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THREE PROBLEMS IN HARMONIC ANALYSIS AND APPROXIMATION THEORY

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Chapter 1

Introduction

In this dissertation, we deal with three problems in harmonic analysis and approximation theory. The first problem concerns the Hardy-Littlewood relations for Fourier coefficients in the two-dimensional setting, the second one is related to estimates of the coefficients of a trigonometric polynomial in different bases, and the third one refers to multidimensional integer partitions.

In Chapter 2, we study the relations between integrability of functions and summability of their Fourier coefficients. Assuming that a function is square-integrable we have the Parseval's identity, which enables us to reduce a wide class of problems concerning functions to those concerning their Fourier series, and vice versa. We would like to obtain analogues of this relation in the spaces L_p, $p \neq 2$, establishing equivalences of norms of functions and norms of their Fourier series under, of course, some additional requirements. Results of this kind are important, in the first place, due to the fact that once such a relation is found, one becomes free to choose if it is handy to deal with functions or with coefficients in this or that case, as if having Parseval's identity (see e.g. [18, Chs. 4–6, 12–13] and [34, Sec. 7] for applications).

Before we give precise formulations, let us introduce some notations that we are going to use throughout the dissertation. For two functions f and g, the relation $f \gtrsim g$ (or $g \lesssim f$) will mean that there exists a constant C such that $f(x) \geq Cg(x)$ for all x, and the relation $f \asymp g$ is equivalent to $f \gtrsim g \gtrsim f$. If we write $f \gtrsim_a g$, this means that the corresponding constant is allowed to depend on a, however, in what follows we will usually omit the dependence of the implicit constants on the integrability parameters p and q, so that this dependence will be taken for granted.

The following result by Paley [63] can be considered the starting point for the research in this direction (note that the same result for the trigonometric system was obtained several years before by Hardy and Littlewood [37]).

Theorem A (Paley, 1931). *Let $\{\phi_n(x)\}$ be an orthonormal system on $[a, b]$ with $|\phi_n(x)| \leq M$ for all $x \in [a, b]$ and $n \in \mathbb{N}$. Then*
a) If $p \in (1, 2]$, then for any $f \in L_p(a, b)$ with Fourier coefficients $\{c_n\}$ there holds

$$\sum_{n=1}^{\infty} |c_n|^p n^{p-2} \lesssim_{p,M} \|f\|_p^p. \tag{1.1}$$

b) If $p \in [2, \infty)$, then, for any sequence $\{c_n\}$ with $\sum_{n=1}^{\infty} |c_n|^p n^{p-2} < \infty$, there exists a *function* $f \in L_p(a,b)$ *that has* $\{c_n\}$ *as its Fourier coefficients and*

$$\sum_{n=1}^{\infty} |c_n|^p n^{p-2} \gtrsim_{p,M} \|f\|_p^p. \tag{1.2}$$

From now on, we focus only on Fourier series with respect to the trigonometric system.

The ranges of p in Theorem A are sharp, therefore to have both (1.1) and (1.2) true for all $p \in (1,\infty)$, one has to impose some additional requirements. Hardy and Littlewood [38] showed that if we restrict ourselves to sine or cosine series with monotone tending to zero coefficients, then both relations (1.1) and (1.2) hold for all $p \in (1,\infty)$. In this regard, a natural question to ask was: how much can we release the requirement of monotonicity to have

$$\sum_{n=1}^{\infty} |c_n|^p n^{p-2} \asymp_p \|f\|_p^p \tag{1.3}$$

still true? This question in turn motivated creation of various extentions of the class of monotone sequences satisfying (1.3). In particular, the class of general monotone sequences $\{c_n\}$ obeying

$$\sum_{k=n}^{2n} |c_k - c_{k+1}| \lesssim |c_n|$$

for all n was shown [69] to fulfil (1.3). Moreover, Hardy-Littlewood type relations were proved for functions with general monotone Fourier coefficients not only in the Lebesgue spaces L_p, but also in weighted Lebesgue and Lorentz spaces.

A powerful application of the Hardy-Littlewood relation is that the best approximation $E_n(f)_p$ and the modulus of smoothness $\omega_k(f,\delta)_p$ of a function f can be expressed in terms of its coefficients $\{a_n\}$, provided their general monotonicity, in the following way:

$$\Big(\sum_{m=2n}^{\infty} a_m^p m^{p-2}\Big)^{\frac{1}{p}} \lesssim_p E_n(f)_p \lesssim_p a_n n^{\frac{p-1}{p}} + \Big(\sum_{m=n}^{\infty} a_m^p m^{p-2}\Big)^{\frac{1}{p}}$$

and

$$\omega_k(f,2^{-n})_p \asymp_{p,k} 2^{-nk}\Big(\sum_{m=0}^{n} a_{2^m}^p 2^{m(kp+p-1)}\Big)^{\frac{1}{p}} + \Big(\sum_{m=n}^{\infty} a_{2^m}^p 2^{m(p-1)}\Big)^{\frac{1}{p}}.$$

These inequalities, in particular, allow one to characterize the Besov spaces for trigonometric series with general monotone coefficients (see, for instance, [6]).

Our goal will be to prove that the two-dimensional version of relation (1.3) is true for functions whose Fourier coefficients belong to some classes of general monotone and, impotantly, not necessarily non-negative sequences. Moreover, we will show that for a slightly wider class, the Hardy-Littlewood relation fails for $p > 2$. We note also that these results will be proved in a more general setting of weighted Lebesgue spaces.

Chapter 3 is devoted to the following question. Suppose we are given a cosine polynomial $\sum_{k=0}^{n} a_k \cos kx,\ a_k \in \mathbb{R}$. With the help of Chebyshev polynomials $T_k(x)$, according to the equality $\cos kx = T_k(\cos x)$, we can rewrite it as $\sum_{k=0}^{n} b_k \cos^k x$, an algebraic polynomial in $\cos x$. Conversely, any algebraic polynomial in $\cos x$ can be represented as a trigonometric one. So one can pass from one of these representations to another choosing the suitable basis: $\{\cos kx\}_{k=0}^{\infty}$ or $\{\cos^k x\}_{k=0}^{\infty}$. But what if we look at all the cosine polynomials whose certain coordinates with respect to the basis $\{\cos^k x\}_{k=0}$ are fixed? In other words, if we fix some $K \subset \mathbb{N}$, some numbers $\{c_k\}_{k\in K}$ and consider

$$A(K, \{c_k\}) := \left\{\{b_k\}_{k=0}^{n},\ n \in \mathbb{N} : \sum_{k=0}^{n} b_k \cos kx \equiv \sum_{k=0}^{n} a_k \cos^k x,\ a_k = c_k \text{ for } k \in K\right\},$$

what can we say about $\sum_{k=0}^{n} |b_k|$ if we know that $\{b_k\}_{k=0}^{n} \in A(K, \{c_k\})$? In more detail, can we find a trigonometric polynomial belonging to $A(K, \{c_k\})$ with "small" l_1-norm of the coefficients? We show that the answer is indeed positive, which yields that it is possible to adjust any $\sum_{k\in K} c_k \cos^k x$ by adding a trigonometric polynomial with small l_1-norm of the coefficients so that the coefficients of our sum at $\cos^k x,\ k \in K$, are equal to zero.

The principle motivation for posing such a question is the problem of estimating the value of a trigonometric polynomial at some point x, $|\cos x| = \delta < 1$, under some special conditions. Indeed, once a result about the existence of $\{b_k\}_{k=0}^{n} \in A(K, \{c_k\})$ with small l_1-norm of the coefficients is established, one can rewrite the trigonometric polynomial as the algebraic one and adjust it by means of a trigonometric polynomial with small l_1-norm of its coefficients so that its first, say, k coefficients become zero. Then the value at x does not exceed δ^k multiplied by the sum of absolute values of the coefficients of the obtained polynomial plus something small that comes from the adjustment. An argument of this type enables us to construct a nondegenerate double trigonometric series that converges to zero by a subsequence of squares everywhere in such a way that we can control the sizes of these squares and have explicit estimates both for the rate of convergence and for the perturbations in the intermediate steps at every point. The problem of constructing such series is closely related to that of finding universal trigonometric series (see [68] and the references therein).

Another application comes from the fact that, for $\{b_k\}_{k=1}^{r}$ in $A(\{0, 1, ..., p-1\}, \{c_t\}_{t=0}^{p-1})$, there holds

$$\sum_{k=1}^{r} b_k T_k(y) - y^p g(y) \equiv \sum_{t=0}^{p-1} c_t y^t,$$

for some polynomial g, so the result can be applied to the study of Chebyshev polynomials and Chebyshev series [53], as series of Chebyshev polynomials are known to have properties of fast convergence among other their advantages in approximation theory and numerical analysis (see, for instance, [13]).

To prove the mentioned result, we consider the matrix $\mathbf{T} = (t_m^k)_{m,k=0}^{\infty}$ whose entry t_m^k is the coefficient at x^m of the Chebyshev polynomial $T_k(x)$, and derive an explicit formula for the inverse of a square submatrix of $\mathbf{T}$. This allows us to determine the coefficients with respect to the basis $\{\cos kx\}_{k=0}^{\infty}$ of an algebraic polynomial in $\cos x$. In the course of the proof of Theorem 3.1, we also give some useful estimates (see Lemma 3.6) on sums of products of binomial coefficients appearing in the expression for entries of the pseudoinverse of a Vandermonde matrix in [29] (see [7] for a substantive survey of generalized inverses and also [64] and [71] for algebraic properties of generalized inverses of Vandermonde matrices).

In Chapter 4, we study multidimensional partitions or, equivalently, lower sets and establish estimates for the number of d-dimensional lower sets with fixed cardinality.

For a given d, we call a set $S \subset \mathbb{Z}_+^d$ *a lower set* if for any $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{Z}_+^d$ the condition $\mathbf{x} \in S$ implies $\mathbf{x}' = (x'_1, ..., x'_d) \in S$ for all $\mathbf{x}' \in \mathbb{Z}_+^d$ with $x'_i \leq x_i$, $1 \leq i \leq d$. There is a one-to-one correspondence between d-dimensional lower sets of cardinality n and $(d-1)$-dimensional partitions of n, that is, representations of the form

$$n = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} ... \sum_{i_{d-1}=1}^{\infty} n_{i_1 i_2 ... i_{d-1}}, \quad n_{i_1 i_2 ... i_{d-1}} \in \mathbb{Z}_+,$$

where $n_{i_1 i_2 ... i_{d-1}} \geq n_{j_1 j_2 ... j_{d-1}}$ if $j_k \geq i_k$ for all $k = 1, 2, ..., d-1$. Thus, lower sets represent a geometric interpretation of integer partitions. By $p_d(n)$ we denote the number of lower sets in $\mathbb{Z}_+^d$ containing exactly n points[1].

Importantly, the theory of partitions has many applications in physics, as there are a lot of physical structures resembling that of multidimensional integer partitions. In particular, integer partitions are used to estimate the energy levels for a heavy nucleus [10] and to study the shape of crystal growth [67]. Another direction of research is based on the existence of a one-to-one correspondence between partitions of an integer and microstates of a gas particles stored in a harmonic oscillator, not only in two-dimensional case [4, 72] but also in multidimensional setting [56].

Furthermore, certain classes of trigonometric polynomials with harmonics in lower sets have recently turned out to be a powerful tool in multivariate approximation (see [11, 15, 16] and references therein).

It is known that the two-sided inequality

$$C_1(d) n^{1-1/d} < \log p_d(n) < C_2(d) n^{1-1/d}$$

is always true and that $C_1(d) > 1$ whenever $\log n > 3d$. However, establishing the "right" dependence of C_2 on d remained an open problem. We will show that if d is sufficiently small with respect to n, then C_2 does not depend on d, which means that $\log p_d(n)$ is up to an absolute constant equal to $n^{1-1/d}$. Besides, we provide estimates of $p_d(n)$ for different ranges of d in terms of n, which give the asymptotics of $\log p_d(n)$ in each case.

The results of Chapters 2 and 3 are published in [60] and [61], while those of Chapter 4 can be found in [62].

[1]In some sources the same value is denoted by $p_{d-1}(n)$.

Chapter 2

Hardy-Littlewood theorem in two dimensions

2.1 Concepts of monotonicity and known results

One of the classes of sequences such that functions with Fourier coefficients belonging to this class still obey (1.3), is the so-called general monotone or just GM class [69, Th. 4.2]. It consists of all sequences $\{a_n\}$ fulfilling the condition

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \lesssim |a_n| \tag{2.1}$$

for all n. Thus, now we dropped not only the monotonicity condition but even the basic requirement of positivity, keeping though some regularity of our sequences. One can see that GM class can yet be generalized (see [70, Th. 6.2(B)] and [76, Th. 1]) by putting a mean value on the right-hand side of (2.1) instead of $|a_n|$ as follows:

$$\sum_{k=n}^{2n} |a_k - a_{k+1}| \lesssim \sum_{k=\frac{n}{\lambda}}^{\lambda n} \frac{|a_k|}{k} \tag{2.2}$$

with some $\lambda > 1$ (see also [32] for some properties of such sequences). Note that these classes and several other ones, defined as (2.1) but with some other majorants on the right-hand side, in different sources can be also called GM. For a comprehensive survey on the concept of general monotonicity, we refer the reader to [48].

One more direction of extending the obtained results (see [1, 41, 76]) is proving them for weighted spaces. Define the weighted Lebesgue spaces $L^q_{w(p,q)}$, $p, q \in (0, \infty]$, on $[-\pi, \pi]$, as the set of all measurable functions f with finite norm

$$\|f\|_{L^q_{w(p,q)}} := \begin{cases} \left(\int\limits_{-\pi}^{\pi} |t|^{\frac{q}{p}-1} |f(t)|^q dt \right)^{\frac{1}{q}}, & \text{if } 0 < p, q < \infty, \\ \operatorname*{ess\,sup}\limits_{t \in [-\pi,\pi]} |t^{\frac{1}{p}} f(t)|, & \text{if } 0 < p \le \infty,\ q = \infty. \end{cases}$$

The discrete weighted Lebesgue space $l^q_{w(p,q)}$ is to be defined in the same way.

Now, a weighted version of relation (1.3) is given by

$$\|\{c_n\}\|^q_{l^q_{w(p',q)}} := \sum_{n=1}^{\infty} |c_n|^q n^{\frac{q}{p'}-1} \asymp \|f\|^q_{L^q_{w(p,q)}}, \tag{2.3}$$

where p' stands for the conjugate to p, that is, $1/p + 1/p' = 1$. Note that if we put $q = p$, we get the standard Hardy-Littlewood relation (1.3). The following theorem for weighted Lebesgue spaces was obtained by Sagher [66].

Theorem A (Sagher, 1976)**.** *If the sequences $\{a_n\}$ and $\{b_n\}$ are monotone and vanishing at infinity and the function f has the Fourier series*

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

then for $p \in (1,\infty)$, $q \in [1,\infty]$, there holds

$$\|f\|_{L^q_{w(p,q)}} \asymp \|\{a_n\}\|_{l^q_{w(p',q)}} + \|\{b_n\}\|_{l^q_{w(p',q)}}.$$

It turns out that the same holds if we release the monotonicity condition in the theorem above to (2.2), thus withdrawing the requirement of positivity. This result, along with the similar statement proved for Lorentz spaces, was given by Dyachenko, Mukanov and Tikhonov [23].

So, in the one-dimensional case we have quite a complete picture.

The whole scenario becomes more complicated if we step out from the one-dimensional setting to the multidimensional one, and the first question we face is to determine what we should mean by monotonicity if we deal with multiple sequences. The usual one-dimensional monotonicity is characterized by the inequalities $a_n \geq a_{n+1}$, or equivalently, $\Delta a_n := a_n - a_{n+1} \geq 0$. These two ways of writing the same property give rise to the following fundamentally different multidimensional monotonicity concepts. Our focus will be on the two-dimensional case.

2.1.1 Monotonicity in each variable

Likewise $a_n \geq a_{n+1}$ in one dimension, we can require coordinatewise monotonicity, that is, in two-dimensional case the condition will be

$$a_{mn} \leq a_{m'n'}, \quad \text{for all } \ m \geq m',\ n \geq n'. \tag{2.4}$$

It turns out, however, that for such sequence the Hardy-Littlewood relation (1.3) does not hold for some values of $p > 1$, namely, we have the following result proved by Dyachenko [20, 22].

Theorem B (Dyachenko, 1986)**.** *a) [20, Th. 1] If $\{a_{mn}\}_{m,n=1}^{\infty}$ satisfying (2.4) and*

$$a_{mn} \to 0, \quad \text{as } m+n \to \infty, \tag{2.5}$$

is the sequence of the Fourier coefficients with respect to one of the orthonormal systems $\{e^{inx}e^{imy}\}_{m,n=1}^{\infty}$, $\{\sin nx \sin my\}_{m,n=1}^{\infty}$, and $\{\cos nx \cos ny\}_{m,n=1}^{\infty}$, of a function f, then for any $p \in (1,\infty)$,

$$\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} \lesssim \|f\|_p^p.$$

b) [22, Cor. 2] Let $p > 4/3$ and the sequence $\{a_{mn}\}$ satisfy (2.4) and $\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} < \infty$ (therefore, (2.5) as well). Then, for any of the systems above, there exists a function f having $\{a_{mn}\}$ as its Fourier coefficients and satisfying

$$\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} \gtrsim \|f\|_p^p. \tag{2.6}$$

c) [20, Ths. 8, 8'] For $p \in (1, 4/3]$, there exists a sequence $\{a_{mn}\}$ satisfying (2.4) and (2.5) with $\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} < \infty$ such that the corresponding trigonometric series diverges by squares almost everywhere on $(0, 2\pi)^2$.

Note that it was shown by Fefferman [31] that for any $p > 1$ and any $f \in L_p(0, 2\pi)^2$, the Fourier series of f converges by squares almost everywhere on $(0, 2\pi)^2$, thus, the third part of the theorem means that (2.6) is no longer true for the integrability parameter $p \in (1, 4/3)$. We also remark that in general d-dimensional case the critical value is $2d/(d+1)$ (see [21, Th. 1, Th. 4] and [22, Cor. 2]) and that the d-dimensional part c) of Theorem B for $p = 2d/(d+1)$ was proved in [25].

2.1.2 Monotonicity in the sense of Hardy

The next approach to the multiple concept of monotonicity is to consider the so-called monotonicity in the sense of Hardy (or Hardy-Krause, see [36] and [45], where this concept initially arises). In more detail, define the differences

$$\Delta^{10} a_{mn} := a_{mn} - a_{m+1,n}, \qquad \Delta^{01} a_{mn} := a_{mn} - a_{m,n+1},$$
$$\Delta^{11} a_{mn} := \Delta^{01}(\Delta^{10} a_{mn}) = \Delta^{10}(\Delta^{01} a_{mn}) = a_{mn} - a_{m+1,n} - a_{m,n+1} + a_{m+1,n+1},$$

and recalling the one-dimentional condition $\Delta a_n \geq 0$, one generalizes it in the following way

$$\Delta^{11} a_{mn} \geq 0 \quad \text{for all } m, n. \tag{2.7}$$

Note that under the natural requirement (2.5), condition (2.7) implies

$$a_{mn} \geq 0, \quad \Delta^{10} a_{mn} \geq 0, \quad \Delta^{01} a_{mn} \geq 0.$$

Here comes the result obtained by Móricz [54, Th. 1,2, Cor. 1].

Theorem C (Móricz, 1990)**.** *Let $p \geq 1$ and the sequence $\{a_{mn}\}$ satisfy (2.5) and (2.7).*

a) If $\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} < \infty$, then the double sine or cosine series with coefficients $\{a_{mn}\}$ is the Fourier series of its sum f and

$$\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} \gtrsim \|f\|_p^p.$$

b) If $\{a_{mn}\}$ is the sequence of double sine or cosine Fourier coefficients of $f \in L_p$, then

$$\sum_{m,n=1}^{\infty} a_{mn}^p (mn)^{p-2} \lesssim \|f\|_p^p.$$

The reader can find Theorem C proved for Vilenkin systems (and hence for the Walsh system) in [73, Sec. 6.3] and [74] (see also [27, Sec. 4]).

Condition (2.7) is quite restrictive and one of the closest generalizations of it in, say, GM spirit is the following one

$$\sum_{m=k}^{\infty}\sum_{n=l}^{\infty}|\Delta^{11}a_{mn}|\lesssim|a_{mn}|.$$

Note that if the sequence satisfies (2.7), then the left-hand side above becomes just equal to a_{mn}. The next result [26, Th. 6B] (see [27] for the proof) extends the one of Móricz.

Theorem D (Dyachenko, Tikhonov, 2007). *If a nonnegative sequence $\{a_{mn}\}$ satisfy (2.5) and the so-called GM^2 condition*

$$\sum_{m=k}^{\infty}\sum_{n=l}^{\infty}|\Delta^{11}a_{mn}|\lesssim|a_{kl}|+\sum_{m=k}^{\infty}\frac{|a_{ml}|}{m}+\sum_{n=l}^{\infty}\frac{|a_{kn}|}{n}+\sum_{m=k}^{\infty}\sum_{n=l}^{\infty}\frac{|a_{mn}|}{mn}, \tag{2.8}$$

then the corresponding double sine, cosine, or exponential series converges everywhere on $(0,2\pi)^2$ and is the Fourier series of its sum. Besides, for any $p\in(1,\infty)$,

$$\sum_{m,n=1}^{\infty}a_{mn}^p(mn)^{p-2}\asymp\|f\|_p^p.$$

It is worth mentioning that the $\gtrsim$ part was proved without assuming $a_{mn}\geq 0$, moreover, it was shown that if $\sum_{m=k}^{\infty}\sum_{n=l}^{\infty}|\Delta^{11}a_{mn}|\lesssim\beta_{kl}$, then $\sum_{m,n=1}^{\infty}\beta_{mn}^p(mn)^{p-2}\gtrsim\|f\|_p^p$. However, in the proof of the counterpart the requirement of nonnegativity plays a crucial role. It was noted in [28, Th. 4.1] that following the lines of this proof one can adapt it for a more general class of sequences for which the right-hand side of (2.8) is replaces by

$$\sum_{m=\lceil k/\lambda\rceil}^{\infty}\sum_{n=\lceil l/\lambda\rceil}^{\infty}\frac{|a_{mn}|}{mn}$$

with $\lambda>1$.

Further, it was shown in [77] that some other GM type nonnegative sequences happen to obey the two-sided Hardy-Littlewood relation. We present the result from [77] for weighted spaces.

Theorem E (Yu, Zhou, Zhou, 2012). *Let $\{a_{mn}\}$ be a nonnegative sequence satisfying (2.5) and the following GM type conditions*

$$\sum_{m=k}^{2k}|\Delta a_{ml}|\lesssim\sum_{m=\lfloor\lambda^{-1}k\rfloor}^{\lfloor\lambda k\rfloor}\frac{|a_{ml}|}{m},\qquad\sum_{n=l}^{2l}|\Delta a_{kn}|\lesssim\sum_{n=\lfloor\lambda^{-1}l\rfloor}^{\lfloor\lambda l\rfloor}\frac{|a_{kn}|}{n},$$

$$\sum_{m=k}^{2k}\sum_{n=l}^{2l}|\Delta a_{mn}|\lesssim\sum_{m=\lfloor\lambda^{-1}k\rfloor}^{\lfloor\lambda k\rfloor}\sum_{n=\lfloor\lambda^{-1}l\rfloor}^{\lfloor\lambda l\rfloor}\frac{|a_{mn}|}{mn}$$

for some $\lambda\geq 2$, and let $f(x,y):=\sum_{m,n=1}^{\infty}a_{mn}\sin mx\sin ny$. Then, for any $p\in[1,\infty)$, for any function $\phi\in\Phi$ with either $\phi^{-\frac{1}{p-1}}\in L$ if $p>1$, or $\phi^{-1}\in L_\infty$, if $p=1$, we have

$$\phi|f|^p\in L\Leftrightarrow\sum_{m,n=1}^{\infty}a_{mn}^p\phi(1/m,1/n)(mn)^{p-2}<\infty.$$

In the above result Φ stands for some class of power-like positive functions, which we are not going to specify here. A similar result with a more general GM type positive sequences and some other (not comparable) class of power-like functions was obtained in [19].

Similar results for the Fourier transform are also well known in the literature, see e.g. [12, 24, 33, 59, 66].

2.2 New results for two-dimensional case

The main purpose of this chapter is to show that for some kinds of double GM sequences we can prove the Hardy-Littlewood theorem without restricting ourselves only to positive sequences. We present two GM type classes for which the two-sided Hardy-Littlewood inequality holds true.

We write that $\{a_{mn}\} \in GM_1^c$ if it satisfies (2.5) and

$$\sum_{m=k}^{2k}\sum_{n=l}^{\infty}|\Delta^{11}a_{mn}| + \sum_{m=k}^{\infty}\sum_{n=l}^{2l}|\Delta^{11}a_{mn}| \le C|a_{kl}|, \tag{2.9}$$

and $\{a_{mn}\} \in GM_2^c$, if it satisfies (2.5) and

$$\sum_{m=k}^{2k}\sum_{n=l}^{\infty}|\Delta^{11}a_{mn}| + \sum_{m=k}^{\infty}\sum_{n=l}^{2l}|\Delta^{11}a_{mn}| \le C|a_{2k,l}|, \tag{2.10}$$

for all $k, l \in \mathbb{N}$ and some constant C depending only on the sequence $\{a_{mn}\}$. We remark that the letter c in GM^c comes from the word "corner", since a set of the kind $[k, 2k] \times [l, \infty) \cup [k, \infty) \times [l, 2l]$ generates a corner on the plane. Note that GM_1^c sequences obey the one-dimensional GM conditions (2.1) in each variable (see (2.11) in the proof of Lemma 2.2), while GM_2^c in one variable satisfy (2.1), and in another one, the "backward" GM condition.

Note that for $[-\pi,\pi]^2$ the $L^q_{w(p,q)}$-norms take the form

$$\|f\|_{L^q_{w(p,q)}} := \begin{cases} \left(\int\limits_{-\pi}^{\pi}\int\limits_{-\pi}^{\pi} |ts|^{\frac{q}{p}-1}|f(t,s)|^q dt\, ds\right)^{\frac{1}{q}}, & \text{if } 0<p,q<\infty, \\ \operatorname*{ess\,sup}\limits_{(t,s)\in[-\pi,\pi]^2} |(ts)^{\frac{1}{p}} f(t,s)|, & \text{if } 0<p\le\infty,\ q=\infty. \end{cases}$$

From now on, for convenience, we adopt the following notation: using that $(\sin x)^{(1)} = (\sin x)' = \cos x$ and $(\sin x)^{(0)} = \sin x$, we will write a two-dimensional trigonometric series as

$$\sum_{i,j=0}^{1}\sum_{m,n=0}^{\infty} a_{mn}^{ij} \sin^{(i)} mx \sin^{(j)} ny$$

and we will say that $\{a_{mn}^{ij}\}_{m,n=1}^{\infty}$, $i,j = 0,1$, is the sequence of its coefficients.

Theorem 2.1. *Let* $p \in (1,\infty)$, $q \in [1,\infty]$, *and let each of the sequences* $\{a_{mn}^{ij}\}_{m,n=1}^{\infty}$, $i,j = 0,1$, *belong either to* GM_1^c *or to* GM_2^c.

a) If $\{a_{mn}^{ij}\}_{m,n=1}^{\infty}$, $i,j=0,1$, *is the sequence of Fourier coefficients of* $f \in L(-\pi,\pi)^2$, *then*

$$\|f\|_{L^q_{w(p,q)}} \gtrsim \sum_{i,j=0}^{1} \|\{a_{mn}^{ij}\}\|_{l^{q'}_{w(p,q')}}.$$

b) If $\sum_{i,j=0}^{1} \|\{a_{mn}^{ij}\}\|_{l^{q'}_{w(p,q')}} < \infty$, *then the corresponding trigonometric series converges everywhere on* $(0,2\pi)^2$ *and is the Fourier series of its sum* f, *moreover,*

$$\|f\|_{L^q_{w(p,q)}} \lesssim \sum_{i,j=0}^{1} \|\{a_{mn}^{ij}\}\|_{l^{q'}_{w(p,q')}}.$$

Sharpness of Theorem 2.1 for GM_2^c sequences will be provided by a counterexample in Theorem 2.8, which shows that if we restrict the sum on the left-hand side of (2.10) to the rectangle (that is, to the intersection and not the union of the two corresponding strips), which is one of the most natural generalizations of the left-hand side of the GM condition (2.1), then the $\gtrsim$ part fails for $p > 2$ and $q \geq p$.

2.3 Proof of the Hardy-Littlewood theorem for GM^c sequences

For a sequence $\{a_{mn}\}_{m,n=1}^{\infty}$, we define

$$A_{mn} := \max_{(k,l)\in Q_{m,n}} |a_{kl}| := \max_{(k,l)\in[2^m,2^{m+1}]\times[2^n,2^{n+1}]} |a_{kl}|.$$

Lemma 2.2. *a) For any sequence* $\{a_{kl}\}_{k,l=1}^{\infty} \in GM_1^c$, *there exist* $c, v > 0$ *such that for any* (m,n) *with* $A_{m-1,n-1} \leq TA_{m,n}$ *there exist a rectangle* $Q'_{m-1,n-1} \subset Q_{m-1,n-1}$ *of size* $2^{m-v} \times 2^{n-v}$ *satisfying*

$$\left| \sum_{k,l\in\, Q'_{m-1,n-1}} a_{kl} \right| > c2^{m+n} A_{mn},$$

where c *and* v *depend only on* C *and* T.

b) For any sequence $\{a_{kl}\}_{k,l=1}^{\infty} \in GM_2^c$, *there exist* $c, v > 0$ *such that for any* (m,n) *with* $A_{m+1,n-1} \leq TA_{m,n}$ *there exist a rectangle* $Q'_{m+1,n-1} \subset Q_{m+1,n-1}$ *of size* $2^{m-v} \times 2^{n-v}$ *satisfying*

$$\left| \sum_{k,l\in\, Q'_{m+1,n-1}} a_{kl} \right| > c2^{m+n} A_{mn},$$

where c *and* v *depend only on* C *and* T.

Proof. Note that (2.5) and (2.9) imply that

$$\sum_{m=k}^{2k} |\Delta^{10} a_{mt}| + \sum_{n=l}^{2l} |\Delta^{01} a_{sn}| \leq C|a_{k,l}| \tag{2.11}$$

for any $k, l \in \mathbb{N}$ and $(s,t) \in [k,2k] \times [l,2l]$. Similarly, (2.5) along with (2.10) imply (2.11) with $a_{2k,l}$ instead of $a_{k,l}$ on the right-hand side. In particular, (2.11) yields that

$$|a_{s,t}| - |a_{k,l}| = |a_{s,t}| - |a_{k,t}| + |a_{k,t}| - |a_{2k,l}| \leq C|a_{k,l}|,$$

so

$$|a_{s,t}| \leq (C+1)|a_{k,l}| \leq (C+1)^2 |a_{s't'}|$$

for any $(s',t') \in [0.5k, k] \times [0.5l, l]$. Considering $k = 2^m$, $l = 2^n$, we get for any $(s,t) \in Q_{m-1,n-1}$

$$|a_{st}| \geq (C+1)^{-2} A_{mn} =: \alpha A_{mn}. \tag{2.12}$$

Under conditions (2.5) and (2.10), the same arguments give

$$|a_{s,t}| - |a_{2k,l}| = |a_{s,t}| - |a_{2k,t}| + |a_{2k,t}| - |a_{2k,l}| \leq C|a_{2k,l}|,$$

and

$$|a_{s,t}| \leq (C+1)|a_{2k,l}| \leq (C+1)^2 |a_{s't'}|$$

for any $(s',t') \in [2k, 4k] \times [0.5l, l]$. Once more, considering $k = 2^m$, $l = 2^n$, we get (2.12) for $(s,t) \in Q_{m+1,n-1}$ instead of $Q_{m-1,n-1}$.

Thus, any sequence $\{a_{kl}\} \in GM_1^c$ satisfies $|a_{kl}| \leq (C+1)|a_{k'l'}|$ for $(k',l') \in [0.5k, k] \times [0.5l, l]$ as well as any $\{a_{kl}\} \in GM_2^c$ does for $(k',l') \in [k, 2k] \times [0.5l, l]$.

In Lemma 2.2a), due to condition (2.11) and inequality (2.12), for any $(k,l) \in Q_{m-1,n-1}$, each one of the sequences $a_{2^{m-1},l}, a_{2^{m-1}+1,l}, \ldots, a_{2^m,l}$ and $a_{k,2^{n-1}}, a_{k,2^{n-1}+1}, \ldots, a_{k,2^n}$ can have at most

$$\frac{C \max\limits_{(k,l) \in Q_{m-1,n-1}} |a_{kl}|}{2\alpha A_{mn}} = \frac{C A_{m-1,n-1}}{2\alpha A_{mn}} \leq \frac{CT}{2\alpha} =: b \tag{2.13}$$

changes of sign.

The same holds for $Q_{m+1,n-1}$ in place of $Q_{m-1,n-1}$ in Lemma 2.2b).

Focus now on Lemma 2.2a). Consider the rectangle $R := Q_{m-1,n-1} = [2^{m-1}, 2^m] \times [2^{n-1}, 2^n]$ on the plane and draw all the segments $[(k,l),(k+1,l)]$ such that $a_{k,l-1}$ and $a_{k,l}$ have different signs and all the segments $[(k,l),(k,l+1)]$ such that $a_{k-1,l}$ and $a_{k,l}$ have different signs (call them *marked* segments). Then our rectangle R is divided by the marked segments into several connected parts corresponding to the terms of $\{a_{kl}\}$ of the same sign. The interior part of the union of their boundaries has at most $b2^{n-1}$ vertical marked segments and at most $b2^{m-1}$ horizontal ones. Take a positive integer u such that

$$2^u > 8b\tau, \tag{2.14}$$

where $\tau := 4\sqrt{T(C+1)^2 + 1}$. Divide R into 2^{2u} equal rectangles of size $2^{m-1-u} \times 2^{n-1+u}$ and consider a half of them in a checkerboard pattern. Suppose that there is no rectangle among them containing at most $2^{n-1-u}/\tau$ vertical marked segments and at most $2^{m-1-u}/\tau$ horizontal ones. Then we must have

$$2^{2u-1} \leq \frac{b2^{m-1}\tau}{2^{m-1-u}} + \frac{b2^{n-1}\tau}{2^{n-1-u}} = 2^{u+2} b\tau \leq 4b\tau 2^u,$$

which contradicts (2.14). So, there is a rectangle $r = [\alpha_1, \alpha_2] \times [\beta_1, \beta_2]$ of size $2^{m-1-u} \times 2^{n-1-u}$ with at most $2^{n-1-u}/\tau$ vertical marked segments and at most $2^{m-1-u}/\tau$ horizontal ones inside it. Consider the parts corresponding to the terms of $\{a_{kl}\}$ of the same sign inside r. Call the parts whose boundaries intersect the boundary of r by A-parts, the other ones, by B-parts. Note that there is no marked segment of an A-part inside the rectangle $r' := [\frac{3\alpha_1+\alpha_2}{4}, \frac{\alpha_1+3\alpha_2}{4}] \times [\frac{3\beta_1+\beta_2}{4}, \frac{\beta_1+3\beta_2}{4}]$. Indeed, otherwise there would exist a broken line of marked segments with either at least $0.25(\alpha_2 - \alpha_1) = 2^{m-3-u}$ horizontal segments or at least $0.25(\beta_2 - \beta_1) = 2^{n-3-u}$ vertical ones. But this is impossible, since $\tau > 4$. The area of all B-parts does not exceed $2^{m+n-2-2u}/\tau^2$. Thus, there are at least $2^{m+n-4-2u}(1 - 4\tau^{-2})$ terms of the same sign in r', so the absolute value of the sum of the terms $\{a_{kl}\}$ in r' is at least

$$2^{n+m-2u-4}\Big(1 - \frac{4}{\tau^2} - \frac{4}{\tau^2}T(C+1)^2\Big)\alpha A_{mn} > 2^{n+m-2u-5}\alpha A_{mn},$$

which concludes the proof of Lemma 2.2a) with $c := 2^{-2u-5}\alpha$ and $v := u+1$.

A similar argument is valid for $Q_{m+1,n-1}$ in Lemma 2.2b), which completes the proof. □

Remark 2.3. In the proof of Lemma 2.2, for GM_1^c class we only used its one-dimensional GM properties (2.11), and for GM_2^c, the corresponding nonsymmetric relations (namely, (2.11) with $a_{2k,l}$ in place of $a_{k,l}$).

Remark 2.4. The claim of Lemma 2.2a) is no longer true if we substitute the GM_1^c condition (2.9) for

$$\sum_{m=k}^{2k}\sum_{n=l}^{2l} |\Delta^{11} a_{mn}| \le C|a_{kl}|. \tag{2.15}$$

Proof. Indeed, consider the sequence

$$a_{mn} := \frac{(-1)^m}{m} f_m(n),$$

where $f_m(n)$ we define as follows:

$$f_m(n) = \begin{cases} 2^{-m+1}, & \log_2 n < \frac{m(m+1)}{2}, \\ 2^{-m-t}, & \frac{(m+t)^2+m-t}{2} \le \log_2 n < \frac{(m+t+1)^2+m-t-1}{2}, \quad t \in \mathbb{Z}_+. \end{cases}$$

For such a sequence, condition (2.5) obviously holds. Consider a rectangle S_{mn} of the form $[m, 2m) \times [n, 2n)$. The only nonzero $\Delta^{11} a_{kl}$ in this rectangle are $\Delta^{11} a_{m'-1,n'}$ and $\Delta^{11} a_{m'n'}$, where $n' \in [n, 2n) : \lfloor \log_2(n') \rfloor = \lfloor \log_2(n'-1) \rfloor + 1$, i.e. n' is a power of two, and

$$m' := \min\Big\{ m \in \mathbb{N} : \ m = \log_2 n' - \frac{k(k+1)}{2}, \ k \in \mathbb{Z}_+ \Big\}.$$

Note that $|a_{kl}| \le |a_{mn}|$ for $k \ge m$, $l \ge n$, so $|\Delta^{11} a_{m'n'}| \le |a_{m'n'}| + |a_{m'+1,n'}| \le 2|a_{mn}|$, which yields condition (2.15) with $C = 2$.

Assume that the assertion of Lemma 2.2 holds. Then there must exist a constant c such that for at least cmn squares $[k, k+2) \times [l, l+2)$ in any S_{mn} there holds

$$|a_{kl} + a_{k,l+1} + a_{k+1,l} + a_{k+1,l+1}| \ge c|a_{kl}|. \tag{2.16}$$

Consider a rectangle S_{mn} with

$$\frac{t(t+1)}{2} + 2m \le \log_2 n \le \frac{(t+1)(t+2)}{2} - 2,$$

where $t > 4m$ is a positive integer. For any a_{kl} in S_{mn}, we have

$$a_{kl} = 2^{-t-1}\frac{(-1)^k}{k},$$

whence for any 2×2 square $[k, k+2) \times [l, l+2) \subset S_{mn}$

$$|a_{kl} + a_{k,l+1} + a_{k+1,l} + a_{k+1,l+1}| = 2^{-t-1} \cdot 2\Big(\frac{1}{k} - \frac{1}{k+1}\Big) = \frac{2}{k+1}|a_{kl}| < \frac{2}{m}|a_{kl}| = o(|a_{kl}|),$$

as $m \to \infty$, which leads to a contradiction. □

Lemma 2.5. *For a function $f \in L(-\pi, \pi)^2$, given the representation*

$$f(x,y) = \sum_{i,j=0}^{1} f^{ij}(x,y), \quad f^{ij}(-x,y) = (-1)^i f^{ij}(x,y),\ f^{ij}(x,-y) = (-1)^j f^{ij}(x,y),$$

for any $p \in (1,\infty)$, $q \in [1,\infty]$, we have

$$\|f\|_{L^q_{w(p,q)}} \asymp \sum_{i,j=0}^{1} \|f^{ij}\|_{L^q_{w(p,q)}}.$$

Proof. The $\lesssim$ part is clear, so we have to prove the reverse.
We start with the case $q < \infty$. Noting that for any pair of functions g_1, g_2 there always holds $|g_1|^q + |g_2|^q \lesssim |g_1 + g_2|^q + |g_1 - g_2|^q$ and recalling that the weight is an even in each variable function, we obtain

$$\begin{aligned}\|f^{i0}(x,\cdot)\|^q_{L^q_{w(p,q)}} + \|f^{i1}(x,\cdot)\|^q_{L^q_{w(p,q)}} &\lesssim \|(f^{i0}+f^{i1})(x,\cdot)\|^q_{L^q_{w(p,q)}} + \|(f^{i0}-f^{i1})(x,\cdot)\|^q_{L^q_{w(p,q)}} \\ &\asymp \|(f^{i0}+f^{i1})(x,\cdot)\|^q_{L^q_{w(p,q)}}\end{aligned}$$

for $i = 0, 1$. Similarly,

$$\begin{aligned}\sum_{i,j=0}^{1} \|f^{ij}\|^q_{L^q_{w(p,q)}} &\lesssim \|f^{00}+f^{01}+f^{10}+f^{11}\|^q_{L^q_{w(p,q)}} + \|f^{00}+f^{01}-f^{10}-f^{11}\|^q_{L^q_{w(p,q)}} \\ &\asymp \Big\|\sum_{i,j=0}^{1} f^{ij}\Big\|^q_{L^q_{w(p,q)}} = \|f\|^q_{L^q_{w(p,q)}}.\end{aligned}$$

For $q = \infty$, the claim follows from the equalities

$$4f^{ij}(x,y) \equiv f(x,y) + (-1)^i f(-x,y) + (-1)^j f(x,-y) + (-1)^{i+j} f(-x,-y).$$

□

Next we prove a two-dimensional analogue of [23, L. 2.2] (see also the one-dimensional result [66, Th. 2.4] for the Lorentz spaces). Note that similar multidimensional results for Lorentz spaces were obtained in [57] and [58].

Lemma 2.6. *Let* $\{a_{mn}^{ij}\}_{m,n=1}^{\infty}$, $i,j=0,1$, *be the sequence of Fourier coefficients of* $f \in L(-\pi,\pi)^2$. *Then for any* $p\in(1,\infty)$, $q\in[1,\infty]$, *there holds*

$$\sum_{i,j=0}^{1}\Big(\sum_{m,n=1}^{\infty}\Big(\sup_{k\geq m,\, l\geq n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big)^{q}(mn)^{\frac{q}{p'}-1}\Big)^{\frac{1}{q}}\lesssim\|f\|_{L^q_{w(p,q)}}.$$

Proof of Lemma 2.6. Note that if we prove the statement of the lemma for odd in each variable functions $f\in L(-\pi,\pi)^2$, then it will be true for any integrable f. Indeed, the relation for such functions implies the same for all functions that are either odd or even in each variable due to the boundedness of the Hilbert transform in the weighted Lebesgue spaces under our assumptions on weights (see e.g. [42]). The general case follows then by Lemma 2.5. Thus, we can assume that $a_{mn}^{ij}=0$ if $(i,j)\neq(0,0)$ and omit the upper indices of a_{mn}^{00}.

According to [23, (2.4), (2.7)], for any $1<p<\infty$, $1\leq q\leq\infty$, and $m\in\mathbb{N}$, for

$$I_m(x):=\frac{\cos\frac{x}{2}(1-\cos mx)}{m\sin\frac{x}{2}}+\frac{\sin mx}{m},$$

there holds

$$\|I_m(x)\|_{l_{p,q}}\lesssim m^{-\frac{1}{p}}.$$

Therefore, for any $1<p_1,p_2<\infty$, $1<q\leq\infty$, and $m,n\in\mathbb{N}$, by Hölder's inequality

$$\begin{aligned}\frac{1}{mn}\Big|\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}\Big| &\leq \int_0^{\pi}\int_0^{\pi}|f(x,y)I_m(x)I_n(y)|dxdy\\
&\leq \int_0^{\pi}|I_n(y)|\Big(\int_0^{\pi}x^{\frac{q}{p_1}-1}|f(x,y)|^q dx\Big)^{\frac{1}{q}}\Big(\int_0^{\pi}x^{\frac{q'}{p_1'}}|I_m(x)|^{q'}dx\Big)^{\frac{1}{q'}}dy\\
&\lesssim m^{-\frac{1}{p_1'}}\int_0^{\pi}|I_n(y)|\Big(\int_0^{\pi}x^{\frac{q}{p_1}-1}|f(x,y)|^q dx\Big)^{\frac{1}{q}}dy\\
&\leq m^{-\frac{1}{p_1'}}\Big(\int_0^{\pi}\int_0^{\pi}x^{\frac{q}{p_1}-1}y^{\frac{q}{p_2}-1}|f(x,y)|^q dxdy\Big)^{\frac{1}{q}}\Big(\int_0^{\pi}y^{\frac{q'}{p_2'}-1}|I_n(y)|dy\Big)^{\frac{1}{q'}}\\
&\lesssim m^{-\frac{1}{p_1'}}n^{-\frac{1}{p_2'}}\Big(\int_0^{\pi}\int_0^{\pi}x^{\frac{q}{p_1}-1}y^{\frac{q}{p_2}-1}|f(x,y)|^q dxdy\Big)^{\frac{1}{q}}\\
&=: m^{-\frac{1}{p_1'}}n^{-\frac{1}{p_2'}}\|f\|_{L^q_{w((p_1,p_2),q)}}. \qquad (2.17)\end{aligned}$$

Similarly, if $q=1$,

$$\begin{aligned}\frac{1}{mn}\Big|\sum_{k=1}^{m}\sum_{l=1}^{n}a_{kl}\Big| &\leq \int_0^{\pi}\int_0^{\pi}|f(x,y)I_m(x)I_n(y)|dxdy\\
&\leq \sup_{x\in[0,\pi]}x^{\frac{1}{p_1'}}|I_m(x)|\cdot\sup_{y\in[0,\pi]}y^{\frac{1}{p_2'}}|I_n(y)|\cdot\int_0^{\pi}\int_0^{\pi}x^{\frac{1}{p_1}-1}y^{\frac{1}{p_2}-1}|f(x,y)|\,dxdy\\
&\lesssim m^{-\frac{1}{p_1'}}n^{-\frac{1}{p_2'}}\|f\|_{L^1_{w((p_1,p_2),1)}}.\end{aligned}$$

Thus, for any $1 < p_1, p_2 < \infty,\ 1 \le q \le \infty$, and $m \in \mathbb{N}$, we obtain

$$m^{\frac{1}{p_1'}} \sup_{n\in\mathbb{N}} n^{\frac{1}{p_2'}} \sup_{k\ge m,\, l\ge n} \frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l} a_{st}\Big| \le C\|f\|_{L^q_{w((p_1,p_2),q)}}, \tag{2.18}$$

with the constant C independent of m.

Now, in order to prove the desired inequality, we will invoke interpolation theory. Recall that the norm of a sequence $\mathbf{c} := \{c_k\}_{k=1}^{\infty}$ in the discrete Lorentz space $l_{p,q}$, for $p \in (1,\infty)$ and $q \in (0,\infty]$, is defined as follows

$$\|\mathbf{c}\|_{l_{p,q}} := \begin{cases} \left(\sum_{k=1}^{\infty} k^{\frac{q}{p}-1}|c_k^*|^q\right)^{\frac{1}{q}}, & \text{if } q < \infty, \\ \sup\limits_{k\ge 1} k^{\frac{1}{p}}|c_k^*|, & \text{if } q = \infty, \end{cases}$$

where $\{c_k^*\}$ stands for the decreasing rearrangement of $\mathbf{c}$. It follows from [8, Th. 5.3.1] that for $\theta \in (0,1)$ and $q \in (0,\infty]$, for the discrete Lorentz spaces $l_{p_1,\infty}$ and $l_{p_2,\infty}$, $0 < p_1 < p_2 \le \infty$, with $\theta/p_1 + (1-\theta)/p_2 = 1/p$, we have

$$(l_{p_1,\infty}, l_{p_2,\infty})_{\theta,q} = l_{p,q}. \tag{2.19}$$

For the Lebesgue spaces $L^q_{w((p_{11},p_{21}),q)}$ and $L^q_{w((p_{21},p_{22}),q)}$, $q \in (0,\infty]$, (see (2.17)), with

$$\frac{\theta}{p_{11}} + \frac{1-\theta}{p_{12}} = \frac{1}{p_1}, \qquad \frac{\theta}{p_{21}} + \frac{1-\theta}{p_{22}} = \frac{1}{p_2},$$

[8, Th. 5.4.1] gives

$$(L^q_{w((p_{11},p_{21}),q)}, L^q_{w((p_{12},p_{22}),q)})_{\theta,q} = L^q_{w((p_1,p_2),q)}. \tag{2.20}$$

For any fixed $m_0 \in \mathbb{N}$, in light of the monotonicity of $\sup\limits_{k\ge m_0,\, l\ge n} \frac{1}{kl}\Big|\sum\limits_{s=1}^{k}\sum\limits_{t=1}^{l} a_{st}\Big|$ in n, (2.18) is equivalent to

$$m_0^{\frac{1}{p_1'}} \left\|\left\{ \sup_{k\ge m_0,\, l\ge n} \frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l} a_{st}\Big|\right\}_{n=1}^{\infty}\right\|_{l_{p_2',\infty}} \le C\|f\|_{L^q_{w((p_1,p_2),q)}}. \tag{2.21}$$

Fix now $p_1, p_2 \in (1,\infty)$ and $q \in [1,\infty]$. Take $\theta \in (0,1)$ and $p_{11} < p_{12}$, $p_{21} < p_{22}$ such that $\theta/p_{11} + (1-\theta)/p_{12} = 1/p_1$ and $\theta/p_{21} + (1-\theta)/p_{22} = 1/p_2$. Note that, for any fixed m_0, the operator

$$T_{m_0} f = \left\{ \sup_{k\ge m_0,\, l\ge n} \frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l} a_{st}\Big|\right\}_{n=1}^{\infty}$$

is sublinear and that due to (2.21)

$$T_{m_0} : L^q_{w((p_1,p_{21}),q)} \to l_{p_{21}',\infty} \quad \text{and} \quad T_{m_0} : L^q_{w((p_1,p_{22}),q)} \to l_{p_{22}',\infty},$$

where the involved constants do not depend on m_0. Then it follows from [52, Th. 6], (2.19), and (2.20) that

$$T_{m_0} : L^q_{w((p_1,p_2),q)} = (L^q_{w((p_1,p_{21}),q)}, L^q_{w((p_1,p_{22}),q)})_{\theta,q} \to (l_{p_{21}',\infty}, l_{p_{22}',\infty})_{\theta,q} = l_{p_2',q},$$

so we arrive at

$$m^{\frac{1}{p_1'}}\Big\|\Big\{\sup_{k\ge m,\,l\ge n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big\}_{n=1}^{\infty}\Big\|_{l_{p_2,q}}\lesssim\|f\|_{L^q_{w((p_1,p_2),q)}},\tag{2.22}$$

for any m. Now we note that

$$\Big\|\Big\{\sup_{k\ge m,\,l\ge n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big\}_{n=1}^{\infty}\Big\|_{l_{p_2,q}}=\Big(\sum_{n=1}^{\infty}n^{\frac{q}{p_2}-1}\Big(\sup_{k\ge m,\,l\ge n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big)^q\Big)^{1/q}$$

is decreasing in m for any $p_2\in(1,\infty)$ and that the operator

$$Tf=\Big\{\Big(\sum_{n=1}^{\infty}n^{\frac{q}{p_2}-1}\Big(\sup_{k\ge m,\,l\ge n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big)^q\Big)^{1/q}\Big\}_{m=1}^{\infty}$$

is sublinear. Since according to (2.22) we have

$$T:L^q_{w((p_{11},p_2),q)}\to l_{p_{11}',\infty}\quad\text{and}\quad T:L^q_{w((p_{12},p_2),q)}\to l_{p_{12}',q},$$

we can once again apply [52, Th. 6] and obtain

$$T:L^q_{w((p_1,p_2),q)}=(L^q_{w((p_{11},p_2),q)},L^q_{w((p_{12},p_2),q)})_{\theta,q}\to(l_{p_{11}',\infty},l_{p_{12}',\infty})_{\theta,q}=l_{p_1',q}.$$

The latter means that

$$\Big\|\Big\{\Big(\sum_{n=1}^{\infty}n^{\frac{q}{p_2}-1}\Big(\sup_{k\ge m,\,l\ge n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big)^q\Big)^{\frac{1}{q}}\Big\}_{m=1}^{\infty}\Big\|_{l_{p_1,q}}\lesssim\|f\|_{L^q_{w((p_1,p_2),q)}},$$

whence the claim follows by putting $p_1=p_2=p$. □

Proof of Theorem 2.1. In light of Lemma 2.5 it suffices to prove the theorem only for either odd or even in each variable functions, omitting therefore the upper indices of a_{mn}.

We start with the part a). Due to Lemma 2.6, for $q<\infty$, there holds

$$\begin{aligned}\|f\|^q_{L^q_{w(p,q)}}&\gtrsim\sum_{m,n=1}^{\infty}\Big(\sup_{k\ge m,\,l\ge n}\frac{1}{kl}\Big|\sum_{s=1}^{k}\sum_{t=1}^{l}a_{st}\Big|\Big)^q(mn)^{\frac{q}{p'}-1}\\&\asymp\sum_{m,n=0}^{\infty}2^{(m+n)\frac{q}{p'}}\Big(\sup_{k\ge 2^m,\,l\ge 2^n}\frac{1}{kl}\Big|\sum_{i=1}^{k}\sum_{j=1}^{l}a_{ij}\Big|\Big)^q=:\sum_{m,n=0}^{\infty}P_{mn}.\end{aligned}\tag{2.23}$$

Denote

$$W_{mn}:=\sum_{k=2^m}^{2^{m+1}-1}\sum_{l=2^n}^{2^{n+1}-1}|a_{kl}|^q(kl)^{\frac{q}{p'}-1}.$$

First, we consider GM_1^c sequences. Let us fix some $T>1$. We call a pair (m,n) *good* (we write $(m,n)\in G$), if either $mn=0$ or

$$A_{m-1,n-1}\le TA_{mn}.$$

We have

$$\begin{aligned}\sum_{k,l=1}^{\infty}|a_{kl}|^q(kl)^{\frac{q}{p'}-1} &= \sum_{m,n=0}^{\infty}W_{mn}\\ &\le \sum_{m=0}^{\infty}W_{m0}+\sum_{n=0}^{\infty}W_{0n}+\sum_{(m,n)\in G\cap\mathbb{N}^2}W_{mn}+\sum_{(m,n)\in G}\sum_{(k,l)\in B_{mn}}W_{kl}\\ &=: J_1+J_2+J_3+J_4,\end{aligned}$$

where $B_{mn},\ (m,n)\in G$, stands for the set of all pairs $(k,l)\notin G$ such that $k=m+t,\ l=n+t$ for some $t\in\mathbb{N}$.

According to the one-dimentional Hardy-Littlewood theorem for GM sequences [23, Th. 1.2], we obtain

$$\begin{aligned}J_1=\sum_{m=0}^{\infty}W_{m0}=\sum_{k=1}^{\infty}|a_{k1}|^qk^{\frac{q}{p'}-1}\lesssim \|g\|^q_{L^q_{w(p,q)}} &= \int_{-\pi}^{\pi}x^{\frac{q}{p}-1}\Big|\int_{-\pi}^{\pi}f(x,y)\sin y dy\Big|^q dx\\ &\le \int_{-\pi}^{\pi}x^{\frac{q}{p}-1}\int_{-\pi}^{\pi}|f(x,y)|^q|y|^q dydx\lesssim \|f\|^q_{L^q_{w(p,q)}},\end{aligned}\tag{2.24}$$

where $g(x):=\int_{-\pi}^{\pi}f(x,y)\sin y\ dy$, since $|y|^q\lesssim|y|^{q/p-1}$ in $[-\pi,\pi]$. A similar estimate is valid for J_2.

Consider a pair $(m,n)\in G\cap\mathbb{N}^2$. Denote the rectangles we constructed in Lemma 2.2a) by $[s^1_{mn},s^2_{mn}]\times[t^1_{mn},t^2_{mn}]$, so that applying this lemma we have

$$\begin{aligned}&P_{m-1,n-1}=2^{(m+n-2)\frac{q}{p'}}\Big(\sup_{k\ge 2^{m-1},\ l\ge 2^{n-1}}\frac{1}{kl}\Big|\sum_{i=1}^{k}\sum_{j=1}^{l}a_{ij}\Big|\Big)^q\\ &\gtrsim 2^{(m+n)\frac{q}{p'}-(m+n)q}\left(\Big|\sum_{i=1}^{s^1_{mn}-1}\sum_{j=1}^{t^1_{mn}-1}a_{ij}\Big|^q+\Big|\sum_{i=1}^{s^1_{mn}-1}\sum_{j=1}^{t^2_{mn}}a_{ij}\Big|^q+\Big|\sum_{i=1}^{s^2_{mn}}\sum_{j=1}^{t^1_{mn}-1}a_{ij}\Big|^q+\Big|\sum_{i=1}^{s^2_{mn}}\sum_{j=1}^{t^2_{mn}}a_{ij}\Big|^q\right)\\ &\gtrsim 2^{(m+n)\frac{q}{p'}-(m+n)q}\Big|\sum_{i=s^1_{mn}}^{s^2_{mn}}\sum_{j=t^1_{mn}}^{t^2_{mn}}a_{ij}\Big|^q\gtrsim 2^{(m+n)\frac{q}{p'}}A^q_{mn}\gtrsim W_{mn}.\end{aligned}$$

Here we used the inequality

$$|x+y+z+t|+|x+y|+|x+z|+|x|\ge|z+t|+|z|\ge|t|,$$

which is valid for any $x,y,z,t\in\mathbb{C}$.
Hence, using (2.23), we obtain

$$J_3=\sum_{(m,n)\in G\cap\mathbb{N}^2}W_{mn}\lesssim\sum_{(m,n)\in G\cap\mathbb{N}^2}P_{m-1,n-1}\le\|f\|^q_{L^q_{w(p,q)}}.\tag{2.25}$$

Finally, combining (2.24) with the analogous estimate for J_2 and with (2.25), we derive

$$J_4 \le \sum_{(m,n)\in G} W_{mn} \sum_{j=1}^{\infty} T^{-j} \le \frac{1}{1-T^{-1}}(J_1+J_2+J_3) \lesssim \|f\|^q_{L^q_{w(p,q)}},$$

which concludes the proof of the first part for the case of GM_1^c.

A simplified version of the argument above yields the result for $q=\infty$.

If we replace GM_1^c by GM_2^c, i.e. (2.9) by (2.10), we change the definition of a good pair of numbers to the following one: we call a pair (m,n) good, if either $mn=0$ or $A_{m+1,n-1} \le TA_{mn}$. The rest of the proof is the same in light of Lemma 2.2b) with the only changes: now B_{mn}, $(m,n)\in G$, stands for the set of all pairs $(k,l)\notin G$ such that $k=m-t$, $l=n+t$ for some $t\in\mathbb{N}$ and $P_{m-1,n-1}$ in (2.25) becomes $P_{m+1,n-1}$.

Turn now to the part b). Note that if $\{a_{mn}\}\in GM_1^c\cup GM_2^c$ and $\sum_{m,n=1}^{\infty}|a_{mn}|^q(mn)^{\frac{q}{p'}-1} < \infty$, then we have $\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|\Delta^{11}a_{kl}| < \infty$, which implies that the corresponding trigonometric series converges in the Pringsheim sense (that is, by rectangles) everywhere on $(0,2\pi)^2$ and is the Fourier series of its sum (see [20, L. 4]). Indeed, under condition (2.9) we have by (2.12) and Hölder's inequality

$$\begin{aligned}\sum_{k,l=1}^{\infty}|\Delta^{11}a_{kl}| &\lesssim \sum_{k=0}^{\infty}|a_{2^k,2^k}| \lesssim \sum_{k=0}^{\infty}|a_{2^k,2^k}| \sum_{m=2^{k-1}}^{2^k}\sum_{n=2^{k-1}}^{2^k}(mn)^{-1} \\ &\lesssim \sum_{m,n=1}^{\infty}|a_{mn}|(mn)^{-1} = \sum_{m,n=1}^{\infty}|a_{mn}|(mn)^{\frac{1}{p'}-\frac{1}{q}}(mn)^{-\frac{1}{p'}-\frac{1}{q'}} \\ &\lesssim \Big(\sum_{m,n=1}^{\infty}|a_{mn}|^q(mn)^{\frac{q}{p'}-1}\Big)^{\frac{1}{q}}\Big(\sum_{m,n=1}^{\infty}(mn)^{-\frac{q'}{p'}-1}\Big)^{\frac{1}{q'}} < \infty,\end{aligned}$$

and similarly under (2.10),

$$\begin{aligned}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}|\Delta^{11}a_{kl}| \lesssim \sum_{k=0}^{\infty}|a_{2^{k+1},2^k}| &\lesssim \sum_{k=0}^{\infty}|a_{2^{k+1},2^k}|\sum_{m=2^{k+1}}^{2^{k+2}}\sum_{n=2^{k-1}}^{2^k}(mn)^{-1} \\ &\lesssim \sum_{m,n=1}^{\infty}|a_{mn}|(mn)^{-1} < \infty.\end{aligned}$$

We will provide the proof only for the system $\{\sin mx, \sin ny\}$, the other cases will follow then from boundedness of Hilbert transform in the weighted Lebesgue spaces.

For $(x,y)\in(\frac{\pi}{m+1},\frac{\pi}{m}]\times(\frac{\pi}{n+1},\frac{\pi}{n}]$, we have

$$\begin{aligned}|f(x,y)| &= \Big|\sum_{k=1}^{\infty}\sum_{l=1}^{\infty} a_{kl}\sin kx\sin ly\Big| \le xy\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| \\ &+ x\sum_{k=1}^{m} k\sum_{l=n}^{\infty}|a_{kl}-a_{k,l+1}||\tilde{D}_l(y)-\tilde{D}_n(y)| \\ &+ y\sum_{l=1}^{n} l\sum_{k=m}^{\infty}|a_{kl}-a_{k+1,l}||\tilde{D}_k(x)-\tilde{D}_m(x)| \\ &+ \sum_{k=m}^{\infty}\sum_{l=n}^{\infty}|\Delta^{11}a_{kl}|\cdot|(\tilde{D}_k(x)-\tilde{D}_m(x))(\tilde{D}_l(y)-\tilde{D}_n(y))| \\ &\lesssim \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| + \frac{n}{m}\sum_{k=1}^{m} k\sum_{l=n}^{\infty}|a_{kl}-a_{k,l+1}| \\ &+ \frac{m}{n}\sum_{l=1}^{n} l\sum_{k=m}^{\infty}|a_{kl}-a_{k+1,l}| + mn\sum_{k=m}^{\infty}\sum_{l=n}^{\infty}|\Delta^{11}a_{kl}|.\end{aligned}$$

Applying condition (2.9), we derive

$$\begin{aligned}|f(x,y)| &\lesssim \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| + \frac{n}{m}\sum_{k=1}^{m} k\sum_{t=0}^{\infty}|a_{k,2^t n}| + \frac{m}{n}\sum_{l=1}^{n} l\sum_{t=0}^{\infty}|a_{2^t m,l}| + mn\sum_{t=0}^{\infty}|a_{2^t m,2^t n}| \\ &\lesssim \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| + \frac{n}{m}\sum_{k=1}^{m} k\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{l} \\ &+ \frac{m}{n}\sum_{l=1}^{n} l\sum_{k=\lceil m/2\rceil}^{\infty}\frac{|a_{kl}|}{k} + mn\sum_{k=\lceil m/2\rceil}^{\infty}\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{kl}.\end{aligned}$$

In turn, (2.10) yields

$$\begin{aligned}|f(x,y)| &\lesssim \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| + \frac{n}{m}\sum_{k=1}^{m} k\sum_{t=0}^{\infty}|a_{k,2^t n}| \\ &+ \frac{m}{n}\sum_{l=1}^{n} l\sum_{t=0}^{\infty}|a_{2^{t+1}m,l}| + mn\sum_{t=0}^{\infty}|a_{2^{t+1}m,2^t n}| \\ &\lesssim \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| + \frac{n}{m}\sum_{k=1}^{m} k\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{l} \\ &+ \frac{m}{n}\sum_{l=1}^{n} l\sum_{k=2m}^{\infty}\frac{|a_{kl}|}{k} + mn\sum_{k=2m}^{\infty}\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{kl}.\end{aligned}$$

Hence, in both cases we get

$$\begin{aligned}|f(x,y)| &\lesssim \frac{1}{mn}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}| + \frac{n}{m}\sum_{k=1}^{m} k\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{l} \\ &+ \frac{m}{n}\sum_{l=1}^{n} l\sum_{k=\lceil m/2\rceil}^{\infty}\frac{|a_{kl}|}{k} + mn\sum_{k=\lceil m/2\rceil}^{\infty}\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{kl} \\ &=: I^1_{m,n} + I^2_{m,n} + I^3_{m,n} + I^4_{m,n}. \qquad (2.26)\end{aligned}$$

Thus, for $q<\infty$, denoting $\alpha := 1-q/p$, we obtain

$$\begin{aligned}\|f\|^q_{L^q_{p,q}} &\asymp \int_0^\pi\int_0^\pi (xy)^{-\alpha}|f(x,y)|^q\,dxdy\\ &\lesssim \sum_{m=1}^\infty\sum_{n=1}^\infty \int_{\frac{\pi}{m+1}}^{\frac{\pi}{m}}\int_{\frac{\pi}{n+1}}^{\frac{\pi}{n}} (xy)^{-\alpha}(I^1_{m,n}+I^2_{m,n}+I^3_{m,n}+I^4_{m,n})^q\,dxdy\\ &\asymp \sum_{m=1}^\infty\sum_{n=1}^\infty (mn)^{\alpha-2}((I^1_{m,n})^q+(I^2_{m,n})^q+(I^3_{m,n})^q+(I^4_{m,n})^q).\end{aligned}$$

Recall the Hardy-type inequalities for power weights (see, for instance, [46, (0.6), (0.10), (1.102)]) for $q\ge 1$:

$$\sum_{n=1}^\infty n^\gamma\Big(\sum_{k=1}^n a_k\Big)^q \lesssim_q \sum_{n=1}^\infty n^{\gamma+q}a_n^q, \quad \text{for } \gamma<-1, \tag{2.27}$$

and its dual,

$$\sum_{n=1}^\infty n^\gamma\Big(\sum_{k=n}^\infty a_k\Big)^q \lesssim_q \sum_{n=1}^\infty n^{\gamma+q}a_n^q, \quad \text{for } \gamma>-1. \tag{2.28}$$

Using (2.27) in each variable we arrive at

$$\begin{aligned}\sum_{m=1}^\infty\sum_{n=1}^\infty (mn)^{\alpha-2}(I^1_{m,n})^q &= \sum_{m=1}^\infty m^{\alpha-2-q}\sum_{n=1}^\infty n^{\alpha-2-q}\Big(\sum_{l=1}^n l\sum_{k=1}^m k|a_{kl}|\Big)^q\\ \lesssim \sum_{n=1}^\infty n^{\alpha-2+q}\sum_{m=1}^\infty m^{\alpha-2-q}\Big(\sum_{k=1}^m k|a_{kn}|\Big)^q &\lesssim \sum_{m=1}^\infty\sum_{n=1}^\infty (mn)^{\alpha-2+q}|a_{mn}|^q\end{aligned}$$

and

$$\begin{aligned}\sum_{m=1}^\infty\sum_{n=1}^\infty (mn)^{\alpha-2}(I^2_{m,n})^q &= \sum_{m=1}^\infty m^{\alpha-2-q}\sum_{n=1}^\infty n^{\alpha-2+q}\Big(\sum_{l=\lceil n/2\rceil}^\infty \frac{1}{l}\sum_{k=1}^m k|a_{kl}|\Big)^q\\ &\asymp \sum_{m=1}^\infty m^{\alpha-2-q}\sum_{n=1}^\infty n^{\alpha-2+q}\Big(\sum_{l=n}^\infty \frac{1}{l}\sum_{k=1}^m k|a_{kl}|\Big)^q\\ &\lesssim \sum_{n=1}^\infty n^{\alpha-2+q}\sum_{m=1}^\infty m^{\alpha-2-q}\Big(\sum_{k=1}^m k|a_{kn}|\Big)^q\\ &\lesssim \sum_{m=1}^\infty\sum_{n=1}^\infty (mn)^{\alpha-2+q}|a_{mn}|^q,\end{aligned}$$

where we used inequality (2.12). The similar estimate holds for $I^3_{m,n}$. And finally, due to

(2.28), we have

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\alpha-2}(I^4_{m,n})^q = \sum_{m=1}^{\infty} m^{\alpha-2+q}\sum_{n=1}^{\infty} n^{\alpha-2+q}\Big(\sum_{l=\lceil n/2\rceil}^{\infty}\frac{1}{l}\sum_{k=\lceil m/2\rceil}^{\infty}\frac{|a_{kl}|}{k}\Big)^q$$
$$\asymp \sum_{m=1}^{\infty} m^{\alpha-2+q}\sum_{n=1}^{\infty} n^{\alpha-2+q}\Big(\sum_{l=n}^{\infty}\frac{1}{l}\sum_{k=m}^{\infty}\frac{|a_{kl}|}{k}\Big)^q$$
$$\lesssim \sum_{m=1}^{\infty} m^{\alpha-2+q}\sum_{n=1}^{\infty} n^{\alpha-2+q}\Big(\sum_{k=m}^{\infty}\frac{|a_{kn}|}{k}\Big)^q$$
$$\lesssim \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}(mn)^{\alpha-2+q}|a_{mn}|^q,$$

which completes the proof for the case $q\in[1,\infty)$. For $q=\infty$, using (2.26) we can write

$$\sup_{(x,y)\in(\frac{\pi}{m+1},\frac{\pi}{m}]\times(\frac{\pi}{n+1},\frac{\pi}{n}]}(xy)^{\frac{1}{p}}|f(x,y)| \le (mn)^{-\frac{1}{p}}(I^1_{m,n}+I^2_{m,n}+I^3_{m,n}+I^4_{m,n}).$$

Next,

$$(mn)^{-\frac{1}{p}}I^1_{m,n} = (mn)^{-\frac{1}{p}-1}\sum_{k=1}^{m}\sum_{l=1}^{n} kl|a_{kl}|$$
$$\le (mn)^{-\frac{1}{p}-1}\sum_{k=1}^{m}\sum_{l=1}^{n}(kl)^{\frac{1}{p}}\sup_{k,l}\Big((kl)^{\frac{1}{p'}}|a_{kl}|\Big) \lesssim \sup_{k,l}\Big((kl)^{\frac{1}{p'}}|a_{kl}|\Big).$$

We also have

$$(mn)^{-\frac{1}{p}}I^2_{m,n} = (mn)^{-\frac{1}{p}}\frac{n}{m}\sum_{k=1}^{m}\sum_{l=\lceil n/2\rceil}^{\infty}\frac{k}{l}|a_{kl}| \lesssim \sup_{k,l}\Big((kl)^{\frac{1}{p'}}|a_{kl}|\Big),$$

and the similar estimate for $I^3_{m,n}$. Finally,

$$(mn)^{-\frac{1}{p}}I^4_{m,n} = (mn)^{-\frac{1}{p}}mn\sum_{k=\lceil m/2\rceil}^{\infty}\sum_{l=\lceil n/2\rceil}^{\infty}\frac{|a_{kl}|}{kl} \lesssim \sup_{k,l}\Big((kl)^{\frac{1}{p'}}|a_{kl}|\Big),$$

which completes the proof of the theorem. □

Remark 2.7. For the spaces $L^q_{w(p,q)}(0,2\pi)^2$ in place of $L^q_{w(p,q)}(-\pi,\pi)^2$, the assertion of Theorem 2.1 still holds for $q\le p$ but fails for $q>p$.

Indeed, for $q>p$ it suffices to consider the one-dimensional sine series

$$f(x) := \sum_{k=1}^{\infty} k^{-\frac{1}{p'}}\log^{-\frac{1}{p}}(k+2)\sin kx =: \sum_{k=1}^{\infty} a_k\sin kx.$$

We have $\sum|a_k|^p k^{p-2} = \sum k^{-1}\log^{-1}(k+2) = \infty$, so by the Hardy-Littlewood theorem $f\notin L_p$, whence $\|f\|_{L^q_{w(p,q)}(0,2\pi)} \gtrsim \|f\|_{L_p(\pi,2\pi)} = \infty$. On the other hand,

$$\|f\|_{L^q_{w(p,q)}(-\pi,\pi)} \asymp \sum|a_k|^q k^{\frac{q}{p'}-1} = \sum k^{-1}\log^{-\frac{q}{p}}(k+2) < \infty.$$

However, for $q \le p$, there holds $x^{q/p-1} \gtrsim 1$, so that

$$\|f\|_{L^q_{w(p,q)}(0,2\pi)} \asymp \|f\|_{L^q_{w(p,q)}(0,\pi)} + \|f\|_{L^q_{w(p,q)}(\pi,2\pi)} \asymp \|f\|_{L^q_{w(p,q)}(0,\pi)} \asymp \|f\|_{L^q_{w(p,q)}(-\pi,\pi)}.$$

The reason of the failure of the Hardy-Littlewood relation here is that the function in case is supposed to be periodic, while a power weight is not. Thus, if one deals with weighted Lebesgue spaces on $[0,2\pi]^2$, it makes more sense to consider a weight of the type $|\sin x|^\alpha$ in place of $|x|^\alpha$, which was in fact done by many authors. Note that for a power weight, weighted integrability at 2π is equivalent to integrability at zero without weight, so, as in the example above, one has to additionally check integrability at zero.

2.4 Sharpness of the result

Theorem 2.8. *For $p > 2$, $q \ge p$, the claim of Theorem 2.1a) does not hold if we replace the GM_2^c condition (2.10) by*

$$\sum_{m=k}^{2k} \sum_{n=l}^{2l} |\Delta^{11} a_{mn}| \le C|a_{2k,l}|. \tag{2.29}$$

Proof. Assume that $p > 2$ and consider the sequence

$$a_{mn} := \frac{(-1)^{\delta_m}}{m^\gamma} g_m(n),$$

where $\gamma > 0$ and $\delta_m \in \{0,1\}$ are to be chosen later, and $g_m(n) = g_m(n,p')$ is defined as follows

$$g_m(n) := \begin{cases} (-1)^{\delta_m} m^{-3} n^{-\frac{1}{p'}}, & \log_2 n < m(m+1)p', \\ 2^{-(m+t)^2-3(m+t)}, & ((m+t)^2 + m - t)p' \le \log_2 n < ((m+t)^2 + 3m + t)p', \quad t \in \mathbb{Z}_+. \end{cases}$$

In other words, the functions g_m are constructed in the following way. First, we divide $[1,\infty)$ into the intervals I_j, $j = 0,1,...$, so that $I_j := \{x : 2p'j \le \log_2 x < 2p'(j+1)\}$. After that consider the lower-triangular infinite down and to the right matrix that is filled by all positive integers in increasing order going down and to the right.

$$\begin{matrix} 1 \\ 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 & 10 \\ \vdots & \vdots & \vdots & \vdots \end{matrix}$$

Next, for any j we asign it the integer $i = i(j)$ if it is ith column that contains the element j. Fix some m and consider the values $g_m(1), g_m(2), ...$. While $i(j) \ne m$, we have

$$g_m(n) = (-1)^{\delta_m} m^{-3} n^{-\frac{1}{p'}}$$

for $n \in I_j$. Once $i(j)$ becomes equal to m for the first time, that is, when $\log_2 n \ge m(m+1)p'$ for the first time, we get $g_m(n) = 2^{-m^2-3m}$ and this value does not change till $i(j)$ becomes equal to m again and $n \in I_j$. When $i(j)$ becomes equal to m for the

$(s+1)$th time, the value $g_m(n)$ changes for $2^{-(m+s)^2-3(m+s)}$ (see Figure 2.1 for a scheme of changes of absolute values of $g_m(n)$).

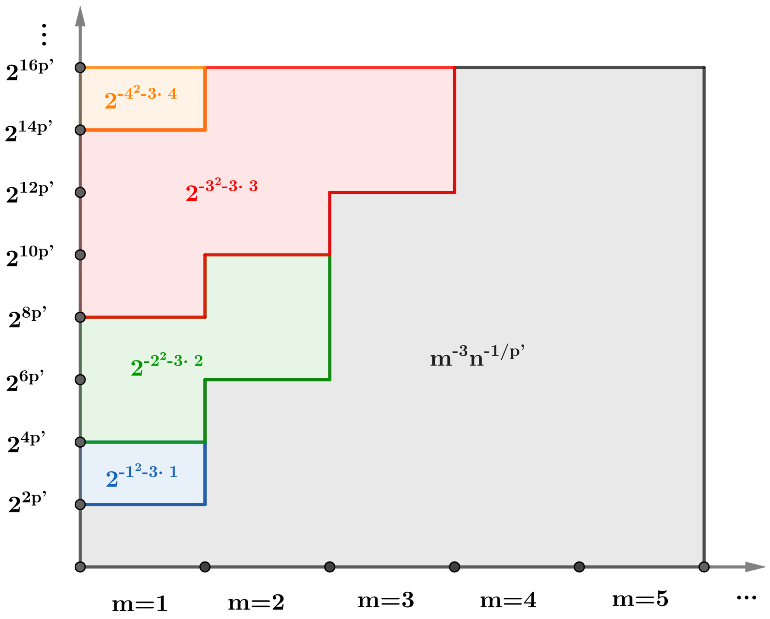

Figure 2.1:

Fix $n \in I_j$ for some j and consider $g_1(n), g_2(n), \ldots$ Let k be such that $\log_2 n < m(m+1)p'$ if $1 \leq m \leq k$ and $\log_2 n \geq m(m+1)p'$ if $m \geq k+1$. Then

$$|g_m(n)| \lesssim |g_{m'}(n)|, \quad \text{for } k+1 \leq m < m' \leq 2m. \tag{2.30}$$

Denote $m_0 := i(j+1)$. If $m_0 = k+1$, then $g_1(n) = g_2(n) = \ldots = g_k(n) = 2^{-(k+1)^2-3(k+1)}$, otherwise, $g_m(n) = 2^{-(k+1)^2-3(k+1)}$ for $m \leq m_0 - 1$ and $g_m(n) = 2^{-k^2-3k}$ for $m_0 \leq m \leq k$. Let us compare $g_k(n)$ and $g_{k+1}(n)$. There are two cases.

Case 1. $m_0 = i(j+1) = k+1$. Then

$$|g_{k+1}(n)| = (k+1)^{-3} n^{-\frac{1}{p'}} \gtrsim (k+1)^{-3} 2^{-(k+1)(k+2)} \gtrsim 2^{-(k+1)^2-3(k+1)} = g_k(n).$$

Case 2. $m_0 = i(j+1) < k+1$. Then

$$|g_{k+1}(n)| = (k+1)^{-3} n^{-\frac{1}{p'}} \gtrsim (k+1)^{-3} 2^{-k(k+1)-m_0} \gtrsim 2^{-k^2-3k} = g_k(n).$$

Thus, in both cases we obtain $0 < g_1(n) \leq g_2(n) \leq \ldots \leq g_k(n) \lesssim |g_{k+1}(n)|$, whence in light of (2.30),

$$|g_m(n)| \lesssim |g_{m'}(n)|, \quad \text{for all } m < m' \leq 2m. \tag{2.31}$$

It remains to note that for a fixed m, we have for $n_m := \lceil 2^{m(m+1)p'} \rceil - 1$ that

$$|g_m(n_m)| = m^{-3} n_m^{-\frac{1}{p'}} \asymp m^{-3} 2^{-m(m+1)} \gtrsim 2^{-m^3-3m} = g_m(n_m+1)$$

and for other n there holds $g_m(n) \geq g_m(n+1)$. So, over all $|a_{mn}|$ in $r_{kl} := [k, 2k] \times [l, 2l]$, the maximal is up to a constant $|a_{2k,l}|$.

Further we note that the constructed sequence clearly satisfies (2.5).

To prove that our sequence belongs to GM_2^c, let us estimate $\sum_{m=k}^{2k} \sum_{n=l}^{2l} |\Delta^{11} a_{mn}|$. Consider a quadruple

$$\begin{matrix} a_{m,n+1} & a_{m+1,n+1} \\ a_{mn} & a_{m+1,n} \end{matrix}$$

with $(m, n) \in r_{kl}$. Note that it can be only of the following five types

$$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix} \qquad \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}$$

where 0 stands for the terms with $\log_2 n < m(m+1)p'$, while 1, for those with $\log_2 n \geq m(m+1)p'$. We will write $(m, n) \in T_i, \ i = 1, ..., 5$, if the corresponding quadruple is of the ith type. Note that if $(m, n) \in T_3$, then $(m-1, n) \in T_1$ and $(m+1, n) \in T_2$, while if $(m, n) \in T_4$, then $(m-1, n) \in T_2$ and $(m+1, n) \in T_5$. By the construction, quadruples of the three last types with nonzero $\Delta^{11} a_{mn}$ can appear at most four times in r_{kl}, since any $(m, n) \in T_3 \cup T_4$, as well as $(m, n) \in T_5$ with nonzero $\Delta^{11} a_{mn}$, satisfies $n \in I_j, n+1 \in I_{j+1}$, for some j, which cannot happen twice in $[l, 2l]$. If there exists a quadruple of the first type, then

$$\sum_{(m,n) \in T_1 \cap r_{kl}} |\Delta^{11} a_{mn}| = \sum_{(m,n) \in T_1 \cap r_{kl}} \Delta^{11} a_{mn} < \sum_{m \geq k, \, n \geq l} \Delta^{11} (m^{-3-\gamma} n^{-\frac{1}{p'}})$$
$$= k^{-3-\gamma} l^{-\frac{1}{p'}} \lesssim \max_{(m,n) \in r_{kl}} |a_{mn}|.$$

As for $(m, n) \in T_2 \cap r_{kl}$, they all belong to a strip $[k', k'+1] \times [l, 2l]$ for some k'. Indeed, otherwise there are m_1 and $m_2 \geq m_1 + 2$ belonging to $[k, 2k]$, and $n_1, n_2 \in [l, 2l]$ such that $(m_1, n_1), (m_2, n_2) \in T_2$. But it follows from $(m_1, n_1) \in T_2$ that $a_{m_1+1,k}$, and hence $a_{m_2,k}$, has type 0, while $(m_2, n_2) \in T_2$ implies that $a_{m_2,2k}$, and hence $a_{m_1+1,2k}$, has type 1. Thus, there exist two pairs of the form $(n, n+1)$ inside $[l, 2l]$ such that $n \in I_j, n+1 \in I_{j+1}$, for some j, which cannot be true. Therefore, all $(m, n) \in T_2 \cap r_{kl}$ do belong to a strip $[k', k'+1] \times [l, 2l]$, whence using

$$|\Delta^{11} a_{mn}| \leq |\Delta^{01} a_{mn}| + |\Delta^{01} a_{m+1,n}| = \Delta^{01} |a_{mn}| + \Delta^{01} |a_{m+1,n}|,$$

which is true as long as $(m, n) \in T_2 \cap r_{kl}$, we deduce that the sum of $|\Delta^{11} a_{mn}|$ over $(m, n) \in T_2 \cap r_{kl}$ is bounded above by four times the maximal $|a_{mn}|$ in r_{kl}. Combining the observations above, we arrive at

$$\sum_{m=k}^{2k} \sum_{n=l}^{2l} |\Delta^{11} a_{mn}| \lesssim \max_{(m,n) \in r_{kl}} |a_{mn}| \lesssim |a_{2k,l}|,$$

which proves (2.29).

Further, for any $q > 0$,

$$\sum_{m,n=1}^{\infty} |a_{mn}|^q (mn)^{\frac{q}{p'}-1} \gtrsim \sum_{m=1}^{\infty} m^{\frac{q}{p'}-1-\gamma q} \times \sum_{t=0}^{\infty} 2^{-((m+t)^2+3m+3t)q} 2^{((m+t)^2+3m+t)p'(\frac{q}{p'}-1)} 2^{((m+t)^2+3m+t)p'} \gtrsim \sum_{m=1}^{\infty} m^{\frac{q}{p'}-1-\gamma q} = \infty,$$

if we set $\gamma = 1/p'$.

Note that our sequence generates the Fourier sine (or cosine) series of an odd (or even) function f that converges in the Pringsheim sense everywhere on $(0, 2\pi)^2$ to f according to [20, L. 4]. To prove this, since the sequence fulfils (2.5), it suffices to show that the following sum is finite

$$\begin{aligned}\sum_{m,n=1}^{\infty} |\Delta^{11} a_{mn}| &\leq \sum_{(m,n)\in T_1} \Delta^{11} a_{mn} + \sum_{(m,n)\in T_2 \cup T_5} (|\Delta^{01} a_{mn}| + |\Delta^{01} a_{m+1,n}|) \\ &+ \sum_{(m,n)\in T_3 \cup T_4} (|a_{mn}| + |a_{m,n+1}| + |a_{m+1,n}| + |a_{m+1,n+1}|) \\ &\lesssim 1 + \sum_{(m,n)\in T_2 \cup T_5} (\Delta^{01} a_{mn} + \Delta^{01} a_{m+1,n}) + \sum_{m=1}^{\infty} m^{-3-\gamma} 2^{-m(m+1)} \\ &\lesssim 1 + \sum_{m=1}^{\infty} \sum_{t=0}^{\infty} 2^{-(m+t)^2 - 3(m+t)} + \sum_{(m,n)\in T_2} \Delta^{01} a_{m+1,n} \\ &\lesssim 1 + \sum_{m=1}^{\infty} m^{-3-\gamma} 2^{-m(m-1)} < \infty.\end{aligned}$$

Let us stick to the case of an odd f, as for cosine series the argument is exactly the same. Denote for $m, n \geq 1$,

$$c_{mn} := \begin{cases} a_{mn}, & \text{if } \log_2 n \geq m(m+1)p', \\ 0, & \text{otherwise,} \end{cases},$$

and $b_{mn} := a_{mn} - c_{mn}$. Then

$$\|f\|_{L^q_{w(p,q)}} \leq \Big\| \sum_{m,n=1}^{\infty} b_{mn} \sin mx \sin ny \Big\|_{L^q_{w(p,q)}} + \Big\| \sum_{m,n=1}^{\infty} c_{mn} \sin mx \sin ny \Big\|_{L^q_{w(p,q)}}.$$

Note that

$$\sum_{m=1}^{M}\sum_{n=1}^{N} b_{mn}\sin mx\sin ny = \sum_{m=1}^{M}\sin mx\Big(\sum_{n=1}^{N-1}\Delta^{01}b_{mn}D_n(y)+b_{mN}D_N(y)\Big)$$
$$= \sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\Delta^{11}b_{mn}D_m(x)D_n(y)+\sum_{n=1}^{N-1}\Delta^{01}b_{Mn}D_M(x)D_n(y)$$
$$+\sum_{m=1}^{M-1}\Delta^{10}b_{mN}D_m(x)D_N(y)+b_{MN}D_M(x)D_N(y)$$
$$=: \sum_{m=1}^{M-1}\sum_{n=1}^{N-1}\Delta^{11}b_{mn}D_m(x)D_n(y)+A_1+A_2+A_3.$$

Since $\|D_k\|^q_{L^q_{w(p,q)}} \asymp \sum_{l=1}^{k} l^{\frac{q}{p'}-1} \asymp k^{\frac{q}{p'}}$ by Theorem A, we have for $N_0 := \max(N-1, \lceil 2^{M(M+1)p'}\rceil - 1)$,

$$\|A_1\|_{L^q_{w(p,q)}} \lesssim \sum_{n=1}^{N_0} M^{-3-\gamma}n^{-1-\frac{1}{p'}}(Mn)^{\frac{1}{p'}} + M^{-3-\gamma}N_0^{-\frac{1}{p'}}(MN_0)^{\frac{1}{p'}} \lesssim M^{-1-\gamma}\to 0$$

as $M\to\infty$. For $M_0 := \min\{m : m(m+1)p' \geq N\}$,

$$\|A_2\|_{L^q_{w(p,q)}} \lesssim \sum_{m=M_0}^{M-1} m^{-4-\gamma}N^{-\frac{1}{p'}}(mN)^{\frac{1}{p'}} + M_0^{-3-\gamma}N^{-\frac{1}{p'}}(M_0N)^{\frac{1}{p'}} \to 0$$

as $N\to\infty$. And finally,

$$\|A_3\|_{L^q_{w(p,q)}} \lesssim M^{-3-\gamma}N^{-\frac{1}{p'}}(MN)^{\frac{1}{p'}} \to 0$$

as $M\to\infty$. Thus,

$$\Big\|\sum_{m,n=1}^{\infty} b_{mn}\sin mx\sin ny\Big\|_{L^q_{w(p,q)}} = \Big\|\sum_{m,n=1}^{\infty}\Delta^{11}b_{mn}D_m(x)D_n(y)\Big\|_{L^q_{w(p,q)}}. \tag{2.32}$$

Besides,

$$\sum_{m=1}^{M}\sum_{n=1}^{N} c_{mn}\sin mx\sin ny = \sum_{m=1}^{M}\sin mx\Big(\sum_{n=1}^{N-1}\Delta^{01}c_{mn}D_n(y)+c_{mN}D_N(y)\Big),$$

where in light of the inequalities $0 < g_1(n) \leq ... \leq g_{M_0}(n)$ for M_0 defined as above

$$\Big\|\sum_{m=1}^{M} c_{mN}\sin mxD_N(y)\Big\|_{L^q_{w(p,q)}} \lesssim \sum_{m=1}^{M_0}|c_{mN}|N^{\frac{1}{p'}} \leq M_0 g_{M_0}(N)N^{\frac{1}{p'}} \lesssim M_0^{-2}\to 0$$

as $N\to\infty$. Hence,

$$\Big\|\sum_{m,n=1}^{\infty} c_{mn}\sin mx\sin ny\Big\|_{L^q_{w(p,q)}} = \Big\|\sum_{m,n=1}^{\infty}\Delta^{01}c_{mn}\sin mxD_n(y)\Big\|_{L^q_{w(p,q)}}. \tag{2.33}$$

Combining (2.32) and (2.33) we arrive at

$$\|f\|_{L^q_{w(p,q)}} \le \Big\| \sum_{m,n=1}^{\infty} \Delta^{11} b_{mn} D_m(x) D_n(y) \Big\|_{L^q_{w(p,q)}} + \Big\| \sum_{m,n=1}^{\infty} \Delta^{01} c_{mn} \sin mx D_n(y) \Big\|_{L^q_{w(p,q)}}$$
$$=: S_1 + S_2.$$

First, for $n_m = \lceil 2^{m(m+1)p'} \rceil - 1$, we see that $\log n_m \asymp m^2$ and $\log n_{m+1} - \log n_m \asymp m$, so

$$S_1 \lesssim \sum_{m=1}^{\infty} m^{\frac{1}{p'}} \Big(\sum_{n=1}^{n_m - 1} \Delta^{11} (m^{-3-\gamma} n^{-\frac{1}{p'}}) n^{\frac{1}{p'}} + \sum_{n=n_m}^{n_{m+1}-1} \Delta^{01} ((m+1)^{-3-\gamma} n^{-\frac{1}{p'}}) n^{\frac{1}{p'}}$$
$$+ (m^{-3-\gamma} n_m^{-\frac{1}{p'}}) n_m^{\frac{1}{p'}} \Big)$$
$$\lesssim \sum_{m=1}^{\infty} m^{\frac{1}{p'}} \Big(\sum_{n=1}^{n_m-1} m^{-4-\gamma} n^{-1} + \sum_{n=n_m}^{n_{m+1}-1} m^{-3-\gamma} n^{-1} + m^{-3-\gamma} \Big) \lesssim \sum_{m=1}^{\infty} m^{\frac{1}{p'} - 2 - \gamma} < \infty.$$

Second, denoting $n_{mt} := \lceil 2^{((m+t)^2 + 3m + t)p'} \rceil - 1$, using $c_{mn} = (-1)^{\delta_m} |c_{mn}|$ and the fact that $\Delta^{01} c_{mn} \ne 0$ only if $n = n_{mt}$ for $t \ge -1$, we get for $q \ge p$,

$$S_2^q = \Big\| \sum_{m=1}^{\infty} (-1)^{\delta_m} \sin mx \sum_{t=-1}^{\infty} \Delta^{01} |c_{m,n_{mt}}| D_{n_{mt}}(y) \Big\|^q_{L^q_{w(p,q)}}$$
$$= \int_{-\pi}^{\pi} |y|^{\frac{q}{p}-1} \int_{-\pi}^{\pi} |x|^{\frac{q}{p}-1} \Big| \sum_{m=1}^{\infty} (-1)^{\delta_m} \sin mx \sum_{t=-1}^{\infty} \Delta^{01} |c_{m,n_{mt}}| D_{n_{mt}}(y) \Big|^q dx dy$$
$$\le \int_{-\pi}^{\pi} |y|^{\frac{q}{p}-1} \int_{-\pi}^{\pi} \Big| \sum_{m=1}^{\infty} (-1)^{\delta_m} \sin mx \sum_{t=-1}^{\infty} \Delta^{01} |c_{m,n_{mt}}| D_{n_{mt}}(y) \Big|^q dx dy. \tag{2.34}$$

Recall the Khintchine inequality (see e.g. [2, Rem. 1.4]): for any real sequence $\{s_k\} \in l_2$ and the system of Rademacher functions $\{r_n(t)\}$, we have

$$\int_0^1 \Big| \sum_{k=1}^{\infty} s_k r_k(t) \Big|^q \asymp_q \Big(\sum_{k=1}^{\infty} s_k^2 \Big)^{\frac{q}{2}}.$$

Hence,

$$\int_0^1 \int_{-\pi}^{\pi} \Big| \sum_{m=1}^{\infty} r_m(t) \sin mx \sum_{t=-1}^{\infty} \Delta^{01} |c_{m,n_{mt}}| D_{n_{mt}}(y) \Big|^q dx dt$$
$$\lesssim \int_0^1 \Big| \sum_{m=1}^{\infty} r_m(t) \sum_{t=-1}^{\infty} \Delta^{01} |c_{m,n_{mt}}| D_{n_{mt}}(y) \Big|^q dt$$
$$\lesssim \Big(\sum_{m=1}^{\infty} \Big(\sum_{t=-1}^{\infty} \Big| \Delta^{01} |c_{m,n_{mt}}| D_{n_{mt}}(y) \Big| \Big)^2 \Big)^{\frac{q}{2}}, \tag{2.35}$$

whenever the series on the right-hand side converges. Observe that by the Minkowski inequality and the fact that $\|D_{n_{mt}}\|_{L^q_{w(p,q)}} \asymp 2^{((m+t)^2+3m+t)p'\frac{1}{p'}}$, we have

$$\begin{aligned}
&\int_{-\pi}^{\pi} |y|^{\frac{q}{p}-1}\Big(\sum_{m=1}^{\infty}\Big(\sum_{t=-1}^{\infty}\Big|\Delta^{01}|c_{m,n_{mt}}|D_{n_{mt}}(y)\Big|\Big)^2\Big)^{\frac{q}{2}}dy \\
&\qquad\asymp \Big\|\sum_{m=1}^{\infty}\Big(\sum_{t=-1}^{\infty}\Big|\Delta^{01}|c_{m,n_{mt}}|D_{n_{mt}}(y)\Big|\Big)^2\Big\|^{\frac{q}{2}}_{L^{q/2}_{w(p/2,q/2)}} \\
&\qquad\lesssim \Big(\sum_{m=1}^{\infty}\Big\|\sum_{t=-1}^{\infty}\Big|\Delta^{01}|c_{m,n_{mt}}|D_{n_{mt}}(y)\Big|\Big\|^2_{L^q_{w(p,q)}}\Big)^{\frac{q}{2}} \\
&\qquad\lesssim \Big(\sum_{m=1}^{\infty} m^{-2\gamma}\Big(2^{-m^2-3m}2^{m(m+1)}+\sum_{t=0}^{\infty}2^{-((m+t)^2+3(m+t))}2^{((m+t)^2+3m+t)}\Big)^2\Big)^{\frac{q}{2}} \\
&\qquad\lesssim \Big(\sum_{m=1}^{\infty} m^{-\frac{2}{p'}}\Big)^{\frac{q}{2}} < \infty. \qquad (2.36)
\end{aligned}$$

Thus, by (2.35) and (2.36), for almost all t, the sum

$$\sum_{m=1}^{\infty} r_m(t)\sin mx \sum_{t=-1}^{\infty}\Delta^{01}|c_{m,n_{mt}}|D_{n_{mt}}(y)$$

converges for almost all y uniformly in x, and moreover, (2.35) and (2.36) imply that

$$\int_{-\pi}^{\pi}|y|^{\frac{q}{p}-1}\int_{-\pi}^{\pi}\Big|\sum_{m=1}^{\infty} r_m(t)\sin mx\sum_{t=-1}^{\infty}\Delta^{01}|c_{m,n_{mt}}|D_{n_{mt}}(y)\Big|^q dxdy < \infty$$

for almost all t (denote this set by $E \subset (0,1)$). Taking any $t_0 \in E \setminus \{k2^{-l}\}_{k,l\in\mathbb{N},k<2^l}$, so that $r_m(t_0) = \pm 1$ for all m, and setting $\{\delta_m\}$ according to the equality $(-1)^{\delta_m} = r_m(t_0)$, we obtain in light of (2.34) that $S_2 < \infty$. □

Chapter 3

Cosine polynomials with restrictions on their algebraic representation

In this chapter, we show that for any $\varepsilon > 0$ one can find a trigonometric polynomial with the l_1-norm of its coefficients less than ε and with the desired first p coefficients with respect to the basis $\{\cos^{2k} x\}_{k=0}^{\infty}$.

Theorem 3.1. *Let $p, s \in \mathbb{N}$ and $(a_0, a_1, ..., a_{p-1}) \in \mathbb{R}^p$. Then for $r \geq C_1(p, s)$ there exist a vector of coefficients $(b_s, b_{s+1}, ..., b_r) \in \mathbb{R}^{r-s+1}$ and a polynomial $g(x)$, $\deg g = 2r - 2p$, such that*

$$\sum_{k=s}^{r} b_k \cos 2kx - (\cos x)^{2p} g(\cos x) \equiv \sum_{t=0}^{p-1} a_t \cos^{2t} x \tag{3.1}$$

and

$$\sum_{k=s}^{r} |b_k| < \frac{C_2(p,s)}{r} \sum_{t=0}^{p-1} |a_t|, \tag{3.2}$$

where $C_1(p, s) := \max(16p^2 s^{4p-1}, 8L^{2p-1}p^3)$, $L = 4.56...$, and $C_2(p, s) := 2^{16} p^{4p+9} s^{4p-1}$.

3.1 Inverse of a matrix containing coefficients of Chebyshev polynomials

Let $\mathbf{T_n}$ be a square $n \times n$-matrix whose entry t_m^k in the mth row and kth column is the coefficient at x^m of the kth Chebyshev polynomial $T_k(x)$ (we enumerate rows and columns of $\mathbf{T_n}$ beginning from 0). It is clear that $\mathbf{T_n}$ is upper triangular with nonzero entries along the main diagonal. For t_m^k, an explicit formula is known (see, for instance, [43, (4.5.26)]):

$$t_m^k = \begin{cases} 0, & \text{if } m > k \text{ or } k - m \equiv 1 \pmod 2, \\ (-1)^{\frac{k-m}{2}} \frac{k}{k+m} 2^m \binom{\frac{k+m}{2}}{m}, & \text{otherwise.} \end{cases}$$

Denote by $\mathbf{T_{k,l}}$ the $l \times l$-matrix whose entry in the ith row and jth column is equal to t_i^{k+j}.

Lemma 3.2. *Let $l \in \mathbb{N}$ and let k be an even positive integer. The entry g_i^j of the matrix $\mathbf{T_{k,l}^{-1}}$ is equal to* 0 *if $i+j \equiv 1 \pmod 2$, otherwise there hold*

$$g_{2i}^{2j} = \frac{(-1)^{\alpha+j+\frac{k}{2}}(2j)!(k+i-1)!(k+2i)}{4^j i!(\alpha-i)!(\alpha+k+i)!} \sum_{b=0}^{j} \frac{\prod_{d=0,\, d\neq i}^{\alpha} (b^2 - (\frac{k}{2}+d)^2)}{\prod_{d=0,\, d\neq b}^{j} (b^2 - d^2)}$$

and

$$g_{2i+1}^{2j+1} = \frac{(-1)^{\beta+j+\frac{k}{2}}(2j+1)!(k+i)!}{4^\beta i!(\beta-i)!(\beta+k+i+1)!} \sum_{b=0}^{j} \frac{\prod_{d=0,\, d\neq i}^{\beta} ((2b+1)^2 - (k+2d+1)^2)}{\prod_{d=0,\, d\neq b}^{j} ((2b+1)^2 - (2d+1)^2)},$$

where $\alpha := \lceil l/2 \rceil - 1,\ \beta := \lfloor l/2 \rfloor - 1$.

Proof. Note that the entries of $\mathbf{T_{k,l}}$ belonging to a row and a column of different parities are zero, i.e. $g_i^j = 0$ for $2 \nmid i+j$. Fix some $j,\ 0 \leq j < l$, and consider the jth column of $\mathbf{T_{k,l}^{-1}}$. Its entries must satisfy

$$\sum_{i=0}^{l-1} g_i^j T_{k+i}(x) \equiv x^j + x^l g(x),$$

where $g(x)$ is some polynomial. Rewriting this, we get

$$\sum_{i=0}^{l-1} g_i^j \cos(k+i)x \equiv \cos^j x + \cos^l x\ g(\cos x). \tag{3.3}$$

We start with the case of an even j. Note that for any positive integer q we have

$$(\cos^q x)'' = (-q \sin x \cos^{q-1} x)' = q(q-1)\cos^{q-2} x - q^2 \cos^q x.$$

So, after taking the pth derivative of (3.3) for $p = 0, 1, ..., \lceil l/2 \rceil - 1 =: \alpha$, in each case we obtain at both sides polynomials in $\cos x$. As their constant terms match, we infer that

$$\sum_{i=0}^{l-1} (-(k+i)^2)^p g_i^j t_0^{k+i} = y_p^j,$$

where y_p^j stands for the constant term of $(\cos^j x)^{(2p)}$ (as of a polynomial in $\cos x$). So we have

$$\begin{pmatrix} 1 & 1 & ... & 1 \\ k^2 & (k+2)^2 & ... & (k+2\alpha)^2 \\ \vdots & \vdots & \vdots & \vdots \\ k^{2\alpha} & (k+2)^{2\alpha} & ... & (k+2\alpha)^{2\alpha} \end{pmatrix} \operatorname{diag}\left\{t_0^k, t_0^{k+2}, ..., t_0^{k+2\alpha}\right\} \begin{pmatrix} g_0^j \\ g_2^j \\ \vdots \\ g_{2\alpha}^j \end{pmatrix} = \begin{pmatrix} y_0^j \\ -y_1^j \\ \vdots \\ (-1)^\alpha y_\alpha^j \end{pmatrix}. \tag{3.4}$$

Let us find y_p^j for all p. Note that

$$y_p^{2\alpha} = \begin{cases} 0, & p < \alpha, \\ (2\alpha)!, & p = \alpha, \end{cases} \tag{3.5}$$

and that

$$\cos^j x \equiv \sum_{t=0}^{j/2} \eta_{2t} \cos 2tx.$$

The coefficients η_{2t} can be found from the following relation:

$$\left(\eta_0, 0, \eta_2, 0, ..., 0, \eta_j\right)^T = \mathbf{T_{j+1}^{-1}} \left(0, 0, ..., 0, 1\right)^T.$$

Applying equality (3.4) to the matrix $\mathbf{T_{j+1}^{-1}}$ and taking into account (3.5), we derive

$$\begin{pmatrix} \eta_0 \\ \eta_2 \\ \vdots \\ \eta_j \end{pmatrix} = \operatorname{diag}\left\{\frac{1}{t_0^0}, \frac{1}{t_0^2}, ..., \frac{1}{t_0^j}\right\} \begin{pmatrix} 1 & 1 & ... & 1 \\ 0^2 & 2^2 & ... & j^2 \\ \vdots & \vdots & \vdots & \vdots \\ 0^j & 2^j & ... & j^j \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{\frac{j}{2}} j! \end{pmatrix}. \tag{3.6}$$

Further,

$$(\cos^j x)^{(2p)} \equiv \sum_{t=0}^{j/2} (-4t^2)^p \eta_{2t} \cos 2tx,$$

whence in light of (3.6),

$$y_p^j = \sum_{m=0}^{j/2} (-4m^2)^p \eta_{2m} t_0^{2m} = \left((-0^2)^p t_0^0, (-2^2)^p t_0^2, ..., (-j^2)^p t_0^j\right) \begin{pmatrix} \eta_0 \\ \eta_2 \\ \vdots \\ \eta_j \end{pmatrix}$$

$$= \left((-0^2)^p, (-2^2)^p, ..., (-j^2)^p\right) \begin{pmatrix} 1 & 1 & ... & 1 \\ 0^2 & 2^2 & ... & j^2 \\ \vdots & \vdots & \vdots & \vdots \\ 0^j & 2^j & ... & j^j \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{\frac{j}{2}} j! \end{pmatrix}. \tag{3.7}$$

According to [51], the tth element of the last column of the inverse of the Vandermonde matrix of size m with the parameters $\lambda_0, ..., \lambda_{m-1}$ is equal to

$$\prod_{l=0,\, l \neq t}^{m-1} (\lambda_t - \lambda_l)^{-1}.$$

Thus, we obtain

$$\begin{pmatrix} g_0^j \\ g_2^j \\ \vdots \\ g_{2\alpha}^j \end{pmatrix} = (-1)^{\frac{j}{2}} j! \operatorname{diag}\left\{\frac{1}{t_0^k}, ..., \frac{1}{t_0^{k+2\alpha}}\right\} \begin{pmatrix} 1 & 1 & ... & 1 \\ k^2 & (k+2)^2 & ... & (k+2\alpha)^2 \\ \vdots & \vdots & \vdots & \vdots \\ k^{2\alpha} & (k+2)^{2\alpha} & ... & (k+2\alpha)^{2\alpha} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} (-0^2)^0 & (-2^2)^0 & ... & (-j^2)^0 \\ -(-0^2)^1 & -(-2^2)^1 & ... & -(-j^2)^1 \\ \vdots & \vdots & \vdots & \vdots \\ (-1)^\alpha(-0^2)^\alpha & (-1)^\alpha(-2^2)^\alpha & ... & (-1)^\alpha(-j^2)^\alpha \end{pmatrix} \begin{pmatrix} \prod_{t=0,\, t\neq 0}^{j/2} (-(2t)^2)^{-1} \\ \prod_{t=0,\, t\neq 1}^{j/2} (2^2-(2t)^2)^{-1} \\ \vdots \\ \prod_{t=0,\, t\neq j/2}^{j/2} (j^2-(2t)^2)^{-1} \end{pmatrix}$$

$$= (-1)^{\frac{j}{2}} j! \operatorname{diag}\left\{\frac{1}{t_0^k}, ..., \frac{1}{t_0^{k+2\alpha}}\right\} \begin{pmatrix} 1 & 1 & ... & 1 \\ k^2 & (k+2)^2 & ... & (k+2\alpha)^2 \\ \vdots & \vdots & \vdots & \vdots \\ k^{2\alpha} & (k+2)^{2\alpha} & ... & (k+2\alpha)^{2\alpha} \end{pmatrix}^{-1} \begin{pmatrix} 0^0 & 2^0 & ... & j^0 \\ 0^2 & 2^2 & ... & j^2 \\ \vdots & \vdots & \vdots & \vdots \\ 0^{2\alpha} & 2^{2\alpha} & ... & j^{2\alpha} \end{pmatrix}$$

$$\times \begin{pmatrix} \prod_{t=0,\, t\neq 0}^{j/2} (-(2t)^2)^{-1} \\ \vdots \\ \prod_{t=0,\, t\neq j/2}^{j/2} (j^2-(2t)^2)^{-1} \end{pmatrix} =: (-1)^{\frac{j}{2}} j! \operatorname{diag}\left\{\frac{1}{t_0^k}, ..., \frac{1}{t_0^{k+2\alpha}}\right\} \mathbf{J_0} \begin{pmatrix} \prod_{t=0,\, t\neq 0}^{j/2} (-(2t)^2)^{-1} \\ \vdots \\ \prod_{t=0,\, t\neq j/2}^{j/2} (j^2-(2t)^2)^{-1} \end{pmatrix}.$$

The matrix $\mathbf{J_0}$ is of size $(\alpha+1) \times (j/2+1)$ and its entries are

$$j_a^b = \frac{\prod_{d=0,\, d\neq a}^{\alpha} ((2b)^2 - (k+2d)^2)}{\prod_{d=0,\, d\neq a}^{\alpha} ((k+2a)^2 - (k+2d)^2)},$$

since the entry v_i^j of the square Vandermonde matrix with the parameters $k^2, (k+2)^2, ..., (k+2\alpha)^2$ is equal to

$$v_i^j = \frac{\left[\prod_{d=0,\, d\neq i}^{\alpha} (x-(k+2d)^2)\right]_j}{\prod_{d=0,\, d\neq i}^{\alpha} ((k+2i)^2 - (k+2d)^2)},$$

where $[P(x)]_j$ stands for the coefficient at x^j of the polynomial $P(x)$. Hence, recalling that $\alpha = \lceil l/2 \rceil - 1$, we have

$$g_{2a}^j = \frac{(-1)^{\frac{j}{2}+a+\frac{k}{2}} j!}{\prod_{d=0,\, d\neq a}^{\lceil l/2\rceil-1} ((k+2a)^2 - (k+2d)^2)} \sum_{b=0}^{j/2} \frac{\prod_{d=0,\, d\neq a}^{\lceil l/2\rceil-1} ((2b)^2 - (k+2d)^2)}{\prod_{d=0,\, d\neq b}^{j/2} ((2b)^2 - (2d)^2)}. \tag{3.8}$$

Turn now to the case of an odd j. Once more, taking the pth derivative of (3.3) for $p = 0, 1, ..., \lfloor l/2 \rfloor - 1 =: \beta$ and obtaining the same coefficients at $\cos x$, we get

$$\sum_{i=0}^{l-1} (-(k+i)^2)^p g_i^j t_1^{k+i} = z_p^j,$$

where z_p^j is the coefficient at $\cos x$ of $(\cos^{j+1} x)^{(2p)}$ (as of a polynomial in $\cos x$). We have

$$\begin{pmatrix} 1 & 1 & \dots & 1 \\ (k+1)^2 & (k+3)^2 & \dots & (k+1+2\beta)^2 \\ \vdots & \vdots & \vdots & \vdots \\ (k+1)^{2\beta} & (k+3)^{2\beta} & \dots & (k+1+2\beta)^{2\beta} \end{pmatrix} \operatorname{diag}\left(t_1^{k+1}, ..., t_1^{k+1+2\beta}\right) \begin{pmatrix} g_1^j \\ g_3^j \\ \vdots \\ g_{2\beta+1}^j \end{pmatrix} = \begin{pmatrix} z_0^j \\ -z_1^j \\ \vdots \\ (-1)^\beta z_\beta^j \end{pmatrix}. \tag{3.9}$$

Noting that

$$z_p^{2\beta} = \begin{cases} 0, & p < \beta, \\ (2\beta+1)!, & p = \beta, \end{cases} \tag{3.10}$$

and that

$$\cos^j x \equiv \sum_{t=0}^{(j-1)/2} \eta_{2t+1} \cos(2t+1)x,$$

we derive

$$(0, \eta_1, 0, \eta_3, ..., 0, \eta_j)^T = \mathbf{T_{j+1}^{-1}} (0, 0, ..., 0, 1)^T.$$

Applying (3.9) to $\mathbf{T_{j+1}^{-1}}$ and using (3.10), we obtain

$$\begin{pmatrix} \eta_1 \\ \eta_3 \\ \vdots \\ \eta_j \end{pmatrix} = \operatorname{diag}\left\{\frac{1}{t_1^1}, ..., \frac{1}{t_1^j}\right\} \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1^2 & 3^2 & \dots & j^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1^j & 3^j & \dots & j^j \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{\frac{j-1}{2}} j! \end{pmatrix}.$$

Further,

$$(\cos^j x)^{(2p)} \equiv \sum_{t=0}^{j/2} (-(2t+1)^2)^p \eta_{2t+1} \cos(2t+1)x,$$

whence

$$z_p^j = \sum_{m=0}^{\frac{j-1}{2}} (-(2m+1)^2)^p \eta_{2m+1} t_1^{2m+1} = \left((-1^2)^p t_1^1, (-3^2)^p t_1^3, ..., (-j^2)^p t_1^j\right) \begin{pmatrix} \eta_1 \\ \eta_3 \\ \vdots \\ \eta_j \end{pmatrix}$$

$$= \left((-1^2)^p, (-3^2)^p, ..., (-j^2)^p\right) \begin{pmatrix} 1 & 1 & \dots & 1 \\ 1^2 & 3^2 & \dots & j^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1^j & 3^j & \dots & j^j \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ (-1)^{\frac{j-1}{2}} j! \end{pmatrix}.$$

Finally, as before

$$\begin{pmatrix} g_1^j \\ g_3^j \\ \vdots \\ g_{2\beta+1}^j \end{pmatrix} = (-1)^{\frac{j-1}{2}} j!\ \mathrm{diag}\left\{\frac{1}{t_1^{k+1}}, ..., \frac{1}{t_1^{k+1+2\beta}}\right\}$$

$$\times \begin{pmatrix} 1 & 1 & ... & 1 \\ (k+1)^2 & (k+3)^2 & ... & (k+1+2\beta)^2 \\ \vdots & \vdots & \vdots & \vdots \\ (k+1)^{2\beta} & (k+3)^{2\beta} & ... & (k+1+2\beta)^{2\beta} \end{pmatrix}^{-1}$$

$$\times \begin{pmatrix} 1 & 1 & ... & 1 \\ 1^2 & 3^2 & ... & j^2 \\ \vdots & \vdots & \vdots & \vdots \\ 1^{2\beta} & 3^{2\beta} & ... & j^{2\beta} \end{pmatrix} \begin{pmatrix} \prod\limits_{t=0,\ t\neq 0}^{(j-1)/2} (1^2-(2t+1)^2)^{-1} \\ \prod\limits_{t=0,\ t\neq 1}^{(j-1)/2} (3^2-(2t+1)^2)^{-1} \\ \vdots \\ \prod\limits_{t=0,\ t\neq (j-1)/2}^{(j-1)/2} (j^2-(2t+1)^2)^{-1} \end{pmatrix}$$

$$=: (-1)^{\frac{j-1}{2}} j!\ \mathrm{diag}\left\{\frac{1}{t_1^{k+1}}, ..., \frac{1}{t_1^{k+1+2\beta}}\right\} \mathbf{J_1} \begin{pmatrix} \prod\limits_{t=0,\ t\neq 0}^{(j-1)/2} (1^2-(2t+1)^2)^{-1} \\ \vdots \\ \prod\limits_{t=0,\ t\neq (j-1)/2}^{(j-1)/2} (j^2-(2t+1)^2)^{-1} \end{pmatrix}.$$

The matrix $\mathbf{J_1}$ is of size $(\beta+1)\times((j-1)/2+1)$ and its entries are

$$j_a^b = \frac{\prod\limits_{d=0,\ d\neq a}^{\beta} ((2b+1)^2-(k+2d+1)^2)}{\prod\limits_{d=0,\ d\neq a}^{\beta} ((k+2a+1)^2-(k+2d+1)^2)}.$$

Hence,

$$g_{2a+1}^j = \frac{(-1)^{\frac{j-1}{2}+a+\frac{k}{2}} j!}{(k+2a+1) \prod\limits_{d=0,\ d\neq a}^{\lfloor l/2\rfloor -1} ((k+2a+1)^2-(k+2d+1)^2)}$$
$$\times \sum_{b=0}^{(j-1)/2} \frac{\prod\limits_{d=0,\ d\neq a}^{\lfloor l/2\rfloor -1} ((2b+1)^2-(k+2d+1)^2)}{\prod\limits_{d=0,\ d\neq b}^{(j-1)/2} ((2b+1)^2-(2d+1)^2)}, \tag{3.11}$$

and the claim follows. □

Remark 3.3. Following the ideas of the proof of Lemma 3.2, one can establish an explicit formula for the elements of the inverse of any submatrix $\mathbf{T_{k,l,m}} := (t_{m+i}^{k+j})_{i,j=0}^{l-1}$ of $\mathbf{T_n}$ with even k and m.

Remark 3.4. For any $n \in \mathbb{N}$, the entry h_i^j of the matrix $\mathbf{T_n^{-1}}$ is zero if $2 \nmid i+j$ or $i > j$, otherwise can be found by

$$h_{2i}^{2j} = 2^{\delta_i - 2j}\binom{2j}{j-i}, \qquad h_{2i+1}^{2j+1} = 2^{\delta_i - 2j}\binom{2j+1}{j-i},$$

where

$$\delta_i := \begin{cases} 0, & \text{if } \ i = 0, \\ 1, & \text{if } \ i \neq 0. \end{cases}$$

Proof. Noting that, for $b \neq a$,

$$\prod_{d=0,\ d\neq a}^{\lceil n/2\rceil - 1} ((2b)^2 - (2d)^2) = 0,$$

we obtain $h_{2i}^{2j} = 0$, for $i > j$, and otherwise due to (3.8),

$$h_0^{2j} = \frac{(2j)!}{((2j)!!)^2} = 2^{-2j}\binom{2j}{j},$$

and

$$h_{2i}^{2j} = \frac{(-1)^{j+i}(2j)!}{\prod\limits_{d=0,\ d\neq i}^{\lceil n/2\rceil - 1}((2i)^2 - (2d)^2)} \frac{\prod\limits_{d=0,\ d\neq i}^{\lceil n/2\rceil - 1}((2i)^2 - (2d)^2)}{\prod\limits_{d=0,\ d\neq i}^{j}((2i)^2 - (2d)^2)} = \frac{(-1)^{j+i}(2j)!}{(2i)!!(-1)^{j-i}(2j-2i)!!\frac{(2j+2i)!!}{(2i-2)!!4i}}$$

$$= \frac{2^{1-2j}(2j)!}{(j-i)!(j+i)!} = 2^{1-2j}\binom{2j}{j-i},$$

if $i > 0$.

For odd entries, once more we get $h_{2i+1}^{2j+1} = 0$ for $i > j$, otherwise from (3.11),

$$h_1^{2j+1} = \frac{(2j+1)!}{(2j+2)!!(2j)!!} = 2^{-2j}\binom{2j+1}{j},$$

and

$$h_{2i+1}^{j} = \frac{(-1)^{j+i}(2j+1)!(-1)^{j-i}(2i)!!(4i+2)}{(2i+1)(2i)!!(2j-2i)!!(2i+2j+2)!!} = 2^{1-2j}\binom{2j+1}{j-i},$$

if $i > 0$, so the proof is complete. □

Corollary 3.5. *There holds*

$$(\cos^{2j})^{(2p)}|_{x=\pi/2} =: y_p^{2j} = (-4)^{p-j}\sum_{k=0}^{2j}(-1)^k\binom{2j}{k}(j-k)^{2p}.$$

Proof. It follows from (3.7) that

$$y_p^{2j} = (-1)^j (2j)! \left((-0^2)^p, \quad (-2^2)^p, \quad ..., \quad (-(2j)^2)^p\right) \begin{pmatrix} \prod_{t=0,\, t\neq 0}^{j} (-(2t)^2)^{-1} \\ \prod_{t=0,\, t\neq 1}^{j} (2^2-(2t)^2)^{-1} \\ \vdots \\ \prod_{t=0,\, t\neq j}^{j} ((2j)^2-(2t)^2)^{-1} \end{pmatrix}$$

$$= (-1)^{p+j}(2j)! \sum_{a=0}^{j} \frac{(2a)^{2p}}{\prod_{t=0,\, t\neq a}^{j} ((2a)^2-(2t)^2)} = (-4)^p 2^{-2j} \sum_{a=0}^{j} (-1)^a \binom{2j}{j-a} a^{2p} 2^{\delta_a}$$

$$= (-4)^{p-j} \sum_{k=0}^{2j} (-1)^k \binom{2j}{k} (j-k)^{2p},$$

where δ_a is as in Remark 3.4, and we are done. □

3.2 Proof of Theorem 3.1

Now we are ready to prove the main theorem.

Proof of Theorem 3.1. For the sake of clarity, let us split the proof into three main parts.

3.2.1 Finding a sufficient condition for (3.1) to hold

First we note that (3.1) is equivalent to

$$\begin{pmatrix} t_0^{2s} & t_0^{2s+2} & \dots & t_0^{2r} \\ t_2^{2s} & t_2^{2s+2} & \dots & t_2^{2r} \\ \vdots & \vdots & \vdots & \vdots \\ t_{2p-2}^{2s} & t_{2p-2}^{2s+2} & \dots & t_{2p-2}^{2r} \end{pmatrix} \begin{pmatrix} b_s \\ b_{s+1} \\ \vdots \\ b_r \end{pmatrix} = \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_{p-1} \end{pmatrix}.$$

Pick some $k \in \{s, s+1, ..., r\}$ and take the $(2q)$th derivative of the equality

$$\cos 2kx \equiv \sum_{l=0}^{k} t_{2l}^{2k} \cos^{2l} x,$$

where $q \in \{0, 1, ..., p-1\}$, at the point $\pi/2$. What we get is

$$(-1)^q (2k)^{2q} t_0^{2k} = \sum_{l=0}^{k} t_{2l}^{2k} (\cos^{2l} x)^{(2q)}|_{x=\pi/2},$$

which is equivalent to

$$(-1)^{q+k}(2k)^{2q} = \sum_{l=0}^{k} t_{2l}^{2k} y_q^{2l}.$$

From the relations above for all k and q in the mentioned ranges, we derive

$$\begin{pmatrix} y_0^0 & y_0^2 & \dots & y_0^{2p-2} \\ y_1^0 & y_1^2 & \dots & y_1^{2p-2} \\ \vdots & \vdots & \vdots & \vdots \\ y_{p-1}^0 & y_{p-1}^2 & \dots & y_{p-1}^{2p-2} \end{pmatrix} \begin{pmatrix} t_0^{2s} & t_0^{2s+2} & \dots & t_0^{2r} \\ t_2^{2s} & t_2^{2s+2} & \dots & t_2^{2r} \\ \vdots & \vdots & \vdots & \vdots \\ t_{2p-2}^{2s} & t_{2p-2}^{2s+2} & \dots & t_{2p-2}^{2r} \end{pmatrix}$$
$$= N_p \begin{pmatrix} (2s)^0 & (2s+2)^0 & \dots & (2r)^0 \\ (2s)^2 & (2s+2)^2 & \dots & (2r)^2 \\ \vdots & \vdots & \vdots & \vdots \\ (2s)^{2p-2} & (2s+2)^{2p-2} & \dots & (2r)^{2p-2} \end{pmatrix} N_{r-s+1},$$

where N_j is a square diagonal matrix of size j with its entries belonging to the even columns equal 1, while the entries belonging to the odd ones, equal -1. Thus,

$$\begin{pmatrix} t_0^{2s} & t_0^{2s+2} & \dots & t_0^{2r} \\ t_2^{2s} & t_2^{2s+2} & \dots & t_2^{2r} \\ \vdots & \vdots & \vdots & \vdots \\ t_{2p-2}^{2s} & t_{2p-2}^{2s+2} & \dots & t_{2p-2}^{2r} \end{pmatrix} = \begin{pmatrix} y_0^0 & y_0^2 & \dots & y_0^{2p-2} \\ \frac{y_1^0}{4} & \frac{y_1^2}{4} & \dots & \frac{y_1^{2p-2}}{4} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{y_{p-1}^0}{2^{2p-2}} & \frac{y_{p-1}^2}{2^{2p-2}} & \dots & \frac{y_{p-1}^{2p-2}}{2^{2p-2}} \end{pmatrix}^{-1}$$
$$\times N_p \begin{pmatrix} s^0 & (s+1)^0 & \dots & r^0 \\ s^2 & (s+1)^2 & \dots & r^2 \\ \vdots & \vdots & \vdots & \vdots \\ s^{2p-2} & (s+1)^{2p-2} & \dots & r^{2p-2} \end{pmatrix} N_{r-s+1}. \tag{3.12}$$

Let Y be a square matrix of size $2p$ such that the matrix generated by the odd rows and the odd columns of Y (as before, we start enumerating from zero) is the identity matrix, an entry belonging to the $(2u)$th column and $(2k)$th row is equal to $2^{-2k}y_k^{2u}$, the other entries are zeros. Then Y is invertible due to invertibility of the identity matrix and that of the matrix $(2^{-2i}y_i^{2j})_{i,j=0}^{p-1}$. Therefore, it follows from (3.12) that there exist τ_{2i+1}^{2j}, $i = 0, ..., p-1$, $j = s, s+1, ..., r$, such that

$$T := \begin{pmatrix} t_0^{2s} & t_0^{2s+2} & \dots & t_0^{2r} \\ \tau_1^{2s} & \tau_1^{2s+2} & \dots & \tau_1^{2r} \\ t_2^{2s} & t_2^{2s+2} & \dots & t_2^{2r} \\ \tau_3^{2s} & \tau_3^{2s+2} & \dots & \tau_3^{2r} \\ \vdots & \vdots & \vdots & \vdots \\ t_{2p-2}^{2s} & t_{2p-2}^{2s+2} & \dots & t_{2p-2}^{2r} \\ \tau_{2p-1}^{2s} & \tau_{2p-1}^{2s+2} & \dots & \tau_{2p-1}^{2r} \end{pmatrix} = Y^{-1}\tilde{N}_p \begin{pmatrix} s^0 & (s+1)^0 & \dots & r^0 \\ s^1 & (s+1)^1 & \dots & r^1 \\ s^2 & (s+1)^2 & \dots & r^2 \\ s^3 & (s+1)^3 & \dots & r^3 \\ \vdots & \vdots & \vdots & \vdots \\ s^{2p-2} & (s+1)^{2p-2} & \dots & r^{2p-2} \\ s^{2p-1} & (s+1)^{2p-1} & \dots & r^{2p-1} \end{pmatrix} N_{r-s-1}$$
$$=: Y^{-1}\tilde{N}_p V N_{r-s+1}. \tag{3.13}$$

Here $\tilde{N}_j$ stands for a square matrix of size $2j$ having N_j in the intersection of the even columns and the even rows, E_j in the intersection of the odd columns and the odd rows, and the other entries equal zero.

Note that if we make $\mathbf{b} := (b_s, ..., b_r)^T$ satisfy the equality

$$T\mathbf{b} = \mathbf{a},$$

where $\mathbf{a} := (a_0, 0, a_1, 0, ..., a_{p-1}, 0)^T$, then condition (3.1) will be fulfilled.

3.2.2 Constructing a vector of coefficients

Let

$$\mathbf{b} := T^*(TT^*)^{-1}\mathbf{a}.$$

Since using (3.13) we have

$$\begin{aligned} T^\dagger := (TT^*)^{-1}T &= (Y^{-1}\tilde{N}_p V N_{r-s+1} N^*_{r-s+1} V^* \tilde{N}^*_p (Y^{-1})^*)^{-1} Y^{-1}\tilde{N}_p V N_{r-s+1} \\ &= Y^*\tilde{N}_p (VV^*)^{-1} V N_{r-s+1}, \end{aligned}$$

the definition of $\mathbf{b}$ is equivalent to

$$\begin{aligned} \big(b_s, ..., b_r\big) &= \big(a_0, 0, ..., a_{p-1}, 0\big)\, Y^*\tilde{N}_p (VV^*)^{-1} V N_{r-s+1} \\ &=: \big(a_0, 0, ..., a_{p-1}, 0\big)\, Y^*\tilde{N}_p V^\dagger N_{r-s+1}, \end{aligned} \tag{3.14}$$

where $V^\dagger$ is the pseudoinverse for the Vandermonde matrix V. Note that

$$VV^* = WW^* - ZZ^*,$$

where $W = (w_i^j)$, $w_i^j := (j+1)^i$, $j = 0, ..., r-1, i = 0, ..., 2p-1$, $Z = (z_i^j)$, $z_i^j := (j+1)^i$, $j = 0, ..., s-2, i = 0, ..., 2p-1$. According to [29, (10)], the condition number of WW^* is

$$\hat{\kappa} = \frac{(2p)^2}{4p-1} r^{4p-2}.$$

The maximal entry of WW^* is greater than $r^{4p-1}/(4p-1)$, therefore the l^2-norm of this matrix exceeds this value. Thus, $\|(WW^*)^{-1}\|_2 < \hat{\kappa}(4p-1)/r^{4p-1} < 8p^2/r$. In turn, $\|ZZ^*\|_2 < (s-1)^{4p-1}$, which yields $\|(WW^*)^{-1}\|_2 \|ZZ^*\|_2 \leq 8p^2 s^{4p-1}/r \leq 0.5$. So, we have the following representation

$$(VV^*)^{-1} = (WW^* - ZZ^*)^{-1} = \sum_{k=0}^{\infty} ((WW^*)^{-1} ZZ^*)^k (WW^*)^{-1} =: (E_{2p} + X)(WW^*)^{-1}, \tag{3.15}$$

where E_{2p} is the identity matrix of size $2p$ and

$$\|X\|_2 < \frac{8p^2 s^{4p-1}}{r - 8p^2 s^{4p-1}}. \tag{3.16}$$

Due to [29, relation before Prop. 3], for entries of $W^\dagger = (WW^*)^{-1}W$ we have (taking into account that we enumerate from zero)

$$(W^\dagger)_{q,k} = (-1)^q \sum_{w=q}^{2p-1} \frac{1}{w!} s(w+1,q+1) \sum_{t=w}^{2p-1} \frac{\binom{t+w}{w}\binom{r-w-1}{r-t-1}}{\binom{2t}{t}\binom{r+t}{2t+1}} \times \sum_{j=0}^{\min(t,k)} (-1)^{j+1} \binom{k}{j}\binom{j+t}{j}\binom{r-j-1}{r-t-1}, \tag{3.17}$$

where $s(w+1,q+1)$ is the Stirling number of the first kind. Further, we have from Corollary 3.5

$$y_q^{2u} = (-4)^{q-u} \sum_{v=0}^{2u} (-1)^v \binom{2u}{v} (u-v)^{2q}, \tag{3.18}$$

hence, there holds

$$(Y^* \tilde{N}_p W^\dagger)_{2u,k} = (-4)^{-u} \sum_{v=0}^{2u} (-1)^v \binom{2u}{v} \sum_{w=0}^{2p-1} \frac{1}{w!} \sum_{q=0}^{\lfloor w/2 \rfloor} (-1)^q (u-v)^{2q} s(w+1,2q+1) \times \sum_{t=w}^{2p-1} \frac{\binom{t+w}{w}\binom{r-w-1}{r-t-1}}{\binom{2t}{t}\binom{r+t}{2t+1}} \sum_{j=0}^{\min(t,k)} (-1)^{j+1} \binom{k}{j}\binom{j+t}{j}\binom{r-j-1}{r-t-1}. \tag{3.19}$$

3.2.3 Estimating $T^\dagger$

First, by the definition of Stirling numbers, $s(w+1,2q+1)$ is the coefficient at x^{2q+1} of the polynomial $x(x+1)...(x+w)$, which is the same as to be the coefficient at x^{2q} of the polynomial $(x+1)...(x+w)$. Thus,

$$\sum_{q=0}^{\lfloor w/2 \rfloor} s(w+1,2q+1)(-1)^q (u-v)^{2q} = \mathrm{Re}(i(u-v)+1)...(i(u-v)+w) < \frac{(|u-v|+w)!}{|u-v|!}. \tag{3.20}$$

To obtain upper bounds for the sum

$$A(t,k,r) := \sum_{j=0}^{\min(k,t)} (-1)^j \binom{k}{j}\binom{j+t}{j}\binom{r-j-1}{r-t-1}$$

that appears in (3.19), we need the following

Lemma 3.6. *Let $t,q,k,r \in \mathbb{N}$ be such that $q \geq t \geq 2$.*
a) If $r \geq q + 2q^2$ and $r-q-1 \geq k \geq q$, then

$$|A(t,k,r)| = \left| \sum_{j=0}^{\min(k,t)} (-1)^j \binom{k}{j}\binom{j+t}{j}\binom{r-j-1}{r-t-1} \right| \leq 4 \left(\frac{q}{r-1-q} \right)^{q-t} \binom{r-1}{t}. \tag{3.21}$$

b) If $r \geq 2t^3 + t$ and $k < t$, then

$$|A(t,k,r)| < \binom{r-1}{t}. \tag{3.22}$$

c) If $r \geq 2L^t t^{1.5}$, where $L := (\sqrt{2}+1)^{1+\frac{1}{\sqrt{2}}}(\sqrt{2}-1)^{-1+\frac{1}{\sqrt{2}}}2^{-\frac{1}{2\sqrt{2}}}$, and $t \leq k \leq r-1$, then

$$|A(t,k,r)| < 3\binom{r-1}{t}. \tag{3.23}$$

Proof. a) We begin with estimate (3.21) and the proof will be divided into several steps.

Step 1. Algebraic representation of $A(t,k,r)$. It turns out that the sum of products of binomial coefficients $A(t,k,r)$ has the following algebraic meaning: it represents the coefficient at $x^t y^k$ of the Taylor expansion at zero of the function

$$G(x,y) := \frac{(1+y)^k}{(1+xy)^{t+1}(1-x)^{r-t}}.$$

Indeed,

$$\binom{r-j-1}{r-t-1} = \binom{r-j-1}{t-j} = \frac{(r-j-1)(r-j-2)...(r-t)}{(t-j)!}$$
$$= (-1)^{t-j}\frac{(-(r-t))(-(r-t+1))...(-(r-j-1))}{(t-j)!} = (-1)^{t-j}\binom{-r+t}{t-j},$$

so we have for $k \geq q \geq t$

$$A(t,k,r) = \sum_{j=0}^{\min(k,t)} (-1)^j \binom{k}{j}\binom{j+t}{j}\binom{r-j-1}{r-t-1}$$
$$= \sum_{j=0}^{\min(k,t)} \binom{k}{j}\binom{-t-1}{j}(-1)^{t-j}\binom{-r+t}{t-j},$$

which corresponds to the mentioned coefficient.

Take some $\varepsilon \in (0,1)$ and let $\delta = t/r$. By Cauchy's formulas,

$$A(t,k,r) = \frac{1}{(2\pi i)^2}\int_{|y|=\varepsilon}\int_{|x|=\varepsilon} G(x,y)x^{-t-1}y^{-k-1}dx\,dy = \frac{1}{(2\pi i)^2}\int_{|y|=\varepsilon}\frac{(1+y)^k}{y^{k+1}}$$
$$\times\left(\int_{|x|=\varepsilon^{\delta-1}}\frac{1}{(1+xy)^{t+1}(1-x)^{r-t}x^{t+1}}dx - 2\pi i\,\mathrm{res}_{x=1}\frac{1}{(1+xy)^{t+1}(1-x)^{r-t}x^{t+1}}\right)dy$$
$$=: S_1 + S_2.$$

Step 2. Estimating S_2. We have

$$\mathrm{res}_{x=1}\frac{1}{(1+xy)^{t+1}(1-x)^{r-t}x^{t+1}} = \frac{(-1)^{r-t}}{(r-t-1)!}\left(\frac{1}{(1+xy)^{t+1}x^{t+1}}\right)^{(r-t-1)}\Bigg|_{x=1}$$
$$= \frac{(-1)^{r-t}}{(r-t-1)!}\sum_{l=0}^{r-t-1}\left(\frac{1}{x^{t+1}}\right)^{(l)}\left(\frac{1}{(1+xy)^{t+1}}\right)^{(r-t-1-l)}\Bigg|_{x=1}$$
$$= \frac{-1}{(r-t-1)!}\sum_{l=0}^{r-t-1}\frac{(t+l)!}{t!}\frac{(r-l-1)!}{t!}\frac{y^{r-t-1-l}}{(1+y)^{r-l}},$$

hence, S_2 is the coefficient at y^0 of the Laurent expansion of the function

$$\begin{aligned}&\frac{(1+y)^k}{y^k}\frac{-1}{(r-t-1)!}\sum_{l=0}^{r-t-1}\frac{(t+l)!}{t!}\frac{(r-l-1)!}{t!}\frac{y^{r-t-1-l}}{(1+y)^{r-l}}\\ =&\frac{-1}{(r-t-1)!(t!)^2}\sum_{l=0}^{r-t-1}(t+l)!(r-l-1)!y^{r-t-1-l-k}\sum_{j=0}^{\infty}\binom{-r+l+k}{j}y^j.\end{aligned}$$

Note that for l satisfying $r-t-1-l-k>0$, the coefficient of the corresponding term at y^0 is zero, therefore it suffices to consider just $l \geq r-t-1-k$. At the same time, if $-r+l+k \geq 0$, then $-(r-t-1-l-k) = -r+t+1+l+k > -r+l+k$, so $\binom{-r+l+k}{-(r-t-1-l-k)} = 0$, which means that for $l \geq r-k$ the corresponding term is zero. Hence,

$$\begin{aligned}S_2 &= \frac{-1}{(r-t-1)!(t!)^2}\sum_{l=r-t-1-k}^{r-k-1}(t+l)!(r-l-1)!\binom{-r+l+k}{-r+t+1+l+k}\\ &= -\sum_{l=r-t-1-k}^{r-k-1}(-1)^{r-t-1-l-k}\frac{(t+l)!(r-l-1)!(r-l-k-1-r+t+1+l+k)!}{(r-t-1)!(t!)^2(-r+t+1+l+k)!(r-l-k-1)!}\\ &= -\sum_{m=0}^{t}(-1)^m\frac{(r-1-k+m)!(t+k-m)!}{(r-t-1)!t!m!(t-m)!} =: -\sum_{m=0}^{t}(-1)^m D_m(r,t,k).\end{aligned}$$

Step 2.1. Estimating D_m. Note that

$$\frac{D_m(r,t,k+1)}{D_m(r,t,k)} = \frac{t+k+1-m}{r-1-k+m},$$

and since $q < \frac{r-t}{2} - 1 + m < r-q-1$, the maximum of the above expression is attained either at $k=q$ or at $k=r-1-q$.

For $k=q$, we have

$$\frac{D_{m+1}(r,t,q)}{D_m(r,t,q)} = \frac{(r-q+m)(t-m)}{(m+1)(t+q-m)} = \frac{rt-qt+mt-rm+qm-m^2}{mt+t+qm+q-m^2-m} > 1,$$

since

$$rt-qt-rm \geq r-qt \geq t+q \geq t+q-m.$$

Thus, $D_m(r,t,q)$ is maximal at $m=t$ and

$$D_t(r,t,q) = \frac{(r-1-q+t)!q!}{(r-t-1)!(t!)^2} \leq \left(\frac{q}{r-1-q}\right)^{q-t}\binom{r-1}{t}.$$

For $k=r-1-q$, we have

$$\begin{aligned}\frac{D_{m+1}(r,t,r-1-q)}{D_m(r,t,r-1-q)} &= \frac{(q+m+1)(t-m)}{(m+1)(r-1+t-q-m)}\\ &= \frac{qt+mt+t-qm-m^2-m}{rm+r-2m-1+tm+t-qm-q-m^2} < 1,\end{aligned}$$

in light of

$$rm + r - m - 1 + t - q \geq r - 1 - q \geq qt + t.$$

Thus, $D_m(r, t, r-1-q)$ is maximal at $m = 0$ and

$$D_0(r, t, r-1-q) = \frac{q!(r-1+t-q)!}{(r-t-1)!(t!)^2} \leq \left(\frac{q}{r-1-q}\right)^{q-t} \binom{r-1}{t}.$$

Finally,

$$|S_2| \leq 4\left(\frac{q}{r-1-q}\right)^{q-t} \binom{r-1}{t}.$$

Step 3. Estimating S_1. Observe that for small enough ε and $|y| = \varepsilon$, $|x| = \varepsilon^{\delta-1}$, the function $|G(x,y)x^{-t-1}y^{-k-1}|$ is equivalent to

$$\varepsilon^{-(\delta-1)(r-t)}\varepsilon^{-k-1}\varepsilon^{-(\delta-1)(t+1)},$$

then

$$|S_1| \lesssim \varepsilon\varepsilon^{\delta-1}\varepsilon^{-(\delta-1)(r-t)-k-1-(\delta-1)(t+1)} = \varepsilon^{r-t-k} \leq \varepsilon^{q+1-t} \underset{\varepsilon\to 0}{\to} 0,$$

whence we get (3.21).

b) Turn now to (3.22) for $r \geq 2t^3 + t$ and $k < t$. We have

$$\begin{aligned} A(t,k,r) &= \sum_{j=0}^{\min(k,t)} (-1)^j \frac{k!}{j!(k-j)!} \frac{(j+t)!}{j!t!} \frac{(r-j-1)!}{(t-j)!(r-t-1)!} \\ &= \frac{(r-1)!}{t!(r-t-1)!} \sum_{j=0}^{t} (-1)^j \binom{t}{j}\binom{t+j}{j} \frac{k(k-1)...(k-j+1)}{(r-1)(r-2)...(r-j)}. \end{aligned}$$

For all j, there holds

$$\binom{t}{j}\binom{t+j}{j}\frac{k(k-1)...(k-j+1)}{(r-1)(r-2)...(r-j)} < t^j(2t)^j\frac{t^j}{(r-t)^j} \leq 1,$$

since $r \geq 2t^3 + t$. Note that the expression above decreases. Indeed, going from j to $j+1$ we get our value changed by

$$\frac{(j+t+1)(k-j)(t-j)}{(r-j-1)(j+1)^2} < \frac{2t \cdot t^2}{r-t} \leq 1.$$

Therefore, we derive

$$A(t,k,r) < \binom{r-1}{t}.$$

c) Now we have only to prove (3.23) under the mentioned conditions. Divide our sum into two sums in the following way:

$$\begin{aligned} A(t,k,r) &= \binom{r-1}{t}\sum_{j=0}^{t}(-1)^j\binom{t}{j}\binom{t+j}{j}\Big(\frac{k+1}{r}\Big)^j \\ &+ \binom{r-1}{t}\sum_{j=1}^{t}(-1)^j\binom{t}{j}\binom{t+j}{j}\Big(\frac{k...(k-j+1)}{(r-1)...(r-j)}-\Big(\frac{k+1}{r}\Big)^j\Big) \\ &=: \binom{r-1}{t}(S_3+S_4). \end{aligned}$$

Step 1. Estimating S_3. Since

$$\begin{aligned} \binom{t+j}{j} = \frac{(t+j)(t+j-1)...(t+1)}{j!} &= (-1)^j\frac{(-t-1)(-t-2)...(-t-j)}{j!} \\ &= (-1)^j\binom{-t-1}{j}, \end{aligned}$$

we have

$$S_3 = \sum_{j=0}^{t}\binom{t}{t-g}\binom{-t-1}{j}\Big(\frac{k+1}{r}\Big)^j = \frac{1}{t!}\Big((1+x)^t\Big(1+\Big(\frac{k+1}{r}\Big)x\Big)^{-t-1}\Big)^{(t)}\Big|_{x=0}.$$

To estimate this value, we will need the following

Lemma 3.7. *For any positive integer n and any $\gamma\in(0,1)$, there holds*

$$|c_n^\gamma| := \frac{1}{n!}\Big|\Big((1+x)^n(1+\gamma x)^{-n-1}\Big)^{(n)}\Big|_{x=0}\Big| \le 2$$

and $|c_n^0| = |c_n^1| = 1$.

Proof. Fix some $n\in\mathbb{N}$. We have

$$\begin{aligned} h(x) := (1+x)^n(1+\gamma x)^{-n-1} &= \left(\frac{1}{\gamma}+\frac{1-\frac{1}{\gamma}}{1+\gamma x}\right)^n\frac{1}{1+\gamma x} \\ &= \sum_{g=0}^{n}\binom{n}{g}\Big(\frac{1}{\gamma}\Big)^g\frac{\Big(1-\frac{1}{\gamma}\Big)^{n-g}}{(1+\gamma x)^{n-g+1}}, \end{aligned}$$

whence

$$\frac{1}{n!}h^{(n)}(x) = \frac{1}{n!}\sum_{g=0}^{n}\binom{n}{g}\frac{\Big(1-\frac{1}{\gamma}\Big)^{n-g}}{\gamma^g}\frac{(-1)^n\gamma^n}{(1+\gamma x)^{2n-g+1}}\frac{(2n-g)!}{(n-g)!}.$$

Making the change of variable $r = n - g$, we obtain

$$\begin{aligned}\frac{1}{n!}h^{(n)}(0) &= \frac{(-1)^n}{n!}\sum_{r=0}^{n}(\gamma-1)^r\binom{n}{n-r}\frac{(n+r)!}{r!}\\ &= (-1)^n\sum_{r=0}^{n}(1-\gamma)^r(-1)^r\frac{(n+r)!}{(n-r)!(r!)^2}\\ &= (-1)^n\sum_{r=0}^{n}(1-\gamma)^r(-1)^r\binom{n+r}{r}\binom{n}{n-r}\\ &= (-1)^n\sum_{r=0}^{n}(1-\gamma)^r\binom{-n-1}{r}\binom{n}{n-r}\\ &= (-1)^n\frac{1}{n!}\Big((1+x)^n(1+(1-\gamma)x)^{-n-1}\Big)^{(n)}\Big|_{x=0}.\end{aligned}$$

Hence, $c_n^\gamma = (-1)^n c_n^{1-\gamma}$. Therefore, it is enough to prove the claim only for $\gamma \in (0, 1/2]$, and separately, for $\gamma = 0$.

Let $\gamma \in (0, 1/2]$. Note that the function $h(x) := (1+x)^n(1+\gamma x)^{-n-1}$ is analytic inside the circle of radius $1/\sqrt{\gamma}$. Let $x = (a+bi)/\sqrt{\gamma}$, $a^2 + b^2 = 1$, then

$$\left|\frac{1+x}{1+\gamma x}\right|^2 = \frac{1+\frac{a^2}{\gamma}+\frac{2a}{\sqrt{\gamma}}+\frac{b^2}{\gamma}}{1+\gamma a^2+2\sqrt{\gamma}a+\gamma b^2} = \frac{1}{\gamma}.$$

Thus, the maximum of the function $h(x)$ on the circle $|x| = 1/\sqrt{\gamma}$ cannot exceed $\gamma^{-n/2}/(1-\gamma)$ and due to Cauchy's inequalities,

$$|c_n^\gamma| \le (1/\sqrt{\gamma})^{-n}\gamma^{-n/2}/(1-\gamma) = (1-\gamma)^{-1} \le 2.$$

For $\gamma = 0$, we have $h(x) = (1+x)^n = \sum_{k=0}^{n}\binom{n}{k}x^k$, whence $c_n = 1$. □

Thus, Lemma 3.7 gives

$$|S_3| \le 2. \tag{3.24}$$

Step 2. Estimating S_4. For any $j = 1, ..., t$, there holds

$$0 < \Big(\frac{k+1}{r}\Big)^j - \frac{k...(k-j)}{(r-1)...(r-j)} < \Big(\frac{k+1}{r}\Big)^j - \Big(\frac{k+1-t}{r}\Big)^j < \frac{j(k+1)^{j-1}}{r^j} \le \frac{1}{r}. \tag{3.25}$$

By Stirling's formula,

$$\begin{aligned}\binom{t}{j}\binom{t+j}{j} = \frac{(t+j)!}{(t-j)!(j!)^2} < \sqrt{t+j}\,\frac{(t+j)^{t+j}}{(t-j)^{t-j}j^{2j}} &\le \sqrt{t+\frac{t}{\sqrt{2}}}\left(\frac{(1+\frac{1}{\sqrt{2}})^{1+\frac{1}{\sqrt{2}}}}{(1-\frac{1}{\sqrt{2}})^{1-\frac{1}{\sqrt{2}}}2^{-\frac{1}{2\sqrt{2}}}}\right)^t\\ &= \sqrt{t+\frac{t}{\sqrt{2}}}L^t.\end{aligned} \tag{3.26}$$

Combining (3.25) and (3.26), we get

$$|S_4| \le t \cdot \sqrt{t + \frac{t}{\sqrt{2}} L^t} \cdot \frac{1}{r} < 2L^t \frac{t^{1.5}}{r} < 1$$

for $r \ge 2L^t t^{1.5}$, which along with (3.24) gives us relation (3.23). □

Let us turn back to the entries of the matrix $T^\dagger$. In view of (3.20), we derive from (3.19) and Lemma 3.6

$$\begin{aligned}
|(Y^* \tilde{N}_p W^\dagger)_{2u,k}| &< 2 \cdot 4^{-u} \sum_{v=0}^{u} \binom{2u}{u-v} \sum_{w=0}^{2p-1} \frac{(v+w)!}{v!w!} \sum_{t=w}^{2p-1} \frac{\binom{t+w}{w}\binom{r-w-1}{r-t-1}}{\binom{2t}{t}\binom{r+t}{2t+1}} 4 \binom{r-1}{t} \tau(k) \\
&= 8(r-1)! \sum_{v=0}^{u} \sum_{w=0}^{2p-1} \sum_{t=w}^{2p-1} \frac{(u!)^2 (v+w)!(t+w)!(r-w-1)!(2t+1)}{(u-v)!(u+v)!v!(w!)^2(r-t-1)!(t-w)!(r+t)!} \tau(k),
\end{aligned}$$

where

$$\tau(k) = \begin{cases} 1, & \text{if } k \le 2p-1 \text{ or } k \ge r-2p, \\ r^{\frac{-q+2p-1}{2}}, & \text{if } k = q \text{ or } k = r-1-q,\ 2p-1 \le q \le \frac{\sqrt{r}}{2}, \\ r^{\frac{-\sqrt{r}/2+2p-1}{2}}, & \text{if } \frac{\sqrt{r}}{2} < k < r-1-\frac{\sqrt{r}}{2}. \end{cases}$$

Going from $w-1$ to w, the corresponding product changes by

$$\frac{(v+w)(t+w)(t-w+1)}{(r-w)w^2} < \frac{(4p)^3}{r-2p} < 1,$$

hence, the maximum is attained at $w = 0$. So,

$$\begin{aligned}
|(Y^* \tilde{N}_p W^\dagger)_{2u,k}| &< 8(r-1)!\ 2p \sum_{v=0}^{u} \sum_{t=0}^{2p-1} \frac{(u!)^2 (r-1)!(2t+1)}{(u-v)!(u+v)!(r-t-1)!(r+t)!} \tau(k) \\
&< 16p \cdot 4p \cdot (u+1) \sum_{t=0}^{2p-1} \frac{((r-1)!)^2}{(r-t-1)!(r+t)!} \tau(k) \\
&< 16p \cdot 4p \cdot p \cdot 2p \frac{1}{r} \tau(k) = \frac{128 p^4 \tau(k)}{r}.
\end{aligned}$$

Similarly, from (3.17) we have

$$\begin{aligned}
|(W^\dagger)_{qk}| &\le \sum_{w=q}^{2p-1} (w+1) \sum_{t=w}^{2p-1} \frac{\binom{t+w}{w}\binom{r-w-1}{r-t-1}}{\binom{2t}{t}\binom{r+t}{2t+1}} 4 \binom{r-1}{t} \tau(k) \\
&= \sum_{w=q}^{2p-1} (w+1)(r-1)! \sum_{t=w}^{2p-1} \frac{(t+w)!(r-w-1)!(2t+1)}{w!(r-t-1)!(t-w)!(r+t)!} \tau(k).
\end{aligned}$$

Here going from $w-1$ to w our term changes by $(t+w)(t-w+1)/w(r-w) < 2(2p)^2/(r-2p) < 1$, so,

$$|(W^\dagger)_{qk}| \le 2p \cdot 2p \cdot 4p \sum_{t=0}^{2p-1} \frac{((r-1)!)^2}{(r-t-1)!(r+t)!} \tau(k) < \frac{32 p^4 \tau(k)}{r}. \tag{3.27}$$

Since an entry of Y^* does not exceed in absolute value $2p \cdot p^{4p-2}$ (see (3.18)), then an entry of $Y^*\tilde{N}_pX$ is less than or equal in absolute value to $2p \cdot 2p^{4p-1} \cdot 8p^2s^{4p-1}/(r-8p^2s^{4p-1})$ (here we used estimate (3.16)). So an entry of $Y^*\tilde{N}_pXW^\dagger$ does not exceed

$$2p \cdot 32p^{4p+2}\frac{8p^2s^{4p-1}}{r-8p^2s^{4p-1}} \cdot \frac{32p^4}{r}.$$

Thus, for $r-1 \geq k > s-1$, according to (3.15) we have

$$\begin{aligned}|(Y^*\tilde{N}_pV^\dagger)_{2u,k-s}| = |(Y^*\tilde{N}_p(E+X)W^\dagger)_{2u,k}| &\leq |(Y^*\tilde{N}_pW^\dagger)_{2u,k}| + |(Y^*\tilde{N}_pXW^\dagger)_{uk}| \\ &< \frac{2^7p^4\tau(k)}{r} + \frac{2^{15}p^{4p+9}s^{4p-1}}{(r-8p^2s^{4p-1})r}. \end{aligned} \tag{3.28}$$

For $r-1 \geq k > s-1$ and an odd u, due to (3.27)

$$|(Y^*\tilde{N}_pV^\dagger)_{u,k-s}| \leq \max_i |(V^\dagger)_{i,k-s}| \leq 2\max_i |(W^\dagger)_{ik}| < 2\frac{32p^4\tau(k)}{r}. \tag{3.29}$$

From (3.28) and (3.29) we finally get

$$\begin{aligned}\|Y^*\tilde{N}_pV^\dagger\|_\infty \leq r \cdot &\frac{2^{15}p^{4p+9}s^{4p-1}}{(r-8p^2s^{4p-1})r} \\ &+ \frac{2^7p^4}{r}\Big(\sum_{k=0}^{2p-1} + \sum_{k=r-2p}^{r-1} + \sum_{k=2p}^{\lfloor\sqrt{r}/2\rfloor} + \sum_{k=r-1-\lfloor\sqrt{r}/2\rfloor}^{r-2p-1} + \sum_{k=\lfloor\sqrt{r}/2\rfloor+1}^{r-\lfloor\sqrt{r}/2\rfloor-2}\Big)\tau(k) \\ &< \frac{2^{15}p^{4p+9}s^{4p-1}}{r-8p^2s^{4p-1}} + \frac{2^7p^4}{r}(2p+2p+1+1+r\cdot r^{-\frac{\sqrt{r}}{4}+p-\frac{1}{2}}) \\ &< \frac{2^{16}p^{4p+9}s^{4p-1}}{r},\end{aligned}$$

whence in light of (3.14) condition (3.2) follows, and the needed is proved. □

Chapter 4

Number of lower sets with fixed cardinality

For a given d, we call a set $S \subset \mathbb{Z}_+^d$ *a lower set* (or *a downward closed set*) if for any $\mathbf{x} = (x_1, ..., x_d) \in \mathbb{Z}_+^d$ the condition $\mathbf{x} \in S$ implies $\mathbf{x}' = (x'_1, ..., x'_d) \in S$ for all $\mathbf{x}' \in \mathbb{Z}_+^d$ with $x'_i \leq x_i$, $1 \leq i \leq d$. By $p_d(n)$ we denote the number of lower sets in $\mathbb{Z}_+^d$ containing exactly n points.

There is a one-to-one correspondence between d-dimensional lower sets of cardinality n and $(d-1)$-dimensional partitions of n, that is, representations of the form

$$n = \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} ... \sum_{i_{d-1}=1}^{\infty} n_{i_1 i_2 ... i_{d-1}}, \quad n_{i_1 i_2 ... i_{d-1}} \in \mathbb{Z}_+,$$

where $n_{i_1 i_2 ... i_{d-1}} \geq n_{j_1 j_2 ... j_{d-1}}$ if $j_k \geq i_k$ for all $k = 1, 2, ..., d-1$. Thus, lower sets represent a geometric interpretation of multidimensional integer partitions. In particular, two-dimensional lower sets with n elements visualize integer partitions of the number n, i.e. its representations as a sum $n = n_1 + n_2 + ... n_k$, $n_1 \geq n_2 \geq ... \geq n_k$ via the so-called Young diagrams, which consist of n cells placed in k rows and n_1 columns so that the ith row contains n_i cells and the first cell in each row belongs to the first column.

4.1 History of the problem

4.1.1 Small dimensional lower sets

The history begins with finding the number $p_2(n)$ of integer partitions of a positive integer n, i.e. of representations of n as a sum of nonincreasing positive integers, and evidently goes back to Leibniz [47]. However, the first significant results in the partition theory were obtained much later by Euler [30]. Another way to understand $p_2(n)$ is considering the following generating function [50, Vol. 2, p. 1]

$$\prod_{k=1}^{\infty} (1 - x^k)^{-1} = \sum_{n=0}^{\infty} p_2(n) x^n,$$

where we assume $p_d(0) = 1$ for any d. In 1917, Hardy and Ramanujan revealed the asymptotic behaviour of the function $p_2(n)$ (see [39, (3)] or [40, (1.4)]):

$$p_2(n) \sim \frac{e^{\sqrt{\frac{2n}{3}}\pi}}{4\sqrt{3}n}.$$

Later, Rademacher [65, (1.8)] found an expansion of $p_2(n)$ as a convergent series.

In the case $d = 3$, for the so-called plane partitions, the generating function was given by MacMahon [49]:

$$\prod_{k=1}^{\infty}(1-x^k)^{-k} = \sum_{n=0}^{\infty} p_3(n)x^n$$

(see [14] for a simpler proof). The asymptotics of $p_3(n)$ was obtained by Wright [75, (2.21)], namely,

$$p_3(n) \sim \frac{(2\zeta(3))^{\frac{7}{36}} e^{\zeta'(-1)}}{\sqrt{2\pi} n^{\frac{25}{36}}} e^{3(\zeta(3))^{\frac{1}{3}} 2^{-\frac{2}{3}} n^{\frac{2}{3}}},$$

where

$$\zeta'(-1) = 2\int_0^{\infty} \frac{y \log y}{e^{2\pi y}-1}\, dy \approx -0.165421.$$

For the cases $d > 3$, no generating functions are known so far, although MacMahon conjectured that the function

$$\prod_{k=1}^{\infty}(1-x^k)^{-\binom{d+n-2}{n-1}} \tag{4.1}$$

should generate $p_d(n)$ for every d, but this turned out to be wrong. On the other hand, some relations between the numbers $p_d(n)$ and the so-called MacMahon's numbers generated by (4.1), as well as some numerical values of $p_d(n)$, can be found in [3]. Besides, it was conjectured in [55] that MacMahon's numbers give the asymptotics of $\log p_d(n)/n^{1-1/d}$ for solid partitions, i.e. for $d = 4$, and the hyposesis was accompanied by the exact values of $p_4(n)$, $n \leq 50$, and Monte Carlo simulations (see also [5] for related numerical results in higher dimensions). However, the computations in [17] make this conjecture unlikely to be true for $d = 4$.

It is worth mentioning that an effective method for evaluating $p_4(n)$ is suggested in [44]. Moreover, there is an algorithm that enables one to compute numbers of partitions for $n \leq 26$ in any dimension (see [35]).

Importantly, the partition theory has many applications in physics, as there are a lot of physical structures resembling that of multidimensional integer partitions. In particular, integer partitions are used to estimate the energy levels for a heavy nucleus [10] and to study the shape of crystal growth [67]. Another direction of research is based on the existence of a one-to-one correspondence between partitions of an integer and microstates of a gas particles stored in a harmonic oscillator, not only in two-dimensional case [4, 72] but also in multidimensional setting [56].

Furthermore, the spaces of polynomials associated with lower sets have recently turned out to be a powerful tool in multivariate approximation (see [11, 15, 16] and references therein).

In the problem of estimating $p_d(n)$, the important relation

$$C_1(d) \leq \frac{\log p_d(n)}{n^{1-\frac{1}{d}}} \leq C_2(d) \tag{4.2}$$

was established by Bhatia, Prasad and Arora [9, (12), (16)], however the exact dependence of the constants on d remained an open problem. Explicit values of C_1 and C_2 have recently been suggested in [16, Th. 1.5], according to which (4.2) holds with

$$C_1(d) = 0.9\frac{d}{(d!)^{\frac{1}{d}}}\log 2, \quad C_2(d) = \pi\sqrt{\frac{2}{3}}d^{\log d}, \tag{4.3}$$

where the upper bound holds for any $n \in \mathbb{N}$ and the lower bound is valid for $n > 55^d$. Note that in this case $C_1(d)$ is uniformly bounded from below since Stirling's formula gives

$$d! < \sqrt{2\pi d}\left(\frac{d}{e}\right)^d e^{\frac{1}{12d}}$$

for all $d \geq 1$, and consequently, we have for $d \geq 3$,

$$C_1(d) \geq \frac{0.9e\log 2}{(2\pi d e^{\frac{1}{6d}})^{\frac{1}{2d}}} \geq \frac{0.9e\log 2}{(6\pi e^{\frac{1}{18}})^{\frac{1}{6}}} > 1.$$

So, for $n > 55^d$, we have $\log p_d(n) > n^{1-1/d}$ (see [40, Sec. 2] for the case $d = 2$).

4.1.2 High dimensional lower sets

If we do not restrict ourselves to the case of a fixed (or relatively small) dimension d and assume that d grows somehow significantly along with n, then the general structure of lower sets changes, and the two-sided estimates in Theorem 4.1 are no longer true. Besides, estimate (4.2) with C_1 and C_2 from (4.3) becomes quite rough if we just allow d to be of order $\log n$. Somewhat better bounds for this setting were obtained in [15, (24), (31)]:

$$p_d(n) \leq 2^{dn} \quad \text{and} \quad p_d(n) \leq d^{n-1}(n-1)!$$

for any positive integers d and n. The latter inequality was strengthened and complemented by a lower bound in [16, Th. 1.4]

$$\binom{d+n-2}{n-1} \leq p_d(n) \leq d^{n-1}.$$

Note that $\binom{d+n-2}{n-1} > d^{n-1}/(n-1)!$

4.2 New bounds

We show that if the dimension d is sufficiently small with respect to n, then C_2 in (4.2) is also independent of d.

Theorem 4.1. *For any $d \geq 2$ and any $n \geq (30d)^{2d^2}$, there holds*

$$1 < \frac{\log p_d(n)}{n^{1-\frac{1}{d}}} < 7200.$$

If n satisfies the weaker condition $n \geq d^{12d\log d}$, then

$$1 < \frac{\log p_d(n)}{n^{1-\frac{1}{d}}} < d^2.$$

The case of high dimensional lower sets is treated in the following theorem, which gives asymptotics of $\log p_d(n)$ for different orders of growth of d provided that $d \gtrsim n/\log^{\gamma} n$ for some γ.

Theorem 4.2. *(a) If $d > n^3/2$, then*

$$1 \leq \frac{p_d(n)}{\binom{d+n-2}{d-1}} < \frac{1}{1-\frac{n^3}{2d}}.$$

(b) If $dn^{-2} \to \infty$ as $n \to \infty$, then

$$\log p_d(n) = (n-1)(\log d - \log n + 1) + o(n).$$

(c) If d satisfies $cn^2 \leq d \leq Cn^2$ for some constants c and C, then

$$\log p_d(n) = n\log n + O(n).$$

(d) If $dn^{-2} \to 0$ and $\log d \geq \log n + o(\log n)$ as $n \to \infty$, then

$$\log p_d(n) = n\log n + o(n\log n).$$

In particular, combining the estimates that lead us to the result above and applying them to the power–logarithmic scale of d in terms of n, we come to

Corollary 4.3. *If $cn^{\alpha}\log^{\gamma} n \leq d \leq Cn^{\alpha}\log^{\gamma} n$ for some $\alpha \geq 1$, $\gamma \in \mathbb{R}$, and positive constants c and C, then*

$$p_d(n) = \begin{cases} \binom{d+n-2}{d-1}\theta(d,n), & \text{if } \alpha > 3, \text{ or } \alpha = 3, \ \gamma > 0, \\ e^n n^{(\alpha-1)n}\log^{\gamma n} n \; e^{O(n^{3-\alpha}\log^{-\gamma} n + \log n)}, & \text{if } 2 \leq \alpha \leq 3, \\ n^n e^{O(n\log\log n)}, & \text{if } 1 \leq \alpha < 2. \end{cases}$$

Here the function $\theta(d,n) \geq 1$ is bounded above by a constant that depends only on α and γ.

Remark 4.4. Note that the case $\alpha = 3$, $\gamma = 0$, $c > 0.5$, is covered by Theorem 4.2 (a).

4.3 Lower sets in small dimensional spaces

In this section, we prove Theorem 4.1 and in the course of the proof reveal some features of the nature of lower sets.

From now on we associate any point $q = (q_1, ..., q_d) \in \mathbb{Z}_+$ of a lower set with a unit cube having its center at that point. So, we will stick to this visualization of a lower set as a set of cubes leaning on one another. In Figure 4.1, we give an example of such a visualization of a plane partition of $n = 15$ with

$$n_{11} = 4,\ n_{12} = 3,\ n_{13} = 2,\ n_{14} = 1,\ n_{21} = 3,\ n_{22} = 1,\ n_{31} = 1,$$

so that the lower layer by itself represents a partition of $n_{11} + n_{12} + n_{13} + n_{14} = 10$, while the next one, a partition of $n_{21} + n_{22} = 4$.

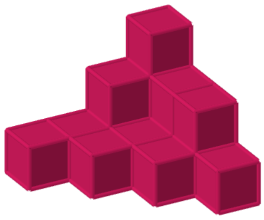

Figure 4.1:

For two cubes $q = (q_1, ..., q_d)$ and $q' = (q'_1, ..., q'_d)$, we write $q \succ q'$ if $q_i \geq q'_i$ for all $i = 1, ..., d$. If there holds either $q \succ q'$ or $q' \succ q$, we say that q and q' are *comparable.*

In the first place, we will be interested in the "top" subsets of lower sets, which will play a crucial role in our further analysis. To be more specific, we need the following

Definition. We call a subset Q' of a lower set Q *available* if for any $q' \in Q'$ there is no $q \in Q \setminus \{q'\}$ such that $q \succ q'$.

In other words, it is such a subset that we can take its elements out in any order without breaking the lower set structure in any step. Denote by $M(Q)$ the maximal available subset of Q. The lemma below delivers the bound on $|M(Q)|$ that in some sense we would expect basing on our intuition: it seems that the more concentrated the lower set is, the richer its available subset can be (see Remark 4.6).

Lemma 4.5. *For any $d \geq 2$ and $n \geq d^{6d \log d}$, for any d-dimensional lower set Q, $|Q| = n$, there holds*

$$|M(Q)| \leq \prod_{k=1}^{d-1} \left(1 + \frac{1}{k^2}\right) n^{1-\frac{1}{d}} < \frac{\sinh \pi}{\pi} n^{1-\frac{1}{d}}.$$

Remark 4.6. Define the lower sets Q_k in d-dimensional space by the condition $q \in Q_k \Leftrightarrow q_1 + ... + q_d \leq k$. Then it follows from Lemma 4.5 that the sets Q_k, for large enough k, are optimal in the sense that their maximal available subsets are the largest possible up to an absolute constant.

Proof of Lemma 4.5. We prove the left-hand side inequality by induction on d, which will yield the assertion of the lemma. In the case $d = 2$ we have $Q = Q_1 \cup Q_2$, where every

$q = (q_1, q_2) \in Q_i$ satisfies $q_i \le \sqrt{n} - 1,\ i = 1, 2$. Since $M(Q)$ cannot have more than one cube with a fixed q_1 or q_2, we can write $|M(Q)| \le 2\sqrt{n}$.

Suppose now that we proved the inequality for the dimensions $2, 3, ..., d-1$, and let us prove it for $d \ge 3$. For simplicity we denote

$$K_d := \prod_{k=1}^{d-1} (1 + k^{-2}) < 4.$$

Consider all the nonempty subsets $Q_0, ..., Q_m,\ m \le n-1$, being the intersections of Q with the hyperplanes $q_1 = 0, ..., m$, respectively. They are lower sets themselves and if for some j we have $q = (j, q_2, ..., q_d) \in Q_j \cap M(Q)$, then $q = (j+s, q_2 ..., q_d) \notin Q_{j+s}$ for all $s \le m - j$. Let

$$n_i := |Q_i|,\ 0 \le i \le m.$$

Note that we can apply the induction assumption to Q_i with $n_i \ge (d-1)^{6(d-1)\log(d-1)}$. Now, taking into account that

$$(d-1)^{6(d-1)\log(d-1)} < d^{6(d-1)(\log(d-1)-\log d)} n^{1-\frac{1}{d}} \le d^{-3} n^{1-\frac{1}{d}}, \tag{4.4}$$

we have (assuming $n_{m+1} = 0$)

$$\begin{aligned} |M(Q)| &\le \sum_{i=0}^{m} \min\{n_i - n_{i+1}, K_{d-1} n_i^{1-\frac{1}{d-1}}\} + \sum_{n_i \le d^{-3} n^{1-\frac{1}{d}}} n_i - n_{i+1} \\ &\le \sum_{i=0}^{m} \min\{n_i - n_{i+1}, K_{d-1} n_i^{1-\frac{1}{d-1}}\} + \frac{n^{1-\frac{1}{d}}}{d^3} \\ &=: \sum_{i=0}^{m} \min\{\Delta_i, \Gamma_i\} + \frac{n^{1-\frac{1}{d}}}{d^3} =: \sum_{i=0}^{m} M_i + \frac{n^{1-\frac{1}{d}}}{d^3} \\ &=: F(n_0, ..., n_m) + \frac{n^{1-\frac{1}{d}}}{d^3}. \end{aligned} \tag{4.5}$$

We will maximize $F(n_0, ..., n_m)$ over all $(n_0, ..., n_m)$ in the set

$$S_n := \{(n_0, ..., n_m) \in \mathbb{R}^+ : n_0 \ge ... \ge n_m,\ n_0 + ... + n_m = n\},$$

permiting thereby n_i to take noninteger values. Take a point $(n_0, ..., n_m)$ where this maximum is attained. Assume that for some $i,\ 0 \le i \le m-1$, such that

$$n_i > 8^{d-1}$$

we have $\Delta_i > \Gamma_i$. If we substitute the pair (n_i, n_{i+1}) by $(n_i - x, n_{i+1} + x)$ for sufficiently small positive x, the new point will still be in S_n with $M_j,\ j \ne i-1, i, i+1$, and Γ_{i-1} remaining unchanged. At the same time, Δ_{i-1} will increase, which means that M_{i-1} will not decrease. Moreover, choosing x small enough we can keep either the relation $\Delta_{i+1} \ge \Gamma_{i+1}$ or $\Delta_{i+1} \le \Gamma_{i+1}$ true. Consider the two cases.

Case 1. $\Delta_{i+1} \ge \Gamma_{i+1}$.
Then $M_i + M_{i+1} = \Gamma_i + \Gamma_{i+1}$ and the sum $\Gamma_i + \Gamma_{i+1}$ increases as n_i and n_{i+1} become closer to each other while keeping their sum constant. Thus, we increase $F(n_0, ..., n_m)$, which contradicts the definition of $(n_0, ..., n_m)$.

Case 2. $\Delta_{i+1} \le \Gamma_{i+1}$.
The value $\Delta_{i+1} + \Gamma_i$ changes in

$$x - K_{d-1}\big(n_i^{1-\frac{1}{d-1}} - (n_i - x)^{1-\frac{1}{d-1}}\big) \ge x\Big(1 - K_{d-1}\frac{d-2}{d-1}(n_i - x)^{-\frac{1}{d-1}}\Big) > 0,$$

since $n_i \ge 8^{d-1} > K_{d-1}^{d-1}$. Hence, $M_i + M_{i+1}$ increases. Thus, we increase $F(n_0, ..., n_m)$, which once again contradicts the definition of $(n_0, ..., n_m)$.

The fact that both cases led us to contradictions means that there holds

$$\Delta_i \le \Gamma_i, \qquad 0 \le i \le \min\{p, m-1\}, \tag{4.6}$$

where p is the maximal index satisfying $n_p \ge 8^{d-1}$.

If $m \le n^{1/d} - 1$, we have

$$|M(Q)| \le \max_{n_0 + \ldots n_m = n} \sum_{i=0}^{m} K_{d-1} n_i^{1-\frac{1}{d-1}} + \frac{n^{1-\frac{1}{d}}}{d^3} < K_d n^{1-\frac{1}{d}} \tag{4.7}$$

and there is nothing to prove. Thus, from now on, we can assume that

$$m \ge n^{\frac{1}{d}} \qquad \text{and} \qquad n_m \le n^{1-\frac{1}{d}}. \tag{4.8}$$

The rest of the proof we divide into two cases: the case of "large" and the case of "small" values n_0. We will see that n_0 cannot be large at a point of the maximum of $F(n_0, ..., n_m)$.

Case a. $n_0 > 2K_{d-1}n^{1-1/d}$.
Define the sequence $\{a_i\}_{i=0}^{\infty}$ in the following way:

$$a_0 := n_0 \quad \text{and} \quad a_i := a_{i-1} - K_{d-1} a_{i-1}^{1-\frac{1}{d-1}} \text{ for } i \ge 1. \tag{4.9}$$

Let us estimate the maximal number k such that

$$a_k > \frac{K_{d-1} n^{1-\frac{1}{d}}}{2} \quad \text{and} \quad \sum_{i=0}^{k} a_i \le n.$$

From the definition of a_i we see that the ratio a_i/a_{i+1} increases along with i. So,

$$\frac{a_{k/2}}{0.5K_{d-1}n^{1-\frac{1}{d}}} > \frac{a_{k/2}}{a_k} > \frac{a_0}{a_{k/2}} > \frac{2K_{d-1}n^{1-\frac{1}{d}}}{a_{k/2}},$$

where in the case of odd k we understand $a_{k/2}$ as $(a_{(k-1)/2} + a_{(k+1)/2})/2$. Hence, $a_{k/2} > K_{d-1}n^{1-1/d}$. Since $a_i - a_{i+1}$ decreases, we have $(k+1)a_{k/2} \le \sum_{i=0}^{k} a_i \le n$, which yields

$$k + 1 < n^{\frac{1}{d}} K_{d-1}^{-1}.$$

So,

$$a_0 - a_{k+1} = \sum_{i=0}^{k} K_{d-1} a_i^{1-\frac{1}{d-1}} \le K_{d-1} \frac{n^{\frac{1}{d}}}{K_{d-1}} \left(\frac{nK_{d-1}}{n^{1/d}}\right)^{1-\frac{1}{d-1}} = K_{d-1}^{1-\frac{1}{d-1}} n^{1-\frac{1}{d}} < K_{d-1} n^{1-\frac{1}{d}},$$

whence $a_{k+1} > K_{d-1}n^{1-1/d} > 0.5K_{d-1}n^{1-1/d}$. Thus, the sum of a_i becomes equal to n before a_i reaches $0.5K_{d-1}n^{1-1/d}$. Therefore, according to (4.6), since n_i for $i \le p$ decreases slower than a_i does, we obtain that $p = m$ and

$$m + 1 \le k + 1 < n^{\frac{1}{d}} K_{d-1}^{-1} < n^{\frac{1}{d}},$$

which contradicts (4.8).

Case b. $n_0 \le 2K_{d-1}n^{1-1/d}$.
Assume first that

$$n_0 - n_p > L_d n^{1-\frac{1}{d}}, \quad L_d := K_{d-1}\Big(1 + \frac{2}{3(d-1)^2}\Big). \tag{4.10}$$

Considering the sequence $\{a_i\}$ given by (4.9), denote by q the maximal index such that

$$a_q \ge 8^{d-1} \quad \text{and} \quad \sum_{i=0}^{q} a_i \le n.$$

We divide the interval $(a_q + \varepsilon, a_0]$ into

$$I_j := (\nu_j, \mu_j] := (A_j n^{1-\frac{1}{d}}, (A_j + n^{-\frac{1}{2d}})n^{1-\frac{1}{d}}], \quad A_j := A_{j+1} + n^{-\frac{1}{2d}},$$

where $\varepsilon \in [0, n^{1-3/(2d)})$ is chosen so that $(a_0 - a_q - \varepsilon)n^{-1+3/(2d)} \in \mathbb{Z}$. Note that $|I_j| = n^{1-3/(2d)}$ for all j. Denote the number of a_i's belonging to I_j by k_{A_j} and let a_{i_j} be the greatest of a_i that belongs to I_j.

Now, in order to prove that the assumption (4.10) cannot hold, we are going to show that each I_j contains significantly many terms a_i, and this will yield that a_i, and therefore n_i, cannot decrease considerably until the sum of its first terms becomes equal to n. The fact that $a_0 = n_0$ is not very large implies certain regularity of a_i, namely, it will ensure that the leaps between a_i are small enough to get appropriate estimates on k_{A_j} for each of the intervals I_j.

Fix some j and suppress for simplicity the index j in A_j. Suppose that

$$k_A < \frac{A^{\frac{1}{d-1}-1} n^{\frac{1}{2d}}}{L_d}. \tag{4.11}$$

Since the ratio a_i/a_{i+1} increases and $a_{i_{j+1}-1} > \nu_j = A_j n^{1-1/d}$, we have

$$\begin{aligned}
\Big(\frac{1}{1 - K_{d-1}A^{-\frac{1}{d-1}}n^{-\frac{1}{d}}}\Big)^{k_A} &\ge \Big(\frac{1}{1 - K_{d-1}a_{i_{j+1}-1}^{-\frac{1}{d-1}}}\Big)^{k_A} \ge \Big(\frac{a_{i_{j+1}-1}}{a_{i_{j+1}}}\Big)^{k_A} \ge \frac{a_{i_j}}{a_{i_{j+1}}} \\
&\ge \frac{\mu_j - K_{d-1}a_{i_j-1}^{1-\frac{1}{d-1}}}{\mu_{j+1}} \ge \frac{(A + n^{-\frac{1}{2d}})n^{1-\frac{1}{d}} - K_{d-1}a_0^{1-\frac{1}{d-1}}}{An^{1-\frac{1}{d}}} \\
&\ge 1 + \frac{n^{-\frac{1}{2d}}}{A} - \frac{K_{d-1}^2 2n^{-\frac{1}{d}}}{A}.
\end{aligned} \tag{4.12}$$

At the same time, since $(k_A + 1 + x)/(2 + x) \le (k_A + 1)/2$ for $x \ge 0$, the ratio between consequent terms in the binomial expansion of

$$(1 - K_{d-1}A^{-\frac{1}{d-1}}n^{-\frac{1}{d}})^{-k_A}$$

is less than

$$\frac{k_A+1}{2}K_{d-1}A^{-\frac{1}{d-1}}n^{-\frac{1}{d}} < \frac{K_{d-1}A^{-\frac{1}{d-1}}n^{-\frac{1}{d}}}{2} + \frac{A^{-1}n^{-\frac{1}{2d}}K_{d-1}}{2L_d}$$
$$\leq \frac{K_{d-1}(d-1)^{\frac{2}{d-1}}n^{-\frac{1}{d}}}{2} + \frac{(d-1)^2 n^{-\frac{1}{2d}}K_{d-1}}{2L_d} \leq (d-1)^2 n^{-\frac{1}{2d}}.$$

This implies

$$(1-K_{d-1}A^{-\frac{1}{d-1}}n^{-\frac{1}{d}})^{-k_A} < 1 + k_A K_{d-1}A^{-\frac{1}{d-1}}n^{-\frac{1}{d}}\frac{1}{1-(d-1)^2 n^{-\frac{1}{2d}}}$$
$$< 1 + \frac{n^{-\frac{1}{2d}}}{A}\frac{1}{\left(1+\frac{2}{3(d-1)^2}\right)(1-(d-1)^2 n^{-\frac{1}{2d}})}.$$

Combining this with (4.12), we obtain

$$\frac{1}{\left(1+\frac{2}{3(d-1)^2}\right)(1-(d-1)^2 n^{-\frac{1}{2d}})} > 1 - 2n^{-\frac{1}{2d}}K_{d-1}^2,$$

which yields

$$0 > -2n^{-\frac{1}{2d}}K_{d-1}^2 + \frac{2}{3(d-1)^2} - \frac{4n^{-\frac{1}{2d}}K_{d-1}^2}{3(d-1)^2} - (d-1)^2 n^{-\frac{1}{2d}}$$
$$+ 2(d-1)^2 n^{-\frac{1}{d}}K_{d-1}^2 - \frac{2n^{-\frac{1}{2d}}}{3} + \frac{4n^{-\frac{1}{d}}K_{d-1}^2}{3}$$
$$\geq \frac{2}{3(d-1)^2} - \frac{68}{3}n^{-\frac{1}{2d}}(d-1)^2 \geq 0,$$

since $n > d^{18d} > 34^{2d}(d-1)^{8d}$. This contradiction disproves (4.11), whence

$$k_{A_j} \geq \frac{A_j^{\frac{1}{d-1}-1}n^{\frac{1}{2d}}}{L_d}.$$

Summing up this inequality over all j, we derive

$$n \geq \sum_{a_i \in \cup I_j} a_i \geq \sum_j k_{A_j}A_j n^{1-\frac{1}{d}} \geq \sum_j \frac{n^{1-\frac{1}{2d}}A_j^{\frac{1}{d-1}}}{L_d} > nL_d^{-1}\int_{a_q n^{\frac{1}{d}-1}+n^{-\frac{1}{2d}}}^{a_0 n^{\frac{1}{d}-1}-n^{-\frac{1}{2d}}} x^{\frac{1}{d-1}}\,dx$$
$$=: nL_d^{-1}\int_y^{y+z-2n^{-\frac{1}{2d}}} x^{\frac{1}{d-1}}\,dx,$$

where $z \geq L_d$ by the assumption (4.10). This means that there holds

$$1 > L_d^{-1}\int_y^{y+L_d-2n^{-\frac{1}{2d}}} x^{\frac{1}{d-1}}\,dx \geq L_d^{-1}\frac{d-1}{d}(L_d-2n^{-\frac{1}{2d}})^{\frac{d}{d-1}}$$
$$\geq (1-2n^{-\frac{1}{2d}})L_d^{\frac{1}{d-1}}\frac{d-1}{d} \geq \frac{L_d^{\frac{1}{d-1}}\frac{d-1}{d}}{1+3n^{-\frac{1}{2d}}} \geq 1. \tag{4.13}$$

Let us prove the latter inequality. It suffices to show that

$$L_d \geq \Big(1 + \frac{1}{d-1}\Big)^{d-1} (1 + 3n^{-\frac{1}{2d}})^{d-1}.$$

This, in turn, will follow from

$$1 + \frac{2}{3(d-1)^2} \geq (1 + 3n^{-\frac{1}{2d}})^{d-1} \Big(1 + \frac{1}{2(d-1)^2}\Big) \tag{4.14}$$

and

$$K_{d-1} \Big(1 + \frac{1}{2(d-1)^2}\Big) \geq \Big(1 + \frac{1}{d-1}\Big)^{d-1}. \tag{4.15}$$

Firstly, by the assumption of the lemma we have $n > 6^{2d}(d-1)^{2d}$, so

$$\begin{aligned} 1 + \frac{2}{3(d-1)^2} &> 1 + \frac{1}{2(d-1)^2} + 6(d-1)n^{-\frac{1}{2d}} + \frac{3n^{-\frac{1}{2d}}}{d-1} \\ &= (1 + 6(d-1)n^{-\frac{1}{2d}}) \Big(1 + \frac{1}{2(d-1)^2}\Big) \\ &\geq (1 + 3n^{-\frac{1}{2d}})^{d-1} \Big(1 + \frac{1}{2(d-1)^2}\Big), \end{aligned}$$

which proves (4.14). Secondly, note that for $d = 3$ both sides of (4.15) are equal 2.25 and for $d = 4$ inequality (4.15) becomes $95/36 \geq (4/3)^3$. For $d \geq 5$, (4.15) follows from the fact that the left-hand side is greater than e.

The contradiction in (4.13) along with the fact that a_i's decrease faster than n_i shows that (4.10) does not hold and therefore

$$n_0 - n_p \leq L_d n^{1-\frac{1}{d}}. \tag{4.16}$$

In addition, we note that if $p \neq m$, then by (4.6)

$$n_{p+1} \geq n_p (1 - K_{d-1} n_p^{-\frac{1}{d-1}}) > n_p \Big(1 - K_{d-1}(8^{d-1}))^{-\frac{1}{d-1}}\Big) > 0.5 n_p,$$

whence $n_p < 2 \cdot 8^{d-1}$.

Finally, in light of (4.5), (4.7), and (4.16), we obtain

$$\begin{aligned} |M(Q)| &\leq \frac{n^{1-\frac{1}{d}}}{d^3} + \sum_{i=0}^{p-1} (n_i - n_{i+1}) + \sum_{i=p}^{m} M_i \\ &< \frac{n^{1-\frac{1}{d}}}{d^3} + (n_0 - n_p) + 2 \cdot 8^{d-1} + K_{d-1} n_m^{1-\frac{1}{d-1}} \\ &< \frac{n^{1-\frac{1}{d}}}{d^3} + L_d n^{1-\frac{1}{d}} + 2 \cdot 8^{d-1} + K_{d-1} n^{1-\frac{2}{d}} \\ &< K_{d-1} n^{1-\frac{1}{d}} \Big(\frac{1}{2d^3} + \Big(1 + \frac{2}{3(d-1)^2}\Big) + \frac{8^{d-1}}{n^{1-\frac{1}{d}}} + n^{-\frac{1}{d}} \Big) \\ &\leq K_{d-1} n^{1-\frac{1}{d}} \Big(1 + \frac{2}{3(d-1)^2} + \frac{1}{2d^3} + d^{-10(d-1)\log d} + n^{-\frac{1}{d}}\Big) < K_d n^{1-\frac{1}{d}}, \end{aligned}$$

since $n > 12^d (d-1)^{2d}$, and the proof of the lemma is complete. □

Remark 4.7. Note that under the assumptions of Lemma 4.5, the argument above gives

$$\max_{(n_0,\dots,n_m)\in S_n} \sum_{i=0}^{m} \min\{n_i - n_{i+1}, K_{d-1} n_i^{1-\frac{1}{d-1}}\}$$
$$\leq K_{d-1} n^{1-\frac{1}{d}} \Big(1 + \frac{2}{3(d-1)^2} + d^{-10(d-1)\log d} + n^{-\frac{1}{d}}\Big). \qquad (4.17)$$

Remark 4.8. Without any restriction on d and n we can straightforwardly show that there always holds

$$|M(Q)| \leq d n^{1-\frac{1}{d}}.$$

Proof. Proceeding by induction as in the proof of Lemma 4.5 we obtain

$$|M(Q)| \leq \max_{n_0+\dots+n_m=n} \sum_{i=0}^{m} \min\{n_i - n_{i+1}, (d-1) n_i^{1-\frac{1}{d-1}}\}.$$

As long as Q is a lower set, there holds $n_i \geq n_{i+1}$ for any $i = 1, \dots, m-1$, so $n_{\lfloor n^{1/d} \rfloor} \leq n^{1-1/d}$. Thus,

$$\sum_{k \geq \lfloor n^{1/d} \rfloor} (n_k - n_{k+1}) \leq n^{1-\frac{1}{d}}$$

and

$$|M(Q)| \leq n^{1-\frac{1}{d}} + (d-1) \max_{n_0+\dots+n_{\lfloor n^{1/d} \rfloor -1} \leq n} \sum_{k=0}^{\lfloor n^{\frac{1}{d}} \rfloor -1} n_k^{1-\frac{1}{d-1}}$$
$$\leq n^{1-\frac{1}{d}} + (d-1) n^{\frac{1}{d}} \Big(\frac{n}{n^{1/d}}\Big)^{1-\frac{1}{d-1}} = d n^{1-\frac{1}{d}}.$$

□

Now, as we already have the bound for the cardinalities of the available subsets, we are able to obtain needed estimates for the number of lower subsets of a lower set. Define

$$T(n) := \max_{\text{lower sets } Q:\ |Q|=n} |M(Q)|.$$

Lemma 4.9. *For the number $C(Q, k, d)$ of all lower subsets Q', $|Q'| \geq n - k$, of a lower set Q, $|Q| = n$, in d-dimensional space there holds*

$$C(Q, k, d) < \max\Big\{8, \frac{4eT(n)}{k}\Big\}^k.$$

Proof. First we show that every lower subset Q' of a lower set Q can be constructed by successively discarding cubes of Q one by one so that in any step the current set remains being a lower set.

Indeed, let us list all the cubes we have to discard from Q in a sequence in an arbitrary order. By a *disorder* we call a pair (q, q') of cubes in this sequence such that q goes after q' in it, but $q \succ q'$. Now, if there is a disorder (q, q') in the sequence, we simply swap q and q' eleminating thereby the disorder and not creating any new one. Thus, we can rearrange the sequence so that there is no disorder in it.

Consider the part of the sequence that starts at the beginning and ends right before a comparable pair of cubes appears. Then the cubes of this part belong to $M(Q)$, while the subsequent cube does not. By repeating this process, we observe that each lower subset of Q can be constructed as follows. First we discard some cubes (call this set R_1) from $M(Q) =: M(Q_1)$. After that we remove a set R_2 of cubes from $M(Q_1 \setminus R_1) \setminus M(Q_1) =: M(Q_2) \setminus M(Q)$, and so on. In doing so, the number of ways to take away cubes in the first step is $\binom{|M(Q)|}{|R_1|}$, and in the ith step, for $i > 1$, is $\binom{|M(Q_i)|-(|M(Q_{i-1})|-|R_{i-1}|)}{|R_i|}$. Denoting $k_i := T(|Q_i|) - |M(Q_i)| + |R_i|$, we have $\binom{|M(Q)|}{|R_1|} \le \binom{T(|Q|)}{k_1}$ and for $i > 1$,

$$\begin{aligned}\binom{|M(Q_i)| - (|M(Q_{i-1})| - |R_{i-1}|)}{|R_i|} &= \binom{|M(Q_i)| - (T(|Q_{i-1}|) - k_{i-1})}{|R_i|} \\ &\le \binom{T(|Q_i|) - (T(|Q_{i-1}|) - k_{i-1})}{k_i} \le \binom{k_{i-1}}{k_i}.\end{aligned}$$

Hence, the number of ways to construct a lower subset of Q with a fixed sequence of $|R_i|$ is at most

$$\binom{T(n)}{k_1}\binom{k_1}{k_2}\binom{k_2}{k_3}\ldots\binom{k_{l-1}}{k_l} \le \binom{T(n)}{k_1} 2^{k_1+k_2+\ldots+k_{l-1}} < \binom{T(n)}{k_1} 2^k, \tag{4.18}$$

where l is the number of steps. If $T(|Q|) \le k$, the right-hand side is bounded by 2^{2k}. Otherwise, according to Stirling's formula,

$$\binom{T(n)}{k_1} 2^k < \left(\frac{eT(n)}{k_1}\right)^{k_1} 2^k \le \left(\frac{2eT(n)}{k}\right)^k.$$

Finally,

$$C(Q,k,d) \le \sum_{|R_1|+\ldots+|R_l|\le k} \max\left\{4, \frac{2eT(n)}{k}\right\}^k < \max\left\{8, \frac{4eT(n)}{k}\right\}^k. \tag{4.19}$$

□

Corollary 4.10. *If $n \ge d^{6d\log d}$, there holds*

$$C(Q,k,d) < \left(e^4 \max\left\{1, \frac{n^{1-\frac{1}{d}}}{k}\right\}\right)^k \tag{4.20}$$

and

$$C(Q,k,d) < 2^{2k+4n^{1-\frac{1}{d}}}. \tag{4.21}$$

Proof. Inequality (4.20) follows immediately from Lemmas 4.5 and 4.9 (note that $2\sinh\pi/\pi < e^2$). The second estimate is valid due to Lemma 4.5 and relation (4.18) in the same fashion as (4.19). □

Now we are in a position to prove our main result.

Proof of Theorem 4.1. We start with the second part of the theorem. Let us first prove by induction on d that

$$\log p_d(n) < U_d n^{1-\frac{1}{d}} \quad \text{for } n > d^{12d\log d}, \tag{4.22}$$

where $U_2 := 2\sqrt{2}$ and, for $d \geq 3$,

$$U_d := U_{d-1} d^{\frac{1}{d-1}} + 1.$$

The basis $d = 2$ follows from the estimate $p_2(k) < e^{2\sqrt{2k}}$ (see at the end of [40, Sec. 2]). Assuming that (4.22) holds for $2, 3, ..., d-1$ we will prove it for $d \geq 3$. Take a lower set Q, $|Q| = n$, put $k := \lfloor n^{1/d} \rfloor$, and for a fixed p, $p = 1, ..., d$, consider the following "slices" of Q:

$$Q_i^p := \{Q \cap \{q_p = i\}\} \setminus \bigcup_{0<t<p,\, 0\leq j<k} Q_j^t, \qquad i = 0, ..., k-1,$$

of cardinalities $n_0^p \geq n_1^p \geq ... \geq n_{k-1}^p$. Note that $Q = \bigcup_{0\leq i<k,\, 1\leq p\leq d} Q_i^p$, since otherwise there would exist a cube $q \in Q$ with $q_i > n^{1/d} - 1$ for all $i = 1, ..., d$, and, by the definition of lower sets, the cardinality of Q would exceed $(n^{1/d})^d$, which is not true. So,

$$\sum_{p=1}^{d} l_p = n, \quad l_p := n_0^p + ... + n_{k-1}^p.$$

In addition, any Q_i^p, $i > 0$, is a lower subset of Q_{i-1}^p, so once Q_{i-1}^p is constructed, then if $n_i^p \geq (d-1)^{12(d-1)\log(d-1)}$, the number of possible Q_i^p (with a fixed n_i^p) can be estimated either by (4.21) or by the induction assumption. As in (4.4), one can show that $(d-1)^{12(d-1)\log(d-1)} < d^{-6} n^{1-\frac{1}{d}}$. Thus, combining (4.20), (4.21), and the induction assumption we obtain the following bound for the logarithm of the number of slices of fixed cardinalities n_i^p (for simplicity, we omit the upper indexes p for n_i^p):

$$\begin{aligned}\max_{n_0+...+n_{k-1}=l_p} &\Big\{ U_{d-1} n_0^{1-\frac{1}{d-1}} + \sum_{i=0}^{k-2} \min\{2(n_i - n_{i+1}) + 4n_i^{1-\frac{1}{d-1}}, U_{d-1} n_{i+1}^{1-\frac{1}{d-1}}\} \Big\} \\ &+ \sum_{i:\, n_i < d^{-6} n^{1-\frac{1}{d}}} (n_i - n_{i+1})(4 + \log n_i) \\ &=: G(l_p, d) + \sum_{i:\, n_i < d^{-6} n^{1-\frac{1}{d}}} (n_i - n_{i+1})(4 + \log n_i). \qquad (4.23)\end{aligned}$$

Note that

$$\begin{aligned}\sum_{i:\, n_i < d^{-6} n^{1-\frac{1}{d}}} (n_i - n_{i+1})(4 + \log n_i) &\leq \frac{n^{1-\frac{1}{d}}}{d^6} \big(4 + \log (d-1)^{12(d-1)\log(d-1)}\big) \\ &< \frac{24 n^{1-\frac{1}{d}}}{d^4}. \qquad (4.24)\end{aligned}$$

Then, taking into account (4.23), (4.24), and the inequality $p_2(n) \le e^{2\sqrt{2n}}$, we derive

$$\begin{aligned}
\log p_d(n) &\le \log \left| \left\{ \{n_i^p\},\ 1 \le i \le k,\ 1 \le p \le d : \sum_{i,p} n_i^p = n \right\} \right| + \sum_{p=1}^{d} G(l_p, d) + d\frac{24n^{1-\frac{1}{d}}}{d^4} \\
&\le \log \binom{n+d-2}{d-1} + 2d\sqrt{2n} + \frac{8n^{1-\frac{1}{d}}}{9} + \max_{\substack{l_1+\ldots+l_d=n \\ n_0^p+\ldots+n_{k-1}^p=l_p}} \sum_{p=1}^{d} \sum_{i=0}^{k-1} U_{d-1} n_i^{1-\frac{1}{d-1}} \\
&\le d\log 2n + 2d\sqrt{2n} + \frac{8n^{1-\frac{1}{d}}}{9} + U_{d-1} \max_{l_1+\ldots+l_d=n} \sum_{p=1}^{d} n^{\frac{1}{d(d-1)}} l_p^{1-\frac{1}{d-1}} \\
&\le 3d\sqrt{2n} + \frac{8n^{1-\frac{1}{d}}}{9} + U_{d-1} d^{\frac{1}{d-1}} n^{1-\frac{1}{d}} \\
&< U_d n^{1-\frac{1}{d}},
\end{aligned}$$

completing the proof of (4.22).

Further, as $U_d < d^2$ for $d < 8$, it suffices to prove the second part of the theorem by induction on $d \ge 8$. As above, the induction assumption holds for Q_i^p with $n_i^p \ge (d-1)^{12(d-1)\log(d-1)}$. Likewise in (4.23), by (4.24), the logarithm of the number of slices of fixed cardinalities n_i^p does not exceed

$$\tilde{G}(l_p, d) + \sum_{i:\, n_i < d^{-6} n^{1-\frac{1}{d}}} (n_i - n_{i+1})(4 + \log n_i) \le \tilde{G}(l_p, d) + \frac{24n^{1-\frac{1}{d}}}{d^4}$$

with

$$\begin{aligned}
\tilde{G}(l_p, d) :&= \max_{n_0+\ldots+n_{k-1}=l_p} \Big\{ (d-1)^2 n_0^{1-\frac{1}{d-1}} \\
&\quad + \sum_{i=0}^{k-2} \min\{2(n_i - n_{i+1}) + 4n_i^{1-\frac{1}{d-1}}, (d-1)^2 n_{i+1}^{1-\frac{1}{d-1}}\} \Big\} \\
&\le 4k\left(\frac{l_p}{k}\right)^{1-\frac{1}{d-1}} \\
&\quad + \max_{n_0+\ldots+n_{k-1}=l_p} \Big\{ B_{d-1} n_0^{1-\frac{1}{d-1}} + \sum_{i=0}^{k-1} \min\{2(n_i - n_{i+1}), B_{d-1} n_{i+1}^{1-\frac{1}{d-1}}\} \Big\},
\end{aligned}$$

where $B_x := x^2 - 4,\ x \ge 8$. Noting that

$$\begin{aligned}
B_{d-1} n_0^{1-\frac{1}{d-1}} &+ \sum_{i=0}^{k-1} \min\{2(n_i - n_{i+1}), B_{d-1} n_{i+1}^{1-\frac{1}{d-1}}\} \\
&\le B_{d-1} \sum_{i=0}^{\lfloor 2d^{-1} n^{1/d} \rfloor - 1} n_i^{1-\frac{1}{d-1}} + 2n_{\lfloor 2d^{-1} n^{1/d} \rfloor - 1} \\
&\le B_{d-1} 2^{\frac{1}{d-1}} d^{-\frac{1}{d-1}} l_p^{1-\frac{1}{d}} + \frac{2l_p}{2d^{-1} n^{\frac{1}{d}} - 1} \\
&< B_{d-1} 2^{\frac{1}{d-1}} d^{-\frac{1}{d-1}} l_p^{1-\frac{1}{d}} + d l_p n^{-\frac{1}{d}} + 2d^2 l_p n^{-\frac{2}{d}}
\end{aligned}$$

and that

$$4k\Big(\frac{l_p}{k}\Big)^{1-\frac{1}{d-1}} \leq 4n^{\frac{1}{d(d-1)}} l_p^{1-\frac{1}{d-1}},$$

we have

$$\sum_{p=1}^{d} \tilde{G}(l_p, d) \leq 4d^{\frac{1}{d-1}} n^{1-\frac{1}{d}} + B_{d-1} 2^{\frac{1}{d-1}} n^{1-\frac{1}{d}} + dn^{1-\frac{1}{d}} + 2d^2 n^{1-\frac{2}{d}}.$$

Hence, using again $p_2(n) \leq e^{2\sqrt{2n}}$, we obtain for $d \geq 8$

$$\begin{aligned}
\log p_d(n) &\leq \log \Big|\Big\{\{n_i^p\},\ 1 \leq i \leq k,\ 1 \leq p \leq d : \sum_{i,p} n_i^p = n\Big\}\Big| + \sum_{p=1}^{d} \tilde{G}(l_p, d) + \frac{24n^{1-\frac{1}{d}}}{d^3} \\
&< 3d\sqrt{2n} + 4d^{\frac{1}{d-1}} n^{1-\frac{1}{d}} + B_{d-1} 2^{\frac{1}{d-1}} n^{1-\frac{1}{d}} + dn^{1-\frac{1}{d}} + 2d^2 n^{1-\frac{2}{d}} + \frac{24n^{1-\frac{1}{d}}}{d^3} \\
&< 3d\sqrt{2n} + 2d^2 n^{1-\frac{2}{d}} + \Big(d^2 - \frac{d}{16}\Big) n^{1-\frac{1}{d}} \\
&< d^2 n^{1-\frac{1}{d}},
\end{aligned}$$

where in the last step we used that $n > \max\{(64d)^d, (96\sqrt{2})^6\}$. This concludes the proof of the second part of the theorem.

We now turn to the first part. Our aim will be to prove the stronger inequality

$$p_d(n) < 1800 K_d Y_d n^{1-\frac{1}{d}} =: C K_d Y_d n^{1-\frac{1}{d}}, \tag{4.25}$$

where $Y_d := \prod_{k=3}^{d-1}(1 + 2^{-k})^{1/k}$ and $K_d = \prod_{k=1}^{d-1}(1 + k^{-2})$ as above. Inequality (4.25) implies the needed estimate $p_d(n) < 7200 n^{1-1/d}$, since $K_d < \sinh \pi/\pi$, $\log Y_d < 1/12$, and $e^{1/12} \sinh \pi/\pi < 4$.

Let us prove (4.25) by induction on d. The cases $d \leq 59$ follow from the second part of the theorem, so we prove the claim for $d \geq 60$ assuming that it holds for $2, 3, ..., d-1$. We divide the proof into several steps.

Step 1. Partition into slices. Take a lower set Q, $|Q| = n$, and put

$$k_p := \begin{cases} \lfloor 2^{-(d-1)} d^{-1} n^{1/d} \rfloor =: k, & 0 \leq p \leq d-1, \\ \lfloor 2^{d(d-1)} d^{d-1} n^{1/d} \rfloor, & p = d. \end{cases}$$

For a fixed p, $p = 1, ..., d$, consider the slices

$$Q_i^p := \{Q \cap \{q_p = i\}\} \setminus \bigcup_{0<t<p,\ 0 \leq j < k_t} Q_j^t, \qquad i = 0, ..., k_p - 1$$

of cardinalities $n_0^p \geq n_1^p \geq ... \geq n_{k_p-1}^p$. Note that $\cup_{i,p} Q_i^p = Q$, since otherwise there exists a cube q in Q with $q_p \geq k_p$ for all p, so all the cubes with pth coordinate at most k_p belong to Q as well, which contradicts the fact that $\prod_p k_p > n$.

The idea is to split our lower set Q into two subsets: the union of the chosen slices of the first $d-1$ directions $\bigcup_{1 \leq p \leq d-1,\ 0 \leq i < k_p} Q_i^p$ and the complement subset $Q' := \bigcup_{0 \leq i < k_d} Q_i^d$. Both $Q \setminus Q'$ and Q' consist of slices that are lower sets themselves. The number of lower subsets of the former can be well estimated using the induction assumption, as the number of slices is small enough. The number of slices in the remaining set Q' is not that small, however we will see that the number of them is still well bounded, so that Lemma 4.9 can come into play "prohibiting" the cardinalities of slices being close to each other. Denote $|Q'| =: l$, $|Q \setminus Q'| =: t$, so $l + t = n$.

Step 2. Dealing with small slices. Observe that each of the chosen slices is a lower set and for each p and $i < k_p - 1$ the set Q^p_{i+1} is a subset of Q^p_i. We can apply the induction assumption to Q^p_i with $n^p_i \geq (30(d-1))^{2(d-1)^2}$. Noting that

$$(30(d-1))^{2(d-1)^2} < \left(\frac{d}{d-1}\right)^{2(d-1)^2} n^{\frac{(d-1)^2}{d^2}} \leq 2^{-2(d-1)} n^{\frac{(d-1)^2}{d^2}},$$

for a fixed p we have

$$\sum_{n^p_i \leq 2^{-2(d-1)} n^{\frac{(d-1)^2}{d^2}}} (n^p_i - n^p_{i+1})(4 + \log n) < 2^{-2d+3} n^{\frac{(d-1)^2}{d^2}} \log n. \tag{4.26}$$

Step 3. Estimating the number of possible $\mathbf{Q \setminus Q'}$. Using the induction assumption and the bound of the number of lower subsets given by (4.21) along with (4.26), we obtain that the logarithm of the number of possible $Q \setminus Q'$ with fixed n^p_i is less than

$$\begin{aligned}
&\sum_{p=1}^{d-1} \sum_{i=0}^{k-1} C K_{d-1} Y_{d-1} (n^p_i)^{1-\frac{1}{d-1}} + (d-1) \cdot 2^{-2d+3} n^{\frac{(d-1)^2}{d^2}} \log n \\
&\quad < C K_{d-1} Y_{d-1} ((d-1)k)^{\frac{1}{d-1}} t^{1-\frac{1}{d-1}} + n^{\frac{(d-1)^2}{d^2}} \log n \\
&\quad \leq 0.5 C K_{d-1} Y_{d-1} n^{\frac{1}{d(d-1)}} t^{1-\frac{1}{d-1}} + n^{\frac{(d-1)^2}{d^2}} \log n.
\end{aligned} \tag{4.27}$$

Step 4. Obtaining a general bound for the number of possible $\mathbf{Q'}$. Now we estimate the number of possible Q' with fixed $n_i := n^d_i$. For the sake of simplicity, let $m := k_d$, $\Delta_i := n_i - n_{i+1}$, $\Gamma_i = n_i^{1-1/(d-1)}$ (note that the last notation slightly differs from that of the proof of Lemma 4.5). Combining the induction assumption with (4.20) and keeping in mind (4.26), we see that the logarithm of the number of lower sets with fixed n_i cannot exceed the following sum (assuming $n_{m+1} = 0$)

$$\begin{aligned}
&C K_{d-1} Y_{d-1} n_0^{1-\frac{1}{d-1}} + 2^{-2d+3} n^{\frac{(d-1)^2}{d^2}} \log n \\
&\quad + \sum_{i=0}^{m} \min\left\{(n_i - n_{i+1})\left(4 + \log^+ \frac{n_i^{1-\frac{1}{d-1}}}{n_i - n_{i+1}}\right), C K_{d-1} Y_{d-1} n_{i+1}^{1-\frac{1}{d-1}}\right\} \\
&\quad < 4 C n_0^{1-\frac{1}{d-1}} + n^{\frac{(d-1)^2}{d^2}} \log n + \sum_{i=0}^{m} \min\left\{\Delta_i\left(C + \log^+ \frac{\Gamma_i}{\Delta_i}\right), C K_{d-1} Y_{d-1} \Gamma_i\right\}.
\end{aligned} \tag{4.28}$$

We can bound the latter sum of minima by

$$\begin{aligned}
\sum_{i:\, \Delta_i \leq \Gamma_i} \Delta_i \log \frac{e^C \Gamma_i}{\Delta_i} + \sum_{i:\, \Delta_i > \Gamma_i} C K_{d-1} \Gamma_i &=: \sum_{i=0}^{s-1} M_i + \sum_{i=s}^{m} M_i \\
&=: G(n_0, ..., n_s) + H(n_s, ..., n_m),
\end{aligned} \tag{4.29}$$

where s is the first index i such that $n_i \leq l^{1-1/d}/4d^3$. Note that in (4.28) we can assume that $n_m \geq (30(d-1))^{2(d-1)^2}$, as the other n_i are already taken into account.

Step 5. Estimating $\mathbf{G(n_0, ..., n_s)}$. Take a tuple $(n_0, ..., n_s)$ that delivers the maximum of the function G over all the tuples in

$$S_l' := \Big\{(n_0, ..., n_s) \in \mathbb{Z}^+ :$$
$$n_0 \geq ... \geq n_s \geq 0,\ n_{s-1} \geq \max\left\{\frac{l^{1-\frac{1}{d}}}{4d^3}, (30(d-1))^{2(d-1)^2}\right\},\ n_0 + ... + n_s = l\Big\}.$$

Note that $\Delta_{s-1} > \Gamma_{s-1}$, since otherwise $n_s > 0$ and we can decrease n_s so that $\Delta_{s-1} > \Gamma_{s-1}$ increasing thereby the value of G.

Assume that for some i, $0 \leq i \leq s-1$, we have

$$\Delta_i > \Gamma_i + 2.$$

Then $\Delta_i > 3$ and if we substitute the pair (n_i, n_{i+1}) by $(n_i - 1, n_{i+1} + 1)$, the tuple will still be in S_l' with M_j, $j \neq i-1, i, i+1$, and Γ_{i-1} unchanged. At the same time Δ_{i-1} increases, which means that M_{i-1} does not decrease. Denote by Δ_j', Γ_j' and M_j' the corresponding values after the substitution. Consider the three cases.

Case 1. $\Delta_{i+1} > \Gamma_{i+1}$.
Then $\Delta_{i+1}' > \Gamma_{i+1}'$ and

$$(M_i' + M_{i+1}') - (M_i + M_{i+1}) = CK_{d-1}((\Gamma_i' + \Gamma_{i+1}') - (\Gamma_i + \Gamma_{i+1})) > 0.$$

Case 2. $\Delta_{i+1} \leq \Gamma_{i+1}$, $\Delta_{i+1}' \leq \Gamma_{i+1}'$.
We have

$$\begin{aligned}
&(M_i' + M_{i+1}') - (M_i + M_{i+1}) \\
&= (\Delta_{i+1}+1)\log\frac{e^C(n_{i+1}+1)^{1-\frac{1}{d-1}}}{\Delta_{i+1}+1} - \Delta_{i+1}\log\frac{e^C n_{i+1}^{1-\frac{1}{d-1}}}{\Delta_{i+1}} + CK_{d-1}\big((n_i-1)^{1-\frac{1}{d-1}} - n_i^{1-\frac{1}{d-1}}\big) \\
&\qquad \geq C - \Delta_{i+1}\log\frac{\Delta_{i+1}+1}{\Delta_{i+1}} - CK_{d-1}(0.5n_i)^{-\frac{1}{d-1}} > C - 1 - 8Cn_i^{-\frac{1}{d-1}} > 0,
\end{aligned}$$

since $n_i > 16^{d-1}$.

Case 3. $\Delta_{i+1} \leq \Gamma_{i+1}$, $\Delta_{i+1}' > \Gamma_{i+1}'$.
Then

$$\Delta_{i+1} + 1 = \Delta_{i+1}' > \Gamma_{i+1}' > \Gamma_{i+1} > 2,$$

so $\Delta_{i+1} \geq 0.5\Gamma_{i+1}$, and

$$\begin{aligned}
(M_i' + M_{i+1}') - (M_i + M_{i+1}) &= CK_{d-1}(n_{i+1}+1)^{1-\frac{1}{d-1}} - \Delta_{i+1}\log\frac{e^C n_{i+1}^{1-\frac{1}{d-1}}}{\Delta_{i+1}} \\
&\quad + CK_{d-1}\big((n_i-1)^{1-\frac{1}{d-1}} - n_i^{1-\frac{1}{d-1}}\big) \\
&\geq CK_{d-1}(n_{i+1}+1)^{1-\frac{1}{d-1}} - \Delta_{i+1}\log 2e^C - CK_{d-1}(0.5n_i)^{-\frac{1}{d-1}} \\
&\geq 0.25CK_{d-1}(1 - 4(0.5n_i)^{-\frac{1}{d-1}}) > 0,
\end{aligned}$$

since $n_i > 4^d$.

Thus, in all the cases G increases, and we come to a contradiction, which yields that

$$\Delta_i \leq \Gamma_i + 2 < 1.25\Gamma_i, \qquad 0 \leq i \leq s-1, \tag{4.30}$$

as $n_i = \Gamma_i^{\frac{d-1}{d-2}} > 8^{\frac{d-1}{d-2}}$.

Note that for $s \leq l^{1/d}$ we straightforwardly have an appropriate bound

$$G(n_0, ..., n_s) \leq \max_{n_0+...n_{s-1}=l} \sum_{i=0}^{s-1} CK_{d-1} n_i^{1-\frac{1}{d-1}} = CK_{d-1} l^{1-\frac{1}{d}}, \tag{4.31}$$

so from now on we assume

$$s - 1 \geq l^{\frac{1}{d}} \qquad \text{and} \qquad n_{s-1} \leq l^{1-\frac{1}{d}}. \tag{4.32}$$

If $n_0 > 2.5 l^{1-1/d}$, then considering the sequence

$$a_0 := n_0 \quad \text{ and } \quad a_i = a_{i-1} - 1.25 a_{i-1}^{1-\frac{1}{d-1}} \text{ for } i \geq 1,$$

similarly as in the proof of Lemma 4.5 (see Case a), we see that the sum of a_i becomes equal to l before a_i reaches $l^{1-1/d}$, so the same holds for n_i (since n_i decreases slower than a_i, cf. (4.30)). This contradicts (4.32). Thus,

$$n_0 \leq 2.5 l^{1-\frac{1}{d}}.$$

Now, when we have the ratio n_0/n_{s-1} bounded by $10d^3$, we are going to show that Δ_i must be greater that Γ_i for all $i = 0, ..., s-1$. Assume the contrary, that is, for some $0 < i < s$ there holds $\Delta_{i-1} \leq \Gamma_{i-1}$. Then we consider a new tuple $(n_0+n_i, n_1, ..., n_{i-1}, n_{i+1}, ..., n_s, 0)$ instead of $(n_0, n_1, .., n_s)$ and estimate the difference between the values of G at these points. First, let us estimate the difference $M_0' - M_0$.

Case a. $\Delta_0 > \Gamma_0$.

We have

$$\begin{aligned} M_0' - M_0 &= CK_{d-1}((n_0+n_i)^{1-\frac{1}{d-1}} - n_0^{1-\frac{1}{d-1}}) \\ &\geq \frac{d-2}{d-1} CK_{d-1}\Gamma_i \Big(\frac{n_i}{n_0+n_i}\Big)^{\frac{1}{d-1}} \geq CK_{d-1}\Gamma_i (10d^3)^{-\frac{1}{d-1}} \\ &=: W, \end{aligned}$$

as $n_i \geq l^{1-1/d}/4d^3$.

Case b. $\Delta_0 \leq \Gamma_0$, $\Delta_0' \leq \Gamma_0'$.

Then

$$\begin{aligned} M_0' - M_0 &= (\Delta_0 + n_i) \log \frac{e^C (n_0+n_i)^{1-\frac{1}{d-1}}}{(\Delta_0+n_i)} - \Delta_0 \log \frac{e^C n_0^{1-\frac{1}{d-1}}}{\Delta_0} \\ &\geq Cn_i - \Delta_0 \log \frac{\Delta_0 + n_i}{\Delta_0} \geq (C-1)n_i > W. \end{aligned}$$

Case c. $\Delta_0 \leq \Gamma_0$, $\Delta_0' > \Gamma_0'$.

In this case

$$\begin{aligned} M_0' - M_0 &= CK_{d-1}(n_0+n_i)^{1-\frac{1}{d-1}} - \Delta_0 \log \frac{e^C n_0^{1-\frac{1}{d-1}}}{\Delta_0} \\ &\geq 2C(n_0+n_i)^{1-\frac{1}{d-1}} - C\Delta_0 \frac{n_0^{1-\frac{1}{d-1}}}{\Delta_0} \geq W. \end{aligned}$$

Now let us turn to estimating the difference $M'_{i-1} - (M_{i-1} + M_i)$.

Case a'. $\Delta_i > \Gamma_i,\ \Delta_{i-1} + \Delta_i \le \Gamma_{i-1}$.
Then

$$\begin{aligned} M'_{i-1} - (M_{i-1} + M_i) &= (\Delta_{i-1} + \Delta_i) \log \frac{e^C \Gamma_{i-1}}{\Delta_{i-1} + \Delta_i} - \Delta_{i-1} \log \frac{e^C \Gamma_{i-1}}{\Delta_{i-1}} - CK_{d-1}\Gamma_i \\ &\ge C\Delta_i - \Delta_i - CK_{d-1}\Gamma_i \ge (C - 1 - CK_{d-1})\Gamma_i \\ &=: V. \end{aligned}$$

Case b'. $\Delta_i > \Gamma_i,\ \Delta_{i-1} + \Delta_i > \Gamma_{i-1}$.
Note that

$$\Gamma_i = (\Gamma_{i-1}^{\frac{d-1}{d-2}} - \Delta_{i-1})^{\frac{d-2}{d-1}} > (\Gamma_{i-1}^{\frac{d-1}{d-2}} - \Gamma_{i-1})^{\frac{d-2}{d-1}} = \Gamma_{i-1}(1 - \Gamma_{i-1}^{-\frac{1}{d-2}})^{\frac{d-2}{d-1}} \ge \frac{C}{C+1}\Gamma_{i-1},$$

since $n_{i-1} \ge (30(d-1))^{2(d-1)^2} > 60^{4(d-1)} > 1801^{d-1}$. So,

$$\begin{aligned} M'_{i-1} - (M_{i-1} + M_i) &= CK_{d-1}\Gamma_{i-1} - \Delta_{i-1} \log \frac{e^C \Gamma_{i-1}}{\Delta_{i-1}} - CK_{d-1}\Gamma_i \\ &> -C\Gamma_{i-1} \ge (-1 - C)\Gamma_i \ge V. \end{aligned}$$

Case c'. $\Delta_i \le \Gamma_i$.
We have

$$\begin{aligned} M'_{i-1} - (M_{i-1} + M_i) &= (\Delta_{i-1} + \Delta_i) \log \frac{e^C \Gamma_{i-1}}{\Delta_{i-1} + \Delta_i} - \Delta_{i-1} \log \frac{e^C \Gamma_{i-1}}{\Delta_{i-1}} - \Delta_i \log \frac{e^C \Gamma_i}{\Delta_i} \\ &\ge -\Delta_i \log \frac{e^C \Gamma_i}{\Delta_i} \ge -C\Gamma_i > V. \end{aligned}$$

Hence, in all the cases

$$\begin{aligned} G(n_0 + n_i, ..., n_m, 0) - G(n_0, .., n_m) &\ge W + V \\ &= CK_{d-1}\Gamma_i(10d^3)^{-\frac{1}{d-1}} + (C - 1 - CK_{d-1})\Gamma_i \\ &> CK_{d-1}\Gamma_i\Big((10d^3)^{-\frac{1}{d-1}} - \frac{3C+1}{4C}\Big) > 0 \end{aligned}$$

for $d \ge 60$. This means that we come to a contradiction that ensures

$$\Delta_i > \Gamma_i, \qquad 0 \le i \le s-1. \tag{4.33}$$

With (4.33) and (4.31) in hand, we obtain

$$\begin{aligned} G(n_0, ..., n_s) &\le \sum_{0 \le i < s:\ \Delta_i \le \Gamma_i} \Delta_i \log \frac{e^C(\Delta_i + \Delta_{i+1})}{\Delta_i} + \sum_{0 \le i < s:\ \Delta_i > \Gamma_i} CK_{d-1}\Gamma_i \\ &= \sum_{0 \le i < s:\ \Delta_i > \Gamma_i} CK_{d-1}\Gamma_i \\ &= \sum_{0 \le i < s:\ \Delta_i \le \Gamma_i} C\Delta_i + \sum_{0 \le i < s:\ \Delta_i > \Gamma_i} CK_{d-1}\Gamma_i \\ &< CK_{d-1} n^{1-\frac{1}{d}} \Big(1 + \frac{2}{3(d-1)^2} + d^{-10(d-1)\log d} + l^{-\frac{1}{d}}\Big), \end{aligned} \tag{4.34}$$

where the last inequality is due to (4.17).

Step 6. Estimating $\mathbf{H(n_s,...,n_m)}$. Let us split $H(n_s,...,n_m)$ (see (4.29)) into two sums

$$H(n_s,...,n_m) = \sum_{s\le i:\ \Delta_i\le 2^{-2d}d^{-8}\Gamma_i} M_i + \sum_{s\le i:\ \Delta_i> 2^{-2d}d^{-8}\Gamma_i} M_i =: H_1 + H_2,$$

corresponding to, roughly speaking, big and small ratios Γ_i/Δ_i. For H_2 and $d \ge 60$, we have the bound

$$H_2 \le n_s(4+\log 2^{2d}d^8) < 2dn_s \le \frac{l^{1-\frac{1}{d}}}{2d^2}.$$

Further, for i satisfying $\Delta_i \le 2^{-2d}d^{-8}\Gamma_i$, we obtain

$$\log\frac{e^4\Gamma_i}{\Delta_i} < e^2\sqrt{\frac{\Gamma_i}{\Delta_i}} \le e^2 2^{-d}d^{-4}\frac{\Gamma_i}{\Delta_i},$$

whence

$$H_1 \le \sum_{i=s}^{m}\frac{e^2}{2^d d^4}\Gamma_i \le \frac{e^2}{2^d d^4}k_d^{\frac{1}{d-1}}l^{1-\frac{1}{d}} \le \frac{e^2 l^{1-\frac{1}{d}}}{d^3}.$$

Thus,

$$H(n_s,...,n_m) = H_1 + H_2 \le \frac{l^{1-\frac{1}{d}}}{2d^2} + \frac{e^2 l^{1-\frac{1}{d}}}{d^3} < \frac{l^{1-\frac{1}{d}}}{d^2}. \tag{4.35}$$

Step 7. Combining all the estimates together. Note that the number of different n_i^p is less than

$$\binom{n+d-2}{d-1}(e^{2\sqrt{2n}})^d < e^{3d\sqrt{2n}}.$$

Therefore, recalling (4.27), (4.28), (4.29), (4.34), and (4.35), we infer

$$\begin{aligned}\log p_d(n) \le\ & 0.5CK_{d-1}Y_{d-1}n^{\frac{1}{d(d-1)}}t^{1-\frac{1}{d-1}} + 4Cn^{1-\frac{1}{d-1}} + 2n^{\frac{(d-1)^2}{d^2}}\log n + 3d\sqrt{2n}\\ &+ CK_{d-1}Y_{d-1}l^{1-\frac{1}{d}}\Big(1+\frac{2}{3(d-1)^2} + d^{-10(d-1)\log d} + l^{-\frac{1}{d}} + \frac{1}{2Cd^2}\Big).\end{aligned}$$

Note that

$$2n^{\frac{(d-1)^2}{d^2}}\log n < Cn^{1-\frac{2}{d}+\frac{1}{d^2}}\log n \le Cn^{1-\frac{2}{d}+\frac{1}{2d(d-1)}+\frac{1}{d^2}} < Cn^{1-\frac{1}{d-1}}, \tag{4.36}$$

since $n^{\frac{1}{2d(d-1)}} \ge n^{\frac{\log\log n}{\log n}} = \log n$. Indeed, if the latter does not hold, then $d^2 > \log n/2\log\log n$ and

$$d^{8d^2} > \Big(\frac{\log n}{2\log\log n}\Big)^{\frac{2\log n}{\log\log n}} \ge (\log n)^{\frac{\log n}{\log\log n}} = n,$$

which contradicts the conditions of the theorem. So, using (4.36) we come to

$$\begin{aligned}\log p_d(n) &< 0.5CK_{d-1}Y_{d-1}n^{\frac{1}{d(d-1)}}t^{1-\frac{1}{d-1}} + 5Cn^{1-\frac{1}{d-1}} + 3d\sqrt{2n} \\ &+ CK_{d-1}Y_{d-1}l^{1-\frac{1}{d}}\Big(1 + \frac{3}{4(d-1)^2} + l^{-\frac{1}{d}}\Big) \\ &< CK_{d-1}Y_{d-1}n^{\frac{1}{d(d-1)}}\Big(0.5t^{1-\frac{1}{d-1}} + l^{1-\frac{1}{d-1}}\Big(1 + \frac{3}{4(d-1)^2} + l^{-\frac{1}{d}}\Big)\Big) \\ &+ 5Cn^{1-\frac{1}{d-1}} + 3d\sqrt{2n}.\end{aligned} \tag{4.37}$$

Let us estimate the expression in brackets in (4.37). If $l \le n/2^d$, then

$$0.5t^{1-\frac{1}{d-1}} + l^{1-\frac{1}{d-1}}\Big(1 + \frac{3}{4(d-1)^2} + l^{-\frac{1}{d}}\Big) < n^{1-\frac{1}{d-1}}.$$

Otherwise,

$$0.5t^{1-\frac{1}{d-1}} + l^{1-\frac{1}{d-1}}\Big(1 + \frac{3}{4(d-1)^2} + l^{-\frac{1}{d}}\Big) < (0.5t^{1-\frac{1}{d-1}} + l^{1-\frac{1}{d-1}})\Big(1 + \frac{3}{4(d-1)^2} + 2n^{-\frac{1}{d}}\Big).$$

Note that for any $0 < a < b$, $\gamma \in (0,1)$, there holds

$$0.5a^\gamma + (b-a)^\gamma \le b^\gamma(1 + 2^{-\frac{1}{1-\gamma}})^{1-\gamma},$$

which in our case with $\gamma := 1 - 1/(d-1)$ gives

$$0.5t^{1-\frac{1}{d-1}} + l^{1-\frac{1}{d-1}} \le n^{1-\frac{1}{d-1}}(1 + 2^{-d+1})^{\frac{1}{d-1}}.$$

Therefore, in both cases we get

$$\begin{aligned}Y_{d-1}n^{\frac{1}{d(d-1)}}&\Big(0.5t^{1-\frac{1}{d-1}} + l^{1-\frac{1}{d-1}}\Big(1 + \frac{3}{4(d-1)^2} + l^{-\frac{1}{d}}\Big)\Big) \\ &\le Y_d n^{1-\frac{1}{d}}\Big(1 + \frac{3}{4(d-1)^2} + 2n^{-\frac{1}{d}}\Big).\end{aligned} \tag{4.38}$$

Finally, (4.37) and (4.38) together imply

$$\begin{aligned}\log p_d(n) &\le CK_{d-1}Y_d n^{1-\frac{1}{d}}\Big(1 + \frac{3}{4(d-1)^2} + 2n^{-\frac{1}{d}}\Big) + 5Cn^{1-\frac{1}{d-1}} + 3d\sqrt{2n} \\ &< CK_dY_dn^{1-\frac{1}{d}},\end{aligned}$$

where the latter inequality follows from

$$\max\{2n^{-\frac{1}{d}}, 2.5n^{-\frac{1}{d(d-1)}}, dn^{\frac{1}{2}-1+\frac{1}{d}}\} = 2.5n^{-\frac{1}{d(d-1)}} \le \frac{1}{12(d-1)^2},$$

as $n \ge (30d)^{2d^2}$. Thus, Theorem 4.1 is proved. □

4.4 Lower sets in high dimensions

In cases of high dimensions, the situation is quite different. In the first place, the trivial lower bound $p_d(n) \geq \binom{d+n-2}{d-1}$ becomes much more reasonable, as configurations of lower sets, in general, become more sparse. We start by considering the case of a very large dimension d.

Proof of Theorem 4.2 (a). Put the first cube into the origin and, for a fixed j, $0 \leq j \leq n-1$, spread j cubes along the axes. To complete a lower set, we have to add more $n-1-j$ cubes and we will do it stepwise. Note that any cube we now place is not aligned along an axis, so it has at least two nonzero coordinates. This means that in any subsequent step the current cube must be adjacent to at least two faces of two previously placed cubes. Since every pair of cubes can have at most one pair of their faces on which we can place a cube leaning, we come to the following estimate

$$p_d(n) \leq \sum_{j=0}^{n-1} \binom{d-1+j}{d-1} \prod_{k=j}^{n-2} \binom{k}{2} =: \sum_{j=0}^{n-1} A_j.$$

Noting that

$$\frac{A_{j+1}}{A_j} = \frac{2(d+j)}{(j+1)j(j-1)} \geq \frac{2d}{n^3},$$

we obtain

$$\frac{p_d(n)}{\binom{d+n-2}{d-1}} = \frac{\sum_{j=0}^{n-1} A_j}{\binom{d+n-2}{d-1}} \leq \frac{1}{1-\frac{n^3}{2d}} \cdot \frac{A_{n-1}}{\binom{d+n-2}{d-1}} = \frac{1}{1-\frac{n^3}{2d}}.$$

The estimate from below is given by

$$p_d(n) \geq A_{n-1} = \binom{d+n-2}{d-1}.$$

□

The next result provides a more delicate estimate from above by dealing with a similar construction as in the proof of Theorem 4.2 (a).

Lemma 4.11. *There holds*

$$p_d(n) \leq \sum_{m=2}^{n} \frac{e^m}{2^m} \sum_{t=1}^{m-1} (2\pi)^{-\frac{t+1}{2}} \sum_{\substack{s_0+\ldots+s_t=m \\ s_i \geq 2,\ 0 \leq i < t}} \frac{1}{\sqrt{s_0 s_1 \ldots s_t}} (2d)^{s_0} s_0^{2s_1-s_0} s_1^{2s_2-s_1} \ldots s_{t-1}^{2s_t-s_{t-1}} s_t^{-s_t}.$$

Proof. Observe that every lower set can be constructed in the following way. First we put a cube into the origin. After that we choose some axes to put a cube along each of them, we call this zero step. Then, inductively, as we have completed the $(k-1)$th step, we have a lower set whose cubes have the sum of the coordinates less or equal to k. In the kth step we add some cubes to our set so that the following two conditions hold: any cube we put now has the sum of its coordinates equal to $k+1$ and the set we construct remains to be a lower set.

Let us estimate the number of choices to put s_k cubes in the kth step. Note that these s_k cubes must lean only on s_{k-1} cubes that we put in the previous step.

When $k=1$, each pair of s_0 cubes from the previos step generates a place for a new cube and there are also s_0 possibilities to put a cube along an axis. So, the total number of possible places in this case is $s_0(s_0+1)/2 < (s_0+1)^2/2$.

Turn now to the cases of $k>1$. Suppose that there are l cubes among these s_{k-1} ones that lie along some axes, that is, they have all the coordinates except one equal to zero. Then the only two ways to lean a new cube on any of these l cubes are either to continue going along the corresponding axes or to lean it on one of these l cubes and on one of the remaining $s_{k-1}-l$ ones. If a new cube does not lean on those l cubes, then it has more than one nonzero coordinate, thus must lean on at least two cubes from the other $s_{k-1}-l$ ones from the previous step. As we have already noted, each pair of cubes generates at most one place for a new cube to lean on both of them. Summing up, the number of places to put cubes in the kth step is 1 in the case $s_{k-1}=l=1$ and

$$l + l(s_{k-1}-l) + \binom{s_{k-1}-l}{2} \le \frac{s_{k-1}^2}{2},$$

otherwise. In the case $s_{k-1}=l=1$ all the remaining steps must have $s_i=1,\ i\ge k$. We come to the estimate

$$p_d(n) \le \sum_{m=1}^{n-1}\sum_{t=1}^{m}\sum_{\substack{s_0+\ldots+s_t=m\\ s_i\ge 2,\ 1\le i<t}} \binom{d}{s_0}\binom{\frac{(s_0+1)^2}{2}}{s_1}\binom{\frac{s_1^2}{2}}{s_2}\ldots\binom{\frac{s_{t-1}^2}{2}}{s_t}.$$

Using Stirling's formula we see that

$$\binom{a}{b} \le \frac{a^b}{b!} \le \frac{1}{\sqrt{2\pi b}}\left(\frac{ae}{b}\right)^b$$

for any $a\ge b\ge 1$, so we finally obtain

$$\begin{aligned} p_d(n) \le &\sum_{m=1}^{n-1}\sum_{t=1}^{m}\sum_{\substack{s_0+\ldots+s_t=m\\ s_i\ge 2,\ 1\le i<t}} e^{s_0}\left(\frac{e}{2}\right)^{m-s_0}(2\pi)^{-\frac{t+1}{2}} \\ &\times \frac{1}{\sqrt{s_0s_1s_2\ldots s_t}} d^{s_0}s_0^{-s_0}(s_0+1)^{2s_1}s_1^{-s_1}s_1^{2s_2}s_2^{-s_2}\ldots s_{t-1}^{2s_t}s_t^{-s_t} \\ = &\sum_{m=2}^{n}\frac{e^m}{2^m}\sum_{t=1}^{m-1}(2\pi)^{-\frac{t+1}{2}}\sum_{\substack{s_0+\ldots+s_t=m\\ s_i\ge 2,\ 0\le i<t}}\frac{1}{\sqrt{s_0s_1\ldots s_t}}(2d)^{s_0}s_0^{2s_1-s_0}s_1^{2s_2-s_1}\ldots s_{t-1}^{2s_t-s_{t-1}}s_t^{-s_t}. \end{aligned}$$

□

Lemma 4.11 will be our main tool for further upper estimates of $p_d(n)$. The first one is of interest when $d=o(n^2)$ as $n\to\infty$.

Proposition 4.1. *For $d\le n^2/4$, there holds*

$$p_d(n) < 4e^{cn}n^{n+2\sqrt{d}}\max\{2^{-n},(2n)^{-\sqrt{d}}\} \quad \text{with } c=\frac{3}{2e}+1,$$

which in case $dn^{-2}\to 0$ as $n\to\infty$ yields

$$p_d(n) \le n^{n+o(n)}.$$

Proof. Consider a tuple $(s_0, s_1, ..., s_t) \in \mathbb{N}^{t+1}$ such that $s_i \geq 2$ for $0 < i < t$ and $s_0 + ... + s_t = m$. Note that

$$(s_0 s_1 ... s_t)^{\frac{3}{2}} \leq \left(\frac{m}{t}\right)^{\frac{3t}{2}} \leq e^{\frac{3m}{2e}}. \tag{4.39}$$

Suppose that there is no s_i, $i \geq 1$, such that $s_i \geq \sqrt{d}$. Then, using (4.39), we have

$$\begin{aligned} F(d, s_0, ..., s_t) :&= (s_0 s_1 ... s_t)^{\frac{3}{2}} (2d)^{s_0} s_0^{2s_1 - s_0} s_1^{2s_2 - s_1} ... s_{t-1}^{2s_t - s_{t-1}} s_t^{-s_t} \\ &\leq e^{\frac{3m}{2e}} (2d)^{s_0} s_0^{2s_1 - s_0} s_1^{s_2} s_2^{s_3} ... s_{t-1}^{s_t} s_t^{-s_1} \\ &\leq e^{\frac{3m}{2e}} (2d)^{s_0} s_0^{2\sqrt{d} - s_0} d^{\frac{m - s_0}{2}} \\ &= e^{\frac{3m}{2e}} d^{\frac{m + s_0}{2}} 2^{s_0} s_0^{2\sqrt{d} - s_0} \\ &< e^{\frac{3m}{2e}} d^{\frac{m}{2}} m^{2\sqrt{d}} 2^{s_0} d^{\frac{s_0}{2}} s_0^{-s_0}. \end{aligned} \tag{4.40}$$

Here we used the inequality $s_1^{s_1} s_2^{s_2} ... s_t^{s_t} \geq s_1^{s_2} s_2^{s_3} ... s_t^{s_1}$, which is true since for any $k \in \mathbb{N}$, any positive integers $a_1, ..., a_k$, and any permutation σ from the symmetric group $\mathfrak{S}_k$, there holds

$$\prod_i a_i^{a_{\sigma(i)}} \leq \prod_i a_i^{a_i}. \tag{4.41}$$

The maximum of the right-hand side of (4.40) is attained at $s_0 = 2\sqrt{d}/e$, so in this case

$$F(d, s_0, ..., s_t) e^{-\frac{3m}{2e}} \leq d^{\frac{m}{2}} (e^{\frac{1}{e}} m)^{2\sqrt{d}} \leq n^n 2^{-n} e^{\frac{n}{2}} n^{2\sqrt{d}} < n^{n + 2\sqrt{d}}. \tag{4.42}$$

If there exists $s_i \geq \sqrt{d}$, $i \geq 1$, then choosing the maximal such index i and using twice inequality (4.41) along with (4.39) we obtain

$$\begin{aligned} F(d, s_0, ..., s_t) e^{-\frac{3m}{2e}} &\leq 2^{s_0} (d^{s_0} s_0^{2s_1 - s_0} ... s_{i-1}^{2s_i - s_{i-1}} s_i^{-s_i}) m^{2s_{i+1}} (s_{i+1}^{2s_{i+2} - s_{i+1}} ... s_{t-1}^{2s_t - s_{t-1}} s_t^{-s_t}) \\ &\leq 2^{s_0} (s_0^{2s_1 - s_0} ... s_{i-1}^{2s_i - s_{i-1}} s_i^{2s_0 - s_i}) m^{2s_{i+1}} (s_{i+1}^{s_{i+2}} ... s_{t-1}^{s_t} s_t^{-s_{i+1}}) \\ &\leq 2^{s_0} (s_0^{s_1} ... s_{i-1}^{s_i} s_i^{s_0}) m^{2s_{i+1}} m^{s_{i+2} + ... + s_t} \\ &\leq 2^{s_0} m^{s_0 + ... + s_i + 2s_{i+1} + s_{i+2} + ... + s_t} \\ &< 2^{m - \sqrt{d}} m^{m + \sqrt{d}} \leq 2^{n - \sqrt{d}} n^{n + \sqrt{d}}. \end{aligned} \tag{4.43}$$

According to Lemma 4.11, it remains only to estimate the sum

$$\sum_{m=2}^{n} \frac{e^m}{2^m} \sum_{t=1}^{m-1} (2\pi)^{-\frac{t+1}{2}} \sum_{\substack{s_0 + ... + s_t = m \\ s_i \geq 2,\ 0 \leq i < t}} \frac{1}{(s_0 s_1 ... s_t)^2}.$$

Note that the right sum is equal to the coefficient at x^m of the polynomial

$$P(x) := \Big(\sum_{j=1}^{m} \frac{x^j}{j^2}\Big) \Big(\sum_{j=2}^{m} \frac{x^j}{j^2}\Big)^t.$$

We have $P(1) < (\pi^2/6)^{t+1}$, therefore,

$$\sum_{m=2}^{n} \frac{e^m}{2^m} \sum_{t=1}^{m-1} (2\pi)^{-\frac{t+1}{2}} \sum_{\substack{s_0+...+s_t=m \\ s_i \geq 2,\ 0\leq i<t}} \frac{1}{(s_0 s_1 ... s_t)^2} < \sum_{m=2}^{n} \frac{e^m}{2^m} \sum_{t=1}^{m-1} \left(\frac{\pi^2}{6\sqrt{2\pi}}\right)^{t+1}$$
$$< \frac{4}{9} \frac{1}{1-\frac{2}{3}} \sum_{m=2}^{n} \frac{e^m}{2^m} = \frac{4}{3} \sum_{m=2}^{n} \frac{e^m}{2^m}. \tag{4.44}$$

Thus, combining (4.44) with (4.42) and (4.43), we finally derive

$$p_d(n+1) \leq n^{n+2\sqrt{d}} \max\left\{1, \frac{2^n}{(2n)^{\sqrt{d}}}\right\} \frac{4}{3} \sum_{m=1}^{n} \left(\frac{e^{\frac{3}{2e}+1}}{2}\right)^m$$
$$< 4e^{\frac{3n}{2e}+n} n^{n+2\sqrt{d}} \max\{2^{-n}, (2n)^{-\sqrt{d}}\},$$

which concludes the proof. □

The complementary lower bound for the case $d = o(n^2)$ will be as follows.

Proposition 4.2. *If* $d \geq n/\psi(n)$ *for some positive function* $\psi(n) \geq 1$ *such that* $\psi(n) \to \infty$ *and* $\log \psi(n)/\log n \to 0$ *as* $n \to \infty$*, then there holds*

$$p_d(n+1) \geq n^{\left(n + \frac{n \log d}{\psi(n) \log n}\right)(1+o(1))},$$

and, consequently, if $\frac{\log d}{\psi(n) \log n} \to 0$ *as* $n \to \infty$*, we have*

$$p_d(n) \geq n^{n+o(n)}.$$

Proof. Let us count only the lower sets whose cubes have at most two nonzero coordinates and these coordinates are at most 1. First, we place a cube into the origin. After this we fix some number i, $i = 1, ..., n$, and choose i axes to put cubes along them. The remaining cubes will lie on some of the two-dimensional hyperplanes generated by the chosen axes. So the number of such lower sets is

$$\sum_{i=1}^{\min\{d,n\}} \binom{d}{i} \binom{\frac{i(i-1)}{2}}{n-i} =: \sum_{i=1}^{\min\{d,n\}} B_i \geq B_{\lfloor n/\psi(n) \rfloor}.$$

To estimate the latter value, we note that by Stirling's formula

$$\sqrt{2\pi a}\left(\frac{a}{e}\right)^a \leq a! \leq e^{\frac{1}{12}} \sqrt{2\pi a}\left(\frac{a}{e}\right)^a$$

for all $a \geq 1$, so this implies the inequality

$$\binom{a}{b} \geq \frac{\sqrt{a} a^a}{e^{\frac{1}{6}} \sqrt{2\pi(a-b)b}(a-b)^{a-b} b^b} > \frac{1}{2\sqrt{a}} \left(\frac{a}{b}\right)^b$$

for all $a > b \geq 1$. Now, assuming that $\psi(n) \leq n/6$, we can estimate $B_{\lfloor n/\psi(n) \rfloor}$ as follows (writing just ψ in place of $\psi(n)$ for the sake of simplicity)

$$\begin{aligned} B_{\lfloor n/\psi \rfloor} &= \binom{d}{\lfloor \frac{n}{\psi} \rfloor} \binom{\frac{1}{2}\lfloor \frac{n}{\psi} \rfloor^2 - \frac{1}{2}\lfloor \frac{n}{\psi} \rfloor}{n - \lfloor \frac{n}{\psi} \rfloor} \\ &> \frac{1}{2\sqrt{d}} \Big(\frac{d\psi}{n}\Big)^{\frac{n}{\psi}-1} \frac{\psi}{\sqrt{2n}} \Big(\frac{n}{4\psi^2}\Big)^{n-\frac{n}{\psi}} \\ &> d^{\frac{n}{\psi}-\frac{3}{2}} n^{n-2\frac{n}{\psi}} \psi^{-2n+3\frac{n}{\psi(n)}} 2^{-2n+\frac{2n}{\psi}-\frac{3}{2}} \\ &= \exp\Big(n \log n - \frac{2n\log n}{\psi} + \frac{n \log d}{\psi} - \frac{3 \log d}{2} - 2n \log \psi \\ &\qquad + \frac{3n \log \psi}{\psi} - 2n \log 2 + \frac{2n \log 2}{\psi} - \frac{3 \log 2}{2}\Big) \qquad (4.45) \\ &= n^{\left(n + \frac{n \log d}{\psi \log n}\right)(1+o(1))}, \end{aligned}$$

which in case $\log d = o(\psi(n) \log n)$ yields

$$B_{\lfloor n/\psi(n) \rfloor} \geq n^{n+o(n)}.$$

□

Remark 4.12. For $d \geq n/\log n$ and $\psi(n) := \log n$, inequality (4.45) gives

$$p_d(n+1) > n^{n - \frac{6n \log\log n}{\log^2 n}}.$$

Proof of Theorem 4.2 (d). The relation follows straightforwardly from the corresponding parts of Propositions 4.1 and 4.2. □

Now we give a more general estimate, which will imply the sharp exponential order of $p_d(n)$ in case of $n^2 = O(d)$.

Proposition 4.3. *If $d \geq \xi n^2$ for some $\xi = \xi(n) \geq 2n^{-1}$, then*

$$p_d(n) < 3a^{2n} e^{\frac{125n}{\xi}} n^2 e^n \frac{d^n}{n^n} \quad \text{with } a = \max\{2e^{3.5}\xi^{-1}, 1\}.$$

In particular, if $\xi(n) \to \infty$ as $n \to \infty$, then

$$p_d(n) = e^n \frac{d^n}{n^n} e^{o(n)}.$$

Proof. For a tuple $(s_0, ..., s_t) \in \mathbb{N}^{t+1}$ with $s_0 + ... + s_t = m$, as before let

$$F(d, s_0, ..., s_t) := (s_0 s_1 ... s_t)^{\frac{3}{2}} (2d)^{s_0} s_0^{2s_1 - s_0} s_1^{2s_2 - s_1} ... s_{t-1}^{2s_t - s_{t-1}} s_t^{-s_t}.$$

We will prove by induction on t that

$$F(d, s_0, ..., s_t) \leq a^{2m - s_0} \exp\Big(- \int_{\min\{\frac{125m}{\xi}, s_1\}}^{\frac{125m}{\xi}} \log \frac{\xi x}{125m} \, dx \Big) m^2 \frac{(2d)^m}{m^m}. \qquad (4.46)$$

Note that for any $\alpha \le \beta$ and γ,

$$W(\alpha,\beta,\gamma) := \int_{\gamma\alpha}^{\gamma\beta} \log \frac{x}{\gamma}\, dx = \gamma(x\log x - x)\Big|_{\alpha}^{\beta}. \tag{4.47}$$

The case $t = 0$ is clear (we assume $s_1 = 0$). In the case $t = 1$ we have

$$\begin{aligned}(\log F(d, s_0, m-s_0))'_{s_0} &= \Big(\log\Big((2d)^{s_0} s_0^{2m-3s_0+1.5}(m-s_0)^{-m+s_0+1.5}\Big)\Big)'_{s_0}\\ &= \log\frac{2d(m-s_0)}{s_0^3} + \frac{2(m-s_0)+1.5}{s_0} - \frac{1.5}{m-s_0}\\ &> \log\frac{2\xi(m-s_0)}{e^{1.5}m} > \log\frac{\xi(m-s_0)}{3m}.\end{aligned}$$

If $s_1 \ge 6m/\xi$, then applying the inequality above and (4.47), we see that

$$F(d, s_0, s_1) \le \exp\big(-W(0,2,3m\xi^{-1})\big)F(d,m,0) \le F(d,m,0) = \frac{(2d)^m}{m^m},$$

which is less then the right-hand side of (4.46). Otherwise, $s_1 < 6m/\xi < 125m/2\xi$ and

$$F(d, s_0, s_1) \le \exp\big(-W(0,1,3m\xi^{-1})\big)F(d,m,0) = e^{\frac{3m}{\xi}}\frac{(2d)^m}{m^m},$$

while at the right-hand side of (4.46) we get at least

$$\exp\big(-W(0.5,1,125m\xi^{-1})\big)\frac{(2d)^m}{m^m} = e^{\frac{125m}{2\xi}}\frac{(2d)^m}{m^m},$$

which completes the proof of (4.46) for $t = 1$.

Assume now that $t > 1$ and (4.46) is proved for all m and for $1, ..., t-1$. Let us prove it for t.

Consider a tuple $(s_0, s_1, ..., s_t) \in \mathbb{N}^{t+1}$ such that $s_0 + ... + s_t = m$ and suppose that $s_t > s_{t-1}/2$. Fix $s_1, ..., s_{t-1}$ and $s_0 + s_t =: y$ and see what occures if we increase $s_0 =: x$. We have

$$\begin{aligned}&\Big(\log F(d, x, s_1, ..., s_{t-1}, y-x)\Big)'_x\\ &\quad= \big(x\log 2d + (2s_1 - x + 1.5)\log x + (2y - 2x - s_{t-1} + 1.5)\log s_{t-1}\\ &\qquad\qquad + (x - y + 1.5)\log(y-x)\big)'_x\\ &\quad> \log\frac{2d(y-x)}{xs_{t-1}^2} - \frac{1.5}{y-x} \ge \log\frac{2ds_t}{s_0s_{t-1}^2} - 1.5 > \log\frac{d}{s_0s_{t-1}} - 1.5\\ &\quad> \log\frac{d}{m^2} - 1.5 \ge \log\xi - 1.5.\end{aligned}$$

This means that we can increase s_0 by 1 and decrease s_t keeping their sum constant until either $s_t \le s_{t-1}/2$ or $s_t = 1$ so that in every step of this process the value of $F(d, s_0, ..., s_t)$ changes by at least $\exp(\log\xi - 1.5) = \xi e^{-1.5}$.

Suppose now that for some i, $1 < i \le t-1$ and $j \ge i$, we have

$$s_l \ge 2s_{l+1} \text{ for } i \le l < j \quad\text{and}\quad s_j = ... = s_t = 1. \tag{4.48}$$

Fix $s_1, ..., s_{i-1}, s_{i+1}, ..., s_t$ and $s_0 + s_i =: y$ and observe what occures if we start increasing $s_0 =: x$. We see that

$$\Big(\log F(d, x, s_1, ..., s_{i-1}, y - x, s_{i+1}, ..., s_t)\Big)'_x$$
$$= \big(x \log 2d + (2s_1 - x + 1.5) \log x + (2y - 2x) \log s_{i-1}$$
$$+ (2s_{i+1} + x - y + 1.5) \log(y - x)\big)'_x$$
$$> \log \frac{2d(y - x)}{x s_{i-1}^2} - \frac{2s_{i+1} + 1.5}{y - x} = \log \frac{2ds_i}{s_0 s_{i-1}^2} - \frac{2s_{i+1} + 1.5}{s_i} \geq \log \xi - 3.5,$$

while $s_i \geq s_{i+1}$ (which is true under (4.48)) and $s_i \geq s_{i-1}/2$. So we can decrease s_i and increase s_0 with their sum constant, changing $F(d, s_0, ..., s_t)$ by at least $\xi e^{-3.5}$ in each step, until one of the following situations happens.

Case 1. $s_i = s_{i+1} = ... = s_t = 1$.

Then we accumulated at most the extra factor $(e^{3.5}\xi^{-1})^{\Delta s_0}$, where by Δs_0 we denote the number of steps we made increasing s_0, which is exactly the difference between the value of s_0 in the end and in the beginning of the process. So, we come to (4.48) with $i - 1$ in place of i and proceed inductively with $i - 1$ instead of i.

Case 2. $s_{i-1} \geq 2s_i \geq 4s_{i+1}$.

Then we come again to (4.48) with $i - 1$ in place of i with the same accumulated factor.

Case 3. $s_{i-1} < 2s_i = 4s_{i+1}$.

Then $s_i = 2s_{i+1} =: 2x$ and we merge $s_i = 2x$ and $s_{i+1} = x$ into one single variable equal to $2x$. Let us compare the new value of F with the original one:

$$\frac{F(s_0, ..., s_{i-1}, 2x, s_{i+2}, ..., s_t)}{F(s_0, ..., s_t)} = \frac{s_{i-1}^{4x - s_{i-1} + 1.5}(2x)^{2s_{i+2} - 2x + 1.5}}{s_{i-1}^{4x - s_{i-1} + 1.5}(2x)^{2x - 2x + 1.5} x^{2s_{i+2} - x + 1.5}} \geq (4x)^{-x - 1.5}.$$

At the same time the sum of all the variables $s_0, s_1, ..., s_{i-1}, 2x, s_{i+2}, ..., s_t$ becomes $m - x$ instead of m. So, by the induction assumption we have

$$F(d, s_0, ..., s_t) \leq \max_x (4x)^{x+1.5} a^{2m - 2x - s_0} \exp\Bigg(-\int_{\min\{\frac{125(m-x)}{\xi}, s_1\}}^{\frac{125(m-x)}{\xi}} \log \frac{\xi y}{125(m - x)}\, dy\Bigg)$$
$$\times (m - x)^2 \frac{(2d)^{m-x}}{(m - x)^{m-x}}$$
$$\leq \max_x (4x)^{x+1.5} a^{2m - 2x - s_0} \exp\Bigg(-\int_{\min\{\frac{125m}{\xi}, s_1\}}^{\frac{125m}{\xi}} \log \frac{\xi y}{125m}\, dy\Bigg) m^2 \frac{(2d)^{m-x}}{(m - x)^{m-x}}.$$

Since

$$\Big((x + 1.5) \log 4x + (m - x)(\log 2d - \log(m - x))\Big)'_x$$
$$= \log 4x + 1 - \log 2d + \frac{1.5}{x} + \log(m - x) + 1$$
$$= \log \frac{4e^{3.5}x(m - x)}{2d} < \log \frac{2e^{3.5}}{\xi},$$

we finally have the desired inequality (4.46), as

$$F(d,s_0,...,s_t)\le \max_x\Big(\max\Big\{\frac{2e^{3.5}}{\xi},1\Big\}\Big)^x a^{2m-2x-s_0}$$
$$\times\exp\Big(-\int_{\min\{\frac{125m}{\xi},s_1\}}^{\frac{125m}{\xi}}\log\frac{\xi y}{125m}\,dy\Big)m^2\frac{(2d)^m}{m^m}$$
$$=a^{2m-s_0}\exp\Big(-\int_{\min\{\frac{125m}{\xi},s_1\}}^{\frac{125m}{\xi}}\log\frac{\xi y}{125m}\,dy\Big)m^2\frac{(2d)^m}{m^m}.$$

This way we either merged two variables in some step and obtained the needed inequality using the induction assumption or reach the situation $s_1\ge 2s_2\ge 2^{i-1}s_i,\ s_{i+1}=...=s_t=1$, for some $1\le i\le t$. In the latter occasion, considering $s_0+s_1=:y$ to be constant and changing $s_0=:x$ we see that

$$\Big(\log F(d,x,y-x,s_2,...,s_t)\Big)'_x$$
$$=(x\log 2d+(2y-3x+1.5)\log x+(2s_2-y+x+1.5)\log(y-x))'_x$$
$$\ge\log\frac{2d(y-x)}{x^3}-2-\frac{2s_2+1.5}{y-x}\ge\log\frac{2ds_1}{s_0^3}-\frac{2s_2}{s_1}-3.5.$$

Thus, while $s_1\ge s_2$, there holds

$$\Big(\log F(d,x,y-x,s_2,...,s_t)\Big)'_x\ge\log\frac{2ds_1}{e^{5.5}s_0^3}>\log\frac{\xi s_1}{125m}.$$

So, we can decrease s_1 and increase s_0 with s_0+s_1 constant so that $F(d,s_0,...,s_t)$ in this process increases at least by $\exp(\int_{s_1'}^{s_1^*}\log(\xi x/125m)\,dx)$, (where s_1^* stands for the value of s_1 that we started from and s_1', for the value where we stopped), until one of the following situations happens.

Case a. $s_1=s_2=...=s_t=1$. Then

$$F(d,s_0,...,s_t)\le\exp\Big(-\int_{s_1}^{s_1^*}\log\frac{\xi x}{125m}\,dx\Big)\max_{s_0}\frac{s_0^2(2d)^{s_0}}{s_0^{s_0-2}}$$
$$\le\exp\Big(-\int_{\min\{\frac{125m}{\xi},s_1\}}^{\frac{125m}{\xi}}\log\frac{\xi x}{125m}\,dx\Big)\frac{(2d)^m}{m^{m-2}},$$

and the needed inequality is proved.

Case b. $s_1=2s_2$. Then we merge s_1 and s_2 into s_1 as above and use the induction assumption. The only difference is that we have to take into account the factor $\exp(\int_{s_1'}^{s_1^*}\log(\xi x/125m)\,dx)$ that we accumulated while making s_1 decrease.

Thus, in all cases we obtained (4.46) for all m.

Hence, we have

$$F(d, s_0, ..., s_t) \le a^{2m} \exp\left(- \int_0^{\frac{125m}{\xi}} \log \frac{\xi x}{125m}\, dx \right) m^2 \frac{(2d)^m}{m^m},$$

which in light of equality (4.47) implies

$$F(d, s_0, ..., s_t) \le a^{2m} e^{\frac{125m}{\xi}} m^2 \frac{(2d)^m}{m^m}.$$

Finally, taking into account Lemma 4.11 and estimate (4.44), we obtain

$$p_d(n) < a^{2n} e^{\frac{125n}{\xi}} n^2 \frac{d^n}{n^n} \frac{4}{3} \sum_{m=2}^{n} e^m < 3a^{2n} e^{\frac{125n}{\xi}} n^2 \frac{d^n}{n^n} e^n,$$

which concludes the proof. □

Proof of Theorem 4.2 (b), (c). The claim follows from Proposition 4.3 and the simple estimate $p_d(n) \ge \binom{d+n-2}{d-1}$. □

Proof of Corollary 4.3. The first case readily follows from Theorem 4.2 (a).

Let $\alpha \ge 2$. If $\alpha > 2$ or $\alpha = 2$, $\gamma > 0$, for n satisfying $n > 2\pi e^3 + 1$ and $n^{\alpha-2} \log^\gamma n \ge 2e^{3.5}$, invoking Proposition 4.3 with $\xi(n) := n^{\alpha-2} \log^\gamma n$, we can write

$$\begin{aligned} e^n \frac{d^n}{n^n} \cdot \frac{1}{d} &< e^{n-1} \frac{d^{n-1}}{e^{\frac{1}{12}} \sqrt{2\pi} (n-1)^{n-0.5}} < \binom{d+n-2}{d-1} \\ &\le p_d(n) \\ &\le 3e^{\frac{125n^{3-\alpha}}{c \log^\gamma n}} n^2 e^n \frac{d^n}{n^n} = e^n \frac{d^n}{n^n} e^{O(n^{3-\alpha} \log^{-\gamma} n + \log n)}, \end{aligned} \tag{4.49}$$

which gives a sharp estimate up to $e^{o(n)}$. Otherwise, when $\alpha = 2$, $\gamma \le 0$, we obtain an extra $e^{O(n+\gamma n \log\log n)}$ factor at the right-hand side of (4.49), which is still $e^{O(n^{3-\alpha} \log^{-\gamma} n)}$.

Turn now to the case $\alpha < 2$. In light of inequality (4.45), for any $\psi = \psi(n)$ fulfilling the conditions

$$1 \le \psi(n) \le \frac{n}{6},$$

we have

$$\begin{aligned} \log p_d(n+1) \ge\ & n \log n - \frac{n(2-\alpha)\log n}{\psi(n)} + \frac{n\gamma \log\log n}{\psi(n)} + \frac{n \log c}{\psi(n)} \\ & - \frac{3\alpha \log n}{2} - \frac{3\gamma \log\log n}{2} - \frac{3 \log C}{2} - 2n \log \psi(n) + \frac{3n \log \psi(n)}{\psi(n)} \\ & - 2n \log 2 + \frac{2n \log 2}{\psi(n)} - \frac{3 \log 2}{2}. \end{aligned} \tag{4.50}$$

Taking $\psi(n) = \log^\delta n := \log^{\max\{1,-\gamma\}} n$ and plugging this into (4.50), we obtain

$$\begin{aligned}\log p_d(n+1) \geq n\log n - n(2-\alpha)\log^{1-\delta} n + \frac{n\gamma\log\log n}{\log^\delta n} + \frac{n\log c}{\log^\delta n} \\ - \frac{3\alpha\log n}{2} - \frac{3\gamma\log\log n}{2} - \frac{3\log C}{2} - 2\delta n\log\log n + \frac{3\delta n\log\log n}{\log^\delta n} \\ - 2n\log 2 + \frac{2n\log 2}{\log^\delta n} - \frac{3\log 2}{2},\end{aligned} \tag{4.51}$$

which yields

$$\log p_d(n+1) > n\log n + O(n\log\log n).$$

One can see that estimate (4.51) is up to a constant optimal with respect to an appropriate choice of a function ψ. Indeed, we need to counterbalance the two main terms of (4.50), namely, $n\log n/\psi$ and $n\log\psi$. They are equal when $\psi = \log n/W(\log n)$, where $W(x)$ stands for the W–Lambert function, i.e. the inverse function for ye^y. The fact that $W(x)\log^{-1} x \to 1$ as $x\to\infty$, yields $\psi(n) \sim \log n/\log\log n$ and the estimate we obtain by means of such ψ is up to a constant the same as the one for $\psi(n) = \log^\delta n$.

At the same time, according to Proposition 4.1, there holds

$$\log p_d(n) \leq n\log n + O(n).$$

Summing up,

$$n^n e^{O(n\log\log n)} \leq p_d(n) \leq n^n e^{O(n)}.$$

□

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