right)$ is the jth largest $\text{LAC}\left(s_{x_i}, s_{y_i}\right)$ value of the Quasi-LOWAAC pair $\left(s_{x_i}, s_{y_i}\right)$ and g is a strictly continuous monotonic function.

The Quasi-ILOWAAC operator

The Quasi-ILOWAAC operator is described as an extension of the Quasi-LOWAAC operator that employs order-inducing variables. For two linguistic sets $X = \left\{s_{x_1}, \dots, s_{x_n}\right\}$ and $Y = \left\{s_{y_1}, \dots, s_{y_n}\right\}$, the Quasi-ILOWAAC operator is mathematically defined as follows.

Definition 17. A Quasi-ILOWAAC operator of dimension n is a mapping $\text{Quasi} - \text{ILOWAAC}: R^n \times [s_0, s_1]^n \times [s_0, s_1]^n \rightarrow [s_0, s_1]$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{Quasi} - \text{ILOWAAC}(\langle u_1, s_{x_1}, s_{y_1} \rangle, \ldots, \langle u_n, s_{x_n}, s_{y_n} \rangle) = g^{-1}\left(\sum_{j=1}^{n} w_j \, g\left(\text{LAC}\left(s_{x_j}, s_{y_j}\right)\right)\right), \quad (20)$$

where $\text{LAC}\left(s_{x_j}, s_{y_j}\right)$ is the $\text{LAC}(s_{x_i}, s_{y_i})$ value of the Quasi-ILOWAAC triplet $\langle u_i, s_{x_i}, s_{y_i} \rangle$ having the jth largest u_i, u_i is the order-inducing variable, and g is a strictly continuous monotonic function.

7. Illustrative example

In this section, a conducted illustrative example showcases how to use this new approach within the football industry's human resource management. Note that many other decision-making applications could apply the LOWAAC operator and its variants, e.g., selection of investments, marketing decision-making, consumer decision-making, and more.

Suppose that a football club needs to sign a new defensive midfielder and there are six potential candidates to occupy this position. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ be the set of the six candidates. Assume that the management of the football club has assembled a high-quality group of three experts, so that they provide an evaluation of the candidates according to nine different key skills, which are: $c_1 = ball\ control$, $c_2 = passing$, $c_3 = shooting$, $c_4 = heading$, $c_5 = dribbling$, $c_6 = tackling$, $c_7 = game\ intelligence$, $c_8 = speed$, and $c_9 = strength$. Also, consider that the group of experts use the following linguist term set S to independently evaluate the candidates:

$$S = \{s_0 = absolutely\ poor;\ s_{0.1} = very\ poor;\ s_{0.2} = poor;\ s_{0.3} = rather\ poor;\ s_{0.4} = fairly\ poor;\ s_{0.5} = medium;\ s_{0.6} = fairly\ good;\ s_{0.7} = rather\ good;\ s_{0.8} = good;\ s_{0.9} = very\ good;\ s_1 = absolutely\ good\}.$$

The group of experts, jointly with the management team of the football club, agree to establish the required levels of each characteristic of the ideal defensive midfielder player I, as shown in Table 3.1. Next, each expert individually assesses the real levels of each attribute for all the candidates contemplated for the position. Presented in Tables 3.2-3.4 are the results. Afterwards, the evaluations provided by the three experts are unified into a collective matrix using the linguistic averaging (LA) operator (see Table 3.5), as they are considered equally important.

Table 3.1. Assessments of the ideal candidate

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
I	$s_{0.8}$	$s_{0.8}$	$s_{0.6}$	$s_{0.5}$	$s_{0.7}$	$s_{0.9}$	$s_{0.8}$	$s_{0.6}$	$s_{0.8}$

Table 3.2. Assessments of the real candidates by Expert 1

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
x_1	$s_{0.6}$	$s_{0.7}$	$s_{0.7}$	$s_{0.4}$	$s_{0.7}$	$s_{0.8}$	$s_{0.6}$	$s_{0.9}$	$s_{0.7}$
x_2	$s_{0.8}$	$s_{0.8}$	$s_{0.8}$	$s_{0.3}$	$s_{0.6}$	$s_{0.6}$	$s_{0.8}$	$s_{0.5}$	$s_{0.6}$
x_3	$s_{0.7}$	$s_{0.7}$	$s_{0.7}$	$s_{0.6}$	$s_{0.6}$	$s_{0.7}$	$s_{0.6}$	$s_{0.7}$	$s_{0.9}$
x_4	$s_{0.6}$	$s_{0.6}$	$s_{0.7}$	$s_{0.8}$	$s_{0.5}$	$s_{0.9}$	$s_{0.5}$	$s_{0.7}$	$s_{0.8}$
x_5	$s_{0.9}$	$s_{0.9}$	$s_{0.8}$	$s_{0.4}$	$s_{0.5}$	$s_{0.6}$	$s_{0.9}$	$s_{0.6}$	$s_{0.8}$
x_6	$s_{0.6}$	$s_{0.6}$	$s_{0.6}$	$s_{0.8}$	$s_{0.6}$	$s_{0.9}$	$s_{0.4}$	$s_{0.7}$	$s_{0.9}$

Table 3.3. Assessments of the real candidates by Expert 2

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
x_1	$s_{0.7}$	$s_{0.7}$	$s_{0.7}$	$s_{0.3}$	$s_{0.7}$	$s_{0.9}$	$s_{0.8}$	$s_{0.9}$	$s_{0.8}$
x_2	$s_{0.9}$	$s_{0.9}$	$s_{0.8}$	$s_{0.3}$	$s_{0.7}$	$s_{0.7}$	$s_{0.8}$	$s_{0.6}$	$s_{0.6}$
x_3	$s_{0.7}$	$s_{0.7}$	$s_{0.6}$	$s_{0.7}$	$s_{0.6}$	$s_{0.8}$	$s_{0.5}$	$s_{0.7}$	$s_{0.8}$
x_4	$s_{0.6}$	$s_{0.6}$	$s_{0.6}$	$s_{0.6}$	$s_{0.6}$	$s_{0.7}$	$s_{0.4}$	$s_{0.8}$	$s_{0.9}$
x_5	$s_{0.9}$	$s_{0.9}$	$s_{0.9}$	$s_{0.4}$	$s_{0.6}$	$s_{0.7}$	$s_{0.9}$	$s_{0.5}$	$s_{0.7}$
x_6	$s_{0.5}$	$s_{0.5}$	$s_{0.6}$	$s_{0.9}$	$s_{0.5}$	$s_{0.9}$	$s_{0.5}$	$s_{0.6}$	$s_{0.9}$

Table 3.4. Assessments of the real candidates by Expert 3

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
x_1	$s_{0.5}$	$s_{0.5}$	$s_{0.6}$	$s_{0.4}$	$s_{0.6}$	$s_{0.8}$	$s_{0.3}$	$s_{0.8}$	$s_{0.7}$
x_2	$s_{0.8}$	$s_{0.9}$	$s_{0.7}$	$s_{0.4}$	$s_{0.7}$	$s_{0.5}$	$s_{0.9}$	$s_{0.4}$	$s_{0.5}$
x_3	$s_{0.6}$	$s_{0.6}$	$s_{0.6}$	$s_{0.5}$	$s_{0.6}$	$s_{0.7}$	$s_{0.7}$	$s_{0.7}$	$s_{0.8}$
x_4	$s_{0.5}$	$s_{0.6}$	$s_{0.5}$	$s_{0.7}$	$s_{0.5}$	$s_{0.9}$	$s_{0.5}$	$s_{0.7}$	$s_{0.9}$
x_5	$s_{0.9}$	$s_{0.8}$	$s_{0.9}$	$s_{0.3}$	$s_{0.6}$	$s_{0.6}$	s_1	$s_{0.6}$	$s_{0.7}$
x_6	$s_{0.6}$	$s_{0.6}$	$s_{0.7}$	$s_{0.8}$	$s_{0.5}$	s_1	$s_{0.2}$	$s_{0.6}$	$s_{0.9}$

Table 3.5. Collective assessments of the real candidates

	c_1	c_2	c_3	c_4	c_5	c_6	c_7	c_8	c_9
x_1	$s_{0.6}$	$s_{0.633}$	$s_{0.667}$	$s_{0.367}$	$s_{0.667}$	$s_{0.833}$	$s_{0.567}$	$s_{0.867}$	$s_{0.733}$
x_2	$s_{0.833}$	$s_{0.867}$	$s_{0.767}$	$s_{0.333}$	$s_{0.667}$	$s_{0.6}$	$s_{0.833}$	$s_{0.5}$	$s_{0.567}$
x_3	$s_{0.667}$	$s_{0.667}$	$s_{0.633}$	$s_{0.6}$	$s_{0.6}$	$s_{0.733}$	$s_{0.6}$	$s_{0.7}$	$s_{0.833}$
x_4	$s_{0.567}$	$s_{0.6}$	$s_{0.6}$	$s_{0.7}$	$s_{0.533}$	$s_{0.833}$	$s_{0.467}$	$s_{0.733}$	$s_{0.867}$
x_5	$s_{0.9}$	$s_{0.867}$	$s_{0.867}$	$s_{0.367}$	$s_{0.567}$	$s_{0.633}$	$s_{0.933}$	$s_{0.567}$	$s_{0.733}$
x_6	$s_{0.567}$	$s_{0.567}$	$s_{0.633}$	$s_{0.833}$	$s_{0.533}$	$s_{0.933}$	$s_{0.367}$	$s_{0.633}$	$s_{0.9}$

Now, assume that the three experts agree on using the weighting vector $W = (0.2,0.15,0.15,0.15,0.075,0.05,0.05,0.025)$, the order-inducing vector $U = (9,9,7,6,8,10,9,7,9)$, and a parameter $\lambda = 2$. With this information, the group of specialists perform the aggregations by using the LWAC, LOWAAC, ALOWAAC, ILOWAAC, and LOWQAAC operators. The results are shown in Table 3.6, wherein the closer the candidate is to s_1, the better.

Table 3.6. Aggregated results of the LACs

	LWAC	LOWAAC	ALOWAAC	ILOWAAC	LOWQAAC
x_1	$s_{0.892}$	$s_{0.941}$	$s_{0.862}$	$s_{0.881}$	$s_{0.943}$
x_2	$s_{0.937}$	$s_{0.96}$	$s_{0.86}$	$s_{0.893}$	$s_{0.963}$
x_3	$s_{0.916}$	$s_{0.955}$	$s_{0.88}$	$s_{0.889}$	$s_{0.957}$
x_4	$s_{0.877}$	$s_{0.948}$	$s_{0.833}$	$s_{0.859}$	$s_{0.951}$
x_5	$s_{0.937}$	$s_{0.97}$	$s_{0.892}$	$s_{0.922}$	$s_{0.972}$
x_6	$s_{0.872}$	$s_{0.953}$	$s_{0.818}$	$s_{0.853}$	$s_{0.959}$

In Table 3.7, we can see that depending on the linguistic aggregation operator used, the ranking of the candidates may change. With the LWAC operator, the best candidate for the defensive midfielder position is x_2 together with x_5, while the less preferred is x_6. With the LOWAAC and LOWQAAC operators, the best football player is x_5, by contrast, the worst one is x_1. With the ALOWAAC and ILOWAAC operators, the best candidate to sign is x_5 and the poorest one is x_6.

Table 3.7. Ordering of the candidates

	Ordering
LWAC	$x_2 = x_5 \succ x_3 \succ x_1 \succ x_4 \succ x_6$
LOWAAC	$x_5 \succ x_2 \succ x_3 \succ x_6 \succ x_4 \succ x_1$
ALOWAAC	$x_5 \succ x_3 \succ x_1 \succ x_2 \succ x_4 \succ x_6$
ILOWAAC	$x_5 \succ x_2 \succ x_3 \succ x_1 \succ x_4 \succ x_6$
LOWQAAC	$x_5 \succ x_2 \succ x_6 \succ x_3 \succ x_4 \succ x_1$

8. Comparison analysis

The following section compares the presented aggregation operators in this article with existing ones. For this comparison, the linguistic weighted Hamming distance (LWHD) operator, the LOWAD operator, the ascending LOWAD (ALOWAD) operator, the induced LOWAD (LIOWAD) operator (Cheng & Zeng, 2012; Zeng et al., 2013), and the linguistic ordered weighted quadratic averaging distance (LOWQAD) operator, are calculated using the same input information as in Section 7 (see Table 3.8). Presented in Table

3.9 are the order of the candidates according to these operators. Observe that, in this case, the lower the value of the subscript, the better the candidate is.

Table 3.8. Aggregated results of the Hamming distances

	LWHD	LOWAD	ALOWAD	LIOWAD	LOWQAD
x_1	$s_{0.132}$	$s_{0.176}$	$s_{0.098}$	$s_{0.136}$	$s_{0.19}$
x_2	$s_{0.107}$	$s_{0.169}$	$s_{0.082}$	$s_{0.135}$	$s_{0.191}$
x_3	$s_{0.11}$	$s_{0.135}$	$s_{0.087}$	$s_{0.125}$	$s_{0.143}$
x_4	$s_{0.162}$	$s_{0.203}$	$s_{0.11}$	$s_{0.163}$	$s_{0.22}$
x_5	$s_{0.14}$	$s_{0.168}$	$s_{0.098}$	$s_{0.137}$	$s_{0.185}$
x_6	$s_{0.19}$	$s_{0.243}$	$s_{0.113}$	$s_{0.181}$	$s_{0.275}$

Table 3.9. Ordering of the candidates

	Ordering
LWHD	$x_2 \succ x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_6$
LOWAD	$x_3 \succ x_5 \succ x_2 \succ x_1 \succ x_4 \succ x_6$
ALOWAD	$x_2 \succ x_3 \succ x_1 \succ x_5 \succ x_4 \succ x_6$
LIOWAD	$x_3 \succ x_2 \succ x_1 \succ x_5 \succ x_4 \succ x_6$
LOWQAD	$x_3 \succ x_5 \succ x_1 \succ x_2 \succ x_4 \succ x_6$

It can be seen that the results differ from the ones obtained with the LOWAAC operator and its extensions. E.g., for the LOWQAAC operator, the best choice is x_5. However, for the LOWQAD operator, the preferred option is x_3. By employing classical distance models, the outcomes may be inconsistent. For instance, candidate x_5 is penalized for being too good at ball control, which makes no sense.

9. Conclusions

In this article, the LAC, a new deviation measure, has been introduced. Furthermore, based on the LAC and the well-known OWA operator, the LOWAAC operator has been proposed as an original linguistic aggregation operator for ranking a finite set of alternatives in complex decision-making problems.

On top of this, this article has also presented some extensions and generalizations of the LOWAAC operator. Firstly, the ILOWAAC operator, an extension of the LOWAAC operator that uses order-inducing variables in the reordering step. Secondly, the GLOWAAC operator, a generalization of the LOWAAC operator that incorporates generalized means. Thirdly, the GILOWAAC operator, a generalization of the LOWAAC operator that utilizes order-inducing variables and generalized means. Fourthly, the Quasi-LOWAAC operator, which further generalizes the LOWAAC operator by employing quasi-arithmetic means. Last but not least, the Quasi-ILOWAAC operator, which is an extension of the Quasi-LOWAAC operator that embeds order-inducing variables.

In decision-making, these operators offer significant advantages over more traditional methods in specific situations. One of them is that they are advantageous for calculating the differences between a set of alternatives and an ideal, while considering a threshold from which the results are always the same. Another interesting advantage is their capacity to assess uncertain problems in which the available information is not representable with numerical values, but it is possible to use linguistic values. Also, these operators allow to consider the attitudinal character of the decision-maker when the information is fused, thus providing greater flexibility.

To help illustrate this new approach, an example of a MEDM problem regarding the selection of a football player has also been developed. Additionally, the illustrative example has been further developed with a comparative analysis between the new approach and some existing methods, where we can observe that the outcomes are different.

In the future, we intend to create further extensions of the LOWAAC operator. E.g., with the use of basic uncertain information (BUI) (Jin et al., 2018; Mesiar et al., 2018). We also expect to analyze the utilization of unbalanced linguistic information in the LOWAAC operator. Lastly, we look to study different applications of the LOWAAC operator in decision-making problems under uncertainty, such as international market selection for firms.

10.References

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3.2. The uncertain ordered weighted averaging adequacy coefficient operator

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Abstract

This article introduces the uncertain ordered weighted averaging adequacy coefficient (UOWAAC) operator. This novel operator uses the ordered weighted averaging (OWA) operator, the adequacy coefficient, and the interval numbers in a single formulation. This article also extends the UOWAAC operator by using order-inducing variables in the reordering process of the input arguments. This new extension is called the uncertain induced ordered weighted averaging adequacy coefficient (UIOWAAC) operator. The article also presents an application of the new approach in a multi-criteria group decision-making (MCGDM) problem about international expansion. In addition, a comparative analysis is conducted with the purpose of demonstrating the superiority of the UOWAAC and UIOWAAC aggregation operators in specific situations. Likewise, the use of basic uncertain information (BUI) is discussed. The results show the usefulness of these new aggregation operators in real-life decision-making problems under uncertainty, particularly when the decision-maker wants to compare different alternatives with an ideal but without giving any penalty or reward in the case that the ideal levels are exceeded.

Keywords: Adequacy coefficient, aggregation operator, business decision-making, interval number, uncertainty, OWA operator.

1. Introduction

Aggregation operators (also referred as aggregation functions) are commonly used in decision-making procedures in order to combine several sources of information into a single result. An increasingly popular aggregation operator (Blanco-Mesa et al., 2019) is the ordered weighted averaging (OWA), which was first presented by Yager (1988). This operator provides a parameterized family of aggregation operators between the minimum and the maximum. Since its introduction, several applications have been studied, including sales forecasting (Linares-Mustarós et al., 2015; Merigó et al., 2015), portfolio selection (Laengle et al., 2015), retirement planning (Figuerola-Wischke et al., 2021), government transparency (Perez-Arellano et al., 2020), agricultural product prices (León-Castro et al., 2021), and many others (Kacprzyk et al., 2019).

In the literature we can find a wide range of aggregation operators that extend the OWA operator. Some of the most important are the induced OWA (IOWA) (Yager & Filev, 1999) and the IOWA in the expression of weighted averaging (WA) functions (Jin et al., 2021), the generalized OWA (GOWA) (Yager, 2004), the quasi OWA (QOWA) (Fodor et al., 1995), the probabilistic OWA (POWA) (Merigó, 2012), the linguistic OWA (LOWA) (Herrera et al., 1995; Herrera & Martínez, 2000; Xu, 2004), the fuzzy OWA (FOWA) (Chen & Chen, 2003), the uncertain OWA (UOWA) (Xu & Da, 2002), and the OWA distance (OWAD) (Merigó & Gil-Lafuente, 2010). Also, recently, Jin, Mesiar, and Yager (2019) proposed an OWA weight allocation method to deal with convex partially ordered sets (posets).

Furthermore, another extension that received much attention is the OWA adequacy coefficient (OWAAC) operator (Merigó & Gil-Lafuente, 2008, 2010), which as its name indicates, uses the OWA operator with the adequacy coefficient (Kaufmann & Gil-Aluja, 1986, 1987) in a single formulation. This operator is mainly used to compare an ideal set with a real one, but in contrast to other operators, such as the OWAD, it does not penalize the result when the ideal levels are exceeded. However, this aggregation operator only considers exact numbers, which is not always possible, especially when the environment is highly uncertain and complex. In this case, an adequate alternative may be the use of interval numbers (Moore, 1966).

Thus, the aim of this article is to develop a new extension of the OWAAC operator for situations with a high degree of uncertainty. To do so, interval numbers are used instead of exact numbers. It is called the uncertain ordered weighted averaging adequacy coefficient (UOWAAC) operator. Moreover, the UOWAAC operator is extended by using order-inducing variables. As a result, the uncertain induced ordered weighted averaging adequacy coefficient (UIOWAAC) operator is obtained. Lastly, another objective of this article is to demonstrate the utility of these new aggregation operators in real-world situations. To achieve this, the applicability of these novel operators in a multi-criteria group decision-making (MCGDM) problem regarding the international expansion of a business is studied.

This article is arranged as follows. Section 2 conducts a review of the OWA operator, the interval numbers, the UOWA operator, the adequacy coefficient, and the OWAAC operator. Section 3 presents the UOWAAC operator, analyze its properties, and discuss its families. Section 4 extends the UOWAAC operator through order-inducing variables. Section 5 provides an illustrative example of the new approach in order to demonstrate its practicability. Section 6 presents a comparison of the developed aggregation operators with the existing ones. Section 7 summarizes the main conclusions of the article and indicates opportunities for future research.

2. Some preliminary concepts

The following section briefly reviews some basic but necessary concepts, which are the OWA operator, the interval numbers, the UOWA operator, the adequacy coefficient, and the OWAAC operator.

The OWA operator

The OWA operator is an aggregation operator introduced by Yager (1988) and it provides a parameterized family of aggregation operators that include among others the minimum, the maximum, and the arithmetic mean decision criteria. A fundamental characteristic of this operator is found in the reordering step of the input arguments in which it is carried out in a descending way. This operator can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping OWA: $R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j \,, \tag{1}$$

where b_j is the jth largest element of the arguments $a_1, \ldots, a_n$.

Additionally, the OWA operator is commutative, monotonic, bounded, and idempotent.

Interval numbers

In decision-making, uncertainty is often an unavoidable problem. A practical way for handling uncertainty is through the use of interval numbers (Moore, 1966). An interval number can be described as an ordered pair of real numbers or also as a set of the real line R. Mathematically it is defined as follows.

Definition 2. Let $\tilde{a}$ be an interval number. Then $\tilde{a} = [a^L, a^U]$ with $a^L, a^U \in R$ and $a^L \leq a^U$. In the particular case $a^L = a^U$, one can see that $\tilde{a}$ is reduced to a real number, which is known as degenerate interval number.

The UOWA operator

The UOWA operator was developed by Xu and Da (2002) and it is an extension of the OWA operator for uncertain environments where the available information can only be assessed with the use of interval numbers. This aggregation function can be defined as follows.

Definition 3. Let Ω be a set of interval numbers. An UOWA operator of dimension n is a mapping UOWA: $\Omega^n \rightarrow \Omega$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{UOWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{b}_j \,, \tag{2}$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$, and $\tilde{a}_i$ is the argument variable represented in the form of interval numbers.

The UOWA operator satisfies the mathematical properties of commutativity, monotonicity, boundedness, and idempotency.

The adequacy coefficient

The adequacy coefficient (Kaufmann & Gil-Aluja, 1986, 1987) is an index used for calculating the differences between two real numbers in a more effective manner. It can be defined as follows.

Definition 4. Let x and y be two real numbers such that $x, y \in [0,1]$. Then, the adequacy coefficient between x and y is obtained by applying the following formula:

$$\mathrm{AC}(x, y) = [1 \wedge (1 - x + y)]. \tag{3}$$

Note that the symbol $\wedge$ is used to indicate the lower value between 1 and $(1 - x + y)$.

Also, it is noteworthy that the adequacy coefficient is similar to the Hamming distance (Hamming, 1950) but with the difference that it neutralizes the result when $x < y$.

The OWAAC operator

In (Merigó & Gil-Lafuente, 2008, 2010), the authors presented the OWAAC operator, which uses the adequacy coefficient and the OWA operator in the same formulation. This operator is used for complex comparisons between two sets, normally between an ideal set (X) and a real one (Y).

For two sets $X = \{x_1, \dots, x_n\}$ and $Y = \{y_1, \dots, y_n\}$, the OWAAC operator is defined as follows.

Definition 5. An OWAAC operator of dimension n is a mapping $\text{OWAAC}: [0,1]^n \times [0,1]^n \rightarrow [0,1]$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWAAC}(\langle x_1, y_1 \rangle, \dots, \langle x_n, y_n \rangle) = \sum_{j=1}^{n} w_j \, K_j, \tag{4}$$

where K_j is the jth largest of the $[1 \wedge (1 - x_i + y_i)]$, and x_i and y_i are the ith arguments of the sets X and Y.

The OWAAC operator is commutative, monotonic, bounded, idempotent, nonnegative and reflexive.

Remark. Some readers may think that an ideal set should always have the highest level in all characteristics. But in the real world, this is not true, in some situations, if the ideal set is fixed at the maximum, when performing the comparison, the decision-maker could erroneously exclude the most convenient option. One reason is that a high value in a characteristic may compensate for a low value in another characteristic and consequently obtain an inconsistent result.

For example, imagine that a Spanish cellar wants to offer guided tours for international tourists and therefore needs to hire a professional who can speak English (C_1) and German (C_2), and also who has a general knowledge of wine culture (C_3). Suppose that there are two candidates, Y_1 and Y_2. Assume that the candidates are evaluated for each competence as follows: $Y_1 = \{C_1 = 1, C_2 = 0.3, C_3 = 0.8\}$ and $Y_2 = \{C_1 = 0.7, C_2 = 0.6, C_3 = 0.7\}$, where 0 is the worst evaluation and 1 the best.

If the cellar assumes that the ideal candidate should have the maximum level in all three competences, i.e., $X = \{C_1 = 1, C_2 = 1, C_3 = 1\}$, then, using the OWAAC operator with $W = \{w_1 = 1/3\,, w_2 = 1/3\,, w_3 = 1/3\}$, the best candidate would be Y_1.

By contrast, if the cellar establishes the optimal levels of the ideal worker as $X = \{C_1 = 0.6, C_2 = 0.6, C_3 = 0.6\}$, then, the hired worker would be Y_2, which makes more sense. Candidate Y_2 may not be perfect in a specific skill however meets all the requirements. On the other hand, candidate Y_1 is excellent in English but at the same time incapable to do a proper tour in German. Thus, if the hired candidate is Y_1, the cellar probably will have to seek for an additional employee.

3. The UOWAAC operator

The following section first defines the UOWAAC operator, then, analyzes its properties, and lastly, studies its different families.

Definition of the UOWAAC operator

The UOWAAC operator can be described as an extension of the OWAAC operator that uses interval numbers instead of exact numbers. This new operator is very complete as it offers numerous advantages over traditional aggregation operators. For example, by using interval numbers the decision-maker is able to deal with uncertainty. Likewise, as it is built under the OWA operator, it allows to consider the attitudinal character of the decision-maker when the information is fused. Furthermore, through the adequacy coefficient, this operator can be used for comparing a set of available alternatives with an ideal, while at the same time establish a threshold from which the results are always the same. Hence, it only penalizes when the optimal levels are not reached.

Definition 6. Let $\tilde{x}$ and $\tilde{y}$ be two interval numbers. An uncertain adequacy coefficient (UAC) is a similarity measure, such that:

$$\mathrm{UAC}(\tilde{x}, \tilde{y}) = \frac{1}{2}\left([1 \wedge (1 - x^L + y^L)] + [1 \wedge (1 - x^U + y^U)]\right), \quad (5)$$

where $x^L, x^U \in [0,1]$ are the lower and upper values of the interval number $\tilde{x}$, and $y^L, y^U \in [0,1]$ are the lower and upper values of the interval number $\tilde{y}$.

Let $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_n)$ and $\tilde{Y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ be two sets of interval numbers, then, the uncertain weighted adequacy coefficient (UWAC) operator and the UOWAAC operators can be defined respectively as follows.

Definition 7. Let Ω be a set of interval numbers. An UWAC operator of dimension n is a mapping $\mathrm{UWAC}: \Omega^n \times \Omega^n \to R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_i \in [0,1]$ and $\sum_{i=1}^{n} w_i = 1$, in which:

$$\mathrm{UWAC}(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) = \sum_{i=1}^{n} w_i \mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)\,, \tag{6}$$

where $\tilde{x}_i$ and $\tilde{y}_i$ are the ith arguments of the sets $\tilde{X}$ and $\tilde{Y}$, and $\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)$ is the adequacy coefficient between $\tilde{x}_i = [x_i^L, x_i^U]$ and $\tilde{y}_i = [y_i^L, y_i^U]$, with $0 \leq x_i^L \leq x_i^U \leq 1$ and $0 \leq y_i^L \leq y_i^U \leq 1$.

Definition 8. Let Ω be a set of interval numbers. An UOWAAC operator of dimension n is a mapping $\mathrm{UOWAAC}: \Omega^n \times \Omega^n \to R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{UOWAAC}(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}\big(\tilde{x}_j, \tilde{y}_j\big)\,, \tag{7}$$

where $\mathrm{UAC}\big(\tilde{x}_j, \tilde{y}_j\big)$ is the jth largest $\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)$ value of the UOWAAC pair $\langle \tilde{x}_i, \tilde{y}_i \rangle$, and $\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)$ is the adequacy coefficient between $\tilde{x}_i = [x_i^L, x_i^U]$ and $\tilde{y}_i = [y_i^L, y_i^U]$, with $0 \leq x_i^L \leq x_i^U \leq 1$ and $0 \leq y_i^L \leq y_i^U \leq 1$.

Note that for the previous definitions $\tilde{X}$ represents the ideal set in the comparison, thus, a higher UOWAAC value is preferred.

Next, a simple example will be carried out in order to correctly understand how to calculate the UOWAAC operator.

Example 1. Assume two sets of interval numbers $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = ([0.7,0.8], [0.6,0.7], [0.8,0.9], [0.5,0.7])$ and $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4) = ([0.4,0.6], [0.5,0.9], [0.7,0.9], [0.8,0.9])$. If the weighting vector is $W = (w_1, w_2, w_3, w_4) = (0.5,0.3,0.1,0.1)$, then, the UOWAAC operator is obtained as follows.

First, it is necessary to calculate the UAC for each pair of interval numbers using Eq. (5):

$$\text{UAC}(\tilde{x}_1, \tilde{y}_1) = \frac{1}{2}([1 \wedge (1 - 0.7 + 0.4)] + [1 \wedge (1 - 0.8 + 0.6)]) = 0.75.$$

Following the same procedure, the remaining outcomes are achieved:

$\text{UAC}(\tilde{x}_2, \tilde{y}_2) = 0.95$, $\text{UAC}(\tilde{x}_3, \tilde{y}_3) = 0.95$, and $\text{UAC}(\tilde{x}_4, \tilde{y}_4) = 1$.

Next, with Eq. (7) the aggregation is performed:

$$\begin{aligned}\text{UOWAAC}(\tilde{X}, \tilde{Y}) &= 0.5 \times 1 + 0.3 \times 0.95 + 0.1 \times 0.95 + 0.1 \times 0.75 \\ &= 0.955.\end{aligned}$$

From a generalized perspective of the reordering step, it is possible to discriminate between the descending UOWAAC (UDOWAAC) operator and the ascending UOWAAC (UAOWAAC) operator. Specifically, the weights of both operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the UDOWAAC (or UOWAAC) operator and w^*_{n-j+1} the jth weight of the UAOWAAC operator.

Properties of the UOWAAC operator

The UOWAAC operator is commutative, monotonic, bounded, idempotent, nonnegative, and reflexive. These properties can be proven with the following theorems:

Theorem 1. Commutativity. Assume f is the UOWAAC operator. Then:

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle) = f(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle), \qquad (8)$$

where $(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle)$ is any permutation of $(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle)$.

Proof. Let

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j,\tilde{y}_j),$$

$$f(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j^*,\tilde{y}_j^*).$$

As $(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle)$ is any permutation of $(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle)$, we have $\mathrm{UAC}(\tilde{x}_j,\tilde{y}_j) = \mathrm{UAC}(\tilde{x}_j^*,\tilde{y}_j^*)$, for all j, and as a result:

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle) = f(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle).$$

Theorem 2. Monotonicity. Assume f is the UOWAAC operator. If $\mathrm{UAC}(\tilde{x}_i,\tilde{y}_i) \geq \mathrm{UAC}(\tilde{x}_i^*,\tilde{y}_i^*)$, for all i, then:

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle) \geq f(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle). \qquad (9)$$

Proof. Let

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j,\tilde{y}_j),$$

$$f(\langle\tilde{x}_1^*,\tilde{y}_1^*\rangle,\dots,\langle\tilde{x}_n^*,\tilde{y}_n^*\rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j^*,\tilde{y}_j^*).$$

Since $\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i) \geq \mathrm{UAC}(\tilde{x}_i^*, \tilde{y}_i^*)$, for all i, it follows that $\mathrm{UAC}\left(\tilde{x}_j, \tilde{y}_j\right) \geq \mathrm{UAC}\left(\tilde{x}_j^*, \tilde{y}_j^*\right)$. Therefore:

$$f(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) \geq f(\langle \tilde{x}_1^*, \tilde{y}_1^* \rangle, \dots, \langle \tilde{x}_n^*, \tilde{y}_n^* \rangle).$$

Theorem 3. Boundedness. Assume f is the UOWAAC operator. Then:

$$\mathrm{Min}\{\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)\} \leq f(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) \leq \mathrm{Max}\{\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)\}. \quad (10)$$

Proof. Consider $\mathrm{Min}\{\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)\} = z$ and $\mathrm{Max}\{\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i)\} = g$. Subsequently:

$$f(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}\left(\tilde{x}_j, \tilde{y}_j\right) \geq \sum_{j=1}^{n} w_j \, z = z \sum_{j=1}^{n} w_j$$

$$f(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}\left(\tilde{x}_j, \tilde{y}_j\right) \leq \sum_{j=1}^{n} w_j \, g = g \sum_{j=1}^{n} w_j \, .$$

As $\sum_{j=1}^{n} w_j = 1$, we get:

$$z \leq f(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) \leq g.$$

Consequently, we can confirm that the UOWAAC operator is bounded.

Theorem 4. Idempotency. Assume f is the UOWAAC operator. If $\mathrm{UAC}(\tilde{x}_i, \tilde{y}_i) = \mathrm{UAC}(\tilde{x}, \tilde{y})$, for all i, then:

$$f(\langle \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle \tilde{x}_n, \tilde{y}_n \rangle) = \mathrm{UAC}(\tilde{x}, \tilde{y}). \quad (11)$$

Proof. Since $\frac{1}{2}([1 \wedge (1 - x_i^L + y_i^L)] + [1 \wedge (1 - x_i^U + y_i^U)]) = \frac{1}{2}([1 \wedge (1 - x^L + y^L)] + [1 \wedge (1 - x^U + y^U)])$, for all i, we have:

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle)=\sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j,\tilde{y}_j)$$

$$=\sum_{j=1}^{n} w_j \frac{1}{2}([1\wedge(1-x^L+y^L)]+[1\wedge(1-x^U+y^U)])$$

$$=\frac{1}{2}([1\wedge(1-x^L+y^L)]+[1\wedge(1-x^U+y^U)])\sum_{j=1}^{n} w_j\,.$$

Knowing that $\sum_{j=1}^{n} w_j=1$, we obtain:

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle)=\frac{1}{2}([1\wedge(1-x^L+y^L)]+[1\wedge(1-x^U+y^U)])$$
$$=\mathrm{UAC}(\tilde{x},\tilde{y}).$$

Theorem 5. Nonnegativity. Assume f is the UOWAAC operator. Then:

$$f(\langle\tilde{x}_1,\tilde{y}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{y}_n\rangle)\geq 0. \tag{12}$$

Proof. Since $0\leq x_i^L\leq x_i^U\leq 1$ and $0\leq y_i^L\leq y_i^U\leq 1$, the aggregated value will be always positive.

Theorem 6. Reflexivity. Assume f is the UOWAAC operator. Then:

$$f(\langle\tilde{x}_1,\tilde{x}_1\rangle,\dots,\langle\tilde{x}_n,\tilde{x}_n\rangle)=1. \tag{13}$$

Proof. Since $\tilde{x}_i=\tilde{x}_i$, for all i, we have:

$$\frac{1}{2}([1\wedge(1-x_i^L+x_i^L)]+[1\wedge(1-x_i^U+x_i^U)])=1.$$

Thus, we can say that the UOWAAC operator is reflexive.

Additionally, an interesting issue to consider is the measures for characterizing the weighting vector W of the UOWAAC operator. In the following, the measures of attitudinal character (Yager, 1988), entropy of dispersion (Yager, 1988), balance operator (Yager, 1996), and divergence of W (Yager, 2002) are shortly analyzed.

The first measure is the attitudinal character or degree of orness, and it can be defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{14}$$

The second measure is the entropy of dispersion and it shows the amount of information being used. It can be defined as follows:

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{15}$$

The balance operator is another interesting measure which evaluates the tendency to the minimum or to the maximum. It can be defined as follows:

$$\mathrm{Bal}(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1}\right). \tag{16}$$

Lastly, the divergence of W is a measure that quantifies the divergence of the weights against the attitudinal character measure. It can be defined as follows:

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \tag{17}$$

Families of the UOWAAC operator

One appealing aspect of the UOWAAC operator is that it includes a wide range of particular cases, which can be found by using different manifestations of the weighting vector W. Some interesting particular cases of this operator are the following:

- The minimum UAC (Min-UAC) is found when $w_n = 1$ and $w_j = 0$, for all $j \neq n$. It corresponds to the pessimistic criteria.
- The maximum UAC (Max-UAC) is found when $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$. It corresponds to the optimistic criteria.
- The median UAC (Med-UAC). If n is an odd number, then, the Med-UAC is obtained when $w_{(n+1)/2} = 1$ and $w_j = 0$, for all $j \neq (n+1)/2$. If n is an even number, then, the Med-UAC is obtained when $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_j = 0$, for all $j \neq n/2, (n/2)+1$.
- The normalized UAC (UNAC) is found when $w_j = 1/n$, for all j. It corresponds to the Laplace criteria.
- The Hurwicz-UOWAAC is found when $w_1 = \alpha$, $w_n = (1-\alpha)$, and $w_j = 0$, for all $j \neq 1, n$.
- The Olympic-UOWAAC is found when $w_1 = w_n = 0$ and $w_j = 1/(n-2)$, for all $j \neq 1, n$.
- The Step-UOWAAC is found when $w_k = 1$ and $w_j = 0$, for all $j \neq k$.

4. The UIOWAAC operator

The following section studies the UIOWAAC operator, which can be described as an extension of the UOWAAC operator that uses order-inducing variables in the reordering step of the argument variables. Thus, the reordering process does not depend on the values of the argument variables. This feature is very useful as it allows to represent more complex situations.

Let $\tilde{X} = (\tilde{x}_1, \dots, \tilde{x}_n)$ and $\tilde{Y} = (\tilde{y}_1, \dots, \tilde{y}_n)$ be two sets of interval numbers, then, the UIOWAAC operator can be defined as follows.

Definition 9. Let Ω be a set of interval numbers. An UIOWAAC operator of dimension n is a mapping UIOWAAC: $R^n \times \Omega^n \times \Omega^n \to R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{UIOWAAC}(\langle u_1, \tilde{x}_1, \tilde{y}_1 \rangle, \dots, \langle u_n, \tilde{x}_n, \tilde{y}_n \rangle) = \sum_{j=1}^{n} w_j \mathrm{UAC}(\tilde{x}_j, \tilde{y}_j), \quad (18)$$

where $\text{UAC}(\tilde{x}_j, \tilde{y}_j)$ is the $\text{UAC}(\tilde{x}_i, \tilde{y}_i)$ value of the UIOWAAC triplet $\langle u_i, \tilde{x}_i, \tilde{y}_i \rangle$ having the jth largest u_i value, u_i is the order-inducing variable, and $\text{UAC}(\tilde{x}_i, \tilde{y}_i)$ is the adequacy coefficient between $\tilde{x}_i = [x_i^L, x_i^U]$ and $\tilde{y}_i = [y_i^L, y_i^U]$, with $0 \leq x_i^L \leq x_i^U \leq 1$ and $0 \leq y_i^L \leq y_i^U \leq 1$.

Like the UOWAAC operator, the UIOWAAC operator is commutative, monotonic, bounded, idempotent, nonnegative, and reflexive. The theorems and proofs of the mathematical properties of this operator are omitted as they are quite similar to the ones of the UOWAAC operator and thereby repetitive.

In order to understand numerically the UIOWAAC operator, a simple example is presented below.

Example 2. Assume two sets of interval numbers $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3, \tilde{x}_4) = ([0.7,0.8], [0.6,0.7], [0.8,0.9], [0.5,0.7])$ and $\tilde{Y} = (\tilde{y}_1, \tilde{y}_2, \tilde{y}_3, \tilde{y}_4) = ([0.4,0.6], [0.5,0.9], [0.7,0.9], [0.8,0.9])$. If the weighting vector is $W = (w_1, w_2, w_3, w_4) = (0.5,0.3,0.1,0.1)$ and the inducing vector $U = (u_1, u_2, u_3, u_4) = (9,7,3,5)$, then, the UIOWAAC operator is obtained as follows.

As in Example 1, first of all the UAC for each pair of interval numbers needs to be calculated. By doing this, the following values are obtained:

$\text{UAC}(\tilde{x}_1, \tilde{y}_1) = 0.75$, $\text{UAC}(\tilde{x}_2, \tilde{y}_2) = 0.95$, $\text{UAC}(\tilde{x}_3, \tilde{y}_3) = 0.95$, and $\text{UAC}(\tilde{x}_4, \tilde{y}_4) = 1$.

Afterwards, with Eq. (18) the aggregation is carried out:

$$\begin{aligned} \text{UIOWAAC}(\tilde{X}, \tilde{Y}) &= 0.5 \times 0.75 + 0.3 \times 0.95 + 0.1 \times 1 + 0.1 \times 0.95 \\ &= 0.855. \end{aligned}$$

5. Applications of the UOWAAC operator

The following section presents an illustrative example of the new approach in a MCGDM problem regarding the international expansion of a company.

Nevertheless, other applications could be developed, for example in sport management, human resources, asset management, and many others.

Suppose that a company based in Germany and devoted to the production of car components wants to expand internationally in order to increase revenue potential. Therefore, three experts are requested by the company for choosing the most appropriate country to expand in among five options, which are:

- A_1: France.
- A_2: Italy.
- A_3: Portugal.
- A_4: Romania.
- A_5: Spain.

Also, the company considers five different characteristics as key for the assessments, which are:

- C_1: Customer base.
- C_2: Regulatory environment.
- C_3: Economic performance.
- C_4: Skilled labor force.
- C_5: Competitive landscape.

First, the interval numbers of the ideal set I are defined by the company as it is shown in Table 3.10. Then, each expert evaluates the characteristics of the candidate countries one by one and based on a scale from 0 to 1, where 0 is the worst result and 1 the best. The individual evaluations can be seen in Tables 3.11, 3.12, and 3.13.

In order to obtain a unified payoff matrix, the company aggregates the assessments conducted by the three experts. As the company assumes that the evaluations of each expert are not equally important, the uncertain weighted average (UWA) with $W = (0.4,0.4,0.2)$ is used to build this matrix. The results are presented in Table 3.14.

Next, in order to rank the countries according to the collective assessments, the company decides to use the UNAC, UWAC, UOWAAC, UAOWAAC, and UIOWAAC aggregation operators. To do so, the company has agreed to use the weighting vector $W = (0.4,0.2,0.2,0.1,0.1)$ and order-inducing vector $U = (10,5,6,7,8)$. The aggregated results are shown in Table 3.15. It should be taken into account that the preferred alternative will be the one with the highest value.

Table 3.10. Ideal country

	C_1	C_2	C_3	C_4	C_5
I	[0.9,1]	[0.7,0.8]	[0.7,0.8]	[0.8,0.9]	[0.8,0.9]

Table 3.11. Assessments of Expert 1

	C_1	C_2	C_3	C_4	C_5
A_1	[0.7,0.8]	[0.8,0.9]	[0.8,0.9]	[0.9,1]	[0.35,0.45]
A_2	[0.6,0.7]	[0.45,0.55]	[0.7,0.8]	[0.85,0.95]	[0.5,0.6]
A_3	[0.35,0.45]	[0.6,0.7]	[0.45,0.5]	[0.7,0.8]	[0.9,1]
A_4	[0.5,0.6]	[0.5,0.6]	[0.5,0.6]	[0.6,0.7]	[0.8,0.9]
A_5	[0.8,0.9]	[0.85,1]	[0.6,0.7]	[0.8,0.9]	[0.25,0.35]

Table 3.12. Assessments of Expert 2

	C_1	C_2	C_3	C_4	C_5
A_1	[0.75,0.85]	[0.7,0.9]	[0.8,0.95]	[0.8,0.95]	[0.35,0.45]
A_2	[0.75,0.85]	[0.5,0.65]	[0.7,0.8]	[0.8,0.95]	[0.4,0.55]
A_3	[0.4,0.5]	[0.6,0.75]	[0.6,0.7]	[0.7,0.85]	[0.85,1]
A_4	[0.6,0.65]	[0.5,0.65]	[0.5,0.65]	[0.6,0.75]	[0.8,0.95]
A_5	[0.75,0.9]	[0.7,0.9]	[0.65,0.75]	[0.8,0.95]	[0.3,0.45]

Table 3.13. Assessments of Expert 3

	C_1	C_2	C_3	C_4	C_5
A_1	[0.75,0.9]	[0.7,0.8]	[0.85,0.9]	[0.8,1]	[0.25,0.4]
A_2	[0.65,0.8]	[0.65,0.7]	[0.7,0.8]	[0.7,0.85]	[0.3,0.45]
A_3	[0.3,0.4]	[0.65,0.75]	[0.6,0.7]	[0.75,0.9]	[0.75,0.95]
A_4	[0.45,0.65]	[0.6,0.7]	[0.6,0.7]	[0.7,0.8]	[0.7,0.85]
A_5	[0.8,1]	[0.7,0.85]	[0.7,0.75]	[0.7,0.85]	[0.2,0.4]

Table 3.14. Collective results

	C_1	C_2	C_3	C_4	C_5
A_1	[0.73,0.84]	[0.74,0.88]	[0.81,0.92]	[0.84,0.98]	[0.33,0.44]
A_2	[0.67,0.78]	[0.51,0.62]	[0.7,0.8]	[0.8,0.93]	[0.42,0.55]
A_3	[0.36,0.46]	[0.61,0.73]	[0.54,0.62]	[0.71,0.84]	[0.85,0.99]
A_4	[0.53,0.63]	[0.52,0.64]	[0.52,0.64]	[0.62,0.74]	[0.78,0.91]
A_5	[0.78,0.92]	[0.76,0.93]	[0.64,0.73]	[0.78,0.91]	[0.26,0.4]

Table 3.15. Aggregated results

	UNAC	UWAC	UOWAAC	UAOWAAC	UIOWAAC
A_1	0.874	0.888	0.937	0.781	0.841
A_2	0.845	0.837	0.904	0.772	0.819
A_3	0.827	0.727	0.898	0.727	0.744
A_4	0.822	0.766	0.874	0.766	0.782
A_5	0.861	0.894	0.923	0.758	0.848

Table 3.16. Ordering of the countries

	Ordering
UNAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
UWAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
UOWAAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
UAOWAAC	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$
UIOWAAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$

As we can see in Table 3.16, depending on the aggregation operator used, the order of preference may be different. For example, with the UNAC operator, the UOWAAC operator, and the UAOWAAC operator, the best country to expand in is France. However, with the UWAC operator and the UIOWAAC operator, the best country to expand in is Spain. This allows the decision-maker to get a more complete view of the problem and thereby make better decisions.

Uncertainty is a major factor that affects the decision-making of companies on international expansion (Sniazhko, 2019). In particular, this uncertainty arises when the information considered in the analysis is incomplete, imprecise, or vague. In this context, the aggregation operators adopted in this practical example provide an effective solution for dealing with uncertainty associated to business internationalization.

6. Comparative analysis

The purpose of this section is to perform a comparative study of the presented aggregation operators with existing aggregation operators. Other uncertainties are also contemplated.

Existing aggregation operators

In concrete situations the UOWAAC operator and its families are more appropriate than others based on the Hamming distance. To prove this point, a comparative analysis between these two approaches is studied. To do so, the uncertain normalized distance (UND), the uncertain weighted distance (UWD), the uncertain ordered weighted averaging distance (UOWAD) (Merigó et al., 2009), the ascending UOWAD (UAOWAD), and the induced UOWAD (UIOWAD) (Zeng et al., 2013) operators are calculated, considering the same information used in the international expansion MCGDM problem. The comparison results are presented in Table 3.17. Note that the outcomes of the UNAC, UWAC, UOWAAC, UAOWAAC, and UIOWAAC operators are taken from the example conducted in Section 5.

Table 3.17. Comparison with established aggregation operators

	Ordering		Ordering
UNAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$	UND	$A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$
UWAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$	UWD	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$
UOWAAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$	UOWAD	$A_2 \succ A_4 \succ A_1 \succ A_5 \succ A_3$
UAOWAAC	$A_1 \succ A_2 \succ A_4 \succ A_5 \succ A_3$	UAOWAD	$A_2 \succ A_5 \succ A_1 \succ A_4 \succ A_3$
UIOWAAC	$A_5 \succ A_1 \succ A_2 \succ A_4 \succ A_3$	UIOWAD	$A_5 \succ A_2 \succ A_1 \succ A_4 \succ A_3$

We can see that the order of preference given by the UOWAAC operator differs from the UOWAD one. The same happens with the other compared aggregation operators, except for the UWAC against the UWD. Obviously, if a candidate country presents better results than the ideal country, it makes no sense to penalize as it happens with the aggregation operators that are built under the Hamming distance. Thus, by using the UOWAAC operators as well as its families and extensions, the decision-maker only penalizes when the level of the ideal country is not attained, but he/she neither penalizes nor rewards when the level of the ideal country is exceeded.

Basic uncertain information

So far, the illustrative example assumed that the assessments are represented directly by interval numbers. However, the experts could also use basic uncertain information (BUI) (Jin et al., 2018; Mesiar et al., 2018) for the evaluations and afterwards transform it into interval numbers.

BUI is a quite recent concept that was introduced to generalize a wide variety of uncertainties. Specifically, it allows to consider the level of certainty that the decision-maker has on the input data. A BUI is a real pair $\tilde{x} = < x; c >$, where $x (x \in [0,1])$ is the input value and $c (c \in [0,1])$ the certainty degree of x. A BUI can be transformed into a closed interval $[a, b]$, where $a = cx$ and $b = cx + 1 - c$.

In order to enrich this paper, the same example as in Section 5 will be conducted but, in this case, considering BUI assessments. By using BUI, the experts are able to exhibit the amount of confidence that they have in their own assessments. The BUI pair matrix of each expert can be seen in Tables 3.18, 3.19, and 3.20. Likewise, the aggregated results and the ranking of the alternatives can be seen in Tables 3.21 and 3.22. Take into account that the weighting vectors and inducing vector used are the same as in Section 5.

Table 3.18. BUI pairs of Expert 1

	C_1	C_2	C_3	C_4	C_5
A_1	$< 0.75; 0.8 >$	$< 0.85; 0.8 >$	$< 0.85; 0.8 >$	$< 0.95; 0.8 >$	$< 0.4; 0.8 >$
A_2	$< 0.65; 0.8 >$	$< 0.5; 0.8 >$	$< 0.75; 0.8 >$	$< 0.9; 0.8 >$	$< 0.55; 0.8 >$
A_3	$< 0.4; 0.9 >$	$< 0.65; 0.9 >$	$< 0.48; 0.9 >$	$< 0.75; 0.9 >$	$< 0.95; 0.9 >$
A_4	$< 0.55; 0.7 >$	$< 0.55; 0.7 >$	$< 0.55; 0.7 >$	$< 0.65; 0.7 >$	$< 0.85; 0.7 >$
A_5	$< 0.85; 0.9 >$	$< 0.93; 0.9 >$	$< 0.65; 0.9 >$	$< 0.85; 0.9 >$	$< 0.3; 0.9 >$

Table 3.19. BUI pairs of Expert 2

	C_1	C_2	C_3	C_4	C_5
A_1	$< 0.8; 0.7 >$	$< 0.8; 0.95 >$	$< 0.88; 0.75 >$	$< 0.88; 0.8 >$	$< 0.4; 0.7 >$
A_2	$< 0.8; 0.7 >$	$< 0.58; 0.95 >$	$< 0.75; 0.75 >$	$< 0.88; 0.8 >$	$< 0.48; 0.7 >$
A_3	$< 0.45; 0.7 >$	$< 0.68; 0.95 >$	$< 0.65; 0.75 >$	$< 0.78; 0.8 >$	$< 0.93; 0.7 >$
A_4	$< 0.63; 0.7 >$	$< 0.58; 0.95 >$	$< 0.58; 0.75 >$	$< 0.68; 0.8 >$	$< 0.88; 0.7 >$
A_5	$< 0.83; 0.7 >$	$< 0.8; 0.95 >$	$< 0.7; 0.75 >$	$< 0.88; 0.8 >$	$< 0.38; 0.7 >$

Table 3.20. BUI pairs of Expert 3

	C_1	C_2	C_3	C_4	C_5
A_1	$< 0.83; 0.95 >$	$< 0.75; 0.65 >$	$< 0.88; 0.95 >$	$< 0.9; 0.95 >$	$< 0.33; 0.95 >$
A_2	$< 0.73; 0.85 >$	$< 0.68; 0.6 >$	$< 0.75; 0.85 >$	$< 0.78; 0.85 >$	$< 0.38; 0.85 >$
A_3	$< 0.35; 0.85 >$	$< 0.7; 0.6 >$	$< 0.65; 0.85 >$	$< 0.83; 0.85 >$	$< 0.85; 0.85 >$
A_4	$< 0.55; 0.85 >$	$< 0.65; 0.6 >$	$< 0.65; 0.85 >$	$< 0.75; 0.85 >$	$< 0.78; 0.85 >$
A_5	$< 0.9; 0.85 >$	$< 0.78; 0.6 >$	$< 0.73; 0.85 >$	$< 0.78; 0.85 >$	$< 0.3; 0.85 >$

Table 3.21. Aggregated results

	BUI-UNAC	BUI-UWAC	BUI-UOWAAC	BUI-UAOWAAC	BUI-UIOWAAC
A_1	0.859	0.861	0.926	0.772	0.816
A_2	0.810	0.796	0.863	0.748	0.779
A_3	0.807	0.717	0.873	0.715	0.732
A_4	0.784	0.737	0.824	0.734	0.743
A_5	0.838	0.860	0.903	0.745	0.815

Table 3.22. Ordering of the countries

	Ordering
BUI-UNAC	$A_1 \succ A_5 \succ A_2 \succ A_3 \succ A_4$
BUI-UWAC	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$
BUI-UOWAAC	$A_1 \succ A_5 \succ A_3 \succ A_2 \succ A_4$
BUI-UAOWAAC	$A_1 \succ A_2 \succ A_5 \succ A_4 \succ A_3$
BUI-UIOWAAC	$A_1 \succ A_5 \succ A_2 \succ A_4 \succ A_3$

Within each operator, the optimal choice is the alternative with the highest aggregated result. Stated another way, the best option is the one with the closest aggregated result to 1. In all cases it is A_1, i.e., France. Conversely,

A_3 and A_4, i.e., Portugal and Romania, are the less preferred options for the company.

7. Conclusions

The UOWAAC operator is a new aggregation operator that uses the adequacy coefficient, the interval numbers, and the OWA operator in the same formulation. As a result, this comprehensive operator presents several advantages. First, it provides a parametrized family of aggregation operators that includes among others the Min-UAC, the Max-UAC, the Med-UAC, the UNAC, the Hurwicz-UOWAAC, the Olympic-UOWAAC, and the Step-UOWAAC. Second, it can aggregate the information according to the attitudinal character of the decision-maker. Third, it is practical for dealing with uncertainty, especially when the information cannot be represented with exact numbers, but it is possible to use interval numbers. Fourth, it can be used to compare an ideal with the available alternatives while establishing a threshold from which the results are always the same.

Moreover, the UIOWAAC operator is an extension of the UOWAAC operator. The core difference between these two operators is that the UIOWAAC operator uses order-inducing variables, thus allowing the decision-maker to deal with more complex reordering processes of the available information.

An illustrative example of the new approach has been presented in a MCGDM problem regarding the selection of the most suitable country for international business expansion. The adopted aggregation operators permit to consider the judgements of the experts in the form of interval numbers. Moreover, they overcome the limitations of some traditional comparison methods. The numerical results show that depending on the type of aggregation operator used the preference order of the candidate countries may change.

For future research, the proposed operators can be applied to other interesting fields, such as personnel selection, sport management, selection of financial products, and risk management. Also, we suggest investigating new extensions of the UOWAAC operator and the UIOWAAC operator, for example, by including generalized means, quasi arithmetic means, or probabilities.

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3.3. OWA operators in pensions

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Abstract

The public pension system crisis, arising mainly from the changing demographic, has hit different countries worldwide. For governments and citizens, it is very important to have reliable information regarding pensions in order to make decisions with a maximum degree of effectiveness and to ensure a decent income in retirement. This study presents a new method for optimizing forecasts of the average pension by using the ordered weighted averaging (OWA) operator, the induced ordered weighted averaging (IOWA) operator, the generalized ordered weighted averaging (GOWA) operator, the induced generalized ordered weighted averaging (IGOWA) operator, and particular forms of the probabilistic ordered weighted averaging (POWA) operator and the quasi-arithmetic ordered weighted averaging (Quasi-OWA) operator. It also accounts for inflation or deflation, providing a more realistic assessment of the average pension. The main advantage of this approach is the possibility to include the attitudinal character of experts or decision-makers into the calculation. The study also presents an illustrative example of how to forecast the real average pension for all autonomous communities of Spain by using this new approach.

Keywords: Decision-making, forecasting, aggregation operator, OWA operator, average pension, COVID-19.

1. Introduction

As stated in Chapter 3 of the European Pillar of Social Rights (European Commission [EC], 2017), “everyone in old age has the right to resources that ensure living in dignity” (p. 58). But can governments ensure it? The continuous increase in life expectancy and the low rates of fertility implies a declining ratio of workers to pensioners. Consequently, countries find it very difficult to guarantee the long-term financial sustainability of their pension systems. Japan is the most affected country by these demographic changes: in 2019 there were 47 older dependents per 100 in the working-age population (World Bank, 2019). Despite reforms in some countries, for example Spain is increasing gradually the statutory retirement age from 65 to 67 by the year 2027 (España, Cortes Generales, 2011), they are not enough to solve the problem. Also, many of these reforms lead to a reduction in the level of individual public pension income. Additional factors that have a significant influence on the sustainability of public pension systems are those related to the economic growth and employment, among others (Organization for Economic Cooperation and Development [OECD], 2019, 2020).

Especially in the current situation, it is important for governments to have reliable and accurate pension forecasts in realistic terms in order to conduct the best pension policy decision-making. Also, it is important for citizens, so that they can plan properly their retirement and therefore reduce the risk of poverty when they retire.

The aim of this paper is to optimize average pensions forecasts. To accomplish this, a new method called the ordered weighted averaging real average pension (OWARAP) is presented. This new method is built under the ordered weighted averaging (OWA) operator (Yager, 1988) and it includes the consumer price index (CPI) to adjust average pensions forecasts for inflation. The OWA operator introduced by Yager is an increasing popular aggregation method (Blanco-Mesa et al., 2019; Emrouznejad & Marra, 2014; He et al., 2017) that aggregates the information underestimating or overestimating it according to the attitudinal character of the decision-maker. The advantage of using the OWA operator is the possibility to add in a more flexible way the opinion that the decision-maker has about future scenarios. Thus, this operator is extremely helpful when dealing with uncertain environments.

This paper also considers other extensions of the OWA operator to optimize pensions forecasts. These extensions are the induced ordered weighted averaging (IOWA) operator (Yager & Filev, 1999), the generalized ordered weighted averaging (GOWA) operator (Yager, 2004), the induced generalized ordered weighted averaging (IGOWA) operator (Merigó & Gil-Lafuente, 2009), and families of the probabilistic ordered weighted averaging (POWA) operator (Merigó, 2012). The main characteristic of the IOWA operator is that the reordering of the arguments is carried out by another variable that Yager and Filev called order-inducing variable. The main characteristic of the GOWA operator is the addition of a parameter controlling the power to which the argument values are raised. The IGOWA operator was introduced by Merigó and Gil-Lafuente and combines the main characteristics of the IOWA and the GOWA operator, so it uses generalized means and order-inducing variables. The main feature of the POWA operator is the possibility of unifying the attitudinal character and the probabilistic information under the same formulation.

Throughout the literature we can find a large variety of forecasting models that incorporate the family of OWA operators. Among them: Merigó et al. (2015) apply the OWA and the unified aggregation operator (UAO) for sales forecasting. Cheng et al. (2013) propose a forecasting model that incorporates OWA and adaptive network-based fuzzy inference systems (ANFIS) and which is utilized for predicting stock prices. Huang and Cheng (2008) developed an OWA based time series model to predict air quality. Flores-Sosa et al. (2020) estimates exchange rates by unifying the IOWA operator with linear regression in a single formulation.

The remaining of this paper is organized as follows. Section 2 briefly reviews basic preliminaries regarding aggregation operators. Section 3 introduces the OWARAP operator. Section 4 studies some new extensions of the OWARAP operator. Section 5 proposes an algorithm to calculate the real average pension, develops an illustrative example, and compares the new approach with traditional methods. Section 6 discusses the applicability of aggregation operators to address the COVID-19 impacts on pensions. Section 7 summarizes the finding and conclusions of the study.

2. Preliminaries

This section briefly defines the concept of aggregation and aggregation operator and reviews the OWA operator, the IOWA operator, the GOWA operator, and the IGOWA operator.

Aggregation is the process of combining several numerical values into a single representative value, and an aggregation operator (also called aggregation function) performs this operation (Grabisch et al., 2009). Aggregation operators have been applied to a wide range of fields such as social choice and voting (Kacprzyk & Zadrożny, 2009), portfolio selection (Laengle et al., 2015, 2017), inflation calculations (Espinoza-Audelo et al., 2020; León-Castro et al., 2020), retirement planning (Figuerola-Wischke et al., 2021), and many others (Kacprzyk et al., 2019). The most common aggregation operators are the arithmetic average (simple average), the weighted average, and the OWA.

The OWA operator from Yager (1988) provides a parameterized class of mean type aggregation operators that lies between the minimum and the maximum. The OWA operator can be defined as follows.

Definition 1. An OWA operator of dimension n is a function $F: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{OWA}(a_1, \dots, a_n) = \sum_{j=1}^{n} w_j b_j \,, \tag{1}$$

where b_j is the jth largest element of the arguments $a_1, \dots, a_n$, that is $(b_1, \dots, b_n)$ is $(a_1, \dots, a_n)$ reordered from largest to smallest.

An interesting characteristic of this type of operator is that it includes the classical methods for decision-making into a single formulation. This can be achieved through choosing different manifestations of the weighting vector W. The optimistic criterion or maximax criterion selects the most favorable result of each alternative, that is $w_1 = 1$ and $w_j = 0$ for $\forall\ j \neq 1$. The pessimistic criterion or maximin criterion selects the most unfavorable result of each alternative, that is $w_j = 0$ and $w_n = 1$ for $\forall\ j \neq n$. The Laplace criterion is obtained when $w_j = 1/n$ for $\forall\ j$, so it is assumed that all alternatives have equal probability to occur. The Hurwicz criterion is found when $w_1 = \alpha$, $w_n = 1 - \alpha$, and $w_j = 0$ for $\forall\ j \neq 1, n$, so it takes into account both the best and the worst alternative.

Note that if the reordering process is carried out from smallest to largest, we get the ascending ordered weighted averaging (AOWA) operator (Yager, 1992).

One appealing extension of the OWA operator is the IOWA operator developed by Yager and Filev (1999). In this operator, the step of reordering is carried out using order-inducing variables. This allows to consider other factors in the reordering process and not only to the degree of optimism. It can be defined as follows.

Definition 2. An IOWA operator of dimension n is a function $F: R^n \times R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{IOWA}(\langle u_1, a_1 \rangle, \ldots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j\,, \qquad (2)$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and a_i as the argument variable.

Another interesting extension is the GOWA operator. The GOWA operator was introduced by Yager (2004) and it combines the OWA operator with generalized means (Dyckhoff & Pedrycz, 1984). This operator is defined as follows.

Definition 3. A GOWA operator of dimension n is a function $F: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{GOWA}(a_1, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{3}$$

where λ is a parameter such that $\lambda \in (-\infty, +\infty)$ and b_j is the jth largest of the argument variable a_i.

Note that if $\lambda = -1$ we obtain the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004), if $\lambda = 0$ the ordered weighted geometric (OWG) operator (Chiclana et al., 2000, 2002), if $\lambda = 1$ the OWA operator, and if $\lambda = 2$ the ordered weighted quadratic averaging (OWQA) operator (Yager, 2004).

Merigó and Gil-Lafuente (2009) introduced the IGOWA operator. This operator uses the main characteristics of the IOWA operator and the GOWA operator. The IGOWA operator is defined as follows.

Definition 4. An IGOWA operator of dimension n is a function $F: R^n \times R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{IGOWA}(\langle u_1, a_1 \rangle, \dots, \langle u_n, a_n \rangle) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{4}$$

where b_j is the a_i value of the IGOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable, a_i as the argument variable, and λ is a parameter such that $\lambda \in (-\infty, +\infty)$.

All the above-mentioned operators satisfy the conditions of monotonicity, commutativity, boundedness, and idempotency. For a detailed proof consult (Merigó & Gil-Lafuente, 2009; Yager, 1988, 2004; Yager & Filev, 1999).

3. The ordered weighted averaging real average pension

In this section the OWARAP operator, the induced ordered weighted averaging real average pension (IOWARAP) operator, and the generalizations of these two are presented.

The OWARAP operator

The OWARAP operator is a new aggregation function that measures the future average pension adjusted for price changes and which is built under the family of the OWA operator. This operator provides a parametrized family of aggregation operators between the minimum and the maximum real average pension.

An important feature of the OWARAP operator is its possibility of unifying different opinions of a set of experts or decision-makers into a collective result without losing any information. Another interesting advantage is the capability to consider a wide range of situations and alternatives, thus providing a better understanding of the problem.

Most countries publish information regarding pensions in current prices. However, people can obtain a true picture of the average pension by using the OWARAP operator. In this sense, citizens, governments, and companies can know if there will be a purchasing power reduction in the retirement income, thus improving the quality and effectiveness of decision-making. The OWARAP operator can be defined as follows.

Definition 5. An OWARAP operator of dimension n is a function $F: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{OWARAP}(p_1, \ldots, p_n) = \left(\frac{100}{CPI}\right) \sum_{j=1}^{n} w_j P_j \,, \tag{5}$$

where P_j is the jth largest element of a set of nominal average pensions $p_1, \dots, p_n$ (referred as arguments) and CPI is the consumer price index.

Example 1. Assume the following collection of nominal pensions ($p_1 = 920, p_2 = 890, p_3 = 950, p_4 = 980$) and weighting vector $W = (0.40, 0.25, 0.25, 0.10)$. If we assume that $CPI = 105$, then the aggregation process is solved as follows:

$$\left(\frac{100}{105}\right) \times (0.40 \times 980 + 0.25 \times 950 + 0.25 \times 920 + 0.10 \times 890) = 903.33.$$

The OWARAP operator is a mean operator that satisfies the properties of monotonicity, commutativity, and boundedness. These properties are expressed in the following theorems:

Theorem 1. Monotonicity. Let F be the OWARAP operator. If $p_i \geq \hat{p}_i$ for all i, then, $F(p_1, \dots, p_n) \geq F(\hat{p}_1, \dots, \hat{p}_n)$.

Theorem 2. Commutativity (symmetry). In the sense that the initial indexing of the arguments does not meter. So, if F is the OWARAP operator, then, $F(p_1, \dots, p_n) = F(\hat{p}_1, \dots, \hat{p}_n)$, where $(\hat{p}_1, \dots, \hat{p}_n)$ is any permutation of $(p_1, \dots, p_n)$.

Theorem 3. Boundedness. Since the aggregation is delimited. Let F be the OWARAP operator. Then, $\left(\frac{100}{CPI}\right) Min\{p_i\} \leq F(p_1, \dots, p_n) \leq \left(\frac{100}{CPI}\right) Max\{p_i\}$.

Another noteworthy aspect is the measures for characterizing the weighting vector and the type of aggregation it performs. This work focuses on four characterizing features introduced by Yager: the *alpha* value of W (degree of or-ness measure) (Yager, 1988), the dispersion measure (Yager, 1988), the balance operator (Yager, 1996), and the divergence of W (Yager, 2002).

The first measure (degree of or-ness) refers to the attitudinal character associated with a weighting vector and it is denoted as $\alpha(W)$ or also as $\mathrm{AC}(W)$ (Yager & Alajlan, 2014). It can be defined as follows:

$$\alpha(W)=\sum_{j=1}^{n} w_j\left(\frac{n-j}{n-1}\right). \tag{6}$$

As we can see, $\alpha(W) \in [0,1]$. The closer $\alpha(W)$ is to 1, the higher the level of preference for larger values in the aggregation.

The second measure is the measure of dispersion or entropy and it is denoted as $\mathrm{H}(W)$ or also as $\mathrm{Disp}(W)$ (Yager & Alajlan, 2014). Its definition is as follows:

$$\mathrm{H}(W)=-\sum_{j=1}^{n} w_j \ln\left(w_j\right). \tag{7}$$

It can be shown that $\mathrm{H}(W)$ has a value between 0 and the natural logarithm of n. That is $\mathrm{H}(W) \in [0, \ln(n)]$.

The third, is the balance operator $\mathrm{Bal}(W)$, which measures the degree of favoritism towards higher values (optimistic values) or lower values (pessimistic values). Its formula is as follows:

$$\mathrm{Bal}(W)=\sum_{j=1}^{n} w_j\left(\frac{n+1-2j}{n-1}\right). \tag{8}$$

The balance operator can range from –1 to 1, that is $\mathrm{Bal}(W) \in [-1,1]$. For values of $\mathrm{Bal}(W)$ close to –1 the aggregation emphasizes the lower values. For values of $\mathrm{Bal}(W)$ close to 1 the aggregation emphasizes the higher values.

The fourth, is the measure of divergence $\mathrm{Div}(W)$. It measures the divergence of the weights against the degree of or-ness measure. It can be defined by using the following expression:

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2 . \tag{9}$$

In Table 3.23, we can see when the special cases of the pessimistic, Laplace, and optimistic criterion are met according to the measure outcome.

Table 3.23. Particular cases of measures for characterizing a weighting vector

Measure	Criterion		
	Pessimistic	Laplace	Optimistic
α(W)	0	0.5	1
H(W)	0	$\ln(n)$	0
Bal(W)	–1	0	1
Div(W)	0	$\frac{n+1}{12(n-1)}$	0

The IOWARAP operator

A new extension of the OWARAP operator is developed by using order-inducing variables, called the IOWARAP operator. It can be defined as follows.

Definition 6. An IOWARAP operator of dimension n is a function $F: R^n \times R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{IOWARAP}(\langle u_1, p_1\rangle, \dots, \langle u_n, p_n\rangle) = \left(\frac{100}{CPI}\right) \sum_{j=1}^{n} w_j P_j , \tag{10}$$

where P_j is the p_i value of the IOWARAP pair $\langle u_i, p_i\rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and p_i as the nominal average pension variable. CPI is the consumer price index.

Hence, the main difference between the OWARAP operator and the IOWARAP operator resides in the process of reordering the set of values $p = (p_1, \dots, p_n)$. In the case of the OWARAP, the reordering is made based on the magnitude of the values to be aggregated. By contrast, the reordering step of the IOWARAP depends upon the values of their associated order-inducing variables.

Example 2. Assume we have the following four IOWARAP pairs $\langle u_i, p_i \rangle$: $\langle 7{,}920 \rangle$, $\langle 3{,}890 \rangle$, $\langle 9{,}950 \rangle$, $\langle 5{,}980 \rangle$. Assume that the weighting vector is $W = (0.40{,}0.25{,}0.25{,}0.10)$ and $CPI = 105$. We get the following calculation result:

$$\left(\frac{100}{105}\right) \times (0.40 \times 950 + 0.25 \times 920 + 0.25 \times 980 + 0.10 \times 890)$$
$$= 899.05.$$

Generalized aggregation operators

By using generalized means in the OWARAP operator we obtain the generalized ordered weighted averaging real average pension (GOWARAP) operator. The analyst can obtain a wide range of particular cases of the GOWARAP operator by using different values of the parameter *lambda*. Its definition would be the following.

Definition 7. A GOWARAP operator of dimension n is a function $F: R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{GOWARAP}(p_1, \dots, p_n) = \left(\frac{100}{CPI}\right)\left(\sum_{j=1}^{n} w_j P_j^{\lambda}\right)^{1/\lambda}, \qquad (11)$$

where λ is a parameter such that $\lambda \in (-\infty, +\infty)$, P_j is the jth largest of the nominal average pension variable p_i, and CPI is the consumer price index.

By analyzing the parameter λ, we can see that when $\lambda = -1$ the ordered weighted harmonic averaging real average pension (OWHARAP) operator is obtained. If $\lambda = 0$ we form the ordered weighted geometric real average pension (OWGRAP) operator. With $\lambda = 1$ we obtain the OWARAP operator. When $\lambda = 2$ we form the ordered weighted quadratic averaging real average pension (OWQARAP) operator.

Note that when $\lambda = -\infty$ and $w_n \neq 0$ we get the smallest argument adjusted for inflation as the aggregated value, that is $(100/CPI)P_n$. However, if $\lambda = -\infty$ and $w_1 = 1$ (that is $w_j = 0$ for all $j \neq 1$) we get the largest argument of the collection p_i adjusted for inflation, which is $(100/CPI)P_1$. In the case where $\lambda = +\infty$ and $w_1 \neq 0$ we get the largest argument adjusted for inflation as the value of the resulting aggregation, that is $(100/CPI)P_1$. Otherwise, when $\lambda = +\infty$ and $w_n = 1$ (that is $w_j = 0$ for $\forall\ j \neq n$), we obtain the smallest argument adjusted for inflation, which is $(100/CPI)P_n$.

Other families of the GOWARAP operator could be developed by choosing different values of the parameter λ and weighting vector W.

Example 3. Consider the same collection of arguments, weighting vector, and CPI as in Example 1. If we assume $\lambda = -1$, then, the aggregation result is:

$$\left(\frac{100}{105}\right) \times (0.40 \times 980^{-1} + 0.25 \times 950^{-1} + 0.25 \times 920^{-1} + 0.10 \times 890^{-1})^{1/-1} = 902.37.$$

Using generalized means and order-inducing variables we obtain the induced generalized ordered weighted averaging real average pension (IGOWARAP) operator. Its definition is as follows.

Definition 8. An IGOWARAP operator of dimension n is a function $F: R^n \times R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{IGOWARAP}(\langle u_1, p_1 \rangle, \dots, \langle u_n, p_n \rangle) = \left(\frac{100}{CPI}\right) \left(\sum_{j=1}^{n} w_j P_j^{\lambda}\right)^{1/\lambda}, \qquad (12)$$

where P_j is the p_i value of the IGOWARAP pair $\langle u_i, p_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable, p_i as the nominal average pension variable, λ is a parameter such that $\lambda \in (-\infty, +\infty)$, and CPI is the consumer price index.

Example 4. If we take the IOWARAP pairs from Example 2 and a *lambda* equal to -1, then, we get the following aggregation result:

$$\left(\frac{100}{105}\right) \times (0.40 \times 950^{-1} + 0.25 \times 920^{-1} + 0.25 \times 980^{-1} + 0.10 \times 890^{-1})^{1/-1} = 898.26.$$

4. Some other extensions of the OWARAP operator

In this section the use of probability in the OWARAP operator is analyzed. To do so, the POWA operator and the probabilistic induced ordered weighted averaging (PIOWA) operator are used. Also, the particular generalization of the quasi-arithmetic ordered weighted averaging (Quasi-OWA) operator is studied.

The probabilistic aggregation (PA) operator is an aggregation function where the aggregation process is done according to the probability associated to each argument. A PA operator is defined as follows.

Definition 9. A PA operator of dimension n is a function $F: R^n \to R$ such that:

$$\text{PA}(a_1, \dots, a_n) = \sum_{i=1}^{n} v_i a_i, \qquad (13)$$

where a_i is the ith argument variable and each argument a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$.

The POWA operator introduced by Merigó (2012), is an aggregation function that unifies the probability and the OWA operator (attitudinal character) in the same formulation and according to the degree of importance of these two concepts in the aggregation process. Therefore, it provides a unified framework between decision-making problems under uncertainty and under risk. The POWA operator can be defined as follows.

Definition 10. A POWA operator of dimension n is a function $F: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{POWA}(a_1, \dots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j = \beta \sum\nolimits_{j=1}^{n} w_j b_j + (1-\beta) \sum\nolimits_{i=1}^{n} v_i a_i \,, \quad (14)$$

where b_j is the jth largest of the a_i, each argument a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to b_j, that is, according to the jth largest of a_i.

Note that if the parameter β is equal to 1, we obtain the normal OWA operator, and if β is equal to 0, we get the PA operator. Then, by taking into consideration equation 1 and 13, the POWA operator can be formulated alternatively as POWA $= \beta(\text{OWA}) + (1-\beta)$PA.

If the reordering process is carried out whit order-inducing variables, rather than based on the magnitude of the arguments, we get the PIOWA operator. This operator can be defined as follows.

Definition 11. A PIOWA operator of dimension n is a function $F: R^n \times R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{PIOWA}(\langle u_1, a_1 \rangle, \ldots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} \hat{v}_j b_j$$
$$= \beta \sum_{j=1}^{n} w_j b_j + (1 - \beta) \sum_{i=1}^{n} v_i a_i \,, \qquad (15)$$

where b_j is the a_i value of the PIOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to b_j, that is, according to the jth largest of u_i.

An interesting generalization of the OWA operator is the Quasi-OWA operator, presented by Fodor et al. (1995). By using quasi-arithmetic means, the Quasi-OWA operator provides a more general formulation, including a wide range of particular cases that are not considered in the GOWA operator. The Quasi-OWA operator can be defined as follows.

Definition 12. A Quasi-OWA operator of dimension n is a function $F: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{Quasi} - \text{OWA}(a_1, \ldots, a_n) = g^{-1}\left(\sum_{j=1}^{n} w_j g(b_j)\right), \qquad (16)$$

where b_j is the jth largest of the a_i and $g(b)$ is a strictly continuous monotonic function.

Note that if $g(b) = b$ we get the OWA operator and if $g(b) = b^{\lambda}$ the GOWA operator.

By adding probabilities, the Quasi-OWA operator can be extended to a quasi-arithmetic probabilistic ordered weighted averaging (Quasi-POWA) operator, which can be defined as follows.

Definition 13. A Quasi-POWA operator of dimension n is a function $F: R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{Quasi} - \text{POWA}(a_1, \dots, a_n) = g^{-1}\left(\sum_{j=1}^{n} \hat{v}_j g(b_j)\right), \quad (17)$$

where b_j is the jth largest of the a_i, each argument a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to b_j, that is, according to the jth largest of a_i, and $g(b)$ is a strictly continuous monotonic function.

Another interesting operator is the quasi-arithmetic probabilistic induced ordered weighted averaging (Quasi-PIOWA). It is an extension of the Quasi-POWA operator that uses order-inducing variables in the reordering step. It is defined as follows.

Definition 14. A Quasi-PIOWA operator of dimension n is a function $F: R^n \times R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{Quasi} - \text{PIOWA}(\langle u_1, a_1 \rangle, \dots, \langle u_n, a_n \rangle) = g^{-1}\left(\sum_{j=1}^{n} \hat{v}_j g\left(b_j\right)\right), \quad (18)$$

where b_j is the a_i value of the Quasi-PIOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to

b_j, that is, according to the jth largest of u_i, and $g(b)$ is a strictly continuous monotonic function.

The probabilistic ordered weighted averaging real average pension (POWARAP) operator is an extension of the OWARAP operator that includes the probability in the aggregation process. To do so, it is assigned a probability of occurrence to the different values of the average pension of each scenario. So, this operator provides a parametrized family of aggregation operators between the probabilistic minimum and probabilistic maximum average pension. The POWARAP operator can be defined as follows.

Definition 15. A POWARAP operator of dimension n is a function $F: R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{POWARAP}(p_1, \dots, p_n) = \left(\frac{100}{CPI}\right)\left(\sum_{j=1}^{n} \hat{v}_j P_j\right)$$
$$= \left(\frac{100}{CPI}\right)\left(\beta \sum_{j=1}^{n} w_j P_j + (1-\beta) \sum_{i=1}^{n} v_i p_i\right), \quad (19)$$

where P_j is the jth largest value of a set of nominal average pensions p_i. Each p_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to P_j, that is, according to the jth largest of p_i. CPI is the consumer price index.

Example 5. Suppose we have the following vector of arguments ($p_1 = 920, p_2 = 890, p_3 = 950, p_4 = 980$), weighting vector $W = (0.40, 0.25, 0.25, 0.10)$, and probabilistic vector $V = (0.4, 0.10, 0.40, 0.10)$. If we assume that the weighting vector W has a degree of importance of 30% and a CPI of 105 points, then, the POWARAP aggregation can be calculated as follows:

$$\hat{v}_1 = 0.30 \times 0.40 + 0.70 \times 0.10 = 0.19.$$

$$\hat{v}_2 = 0.30 \times 0.25 + 0.70 \times 0.40 = 0.355.$$
$$\hat{v}_3 = 0.30 \times 0.25 + 0.70 \times 0.40 = 0.355.$$
$$\hat{v}_4 = 0.30 \times 0.10 + 0.70 \times 0.10 = 0.10.$$

$$\text{POWARAP} = \left(\frac{100}{105}\right) \times (0.19 \times 980 + 0.355 \times 950 + 0.355 \times 920 + 0.10 \times 890) = 894.33.$$

Alternatively, it can be calculated as follows:

$$\text{POWARAP} = \left(\frac{100}{105}\right)(0.30 \times (0.40 \times 980 + 0.25 \times 950 + 0.25 \times 920 + 0.10 \times 890) + 0.70 \times (0.40 \times 920 + 0.10 \times 890 + 0.40 \times 950 + 0.10 \times 980)) = 894.33.$$

If we use order-inducing variables in the reordering step we form the probabilistic induced ordered weighted averaging real average pension (PIOWARAP) extension, which can be defined as follows.

Definition 16. A PIOWARAP operator of dimension n is a function $F: R^n \times R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{PIOWARAP}(\langle u_1, p_1\rangle, \dots, \langle u_n, p_n\rangle) = \left(\frac{100}{CPI}\right)\left(\sum_{j=1}^{n} \hat{v}_j P_j\right) = \left(\frac{100}{CPI}\right)\left(\beta \sum_{j=1}^{n} w_j P_j + (1-\beta) \sum_{i=1}^{n} v_i p_i\right), \quad (20)$$

where P_j is the p_i value of the PIOWARAP pair $\langle u_i, p_i\rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and p_i as the nominal average pension variable. p_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to P_j, that is, according to the jth largest of u_i. CPI is the consumer price index.

The quasi-arithmetic probabilistic ordered weighted averaging real average pension (Quasi-POWARAP) operator is an extension of the POWARAP operator that uses quasi-arithmetic means. It is defined as follows.

Definition 17. A Quasi-POWARAP operator of dimension n is a function $F: R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{Quasi} - \text{POWARAP}(p_1, \dots, p_n) = \left(\frac{100}{CPI}\right)\left(g^{-1}\left(\sum_{j=1}^{n} \hat{v}_j g(P_j)\right)\right), \quad (21)$$

where P_j is the jth largest value of a set of nominal average pensions p_i. Each p_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1 - \beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to P_j, that is, according to the jth largest of p_i, and $g(b)$ is a strictly continuous monotonic function. CPI is the consumer price index.

The quasi-arithmetic probabilistic induced ordered weighted averaging real average pension (Quasi-PIOWARAP) operator is an aggregation operator that unifies the PIOWARAP operator with the Quasi-OWA operator. This operator is defined as follows.

Definition 18. A Quasi-PIOWARAP operator of dimension n is a function $F: R^n \times R^n \to R$ that has associated a weighting vector W of dimension n $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\begin{aligned} &\text{Quasi} - \text{PIOWARAP}(\langle u_1, p_1 \rangle, \dots, \langle u_n, p_n \rangle) \\ &= \left(\frac{100}{CPI}\right)\left(g^{-1}\left(\sum_{j=1}^{n} \hat{v}_j g(P_j)\right)\right), \end{aligned} \quad (22)$$

where P_j is the p_i value of the Quasi-PIOWARAP pair $\langle u_i, p_i \rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable and p_i as the nominal average pension variable. p_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to P_j, that is, according to the jth largest of u_i, and $g(b)$ is a strictly continuous monotonic function. CPI is the consumer price index.

5. Real average pensions forecasting with OWARAP operators

Proposed methodology

In contrast to traditional forecasting methodologies, the OWARAP operator and its extensions provide greater flexibility allowing to underestimate or overestimate the information according to the opinion and knowledge of one or more experts (multi-expert). Thus, they enable to optimize pension forecasts in complex environments and consequently help governments to improve its assessments of policy decisions. This new formulation and measurement can be applied to any country and region.

In order to forecast the real average pension through the use of the OWARAP operator and its extensions, it is proposed the following algorithm:

Step 1. Data collection. The first step consists in gathering data regarding the average pension in current prices.

Step 2. Scenario-based forecasting. This step consists in generate forecasts of the average pension in current prices for the different scenarios that may occur in the future $S = (S_1, \dots, S_n)$. These forecasts shall be referred as the argument variables $p = (p_1, \dots, p_n)$.

Step 3. Reordering of the arguments. That is building the vector $P_d = (P_1, \dots, P_n)$ for each expert d. For the OWARAP, GOWARAP, POWARAP, and Quasi-POWARAP operators, the reordering process is based on the values of the arguments. By contrast, for the IOWARAP, IGOWARAP, PIOWARAP, and Quasi-PIOWARAP operators, the reordering step is carried out using the order-inducing variables vector $U_d = (u_1, \dots, u_n)$ which the experts must have determined previously.

Step 4. Definition of the weighting weights and aggregation parameters. In this step each expert shall determine its weighting vector $W_d = (w_1, \dots, w_n)$ according to their attitudinal character. Additionally, for the probabilistic operators, experts shall define the probability vector $V_d = (v_1, \dots, v_n)$ and the parameter β.

Step 5. Sub-aggregation. Forecasts calculated in Step 2 and reordered in Step 3 are individually aggregated for each expert based on their weighting vector and probability vector determined in Step 4. This is done using the OWA, IOWA, GOWA, IGOWA, POWA, PIOWA, Quasi-POWA, and Quasi-PIOWA operators.

Step 6. Final aggregation. All the results obtained in Step 5 are aggregated into a single outcome. To do so, the most suitable aggregation function shall be used according to the degree of importance that is given to the assessments made by each expert.

Step 7. Inflation adjustment. Finally, the effect of price inflation or deflation is removed from the results obtained in Step 6, thereby getting the OWARAP and its extensions. This is done by dividing the data by the corresponding CPI and multiplying by 100.

Illustrative example

In the following section an illustrative multi-person example is presented. This example will calculate the OWARAP and the extensions IOWARAP, GOWARAP, IGOWARAP, POWARAP, and PIOWARAP for all the regions of Spain for the year 2023. Also, it will consider only contributory retirement pensions. The following explains the main steps necessary to calculate the future average pension in real terms using the OWARAP operator and its extensions.

Step 1. First, we collect historical monthly data regarding the average pension and the CPI. Our data was gathered from the National Social Security

Institute (INSS is its acronym in Spanish) and the National Statistics Institute (INE is its acronym in Spanish). The most recent data available at date of preparation of this study is December 2020.

Step 2. Once we have collected all the data, we generate the estimations of the current average pension for three different scenarios. The first scenario S_1 contemplates economic growth, the second scenario S_2 assumes the economy will remain unchanged, and the third scenario S_3 considers an economic downturn. Table 3.24 shows the values for each situation. For comparative purposes, in the same table we add the results of applying some of the classical decision approaches, which are the optimistic criterion, the pessimistic criterion, and the Laplace criterion, referred as OC, PC, and LC, respectively. Also, with the OWA and AOWA operators.

Step 3. Afterwards, we proceed to reorder the pension estimates obtained in the previous step. In order to determine the inducing vector a group of three experts is selected. To do this, the experts assess each scenario on a scale of 1 to 10. The different vectors are as follows:

- Expert 1. $U_1 = (6,9,7)$.
- Expert 2. $U_2 = (5,7,9)$.
- Expert 3. $U_3 = (5,8,7)$.

Step 4. We continue with the construction of the weighting vector W and probability vector V. Note that the experts are considering subjective probabilities. On these, we place a level of importance of 35% and 65%, respectively. That is a parameter β equal to 0.35. The weighting and probability vectors are as follows:

- Expert 1. $W_1 = (0.20,0.40,0.40)$ and $V_1 = (0.30,0.50,0.20)$.
- Expert 2. $W_2 = (0.15,0.30,0.55)$ and $V_2 = (0.30,0.35,0.35)$.
- Expert 3. $W_3 = (0.25,0.30,0.45)$ and $V_3 = (0.30,0.40,0.30)$.

Step 5. Through the OWA operator, the IOWA operator, the OWHA operator, the OWQA operator, the induced ordered weighted harmonic averaging (IOWHA) operator, the induced ordered weighted quadratic averaging (OWQA) operator, the POWA operator, and the PIOWA operator we add the estimations of the different scenarios according to the opinion and knowledge of the experts (Tables 3.25, 3.26, and 3.27).

Step 6. Next, we fuse the aggregated results of the experts into a single outcome (see Table 3.28). To do so, we use the arithmetic mean because we consider the assessments of each expert equally important. However, if we believe that the judgements of each expert are not equally relevant, we could employ the OWA operator, the IOWA operator, and many more.

Step 7. Finally, we perform the inflation adjustment to obtain the OWARAP, IOWARAP, OWHARAP, OWQARAP, induced ordered weighted harmonic averaging real average pension (IOWHARAP), induced ordered weighted quadratic averaging real average pension (IOWQARAP), POWARAP, and PIOWARAP operators. The CPIs employed and the final aggregated results can be seen in Tables 3.29 and 3.30, respectively.

Table 3.24. Spain's nominal average pensions 2023 scenario forecasts by region and aggregated results (values in euro)

Region	S1	S2	S3	OC	PC	LC	OWA	AOWA
Andalusia	1,173	1,118	1,062	1,173	1,062	1,118	1,103	1,132
Aragon	1,369	1,304	1,239	1,369	1,239	1,304	1,287	1,321
Balearic Islands	1,207	1,150	1,092	1,207	1,092	1,150	1,134	1,165
Basque Country	1,597	1,521	1,445	1,597	1,445	1,521	1,501	1,541
Canary Islands	1,206	1,148	1,091	1,206	1,091	1,148	1,133	1,163
Cantabria	1,392	1,326	1,260	1,392	1,260	1,326	1,308	1,343
Castile and Leon	1,295	1,234	1,172	1,295	1,172	1,234	1,217	1,250
Castile La Mancha	1,211	1,153	1,095	1,211	1,095	1,153	1,138	1,168
Catalonia	1,327	1,264	1,201	1,327	1,201	1,264	1,247	1,281
Ceuta	1,407	1,340	1,273	1,407	1,273	1,340	1,322	1,358
Community of Navarre	1,478	1,407	1,337	1,478	1,337	1,407	1,388	1,426
Community of Madrid	1,518	1,445	1,373	1,518	1,373	1,445	1,426	1,465
Extremadura	1,079	1,027	976	1,079	976	1,027	1,014	1,041
Galicia	1,102	1,050	997	1,102	997	1,050	1,036	1,064
La Rioja	1,252	1,192	1,133	1,252	1,133	1,192	1,176	1,208
Melilla	1,366	1,301	1,236	1,366	1,236	1,301	1,283	1,318
Principality of Asturias	1,563	1,489	1,414	1,563	1,414	1,489	1,469	1,509
Region of Murcia	1,162	1,106	1,051	1,162	1,051	1,106	1,092	1,121
Valencian Community	1,195	1,138	1,081	1,195	1,081	1,138	1,123	1,153

Table 3.25. Expert 1 aggregated results of the 2023 nominal average pensions of Spain's regions (values in euro)

Region	OWA_1	$IOWA_1$	$OWHA_1$	$OWQA_1$	$IOWHA_1$	$IOWQA_1$	$POWA_1$	$PIOWA_1$
Andalusia	1,106	1,118	1,105	1,107	1,115	1,119	1,117	1,121
Aragon	1,291	1,304	1,289	1,292	1,301	1,305	1,304	1,308
Balearic Islands	1,138	1,150	1,136	1,139	1,147	1,151	1,149	1,153
Basque Country	1,506	1,521	1,504	1,507	1,518	1,523	1,521	1,526
Canary Islands	1,137	1,148	1,135	1,137	1,146	1,149	1,148	1,152
Cantabria	1,313	1,326	1,311	1,313	1,323	1,327	1,325	1,330
Castile and Leon	1,221	1,234	1,220	1,222	1,231	1,235	1,233	1,238
Castile La Mancha	1,142	1,153	1,140	1,142	1,151	1,154	1,153	1,157
Catalonia	1,252	1,264	1,250	1,253	1,262	1,266	1,264	1,268
Ceuta	1,327	1,340	1,325	1,328	1,338	1,342	1,340	1,345
Community of Navarre	1,393	1,407	1,391	1,394	1,404	1,409	1,407	1,412
Community of Madrid	1,431	1,445	1,429	1,432	1,443	1,447	1,445	1,450
Extremadura	1,017	1,027	1,016	1,018	1,025	1,028	1,027	1,031
Galicia	1,039	1,050	1,038	1,040	1,048	1,051	1,050	1,053
La Rioja	1,180	1,192	1,179	1,181	1,190	1,193	1,192	1,196
Melilla	1,288	1,301	1,286	1,288	1,298	1,302	1,300	1,305
Principality of Asturias	1,474	1,489	1,472	1,475	1,486	1,490	1,488	1,494
Region of Murcia	1,095	1,106	1,094	1,096	1,104	1,108	1,106	1,110
Valencian Community	1,126	1,138	1,125	1,127	1,135	1,139	1,137	1,141

Table 3.26. Expert 2 aggregated results of the 2023 nominal average pensions of Spain's regions (values in euro)

Region	OWA_2	$IOWA_2$	$OWHA_2$	$OWQA_2$	$IOWHA_2$	$IOWQA_2$	$POWA_2$	$PIOWA_2$
Andalusia	1,095	1,140	1,094	1,096	1,138	1,141	1,108	1,124
Aragon	1,278	1,330	1,276	1,279	1,328	1,331	1,293	1,311
Balearic Islands	1,127	1,173	1,125	1,127	1,171	1,173	1,140	1,156
Basque Country	1,491	1,551	1,489	1,492	1,549	1,552	1,508	1,529
Canary Islands	1,125	1,171	1,124	1,126	1,170	1,172	1,138	1,154
Cantabria	1,299	1,352	1,298	1,300	1,351	1,353	1,314	1,333
Castile and Leon	1,209	1,258	1,207	1,210	1,257	1,259	1,223	1,240
Castile La Mancha	1,130	1,176	1,128	1,131	1,175	1,177	1,143	1,159
Catalonia	1,239	1,290	1,237	1,240	1,288	1,290	1,253	1,271
Ceuta	1,313	1,367	1,312	1,314	1,365	1,368	1,329	1,348
Community of Navarre	1,379	1,435	1,377	1,380	1,433	1,436	1,395	1,415
Community of Madrid	1,416	1,474	1,415	1,417	1,472	1,475	1,433	1,453
Extremadura	1,007	1,048	1,006	1,008	1,047	1,049	1,019	1,033
Galicia	1,029	1,071	1,027	1,030	1,069	1,072	1,041	1,055
La Rioja	1,168	1,216	1,167	1,169	1,214	1,217	1,182	1,199
Melilla	1,275	1,327	1,273	1,275	1,325	1,327	1,289	1,308
Principality of Asturias	1,459	1,519	1,457	1,460	1,516	1,519	1,476	1,497
Region of Murcia	1,084	1,129	1,083	1,085	1,127	1,129	1,097	1,112
Valencian Community	1,115	1,161	1,113	1,116	1,159	1,161	1,128	1,144

Table 3.27. Expert 3 aggregated results of the 2023 nominal average pensions of Spain's regions (values in euro)

Region	OWA_3	$IOWA_3$	$OWHA_3$	$OWQA_3$	$IOWHA_3$	$IOWQA_3$	$POWA_3$	$PIOWA_3$
Andalusia	1,106	1,126	1,105	1,107	1,124	1,127	1,114	1,121
Aragon	1,291	1,314	1,289	1,292	1,311	1,315	1,299	1,307
Balearic Islands	1,138	1,158	1,136	1,139	1,156	1,159	1,146	1,153
Basque Country	1,506	1,532	1,503	1,507	1,530	1,534	1,516	1,525
Canary Islands	1,137	1,157	1,135	1,138	1,155	1,158	1,144	1,151
Cantabria	1,313	1,336	1,310	1,314	1,333	1,337	1,321	1,329
Castile and Leon	1,221	1,243	1,219	1,222	1,241	1,244	1,229	1,237
Castile La Mancha	1,142	1,162	1,140	1,143	1,160	1,163	1,149	1,156
Catalonia	1,252	1,274	1,250	1,253	1,271	1,275	1,260	1,268
Ceuta	1,327	1,350	1,325	1,328	1,348	1,352	1,336	1,344
Community of Navarre	1,393	1,418	1,391	1,394	1,415	1,419	1,402	1,411
Community of Madrid	1,431	1,456	1,429	1,432	1,454	1,458	1,440	1,449
Extremadura	1,017	1,035	1,016	1,018	1,033	1,036	1,024	1,030
Galicia	1,039	1,058	1,038	1,040	1,056	1,059	1,046	1,053
La Rioja	1,180	1,201	1,178	1,181	1,199	1,202	1,188	1,195
Melilla	1,288	1,310	1,285	1,289	1,308	1,311	1,296	1,304
Principality of Asturias	1,474	1,500	1,471	1,475	1,497	1,501	1,484	1,493
Region of Murcia	1,095	1,115	1,094	1,096	1,113	1,116	1,103	1,109
Valencian Community	1,126	1,146	1,125	1,127	1,144	1,147	1,134	1,141

Table 3.28. Absolute aggregated results of the 2023 nominal average pensions of Spain's regions (values in euro)

Region	OWA	IOWA	OWHA	OWQA	IOWHA	IOWQA	POWA	PIOWA
Andalusia	1,103	1,128	1,101	1,104	1,126	1,129	1,113	1,122
Aragon	1,287	1,316	1,285	1,287	1,314	1,317	1,299	1,309
Balearic Islands	1,134	1,160	1,133	1,135	1,158	1,161	1,145	1,154
Basque Country	1,501	1,535	1,498	1,502	1,532	1,536	1,515	1,527
Canary Islands	1,133	1,159	1,131	1,134	1,157	1,160	1,143	1,152
Cantabria	1,308	1,338	1,306	1,309	1,336	1,339	1,320	1,331
Castile and Leon	1,217	1,245	1,215	1,218	1,243	1,246	1,229	1,238
Castile La Mancha	1,138	1,164	1,136	1,139	1,162	1,165	1,148	1,157
Catalonia	1,247	1,276	1,246	1,248	1,274	1,277	1,259	1,269
Ceuta	1,322	1,353	1,320	1,323	1,350	1,354	1,335	1,345
Community of Navarre	1,388	1,420	1,386	1,390	1,418	1,421	1,401	1,413
Community of Madrid	1,426	1,459	1,424	1,427	1,456	1,460	1,439	1,451
Extremadura	1,014	1,037	1,012	1,015	1,035	1,038	1,023	1,031
Galicia	1,036	1,059	1,034	1,037	1,058	1,060	1,046	1,054
La Rioja	1,176	1,203	1,175	1,177	1,201	1,204	1,187	1,197
Melilla	1,283	1,312	1,281	1,284	1,310	1,314	1,295	1,305
Principality of Asturias	1,469	1,502	1,467	1,470	1,500	1,504	1,483	1,494
Region of Murcia	1,092	1,117	1,090	1,093	1,115	1,118	1,102	1,111
Valencian Community	1,123	1,148	1,121	1,123	1,146	1,149	1,133	1,142

Table 3.29. Estimations of the 2023 CPIs of Spain's regions

Region	CPI
Andalusia	108
Aragon	107
Balearic Islands	108
Basque Country	108
Canary Islands	108
Cantabria	108
Castile and Leon	108
Castile La Mancha	108
Catalonia	108
Ceuta	106
Community of Navarre	108
Community of Madrid	108
Extremadura	107
Galicia	108
La Rioja	108
Melilla	107
Principality of Asturias	107
Region of Murcia	107
Valencian Community	107

Table 3.30. Absolute aggregated results of the 2023 real average pensions of Spain's regions (values in euro)

Region	OWA-RAP	IOWA-RAP	OWHA-RAP	OWQA-RAP	IOWHA-RAP	IOWQA-RAP	POWA-RAP	PIOWA-RAP
Andalusia	1,026	1,049	1,024	1,026	1,047	1,050	1,035	1,043
Aragon	1,198	1,225	1,196	1,199	1,223	1,226	1,209	1,219
Balearic Islands	1,055	1,079	1,053	1,056	1,077	1,080	1,065	1,073
Basque Country	1,387	1,419	1,385	1,389	1,417	1,420	1,400	1,412
Canary Islands	1,049	1,073	1,048	1,050	1,071	1,074	1,059	1,067
Cantabria	1,214	1,242	1,212	1,215	1,239	1,243	1,225	1,235
Castile and Leon	1,128	1,154	1,127	1,129	1,152	1,155	1,139	1,148
Castile La Mancha	1,057	1,081	1,055	1,058	1,079	1,082	1,067	1,075
Catalonia	1,153	1,179	1,151	1,153	1,177	1,180	1,163	1,173
Ceuta	1,249	1,277	1,247	1,250	1,275	1,278	1,261	1,270
Community of Navarre	1,290	1,320	1,289	1,291	1,318	1,321	1,303	1,313
Community of Madrid	1,323	1,353	1,321	1,324	1,351	1,354	1,335	1,346
Extremadura	945	966	943	945	965	967	953	961
Galicia	962	984	960	962	982	984	971	978
La Rioja	1,094	1,119	1,093	1,095	1,117	1,120	1,104	1,113
Melilla	1,205	1,232	1,203	1,206	1,230	1,233	1,216	1,226
Principality of Asturias	1,373	1,405	1,371	1,374	1,402	1,406	1,386	1,397
Region of Murcia	1,022	1,046	1,021	1,023	1,044	1,047	1,032	1,040
Valencian Community	1,045	1,068	1,043	1,045	1,067	1,069	1,054	1,063

In Tables 3.25, 3.26, and 3.27 we can see different scenarios of the estimated nominal average pension of Spain's regions based on the attitudinal character of each expert. Note that depending on the weights and order-inducing variables that the expert chooses for the aggregation, the average pension can change considerably. Table 3.28 unifies the results obtained by the group of experts into a single outcome, which afterwards is expressed in real prices in Table 3.30. With all this information, the decision-maker gain a better understanding of the pension development and can make better decision.

As may be seen, different results are obtained depending on the type of aggregation operator used. For example, in the Community of Madrid the real average pension for the year 2023 ranges from 1,321 euro to 1,354 euro. We can also see that the lowest estimations are obtained with the OWHARAP operator and the highest with the IOWQARAP operator. Moreover, the operators that include order-inducing variables in the reordering step produce greater forecasts. This can be summarized as follows (from less to more):

$$\mathrm{OWHARAP} \prec \mathrm{OWARAP} \prec \mathrm{OWQARAP} \prec \mathrm{POWARAP} \prec \mathrm{PIOWARAP} \\ \prec \mathrm{IOWHARAP} \prec \mathrm{IOWARAP} \prec \mathrm{IOWQARAP}$$

As can be seen, inflation is an important element to consider when analyzing the pension adequacy. For example, the estimated average pension of Catalonia in current prices (OWA) stands at 1,247 euro, however, if we use constant prices (OWARAP) the average pension decreases dramatically to 1,153 euro. Therefore, by using the OWARAP operator and its extensions, people and governments are able to better understand old-age pension changes and thereby improve their decision-making process.

Finally, other scenarios with different assumptions could be considered. For example, one can develop a set of scenarios based on different pension revaluation rates that the government may impose in the future.

Comparison of forecasting methods

In the following, a comparison between the OWARAP operator and some traditional forecasting methods is conducted. To do so, the real average pension of Spain at a national level is estimated for December 2020 using the

OWARAP operator, Linear Trend (LT), Double Moving Average (DMA), and Holt's Exponential Smoothing (HES) (Holt, 2004) forecasting methods. Also, data until December 2017 is included in the training set. Table 3.31 shows the outcomes and the corresponding absolute values of the forecast errors. Results show that the OWARAP operator offers potentially better performance in comparison with the traditional methods.

Table 3.31. Evaluation of forecasting methods

Indicator	Real value	OWARAP	LT	2-month DMA	3-month DMA	HES
Real average pension	1,118	1,123	1,107	989	901	1,064
Absolute error	0	5	11	129	217	54

6. The COVID-19 crisis on pensions: applicability of the OWA operators

The outbreak of the coronavirus (COVID-19) pandemic has caused an economic downturn across the world. Some international organizations, such as the OECD (2020) and the International Monetary Fund (IMF) (Feher & de Bidegain, 2020), warn about the impact of the pandemic on the future of pensions. Prior to the COVID-19, pension systems were already facing financial sustainability problems, mainly driven by the ageing of the population.

This crisis has entailed high levels of unemployment, which leads to a reduction of government revenues from contributions and consequently makes public pension systems more unsustainable. Moreover, the declining employment among elderly people, makes it more difficult for them to find a new job because of age discrimination. Consequently, it is more likely that this population group retire early, leading to lower pension benefits. Therefore, policy makers should adopt new strategies. However, the uncertainty associated to the length of the current economic crisis and the real impact on pensions in the long-term, makes it more difficult for policy makers to choose and implement the best policy response.

In this complex environment, OWA operators can be very useful when generating forecasts of pension indicators, because they use the attitudinal character of the decision-maker. Since the COVID-19 outbreak, the pessimistic attitude towards future public pensions has further increased, and by using the OWA weights the decision-maker is able to reflect it. This method can be also helpful for assessing pension policy responses alternatives. Likewise, to address the problem POWA operators can be very useful as well, given that they allow to consider probabilistic information with the OWA operator.

7. Conclusions

The aim of this study is to provide a new tool to improve the quality of pension information. In view of the present and future situation, it is necessary and extremely important to provide recurrent, accurate, representative, and useful data on pension adequacy for governments and citizens of a country. With this information, governments are able to improve significantly its policy decisions and people can plan properly their retirement, in an attempt to prevent old-age poverty.

This paper proposes a new method to optimize forecasts of the average pension using the OWA operator, the IOWA operator, the GOWA operator, and the IGOWA operator. Moreover, an inflation adjustment is added in order to provide a more realistic value of the average pension. The main advantage of using this method is the possibility to aggregate different estimations according to the attitudinal character of the decision-maker without losing information. As a result, forecasts are more representative and accurate. The POWA operator is also used, which shows to be very useful for situations where we find probabilistic information, and at the same time, we need to consider the attitudinal character.

The study also develops an illustrative example regarding the calculations of the projected real average pension of Spain at a regional level by using the OWARAP operator, the IOWARAP operator, the GOWARAP operator, the IGOWARAP operator, the POWARAP operator, and the PIOWARAP operator. This allows to aggregate different scenarios according to the expectations and knowledge of a selected group of experts. The information given by these operators provide a more comprehensive analysis that helps the decision-maker to deal with uncertainty.

In future works we aim to analyze the use of the OWARAP operator in other countries, like United States or Canada. Moreover, we expect to apply this new method into other pension indicators, such as the aggregate replacement ratio for pensions. Lastly, we suggest the use of other extensions of the OWA operator in the field of pensions, such as the uncertain ordered weighted averaging (UOWA) operator (Xu & Da, 2002).

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3.4. Forecasting retirement benefits in the United States using OWA operators

At the date of this thesis, the following research paper has been submitted to the Q1 journal *Technological and Economic Development of Economy*. According to the Journal Citation Reports (JCR), published by Clarivate Analytics, the Impact Factor in 2021 is 5.656. Moreover, based on the Scopus database, the CiteScore metric of the journal in 2021 is 8.0.

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Abstract

The issue of pensions has become increasingly topical. This paper presents the ordered weighted averaging real average pension (OWARAP) operator. The OWARAP operator is based on the ordered weighted averaging (OWA) operator and measures the future average retirement benefit adjusted for inflation. Moreover, this work extends the OWARAP operator by using order-inducing variables, generalized means, and probabilities. This paper ends by analyzing the applicability of the OWARAP operator and its extensions in forecasting the real average Social Security benefits for retired workers in each state of the United States (U.S.). The results demonstrate the usefulness of the proposed approach in retirement decision-making.

Keywords: Aggregation operator, forecasting, inflation, OWA operator, retirement benefit, Social Security.

1. Introduction

The continuous growth in life expectancy, partly driven by an improvement in healthcare systems, and the low fertility rates imply an increase in the old-age dependency ratio (that is, the number of elderly people compared to those at working age). Consequently, retirement systems become more unsustainable (Organization for Economic Cooperation and Development [OECD], 2019; Peris-Ortiz et al., 2020), which also has a negative impact on the retirement income adequacy. For example, in 1990, life expectancy at age 65 was 18.9 years for women and 15.1 years for men in the United States (U.S.) (OECD, 2023b). Thirty years later, in 2020, life expectancy at age 65 increased considerably to 19.8 years for women and 17.0 years for men. Also, by looking at the fertility rates for the same country, one can see that the fertility rate declined from 2.08 children per woman in 1990 to 1.64 in 2020 (OECD, 2023a). However, these demographic changes are not the only ones that adversely affect the financial health of retirement systems and, consequently, the adequacy levels. There are other factors related to economic growth, the labor market, and the design of the retirement system (OECD, 2019; Peris-Ortiz et al., 2020), among others.

In this complex context, it is very important that citizens are provided with sufficient and recurrent retirement information (Basiglio & Oggero, 2020). For instance, individuals need to be aware of their future retirement income as accurately as possible and in real prices (Bongini & Cucinelli, 2019; O'Neill et al., 2017) so that they can properly plan their retirement and avoid a reduction in purchasing power. Likewise, governments need access to precise and helpful information on the future trend of public retirement benefits and other related indicators to conduct effective policy decision-making and thereby reduce the risk of poverty among older people.

With the purpose of helping individuals to have an adequate amount of savings for their retirement, and also governments to make good decisions, this paper presents the ordered weighted averaging real average pension (OWARAP) operator. The OWARAP operator can be seen as an optimized retirement index. It is built under the ordered weighted averaging (OWA) operator from Yager (1988) while considering the effect of inflation. The OWA operator is an increasingly popular aggregation operator used for unifying numerical information according to the attitudinal character of the decision-maker (Blanco-Mesa et al., 2019; Emrouznejad & Marra, 2014; He et al., 2017). Thus, this approach allows to overestimate or underestimate the real average retirement benefit according to the opinion of the decision-maker, which is very useful for dealing with demographic, economic, and pension policy uncertainties.

Moreover, different extensions of the OWARAP operator are considered, which are the induced OWARAP (IOWARAP) operator, the generalized OWARAP (GOWARAP) operator, and the probabilistic OWARAP (POWARAP) operator. The IOWARAP operator uses order-inducing variables; the GOWARAP operator generalized means (Dyckhoff & Pedrycz, 1984); and the POWARAP operator probabilistic information. This allows the decision-maker to contemplate a diverse range of aggregation operators and adopt the one that best suits with his/her needs and preferences.

In the literature, we can find different authors that apply the OWA operator and extensions of this operator in economic indicators, including exchange rates (Flores-Sosa et al., 2020; León-Castro et al., 2016, 2018), inflation rates (Espinoza-Audelo et al., 2020; León-Castro et al., 2020), and prosperity (Amin & Siddiq, 2019). However, they have not yet been applied to the average retirement benefit. Therefore, the study's novelty consists of using the characteristics of the OWA operator to forecast the average retirement benefit adjusted for inflation.

This paper is organized as follows. Section 2 briefly reviews some basic but necessary concepts. Section 3 explains the mathematical approach used in this work. Section 4 develops a numerical example of the proposed approach, which consists in forecasting the real average Social Security retirement benefit of each state of the U.S. Lastly, Section 5 summarizes the main conclusions of the paper and makes some general recommendations for future research.

2. Preliminaries

The following section briefly reviews the OWA operator, the induced OWA (IOWA) operator, the generalized OWA (GOWA) operator, and the probabilistic OWA (POWA) operator.

The OWA operator

The OWA operator was presented by Yager (1988) and it provides a parameterized class of mean type aggregation operators that lies between the minimum and the maximum. This operator has been successfully applied in a large variety of fields (Kacprzyk et al., 2019). The OWA operator can be defined as follows.

Definition 1. An OWA operator of dimension n is a mapping $\text{OWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{OWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} w_j b_j , \quad (1)$$

where b_j is the jth largest element of the arguments $a_1, \ldots, a_n$, namely $(b_1, \ldots, b_n)$ is $(a_1, \ldots, a_n)$ reordered in a descending way.

The parameterization is carried out by choosing different formations of the weighting vector W. For example, if $w_j = 1/n$, for all j, the Laplace criteria (also knowns as arithmetic mean) is formed. Furthermore, another aspect worth highlighting is that when the reordering process is conducted in an ascending manner, then we get the ascending OWA (AOWA) operator (Yager, 1992).

The IOWA operator

A remarkable extension of the OWA operator is the IOWA operator (Yager & Filev, 1999). The main difference between this operator and the classical OWA operator is that the reordering step is carried out with order-inducing variables. This is why a major advantage of the IOWA operator is that it can consider the complex attitudes of the decision-maker. This operator can be defined as follows.

Definition 2. An IOWA operator of dimension n is a mapping $\text{IOWA}: R^n \times R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{IOWA}(\langle u_1, a_1 \rangle, \ldots, \langle u_n, a_n \rangle) = \sum_{j=1}^{n} w_j b_j , \quad (2)$$

where b_j is the a_i value of the IOWA pair $\langle u_i, a_i \rangle$ having the jth largest u_i value, u_i is referred as the order-inducing variable, and a_i is the argument variable.

The GOWA operator

The GOWA operator was introduced by Yager (2004), combining the OWA operator with generalized means (Dyckhoff & Pedrycz, 1984). Specifically, this operator incorporates a parameter that allows control of the power to which the argument values are raised in the aggregation. The GOWA operator can be defined as follows.

Definition 3. A GOWA operator of dimension n is a mapping $\text{GOWA}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$, with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{GOWA}(a_1, \dots, a_n) = \left(\sum_{j=1}^{n} w_j b_j^{\lambda} \right)^{1/\lambda}, \tag{3}$$

where λ is a parameter such that $\lambda \in (-\infty, \infty)$ and b_j is the jth largest of the argument variable a_i.

Note that: if $\lambda = -1$, the ordered weighted harmonic averaging (OWHA) operator (Yager, 2004) is found; if $\lambda = 0$, the ordered weighted geometric (OWG) operator (Chiclana et al., 2000, 2002); if $\lambda = 1$, the ordinary OWA operator; and if $\lambda = 2$, the ordered weighted quadratic averaging (OWQA) operator (Yager, 2004).

The POWA operator

The POWA operator was introduced by Merigó (2012). It is an aggregation function that unifies the probability and the OWA operator (attitudinal character) in the same formulation and according to the degree of importance of these two concepts in the aggregation process. Therefore, it provides a unified framework between decision-making problems under risk and uncertainty. The POWA operator can be defined as follows.

Definition 4. A POWA operator of dimension n is a mapping POWA: $R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{POWA}(a_1, \ldots, a_n) = \sum_{j=1}^{n} \hat{v}_j b_j = \beta \sum_{j=1}^{n} w_j b_j + (1-\beta) \sum_{i=1}^{n} v_i a_i \,, \qquad (4)$$

where b_j is the jth largest of the argument a_i, each argument a_i has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to b_j, that is, based on the jth largest of the argument a_i.

3. OWA operators in the real average pension benefit

In this section, the OWARAP operator and some of its extensions and generalizations will be defined and analyzed.

The OWARAP operator

The OWARAP operator is an aggregation operator based on Yager's OWA operator. In particular, it aggregates the information of a set of nominal average retirement benefits and another one with inflations while considering the attitude, judgment, or knowledge of the decision-maker. This feature makes the OWARAP operator an attractive method for forecasting the real average retirement benefit under uncertainty, as the decision-maker is capable of overestimating or underestimating the projections. Also, by making inflation adjustments, individuals are able to control if their retirement benefits will or will not grow in the future at the same pace as inflation does. This operator is defined as follows.

Definition 5.1. An OWARAP operator of dimension n is a mapping $\text{OWARAP}: R^n \times R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{OWARAP}(\langle CPI_1, p_1 \rangle, \dots, \langle CPI_n, p_n \rangle) = \sum_{j=1}^{n} w_j P_j \,, \qquad (5.1)$$

where P_j is the jth largest of the $\left(\frac{100}{CPI_i}\right) p_i$, p_i is the ith argument of a set of nominal average retirement benefits, and CPI_i is the ith argument of a set of consumer price indices.

Note that Definition 5.1 contemplates different CPI input values in the aggregation. However, if the decision-maker wants to consider a single CPI input value, then the mathematical expression of the OWARAP operator can be rewritten as follows.

Definition 5.2. An OWARAP operator of dimension n is a mapping $\text{OWARAP}: R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{OWARAP}(p_1, \dots, p_n) = \left(\frac{100}{CPI}\right) \sum_{j=1}^{n} w_j P_j \,, \qquad (5.2)$$

where P_j is the jth largest argument of a set of nominal average retirement benefits $p_1, \dots, p_n$ and CPI is the consumer price index.

Henceforth, the study assumes that several scenarios for the CPI may be possible.

Example 1. Consider the following set of nominal retirement benefits ($p_1 = 1{,}300, p_2 = 1{,}250, p_3 = 1{,}350, p_4 = 1{,}400$) and weighting vector ($w_1 = 0.4, w_2 = 0.3, w_3 = 0.2, w_4 = 0.1$). If the collection of consumer prices indices is ($CPI_1 = 190, CPI_2 = 180, CPI_3 = 200, CPI_4 = 210$), then, the aggregation process through the OWARAP operator, that is, Eq. (5.1), is solved as follows:

$$0.4 \times \frac{100}{180} \times 1{,}250 + 0.3 \times \frac{100}{190} \times 1{,}300 + 0.2 \times \frac{100}{200} \times 1{,}350 + 0.1 \times \frac{100}{210} \times 1{,}400 = 684.7.$$

The OWARAP operator is a mean operator that satisfies the properties of monotonicity, commutativity (also referred to as symmetry or anonymity), boundedness, and idempotency (also called agreement or unanimity). These properties are explained below with their corresponding theorems. Take into account that most of the proofs are omitted as they are considered trivial and repetitive.

Theorem 1. Monotonicity. It states that when an argument increases, the final aggregation remains equal or increases, but in no case decreases. Let F be the OWARAP operator. If $\left(\frac{100}{CPI_i}\right) p_i \geq \left(\frac{100}{\widehat{CPI}_i}\right) \hat{p}_i$, for all i, then, $F(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) \geq F\left(\langle \widehat{CPI}_1, \hat{p}_1\rangle, \dots, \langle \widehat{CPI}_n, \hat{p}_n\rangle\right)$.

Theorem 2. Commutativity. Meaning that the initial indexing of the arguments is completely irrelevant. So, if F is the OWARAP operator, then, $F(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) = F\left(\langle \widehat{CPI}_1, \hat{p}_1\rangle, \dots, \langle \widehat{CPI}_n, \hat{p}_n\rangle\right)$, where $\left(\langle \widehat{CPI}_1, \hat{p}_1\rangle, \dots, \langle \widehat{CPI}_n, \hat{p}_n\rangle\right)$ is any permutation of $(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle)$.

Theorem 3. Boundedness. In the sense that the aggregation is delimited. Let F be the OWARAP operator. Then, $Min\left\{\left(\frac{100}{CPI_i}\right) p_i\right\} \leq F(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) \leq Max\left\{\left(\frac{100}{CPI_i}\right) p_i\right\}$.

Theorem 4. Idempotency. It signifies that if all the input arguments are the same, then the aggregated output should match with them. Let F be the OWARAP operator. If $\left(\frac{100}{CPI_i}\right) p_i = \left(\frac{100}{CPI}\right) p$, for all i, then, $F(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) = \left(\frac{100}{CPI}\right) p$.

Proof. Since $p_i = p$ and $CPI_i = CPI$, for all i, we have

$$F(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) = \sum_{j=1}^{n} w_j P_j = \sum_{j=1}^{n} w_j \left(\frac{100}{CPI}\right) p$$
$$= \left(\frac{100}{CPI}\right) p \sum_{j=1}^{n} w_j \,.$$

Since $\sum_{j=1}^{n} w_j = 1$, we get

$$F(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) = \left(\frac{100}{CPI}\right) p.$$

This operation can be carried out multiple times without changing the result, therefore, it can be stated that the OWARAP operator is idempotent.

Furthermore, to determine the values of the weighting vector W of the OWARAP operator, it is possible to use the well-known characterizing measures presented by Yager and Alajlan. These measures are the degree of orness (Yager, 1988), the entropy of dispersion (Yager, 1988), the balance operator (Yager, 1996), the divergence (Yager, 2002), and the focus (Yager & Alajlan, 2014).

The degree of orness measure, also referred to as the attitudinal character, can be defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{6}$$

The entropy of dispersion measure can be defined as follows:

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln\left(w_j\right). \tag{7}$$

The balance operator measure can be defined as follows:

$$\mathrm{Bal}(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1}\right). \tag{8}$$

The divergence measure can be defined as follows:

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \tag{9}$$

And the focus measure can be defined as follows:

$$\mathrm{Focus}(W) = 1 - \frac{2}{n}\sum_{j=1}^{n} w_j |m-j|, \tag{10}$$

where $m = n\big(1-\alpha(W)\big) + \alpha(W)$.

Families of the OWARAP operator

Different families of the OWARAP operator can be obtained by choosing different manifestations of the weighting vector W. In the following, some of these families are presented.

- When $w_1 = 1$ and $w_j = 0$, for all $j \neq 1$, the maximum OWARAP operator is found, which corresponds to the optimistic decision criterion.
- When $w_n = 1$ and $w_j = 0$, for all $j \neq n$, the minimum OWARAP operator is found, which corresponds to the pessimistic decision criterion.
- If n is an odd number, then, when $w_{(n+1)/2} = 1$ and $w_j = 0$, for all $j \neq (n+1)/2$, the median OWARAP operator is formed. Otherwise, in the case that n is even, the median OWARAP operator is obtained when $w_{n/2} = w_{(n/2)+1} = 0.5$ and $w_j = 0$, for all $j \neq n/2, (n/2)+1$.

- When $w_j = 1/n$, for all j, the normalized OWARAP operator is found, which corresponds to the Laplace decision criterion, that is, the arithmetic mean.
- When $w_1 = \alpha$, $w_n = 1 - \alpha$, and $w_j = 0$, for all $j \neq 1, n$, the Hurwicz OWARAP operator is found.
- When $w_1 = w_n = 0$ and $w_j = 1/(n-2)$, for all $j \neq 1, n$, the Olympic OWARAP operator is found.
- When $w_k = 1$ and $w_j = 0$, for all $j \neq k$, the step OWARAP operator is found.

Extensions and generalizations of the OWARAP operator

An interesting extension of the OWARAP operator is the IOWARAP operator, which uses order-inducing variables in the process of reordering the set of values $(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle)$. Thus, the reordering step does not depend on the values of the arguments p_i and CPI_i. This is why the main advantage of this extension is the possibility to consider more complex attitudes of the decision-maker. The IOWARAP operator is defined as follows.

Definition 6. An IOWARAP operator of dimension n is a mapping IOWARAP: $R^n \times R^n \times R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{IOWARAP}(\langle u_1, CPI_1, p_1\rangle, \dots, \langle u_n, CPI_n, p_n\rangle) = \sum_{j=1}^{n} w_j P_j\,, \qquad (11)$$

where P_j is the $\left(\frac{100}{CPI_i}\right) p_i$ value of the IOWARAP triplet $\langle u_i, CPI_i, p_i\rangle$ having the jth largest u_i value. u_i is referred as the order-inducing variable, p_i as the nominal average retirement benefit variable, and CPI_i as the consumer price index variable.

Moreover, by using generalized means in the OWARAP operator, the GOWARAP operator is obtained. Specifically, it adds a parameter controlling the power to which the argument values are raised. Thus, this operator comprises an extensive range of aggregation operators, including the OWARAP operator and its particular cases, among others. The GOWARAP operator is defined as follows.

Definition 7. A GOWARAP operator of dimension n is a mapping $\text{GOWARAP}: R^n \times R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{GOWARAP}(\langle CPI_1, p_1 \rangle, \ldots, \langle CPI_n, p_n \rangle) = \left(\sum_{j=1}^{n} w_j P_j^{\lambda} \right)^{1/\lambda}, \qquad (12)$$

where P_j is the jth largest of the $\left(\frac{100}{CPI_i}\right) p_i$ and λ is a parameter such that $\lambda \in (-\infty, +\infty)$. p_i is the nominal average retirement benefit variable and CPI_i the consumer price index variable.

By giving different values to the parameter λ, it is possible to obtain particular cases of the GOWARAP operator, among which the following are noteworthy:

- When $\lambda = -1$, the harmonic OWARAP (OWHARAP) operator is formed.
- When $\lambda = 0$, the geometric OWARAP (OWGRAP) operator is formed.
- When $\lambda = 1$, the OWARAP operator is formed.
- When $\lambda = 2$, the quadratic OWARAP (OWQARAP) operator is formed.

Likewise, by analyzing the weighting vector W and the parameter λ jointly, it can be summarized that:

- When $\lambda = -\infty$ and $w_n \neq 0$, the smallest $(100/CPI_i)p_i$ value of the collection is obtained, which is P_n.
- When $\lambda = -\infty$ and $w_1 = 1$, that is, $w_j = 0$, for all $j \neq 1$, the largest $(100/CPI_i)p_i$ value of the collection is achieved, which is P_1.

- When $\lambda = \infty$ and $w_1 \neq 0$, the largest $(100/CPI_i)p_i$ value of the collection is obtained, which is P_1.
- When $\lambda = \infty$ and $w_n = 1$, that is, $w_j = 0$, for all $j \neq n$, the smallest $(100/CPI_i)p_i$ value of the collection is achieved, which is P_n.

Another appealing aggregation operator is the POWARAP operator, which unifies the probability and the OWARAP operator into a single formulation. Hence, it adds more information to the final outcome. With this operator, it is possible to overestimate or underestimate the probabilities based on the attitudinal character of the decision-maker. The POWARAP operator is defined as follows.

Definition 8. A POWARAP operator of dimension n is a mapping $\text{POWARAP}: R^n \times R^n \rightarrow R$ that has associated a weighting vector $W = (w_1, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{POWARAP}(\langle CPI_1, p_1\rangle, \dots, \langle CPI_n, p_n\rangle) = \sum_{j=1}^{n} \hat{v}_j P_j$$
$$= \beta \sum_{j=1}^{n} w_j P_j + (1-\beta) \sum_{i=1}^{n} v_i \left(\frac{100}{CPI_i}\right) p_i \,, \tag{13}$$

where P_j is the jth largest of the $\left(\frac{100}{CPI_i}\right) p_i$, p_i is referred as the nominal average retirement benefit variable, and CPI_i is the consumer price index variable. Each $\left(\frac{100}{CPI_i}\right) p_i$ has associated probability v_i with $\sum_{i=1}^{n} v_i = 1$ and $v_i \in [0,1]$, $\hat{v}_j = \beta w_j + (1-\beta) v_j$ with $\beta \in [0,1]$, and v_j is the probability v_i ordered according to P_j, that is, based on the jth largest of the $\left(\frac{100}{CPI_i}\right) p_i$.

Observe that with the parameter β, the decision-maker can represent the degree of importance that the OWARAP operator and the probability have in the aggregation process. For example, if the parameter β is equal to 1, the OWARAP operator is obtained, which means that the decision-maker does not consider probabilistic information. Conversely, if β is equal to 0, the expected value is gotten, meaning that full importance is given to the probability.

4. Forecasting the U.S. real average Social Security retirement benefit

Retirement income in the U.S. is based upon three pillars (Kintzel, 2017): Social Security, employer-sponsored plans, and personal savings. The following section focuses solely on the first one. However, it is worth to briefly review each of them in order to get a general idea of the U.S. retirement system.

The Old-Age, Survivors, and Disability Insurance (OASDI), or simply known as Social Security, is a program run by the federal government of the U.S., more specifically the Social Security Administration (SSA). It is financed through payroll taxes on employers, employees, and self-employed. Social Security provides different types of benefits, although the largest part is dedicated to the payment of retirement benefits to retired workers. Moreover, OASDI benefits are annually adjusted for inflation based on the CPI for urban wage earners and clerical workers (CPI-W) not seasonally adjusted (NSA) (SAA, 2021). This is known as cost-of-living adjustment (COLA).

We can distinguish between two types of employer-sponsored plans. On the one hand, there are defined benefit (DB) plans. On the other hand, there are defined contribution (DC) plans, where the most popular type is the 401(k). Furthermore, over the last decades, there has been a significant shift from DB to DC plans (Altman & Kingson, 2021; Rauh et al., 2020). However, DC plans are less secure than DB ones because the investment risk is placed on the individuals.

Additionally, workers can also individually save for their retirement. One common way of doing this is through an Individual Retirement Account (IRA), which can be simply described as an investment account with tax advantages.

In the following, an illustrative example of the explained approach is developed for forecasting the December 2025 real average Social Security retirement benefit paid to a retired worker in each state of the U.S. Moreover, a multi-expert analysis will be adopted to provide a more representative view of the problem. This numerical example is divided into five explanatory steps and the assessment of the final results.

Step 1. First, historical data regarding the number of Social Security beneficiaries and the amount of Social Security benefits paid to retired workers in the U.S. by state is collected. Afterward, the amount of benefits is divided by the number of beneficiaries in order to obtain the average benefit. These data were extracted from the SSA database; however, with the limitation that only annual data as value at end of period, that is, December, was available. Similarly, historical COLA data was retrieved from the same source.

Data about the CPI for all urban consumers (CPI-U) NSA with base period 1982-1984 was also gathered, but in this case, from the U.S. Bureau of Labor Statistics (BLS). Note that to calculate the average benefits in real terms, the CPI-U is used instead of the CPI-W. The CPI for the elderly (CPI-E) is not used either. The nominal and real average benefits for December 2021 (latest available data) can be seen in Table 3.32.

Step 2. Once all the data has been collected, three experts (e_1, e_2, e_3) are asked to provide their individual estimations of the COLA and CPI-U NSA development for the years 2023 to 2025. Table 3.33 shows this information.

Step 3. Then, the nominal average Social Security benefits for retired workers can be forecasted through the application of a simple linear regression with the COLA as the independent variable. For each state the coefficient of determination (R^2) is greater than 0.9, meaning that the model has a highly good fit. Three different forecast scenarios (S_1, S_2, S_3) are obtained based on the inputs provided by the experts (see Table 3.34).

Step 4. Next, the weighting vector W, inducing vector U, and probabilistic vector V are defined as follows: $W = (0.7, 0.2, 0.1)$, $U = (7, 9, 8)$, and $V =$

$(0.2, 0.5, 0.3)$. Note that subjective probabilities are considered. Likewise, the parameter β is determined as follows: $\beta = 0.5$.

Step 5. Lastly, the forecast scenarios calculated in Step 3 are aggregated into a single outcome. To do so, the OWARAP operator, the AOWARAP operator, the IOWARAP operator, the OWHARAP operator, the OWQARAP operator, and the POWARAP operator are used. Table 3.35 presents the final results.

Table 3.32. December 2021 nominal and real average Social Security benefits for retired worker (benefits in dollars)

	State	Total beneficiaries	Total benefits (thousands)	Avg. benefits (nominal)	Avg. benefits (real)
NR	Connecticut	532,298	975,916	1,833	658
	Maine	258,610	405,736	1,569	563
	Massachusetts	939,694	1,625,073	1,729	620
	New Hampshire	236,601	426,404	1,802	646
	New Jersey	1,244,222	2,283,490	1,835	658
	New York	2,684,406	4,583,696	1,708	612
	Pennsylvania	2,088,154	3,591,417	1,720	617
	Rhode Island	167,529	286,073	1,708	612
	Vermont	116,636	197,566	1,694	608
MR	Illinois	1,676,914	2,828,644	1,687	605
	Indiana	987,268	1,694,505	1,716	616
	Iowa	503,615	840,728	1,669	599
	Kansas	422,971	728,195	1,722	618
	Michigan	1,600,554	2,795,609	1,747	626
	Minnesota	829,789	1,444,723	1,741	624
	Missouri	936,580	1,525,269	1,629	584
	Nebraska	270,312	453,226	1,677	601
	North Dakota	105,753	170,049	1,608	577
	Ohio	1,689,343	2,740,915	1,622	582
	South Dakota	146,407	234,267	1,600	574
	Wisconsin	976,275	1,662,791	1,703	611
SR	Alabama	760,698	1,233,110	1,621	581
	Arkansas	468,117	732,407	1,565	561
	Delaware	175,408	317,406	1,810	649
	District of Columbia	60,292	99,154	1,645	590
	Florida	3,720,938	6,132,177	1,648	591
	Georgia	1,359,691	2,206,254	1,623	582
	Kentucky	647,855	1,019,674	1,574	565
	Louisiana	583,793	899,530	1,541	553
	Maryland	777,516	1,379,246	1,774	636
	Mississippi	446,981	688,320	1,540	552

	State	Total beneficiaries	Total benefits (thousands)	Avg. benefits (nominal)	Avg. benefits (real)
	North Carolina	1,611,146	2,674,043	1,660	595
	Oklahoma	561,018	908,245	1,619	581
	South Carolina	883,812	1,482,113	1,677	601
	Tennessee	1,044,660	1,717,632	1,644	590
	Texas	3,149,545	5,124,554	1,627	584
	Virginia	1,174,814	2,018,548	1,718	616
	West Virginia	302,162	487,226	1,612	578
WR	Alaska	81,718	130,347	1,595	572
	Arizona	1,105,267	1,874,684	1,696	608
	California	4,636,107	7,534,273	1,625	583
	Colorado	709,963	1,200,924	1,692	607
	Hawaii	230,841	382,062	1,655	594
	Idaho	282,455	461,090	1,632	586
	Montana	189,757	299,295	1,577	566
	Nevada	438,116	706,383	1,612	578
	New Mexico	326,068	510,837	1,567	562
	Oregon	695,077	1,156,068	1,663	597
	Utah	319,644	549,683	1,720	617
	Washington	1,068,554	1,872,199	1,752	628
	Wyoming	91,386	154,964	1,696	608

Abbreviations: NR = Northeast Region; MR = Midwest Region; SR = South Region; WR = West Region; Avg. = Average.

Table 3.33. COLA and CPI-U NSA determined by each expert

Indicator	e_1	e_2	e_3
COLA 2023	1.9	3.1	3.6
COLA 2024	1.8	2.0	2.5
COLA 2025	1.8	2.5	3.1
CPI-U NSA Dec-2025	315.729	322.914	328.287

Table 3.34. December 2025 scenario forecasts of the nominal and real average Social Security benefits for retired worker (values in dollars)

	State	Nominal values			Real values		
		S_1	S_2	S_3	S_1	S_2	S_3
NR	Connecticut	2,199	2,245	2,281	696.5	695.3	694.7
	Maine	1,888	1,927	1,958	597.9	596.9	596.4
	Massachusetts	2,085	2,128	2,162	660.3	659.1	658.6
	New Hampshire	2,185	2,230	2,266	691.9	690.7	690.2
	New Jersey	2,201	2,247	2,283	697.0	695.9	695.4
	New York	2,034	2,077	2,110	644.3	643.3	642.9
	Pennsylvania	2,062	2,106	2,139	653.2	652.1	651.6
	Rhode Island	2,056	2,099	2,132	651.2	650.0	649.5
	Vermont	2,041	2,084	2,117	646.4	645.2	644.7

	State	Nominal values			Real values		
		S_1	S_2	S_3	S_1	S_2	S_3
MR	Illinois	2,008	2,050	2,082	635.9	634.8	634.3
	Indiana	2,051	2,095	2,129	649.7	648.8	648.4
	Iowa	2,000	2,042	2,074	633.6	632.4	631.8
	Kansas	2,066	2,108	2,141	654.2	652.9	652.3
	Michigan	2,088	2,133	2,168	661.2	660.5	660.3
	Minnesota	2,109	2,153	2,188	667.9	666.9	666.5
	Missouri	1,951	1,992	2,024	617.8	616.8	616.5
	Nebraska	2,016	2,058	2,090	638.6	637.3	636.7
	North Dakota	1,934	1,973	2,003	612.5	611.0	610.2
	Ohio	1,930	1,971	2,003	611.2	610.4	610.1
	South Dakota	1,934	1,975	2,006	612.7	611.6	611.1
	Wisconsin	2,040	2,083	2,117	646.3	645.2	644.8
SR	Alabama	1,956	1,997	2,029	619.5	618.5	618.1
	Arkansas	1,883	1,923	1,954	596.5	595.6	595.2
	Delaware	2,184	2,230	2,266	691.7	690.6	690.2
	District of Columbia	2,029	2,071	2,103	642.8	641.4	640.7
	Florida	1,973	2,014	2,047	624.8	623.8	623.4
	Georgia	1,955	1,998	2,031	619.4	618.7	618.5
	Kentucky	1,891	1,931	1,962	598.8	597.9	597.6
	Louisiana	1,843	1,882	1,912	583.9	582.8	582.4
	Maryland	2,148	2,192	2,226	680.2	678.8	678.2
	Mississippi	1,857	1,896	1,927	588.1	587.2	586.8
	North Carolina	2,004	2,047	2,081	634.9	634.1	633.8
	Oklahoma	1,946	1,986	2,017	616.2	615.0	614.4
	South Carolina	2,029	2,072	2,105	642.6	641.6	641.2
	Tennessee	1,985	2,027	2,060	628.6	627.8	627.6
	Texas	1,952	1,993	2,024	618.3	617.1	616.6
	Virginia	2,084	2,128	2,162	660.1	659.0	658.5
	West Virginia	1,923	1,962	1,993	609.0	607.6	606.9
WR	Alaska	1,905	1,943	1,973	603.2	601.8	601.1
	Arizona	2,035	2,078	2,112	644.6	643.6	643.3
	California	1,936	1,977	2,009	613.2	612.3	611.9
	Colorado	2,043	2,086	2,120	647.1	646.1	645.7
	Hawaii	1,987	2,029	2,061	629.2	628.2	627.8
	Idaho	1,960	2,002	2,035	620.9	620.0	619.8
	Montana	1,885	1,924	1,955	596.9	595.9	595.5
	Nevada	1,920	1,962	1,993	608.2	607.4	607.2
	New Mexico	1,879	1,918	1,948	595.1	594.0	593.5
	Oregon	1,987	2,029	2,061	629.4	628.3	627.8
	Utah	2,069	2,112	2,145	655.3	654.0	653.3
	Washington	2,104	2,148	2,181	666.3	665.0	664.5
	Wyoming	2,037	2,079	2,111	645.2	643.7	643.0

Abbreviations: NR = Northeast Region; MR = Midwest Region; SR = South Region; WR = West Region.

Table 3.35. December 2025 aggregated results of the of the real average Social Security benefits for retired worker (values in dollars)

	State	OWA-RAP	AOWA-RAP	IOWA-RAP	OWHA-RAP	OWQA-RAP	POWA-RAP
NR	Connecticut	696.1	695.0	695.3	696.1	696.1	695.7
	Maine	597.6	596.7	596.9	597.6	597.6	597.3
	Massachusetts	659.9	658.8	659.1	659.9	659.9	659.5
	New Hampshire	691.5	690.5	690.7	691.5	691.5	691.1
	New Jersey	696.6	695.7	695.9	696.6	696.6	696.3
	New York	643.9	643.1	643.3	643.9	643.9	643.6
	Pennsylvania	652.8	651.9	652.1	652.8	652.8	652.5
	Rhode Island	650.8	649.8	650.0	650.8	650.8	650.5
	Vermont	646.0	645.0	645.2	646.0	646.0	645.6
MR	Illinois	635.5	634.5	634.8	635.5	635.5	635.2
	Indiana	649.4	648.6	648.8	649.4	649.4	649.1
	Iowa	633.2	632.1	632.4	633.2	633.2	632.8
	Kansas	653.8	652.6	652.9	653.8	653.8	653.4
	Michigan	661.0	660.4	660.5	661.0	661.0	660.8
	Minnesota	667.6	666.7	666.9	667.6	667.6	667.3
	Missouri	617.5	616.7	616.9	617.5	617.5	617.2
	Nebraska	638.1	637.0	637.3	638.1	638.1	637.8
	North Dakota	612.0	610.6	611.0	612.0	612.0	611.5
	Ohio	610.9	610.3	610.4	610.9	610.9	610.7
	South Dakota	612.3	611.4	611.6	612.3	612.3	612.0
	Wisconsin	645.9	645.0	645.2	645.9	645.9	645.6
SR	Alabama	619.1	618.3	618.5	619.1	619.1	618.9
	Arkansas	596.2	595.4	595.6	596.2	596.2	595.9
	Delaware	691.4	690.4	690.7	691.4	691.4	691.0
	District of Columbia	642.3	641.0	641.4	642.3	642.3	641.9
	Florida	624.5	623.6	623.8	624.5	624.5	624.2
	Georgia	619.1	618.6	618.7	619.1	619.1	619.0
	Kentucky	598.5	597.8	597.9	598.5	598.5	598.3
	Louisiana	583.5	582.6	582.8	583.5	583.5	583.2
	Maryland	679.7	678.5	678.8	679.7	679.7	679.3
	Mississippi	587.8	587.0	587.2	587.8	587.8	587.5
	North Carolina	634.6	634.0	634.1	634.6	634.6	634.4
	Oklahoma	615.8	614.7	615.0	615.8	615.8	615.4
	South Carolina	642.2	641.4	641.6	642.2	642.2	642.0
	Tennessee	628.3	627.7	627.8	628.3	628.3	628.1
	Texas	617.9	616.9	617.1	617.9	617.9	617.5
	Virginia	659.7	658.7	659.0	659.7	659.7	659.4
	West Virginia	608.5	607.3	607.6	608.5	608.5	608.1
WR	Alaska	602.7	601.4	601.8	602.7	602.7	602.3
	Arizona	644.2	643.5	643.6	644.2	644.2	644.0
	California	612.9	612.1	612.3	612.9	612.9	612.6
	Colorado	646.7	645.9	646.1	646.7	646.7	646.4
	Hawaii	628.9	628.0	628.2	628.9	628.9	628.6
	Idaho	620.6	619.9	620.1	620.6	620.6	620.3

	State	OWA-RAP	AOWA-RAP	IOWA-RAP	OWHA-RAP	OWQA-RAP	POWA-RAP
	Montana	596.6	595.7	595.9	596.6	596.6	596.3
	Nevada	608.0	607.3	607.5	608.0	608.0	607.7
	New Mexico	594.7	593.7	594.0	594.7	594.7	594.4
	Oregon	629.0	628.1	628.3	629.0	629.0	628.7
	Utah	654.8	653.7	654.0	654.8	654.8	654.4
	Washington	665.8	664.8	665.1	665.8	665.8	665.5
	Wyoming	644.7	643.3	643.7	644.6	644.7	644.2

Abbreviations: NR = Northeast Region; MR = Midwest Region; SR = South Region; WR = West Region.

Table 3.36. Comparison between nominal and real growth

	State	Nominal values			Real values		
		Dec-2021	Dec-2025 IOWA	Growth	Dec-2021	Dec-2025 IOWA-RAP	Growth
NR	Connecticut	1,833	2,248	23%	658	695	6%
	Maine	1,569	1,930	23%	563	597	6%
	Massachusetts	1,729	2,131	23%	620	659	6%
	New Hampshire	1,802	2,233	24%	646	691	7%
	New Jersey	1,835	2,250	23%	658	696	6%
	New York	1,708	2,080	22%	612	643	5%
	Pennsylvania	1,720	2,108	23%	617	652	6%
	Rhode Island	1,708	2,101	23%	612	650	6%
	Vermont	1,694	2,086	23%	608	645	6%
MR	Illinois	1,687	2,052	22%	605	635	5%
	Indiana	1,716	2,097	22%	616	649	5%
	Iowa	1,669	2,044	22%	599	632	6%
	Kansas	1,722	2,111	23%	618	653	6%
	Michigan	1,747	2,135	22%	626	661	5%
	Minnesota	1,741	2,156	24%	624	667	7%
	Missouri	1,629	1,994	22%	584	617	6%
	Nebraska	1,677	2,060	23%	601	637	6%
	North Dakota	1,608	1,975	23%	577	611	6%
	Ohio	1,622	1,973	22%	582	610	5%
	South Dakota	1,600	1,977	24%	574	612	7%
	Wisconsin	1,703	2,086	22%	611	645	6%
SR	Alabama	1,621	1,999	23%	581	619	6%
	Arkansas	1,565	1,925	23%	561	596	6%
	Delaware	1,810	2,233	23%	649	691	6%
	District of Columbia	1,645	2,073	26%	590	641	9%
	Florida	1,648	2,017	22%	591	624	6%
	Georgia	1,623	2,000	23%	582	619	6%
	Kentucky	1,574	1,933	23%	565	598	6%
	Louisiana	1,541	1,884	22%	553	583	5%
	Maryland	1,774	2,194	24%	636	679	7%
	Mississippi	1,540	1,898	23%	552	587	6%

	State	Nominal values			Real values		
		Dec-2021	Dec-2025 IOWA	Growth	Dec-2021	Dec-2025 IOWA-RAP	Growth
	North Carolina	1,660	2,050	24%	595	634	7%
	Oklahoma	1,619	1,988	23%	581	615	6%
	South Carolina	1,677	2,074	24%	601	642	7%
	Tennessee	1,644	2,030	23%	590	628	6%
	Texas	1,627	1,995	23%	584	617	6%
	Virginia	1,718	2,130	24%	616	659	7%
	West Virginia	1,612	1,964	22%	578	608	5%
WR	Alaska	1,595	1,945	22%	572	602	5%
	Arizona	1,696	2,081	23%	608	644	6%
	California	1,625	1,979	22%	583	612	5%
	Colorado	1,692	2,089	23%	607	646	6%
	Hawaii	1,655	2,031	23%	594	628	6%
	Idaho	1,632	2,004	23%	586	620	6%
	Montana	1,577	1,926	22%	566	596	5%
	Nevada	1,612	1,964	22%	578	607	5%
	New Mexico	1,567	1,920	23%	562	594	6%
	Oregon	1,663	2,031	22%	597	628	5%
	Utah	1,720	2,114	23%	617	654	6%
	Washington	1,752	2,150	23%	628	665	6%
	Wyoming	1,696	2,081	23%	608	644	6%

Abbreviations: NR = Northeast Region; MR = Midwest Region; SR = South Region; WR = West Region.

In Table 3.35, we can see that New Jersey, Connecticut, and New Hampshire are the three states of the U.S. with the highest results. By contrast, Louisiana, Mississippi, and New Mexico have obtained the lowest amounts. Moreover, the differences between these states are quite significant. For example, the gap between New Jersey and Louisiana is 113.1 dollars for the OWARAP operator.

Likewise, if we look at the results on a regional level, we observe that, on average, the Northeast Region has the highest estimated real average Social Security benefit for retired workers, compared to the South Region, which has the smallest one.

Furthermore, it is interesting to analyze the effect of the inflation adjustment on the outcomes. For example, by looking at Table 3.36, we can see that in North Carolina, the average Social Security benefit for retired workers in real prices (based on the IOWARAP operator) is expected to increase by 7%. However, if we conduct the same calculations without considering inflation, then it is estimated to increase by 24%. In this case, the real growth is not in line with the current growth, which translates to a considerable loss in the purchasing power of the future beneficiaries of this state. This demonstrates the importance of having information regarding retirement benefits in real prices.

5. Conclusions

The OWARAP operator is an aggregation operator used for calculating the future average retirement benefit adjusted for inflation. The OWARAP operator is based on the OWA operator. Thus, it provides a parametrized family of aggregation operators between the minimum and the maximum real average retirement benefit. The OWARAP operator can be extended by using order-inducing variables, generalized means, and also probabilities. In the first case, the IOWARAP operator is obtained; in the second case, the GOWARAP operator; and in the last case, the POWARAP operator.

This paper also develops a multi-expert analysis of the use of the OWARAP operator and its extensions in calculating the future average Social Security benefit adjusted for inflation of a retired worker in each state of the U.S. This analysis shows that with the use of these new operators, it is possible to underestimate or overestimate the results according to the attitudinal character of the analyst as well as its preferences. Furthermore, it demonstrates the importance of removing the effect of price inflation in order to obtain a true picture of the future average Social Security benefits for retired workers. By using the new approach, individuals can plan their retirement more properly and thereby maintain their standard of living.

In order to continue developing this idea, in future research, it is proposed to study further extensions of the OWARAP, IOWARAP, GOWARAP, and POWARAP operators. Also, apply these aggregation operators in other countries, such as France or Canada. Lastly, it is suggested to develop new algorithms for forecasting retirement indicators.

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3.5. Decision-making methods for retirement financial planning

The following research paper is a conference paper of the International Congress Getting Older: Challenges and Opportunities of an Unusual Demography, held on 19 and 20 November 2020 in Barcelona (Spain). The paper was published in the journal *Cuadernos del CIMBAGE*, indexed in Latindex, DOAJ, CLASE of the National Autonomous University of Mexico (UNAM), Erih Plus, Digital Repository of the University of Buenos Aires (UBA), Dialnet, Redalyc, Emerging Sources Citation Index (ESCI), Biblat, Redib, and EconBiz. It should be taken into account that the original version of the following research paper was written and published in Spanish.

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Abstract

The unfavorable development of the demographic and economic variables has a negative impact on the sustainability of the public pension system of Spain and consequently on the pension adequacy. Therefore, it encourages workers to invest in alternative savings products in order to supplement the state pension and thereby ensure an adequate retirement income. This study suggests different methods that may help citizens to choose the most suitable product for supplementing the state pension when they retire. These methods are based on the use of the linguistic ordered weighted averaging (LOWA) operator, the induced linguistic ordered weighted averaging (ILOWA) operator, the linguistic ordered weighted averaging distance (LOWAD) operator and the linguistic induced ordered weighted averaging distance (LIOWAD) operator. At the end of the work, an illustrative example will be developed using the LOWA, ILOWA, LOWAD and LIOWAD aggregation operators for retirement financial planning. The results show the usefulness of this type of linguistic aggregation operators in retirement decision-making.

Keywords: Decision-making, OWA operators, linguistic aggregation operators, Hamming distance, pension systems.

JEL Code: C44, D81, J32.

1. Introduction

The Spanish public pension system is becoming less and less sustainable (Hernández de Cos, 2021; Hernández de Cos et al., 2018). This is due to multiple factors. One of the most important factors is the demographic. In Spain, the dependency ratio (i.e., population aged 65 years and older as a proportion of the population aged 15 to 64) has strikingly increased over the time. According to the Eurostat (2021), the dependency ratio in Spain increased from 24.1% in 2009 to 29.7% in 2020.

The labor market situation is another factor that impacts on the sustainability of the public pension system. Indeed, the current economic crisis caused by the COVID-19 pandemic has entailed significant job losses. Specifically, when the unemployment rate increases, the Social Security's income used to pay pension benefits is reduced. According to the Instituto Nacional de Estadística (INE, 2021), the unemployment rate in Spain raised from 13.8% in the fourth quarter of 2019 to 16.1% in the fourth quarter of 2020.

In this context, the adequacy level of state pensions is not guaranteed. That is why many citizens seek to supplement their state pension with private savings products and thus avoid a significant loss in their purchasing power at the time of retirement. Currently there are a wide range of products to invest in for retirement, such as the systematic individual savings plan (known as PIAS) or the unit linked.

The aim of this work is to show alternative mathematical tools to improve decision-making processes with regard to the selection of retirement products for supplementing the state pension and thereby guarantee an adequate level of income during old age. In particular, these tools are based on the use of the ordered weighted averaging (OWA) operator (Yager, 1988) and the linguistic variables.

The OWA operator is one of the most popular aggregation functions for aggregating numerical information. Since its appearance, several authors have developed new extensions. Some prominent extensions are the induced ordered weighted averaging (IOWA) (Yager & Filev, 1999) and the linguistic ordered weighted averaging (LOWA) (Herrera et al., 1995). The IOWA operator is characterized by using order-inducing variables. The LOWA operator is very useful for those situations where the available information cannot be assessed with numerical values. Moreover, Xu (2006b) developed the induced linguistic ordered weighted averaging (ILOWA) operator, which extends the LOWA operator by using order-inducing variables in the reordering step of the linguistic arguments.

The authors Merigó and Gil-Lafuente (2010) developed a new approach for selecting financial products based on the use of the OWA operator and the ordered weighted averaging distance (OWAD) and the ordered weighted averaging adequacy coefficient (OWAAC) extensions. The OWAD operator uses the OWA operator in the Hamming distance (Hamming, 1950).

The linguistic ordered weighted averaging distance (LOWAD) operator was introduced in (Merigó & Casanovas, 2010) and it is an extension of the OWAD operator that uses linguistic variables. The LOWAD operator is very useful for those situations where the available information is uncertain and cannot be represented by numerical variables. Furthermore, if we use order-inducing variables in the reordering step, the linguistic induced ordered weighted averaging distance (LIOWAD) operator (Cheng & Zeng, 2012; Zeng et al., 2013) is obtained.

The current paper is structured as follows. First, the basic definitions of the OWA, LOWA, ILOWA, LOWAD, and LIOWAD operators are reviewed. Second, the steps of the proposed algorithm for the selection of savings products for supplementing the state retirement pension are described in detail. Third, an illustrative example of the application of the proposed algorithm is given. Finally, the main conclusions of the work are summarized.

2. Methodology

The following section reviews the main characteristics of the theory of the OWA, LOWA, ILOWA, LOWAD, and LIOWAD aggregation operators.

The OWA operator

The OWA operator was proposed by Yager (1988) and it provides a parameterized family of aggregation operators. The OWA operator has been successfully applied to various fields, including economics and business management (Kacprzyk et al., 2019). The OWA operator can be defined as follows.

Definition 1. An OWA operator of dimension n is a function $\mathrm{OWA}: R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\mathrm{OWA}(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j \, , \tag{1}$$

where b_j is the jth largest element of the arguments $a_1, a_2, \dots, a_n$, that is $(b_1, b_2, \dots, b_n)$ is $(a_1, a_2, \dots, a_n)$ reordered from largest to smallest.

Note that if the reordering step of the arguments is carried out in an ascending way, rather than in a descending one, the ascending ordered weighted averaging (AOWA) operator is obtained, which was introduced in (Yager, 1992). Moreover, the OWA and AOWA operators are related through $w_j = w^*_{n+1-j}$, where w_j is the jth weight of the OWA operator and w^*_{n+1-j} the jth weight of the AOWA operator.

Example 1. Consider the following collection of arguments: $a_1 = 6, a_2 = 4, a_3 = 9$. If the weighting vector is $W = (0.2, 0.3, 0.5)$, then, the OWA operator can be calculated as follows:

$$\mathrm{OWA} = 0.2 \times 9 + 0.3 \times 6 + 0.5 \times 4 = 5.6.$$

The OWA operator is monotonic, commutative (symmetrical), idempotent, and bounded. These properties can be expressed in the following manner:

- Monotonicity. For any OWA operator, if $a_i \geq \hat{a}_i$ for all i, then, $\mathrm{OWA}(a_1, a_2, \dots, a_n) \geq \mathrm{OWA}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)$.
- Commutativity (symmetry). In the sense that the same result is obtained for any permutation of the arguments. I.e., $\mathrm{OWA}(a_1, a_2, \dots, a_n) = \mathrm{OWA}(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)$, where $(\hat{a}_1, \hat{a}_2, \dots, \hat{a}_n)$ is any permutation of $(a_1, a_2, \dots, a_n)$.
- Boundedness. In the sense that the OWA operator is delimited between the maximum and the minimum. I.e., $\mathrm{Min}\{a_i\} \leq \mathrm{OWA}(a_1, a_2, \dots, a_n) \leq \mathrm{Max}\{a_i\}$.
- Idempotency. For any OWA operator, if $a_i = a$ for all i, then, $\mathrm{OWA}(a_1, a_2, \dots, a_n) = a$.

The OWA operator includes the classical decision-making methods. When $w_1 = 1$ and $w_j = 0$ for all $j \neq 1$, the optimistic criterion is found. When $w_n = 1$ and $w_j = 0$ for all $j \neq n$, the pessimistic criterion is found. The Laplace criterion, which is based on the arithmetic mean, is found when $w_j = 1/n$ for all j. Finally, the Hurwicz criterion is obtained when $w_1 = \alpha$, $w_n = 1 - \alpha$, and $w_j = 0$ for all $j \neq 1, n$.

Another aspect worth discussing is the measures for characterizing the weighting vector W and the type of aggregation it performs (Yager, 1988, 1996, 2002). The most significant are the attitudinal character, the dispersion measure, the balance operator, and the divergence measure.

The first measure mentioned above was introduced in (Yager, 1988) and it refers to the attitudinal character of the decision-maker. This measure can be defined as follows:

$$\alpha(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1}\right). \tag{2}$$

The second measure is the dispersion or entropy of the vector W (Yager, 1988) and it is defined as follows:

$$\mathrm{H}(W) = -\sum_{j=1}^{n} w_j \ln(w_j). \tag{3}$$

The balance operator is the third measure, and it was introduced in (Yager, 1996). The balance operator is used for measuring the degree of favoritism towards optimistic or pessimistic values, and it is defined as:

$$\mathrm{Bal}(W) = \sum_{j=1}^{n} w_j \left(\frac{n+1-2j}{n-1}\right). \tag{4}$$

The fourth measure was introduced in (Yager, 2002) and it refers to the degree of divergence of the vector W. This measure can be defined in the following way:

$$\mathrm{Div}(W) = \sum_{j=1}^{n} w_j \left(\frac{n-j}{n-1} - \alpha(W)\right)^2. \tag{5}$$

The LOWA operator

The first linguistic version of the OWA operator was presented by Herrera et al. (1995). Since then, several authors have developed new models (Herrera & Herrera-Viedma, 1997; Martínez & Herrera, 2000; Merigó et al., 2012; Merigó & Gil-Lafuente, 2008; Xu, 2004a, 2004b, 2006a, 2006b). The present work focuses on the one from Xu (2004a, 2004b). The LOWA operator is an extension of the OWA operator that uses linguistic variables for assessing the information. This operator is also known as extended ordered weighted averaging (EOWA). The LOWA operator can be defined in the following manner.

Definition 2. A LOWA operator of dimension n is a function LOWA: $S^n \rightarrow S$ that has associated a weighting vector W of dimension n $W = (w_1, w_2, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, such that:

$$\text{LOWA}(s_{\alpha_1}, s_{\alpha_2}, \ldots, s_{\alpha_n}) = \sum_{j=1}^{n} w_j s_{\beta_j}, \tag{6}$$

where s_{β_j} is the jth largest element of the s_{α_i}.

Example 2. Consider the following linguistic term set with three labels: $s_1 = \text{bad}, s_2 = \text{medium}, s_3 = \text{good}$. If the collection of linguistic arguments is $S = (s_2, s_2, s_3)$ and the weighting vector $W = (0.2, 0.3, 0.5)$, then, the LOWA operator can be calculated as follows:

$$\text{LOWA} = 0.2 \times s_3 + 0.3 \times s_2 + 0.5 \times s_2 = s_{2.2}.$$

The monotonic, commutative, bounded, and idempotent properties are also applicable to the LOWA operator.

The ILOWA operator

An interesting extension of the LOWA operator is the ILOWA operator (Xu, 2006b). The main difference between the ILOWA operator and the LOWA operator is that the reordering of the s_{α_i} is carried out through the so-called order-inducing variables u_i. Thus, the reordering of the arguments does not depend on their values.

Definition 3. An ILOWA operator of dimension n is a function ILOWA: $R^n \times S^n \rightarrow S$ that has associated a weighting vector W of dimension n $W = (w_1, w_2, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{ILOWA}(\langle u_1, s_{\alpha_1}\rangle, \langle u_2, s_{\alpha_2}\rangle, \ldots, \langle u_n, s_{\alpha_n}\rangle) = \sum_{j=1}^{n} w_j s_{\beta_j}, \tag{7}$$

where s_{β_j} is the s_{α_i} value of the ILOWA pair $\langle u_i, s_{\alpha_i}\rangle$ having the jth largest u_i, u_i is the order-inducing variable, and s_{α_i} is the linguistic argument variable.

Example 3. Consider the same linguistic term set as in the Example 2. Also, consider the following collection of linguistic arguments with their respective order-inducing variables $\langle u_i, s_{\alpha_i}\rangle$: $\langle 4, s_2\rangle$, $\langle 9, s_2\rangle$, and $\langle 6, s_3\rangle$. If the weighting vector is $W = (0.2{,}0.3{,}0.5)$, then, the ILOWA operator can be calculated as:

$$\text{ILOWA} = 0.2 \times s_2 + 0.3 \times s_3 + 0.5 \times s_2 = s_{2.3}.$$

Like the LOWA operator, the ILOWA operator is monotonic, commutative, bounded, and idempotent.

The Hamming distance

The Hamming distance d_H (Hamming, 1950) is a very popular technique used for calculating the differences between two elements, two sets, or two fuzzy sets, among others.

Definition 4. Consider two sets $X = \{x_1, x_2, \dots, x_n\}$ and $Y = \{y_1, y_2, \dots, y_n\}$. Then, in the discrete scope, the Hamming distance of dimension n is defined as follows:

$$d_H(X, Y) = \sum_{j=1}^{n} \left| x_j - y_j \right| . \tag{8}$$

Furthermore, if weights are used for aggregating the differences, the weighted Hamming distance d_{WH} is obtained, which is defined as follows.

Definition 5. A weighted Hamming distance of dimension n is a function $d_{WH}: R^n \times R^n \rightarrow R$ that has associated a weighting vector W of dimension n $W = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$d_{WH}(X,Y) = \sum_{j=1}^{n} w_j \left| x_j - y_j \right| , \tag{9}$$

where x_j and y_j are the jth arguments of the sets X and Y, respectively.

Note that if $w_j = 1/n$ for all j, the normalized Hamming distance d_{NH} is obtained.

The LOWAD operator

The LOWAD operator (Merigó & Casanovas, 2010) is an aggregation operator that uses linguistic variables in the Hamming distance. The main advantage of this operator is that it provides a more complete view of the decision-making problem. For two sets $X = \{s_{x_1}, s_{x_2}, \dots, s_{x_n}\}$ and $Y = \{s_{y_1}, s_{y_2}, \dots, s_{y_n}\}$, the LOWAD operator can be defined as follows.

Definition 6. A LOWAD operator of dimension n is a function $\mathrm{LOWAD}: S^n \times S^n \rightarrow S$ that has associated a weighting vector W of dimension n $W = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\mathrm{LOWAD}(X,Y) = \sum_{j=1}^{n} w_j s_{\beta_j} , \tag{10}$$

where s_{β_j} is the jth largest of the $\left| s_{x_i} - s_{y_i} \right|$, and $\left| s_{x_i} - s_{y_i} \right|$ is the argument variable represented in the form of an individual linguistic distance.

Note that it is also possible to distinguish between descending and ascending orders of the linguistic arguments. The first case corresponds to the LOWAD operator, and the second one to the ascending linguistic ordered weighted averaging distance (ALOWAD) operator. Moreover, the weighting vectors of the LOWAD and ALOWAD operators are symmetric to each other. Specifically, the weights of these operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the LOWAD operator and w^*_{n-j+1} is the jth weight of the ALOWAD operator.

The LOWAD operator is monotonic, commutative, bounded, idempotent, reflexive, and nonnegative. These properties can be explained as follows:

- Monotonicity. For any LOWAD operator, if $\left|s_{x_i} - s_{y_i}\right| \geq \left|s_{z_i} - s_{g_i}\right|$ for all i, then, $\text{LOWAD}(X, Y) \geq \text{LOWAD}(Z, G)$.
- Commutativity (symmetry). In the sense that the same result is obtained for any permutation of the arguments. I.e., $\text{LOWAD}(X, Y) = \text{LOWAD}(Z, G)$, where (Z, G) is any permutation of (X, Y).
- Boundedness. In the sense that the LOWAD operator is delimited between the maximum and minimum. I.e., $\text{Min}\{\left|s_{x_i} - s_{y_i}\right|\} \leq \text{LOWAD}(X, Y) \leq \text{Max}\{\left|s_{x_i} - s_{y_i}\right|\}$.
- Idempotency. For any LOWAD operator, if $\left|s_{x_i} - s_{y_i}\right| = s_\alpha$ for all i, then, $\text{LOWAD}(X, Y) = s_\alpha$.
- Reflexivity. The LOWAD operator is reflexive because $\text{LOWAD}(X, X) = s_0$.
- Nonnegativity. The LOWAD operator is nonnegative because $\text{LOWAD}(X, Y) \geq s_0$.

The LIOWAD operator

The LIOWAD operator (Cheng & Zeng, 2012; Zeng et al., 2013) is a distance operator similar to the LOWAD but with the difference that the reordering step is carried out with order-inducing variables. For two sets $X = \{s_{x_1}, s_{x_2}, \dots, s_{x_n}\}$ and $Y = \{s_{y_1}, s_{y_2}, \dots, s_{y_n}\}$, the LIOWAD operator can be defined as follows.

Definition 7. A LIOWAD operator of dimension n is a function LIOWAD: $R^n \times S^n \times S^n \rightarrow S$ that has associated a weighting vector W of dimension n $W = (w_1, w_2, \dots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{LIOWAD}\left(\langle u_1, s_{x_1}, s_{y_1}\rangle, \dots, \langle u_n, s_{x_n}, s_{y_n}\rangle\right) = \sum_{j=1}^{n} w_j s_{\beta_j}, \tag{11}$$

where s_{β_j} is the $\left|s_{x_i} - s_{y_i}\right|$ value of the LIOWAD triplet $\langle u_i, s_{x_i}, s_{y_i}\rangle$ with the jth largest u_i, u_i is the order-inducing variable, and $\left|s_{x_i} - s_{y_i}\right|$ is the argument variable represented in the form of an individual linguistic distance.

Example 4. Given the same the same linguistic term set as in the Example 2, consider the following two collection of linguistic arguments: $X = \{s_{x_1}, s_{x_2}, s_{x_3}\} = \{s_2, s_2, s_3\}$ and $Y = \{s_{y_1}, s_{y_2}, s_{y_3}\} = \{s_3, s_1, s_1\}$. If the vector with the order-inducing variables is $U = (4,9,6)$, and the weighting vector is $W = (0.2,0.3,0.5)$, then, the LIOWAD operator can be calculated as follows:

$$\text{LIOWAD} = 0.2 \times |s_2 - s_1| + 0.3 \times |s_3 - s_1| + 0.5 \times |s_2 - s_3| = s_{1.3}.$$

Similar to the LOWAD operator, the LIOWAD operator is monotonic, commutative, bounded, idempotent, reflexive, and nonnegative.

3. Proposed algorithm

The following section explains in detail the steps to follow in order to select the most appropriate product for supplementing the state pension for retirement by using the previously defined linguistic aggregation operators.

LOWA and ILOWA algorithm

The first option is based on applying the LOWA operator and the ILOWA operator.

Step 1. Determine the different possible alternatives A_k for supplementing the public old-age pension. Through this, the set $A = \{A_1, A_2, \dots, A_m\}$ is obtained.

Step 2. The expert has to determine the factors, singularities, or characteristics $C = \{C_1, C_2, \dots, C_n\}$ to be considered in the analysis and assessment. The expert should consider different characteristics in order to know which supplementary product is the most convenient, such as the age and the risk that the decision-maker is willing to take.

Step 3. The expert needs to define the set of linguistic labels S, and then individually assess the characteristics for each alternative.

Step 4. The expert has to establish the values of the weighting vector $W = (w_1, w_2, \dots, w_n)$ as well as the inducing-variables vector $U = (u_1, u_2, \dots, u_n)$.

Step 5. Carry out the aggregation of the assessment results obtained in Step 3. To do so, the LOWA operator and the ILOWA operator are used.

LOWAD and LIOWAD algorithm

The second option consists of using the Hamming distance through the LOWAD operator and the LIOWAD operator.

Step 1. Determine the different possible alternatives A_k for supplementing the public old-age pension. Through this, the set $A = \{A_1, A_2, \dots, A_m\}$ is obtained.

Step 2. The expert has to determine the factors, singularities, or characteristics $C = \{C_1, C_2, \dots, C_n\}$ to be considered in the analysis and assessment. The expert should consider different characteristics in order to know which supplementary product is the most convenient, such as the age and the risk that the decision-maker is willing to take.

Step 3. The expert needs to define the set of linguistic labels S, and then individually assess the characteristics for each alternative.

Step 4. The expert needs to define the ideal set $I = \{s_{I_1}, s_{I_2}, \dots, s_{I_n}\}$. Also, the expert has to calculate the Hamming distances between the ideal set and the different alternatives A_k considered.

Step 5. The expert has to establish the values of the weighting vector $W = (w_1, w_2, \dots, w_n)$ as well as the inducing-variables vector $U = (u_1, u_2, \dots, u_n)$.

Step 6. Carry out the aggregation of the distances obtained in Step 4. To do so, the LOWAD operator and the LIOWAD operator are used.

4. Illustrative example

The following section develops an illustrative example in a decision-making problem concerning the selection of retirement savings instruments through the utilization of the LOWA, ILOWA, LOWAD, and LIOWAD linguistic aggregation operators.

Suppose that a 38-year-old person and resident in Spain, contacts with a financial expert for advice on which savings product for supplementing the public retirement pension is best to invest in.

Implementation of the LOWA and ILOWA algorithm

Step 1. Suppose that the expert considers the following alternatives:

- A_1 = individual pension plan (also known as PPI).
- A_2 = insured pension plan (also known as PPA).
- A_3 = investment fund.
- A_4 = systematic individual savings plan (also known as PIAS).
- A_5 = unit linked.
- A_6 = reverse mortgage.

Step 2. Also, the expert considers the following characteristics as key for the analysis:

- C_1 = profile.
- C_2 = tax advantages.
- C_3 = liquidity.
- C_4 = commissions.
- C_5 = contribution limit.

Step 3. Additionally, assume that the following set with three linguistic terms is defined: $S = \{s_1 = \text{bad}, s_2 = \text{regular}, s_3 = \text{good}\}$. Then, suppose that the expert individually evaluates the products for each of the above-mentioned characteristics C_i, thus obtaining the results shown in Table 3.37.

For example, individual pension plans and insured pension plans allow to reduce the taxable base of the personal income tax. However, both are also characterized for having very low liquidity. Furthermore, since 2021 contributions are subject to a limit of 2,000 euros per annum.

Table 3.37. Initial assessments matrix

	C_1	C_2	C_3	C_4	C_5
A_1	s_3	s_3	s_1	s_2	s_1
A_2	s_2	s_3	s_1	s_2	s_1
A_3	s_3	s_1	s_3	s_2	s_3
A_4	s_3	s_3	s_2	s_2	s_2
A_5	s_3	s_1	s_3	s_2	s_3
A_6	s_1	s_3	s_1	s_1	s_3

Step 4. Consider that the expert decides to use the following weighting vector for the aggregation: $W = (w_1 = 0.3, w_2 = 0.3, w_3 = 0.2, w_4 = 0.1, w_5 = 0.1)$. For the ILOWA operator, the expert decides to use the following order-inducing variables vector: $U = (u_1 = 9, u_2 = 6, u_3 = 7, u_4 = 5, u_5 = 8)$.

Step 5. Finally, assume that the expert uses the LOWA and ILOWA operators in order to aggregate the linguistic assessments and thereby obtain a single representative value for each alternative. In order to obtain a more complete picture of the situation, the expert also calculates the linguistic weighted averaging (LWA) operator. As can be deduced from the name, the LWA operator is obtained by applying the weighted average. The aggregated results are shown in Table 3.38.

Table 3.38. LWA, LOWA, and ILOWA aggregated results

	LWA	LOWA	ILOWA
A_1	$s_{2.3}$	$s_{2.4}$	$s_{1.9}$
A_2	s_2	$s_{2.1}$	$s_{1.6}$
A_3	$s_{2.3}$	$s_{2.7}$	$s_{2.7}$
A_4	$s_{2.6}$	$s_{2.6}$	$s_{2.4}$
A_5	$s_{2.3}$	$s_{2.7}$	$s_{2.7}$
A_6	$s_{1.8}$	$s_{2.2}$	$s_{1.8}$

Implementation of the LOWAD and LIOWAD algorithm

Step 1. Consider the same set of alternatives as in the LOWA and ILOWA algorithm illustrative example.

Step 2. Consider the same set of characteristics as in the LOWA and ILOWA algorithm illustrative example.

Step 3. Consider the same set of linguistic terms and individual assessments as in the LOWA and ILOWA algorithm illustrative example.

Step 4. Suppose that the expert defines the ideal product $I = \{s_3, s_3, s_3, s_3, s_3\}$ in order to be able to calculate the Hamming distances. Table 3.39 shows the results of the distances between the ideal product and the different alternatives considered.

Table 3.39. Hamming distances matrix

	C_1	C_2	C_3	C_4	C_5
$d_H(I, A_1)$	s_0	s_0	s_2	s_1	s_2
$d_H(I, A_2)$	s_1	s_0	s_2	s_1	s_2
$d_H(I, A_3)$	s_0	s_2	s_0	s_1	s_0
$d_H(I, A_4)$	s_0	s_0	s_1	s_1	s_1
$d_H(I, A_5)$	s_0	s_2	s_0	s_1	s_0
$d_H(I, A_6)$	s_2	s_0	s_2	s_2	s_0

Step 5. Consider that the expert decides to use the same weighting vector and order-inducing variables as in the LOWA and ILOWA algorithm illustrative example.

Step 6. Lastly, assume that the expert employs the ALOWAD, LOWAD, and LIOWAD operators in order to aggregate the distances and thereby produce a single representative value for each alternative. The aggregated results are shown in Table 3.40.

Table 3.40. ALOWAD, LOWAD, and LIOWAD aggregated results

	ALOWAD	LOWAD	LIOWAD
A_1	$s_{0.6}$	$s_{1.4}$	$s_{1.1}$
A_2	$s_{0.9}$	$s_{1.5}$	$s_{1.4}$
A_3	$s_{0.3}$	$s_{0.9}$	$s_{0.3}$
A_4	$s_{0.4}$	$s_{0.8}$	$s_{0.6}$
A_5	$s_{0.3}$	$s_{0.9}$	$s_{0.3}$
A_6	$s_{0.8}$	$s_{1.6}$	$s_{1.2}$

Table 3.41 displays the order of preference for the different alternatives according to the linguistic aggregation operator that has been used. Note that for the LWA, LOWA, and ILOWA operators it is preferable to obtain a high result; by contrast, for the distance operators a low result is much preferable. In this table, we can see that the least attractive options for the customer are signing up for an insured pension plan or a reverse mortgage. Instead, the most interesting and preferred alternatives for the customer are buying an investment fund, a unit linked, or a systematic individual savings plan.

Table 3.41. Ordering of the results obtained with the operators

Operator	Ordering
LWA	$A_4 \succ A_1 = A_3 = A_5 \succ A_2 \succ A_6$
LOWA	$A_3 = A_5 \succ A_4 \succ A_1 \succ A_6 \succ A_2$
ILOWA	$A_3 = A_5 \succ A_4 \succ A_1 \succ A_6 \succ A_2$
ALOWAD	$A_3 = A_5 \succ A_4 \succ A_1 \succ A_6 \succ A_2$
LOWAD	$A_4 \succ A_3 = A_5 \succ A_1 \succ A_2 \succ A_6$
LIOWAD	$A_3 = A_5 \succ A_4 \succ A_1 \succ A_6 \succ A_2$

5. Conclusions

In this paper the use of the LOWA, ILOWA, LOWAD, and LIOWAD operators for the optimization of decision-making regarding pensions has been studied. The main advantage of these operators is that they allow to aggregate linguistic information taking into account the attitudinal character of the decision-maker. Furthermore, linguistic aggregation operators can be very useful in those situations where the available information is uncertain and cannot be evaluated with exact numerical values.

An illustrative example has also been developed by using linguistic aggregation operators to select the most convenient savings product for supplementing the state retirement pension of a given individual. The results demonstrate the usefulness of the LOWA and ILOWA algorithm as well as the LOWAD and LIOWAD algorithm, since they allow the decision-maker to aggregate heterogeneous and ambiguous information and according to his/her degree of optimism and pessimism. Specifically, the application of these algorithms enhances the process of decision-making related to saving for retirement and thereby avoid a reduction in the purchasing power at the time of retirement.

As future research lines, other types of aggregation functions should be used in the field of supplementary pensions, such as the linguistic generalized ordered weighted averaging (LGOWA) operator (Merigó & Gil-Lafuente, 2008) and its extensions.

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4. Conclusions

4.1. Main conclusions

All aspects of life revolve around making decisions. From the very basic, such as choosing each morning what clothes to wear, to the most transcendental, as might be deciding what bachelor's degree to study. Also, to a greater or lesser extent, all decisions are subject to different types of uncertainties. In recent decades, significant theoretical advances have taken place in the field of decision-making under uncertainty. Moreover, a large variety of applications. However, there are still outstanding literature gaps. In the present doctoral thesis, some of these missing pieces have been successfully addressed.

Each of the five objectives set at the beginning of this doctoral thesis has been successfully achieved.

The first objective, i.e., to develop the state of the art of OWA aggregation operators through a bibliometric analysis, was accomplished through an extensive analysis of the scientific production and impact regarding the OWA operator during the last 34 years (from 1988 to 2021). The data was obtained through the WoS Core Collection database. Additionally, the VOS viewer software was used to build bibliometric networks. The key conclusions drawn are the following:

- The most cited document in OWA is "On ordered weighted averaging aggregation operators in multicriteria decisionmaking", written by Yager and published in 1988 by the *IEEE Transactions on Systems, Man, and Cybernetics* scientific journal.
- Yager is the most productive and influential author in OWA.
- The Iona College, located in the United States of America, is the leading institution in OWA production. Noteworthy is the high number of Chinese institutions in the top 50 most productive institutions in OWA.
- China is, by far, the country that has the largest number of published and cited documents in OWA.
- The most productive journal in OWA is the *International Journal of Intelligent Systems*, currently a Wiley-Hindawi journal. In particular,

this journal has managed to publish more than 200 documents related to OWA.
- Computer science is the leading research area in OWA.

A review of the mathematical theories used for decision-making in uncertain environments was the second objective of this thesis work. The concept of decision-making, the theory of confidence intervals, the theory of fuzzy sets (including classical sets, fuzzy sets, fuzzy arithmetic, linguistic variables, and intuitionistic fuzzy sets), the recent idea of BUI, the similarity measures, and above all, the aggregation operators were looked. This served as a basis for all the investigations carried out in this doctoral thesis.

Also, in a complementary manner, a study of the current knowledge about pensions was provided. On the one hand, an overview of the main theoretical concepts was undertaken. On the other hand, a bibliometric analysis for the period 1990-2022 was conducted using the WoS Core Collection data source.

The third objective set was to analyze new extensions of the OWA operator. This objective was fulfilled with the presentation of the UOWAAC operator and the LOWAAC operator. The first one is an extension of the OWA operator that uses the adequacy coefficient with interval numbers. The second one is an extension of the OWA operator that uses the adequacy coefficient, but this time with linguistic variables. These novel operators address gaps in the field of decision-making under uncertainty. In particular, when the decision-maker wants to calculate the differences between a set of alternatives and an ideal while considering a threshold from which the results are always the same and where the available information cannot be assessed with exact numerical values. With some illustrative and comparative examples, the usefulness and superiority of the UOWAAC and LOWAAC operators are properly demonstrated. Additionally, the UOWAAC is extended by employing order-inducing variables. Likewise, the LOWAAC is extended by integrating order-inducing variables, generalized means, and quasi-arithmetic means.

The fourth objective was to analyze new applications of the OWA operator and its extensions in the field of pensions, which was also achieved. Pension plans are experiencing significant demographic and economic pressures.

Within this context, it is extremely important to make sound decisions. Therefore, three different works have been conducted. Based on the application of the LOWA, ILOWA, LOWAD, and LIOWAD aggregation operators, the first study consisted of designing a new algorithm for selecting the optimal savings product for supplementing the public pension among a pool of potential alternatives. Another study utilized the OWA operator and some of its extensions with an inflation adjustment mechanism for forecasting Spanish pensions in real terms. Similarly, a third work also used OWA operators and some of its extensions while considering the effect of inflation, but in this case, to forecast U.S. pensions. The results of all these investigations show that the OWA operator is an effective tool for pension decision-making processes. Accordingly, this compendium of studies seeks to impact the lives of current and future retirees positively.

If we look at the fifth and last objective, i.e., make scientific contributions through international publications, it can be concluded that it has been completed. Six research articles have been written, of which three were already published in scientific journals or book series indexed in the WoS database. "The Uncertain ordered weighted averaging adequacy coefficient operator" has been published in the *International Journal of Approximate Reasoning*, which to date is classified as a Q1 journal. The "OWA operators in pensions" work has been published in the well-known book series *Studies in Computational Intelligence*. The "Decision-making methods for retirement financial planning" has been published in the emerging journal *Cuadernos del CIMBAGE*. At the time of writing this doctoral thesis, two of the three remaining research articles were sent to leading journals and are under review. The "A bibliometric analysis of the OWA operator from 1988 to 2021" study has been sent to the reputed *Fuzzy Sets and Systems* journal. The "Forecasting retirement benefits in the United States using OWA operators" work has been submitted to the *Technological and Economic Development of Economy* journal, which to date is categorized as a Q1 journal. Furthermore, the "The linguistic OWA adequacy coefficient operator and its application to decision-making" work will be submitted to the *Expert Systems with Applications* journal, which to date is classified as a Q1 journal. Moreover, with the publication of this doctoral thesis in the digital repository of the University of Barcelona.

Also, one additional research contribution has been made, called “Risk assessment: An approach based on basic uncertain information”. This contribution was presented at the II International Congress on Accounting and Business Research and published as a book chapter. The major objective of this work is to provide a tool that allows considering the amount of certainty of the risk assessments provided by an expert or group of experts when comparing a range of identified risks within a given organization. To this end, the use of BUI together with the UOWA operator is proposed. The results have direct implications on the risk prioritization outcome and, consequently, in the achievement of an organization’s objectives.

4.2. Future research lines

Bearing in mind the limitations of the OWA bibliometric study, a first recommendation for future work is to perform a complete bibliometric analysis of the OWA operator using not only WoS but also Elsevier's Scopus, Google Scholar, Microsoft Academic, and Dimensions, among others. Furthermore, conducting comparative work between all these data sources would be very interesting to reveal the strengths and weaknesses. Likewise, consider other types of documents, such as proceeding papers.

Moreover, to gain additional empirical evidence, future research lines should replicate for other countries the analysis conducted in the research papers "OWA operators in pensions" and "Forecasting public pensions in the United States using OWA operators". Some of these countries could be Canada or France.

Similarly, it is necessary to investigate the use of the OWA operator in other retirement-related indices, such as the old-age dependency ratio or the pension replacement ratio. In this regard, there are opportunities for future studies.

Additionally, future research could examine new applications of the UOWAAC and LOWAAC operators, such as insurance pricing or equipment acquisition for businesses. Also, future research could address the development of additional extensions of the LOWAAC operator, e.g., with BUI, and the UOWAAC operator, e.g., with quasi-arithmetic means.

As verified in this doctoral thesis, there are many tools for decision-making under uncertainty and with a vast range of potential applications. Also, exploring new tools is workable.

5. Annexes

In this section, other research contributions are provided.

5.1. Risk assessment: An approach based on basic uncertain information

The following research article was presented at the II International Congress on Accounting and Business Research, which was held in Barcelona (Spain) from 13 to 14 October 2022. The article was published in a book titled Economia, Empresa, Contabilidad y Sociedad (volume 3), whit ISIN 978-84-19282-57-6.

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Abstract

An appropriate risk management is necessary more than ever to assure the accomplishment of the objectives of any organization. This study proposes a novel risk assessment and prioritization approach based on the use of basic uncertain information (BUI). This approach allows to consider the reliability of the risk assessments provided by an expert or group of experts when comparing a range of identified risks. Furthermore, the use of the uncertain ordered weighted averaging (UOWA) operator is considered. The study ends with an illustrative example. The results show the possibility of use of BUI and the UOWA operator to assess and prioritize risks in a more complete way.

Keywords: Basic uncertain information (BUI), certainty, enterprise risk management (ERM), risk assessment, uncertain ordered weighted averaging (UOWA) operator.

1. Research purpose

Many organizations are facing an increasingly uncertain environment. For example, the arrival of the coronavirus pandemic (COVID-19) has forced many companies to a digital transformation and as a consequence new types of threatening risks have quickly emerged, but also new opportunities. In this context, enterprise risk management (ERM) has become more important.

Risk assessment is a key process within ERM, which consists in assessing the severity of each identified risk. Normally, two measures are used to assess the severity of a risk, which are likelihood and impact (Hunziker, 2019). Likelihood can be defined as the possibility or probability of a risk event occurring. Impact refers to the effects of this risk event occurring. Nonetheless, it is possible to consider additional measures when assessing a risk, for example velocity, also referred as speed of onset. Moreover, it is possible to broke down one measure into different dimensions. For example, impact can be disaggregated into financial impact and reputational impact.

Furthermore, some types of risk are assessed based solely on the judgement of an expert or group of experts. This is usually the case when it is difficult to obtain data. In this context, the expert may exhibit only some amount of certainty in his/her judgments. The purpose of this study is to reflect the certainty degree given by the expert in their risk assessments. To achieve this, basic uncertain information (BUI) (Jin et al., 2018; Mesiar et al., 2018) is employed. This allows to improve the process of comparison and prioritization of risks, which is key to implement optimal risk responses.

Likewise, the uncertain ordered weighted averaging (UOWA) operator (Z. S. Xu & Da, 2002) is used to aggregate the different assessments, as it offers many advantages. For example, it allows to consider the attitudinal character of the decision-maker when the aggregation is carried out. Also, it includes all the classical decision criteria, which are the optimistic criteria, the pessimistic criteria, the Laplace criteria, and the Hurwicz criteria.

The rest of this paper is organized as follows. Section 2 provides a briefly literature review. Section 3 explains the applied methodology. Section 4 presents an illustrative example of the new risk assessment approach. Finally, the main conclusions of the study are summarized in Section 5.

2. State of art

The next section reviews the main existing literature of ERM, BUI, and the UOWA operator, respectively.

ERM has emerged in the early 1990s as a concept and as a management function within organizations (Dickinson, 2001). An effective ERM implementation may reduce risk exposure and improve performance (Florio & Leoni, 2017; Shad et al., 2019). At the present, the Committee of Sponsoring Organizations of the Treadway Commission (COSO, 2017) and the International Organization for Standardization (ISO, 2018) provide two of the most popular ERM frameworks implemented by organizations (Hunziker, 2019).

BUI is a recently introduced concept that can handle different types of uncertainties. Since its introduction, BUI has been studied and applied by several authors (Chen et al., 2022; Figuerola-Wischke et al., 2022; Jin et al., 2020, 2021, 2022; Li et al., 2022; Tao et al., 2020; Y. Xu et al., 2022; Yang et al., 2020). However, it has not yet been used to assess risks within ERM.

The UOWA operator is an extension of the OWA operator (Yager, 1988) that uses interval numbers instead of singletons. OWA operators are one of the most popular methods for aggregating data (Blanco-Mesa et al., 2019; Shu, 2022). In the literature, there are a few studies that apply the OWA operator and its extensions to ERM (Blanco-Mesa et al., 2018; Tian et al., 2018).

3. Methodology

This section encompasses the detailed definitions of BUI and the UOWA operator.

BUI is a newly introduced concept that allows to represent and generalize different types of uncertain information. A BUI can be defined as follows.

Definition 1. A BUI is a real pair $\tilde{x} = < x; c >$, where $x(x \in [0,1])$ is the input value and $c(c \in [0,1])$ the certainty degree of x.

Following this definition, $c = 1$ specifys full certainty over x. Conversely, $c = 0$ indicates full uncertainty over x, meaning that x could take any value between 0 and 1.

Likewise, a BUI can be transformed into a closed interval $[a, b]$. To do this, different techniques exist. This study implements the one proposed in (Jin et al., 2018; Mesiar et al., 2018), where $a = cx$ and $b = cx + 1 - c$. Nevertheless, if $c = 1$, the interval number would be degenerated to a real crisp number.

The UOWA operator is an aggregation function that uses uncertain information in the form of interval numbers. This operator can be defined as follows.

Definition 2. Let Ω be a set of interval numbers. An UOWA operator of dimension n is a mapping UOWA: $\Omega^n \rightarrow \Omega$ that has associated a weighting vector $W = (w_1, \ldots, w_n)$ with $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$, in which:

$$\text{UOWA}(\tilde{a}_1, \ldots, \tilde{a}_n) = \sum_{j=1}^{n} w_j \tilde{b}_j \,, \tag{1}$$

where $\tilde{b}_j$ is the jth largest of the $\tilde{a}_i$, and $\tilde{a}_i$ is the argument variable represented in the form of interval numbers.

Moreover, the UOWA operator satisfies the mathematical properties of commutativity, monotonicity, boundedness, and idempotency.

Additionally, from a generalized perspective of the reordering step, it is possible to discriminate between the descending UOWA (UDOWA) operator and the ascending UOWA (UAOWA) operator. Specifically, the weights of both operators are related by $w_j = w^*_{n-j+1}$, where w_j is the jth weight of the UDOWA (or simply UOWA) operator and w^*_{n-j+1} the jth weight of the UAOWA operator.

Also, an interesting particular type of the UOWA operator is the uncertain weighted average (UWA) operator, which is obtained when the ordered position of $\tilde{a}_i$ is the same as the ordered position of $\tilde{b}_j$.

4. Illustrative example

The following section presents an illustrative example of a multi-expert risk assessment problem through the use of BUI and the UOWA operator.

Consider the following four risks of a given organization:

- y_1: Business interruption because of IT failure.
- y_2: Wrong interpretation of advertising campaigns.
- y_3: Non-compliance with data protection regulations.
- y_4: Inaccurate financial statements.

Next, three experts assess the likelihood and the impact of each identified with the corresponding certainties. To assess the likelihood (x_1), the experts use the scale shown in Table 5.1. The impact (x_4) is analyzed based on two aspects, which are financial (x_2) and reputational (x_3). The assessments and certainties provided by each expert, i.e., BUI pairs, are summarized in Table 5.2.

Afterwards, the assessment values are normalized (see Table 5.3) using the following mathematical formula:

$$x' = \frac{x - \min(x)}{\max(x) - \min(x)}. \quad (2)$$

Once the assessment values are normalized, the normalized BUI pairs are transformed into closed interval numbers. The results are presented in Tables 5.4-5.6, respectively.

Then, the UWA operator is used to aggregate these interval numbers into a unified payoff matrix. As expert 1 and 2 are considered to be more important, the weights used for the aggregation are: $W = (w_1 = 0.4, w_2 = 0.4, w_3 = 0.2)$. The output is displayed in Table 5.7.

Lastly, the UWA and UOWA operators are utilized to aggregate the financial and reputational impacts. To do so, the following weighting vector is adopted: $W = (w_1 = 0.65, w_2 = 0.35)$.

Table 5.1. Likelihood and impact scale equivalences

Rating	x_1	x_2	x_3
1	<10%	<50K	Very low impact on stakeholders
2	10%-30%	50K-100K	Low impact on stakeholders
3	30%-60%	100K-500K	Medium impact on stakeholders
4	60%-90%	500K-1,000K	High impact on stakeholders
5	>90%	>1,000K	Very high impact on stakeholders

Table 5.2. Initial BUI assessments of the experts

	Expert 1			Expert 2			Expert 3		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
y_1	⟨2; 0.6⟩	⟨2; 0.8⟩	⟨3; 0.8⟩	⟨2; 0.7⟩	⟨3; 0.6⟩	⟨3; 0.7⟩	⟨1; 0.6⟩	⟨3; 0.6⟩	⟨4; 0.8⟩
y_2	⟨2; 0.7⟩	⟨3; 0.8⟩	⟨4; 0.8⟩	⟨2; 0.7⟩	⟨4; 0.6⟩	⟨4; 0.8⟩	⟨1; 0.7⟩	⟨3; 0.8⟩	⟨4; 0.7⟩
y_3	⟨2; 0.6⟩	⟨2; 0.7⟩	⟨5; 0.8⟩	⟨3; 0.6⟩	⟨3; 0.6⟩	⟨4; 0.7⟩	⟨2; 0.8⟩	⟨3; 0.7⟩	⟨4; 0.7⟩
y_4	⟨3; 0.7⟩	⟨2; 0.7⟩	⟨3; 0.7⟩	⟨2; 0.6⟩	⟨2; 0.8⟩	⟨2; 0.7⟩	⟨2; 0.7⟩	⟨2; 0.9⟩	⟨3; 0.6⟩

Table 5.3. Normalized BUI assessments of the experts

	Expert 1			Expert 2			Expert 3		
	x_1	x_2	x_3	x_1	x_2	x_3	x_1	x_2	x_3
y_1	⟨0.25; 0.6⟩	⟨0.25; 0.8⟩	⟨0.5; 0.8⟩	⟨0.25; 0.7⟩	⟨0.5; 0.6⟩	⟨0.5; 0.7⟩	⟨0; 0.6⟩	⟨0.5; 0.6⟩	⟨0.75; 0.8⟩
y_2	⟨0.25; 0.7⟩	⟨0.5; 0.8⟩	⟨0.75; 0.8⟩	⟨0.25; 0.7⟩	⟨0.75; 0.6⟩	⟨0.75; 0.8⟩	⟨0; 0.7⟩	⟨0.5; 0.8⟩	⟨0.75; 0.7⟩
y_3	⟨0.25; 0.6⟩	⟨0.25; 0.7⟩	⟨1; 0.8⟩	⟨0.5; 0.6⟩	⟨0.5; 0.6⟩	⟨0.75; 0.7⟩	⟨0.25; 0.8⟩	⟨0.5; 0.7⟩	⟨0.75; 0.7⟩
y_4	⟨0.5; 0.7⟩	⟨0.25; 0.7⟩	⟨0.5; 0.7⟩	⟨0.25; 0.6⟩	⟨0.25; 0.8⟩	⟨0.25; 0.7⟩	⟨0.25; 0.7⟩	⟨0.25; 0.9⟩	⟨0.5; 0.6⟩

Table 5.4. Transformed BUI assessments of the expert 1

	x_1	x_2	x_3
y_1	[0.15,0.55]	[0.2,0.4]	[0.4,0.6]
y_2	[0.175,0.475]	[0.4,0.6]	[0.6,0.8]
y_3	[0.15,0.55]	[0.175,0.475]	[0.8,1]
y_4	[0.35,0.65]	[0.175,0.475]	[0.35,0.65]

Table 5.5. Transformed BUI assessments of the expert 2

	x_1	x_2	x_3
y_1	[0.175,0.475]	[0.3,0.7]	[0.35,0.65]
y_2	[0.175,0.475]	[0.45,0.85]	[0.6,0.8]
y_3	[0.3,0.7]	[0.3,0.7]	[0.525,0.825]
y_4	[0.15,0.55]	[0.2,0.4]	[0.175,0.475]

Table 5.6. Transformed BUI assessments of the expert 3

	x_1	x_2	x_3
y_1	[0,0.4]	[0.3,0.7]	[0.6,0.8]
y_2	[0,0.3]	[0.4,0.6]	[0.525,0.825]
y_3	[0.2,0.4]	[0.35,0.65]	[0.525,0.825]
y_4	[0.175,0.475]	[0.225,0.325]	[0.3,0.7]

Table 5.7. Collective assessments

	x_1	x_2	x_3
y_1	[0.13,0.49]	[0.26,0.58]	[0.42,0.66]
y_2	[0.14,0.44]	[0.42,0.7]	[0.585,0.805]
y_3	[0.22,0.58]	[0.26,0.6]	[0.635,0.895]
y_4	[0.235,0.575]	[0.195,0.415]	[0.27,0.59]

Table 5.8. Final results and aggregation of the impact dimensions

	x_1	x_4	
		UWA	UOWA
y_1	[0.13,0.49]	[0.316,0.608]	[0.364,0.632]
y_2	[0.14,0.44]	[0.4778,0.7368]	[0.5273,0.7683]
y_3	[0.22,0.58]	[0.3913,0.7033]	[0.5038,0.7918]
y_4	[0.235,0.575]	[0.2213,0.4763]	[0.2438,0.5288]

In Table 5.8, we can see the final results. The higher the value, the greater the estimated severity. Thus, the two risks with a higher possibility of occurrence are y_3 and y_4. Likewise, the two risks with a higher overall impact are y_2 and y_3. Moreover, the confidence in the assessments of the experts is reflected through the length of the interval numbers.

5. Results and conclusions

This paper presented a new approach for assessing the risks of an organization and prioritize them in order to more efficiently allocate its resources and successfully achieve its objectives. For doing so, BUI is used in combination with the UOWA operator. This allows to consider the certainty of the experts with their assessments and fuse the information in a more flexible way. As a result, an additional useful indicator is established for the risk analysis.

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