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Poisson structures on moduli spaces and group actions

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Summary

In this thesis, we study Poisson Structures on Moduli Spaces and Group actions. In particular, we focus on b^m-symplectic structures that can be seen both as symplectic structures with singularities and a particular type of Poisson structures. We also study Poisson structures on the character varieties associated with Fuchsian differential equations in relation with Riemann-Hilbert correspondence and how they can be transformed in the case of higher-order singularities.

For b^m-symplectic manifolds, we consider different classes of group actions starting with b^m-Hamiltonian actions, a natural generalization of Hamiltonian moment maps for the singular symplectic setting. Then, we further generalize this notion as singular quasi-Hamiltonian group actions. The last generalization is motivated by those group actions that preserve a b^m-symplectic structure on the manifold but do not admit a conventional moment map. We use both moment maps (b^m-Hamiltonian and singular quasi-Hamiltonian) to prove a corresponding generalization of the Marsden-Weinstein reduction theorem, showing that in the singular setting, the reduction procedure eliminates the singularity. We prove a singular slice theorem as a first step for the reduction proof. We show that the singular Marsden-Weinstein reduction admits reduction "by stages" and commutes with the desingularization procedure.

In the second part of this thesis, we turn to Poisson structures on moduli spaces of flat connections and monodromy data related by the Riemann-Hilbert correspondence. First, we consider several cases when the Riemann-Hilbert correspondence can be solved explicitly on an elliptic curve. Then we turn to the case of Painlevé transcendents on the Riemann sphere. In particular, the Okamoto Hamiltonian for the second Painlevé equation carries a natural b-symplectic structure. For the rest of the equations, the structure is more complicated. We start with considering the Poisson structures on the moduli space of flat connections and character varieties corresponding to

Fuchsian equations, where all the singularities are simple poles (in particular, Painlevé VI). We consider Poisson structures for which the Riemann-Hilbert correspondence is a Poisson morphism. We also study Poisson structures related to the Painlevé V equation (3 poles: one of order 2 and two simple poles).

Resum

En aquesta tesi s'estudien les estructures de Poisson en espais de moduli i en accions de grups. En particular, ens centrem en les estructures b^m-simplèctiques, que es poden veure com a estructures simplèctiques amb singularitats o també com un tipus particular d'estructures de Poisson. També estudiem les estructures de Poisson en varietats de caràcters associades a les equacions diferencials fuchsianes i el comportament d'aquestes estructures de Poisson sota la confluència de singularitats.

En el cas de les varietats b^m-simplèctiques, considerem diverses classes d'accions de grups, començant amb b^m-accions hamiltonianes, una generalització natural de les funcions de moment hamiltonianes en context simplèctic singular. Després generalitzem encara més aquesta noció a accions de grup quasi hamiltonianes singulars. Aquesta darrera generalització està motivada per aquelles accions de grup que conserven una estructura b^m-simplèctica a la varietat però no admeten una funció de moment convencional. Utilitzem ambdues funcions de moment (b^m-Hamiltoniana i quasi-Hamiltoniana singular) per demostrar una generalització corresponent del teorema de reducció de Marsden-Weinstein, demostrant que en l'entorn singular, el procediment de reducció elimina la singularitat. Demostrem un teorema de slice singular com a primer pas per a la demostració de la reducció. Mostrem que la reducció singular de Marsden-Weinstein admet la reducció "per etapes" i commuta amb el procediment de desingularització.

A la segona part d'aquesta tesi tractem les estructures de Poisson sobre els espais moduli de connexions planes i les dades de monodromia relacionades per la correspondència de Riemann-Hilbert. En primer lloc, considerem diversos casos en què la correspondència de Riemann-Hilbert es pot resoldre explícitament en una corba el·líptica. A continuació, passem al cas dels transcendents de Painlevé sobre l'esfera de Riemann. En particular, el Hamiltonià d'Okamoto per a la segona equació de Painlevé té una estructura b-simplectica natural. Per a la resta d'equacions, l'estructura és més compli-

cada. Comencem considerant les estructures de Poisson a l'espai de moduli de connexions planes i varietats de caràcters corresponents a equacions fuchsianes, on totes les singularitats són pols simples (en particular, Painlevé VI). Considerem estructures de Poisson per a les quals la correspondència de Riemann-Hilbert és un morfisme de Poisson. També estudiem estructures de Poisson relacionades amb l'equació Painlevé V (3 pols: un d'ordre 2 i dues pols simples).

Chapter 1

Introduction

Symplectic and Poisson geometry fields arise at the intersection of geometry and physics. Motivated by understanding the dynamics of mechanical systems, they consider the phase space of such a system as a manifold with a prescribed geometric structure. Understanding the geometric properties of these manifolds brings insights into mechanical systems' behavior. Symplectic structures cover a large part of the examples coming from classical mechanics and provide very applied techniques. Poisson manifolds, more general, can be viewed through the prism of the symplectic foliation. One of the good examples where symplectic methods shine at their best is the problem of finding periodic orbits (if they exist). Another splendid application comes from the simple idea that any symmetry of a system reduces the number of its degrees of freedom, simplifying the system itself. In physical language, this would be formulated as conservation laws and first integrals. In geometric language, this concept can be encoded as a reduction theorem. The celebrated Marsden-Weinstein reduction reveals an exciting phenomenon that for a group of dimension k, the reduction can be doubled: the system can be simplified by $2k$ degrees of freedom.

Marsden-Weinstein quotients are naturally connected to certain moduli spaces. In their seminal article [AB83], Michael Atiyah and Raoul Bott unveiled the symplectic structure on the space of flat connections. The Riemann-Hilbert problem explores the correspondence between the moduli space of flat connections of Fuchsian systems (i.e., differential systems with simple poles) on a sphere and the monodromy data's moduli space (i.e., representations of the fundamental group of a punctured sphere). There are few cases where the solution can be constructed explicitly. For Riemann spheres, positive results of a classical Riemann-Hilbert problem are usually existence theorems. In that case, Riemann-Hilbert correspondence turns out to be a Poisson morphism.

In recent years, there has been an increasing interest in b-symplectic (together with more general b^m- and E-symplectic) geometry. The corresponding manifolds can be viewed as stepping out of the symplectic category toward Poisson, allowing certain types of singularities in the 2-form, which is no longer symplectic. This approach enables a careful transfer of symplectic techniques to larger classes of Poisson structures while tracking which properties break or change.

We study the analog of Marsden-Weinstein reduction in the context of singular symplectic and singular quasi-Hamiltonian structures taking as a motivating example a singular version of the Atiyah-Bott structure on the moduli space of flat connections.

It turned out recently that another interesting example of non-autonomous b-symplectic structures appears in the context of Painlevé transcendents [BM21]. Sigma-coordinates for Okamoto Hamiltonian of the second Painlevé equation leads to a natural b-symplectic structure. For other Painlevé equations $P_{III}-P_{VI}$, the Poisson structure takes more complex form. We consider Poisson structure on moduli spaces of flat connections and monodromy data related by the Riemann-Hilbert correspondence for P_{VI} and P_{V}. For P_{VI} and the other Fuchsian equations, we explicitly construct such a structure on the corresponding character variety. This construction leads us to a conjecture for P_{V} which can be seen as a counterpart of the same structure on flat connections and coincides with obtained in [CMR18].

A more detailed outline of this thesis is provided below.

1.1 Structure and Results of This Thesis

1.1.1 Chapter 2: Preliminaries

This chapter provides a basic introduction to the topics studied in this thesis. It includes a brief overview of symplectic, Poisson, and singular symplectic geometry together with reduction theory.

1.1.2 Chapter 3: A b^m-Slice Theorem

In this section, we prove a b^m-symplcetic slice theorem, which describes a b^m-symplectic group action in the neighborhood of an orbit. This result will be used in further chapters as the first step of reduction.
Here and further, we use the following result of Proposition 5.1 describing a general form of groups preserving b^m-symplectic structure:

Proposition (5.1)**.** *Let (M, G, σ) be a closed quasi-Hamiltonian space of b^m-type, and let Z be its critical set. Then,*

- *Z fibers over a circle S^1.*
- *If the group G acts transversally on the fibers of then the group then G is either of the form $S^1 \times H$ or $S^1 \times H \mod \Gamma$, where $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ and $\mathbb{Z}_k$ is a non-trivial cyclic subgroup of H.*

Having this description, we can state main result of this chapter:

Theorem (3.2 A b^m-slice theorem)**.** *Let G be a compact group acting on a b^m-symplectic manifold (M, Z, ω) by b^m-symplectomorphisms such that the highest modular weight is non-vanishing. Let $\mathcal{O}_z$ be an orbit of the group contained in the critical set of M. Then there is a neighborhood of the zero section of an associated bundle ${}^{b^m}T^*G \times_{H_z \times \mathbb{Z}} V_z$ equipped with the b^m-symplectic model*

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H),$$

where t is a defining function for Z, π is the projection $\pi : T^*S^1 \times T^*H \times_{H_z} V_z \to T^*H \times_{H_z} V_z$ and ω_H is the symplectic form on $T^*H \times_{H_z} V_z$ given by the symplectic slice theorem.
The moment map for such action is given by

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} + \mu_0(x, y).$$

Results of this chapter are published in the preprint [MM22].

1.1.3 Chapter 4: A b^m-Marsden-Weinstein Reduction

In this chapter, we proceed with a generalization of the Marsden-Weinstein reduction for b^m-symplectic manifolds equipped with b^m-Hamiltonian group action and arrive at the statement of the theorem valid under three conditions described below. This reduction carries the same important property as Marsden-Weinstein reduction by simplifying the system by $2 \dim G$ degrees of freedom. Another remarkable property of this reduction explained in Chapter 4 is that it removes the singularity.
Notice that under Proposition 5.1, the group G has to be of the form $(S^1 \times H)/\Gamma$ that can as well be seen as $S^1 \times H$ on the universal cover of M.The reduction theorem is proved under the following assumptions:

· The induced action of H is locally free.

· The action of S^1 on the covering model associated with the finite group Γ is free.

· 0 is a regular value for μ_0 (by abuse of notation, we will then say that $\mathbf{0}$ is a regular point of μ).

Theorem (4.2 The b^m-Marsden-Weinstein reduction)**.** *Given a b^m-Hamiltonian (locally) free action of a Lie group G on a b^m-symplectic manifold M^{2n}, assume that the highest modular weight is non-vanishing. The preimage of a* *regular point $\mu^{-1}(\boldsymbol{0})$ is a b^m-presymplectic manifold that has an induced action of G. The space of orbits of the induced action $M//G$ is a symplectic orbifold. This reduced symplectic orbifold is symplectically isomorphic to the standard symplectic reduction of a symplectic leaf on Z by a Lie subgroup of G.*

Another important result of this chapter considers the interaction between reduction and desingularization procedure:

Theorem (4.4)**.** *The desingularization procedure commutes with the b^m-Hamiltonian reduction.*

This immediately leads us to the corollary that b^m-Hamiltonian reduction can be done by stages. As studied by the authors of [MMeO+07] in the Hamiltonian setting, for a group product $G_1 \times G_2$ reductions with respect to each component commute with each other. We prove the same statement for b^m-Hamiltonian reduction:

Corollary (4.1)**.** *The b^m-Hamiltonian G-action admits a* reduction by stages *procedure.*

Results of this chapter are published in the preprint [MM22].

1.1.4 Chapter 5: Singular Quasi-Hamiltonian Reduction and Fusion Products

This section is devoted to a more general variation of Marsden-Weinstein reduction which can be performed for those of b^m-symplectic group actions that are not b^m-Hamiltonian. To do this, we turn to a concept of quasi-Hamiltonian spaces and introduce the notion of singular quasi-Hamiltonian space.

Definition (5.1)**.** *A **singular quasi-Hamiltonian G-space of b^m-type** is a b-manifold (M, Z) with a G-action ρ, an invariant 2-form $\sigma \in^{b^m} \Omega(M)$ and an equivariant moment map $\Phi : M \to G$ such that:*

(i) σ *is equivariantly closed:* $d\sigma = -\Phi^*\chi$,

(ii) *the moment map condition is satisfied:* $\iota(\upsilon_\xi)\sigma = \frac{1}{2}\Phi^*\left(\theta^l + \theta^r, \xi\right)$,

(iii) σ *is weakly non-degenerate:*

$$\ker \sigma \cap \ker d\Phi = 0.$$

This setting allows us to consider any b^m-symplectic action, including non-Hamiltonian and more general examples of singular quasi-Hamiltonian spaces but not b^m-symplectic. An important step in the proof of the reduction theorem is the following splitting statement:

Corollary 1.1 (5.1)**.** *In a neighbourhood of the critical set* Z*, the* b^m*-form* σ *can be written as* $d\theta \wedge \frac{dt}{t^m} + \beta$*, where* θ *is coordinate on* S^1*. The corresponding* S^1*-action on the covering of* M *is* b^m*-Hamiltonian.*

This result, together with a generalization of the quasi-Hamiltonian reduction theorem proved in [AMM98] will allow us to proceed with singular quasi-Hamiltonian reduction as reduction by stages.

Theorem (5.4, singular quasi-Hamiltonian reduction)**.** *Let* M *be a singular quasi-Hamiltonian* $G_1 \times G_2$*-space with non-vanishing highest modular weight (i.e. one of the components of the product includes transverse* S^1*-action), a singularity of* b^m*-type and the moment map* $(\Phi_1, \Phi_2) : M \to G_1 \times G_2$*. Let* $f \in G_1$ *be a regular value of the moment map* $\Phi_1 : M \to G_1$ *and* $Z_f \subset G_1$ *be its centralizer. Then the pull-back of the* 2*-form* $\sigma \to \Phi_1^{-1}(f)$ *descends to the reduced space*

$$M_f = \Phi_1^{-1}(f)/Z_f$$

and makes it into quasi-Hamiltonian G_2*-space. If* (M, σ, Φ, G) *satisfies all conditions from the definition of a quasi-Hamiltonian space except the weakly non-degeneracy condition, so does the resulting reduced space. In particular, if* G_2 *is abelian then* M_f *is symplectic.*

Considering the first reduction with respect to S^1-component, we remove the singularity and arrive to a quasi-Hamiltonian space for which results of [BTW04]. This leads us to the main result of this chapter:

Theorem (5.6)**.** *Given a singular quasi-Hamiltonian space with a b^m-type singularity (M, σ, Z) and a transverse G-action with group-valued moment map Φ. If the highest modular weight for the S^1-component of the G-action is non-zero, the preimage of a regular point $\Phi^{-1}(\mathbf{f_0})$ admits an induced action of G. The space of orbits of the induced action $M//G$ is quasi-Hamiltonian.*

Results of this chapter are published in the preprint [MM22].

1.1.5 Chapter 6: Riemann-Hilbert Correspondence for Hypergeometric Equation and Poisson structures

This chapter focuses on two specific cases of the Riemann-Hilbert problem. For Riemann spheres, positive results of a classical Riemann-Hilbert problem are usually existence theorems. There are few cases where the solution can be constructed explicitly. We consider the problem on an elliptic curve. In particular, we study two cases for which the Riemann-Hilbert correspondence not only exists but can be written explicitly: The Riemann problem in rank 1 and rank 2 with three Fuchsian singularities.

There are different approaches to generalizing the problem on surfaces other than Riemann sphere compact Riemann surfaces, see [Bol02, EV99, GS99a]. We follow the formulation proposed in [EV99], applying a geometric approach to the problem. A Fuchsian system on a sphere can be considered as a logarithmic connection in a trivial vector bundle on a sphere. It appears that by using this approach, the essential properties of the trivial vector bundle are the semistability and equality of its degree to zero [GP08]. The trivial bundle appears here since a Riemann sphere in any dimension is the only holomorphic semistable vector bundle of degree zero. For this reason, we consider generalization given an elliptic curve, a set of points on it, and a representation of a fundamental group to construct over that curve a semistable

vector bundle of degree zero equipped with a logarithmic connection having prescribed singular points and required monodromy representation. We show that the modular parameter λ, together with the degree k, entirely and uniquely determines the semistable line bundle $\mathcal{O}(k)$.
An explicit solution for the rank 1 problem can be stated as the following theorem:

Theorem (6.1)**.** *The rank* 1 *Riemann problem for given elliptic curve* Λ_τ*, singular points* $\{a_1, \dots, a_n\}$ *and monodromy data* $g_1, \dots, g_n, \lambda$ *is solved positively in a trivial bundle if and only if*

$$\lambda = \sum_{k=1}^{n} \alpha_k a_k + p + q\tau$$

for some integers p *and* q *and normalized* n*-tuple* $\alpha_1, \dots, \alpha_n$*, where* $e^{2\pi\imath\alpha k} = g_k$*.*
The corresponding connection form in the bundle $\mathcal{O}_\lambda(0)$ *is*

$$\omega_\lambda(z) = \sum_{k=1}^{n} \alpha_k \frac{\theta'(z-a_k)}{\theta(z-a_k)} dz.$$

Otherwise, the same connection form solves the problem in $\mathcal{O}_{\sum_{k=1}^{n} \alpha_k a_k - \lambda}(0)$ *and there exist no other solutions.*

Here θ is a quasi-periodic function defined on a complex plane as

$$\theta(z) = \theta_1(z|\tau) = \imath \sum_{m\in\mathbb{Z}} (-1)^m q^{(m-\frac{1}{2})^2} e^{(m-\frac{1}{2})2\pi\imath z},$$

where $q(\tau) = e^{\imath\pi\tau} = e^{\imath\pi x - \pi y}$ sets the mapping of the upper half-plane $H = \{\tau \in \mathbb{C} | \Im\tau > 0\}$ into the unit circle $D = \{q \in \mathbb{C} \,|\, |q| \leq 1\}$.
The next version of the Riemann-Hilbert in this chapter considers systems of rank two with three singularities. To find an explicit solution, we first need to construct on elliptic curve Λ_τ a vector bundle $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}$ equipped

with logarithmic connection with prescribed monodromy representation and singular points location. The following theorem describes this bundle:

Theorem (6.3)**.** *Consider* $\{a_1, \ldots, a_n\} \in \Lambda_\tau$*,* $a_i \neq a_j$ *and complex* $\alpha_i, \beta_i, \gamma_i, \delta_i$*,* $i = 1, \ldots, n$ *such that*

$$\sum_{i=1}^{n} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta \end{pmatrix} = 0.$$

Then matrix 1*-form*

$$\Omega(z) = \sum_{i=1}^{n} \frac{\begin{pmatrix} \alpha_i \theta'(z-a_i) & \beta_i \frac{\theta'(0)}{\theta(-2\lambda)} \theta(z-a_i-2\lambda) \\ \gamma_i \frac{\theta'(0)}{\theta(2\lambda)} \theta(z-a_i+2\lambda) & -\delta_i \theta'(z-a_i) \end{pmatrix}}{\theta(z-a_i)} dz,$$

defines a logarithmic connection on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ *with residues*

$$\operatorname*{Res}_{z=a_i} \Omega(z) = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}.$$

This allows us to formulate the main result of this chapter.

Theorem (6.4)**.** *Consider an irreducible representation*

$$\chi_0 : \pi_1\left(\mathbb{C}\mathrm{P}^1 \setminus \{d_1, d_2, d_3\}\right) \to \mathrm{SL}(2, \mathbb{C}).$$

The Riemann-Hilbert problem for χ_0 *can be solved explicitly. Consider* (B_1, B_2, B_3) *any triple of residues giving the solution.*
Then 1-form $\widetilde{\Omega}(z)$ *constructed following theorem 6.3 with the use of triple* (B_1, B_2, B_3) *and arbitrary parameter* λ *defines a logarithmic connection* $\widetilde{\nabla} = d - \widetilde{\Omega}(z)$ *in semistable vector bundle* $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ *with singular points* $\{a_1, a_2, a_3\}$ *and monodromy representation*

$$\chi : \pi_1\left(\Lambda_\tau \setminus \{a_1, a_2, a_3\}\right) \to \mathrm{SL}(2, \mathbb{C}),$$

such that

$$\chi_{\text{ind}} = \chi_0, \quad \chi(\gamma_a) = 1$$

and

$$\chi(\gamma_b) \sim \exp\left(2\pi i \int_0^\tau \widetilde{\Omega}(z)\right).$$

Results of this chapter are published in the papers [MP17a, MP17b].

1.1.6 Chapter 7: Poisson Structures Associated to Moduli Spaces of Flat Connections

In this chapter, we describe the Poisson structures on the moduli spaces of flat connections and corresponding structures on character varieties of bordered Riemann surfaces that can be seen as representation spaces of the monodromy data associated to the given connections. We follow the references [KS97], [Sak01] and [CMR18].

This chapter was originally devoted to studying behaviour of Poisson structure on moduli space of monodromy data under the confluence of singularities. The idea was to start with a known structure on the monodromy data of Fuchsian equations (see [FT07, KS97]) that can be explicitly obtained as a counterpart under the Riemann-Hilbert correspondence of a classical bracket on the connections

$$\{A(\lambda_1) \otimes I, I \otimes A(\lambda_2)\} = [r(\lambda_1 - \lambda_2), A(\lambda_1) \otimes I + I \otimes A(\lambda_2)]$$

with $r(\lambda) = \frac{\Omega}{\lambda}$ being classical r-matrix, i.e. a solution of the classical Yang–Baxter equation and $\Omega = \sum_{i,j} e_{ij} \otimes e_{ji}$ the permutation operator, λ – spectral parameter. Starting from this structure, Korotkin and Samtleben produced a bracket on the space of monodromy matrices.

The question we tried to address in this chapter is how to construct the analogous Poisson structure when there are irregular singular points. Specifically, we were interested in the isomonodromic problem associated to the fifth Painlevé equation. We were hoping to use the confluence approach to extend the Poisson structure described above to higher-order singularities. However, this program has not been completely successful. We have reached a conjecture about the expected Poisson structure in Section 7.1.2 and Section 7.1.3, we explain how to obtain the Poisson brackets on the sixth Painlevé monodromy data following Faddeev-Takhtadjan approach. The obtained formulae show that, at least at the level of monodromy matrices, our conjecture is correct.

From here, we denote for any matrix X,

$$\overset{1}{X} = X \otimes \mathbb{I} \qquad \text{and} \qquad \overset{2}{X} = \mathbb{I} \otimes X.$$

Conjecture (7.1). *Let $\{M_1, M_2, S_1, S_2\}$ be monodromy data corresponding to P_V, then there is Poisson structure on these data that can be written in the following form:*

$$\begin{aligned}
\{S_k \underset{,}{\otimes} S_k\} &= \overset{1}{S_k}(r - r^T)\overset{2}{S_k} + \overset{2}{S_k}(r - r^T)\overset{1}{S_k} + 2r\overset{1}{S_k}\overset{2}{S_k} - 2\overset{1}{S_k}\overset{2}{S_k}r^T, \\
\{S_1 \underset{,}{\otimes} S_2\} &= r^T\overset{1}{S_1}\overset{2}{S_2} + \overset{2}{S_2}\overset{1}{S_1}r^T - \overset{1}{S_1}r^T\overset{2}{S_k} - \overset{2}{S_2}r\overset{1}{S_1}, \\
\{M_2 \underset{,}{\otimes} M_1\} &= \overset{1}{M_1}\overset{2}{M_2}r + r\overset{2}{M_2}\overset{1}{M_1} - \overset{1}{M_1}r\overset{2}{M_2} - \overset{2}{M_2}r\overset{1}{M_1}.
\end{aligned}$$

This structure can be seen as an image of the following bracket on the moduli space of flat connections under the Riemann-Hilbert correspondence:

$$\left\{A(\lambda_1) \underset{,}{\otimes} A(\lambda_2)\right\} = \left[r(\lambda_1 - \lambda_2), \overset{1}{A}(\lambda_1) + \overset{2}{A}(\lambda_2)\right],$$

where r is Kulish-Sklyanin r-matrix and λ - spectral parameter.

Theorem (7.2). *Let $\{M_i\}$ be monodromy data corresponding to a Fuchsian system. Then there is Poisson structure on the corresponding character variety that can be written in the following form:*

$$\begin{aligned}
\left\{M_i \underset{,}{\otimes} M_j\right\} &= \overset{1}{M}_i r \overset{2}{M}_j + \overset{2}{M}_j r \overset{1}{M}_i - r \overset{2}{M}_j \overset{1}{M}_i - \overset{1}{M}_i \overset{2}{M}_j r,\\
\left\{M_i \underset{,}{\otimes} M_i\right\} &= \overset{2}{M}_i r \overset{1}{M}_i + \overset{2}{M}_i r^T \overset{2}{M}_i - r^T \overset{1}{M}_i \overset{2}{M}_j - \overset{1}{M}_i \overset{2}{M}_j r,\\
\left\{M_\infty \underset{,}{\otimes} M_i\right\} &= \overset{2}{M}_i r \overset{1}{M}_\infty + \overset{2}{M}_\infty r^T \overset{2}{M}_i - r \overset{1}{M}_\infty \overset{2}{M}_j - \overset{1}{M}_\infty \overset{2}{M}_j r^T,
\end{aligned}$$

where $i < j$. This structure can be seen as an image of the following bracket on the moduli space of flat connections under the Riemann-Hilbert correspondence:

$$\left\{A(\lambda_1) \underset{,}{\otimes} A(\lambda_2)\right\} = \left[r(\lambda_1 - \lambda_2), A(\overset{1}{\lambda_1}) + A(\overset{2}{\lambda_2})\right],$$

where r is Kulish-Sklyanin r-matrix and λ - spectral parameter.

1.2 Publications Resulting from This Thesis

Results of this thesis can be found in the following papers:

· *Reduction theory for singular symplectic manifolds and singular forms on moduli spaces*, joint with Eva Miranda, arXiv:2205.12919

· *Two-dimensional Riemann problem for rigid representations on an elliptic curve*, joint with Vladimir Poberezhny, Journal of Geometry and Physics 114(206):384-393

· *The one-dimensional Riemann problem on an elliptic curve*, joint with Vladimir Poberezhny, Mathematical Notes 101(1-2):115-122

· *Stokes phenomenon arising in the confluence of two simple poles and corresponding Poisson structures*, joint with Marta Mazzocco and Volodya Rubtsov, in preparation

Chapter 2

Preliminaries

2.1 Symplectic Manifolds

This section briefly introduces symplectic geometry, providing a definition, some examples, and a local normal form theorem, the generalization of which for singular structures will be used extensively throughout this thesis.

Definition 2.1. *A* ***symplectic manifold*** (M, ω) *is a manifold* M *equipped with a closed non-degenerate* 2*-form* $\omega \in \Omega^2(M)$*.*

Let us provide a few classic examples of symplectic manifolds.

Example 2.1. *Any orientable surface* Σ *of genus* g *with an area form* ω *is a symplectic manifold.*

Example 2.2. *Any even dimensional Euclidean space* $\mathbb{R}^{2n}$ *with coordinates* $\{x_1, \ldots, x_n, y_1, \ldots y_n\}$ *and a* 2*-form* $\omega = \sum_{i=1}^{n} dx_i \wedge dy_i$ *is a symplectic manifold.*

In fact, any symplectic manifold M can only have an even dimension since ω is non-degenerate. It also means that $\omega^n \neq 0$ is a volume form on M, so any symplectic manifold should also be orientable. Locally, the only invariant of a symplectic manifold is its dimension:

Theorem 2.1 (Darboux)**.** *Let (M,ω) be a symplectic manifold of dimension $2n$. Then for any point $p \in M$ there is an open neighbourhood $\mathcal{U}(p) \subset M$ with coordinates $\{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ such that in this coordinates*

$$\omega|_{\mathcal{U}} = \sum_{i=i}^{n} dx_i \wedge dy_i.$$

2.2 Poisson Structures

We now consider manifolds of any dimension that share some properties with symplectic manifolds. These manifolds are endowed with a Poisson bracket.

Definition 2.2. ***Poisson bracket*** *on smooth real manifold M is given by $\mathbb{R}$-bilinear map $\{,\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M)$ satisfying following conditions:*

- *anti-symmetry:*

$$\{f, g\} = -\{g, f\},$$

- *Leibniz rule:*

$$\{f, g \cdot h\} = \{f, g\} \cdot h + g \cdot \{f, h\},$$

- *Jacobi identity:*

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0.$$

Manifold M endowed with Poisson bracket $\{\cdot,\cdot\}$ is called ***Poisson manifold*** *and is denoted by $(M, \{\cdot,\cdot\})$.*

Definition 2.3. *Let $M, \{\cdot,\cdot\}_M$ and $N, \{\cdot,\cdot\}_N$ be a pair of Poisson manifolds. $F : M \to N$ is a smooth map, and $F^* : C^\infty(N) \to C^\infty(M)$ is the induced one. Then F is a* ***Poisson morphism*** *if*

$$F^*\{f, g\}_N = \{F^*f, F^*g\}_M.$$

One can mention dual Lie algebra with Poisson-Lie structure and symplectic manifold with naturally induced Poisson structure as the simplest examples of Poisson manifolds.

Definition 2.4. *Let* $(M, \{\cdot, \cdot\})$ *be a Poisson manifold and* $f \in C^{\infty}(M)$*. Then vector field* X_f *defined by*

$$X_f(g) = \{f, g\}$$

is called ***Hamiltonian vector field*** *associated with the Hamiltonian f.*

Definition 2.5. *Let* $(M, \{\cdot, \cdot\})$ *be a Poisson manifold. Action of* ***Hamiltonian*** *is an action* $J : T^*M \to TM$ *defined by:*

$$Jdf = X_f$$

i.e. $J(d_x f) = X_f(x)$.

In coordinates Poisson bi-vector J has the form

$$J(dx_i) = \sum_{j=1}^{n} X_{x_i}(x_j) \frac{\partial}{\partial x_j} = \sum_{j=1}^{n} \{x_i, x_j\} \frac{\partial}{\partial x_j},$$

therefore

$$J_{ij} = \{x_i, x_j\},$$

$$X_f = \sum_{i,j=1}^{n} J_{ij} \frac{\partial f}{\partial x_i} \frac{\partial}{\partial x_j}.$$

So for two smooth functions f and g on M, Poisson bracket can be also defined as

$$\{f, g\} = \sum_{i,j=1}^{n} J_{ij} \frac{\partial f}{\partial x_i} \frac{\partial g}{\partial x_j}.$$

Any symplectic manifold (M, ω) can be considered as an example of Poisson manifold with the bracket given by $\{f, g\} = \omega(X_f, X_g)$. The associated vector field can be denoted as $\Pi = \omega^{-1}$.
We will often use the vector field Π to define Poisson structure on the manifold and in this case we denote Poisson manifold as (M, Π).

Theorem 2.2 (Weinstein Splitting Theorem). *Let (M, Π) be a Poisson manifold of dimension n and let the rank be $2k$ at the point $p \in M$. Then on a neighborhood of p there exists a coordinate system*

$$(x_1, \dots, x_k, y_1, \dots, y_k, z_1, \dots, z_{N-2k})$$

such that the Poisson structure can be written as

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{i,j=1}^{n-2k} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j},$$

where f_{ij} are functions that depend only on the variables $(z_1, \dots, z_{N-2k})$ and vanish at the origin.

2.3 Singular Symplectic Structures

Throughout a major part of this thesis, we take symplectic techniques as motivating and guiding examples. In this section, we describe important groups of manifolds and geometric structures that can be seen as symplectic with singularities.

2.3.1 b-Symplectic and b-Poisson Manifolds

The letter "b" on b-symplectic theory is a reminiscent from the b in boundary in Melrose's work [Mel93] to extend the proof of the index theorem by Atiyah-Singer to manifolds with boundary. However, in the theory of b-symplectic manifold the notion is extended to consider a hypersurface which,

a posteriori, will turn out to be the critical hypersurface Z of the generalized symplectic structure.
b-Manifolds were first introduced by Melrose in his book [Mel93] to give proof of the Atiyah-Patodi-Singer theorem using the same conceptual proof as in the Atiyah-Singer theorem for manifolds with boundary. In [GMP14] this framework was extended, and Poisson structures were associated to b-forms of degree 2 as bivector fields which vanish along a critical hypersurface. These vector fields can be seen as dual to a two-form with singularity along the hypersurface. b-Symplectic forms were defined and extensively studied in the works [NT96,Mel93,GMP11,GMP14]. We briefly remind the main definitions and concepts of b-symplectic geometry to use the mentioned framework.
In this section we recall the notion of b-symplectic and b-Poisson manifolds defined in [GMP14] as generalization of symplectic manifolds motivated by the notion of symplectic manifolds with boundary extending them to the notion of symplectic structures on manifolds with a hypersurface and subspace in space of Poisson manifolds respectively. We provide definitions of b-symplectic and b-Poisson structures together with b-analogues of Darboux theorem and Weinstein splitting theorem.

Definition 2.6. *Let (M^{2n}, Π) be an oriented Poisson manifold such that the map $p \in M \mapsto (\Pi(p))^n \in \Lambda^{2n}(TM)$ is transverse to the zero section, then $Z = p \in M|(\Pi(p))^n = 0$ is a hypersurface and we say that Π is a* ***b-Poisson structure*** *on (M, Z) and (M, Z) is a* ***b-Poisson manifold****.*

As shown in the paper [GMP14], b-symplectic and b-Poisson structures are the same.

Theorem 2.3. *A two-form ω on a b-manifold (M, Z) is* ***b-symplectic*** *if and only if its dual bi-vector field Π is a b-Poisson structure.*

Definition 2.7. *A* ***b-manifold*** *(M, Z) is an oriented manifold M with an oriented hypersurface Z.*

Definition 2.8. *A* ***b-map*** *is a map $f : (M_1, Z_1) \to (M_2, Z_2)$ so that f is transverse to Z_2 and $f^{-1}(Z_2) = Z_1$.*

Definition 2.9. *A* b-***vector field*** *is a vector field on* (M, Z) *which is everywhere tangent to* Z*.* Z *is called an exceptional hypersurface of* (M, Z)*.*

Let us now define b-tangent bundle using local description of its sections.

Consider an open neighborhood U of a point $p \in Z$ and assume that Z is locally given by the level set of a locally defined function f. We refer to f as a **defining function**. The vector field $f\frac{\partial}{\partial f}$ is tangent to Z. Take a coordinate chart on U of the form $(f, x_2, \ldots, x_n)$ for which the b-vector fields restricted to U form a free C^∞-module with a finite basis given by $f\frac{\partial}{\partial f}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_n}$. According to Serre-Swan [Swa62], there exist a unique vector bundle having the b-vector fields as its sections. This vector bundle is called **b-tangent bundle** and is denoted it by ${}^bTM$. At the points $p \in M \setminus Z$, the b-tangent space coincides with the tangent space, i.e. ${}^bT_pM = T_pM$. At points $p \in Z$, the restriction of a b-vector field on Z yields a vector field on Z. The vector bundle morphism

$${}^bTM|Z \to TM|Z$$

is surjective and the kernel is the line bundle generated by $f\frac{\partial}{\partial f}$, which is called the **normal b-bundle**.

The **b-cotangent bundle** ${}^bT^*M$ of a b-manifold is defined as the dual of ${}^bTM$ and its local basis is given by

$$\left(\frac{df}{f}, dx_2, \ldots, dx_n\right),$$

where the form $\frac{df}{f}$ is well-defined on the b-cotangent bundle. Differential forms for this vector bundle are called **b-forms**. A b-form of degree k is defined as section of the vector bundle ${}^b\Omega^k(M) = \Lambda^k({}^bT^*M)$. Fixing a defining function f, every b-form of degree k can be decomposed as follows:

$$\alpha \wedge \frac{df}{f} + \beta, \text{ where } \alpha \in \Omega^{k-1}(M), \ \beta \in \Omega^k(M).$$

This decomposition makes it possible to define an exterior derivative by

$$d\left(\alpha \wedge \frac{df}{f} + \beta\right) = d\alpha \wedge \frac{df}{f} + d\beta.$$

The notion of closed and exact forms are naturally extended to the b-setting, which allows to extend the de Rham cohomology in a similar manner. By Mazzeo–Melrose [Mel93] theorem, b-cohomology of (M, Z) can be computed in terms of ordinary cohomology of M and Z:

Theorem 2.4 (Mazzeo-Melrose)**.** ${}^bH^*(M, Z) \cong H^*(M) \bigoplus H^{*-1}(Z)$.

In general, b-tangent bundle is not isomorphic to tangent. In general, it is nontrivial to show for an arbitrary surface since b-tangent bundle was defined by describing all its sections. The set of sections uniquely define the bundle though it can be not easy to describe it explicitly. For a particular example of circle S^1 with one marked point as a boundary or torus T^2 with a marked circle we proved it this year with Joaquim Brugues by showing that in both these cases the b-tangent bundles are non-orientable. In case of circle with the marked point the tangent bundle is trivial and the b-tangent bundle is Mobius stripe.

The notion of b-forms allows us to study b-symplectic structure as a generalization of symplectic structure for manifolds with boundaries. Notice that despite b-symplectic structure is much richer than symplectic, there are many constraints that can prohibit

- a manifold M (possibly symplectic) to be b-symplectic,
- a hypersurface Z to be an exceptional hypersurface of any b-symplectic manifold,
- a pair of a manifold and a hypersurface (M, Z) to be a b-symplectic manifold even though M can admit a b-symplectic structure and Z can be an exceptional hypersurface of some b-symplectic manifold.

Here we will mention some of them.

Theorem 2.5 (Guillemin-Miranda-Weitsman [GMW19]).
Any b-symplectic manifold is also folded symplectic.

Theorem 2.6 (Marcut-Osorno-Torres [MOT14]).
If M^{2n} is compact b-symplectic manifold then there exists a cohomology class $\alpha \in H^2(M, \mathbb{R})$ such that $\alpha^{n-1} \neq 0$.

Theorem 2.7 (Cavalcanti [Cav17]).
If M is compact orientable b-symplectic manifold then there exists a non-trivial cohomology class $\beta \in H^2(M, \mathbb{R})$ that squares to zero $\beta^2 = 0$.

These two theorems, for example, show us that S^4 and $\mathbb{C}P^2$ do not admit a b-symplectic structure.
Another important example showing the difference of b-geometry of leads us to statement of b-version of Poincaré-Hopf theorem. The hairy ball theorem doesn't hold for (S^2, S^1) Indeed, if you take $\upsilon = h\frac{\partial}{\partial h}$ as a vector field, you can choose your circle S^1 in such a way that it passes through its zeroes. In this case υ is nowhere vanishing b-vector field. The same is not true if you take υ as a vector field on S^2 without exceptional hypersurface.

Conjecture 2.1 (b-Poincaré-Hopf). *The Euler class of b-tangent bundle $({}^bTM)$ is equal to b-index of the vector field.*

Definition 2.10. *Let M be a Poisson manifold and Ω be a volume form on M. The associated **modular vector field** is defined as the derivation:*

$$\upsilon_{mod} : f \to \frac{\mathcal{L}_{X_f}\Omega}{\Omega}.$$

The existence of this transverse vector field implies cosymplectic structure on Z: the one form, dual to this vector field together with the symplectic form on the leaves of the symplectic foliation defines the cosymplectic structure.

Definition 2.11. *A **cosymplectic manifold** is a manifold M^{2n+1} together with a closed one-form η and a closed two-form ω such that*

$$\eta \wedge \omega^n$$

is a volume form.

2.3.2 b^m-Symplectic Manifolds

This section concentrates on a rich family of singular symplectic forms called b^m-symplectic, where a generalized transversality condition is imposed.

Definition 2.12. *A b^m**-vector field** is a vector field v on M, such that it vanishes to order m at Z. A b^m**-form** is a differential form dual to a b^m-vector field.*

One can also think of b^m-vector fields and b^m-forms as locally generated by $\{x_1^m \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \dots, \frac{\partial}{\partial x_n}\}$ and $\{\frac{dx_1}{x_1^m}, \frac{dx_2}{x_2}, \dots, \frac{dx_n}{x_n}\}$, respectively. Due to the Serre-Swan theorem [Swa62], given a b-manifold (M, Z), there exists a unique vector bundle ${}^{b^m}TM$ all whose smooth sections are b^m-vector fields. Such a bundle is called a b^m**-tangent bundle**. Analogously, a b^m**-cotangent bundle** can be defined either as dual to the tangent one:

$$ {}^{b^m}T^*M = ({}^{b^m}TM)^*, $$

or as a bundle, all smooth sections of which are b-forms.
This allows introducing ${}^{b^m}\Omega^k(M)$ as $\bigwedge^k({}^{b^m}T^*M)$ and the associated b^m-**cohomology** ${}^{b^m}H^*(M)$. The following theorem relates b^m-cohomology to de Rham cohomology [Sco16].

Theorem 2.8 (The b^m-Mazzeo-Melrose)**.** ${}^{b^m}H^*(M) \cong H^*(M) \oplus (H^{*-1}(Z))^m$.

Among b^m-forms, we will focus on forms of degree two that resemble symplectic forms in the de Rham complex.

Definition 2.13. *Let (M^{2n}, Z) be a b-manifold, where Z is the critical hypersurface as in 2.7. Let $\omega \in {}^{b^m}\Omega^2(M)$ be a closed b^m-form. We say that ω is b^m**-symplectic** if ω_p is of maximal rank as an element of $\Lambda^2\left({}^{b^m}T_p^*M\right)$ for all $p \in M$. We call a b^m**-symplectic manifold** a triple (M, Z, ω).*

It is possible to describe these forms more precisely in a neighbourhood U of the critical set Z. Inside $U = Z \times (-\epsilon, \epsilon)$, ω may be written as

$$\omega = \sum_{j=1}^{m} \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta,$$

where the α_j are closed one forms on Z, β is a closed 2-form on U, and $\pi : U \longrightarrow Z$ is the projection. Non-degeneracy of the form ω implies that $\beta|_Z$ is of maximal rank and α_m is nowhere vanishing. α_m defines the symplectic foliation of the Poisson structure associated with ω, and β gives the symplectic form on the leaves of this foliation.
If we look for local invariants rather than semilocal invariants: as happens with symplectic manifolds, the only local invariant for b^m-symplectic forms is the dimension.

Theorem 2.9 (b^m-Darboux)**.** *Let ω be a b^m-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \dots, x_n, y_n)$ centered at p such that hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = dx_1 \wedge \frac{dy_1}{y_1^m} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

The proof of this local normal form relies on the path method.
The b^m-analogue of the Moser theorem for symplectic manifolds is convenient for analyzing other invariants (local, semilocal, global) and is proved in [GMP14].

Theorem 2.10 (b^m**-Moser Theorem).** *Let ω_0 and ω_1 be two b^m-symplectic forms on (M, Z) defining the same b^m-cohomology class $[\omega_0] = [\omega_1]$ on (M^{2n}, Z) with M^{2n} closed and orientable then there exist a b^m-symplectomorphism*

$$\varphi : \left(M^{2n}, Z\right) \to \left(M^{2n}, Z\right),$$

such that $\varphi^*(\omega_1) = \omega_0$.

The equivariance of the path method yields the following generalization of the Moser path method [MP18] under the additional structure of a group action.

Theorem 2.11 (Equivariant b^m-Moser Theorem). *Let ω_0 and ω_1 be two b^m-symplectic forms on (M, Z). If they induce on Z the same corank one Poisson structure and their modular vector fields differ on Z by a Hamiltonian vector field, then there exist neighbourhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.*
If (M, Z) admits b^m-symplectic action of a compact Lie group G, then γ can be chosen equivariant with respect to the G-action.

b^m-Symplectic manifolds are dual to b^m-Poisson, which allows us to describe these objects in two different languages using either bi-vector fields or differential forms. It is then possible to introduce invariants native to Poisson geometry, such as the *modular vector field*.
Even though the only local invariant of a b^m-symplectic manifolds is the dimension, it turns out that the geometry induced by the Poisson structure on the critical set Z yields new semilocal invariants. The structure induced on Z is indeed *cosymplectic*.
Using the flow of the modular vector field, we can define a symplectic mapping torus structure of Z, as proved in [GMP11]). This mapping group structure is also present on the critical set of a b^m-symplectic manifold.

Definition 2.14. *Let (M, Z) be a b^m-symplectic manifold and suppose that Z is compact and connected and that its symplectic foliation has a compact leaf $\mathcal{L}$. Then the critical set Z is a mapping torus which can be explicitly described as follows: taking any modular vector field v_{mod}, there exists a number $c > 0$ such that*

$$Z \cong \frac{[0, c] \times \mathcal{L}}{(0, x) \sim (c, \phi(x))},$$

where the time t-flow of v_{mod} corresponds to the translation by t in the first coordinate. In particular, ϕ is the time c-flow of v_{mod}.

*The number $c > 0$ above is called the **modular period** of Z and does not depend on the choice of the modular vector field v_{mod}.*

2.3.3 Folded Symplectic Manifolds

A symplectic form ω on a manifold M induces a natural volume form on the manifold ω^n, sometimes called the Liouville volume. The next level of sophistication is to consider forms ω such that ω^n might vanish at some points but with *good transversality properties.* This is precisely the notion of *folded symplectic structures.*

Definition 2.15. *Let (M^{2n}, ω) be a manifold with ω a closed 2-form such that the map*

$$p \in M \mapsto (\omega(p))^n \in \Lambda^{2n}(T^*M)$$

*is transverse to the zero section, then $Z = \{p \in M | \, (\omega(p))^n = 0\}$ is a hypersurface and we say that ω defines a **folded symplectic structure** on (M, Z) if additionally its restriction to Z has maximal rank. We call the hypersurface Z **folding hypersurface** and the pair (M, Z) is a **folded symplectic manifold**.*

For simplicity, further, we use the normal form for the folded symplectic structures first described by Martinet in [Mar70].

Theorem 2.12 (Folded Darboux)**.** *Let ω be a folded symplectic form on (M^{2n}, Z) and $p \in Z$. Then we can find a local coordinate chart $(x_1, y_1, \ldots, x_n, y_n)$ centered at p such that the hypersurface Z is locally defined by $y_1 = 0$ and*

$$\omega = y_1 dx_1 \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

2.3.4 Relation Between b^m-Symplectic, Symplectic and Folded Symplectic Manifolds

To relate singular symplectic manifolds to either symplectic or folded symplectic ones, we recall the desingularization theorem first formulated in [GMW19]. Observe that the behaviour of desingularization depends on the degree of the singularity.

Theorem 2.13. *Let ω be a b^m-symplectic structure on a compact manifold M and let Z be its critical hypersurface.*

- *If $m = 2k$ is **even**, there exists a family of **symplectic** forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*

- *If m is **odd**, there exists a family of **folded symplectic** forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighborhood of Z.*

An immediate consequence of this theorem is that b^{2k}-symplectic manifold also has to admit symplectic structure.

Following [GMW19], we will look at the desingularization process in more detail and explicitly write the desingularizing function for even degree as we use it to construct a vital example later on in this paper.

Recall that one can write a Laurent series of a closed b^m-form in a tubular neighbourhood U of Z:

$$\omega = \frac{dx_1}{x_1^m} \wedge \left(\sum_{i=0}^{m-1} \pi^*(\alpha_i) x^i \right) + \beta, \tag{2.3.1}$$

where $\pi : U \to Z$ is the projection of the tubular neighborhood onto Z, α_i is a closed smooth de Rham form on Z, and β is a de Rham form on M.

Due to formula 2.3.1, the b^{2k}-form can be written as

$$\omega = \frac{dx_1}{x_1^{2k}} \wedge \sum_{i=0}^{2k-1} \left(x^i \alpha_i\right) + \beta \tag{2.3.2}$$

on a tubular ε-neighbourhood of a given connected component of Z.
More details, including the desingularization function (together with its explicit form), will be provided further in Section 3.1.4.

2.3.5 E-Symplectic Manifolds

As a last group of singular symplectic manifolds, in this section, we will briefly discuss yet another generalizing step, covering even larger group of Poisson manifolds and still allowing to treat them in somewhat symplectic framework. E-symplectic structures studied in details in [MS21] allow wider class of singularities such as normal crossings. A simple example of an E-symplectic form not fitting into b- or b^m-symplectic category would be $\omega = \frac{1}{xy} dx \wedge dy$.
The main trick and beauty of b- and b^m-symplectic geometry is they you first define a proper bundle (${}^bTM$ and ${}^{b^b}TM$ respectively), then you can construct ${}^b\Omega(M)$ and ${}^{b^m}\Omega(M)$. In this setting, singular symplectic forms can be defined just as symplectic, i.e. closed and non-degenerate but taken in a proper bundle. One can extend a set of "allowed" forms further.
Take E to be a locally free involutive submodule of $\mathfrak{X}(M)$. We can define an E-tangent bundle ${}^ETM$ (together with E-cotangent ${}^ET^*M$) as before, using Serre-Swan theorem as a bundle whose all sections are given by E. Using the involutive property of E and applying Cartan formula, we can extend the exterior product of to ${}^E\Omega^*(M)$.

Definition 2.16. *For a locally free involutive submodule of $\mathfrak{X}(M)$, a closed non-degenerate 2-form $\omega \in^E \Omega(M)$ is called an **E-symplectic form**. The manifold (M, E, ω) is an **E-symplectic manifold**.*

2.4 Group Actions and Moment Maps

This section describes different frameworks to study group actions on the manifolds described above. We start with reviewing Hamiltonian group actions on symplectic manifolds, then proceed with a natural generalization of b- and b^m-Hamiltonian group actions and describe corresponding moment maps. We also give a brief review of quasi-Hamiltonian spaces that allows to study wider class of group actions.
In this section we follow references [CdS01], [GS99b], [DH82], [AB84], [MW74], [Mey73], [Kir84a], [Wei93], [McD88], [BKM18b], [AMM98], [Jef94], [CDM88], [OR06], [GHJW97], [AMW02], [BTW04], [Sco16], [GMW19], [GMW18a], [CdSGP11] and [FR99].

2.4.1 Hamiltonian Spaces

Definition 2.17. *A* ***Hamiltonian** G**-space*** $(M, \mathcal{A}, \sigma, \varphi)$ *is a* $2n$*-dimensional manifold* M *with* G*-action* $\mathcal{A}$*, invariant* 2*-form* $\sigma \in \Omega^2(M)$ *and an equivariant moment map* $\varphi : M \to \mathfrak{g}^*$ *such that*

(i) σ *is closed:* $d\sigma = 0$*,*

(ii) moment map condition: $\iota(v_\xi)\sigma = d\left\langle \varphi, \xi \right\rangle, \quad \forall \xi \in \mathfrak{g}$*,*

(iii) σ *is non-degenerate,*

where $\langle , \rangle$ *is natural pairing identifying* $\mathfrak{g}$ *and* $\mathfrak{g}^*$*, and* v_ξ *is generating vector field on* M*.*

For closed G-invariant 2-form σ the contraction $\iota(v_\xi)\omega$ is always closed:

$$d\iota(v_\xi)\sigma = \mathcal{L}_{v_\xi}\sigma - \iota(v_\xi)\sigma = 0.$$

Condition (ii) also requires it to be exact.

Example 2.3. *For any orientable surface, any function is a moment map for* $\mathbb{R}$*-action*

Another way to define Hamiltonian G-spaces is by using language of equivariant de Rham cohomologies [GS99b]. We consider complex of equivariant differential forms

$$\omega_G^*(M) = \bigoplus_k \Omega_G^k(M),$$

where

$$\Omega_G^k(M) = \bigoplus_{2l+j=k} (\omega^j(M) \otimes S^l \mathfrak{g}^*)^G$$

and the equivariant differential is

$$(d_G\alpha)\xi = d(\alpha(\xi)) - \iota(v_\xi)\alpha(\xi).$$

First two conditions (i) and (ii) from the definition of a Hamiltonian G-space can be replaced by asking the form $\omega_g(\xi) = \omega + \langle \mu, \xi \rangle$ to be equivarintly closed $d_G\omega_G = 0$:

$$d_G\omega(\xi) = d\omega - \iota(v_\xi)\omega = -d\langle \mu, \xi \rangle = \mu^*\chi_G,$$

where $\chi_G \in \Omega_G^3$ is an equivariant 3-form defined as $\chi_G(\xi) = \langle e, \xi \rangle$ with $e : \mathfrak{g}^* \to \mathfrak{g}^*$ the identity map.
We denote by $H_G^*(M)$ the equivariant de Rham cohomology ring of M.
For Hamiltonian spaces there exists measure on the Lie algebra dual first described by Duistermaat and Heckman in [DH82].
$M//_pG$ denotes the symplectic quotient of M by G at a regular central value $p \in Z(\mathfrak{g}^*)$ of φ, $M//_pG = \varphi^{-1}(p)/G$.

Definition 2.18. *The pair* (M_{red}, ω_{red}) *is called the reduction of* (M, ω) *with respect to* G, φ *or the reduced space, or the symplectic quotient.*

A transverse section S to the G-orbit through p $\mathcal{O}_p$ is called a slice. Choose a coordinate system $(x_1, \dots, x_n)$ centered at p such that

$$\mathcal{O}_p \simeq G : x_1 = \dots = x_k = 0$$

$$S : x_{k+1} = \ldots = x_n = 0.$$

Let $S_\epsilon = S \cap B_\epsilon(0, \mathbb{R}^n)$. Let $\eta : G \times S \to M, \eta(g, s) = gs$. Then there is a following equivariant tubular neighborhood theorem.

Theorem 2.14 (Slice Theorem)**.** *Let G be a compact Lie group acting on a manifold M such that G acts freely at $p \in M$. For sufficiently small $\epsilon, \eta : G \times S_\epsilon \to M$ maps $G \times S_\epsilon$ diffeomorphically onto a G-invariant neighborhood $\mathcal{U}$ of the G-orbit through p.*

2.4.2 b^m-Hamiltonian Group Actions

In order to describe the group actions on singular symplectic manifolds and the corresponding moment maps, we recall the results of two papers: [BKM18a] and [GMPS15] (check the preprint version for completeness *arXiv:1309.1897v1*).
As a Poisson manifold, b^m-symplectic manifolds admits an induced symplectic foliation. The connected components of $M \setminus Z$ are open symplectic leaves of dimension $2n$, and the critical hypersurface Z admits a co-rank 1 Poisson (cosymplectic) structure.

Theorem 2.15 (Braddell, Kiesenhofer, Miranda)**.** *Let G be a compact Lie group acting on a compact b-symplectic manifold. Then G is either of the form $S^1 \times H$ or $S^1 \times H \mod \Gamma$, where $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ and $\mathbb{Z}_k$ is a non-trivial cyclic subgroup of H.*

Definition 2.19. *The action of G on a b^m-symplectic manifold (M, Z, ω) is called b^m**-Hamiltonian** if there exists a moment map $\mu \in^{b^m} \mathcal{C}^\infty(M) \otimes \mathfrak{g}^*$ with*

$$\iota(\upsilon_\xi)\omega = \langle d\mu, \xi \rangle ,$$

where υ_ξ is the fundamental vector field generated by ξ and the set of b^m-functions is ${}^{b^m}\mathcal{C}^\infty(M) = \left(\bigoplus_{i=1}^{m-1} t^{-i}\mathcal{C}^\infty(t) \right) \oplus {}^b\mathcal{C}^\infty(M)$ and ${}^b\mathcal{C}^\infty(M) = \{a \log |t| + g, g \in \mathcal{C}^\infty(M)\}$.

In other words, the action is b^m-Hamiltonian if it preserves the b^m-symplectic form and $\iota(\upsilon_\xi)\omega$ is exact.
For simplicity, let us consider the case of b-surfaces (for the general case, check [GMPS15]). One can notice that outside the critical hypersurface, locally, the image of the b-moment map is just $\mathbb{R}$, and on the hypersurface, it blows up. To prescribe a smooth structure on the image of the moment map, in [GMPS15] the authors use $\mathbb{R}_{>0}$-valued labels ("weights") on the points at infinity. Thus, the image of the moment map forms what is called a b-line (or b-circle) constructed by gluing copies of the extended real line $\overline{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\}$ together in a zig-zag pattern where points are *at infinity* are glued together.

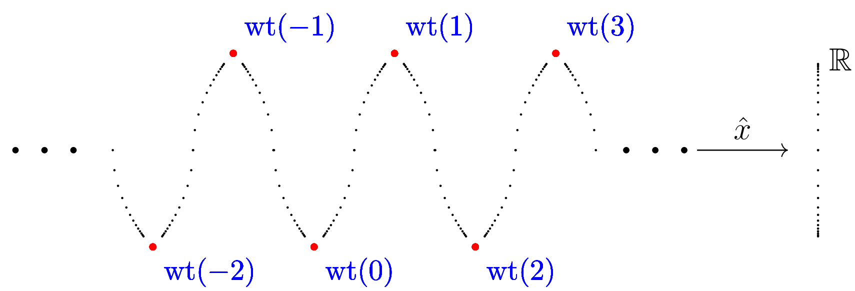


Figure 2.1: A weighted b-line with $I = \mathbb{Z}$.

To explore the difference between Hamiltonian and b-Hamiltonian actions and the corresponding moment maps, let us consider the following example:

1. A Hamiltonian S^1-action on the sphere by rotation around the vertical axis with $\mathbb{R}$ being the image of the moment map. (Fig. 2.2)

2. A b-Hamiltonian S^1-action on the torus by the same rotation with S^1 being the image of b-moment map. (Fig. 2.3)

As explained above, the moment map, in this case, contains a *log*-component that explodes at $h = 0$. To work with this object, one needs to introduce the notion of b-line and b-circle. This notion can be generalized using the language of b-Lie groups as introduced in [BKM22].

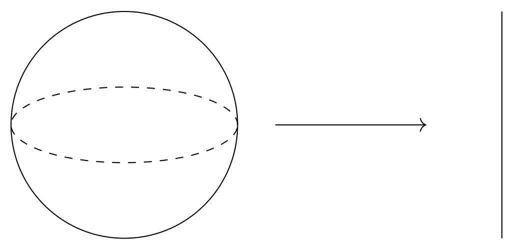


Figure 2.2: Moment map for circle action on S^2

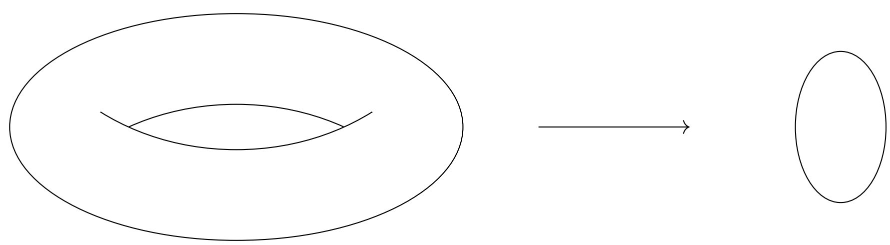


Figure 2.3: Moment map for circle action on T^2

Definition 2.20. *A b-manifold (G, H), where G is a Lie group and $H \subset G$ is a closed co-dimension one subgroup*[1] *is called a* b-**Lie group**.

For more details and examples of b-Lie groups, see [BKM22]. It is clear that the b-line and the b-circle are themselves examples of b-Lie groups, with the action given by translations and rotations, respectively.

In this article, we consider the case of general b^m-symplectic actions. It was proved in [BKM18a] that a group which acts transversally to the symplectic foliation inside Z decomposes as $G = S^1 \times H$ mod by a discrete group. By considering the restriction of ρ to the S^1-component $\rho|_{S^1}$, we obtain a torus action on the b^m-symplectic manifold.

We follow [GMW18a] and [GMW18b] where the notion of modular weights of a torus action is defined and studied.

In a neighborhood U of the critical set Z, $U = Z \times (-\epsilon, \epsilon)$, write ω as in equation 2.3.2:

[1]This is equivalent to H being an embedded Lie subgroup.

$$\omega = \sum_{j=1}^{m} \frac{df}{f^j} \wedge \pi^*(\alpha_j) + \beta.$$

Assume that there exists a moment map $\mu \in {}^{b^m}\mathcal{C}^\infty(M) \otimes \mathfrak{t}$ with

$$\langle d\mu, \xi \rangle = i_{\xi^M} \omega$$

for any $\xi \in \mathfrak{t}$ and where ξ^M stands for the fundamental vector field generated by ξ;

Definition 2.21. *The **modular weights** $a_1, \ldots, a_m \in \mathfrak{t}^*$ in each connected component of Z are given by*

$$a_j(\xi) = \alpha_j(\xi^M).$$

In [GMW18a] it is shown that these are constants.

Definition 2.22. *The modular weights of ρ are the modular weights of the induced S^1-action $\rho|_{S^1}$.*

2.4.3 quasi-Hamiltonian Spaces

Quasi-Hamiltonian spaces (confer [AMM98, Boa07, AMW02, HJS06]) provide a natural generalization of Hamiltonian spaces and understanding their properties can be revealing in terms of representation theory. As shown in [AMM98], The category of G-quasi-Hamiltonian spaces is equivalent to a subcategory of the category of infinite-dimensional symplectic manifolds with Hamiltonian actions of the loop group of G. Thus exploring this extension allows us gain understanding of infinite-dimensional analogues as in [DR20]. From now on, the following notation is used: θ^l and θ^r stand for the left- and right-invariant Maurer-Cartan forms, $(\cdot, \cdot)$ denotes a choice of an invariant positive definite inner product on $\mathfrak{g}$ and $\chi \in \Omega^3(G)$ is a canonical closed bi-invariant 3-form $\chi = \frac{1}{12}(\theta^l, [\theta^l, \theta^l]) = \frac{1}{12}(\theta^r, [\theta^r, \theta^r])$.

Definition 2.23. *A **quasi-Hamiltonian** G**-space** is a manifold M with a G-action ρ, an invariant 2-form σ and an equivariant **group-valued moment map** $\Phi : M \to G$ such that:*

(i) σ is equivariantly closed: $d\sigma = -\Phi^*\chi$,

(ii) the moment map condition is satisfied: $\iota(\upsilon_\xi)\sigma = \frac{1}{2}\Phi^*\left(\theta^l + \theta^r, \xi\right)$,

(iii) σ is weakly non-degenerate:

$$\ker \sigma_x \cap \ker d\Phi = 0.$$

Remark 2.1. *The manifold is not necessarily symplectic. For instance S^4 is an $SU(2)$-quasi-Hamiltonian space and the with moment map $\Phi : S^4 \to SU(2) \cong S^3$ the suspension of the Hopf fibration $S^3 \to S^2$ (see Appendix A in [AMW02]). More generally, the spin spheres S^{2n} admit a quasi-Hamiltonian structure (confer [HJS06]). Other classical non-symplectic examples are contained in the seminal article [AMM98]. For instance, $D(G)$ the double of a Lie group is not symplectic if the group is compact and simply connected (as its second cohomology group vanishes).*

Example 2.4. *Conjugacy classes of a Lie group G provide basic examples of quasi-Hamiltonian spaces. Let $\mathcal{C} \subset G$ be a conjugacy class with the conjugation action of G. Then $\mathcal{C} \subset G$ is a quasi-Hamiltonian space with moment map Φ given by the inclusion map into G. As observed in [AMW02] these include all compact symmetric spaces (up to finite covers).*

Remark 2.2. *If the Lie group G is abelian then these conditions imply that the two-form ω is automatically a symplectic form (see for instance [Boa07] and [HJS06]).*

Lemma 2.1 (Jeffrey [Jef94])**.**
For $s \in \mathbb{R}$ let $\exp_s : \mathfrak{g} \to G$ be defined by $\exp_s(\eta) = \exp(s\eta)$. The 2-form on

the Lie algebra $\mathfrak{g}$ *given by*

$$\varpi = \frac{1}{2}\int_0^1 (\exp_s^* \theta^r, \frac{\partial}{\partial s}\exp_s^* \theta^r)ds$$

is G*-invariant and satisfies* $d\varpi = -\exp^* \chi$*. If* υ_ξ *is a fundamental vector field for the adjoint* G*-action on* $\mathfrak{g}$ *we have*

$$\iota(\upsilon_\xi)\varpi = -d(\cdot, \xi) + \frac{1}{2}\exp^*(\theta^l + \theta^r, \xi).$$

Theorem 2.16 (Alekseev-Malkin-Meinrenken [AMM98])**.**
Let $(M, \mathcal{A}, \sigma, \varphi)$ *be a Hamiltonian* G*-space. Then* M *with* 2*-form* $\omega = \sigma + \varphi^*\varpi$ *and moment map* $\mu = \exp(\varphi)$ *satisfies all axioms of a quasi-Hamiltonian* G*-space except possibly the non-degeneracy condition, (iii). If the differential* $d_\xi \exp$ *is bijective for all* $\xi \in \varphi(M)$*, (iii) is satisfied as well and* $(M, \mathcal{A}, \omega, \mu)$ *is a quasi-Hamiltonian* G*-space.*

As shown in [CDM88] and [OR06] any symplectic non-Hamiltonian action can be seen as a quasi-Hamiltonian space.

2.5 Slice Theorems and Reduction

Reduction theory is crucial in geometry and physics. It reconciles the abstract concept of symmetry of a system with the practical implementation of changes of variables to simplify the system. The intuition that the number of degrees of freedom reduces under the existence of a group symmetry can be encoded as a reduction theorem. The first ones to observe this were probably Emmy Noether [Noe71] and Sofia Kovalevskaya who applied the idea of symmetry to actual mechanical systems (see, for instance, [Kow02]).
The idea can be taken to different levels of sophistication. When reduction is applied to symplectic geometry, an interesting phenomenon occurs: For a group of dimension k, the reduction can be doubled and the system can

be simplified by $2k$ degrees of freedom. This fact is known in the literature as Marsden-Weinstein reduction [MW74]. In symplectic geometry, the existence of symmetries is very special. Locally any symplectic manifold is a cotangent bundle, and the existence of symmetries on the base manifold M lifts certain actions (Hamiltonian) to the cotangent bundle. In this line of thought, the idea of symplectic reduction reduces by $2\dim G$ the number of degrees of freedom and produces an actual symplectic manifold of dimension $2n - 2\dim G$ when the action is free. For non-free actions, the structure of the reduced space is that of a stratified manifold (see [SL91]), and symplectic orbifolds are obtained for locally free actions (see [GGK02]).
The celebrated Marsden-Weinstein theorem [MW74] endows the reduced manifold determined by a fixed-energy level and its symmetries with a symplectic structure. Marsden-Weinstein quotients are closely related to several moduli spaces in geometry and, more concretely, to Geometric Invariant Theory. Frances Kirwan [Kir84b] related classical Geometric Invariant Theory to symplectic quotients. Symplectic quotients are naturally connected to certain moduli spaces. Michael Atiyah and Raoul Bott unveiled the symplectic structure on the space of flat connections in their celebrated article [AB83]. This was just the commencement of a brave new world [Hit79, BGPH19, BGP99] building bridges between the geometry and physics community.

2.5.1 Symplectic Slice Theorem

This section revisits different versions of slice theorems that are widely used in geometry. We will start with the general case of a slice theorem formulated by Palais in [Pal60, Pal61] for a compact Lie group action on an abstract smooth manifold and explain the notion of a slice. Then we will consider the Guillemin-Sternberg symplectic slice theorem formulated for Hamiltonian group actions on a symplectic manifold. This theorem gives a normal form theorem providing a semi-global description of the moment map using the slice representation. We finish this section with a short review of possible generalizations.

A general slice theorem for smooth actions

In this section, we explain the notion of slice and state the most basic slice theorem, first formulated by Palais [Pal60, Pal61] for the group action on a general manifold and symplectic group action by Marle in [Mar85].
Consider compact Lie group G and a smooth manifold W such that G acts on W by diffeomorphisms. For now, we do not put any restrictions on the geometric structure of W. It is well known that the orbits of G-action are submanifolds of W. The slice theorem allows us to describe the action in a tubular neighbourhood of an obit. We denote the orbit of point $x \in W$ by $\mathcal{O}_x = \{y \in W | y = g \cdot x \text{ for some } g \in G\}$ and its stabilizer by $G_x = \{g \in G | g \cdot x = x\}$.
A map $f_x : G \longrightarrow W$ such that $g \longmapsto g \cdot x$ is called an orbit map. The quotient vector space $V_x = T_xW/T_x\mathcal{O}_x$ is called slice in point x under the G-action. Slice theorem shows that the following diagram commutes

$$\begin{array}{ccc} G/G_x & \longrightarrow & G \times_{G_x} V_x \\ \downarrow f_x & & \downarrow \bar{f}_x \\ \mathcal{O}_x & \longrightarrow & W \end{array}$$

This can be stated as a theorem:

Theorem 2.17 (Slice theorem)**.** *There exists an equivariant diffeomorphism from an equivariant open neighborhood of the zero section in $G \times_{G_x} V_x$ to an open neighborhood of $\mathcal{O}_x$ in W, which sends the zero section G/G_x onto the orbit $\mathcal{O}_x$ by the natural map f_x.*

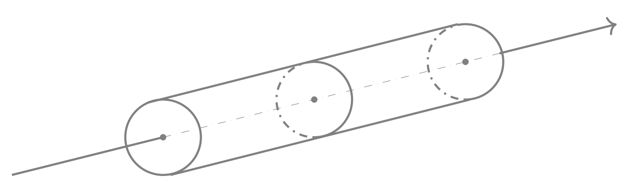


Figure 2.4: A slice of an orbit

A slice theorem for Hamiltonian actions

In this section, we deal with the symplectic (and at the same time Hamiltonian) slice theorem formulated by Guillemin-Sternberg in [GS84] and independently by Marle in [Mar85]. Even more interestingly for us, we recall the Guillemin-Sternberg local normal form theorem [GS90] that also gives a semi-global normal form for the moment map of the slice representation.
First, we start by reminding the definition of a Hamiltonian space.

Definition 2.24. *An action of a group G on a symplectic manifold (M, ω) is called* ***Hamiltonian*** *if it preserves the symplectic structure and admits an equivariant moment map $\mu : M \to \mathfrak{g}^*$ such that*

$$\iota(\upsilon_\xi)\omega = d\left\langle \phi, \xi \right\rangle, \forall \xi \in \mathfrak{g}.$$

Here $\langle , \rangle$ is natural pairing identifying $\mathfrak{g}$ and $\mathfrak{g}^*$ and υ_ξ is generating vector field on M.

Theorem 2.18 (Guillemin-Sternberg, Marle)**.** *Let (M, ω, G) be a symplectic manifold together with Hamiltonian group action. Let p be a point in M such that $\mathcal{O}_p$ is contained in the zero level set of the moment map. Denote G_p the stabilizer and $\mathcal{O}_p$ the orbit of p. There is a G-equivariant symplectomorphism from a neighbourhood of the zero section of the bundle $T^*G \times_{G_p} V_p$ equipped with a symplectic model to a neighbourhood of the orbit $\mathcal{O}_p$.*

The action of the transversal element of $T^*G \times_{G_p} V_p$ is given by a cotangent lift which we explain in details in Section 3.1.2.
Besides the symplectic slice theorem, we are also going to refer to the following theorem by Guillemin and Sternberg [GS90] that gives not only a semilocal description of the group action in the neighbourhood of an orbit but also a normal form for the corresponding moment map.

Theorem 2.19 (Guillemin-Sternberg)**.** *Let (M, ω, μ) be a Hamiltonian G-space. For any $p \in M$, let $H = Stab(p)$, let $K = Stab(\mu(p))$, and let V*

be the symplectic slice at p. We denote by $\mathfrak{h}$ the Lie algebra of H and by $\mathfrak{h}^0$ its annihilator. There exists a neighbourhood of the orbit $G \cdot p$ which is equivariantly diffeomorphic to a neighborhood of the orbit $G \cdot [e, 0, 0]$ in

$$Y := G \times_H ((\mathfrak{h}^0 \cap \mathfrak{k}^*) \times V).$$

In terms of this diffeomorphism, the moment map $\mu : M \to \mathfrak{g}^*$ may be written as

$$\mu([g, \gamma, v]) = Ad_g^*(\mu(p) + \gamma + \phi(v)),$$

where $\phi : V \to \mathfrak{h}^*$ is the moment map for the slice representation.

2.5.2 Marsden-Weinstein Reduction

Now we recall the statement of symplectic reduction proved by Marsden and Weinstein for free actions in [MW74] (check [GGK02] for the proof in the more general case of locally free actions that we include below).

Theorem 2.20 (Symplectic Reduction)**.** *Let (M, ω, μ) be a Hamiltonian G-space. Suppose that $\alpha \in (\mathfrak{g}^*)^G$ is a regular value for μ; or, more generally, that the level set $\mu^{-1}(\alpha)$ is a manifold and G acts on it (locally) freely. Then the topological space of orbits M_α of the level-set $\mu^{-1}(\alpha)$ is a manifold (orbifold resp.); and there exists a unique closed two-form ω_α on M_α such that $\pi^*\omega_\alpha = i^*\omega$, where $\pi : Z \to M_\alpha$ is the quotient map and $i : Z \to M$ is the inclusion map. The reduced form ω_α is non-degenerate on M_α if and only if the form ω is non-degenerate on M at the points of $\mu^{-1}(\alpha)$.*

Remark 2.3. *Discrete isotropy groups appear naturally in Hamiltonian Dynamics (check, for instance, the twisted models for hyperbolic singularities of integrable systems in [MZ04]). This obliges us to consider actions that are not free but locally free.*

2.5.3 b-Symplectic Slice Theorem

Theorem 2.21 (*b*-symplectic slice theorem, Braddell-Kiesenhofer-Miranda). *[BKM18b]*
Let $H \times S^1$ be a compact group acting on a b-symplectic manifold (M, ω) transverse to the symplectic foliation. Let $\mathcal{O}_z$ be an orbit of the group action contained in the critical set of M. Then there is a neighbourhood U of $\mathcal{O}_z \cong S^1 \times H/H_z$ which is b-symplectomorphic to a neighbourhood of the zero section of an associated bundle $T^*(S^1 \times H) \times_{(H_z \times \mathbb{Z}_d)} V_z$ equipped with the b-symplectic model

$$\omega = \frac{dt}{t} \wedge d\theta + \pi^*(\omega_H),$$

where t is a defining function for Z, π is the projection $\pi : T^*S^1 \times T^*H \times_{H_z} V_z \to T^*H \times_{H_z} V_z$ and ω_H is the symplectic form on $T^*H \times_{H_z} V_z$ given by the symplectic slice theorem.

2.5.4 Quasi-Hamiltonian Local Normal Form Theorem and Reduction

Theorem 2.22 (quasi-Hamiltonian Reduction).
Let M be a quasi-Hamiltonian $G_1 \times G_2$-space and let $f \in G_1$ be a regular value of the moment map $\mu_1 : M \to G_1$. Then the pull-back of the 2-form ω to $\mu_1^{-1}(f)$ descends to the reduced space

$$M_f = \mu_1^{-1}(f)/Z_f$$

and makes it into a quasi-Hamiltonian G_2-space. In particular, if $G_2 = \{e\}$ is trivial, then M_f is a symplectic orbifold.

Theorem 2.23 (quasi-Hamiltonian Slice Theorem [BTW04]). *Let $(M, \mathcal{A}, \omega, \mu)$ be a quasi-Hamiltonian G-space. For any $p \in M$, let $H = Stab(p)$, $K = Stab(\mu(p))$, and V be the symplectic slice at p. There exists a neighbourhood of the orbit $G \cdot p$ which is equivariantly diffeomorphic to a neighborhood of* *the orbit* $G \cdot [e, 0, 0]$ *in*

$$Y := G \times_H ((\mathfrak{f}^\perp \cap \mathfrak{k}) \times V).$$

In terms of this diffeomorphism, the G*-valued moment map* $\mu : M \to G$ *may be written as*

$$\mu([g, \gamma, v]) = \mathrm{Ad}_g(\mu(p)\exp(\gamma + \varphi(v))),$$

where $\varphi : V \to h^* \simeq h$ *is the moment map for the slice representation.*

2.6 Riemann-Hilbert Problem, Flat Connections and Character Varieties

2.6.1 Refined Riemann-Hilbert Problem and Flat Connections

Let $\Sigma_{0,n} = \mathbb{P}^1 \setminus \{z_1 \dots z_n\}$, where z_i are marked points on the Riemann sphere (z_i might be equal to ∞). Then the fundamental group $\pi_1(\Sigma_{0,n})$ has n generators $\gamma_1, \dots \gamma_n$ subjected one relation $\gamma_1 \circ \dots \circ \gamma_n = 1$. The representation $\chi : \pi_1(\Sigma_{0,n}) \to SL_2(\mathbb{C})$ are specified by collection of matrices $M_k := \chi(\gamma_k) \in SL_2(\mathbb{C})$ satisfying $M_n \circ \dots \circ M_1 = \mathbb{I}_2$ up to overall conjugation by $SL_2(\mathbb{C})$, $1 \le k \le n$. This representation is often called the monodromy representation.

We will be interested in the case when M_k are diagonal with fixed eigenvalues $e^{\pm 2\pi i \lambda_k}$. We denote by Rep $\pi_1(\Sigma_{0,\mathrm{n}})$ the quotient $\mathrm{Hom}(\pi_1(\Sigma_{0,\mathrm{n}}), \mathrm{SL}_2\mathbb{C})/\mathrm{SL}_2(\mathbb{C})$. It is well-known that dimension of Rep $\pi_1(\Sigma_{0,\mathrm{n}})$ equals to $2(n-3)$.

Let us choose the point z_0 (different from $z_1, \dots, z_n$) as a base point in $\mathbb{P}_1$.

A classical Riemann-Hilbert problem concern to a searching of a multi-valued analytic matrix function $Y(z)$ such that the monodromy M_k along γ_k is represented by analytic continuation $Y(\gamma_k, z) = Y(z)M_k$. A solution of the problem is unique up to a left multiplication by a single-valued matrix function. To fix this ambiguity we will consider the *refined* Riemann-Hilbert problem which means that we want to find a matrix-valued function $Y(z)$ such that

- $Y(z_0) = \mathbb{I}_2$,
- $Y(z)$ is multi-valued, analytic and invertible on $\Sigma_{0,n}$,
- there exists (for each k) a neighbourhood of z_k such that

  $$Y(z) = \hat{Y}_k(z - z_k)^{M_k},$$

  where $M_k = e^{2\pi i \mu_k}$, $\begin{pmatrix} \mu_k & 0 \\ 0 & -\mu_k \end{pmatrix} \in sl_2(\mathbb{C})$. $\hat{Y}_k$ is holomorphic and invertible at $z = z_k$.

If such $Y(z)$ exists, it is uniquely defined by the monodromy data μ_k, $1 \leq k \leq n$. Such refined Riemann-Hilbert problem arises naturally in the study of rank r flat connections on the Riemann sphere with n marked points $\Sigma_{0,n}$. Any flat connection on $\Sigma_{0,n}$ is gauge equivalent to a holomorphic connection $\partial_z - A(z)$, where $A(z) = \sum\limits_1^n \frac{a_k}{z-z_k}$, $A_k \in sl_2(\mathbb{C})$ and $\sum\limits_1^n A_k = 0$.
Let $Y(z)$ be a fundamental matrix solution $\frac{\partial}{\partial z}Y(z) - A(z)Y(z)$ normalizing $Y(z_0) = \mathbb{I}_2$. It automatically satisfies conditions 2 and 3 for certain monodromy data $\{\mu_k\}$, $1 \leq k \leq n$ provided by the eigenvalues eigenvalues $\pm\lambda_k$ of A_k satisfying the condition $2\lambda_k \notin \mathbb{Z}$.
Any representation of $\pi_1(\Sigma_{0,n})$ of $SL_2(\mathbb{C})$ can be realized as a monodromy representation of such a Fuchsian system, which means that solution of Riemann-Hilbert problem generically exists.
The Riemann-Hilbert correspondence between flat connections

$$\partial_z Y = A(z)Y$$

and

$$\chi : \pi_1\Sigma_{0,n} \to SL_2(\mathbb{C})$$

allows us to identify the moduli space of SL_2-flat connections on the Riemann sphere with n marked points with $\mathrm{Rep}\pi_1(\Sigma_{0,n}) = \mathrm{Hom}\pi_1(\Sigma_{0,n}, \mathrm{SL}_2(\mathbb{C}))/\mathrm{SL}_2(\mathbb{C})$. We will call such varieties $SL_2(\mathbb{C})$-character variety of $\pi_1(\Sigma_{0,n})$.

2.6.2 Case of $n = 3$: Hypergeometric

In what follows, we will be interested in Poisson structures on *decorated* character varieties, i.e. on some specific generalisation of above considered group representation moduli spaces. We have seen that a usual notion of character varieties arised as the representation spaces of punctured Riemann surface fundamental groups. Such Riemann surfaces play a significant role in the theory of complex differential equations; namely, in studies of singular solution properties. This topic has been actively studied since 19th century in works of Gauss, Fuchs, Painlevé [Pai02], their students and their followers in the context of attempts to classify equations satisfying certain conditions on the type of singularities and their asymptotic behavior.

The most widely known condition of this type is the famous Painlevé property which requires the differential equations to have simple poles as the only movable singularities. For linear matrix systems associated with 2nd order ordinary differential equations satisfying Painlevé property, the notion of character varieties arises very naturally.

Let us consider the simplest example [Gol84] of second order regular linear differential equation, where trivial character variety appears:

$$\frac{d^2 f}{dz^2} + p(z)\frac{df}{dz} + q(z)f = 0,$$

where p and q are meromorphic functions with poles of order one and two respectively in three points on Riemann sphere. Typical example of such an equation is the Euler's hypergeometric equation:

$$-z(z-1)\frac{d^2 f}{dz^2} - ((a+b+1)z - c)\frac{df}{dz} - abf = 0, \qquad a, b, c \in \mathbb{C}$$

In this case solution f is equal to Gauss hypergeometric function

$$F(a,b,c,z) = \int \frac{\lambda^{a-c}(1-\lambda)^{c-b-1}}{(z-\lambda)^a} d\lambda.$$

Under some (not very restrictive) conditions this equation is equivalent to a system

$$\frac{\partial Y(z)}{\partial z} = \left(\frac{A_0(z)}{z} + \frac{A_1(z)}{z-1} \right) Y(z).$$

In the point ∞ this system also has singularity with residue $A_\infty = -(A_0 + A_1)$ with $A_i \in \mathfrak{gl}_2(\mathbb{C})$. In case of $A_0 = A_1 = 0$, Y is meromorphic on $\bar{\mathbb{C}}$ with $\frac{\partial Y}{\partial z} = 0$ and has poles in points $0, 1$ and ∞. Hence, the local solution will be $Y = \mathit{const}$ but in general globally $Y(z)$ is multivalued and has monodromies corresponding to continuation of local solutions along the loops encircling singular points. This monodromy matrices naturally define representation of fundamental group of Riemann sphere with 3 punctures (the "pants"):

$$\chi : \pi_1(\Sigma_{0,3} z_0) \longrightarrow \mathrm{GL}_2(\mathbb{C}).$$

In general setting representation χ is rigid (i.e. it can be uniquely reconstructed by eigenvalues of monodromy matrices) which means that $GL_2(\mathbb{C})$-representation variety of the pants ($\Sigma_{0,3}$ - sphere with three punctures) is just one point. We will see that the situation with decorated character varieties is much more complicated. More over, in some of the definitions, we will have to replace the the very fundamental group notion by a notion of a fundamental groupoid.

2.7 Fundamental groupoid

2.7.1 Teichmuller spaces

Definition 2.25. *The Teichmuller space $\mathcal{T}$ of some Riemann surface Σ is the space of all conformal classes of metrics on Σ up to diffeomorhisms con-*

tinuously connected to the identyty.

The real slice of the $SL_2(\mathbb{C})$-character is Teichmuller space, shear coordinates on which after complexification give us coordinates on the character variety. Confluence procedure on the punctures Riemann surface decorating character variety correspond to certain asymptotics in the shear coordinates.

Poisson structure on the Teichmuller space can be described in different terms and the one most appropriate for our needs, was suggested by Goldman.

2.7.2 The Goldman bracket

In 1986, W. Goldman had constructed [Gol84] Poisson structure on character varieties using parametrization by traces of the monodromy matrices.

For the Riemann sphere with 4 punctures $C_{0,4}$, the character variety can be described the an affine cubic surface in $\mathbb{C}^3$, given by the relation

$$\begin{aligned} W(p_{01}, p_{1t}, p_{t0}; p_0, p_1, p_t, p_\infty) = p_{01}p_{1t}p_{t1} - [p_{01}^2 + p_{1t}^2 + p_{t1}^2 + \\ (p_0p_1 + p_tp_\infty)p_{01} + (p_0p_t + p_1p_\infty)p_{t0} + (p_0p_\infty + p_1p_t)p_{1t} \\ + p_0^2 + p_1^2 + p_t^2 + p_\infty^2 + p_0p_1p_tp_\infty - 4] = 0, \end{aligned} \tag{2.7.1}$$

where $p_{ij} = \operatorname{Tr} M_iM_j$ and $p_i = \operatorname{Tr} M_i$.

This parametrization is known since XIX century and the cubic 2.7.1 is sometimes reffered as the Fricke-Klein cubic.

The Goldman Poisson bracket in this trace coordinates has the following Jacobian form:

$$\{p_i, p_j\} = 0, \qquad \{p_{12}, p_j\} = \{p_{31}, p_j\} = \{p_{23}, p_j\} = 0,$$

and

$$\{p_{12}, p_{31}\} = p_{12}p_{31} - 2p_{23} - (p_1p_4 + p_2p_3) = \frac{\partial W}{\partial p_{23}}$$
$$\{p_{31}, p_{23}\} = p_{31}p_{23} - 2p_{12} - (p_1p_2 + p_3p_4) = \frac{\partial W}{\partial p_{12}}$$
$$\{p_{23}, p_{12}\} = p_{12}p_{23} - 2p_{31} - (p_1p_3 + p_2p_4) = \frac{\partial W}{\partial p_{31}}.$$

2.7.3 Generalized Goldman bracket

Definition 2.26. *For Riemann surface $\Sigma_{g,s}$ of genus g with s punctures*

$$\mathrm{Hom}\left(\pi_1(\Sigma_{g,s}) \to SL_2(\mathbb{C})\right)/_{SL_2(\mathbb{C})}$$

is called character variety.

If we want to generalize this notion for the case of Riemann surfaces with cusps, instead of $\Sigma_{g,s}$ we should consider $\Sigma_{g,s,n}$, where n is number of bordered cusps.
We should also replace $\pi_1(\Sigma_{g,s})$ by *the fundamental groupoid of arcs:*

$$\pi_{\mathfrak{a}}(\Sigma_{g,s,n}) := \{\mathfrak{a}_{ij} : [0,1] \to \Sigma_{g,s,n} | \, \mathfrak{a}_{ij}(0) = c_i, \; \mathfrak{a}_{ij}(1) = c_j\}/_{\text{homotopy}}.$$

Definition 2.27. *Decorated character variety:*

$$\mathrm{Hom}\left(\pi_{\mathfrak{a}}(\Sigma_{g,s,n}), SL_2(\mathbb{C})\right)/_{\prod_{j=1}^{n} U_j}$$

Chekhov and Mazzocco in their paper [CM18] have explicitly computed Goldman bracket in the extended shear coordinates.

2.7.4 Painlevé equations and linear differential systems

One of the main examples of systems of type (7.0.1) and their non-Fuchsian generalizations are linear differential systems associated to the six famous

Painlevé transcendents which are second order differential equations whose only moveable singularities are ordinary poles. We remind here this setting.

List of Painlevé equations

$$(PVI)\quad \begin{aligned}\frac{d^2y}{dt^2} = \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}+\\ +\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right)\end{aligned}$$

$$(PV)\quad \begin{aligned}\frac{d^2y}{dt^2} = \left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt}+\\ +\frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)+\gamma\frac{y}{t}+\delta\frac{y(y+1)}{y-1}\end{aligned}$$

$$(PIV)\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2 + \frac{3}{2}y^3 + 4ty^2 + 2(t^2-\alpha)y + \frac{\beta}{y}$$

$$(PIII)\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2 - \frac{1}{t}\frac{dy}{dt} + \frac{y^2}{4t^2}\left(\alpha+\frac{\beta t}{y^2}+\gamma y+\frac{\delta t^2}{4y^3}\right)$$

$$(PII)\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha$$

$$(PI)\frac{d^2y}{dt^2} = 6y^2 + t,$$

where $\alpha, \beta, \gamma, \delta$ are complex constants. This list of equations can be extracted from classical work of Painlevé [Pai02].

Painlevé equations as compatibility conditions

As it was briefly explained in the Introduction, each Painlevé equation can be considered as a compatibility condition to some system of linear differential equations. Particularly, we are interested in such systems for P_{VI} and P_{V} equations.

Painlevé VI equation can be considered as a compatibility condition for the following system

$$\begin{cases} \frac{\partial Y_6}{\partial \lambda_6} = \left(\frac{A_{06}(t_6)}{\lambda_6} + \frac{A_{t6}(t_6)}{\lambda_6 - t_6} + \frac{A_{16}(t_6)}{\lambda_6 - 1} \right) Y_6 \\ \frac{\partial Y_6}{\partial t_6} = \left(\frac{-A_{t6}(t_6)}{\lambda_6 - t_6} \right) Y_6 \end{cases},$$

with

$$A_{k6}(t_6) = \begin{pmatrix} \frac{\Theta_{k6}}{2} + z_{k6} & -u_{k6} z_{k6} \\ \frac{\Theta_{k6} + z_{k6}}{u_{k6}} & -\frac{\Theta_{k6}}{2} - z_{k6} \end{pmatrix}, \qquad k = 0, 1, t,$$

where u_{k6} and z_{k6} depend on t, and Θ-s are parameters.
This system is of Fuchsian type as the only singularities it has are 4 poles of order 1 on Riemann sphere in points $0, 1, t$ and ∞. As it was mentioned in the example above, monodromy representation of such a system corresponds to the representation of fundamental group of sphere with 4 punctures.

$$\chi : \pi_1(\bar{\mathbb{C}} \setminus \{a_1, a_2, a_3, a_4\}, z_0) \longrightarrow \mathrm{GL}_2(\bar{\mathbb{C}})$$

Definition 2.28. *Set of matrices: $\{M_{06}, M_{16}, M_{t6}, M_{\infty 6}, \}$ is called monodromy data of P_{VI} equation.*

Further, instead of usual monodromy data it will be more convenient for us to use equivalent set of transition matrices $\{C_6^{0\infty}, C_6^{t\infty}, C_6^{1\infty}\}$, defined above through equation 7.0.2.
Percise relation between monodromy matrices M_{i6} and transition matrices C_6^{kj} in the following way:

$$M_{k6} = C_6^{\infty k} e^{i\pi\Theta_{k6}\sigma_3} C_6^{k\infty}, \text{ for } k = 0, t, 1 \text{ and } M_{\infty 6} = e^{i\pi\Theta_{\infty 6}\sigma_3}.$$

Each monodromy matrix is exponent of Pauli matrix σ_3 conjugated by transition matrix of the path between corresponding singularity and infinity point.

Painleve V equation can be also considered as a compatibility condition of the following system:

$$\begin{cases} \frac{\partial Y_5}{\partial \lambda_5} = \left(\frac{\sigma_3}{2} + \frac{A_{05}(t_5)}{\lambda_5} + \frac{A_{t5}(t_5)}{\lambda_5 - t_5}\right) Y_5 \\ \frac{\partial Y_5}{\partial t_5} = \left(\frac{-A_{t5}(t_5)}{\lambda_5 - t_5}\right) Y_5 \end{cases}, \tag{2.7.2}$$

where

$$A_{05}(t_5) = \begin{pmatrix} z_5 + \frac{\Theta_0 6}{2} & -u_5(z_5 + \Theta_{05}) \\ \frac{z_5}{u_5} & -z_5 - \frac{\Theta_0 5}{2} \end{pmatrix},$$

$$A_{t5}(t_5) = \begin{pmatrix} -z_5 - \frac{\Theta_{05} + \Theta_{\infty 5}}{2} & u_5 y_5 (z_5 + \frac{\Theta_{05} + \Theta_{t5} + \Theta_{\infty 5}}{2}) \\ -(z_5 + \frac{\Theta_{05} + \Theta_{t5} + \Theta_{\infty 5}}{2}) u_5^{-1} y_5^{-1} & z_5 + \frac{\Theta_{05} + \Theta_{\infty 5}}{2} \end{pmatrix},$$

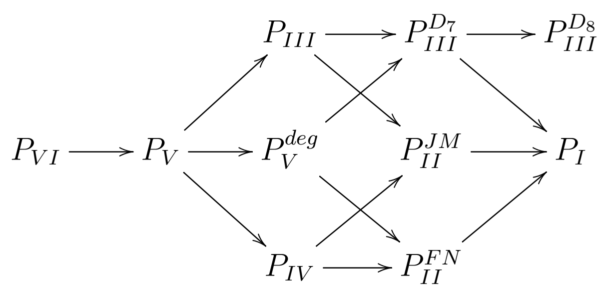
where u_5, y_5 and z_5 depend on t_5 and Θ_{05}, Θ_{t5} and $\Theta_{\infty 5}$ are parameters.

Unlike the system corresponding to the Sixth Painlevé equation, this system is not of the Fuchsian type: it has two poles of order 1 and one pole of order 2 at infinity. In the neighborhood of ∞ due to irregularity of singularity, Stokes phenomena arises. Monodromy data in this case will consist not only of monodromy matrices $M_{05}, M_{t5}, M_{\infty 5}$ but will contain also Stokes data S_{-1}, S_0.

In the monodromy representation of (2.7.2), Stokes matrices will not correspond to any closed path on the punctured sphere so instead of considering fundamental group of punctured sphere we should pass to the representation of fundamental groupoid of arcs triangulating Riemann sphere with two punctures and two cusps.

Confluence of singularities

There is a well known confluence scheme for the Painlevé equations which goes back to Sakai [Sak01]:



in which we confine our attention to the confluence of singularities in linear system associated to P_{VI} leading to the system associated to P_V.

As it was already mentioned, axillary system of P_V can be regarded as a result of confluence procedure of the correponding system associated to P_{VI} equation when singular point 1 tends to ∞. As after the confluence there arises Stokes phenomena, in the neighborhood of ∞ there are two Stokes rays in two sectors respectively corresponding to two different asymptotic of local solutions. Geometrically this corresponds to appearence of two cusps on the border of confluenced singularity.

Relation between confluence and Poisson structure for flat connections in any rank and any multiplicity of singularities is studied in detail in [GMR21].

2.8 Poisson structure on character variety associated to P_V

Even for slightly more complicated case of second order differential equation with 4 simple poles character variety becomes interesting and nontrivial. Linear 2×2 system, corresponding to Painlevé VI equations has simple poles in points $0, 1, t$, and ∞

$$\left(\frac{d}{dz} + \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}\right) Y(z) = 0,$$

with all traces of $A_i = 0$ and corresponding monodromy data $M_1, M_2, M_3, M_4 \in SL_2(\mathbb{C})$, satisfying $M_1 M_2 M_3 M_4 = 1$.
As noted above, the character variety corresponding to P_{VI} is the Fricke-Klein cubic 2.7.1 $W(p_{01}, p_{1t}, p_{t0}; p_0, p_1, p_t, p_\infty) = 0$ in $\mathbb{C}^3$, on which traces of monodromies give coordinates $p_{ij} = \operatorname{Tr} M_i M_j$ and $p_i = \operatorname{Tr} M_i$. For P_{VI}, the character variety equation can be written as follows [Hik19]:

$$\begin{aligned} W(p_{01}, p_{1t}, p_{t0}; p_0, p_1, p_t, p_\infty) = p_{01} p_{1t} p_{t1} - [p_{01}^2 + p_{1t}^2 + p_{t1}^2 + \\ (p_0 p_1 + p_t p_\infty) p_{01} + (p_0 p_t + p_1 p_\infty) p_{t0} + (p_0 p_\infty + p_1 p_t) p_{1t} \\ + p_0^2 + p_1^2 + p_t^2 + p_\infty^2 + p_0 p_1 p_t p_\infty - 4] = 0, \end{aligned}$$

Poisson structures arising in this case for monodromy data of this system were studied in works [KS97] [CM18].
Unfortunately, for other Painlevé equations and their related linear systems we can not match character varieties in a similar way because they hfave irregular singularities and the Stokes phenomenon blocks the above monodromy representation constructions. The Stokes phenomenon arises in the neighborhood of irregular singularities of these systems. In algebraic sense it means that (in addition to formal monodromies) the system now has corresponding Stokes matrices as part of its (sectorial) monodromy data. Geometrically this system corresponds not just to a punctured Riemann surface but to a "decorated" punctured Riemann surface with cusps (each cusp correspond to a Stokes matrix). Monodromy representation now is no more a representation of the fundamental group of this surface because the Stokes matrices correspond to non-closed paths ("arcs"). Instead of the fundamental group in this case one may consider a fundamental groupoid of arcs triangulating the surface. Analogously to regular case the decorated character variety [CMR17] is defined as a representation space of this fudamental groupoid of bordered

cusped Riemann surfaces.

Chapter 3

A b^m-slice theorem

The next two chapters are devoted to extending the concept of Marsden-Weinstein symplectic reduction to include symplectic manifolds with singular structures and extend the admissible Hamiltonian functions beyond smooth functions. We do this for a class of Poisson manifolds that have been recently closely examined: including b-symplectic or log-symplectic (and b^m-symplectic) manifolds and *certain folded symplectic manifolds*. Several authors considered group actions on these manifolds (see, for instance, [GMPS15,GLPR17,GMW18a,KM16,KMS16,KM17,BKM18a,BKM22,GMPS17]). In [GMPS15, GMW18a, GSW00], Delzant type polytopes were investigated for toric actions on manifolds endowed with symplectic structures with singularities (of b or folded type). Toric symmetries have also been used in the study of formal geometric quantization [GMW18b, GMW21] where the set of Hamiltonian functions extends to b^m-functions. However, a typical picture where the reduced manifolds are analyzed is missing in the literature.

Our motivating example is a moduli space of flat connections on a symplectic surface. It is possible to associate a geometrical template of identified polygons to such a problem (this is classical; see, for instance, the clear exposition in [Mic13]). We start with a symplectic template and use the desingularization technique of [GMW19] to obtain a singular toy model (of b^m-type for even m) by an *ad-hoc* construction from a symplectic template. This moduli space which is symplectic can then be also seen as a reduction obtained from a singular model. This toy example inspires us to extend the identification between moduli space and symplectic reduction to the singular realm and formally define the symplectic reduction for arbitrary Lie groups for b^m-symplectic manifold. Other motivating examples come from Yang-Mills fields theories on manifolds with boundary (see [MMN22]).

In order to define the Marsden-Weinstein reduction in the singular realm, we first need to refine the slice theorem for group actions to consider these singularities in the underlying geometrical structure. As the slice theorem gives a normal form for the geometrical structure, it yields a proper structure and group action on the set of orbits induced on the pre-image of a regular point by the moment map. In particular, this defines a reduced space which is symplectic whenever the highest modular weight of the transverse S^1-action is non-vanishing. So, in this case, the reduction procedure eliminates the singularity from the original symplectic structure.

The philosophical approach to the reduction theory in that article is that of simplifying not only the symmetries of the system but also the singularities of the symplectic structure, as we prove when the highest modular weight is non-vanishing.

Other approaches to the removal, blow-up or desingularization of singularities in this theory have been developed by Guillemin-Miranda-Weitsman in [GMW19]. This desingularization technique in [GMW19] will be a close ally in our endeavour as it puts the reduction and the slice theorem for these different singular symplectic manifolds on equal footing. In particular, it allows us to extend the notion of *reduction by stages* to the new category of singular symplectic manifolds and more general Hamiltonian functions.

The idea of reduction also prevails outside the symplectic realm. Quasi-Hamiltonian spaces (confer [AMM98, Boa07, AMW02, HJS06]) provide a natural generalization of Hamiltonian spaces and understanding their properties can be revealing in terms of representation theory. As shown in [AMM98], The category of G-quasi-Hamiltonian spaces is equivalent to a subcategory of the category of infinite-dimensional symplectic manifolds with Hamiltonian actions of the loop group of G. Thus exploring this extension allows us gain understanding of infinite-dimensional analogues as in [DR20]. By the same token, we consider quasi-Hamiltonian actions and reduction as a natural completion of the picture, specially guided by our motivating example. In doing so, we also extend the reduction scheme "by stages" to the singular quasi-Hamiltonian realm. We obtain new examples of quasi-Hamiltonian spaces by combining classical quasi-Hamiltonian constructions with techniques native to b^m-Hamiltonian spaces using the fusion product. These structures can be generalized further to E-quasi-Hamiltonian spaces.

The reduction removes the singularity from the symplectic structure. So, as a motto *the reduction entails a desingularization*. In section 6, we also prove that this reduction procedure commutes with the desingularization procedure in [GMW19] and observe that reduction can be done by stages. In the last section, we discuss several generalizations of these ideas to more general notions of moment map on b^m-manifolds which are not necessarily symplectic and focus on the quasi-Hamiltonian reduction in the singular framework. The mnemonics and removal of the singularity works in the singular quasi-Hamiltonian case as the b^m-Hamiltonian case. Reduction with the appropriate group gets rid of the singularity of the form. Our constructions via the fusion product provide brand-new examples of non-trivial singular quasi-Hamiltonian spaces.

3.1 A b^m-symplectic slice theorem

In this section we prove the b^m-symplectic version of the slice theorem 3.2.

Let us recall basic notions concerning slice theorems in the smooth and symplectic categories.

3.1.1 Group actions on b^m-symplectic manifolds

We start by reminding the main definitions and statements from b^m-symplectic geometry, including the generalization of b^m-cotangent lift, which will be required for the proof of theorem 3.2.

As proved in [MP18], the equivariant b^m-Moser theorem in the case of surfaces lets us visualize the b^m-symplectic manifold $S^1 \times (-\epsilon, \epsilon)$ as a neighbourhood of the zero section of the cotangent bundle $T^*S^1 \cong S^1 \times \mathbb{R}$ with b^m-symplectic form given by the formula

$$\omega_c := c d\theta \wedge \frac{dt}{t}.$$

This serves as one of the building blocks in our b^m-symplectic model for group actions. We start by revising the cotangent model for b^m-symplectic actions.

3.1.2 The b^m-symplectic cotangent lift

The cotangent lift is one of the essential tools in symplectic geometry and the theory of integrable systems. It allows lifting group actions from a manifold to automatically Hamiltonian actions on the cotangent bundles. This leads to many examples of integrable systems. We will consider a generalization of cotangent lift that lifts actions on a b^m-manifold to b^m-Hamiltonian actions on the b^m-cotangent bundle.

Given an action ρ of a Lie group G on a b-manifold (M, Z), one can lift it to the b^m-Hamiltonian action $\hat{\rho}$ of G on the b^m-cotangent bundle ${}^{b^m}T^*M$. The lifted action $\hat{\rho}$ is given by $\hat{\rho}_g := \rho_{g^{-1}}$ and π is a canonical projection from ${}^{b^m}T^*M$ to M. The following diagram commutes:

$$\begin{array}{ccc} {}^{(b^m)}T^*M & \xrightarrow{\hat{\rho}_g} & {}^{(b^m)}T^*M \\ \downarrow \pi & & \downarrow \pi \\ M & \xrightarrow{\rho_g} & M \end{array}$$

Having fixed the action $S^1 \times H \curvearrowright^{b^m} T^*\mathbb{S}^1 \times^{b^m} T^*H$ we consider the coordinates $(a, \theta, x_1, \ldots, x_n, y_1, \ldots, y_n)$ with $\theta \in S^1, \{x_i\} \in H$ and $a, \{y_i\} \in \mathbb{R}$. Here H is itself an $(n-1)$-dimensional manifold and T^*H is equipped with standard Liouville one-form λ_H.

$$L = \sum_{1}^{m-1} c_{i+1} \frac{d\theta}{t^i} + c_1 \log t d\theta + \sum_{1}^{n-1} y_j dx_j$$

For T^*S^1 we consider the twisted b^m-cotangent lift with the moment map:

$$\mu_S = c_1 \log |a| + \sum_{i=1}^{m-1} c_{i+1} \frac{a^{-i}}{i}$$

and the twisted form

$$\tilde{\omega_S} = \sum_{1}^{m} \frac{\tilde{c_i}}{t_1^i} d\theta \wedge dt.$$

Notice that the new constants $\{\tilde{c_i}\}$ equal the product of the former constants c_i with the modular weight of the connected component of Z. However, for ease of notation we often omit this distinction.

The action of $S^1 \times H$ on its cotangent bundle is Hamiltonian with the moment map, given by contraction of λ with the fundamental vector field:

$$\langle \mu(p), X \rangle := \left\langle L_p, X^{\#}|_p \right\rangle$$

We should prove that the Liouville form is invariant under this action. L splits in two: λ_H and λ. One has two show that λ_H is invariant under S^1-action, and for λ, we already have it proven from the standard symplectic cotangent lift.

The moment map then is given by

$$\mu = c_1 \log |a| + \sum_{i=1}^{m-1} c_{i+1} \frac{a^{-i}}{i} + \mu_0(x, y)$$

with the twisted form $\tilde{\omega}$ defined as follows

$$\tilde{\omega} = \sum_{1}^{m} \frac{\tilde{c}_i}{t_1^i} d\theta_1 \wedge dt_1 + \sum_{2}^{n} dx_j \wedge dy_j.$$

3.1.3 The Slice theorem

Before proceeding to the proof of the b^m-symplectic slice theorem, we need to prove some preliminary material.
We start proving the following lemma in the context of b^m-symplectic manifolds (the result for b-symplectic manifolds can be found in Section 5 in [GMW18b]). Here we provide the proof for general b^m-symplectic manifolds following mutatis mutandis [GMW21] and also consider its equivariant version, which will be needed in the proof.
As we will see in the proof of the b^m-symplectic slice theorem, it will be sufficient to consider the case in which the critical set Z is a trivial mapping torus. So each connected component Z_i of the critical set $Z_i = S^1 \times L$ where L denotes a symplectic leaf of the cosymplectic manifold Z.

Lemma 3.1. *Let (M, Z, ω) be a b^m-symplectic manifold endowed with an S^1-action with non-vanishing highest modular weight. The b^m-symplectic form on $Z \times (-\epsilon, \epsilon)$ can be taken to be the two-form*

$$\omega = -d\theta \wedge (\sum_{i=1}^{m} c_i \frac{dt}{t^i}) + \gamma_L \tag{3.1.1}$$

where γ_L is the symplectic form on the symplectic leaf L. In case there is an action of a group G by b^m-symplectic diffeomorphism, this decomposition can be achieved equivariantly. So, we may assume that the form γ_L is G-invariant.

Proof. By using the Laurent decomposition given by equation 2.3.2 of the b^m-form and denoting by θ the angular coordinate on S^1, we can assume that the b^m-symplectic form is written as,

$$-d\theta \wedge (\sum_{i=1}^{m} c_i \frac{dt}{t^i}) + \gamma_L + d\theta \wedge \beta$$

with γ_L, the symplectic form on the symplectic leaf L (so, a priori, possibly depending on t) and β, a one-form depending on all the coordinates. Denote by g the function $g = \iota_{\frac{\partial}{\partial \theta}} \beta$.
Now replace β by a new β equal to $\beta - g\ d\theta$ in such a way that $\iota_{\frac{\partial}{\partial \theta}} \beta = 0$.
On the other hand, as the action of S^1 is b^m-Hamiltonian. There exist a smooth function $h \in C^\infty(M)$ such that

$$\iota_{\frac{\partial}{\partial \theta}} \omega = d(-\sum_{i=2}^{m} \left(c_i \frac{1}{it^{i-1}} \right) - c_1 \log |t| + h),$$

and hence plugging on the equation above, proves that the one form β is indeed exact.

$$\beta = dh$$

Now we are going to apply Moser's trick. For that, we take the one-parameter family of forms for $0 \leq s \leq 1$:

$$\omega_s = -d\theta \wedge (\sum_{i=1}^{m} c_i \frac{dt}{t^i}) + \gamma_L + sd(hd\theta). \tag{3.1.2}$$

For $s = 1$ this form is ω, and for $s = 0$ the simplified form (3.1.1). In order to apply Moser's trick, we need to check that ω_s is a path of b^m-symplectic forms. Observe that for small ϵ on an ϵ-neighbourhood, the first term of (3.1.2) is much larger than the third. So we conclude that the form (3.1.2) is b^m-symplectic and for all s. By construction, their class in b^m-cohomology coincides:

$$[\omega_s] = [\omega_0].$$

We are now ready to apply the b^m-Moser theorem (Theorem 2.11) to conclude that ω_0 and ω_1 are equivariantly b^m-symplectomorphic.
A final remark: As done in Section 5 in [GMW18b], the 2-form, γ_L, which restricts to the symplectic form along L depends in principle on t.
However, the inclusion map

$$i : L \to L \times (-\epsilon, \epsilon),\ \ p \to (p, 0)$$

and the projection map

$$\pi : L \times (-\epsilon, \epsilon) \to L, (p, e) \to p$$

induce isomorphisms on cohomology. In other words, $[\gamma_L] = [\pi^* i^* \gamma_L]$. Therefore, by using the Moser path method again, we can deform the (possibly t-dependent) form γ_L to $\pi^* i^* \gamma_L$. Observe that this deformation can be done in an equivariant fashion by virtue of the equivariant Moser theorem for b^m-symplectic manifolds (Theorem 2.11). Thus we can assume that $\gamma_L = \pi^* i^* \gamma_L$ and the new γ_L is G-invariant and does no longer depend on t and it is just a symplectic 2-form on L.
The G-action preserves the b^m-symplectic structure on (M, Z): $g \cdot \omega = \omega$ for any $g \in G$. We need to show that $f(g \cdot (\alpha, \beta) = g \cdot f(\alpha, \beta)) = g \cdot \omega = \omega$. The G-action splits into a direct product of S^1- and H-action where the S^1-action preserves cosymplectic structure on Z.
In other words the following diagram commutes:

$$\begin{array}{ccc} (M, Z) & \xrightarrow{G} & (M, Z) \\ \uparrow f & & \uparrow f \\ Z & \xrightarrow{G} & Z \end{array}$$

This ends the proof of the lemma. □

Theorem 3.1 (Braddell, Kiesenhofer, Miranda)**.** *Let Z be a cosymplectic manifold and suppose Z has a transverse S^1-action preserving the cosym-*

plectic structure. Then Z has a finite cover $\tilde{Z} := S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the foliation, equipped with an S^1 action given by translation in the first coordinate for which the projection $p : S^1 \times \mathcal{L} \to Z$ is equivariant. To get a cosymplectic structure on the cover, one lifts the associated defining one- and two-forms.The cosymplectic structure on Z is given by the quotient of a cosymplectic structure on $\tilde{Z} = S^1 \times \mathcal{L}$ by the action of a finite cyclic group $\mathbb{Z}_k$.

Now we are ready to prove the b^m-symplectic version of the slice theorem.

Theorem 3.2 (A b^m-slice theorem)**.** *Let G be a compact group acting on a b^m-symplectic manifold (M, Z, ω) by b^m-symplectomorphisms such that the highest modular weight is non-vanishing. Let $\mathcal{O}_z$ be an orbit of the group contained in the critical set of M. Then there is a neighbourhood of the zero section of an associated bundle ${}^{b^m}T^*G \times_{H_z \times \mathbb{Z}} V_z$ equipped with the b^m-symplectic model*

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H),$$

where t is a defining function for Z, π is the projection $\pi : T^*S^1 \times T^*H \times_{H_z} V_z \to T^*H \times_{H_z} V_z$ and ω_H is the symplectic form on $T^*H \times_{H_z} V_z$ given by the symplectic slice theorem.
The moment map for such action is given by

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} + \mu_0(x, y).$$

Remark 3.1. *Below, we prove the b^m-symplectic slice theorem when the group action is b^m-Hamiltonian. Nevertheless, the statement holds for actions that preserve the b^m-symplectic structures. This would be a particular type of quasi-Hamiltonian structure. The proof in this set-up can be found in the section 5.*

Remark 3.2. *The slice theorem for b-symplectic manifolds has been investigated by the second author of this article in [BKM18a]. In this article, we* *give a fresh new proof for b^m-symplectic manifolds, which differs from the one contained in [BKM18a].*

Proof. Without loss of generality, since the isotropy group Γ is discrete, we can pass from the action of $(H \times S^1)/\Gamma$ to the free action of $H \times S^1$ on the finite cover of (M, Z) and then we apply equivariance as in the previous theorem to conclude. In view of the Lemma 3.1 the form ω splits into two parts, where $\alpha = -d\theta \wedge (\sum_{i=1}^{m} c_i \frac{dt}{t^i})$ is a b^m-symplectic form on $S^1 \times (-\epsilon, \epsilon)$ and β is the symplectic form on the leaf $\mathcal{L}$. First, we consider the Hamiltonian action of H separately. It is a Hamiltonian induced on the leaves on Z. Thus one can apply the symplectic slice theorem (2.18) and there is an H-equivariant neighbourhood U_H of $\mathcal{O}_p^H$ which is equivariantly symplectomorphic to $T^*H \times_{H_p} V_p$ with the symplectic form ω_H on $T^*H \times_{H_p} V_p$.
Consider the b^m-symplectic form on $T^*S^1 \times T^*H \times_{H_p} V_p$ given by

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \omega_H,$$

where t is a defining function for Z.
Take the quotient b^m-Poisson structure on $T^*(S^1 \times H) \times_{H_p \times \mathbb{Z}_d} V_p$ where $\mathbb{Z}_d$ acts on T^*S^1 as the twisted b^m-cotangent lift of $\mathbb{Z}_d$ acting by translations on S^1 and by linear symplectomorphisms on V_p and H_p acts on T^*H by the cotangent lift of H_p acting on H by translations and by linear symplectomorphisms on V_p.
The last step of the proof is to do the projection from the universal cover of M back onto the base. For that, we use that we can assume that the *linear* symplectic form ω_H on $\mathcal{L}$ is invariant as proved in Lemma 3.1. This ends the proof of the theorem. □

Remark 3.3. *We call a* ***normal form for the slice*** *the collection of b-manifold (M, Z), associated bundle ${}^{b^m}T^*G \times_{(H_z \times \mathbb{Z}_d)} V_z$, b^m-symplectic model $\omega = \sum_1^m c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H)$, and the group action ρ as described in Theorem 3.2, which is linear on the slice. We denote it as a triple $({}^{b^m}T^*G \times_{(H_z \times \mathbb{Z}_d)}$*

$V_z, \omega, \rho)$.

3.1.4 Desingularization and slices

We can now compare the b^m-symplectic slice theorem with its symplectic analogue. This will be needed to prove that desingularization commutes with reduction.
Let us recall how to construct *the desingularization function* as done in [GMW19].

Definition 3.1. *Let (S, Z, x), be a b^{2k}-manifold, where S is a closed orientable manifold and let ω be a b^{2k}-symplectic form. Consider the decomposition given by the expression 2.3.2 on an ε-tubular neighborhood U_ε of a connected component of Z.*
Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be an odd smooth function satisfying $f'(x) > 0$ for all $x \in [-1, 1]$ and satisfying outside that

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for } x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for } x > 1 \end{cases} .$$

Let f_ε be defined as $\varepsilon^{-(2k-1)} f(x/\varepsilon)$.
*The f_ε-**desingularization** ω_ε is a form that is defined on U_ε by the following expression:*

$$\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=1}^{2k} x^i \alpha_i \right) + \beta. \tag{3.1.3}$$

As ω_ε can be trivially extended to the whole manifold S so that it coincides with ω outside U_ε, we further refer to it as a form on S.

Definition 3.2. *Let (S, Z, x), be a b^{2k+1}-manifold, where S is a closed orientable manifold and let ω be a b^{2k+1}-symplectic form. Consider the decomposition given by the expression 2.3.2 on an ε-tubular neighborhood U_ε of a connected component of Z.*

Let $f \in \mathcal{C}^\infty(\mathbb{R})$ be a positive even smooth function satisfying $f'(x) > 0$ for $x < 0$ and $f(x) = -x^2 + 2$ if $x \in [-1, 1]$ satisfying outside $[-2, 2]$ that

$$f(x) = \begin{cases} \frac{-1}{(2k+2)x^{2k+2}} & \text{for } k > 0 \\ \log(|x|) & \text{for } k = 0 \end{cases}.$$

Let f_ε be defined as $\varepsilon^{-(2k)} f(x/\varepsilon)$.
*The f_ε-**desingularization** ω_ε is a form that is defined on U_ε by the following expression:*

$$\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=0}^{2k} \pi^*(\alpha_i) x^i \right) + \beta, \tag{3.1.4}$$

where $\pi : U \to Z$ is the projection.

Notice that in both odd and even cases, the desingularization function can be chosen invariant under the group action, as the following lemma proves:

Lemma 3.2. *Given any desingularization function f_ε one can always find an invariant desingularization f_ε^G by averaging over the group action*

$$f_\varepsilon^G = \int_G f_\varepsilon d\mu,$$

for μ a Haar measure.

As an immediate consequence of Lemma 3.2 the b^m-symplectic slice is the symplectic slice of the symplectic slice theorem applied to the desingularized form. In the odd case, we can apply the desingularization procedure to prove a *folded symplectic slice theorem.*

Theorem 3.3 (Desingularization of normal form, even case)**.** *Given a b^m-symplectic manifold (M, Z, ω) we consider an orbit of a point p under the G-action ρ. Let $({}^{b^m}T^*G \times_{H_z \times \mathbb{Z}_d} V_z, \omega, \rho)$ be a normal form for the b^m-slice. The b^m-symplectic slice V_p is also symplectic slice for the desingularized symplectic structure and G-action with respect to the invariant desingularization* *function f_ε^G. Then the triple $({}^{b^m}T^*G \times_{H_z \times \mathbb{Z}_d} V_z, \omega_\varepsilon, \rho)$ is the normal form for desingularized action where*

$$\omega_\varepsilon = df_\varepsilon^G \wedge \left(\sum_{i=1}^{2k} x^i \alpha_i \right) + \beta$$

defines symplectic structure in the neighbourhood of zero section of the associated bundle $T^*G \times_G V_p$.

For the folded symplectic case, we provide a weaker statement. A general folded symplectic slice theorem is not written in the literature to the authors' knowledge. Though, for the case of folded symplectic manifolds, which an invariant desingularization procedure can obtain, the b^m-symplectic slice theorem yields a folded symplectic slice theorem. In this theorem, the slice V_p remains the same, and the corresponding folded symplectic form is given by the formula 3.1.4. Moreover, due to the Lemma 3.2, this is true for any desingularization as it can always be done in an invariant way.
In particular, this proves:

Corollary 3.1. *Let (M, ω_ε) be a folded symplectic manifold with fold Z whose form can be seen as a desingularization of a b^{2k+1}-symplectic form on (M, Z, ω). Let G be a Lie group that acts on M and preserves the critical hypersurface Z. Then the folded symplectic slice V_p is given by the b^m-symplectic slice theorem, and the corresponding folded symplectic form can be written as*

$$\omega_\varepsilon = df_\varepsilon^G \wedge \left(\sum_{i=0}^{2k} \pi^*(\alpha_i) x^i \right) + \beta.$$

Remark 3.4. *Semilocally, the only constraint for a folded symplectic manifold to admit a b^m-symplectic structure is that the critical set of the folded symplectic manifold should be a cosymplectic manifold.*

Chapter 4

A b^m-Marsden-Weinstein reduction theorem for b^m-Hamiltonian actions

In this section we are going to describe a generalisation of the Marsden-Weinstein reduction for the case of b^m-Hamiltonian group actions. Eventually, we adapt the approach of the book [MMeO+07] for the b^m-Hamiltonian reduction by stages and extend it from the Hamiltonian to the b^m-Hamiltonian set-up.

This reduction theorem allows considering reduction by admissible Hamiltonian functions, which are not smooth. Thus, our reduction scheme supersedes other general reduction schemes such as the ones explored in [CM21] or the ones of symplectic Lie algebroids in [MPRO12].

The main theorem in this section considers the case when the highest modular case is non-vanishing. If the modular weight is non-zero, but the highest modular weight is zero, the b^m-Hamiltonian vector fields generated by the action vanish along Z. This case is not so interesting for reduction purposes and will not be considered in this article.

When the modular weight is zero, the reduction scheme is a consequence of the main result in [MPRO12] as we explain in section 4.3 below.

These reduction schemes are considered at points $\mu(p)$ with $p \in Z$. Away from Z, the standard symplectic reduction scheme is applied.

4.1 Three motivating examples

Let us start with two motivating examples extended from [GMPS15] to the b^m-category (see also [MP18]). In the examples below we examine the image of the moment map and use it to describe the process of reduction in an intuitive manner. Both examples correspond to circle actions on b^m-surfaces (completely classified as b^m-manifolds in [MP18]).

Example 4.1 (The b^m- Hamiltonian S^2)**.** *Consider the sphere S^2 as a b^m-symplectic manifold with critical set the equator:*

$$(S^2, Z = \{h = 0\}, \omega = \frac{dh}{h^m} \wedge d\theta),$$

with $h \in [-1, 1]$ and $\theta \in [0, 2\pi)$.
Take the S^1-action by rotations given by the flow of $\frac{\partial}{\partial\theta}$. Let us check that this action is indeed b^m-Hamiltonian and let us compute the moment map. There are two cases to consider:

- *The case $m = 1$: As $\iota_{\frac{\partial}{\partial\theta}}\omega = -\frac{dh}{h} = -d(\log|h|)$, the moment map on $M\backslash Z$ is $\mu(h,\theta) = \log|h|$.*
- *The case $m > 1$. Then, $\iota_{\frac{\partial}{\partial\theta}}\omega = -\frac{dh}{h^m} = -d(-\frac{1}{(m-1)h^{m-1}})$, the moment map on $M\backslash Z$ is $\mu(h,\theta) = -\frac{1}{(m-1)h^{m-1}}$.*

The image of μ for $m = 1$ is drawn in Figure 4.1 as two superimposed half-lines. Each point in the image has two connected components in its pre-image (one pre-image hemisphere) in contrast with the classical symplectic case.
In both cases, as we explained in section 2.4.2 for the case $m = 1$, the moment map can be understood as a section of ${}^{b^m}C^\infty$ by including points "at infinity".

Let us yet examine another example.

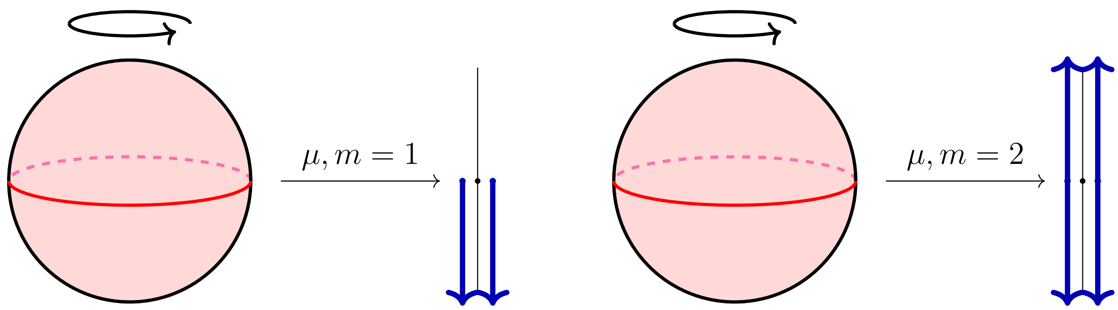


Figure 4.1: The moment map of the S^1-action by rotations on a b^m-symplectic S^2.

Example 4.2. *Consider now as b^2-symplectic manifold the torus*

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin^2\theta_1} \wedge d\theta_2)$$

with standard coordinates: $\theta_1, \theta_2 \in [0, 2\pi)$. The critical hypersurface Z in this example is not connected. It is the union of two disjoint circles. Consider the circle action of rotation on the θ_2-coordinate with fundamental vector field $\frac{\partial}{\partial\theta_2}$. As

$$\iota_{\frac{\partial}{\partial\theta_2}}\omega = -\frac{d\theta_1}{\sin^2\theta_1} = d\left(\frac{\cos\theta_1}{\sin\theta_1}\right).$$

Thus the associated S^1-action has as ${}^{b^2}C^\infty$-Hamiltonian the function $-\frac{\cos\theta_1}{\sin\theta_1}$. The image of this function on $M \backslash Z$ is drawn in Figure 4.2. Each of the two connected components of $M \setminus Z$ is diffeomorphic to an open cylinder and maps to one of these lines. Again, notice that the pre-image of a point in the image consists of two orbits.

Example 4.3. *Similarly, one can consider torus to be a b^m-symplectic manifold for any integer m*

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0, \pi\}\}, \omega = \frac{d\theta_1}{\sin^m\theta_1} \wedge d\theta_2).$$

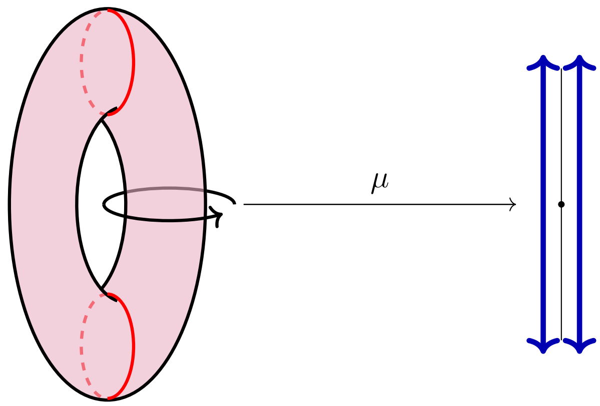


Figure 4.2: An S^1-action on a b-symplectic $\mathbb{T}^2$ and its moment map.

Then

$$\iota_{\frac{\partial}{\partial\theta_2}}\omega = -\frac{d\theta_1}{\sin^m\theta_1} = d\left(\frac{|\cos\theta_1|}{\cos\theta_1}\frac{{}_2F_1\left(\frac{1}{2},\frac{1-m}{2};\frac{3-m}{2};\sin^2(\theta_1)\right)}{(1-m)\sin^{m-1}\theta_1}\right),$$

where ${}_2F_1$ is the hypergeometric function.

Thus the associated S^1-action has as ${}^{b^m}C^\infty$-Hamiltonian the function

$$-\frac{|\cos\theta_1|}{\cos\theta_1}\frac{{}_2F_1\left(\frac{1}{2},\frac{1-m}{2};\frac{3-m}{2};\sin^2(\theta_1)\right)}{(1-m)\sin^{m-1}\theta_1}.$$

In all the examples described above when we fix a value of the moment map we obtain a circle (with the exception of the fixed points) where the initial S^1-action restricts. This circle can be quotiented out by the induced S^1-action to obtain a point. The singular symplectic structure in this process is also reduced to the trivial symplectic structure on the point.

This would be a hands-on example of b^m-symplectic reduction. This reduction reduces the b^m-sphere/torus to a point. In higher dimensions, this reduction yields a non-trivial symplectic structure on the quotient.

Atiyah-Bott space of flat connections on b-symplectic surfaces

One of the motivating examples for this work comes from symplectic structures on the moduli space of flat connections $\mathcal{M}^G$. There are different approaches on how to introduce symplectic structure on $\mathcal{M}^G$ (Narasimhan and Seshadri [NS65] and [Ses67], Goldman [Gol84] and [Gol86], Karshon [Kar92] and Atiyah-Bott [AB83]). Some of these approaches can be generalized from closed oriented Riemann surfaces Σ of genus g to surfaces with boundary. However, this kind of singularities on the surface does not lead to the singularities in the symplectic structure. The question that captured our attention is if and under which condition can the Poisson structure on the moduli space of flat connections become b- or b^m-symplectic.

In this section, we follow the approach of Atiyah and Bott. As a result, we arrive to a family of examples of b^m-symplectic structures on the moduli space of flat connections in trivial bundles over manifolds with boundary.

In order to construct an example of the singular Atiyah-Bott form, we first turn to the usual symplectic case. We remind the approach from [AM95] and [Mic13] on how to explicitly write the Atiyah-Bott form on a compact oriented manifold in Darboux coordinates via holonomies corresponding to the generators of the fundamental group. The main idea underlying this approach is that the integral $\int_\Sigma \alpha \wedge \beta$ taken over the whole surface can be lifted to the universal cover and becomes equal to the integral over the fundamental polygon. The lifted integral itself can be localized to the alternating sum of the values taken in the vertexes of the polygon, that is, the sum of the holonomies corresponding to the edges of the fundamental polygon (i.e., generators of the fundamental group).

Let Σ be a compact orientable surface, and G be a Lie group admitting Ad-invariant bilinear form on its Lie algebra. Let $\Sigma \times G$ be a trivial bundle. Now we can introduce a theorem relating generators of the fundamental group of Σ and corresponding holonomies with the Atiyah-Bott symplectic structure as Darboux coordinates.

Theorem 4.1 (D. Michiels)**.** *Suppose Σ is a compact orientable surface of* *genus $g \geq 1$, and suppose G is an abelian Lie group with a fixed symmetric non-degenerate bilinear form on its Lie algebra. Then the moduli space $\mathcal{M}(\Sigma, G)$ is diffeomorphic to G^{2g}.*

Write $a_1, b_1, \ldots, a_g, b_g$ for the $2g$ projections to the copies of G composed with the inversion map $G \to G$ so that by abuse of notation $a_i \in G$ is the holonomy around the generator a_i of the fundamental group (and analogously for b_i). Then $\theta \circ (da_i) : dHom(\pi_1(\Sigma), G) \to \mathfrak{g}$ and $\theta \circ (db_i) : dHom(\pi_1(\Sigma), G) \to \mathfrak{g}$ are $\mathfrak{g}$-valued 1-forms on $Hom(\pi_1(\Sigma), G)$. Call these 1-form da_i and db_i. Then the symplectic structure on the block $M_{\Sigma \times G}(\Sigma, G)$ of the moduli space is given by

$$\omega_{AB} = \sum_{i=1}^{g} (db_i \wedge da_i)$$

Example 4.4. *Let us consider torus $\mathbb{T}^2$ as an illustrating example. The corresponding fundamental polygon is a square with oriented edges (see Fig. 4.3) and the form $\omega_{AB} = db \wedge da$ as in theorem 4.1.*

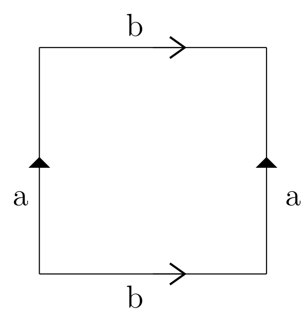


Figure 4.3: Fundamental polygon for $\mathbb{T}^2$

Now we consider the singular analogue of the above example. In order to construct it, we use the desingularization Theorem 2.13 and treat a b^m-symplectic structure as a limit of a family of symplectic ones.

Take the limit of symplectic forms on $\mathbb{T}^2$ as in 3.1.3:

$$\lim_{\varepsilon\to 0}\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=1}^{2k} x^i\alpha_i\right) + \beta$$

and consider f_ε being desingularizing function for a b^2-form $\frac{d\theta}{\sin^2(\theta/2)}$ having singularity along the hypersurface $Z = \{(\theta,\varphi)|\theta = 0\}$. In the case of $\mathbb{T}^2$, $k = 1$ and substitute the coordinates (θ,φ) into the expression, we get that $\beta = 0$ and

$$\lim_{\varepsilon\to 0}\omega_\varepsilon = df_\varepsilon \wedge x\alpha + \beta = \frac{d\theta}{\sin^2(\theta/2)} \wedge d\varphi = \omega.$$

The form on the right hand side is a b^2-symplectic form on $(\mathbb{T}^2, S^1)$ where S^1 is the critical hypersurface $\alpha = 0$ corresponding to the $a-$cycle.

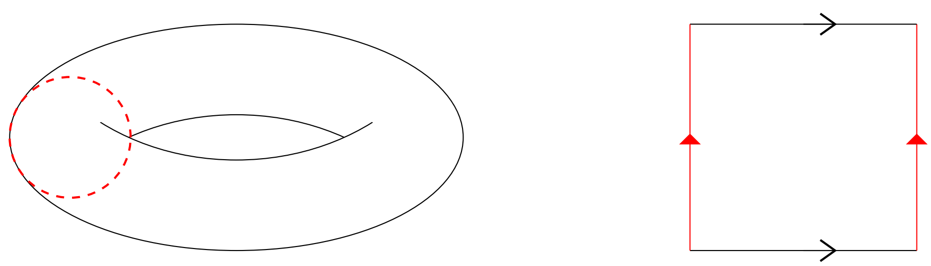


Figure 4.4: b-manifold $(\mathbb{T}^2, S^1)$ and the corresponding fundamental polygon

For the left-hand side of the equality, we can easily juxtapose the corresponding polygon and, therefore, the normal form of ω_{AB}. Note that ω here is a form on the manifold Σ and ω_{AB} is the form on the moduli space of flat connections $\mathcal{M}^G$. The family of polygons would still be squares as in example 4.4. What changes here is the values of the holonomies (i.e., lengths of the edges), with one of them tending to infinity. In the limit, the corresponding fundamental polygon is as on Fig. 4.4 where red edges correspond to the single distinguished circle $\alpha = 0$ that is the critical hypersurface for the form $\omega = \frac{d\alpha}{\alpha} \wedge d\beta$. The corresponding Atiyah-Bott form is then $\omega_{AB} = \frac{db}{b} \wedge da$.

This example is notable because the Darboux coordinates on the surface coincide with the generators of the fundamental group, which is not true in general.
First, consider the following b-manifold: a torus $\mathbb{T}^2$ with one cycle S^1, chosen as a critical hypersurface. As shown in [MP18] such a b-manifold can only admit b^k-symplectic structures for even k. As an example, we choose $\omega = \frac{d\theta}{\sin^2(\theta/2)} \wedge d\varphi$. The polygon corresponding to this surface is a square (see Fig. 4.4b), where the red edges correspond to a single distinguished circle (as in Fig. 4.4a).
Consider the following two b-manifolds:

- S^2 with an equator $Z = S^1$ as an critical hypersurface
- T^2 with two copies of S^1 as an critical hypersurface

The corresponding polygons look as depicted below:

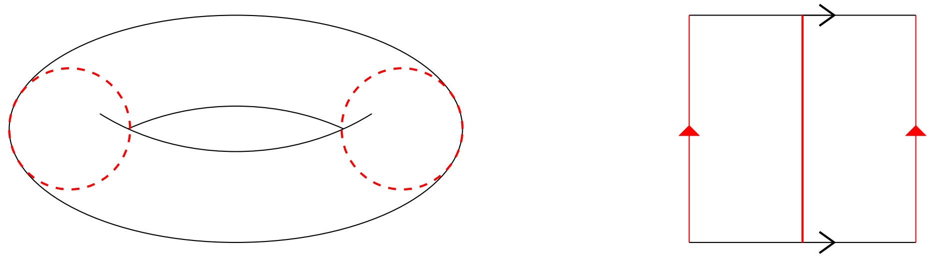


Figure 4.5: A b-manifold $(\mathbb{T}^2, S^1 \times S^1)$ and the corresponding fundamental polygon.

4.2 The case of non-vanishing highest modular weight

Let us now state the b^m-symplectic reduction theorem of a b^m-Hamiltonian action with non-vanishing highest modular weight. The critical outcome of this result is that the *singularity* of the b^m-symplectic structure is cleared

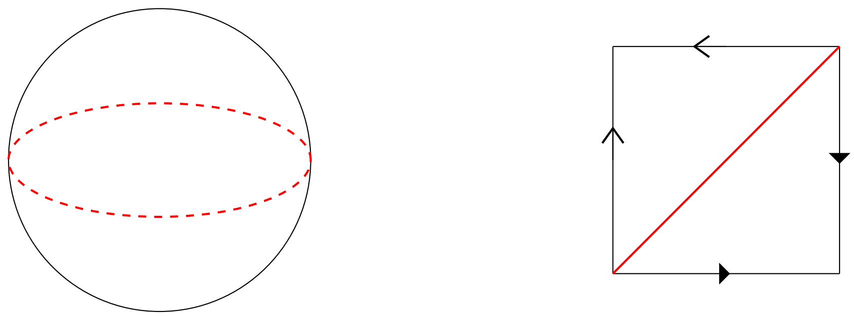


Figure 4.6: A b-manifold $(\mathbb{S}^2, S^1)$ and the corresponding fundamental polygon.

away by the reduction procedure. So we could think that reduction is out of the b^m-symplectic category. However, we observe that reduction sits in the category of E-symplectic manifolds [MS21] where one could more generally formulate the reduction scheme.

From now on, inspired by example from Fig. 2.2 we will introduce a notation to denote the image of the points at infinity as a boldfaced zero $\mathbf{0}$ by this, we mean the point $\mathbf{0} = (p_\infty, 0)$ where the splitting of the moment map is given in view of the slice theorem (Theorem 2.15), in a neighbourhood of the orbit as,

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} + \mu_0(x, y). \tag{4.2.1}$$

with symplectic form:

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H)$$

Thus when we consider $\mu^{-1}(\mathbf{0})$ we mean the intersection of the pre-image of μ_0 in the enlarged model with $t = 0$

We recall once again that the group G is of the form $(S^1 \times H)/\Gamma$ that can as well be seen as $S^1 \times H$ on the universal cover of M. For convenience, we will make two assumptions:

· The induced action of H is locally free.

· The action of S^1 on the covering model associated with the finite group Γ is free.

· 0 is a regular value for μ_0 (by abuse of notation, we will then say that $\mathbf{0}$ is a regular point of μ).

Theorem 4.2 (The b^m-Marsden-Weinstein reduction)**.** *Given a b^m-Hamiltonian (locally) free action of a Lie group G on a b^m-symplectic manifold M^{2n}. Assume that the highest modular weight is non-vanishing, then the pre-image of a regular point $\mu^{-1}(\mathbf{0})$ is a b^m-presymplectic manifold that has an induced action of G. The space of orbits of the induced action $M//G$ is a symplectic orbifold. This reduced symplectic orbifold is symplectically isomorphic to the standard symplectic reduction of a symplectic leaf on Z by a Lie subgroup of G.*

Proof. **Step 1:**The proof starts by setting up a structure (smooth or orbifold type) on the topological quotient. When the action is free, the quotient is a smooth manifold. If the action is locally free, the quotient has an orbifold structure (see [GGK02]). This orbifold structure is well-understood in the symplectic case (see for instance, [GGK02]).
Step 2: Next, using the slice theorem (Theorem 3.2), we describe the induced geometrical structure on the quotient. If the group action is free, then a neighbourhood of the orbit is diffeomorphic to a product of the orbit with a symplectic slice. Otherwise, there is a finite group Γ involved, and by arguing on a covering in a standard way, we can reduce to the product case. We apply the b^m-symplectic slice theorem 3.2 and, more concretely, the normal form for the b^m-symplectic form and the moment map on a neighbourhood of $\mathcal{O}_x$. Due to the b^m-symplectic slice theorem 3.2, the tubular neighbourhood of

the orbit $\mathcal{O}_x$ is equipped with the following symplectic model:

$$\omega = \sum_{i=1}^{m} c_i \frac{dt}{t^i} \wedge d\theta + \pi^*(\omega_H)$$

and the induced moment map along the orbit has the form:

$$\mu = c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1} \frac{t^{-i}}{i} + \mu_0(x, y).$$

Thanks to this expression: As the highest modular weight c_m is non-zero, we can first (for simplicity and clearness) consider the reduction with respect only to the S^1 component (this is possible because of the normal form model in Theorem 3.2). The resulting space is a symplectic manifold which we denote as $M//S^1$ with Hamiltonian H-action and the corresponding induced moment map $\mu_0(x, y)$. This moment map μ_0 is a standard Hamiltonian moment map.

Final step: The H-action on the cover can be seen as a usual Hamiltonian action on a symplectic slice so that the Marsden-Weinstein reduction can be applied directly to the second component, and the reduction $\mu^{-1}(0)/G$ is a symplectic orbifold which is symplectically equivalent to $\mathcal{L}//H$ (where $\mathcal{L}$ is any symplectic leaf on Z). This ends the proof of the theorem. □

Remark 4.1. *The very general reduction scheme explained in [CM21] seems to consider only smooth functions, so our approach is more general in the case of b^m-symplectic manifolds.*

Remark 4.2. *In the classical set-up of the study of Hamiltonian G-spaces, the Kirwan map $\kappa : H^*_G(M) \mapsto H^*(M//G)$ defines a surjection between the equivariant cohomology of the symplectic manifold and the cohomology of the symplectic reduced space.*

Using the b^m-equivariant cohomology and the Mazzeo-Melrose formula, one could define the Kirwan map for b^m-Hamiltonian actions. Properties of this Kirwan map will be studied elsewhere.

4.3 The case of vanishing modular weight

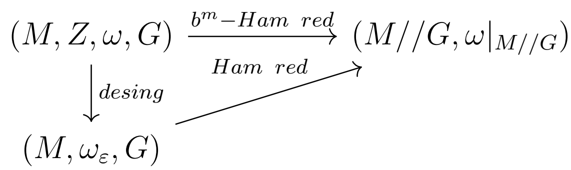
The case of vanishing modular weight is easier to deal with as the action is Hamiltonian, and the b^m-symplectic reduction of a b^m-symplectic manifold is a b^m-symplectic manifold.
For general symplectic Lie algebroids, the general Marsden-Weinstein reduction has been proved by Marrero, Padrón and Rodriguez-Olmos (see Theorem 3.11 in [MPRO12]).
By direct application of this result for the particular Lie algebroid given by the b^m-cotangent bundle, we obtain the following:

Theorem 4.3 (The b^m-Marsden-Weinstein reduction with zero modular weight)**.** *Given a b^m-Hamiltonian (locally) free action of a Lie group G on a b^m-symplectic manifold M^{2n} with vanishing modular weight. Then the pre-image of a point $\mu^{-1}(0)$ is a b^m-presymplectic manifold that has an induced action of G. The space of orbits of the induced action $M//G$ is a b^m-symplectic manifold.*

Remark 4.3. *The reduction scheme in the zero modular weight case for b-symplectic manifolds is Corollary 3.10 in [GZ21]. This corresponds to the case when the b^m-Hamiltonian action is indeed Hamiltonian. So, it is a particular case of our reduction procedure.*

4.4 The desingularization procedure and reduction

In this section, we investigate the desingularization procedure of [GMW19] in more detail to understand how it behaves under group actions on the same manifold. This leads us to prove that *desingularization commutes with reduction.*
We summarize this idea in the following diagram:



Theorem 4.4. *The desingularization procedure commutes with the b^m-Hamiltonian reduction.*

Proof. Notice that by virtue of Lemma 3.2, we can assume that this function is invariant by the G-action ρ. The normal form given by the slice theorem 3.2 contains precisely the associated bundle ${}^{b^m}T^*G \times_{H_z \times \mathbb{Z}} V_z$. Theorem 3.2 also provides the normal form for the moment map

$$\mu = \left(c_1 \log|t| + \sum_{i=1}^{m-1} c_{i+1}\frac{t^{-i}}{i}\right) + \mu_0(x, y).$$

As G-action can be locally seen as a product $S^1 \times H$, the first components of μ (in brackets) correspond to the S^1-action, and the last one μ_0 corresponds to H-action. We will refer to them as S^1- and H-components of the moment map. Notice that the H-component μ_0 is independent of t therefore, once we apply the b^m-Hamiltonian reduction with respect to S^1-action, the S^1-component vanishes and the H-component acts on the symplectic slice. Moreover, $M//S^1$ is, in fact, a symplectic manifold with Hamiltonian H-action and the normal form for the moment map μ_0. Now we can apply the classic symplectic Marsden-Weinstein reduction. □

The critical point to the last proof is that in the moment map normal form given by Theorem 3.2, μ splits onto two orthogonal components automatically leading us to the following corollary.

Corollary 4.1. *The b^m-Hamiltonian G-action admits a* reduction by stages *procedure.*

Proof. As the Marsden-Weinstein reduction commutes with the desingularization and the Marsden-Weinstein reduction for a Hamiltonian action of $G_1 \times G_2$ can be done by stages [MMeO+07], we infer that the b^m-Hamiltonian reduction can be done by stages. □

Chapter 5

Singular quasi-Hamiltonian reduction and fusion products

In this section, we would like to emphasize the universality of the proof of the reduction theorem 4.2. Notice that this proof highly depends on the splitting property of the moment map 4.2.1 and uses Hamiltonian reduction as a *black box* for the corresponding term of the moment map. In this section, we would like to show that different generalizations of the moment map can be used in this proof. Particularly, we focus on the group-valued moment maps [AMM98], and the corresponding reduction theory [BTW04]. We will follow the same approach as in [BTW04], showing throughout the proof that all the statements used can be generalized to a singular version (including b^m-type singularities). Let us first define what a group-valued moment map is. The definition of quasi-Hamiltonian spaces 2.23 can be generalized to consider b^m-type singularities along the hypersurface Z of a b^m-manifold and, more generally, to consider E-manifolds introduced in [NT01] and [MS21].

Definition 5.1. *A* ***singular quasi-Hamiltonian*** G***-space of*** b^m***-type*** *is a b-manifold* (M, Z) *with a* G*-action* ρ*, an invariant* 2*-form* $\sigma \in^{b^m} \Omega(M)$ *and an equivariant moment map* $\Phi : M \to G$ *such that:*

(i) σ *is equivariantly closed:* $d\sigma = -\Phi^*\chi$*,*

(ii) the moment map condition is satisfied: $\iota(v_\xi)\sigma = \frac{1}{2}\Phi^*\left(\theta^l + \theta^r, \xi\right)$,

(iii) σ *is weakly non-degenerate:*

$$\ker \sigma \cap \ker d\Phi = 0.$$

Remark 5.1. *If the Lie group G is abelian then the singular two-form ω is automatically a b^m-symplectic form (or more generally an E-symplectic form if we replace the b^m-functions by E-functions). Then this definition of quasi-Hamiltonian generalizes the investigation of symplectic actions to the b^m-symplectic realm.*

Remark 5.2. *The form $d\sigma$ is a smooth 3-form in $\Omega^3(M)$ rather than a singular form in ${}^{b^m}\Omega^3(M)$.*

Lemma 5.1. *The b^m-form σ can be decomposed as:*

$$\sigma = \alpha \wedge \frac{dt}{t^m} + \beta,$$

where α is a closed smooth one-form and β is a smooth 2-form $\beta \in \Omega^2(M)$.

Proof. We use Proposition 10 in [GMP14], the b^m-form σ can be decomposed as:

$$\sigma = \alpha \wedge \frac{dt}{t^m} + \beta.$$

Now from the first condition in the definition of quasi-Hamiltonian we know that $d\sigma$ is smooth (as $d\sigma = -\Phi^*\chi$, where $\chi \in \Omega^3(G)$).
The form $d\sigma$ equals the form $d\beta$ which is a smooth form $\in \Omega^2(M)$. This automatically implies that $d\alpha = 0$ as otherwise $d\sigma$ would have a singular term.

□

The fact that α is closed allows to conclude that the critical set has a mapping torus structure thanks to Tischler theorem [Tis70]. This mapping torus inherits the quasi-Hamiltonian space structure from the ambient space (in

the same way the critical set of a b^m-symplectic manifold inherits a regular Poisson structure which is cosymplectic). Along the same lines we could prove that the critical set is a quasi-Hamiltonian mapping torus.
In order to conclude our reduction theorem we just need to apply Tischler theorem to the critical set. We recall it here for convenience:

Theorem 5.1. *Let M^n be a closed manifold endowed with a 1-form β which is nowhere vanishing. Then M^n fibers over a circle S^1.*

This leads us to the following proposition:

Proposition 5.1. *Let (M, G, σ) be a closed quasi-Hamiltonian space of b^m-type, and let Z be its critical set. Then,*

- *Z fibers over a circle S^1.*
- *If the group G acts transversally on the fibers of then the group then G is either of the form $S^1 \times H$ or $S^1 \times H \mod \Gamma$, where $\Gamma = \mathbb{Z}_l \times \mathbb{Z}_k$ and $\mathbb{Z}_k$ is a non-trivial cyclic subgroup of H.*

Proof. For the first part of the proof as a consequence of lemma 5.1 the form α is closed. If σ satisfies minimal non-degeneracy conditions, we may assume that α is nowhere vanishing. In view of Tischler's theorem 5.1, M fibers over a circle S^1. The second part of the proposition is proved mutatis mutandis as theorem 2.15 in [BKM18a]. □

In view of this result it is possible to talk about transverse S^1-action in the quasi-Hamiltonian context as we did in former sections for b^m-Hamiltonian actions. From Lemma 5.1 and Proposition 5.1, we get an immediate corollary:

Corollary 5.1. *In a neighbourhood of the critical set Z, the b^m-form σ can be written as $d\theta \wedge \frac{dt}{t^m} + \beta$, where θ is coordinate on S^1. The corresponding S^1-action on the covering of M is b^m-Hamiltonian.*

Remark 5.3. *Notice that singular quasi-Hamiltonian moment map for the G-action on the covering of M splits into two independent components (Φ_S, Φ_H) corresponding to the S^1- and the H-action on the covering respectively.*

Now we can compute the moment map for a b^m-Hamiltonian S^1-action. The ${}^{b^m}\mathbb{R}$- or ${}^{b^m}S^1$-valued moment map of Example 4.1 $\mu_S = \frac{-1}{(m-1)t^{m-1}}$ for $m > 1$ and $\mu_S = \log|t|$ for $m = 1$. We can extend the exponential map from ${}^{b^m}\mathbb{R}$ and ${}^{b^m}S^1$ to points at infinity using a standard compactification procedure. By abuse of notation we will denote this extension by exp.

Lemma 5.2. *The b^m-Hamiltonian moment map for the transverse S^1-action is $\Phi_S = \exp\left(\frac{-1}{(m-1)t^{m-1}}\right)$ for $m > 1$ and $\Phi_S = |t|$ for $m = 1$.*

In particular, from now on we can talk about the S^1-component when referring to quasi-Hamiltonian actions which are *transverse* (in the sense that the action on Z acts transversally to the fibers of proposition 5.1 over S^1). Such a component, in view of 5.1 is automatically b^m-Hamiltonian.
Analogously to the smooth case, one can construct a b^m-quasi Hamiltonian space out of a b^m-Hamiltonian space. To do that, we first provide a relevant construction of a 2-form ϖ that is G-invariant and $d\varpi = -\exp^*\chi$ (see Lemma 3.3 in [AMM98]):

$$\varpi = \frac{1}{2}\int_0^1 \left(exp_s^*\theta^l, \frac{\partial}{\partial s}\exp_s^*\theta^l\right) ds,$$

where for $s \in \mathbb{R}$, the map $\exp_s : \mathfrak{g} \to G$ is defined by $\exp_s(\eta) = \exp(s\eta)$. This form is commonly used in the investigation of the correspondences between Hamiltonian and quasi-Hamiltonian spaces. We provide the b^m-generalizations of the statements together with the references to the original statements:

Lemma 5.3 ([AMM98], proposition 3.4)**.** *Let (M, ρ, ω, Φ) be a Hamiltonian G-space. Then M with 2-form $\sigma = \omega + \mu^*\varpi$ and moment map $\Phi = \exp(\mu)$ satisfies all axioms of a quasi-Hamiltonian G-space except possibly the non-degeneracy condition. If the differential $d_\xi \exp$ is bijective for all $\xi \in \mu(M)$, then the non-degeneracy condition is satisfied as well, and (M, ρ, σ, Φ) is a b^m-quasi Hamiltonian G-space.*

Proof. Equivariance of μ follows from the definition and the fact that ϖ is an invariant form. The proof of the first condition that σ is equivariantly closed is straightforward:

$$d\sigma = d\omega + d\Phi^* \varpi = 0 - d\Phi^* \exp^* \chi = -\mu^* \chi.$$

The moment map condition:

$$\iota(\upsilon_\xi)\sigma = d(\mu, \xi) + \frac{1}{2}\mu^* \exp^*(\theta^l + \theta^r, \xi) - d(\mu, \xi) = \frac{1}{2}\Phi^*(\theta^l + \theta^r, \xi).$$

The proof of the non-degeneracy condition completely follows the original proof with the only minor distinction that υ should be taken in ${}^{b^m}TM \subset TM$. $\square$

5.0.1 Examples

One of the interesting examples of quasi-Hamiltonian spaces comes from the moduli space of flat connections on surfaces. In [AMM98], the authors show that the moduli space of flat connections $\mathcal{M}(\Sigma)$ carries a quasi-Hamiltonian structure. Consider a compact, connected surface Σ of genus g with boundary $\partial\Sigma = S^1$. The space of flat G-connections $\mathcal{A}_{flat}(\Sigma)$ on $\Sigma \times G$ is invariant under the gauge group $\mathcal{G}(\Sigma)$ action. The space of connections $\mathcal{A}(\Sigma) = \Omega^1(\Sigma, \mathfrak{g})$ on the trivialized principal G-bundle $G \times \Sigma \longrightarrow \Sigma$ is a symplectic manifold, equipped with a Hamiltonian action of the gauge group $\mathcal{G}(\Sigma) = Map(\Sigma, G)$. The moment map $\mu : \mathcal{A}(\Sigma) \longrightarrow Lie(\mathcal{G}(\Sigma))^* = \Omega^2(\Sigma, \mathfrak{g})$ is given by $\mu(A) = F_A$, where F_A is the curvature of the connection A. A normal subgroup $\mathcal{G}(\Sigma, \partial\Sigma)$ of the gauge group $\mathcal{G}(\Sigma)$ is defined by $\mathcal{G}(\Sigma, \partial\Sigma) = \gamma \in \mathcal{G}(\Sigma) | \gamma|_{\partial\Sigma} = e$. The reduced space $X := \mathcal{A}(\Sigma)//\mathcal{G}(\Sigma, \partial\Sigma)$ is a quasi-Hamiltonian G-space with proper moment map ϕ. For the b^m-symplectic analogue of this example in the abelian case see 4.1.

Other constructions are directly given in [Boa07] following [AMM98]. In particular, the following theorem characterizes when a product manifold is a quasi-Hamiltonian space with a product group. This will be specially relevant for our purposes.

Theorem 5.2 ([AMM98]). *Let M be a quasi-Hamiltonian $G \times G \times H$-space, with moment map $\Phi = (\Phi_1, \Phi_2, \Phi_3)$. Let the group $G \times H$ act by the diagonal embedding described as $(g, h) \to (g, g, h)$. Then M with the two-form*

$$\hat{\sigma} = \sigma - \frac{1}{2}(\Phi_1^*\theta^l, \Phi_2^*\theta^r)$$

and moment map

$$\hat{\Phi} = (\Phi_1 \cdot \Phi_2, \Phi_3) : M \to G \times H$$

is a quasi-Hamiltonian $G \times H$-space.

Remark 5.4. *We can easily extend this fusion procedure to singular quasi-Hamiltonian spaces.The fusion product enables to construct new quasi-Hamiltonian spaces from two given quasi-Hamiltonian spaces and combine different type of singularities to obtain E-quasi-Hamiltonian spaces. For the sake of simplicity we will only present fusion constructions that stay in the b^m-category bearing in mind that this procedure allows to obtain more general singular examples.*

The b^m-reincarnation of theorem 5.2 yields:

Theorem 5.3. *Let M_1 be a quasi-Hamiltonian space of b^m-type with associated group $G \times H_1$ and M_2 a quasi-Hamiltonian space with group $G \times H_2$. Their fusion product $M_1 \circledast M_2$ is the b^m-singular quasi-Hamiltonian $(G \times H_1 \times H_2)$-space obtained from the quasi-Hamiltonian $(G \times G \times H_1 \times H_2)$-space $M_1 \times M_2$ by fusing the two factors of G.*

We can use this procedure to get new examples of quasi-Hamiltonian spaces by combining classical examples in the theory of quasi-Hamiltonian spaces with singular quasi-Hamiltonian spaces.

Let us consider several examples of this family of new singular fusion constructions in detail.

Example 5.1. *Take a b^m-Hamiltonian space from example 4.1* $(S^2, S^1, \omega_S, \mu_S)$ *and a quasi-Hamiltonian space* $(\mathbb{T}^2, S^1, \sigma_T, \Phi_T)$*, where* $\omega_S = \frac{dh}{h^2} \wedge d\theta$ *and* $\sigma_T = d\theta_1 \wedge d\theta_2 + \varpi$*. The moment map for the* S^1*-action on* S^2 *is* $\mu_S = -\frac{1}{h}$ *and the group-valued moment map for* $\mathbb{T}^2$ *is* $\Phi_T = \theta_1$*. Applying lemma 5.3, we get a* b^2*-quasi Hamiltonian space* $(S^2, S^1, \sigma_S, \Phi_S)$*, where* $\sigma_S = \omega_S + \varpi$*, and the corresponding* S^1*-valued moment map* $\Phi_S = \exp \mu_S$*. Then the fusion product* $S^2 \circledast \mathbb{T}^2$ *is an* S^1 b^2*-quasi Hamiltonian space with the moment map* $\Phi = \Phi_S \cdot \Phi_T = e^{-1/h}\theta_1$*.*

Consider now a similar example stepping out of the b^m-category as mentioned in Remark 5.4:

Example 5.2. *Take two b^m-Hamiltonian spaces from examples (4.1, 4.2)* $(S^2, S^1, \omega_S, \mu_S)$ *and* $(\mathbb{T}^2, S^1, \omega_T, \mu_T)$*, where* $\omega_S = \frac{dh}{h^2} \wedge d\theta$ *and* $\omega_T = \frac{d\theta_1}{\sin^2\theta_1} \wedge d\theta_2$*. The corresponding moment maps for* S^1*-action* $\mu_S = -\frac{1}{h}$ *and* $\mu_T = -\frac{\cos\theta_1}{\sin\theta_1}$*. Applying lemma 5.3, we get two corresponding quasi-Hamiltonian spaces* $(S^2, S^1, \sigma_S, \Phi_S)$ *and* $(\mathbb{T}^2, S^1, \sigma_T, \Phi_T)$*, where* $\sigma_S = \omega_S + \varpi$*,* $\sigma_T = \omega_T + \varpi$ *and the corresponding* S^1*-valued moment maps* $\Phi_S = \exp \mu_S$ *and* $\Phi_T = \exp \mu_T$*. Then the fusion product* $S^2 \circledast \mathbb{T}^2$ *is an* S^1*-E-quasi Hamiltonian space with the moment map* $\Phi = \Phi_S \cdot \Phi_T = e^{-1/h - \cos\theta_1/\sin\theta_1}$*.*

More generally, any fusion product of a b^m-Hamiltonian and quasi-Hamiltonian spaces will lead to a singular quasi-Hamiltonian space.

Example 5.3. *Consider a b^m-Hamiltonian G-space* (M_1, G, ω) *with* ω *a* b^m*-symplectic form. Applying lemma 5.3, we obtain a singular quasi-Hamiltonian space* (M_1, G, σ)*. From theorem 2.15, we can assume that in a finite covering* $G = S^1 \times H_1$*. Take* M_2 *be a quasi-Hamiltonian* $S^1 \times H_2$ *space, then in view of theorem 5.3 the fusion product* $M_1 \circledast M_2$ *is a* $G \times H_2$ b^m*-quasi Hamiltonian space.*

Another particular case comes by considering as M_1 *the former examples 4.1 and 4.3 and as* M_2 *the conjugacy classes example in Example 2.4.*

We can indeed obtain quasi-Hamiltonian spaces with a b^m-type singularity for any prescribed group of type $S^1 \times H$ as the following example shows:

Example 5.4. *Given a Lie group H, Consider a b^m-Hamiltonian $S^1 \times H$-space (M_1, G, ω) with ω a b^m-symplectic form. Take M_2 be any quasi-Hamiltonian H-space, then in view of theorem 5.3 the fusion product $M_1 \circledast M_2$ is a $S^1 \times H$-b^m-quasi Hamiltonian space.*

We can get other constructions of more general singularities via fusion product examples using Remark 5.4.

5.0.2 Reduction for singular quasi-Hamiltonian spaces

In the work [BTW04], the authors prove a reduction theorem for quasi-Hamiltonian spaces using two auxiliary statements from [AMM98].
Now let us provide the singular quasi-Hamiltonian reduction theorem as stated in [AMM98] and [Boa07]:

Theorem 5.4 (singular quasi-Hamiltonian reduction)**.** *Let M be a singular quasi-Hamiltonian $G_1 \times G_2$-space with non-vanishing highest modular weight (i.e. one of the components of the product includes transverse S^1-action), a singularity of b^m-type and the moment map $(\Phi_1, \Phi_2) : M \to G_1 \times G_2$. Let $f \in G_1$ be a regular value of the moment map $\Phi_1 : M \to G_1$ and $Z_f \subset G_1$ be its centralizer. Then the pull-back of the 2-form $\sigma \to \Phi_1^{-1}(f)$ descends to the reduced space*

$$M_f = \Phi_1^{-1}(f)/Z_f$$

and makes it into quasi-Hamiltonian G_2-space. If (M, σ, Φ, G) satisfies all conditions from the definition of a quasi-Hamiltonian space except weakly non-degeneracy condition, so does the resulting reduced space. In particular, if G_2 is abelian then M_f is symplectic.

For the non-singular version see Theorem 5.1 in [AMM98]. We follow the same frame of the proof.

Proof. First, we need to show that $i^*\omega$ is Z_f-basic, where i is the embedding $\mu_1^{-1}(f) \hookrightarrow M$. Take π to be the projection from μ_1 onto M_f. Taking $\xi \in \mathfrak{z}_f$, the Lie algebra of Z_f, we get

$$\iota(v_\xi)i^*\omega = i^*\iota(v_\xi)\omega = i^*\mu_1^*(\theta^l + \theta^r, \xi) = 0.$$

Condition (i): Take χ_1 and χ_2 be the canonical 3-forms for G_1 and G_2 respectively. Then

$$\pi^* d\omega_f = di * \omega = i^* d\omega = -i^*(\Phi_1\chi_1 + \Phi_2\chi_2) = -i^*\Phi_2\chi_2 = -\pi^*(\Phi_2)_f^*\chi_2.$$

Condition (ii): Let $\sigma_f \in^{b^m} \Omega^2(M_f)^{G_2}$ be the unique 2-form such that $\pi^*\sigma_f = i^*\sigma$. The restriction $i^*\Phi_2$ is $Z_f \times G_2$-invariant and descends to an equivariant map $(\Phi_2)_f \in^{b^m} \mathcal{C}^\infty(M_f, G_2)^{G_2}$. □

Now we can provide the last statement needed to finalize our reduction theorem.

Definition 5.2. *Let (M, σ, Φ) be a quasi-Hamiltonian G-space. Given a subspace $W \subset T_xM$, let $W^\sigma \subset T_xM$ denote the subspace of σ orthogonal vectors. The symplectic slice at $p \in M$ is the vector space*

$$V = (T_p\mathcal{O})^\sigma / \left(T_p\mathcal{O} \cap (T_p\mathcal{O})^\sigma\right),$$

where $\mathcal{O} = G \cdot p$ is the G orbit of p.

Notice that, even if M is not a symplectic space, by the axioms for quasi-Hamiltonian G-spaces, the kernel of σ_p is contained entirely in $T_p\mathcal{O}$. Hence V is a symplectic vector space.

Theorem 5.5 (quasi-Hamiltonian Slice Theorem, Bott-Tolman-Weitsman)**.** *Let (M, σ, Φ) be a quasi-Hamiltonian G-space. For any $p \in M$, let $H = Stab(p)$, $K = Stab(\Phi(p))$, and V be the symplectic slice at p. There exists a neighbourhood of the orbit $G \cdot p$ which is equivariantly diffeomorphic to a* *neighborhood of the orbit* $G \cdot [e, 0, 0]$ *in*

$$Y := G \times_H ((\mathfrak{h}^\perp \cap \mathfrak{k}) \times V).$$

In terms of the diffeomorphism, the moment map can be written as

$$\Phi([g, \gamma, v]) = Ad_g(\Phi(p) \exp(\gamma + \varphi(v)),$$

where $\varphi : V \to \mathfrak{h}^* \simeq \mathfrak{h}$ *is the moment map for slice representation.*

Finally, we can use the quasi-Hamiltonian reduction to complete the statement and the proof of the b^m-quasi-Hamiltonian reduction theorem. In definition of b^m-Hamiltonian moment map, the moment map μ is an element in ${}^{b^m}\mathcal{C}^\infty(M) \otimes \mathfrak{g}^*$. In the quasi-Hamiltonian realm, one can consider $\Phi \in^{b^m} \mathcal{C}^\infty(M) \otimes G$ to be a suitable moment map.
This allows us to investigate the case of a group action on a b-manifold when the H-component is not b^m-Hamiltonian. For the quasi-Hamiltonian reduction we will abuse the notation of Marsden-Weinstein reduction even though the reduced space $M//G$ will not be necessarily symplectic (see Theorem 4.2 and Corollary 5.2). It can be seen as a singular quasi-Hamiltonian with the moment map taking values in $S^1 \times H$. Then Theorem 5.4 allows us to do reduction by stages even without having a splitting of the moment map at our disposal as in Theorem 4.2. Notice that the transverse S^1-action is b^m-Hamiltonian in view of Corollary 5.1). Theorem 5.4 declares $M//S^1$ as a quasi-Hamiltonian space with an induced H-action. Bearing this in mind, we consider solely the S^1-action on M and perform the singular Marsden-Weinstein reduction depicted in Theorem 4.2 with respect to this circle action.
As a result, we notice two essential properties of the reduced space: it is a quasi-Hamiltonian space endowed with an H-action, and the singularity has been eliminated from the singular form. This means that the reduction is a honest quasi-Hamiltonian space. This allows us to use the Bott-Tolman-Weitsman quasi-Hamiltonian reduction Theorem 5.5 directly.

We denote by $\mathbf{f_0}$ a regular value of the moment map. In view of Corollary 5.1 as in Remark 5.3, the moment map splits.
This leads us to the final statement of our main theorem under the following assumptions:

- The induced action of H is locally free.
- The action of S^1 on the covering model associated to the finite group Γ is free.
- The first component of $\mathbf{f_0}$ is $\exp(0)$ (a regular value for the induced S^1-action).

Theorem 5.6 (singular quasi-Hamiltonian reduction of b^m-type)**.** *Given a singular quasi-Hamiltonian space with a b^m-type singularity (M, σ, Z) and a transverse G-action with group-valued moment map Φ. If the highest modular weight for the S^1-component of the G-action is non-zero, the pre-image of a regular point $\Phi^{-1}(\mathbf{f_0})$ admits an induced action of G. The space of orbits of the induced action $M//G$ is quasi-Hamiltonian.*

When the group G is abelian (confer [HJS06] and [Boa07] for details in the standard quasi-Hamiltonian case), the theorem above yields a honest symplectic orbifold by reduction:

Corollary 5.2. *If the group G is abelian, the reduced quasi-Hamiltonian space $M//G$ is a symplectic orbifold.*

Chapter 6

Riemann-Hilbert Problem for Hypergeometric Equation and Poisson Structures

6.1 Intermezzo

In the last two chapters of this thesis, we study the Poisson and symplectic geometry of a different type. Thinking in terms of physical configuration and phase space coordinatization, we study the Poisson structures dependent explicitly on a time variable. Such structures are often called non-autonomous. It seems that b- and b^m-symplectic structures are very different from such non-autonomous structures. But in fact, it is not so. The difficulty is hidden as we will see in the following example of the non-autonomous Poisson structure associated with the second Painlevé transcendent

$$\frac{d^2y}{dt^2} = 2y^3 + ty + \alpha.$$

The "naive" Okamoto non-autonomous phase space for P_{II} has 4 coordinates: q, p, H and t. Written in this coordinates, equation P_{II} has the form

$$p_{tt} = 2p^3 - 4pt + 4\left(\alpha + \frac{1}{2}\right).$$

Okamoto proposed the following non-autonomous Hamiltonian:

$$H_{II}(p,q) = 4p^2 + \frac{1}{4}pq^2 + 2pt - \frac{1}{4}q(2\alpha - 1).$$

Then the equation P_{II} is equivalent to the following non-autonomous Hamiltonian system Taking sigma coordinates as derivations of the Hamiltonian itself

$$\begin{cases} \dot{p} = -\frac{\partial H_{II}}{\partial q} = -\frac{1}{2}pq + \frac{1}{4}(2\alpha - 1) \\ \dot{q} = \frac{\partial H_{II}}{\partial p} = 8p + \frac{1}{4}q^2 + 2t. \end{cases}$$

The symplectic 2-form $dp \wedge dq$ can be written as a b-symplectic 2-form using another coordinatization of the Okamoto phase space in terms of σ-coordinates. Let us consider the Hamiltonian H_{II} as a proto coordinate and denote it as σ and the first derivative $\sigma_1 = d\sigma$ and second derivative $\sigma_2 = d\sigma_1$ of σ can be considered as alternative coordinate functions that are related to p and q as follows

$$p = \frac{1}{2}\sigma_1,$$

$$q = 2\frac{-\sigma_2 + \alpha - 1/2}{\sigma_1}.$$

Then the form $dp \wedge dq$ can be written as a b-form in terms of σ

$$dp \wedge dq = d\sigma_1 \wedge d\left(2\frac{-\sigma_2 + \alpha - 1/2}{\sigma_1}\right) = \frac{d\sigma_1}{\sigma_1} \wedge d\sigma_2.$$

The corresponding σ-coordinate presentation on the whole Painlevé II hierarchy is studies in detail in [BM21].

One can show that in the case of P_I, σ-coordinates give just a symplectic form. In the case of other Painlevé transcendents, a good *sigma*-cordinatization appears to be too complicated. Therefore, we investigate alternative Poisson descriptions of linear systems as well as their monodromy data. It is interesting that Goldman-type coordinates for $\Sigma_{0,4}$ for zero values of monodromy traces except coefficient at ∞.

As show by Chekhov, Rubtsov and Mazzocco in [CMR17] and Hikami in [Hik19] using results of Fock-Goncharov [FG06], all the character varieties of Painlevé transcendents can be shown as either b-symplectic or their reductions.

6.2 Rank one Riemann-Hilbert problem on sphere

In rank 1 over the sphere the Riemann-Hilbert problem can be solved explicitly and the solution is quite simple. Namely, locally it is easy to construct a function multiplying by g_0 under analytic continuation around the singular point $z = 0$. For example, from the fact that $y(z) = z^\alpha$ while going around zero becomes $z^\alpha \cdot \exp 2\pi\imath\alpha$, where $\imath$ denotes imaginary unit it follows that $y(z) = z^{\frac{1}{2\pi\imath}\ln g_0}$ satisfies the mentioned condition on ramification for any branch of the complex logarithm. From the set of local solutions of this kind one can trivially get the global solution.

Indeed, take

$$y(z) = \prod_{i=1}^{n} (z - a_i)^{\frac{1}{2\pi\imath}\ln g_i}.$$

where branches of complex logarithms are fixed by the condition $\Im(\ln g_i) \in [0, 2\pi\imath)$.

It is obvious that $y(z)$ after continuation around a_i becomes $y(z)g_i$. Then from $g_1 \ldots g_n = 1$ it follows that $\frac{1}{2\pi\imath}(\ln g_1 + \ldots + \ln g_n) = k$, where $k \in \mathbb{Z}$. In the limit $z \to \infty$ we obtain

$$y(z) \sim z^{\left(\sum_{i=1}^{n} \frac{1}{2\pi\imath} \ln g_i\right)} = z^k$$

Notice that function $\widetilde{y}(z) = \frac{y(z)}{(z-a_1)^k}$ in the complex plane has no singular points different from the singularities of $y(z)$ and ramifies along the loops encircling each of them just as $y(z)$. Therefore $\widetilde{y}(z)$ has given singular points and branching and is holomorphic at infinity. Thus Fuchsian equation with required monodromy and singularities has the form:

$$\frac{d\widetilde{y}(z)}{dz} = \left(\frac{\frac{1}{2\pi\imath}\ln g_1}{z-a_1} + \ldots + \frac{\frac{1}{2\pi\imath}\ln g_n}{z-a_n} - \frac{k}{z-a_1}\right)\widetilde{y}.$$

Notice that the obtained expression can be considered as isomonodromic family. Taking the positions of singular points as parameters of the family, we see that for fixed coefficients of residues, monodromy of the equations in the family doesn't depend on the position of a_i. Therefore all equations of the family have the same monodromy, isomonodromic deformation in rank 1 is trivial.

6.2.1 Line bundles on elliptic curve

The most important objects related to elliptic curves are theta functions. They play a key role for many analytic and geometric constructions on these curves. In this section, following [PS97] we give definition and a short review of basic properties of the theta function we need (θ_1-Jacobi function) and then with the use of all thees we describe and classify linear holomorphic vector bundles on the elliptic curve.

6.2.2 θ-function

Define on the complex plane the function $\theta(z)$ by

$$\theta(z) = \theta_1(z|\tau) = \imath \sum_{m\in\mathbb{Z}} (-1)^m q^{(m-\frac{1}{2})^2} e^{(m-\frac{1}{2})2\pi\imath z},$$

where $q(\tau) = e^{\imath\pi\tau} = e^{\imath\pi x - \pi y}$ sets the mapping of the upper half-plane $H = \{\tau \in \mathbb{C} | \Im\tau > 0\}$ into the unit circle $D = \{q \in \mathbb{C} \,|\, |q| \le 1\}$.

Consider module of ratio of adjacent terms in the expansion of $\theta(z)$. It is equal to $|-q^{2m}e^{2\pi iz}| \le |q|^{2m} e^{2\pi|z|}$. Since $\lim_{m\to\infty} |q|^{2m} = 0$, then θ-function is given by the series of the entire functions of z, converging absolutely and uniformly in any circle centered at zero. Therefore the function $\theta(z)$ itself is entire. Directly from the definition it is easy to check that $\theta(z)$ is an odd function $\theta(z) = -\theta(-z)$ and hence $\theta(0) = 0$.

We shall need an information about branching of $\theta(z)$ and its derivative. Directly from definition we get

$$\begin{aligned} \theta(z+1) &= -\theta(z) \\ \theta(z+\tau) &= -q^{-1} e^{-2\pi iz} \theta(z). \end{aligned}$$

That implies following relations on the derivatives:

$$\begin{aligned} \theta'(z+1) &= -\theta'(z) \\ \theta'(z+\tau) &= q^{-1} e^{-2\pi iz} \big(2\pi i \theta(z) - \theta'(z)\big). \end{aligned}$$

Therefore,

$$\begin{aligned} \frac{\theta'(z+1)}{\theta(z+1)} &= \frac{\theta'(z)}{\theta(z)} \\ \frac{\theta'(z+\tau)}{\theta(z+\tau)} &= \frac{\theta'(z)}{\theta(z)} - 2\pi i. \end{aligned} \tag{6.2.1}$$

We also need the expression for the shifted theta functions. Let a be an arbitrary point of the elliptic curve. Then

$$\begin{aligned} \theta(z-a+1) &= -\theta(z-a) \\ \theta(z-a+\tau) &= -q^{-1}e^{-2\pi\imath z}\theta(z-a)e^{2\pi\imath a} \end{aligned} \tag{6.2.2}$$

The formulas above imply that integral of logarithmic derivative of $\theta(z)$ over the perimeter of fundamental parallelogram equals to $2\pi\imath$. Since $\theta(z)$ has no poles inside the fundamental parallelogram, it has the only simple zero there and as we have already seen it is located in the point $z = 0$. Let us examine how changes the value of $\theta^\alpha(z)$ under analytic continuation along the loop around $z = 0$. Since zero is simple, it branches similar to z^α. Denoting g^* operator of the monodromy around zero, it is

$$g^*\left(\theta^\alpha(z)\right) = \theta^\alpha(z)\cdot e^{2\pi\imath\alpha}. \tag{6.2.3}$$

6.2.3 Line bundles on elliptic curve

Denote Λ_τ an elliptic curve, obtained by factorization of the complex plane by lattice $\{1,\tau\}$, $\mathrm{Im}\tau > 0$. On the curve Λ_τ vector bundle can be set by action of two shifts: by 1 and by τ on sections of the bundle. It suffices to consider sections over the fundamental parallelogram.

Consider a holomorphic line bundle of degree zero E over the elliptic curve Λ_τ and $\varphi(z)$ to be a meromorphic section of E. Since $\deg E = 0$ section $\varphi(z)$ has equal number of zeroes and poles inside the fundamental parallelogram. It also has some monodromy corresponding to a- and b-cycles, or, which is the same corresponding to shifts by 1 and τ. This monodromies are not uniquely defined, one can always set monodromy corresponding to 1-shift to be equal to 1 and monodromy corresponding to τ-shift equal to some constant ν.

In that setting parameter ν still is not uniquely defined. Multiplying the section by $e^{2\pi\imath z}$ preserves its zeroes, poles, invariance under shifting by one and changes ν to $\nu\cdot e^{2\pi\imath\tau}$. Hence ν is defined up to multiplication by an integer power of $e^{2\pi\imath\tau}$. To work with it is more convenient to take parameter

λ connected with ν via the relation $\nu = e^{2\pi\imath\lambda}$. Parameter λ is defined on the complex plane up to shifts along the lattice $\{1, \tau\}$ i.e. parameter λ encoding line bundles on the curve Λ_τ takes values from the curve Λ_τ itself. It is a well known fact that the moduli space of line bundles of fixed degree on an elliptic curve is an elliptic curve itself with the same modular parameter.
Hence considering a section of E we can assume $\varphi(z)$ to be invariant when z shifts by one and multiplied by $e^{2\pi\imath\lambda}$ when z shifts by τ. Such an objects we can effectively investigate using θ-functions defined above.
Consider $\varphi_\lambda(z) = \frac{\theta(z-\lambda)}{\theta(z)}$. From the properties of θ-function given in the previous section follows that

$$\begin{aligned} \varphi_\lambda(z+1) &= \varphi(z) \\ \varphi_\lambda(z+\tau) &= \varphi(z) \cdot e^{2\pi\imath\lambda} \end{aligned} \tag{6.2.4}$$

$\varphi_\lambda(z)$ has exactly one zero and one pole on the elliptic curve. Therefore $\varphi_\lambda(z)$ is a section of a line bundle E of degree zero. Denote this bundle as $\mathcal{O}_\lambda(0)$. Further, ${}^b\varphi_\lambda(z) = \varphi_\lambda(z-b)$ differs from $\varphi_\lambda(z)$ by multiplication on meromorphic function and thus it is also a section of $\mathcal{O}_\lambda(0)$ for any point b on the elliptic curve. Hence the modular parameter λ together with degree k completely define the line bundle $\mathcal{O}_\lambda(k)$, the ratio of two sections with equal λ and k is a meromorphic function on Λ_τ.
Now consider $\varphi(z)$ to be the product of k different sections of the type ${}^{b_i}\varphi_\lambda(z)$. It is a section of $\mathcal{O}_{k\lambda}(0)$. Denoting zeros and poles of this product by a_i we get

$$\varphi(z) = \theta^{k_1}(z-a_1)\cdots\theta^{k_n}(z-a_n),$$

where k_i-integer and

$$\sum_{i=1}^{n} k_i = 0.$$

From the relations (6.2.2) we obtain

$$\begin{aligned} \varphi(z+1) &= (-1)^{\sum k_i}\varphi(z) = \varphi(z) \\ \varphi(z+\tau) &= \varphi(z) \cdot e^{2\pi\imath\sum k_i a_i} \end{aligned} \tag{6.2.5}$$

Since $\varphi(z)$ is a section if the $\mathcal{O}_{k\lambda}(0)$ it implies

$$\sum_{i=1}^{n} k_i a_i = k\lambda.$$

It is easy to see that for any set of points a_i an expression

$$\varphi(z) = \theta^{\alpha_1}(z - a_1) \cdots \theta^{\alpha_n}(z - a_n) \tag{6.2.6}$$

with any complex α_i such that $\sum \alpha_i = 0$, gives a (multivalued) section of the bundle $\mathcal{O}_\lambda(0)$ where

$$\lambda = \sum_{i=1}^{n} \alpha_i a_i \tag{6.2.7}$$

Now let us find in the bundle $\mathcal{O}_\lambda(0)$ the connection form for which φ_λ is a horizontal section. We mention that when we are talking about sections here we mean an entire analytic function by Weierstrass constructed by germ of the local horizontal section.

$$d\varphi(z) = \omega_\lambda(z)\varphi(z)$$

From $d\theta^{\alpha_i}(z - a_i) = \alpha_i \theta'(z - a_i)\theta^{\alpha_i - 1}(z - a_i)dz$ we get

$$d\varphi = \sum_{i=1}^{n} \alpha_i \frac{\theta'(z - a_i)}{\theta(z - a_i)} \varphi dz \tag{6.2.8}$$

$$\omega_\lambda(z) = \sum_{i=1}^{n} \alpha_i \frac{\theta'(z - a_i)}{\theta(z - a_i)} dz \tag{6.2.9}$$

The connection form we get has logarithmic singularities at points a_i. Consider how does this differential form changes under shifts by 1 and by τ.

$$\omega_\lambda(z+1) = \sum_{i=1}^{n} \alpha_i \frac{\theta'(z-a_i+1)}{\theta(z-a_i+1)} dz = \sum_{i=1}^{n} \alpha_i \frac{\theta'(z-a_i)}{\theta(z-a_i)} dz = \omega_\lambda(z)$$

$$\omega_\lambda(z+\tau) = \sum_{i=1}^{n} \alpha_i \frac{\theta'(z-a_i+\tau)}{\theta(z-a_i+\tau)} dz = \sum_{i=1}^{n} \alpha_i \frac{\theta'(z-a_i)}{\theta(z-a_i)} dz - 2\pi\imath \sum_{i=1}^{n} \alpha_i = \omega_\lambda(z)$$

We see that connection form doesn't change when shifting, that is a feature of line bundles. In general case for greater dimensions, connection form conjugates by a matrix dependent on λ.

6.2.4 Rank 1 Riemann problem on elliptic curve

As it was mentioned above, rank 1 Riemann problem on elliptic curve consists in searching for logarithmic connection in semistable bundle of degree zero with given monodromy and singularities. In the rank 1 case we don't need to care about semistability, all line bundles are semistable.

6.2.5 Monodromy data on elliptic curve

Suppose we are given an elliptic curve Λ_τ with logarithmic connection, set of singular points $a_1, \ldots, a_n$ and monodromy representation

$$\chi : \pi_1 \left(\Lambda_\tau \setminus \{a_1, \ldots, a_n\}\right) \to \mathrm{GL}(1, \mathbb{C}) \simeq \mathbb{C}^*$$

Namely we are given a set of numbers $g_1, \ldots, g_n, \nu_1, \nu_2$ by which multiplies local horizontal section when it continues along the $a-$cycle, $b-$cycle and around singular points respectively. Let us see what are the conditions that monodromy should satisfy and which of the forms of encoding it is the most convenient to deal with.

The fundamental group of an elliptic curve satisfy the relation $aba^{-1}b^{-1} = \mathrm{id}$, the loop, encircling the fundamental parallelogram along perimeter can be contracted inside it. Obviously, that for some natural ordering points a_i and

choice of classes of basic loops γ_i encircling them, the sequential bypassing all the punctures is equivalent to bypassing the perimeter of the fundamental parallelogram and hence the relation in the fundamental group of punctured torus is $\gamma_1 \cdots \gamma_n = aba^{-1}b^{-1}$ or $\gamma_1 \cdots \gamma_n bab^{-1}a^{-1} = \mathrm{id}$. It corresponds to the condition $g_1 \cdots g_n \nu_2 \nu_1 \nu_2^{-1} \nu_1^{-1} = 1$ on the monodromy matrices. Since the problem is in rank 1, the monodromy matrices are all 1×1 and commute. Therefore, $\nu_2 \nu_1 \nu_2^{-1} \nu_1^{-1} = 1$ and $g_1 \cdots g_n = 1$ fulfill simultaneously.
Furthermore, as it was already mentioned, monodromy corresponding to the periods is not defined uniquely. The section with the monodromy (ν_1, ν_2) can be transformed by elementary gauge transformation preserving all other properties into section with (ν_1', ν_2') if $\nu_1^\tau/\nu_2 = {\nu'}_1^\tau/{\nu'}_2$. Therefore we can always set one of the cyclic monodromies to be trivial. We choose for convenience $\nu_1 = 1$, and define λ by the ratio $\nu_2 = e^{2\pi i \lambda}$. Order of (ν_1, ν_2) corresponds here to shifts by 1 and τ on the curve Λ_τ. As we have already seen above λ is defined modulo the lattice $\{1, \tau\}$ shifts.
So finally the input data of monodromy for rank 1 Riemann problem on the elliptic curve Λ_τ is a set $g_1, \ldots, g_n, \lambda$ such that $g_1 \cdots g_n = 1$.

6.2.6 Construction of the solution

From the calculations above, it follows that the rank 1 Riemann problem on elliptic curve can be formulated as follows.
For given elliptic curve Λ_τ, set of singular points $\{a_1, \ldots, a_n\}$ and monodromy data $g_1, \ldots g_n, \lambda$ to construct logarithmic connection in some bundle $\mathcal{O}_{\widehat{\lambda}}(0)$ with singularities $\{a_1, \ldots, a_n\}$ and monodromy $g_1, \ldots g_n, \lambda$.
We will show that the problem can be solved explicitly for any given $\Lambda_\tau, \{a_1, \ldots, a_n\}$ and $g_1, \ldots g_n$ if and only if parameter λ belongs to some discrete set, that we describe below.
Let us start from constructing auxiliary section $\psi(z)$ with required branching around singular points, invariant under shift by one and and being a section of some bundle of degree zero.
Define normalized set $\alpha_1, \ldots \alpha_n$ as follows. We choose α_k to satisfy $\exp(2\pi \imath \alpha_k) =$

g_k. Each of α_k is defined up to integer summand, and for any choice from $g_1 \cdots g_n = 1$ should holds $\alpha_1 + \cdots \alpha_n = p, p \in \mathbb{Z}$. We call a set $\alpha_1, \dots \alpha_n$ normalized if the sum of all its elements is equal to zero. Obviously for any initial choice of α_k the set can be normalized by changing at least one of its components.

Consider $\alpha_1, \dots \alpha_n$ to be some normalized set constructed by given $g_1, \dots g_n$ and let

$$\psi(z) = \theta^{\alpha_1}(z - a_1) \cdots \theta^{\alpha_n}(z - a_n)$$

From properties of theta functions (6.2.3) it follows that the monodromy of $\psi(z)$ under continuation around $z = a_k$ is equal to $\exp(2\pi\imath\alpha_k) = g_k$. The behavior of $\psi(z)$ when shifting by 1 and τ is described by relations (6.2.5) hence $\psi(z)$ is a section of the bundle $\mathcal{O}_{\widehat{\lambda}}(0)$, where $\widehat{\lambda} = \sum_{i=1}^{n} \alpha_i a_i$, horizontal for the logarithmic connection (6.2.9)

$$\nabla = d - \sum_{i=1}^{n} \alpha_i \frac{\theta'(z - a_i)}{\theta(z - a_i)} dz$$

So $\psi(z)$ solves the Riemann problem with perhaps slightly different from initial monodromy data. Namely, while all other parameters coincide the $\widehat{\lambda}$ can differ from required λ. Let us show that the Riemann problem can not have positive solution if in initial monodromy data λ is not equal to $\sum_{i=1}^{n} \alpha_i a_i$ for some normalized set α_k.

Assume the contrary, let such a solution exists and $\eta(z)$ to be corresponding horizontal section of $\mathcal{O}_{\lambda}(0)$. Consider the ratio $\xi(z) = \eta(z)/\psi(z)$. Section $\xi(z)$ has in the fundamental parallelogram equal number of zeroes and poles because $\eta(z)$ and $\psi(z)$ are sections of bundles of equal degrees. Furthermore, all the zeros and the poles are located at the points a_i as long as $\eta(z)$ and $\psi(z)$ do not vanish anywhere else. When shifting by one $\xi(z)$ doesn't change, when shifting by τ it multiplies by $e^{2\pi\imath(\lambda - \widehat{\lambda})}$. Therefore $\xi(z)$ is single-valued in the fundamental parallelogram section of the bundle $\mathcal{O}_{\lambda - \widehat{\lambda}}(0)$ with m zeroes and m poles located at the points a_i. Denote k_i the order of zero or pole at

a_i and consider the ratio

$$\widetilde{\xi}(z) = \frac{\xi(z)}{\theta^{k_1}(z-a_1)\cdots\theta^{k_n}(z-a_n)}.$$

Relations (6.2.5) imply the denominator stays invariant when z shifts by 1, and multiplies by $e^{2\pi\imath\sum_{i=1}^{n}k_ia_i}$ when it shifts by τ. Hence, $\widetilde{\xi}(z)$ is a smooth section of $\mathcal{O}_{\widehat{\lambda}-\lambda-\sum_{i=1}^{n}k_ia_i}(0)$ with no zeroes in fundamental parallelogram. That is possible only when $\lambda-\widehat{\lambda}-\sum_{i=1}^{n}k_ia_i=0$. Indeed taking any fixed w in fundamental parallelogram and considering $\widetilde{\xi}(z+w)/\widetilde{\xi}(z)$ we see that it is a holomorphic double-periodic function.

$$\frac{\widetilde{\xi}(z+w)}{\widetilde{\xi}(z)} = \frac{\widetilde{\xi}(z+w+1)}{\widetilde{\xi}(z+1)} = \frac{\widetilde{\xi}(z+w+\tau)}{\widetilde{\xi}(z+\tau)}.$$

Therefore for any fixed w that ratio is a bounded entire function on the complex plane and hence, due to Liouville theorem it is constant

$$\widetilde{\xi}(z+w) = \widetilde{\xi}(z)\cdot C(w).$$

From $\widetilde{\xi}(w) = \widetilde{\xi}(0+w) = \widetilde{\xi}(0)\cdot C(w)$ and $C(w) = \widetilde{\xi}(w)/\widetilde{\xi}(0)$ we get

$$\frac{\widetilde{\xi}(z+w)}{\widetilde{\xi}(0)} = \frac{\widetilde{\xi}(z)}{\widetilde{\xi}(0)}\cdot\frac{\widetilde{\xi}(w)}{\widetilde{\xi}(0)}.$$

But if a holomorphic function $f(x)$ satisfies $f(x+y)=f(x)\cdot f(y)$ for all x,z then $f(x+h)-f(x)=f(x)(f(h)-f(0))$ and sending h to zero we get $f'(x)=f'(0)\cdot f(x)$ and hence $f(x)=e^{f'(0)x}$. Therefore $\widetilde{\xi}(z)=\widetilde{\xi}(0)\cdot e^{\beta z}$. From $\widetilde{\xi}(z)=\widetilde{\xi}(z+1)$ we get $\beta=2\pi\imath$ and finally $\widetilde{\xi}(z)=\widetilde{\xi}(0)\cdot e^{2\pi\imath z}$. The resulting function has uniquely defined behaviour when z shifts by τ, the corresponding multiplier equals $e^{2\pi\imath\tau}$ implying that this function is a section of $\mathcal{O}_\tau(0)$, or, equivalently $\mathcal{O}_0(0)$ as the modular parameter of the bundle is defined modulo lattice $\mathbb{Z}+\tau\mathbb{Z}$ as we have already mentioned.

Therefore for positive solvability of Riemann-Hilbert problem $\lambda-\widehat{\lambda}-\sum_{i=1}^{n}k_ia_i$

must equals to zero modulo $\mathbb{Z}+\tau\mathbb{Z}$. Notice that $\widehat{\lambda}+\sum_{i=1}^{n} k_i a_i = \sum_{i=1}^{n} \alpha_i a_i - \sum_{i=1}^{n} k_i a_i = \sum_{i=1}^{n} \widetilde{\alpha}_i a_i = \widetilde{\lambda}$ where $\widetilde{\alpha}_i = \alpha_i + k_i$ some new normalized set $\widetilde{\alpha}_i$. And hence the modular parameter λ coincide with $\widetilde{\lambda} = \sum_{i=1}^{n} \widetilde{\alpha}_i a_i$ for some normalized set $\widetilde{\alpha}_i$.
Summarizing all above we get the following statement

Theorem 6.1. *The rank* 1 *Riemann problem for given elliptic curve* Λ_τ*, singular points* $\{a_1, \ldots, a_n\}$ *and monodromy data* $g_1, \ldots, g_n, \lambda$ *is solved positively in a trivial bundle if and only if*

$$\lambda = \sum_{k=1}^{n} \alpha_k a_k + p + q\tau$$

for some integers p *and* q *and normalized* n*-tuple* $\alpha_1, \ldots, \alpha_n$*, where* $e^{2\pi\imath\alpha k} = g_k$.
The corresponding connection form in the bundle $\mathcal{O}_\lambda(0)$ *is*

$$\omega_\lambda(z) = \sum_{k=1}^{n} \alpha_k \frac{\theta'(z-a_k)}{\theta(z-a_k)} dz.$$

Otherwise, the same connection form solves the problem in $\mathcal{O}_{\sum_{k=1}^{n} \alpha_k a_k - \lambda}(0)$ *and there exist no other solutions.*

All alike the Riemann sphere case the solution we obtained is (a trivial) isomonodromic family. Taking the connection coefficients to be constant when the pole's positions vary, the monodromy stay invariant while the line bundle's modular parameter λ changes obeying to (6.2.7).

6.3 Rank 2 RHP on Riemann sphere

It it known that in rank 2 on Riemann sphere the Riemann-Hilbert problem has positive solution for any poles positions and monodromy representations. For systems with three singular points the solution can be constructed explicitly. The explicit solvability of Riemann-Hilbert problem in that case is based on explicit solvability of corresponding Deligne-Simpson problem mutually related with the classical Riemann-Hilbert problem. In its multiplicative and additive versions it asks being given a set of orbits $\mathcal{G}_i$ of adjoint action of $\mathrm{GL}(p, \mathbb{C})$ on itself, or, in additive version on $\mathfrak{gl}(2, \mathbb{C})$ to choose an irreducible set $g_i \in \mathcal{G}_i$ satisfying $g_1 \cdot \dots \cdot g_n = 1$ or $g_1 + \dots + g_n = 0$ respectively. It is unsolved in general setting, but $p = 2$, $n = 3$ can be easily solved explicitly. Below we give detailed explanation.

6.3.1 Rigid representations

The representation is called rigid if it can be restored from the spectra of local monodromy. The rank 2 irreducible representations of the fundamental group of a sphere with three punctures are rigid. It is easy to see that for irreducible representations a triple of local monodromy matrices G_1, G_2, G_3 corresponding to loops encircling the punctures at a_1, a_2, a_3 is defined by the spectra $(\lambda_1, \lambda_2), (\mu_1, \mu_2), (\nu_1, \nu_2)$ uniquely up to an overall conjugation. Reducible case can be treated separately, it is not very difficult but need special approach and study for each of a number of degenerate cases. The criterion of representation irreducibility in terms of eigenvalues is well-known, if for all i, j, k there holds $\lambda_i \mu_j \nu_k \neq 1$ then the representation is irreducible. Consider the basis consisting of the vectors e_1 and e_2, the non-collinear eigenvectors of G_1 and G_2 respectively. For irreducible representations such a pair obviously exists. One can always normalize one of basis vectors in a way that G_1 has the form

$$G_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}.$$

By construction G_2 in this basis is lower triangular:

$$G_2 = \begin{pmatrix} \mu_1 & 0 \\ k & \mu_2 \end{pmatrix}.$$

Matrix G_3 can be obtained from the relation $G_1 G_2 G_3 = 1$ or $G_3 = (G_1 G_2)^{-1}$

that gives

$$G_3 = \frac{1}{\lambda_1\lambda_2\mu_1\mu_2}\begin{pmatrix} \lambda_2\mu_2 & -\mu_2 \\ -\lambda_2 k & \lambda_1\mu_1 + k \end{pmatrix}.$$

The only parameter k sets the representation and can be obtained from the relation on trace of G_3:

$$\frac{\lambda_1\mu_1 + \lambda_2\mu_2 + k}{\lambda_1\lambda_2\mu_1\mu_2} = \nu_1 + \nu_2.$$

Therefore the spectra $(\lambda_1, \lambda_2), (\mu_1, \mu_2)$ and (ν_1, ν_2) together with relation $G_1 \cdot G_2 \cdot G_3 = 1$ uniquely define the triple

$$G_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix},\ G_2 = \begin{pmatrix} \mu_1 & 0 \\ (\nu_1+\nu_2)\lambda_1\lambda_2\mu_1\mu_2 - \lambda_1\mu_1 - \lambda_2\mu_2 & \mu_2 \end{pmatrix},\ G_3 = (G_1 {\cdot} G_2)^{-1}$$

modulo an overall conjugation and hence, monodromy representation is fixed and therefore rigid.

Additive version of $p = 2, n = 3$ Deligne-Simpson problem can be solved in completely analogues way. Consider we are given spectra $(\widetilde{\lambda}_1, \widetilde{\lambda}_2)$, $(\widetilde{\mu}_1, \widetilde{\mu}_2)$, $(\widetilde{\nu}_1, \widetilde{\nu}_2)$ such that $\widetilde{\lambda}_i + \widetilde{\mu}_j + \widetilde{\nu}_k \neq 0$ for all i, j, k. Then, taking B_1 to be upper-triangular

$$B_1 = \begin{pmatrix} \widetilde{\lambda}_1 & 1 \\ 0 & \widetilde{\lambda}_2 \end{pmatrix}.$$

B_2 to be lower triangular

$$B_2 = \begin{pmatrix} \widetilde{\mu}_1 & 0 \\ \kappa & \widetilde{\mu}_2 \end{pmatrix}.$$

we get B_3 defined by matrices B_1 and B_2:

$$B_3 = -B_1 - B_2 = -\begin{pmatrix} \widetilde{\lambda}_1 + \widetilde{\mu}_1 & 1 \\ \kappa & \widetilde{\lambda}_2 + \widetilde{\mu}_2 \end{pmatrix}.$$

And the only parameter κ can be computed from the relation on eigenvalues of B_3

$$\det B_3 = \widetilde{\nu}_1\widetilde{\nu}_2$$

$$(\widetilde{\lambda}_1 + \widetilde{\mu}_1)(\widetilde{\lambda}_2 + \widetilde{\mu}_2) - \kappa = \widetilde{\nu}_1\widetilde{\nu}_2$$

$$\kappa = (\widetilde{\lambda}_1 + \widetilde{\mu}_1)(\widetilde{\lambda}_2 + \widetilde{\mu}_2) - \widetilde{\nu}_1\widetilde{\nu}_2.$$

6.3.2 Explicit construction of solution on Riemann sphere

A logarithmic connection ∇ on Riemann sphere has the form $\nabla = d - \omega(z)$, where $\omega(z)$ is a matrix differential one-form having only simple poles as singular points.

Below we enlist some properties of logarithmic connections required for our construction. Proofs can be found for example at [Bol93]. Consider $a_i \neq \infty$ for all i.

Statement 6.1. *A matrix one-form of logarithmic connection with three singular points $\{a_1, a_2, a_3\}$ on Riemann sphere is set by the triple of residue matrices (B_1, B_2, B_3) defined up to an overall conjugation and satisfying $B_1 + B_2 + B_3 = 0$:*

$$\omega(z) = \left(\frac{B_1}{z - a_1} + \frac{B_2}{z - a_2} + \frac{B_3}{z - a_3}\right) dz,$$

If the eigenvalues of B_i do not differ by a natural number the point a_i is called non-resonant.

Statement 6.2. *In non-resonant point, local monodromy of connection is conjugated to the exponent of the corresponding residue multiplied by $2\pi\imath$*

$$G_i \sim \exp\left(2\pi\imath \operatorname*{Res}_{z=a_i} \omega(z)\right) = e^{2\pi\imath B_i}.$$

Statement 6.3. *For logarithmic connection $\nabla = d - \omega(z)$, the eigenvalues of the local monodromy G_i coincide with the eigenvalues of $\exp\big(2\pi\imath \operatorname{Res}_{z=a_i} \omega(z)\big) = \exp(2\pi\imath B_i)$.*

Statement 6.4 (Fuchs relation). *The sum of eigenvalues of $B_i = \mathrm{Res}_{z=a_i}\,\omega(z)$ over all singular points of a logarithmic connection is equal to zero.*

The statements above together with results of section 6.3.1 shows the way to explicit construction of logarithmic connection with three singular points and irreducible monodromy.

Corollary 6.1. *Consider an irreducible representation*

$$\chi : \pi_1(\mathbb{CP}^1 \setminus \{a_1, a_2, a_3\}) \to \mathrm{GL}(2, \mathbb{C})$$

with eigenvalues of $\chi(\gamma_{1,2,3})$ being equal to (λ_1, λ_2), (μ_1, μ_2), (ν_1, ν_2) respectively and fix complex logarithms of these eigenvalues in a way that Fuchs relation

$$\ln \lambda_1 + \ln \lambda_2 + \ln \mu_1 + \ln \mu_2 + \ln \nu_1 + \ln \nu_2 = 0$$

is fulfilled. Then monodormy of logarithmic connection defined by 1-form

$$\omega(z) = \left(\frac{B_1}{z - a_1} + \frac{B_2}{z - a_2} + \frac{B_3}{z - a_3} \right) dz,$$

where the triple of residues (B_1, B_2, B_3) solves additive Deligne-Simpson problem for spectra $(\frac{1}{2\pi i} \ln \lambda_1, \frac{1}{2\pi i} \ln \lambda_2)$, $(\frac{1}{2\pi i} \ln \mu_1, \frac{1}{2\pi i} \ln \mu_2)$ and $(\frac{1}{2\pi i} \ln \nu_1, \frac{1}{2\pi i} \ln \nu_2)$ respectively has monodromy χ.

Corollary 6.2. *If rank 2 representation of $\pi_1(\mathbb{C}P^1 \setminus \{a_1, a_2, a_3\}, z_0)$ is irreducible, the corresponding Riemann-Hilbert problem can be solved explicitly.*

It is easy to see that in the case of $\mathrm{SL}(2, \mathbb{C})$ representations one can impose additional restrictions $\mathrm{Tr}\, B_i = 0$ for $i = 1, 2, 3$ on explicit solution.

6.3.3 Rank 2 vector bundles on elliptic curve and logarithmic connections

In this section we examine rank 2 vector bundles of degree zero over Λ_τ. From the results of previous section it follows that $\mathcal{O}_\lambda(k) \oplus \mathcal{O}_\mu(-k)$ gives an example of such a bundle. The general theory [Ati57] says that there also exist exceptional indecomposable rank 2 vector bundles of degree zero parametrized by λ taking values in Jacobian just like the line bundles are. In our work we shall only consider decomposable bundles.

Definition 6.1. *Vector bundle E is semistable if for any subbundle $F \subset E$ there holds* $\deg F / \operatorname{rk} F \leq \deg E / \operatorname{rk} E$.

Theorem 6.2. *If F is a line sub-bundle of $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_\mu(0)$ then* $\deg F \leq 0$.

Proof. Consider φ to be a meromorphic section of F. Then $\deg F = N_\varphi - P_\varphi$ where N_φ, P_φ are total numbers of zeroes and poles of φ in a fundamental parallelogram. Being a section of F, φ is also a section of $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_\mu(0)$ and hence $\varphi = \varphi_1 \oplus \varphi_2$ where φ_1, φ_2 are some sections of $\mathcal{O}_\lambda(0)$ and $\mathcal{O}_\mu(0)$ respectively. Therefore zeroes of φ are the common zeroes of φ_1 and φ_2 while poles of φ are both poles of φ_1 and poles of φ_2. Hence $N_\varphi \leq \min(N_{\varphi_1}, N_{\varphi_2})$ and $P_\varphi \geq \max(P_{\varphi_1}, P_{\varphi_2})$ implying $N_\varphi - P_\varphi \leq \min(N_{\varphi_i} - P_{\varphi_i}) = 0$. □

Corollary 6.3. *Vector bundle $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_\mu(0)$ is semistable.*

According to our formulation of generalized Riemann-Hilbert problem we need to construct on elliptic curve Λ_τ vector bundle $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}$ equipped with logarithmic connection with prescribed monodromy representation and singular points location.

Let us describe the explicit form of logarithmic connection in that bundle. Consider a canonical base (s_1, s_2) in meromorphic sections of E, taking s_1 to be a section of $\mathcal{O}_\lambda(0)$ and s_2 to be a section of $\mathcal{O}_{-\lambda}(0)$ respectively. Any meromorphic section of E in that base has the form

$$\varphi(z) = \begin{pmatrix} f_1(z) s_1(z) \\ f_2(z) s_2(z) \end{pmatrix} = \begin{pmatrix} \varphi_\lambda(z) \\ \varphi_{-\lambda}(z) \end{pmatrix},$$

where $f_{1,2}(z)$ are meromorphic functions on Λ_τ and $\varphi_{\pm\lambda}(z)$ are the sections of $\mathcal{O}_{\pm\lambda}(0)$ respectively. Section $\varphi(z)$ is horizontal for some meromorphic connection with matrix differential 1-form ω:

$$d\varphi(z) = \omega(z)\varphi(z).$$

From relations 6.2.4 it follows

$$\begin{aligned}\varphi(z+1) &= \varphi(z)\\ \varphi(z+\tau) &= \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda}\end{pmatrix}\varphi(z)\end{aligned}$$

and hence

$$\begin{aligned}\omega(z+1) &= \omega(z)\\ \omega(z+\tau) &= \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda}\end{pmatrix}\omega(z)\begin{pmatrix} e^{-2\pi\imath\lambda} & 0 \\ 0 & e^{2\pi\imath\lambda}\end{pmatrix}.\end{aligned} \tag{6.3.1}$$

Theorem 6.3. *Consider* $\{a_1, \ldots, a_n\} \in \Lambda_\tau$, $a_i \neq a_j$ *and complex* $\alpha_i, \beta_i, \gamma_i, \delta_i$, $i = 1, \ldots, n$ *such that*

$$\sum_{i=1}^{n} \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta \end{pmatrix} = 0.$$

Then matrix 1-form

$$\Omega(z) = \sum_{i=1}^{n} \frac{\begin{pmatrix} \alpha_i\theta'(z-a_i) & \beta_i\frac{\theta'(0)}{\theta(-2\lambda)}\theta(z-a_i-2\lambda) \\ \gamma_i\frac{\theta'(0)}{\theta(2\lambda)}\theta(z-a_i+2\lambda) & -\delta_i\theta'(z-a_i)\end{pmatrix}}{\theta(z-a_i)}dz,$$

defines a logarithmic connection on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ *with residues*

$$\operatorname*{Res}_{z=a_i}\Omega(z) = \begin{pmatrix} \alpha_i & \beta_i \\ \gamma_i & \delta_i \end{pmatrix}.$$

Proof. Relations 6.2.1 and 6.2.4 imply

$$\frac{\theta'(z-a_i+1)}{\theta(z-a_i+1)} = \frac{\theta'(z-a_i)}{\theta(z-a_i)}$$
$$\frac{\theta'(z-a_i+\tau)}{\theta(z-a_i+\tau)} = \frac{\theta'(z-a_i)}{\theta(z-a_i)} - 2\pi\imath.$$

and

$$\frac{\theta(z-a_i\mp 2\lambda+1)}{\theta(z-a_i+1)} = \frac{\theta(z-a_i\mp 2\lambda)}{\theta(z-a_i)}$$
$$\frac{\theta(z-a_i\mp 2\lambda+\tau)}{\theta(z-a_i+\tau)} = \frac{\theta(z-a_i\mp 2\lambda)}{\theta(z-a_i)}e^{\pm 4\pi\imath\lambda}.$$

Therefore from $\sum\alpha_i = \sum\delta_i = 0$ it follows

$$\Omega(z+1) = \Omega(z)$$
$$\Omega(z+\tau) = \begin{pmatrix} e^{2\pi\imath\lambda} & 0 \\ 0 & e^{-2\pi\imath\lambda} \end{pmatrix} \Omega(z) \begin{pmatrix} e^{-2\pi\imath\lambda} & 0 \\ 0 & e^{2\pi\imath\lambda} \end{pmatrix}$$

and $\Omega(z)$ is a 1-form of meromorphic connection on some vector bundle $F \simeq \mathcal{O}_\lambda(k) \oplus \mathcal{O}_{-\lambda}(l)$.

Since $\operatorname{Tr}\Omega(z) = 0$ degree of F equals to zero and therefore $F \simeq \mathcal{O}_\lambda(k) \oplus \mathcal{O}_{-\lambda}(-k)$ for some integer k. Consider a section Φ of bundle F written down as

$$\Phi(z) = \begin{pmatrix} \varphi_\lambda(z) \\ \varphi_{-\lambda(z)} \end{pmatrix}$$

where $\varphi_{\pm\lambda}(z)$ are some sections of $\mathcal{O}_{\pm\lambda}(\pm k)$. Any connection on F maps sections of F to sections of $F \otimes T^*\Lambda_\tau$. For our Ω that imply that in the first row of $\Omega\Phi$

$$\left(\sum_{i=1}^n \alpha_i \frac{\theta'(z-a_i)}{\theta(z-a_i)}\right)\varphi_\lambda(z) + \left(\sum_{i=1}^n \beta_i \frac{\theta'(0)}{\theta(-2\lambda)}\frac{\theta(z-a_i-2\lambda)}{\theta(z-a_i)}\right)\varphi_{-\lambda}(z)$$

should be a section of $\mathcal{O}_\lambda(k)$.

Since $A(z) = \left(\prod_{i=1}^n \theta^{\alpha_i}(z-a_i)\right)'/\left(\prod_{i=1}^n \theta^{\alpha_i}(z-a_i)\right)$ is a ratio of two sections of

$\mathcal{O}_{\sum \alpha_i a_i}(0)$ it is a single-valued function on Λ_τ or a section of $\mathcal{O}_0(0)$. Therefore $A(z)\varphi_\lambda(z)$ is a section of $\mathcal{O}_\lambda(k)$.
Hence, $B(z)\varphi_{-\lambda}(z) = \left(\sum_{i=1}^n \beta_i \frac{\theta'(0)}{\theta(-2\lambda)} \frac{\theta(z-a_i-2\lambda)}{\theta(z-a_i)}\right) \varphi_{-\lambda}(z)$ as a difference of two sections should also be a section of $\mathcal{O}_\lambda(k)$. But by explicit construction $B(z)$ is a section of $\mathcal{O}_{2\lambda}(0)$ and therefore $B(z)\varphi_{-\lambda}(z)$ is a section of $\mathcal{O}_\lambda(-k)$. Hence $k = 0$ and $\Omega(z)$ defines a connection on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$.
Differential 1-form $\Omega(z)$ is holomorphic on $\Lambda_\tau \setminus \{a_1, \ldots, a_n\}$ and since $\theta(z)$ is an entire function and $\theta'(0)$ and $\theta(\pm 2\lambda)$ do not equal to zero $\Omega(z)$ has a simple poles in $z = a_i$. Therefore the connection defined by Ω is logarithmic with prescribed polar locus. Calculation of residues is trivial. □

It is important to notice that unlike to logarithmic connections over the Riemann sphere logarithmic connection over an elliptic curve is not uniquely defined by its residues. Since in the bundle E can exist holomorphic matrix 1-forms one can add them to $\Omega(z)$ and obtain new connection with the same residues. Explore in greater details the construction of such a 1-form $\Upsilon(z)$. Denote

$$\Upsilon(z) = \begin{pmatrix} \Upsilon_1(z) & \Upsilon_2(z) \\ \Upsilon_3(z) & \Upsilon_4(z) \end{pmatrix} dz$$

and consider relations (6.3.1). We get

$$\begin{aligned} \Upsilon_{1,4}(z+1) &= \Upsilon_{1,4}(z) \\ \Upsilon_{1,4}(z+\tau) &= \Upsilon_{1,4}(z) \\ \Upsilon_{2,3}(z+1) &= \Upsilon_{2,3}(z) \\ \Upsilon_{2,3}(z+\tau) &= \Upsilon_{2,3}(z) e^{\mp 4\pi\imath\lambda} \end{aligned} \tag{6.3.2}$$

Since all $\Upsilon_i(z)$ are holomorphic and $\Upsilon_{1,4}(z)$ are double-periodic, $\Upsilon_{1,4}(z)$ are constant. Relations (6.3.2) imply

$$\begin{aligned} \frac{\Upsilon'_{2,3}(z+1)}{\Upsilon_{2,3}(z+1)} &= \frac{\Upsilon'_{2,3}(z)}{\Upsilon_{2,3}(z)} \\ \frac{\Upsilon'_{2,3}(z+\tau)}{\Upsilon_{2,3}(z+\tau)} &= \frac{\Upsilon'_{2,3}(z)}{\Upsilon_{2,3}(z)} \end{aligned}$$

Therefore integral of logarithmic derivative of $\Upsilon_{2,3}(z)$ along the perimeter of

fundamental parallelogram is zero and $\Upsilon_{2,3}(z)$ has equal number of zeroes and poles in the parallelogram. Since $\Upsilon_{2,3}(z)$ is holomorphic it has no poles and hence no zeroes. But as shown in [MP17a] the only entire functions obeying relations (6.3.2) with no zeroes in complex plane are $f(z) = Ce^{2\pi k \imath z}$ with integer k inducing $2\lambda = k\tau$. Since λ is defined modulo $\{1, \tau\}$ it follows that λ equals either zero, or $\tau/2$. The first case corresponds to $\Upsilon_{2,3}(z) = 0$, the second to $\Upsilon_{2,3}(z) = C_{\mp}e^{\mp 2\pi \imath z}$.
Finally, all holomorphic matrix 1-forms $\Upsilon(z)$ on $E \simeq \mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ have the form

$$\Upsilon(z) = \begin{pmatrix} C_1 & C_- e^{-2\pi \imath z} \\ C_+ e^{2\pi \imath z} & C_4 \end{pmatrix} dz$$

with constant $C_1, C_{\mp}, C_4$ and $C_{\mp} = 0$ if $\lambda \neq \tau/2$. Logarithmic connections defined by 1-forms Ω and $\Omega(z) + \Upsilon(z)$ have coinciding residues, but in general different monodromy representations.

6.3.4 Rank 2 RHP on Elliptic curve

Rank 2 Riemann problem on elliptic curve we consider consists in establishing a logarithmic connection with given monodromy and singularities in semistable vector bundle of degree zero over a given elliptic curve. For shortness of explicit expressions and calculations we restrict ourselves to the case of $\mathrm{SL}_2(\mathbb{C})$-monodormy.

Monodromy data

Consider a logarithmic connection on an elliptic curve Λ_τ with singular points $a_1, \ldots, a_n$ and monodromy representation

$$\chi : \pi_1 \left(\Lambda_\tau \setminus \{a_1, \ldots, a_n\}, z_0\right) \to \mathrm{SL}_2\mathbb{C}).$$

Namely we are given a set of matrix multipliers $G_1, \ldots, G_n, G_a, G_b, \widetilde{G}_a, \widetilde{G}_b$ where first $n+2$ matrices correspond to the change of local horizontal sections basis under continuation along the loops encircling singular points and $a-, b-$

cycles while $\widetilde{G}_a, \widetilde{G}_b$ describe trivialisation deformation along $a-, b-$ cycles respectively.

$$\begin{aligned} \gamma_i &: Y(z) \mapsto Y(z) G_i \\ \gamma_{a,b} &: Y(z) \mapsto \widetilde{G}_{a,b} Y(z) G_{a,b} \end{aligned}$$

Let us precise the conditions that monodromy satisfy and the most convenient way to encode it.

The fundamental group of an elliptic curve obey the relation $aba^{-1}b^{-1} = \text{id}$, the loop, encircling the fundamental parallelogram along perimeter can be contracted inside it. Obviously, that for some natural ordering points a_i and choice of classes of basic loops γ_i encircling them, the sequential bypassing all the punctures is equivalent to bypassing the perimeter of the fundamental parallelogram and hence the relation in the fundamental group of punctured torus is $\gamma_1 \cdots \gamma_n = \gamma_a \gamma_b \gamma_a^{-1} \gamma_b^{-1}$ or equivalently $\gamma_1 \cdots \gamma_n \gamma_b \gamma_a \gamma_b^{-1} \gamma_a^{-1} = \text{id}$. It corresponds to the condition

$$\left(\widetilde{G}_a \widetilde{G}_b \widetilde{G}_a^{-1} \widetilde{G}_b^{-1} \right)^{-1} Y(z) \left(G_1 \cdots G_n G_b G_a G_b^{-1} G_a^{-1} \right)^{-1} = Y(z)$$

on monodromy matrices. As we have seen for decomposible bundles there exists a natural trivialization where $\widetilde{G}_a = 1, \widetilde{G}_b = \text{diag}(e^{2\pi i \lambda}, e^{-2\pi i \lambda})$ and therefore the commutator on the left equals to identity implying

$$G_1 \cdots G_n G_b G_a G_b^{-1} G_a^{-1} = 1.$$

Finally the input monodromy data for Riemann problem on elliptic curve Λ_τ in our approach is

$$\left\{ \begin{array}{l} \widetilde{G}_a = 1, \widetilde{G}_b = \text{diag}(e^{2\pi i \lambda}, e^{-2\pi i \lambda}), \\ \left\{ G_1, G_2, G_3, G_a, G_b | G_{1,2,3,a,b} \in \text{SL}(2, \mathbb{C}), G_1 G_2 G_3 = G_a G_b G_a^{-1} G_b^{-1} \right\} \big/ \sim \end{array} \right\},$$

where factorization is taken by diagonal adjoint action.

Explicit construction of solution on elliptic curve

Suppose we are given an irreducible representation

$$\chi_0 : F_2 \to \mathrm{SL}(2, \mathbb{C}),$$

of free group with two generators. Obviously we can interpret it as an irreducible representation

$$\chi_0 : \pi_1 \left(\mathbb{C}\mathrm{P}^1 \setminus \{d_1, d_2, d_3\}\right) \to \mathrm{SL}(2, \mathbb{C}),$$

of fundamental group of three-punctured Riemann sphere with arbitrary location of punctures. As it was shown in section 6.3.1 representation χ is rigid and therefore uniquely defined by eigenvalues of three local monodromies. Below we shall give an explicit construction of logarithmic connection ∇ in a rank two semistable bundle of degree zero over given elliptic curve Λ_τ with prescribed poles positions $\{a_1, a_2, a_3\}$ such that its monodromy χ being reduced to fundamental parallelogram gives χ_0, while monodromy along $a-$ and $b-$cycles also have explicit description.
Following auxiliary statements analogous to 6.2, 6.3, are essentially local and are valid for an elliptic curve as well as for the Riemann sphere. Consider a general form of logarithmic connection on semistable rank two vector bundle over Λ_τ. As explained in section 6.3.3 it is $\nabla = d - \Omega(z) - \Upsilon(z)$ where $\Omega(z)$ is given in theorem 6.3 and $\Upsilon(z)$ equals to dz with some diagonal constant coefficient $\mathrm{diag}(C_1, C_4)$.

Statement 6.5. *The logarithm of local monodromy of connection $\nabla = d - \Omega(z) - \Upsilon(z)$ is conjugated to the exponent of the corresponding residue multiplied by $2\pi\imath$*

$$\ln G_i \sim 2\pi\imath \operatorname*{Res}_{z=a_i} \Omega(z) = \oint_{\gamma_i} \big(\Omega(z) + \Upsilon(z)\big).$$

Statement 6.6. *For logarithmic connection $\nabla = d - \Omega(z) - \Upsilon(z)$, the eigenvalues of the local monodromy G_i coincide with the eigenvalues of* $\exp\bigl(2\pi\imath \operatorname{Res}_{z=a_i} \Omega(z)\bigr) = \exp(2\pi\imath B_i)$.

The relations for monodromies G_a, G_b are obvious analogues of expressions above

Statement 6.7. *The logarithms of monodromy along $a-, b-$cycles of logarithmic connection $\nabla = d - \Omega(z) - \Upsilon(z)$ are conjugated to the corresponding integrals along $a-, b-$cycles*

$$\ln G_a \sim \int_0^1 \bigl(\Omega(z) + \Upsilon(z)\bigr), \quad \ln G_b \sim \int_0^\tau \bigl(\Omega(z) + \Upsilon(z)\bigr)$$

Last statement can be precised due to know structure of 1-form $\Omega(z)$.

Lemma 6.1. *In conditions above*

$$\exp\left(\int_0^1 \Omega(z)\right) = 1$$

and therefore

$$\ln G_a \sim \int_0^1 \Upsilon(z)$$

and do not depend on explicit form of Ω.

Proof. For diagonal terms since $\sum_{i=1}^n \alpha_i = 0$ we have

$$\int_0^1 \sum_{i=1}^n \alpha_i \frac{\theta'(z-a_i)}{\theta(z-a_i)} dz = \int_0^1 d\ln\left(\prod_{i=1}^n \theta^{\alpha_i}(z-a_i)\right) = 2\pi\imath k, \quad k \in \mathbb{Z}.$$

For off-diagonal terms introduce integral

$$I_k(\lambda) = \int_0^1 \frac{\theta'(0)}{\theta(-2\lambda)} \frac{\theta(z - a_k - 2\lambda)}{\theta(z - a_k)} dz$$

and consider an integral

$$\widetilde{I}_k = \oint_\Pi \frac{\theta'(0)}{\theta(-2\lambda)} \frac{\theta(z - a_k - 2\lambda)}{\theta(z - a_k)} dz$$

along the perimeter of fundamental parallelogram. The sum of residues gives $\widetilde{I}_k = 2\pi\imath$ while shift relations 6.2.4 imply $\widetilde{I}_k = \left(1 - e^{4\pi\imath\lambda}\right) I_k(\lambda)$. Therefore

$$I_k(\lambda) = \frac{2\pi\imath}{1 - e^{4\pi\imath\lambda}}$$

do not depend on k and hence

$$\int_0^1 \sum_{i=1}^n \beta_i \frac{\theta'(0)}{\theta(-2\lambda)} \frac{\theta(z - a_k - 2\lambda)}{\theta(z - a_k)} dz = \int_0^1 \sum_{i=1}^n \beta_i I_i(\lambda) = \frac{2\pi\imath}{1 - e^{4\pi\imath\lambda}} \sum_{i=1}^n \beta_i = 0$$

since $\sum_{i=1}^n \beta_i = 0$. Calculating in the same way the second line of integral of $\Omega(z)$ we obtain

$$\int_0^1 \Omega(z) = \begin{pmatrix} 2\pi\imath k & 0 \\ 0 & 2\pi\imath l \end{pmatrix}, \quad k, l \in \mathbb{Z}$$

and therefore

$$\exp\left(\int_0^1 \Omega(z)\right) = 1.$$

□

Now we have a clear straightforward way to obtain the explicit construction of logarithmic connection ∇ with monodromy χ inducing on the parallelogram Π required irreducible monodromy χ_0 defined in the beginning of this section.

For any given χ_0 we can explicitly construct (B_1, B_2, B_3) a triple of residues, solving Riemann-Hilbert problem for χ_0 on the sphere. Define $\widetilde{\Omega}(z)$ to be a 1-form of the structure described in theorem 6.3 with residues (B_1, B_2, B_3) in points $\{a_1, a_2, a_3\}$ respectively and consider $\widetilde{\nabla} = d - \widetilde{\Omega}(z)$ logarithmic connection with holomorphic term $\Upsilon(z) = 0$ and arbitrary modular parameter λ. Consider its monodromy representation

$$\chi : \pi_1 \left(\Lambda_\tau \setminus \{a_1, a_2, a_3\}\right) \to \mathrm{SL}(2, \mathbb{C}).$$

As we have seen monodromy data consists of tuple $(G_1, G_2, G_3, G_a, G_b)$ of $\mathrm{SL}(2, \mathbb{C})$ matrices defined up to an overall conjugation and obeying $G_1 G_2 G_3 = G_a G_b G_a^{-1} G_b^{-1}$.

From lemma 6.1 follows that $\ln G_a \sim 0$, therefore $\ln G_a = 0$ and $G_a = 1$. Monodromy relation $G_1 G_2 G_3 = G_a G_b G_a^{-1} G_b^{-1}$ in that case is degenerated into $G_1 G_2 G_3 = 1$. From statements 6.3 and 6.6 it follows that spectra of G_i coincide with spectra of local monodromy of χ_0. Since χ_0 is irreducible and therefore rigid, as it was shown in section 6.3.1 triple (G_1, G_2, G_3) is defined uniquely up to an overall conjugation and coincides with monodromy data of representation χ_0. Altogether that gives the final theorem.

Theorem 6.4. *Consider an irreducible representation*

$$\chi_0 : \pi_1 \left(\mathbb{CP}^1 \setminus \{d_1, d_2, d_3\}\right) \to \mathrm{SL}(2, \mathbb{C}).$$

The Riemann-Hilbert problem for χ_0 can be solved explicitly, consider (B_1, B_2, B_3) any triple of residues giving the solution.

Then 1-form $\widetilde{\Omega}(z)$ constructed following theorem 6.3 with the use of triple (B_1, B_2, B_3) and arbitrary parameter λ defines a logarithmic connection $\widetilde{\nabla} = d - \widetilde{\Omega}(z)$ in semistable vector bundle $\mathcal{O}_\lambda(0) \oplus \mathcal{O}_{-\lambda}(0)$ with singular points

$\{a_1, a_2, a_3\}$ *and monodromy representation*

$$\chi : \pi_1 (\Lambda_\tau \setminus \{a_1, a_2, a_3\}) \to \mathrm{SL}(2, \mathbb{C}),$$

such that

$$\chi_{\mathrm{ind}} = \chi_0, \quad \chi(\gamma_a) = 1$$

and

$$\chi(\gamma_b) \sim \exp\left(2\pi i \int\limits_0^\tau \widetilde{\Omega}(z)\right).$$

To finalize we can notice that even in that reduced (G_a=1) case there are little chances if any to calculate explicitly entire monodromy representation χ. However from the reasonings above can be easily seen that all logarithmic connections with same induced monodromy χ_0 and $G_a = 1$ and asymptotic behaviour at $\{a_1, a_2, a_3\}$ differ from each other by an overall conjugation of initial triple (B_1, B_2, B_3). In [Bol02] Bolibruch proved positive solvability of generalized Riemann-Hilbert problem for irreducible monodromy representation. Among all irreducible representations of fundamental group of three-punctured torus there is one with induced monodromy χ_0 and $G_a = G_b = 1$. Omitting subtleties related to theta-divisor of deformation problem we can suppose that this representation is realized by connection with the same asymptotic behaviour, from general theory it is known that it always can be achieved by deformation of parameter λ.
The 1-form $\widehat{\Omega}(z)$ of connection realizing this representation should differ from $\widetilde{\Omega}$ by an overall conjugation of residues triple $(\widehat{B}_1, \widehat{B}_2, \widehat{B}_3) = D(B_1, B_2, B_3)D^{-1}$ and satisfy

$$\int_0^\tau \widehat{\Omega}(z) = \begin{pmatrix} 2\pi\imath k & 0 \\ 0 & 2\pi\imath l \end{pmatrix}, \quad k, l \in \mathbb{Z}.$$

That gives a system of quadratic equations on the elements of D with known in advance existence of solution.

Chapter 7

Poisson Structures Associated to Moduli Spaces of Flat Connections

In this chapter, we describe the Poisson structure on the moduli space of flat connections and the corresponding structure on character varieties of bordered Riemann surfaces that can be seen as representation spaces of the monodromy data associated to the given connections. We follow the references [KS97], [Sak01] and [CMR18].

Let us consider Fuchsian system of linear differential equations on Riemann surface Λ written in matrix form:

$$\frac{dY}{dz} = A(z)Y, \tag{7.0.1}$$

here $A(z)$ is 1-form of logarithmic connections in semistable vector bundle over Λ and solutions $Y(z)$ are its horizontal sections.

From here, we denote for any matrix X,

$$\overset{1}{X} = X \otimes \mathbb{I} \quad \text{and} \quad \overset{2}{X} = \mathbb{I} \otimes X.$$

For connections in such a bundle there is a known Poisson bracket [FT07]:

$$\{A(z_1) \underset{,}{\otimes} A(z_2)\} = \frac{1}{z_1 - z_2}[r_{12}, \overset{1}{A}(z_1) + \overset{2}{A}(z_2)],$$

where r_{12} is classical r-matrix and A satisfies

$$\partial_z Y(z) = A(z)Y(z).$$

Poisson structure on sections $Y(z)$ can be obtained from the above bracket on connections. Moreover, from bracket on connections $A(z)$ one can obtain the bracket on transition matrices $C(a,b)$ defined as:

$$Y(b) = Y(a)C(a,b). \tag{7.0.2}$$

Then we can consider fixed point s_0 such that $Y(s_0) = \mathbb{I}$ and express each section $Y(z)$ as a transition matrix $C(s_0, z)$, automatically getting bracket on sections.

The Poisson bracket on transition matrices mentioned above takes the following form [FT07]:

$$\{C(s_0, s_1) \underset{,}{\otimes} C(s_0 s_2)\} = \int_{s_0}^{s_1}\int_{s_0}^{s_2} dz_1 dz_2 (C^{-1}(s_0, z_1) \otimes C^{-1}(s_0, z_2)$$
$$\{A(z_1) \underset{,}{\otimes} A(z_2)\}(C(s_0, z_1) \otimes C(s_0, z_2))$$

Using $A(z)C(s_0, z) = \partial_z C(s_0, z)$ we transform the above formula:

$$\{C(s_0, s_1) \underset{,}{\otimes} C(s_0, s_2)\} = (C(s_0, s_1) \otimes C(s_0, s_2)) \int_{s_0}^{s_1}\int_{s_0}^{s_2} dz_1 dz_2 \Big[\frac{r_{12}}{z_2 - z_1}(\partial_{z_1} + \partial_{z_2})$$
$$\big(C^{-1}(s_0, z_2)C(s_0, z_1) \otimes C^{-1}(s_0, z_1)C(s_0, z_2)\big)\Big]. \tag{7.0.3}$$

The integral paths are considered not intersecting, so the only singularity

is in the point s_0 when $z_1 = z_2 = s_0$ so we consider a limit of two paths beginning in the ε neighborhood of s_0.
We choose two pairs of points s_1, s_2 and s_3, s_4 so that their projections to the sphere coincide, but the points lie on different neighboring leaves of the covering space. Then

$$Y(s_2) = Y(s_1)M_i, \qquad Y(s_4) = Y(s_3)M_j,$$

where M_i and M_j – are mondromies corresponding respectively to the analytic continuation around singularities a_i and a_j.
After some transformations of integration paths from (7.0.3) we obtain Poisson structure on the monodromy data:

$$\{M_i \underset{,}{\otimes} M_i\} = i\pi[r_{12}, M_i M_i \otimes \mathbb{I}],$$

For $i < j$:

$$\{M_i \otimes_, M_j\} = i\pi r_{12}(M_j M_i \otimes \mathbb{I} + \mathbb{I} \otimes M_i M_j - M_i \otimes M_j - M_j \otimes M_i).$$

For an example of this bracket one can consider the following Fuchsian system of rank 2 with four regular singularities $0, 1, t, \infty$ on the Riemann sphere:

$$\partial Y = A(z)Y, \qquad A(z) = \frac{A_0}{z} + \frac{A_1}{z-1} + \frac{A_t}{z-t}$$

with $A_i \in \mathfrak{gl}_2(\mathbb{C})$, $i = 0, 1, t$ and $A_\infty = -(A_0 + A_1 + A_t)$. Corresponding to each singularity this system has 4 monodromy matrices M_0, M_1, M_t, M_∞ satisfying condition $M_0 M_1 M_t M_\infty = 1$. The Schlesinger equations describing the isomonodromic deformations of our Fuchsian system are given by:

$$\frac{dA_0}{dt} = \frac{[A_t, A_0]}{t}, \qquad \frac{dA_1}{dt} = \frac{[A_t, A_1]}{t-1}$$

This system can be considered as an auxiliary system for P_{VI}. This system has only simple poles as singularities, whereas the other equations from the family have higher-order poles.

The question we tried to address in this chapter is how to construct the analogous Poisson structure when there are irregular singular points. Specifically, we were interested in the isomonodromic problem associated to the fifth Painlevé equation.

Let us remind the Painlevé equations.

$$(PVI)\begin{aligned}\frac{d^2y}{dt^2} &= \frac{1}{2}\left(\frac{1}{y}+\frac{1}{y-1}+\frac{1}{y-t}\right)\left(\frac{dy}{dt}\right)^2-\left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{y-t}\right)\frac{dy}{dt}+\\ &+\frac{y(y-1)(y-t)}{t^2(t-1)^2}\left(\alpha+\beta\frac{t}{y^2}+\gamma\frac{t-1}{(y-1)^2}+\delta\frac{t(t-1)}{(y-t)^2}\right)\end{aligned}$$

$$(PV)\begin{aligned}\frac{d^2y}{dt^2} &= \left(\frac{1}{2y}+\frac{1}{y-1}\right)\left(\frac{dy}{dt}\right)^2-\frac{1}{t}\frac{dy}{dt}+\\ &+\frac{(y-1)^2}{t^2}\left(\alpha y+\frac{\beta}{y}\right)+\gamma\frac{y}{t}+\delta\frac{y(y+1)}{y-1}\end{aligned}$$

$$(PIV)\frac{d^2y}{dt^2} = \frac{1}{2y}\left(\frac{dy}{dt}\right)^2+\frac{3}{2}y^3+4ty^2+2(t^2-\alpha)y+\frac{\beta}{y}$$

$$(PIII)\frac{d^2y}{dt^2} = \frac{1}{y}\left(\frac{dy}{dt}\right)^2-\frac{1}{t}\frac{dy}{dt}+\frac{y^2}{4t^2}\left(\alpha+\frac{\beta t}{y^2}+\gamma y+\frac{\delta t^2}{4y^3}\right)$$

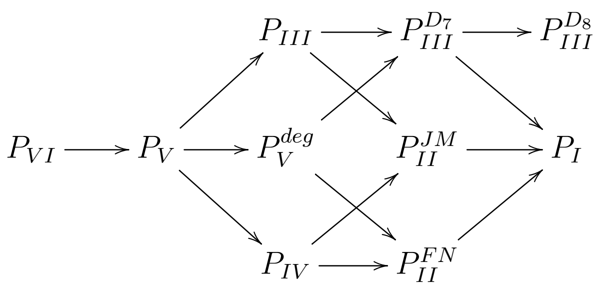
$$(PII)\frac{d^2y}{dt^2} = 2y^3+ty+\alpha$$

$$(PI)\frac{d^2y}{dt^2} = 6y^2+t,$$

where $\alpha, \beta, \gamma, \delta$ are complex constants. This list of equations can be extracted from classical work of Painlevé [Pai02].

The corresponding monodromy manifolds are described in [CMR18].

Painlevé equations can be obtained one from another following the Sakai diagram [Sak01] by the confluence of singularities so that from two simple poles, you can get one of order two, etc.



We hoped to use this approach to extend the Poisson structure described above to higher order singularities. However, this program has not been entirely successful. We have reached a conjecture about the expected Poisson structure in Section 7.1.2 and Section 7.1.3, we explain how to obtain the Poisson brackets on the sixth Painlevé monodromy data following Faddeev-Takhtadjan approach. The obtained formulae show that, at least at the level of monodromy matrices, our conjecture is correct.

7.1 Poisson structure on the space of monodromy data for the Fifth Painlevé equation

In this section, we consider the isomonodromic problem for the fifth Painlevé equation:

$$\frac{\partial \Psi}{\partial \mu} = B(\mu, s)\Psi, \quad \text{with} \quad B(\mu, s) := \frac{1}{2}\sigma_3 + \frac{B_0(s)}{\mu} + \frac{B_s(s)}{\mu - s}, \tag{7.1.1}$$

$$\frac{\partial \Psi}{\partial s} = -\frac{B_s(s)}{\mu - s}\,\Psi, \tag{7.1.2}$$

where σ_3 is the third Pauli matrix, and B_0, B_s are 2×2 matrices such that

$$\operatorname{diag}(B_0 + B_s) = -\frac{1}{2}\Theta_\infty, \tag{7.1.3}$$

and

$$\text{eigen}(B_k) = \frac{1}{2}\Theta_k, \qquad k = 0, s, \tag{7.1.4}$$

for some non resonant diagonal matrices $\Theta_0, \Theta_s, \Theta_\infty$.

7.1.1 Local theory and monodromy data

In this section, we briefly remind the reader about the isomonodromic deformations theory of the system (7.1.1) and fix notations.
Near each Fuchsian singularity $0, s$, we fix analytic local fundamental solutions of equation (7.1.1):

$$\Psi_0(\mu) = \sum_{n=0}^{\infty} H_{n,0}\mu^n \ \mu^{\frac{1}{2}\Theta_0}, \qquad \mu \in \Omega_0,$$
$$\Psi_s(\mu) = \sum_{n=0}^{\infty} H_{n,s}(\mu - s)^n \ (\mu - s)^{\frac{1}{2}\Theta_2}, \qquad \mu \in \Omega_s,$$

where Ω_0 and Ω_s are some open discs with branch cuts, and the leading terms $H_{0,k}$, $k = 0, s$, are the diagonalising matrices of B_k respectively normalised as follows:

$$\text{diag}(H_{0,0}) = \text{diag}(H_{0,s}) = \mathbb{I},$$

and all other terms of each series are determined by certain recursive relations, for example around zero,

$$nH_{n,0} + H_{n,0}\frac{1}{2}\Theta_0 - B_0 H_{n,0} = \frac{\sigma_3}{2}H_{n-1,0} - B_s \sum_{l=0}^{n-1} s^{l-n} H_{l,0}.$$

The recursive relation for $H_{n,s}$ is analogous.
Near ∞ we fix analytic local fundamental solutions of equation (7.1.1) as follows.

Theorem 7.1. *Consider the sectors*

$$\Sigma_\infty^{(k)} = \left\{ \mu : |\mu| > \rho_\infty, \ \frac{\pi}{2} < arg(\mu) + k\pi < \frac{5\pi}{2} \right\},$$

for some real positive $\rho_\infty > 0$, as illustrated in Figure 7.1. For all $k \in \mathbb{Z}$, there exists a unique solution $\Psi_\infty^{(k)}(\mu)$ of equation (7.1.1) analytic in the sector $\Sigma_\infty^{(k)}$ such that,

$$\Psi_\infty^{(k)}(\mu) \sim \sum_{n=0}^{\infty} H_{n,\infty} \mu^{-n} \ \mu^{-\frac{1}{2}\Theta_\infty} e^{\frac{\mu}{2}\sigma_3}, \qquad as\ \mu \to \infty, \ \mu \in \Sigma_\infty^{(k)}, \tag{7.1.5}$$

where $H_{0,\infty} = \mathbb{I}$, and the $H_{n,\infty}$ are determined by the recursive relations

$$\left[H_{1,\infty}, \frac{\sigma_3}{2} \right] = \frac{1}{2}\Theta_\infty + (B_0 + B_s)$$

and for $n \geq 2$,

$$\left[H_{n,\infty}, \frac{\sigma_3}{2} \right] = (n-1)H_{n-1,\infty} + H_{n-1,\infty}\frac{1}{2}\Theta_\infty + (B_0 + B_s)H_{n-1,\infty} + \sum_{l=0}^{n-2} s^{n-1-l} B_s H_{l,\infty}.$$

Proof. This is an instance of a well-known general result whose proof can be found in several texts such as [BJL79, Was87, Sib90]. □

We denote the asymptotic behaviour of true solutions of (7.1.1) at infinity as in (7.1.5) by,

$$\Psi_\infty^{(\text{formal})}(\mu) = \sum_{n=0}^{\infty} H_{n,\infty} \mu^{-n} \ \mu^{-\frac{1}{2}\Theta_\infty} e^{\frac{\mu}{2}\sigma_3}. \tag{7.1.6}$$

We call this a *formal* solution in the sense that the series $\sum_{n=0}^{\infty} H_{n,\infty}\mu^{-n}$ is not convergent in general. The asymptotic relation (7.1.5) means, by definition, for all $m \in \mathbb{N}$ and for all closed subsectors $\Sigma \subset \Sigma_\infty^{(k)}$,

$$\left| \lambda^m \left(\Psi_\infty^{(k)}(\mu)\mu^{\frac{1}{2}\Theta_\infty} e^{-\frac{\mu}{2}\sigma_3} - \sum_{n=0}^{m} H_{n,\infty}\mu^{-n} \right) \right| \to 0, \text{ as } \mu \to \infty, \mu \in \Sigma.$$

From the asymptotic relation (7.1.5), it is clear that the solutions,

$$\Psi_\infty^{(k+2)}(\mu) \text{ and } \Psi_\infty^{(k)}\left(\mu e^{2\pi i}\right) e^{i\pi\Theta_\infty}$$

have the same asymptotic behaviour as $\mu \to \infty$ in the sector $\mu \in \Sigma_\infty^{(k+2)}$. By the last statement of Theorem 7.1, we therefore conclude that,

$$\Psi_\infty^{(k+2)}(\mu) \equiv \Psi_\infty^{(k)}(\mu e^{2\pi i})e^{i\pi\Theta_\infty}, \quad \mu \in \Sigma_\infty^{(k+2)}. \tag{7.1.7}$$

In this sense, all solutions $\Psi_\infty^{(k)}(\mu)$ are categorized into two fundamentally distinct cases, namely, when k is even and when k is odd .

Definition 7.1. *Let $\Psi_\infty^{(k)}(\mu)$ be the fundamental solutions given in Theorem 7.1 and define sectors,*

$$\Pi_\infty^{(k)} := \Sigma_\infty^{(k)} \cap \Sigma_\infty^{(k+1)} \equiv \left\{ \mu : |\mu| > \rho_\infty,\ \frac{\pi}{2} < arg(\mu) + k\pi < \frac{3\pi}{2} \right\},$$

as illustrated in Figure 7.1 below. We define Stokes matrices $S_k \in SL_2(\mathbb{C})$ as follows,

$$\Psi_\infty^{(k+1)}(\mu) = \Psi_\infty^{(k)}(\mu) S_k, \quad \mu \in \Pi_\infty^{(k)}.$$

We can combine Definition 7.1 with the relation (7.1.7) to deduce,

$$e^{i\pi\Theta_\infty} S_{k+1} = S_{k-1} e^{i\pi\Theta_\infty},$$

which shows that equation (7.1.1) has only two types of Stokes matrices S_k which are fundamentally different: one with k odd and one with k even.

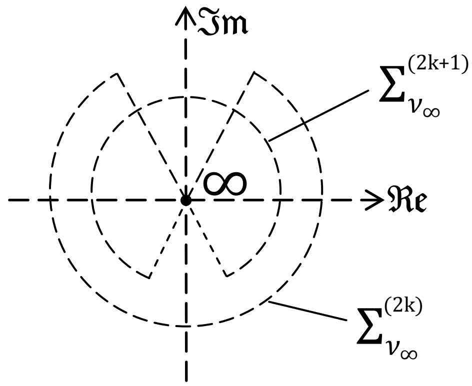


Figure 7.1: Sectors $\Sigma_\infty^{(k)}$ and $\Pi_\infty^{(k)}$, $k \in \mathbb{Z}$, projected onto the plane $\overline{\mathbb{C}}\backslash\{0\}$. The Stokes rays lie on the imaginary axis.

Moreover, from the asymptotic relation (7.1.5), we deduce,

$$\mu^{-\frac{1}{2}\Theta_\infty} e^{\frac{\mu}{2}\sigma_3} S_k e^{-\frac{\mu}{2}\sigma_3} \mu^{\frac{1}{2}\Theta_\infty} \sim I, \quad \text{as } \mu \to \infty,\ \arg(\mu) - k\pi \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right),$$

from which it is easy to see that the matrices $S_\infty^{(2k)}$ are upper triangular, the matrices $S_\infty^{(2k+1)}$ are lower triangular and all Stokes matrices have unit diagonal.

To deal with the global behaviour, we fix the fundamental matrix at infinity to be given by $\Psi_\infty^{(1)}$ and define the connection matrices C_k, $k = 0, s$ by

$$\begin{aligned} \Psi_\infty^{(1)}(\mu) &= \Psi_0(\mu) C_0, \quad \mu \in \Omega_0, \\ &= \Psi_s(\mu) C_s, \quad \mu \in \Omega_s. \end{aligned}$$

We fix a basis γ_0 and γ_s of $\pi_1(\mathbb{C}\backslash\{0,s\})$ as in Figure 7.2. Then, the analytic continuations of $\Psi_\infty^{(1)}(\mu)$ around the curves γ_0 and γ_s define the monodromy anti-representations of the fundamental group

$$p : \pi_1\left(\mathbb{C}\backslash\{0,s\}, \infty\right) \to \mathrm{SL}_2(\mathbb{C}), \quad : [\gamma_k] \mapsto M_k,$$

where

$$M_\infty M_s M_0 = \mathbb{I},$$

and

$$\begin{aligned} M_k &= (C_k)^{-1} e^{i\pi\Theta_k} C_k, \text{ for } k = 0, s, \\ M_\infty &= e^{i\pi\Theta_\infty} S_2^{-1} S_1^{-1}. \end{aligned}$$

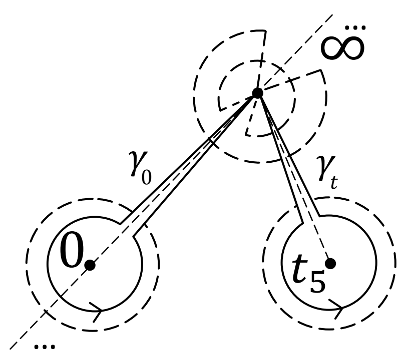


Figure 7.2: Loops, sectors and branch cuts for the P_V linear system.

7.1.2 Geometric description

Chekhov, Mazzocco and Rubtsov [CMR17] proposed a system of basic arcs generating the fundamental groupoid associated to P_V. These are reported in Figure 7.3. In this subsection, following Chekhov et al, we consider the matrices associated to the arcs a, b, d and e. Let us call these matrices M_a, M_b, S_d and S_e respectively.

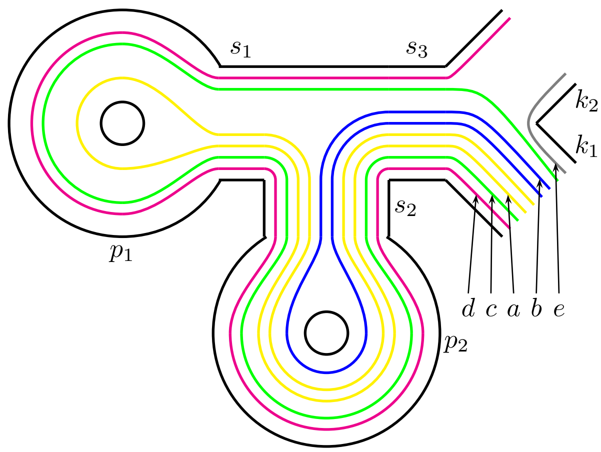


Figure 7.3: System of arcs for the P_V linear system. All arcs are oriented counter–clockwise.

This section aims to compute the quantum relations between the matrices M_a, M_b, S_d, and S_e following the results of [CMR18]. We need to express each of these matrices in terms of "building blocks" at each cusp. At each cusp, we define the building blocks as the matrices corresponding to arcs entering the Riemann surface from the cusp. We enumerate the arcs from right to left, i.e., given two arcs α_i and α_j, we say $i < j$ if going along α_i, the arc α_j is on the left. Then the following relations were proved for the matrices A_i, A_j corresponding to the arcs α_i and α_j respectively:

$$\overset{1}{A}_i\overset{2}{A}_j = \overset{2}{A}_j\overset{1}{A}_i R, \qquad \overset{2}{A}_i\overset{1}{A}_j = \overset{1}{A}_j\overset{2}{A}_i R^T, \qquad i < j, \tag{7.1.8}$$

and

$$R\overset{1}{A}_i\overset{2}{A}_i = \overset{2}{A}_i\overset{1}{A}_i R^T, \tag{7.1.9}$$

where R is the Kulish–Sklyanin R-matrix

$$R = \overset{1}{e}_{11}\overset{2}{e}_{22} + q^{-\frac{1}{2}}\left(q - \frac{1}{q}\right)\overset{1}{e}_{12}\overset{2}{e}_{21}.$$

Using this basic relation, we can deduce the quantum relations between the matrices M_a, M_b, S_d and S_e. In particular, let us observe that S_d and S_e correspond to arcs starting and finishing at different cusps, while M_a and M_b correspond to loops, i.e., arcs starting and finishing at the same cusp.

Using the basic relation (7.1.9), let us first deduce what happens when a matrix corresponds to an open arc starting and finishing at different cusps. Let these arcs be oriented in the same way, i.e. they originate at the same cusp and end at the same cusp. Inverting orientation corresponds to inverting a matrix. Let us denote by an index $^{(k)}$ the building blocks at cusp k, then

$$S_e = (A_i^{(k_2)})^{-1} A_j^{(k_1)}, \qquad S_d = (A_l^{(k_1)})^{-1} A_k^{(k_2)}, \qquad k < i,\, j < l$$

which, by applying (7.1.9), leads to[1]

$$\overset{1}{S}_e R \overset{2}{S}_d = \overset{2}{S}_d R^T \overset{1}{S}_e,$$

and

$$R \overset{1}{S}_e \overset{2}{S}_e = R^T \overset{2}{S}_e \overset{1}{S}_e.$$

In the semi–classical limit, this leads to

$$\{\overset{1}{S}_e \underset{,}{\otimes} \overset{2}{S}_d\} = \overset{2}{S}_d r^T \overset{1}{S}_e - \overset{1}{S}_e r \overset{2}{S}_d,$$

and

$$\{\overset{1}{S}_e \underset{,}{\otimes} \overset{2}{S}_e\} = r^T \overset{2}{S}_e \overset{1}{S}_e - r \overset{1}{S}_e \overset{2}{S}_e,$$

with

$$r = \frac{1}{2} \sum_{i,j} \overset{1}{e}_{ii} \overset{2}{e}_{jj} - \sum_i \overset{1}{e}_{ii} \overset{2}{e}_{ii} - 2 \sum_{j>i} \overset{1}{e}_{ij} \overset{2}{e}_{ji}.$$

Let us now study the matrices M_a, M_b. These start and finish at the same cusp k_1, so that $M_b = (A_1^{(k_1)})^{-1} A_2^{(k_1)}$ and $M_a = (A_3^{(k_1)})^{-1} A_4^{(k_1)}$, hence giving

[1] We can use relation (2.13) in [CMR18] with $S_e^{-1} = M_i^j$ and $M_d = M_k^l$.

rise to the following quantum relations

$$\overset{1}{M}_b R \overset{2}{M}_a R^{-1} = R \overset{2}{M}_a R^{-1} \overset{1}{M}_b.$$

In the semi-classical limit, this leads to

$$\{M_b \underset{,}{\otimes} M_a\} = \overset{1}{M}_a \overset{2}{M}_b r + r \overset{2}{M}_b \overset{1}{M}_a - \overset{1}{M}_a r \overset{2}{M}_b - \overset{2}{M}_b r \overset{1}{M}_a.$$

We now want to link to the P_V monodromy data. Because of the relation $S_e S_d = M_b M_a$, and thanks to the fact that M_a, M_b encircle holes, we claim that $M_b \sim M_0$, $M_a \sim M_s$, $S_e \sim S_1 X$ and $S_d \sim X^{-1} S_2 e^{i\pi\Theta_\infty}$ up to a suitable global conjugation and a choice of the matrix X. This reasoning brings us to conjecture the following Poisson relations for the P_V monodromy data

Conjecture 7.1. *Let $\{M_1, M_2, S_1, S_2\}$ be monodromy data corresponding to P_V, then there is Poisson structure on these data that can be written in the following form:*

$$\{S_k \underset{,}{\otimes} S_k\} = \overset{1}{S}_k (r - r^T) \overset{2}{S}_k + \overset{2}{S}_k (r - r^T) \overset{1}{S}_k + 2r \overset{1}{S}_k \overset{2}{S}_k - 2 \overset{1}{S}_k \overset{2}{S}_k r^T, \quad (7.1.10)$$

$$\{S_1 \underset{,}{\otimes} S_2\} = r^T \overset{1}{S}_1 \overset{2}{S}_2 + \overset{2}{S}_2 \overset{1}{S}_1 r^T - \overset{1}{S}_1 r^T \overset{2}{S}_k - \overset{2}{S}_2 r \overset{1}{S}_1, \quad (7.1.11)$$

$$\{M_2 \underset{,}{\otimes} M_1\} = \overset{1}{M}_1 \overset{2}{M}_2 r + r \overset{2}{M}_2 \overset{1}{M}_1 - \overset{1}{M}_1 r \overset{2}{M}_2 - \overset{2}{M}_2 r \overset{1}{M}_1. \quad (7.1.12)$$

This structure can be seen as an image of the following bracket on the moduli space of flat connections under the Riemann-Hilbert correspondence:

$$\left\{ A(\lambda_1) \underset{,}{\otimes} A(\lambda_2) \right\} = \left[r(\lambda_1 - \lambda_2), \overset{1}{A}(\lambda_1) + \overset{2}{A}(\lambda_2) \right], \quad (7.1.13)$$

where r is Kulish-Sklyanin r-matrix and λ - spectral parameter.

7.1.3 Faddeev-Takhtadjan Poisson structure on the Connection Matrices

We start with recalling some known results of Faddeev-Takhtajan [FT07] and Korotkin-Samtleben [KS97] who obtained Poisson and quasi-Poisson structures on the connection matrices and monodromy matrices for Fuchsian systems respectively.

Consider the differential system

$$dY(z) = A(z)Y(z). \tag{7.1.14}$$

Here the fundamental solutions $Y(z)$ can be seen as sections in trivial bundle on sphere and $A(z)$ respectively as connections in the same bundle. Faddeev and Takhtajan in their book [FT07] start with classical bracket on the connections

$$\left\{A(\lambda_1) \underset{,}{\otimes} A(\lambda_2)\right\} = \left[r(\lambda_1 - \lambda_2), \overset{1}{A}(\lambda_1) + \overset{2}{A}(\lambda_2)\right] \tag{7.1.15}$$

with $r(\lambda) = \frac{\Omega}{\lambda}$ being classical r-matrix, i.e. a solution of the classical Yang–Baxter equation and $\Omega = \sum_{i,j} e_{ij} \otimes e_{ji}$ the permutation operator, λ – spectral parameter. To obtain the bracket on the monodromy data one needs to introduce an auxiliary object which is called connection matrices. If one considers $Y(z)$ at some fixed point z_1 and takes its analytic continuation along the path from z_1 to some point z_2, $Y(z_1)$ will be multiplied by a connection matrix $C_{z_1 z_2}$:

$$Y(z_2) = Y(z_1)C_{z_1 z_2}. \tag{7.1.16}$$

Using the relation (7.1.14) and the definition of connection matrix, one can obtain

$$\frac{\partial C_{z_1 z_2}(\lambda)}{\partial z_2} = A(z_2,\lambda) C_{z_1 z_2}(\lambda) \tag{7.1.17}$$

$$\frac{\partial C_{z_1 z_2}(\lambda)}{\partial z_1} = -A(z_1,\lambda) C_{z_1 z_2}(\lambda). \tag{7.1.18}$$

Faddeev and Takhtajan derive the bracket on the connection matrices from the bracket on connections (7.1.15). Here we will give a sketch of their proof for the bracket between to identical connection matrices since we will use its intermediate steps further. Considering $C_{z_1 z_2}$ as a functional of matrix elements of $A(z)$ and then applying the chain rule one can obtain the following relation (formula 1.37 Section 3 §1 in [FT07]):

$$\left\{ C_{s_0 z_1}(\lambda_1) \underset{,}{\otimes} C_{s_0 z_2}(\lambda_2) \right\} = \int_{s_0}^{z_1} \int_{s_0}^{z_2} \overset{1}{C_{z z_1}}(\lambda_1) \overset{2}{C_{z' z_2}}(\lambda_2) \cdot \left\{ A(z,\lambda_1) \underset{,}{\otimes} A(z',\lambda_2) \right\} \overset{1}{C_{s_0 z}}(\lambda_1) \overset{2}{C_{s_0 z'}}(\lambda_2) dz' dz. \tag{7.1.19}$$

Expanding the bracket inside the integrand, we obtain

$$\left\{ C_{s_0 z_1}(\lambda_1) \underset{,}{\otimes} C_{s_0 z_2}(\lambda_2) \right\} = \int_{s_0}^{z_1} \int_{s_0}^{z_2} \overset{1}{C_{z z_1}}(\lambda_1) \overset{2}{C_{z' z_2}}(\lambda_2) \cdot \left[r(\lambda_1 - \lambda_2), \overset{1}{A}(z,\lambda_1) + \overset{2}{A}(z',\lambda_2) \right] \overset{1}{C_{s_0 z}}(\lambda_1) \overset{2}{C_{s_0 z'}}(\lambda_2) dz' dz. \tag{7.1.20}$$

If the paths $s_0 \to s_1$ and $s_0 \to s_2$ has distinguished end points, they can bi split into intersecting and non-intersecting parts $s_0 \to s_x \to s_1$ and $s_0 \to s_x \to s_2$. Then the connection matrices become products $C_{s_0 s_x} C_{s_x s_1}$ and $C_{s_0 s_x} C_{s_x s_2}$. The Poisson bracket between these products can be split into the sum of four brackets between the multipliers. The only term does not vanish is $\left\{ C_{s_0 s_x} \underset{,}{\otimes} C_{s_0 s_x} \right\} \left(\overset{1}{C}_{s_x s_1} \overset{2}{C}_{s_x s_2} \right)$. Now in order to compute the bracket on any pair of connection matrices we pass to the computation of the bracket on the coinciding ones.

$$\left\{ C_{s_0 s_x}(\lambda_1) \underset{,}{\otimes} C_{s_0 s_x}(\lambda_2) \right\} = \int_{s_0}^{s_x} \overset{1}{C}_{s_0 z}(\lambda_1) \overset{2}{C}_{s_0 z}(\lambda_2) \cdot \left[r(\lambda_1 - \lambda_2), \overset{1}{A}(z, \lambda_1) + \overset{2}{A}(z, \lambda_2) \right] \overset{1}{C}_{z s_x} \overset{2}{C}_{z s_x} dz. \quad (7.1.21)$$

One can notice that the integrand is complete derivative of

$$\overset{1}{C}_{s_0 z}(\lambda_1) \overset{2}{C}_{s_0 z}(\lambda_2) r(\lambda_1 - \lambda_2) \overset{1}{C}_{z s_x}(\lambda_1) \overset{2}{C}_{z s_x}(\lambda_2). \quad (7.1.22)$$

After integration, one obtains the following formula for the case of coinciding connection matrices:

$$\left\{ C_{s_0 s_x}(\lambda_1) \underset{,}{\otimes} C_{s_0 s_x}(\lambda_2) \right\} = \left[r(\lambda_1 - \lambda_2), \overset{1}{C}_{s_0 s_x}(\lambda_1) \overset{2}{C}_{s_0 s_x}(\lambda_2) \right]. \quad (7.1.23)$$

We should consider case of $\lambda_1 = \lambda_2$.

In this setting, points z_1 and z_2 are taken on the Riemann surface. In order to construct the analogues structure on the generalized monodromy matrices, one should pass to considering points on the cover of punctured cusped Riemann surface. In particular, a monodromy matrix M_i corresponding to a simple pole at some point a_i can be seen as a connection matrix $C_{z_1 z_2}$ with z_1 and z_2 coinciding on the Riemann surface but distinguished on the covering.

Regularization of formula 7.1.23

In the Fuchsian case, one can write the connection Adz in the form $\frac{A_i}{z-a_i}$. In the similar way, from the equation $\frac{\partial}{\partial z}C_{s_0z} = AC_{s_0z}$, one can write

$$C_{s_0z} = \Pi(z-a_i)^{A_i} \tag{7.1.24}$$

$$\left\{C_{s_0s_x}(\lambda_1) \underset{,}{\otimes} C_{s_0s_x}(\lambda_2)\right\} = \left[r(\lambda_1-\lambda_2), \Pi(z\overset{1}{-}a_i)^{A_i}\Pi(z\overset{2}{-}a_i)^{A_i}\right] \tag{7.1.25}$$

Following the approach of Hasibul Hassan Chowdhury, starting with Atiyah-Bott bracket, we arrive to the following version of Faddeev-Takhtajan bracket:

$$\{C_{x_1x_2} \underset{,}{\otimes} C_{y_1y_2}\} = \frac{1}{2}\left[C_{x_1O}\otimes C_{y_1,O}\right] r \left[C_{Ox_2}\otimes C_{Oy_2}\right] \tag{7.1.26}$$

where O stand for intersection point of the corresponding paths.

For our purpose we need the right hand side of the formula to be written in terms of the same elements that there are on the left hand side so we rewrite it using $C_{x_1x_2} = C_{x_1O}C_{Ox_2}$ and $C_{y_1y_2} = C_{y_1O}C_{Oy_2}$:

$$\{C_{x_1x_2} \underset{,}{\otimes} C_{y_1y_2}\} = \overset{1}{C}_{x_1O}\overset{2}{C}_{y_1O}\{C_{Ox_2} \underset{,}{\otimes} C_{Oy_2}\} + \overset{1}{C}_{x_1O}\{C_{Ox_2} \underset{,}{\otimes} C_{y_1O}\}\overset{2}{C}_{Oy_2} \tag{7.1.27}$$

$$+ \overset{2}{C}_{y_1O}\{C_{x_1O} \underset{,}{\otimes} C_{Oy_2}\}\overset{1}{C}_{Ox_2} + \{C_{x_1O} \underset{,}{\otimes} C_{y_1O}\}\overset{1}{C}_{Ox_2}\overset{2}{C}_{Oy_2} \tag{7.1.28}$$

and as a result, by opening the bracket, we get the formula:

$$\{C_{x_1x_2} \underset{,}{\otimes} C_{y_1y_2}\} = \overset{1}{C}_{x_1x_2} r \overset{2}{C}_{y_1y_2} + \overset{2}{C}_{y_1y_2} r \overset{1}{C}_{x_1x_2} + r \overset{1}{C}_{x_1x_2} \overset{2}{C}_{y_1y_2} + \overset{1}{C}_{x_1x_2} \overset{2}{C}_{y_1y_2} r \tag{7.1.29}$$

The quantum version of this formula looks like:

$$R^{-1} \overset{1}{C}_{x_1x_2} R \overset{2}{C}_{y_1y_2} = \overset{2}{C}_{y_1y_2} R^{-1} \overset{1}{C}_{x_1x_2} R \tag{7.1.30}$$

which coincides with the formula from [CMR18].

The formula (7.1.26) can be specialised to the case when $x_1 = y_1 = 0$ and $x_2 = x$, $y_2 = y$ leading to

$$\{C_{Ox} \underset{,}{\otimes} C_{Oy}\} = \overset{1}{C}_{Ox} \overset{2}{C}_{Oy} r \tag{7.1.31}$$

which corresponds to the quantum relation

$$\overset{2}{C}_{Oy} \overset{1}{C}_{Ox} R^{-1} = \overset{1}{C}_{Ox} \overset{2}{C}_{Oy}. \tag{7.1.32}$$

Respectively,

$$\{C_{xO} \underset{,}{\otimes} C_{yO}\} = \overset{1}{C}_{xO} \overset{2}{C}_{yO} r \tag{7.1.33}$$

and

$$\overset{1}{C}_{xO} \overset{2}{C}_{yO} R^{-1} = \overset{2}{C}_{yO} \overset{1}{C}_{xO}. \tag{7.1.34}$$

In order to rewrite bracket on $C_{x_1x_2}$ and $C_{y_1y_2}$ we have two remaining building blocks to consider:

$$\{C_{xO} \underset{,}{\otimes} C_{Oy}\} = \overset{1}{C}_{xO} r \overset{2}{C}_{Oy} \tag{7.1.35}$$

with the quantum relation

$$\overset{1}{C}_{xO} R^{-1} \overset{2}{C}_{Oy} = \overset{2}{C}_{Oy} \overset{1}{C}_{xO}. \tag{7.1.36}$$

And the last building block

$$\{C_{Ox} \underset{,}{\otimes} C_{yO}\} = \overset{2}{C}_{yO} r \overset{1}{C}_{Ox} \tag{7.1.37}$$

with the quantum relation

$$\overset{1}{C}_{Ox} \overset{2}{C}_{yO} = \overset{2}{C}_{yO} R \overset{1}{C}_{Ox}. \tag{7.1.38}$$

7.1.4 Poisson bracket on the monodromy matrices of Fuchsian system

Combining the "building blocks" 7.1.31-7.1.37 together, we arrive to the following Poisson structure on the monodromy data of a Fuchsian system. Note that a similar Poisson structure (with a different choice of r-matrix) was obtained by Korotkin and Samtleben in [KS97].

Theorem 7.2. *Let $\{M_i\}$ be monodromy data corresponding to a Fuchsian system. Then there is Poisson structure on the corresponding character variety that can be written in the following form:*

$$\left\{ M_i \underset{,}{\otimes} M_j \right\} = \overset{1}{M_i} r \overset{2}{M_j} + \overset{2}{M_j} r \overset{1}{M_i} - r \overset{2}{M_j} \overset{1}{M_i} - \overset{1}{M_i} \overset{2}{M_j} r \tag{7.1.39}$$

$$\left\{ M_i \underset{,}{\otimes} M_i \right\} = \overset{2}{M_i} r \overset{1}{M_i} + \overset{2}{M_i} r^T \overset{2}{M_i} - r^T \overset{1}{M_i} \overset{2}{M_j} - \overset{1}{M_i} \overset{2}{M_j} r \tag{7.1.40}$$

$$\left\{ M_\infty \underset{,}{\otimes} M_i \right\} = \overset{2}{M_i} r \overset{1}{M_\infty} + \overset{2}{M_\infty} r^T \overset{2}{M_i} - r \overset{1}{M_\infty} \overset{2}{M_j} - \overset{1}{M_\infty} \overset{2}{M_j} r^T, \tag{7.1.41}$$

where $i < j$. This structure can be seen as an image of the following bracket on the moduli space of flat connections under the Riemann-Hilbert correspondence:

$$\left\{ A(\lambda_1) \underset{,}{\otimes} A(\lambda_2) \right\} = \left[r(\lambda_1 - \lambda_2), \overset{1}{A(\lambda_1)} + \overset{2}{A(\lambda_2)} \right], \tag{7.1.42}$$

where r is Kulish-Sklyanin r-matrix and λ - spectral parameter.

This theorem covers the case of P_{VI}. Our conjecture 7.1 for the data of P_V is motivated by this theorem.

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