UNIVERSITAT POLITÈCNICA DE CATALUNYA

DOCTORAL THESIS

Canonical realizations of Bondi-Metzner-Sachs–like symmetries in field theory

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Author:
Víctor M. CAMPELLO

Supervisors:
Dr. Carles BATLLE
Dr. Joaquim GOMIS

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Abstract

Facultat de Matemàtiques i Estadística
Departament de Matemàtiques

Doctor en Matemàtica Aplicada

Canonical realizations of Bondi-Metzner-Sachs–like symmetries in field theory

by Víctor M. CAMPELLO

The BMS group appears as an infinite-dimensional group of isometries of asymptotically flat spacetimes first introduced by Bondi, Metzner, van der Burg, and Sachs in 1962. This group has gained interest recently due to the invariance of the gravitational S-matrix under these transformations and the existence of a connection between Weinberg's soft gravitons theorems and Ward identities of BMS supertranslations, and also due to the relation between flat space holography and BMS.

Despite being originally related to gravitational physics, the BMS group and its Lie algebra can be realized in free flat field theories by means of the Fourier modes of the field. One of these realizations, which we refer to as the canonical realization, can be built for a free scalar field in Minkowski space using a generalization of the usual Poincaré charges.

In this Thesis, we study in detail the canonical realization to uncover the expression of the infinite-dimensional conserved charges associated with BMS transformations in $d = 3$ spacetime. The final expression consists of an integral transformation in terms of derivatives of polyharmonic Green functions. We later explore a particle non-linear realization of BMS using the Maurer-Cartan form to find an infinite set of BMS coordinates that are constrained by gauge transformations. We construct the corresponding Poincaré transformation generators in terms of these infinite-dimensional coordinates and the associated momenta. Finally, we study the extension of BMS with conformal transformations in the massless theory. We conclude that it is possible to extend the algebra to a Weyl-BMS realization by defining new superdilatation operators, but the incorporation of special conformal transformations results in an infinite tower of new operators that need further study.

The work presented in this Thesis could be of some use for the study of flat-space holography since it describes a field theory in three-dimensional space-time that could act as the dual to asymptotic flat gravity theory in the bulk.

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Resum en català

Facultat de Matemàtiques i Estadística
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Doctor en Matemàtica Aplicada

Canonical realizations of Bondi-Metzner-Sachs–like symmetries in field theory

per Víctor M. CAMPELLO

El grup BMS es descobreix com un grup infinit-dimensional d'isometries d'espaitemps asimptòticament plans per part de Bondi, Metzner, van der Burg i Sachs al 1962. Recentment, el grup ha suscitat interès degut a la invariància de la matriu gravitacional S sota les seves transformacions i per l'existència d'una connexió entre els teoremes de Weinberg sobre gravitons dèbils i les identitats de Ward per a les supertranslacions BMS. Així mateix, hi ha un interès en la relació entre l'holografia a l'espai pla i el grup BMS.

Tot i estar relacionat originalment amb la física gravitacional, el grup BMS and la seva àlgebra de Lie es poden realitzar en teories de camps lliures a un espai pla mitjançant els modes de Fourier del camp. Una d'aquestes realitzacions, que nosaltres anomenem la realització canònica, es pot construir per a un camp escalar lliure a l'espai de Minkowski fent servir una generalització de les càrregues de Poincaré. En aquest Tesi, s'estudia en detall la realització canònica per descobrir l'expressió de les infinites càrregues conservades associades a les transformacions BMS en un espaitemps de dimensió $d = 3$. L'expressió final consisteix en una transformació integral en termes de derivades de les funcions de Green poliharmòniques. A continuació, s'estudia una realització no-lineal de partícula del grup BMS fent servir la forma de Maurer-Cartan per acabar trobant un conjunt infinit de coordenades BMS restringides per transformacions de gauge. Així mateix, es construeixen els generadors de les transformacions de Poincaré en funció d'aquestes coordenades i els seus moments associats. Finalment, s'estudia l'extensió del grup BMS amb transformacions conformes en la teoria no massiva. Es conclou que és possible estendre l'àlgebra a una realització Weyl-BMS definint nous operadors anomenats superdilatacions, però la incorporació de les transformacions conformes especials dona com a resultat una torre infinita de nous operadors que requereix un estudi apart.

El treball presentat en aquesta Tesi podria ser rellevant per a l'estudi de l'holografia a l'espai pla, ja que descriu una teoria de camps en un espaitemps tridimensional que podria actuar com el dual d'una teoria gravitatòria asimptòticament plana al bulk.

UNIVERSITAT POLITÈCNICA DE CATALUNYA

Resumen en castellano

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Doctor en Matemàtica Aplicada

Canonical realizations of Bondi-Metzner-Sachs–like symmetries in field theory

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El grupo BMS se decubre como un grupo infinito-dimensional de isometrías de espacio-tiempos asintóticamente planos por Bondi, Metzner, van der Burg y Sachs en el 1962. Recientemente, el grupo ha suscitado interés debido a la invariancia de la matriz gravitacional S bajo estas transformaciones, así como por la existencia de una conexión entre los teoremas de Weinberg sobre gravitones débiles y las identidades de Ward para las supertranslaciones BMS. Así mismo, también existe un interés sobre la relación entre la holografía en un espacio plano y el grupo BMS.

A pesar de estar relacionado originalmente con la física gravitacional, el grupo BMS y su álgebra de Lie se pueden realizar en teorías de campos libres en un espacio plano mediante los modos de Fourier del campo. Una de estas realizaciones, a la que nos referimos como la realización canónica, se puede construir para un campo escalar libre en el espacio de Minkowski utilizando una generalización de las cargas de Poincaré. En esta Tesis se estudia en detalle la realización canónica para descubrir la expresión de las infinitas cargas conservadas asociadas a las transformaciones BMS en un espaciotiempo de dimensión $d = 3$. La expresión final consiste en una transformación integral en función de las derivadas de las funciones de Green poliarmónicas. A continuación, se estudia una realización no-lineal de partícula del grupo BMS mediante la forma de Maurer-Cartan para encontrar un conjunto infinito de coordenadas BMS con restricciones dadas por transfromaciones de gauge. Así mismo, se construyen los generadores de las transformaciones de Poincaré en función de estas coordenadas y sus momentos asociados. Finalmente, se estudia la extensión del grupo BMS con transformaciones conformes en la teoría no masiva. Se concluye que es posible extender el álgebra a una realización Weyl-BMS definiendo nuevos operadores llamados superdilataciones, pero la incorporación de las transformaciones conformes especiales resulta en una torre infinita de nuevos operadores que requieren un estudio más en profundidad.

El trabajo presentado en esta Tesis podría ser relevante para el estudio de la holografía en el espacio plano, ya que describe una teoría de campos en un espaciotiempo tridimensional que podría actuar como el dual de una teoría gravitatoria asintóticamente plana en el bulk.

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I would like to also thank Dr. Antonio Lasanta and Dr. Eduardo S. Villaseñor for inviting me to discuss some of the results presented in this Thesis in Ceuta.

Lastly, to my good friends Javi, Álex, and Guille and my loved ones, my parents, my brother, and Violeta. Thank you for always standing by me. Gracias. Gràcies.

Contents

Chapter 1

Introduction

1.1 The BMS group

There had been the belief for some time that Minkowski space was a good approximation for a four-dimensional asymptotically flat space-time originated by an isolated system emitting gravitational radiation. It turns out that this is not the case and that, in a sense, the asymptotics of a weak gravitational field are not described by Minkowski space.

The first hint about this difference was obtained in 1962 when Bondi, van der Burg, Metzner, and Sachs [19, 46, 47] found out that one does not recover the Poincaré group of symmetries in the boundary but an infinite dimensional extension of it, the BMS (Bondi-Metzner-Sachs) group. This means that when gravity is present, a much richer physics arises at the boundary that must be investigated. They obtained these extra transformations by imposing a specific behavior in the functions associated with the metric when approaching the boundary and investigating the isometries of the asymptotic metric at first order. Thus, these extra symmetries are gauge transformations that leave invariant the asymptotic configuration of the fields.

Despite its origin as an asymptotic symmetry in a gravitational context, it was noticed in [36] that a generalization of the ordinary Poincaré generators of translations yields an infinite set of translation generators, called super-translations, that, together with the Lorentz generators, form an algebra with the same structure than that of the original BMS transformations. These generalized translations were further studied in [37] and [33], and more specific results for the case of $2+1$-dimensional space-time were obtained in [12], where a further enlargement of the algebra, including transformations generalizing the ordinary Lorentz ones, was also considered in this framework for the case of a massless Klein-Gordon field. These new transformations, called super-rotations, had been considered previously [6, 8, 9] in the standard gravitational approach. The extension of some of the results of [12] to the case of $3+1$ spacetime and the non-relativistic limits were studied in [16, 28].

As we will see, the approach initiated by Longhi and Materassi in [36] presents the BMS algebra as a set of symmetries of a scalar field. As shown in [12], the Noether charges associated with the super-translations are non-local in space and are formulated as transformations of the field. The question arises of whether these field transformations correspond to symmetries of the space-time coordinates, in the same sense that ordinary translations $\delta x^\mu = \epsilon^\mu$ yield (functional) field transformations $\delta\phi(x) = \epsilon^\mu \partial_\mu \phi(x)$. The space-time interpretation of the BMS symmetry has also been considered recently in a completely different approach in [50], by analysing some universal geometric structures that appear on infinitesimal tangent light cones.

A related topic is the construction of particle Lagrangians that implement the BMS symmetry, in the same sense that $L = -m\sqrt{-\dot{x}^\mu \dot{x}_\mu}$ is invariant under standard Poincaré transformations. If one can construct such a Lagrangian one should in principle be able to compute explicitly the transformation of the space-time coordinates under a super-translation. BMS particles have also been studied in [40–42] in the formalism of co-adjoint orbits.

Aside from the extension to super-rotations, there have been proposals for the inclusion of other symmetries, such as those associated with the conformal group [34] or supersymmetry in the context of supergravity [3, 10, 11], in the spirit of the original asymptotic approach.

It might be argued that free fields in $2+1$ space-time will give results not relevant for actual physics. However, besides being simpler and yielding expressions that are much easier to analyze and hence to give hints about what to expect in higher dimensions, one can also interpret the $2+1$ free fields in the framework of flat holography [9, 10][4, 5] as fields living in the asymptotic boundary of $3+1$ space-time, and maybe being relevant as dual fields of the gravitational field in the bulk (see also the discussion in [26, 27]). Hence, any result concerning the realizations of BMS symmetry in $2+1$ dimensions is worthy of study, and this is the central guiding principle of this thesis.

1.2 Derivation of the BMS group

The BMS group was derived in the context of an isolated system that radiates, where the form of an asymptotically flat spacetime is introduced. In retarded Bondi coordinates $(u = t - r, r, x^A \equiv (\theta, \varphi))$, for the case of a reflection- and axial-symmetrical spacetime, this is

$$ds^2 = -Vr^{-1}e^{2\beta}du^2 - 2e^{2\beta}dudr + r^2 h_{AB}(dx^A - U^A du)(dx^B - U^B du), \tag{1.1}$$

where

$$h_{AB}dx^A dx^B = e^{2\gamma}d\theta^2 + \sin^2\theta e^{-2\gamma}d\varphi^2, \tag{1.2}$$

and $U^\varphi = 0$. The functions V, β, U^θ and γ have the following behavior for large r

$$\begin{cases} V = & r - 2M(u,\theta) + \mathcal{O}(r^{-1}), \\ \beta = & -|c(u,\theta)|^2 r^{-2} + \mathcal{O}(r^{-2}), \\ U^\theta = & -(\partial_\theta c(u,\theta) + 2c(u,\theta)\cot\theta)r^{-2} + \mathcal{O}(r^{-3}), \\ \gamma = & c(u,\theta)r^{-1} + \mathcal{O}(r^{-3}). \end{cases} \tag{1.3}$$

One can check that for r sufficiently large, flat spacetime is recovered. The first corrections to it are

$$\begin{aligned} ds^2 = & -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \\ & + \frac{2M(u,\theta)}{r}du^2 + 2(\partial_\theta c(u,\theta) + 2c(u,\theta)\cot\theta)dud\theta \\ & + 2rc(u,\theta)(d\theta^2 - \sin^2\theta d\varphi^2). \end{aligned} \tag{1.4}$$

Two functions appear in the next to leading order terms which are commonly called *Bondi mass aspect*, $M(u,\theta)$, and *Bondi news*, $\partial_u c(u,\theta)$. The latter depends on the function $c(u,\theta)$, that describes gravitational waves [48]. The Bondi mass aspect coincides with the mass of the static system, but it is not constant in general. For instance, for the Schwarzschild solution in retarded coordinate $u = t - r - 2m \ln |r - 2m|$,

$$ds^2_{\text{Schw.}} = -du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) + \frac{2m}{r}du^2, \tag{1.5}$$

$M(u,\theta) = m$ and corresponds to the black hole mass, which is constant. The second function is related to the change in the mass of the system. If one defines the average mass

$$m(u) = \frac{1}{2}\int_0^\pi d\theta\, M(u,\theta)\sin\theta, \tag{1.6}$$

then it can be shown that [see 19, Equation (58)]

$$\partial_u m(u) = -\frac{1}{2}\int_0^\pi d\theta\, \left(\partial_u c(u,\theta)\right)^2 \sin\theta. \tag{1.7}$$

Thus a variation in c causes the average mass to diminish. This function is connected to all the other functions through the Bianchi identities, so every new information in the system comes with a change in c, and hence the name.

A natural question arises now: what are the diffeomorphisms that preserve the form of the metric (1.4) to leading order? If one assumes that these diffeomorphisms can be expanded in powers of r the result is, in the general case,

$$\bar{r} = K(\theta,\varphi)r, \quad \bar{u} = K(\theta,\varphi)^{-1}[u + \alpha(\theta,\varphi)], \quad \bar{\theta} = H(\theta,\varphi), \quad \bar{\varphi} = I(\theta,\varphi), \tag{1.8}$$

where K, H and I are the functions of a conformal transformation, *i.e.* such that

$$ds^2 = d\theta^2 + \sin^2\theta d\varphi^2 = K^2(\theta,\varphi)(d\bar{\theta}^2 + \sin^2\bar{\theta} d\bar{\varphi}^2). \tag{1.9}$$

These transformations form a group and its structure is the following. Setting $\alpha = 0$, one recovers the conformal transformations of S^2, *i.e.* the Lorentz group. On the other hand, transformations of the type

$$\bar{r} = r, \quad \bar{u} = u + \alpha(\theta,\varphi), \quad \bar{\theta} = \theta, \quad \bar{\varphi} = \varphi, \tag{1.10}$$

are the so-called *supertranslations*, $\mathcal{T}$. These are a cofactor of the BMS group by conformal transformations. As an invariant subgroup of it, one finds the four ordinary translations

$$\alpha = \epsilon_0 + \epsilon_1 \sin\varphi \sin\theta + \epsilon_2 \cos\varphi \sin\theta + \epsilon_3 \cos\theta. \tag{1.11}$$

Therefore the BMS group, as originally derived, is a semidirect product of the homogeneous Lorentz group $SO(1,3)$ with the supertranslations infinite-dimensional group. The latter has as an invariant group the usual four translations. So finally, the Poincaré group is recovered but including also a generalization of translations. This can be written as

$$\text{BMS} = SO(1,3) \ltimes \mathcal{T}. \tag{1.12}$$

The extension of the first factor of the BMS group to an infinite-dimensional one is proposed in [9] by considering transformations that are singular at some point, contrary to the ones of $SO(1,3)$. These extra transformations are called *superrotations*, $\mathcal{R}$.

Furthermore if $P_{n,m}$ are the infinitesimal generators of supertranslations and $\mathcal{R}_\ell$ the ones of superrotations, the algebra is the following

$$\begin{gathered}\{P_{n,m}, P_{n',m'}\} = 0, \quad \{\mathcal{R}_n, \mathcal{R}_m\} = (n-m)\mathcal{R}_{n+m} \\ \{\overline{\mathcal{R}}_n, \overline{\mathcal{R}}_m\} = (n-m)\mathcal{R}_{n+m}, \quad \{\overline{\mathcal{R}}_n, \mathcal{R}_m\} = 0, \\ \{\mathcal{R}_\ell, P_{n,m}\} = \left(\frac{\ell+1}{2} - n\right) P_{n+\ell,m}, \quad \{\overline{\mathcal{R}}_\ell, P_{n,m}\} = \left(\frac{\ell+1}{2} - m\right) P_{n,m+\ell},\end{gathered} \tag{1.13}$$

$n, m, n', m', \ell \in \mathbb{Z}$, except for possible central extensions. In the three dimensional case one has P_n and $\mathcal{R}_m$ and the algebra is

$$\{P_n, P_m\} = 0, \quad \{\mathcal{R}_n, P_m\} = (n-m)P_{n+m}, \quad \{\mathcal{R}_n, \mathcal{R}_m\} = (n-m)\mathcal{R}_{n+m}, \tag{1.14}$$

$n, m \in \mathbb{Z}$, except for central extensions.

1.3 Scope and structure of the thesis

As explained in more detail in Chapter 2, one can obtain the super-translations as generalizations of the ordinary Poincaré translations for a scalar field. Indeed, the conserved charges associated with a translation can be written in terms of the Fourier modes of the scalar field as

$$\begin{gathered}P^\mu = \int d\widetilde{k}\, a^*(\vec{k}, t) k^\mu a(\vec{k}, t), \\ k^\mu = (k^0, \vec{k}), \quad k^0 = \sqrt{m^2 + \vec{k}^2}, \quad d\widetilde{k} = \frac{d^d k}{(2\pi)^2 2k^0}.\end{gathered} \tag{1.15}$$

It turns out that the functions k^μ that appear in these expressions satisfy the eigenvalue equation[1]

$$\Delta k^\mu = \frac{d-1}{m^2} k^\mu \tag{1.16}$$

where Δ is the Beltrami-Laplace operator of the on-shell manifold $(k^0)^2 - \vec{k}^2 = m^2$, parameterized by the spatial components of k^μ. It was the key observation in the seminal work by Longhi and Materassi [36] that one could compute other functions of $\vec{k}$ satisfying the same equation with the same eigenvalue, and that using these functions instead of k^μ one can construct an infinite set of conserved quantities which, together with the corresponding expressions for the Lorentz generators, yield an infinite-dimensional algebra which is exactly the one uncovered by Bondi, van der Burg, Metzner and Sachs in their asymptotic study of a gravitational field.

As already pointed out in [36] and further developed in [12, 28] and in the Master Thesis of Victor Campello, with special attention to the $2+1$ space-time case, when expressed in terms of the scalar field itself, these extra generators become non-local in space. The recurring idea of this thesis is to get further insight about the geometry of these non-local transformations, and whether they can be extended to include extra symmetries.

The charges computed by this method can be expressed as functionals of the scalar field and its canonical momentum, hence the name of the approach. By using the corresponding Poisson brackets one can compute then the transformations of

[1] In Chapter 2, Δ is scaled by a factor of m^2, and the number of spatial dimensions is d, so that this equation becomes $\Delta k^\mu = d k^\mu$.

the field which turn out to be, as already said before, non-local in space for the extra transformations. While ordinary space-time translation symmetries of a field can be easily understood in terms of the translations of the space-time variables, the non-locality makes this non-obvious for the super-translations. One of the goals of this thesis is thus to try to obtain simplified expressions of the charges, and more specifically for the case of a massless field in $2+1$, to get an interpretation in terms of space-time variables. The development of this idea is largely the contents of Chapter 2.

Another possibility to obtain a space-time interpretation of the super-translations in this formalism is to construct a model of a particle incorporating these symmetries. This is done in Chapter 3 using the non-linear realization approach, and the canonical analysis and the symmetries of the obtained Lagrangian are studied in detail.

Finally, the third block of the thesis, not directly related to the interpretation issues of the first two, deals with the extension of the symmetries of the field in this canonical formalism so as to include conformal transformations. This extension is only partially successful and is the contents of Chapter 4.

Our conclusions are presented in Chapter 5, where the discussion of some open problems is also introduced.

The appendixes contain some detailed computations not included in the main chapters. Appendix A contains the derivation of some of the properties of the polyharmonic functions used in Chapter 2, together with the detailed computation of some Poisson brackets of the charges defined there. Appendix B extends the polyharmonic functions to the $3+1$ space-time, but without discussing their properties at the same level as in the $2+1$ case, since they are not used in the main text. Appendix C contains most of the more involved computations associated with the construction of the particle model of Chapter 3, and in particular the obtention of the relevant term in the Maurer-Cartan form. Appendix D presents quite generally the construction of the charges associated with the space-time symmetries of a scalar field, and in particular the ones corresponding to conformal transformations, and their expression in terms of the Fourier modes, for a d-dimensional space-time, which in Chapter 4 are specialized to the $d = 3$ case. Appendix E contains further attempts to close the conformal algebra extended with BMS transformations, and finally Appendix F derives the relation between the Poisson bracket of conserved charges expressed in terms of Fourier modes and the commutator of the differential operators associated to the charges.

1.4 Publications associated with this thesis

Three journal publications are associated with this Thesis and contain part of the material here presented.

Carles Batlle, Víctor Campello, and Joaquim Gomis. "A canonical realization of the Weyl BMS symmetry". In: *Physics Letters B* 811 (2020), p. 135920. DOI: 10.1016/j.physletb.2020.135920. arXiv: 2008.10290 [hep-th]
JCR IF: 3.6 Q2.

Carles Batlle, Victor Campello, and Joaquim Gomis. "Polyharmonic Green functions and nonlocal Bondi-Metzner-Sachs transformations of a free scalar field". In: *Phys. Rev. D* 107.2 (2023), p. 025010. DOI: 10.1103/PhysRevD.107.025010. arXiv: 2207.12299 [hep-th]
JCR IF: 5.0 Q1.

Carles Batlle, Víctor Campello, and Joaquim Gomis. "Particle realization of Bondi-Metzner-Sachs symmetry in 2+ 1 space-time". In: *Journal of High Energy Physics* 2023.11 (2023), pp. 1–30. DOI: 10.1007/JHEP11(2023)011. arXiv: 2307.13984 [hep-th]
JCR IF: 5.4 Q1.

Additionally, partial results obtained during the development of this Thesis were presented by the author in an invited seminar within the workshop "Statistical Mechanics in General Relativity" organized in Ceuta on the 20th and 21st of October, 2022, by Dr. Eduardo S. Villaseñor and Dr. Antonio Lasanta. A link to the presentation can be found here.

Chapter 2

Canonical realization of BMS symmetries

2.1 Introduction

The canonical realization formalism consists of generalizing the conserved charges associated with a given field by defining new conserved quantities on the Fourier space. To do that, one introduces the differential operator Δ from the Laplace-Beltrami operator, naturally defined on the mass hyperboloid $k^2 + m^2 = 0$, as

$$\frac{1}{m^2}\Delta = \frac{1}{\sqrt{g}}\partial_i\left(\sqrt{g}g^{ij}\partial_j\,\cdot\right), \tag{2.1}$$

with g the metrics on the hyperboloid, which is assumed to be parameterized by the spatial components $\vec{k}$. The factor $1/m^2$ makes the operator Δ dimensionless. This allows us to discuss the massive and massless cases in a unified way, with the same values of the eigenvalues of Δ. For the massless case, one multiplies the expression by m^2 and takes the massless limit to get the corresponding operator for the cone, as will be shown later in the chapter.

As first shown in [36] (see [33] [12] for the details in the massless case), the operator Δ is actually proportional to the quadratic Casimir $M^{\mu\nu}M_{\mu\nu}$ of the Lorentz group, where the Lorentz generators in the Casimir are realized as differential operators in $\vec{k}$.

It turns out that this operator has the momenta k^μ as eigenfunctions with an eigenvalue that depends on the spatial dimension, d [36]. Explicitly,

$$\Delta k^\mu = \lambda k^\mu, \tag{2.2}$$

for $\lambda = d$. Importantly, k^μ are not the only eigenfunctions for this operator. There are, in general, infinite eigenfunctions χ_ℓ^λ, with $\ell = (\ell_1, \ldots, \ell_{d-1})$ a multi-index.

For a space-time of dimension $d+1$, one considers the Laplace-Beltrami operator defined on the $\mathbb{H}_d$ hyperboloid

$$-(k_0)^2 + (k_1)^2 + \cdots (k_d)^2 = -m^2. \tag{2.3}$$

For this manifold, it is convenient to define the following radial coordinates

$$k_0 = \sqrt{\rho^2 + m^2}, \quad k_i = z_i\rho, \quad i = 1, \ldots, d, \tag{2.4}$$

where $\rho \in [0, \infty)$, $k_0 > 0$ (since we will consider the upper sheet of the hyperboloid) and the z_i variables correspond to the spherical parametrization of the unit $(d-1)$-sphere, $\mathbb{S}^{d-1}$, in terms of angular coordinates $\Phi = \{\theta_1, \ldots, \theta_{d-1}\}$. In these coordinates, the Laplace-Beltrami operator takes the form

$$\frac{1}{m^2}\Delta = \left(1+\frac{\rho^2}{m^2}\right)\partial_\rho^2 + \left(\frac{d-1}{\rho}+\frac{d\rho}{m^2}\right)\partial_\rho + \frac{1}{\rho^2}\Delta_{\mathbb{S}^{d-1}}, \tag{2.5}$$

where $\Delta_{\mathbb{S}^{d-1}}$ is the Laplace-Beltrami operator for the $(d-1)$-sphere.

This operator has a continuous spectrum of eigenvalues $\lambda = \Lambda(\Lambda+d-1)$, $\Lambda \in \mathbb{R}$. The spectrum for the sphere is discrete and has eigenvalues $-\ell_{d-1}(\ell_{d-1}+d-2)$, $\ell_{d-1} \in \mathbb{Z}$. If we assume now that the eigenfunctions, χ_ℓ^λ, can be factorized as a product of eigenfunctions in the sphere (spherical harmonics in $d-1$) and a radial function $f(\rho)$, the eigenfunction problem can be reduced to the following equation

$$(1+z^2)f'' + \left(\frac{d-1}{z}+dz\right)f' - \left(\frac{L(L+d-2)}{z^2}+\lambda\right)f = 0, \tag{2.6}$$

where $z = \rho/m$ and $L := \ell_{d-1}$. This equation turns out to be the differential equation for hypergeometrics functions [1, eq. 15.5.1]

$$z\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\alpha\right)\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\beta\right)F = z\frac{\mathrm{d}}{\mathrm{d}z}\left(z\frac{\mathrm{d}}{\mathrm{d}z}+\gamma-1\right)F, \tag{2.7}$$

that can be obtained after the change of variables $f(z) = z^L F(-z^2)$, where

$$\alpha = \frac{1}{4}\left(+\sqrt{(d+1)^2+4(\lambda-d)}+d-1+2L\right) = \frac{1}{2}\left(L+\Lambda+d-1\right), \tag{2.8}$$

$$\beta = \frac{1}{4}\left(-\sqrt{(d+1)^2+4(\lambda-d)}+d-1+2L\right) = \frac{1}{2}\left(L-\Lambda\right), \tag{2.9}$$

$$\gamma = \frac{d}{2}+L. \tag{2.10}$$

The two independent solutions to equation (2.6) are

$$\left(\frac{\rho}{m}\right)^L {}_2F_1\left[\frac{L+\Lambda+d-1}{2},\frac{L-\Lambda}{2};\frac{d}{2}+L;-\frac{\rho^2}{m^2}\right], \tag{2.11}$$

$$\left(\frac{m}{\rho}\right)^{d+L-2} {}_2F_1\left[\frac{\Lambda-L+1}{2},\frac{2-d-L-\Lambda}{2};2-\frac{d}{2}-L;-\frac{\rho^2}{m^2}\right], \tag{2.12}$$

where ${}_2F_1$ is the hypergeometric function. For $d>2$, $L \geq 0$ and the second solution is singular at $\rho = 0$ and is therefore discarded in the general derivation (the particular case $d = 2$ will be studied later). The radial function can be written then as

$$f_L^\lambda(\rho) = \left(\frac{\rho}{m}\right)^L {}_2F_1\left[\frac{L+\Lambda+d-1}{2},\frac{L-\Lambda}{2};\frac{d}{2}+L;-\frac{\rho^2}{m^2}\right]. \tag{2.13}$$

The asymptotic behavior of this function depends on several conditions. If $\lambda < 0$ (equivalently, $\Lambda < 1$), $f_L^\lambda(\rho)$ vanishes for $\rho \to \infty$. For $\lambda > 0$, however, the tendency is

$$f_L^\lambda(\rho) = C\left(\frac{\rho}{m}\right)^\Lambda + \cdots, \tag{2.14}$$

representing a sub-linear behavior for $\Lambda < 1$ ($\lambda < d$) and super-linear for $\Lambda > 1$ ($\lambda > d$). In this Thesis, we consider the special case $\Lambda = 1$ ($\lambda = d$), as it corresponds to the realization of the usual (gravitational) BMS algebra, as we will see next. In this particular case, one recovers the ordinary momenta as

$$\chi^{d}_{00\ldots 0} \propto k_0 = \sqrt{\rho^2 + m^2}, \tag{2.15}$$
$$\chi^{d}_{\ell_1 \ell_2 \ldots 1} \propto Y_{\ell_1 \ell_2 \ldots 1}(z)\rho \sim \text{linear combinations of } k_i, \quad i = 1, \ldots, d. \tag{2.16}$$

With regards to the spherical harmonics (in the unit sphere), one can write them in terms of the Gegenbauer polynomials, for $d > 2$, as [25, Theorem 5.1]

$$Y_\alpha(x) = \mathcal{C}_\alpha g_\alpha(\theta_1) \prod_{j=1}^{d-1} (sin\theta_{d-j})^{|\alpha^{j+1}|} C^{\lambda_j}_{\alpha_j}(\cos\theta_{d-j}), \tag{2.17}$$

where

$$g_\alpha(\theta_1) = \begin{cases} \cos\alpha_{d-1}\theta_1, & \text{for } \alpha_d = 0, \\ \sin\alpha_{d-1}\theta_1, & \text{for } \alpha_d = 1, \end{cases} \tag{2.18}$$

$\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{N}^d_0$ is a multi-index, $|\alpha^j| = \sum_{i=j}^{d-1} \alpha_i$, $\lambda_j = |\alpha^{j+1}| + (d-j-1)/2$ and $\mathcal{C}_\alpha$ is a normalization constant.

Once these eigenfunctions have been found, one can build the generalized charges $\mathcal{P}^\lambda_\ell$ in the Fourier space by reproducing the structure of the charges associated with ordinary momenta, P^μ. To do this, we consider the scalar particle theory with associated Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2, \tag{2.19}$$

whose momenta charges can be written as

$$P^\mu = \int \tilde{dk}\, \bar{a}(\vec{k}) k^\mu a(\vec{k}), \tag{2.20}$$

and generalize them by substituting the momenta with the new general eigenfunctions

$$\mathcal{P}^\lambda_\ell = \int \tilde{dk}\; \bar{a}(\vec{k})\, \chi^\lambda_\ell\, a(\vec{k}) = \int \tilde{dk}\; \bar{a}(\vec{k})\, Y_{\ell_1\ell_2\ldots L} f^\lambda_L\, a(\vec{k}). \tag{2.21}$$

These new charges are commutative operators and interact with the Lorentz charges

$$M^{ij} = -i \int \tilde{dk}\, \bar{a}(\vec{k}) \left(k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) a(\vec{k}), \tag{2.22}$$
$$M^{0j} = tP^j - i \int \tilde{dk}\, \bar{a}(\vec{k}) k^0 \frac{\partial}{\partial k^j} a(\vec{k}), \tag{2.23}$$

to generate the following infinite-dimensional algebra

$$\begin{aligned} \{\mathcal{P}^\lambda_\ell, \mathcal{P}^{\lambda'}_{\ell'}\} &= 0, \\ \{\mathcal{P}^\lambda_\ell, M_{\mu\nu}\} &= \sum_{\alpha \in \mathbb{Z}^{d-1}} c^\alpha_{\mu\nu}(\lambda, \ell) \mathcal{P}^\lambda_{\ell+\alpha}, \\ \{M_{\mu\nu}, M^{\sigma\rho}\} &= 4\delta^{[\sigma}{}_{[\mu} M^{\rho]}{}_{\nu]}, \end{aligned} \tag{2.24}$$

with structure constants, $c^\alpha_{\mu\nu}(\lambda, \ell)$, to be determined [see 28, appendix B, for specific expressions for $d = 3, 4, 5$]. The relations above are obtained using the following bracket for the Fourier modes

$$\{a(\vec{k}), \bar{a}(\vec{q})\} = -i(2\pi)^d 2k^0 \delta^{(d)}(\vec{k} - \vec{q}). \tag{2.25}$$

This general realization has been referred to as the λ-extended BMS algebra and it contains the standard (gravitational) BMS algebra, which contains, in turn, the Poincaré algebra. Formally,

$$\lambda\mathfrak{bms}_{d+1} \supset \mathfrak{bms}_{d+1} \supset \mathfrak{iso}(1,d). \tag{2.26}$$

The extended algebra has the particularity that for $\Lambda \in \mathbb{Z}$ one obtains a closed subalgebra, but none exists for non-integer values of Λ.

Massless case

In the particular setting of the massless theory, the mass-shell hypersurface is a cone $k^2 = 0$, which is a singular manifold and has no metric. However, one can still define the Laplace-Beltrami operator as the limit for $m \to 0$ of the operator in the massive theory (2.5) as follows

$$\hat{\Delta} = \lim_{m\to 0} \Delta = \rho^2 \partial_\rho^2 + d\rho\partial_\rho, \tag{2.27}$$

where now the angular dependence disappears. The new eigenfunctions take therefore the following expression [28]

$$\hat{\chi}_\ell^\lambda = \frac{(d+2\Lambda-3)!(d+2L-2)!!}{(d+2\Lambda-4)!!(d+L+\Lambda-2)!}\rho^\Lambda Y_\ell(\theta). \tag{2.28}$$

In the next sections, we will consider the realization of the gravitational BMS algebra, i.e. the case $\Lambda = 1$ (equivalently, $\lambda = d$), and its extension with superrotations in the cases $d = 2, 3$. Then, in Section 2.5, we will derive an expression of the generalized charges in position space and explore its expression, in Section 2.6, in terms of Green functions of appropriate operators. Using these, the existence of the charges is discussed. Section 2.7 computes the Poisson brackets of the obtained charges, while Section 2.8 considers the BMS transformations in configuration space. Finally, we briefly comment on the nonlocal expression of superrotations for obtaining the extended BMS algebra. Detailed calculations of all the results have been moved to the appendixes.

2.2 Canonical realization of Poincaré

Before we start working with the BMS algebra realization, let us introduce the canonical realization of the Poincaré algebra for the simple case of a free scalar particle defined by the Klein-Gordon Lagrangian in (2.19). The field describing the dynamics of the scalar particle is obtained as the solution of the Klein-Gordon equation and has the following expression, in terms of Fourier modes,

$$\phi(t,\vec{x}) = \int \tilde{dk}\,\left(a(\vec{k})e^{ikx} + \bar{a}(\vec{k})e^{-ikx}\right), \tag{2.29}$$

where the phase space Fourier modes satisfy the Poisson bracket

$$\{a(\vec{k}), \bar{a}(\vec{q})\} = -i(2\pi)^d 2\omega(\vec{k})\delta^{(d)}(\vec{k}-\vec{q}), \tag{2.30}$$

and the Lorentz invariant integration measure is

$$\tilde{dk} = \frac{\mathrm{d}^d k}{(2\pi)^d 2\omega(\vec{k})}, \quad \omega(\vec{k}) = k^0(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \tag{2.31}$$

Using now Noether's theorem, one can write the conserved charges in terms of the energy-momentum tensor and employ the field expression in Fourier space (2.29) to obtain the on-shell form

$$P^\mu = \int \mathrm{d}^d x\, T^{\mu 0} = \int \tilde{dk}\, \bar{a}(\vec{k}) k^\mu a(\vec{k}). \tag{2.32}$$

Analogously, the expressions for the Lorentz charges on-shell are

$$M^{ij} = -i \int \tilde{dk}\, \bar{a}(\vec{k}) \left(k^i \frac{\partial}{\partial k^j} - k^j \frac{\partial}{\partial k^i} \right) a(\vec{k}), \tag{2.33}$$

for rotations, and

$$M^{0j} = t P^j - i \int \tilde{dk}\, \bar{a}(\vec{k}) k^0 \frac{\partial}{\partial k^j} a(\vec{k}), \tag{2.34}$$

for boosts. The Poincaré algebra then has the following form

$$\begin{aligned} &\{P^\mu, P^\nu\} = 0, \quad \{P^\rho, M_{\mu\nu}\} = 2\delta^\rho{}_{[\mu} P_{\nu]}, \\ &\{M_{\mu\nu}, M^{\sigma\rho}\} = 4\delta^{[\sigma}{}_{[\mu} M^{\rho]}{}_{\nu]}. \end{aligned} \tag{2.35}$$

In the following sections, we work with a combination of the Lorentz generators that is convenient to obtain the usual BMS algebra. In the $2+1$-dimensional case, these are given by

$$L_0 = \frac{1}{2i} M^{12}, \quad L_1 = -M^{01} - iM^{02}, \quad L_{-1} = M^{01} - iM^{02} = -(L_1)^*. \tag{2.36}$$

2.3 Canonical realization of the BMS algebra in $d = 2, 3$

We now delve explicitly into two particular cases of the infinite-dimensional realization of the BMS algebra: the cases with spatial dimension $d = 3$ and $d = 2$. The first case represents the Minkowski space in the physical dimensions and is therefore of special relevance, while the case $d = 2$ will serve as the use case along this Thesis where expressions are still simple enough to extract some intuition.

In the $(3+1)$-dimensional case, the supertranslations can be written as

$$\begin{gathered} \chi^3_\ell = Y_{\ell_1 \ell_2}(\theta, \varphi) \left(\frac{\rho^2}{m^2} \right)^{\ell_2} {}_2F_1 \left[\frac{\ell_2 + 3}{2}, \frac{\ell_2 - 1}{2}; \ell_2 + \frac{3}{2}, -\frac{\rho^2}{m^2} \right], \\ \ell_1 \in \mathbb{Z}, \quad \ell_2 \in \mathbb{N}, \quad |\ell_1| \leq \ell_2, \end{gathered} \tag{2.37}$$

where $Y_{\ell_1 \ell_2}$ are the spherical harmonics.

In the $(2+1)$-dimensional case, the eigenfunctions depend only on two coordinates, one radial and one angular. Therefore, the value $L = \ell_1 \in \mathbb{Z}$ may be negative, and we must consider the two independent and regular solutions at the origin. Using the coordinates

$$k^0 = mz, \tag{2.38}$$

$$k^1 = m\sqrt{z^2 - 1} \cos \varphi, \tag{2.39}$$

$$k^2 = m\sqrt{z^2 - 1} \sin \varphi, \tag{2.40}$$

one gets

$$\chi^2_\ell(z,\varphi) = \begin{cases} e^{i\ell\varphi}\left(\frac{z-1}{z+1}\right)^{\frac{\ell}{2}}(\ell+z), & \text{for } \ell \geq 0, \\ e^{i\ell\varphi}\left(\frac{z+1}{z-1}\right)^{\frac{\ell}{2}}(\ell-z), & \text{for } \ell < 0. \end{cases} \tag{2.41}$$

The massless expressions for these generalized charges are far simpler since they preserve the angular part while the radial part depends simply on the radial coordinate, as noted in (2.28). Most of our calculations will be conducted for the massless case for its simplicity when compared to the massive one. In particular, we will use the eigenfunction in 2+1 in polar coordinates

$$\omega_\ell := \hat{\chi}^2_\ell = re^{i\ell\varphi}, \quad \ell \in \mathbb{Z}. \tag{2.42}$$

This expression can be obtained as the limit $m \mapsto 0$ of the massive one above when written in the appropriate coordinates. This is done in general for any dimension in Delmastro [28].

2.4 Canonical realization of the extended BMS algebra

To realize the extended BMS algebra proposed in [7, 8] we follow a similar procedure to generalize the Lorentz charges (rotations and boosts). First, we find the equation satisfied by these operators and then, we look for all the eigenfunctions. Thus, we note that if one writes the generators in the form $\xi^\alpha\partial_\alpha$, the following equations are satisfied

$$D_d\xi^\rho = 0, \quad D_0\xi^{z_i} = 0, \quad \nabla_\alpha\xi^\alpha = 0, \tag{2.43}$$

where $D_\lambda = -m^2\Delta + \lambda$, ∇ is the covariant derivative. See also [12] for an attempt to find a general set of equations that are satisfied by the Lorentz operators and whose eigenfunctions result in their extension to superrotations.

This approach results in an extended BMS algebra realization in 2+1. However, in the 3+1 case, we obtain two Virasoro algebras whose operators commute with each other. The explicit operator in 2+1 spacetime for the massless case is

$$\chi^{(R)}_n = e^{in\varphi}\left(-\partial_\varphi + in\rho\partial_\rho\right). \tag{2.44}$$

The resulting superrotations operators

$$\mathcal{R}_n = \int \tilde{dk}\, \bar{a}(\vec{k})\chi^{(R)}_n a(\vec{k}) \tag{2.45}$$

obey the Witt algebra

$$\{\mathcal{R}_n, \mathcal{R}_m\} = (n-m)\mathcal{R}_{m+n}, \tag{2.46}$$

and together with the supertranslations charges form the extended BMS algebra. The details of the construction of the superrotation generators described here can be found in [12].

2.5 BMS in position space

The operators $\mathcal{P}_\ell$ obtained with the canonical realization approach, together with the Lorentz transformations, form a representation of BMS_3 in configuration space.

However, an expression in position space is still missing despite being the natural way of obtaining field variations.

One can construct the expressions in position space by inverting the Fourier modes in terms of the field ϕ and its canonical momentum π as follows

$$a(\vec{k}) = \int \mathrm{d}^2 x e^{-ikx} \left(\omega\phi(t,\vec{x}) + i\pi(t,\vec{x})\right), \tag{2.47}$$

for $\omega = k^0 = \sqrt{m^2 + \vec{k}^2}$ and with $\bar{a}(\vec{k})$ given by the complex conjugate, from which functional variations can be computed.

For a massless field, the above procedure yields a supertranslation transformation given by

$$\delta_\ell \phi(t,\vec{x}) = \int \mathrm{d}^2 y \, [f_\ell(\vec{x}-\vec{y})\phi(t,\vec{y}) + g_\ell(\vec{x}-\vec{y})\pi(t,\vec{y})], \tag{2.48}$$

$$\delta_\ell \pi(t,\vec{x}) = \int \mathrm{d}^2 y \, [h_\ell(\vec{x}-\vec{y})\phi(t,\vec{y}) + f_\ell(\vec{x}-\vec{y})\pi(t,\vec{y})], \tag{2.49}$$

where integration is all over the two-dimensional space. The functions appearing in the above expressions are given by

$$f_\ell(\vec{x}) = \quad 2\int \widetilde{dk}\, \omega\, \omega_\ell(\vec{k}) \sin(\vec{k}\cdot\vec{x}), \tag{2.50}$$

$$g_\ell(\vec{x}) = \quad 2\int \widetilde{dk}\, \omega_\ell(\vec{k}) \cos(\vec{k}\cdot\vec{x}), \tag{2.51}$$

$$h_\ell(\vec{x}) = -2\int \widetilde{dk}\, \omega^2 \omega_\ell(\vec{k}) \cos(\vec{k}\cdot\vec{x}), \tag{2.52}$$

with ω_ℓ given by (2.42), which takes the following form in terms of the momentum coordinates

$$\omega_\ell = \omega^{1-\ell}(k_1 + ik_2)^\ell, \tag{2.53}$$

and the measure in momenta space as in (2.31). Given the property that $\omega_{-\ell} = \omega_\ell^*$, one can work simply with terms with index $\ell \geq 0$ and take the complex conjugate when the negative indexes are needed. Furthermore, using the parity properties of the eigenfunction, $\omega_\ell(-\vec{k}) = (-1)^\ell \omega_\ell(\vec{k})$, one can derive the following corresponding properties for the integral functions

$$f_{2\ell}(\vec{x}) = 0, \quad g_{2\ell+1}(\vec{x}) = 0, \quad h_{2\ell+1}(\vec{x}) = 0, \quad \ell \in \mathbb{Z}. \tag{2.54}$$

Notice that, in general, the transformations (2.48),(2.49) are nonlocal unless the functions f_ℓ, g_ℓ, h_ℓ are proportional to a delta function or a finite number of its derivatives. As we will see, this happens only for $\ell = 0, \pm 1$, which corresponds to ordinary space-time translations.

Using standard equal-time Poisson brackets and the properties $f_\ell(-\vec{x}) = -f_\ell(\vec{x})$, $g_\ell(-\vec{x}) = g_\ell(\vec{x})$, $h_\ell(\vec{x}) = \nabla^2 g_\ell(\vec{x})$, it can be seen that the field transformations (2.48), (2.49) are generated by the supertranslation charges

$$Q_\ell(t) = \int \mathrm{d}^2x \mathrm{d}^2 y \Big(f_\ell(\vec{x}-\vec{y})\pi(t,\vec{x})\phi(t,\vec{y}) + \frac{1}{2} g_\ell(\vec{x}-\vec{y})\pi(t,\vec{x})\pi(t,\vec{y}) - \frac{1}{2} h_\ell(\vec{x}-\vec{y})\phi(t,\vec{x})\phi(t,\vec{y}) \Big). \tag{2.55}$$

Once the asymptotic behavior of $\phi(t,\vec{x}), \pi(t,\vec{x})$ at spatial infinity is given, see Appendix A.1, and using standard equal-time Poisson brackets, it can be shown (see

Appendix A.2) that these charges have zero Poisson bracket with the Hamiltonian of the massless scalar field

$$H(t) = \int \mathrm{d}^2x \left(\frac{1}{2}\pi^2(t,\vec{x}) + \frac{1}{2}(\vec{\nabla}\phi(t,\vec{x}))^2 \right), \tag{2.56}$$

and are thus conserved. The proof relies solely on the symmetry properties of the functions f_ℓ and g_ℓ and on the relation between g_ℓ and h_ℓ. We will see next that the functions (2.50), (2.51), and (2.52) can be cast in terms of higher-level objects –which turn out to be Green functions– and use their properties to discuss some aspects of the transformations. Moreover, we will show that the algebra of the transformations in terms only of $\phi(t,\vec{x})$ closes only up to an antisymmetric combination of the equations of motion.

2.6 Nonlocal transformation of the fields in terms of polyharmonic functions

Using the explicit expression for ω_ℓ, one can write g_ℓ as

$$g_\ell(\vec{x}) = \frac{1}{(2\pi)^2} \int d^2k\, \omega^{-\ell} (k^1 + ik^2)^\ell \cos(\vec{k}\cdot\vec{x}). \tag{2.57}$$

For ℓ odd, the integrand is antisymmetric in $\vec{k}$ and the integral cancels out, as observed in the previous section. For ℓ even and non-negative, one can write

$$\begin{aligned} g_{2\ell}(\vec{x}) &= \frac{1}{(2\pi)^2} \int d^2k\, \omega^{-2\ell} (k^1 + ik^2)^{2\ell} \cos(\vec{k}\cdot\vec{x}) \\ &= (\partial_{x_1} + i\partial_{x_2})^{2\ell} (-1)^\ell \frac{1}{(2\pi)^2} \int d^2k\, \omega^{-2\ell} \cos(\vec{k}\cdot\vec{x}) \\ &= (\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}), \end{aligned} \tag{2.58}$$

where we have defined a distribution function $G_\ell(\vec{x})$ such that

$$G_\ell(\vec{x}) = (-1)^\ell \frac{1}{(2\pi)^2} \int d^2k\, \omega^{-2\ell} \cos(\vec{k}\cdot\vec{x}). \tag{2.59}$$

It turns out that this function is the polyharmonic Green function since it satisfies the polyharmonic equation

$$(\nabla^2_{\vec{x}})^\ell G_\ell(\vec{x}) = \frac{1}{(2\pi)^2} \int d^2k\, \cos(\vec{k}\cdot\vec{x}) = \delta(\vec{x}), \quad \ell \geq 0. \tag{2.60}$$

For $\ell = 0$, one gets $G_0(\vec{x}) = \delta(\vec{x})$. For $\ell \geq 1$, the Green function can be explicitly obtained as [21, 24]

$$G_\ell(\vec{x},\vec{y}) = G_\ell(\vec{x}-\vec{y}) = \frac{|\vec{x}-\vec{y}|^{2(\ell-1)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi} (\log|\vec{x}-\vec{y}| - H_{\ell-1}), \tag{2.61}$$

where $H_\ell = \sum_{i=1}^{\ell} \frac{1}{i}$ and $H_0 = 0$. These functions satisfy the recurrence relation, easily derived from (2.60),

$$\nabla^2 G_\ell(\vec{x}-\vec{y}) = G_{\ell-1}(\vec{x}-\vec{y}), \quad \ell \geq 1. \tag{2.62}$$

Another important property of these functions is the convolution property, derived in Appendix A.5,

$$\int d^2x G_\ell(\vec{y}-\vec{x})G_m(\vec{z}-\vec{x}) = G_{\ell+m}(\vec{y}-\vec{z}). \tag{2.63}$$

Then, the expressions for f_ℓ and h_ℓ can be directly obtained as a function of g_ℓ by observing that $f_{2\ell+1} = -(\partial_{x_1} + i\partial_{x_2})g_{2\ell}$ and $h_{2\ell} = \nabla^2 g_{2\ell}$, for $\ell \in \mathbb{N}$, and they are zero otherwise. In terms of the polyharmonic Green function, the integral functions have the following clearer expressions

$$g_{2\ell}(\vec{x}) = (\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}), \tag{2.64}$$

$$f_{2\ell+1}(\vec{x}) = -(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x}), \tag{2.65}$$

$$h_{2\ell}(\vec{x}) = (\partial_{x_1} - i\partial_{x_2})(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x}), \tag{2.66}$$

for $\ell \geq 0$. For $\ell = 0$, one has $g_0(\vec{x}) = \delta(\vec{x})$, $f_1(\vec{x}) = -(\partial_{x_1} + i\partial_{x_2})\delta(\vec{x})$ and $h_0(\vec{x}) = \nabla^2\delta(\vec{x})$, which yield the standard space-time translations for the fields.

The supertranslation charges in (2.55) can now be written in terms of the G_ℓ Green functions for $\ell \geq 0$ as

$$\begin{aligned} Q_{2\ell}(t) = \int d^2x\, d^2y \left(\frac{1}{2}\pi(t,\vec{x})\pi(t,\vec{y}) \right. \\ \left. + \frac{1}{2}\vec{\nabla}\phi(t,\vec{x}) \cdot \vec{\nabla}\phi(t,\vec{y})\right) (\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}), \end{aligned} \tag{2.67}$$

$$Q_{2\ell+1}(t) = \int d^2x\, d^2y\, (\partial_{x_1} + i\partial_{x_2})\pi(t,\vec{x})\phi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}) \tag{2.68}$$

$$= \int d^2x\, d^2y\, \phi(t,\vec{x})\pi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x}-\vec{y}) \tag{2.69}$$

$$= -\int d^2x\, d^2y\, (\partial_{x_1} + i\partial_{x_2})\phi(t,\vec{x})\pi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}). \tag{2.70}$$

Using that $\omega_\ell^* = \omega_{-\ell}$, one can derive that the only difference for charges with index $\ell < 0$ is the appearance of the operator $\partial_{x_1} - i\partial_{x_2}$ instead of $\partial_{x_1} + i\partial_{x_2}$. Thus

$$Q_{-2\ell}(t) = Q_{2\ell}^*(t), \quad Q_{-(2\ell+1)}(t) = Q_{2\ell+1}^*(t). \tag{2.71}$$

With these new expressions, the nonlocal supertranslation field transformations are given by

$$\delta_{2\ell}\phi(t,\vec{x}) = \{\phi(t,\vec{x}), Q_{2\ell}(t)\} = \int d^2y\, \pi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}), \tag{2.72}$$

$$\begin{aligned} \delta_{2\ell+1}\phi(t,\vec{x}) = \{\phi(t,\vec{x}), Q_{2\ell+1}(t)\} &= \int d^2z\, \phi(t,\vec{z})(\partial_{z_1} + i\partial_{z_2})^{2\ell+1} G_\ell(\vec{z}-\vec{x}) \\ &= -\int d^2y\, \phi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x}-\vec{y}), \end{aligned} \tag{2.73}$$

where we have used the Poisson bracket $\{\phi(t,\vec{x}), \pi(t,\vec{y})\} = \delta(\vec{x}-\vec{y})$.

In particular, using that $G_0(\vec{x}-\vec{y}) = \delta(\vec{x}-\vec{y})$, one can verify that $Q_0(t) = H(t)$ is the generator of time translations and that

$$Q_{x_1}(t) = \frac{Q_1(t) + Q_{-1}(t)}{2} = -\int \mathrm{d}^2x\, \pi(t,\vec{x})\partial_{x_1}\phi(t,\vec{x}), \tag{2.74}$$

$$Q_{x_2}(t) = \frac{Q_1(t) - Q_{-1}(t)}{2i} = -\int \mathrm{d}^2x\, \pi(t,\vec{x})\partial_{x_2}\phi(t,\vec{x}), \tag{2.75}$$

generate the spatial translations.

Similar expressions can be obtained in terms of 3-dimensional polyharmonic Green functions in the (3+1)-dimensional case but with an extra sum of derivatives with respect to the third coordinate (see Appendix B). One would expect that a similar analysis can be conducted in this case.

Asymptotic behavior of the scalar field

To guarantee the finiteness of the nonlocal charges obtained in (2.67) and (2.68), as well as the existence of the symplectic form, one needs to define the constraints on the field's asymptotic behavior.

To do that, we first consider the kinetic term in the action, which eventually leads to a well-defined Poisson bracket,

$$\int \mathrm{d}^2x\, \dot{\phi}(t,\vec{x})\pi(t,\vec{x}). \tag{2.76}$$

If we assume asymptotic expansions of the form

$$\phi(t,\vec{x}) = \frac{\bar{\phi}_1}{|\vec{x}|} + \frac{\bar{\phi}_2}{|\vec{x}|^2} + \dots, \tag{2.77}$$

$$\pi(t,\vec{x}) = \frac{\bar{\pi}_2}{|\vec{x}|^2} + \frac{\bar{\pi}_3}{|\vec{x}|^3} + \dots, \tag{2.78}$$

where the $\bar{\phi}_1$, $\bar{\phi}_2$, $\bar{\pi}_1$, $\bar{\pi}_2$, ... are functions depending on time and the angular variable, then

$$\int \mathrm{d}^2x\, \dot{\phi}(t,\vec{x})\pi(t,\vec{x}) = \int \mathrm{d}\theta \int r\mathrm{d}r \left(\dot{\bar{\phi}}_1\bar{\pi}_2\frac{1}{r^3} + O(r^{-4}) \right), \tag{2.79}$$

which makes the term well-defined. It follows also from these conditions that the field configuration has then finite energy, and in fact, the conditions cannot be relaxed, for instance by assuming $\phi \sim \log r$ or $\pi \sim 1/r$, if one wants to have finite energy. Notice that this expression is the leading term for the $r \mapsto \infty$ limit and does not imply a singularity at $r = 0$ (the behavior at $r = 0$ is related to the field regularity and not to this asymptotic expansion). Notice that under these conditions no logarithmic divergence appears in (2.79), in contrast with the case in $3+1$ space-time discussed in [35].

On the other side, the leading order behavior of $G_l(\vec{x}-\vec{y})$ for large $r = |\vec{x}-\vec{y}|$ is

$$G_\ell(r) \sim r^{2(\ell-1)}\log r. \tag{2.80}$$

As shown in Appendix A.1, the derivatives of order 2ℓ which appear in (2.67) and (2.70) behave as

$$(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(r) \sim \frac{1}{r^2} \quad \forall \ell \geq 1. \tag{2.81}$$

Taking this into account and comparing the Hamiltonian (2.56) with the supertranslation charges in (2.67) and (2.70), it follows that the supertranslation charges exist for field configurations behaving as in (2.77), (2.78).

To complete the BMS algebra we need, besides the generators of supertranslations, those of the Lorentz symmetries, given by

$$M^{12}(t) = -\int \mathrm{d}^2x\, \pi(t,\vec{x})\,(x_1\partial_{x_2}\phi(t,\vec{x}) - x_2\partial_{x_1}\phi(t,\vec{x})) \tag{2.82}$$

$$M^{0i}(t) = -\int \mathrm{d}^2x\, \left(t\pi(t,\vec{x})\partial_{x_i}\phi(t,\vec{x}) + x_i\mathcal{H}(t,\vec{x})\right), \quad i = 1,2, \tag{2.83}$$

where

$$\mathcal{H}(t,\vec{x}) = \frac{1}{2}(\pi^2(t,\vec{x}) + (\vec{\nabla}\phi(t,\vec{x}))^2) \tag{2.84}$$

is the energy density of the scalar field.

2.7 The BMS algebra in phase space

We can see now that the nonlocal supertranslation charges above (2.67), (2.68), (2.71) provide a realization of the BMS algebra in $2+1$ in terms of the standard equal-time Poisson brackets.

The abstract BMS algebra in $2+1$ is given by

$$[L_n, P_m] = (n-m)P_{m+n}, \quad [P_m, P_{m'}] = 0, \tag{2.85}$$

with $n \in \{-1,0,1\}$ and $m, m' \in \mathbb{Z}$, and with the L_n satisfying $[L_n, L_{n'}] = (n - n')L_{n+n'}$ and yielding the $2+1$ Lorentz algebra. The L_n generators are defined as the following combinations of the Lorentz generators

$$L_0(t) = \frac{1}{2i}M^{12}(t), \tag{2.86}$$

$$L_1(t) = -M^{01}(t) - iM^{02}(t), \tag{2.87}$$

$$L_{-1}(t) = M^{01}(t) - iM^{02}(t) = -(L_1(t))^*. \tag{2.88}$$

The algebra can be extended to $n, n' \in \mathbb{Z}$ by introducing the superrotations L_n, $|n| > 1$ [7, 8].

The proof relies on the general symmetry properties of the Green functions G_ℓ and their derivatives, as well as on a key identity that is proved in Appendix A.4. The brackets between the supertranslation charges are discussed in Appendix A.2 and here we will discuss only those involving the Lorentz generators.

Let us consider first the Poisson bracket $\{L_0(t), Q_{2\ell}(t)\}$, for $\ell \geq 0$. With the notation $\phi(x) = \phi(t,\vec{x})$ and so on, and defining

$$\mathcal{H}(x,y) = \frac{1}{2}(\pi(x)\pi(y) + \vec{\nabla}\phi(x)\cdot\vec{\nabla}\phi(y)), \tag{2.89}$$

one has

$$\{L_0(t), Q_{2\ell}(t)\} = -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\, \{\pi(x)(x_1\partial_{x_2}\phi(x) - x_2\partial_{x_1}\phi(x)), \mathcal{H}(y,z)\}\, (\partial_{y_1} + i\partial_{y_2})^{2\ell} G_\ell(\vec{y} - \vec{z}). \tag{2.90}$$

This bracket is computed in Appendix A.3, and the result is (see (A.20))

$$\{L_0(t), Q_{2\ell}(t)\} = -2\ell Q_{2\ell}(t), \quad \ell \geq 0. \tag{2.91}$$

Using similar steps to those in Appendix A.3, one can also obtain

$$\{L_0(t), Q_{2\ell+1}(t)\} = -(2\ell+1)Q_{2\ell+1}(t). \quad \ell \geq 0. \tag{2.92}$$

The above computations are only valid for $\ell \geq 0$. For $\ell < 0$ one can take the complex conjugate of (2.91) and (2.92), and use (2.71) and also $L_0^*(t) = -L_0(t)$. In this way, one obtains, for $\ell \geq 0$,

$$\{L_0(t), Q_{-2\ell}(t)\} = 2\ell Q_{-2\ell}(t) = -(-2\ell)Q_{-2\ell}(t), \tag{2.93}$$

$$\begin{aligned}\{L_0(t), Q_{-(2\ell+1)}(t)\} &= (2\ell+1)Q_{-(2\ell+1)}(t) \\ &= -(-(2\ell+1))Q_{-(2\ell+1)}(t).\end{aligned} \tag{2.94}$$

Relations (2.91–2.94) give the complete set of BMS algebra relations involving the rotation generator L_0.

Let us proceed now with the brackets involving the boost generators. Consider first

$$\begin{aligned}&\{L_1(t), Q_{2\ell}(t)\} = \\ &= \int d^2x d^2y d^2z \Big\{ t\pi(x)(\partial_{x_1} + i\partial_{x_2})\phi(x) + (x_1 + ix_2)\mathcal{H}(x), \\ &\qquad \mathcal{H}(y,z)\Big\}(\partial_{y_1} + i\partial_{y_2})^{2\ell} G_\ell(\vec{y} - \vec{z}).\end{aligned}$$

As shown in Appendix A.3, this can be seen to be (see equation (A.22))

$$\{L_1(t), Q_{2\ell}(t)\} = (1 - 2\ell)Q_{2\ell+1}(t), \tag{2.95}$$

which is the correct action of L_1 on a supertranslation of order 2ℓ, $\ell > 0$, and can be extended to the trivial (Poincaré) case, $\ell = 0$.

Using similar computations, together with the property $G_\ell = \vec{\nabla}^2 G_{\ell+1}$ one can show that, for $\ell > 0$,

$$\begin{aligned}&\{L_1(t), Q_{2\ell+1}(t)\} = \\ &\quad -2\ell \int d^2x d^2y \mathcal{H}(x,y)(\partial_{x_1} + i\partial_{x_2})^{2\ell+2} G_{\ell+1}(\vec{x} - \vec{y}).\end{aligned} \tag{2.96}$$

Changing now $\ell \to \ell - 1$ one has

$$\begin{aligned}&\{L_1(t), Q_{2\ell-1}(t)\} = \\ &\quad -2(\ell-1) \int d^2x d^2y \mathcal{H}(x,y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x} - \vec{y})\end{aligned} \tag{2.97}$$

$$= \quad -2(\ell-1)Q_{2\ell}(t) = \big(1 - (2\ell-1)\big)Q_{2l}(t), \quad \ell > 1, \tag{2.98}$$

which is the correct relation, and which again can be extended to the Poincaré $\ell = 1$ case.

To get all the relations of the BMS algebra, an extra pair of brackets must be computed. The final results are

$$\begin{aligned}\{L_1(t), Q_{-2\ell}(t)\} &= (2\ell+1)Q_{-2\ell+1}(t) \\ &= (1-(-2\ell))Q_{-2\ell+1}(t), \end{aligned} \tag{2.99}$$

$$\begin{aligned}\{L_1(t), Q_{-(2\ell+1)}(t)\} &= (2\ell+2)Q_{-2\ell}(t) \\ &= (1-(-(2\ell+1)))Q_{-2\ell}(t). \end{aligned} \tag{2.100}$$

In these cases, the computations involve slightly different manipulations but always using (A.25).

The brackets involving L_{-1} can be computed from (2.95), (2.98), (2.99) and (2.100) by complex conjugation and using $L_1(t) = -L^*_{-1}(t)$, and one gets

$$\begin{aligned}\{L_{-1}(t), Q_{-2\ell}(t)\} &= \big(-1+2\ell\big)Q_{-(2\ell+1)}(t) \\ &= \big(-1-(-2\ell)\big)Q_{-(2\ell+1)}(t), \end{aligned} \tag{2.101}$$

$$\begin{aligned}\{L_{-1}(t), Q_{2\ell}(t)\} &= -\big(2\ell+1\big)Q_{2\ell-1}(t) \\ &= \big(-1-2\ell)\big)Q_{2\ell-1}(t), \end{aligned} \tag{2.102}$$

$$\begin{aligned}\{L_{-1}(t), Q_{2\ell+1}(t)\} &= -\big(2\ell+2\big)Q_{2\ell}(t) \\ &= \big(-1-(2\ell+1)\big)Q_{2\ell}(t), \end{aligned} \tag{2.103}$$

$$\begin{aligned}\{L_{-1}(t), Q_{-2\ell-1}(t)\} &= 2\ell Q_{-2(\ell+1)}(t) \\ &= \big(-1-(-2\ell-1)\big)Q_{-2(\ell+1)}(t). \end{aligned} \tag{2.104}$$

This completes the proof that the charges defined by (2.67), (2.68) and (2.71) provide, together with the Lorentz generators, a realization of the $2+1$ BMS algebra.

2.8 The BMS algebra in configuration space

In the previous section, we have shown that the BMS algebra is obtained in phase space using the expression of the supertranslation and Lorentz charges in terms of the fields ϕ, π. We will discuss now the transformations in configuration space, using ϕ and $\dot\phi$ as independent fields. The result is that one obtains a BMS algebra of transformations modulo trivial symmetry transformations, given by skew-symmetric combinations of the equations of motion of ϕ.

To show this, let us consider the specific case of the Lorentz transformation δ_1^B associated with L_1 and the transformation $\delta_{2\ell}$ given by the supertranslation charge $Q_{2\ell}$, $\ell \geq 0$. In configuration space we must substitute $\dot\phi$ by π, and the transformations are

$$\delta_1^B\phi(x) = t(\partial_{x_1} + i\partial_{x_2})\phi(x) + (x_1 + ix_2)\dot\phi(x), \tag{2.105}$$

$$\delta_{2\ell}\phi(x) = \int \mathrm{d}^2y\, \dot\phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}). \tag{2.106}$$

Since these are functional variations, the transformation of $\dot\phi$ is obtained by derivation, and one gets

$$\delta_1^B\dot\phi(x) = (\partial_{x_1} + i\partial_{x_2})\phi(x) + t(\partial_{x_1} + i\partial_{x_2})\dot\phi(x) + (x_1 + ix_2)\ddot\phi(x), \tag{2.107}$$

$$\delta_{2\ell}\dot\phi(x) = \int \mathrm{d}^2y\, \ddot\phi(y)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}). \tag{2.108}$$

Now we can compute the compositions of transformations

$$\begin{aligned}\delta_{2\ell}\delta_1^B\phi(x) &= t(\partial_{x_1}+i\partial_{x_2})\delta_{2\ell}\phi(x)+(x_1+ix_2)\delta_{2\ell}\dot{\phi}(x)\\ &= t\int d^2y\,\dot{\phi}(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}-\vec{y})\\ &\quad+(x_1+ix_2)\int d^2y\,\ddot{\phi}(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}),\end{aligned} \tag{2.109}$$

and

$$\begin{aligned}\delta_1^B\delta_{2\ell}\phi(x) &= \int d^2y\,\delta_1^B\dot{\phi}(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})\\ &= \int d^2y\,(\partial_{y_1}+i\partial_{y_2})\phi(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})\\ &\quad+t\int d^2y\,(\partial_{y_1}+i\partial_{y_2})\dot{\phi}(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})\\ &\quad+\int d^2y\,(y_1+iy_2)\ddot{\phi}(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}).\end{aligned} \tag{2.110}$$

The first term in (2.110) can be manipulated to see that

$$\begin{aligned}&\int d^2y\,(\partial_{y_1}+i\partial_{y_2})\phi(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}) =\\ &\qquad\int d^2y\,\phi(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}-\vec{y}),\end{aligned} \tag{2.111}$$

which is just $-\delta_{2\ell+1}\phi(x)$ and, assembling the remaining terms, the commutator of the two transformations turns out to be

$$\begin{aligned}&[\delta_1^B,\delta_{2\ell}]\phi(x) = -\delta_{2\ell+1}\phi(x) \\ &\quad-\int d^2y\,\big((x_1-y_1)+i(x_2-y_2)\big)\ddot{\phi}(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}).\end{aligned} \tag{2.112}$$

Now we add and subtract $\vec{\nabla}^2\phi$ and obtain

$$\begin{aligned}&[\delta_1^B,\delta_{2\ell}]\phi(x) = -\delta_{2\ell+1}\phi(x)\\ &-\int \mathrm{d}^2y\,\big((x_1-y_1)+i(x_2-y_2)\big)\vec{\nabla}_y^2\phi(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})\\ &-\int \mathrm{d}^2y\,\big((x_1-y_1)+i(x_2-y_2)\big)\big(\ddot{\phi}(y)-\vec{\nabla}_y^2\phi(y)\big)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})\\ &= -\delta_{2\ell+1}\phi(x)\\ &-\int \mathrm{d}^2y\,\big((x_1-y_1)+i(x_2-y_2)\phi(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell}(\partial_{x_1}-i\partial_{x_2})G_\ell(\vec{x}-\vec{y})\\ &-2\int \mathrm{d}^2y\,\phi(y)(\partial_{x_1}+i\partial_{x_2})^{2\ell+1}G_\ell(\vec{x}-\vec{y})-\int \mathrm{d}^2y\,F_\ell(x,y)\big(\ddot{\phi}(y)-\vec{\nabla}_y^2\phi(y)\big),\end{aligned} \tag{2.113}$$

where we have integrated twice by parts the term $\vec{\nabla}^2\phi$ and defined

$$F_\ell(x,y) = \big((x_1-y_1)+i(x_2-y_2)\big)(\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y}). \tag{2.114}$$

The third term in (2.113) is $2\delta_{2\ell+1}\phi(x)$, while the second term, following the same steps that led to (A.21), becomes $2(\ell-1)\delta_{2\ell+1}\phi(x)$. Putting everything together one has

$$[\delta_1^B, \delta_{2\ell}]\phi(x) = (2\ell - 1)\delta_{2\ell+1}\phi(x) \\ - \int \mathrm{d}^2y\, F_\ell(x,y)\big(\ddot{\phi}(y) - \vec{\nabla}_y^2\phi(y)\big). \tag{2.115}$$

Taking into account that, for any transformations generated by charges A, B, one has that $[\delta_A, \delta_B]\phi = -\delta_{\{A,B\}}\phi$, the first term in (2.115) is the one expected from the BMS algebra and, because $F_\ell(x,y) = -F_\ell(y,x)$, the extra term is a skew-symmetric linear combination of the equations of motion,

$$\delta_{\ell,\mathrm{trivial}}\phi(x) = \int \mathrm{d}^2y\, F_\ell(x,y)\big(\ddot{\phi}(y) - \vec{\nabla}_y^2\phi(y)\big), \tag{2.116}$$

which is a trivial symmetry transformation of any system. Notice that, for $\ell = 0$, $F_\ell(x,y) = 0$ and, as it must be, the extra term is not present for the standard commutator of a time translation and a Lorentz boost.

Similar results are obtained for the other commutators of transformations, and hence the algebra closes on-shell in a consistent way.

2.9 The extended BMS algebra

To get the extended BMS algebra one can construct the analogous nonlocal charge for superrotations

$$\mathcal{R}_n = \frac{1}{2}\int d^2y\, d^2x\, [\phi(t,\vec{x})\phi(t,\vec{y})(\tilde{H}_n + i\tilde{F}_n) - i\phi(t,\vec{x})\pi(t,\vec{y})(\tilde{J}_n - i\tilde{I}_n) \\ - i\pi(t,\vec{x})\phi(t,\vec{y})\,(H_n + iF_n) + \pi(t,\vec{x})\pi(t,\vec{y})\,(G_n - iI_n)], \tag{2.117}$$

which depends on the previous polyharmonic Green functions and other similar expressions given by

$$F_n(\vec{x},\vec{y}) = -i\int \tilde{dk}\, 2\omega_n(\vec{k})\Big[in\cos(\vec{k}(\vec{y}-\vec{x})) \\ - (in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\sin(\vec{k}(\vec{y}-\vec{x}))\Big], \tag{2.118}$$

$$J_n(\vec{x},\vec{y}) = i\int \tilde{dk}\, 2\frac{\omega_n(\vec{k})}{\omega}\Big[(in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\cos(\vec{k}(\vec{y}-\vec{x}))\Big], \tag{2.119}$$

$$\tilde{H}_n(\vec{x},\vec{y}) = i\int \tilde{dk}\, 2\omega\omega_n(\vec{k})\Big[in\sin(\vec{k}(\vec{y}-\vec{x})) \\ + (in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\cos(\vec{k}(\vec{y}-\vec{x}))\Big], \tag{2.120}$$

$$\tilde{I}_n(\vec{x},\vec{y}) = i\int \tilde{dk}\, 2\omega_n(\vec{k})\Big[(in\vec{y}\cdot\vec{k} + \vec{y}\times\vec{k})\sin(\vec{k}(\vec{y}-\vec{x}))\Big], \tag{2.121}$$

where $\tilde{F}_n$ and $\tilde{J}_n$ are just like the functions F_n and J_n in (2.118) and (2.119), respectively, but with an extra ω factor under the integral sign, while H_n and I_n are defined as the corresponding functions but with an additional $1/\omega$ factor. A closer look into these functions reveals a more compact expression in terms of the functions found for supertranslations in the previous sections, f_n (2.50) and g_n (2.51), as

$$F_n(\vec{x},\vec{y}) = ng_n(\vec{x}-\vec{y}) + \tilde{I}_n(\vec{x},\vec{y}), \tag{2.122}$$

$$\tilde{H}_n(\vec{x},\vec{y}) = nf_n(\vec{x}-\vec{y}) - \nabla_x^2 J_n(\vec{x},\vec{y}). \tag{2.123}$$

However, we have not been able to write the extra terms in the expressions above in terms of polyharmonic Green functions. One would expect that similar Green functions can also be found in this case with corresponding properties that guarantee the closure of an extended BMS algebra that generalizes (2.85) to any integer value.

2.10 Ansatz for supertranslations for space-time coordinates

Using the expressions in the position space obtained for the functions f_ℓ, g_ℓ and h_ℓ in the previous section, we can now write the field transformation as

$$\delta_{ST,2\ell}\phi = \int d^2y\, \pi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}), \quad \ell \geq 0, \tag{2.124}$$

$$\delta_{ST,-2\ell}\phi = \int d^2y\, \pi(t,\vec{y})(\partial_{x_1} - i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}), \quad \ell > 0, \tag{2.125}$$

$$\delta_{ST,2\ell+1}\phi = -\int d^2y\, (\partial_{y_1} + i\partial_{y_2})\phi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}), \quad \ell \geq 0, \tag{2.126}$$

$$\delta_{ST,-2\ell+1}\phi = -\int d^2y\, (\partial_{y_1} - i\partial_{y_2})\phi(t,\vec{y})(\partial_{x_1} - i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{y}), \quad \ell > 0. \tag{2.127}$$

One can verify that ordinary spacetime translations are recovered for values of ℓ equal to 0 and ± 1. In detail,

$$\delta_{ST,0}\phi = \int d^2y\, \pi(t,\vec{y})\delta^{(2)}(\vec{x}-\vec{y}) = \pi(t,\vec{x}), \tag{2.128}$$

$$\delta_{ST,1}\phi = -\int d^2y\, (\partial_{y_1} + i\partial_{y_2})\phi(t,\vec{y})\delta^{(2)}(\vec{x}-\vec{y}) = -(\partial_{x_1} + i\partial_{x_2})\phi(t,\vec{x}), \tag{2.129}$$

$$\delta_{ST,-1}\phi = -\int d^2y\, (\partial_{y_1} - i\partial_{y_2})\phi(t,\vec{y})\delta^{(2)}(\vec{x}-\vec{y}) = -(\partial_{x_1} - i\partial_{x_2})\phi(t,\vec{x}), \tag{2.130}$$

where the last two transformations for $\ell = \pm 1$ need to be combined to obtain the corresponding translation transformations as follows

$$-\frac{1}{2}(\delta_{ST,1}\phi + \delta_{ST,-1}\phi) = \partial_{x_1}\phi(t,\vec{x}), \tag{2.131}$$

$$\frac{i}{2}(\delta_{ST,1}\phi - \delta_{ST,-1}\phi) = \partial_{x_2}\phi(t,\vec{x}). \tag{2.132}$$

The ordinary infinitesimal translations recovered above have a corresponding transformation in position space for spacetime coordinates so that

$$x^\mu \mapsto x^\mu + a^\mu, \tag{2.133}$$

with a^μ being non-zero for the coordinate being transformed. For instance, for time translations $a^\mu = (\epsilon, 0, 0)$ and the associated transformation for the field ϕ is

$$\delta\phi = \frac{\delta x^\mu}{\delta\epsilon}\partial_\mu\phi = \partial_t\phi. \tag{2.134}$$

A natural question arises then for transformations with $|\ell| \geq 2$ and it is whether one can find equivalent values for a^μ for general supertranslations. The first difference for these higher index supertranslations is that they are non-local, meaning that one can no longer find an expression as in (2.134). Instead, the new variation considers the value of the field in all points in space. For illustration, one can look at the first supertranslation that generalizes the ordinary case, namely $\ell = \pm 2$. For

$\ell = 2$, one obtains

$$\delta_{ST,2}\phi = \int d^2y\, \pi(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^2 G_1(\vec{x}-\vec{y}), \tag{2.135}$$

$$\delta_{ST,-2}\phi = \int d^2y\, \pi(t,\vec{y})(\partial_{x_1} - i\partial_{x_2})^2 G_1(\vec{x}-\vec{y}), \tag{2.136}$$

where this time π is being integrated all over the space.

Since $\pi(t,\vec{y}) = \partial_t\phi(t,\vec{y})$, one may guess that the above expressions contain the information about the possible transformation of x^0 under $\delta_{ST,\pm 2}$, and the following *ansatz*

$$\delta_{ST,2}x^0 = \int d^2y\, (\partial_{x_1} + i\partial_{x_2})^2 G_1(\vec{x}-\vec{y}), \tag{2.137}$$

$$\delta_{ST,-2}x^0 = \int d^2y\, (\partial_{x_1} - i\partial_{x_2})^2 G_1(\vec{x}-\vec{y}), \tag{2.138}$$

which is constructed by simply taking the part of the expression in the integrals different from the $\pi = \partial_t\phi$ and considering it as a kernel function. In this way, one ends up with a non-local transformation of x^0, which is not just a translation in x^0, since it depends on the spatial components of $x^\mu = (x^0, \vec{x})$. Notice, furthermore, that since no spatial derivatives of the field appear inside the integrals for $\delta_{ST,\pm 2}\phi$, one is forced to define

$$\delta_{ST,\pm 2}\vec{x} = 0, \tag{2.139}$$

and this leads to an algebra of transformations inconsistent with BMS. Notice that odd-order supertranslation transformations for $\vec{x}$ could be defined in the same way as the odd-order supertranslation transformations of the field since they contain space derivatives of the field inside the integral.

Indeed, from $[\mathcal{R}_m, \mathcal{P}_\ell] = i(m-\ell)\mathcal{P}_{\ell+m}$, one has, taking $m = -1$ and $\ell = 2$, $[\mathcal{R}_{-1}, \mathcal{P}_2] = -3i\mathcal{P}_1$, from which one should have

$$[\delta_{-1}, \delta_{ST,2}] \sim \delta_{T,1}, \tag{2.140}$$

where $\delta{-1}$ is an ordinary boost and $\delta_{T,1}$ an ordinary space translation. Applying this to x^0 one has

$$\delta_{-1}\delta_{ST,2}x^0 = \delta_{-1}\int d^2y\, (\partial_{x_1} + i\partial_{x_2})^2 G_1(\vec{x}-\vec{y}), \tag{2.141}$$

which applies a boost $\delta_{-1}x_{1,2} \sim x^0$ to the spatial coordinates $\vec{x}$ of $G_1(\vec{x}-\vec{y})$ and yields a non-zero result. On the other hand, $\delta_{-1}x^0 \sim x^1$ and hence

$$\delta_{ST,2}\delta_{-1}x^0 \sim \delta_{ST,2}x^1 = 0, \tag{2.142}$$

due to (2.139). One has then the non-zero result

$$[\delta_{-1}, \delta_{ST,2}]x^0 = \delta_{-1}\int d^2y\, (\partial_{x_1} + i\partial_{x_2})^2 G_1(\vec{x}-\vec{y}), \tag{2.143}$$

while the right-hand side of (2.140) yields $\delta_{T,1}x^0 = 0$, since this is a spatial translation in ordinary space.

Hence, the proposed ansatz does not define a BMS algebra for space-time coordinates, and the interpretation of the non-local transformations of the fields in terms of transformations of the Poincaré space-time variables fails in this framework.

Chapter 3

Non-linear realization of the BMS particle

3.1 Introduction

With the idea of further exploring the BMS symmetry, in this chapter we propose a massive particle realization in $2+1$ dimensions, based on non-linear realizations of symmetry algebras (see [18, 32] and references therein). The particle Lagrangian is constructed in an infinite-dimensional space, generalizing the Minkowski space, that we call BMS space. The infinite number of coordinates are associated with the supertranslations, and we also use two Goldstone coordinates associated with the two broken boost generators.

We want to construct a Lagrangian in terms of the pull-back of the Maurer-Cartan form subject to two conditions: it should have the lowest possible number of derivatives and it should be invariant under the unbroken $SO(2)$ subgroup of the Lorentz group in $2+1$, which is the symmetry that is preserved by the presence of a massive spinless point particle. One can therefore choose any component of the Maurer-Cartan form that is invariant under rotations and we take the component Ω_{P_0} along the generator of time translations P_0. In $2+1$ dimensions one can also consider the term proportional to the generator of rotations, which can be used to construct models of particles with spin (see, for example, [17, 31, 38, 43]), but in this thesis we are only interested in the spinless case. The Lagrangian is then the pull-back of Ω_{P_0} to the world-line of the particle, up to a multiplicative constant that makes the role of the mass. If one does not consider the super-translation degrees of freedom, this method produces the Lagrangian of a standard spinless massive particle in canonical form. More details of the method can be found, for instance, in Section 5.1 of [18].

The Hamiltonian formalism for the resulting Lagrangian is constructed, and the infinite phase-space first-class constraints and set of gauge transformations are analyzed. We also compute the massless limit of the model. We obtain the physical reduced space after eliminating the gauge degrees of freedom by introducing an infinite set of gauge fixing constraints. This space is left only with the degrees of freedom of a standard Poincaré particle. Since in the gauge fixing procedure the rigid symmetries are maintained, we prove that the ordinary relativistic massive Poincaré particle is invariant under a realization of the BMS symmetry that we construct using the compensating gauge transformations, which are necessary to preserve the gauge fixing under supertranslations. In the massless case, we further find an infinite set of superrotations that are symmetries of the massless relativistic particle. It turns out that the infinite set of BMS coordinates associated with the supertranslations are not physical because they can always be gauged away, and therefore the model does not have the so-called soft BMS modes. These results agree with the fact that the quadratic Casimir of BMS algebra coincides with the quadratic Casimir of the Poincaré algebra.

A different approach to the definition of BMS particles in $2+1$ dimensions is based on the coadjoint orbit approach [22, 23, 40, 41]. However, as far as we know, no particle action has been constructed in the literature using this approach. For the relation among the non-linear realization framework and the coadjoint orbit method, see for example [18].

This chapter is organized as follows. In section 3.2, we derive the BMS particle Lagrangian in $2+1$ space-time using the non-linear realization approach. The canonical analysis of this Lagrangian is given in Section 3.3, including the discussion of the constraints, the reduced phase space and the gauge transformations induced by the first-class constraints. Section 3.4 presents the generators that realize the Poincaré symmetry in BMS coordinates, and shows that the theory is indeed invariant under them. The massless limit of the theory is computed in Section 3.5, and it is seen that the set of symmetry generators is extended to include superrotations. The gauge fixing of the theory is presented in Section 3.6, and the physical degrees of freedom of the BMS particle are determined.

The chapter ends with a proposal to connect the first-class constraints of the particle model with the non-local transformations of the scalar field presented in Chapter 2. Detailed proofs of some of the results in this chapter are presented in Appendix C.

3.2 Non-linear realization of the BMS algebra in $2+1$ dimensions

The extended BMS algebra in $2+1$ dimensions [8] without central extensions is given by

$$[L_n, P_m] = i(n-m)P_{n+m}, \tag{3.1}$$
$$[P_n, P_m] = 0, \tag{3.2}$$
$$[L_n, L_m] = i(n-m)L_{n+m}, \tag{3.3}$$

with $m, n \in \mathbb{Z}$. We are interested in the subalgebra BMS_3 formed by the Lorentz generators L_0, $L_{\pm 1}$ and the supertranslations P_n:

$$[L_1, L_{-1}] = 2iL_0, \quad [L_1, L_0] = iL_1, \quad [L_{-1}, L_0] = -iL_{-1}, \tag{3.4}$$
$$[L_{-1}, P_m] = -i(m+1)P_{m-1}, \quad [L_0, P_m] = -imP_m, \quad [L_1, P_m] = -i(m-1)P_{m+1}, \tag{3.5}$$
$$[P_n, P_m] = 0. \tag{3.6}$$

Notice that, besides having this subalgebra, the set of relations (3.1) provide, taking $n, m = -1, 0, 1$, the vector representation of the Lorentz algebra in terms of the P_{-1}, P_0, P_1.

In order to construct a massive BMS particle we should consider the coset $BMS_3/SO(2)$, locally given by

$$g(\{x\}, u, v) = g_0(\{x\})e^{iL_{-1}v}e^{iL_1 u} = g_0(\{x\})U(u, v), \tag{3.7}$$

with

$$g_0(\{x\}) = \prod_{n\in\mathbb{Z}} e^{iP_n x^n}. \tag{3.8}$$

Here $U(u,v)$ is a boost transformation generated by L_1, L_{-1} and parametrized by the Goldstone coordinates u,v and $\{x^n\}_{n\in\mathbb{Z}}$ are the BMS coordinates (with x^0, $x^{\pm1}$ related to ordinary $2+1$ space-time). The BMS coordinates are complex, with x^n and x^{-n} complex conjugate of each other.

The Maurer-Cartan form associated with g is

$$\Omega(g) = -ig^{-1}\mathrm{d}g = -iU^{-1}g_0^{-1}\mathrm{d}g_0\,U - iU^{-1}\mathrm{d}U \tag{3.9}$$

$$= \sum_{n\in\mathbb{Z}} \mathrm{d}x^n\,U^{-1}P_nU - iU^{-1}\mathrm{d}U. \tag{3.10}$$

In the spirit of obtaining spinless particle actions, see for example [18, 32], we are only interested in the terms Ω_{P_0} of Ω proportional to P_0, which can only come from $U^{-1}P_nU$. The detailed computation is presented in Appendix B, and the result is

$$\Omega_{P_0} = \mathrm{d}x^0(1-2uv) + \sum_{n=1}^{\infty}\mathrm{d}x^n(-1)^n v^n(n+1-2uv) + \mathrm{d}x^{-1}2u + \sum_{n=2}^{\infty}\mathrm{d}x^{-n}u^n\frac{n+1-2uv}{(1-uv)^n}. \tag{3.11}$$

Following the standard procedure, we integrate the pullback of Ω_{P_0} to the world-line of the particle

$$S[\{x\},u,v] = -\mu\int \mathrm{d}\tau(\dot{x}^0(1-2uv) - 2\dot{x}^1 v(1-uv) + 2\dot{x}^{-1}u$$
$$+ \sum_{n=2}^{\infty}\dot{x}^n(-1)^n v^n(n+1-2uv) + \sum_{n=2}^{\infty}\dot{x}^{-n}u^n\frac{n+1-2uv}{(1-uv)^n})$$
$$= -\mu\int\mathrm{d}\tau\left(\sum_{n=0}^{\infty}\dot{x}^n(-1)^n v^n(n+1-2uv) + \sum_{n=1}^{\infty}\dot{x}^{-n}u^n\frac{n+1-2uv}{(1-uv)^n}\right), \tag{3.12}$$

and define the BMS particle Lagrangian

$$\mathcal{L} = -\mu\left(\sum_{n=0}^{\infty}\dot{x}^n(-1)^n v^n(n+1-2uv) + \sum_{n=1}^{\infty}\dot{x}^{-n}u^n\frac{n+1-2uv}{(1-uv)^n}\right). \tag{3.13}$$

The constant μ here will be the mass of the particle.

The contribution to (3.12) of the ordinary space-time coordinates, *i.e.* x^0, $x^{\pm1}$, is

$$S_0 = -\mu\int \mathrm{d}\tau\left(\dot{x}^0(1-2uv) + 2\dot{x}^1(-v+v^2u) + 2\dot{x}^{-1}u\right) \equiv \int \mathrm{d}\tau\mathcal{L}_0. \tag{3.14}$$

This action corresponds to an ordinary spinless relativistic particle in flat (2+1) Minkowski space-time, as can be seen by computing the momenta

$$p_0 = \frac{\partial\mathcal{L}_0}{\partial\dot{x}^0} = -\mu(1-2uv), \tag{3.15}$$

$$p_1 = \frac{\partial\mathcal{L}_0}{\partial\dot{x}^1} = -2\mu(-v+v^2u), \tag{3.16}$$

$$p_{-1} = \frac{\partial\mathcal{L}_0}{\partial\dot{x}^{-1}} = -2\mu u, \tag{3.17}$$

and checking the mass-shell condition

$$-p_0^2 + p_1p_{-1} = -\mu^2. \tag{3.18}$$

Actually, if one computes the EOM given by (3.14) for the boost variables,

$$\begin{aligned} -v\dot{x}^0 + v^2\dot{x}^1 + \dot{x}^{-1} &= 0, & (3.19)\\ -u\dot{x}^0 - \dot{x}^1 + 2uv\dot{x}^1 &= 0, & (3.20)\end{aligned}$$

solves them for u, v,

$$u = \frac{\dot{x}^1}{\pm\sqrt{(\dot{x}^0)^2 - 4\dot{x}^1\dot{x}^{-1}}}, \quad v = \frac{\dot{x}^0 \pm \sqrt{(\dot{x}^0)^2 - 4\dot{x}^1\dot{x}^{-1}}}{2\dot{x}^1}, \tag{3.21}$$

makes the change of space variables from the complex ones $x^{\pm 1}$ to the real ones x_1, x_2, given by

$$x^{\pm 1} = \frac{1}{2}(x_1 \pm i x_2), \tag{3.22}$$

and substitutes the resulting expressions for u, v in (3.14), one gets the ordinary space-time action with Lagrangian

$$\mathcal{L}_0^* = \mp\mu\sqrt{(\dot{x}^0)^2 - (\dot{x}_1)^2 - (\dot{x}_2)^2}. \tag{3.23}$$

3.3 Canonical analysis of the BMS particle action

In order to understand the structure of the BMS Lagrangian (3.13) we perform the Hamiltonian analysis. The momenta are given by

$$\begin{aligned} \pi_u &= \frac{\partial\mathcal{L}}{\partial\dot{u}} = 0, & (3.24)\\ \pi_v &= \frac{\partial\mathcal{L}}{\partial\dot{v}} = 0, & (3.25)\\ p_n &= \frac{\partial\mathcal{L}}{\partial\dot{x}^n} = -\mu(-1)^n v^n(n+1-2uv), \;\; n = 0,1,2,\dots, & (3.26)\\ \bar{p}_n &= \frac{\partial\mathcal{L}}{\partial\dot{x}^{-n}} = -\mu u^n\frac{n+1-2uv}{(1-uv)^n}, \;\; n = 1,2,\dots. & (3.27)\end{aligned}$$

Notice that, since the x^{-n} are complex conjugates of the x^n, then the fact that $\mathcal{L}$ is real implies that p_n and $\bar{p}_n$ are also complex conjugates of each other.

From the expressions of the momenta one gets the set of primary constraints $\pi_u = 0$, $\pi_v = 0$ and $\phi_n = 0$, $\bar{\phi}_n = 0$, with

$$\begin{aligned} \phi_n &:= p_n + \mu(-1)^n v^n(n+1-2uv), \;\; n = 0,1,2,\dots & (3.28)\\ \bar{\phi}_n &:= \bar{p}_n + \mu u^n\frac{n+1-2uv}{(1-uv)^n}, \;\; n = 1,2,\dots & (3.29)\end{aligned}$$

The non-zero Poisson brackets between these constraints are

$$\begin{aligned} \{\phi_n, \pi_u\} &= -2\mu(-1)^n v^{n+1}, \; n = 0,1,2,\dots & (3.30)\\ \{\phi_n, \pi_v\} &= \mu(-1)^n v^{n-1}(n(n+1) - 2(n+1)uv), \; n = 0,1,2,\dots & (3.31)\\ \{\bar{\phi}_n, \pi_u\} &= \frac{\mu u^{n-1}}{(1-uv)^{n+1}}(2u^2v^2 - 2nuv + n^2 - 2uv + n), \; n = 1,2,\dots & (3.32)\\ \{\bar{\phi}_n, \pi_v\} &= \frac{\mu u^{n+1}}{(1-uv)^{n+1}}(n-1)(-2uv + n + 2), \; n = 1,2,\dots & (3.33)\end{aligned}$$

Since the Lagrangian is homogeneous of degree one in the velocities, the canonical Hamiltonian is identically zero and one must consider the Dirac Hamiltonian

$$H_D = \lambda_u \pi_u + \lambda_v \pi_v + \sum_{n=0}^{\infty} \lambda_n \phi_n + \sum_{n=1}^{\infty} \bar{\lambda}_n \bar{\phi}_n, \tag{3.34}$$

where the λ are arbitrary functions.

If we order the constraints as $(\pi_u, \pi_v, \phi_1, \bar{\phi}_1, \phi_2, \bar{\phi}_2, \ldots)$, the infinite-dimensional matrix of Poisson brackets between them has the form

$$M_\infty = \begin{pmatrix} 0 & A_1 & A_2 & A_3 & \cdots \\ -A_1^T & 0 & 0 & 0 & \cdots \\ -A_2^T & 0 & 0 & 0 & \cdots \\ -A_3^T & 0 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{3.35}$$

where all the entries are 2×2 blocks and

$$A_i = \begin{pmatrix} \{\pi_u, \phi_i\} & \{\pi_u, \bar{\phi}_i\} \\ \{\pi_v, \phi_i\} & \{\pi_v, \bar{\phi}_i\} \end{pmatrix}, \quad i = 1, 2, 3, \ldots \tag{3.36}$$

Since, for instance,

$$A_1 = 2\mu \begin{pmatrix} -v^2 & -1 \\ 1 - 2uv & 0 \end{pmatrix}, \tag{3.37}$$

has non-zero determinant, provided that $1 - 2uv \neq 0$, it turns out that all the (infinite dimensional) columns to the right of A_1 can be expressed as linear combinations of the columns which contain A_1. Hence, considering also the first two columns of M_∞, one can show that M_∞ has a rank equal to 4. This means that, at most, we can select 4 second-class constraints, including necessarily π_u and π_v, plus another two which allow us to eliminate u and v in terms of two of the momenta p_i and $\bar{p}_i$. Also, notice that the number of first class constraints is infinite.

Notice that, although u, v can be eliminated from any two of the ϕ, $\bar{\phi}$, it is convenient to select ϕ_1, $\bar{\phi}_1$, as it was done for the case of the pure Poincaré particle. The four selected constraints

$$\pi_u, \quad \pi_v, \quad \phi_1, \quad \bar{\phi}_1 \tag{3.38}$$

are second class, with the Poisson bracket matrix

$$M = \begin{pmatrix} 0 & A_1 \\ -A_1^T & 0 \end{pmatrix} = 2\mu \begin{pmatrix} 0 & 0 & -v^2 & -1 \\ 0 & 0 & 1 - 2uv & 0 \\ v^2 & -1 + 2uv & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \tag{3.39}$$

which has determinant $16\mu^4(1 - 2uv)^2$ and inverse

$$M^{-1} = \frac{1}{2\mu} \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{1}{1-2uv} & \frac{v^2}{1-2uv} \\ 0 & \frac{1}{1-2uv} & 0 & 0 \\ -1 & -\frac{v^2}{1-2uv} & 0 & 0 \end{pmatrix}. \tag{3.40}$$

From this, one can define the Dirac bracket

$$\{A, B\}_D = \{A, B\} - \{A, \Psi_i\} M^{-1}_{ij} \{\Psi_j, B\}, \tag{3.41}$$

where $\Psi_j \in \{\pi_u, \pi_v, \phi_1, \bar{\phi}_1\}$.

If one demands the stability of the primary second-class constraints, *i.e.*

$$\dot{\Psi}_i = \{\Psi_i, H_D\} \stackrel{\mathcal{M}}{=} 0, \tag{3.42}$$

where $\mathcal{M}$ is the submanifold defined by the constraints Ψ_i, one can determine the values of the four arbitrary functions λ_u, λ_v, λ_1, $\bar{\lambda}_1$. The result is that $\lambda_u = \lambda_v = 0$, while λ_1 and $\bar{\lambda}_1$ are quite involved functions of all the other λ. However, in the reduced space $\mathcal{M}$ we can set $\phi_1 = \bar{\phi}_1 = 0$, and the reduced Dirac Hamiltonian is

$$H_{\mathcal{M}} = \lambda_0\phi_0 + \sum_{n=2}^{\infty}(\lambda_n\phi_n + \bar{\lambda}_n\bar{\phi}_n), \tag{3.43}$$

where all the constraints are assumed to be computed on $\mathcal{M}$. Using $\phi_1 = 0$ and $\bar{\phi}_1 = 0$ to effectively eliminate u and v in terms of p_1 and $\bar{p}_1$ one has

$$u = -\frac{1}{2\mu}\bar{p}_1, \tag{3.44}$$

$$v = -\frac{\mu}{\bar{p}_1} \pm \frac{1}{\bar{p}_1}\sqrt{\mu^2 + p_1\bar{p}_1}, \tag{3.45}$$

from which it also follows that

$$uv = \frac{1}{2} \mp \frac{1}{2\mu}\sqrt{\mu^2 + p_1\bar{p}_1}. \tag{3.46}$$

Then, on the reduced space,

$$\phi_0 = p_0 \pm \sqrt{\mu^2 + p_1\bar{p}_1}, \tag{3.47}$$

and

$$\phi_n = p_n + \mu(-1)^n\left(-\frac{\mu}{\bar{p}_1} \pm \frac{1}{\bar{p}_1}\sqrt{\mu^2 + p_1\bar{p}_1}\right)^n\left(n \pm \frac{1}{\mu}\sqrt{\mu^2 + p_1\bar{p}_1}\right), \tag{3.48}$$

$$\begin{aligned}\bar{\phi}_n &= \bar{p}_n + \mu(-1)^n\bar{p}_1^n\frac{n \pm \frac{1}{\mu}\sqrt{\mu^2 + p_1\bar{p}_1}}{\left(\mu \pm \sqrt{\mu^2 + p_1\bar{p}_1}\right)^n} \\ &= \bar{p}_n + \mu(-1)^n\left(\frac{\mu}{\bar{p}_1} \pm \frac{1}{\bar{p}_1}\sqrt{\mu^2 + p_1\bar{p}_1}\right)^{-n}\left(n \pm \frac{1}{\mu}\sqrt{\mu^2 + p_1\bar{p}_1}\right),\end{aligned} \tag{3.49}$$

for $n \geq 2$ (as a check, these expressions become identities for $n = 1$). The case $n = 0$ of (3.47) can also be included in either (3.48) or (3.49). Notice that the mass-shell condition can be recovered by taking the square of ϕ_0. This is inherent to the nonlinear realization formalism, since the momenta appear always linearly in the canonical constraints (see [31] for a discussion of this mechanism in the case of p-branes).

The constraints can also be written as

$$\phi_n = p_n + \frac{\mu}{\bar{p}_1^n}f_n^{\pm}(p_1\bar{p}_1), \tag{3.50}$$

$$\bar{\phi}_n = \bar{p}_n + \mu\bar{p}_1^n g_n^{\pm}(p_1\bar{p}_1), \tag{3.51}$$

for $n = 0, 2, 3, \ldots$, where

$$f_n^{\pm}(x) = \left(\mu \mp \sqrt{\mu^2 + x}\right)^n \left(n \pm \frac{1}{\mu}\sqrt{\mu^2 + x}\right), \tag{3.52}$$

$$g_n^{\pm}(x) = \left(-\mu \mp \sqrt{\mu^2 + x}\right)^{-n} \left(n \pm \frac{1}{\mu}\sqrt{\mu^2 + x}\right), \tag{3.53}$$

which satisfy, for $n \geq 1$,

$$\frac{\mathrm{d}}{\mathrm{d}x} f_n^{\pm}(x) = \mp \frac{n+1}{2\sqrt{\mu^2 + x}} f_{n-1}^{\pm}(x), \tag{3.54}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} g_n^{\pm}(x) = \pm \frac{n-1}{2\sqrt{\mu^2 + x}} g_{n+1}^{\pm}(x), \tag{3.55}$$

and also the second-order recurrence relation

$$(n-1) f_{n+1}^{\pm}(x) \pm 2n\sqrt{\mu^2 + x} f_n^{\pm}(x) + (n+1) x f_{n-1}^{\pm}(x) = 0, \quad n \geq 1, \tag{3.56}$$

$$(n+1) g_{n-1}^{\pm}(x) \pm 2n\sqrt{\mu^2 + x} g_n^{\pm}(x) + (n-1) x g_{n+1}^{\pm}(x) = 0, \quad n \geq 1. \tag{3.57}$$

The signs $\pm$ which appear in the above expressions correspond to different sheets of the constraint manifold in reduced space, parametrized by p_1 and $\bar{p}_1$. Notice that constraints $\phi_n, \bar{\phi}_n$ are first class. Therefore we expect that the physical degrees of freedom of the BMS particle will be finite-dimensional.

In the reduced phase space, with the boost variables u, v eliminated in terms of p_1, $\bar{p}_1$, the symmetry between the momenta corresponding to coordinates with positive and negative index is restored, and also that the constraint $\bar{\phi}_n$ is the complex conjugate of ϕ_n, thereby justifying the notation.

It will also be convenient to introduce the functions P_n, $\bar{P}_n$ of the variables P_1, $\bar{P}_1$ defined by

$$P_n = -\frac{\mu}{\bar{p}_1^n} f_n^{\pm}(p_1 \bar{p}_1), \tag{3.58}$$

$$\bar{P}_n = -\mu \bar{p}_1^n g_n^{\pm}(p_1 \bar{p}_1), \tag{3.59}$$

and, in particular,

$$P_0 = \mp\sqrt{\mu^2 + p_1 \bar{p}_1}, \tag{3.60}$$

in terms of which the constraints are $\phi_n = p_n - P_n$, $\bar{\phi}_n = \bar{p}_n - \bar{P}_n$, $\phi_0 = p_0 - P_0$. Notice also that $P_1 = p_1$, $\bar{P}_1 = \bar{p}_1$.

Due to the fact that M^{-1} does not have contributions in the lower-right square, the Dirac brackets of the variables x^n, x^{-n}, p_n, $\bar{p}_n$ do not change with respect to the Poisson ones,

$$\{x^n, p_m\}_D = \{x^n, p_m\} = \delta^n_m, \tag{3.61}$$

$$\{x^{-n}, \bar{p}_m\}_D = \{x^{-n}, \bar{p}_m\} = \delta^n_m. \tag{3.62}$$

First of all, the x^n, x^{-n}, p_n, $\bar{p}_n$ have zero Poisson brackets with all the 4 second-class constraints for $n \geq 2$, so only the cases for $n = 1$ need to be discussed. Since p_1, $\bar{p}_1$ have also zero brackets with all the constraints, all brackets involving either of them remain also unchanged. Finally,

$$\{x^1, x^{-1}\}_D = 0 - \{x^1, \Psi_i\} M^{-1}_{ij} \{\Psi_j, x^{-1}\},$$

but the only non-zero brackets of the $x^{\pm 1}$ are with ϕ_1 and $\bar{\phi}_1$, and this selects the lower-right square of M^{-1}, which is identically zero. The non-trivial Dirac brackets are those involving u, v, π_u, π_v and $x^{\pm 1}$:

$$\{u, x^1\}_D = 0, \quad \{v, x^1\}_D = -\frac{1}{2\mu} \frac{1}{1 - 2uv}, \tag{3.63}$$

$$\{u, x^{-1}\}_D = \frac{1}{2\mu}, \quad \{v, x^{-1}\}_D = \frac{1}{2\mu} \frac{v^2}{1 - 2uv}, \tag{3.64}$$

$$\{\pi_u, u\}_D = \{\pi_u, v\}_D = \{\pi_v, u\}_D = \{\pi_v, v\}_D = 0, \tag{3.65}$$

$$\{\pi_u, x^1\}_D = \{\pi_u, x^{-1}\}_D = \{\pi_v, x^1\}_D = \{\pi_v, x^{-1}\}_D = 0. \tag{3.66}$$

Summing up, in the reduced phase space one has coordinates

$$\{x^n,\ x^{-m},\ p_n,\ \bar{p}_m\}_{\substack{n=0,1,2,\dots, \\ m=1,2,\dots}} \tag{3.67}$$

with Hamiltonian (3.43), where the constraints are given by (3.47), (3.48), (3.49), and with ordinary brackets. The first class constraints $\phi_n, \bar{\phi}_n$ will generate infinite gauge transformations given by the canonical generator

$$G = \epsilon(\tau)\phi_0 + \sum_{m \geq 2} \left(\alpha_m(\tau)\phi_m + \beta_m(\tau)\bar{\phi}_m\right). \tag{3.68}$$

For instance, the re-parametrization associated with $\phi_0 = p_0 \pm \sqrt{\mu^2 + p_1 \bar{p}_1}$ acts only on the standard space-time coordinates, and is given by

$$\delta x^0 = \epsilon(\tau)\{x^0, \phi_0\}_D = \epsilon(\tau), \tag{3.69}$$

$$\delta x^1 = \epsilon(\tau)\{x^1, \phi_0\}_D = \pm\epsilon(\tau) \frac{\bar{p}_1}{2\sqrt{\mu^2 + p_1 \bar{p}_1}} = -\epsilon \frac{\bar{p}_1}{2P_0}, \tag{3.70}$$

$$\delta x^{-1} = \epsilon(\tau)\{x^{-1}, \phi_0\}_D = \pm\epsilon(\tau) \frac{p_1}{2\sqrt{\mu^2 + p_1 \bar{p}_1}} = -\epsilon \frac{p_1}{2P_0}, \tag{3.71}$$

where the function P_0 of $p_1, \bar{p}_1$ has been used to rewrite the final expression. Furthermore, $\delta p_0 = \delta p_1 = \delta \bar{p}_1 = 0$ and hence also $\delta u = \delta v = 0$.

In order to check the action of this symmetry on the Lagrangian $\mathcal{L}$ it suffices to consider the action on $\mathcal{L}_0$, since the re-parametrization acts only on $x^0, x^{\pm 1}$. It is convenient to re-define the arbitrary function $\epsilon(\tau)$ as $\epsilon(\tau)/(2P_0)$, so that

$$\delta x^0 = 2P_0 \epsilon(\tau),\ \delta x^1 = -\epsilon(\tau)\bar{p}_1,\ \delta x^{-1} = -\epsilon(\tau) p_1, \tag{3.72}$$

and also to write the Lagrangian in terms of the momenta $p_1, \bar{p}_1$,[1]

$$\mathcal{L}_0 = \dot{x}^0 P_0 + \dot{x}^1 p_1 + \dot{x}^{-1} \bar{p}_1. \tag{3.73}$$

[1] If one uses the equations of motion for $p_1, \bar{p}_1$ to eliminate these non-dynamical variables, the result is the standard Poincaré Lagrangian in the form (3.23).

One has then, taking into account that the p_1, $\bar{p}_1$ do not transform under this symmetry,

$$\delta\mathcal{L}_0 = 2\frac{\mathrm{d}}{\mathrm{d}\tau}(\epsilon P_0)P_0 - \frac{\mathrm{d}}{\mathrm{d}\tau}(\epsilon\bar{p}_1)p_1 - \frac{\mathrm{d}}{\mathrm{d}\tau}(\epsilon p_1)\bar{p}_1 \tag{3.74}$$

$$= \epsilon(2P_0\dot{P}_0 - \frac{\mathrm{d}}{\mathrm{d}\tau}(p_1\bar{p}_1)) + \dot{\epsilon}(2P_0^2 - 2p_1\bar{p}_1) = \epsilon\frac{\mathrm{d}}{\mathrm{d}\tau}(P_0^2 - p_1\bar{p}_1) + 2\dot{\epsilon}(P_0^2 - p_1\bar{p}_1) \tag{3.75}$$

$$= \frac{\mathrm{d}}{\mathrm{d}\tau}(2\mu^2\epsilon), \tag{3.76}$$

where $P_0^2 - p_1\bar{p}_1 = \mu^2$ has been used.

Similarly, one can study the transformation of the Lagrangian (3.13) under the transformation generated by ϕ_m for $m \geq 2$. We write the Lagrangian (3.13) in the notationally convenient form

$$\mathcal{L} = \sum_{n=0}^{\infty} \dot{x}^n P_n + \sum_{n=1}^{\infty} \dot{x}^{-n} P_{-n}, \tag{3.77}$$

where $P_{-n} = \bar{P}_n$ and all of them are functions of p_1, $p_{-1} = \bar{p}_1$ (or of u and v).

As shown in Appendix C.2, if $G = \alpha_m(\tau)\phi_m$ one has

$$\delta x^n = \{x^n, G\}_D = \begin{cases} \alpha_m & \text{if } n = m, \\ -\alpha_m(m+1)\frac{P_{m-1}}{2P_0} & \text{if } n = 1, \\ \alpha_m(m-1)\frac{P_{m+1}}{2P_0} & \text{if } n = -1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.78}$$

and then

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\epsilon_m(2P_0P_m - (m+1)p_1P_{m-1} + (m-1)\bar{p}_1P_{m+1})\right), \tag{3.79}$$

where the parameter of the transformation has been written as $\alpha_m = 2P_0\epsilon_m$. Notice that the function inside the total derivative depends on p_1, $\bar{p}_1$ and hence, through (3.16), (3.17), on the original Lagrangian variables u, v. Similarly, for $G = \beta_m\bar{\phi}_m$, one has

$$\delta x^n = \{x^n, G\}_D = \begin{cases} \beta_m & \text{if } n = -m, \\ \beta_m(m-1)\frac{\bar{P}_{m+1}}{2P_0} & \text{if } n = 1, \\ -\beta_m(m+1)\frac{\bar{P}_{m-1}}{2P_0} & \text{if } n = -1, \\ 0 & \text{otherwise,} \end{cases} \tag{3.80}$$

and then,

$$\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\epsilon_m(2P_0\bar{P}_m - (m+1)\bar{p}_1\bar{P}_{m-1} + (m-1)p_1\bar{P}_{m+1})\right), \tag{3.81}$$

with $\beta_m = 2P_0\epsilon_m$.

Notice that, in fact, these generators do not yield real transformations of $\mathcal{L}$, since they do not respect the fact that x^n and x^{-n} are complex conjugates, but this can be easily solved by working with the real generators

$$G_m = \epsilon_m(\phi_m + \bar{\phi}_m), \quad \epsilon_m^* = \epsilon_m, \tag{3.82}$$

$$\bar{G}_m = \epsilon_m(\phi_m - \bar{\phi}_m), \quad \epsilon_m^* = -\epsilon_m, \tag{3.83}$$

for $m = 2, 3, \ldots$, and using (3.79) and (3.81) to obtain the corresponding variations of the Lagrangian.

3.4 Realization of Lorentz symmetry in BMS space

From now on we will use the notation $p_{-n} = \bar{p}_n$ in order to obtain more compact expressions. The generators of the Lorentz group in physical $2+1$ space-time with coordinates x^0, x_1, x_2 and corresponding canonical momenta p_0, P_1, P_2 are given by $K_0 = x_1 P_2 - x_2 P_1$ for rotations and $K_i = x^0 P_i + x_i p_0$, $i = 1, 2$, for boosts. The relation with the $x^{\pm 1}$ coordinates is given by $x^{\pm 1} = \frac{1}{2}(x_1 \pm i x_2)$, which induces for the momenta the relation $p_{\pm 1} = P_1 \mp i P_2$. Defining $J = -iK_0$, $K_\pm = K_1 \mp i K_2$, one has, in terms of the coordinates x^0, $x^{\pm 1}$ and their associated canonical momenta p_0, $p_{\pm 1}$,

$$J = x^1 p_1 - x^{-1} p_{-1}, \tag{3.84}$$

$$K_+ = x^0 p_1 + 2 p_0 x^{-1}, \tag{3.85}$$

$$K_- = x^0 p_{-1} + 2 p_0 x^1, \tag{3.86}$$

which obey the Lorentz $SO(2,1)$ algebra $\{K_+, K_-\} = 2J$, $\{J, K_+\} = K_+$, $\{J, K_-\} = -K_-$. These generators can be extended to the BMS space with coordinates x^n, p_n, $n \in \mathbb{Z}$ by defining

$$\begin{aligned} J &= \sum_{n=-\infty}^{+\infty} n x^n p_n \\ &= \cdots - 2x^{-2} p_{-2} \boxed{-x^{-1} p_{-1} + 0 \cdot x^0 p_0 + x^1 p_1} + 2x^2 p_2 + \cdots, \end{aligned} \tag{3.87}$$

$$\begin{aligned} K_+ &= \sum_{n=-\infty}^{+\infty} (1-n) x^n p_{n+1} \\ &= \cdots + 3x^{-2} p_{-1} \boxed{+2x^{-1} p_0 + x^0 p_1} + 0 \cdot x^1 p_2 - x^2 p_3 + \cdots, \end{aligned} \tag{3.88}$$

$$\begin{aligned} K_- &= \sum_{n=-\infty}^{+\infty} (1+n) x^n p_{n-1} \\ &= \cdots - x^{-2} p_{-3} + 0 \cdot x^{-1} p_{-2} \boxed{+x^0 p_{-1} + 2x^1 p_0} + 3x^2 p_1 + \cdots, \end{aligned} \tag{3.89}$$

Notice that, under complex conjugation, $(K_+)^* = K_-$, and that $J^* = -J$, as it must be according to their definition from the real generators K_0, K_1 and K_2.

This set of generators, together with the supertranslation generators

$$\mathcal{P}_n = p_n, \tag{3.90}$$

provides a realization of BMS (Lorentz + supertranslations) in phase space, with $\{x^n, p_m\} = \delta^n_m$. Indeed, one has

$$\{K_+, K_-\} = 2J, \quad \{J, K_+\} = K_+, \quad \{J, K_-\} = -K_-, \tag{3.91}$$
$$\{K_+, \mathcal{P}_n\} = (1-n)\mathcal{P}_{n+1}, \quad \{K_-, \mathcal{P}_n\} = (1+n)\mathcal{P}_{n-1}, \tag{3.92}$$
$$\{J, \mathcal{P}_n\} = n\mathcal{P}_n, \quad \{\mathcal{P}_n, \mathcal{P}_m\} = 0, \tag{3.93}$$

with $n, m \in \mathbb{Z}$. The connection with the abstract algebra (3.4-3.6) is made by means of the identifications $K_+ \mapsto -iL_1$, $K_- \mapsto iL_{-1}$, $J \mapsto iL_0$, $\mathcal{P}_n \mapsto P_n$.

The fact that the extended generators satisfy the correct algebra is not enough to state that we have a realization of BMS symmetry. Indeed, one must prove that the generators K_+, K_-, J, and $\mathcal{P}_n$ are conserved charges of the system. In this case, one must prove that the generators have weakly zero Poisson brackets with all the first-class constraints ϕ_0, ϕ_n, $\bar{\phi}_n$, $n = 2, 3, \ldots$, appearing in the reduced Hamiltonian (3.43). Here, weakly zero means zero up to the constraints, and we will denote this by $\simeq 0$.

Since the constraints do not depend on the coordinates, this condition is trivially satisfied by the generators of supertranslations $\mathcal{P}_n$. For J one has, using that $\{p_n, J\} = -np_n$, $\{p_{-n}, J\} = np_{-n}$, $\{p_1 p_{-1}, J\} = 0$,

$$\begin{aligned}
\{\phi_n, J\} &= \{p_n + \mu p_{-1}^{-n} f_n^{\pm}(p_1 p_{-1}), J\} = -np_n + \mu f_n^{\pm}(p_1 p_{-1})\{p_{-1}^{-n}, J\} \\
&= -np_n - n\mu f_n^{\pm}(p_1 p_{-1}) p_{-1}^{-n-1}\{p_{-1}, J\} = -np_n - n\mu f_n^{\pm}(p_1 p_{-1}) p_{-1}^{-n-1} p_{-1} \\
&= -n\phi_n \simeq 0, \\
\{\bar{\phi}_n, J\} &= \{p_{-n} + \mu p_{-1}^{n} g_n^{\pm}(p_1 p_{-1}), J\} = np_{-n} + \mu g_n^{\pm}(p_1 p_{-1})\{p_{-1}^{n}, J\} \\
&= np_{-n} + n\mu g_n^{\pm}(p_1 p_{-1}) p_{-1}^{n-1}\{p_{-1}, J\} = np_n + n\mu g_n^{\pm}(p_1 p_{-1}) p_{-1}^{n-1} p_1 \\
&= n\bar{\phi}_n \simeq 0.
\end{aligned}$$

For K_+, using that $\{p_n, K_+\} = -(1-n)p_{n+1}$, $\{p_{-n}, K_+\} = -(1+n)p_{-n+1}$, and in particular that $\{p_1, K_+\} = 0$, $\{p_{-1}, K_+\} = -2p_0$,

$$\begin{aligned}
\{\phi_n, K_+\} &= \{p_n + \mu p_{-1}^{-n} f_n^{\pm}(p_1 p_{-1}), K_+\} \\
&= -(1-n)p_{n+1} + \mu f_n^{\pm}(p_1 p_{-1})\{p_{-1}^{-n}, K_+\} + \mu p_{-1}^{-n}\{f_n^{\pm}(p_1 p_{-1}), K_+\} \\
&= -(1-n)p_{n+1} + \mu f_n^{\pm}(p_1 p_{-1})(-np_{-1}^{-n-1}(-2p_0)) + \mu p_{-1}^{-n}(f_n^{\pm})'(p_1 p_{-1}) p_1(-2p_0) \\
&\overset{(3.54)}{=} -(1-n)p_{n+1} + 2\mu n p_0 p_{-1}^{-n-1} f_n^{\pm}(p_1 p_{-1}) \pm \mu(n+1) p_0 p_1 p_{-1}^{-n} \frac{f_{n-1}^{\pm}(p_1 p_{-1})}{\sqrt{\mu^2 + p_1 p_{-1}}} \\
&\overset{\phi_0}{\simeq} -(1-n)p_{n+1} \mp 2\mu n\sqrt{\mu^2 + p_1 p_{-1}}\, p_{-1}^{-n-1} f_n^{\pm}(p_1 p_{-1}) - \mu(n+1) p_1 p_{-1}^{-n} f_{n-1}^{\pm}(p_1 p_{-1}) \\
&\overset{\phi_{n+1}}{\simeq} (1-n)\mu p_{-1}^{-n-1} f_{n+1}(p_1 p_{-1}) \\
&\qquad \mp 2\mu n\sqrt{\mu^2 + p_1 p_{-1}}\, p_{-1}^{-n-1} f_n^{\pm}(p_1 p_{-1}) - \mu(n+1) p_1 p_{-1}^{-n} f_{n-1}^{\pm}(p_1 p_{-1}) \\
&= -\mu p_{-1}^{-n-1}\left((n-1) f_{n+1}(x) \pm 2n\sqrt{\mu^2 + x} f_n^{\pm}(x) + (n+1) x f_{n-1}^{\pm}(x)\right)_{x = p_1 p_{-1}} \overset{(3.56)}{=} 0,
\end{aligned}$$

and

$$\begin{aligned}
\{\bar{\phi}_n, K_+\} &= \{p_{-n} + \mu p_{-1}^n g_n^{\pm}(p_1 p_{-1}), K_+\} \\
&= -(1+n)p_{-n+1} + \mu g_n^{\pm}(p_1 p_{-1})\{p_{-1}^n, K_+\} + \mu p_{-1}^n \{g_n^{\pm}(p_1 p_{-1}), K_+\} \\
&= -(1+n)p_{-n+1} + n\mu g_n^{\pm}(p_1 p_{-1}) p_{-1}^{n-1}(-2p_0) + \mu p_{-1}^n (g_n^{\pm})'(p_1 p_{-1}) p_1 (-2p_0) \\
&\overset{(3.55),\phi_0}{\simeq} -(1+n)p_{-n+1} \mp 2\mu n\sqrt{\mu^2 + p_1 p_{-1}}\, p_{-1}^{n-1} g_n^{\pm}(p_1 p_{-1}) + \mu(n-1) p_1 p_{-1}^n g_{n+1}^{\pm}(p_1 p_{-1}) \\
&\overset{\phi_{n-1}}{\simeq} (1+n)\mu p_{-1}^{n-1} g_{n-1}(p_1 p_{-1}) \\
&\qquad \mp 2\mu n\sqrt{\mu^2 + p_1 p_{-1}}\, p_{-1}^{n-1} g_n^{\pm}(p_1 p_{-1}) + \mu(n-1) p_1 p_{-1}^n g_{n+1}^{\pm}(p_1 p_{-1}) \\
&= \mu p_{-1}^{n-1}\left((n+1) g_{n-1}(x) \pm 2n\sqrt{\mu^2 + x}\, g_n^{\pm}(x) + (n-1) x g_{n+1}^{\pm}(x) \right)_{x = p_1 p_{-1}} \overset{(3.57)}{=} 0.
\end{aligned}$$

Similarly, one can show that $\{\phi_n, K_-\} \simeq 0$, $\{\bar{\phi}_n, K_-\} \simeq 0$. This can be done by direct calculation as above or using that the Poisson bracket structure is real and then

$$\{\phi_n, K_-\}^* = \{\bar{\phi}_n, K_+\} \simeq 0, \tag{3.94}$$

$$\{\bar{\phi}_n, K_-\}^* = \{\phi_n, K_+\} \simeq 0. \tag{3.95}$$

One concludes then that the extended Poincaré generators are conserved charges for our system. A discussion of the Casimirs of the Lorentz and Poincaré groups in BMS space is presented in Appendix C.4, where, in particular, it is shown that the only quadratic Casimir of the BMS group is the standard one Poincaré Casimir, $C_2 = p_0^2 - p_1 p_{-1}$, see appendix D.

One might be tempted to generalize the Lorentz generators to include superrotations (or rather superboosts) by replacing the "1" which appear in (3.88) and (3.89) with arbitrary positive integers m,

$$K_+^m = \sum_{n=-\infty}^{+\infty} (m-n) x^n p_{n+m}, \tag{3.96}$$

$$K_-^m = \sum_{n=-\infty}^{+\infty} (m+n) x^n p_{n-m}, \tag{3.97}$$

for $m = 1, 2, \ldots$ By appropriate identifications, these generators, together with J and the $\mathcal{P}_n$, provide a representation of the extended BMS algebra in terms of Poisson brackets. However, the extended generators obtained for $m = 2, 3, \ldots$ do not commute with all the first class constraints of our system, and hence are not conserved quantities. To be more precise, the constraints ϕ_n, $n = 2, 3, \ldots$, are weakly invariant only under K_+^m, while the $\bar{\phi}_n$ are invariant only under K_-^m. We will see in Section 3.5 that the massless limit of our theory is fully invariant under these generalized transformations.

3.5 Massless limit

Since the Lagrangian (3.13) is proportional to μ one cannot take the massless limit directly in configuration space. This is not a problem in phase space, since the system is in this case entirely defined by the set of constraints, which have a non-trivial limit when $\mu \to 0$. Indeed, performing this limit in (3.50) and its complex conjugate (3.51) one gets the constraints

$$\varphi_n = p_n \pm (\mp 1)^n p_{-1}^{-n} (\sqrt{p_1 p_{-1}})^{n+1}, \quad n = 0, 1, 2, \dots, \tag{3.98}$$
$$\bar{\varphi}_n = p_{-n} \pm (\mp 1)^n p_{-1}^{n} (\sqrt{p_1 p_{-1}})^{-n+1}, \quad n = 1, 2, \dots, . \tag{3.99}$$

Notice that $\varphi_0 = p_0 \pm \sqrt{p_1 p_{-1}}$, and that φ_1 and $\bar{\varphi}_1$ are trivial, as in the massive case.

As shown in Appendix C.3, one has that

$$\{\varphi_n, K_+^m\} \simeq 0, \ \{\varphi_n, K_-^m\} \simeq 0, \ \{\bar{\varphi}_n, K_+^m\} \simeq 0, \ \{\bar{\varphi}_n, K_-^m\} \simeq 0, \tag{3.100}$$

where the superrotation generators $K_\pm^m$, $m = 0, 1, 2, \dots$ are defined as in (3.96) and (3.97), and where the weak equality is now over the manifold defined by $\varphi_n = 0$, $\bar{\varphi}_n = 0$. Thus, $K_\pm^m$ are conserved quantities in the massless limit theory.

The superrotation operators obey the algebra

$$\{K_+^m, K_+^n\} = (m-n) K_+^{m+n}, \tag{3.101}$$
$$\{K_+^m, K_-^n\} = \begin{cases} 2mJ & \text{if } m = n, \\ -(m+n) K_+^{m-n} & \text{if } m > n, \\ (m+n) K_-^{n-m} & \text{if } m < n, \end{cases} \tag{3.102}$$
$$\{K_-^m, K_-^n\} = (m-n) K_-^{m+n}, \tag{3.103}$$

Furthermore, they act on the supertranslation generators as

$$\{K_\pm^m, p_n\} = (m \mp n) p_{n \pm m}, \quad m = 1, 2, \dots, \quad n \in \mathbb{Z}. \tag{3.104}$$

If we now define

$$L_m = \begin{cases} -J & \text{if } m = 0, \\ K_+^m & \text{if } m > 0, \\ -K_-^{-m} & \text{if } m < 0, \end{cases} \tag{3.105}$$

it turns out that $\{L_m, L_n\} = (m-n) L_{m+n}$ and the extended BMS algebra (3.3) is obtained in terms of Poisson brackets of the massless limit BMS particle.

3.6 Gauge fixing

In Section 3.3 it has been shown that, after eliminating the degrees of freedom u, v and its corresponding canonical momenta by means of the primary second-class constraints $\pi_u = 0$, $\pi_v = 0$, $\phi_1 = 0$ and $\bar{\phi}_1 = 0$, the theory still contains an infinite number of primary first-class constraints which generate gauge transformations and indicate the presence of gauge degrees of freedom.

These gauge degrees can be eliminated by converting the first-class constraints to second class, by introducing appropriate gauge fixing conditions. Since all the constraints ϕ_n (resp. $\bar{\phi}_n$) depend linearly on p_n (resp. p_{-n}), a sensible choice is to introduce the constraints

$$\psi_n = x^n, \quad n = \pm 2, \pm 3, \dots, \tag{3.106}$$

so that

$$\{\psi_n, \phi_m\} = \delta_{n,m}, \quad n, m = \pm 2, \pm 3, \dots, \tag{3.107}$$

and define a gauge fixing as

$$GF = \{\psi_m = 0\}_{|m|\geq 2}, \tag{3.108}$$

which allows the consistent elimination of all the extra BMS degrees of freedom in phase space,

$$p_{\pm n} = -\frac{\mu}{p_{\mp 1}^n} f_n^{\pm}(p_1 p_{-1}), \quad x^n = 0, \quad x^{-n} = 0, \quad n = 2,3,\ldots, \tag{3.109}$$

with only the gauge symmetry associated with ϕ_0 remaining. In this way, the physical degrees of freedom of the theory in phase space are reduced to $x^0, p_0, x^{\pm 1}, p_{\pm 1}$, and it can be seen that the Dirac brackets between these remaining variables are the standard Poisson brackets.

Notice that the gauge condition is not invariant under supertranslations and hence one must introduce a compensating gauge transformation so that the total variation of the gauge condition, computed on the gauge condition, is zero. If we consider $|m| \geq 2$ and denote by $\delta_{ST}^n x^m$ the supertranslation of x^m generated by p_n, and by $\epsilon^m(\tau)$ the gauge transformation on x^m, generated by ϕ_m, one has

$$0 = (\epsilon^m(\tau) + \delta_{ST}^n x^m)|_{GF}, \tag{3.110}$$

and, since $\delta_{ST}^n x^m = \epsilon^n \delta_n^m = \epsilon^m$, the compensating gauge transformation associated with the supertranslation along the m coordinate, $|m| \geq 2$, is just

$$\epsilon^m(\tau) = -\epsilon^m. \tag{3.111}$$

Since the generators of the gauge transformations ϕ_m, $|m| \geq 2$, contain the momenta p_1, p_{-1}, it turns out that these compensating gauge transformations induce a residual transformation on the remaining variables $x^{\pm 1}$, given by

$$\delta_{\text{res}}^m x^{\pm 1} = \{x^{\pm 1}, -\epsilon^m \phi_m\}, \quad |m| \geq 2. \tag{3.112}$$

Using $\{x^1, \phi_n\} = -(n+1)P_{n-1}/(2P_0)$, $\{x^{-1}, \phi_n\} = (n-1)P_{n+1}/(2P_0)$, $n = \pm 2, \pm 3, \ldots$, one gets

$$\delta_{\text{res}}^m x^1 = \epsilon^m (m+1)\frac{P_{m-1}}{2P_0}, \tag{3.113}$$

$$\delta_{\text{res}}^m x^{-1} = -\epsilon^m (m-1)\frac{P_{m+1}}{2P_0}, \tag{3.114}$$

where it should be reminded that the several P_n appearing on the right-hand sides are functions of p_1, p_{-1}. That these transformations are a symmetry of the theory is proved at the end of Appendix C.2. Notice that for $m = 1$ and $m = -1$, although no compensating gauge transformation is needed, one formally obtains the standard translations in x^1 and x^{-1}, respectively.

These residual transformations on the physical variables $x^{\pm 1}$ provide, together with the Lorentz transformations, a realization of BMS in the physical reduced space, up to reparametrizations. The need for a reparametrization follows from the fact that x^0 does not transform under δ_{res}^m but, under a boost, transforms into x^1 or x^{-1}. For instance, for K_+ one has $\delta_+ x^0 = \{x^0, K_+\} = 2x^{-1}$ and then (we drop the constant parameters ϵ^m)

$$[\delta_+, \delta_{\text{res}}^m] x^0 = (m-1)\frac{P_{m+1}}{P_0}, \tag{3.115}$$

while $\delta_{\rm res}^{m+1}x^0$, which should appear on the right-hand side in order to have the BMS algebra, is zero. However, since $\{x^0,\phi_0\}=1$, the right-hand side can be interpreted as a reparametrization with parameter

$$\epsilon_+^m=(m-1)\frac{P_{m+1}}{P_0}, \tag{3.116}$$

so that the commutator is indeed a vanishing BMS supertranslation on x^0 plus a reparametrization,

$$[\delta_+,\delta_{\rm res}^m]x^0=(m-1)\cdot 0+\delta_0^{\epsilon_+^m}x^0. \tag{3.117}$$

For this to be consistent, the same reparametrization should appear when one considers the action of the transformations on x^1 and x^{-1},

$$[\delta_+,\delta_{\rm res}^m]x^1=\frac{m+1}{2}\left(\frac{p_1P_{m-1}}{P_0^2}+m\frac{P_{m-2}}{P_0}\right), \tag{3.118}$$

$$[\delta_+,\delta_{\rm res}^m]x^{-1}=-\frac{m-1}{2}\left(\frac{p_1P_{m+1}}{P_0^2}+m\frac{P_{m+2}}{P_0}\right). \tag{3.119}$$

Using that

$$\delta_{\rm res}^{m+1}x^1=(m+2)\frac{P_m}{2P_0},\quad \delta_{\rm res}^{m+1}x^{-1}=-m\frac{P_{m+2}}{2P_0}, \tag{3.120}$$

Notice this residual transformation is no longer a point transformation. Under reparametrizations with parameter ϵ_+^m (3.116),

$$\delta_0^{\epsilon_+^m}x^1=\epsilon_+^m\{x^1,\phi_0\}=-\frac{m-1}{2}\frac{p_{-1}P_{m+1}}{P_0^2}, \tag{3.121}$$

$$\delta_0^{\epsilon_+^m}x^{-1}=\epsilon_+^m\{x^{-1},\phi_0\}=-\frac{m-1}{2}\frac{p_1P_{m+1}}{P_0^2}, \tag{3.122}$$

one can check that (3.118), (3.119) can be rewritten as

$$[\delta_+,\delta_{\rm res}^m]x^1=(m-1)\delta_{\rm res}^{m+1}x^1+\delta_0^{\epsilon_+^m}x^1, \tag{3.123}$$

$$[\delta_+,\delta_{\rm res}^m]x^{-1}=(m-1)\delta_{\rm res}^{m+1}x^{-1}+\delta_0^{\epsilon_+^m}x^{-1}, \tag{3.124}$$

which, together with (3.117) and up to the reparametrization, yield the correct term for the BMS algebra. Similarly, for K_-, the reparametrization parameter is

$$\epsilon_-^m=-(m+1)\frac{P_{m-1}}{P_0}, \tag{3.125}$$

while no reparametrization is necessary to close the commutators of (3.113,3.114) with the transformation given by the rotation generator J.

This is an example of a fact previously reported in the literature [44, eq. (3.11)], *i.e.* the closure of the algebra of rigid symmetries with the help of gauge transformations. In our case, the rigid transformations correspond to Lorentz and supertranslations in physical space, and the gauge transformation is given by the reparametrization invariance associated with the first-class constraint ϕ_0, which has not been fixed. That the reparametrizations might be needed to close the algebra can be also be inferred from the fact that the constraints ϕ_n, $|n|\geq 2$, are weakly Lorentz invariant on the manifold defined by ϕ_0 (see the proof in Section 3.4) and that those ϕ_n are the generators of the residual gauge transformations that give rise to the BMS symmetry in physical space.

Under a Lorentz transformation,

$$\delta_J x^n = \{x^n, J\} = n x^n, \tag{3.126}$$
$$\delta_+ x^n = \{x^n, K_+\} = (2-n) x^{n-1}, \tag{3.127}$$
$$\delta_- x^n = \{x^n, K_-\} = (2+n) x^{n+1}, \tag{3.128}$$

and one has

$$\delta_J x^n|_{GF} = 0, \quad \delta_+ x^n|_{GF} = 0, \quad \delta_- x^n|_{GF} = 0, \tag{3.129}$$

so that the gauge condition is preserved, without the need for a compensating gauge transformation. Notice that the factors $(2-n)$ and $(2+n)$ play a fundamental role for $n = 2$ and $n = -2$, respectively.

In the massless case, where the Lorentz group can be extended so as to include superrotations, one has, for $m = 2, 3, \ldots$,

$$\delta_+^m x^n = \{x^n, K_+^m\} = (2m-n) x^{n-m}, \tag{3.130}$$
$$\delta_-^m x^n = \{x^n, K_-^m\} = (2m+n) x^{n+m}, \tag{3.131}$$

and a compensating gauge transformation must be introduced for $|n| \geq 2$ for the values of m such that the right-hand side of (3.130) or (3.131) are not zero when evaluated on the gauge fixing condition (3.108). As in the case of supertranslations, this will generate a residual gauge transformation for $x^{\pm 1}$, which should provide a realization of superrotations.

In any case, after eliminating the gauge degrees of freedom, the remaining variables are just x^0, $x^{\pm 1}$ and their canonical momenta, together with the first class constraint ϕ_0. This describes a Poincaré particle in $2+1$, with the corresponding reparametrization invariance, with no extra degrees of freedom, and with a realization of the supertranslations, plus superrotations in the massless case, provided by the residual gauge transformations. Summing up, the physical degrees of freedom of the BMS particle do not contain the BMS coordinates for $|n| > 1$.

3.7 Discusion of the possible relation between the canonical constraints of the BMS particle model and the non-local BMS transformations of the scalar field

A possible link with the nonlocal transformations for the massless scalar field presented in Chapter 2 is as follows.

Since the standard field equations can be obtained from the particle mass-shell condition following the world-line approach to field theory, one can consider, for the massless BMS particle, the infinite tower of squares of the constraints

$$\varphi_n \bar{\varphi}_n \simeq p_n p_{-n} - p_1 p_{-1}, \; n = 0, 2, 3, \ldots \tag{3.132}$$

(the case $n = 1$ is trivial). For $n = 0$, $p_0^2 - p_1 p_{-1} = 0$, this yields the standard massless Klein-Gordon equation. Indeed, using the relation with the real coordinates $p_{\pm 1} =$

$P_1 \mp iP_2$ and $p_0 = -i\partial_{x^0}$, $P_j = -i\partial_{x_j}$, $j = 1, 2$, one has

$$(p_0^2 - p_1 p_{-1})\Psi = (-\partial_{x^0}^2 + \nabla^2)\Psi = 0. \tag{3.133}$$

For $n = 2, 3\ldots$, one can introduce similar real coordinates $x_{1,2}^{(n)}$ related to the complex BMS coordinates $x^{\pm n}$, $n \geq 2$, by

$$x^{\pm n} = \frac{1}{2}(x_1^{(n)} \pm i x_2^{(n)}), \quad n = 2, 3, \ldots \tag{3.134}$$

so that

$$p_{\pm n} = P_1^{(n)} \mp P_2^{(n)} = -i\frac{\partial}{\partial x_1^{(n)}} \mp \frac{\partial}{\partial x_2^{(n)}}. \tag{3.135}$$

The imposition of all the conditions on an scalar field $\Psi(x^0, x_1, x_2, x_1^{(2)}, x_2^{(2)}, \ldots)$, depending on all of the BMS coordinates, leads then to the infinite set of equations

$$(p_n p_{-n} - p_1 p_{-1})\Psi = (-\nabla_n^2 + \nabla^2)\Psi = 0, \quad \nabla_n^2 = \frac{\partial^2}{\partial x_1^{(n)2}} + \frac{\partial^2}{\partial x_2^{(n)2}},\ n = 2, 3, \ldots \tag{3.136}$$

together with (3.133).

Notice that $p_n p_{-n} - p_1 p_{-1} = 0$, $n = 0, 2, 3, \ldots$, can be written in the equivalent form

$$p_n p_{-n} - p_{n+1} p_{-(n+1)} = 0,\ n = 0, 1, 2, 3, \ldots, \tag{3.137}$$

which, upon converted to a field equation, yields the standard Klein-Gordon equation plus an infinite tower of equations,

$$\nabla_{n+1}^2 \Psi = \nabla_n^2 \Psi, \quad n = 1, 2, \ldots, \tag{3.138}$$

or, equivalently, the infinite set of Klein-Gordon equations

$$\nabla_n^2 \Psi = \partial_{x^0}^2 \Psi, \quad n = 1, 2, \ldots \tag{3.139}$$

Whether the Lie symmetries of this set of PDE can be related to the nonlocal symmetries of a scalar field depending only on the ordinary Poincaré variables $x^0, x^{(1)}$, which were obtained in Chapter 2 in terms of polyharmonic functions, is a subject worthy of further study.

Chapter 4

BMS extension of conformal transformations

4.1 Introduction

Having seen that the Poincaré symmetry can be extended to a Poincaré BMS symmetry, it is natural to look for a possible infinite-dimensional extension of dilatations and special conformal transformations to obtain some sort of conformal BMS group realization. The extension of Poincaré to conformal transformations is a natural one that results in a conformal field theory. These theories play an important role in physical phenomena and theory, like for studying phase transitions at critical points or for the AdS/CFT correspondence in holography.

We will see that there is a natural generalization of dilatations to superdilatations that together with the supertranslations and superrotations close in an algebra that we call Weyl BMS algebra, as it is a generalization of the standard Weyl (Poincaré plus dilatations) algebra [20, 39, 51]. We will also see that a generalization of special conformal transformations that close with supertranslations, superrotations, and superdilatations faces some problems.

A generalization of the BMS symmetry that includes superdilatations as asymptotic symmetries of a gravitational theory has been considered in [29]. The algebra obtained there agrees with the results presented in this Thesis. A more general version of the algebra has also been found in [2], where general boundary symmetries in $d = 2$ and $d = 3$ are studied.

This chapter is organized as follows. In Section 4.2, we review the conformal transformations and compute the conserved charges for a general scalar theory. In Section 4.4, we construct the generators of the conformal algebra in terms of the Fourier modes of a massless KG field in arbitrary spacetime dimensions. In Section 4.4, we specialize to the case $d = 3$ and give the expression of the BMS supertranslations and superrotations for the same field and introduce the superdilatations. We explain the problems that appear when one considers the full BMS plus conformal algebra.

In this chapter, as well as in Appendix D, d denotes the space-time dimension, in contrast with the general discussion in Chapter 2, where d denotes the number of spatial dimensions. We do not use D for the space-time dimension, so that confusion with the generator of dilatations is avoided, nor do we use $d + 1$, so that factors of the form $(d - 2)$ are preserved in some of the expressions, indicating the special properties of the conformal symmetry in $d = 1 + 1$ space-time.

4.2 Conformal transformations

A conformal transformation is a transformation that leaves the metric invariant up to some constant, *i.e.* (see Lectures in [45])

$$dx'^2 = \Omega(x)^2 dx^2. \tag{4.1}$$

The vector that generates these transformations infinitesimally is

$$\xi_\mu(x) = -a_\mu - \omega_{\mu\nu}x^\nu + \kappa x_\mu + b_\mu x^2 - 2x_\mu b_\nu x^\nu, \quad \omega_{\mu\nu} = -\omega_{\nu\mu}, \tag{4.2}$$

and $\Omega(x) = 1 + \sigma(x)$, for $\sigma = \frac{1}{d}\partial_\mu \xi^\mu$.

A field with spin index I for the rotation group $O(d)$, transforms under conformal transformations as

$$\phi'_I(x') = \Omega(x)^{-\Delta} R_I{}^J(x)\phi_J(x), \tag{4.3}$$

where Δ is the scale dimension of ϕ and $R_I{}^J$ is a matrix of $O(d)$. The infinitesimal tranformation is therefore[1]

$$\delta_\xi \phi = \phi'(x) - \phi(x) = -\xi^\mu \partial_\mu \phi - \sigma\Delta\phi + \frac{1}{2}\hat{\omega}^{\mu\nu}(s_{\mu\nu})_I{}^J \phi_J, \tag{4.4}$$

for $s_{\mu\nu}$ the corresponding spin matrices that satisfy

$$[s_{\mu\nu}, s_{\rho\tau}] = \eta_{\mu\rho}s_{\nu\tau} - \eta_{\mu\tau}s_{\nu\rho} - \eta_{\nu\rho}s_{\mu\tau} + \eta_{\nu\tau}s_{\mu\rho}, \tag{4.5}$$

and

$$\hat{\omega}^{\mu\nu} = \omega^{\mu\nu} - 2(b^\mu x^\nu - b^\nu x^\mu). \tag{4.6}$$

The corresponding generators for infinitesimal transformations are

$$\begin{cases} P_\mu = -i\partial_\mu, \\ M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu), \\ D = -ix^\mu\partial_\mu, \\ K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^2\partial_\mu). \end{cases} \tag{4.7}$$

The algebra associated with these generators is

$$\begin{aligned} [D, P_\mu] &= iP_\mu, \\ [D, K_\mu] &= -iK_\mu, \\ [K_\mu, P_\nu] &= 2i(\eta_{\mu\nu}D - M_{\mu\nu}), \\ [K_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}K_\nu - \eta_{\rho\nu}K_\mu), \\ [P_\rho, M_{\mu\nu}] &= i(\eta_{\rho\mu}P_\nu - \eta_{\rho\nu}P_\mu), \\ [M_{\mu\nu}, M_{\rho\sigma}] &= i(\eta_{\nu\rho}M_{\mu\sigma} + \eta_{\mu\sigma}M_{\nu\rho} - \eta_{\mu\rho}M_{\nu\sigma} - \eta_{\nu\sigma}M_{\mu\rho}). \end{aligned} \tag{4.8}$$

From the point of view of the algebra, the only difference between the generators of translations P and those of special conformal transformations K is the different weight they get in the commutator with D.

The massive theory is invariant under Lorentz transformations, but only in the massless case one has invariance under the full conformal group. Let's see this,

[1] Using $\phi(x') = \phi(x) + \xi^\mu\partial_\mu\phi(x)$.

using that the variation of the Lagrangian has the following expression

$$\delta_\xi \mathcal{L} = \partial_\mu \phi \partial^\mu (\delta_\xi \phi) + m^2 \phi \delta_\xi \phi. \tag{4.9}$$

For $a_\mu \neq 0$ (the only non-zero parameter),

$$\delta_\xi \phi = a^\nu \partial_\nu \phi, \tag{4.10}$$

$$\delta_\xi \mathcal{L} = a^\nu \frac{1}{2} \partial_\nu \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) = \partial_\nu (a^\nu \mathcal{L}), \tag{4.11}$$

obtaining a total derivative for any m.

For $\omega_{\mu\nu}$ not null,

$$\delta_\xi \phi = \eta^{\gamma\alpha} \omega_{\alpha\beta} x^\beta \partial_\gamma \phi, \tag{4.12}$$

$$\delta_\xi \mathcal{L} = \eta^{\gamma\alpha} \omega_{\alpha\beta} \partial_\gamma \left(\frac{1}{2} x^\beta \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} x^\beta m^2 \phi^2 \right) = \partial_\gamma \left(\omega^\gamma{}_\beta x^\beta \mathcal{L} \right), \tag{4.13}$$

which is a total derivative again for any m. Notice that we have used the antisymmetric properties of $\omega_{\mu\nu}$ to find the previous expression.

For $\kappa \neq 0$,

$$\delta_\xi \phi = -\kappa x^\nu \partial_\nu \phi - \sigma \Delta \phi, \tag{4.14}$$

$$\delta_\xi \mathcal{L} = -\kappa \partial_\nu \left(\frac{1}{2} x^\nu \partial_\mu \phi \partial^\mu \phi \right) - m^2 \kappa \phi \left(x^\nu \partial_\nu \phi + \frac{d-2}{2} \phi \right), \tag{4.15}$$

where $\Delta = \frac{d-2}{2}$ and $\sigma = \kappa$, which is again a total derivative provided that $m = 0$.

For b_μ not null,

$$\delta_\xi \phi = -(x^2 b^\nu - 2(b \cdot x) x^\nu) \partial_\nu \phi - \sigma \Delta \phi, \tag{4.16}$$

$$\delta_\xi \mathcal{L} = -\frac{1}{2} \partial_\nu \left((x^2 b^\nu - 2(b \cdot x) x^\nu) \partial^\mu \phi \partial_\mu \phi - (d-2) b^\nu \phi^2 \right) + m^2 \phi \delta_\xi \phi, \tag{4.17}$$

for $\sigma = -2(b \cdot x)$, which is a total derivative for $m = 0$.

4.2.1 Conserved charges

According to Noether's theorem, each continuous symmetry of a Lagrangian has a conserved current associated with it, from which a conserved charge can be computed (see Appendix D).

For the case of spacetime translations, we obtain d conserved currents defined in terms of the classical energy-momentum tensor $T_c^{\mu\nu}$. Therefore, one can construct a set of conserved charges defined globally, assuming the field ϕ falls sufficiently fast at spatial infinity, in the following way

$$P^\mu = \int_{\mathbb{R}^{d-1}} d^{d-1} x T_c^{0\mu}, \qquad T_c^{\mu\nu} = \partial^\mu \phi \partial^\nu \phi - \frac{1}{2} \eta^{\mu\nu} \partial_\rho \phi \partial^\rho \phi. \tag{4.18}$$

Analogously, one can derive $\frac{1}{2}d(d-1)$ conserved charges for the Lorentz transformations, one for dilatations, and d for special conformal transformations. The expressions for them follow

$$M^{\mu\nu} = \int dx\, (I^0)^{\mu\nu}, \quad (I^\lambda)^{\mu\nu} = (x^\mu T_c^{\lambda\nu} - x^\nu T_c^{\lambda\mu}), \tag{4.19}$$

$$D = \int dx\, J_D^0, \quad J_D^\mu = -\left(x^\nu T^\mu_{c\ \nu} + \frac{d-2}{2}\phi\partial^\mu\phi\right), \tag{4.20}$$

$$K^\mu = \int dx\, J_K^{0\mu},$$

$$J_K^{\mu\nu} = -(x^2\eta^{\nu\sigma} - 2x^\nu x^\sigma)T^\mu_{c\ \sigma} - (d-2)\left(-x^\nu\phi\partial^\mu\phi + \frac{1}{2}\eta^{\mu\nu}\phi^2\right). \tag{4.21}$$

Each current is conserved with respect to its first index, *i.e.* $\partial_\lambda(I^\lambda)^{\mu\nu} = \partial_\mu J_D^\mu = \partial_\mu J_K^{\mu\nu} = 0$.

One can also redefine the energy-momentum tensor so that the second term in (4.20) is absorbed by adding a term of the form $\frac{1}{2}\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu}$. This term does not alter the conservation of this current when is defined such that satisfies the following relations

$$\partial_\rho\partial_\lambda X^{\rho\lambda\mu}_\mu = \partial_\mu V^\mu, \quad V^\mu = \partial_\nu\sigma^{\nu\mu}, \quad \sigma^{\mu\nu} = \frac{1}{2}\eta^{\mu\nu}\Delta\phi^2. \tag{4.22}$$

It turns out that

$$X^{\lambda\rho\mu\nu} = \frac{2}{d-2}\left[\eta^{\lambda\rho}\sigma^{\mu\nu} - \eta^{\lambda\mu}\sigma^{\rho\nu} - \eta^{\lambda\nu}\sigma^{\nu\rho} + \eta^{\mu\nu}\sigma^{\lambda\rho} + \frac{1}{d-1}(\eta^{\lambda\rho}\eta^{\mu\nu} - \eta^{\lambda\mu}\eta^{\nu\rho})\sigma^\alpha_\alpha\right] \tag{4.23}$$

satisfies the aforementioned conditions and one can check that the new term is divergenceless in the third index, $\partial_\mu\partial_\lambda\partial_\rho X^{\lambda\rho\mu\nu} = 0$. Thus, the current in terms of the improved energy-momentum tensor $T^{\mu\nu}$ is

$$J^\mu = x^\nu T^\mu{}_\nu + \frac{d-2}{2}\phi\partial^\mu\phi - \frac{1}{2}\partial_\lambda\partial_\rho(x_\nu X^{\lambda\rho\mu\nu}) + \frac{1}{2}(\eta_{\nu\lambda}\partial_\rho + \eta_{\nu\rho}\partial_\lambda)X^{\lambda\rho\mu\nu} = x^\nu T^\mu{}_\nu, \tag{4.24}$$

where the third term is the divergence of an antisymmetric tensor and the second and fourth terms cancel each other.

The new tensor has the following explicit expression in terms of the scalar field

$$T^{\mu\nu} = T_c^{\mu\nu} - \frac{1}{4}\frac{d-2}{d-1}\left(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\rho\partial_\rho\right)\phi^2. \tag{4.25}$$

One can check that this new tensor is both symmetric and traceless, so one can write now the global conserved charges in the following form

$$J^\mu = \int d^{d-1}x\, \xi_\nu T^{\mu\nu}, \tag{4.26}$$

for ξ^μ that of (4.2). Indeed, due to the aforementioned properties of this new energy-momentum tensor

$$\partial_\mu J^\mu = \int d^{d-1}x\, (T^\mu_\mu + \partial_{(\mu}\xi_{\nu)}T^{\mu\nu}) = \int d^{d-1}x\, (1+\sigma)T^\mu_\mu = 0, \tag{4.27}$$

for $\sigma = \kappa - 2b_\mu x^\mu$.

4.3 Conformal algebra realization in terms of a massless free Klein-Gordon field

In this section, we will construct a realization of the conformal group in terms of the Fourier modes of a free massless scalar field. Let us consider a massless scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi, \tag{4.28}$$

The solution of the equations of motion can be written in terms of Fourier modes as

$$\begin{aligned} \phi(t,\vec{x}) &= \int d\widetilde{k}\, \left(a(\vec{k},t)e^{i\vec{k}\cdot\vec{x}} + a^*(\vec{k},t)e^{-i\vec{k}\cdot\vec{x}} \right), \\ d\widetilde{k} &= \frac{d^{d-1}k}{(2\pi)^{(d-1)}(2\omega)}, \quad \omega = k^0 = \sqrt{\vec{k}\cdot\vec{k}}. \end{aligned} \tag{4.29}$$

Following standard procedures (see for example [36, 37, 45]), one can construct the generators of the conformal algebra (Poincaré, dilatations, and special conformal transformations) as integrals over the space of local densities depending on the field and their space-time derivatives. In terms of the Fourier modes $a(\vec{k},t)$, $a^*(\vec{k},t)$ defined in (4.29) they are given by (see Appendix D)

$$P^\mu = \int d\widetilde{k}\, a^*(\vec{k},t)k^\mu a(\vec{k},t), \quad k^\mu = (\omega,\vec{k}), \tag{4.30}$$

$$M^{0j} = tP^j - i\int d\widetilde{k}\, a^*(\vec{k},t)\omega\frac{\partial}{\partial k_j}a(\vec{k},t), \qquad M^{j0} = -M^{0j}, \tag{4.31}$$

$$M^{ij} = -i\int d\widetilde{k}\, a^*(\vec{k},t)\left(k^i\frac{\partial}{\partial k_j} - k^j\frac{\partial}{\partial k_i}\right)a(\vec{k},t), \qquad M^{ji} = -M^{ij}, \tag{4.32}$$

$$D = -tP^0 + i\int d\widetilde{k}\, a^*(\vec{k},t)\left(k^j\frac{\partial}{\partial k^j} + \frac{d-2}{2}\right)a(\vec{k},t), \tag{4.33}$$

$$K^0 = -t^2P^0 - \int d\widetilde{k}\, a^*(\vec{k},t)\left[-\left(\omega\frac{\partial}{\partial k_i} + 2itk^i\right)\frac{\partial}{\partial k^i} - it(d-2)\right]a(\vec{k},t), \tag{4.34}$$

$$K^j = t^2P^j - \int d\widetilde{k}\, a^*(\vec{k},t)\left[-\left(k^j\frac{\partial}{\partial k_i} - 2k^i\frac{\partial}{\partial k_j}\right)\frac{\partial}{\partial k^i} + (2it\omega + (d-2))\frac{\partial}{\partial k_j}\right]a(\vec{k},t). \tag{4.35}$$

The equal-time canonical Poisson brackets between $\phi(t,\vec{x})$ and $\pi(t,\vec{x}) = \dot{\phi}(t,\vec{x})$ lead to the following Poisson brackets for the Fourier modes

$$\{a(\vec{k},t), a^*(\vec{k},t)\} = -i(2\pi)^{(d-1)}(2\omega)\delta^{d-1}(\vec{k}-\vec{q}). \tag{4.36}$$

Using (4.36) one can compute the brackets between the generators of the conformal algebra

$$\{D, P^\mu\} = P^\mu, \tag{4.37}$$

$$\{D, K^\mu\} = -K^\mu, \tag{4.38}$$

$$\{K^\mu, P^\nu\} = 2(\eta^{\mu\nu}D + M^{\mu\nu}), \tag{4.39}$$

$$\{K^\mu, M^{\nu\sigma}\} = \eta^{\mu\sigma}K^\nu - \eta^{\mu\nu}K^\sigma, \tag{4.40}$$

$$\{P^\mu, M^{\nu\sigma}\} = \eta^{\mu\sigma}P^\nu - \eta^{\mu\nu}P^\sigma, \tag{4.41}$$

$$\{M^{\mu\nu}, M^{\sigma\rho}\} = \eta^{\mu\sigma}M^{\nu\rho} + \eta^{\nu\rho}M^{\mu\sigma} - \eta^{\mu\rho}M^{\nu\sigma} - \eta^{\nu\sigma}M^{\mu\rho}, \tag{4.42}$$

with all the remaining brackets equal to zero, and in particular

$$\{K^\mu, K^\nu\} = 0, \tag{4.43}$$

which is non-trivial due to the second-order character of the differential operators associated with K^0, K^i.

The Poisson brackets are related to the commutators of the differential operators in $\vec{k}$ by means of

$$\{P, Q\} = -i \int d\widetilde{k}\, a^*(\vec{k}, t)[\hat{P}, \hat{Q}]a(\vec{k}, t), \tag{4.44}$$

where $\hat{P}$, $\hat{Q}$ are the operators appearing in the integrals of conserved charges P and Q, respectively, at $t = 0$. Explicitly,

$$\hat{P}^\mu = k^\mu, \tag{4.45}$$

$$\hat{M}^{0j} = -i\omega \frac{\partial}{\partial k_j}, \tag{4.46}$$

$$\hat{M}^{ij} = -i\left(k^i \frac{\partial}{\partial k_j} - k^j \frac{\partial}{\partial k_i}\right), \tag{4.47}$$

$$\hat{D} = i\left(k^j \frac{\partial}{\partial k^j} + \frac{d-2}{2}\right), \tag{4.48}$$

$$\hat{K}^0 = \omega \frac{\partial^2}{\partial k_i^2}, \tag{4.49}$$

$$\hat{K}^j = \left(k^j \frac{\partial}{\partial k_i} - 2k^i \frac{\partial}{\partial k_j}\right) \frac{\partial}{\partial k^i} - (d-2)\frac{\partial}{\partial k_j}. \tag{4.50}$$

4.4 Canonical realization of the Weyl-BMS algebra

As discussed in Chapter 2, the generators of supertranslations and superrotations are given in terms of the Fourier modes respectively by

$$\mathcal{P}_\ell = \int d\widetilde{k}\, a^*(\vec{k}, t)\hat{\mathcal{P}}_\ell a(\vec{k}, t),\ m \in \mathbb{Z}, \tag{4.51}$$

$$\mathcal{R}_m = \int d\widetilde{k}\, a^*(\vec{k}, t)\hat{\mathcal{R}}_m a(\vec{k}, t),\ m \in \mathbb{Z}, \tag{4.52}$$

with

$$\hat{\mathcal{P}}_\ell = \omega_\ell = \omega^{1-\ell}(k^1 + ik^2)^\ell, \quad \omega = \sqrt{k_1^2 + k_2^2}, \tag{4.53}$$

$$\hat{\mathcal{R}}_m = \frac{1}{\omega}\omega_m \left((k^2 + imk^1)\frac{\partial}{\partial k^1} - (k^1 - imk^2)\frac{\partial}{\partial k^2}\right). \tag{4.54}$$

These differential operators obey the BMS algebra

$$[\hat{\mathcal{P}}_m, \hat{\mathcal{P}}_\ell] = 0, \tag{4.55}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{P}}_\ell] = i(m-\ell)\hat{\mathcal{P}}_{m+\ell}, \tag{4.56}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{R}}_n] = i(m-n)\hat{\mathcal{R}}_{m+n}. \tag{4.57}$$

For $m = 0, \pm 1$ one obtains a 6-dimensional closed algebra which corresponds to Poincaré in $2+1$. Corresponding expressions for the massive Klein-Gordon field are also presented in Section 2.3, but we will not discuss them here since we are interested in extending this algebra with conformal generators.

We have so far a canonical realization of the conformal algebra in terms of a massless Klein-Gordon obtained in Section 4.3 and the BMS extension of Poincaré, all in the context of a $2+1$-dimensional theory. It is therefore natural to look, in this simple case, for an extension of BMS Poincaré that also accommodates dilatations and special conformal transformations.

The differential operators (4.48), (4.49), (4.50), appearing in the canonical realization of the conformal group are, for $d = 3$,

$$\hat{D} = i\left(k^j \frac{\partial}{\partial k^j} + \frac{1}{2}\right), \tag{4.58}$$

$$\hat{K}^0 = \omega \frac{\partial^2}{\partial k_i^2}, \tag{4.59}$$

$$\hat{K}^j = \left(k^j \frac{\partial}{\partial k_i} - 2k^i \frac{\partial}{\partial k_j}\right) \frac{\partial}{\partial k^i} - \frac{\partial}{\partial k_j}. \tag{4.60}$$

For the dilatations the commutations relations with the Poincaré BMS generators are

$$[\hat{D}, \hat{\mathcal{P}}_\ell] = i\hat{\mathcal{P}}_\ell, \tag{4.61}$$

$$[\hat{D}, \hat{\mathcal{R}}_m] = 0, \tag{4.62}$$

but when acting with the special conformal transformations on the supertranslations one gets new operators not present in the algebra (4.45)

$$-i[\hat{K}^1, \hat{P}_\ell] = -(1-\ell)\ell\frac{1}{\omega}\omega_{\ell+1}\hat{D} + (1+\ell)\ell\frac{1}{\omega}\omega_{\ell-1}\hat{D} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} - (1+\ell)\hat{\mathcal{R}}_{\ell-1}, \tag{4.63}$$

$$[\hat{K}^2, \hat{P}_\ell] = -(1-\ell)\ell\frac{1}{\omega}\omega_{\ell+1}\hat{D} - (1+\ell)\ell\frac{1}{\omega}\omega_{\ell-1}\hat{D} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} + (1+\ell)\hat{\mathcal{R}}_{\ell-1}, \tag{4.64}$$

This algebra can be closed if we introduce an infinite family of superdilatations, given by

$$\hat{D}_\ell = \frac{1}{\omega}\omega_\ell \hat{D}, \tag{4.65}$$

so that

$$-i[\hat{K}^1, \hat{P}_\ell] = -(1-\ell)\ell\hat{D}_{\ell+1} + (1+\ell)\ell\hat{D}_{\ell-1} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} - (1+\ell)\hat{\mathcal{R}}_{\ell-1}, \tag{4.66}$$

$$[\hat{K}^2, \hat{P}_\ell] = -(1-\ell)\ell\hat{D}_{\ell+1} - (1+\ell)\ell\hat{D}_{\ell-1} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} + (1+\ell)\hat{\mathcal{R}}_{\ell-1}. \tag{4.67}$$

Using $\hat{D}_\ell$, the commutator of $\hat{K}^0$ with the supertranslations also closes,

$$[\hat{K}^0, \hat{P}_\ell] = -2i\left((1-\ell^2)\hat{D}_\ell + \ell\hat{\mathcal{R}}_\ell\right). \tag{4.68}$$

These commutators yield Poisson brackets, via (4.44), which generalize (4.39) when the supertranslations are considered.

It remains to study the action of the special conformal transformations on the superrotations. Using $\hat{K}^{\pm}$ instead of $\hat{K}^{1,2}$, given by

$$\hat{K}^{\pm} = \frac{1}{2}\left(\hat{K}^1 \pm i\hat{K}^2\right) \tag{4.69}$$

one gets

$$[\hat{K}^+, \hat{\mathcal{R}}_\ell] = -i\frac{1}{2}\ell(1-\ell)\frac{1}{\omega}\omega_{\ell+1}\hat{K}^0 + i(1-\ell^2)\frac{1}{\omega}\omega_\ell\hat{K}^+ + i\frac{1}{2}\ell(1-\ell^2)\frac{1}{\omega^2}\omega_{\ell+1}k^i\partial_i, \tag{4.70}$$

$$[\hat{K}^-, \hat{\mathcal{R}}_\ell] = -i\frac{1}{2}\ell(1+\ell)\frac{1}{\omega}\omega_{\ell-1}\hat{K}^0 - i(1-\ell^2)\frac{1}{\omega}\omega_\ell\hat{K}^- + i\frac{1}{2}\ell(1-\ell^2)\frac{1}{\omega^2}\omega_{\ell-1}k^i\partial_i, \tag{4.71}$$

$$[\hat{K}^0, \hat{\mathcal{R}}_\ell] = -i\ell(1+\ell)\frac{1}{\omega}\omega_{\ell-1}\hat{K}^+ - i\ell(1-\ell)\frac{1}{\omega}\omega_{\ell+1}\hat{K}^- + i\ell(1-\ell^2)\frac{1}{\omega^2}\omega_\ell k^i\partial_i, \tag{4.72}$$

and new operators appear one more time. One can be tempted, as we did for superdilatations, to introduce a family of superspecial conformal transformations to take into account the first and second terms on the right-hand side of the above commutators. However, the last term cannot be absorbed, unless one introduces yet another, completely new family of operators, and we have not succeeded in obtaining a close algebra that includes both the supertranslations and superrotations and the special conformal transformations. Notice also that one cannot just drop the superrotations, since they appear in the commutator between special conformal transformations and supertranslations. The extra terms in (4.70)-(4.72) disappear for $\ell = 0, \pm 1$ as it must be, since then we have a part of the conformal algebra.

Leaving aside the special conformal transformations, the superdilatations, together with the supertranslations and the superrotations, form a closed algebra, which we call a Weyl BMS algebra, given by

$$[\hat{\mathcal{P}}_m, \hat{\mathcal{P}}_\ell] = 0, \tag{4.73}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{P}}_\ell] = i(m-\ell)\hat{\mathcal{P}}_{m+\ell}, \tag{4.74}$$

$$[\hat{\mathcal{R}}_m, \hat{\mathcal{R}}_n] = i(m-n)\hat{\mathcal{R}}_{m+n}, \tag{4.75}$$

$$[\hat{D}_\ell, \hat{\mathcal{P}}_m] = i\hat{\mathcal{P}}_{\ell+m}, \tag{4.76}$$

$$[\hat{D}_\ell, \hat{D}_m] = 0, \tag{4.77}$$

$$[\hat{D}_\ell, \hat{\mathcal{R}}_m] = i\ell\hat{D}_{\ell+m}. \tag{4.78}$$

To summarize, we have obtained a family of operators, the superdilatations, that appear naturally when considering the action of the special conformal transformations on the supertranslations. The set of supertranslations, superrotations, and superdilatations form a closed infinite dimensional algebra which can be considered a BMS extension of the ordinary Weyl algebra (Poincaré plus dilatations). Trying to include the special conformal transformations leads to the appearance of an infinite tower of new kinds of operators.

From the detailed computations for this problem presented in Appendix D, it turns out that, actually, it is possible to construct families of operators from the special-conformal ones that yield a closed algebra when considered together with supertranslations, superrotations, and superdilatations,

$$[\hat{\mathscr{K}}_m, \hat{\mathscr{K}}_\ell] = 0, \tag{4.79}$$

$$[\hat{\mathscr{K}}_m, \hat{\mathcal{P}}_\ell] = -4i\hat{D}_{\ell+m}, \tag{4.80}$$

$$[\hat{\mathscr{K}}_m, \hat{D}_\ell] = i\hat{\mathscr{K}}_{\ell+m}, \tag{4.81}$$

$$[\hat{\mathscr{K}}_m, \hat{\mathcal{R}}_\ell] = i(\ell+m)\hat{\mathscr{K}}_{\ell+m}, \tag{4.82}$$

where the $\hat{\mathscr{K}}_m$ are linear combinations of the superrotations and the families of operators obtained from $\hat{K}^{0,+,-}$ by using

$$\mathcal{O}_\ell = \frac{1}{\omega}\omega_\ell O \tag{4.83}$$

in the same spirit as the superdilatations $\hat{D}_m$ are obtained from $\hat{D}$. See (E.47) and the computations leading to it for further details. Writing (4.81) as

$$[\hat{D}_\ell, \hat{\mathscr{K}}_m] = -i\hat{\mathscr{K}}_{\ell+m} \tag{4.84}$$

and comparing it to

$$[\hat{D}_\ell, \hat{\mathcal{P}}_m] = i\hat{\mathcal{P}}_{\ell+m} \tag{4.85}$$

one notes that the new operators obtained from the conformal ones and the supertranslation operators have opposite weights with respect to the superdilatations $\hat{D}_\ell$.

The algebra (4.79-4.82) does not contain the conformal algebra because the structure constants do not match; for instance (4.80) is missing the term proportional to the (super)rotations. Furthermore, since the $\hat{\mathscr{K}}_m$ are constructed by linearly combining operators of the form (4.83) for different values of ℓ, it is not possible to recover the ordinary special conformal transformations from them.

Although the above does not yield the kind of BMS extension of the conformal algebra that we are looking for, it must be noticed that, if one considers the subalgebra formed by (4.79) and (4.82), one gets an algebra that corresponds to the $\lambda = +1$ case considered in [49]. Further work is needed to understand this relation.

Chapter 5

Conclusions and outlook

This thesis has developed several aspects of the so-called canonical realization of Bondi-Metzner-Sachs symmetries, first introduced in [36], for the special case of a $2+1$ dimensional space-time.

In Chapter 2, extending the results from [12], a compact expression for the functions appearing in the conserved charges for the super-translations is obtained, in terms of Green functions of the polyharmonic operator of arbitrarily higher order in 2 dimensions.

While the conservation of the charges only depends on the general symmetry properties of the involved functions, the commutative character of the algebra satisfied by these charges relies on a key convolution property of the Green functions. Finally, the algebra relations with the Lorentz generators are obtained by using more specific properties of the polyharmonic Green functions. The fall-off conditions of the field to ensure that the super-translation charges are finite are also studied.

The closure of the transformations in configuration space, that is of the fields themselves, is also discussed, and it turns out that the correct algebra is obtained modulo transformations given by skew-symmetric combinations of the equations of motion, which are trivial symmetry transformations of any system.

A possible interpretation of the super-translation of the field in terms of a transformation of the space-time variables, which exists for the standard Poincaré space-time translations, is also discussed, but the results are inconsistent with the BMS algebra and the problem needs further study.

In Chapter 3 a non-linear realization of a massive particle Lagrangian for the BMS symmetry algebra in $2+1$ space-time is constructed. This Lagrangian depends on an infinite set of BMS coordinates, which include the standard $2+1$ Poincaré ones, together with the Goldstone boost variables.

The canonical analysis of this Lagrangian reveals the existence of a finite set of second-class constraints, which can be eliminated using the standard Dirac bracket construction, together with an infinite set of first-class constraints, which generate a corresponding infinite set of gauge transformations.

The standard Lorentz generators in $2+1$ are extended so that they act on the full set of BMS variables, and the theory is shown to be invariant under them. These extended Lorentz generators can be further generalized to an infinite set, the so-called superrotations, that obey the extended BMS_3 algebra, but it is only in the massless limit of the theory that they are conserved quantities and thus represent a symmetry of the system.

Upon fixing all the gauge degrees of freedom of the theory, except for the standard reparametrization, one obtains a theory whose physical content is that of an ordinary relativistic Poincaré particle, with the standard reparametrization invariance provided by the remaining first-class constraint. However, this gauge fixing procedure results in residual gauge transformations acting on the standard space coordinates x^0, x^1, x^{-1} which, modulo reparametrizations, realize the BMS symmetry. Since the remaining first-class constraint is the standard one, the field equation associated with this particle model is that of a free Klein-Gordon field, in the sense of the world-line approach to field theory.

The interpretation of these transformations for x^1, x^{-1}, which depend on the associated canonical momenta p_1, p_{-1}, is a subject for further study. The appearance of the momenta in a non-polynomial form seems to indicate that, in the field theory associated with this model, the field should transform non-locally, which is in accordance with the results in Chapter 2, but no more specific results have been obtained.

The massless limit has been obtained in the Hamiltonian formalism as the theory defined by the massless limit of the Hamiltonian constraints. An alternative approach should be to obtain the model of a massless particle in the nonlinear realization approach.

The construction of a particle model exhibiting BMS symmetry presented in this thesis could be, in principle, repeated for BMS_4, using, for instance, the stereographic parametrization of BMS_4 [8, 9]. BMS structure constants for BMS_4, BMS_5 and BMS_6 using generalized spherical harmonics parametrizations can also be found in [28], although they are much more involved.

Chapter 4 deals with the extension of the BMS transformations in the canonical formalism for a massless scalar field so as to include conformal transformations. The main result is the introduction of a family of operators, the superdilatations, that appear naturally when considering the action of the special conformal transformations on the supertranslations. The set of supertranslations, superrotations, and superdilatations form a closed infinite dimensional algebra which can be considered a BMS extension of the ordinary Weyl algebra (Poincaré plus dilatations). Trying to include the special conformal transformations leads to the appearance of an infinite tower of new kinds of operators, and a closed algebra can only be obtained at the price of not having the standard conformal algebra as a sublagebra.

Open problems

Some of the problems discussed in the thesis proposal have been addressed, namely the properties of the non-local transformations of the field, the detailed study of a particle model exhibiting BMS symmetry, and the extent to which conformal transformations can be included. However, many questions are still open, for which the results of this thesis have only provided a partial answer or have not been discussed at all:

- Interpretation of the non-local supertranslation transformations of the fields in terms of space-time coordinates.

- Relation between the results for the field in Chapter 2 and the transformations of the extended BMS coordinates of Chapter 3.

- Further study of the super-rotations in configuration space, and their formulation in terms of polyharmonic functions.

- Study of the relation of the particle model obtained in Chapter 3 with the coadjoint approach of [40–42]-

- Inclusion of a fermionic field and whether it is possible to define some kind of generalized spinor obeying a specific equation, in the spirit of the equation obeyed by the components of the momentum in the mass-shell manifold, and see if this allows the introduction of an infinite set of supersymmetry charges, which together with the standard Poincaré and super-translation charges, yield the supersymmetric BMS algebra discussed in [3, 10, 11].

- Extension of the results in this thesis to the *exotic* BMS algebras that, according to the discussion in the first part of Chapter 2, are obtained integer values of Λ higher than 1.

Appendix A

Properties of the polyharmonic functions in 2+1 dimensions

A.1 Asymptotic behavior of the polyharmonic Green functions

The polyharmonic Green function G_ℓ has the form

$$G_\ell(x) = A_\ell^{(0)}|\vec{x}|^{2\ell-2}\log|\vec{x}| + B_\ell^{(0)}|\vec{x}|^{2\ell-2}, \tag{A.1}$$

where the constants $A_\ell^{(0)}$ and $B_\ell^{(0)}$ can be read from (2.61). Successive applications of $\partial_{x_1} + i\partial_{x_2}$ yield expressions of the same form, with decreasing powers of $|\vec{x}|$, until one reaches

$$\begin{aligned}(\partial_{x_1} + i\partial_{x_2})^{\ell-1}G_\ell(\vec{x}) = \\ (x_1 + ix_2)^{\ell-1}\left(A_\ell^{(\ell-1)}\log|\vec{x}| + B_\ell^{(\ell-1)}\right).\end{aligned} \tag{A.2}$$

From this point the derivatives cease to content the log term and one can see that the derivative of order 2ℓ is of the form

$$(\partial_{x_1} + i\partial_{x_2})^{2\ell}G_\ell(\vec{x}) = C^{(2\ell)}(x_1 + ix_2)^{2\ell}\frac{1}{|\vec{x}|^{2\ell+2}}, \tag{A.3}$$

with a constant $C^{(2\ell)}$. This is a rational function of x_1, x_2 with asymptotic behavior

$$(\partial_{x_1} + i\partial_{x_2})^{2\ell}G_\ell(\vec{x}) \sim \frac{1}{|\vec{x}|^2} \quad \forall\ell \geq 1 \quad \text{for } |\vec{x}| \to \infty, \tag{A.4}$$

which is independent of $\ell \geq 1$. This allows us to study the conditions that must be imposed on the fields so that the supertranslation charges are finite. A general supertranslation charge Q_ℓ, $\ell \geq 1$, has integrals of the form

$$Q_\ell(t) = \int \mathrm{d}^2x\mathrm{d}^2y F(t,\vec{x})G(t,\vec{y})(\partial_{x_1} + i\partial_{x_2})^{2\ell}G_\ell(\vec{x} - \vec{y}) \tag{A.5}$$

where F and G are either π or first order derivatives of ϕ. Performing a change of variables $\vec{x} = \vec{y} + \vec{r}$ and using the asymptotic behavior (A.4), the existence of the charge reduces to the existence of the integral

$$\int \mathrm{d}^2r\mathrm{d}^2y F(\vec{y} + \vec{r})G(\vec{y})\frac{1}{r^2}. \tag{A.6}$$

Let us assume now that the fields F and G behave, for large argument, as

$$F(\vec{y}+\vec{r}) \sim \frac{\bar{F}}{|\vec{y}+\vec{r}|^{\alpha}}, \quad G(\vec{y}) \sim \frac{\bar{G}}{|\vec{y}|^{\beta}}, \tag{A.7}$$

with $\bar{F}$, $\bar{G}$ depending on the angular variable and time. This leads to the study of the integral

$$\int \mathrm{d}^2\Omega\, \bar{F}\bar{G} \int r\mathrm{d}r\, y\mathrm{d}y\, \frac{1}{|\vec{y}+\vec{r}|^{\alpha}|\vec{y}|^{\beta}r^2}, \tag{A.8}$$

where $\mathrm{d}^2\Omega$ is the angular measure. Performing a change to polar coordinates in $\mathbb{R}^2_+$ for the $\mathrm{d}r\mathrm{d}y$ measure, one finally gets

$$\int \mathrm{d}^3\Omega\, \bar{F}\bar{G} \int \rho\mathrm{d}\rho\, \frac{1}{\rho^{\alpha+\beta}}, \tag{A.9}$$

with $\mathrm{d}^3\Omega$ including the additional integration over the angular coordinate of the polar change of variables. For this integral to converge it is necessary that

$$\alpha+\beta > 2. \tag{A.10}$$

Considering the forms of F and G for the different supertranslation charges, one concludes that the asymptotic behavior of the fields which guarantees the existence of all the Q_ℓ is the one given in (2.77) and (2.78).

A.2 Brackets between the supertranslation charges

Consider two arbitrary supertranslation charges $Q_\ell(t)$ and $Q_m(t)$. Using standard Poisson brackets, one gets

$$\begin{aligned} \{Q_\ell(t), Q_m(t)\} = & \qquad\qquad \text{(A.11)} \\ & \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z \Big(f_\ell(\vec{x}-\vec{y})f_m(\vec{y}-\vec{z})\pi(x)\phi(z) \\ & -f_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\phi(y)\pi(z) \\ & +f_\ell(\vec{x}-\vec{y})g_m(\vec{y}-\vec{z})\pi(x)\pi(z) \\ & +f_\ell(\vec{x}-\vec{y})h_m(\vec{x}-\vec{z})\phi(y)\phi(z) \\ & -g_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\pi(y)\pi(z) \\ & +g_\ell(\vec{x}-\vec{y})h_m(\vec{x}-\vec{z})\pi(y)\phi(z) \\ & -h_\ell(\vec{x}-\vec{y})f_m(\vec{x}-\vec{z})\phi(y)\phi(z) \\ & -h_\ell(\vec{x}-\vec{y})g_m(\vec{x}-\vec{z})\phi(y)\pi(z)\Big). \qquad \text{(A.12)} \end{aligned}$$

Let us consider first the case $m=0$, that is $Q_0(t)=H(t)$, for which $f_0(\vec{x}-\vec{y})=0$, $g_0(\vec{x}-\vec{y})=\delta(\vec{x}-\vec{y})$ and $h_0(\vec{x}-\vec{y})=\vec{\nabla}^2_x\delta(\vec{x}-\vec{y})$. Integration by parts yields

$$\begin{aligned} \dot{Q}_\ell(t) = \{Q_\ell(t), H(t)\} = & \qquad\qquad \text{(A.13)} \\ = \int \mathrm{d}^2x\mathrm{d}^2y \Big(& f_\ell(\vec{x}-\vec{y})\pi(x)\pi(y) + \vec{\nabla}^2_x f_\ell(\vec{x}-\vec{y})\phi(y)\phi(x) \\ + \; & \vec{\nabla}^2_x g_\ell(\vec{x}-\vec{y})\pi(y)\phi(x) - h_\ell(\vec{x}-\vec{y})\phi(y)\pi(x)\Big). \qquad \text{(A.14)} \end{aligned}$$

The first two terms are zero, each by itself, due to the skew-symmetry of f_ℓ and its even-order derivatives, while the two last terms cancel each other after using $\vec{\nabla}^2(x)g_\ell(\vec{x}-\vec{y}) = h_\ell(\vec{x}-\vec{y})$ and the symmetry of h_ℓ. This shows that the supertranslation charges are conserved under the symmetry properties of f_ℓ, g_ℓ and h_ℓ, and the relation between g_ℓ and h_ℓ, without using the explicit form of these functions in terms of the polyharmonic Green functions.

For general ℓ and m one must consider the different cases separately.

1. ℓ and m odd. In this case $g_\ell = h_\ell = g_m = h_m = 0$ and, after renaming the variables of integration in the first non-zero contribution,

$$\begin{aligned}\{Q_\ell(t), Q_m(t)\} &= \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z \Big(f_\ell(\vec{z}-\vec{x})f_m(\vec{x}-\vec{y}) \\ &- f_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\Big)\phi(y)\pi(z). \end{aligned} \tag{A.15}$$

2. ℓ even and m odd. Now $f_\ell = 0$ and $g_m = h_m = 0$, and the result can be written as

$$\begin{aligned}&\{Q_\ell(t), Q_m(t)\} = \\ &\quad -\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\, g_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\pi(y)\pi(z) \\ &\quad -\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\, h_\ell(\vec{x}-\vec{y})f_m(\vec{x}-\vec{z})\phi(y)\phi(z). \end{aligned} \tag{A.16}$$

3. ℓ and m even. We have $f_\ell = f_m = 0$ and (A.12) boils down to

$$\begin{aligned}\{Q_\ell(t), Q_m(t)\} = \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z \Big(g_\ell(\vec{x}-\vec{z})h_m(\vec{x}-\vec{y}) \\ - h_\ell(\vec{x}-\vec{y})g_m(\vec{x}-\vec{z})\Big)\phi(y)\pi(z). \end{aligned} \tag{A.17}$$

The elementary symmetry properties used up to now are not enough to show that the above expressions are actually zero. To do so, one must use the fact that the functions f_ℓ, g_ℓ, and h_ℓ can be written in terms of Green functions that obey the convolution property (2.63).

Let us prove, for instance, that the first term in (A.16) is zero. Changing $\ell \to 2\ell$ and $m \to 2m+1$ one has, assuming $l > 0$, $m > 0$,

$$\begin{aligned}&-\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\, g_\ell(\vec{x}-\vec{y})f_m(\vec{z}-\vec{x})\pi(y)\pi(z) \to \\ &- \quad \int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\, (\partial_{x_1}+i\partial_{x_2})^{2\ell}G_\ell(\vec{x}-\vec{y})(\partial_{x_1}+i\partial_{x_2})^{2m+1}G_m(\vec{x}-\vec{z})\pi(y)\pi(z) \\ &= \quad -\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\, G_\ell(\vec{x}-\vec{y})G_m(\vec{x}-\vec{z})(\partial_{y_1}+i\partial_{y_2})^{2\ell}\pi(y)(\partial_{z_1}+i\partial_{z_2})^{2m+1}\pi(z) \\ &\overset{(2.63)}{=} \quad -\int \mathrm{d}^2y\mathrm{d}^2z G_{\ell+m}(\vec{y}-\vec{z})(\partial_{y_1}+i\partial_{y_2})^{2\ell}\pi(y)(\partial_{z_1}+i\partial_{z_2})^{2m+1}\pi(z) \\ &= \quad -\int \mathrm{d}^2y\mathrm{d}^2z(\partial_{y_1}+i\partial_{y_2})^{2\ell+2m+1}G_{\ell+m}(\vec{y}-\vec{z})\pi(y)\pi(z) = 0 \end{aligned}$$

due to the skew-symmetry of

$$(\partial_{y_1}+i\partial_{y_2})^{2\ell+2m+1}G_{\ell+m}(\vec{y}-\vec{z}),$$

and the second term of (A.16) can also be shown to be zero using the same manipulations. Notice that the same reasoning can be used for ℓ and/or m negative since this amounts to change some $\partial_{x_1} + i\partial_{x_2}$ to $\partial_{x_1} - i\partial_{x_2}$ and the result, which only depends on the number of derivatives, is the same.

Using the same techniques and the convolution property, the two terms that appear in (A.15) or (A.17) can be shown to be the same, and hence the corresponding brackets are zero. This completes the proof that the supertranslation charges yield a commutative algebra under the Poisson brackets.

A.3 Detailed computation of some Poisson brackets

Consider first

$$\begin{aligned}\{L_0(t), Q_{2\ell}(t)\} &= \\ = & -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\Big\{\pi(x)(x_1\partial_{x_2}\phi(x) - x_2\partial_{x_1}\phi(x)), \\ & \mathcal{H}(y,z)\Big\}(\partial_{y_1} + i\partial_{y_2})^{2\ell} G_\ell(\vec{y}-\vec{z}).\end{aligned}$$

Using that even-order derivatives of an even-symmetric function are even, and the equal-time Poisson brackets $\{\phi(x), \pi(y)\} = \delta(\vec{x}-\vec{y})$, the above expression equals

$$\begin{aligned}&\{L_0(t), Q_{2\ell}(t)\} = \\ &= -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2y\mathrm{d}^2z\Big((x_1\partial_{x_2}\phi(x) - x_2\partial_{x_1}\phi(x))\vec{\nabla}\phi(z)\cdot\vec{\nabla}_y(-\delta(\vec{x}-\vec{y})) \\ &\quad + \pi(x)\pi(z)(x_1\partial_{x_2} - x - 2\partial_{x_1})\delta(\vec{x}-\vec{y})\Big)(\partial_{y_1} + i\partial_{y_2})^{2\ell} G_\ell(\vec{y}-\vec{z}) \\ &= -\frac{1}{2i}\int \mathrm{d}^2x\mathrm{d}^2z\Big(2\mathcal{H}(x,z)\Big)(x_1\partial_{x_2} - x_2\partial_{x_1})(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{z}),\end{aligned} \tag{A.18}$$

where several integrations by parts, assuming the appropriate asymptotic behavior for the fields, have been performed, and the relation $\vec{\nabla}_x G_\ell(\vec{x}-\vec{y}) = -\vec{\nabla}_y G_\ell(\vec{x}-\vec{y})$ has been used. Next, we use the commutator $\left[x_1\partial_{x_2} - x_2\partial_{x_1}, (\partial_{x_1} + i\partial_{x_2})^n\right] = in(\partial_{x_1} + i\partial_{x_2})^n$, $n = 0, 1, \ldots$ to write (A.18) as

$$\begin{aligned}&\{L_0(t), Q_{2\ell}(t)\} = \\ &-\frac{1}{i}\int \mathrm{d}^2x\mathrm{d}^2z\mathcal{H}(x,z)\Big((\partial_{x_1} + i\partial_{x_2})^{2\ell}(x_1\partial_{x_2} - x_2\partial_{x_1}) + i\,2\ell\,(\partial_{x_1} + i\partial_{x_2})^{2\ell}\Big) G_\ell(\vec{x}-\vec{z}).\end{aligned}$$

One has that

$$(x_1\partial_{x_2} - x_2\partial_{x_1}) G_\ell(\vec{x}-\vec{z}) = G'_\ell(|\vec{x}-\vec{z}|)\frac{-x_1z_2 + x_2z_1}{|\vec{x}-\vec{z}|} \tag{A.19}$$

is an odd function under $\vec{x} \leftrightarrow \vec{z}$, and hence its derivatives of even order are also odd. The product with $\mathcal{H}(x,z)$ is also odd and the term vanishes under integration in x and z. Thus one is left only with the term from the commutator and

$$\begin{aligned}&\{L_0(t), Q_{2\ell}(t)\} = \\ &-2\ell\int \mathrm{d}^2x\mathrm{d}^2z\mathcal{H}(x,z)(\partial_{x_1} + i\partial_{x_2})^{2\ell} G_\ell(\vec{x}-\vec{z}) = -2\ell Q_{2\ell}(t), \quad \ell \geq 0.\end{aligned} \tag{A.20}$$

Let us compute now the Poisson bracket of the boost generator $L_1(t)$ with an even-order supertranslation charge,

$$\begin{aligned}
&\{L_1(t), Q_{2\ell}(t)\} = \\
&= \int d^2x d^2y d^2z \Big\{ t\pi(x)(\partial_{x_1} + i\partial_{x_2})\phi(x) \\
&+ (x_1 + ix_2)\mathcal{H}(x), \mathcal{H}(y,z) \Big\} (\partial_{y_1} + i\partial_{y_2})^{2\ell} G_\ell(\vec{y} - \vec{z}).
\end{aligned}$$

After computing the Poisson brackets and using integration by parts and the symmetry of G_ℓ and its derivatives, the terms containing t can be written as

$$2t \int d^2x d^2z \mathcal{H}(x,z)(\partial_{x_1} + i\partial_{x_2}) G_\ell(\vec{x} - \vec{z}),$$

which is zero due to the skew-symmetry of $(\partial_{x_1} + i\partial_{x_2}) G_\ell(\vec{x} - \vec{z})$ under $\vec{x} \leftrightarrow \vec{z}$. Performing the same manipulations, the remaining terms can be written as

$$\begin{aligned}
&\{L_1(t), Q_{2\ell}(t)\} = \\
&= \int d^2x d^2z \Big((x_1 + ix_2)\pi(x)\vec{\nabla}\phi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell} \cdot \vec{\nabla}_x G_\ell(\vec{x} - \vec{z}) \\
&\quad + (x_1 + ix_2)\vec{\nabla}\phi(x)\pi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell} \cdot \vec{\nabla}_x G_\ell(\vec{x} - \vec{z}) \Big).
\end{aligned}$$

The $\vec{\nabla}\phi(z)$ in the first term can be integrated by parts, yielding one term, while the integration by parts of $\vec{\nabla}\phi(x)$ yields two. After a change of variables and using $(\partial_{z_1} + i\partial_{z_2})^{2\ell} \vec{\nabla}_z^2 G_l(\vec{x} - \vec{z}) = (\partial_{x_1} + i\partial_{x_2})^{2\ell} \vec{\nabla}_x^2 G_l(\vec{x} - \vec{z})$, the two terms that are similar can be combined, and the result is

$$\begin{aligned}
&\{L_1(t), Q_{2\ell}(t)\} = \\
&\int d^2x d^2z \Big(((x_1 - z_1) + i(x_2 - z_2))\pi(x)\phi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell} \cdot \vec{\nabla}_x^2 G_\ell(\vec{x} - \vec{z}) \\
&\quad - \phi(x)\pi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell}(\partial_{x_1} + i\partial_{x_2}) G_\ell(\vec{x} - \vec{z}) \Big).
\end{aligned}$$

The second term is just $-Q_{2\ell+1}(t)$ and the integrand in the first one can be re-written, using that $[x_1 + ix_2, \partial_{x_1} + i\partial_{x_2}] = 0$ and $\vec{\nabla}_x^2 = (\partial_{x_1} + i\partial_{x_2})(\partial_{x_1} - i\partial_{x_2})$, as

$$\begin{aligned}
&\{L_1(t), Q_{2\ell}(t)\} = -Q_{2\ell+1}(t) \\
&+ \int d^2x d^2z \Big(\pi(x)\phi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} \Big[(x_1 - z_1) \\
&+ i(x_2 - z_2) \Big] (\partial_{x_1} - i\partial_{x_2}) G_\ell(\vec{x} - \vec{z}) \Big).
\end{aligned} \tag{A.21}$$

Since we are considering $\ell \geq 1$ (the case $\ell = 0$ correspond to the standard Poincaré algebra) and $2\ell + 1 > 2(\ell - 1)$, we can use relation (A.25) to rewrite (A.21) as

$$\begin{aligned}
&\{L_1(t), Q_{2\ell}(t)\} = \\
&- \quad Q_{2\ell+1}(t) \\
&+ \quad 2(\ell-1)\int d^2x d^2z \pi(x)\phi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x} - \vec{z}) \\
&= \quad -Q_{2\ell+1}(t) \\
&- \quad 2(\ell-1)\int d^2x d^2z \phi(x)\pi(z)(\partial_{x_1} + i\partial_{x_2})^{2\ell+1} G_\ell(\vec{x} - \vec{z}) \\
&= \quad -Q_{2\ell+1}(t) - 2(\ell-1)Q_{2\ell+1}(t) \\
&= \quad (1-2\ell)Q_{2\ell+1}(t).
\end{aligned} \tag{A.22}$$

A.4 Some identities satisfied by the polyharmonic Green functions

Assuming $\ell > 1$ and using $H_{\ell-1} = H_{\ell-2} + 1/(\ell-1)$ one can show from (2.61) that

$$\begin{aligned}
&(\partial_{x_1} - i\partial_{x_2})G_\ell(\vec{x} - \vec{y}) = \\
&\quad \frac{1}{2(\ell-1)}((x_1 - y_1) - i(x_2 - y_2))G_{\ell-1}(\vec{x} - \vec{y}) \\
&- \quad \frac{|\vec{x} - \vec{y}|^{2(\ell-2)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi}((x_1 - y_1) - i(x_2 - y_2)),
\end{aligned}$$

which is a recurrence relation valid for $\ell > 1$. Multiplying by $(x_1 - y_1) + i(x_2 - y_2)$ one gets

$$\begin{aligned}
&((x_1 - y_1) + i(x_2 - y_2))(\partial_{x_1} - i\partial_{x_2})G_\ell(\vec{x} - \vec{y}) = \\
&\quad \frac{1}{2(\ell-1)}|\vec{x} - \vec{y}|^2 G_{\ell-1}(\vec{x} - \vec{y}) - \frac{|\vec{x} - \vec{y}|^{2(\ell-1)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi},
\end{aligned} \tag{A.23}$$

which, except for polynomial terms, has the functional dependence of G_ℓ. Indeed, using again the relation between $H_{\ell-1}$ and $H_{\ell-2}$, one obtains

$$\begin{aligned}
&((x_1 - y_1) + i(x_2 - y_2))(\partial_{x_1} - i\partial_{x_2})G_\ell(\vec{x} - \vec{y}) = \\
&\quad 2(\ell-1)G_\ell(\vec{x} - \vec{y}) + \frac{|\vec{x} - \vec{y}|^{2(\ell-1)}}{[(\ell-1)!]^2 2^{2\ell-1}\pi}.
\end{aligned} \tag{A.24}$$

Although (A.24) has been obtained under the assumption that $\ell > 1$ it can be checked by direct computation that it is also valid for $\ell = 1$.

In the computations in Section 2.7 the left-hand side of this identity appears with derivatives $\partial_{x_1} + i\partial_{x_2}$ acting on it. Since the second term in (A.24) is a polynomial of order $2(\ell - 1)$ in the components of $\vec{x}$, it turns out that, for $\ell \geq 1$ and $n > 2(\ell - 1)$,

$$\begin{aligned}
&(\partial_{x_1} + i\partial_{x_2})^n((x_1 - y_1) + i(x_2 - y_2))(\partial_{x_1} - i\partial_{x_2})G_\ell(\vec{x} - \vec{y}) \\
&= 2(\ell-1)(\partial_{x_1} + i\partial_{x_2})^n G_\ell(\vec{x} - \vec{y}).
\end{aligned} \tag{A.25}$$

A.5 Convolution property of the polyharmonic Green functions

One has

$$\begin{aligned}
&\vec{\nabla}_y^{2(\ell+m)} \int \mathrm{d}^2 x \, G_\ell(\vec{y}-\vec{x}) G_m(\vec{z}-\vec{x}) \\
&= \int \mathrm{d}^2 x \, \vec{\nabla}_y^{2(\ell+m)} G_\ell(\vec{y}-\vec{x}) G_m(\vec{z}-\vec{x}) \\
&= \int \mathrm{d}^2 x \, \vec{\nabla}_y^{2\ell} \vec{\nabla}_x^{2m} G_\ell(\vec{y}-\vec{x}) G_m(\vec{z}-\vec{x}) \\
&= \int \mathrm{d}^2 x \, \vec{\nabla}_y^{2\ell} G_\ell \vec{y}-\vec{x} \vec{\nabla}_x^{2m} G_m(\vec{z}-\vec{x}) \\
&= \int \mathrm{d}^2 x \, \delta(\vec{y}-\vec{x}) \delta(\vec{z}-\vec{x}) = \delta(\vec{y}-\vec{z}).
\end{aligned} \tag{A.26}$$

Under standard regularity conditions, the homogeneous polyharmonic problem has only the trivial solution [30], and the above computation proves (2.63).

Appendix B

Polyharmonic functions for BMS in $3+1$

In Chapter 2, a realization of the BMS Lie algebra for both a massive and massless scalar field is constructed. The general eigenfunctions of the Laplace-Beltrami operator with an eigenvalue equal to 3 are

$$\chi_{\ell,m}(r,\theta,\varphi) = R_\ell Y_{\ell,m}(\theta,\varphi), \quad \ell \in \mathbb{N}, \quad m \in \mathbb{Z}, \quad |m| \leq \ell, \tag{B.1}$$

where $R_\ell = r = \omega = \sqrt{k_1^2 + k_2^2 + k_3^2}$ for the massless case and it depends on hypergeometric functions for the massive case (see Equation (2.37)). The functions $Y_{\ell,m}$ correspond to the spherical harmonics and the variables are the spherical coordinates in momentum space. From now on, let us focus on the simpler massless case.

The supertranslation operator is then defined in terms of the Fourier modes as

$$\mathcal{P}_{\ell,m} = \int \widetilde{dk}\, \bar{a}(\vec{k}) \chi_{\ell,m} a(\vec{k}), \qquad \widetilde{dk} = \frac{d^3k}{2\omega(2\pi)^3}, \tag{B.2}$$

and generates a transformation over the field ϕ that resembles the transformation in the (2+1)-dimensional case

$$\delta_{ST}\phi = \int d^3y \, \left[f_{\ell,m}(\vec{x}-\vec{y})\phi(t,\vec{y}) + g_{\ell,m}(\vec{x}-\vec{y})\pi(t,\vec{y}) \right], \tag{B.3}$$

where π is the field momentum and f and g are defined in terms of the supertranslation eigenfunctions as

$$f_{\ell,m}(\vec{x}) = 2 \int \widetilde{dk}\, \chi_{\ell,m} \omega \sin(\vec{k}\cdot\vec{x}), \tag{B.4}$$

$$g_{\ell,m}(\vec{x}) = 2 \int \widetilde{dk}\, \chi_{\ell,m} \cos(\vec{k}\cdot\vec{x}). \tag{B.5}$$

We can observe now several properties for these functions depending on the values of the subindices. First, we recall that the spherical harmonics satisfy the following parity property

$$Y_{\ell,m}(-\vec{r}) = (-1)^\ell Y_{\ell,m}(\vec{r}). \tag{B.6}$$

Then, based on the symmetry properties of the integrands, f is zero for ℓ even while g is zero for ℓ odd. This is fundamental to obtaining the expressions in terms of the polyharmonic Green functions.

For g, we have the following simple expression

$$g_{2\ell,m}(\vec{x}) = \frac{1}{(2\pi)^3}\int d^3k\, Y_{2\ell,m}(\theta,\varphi)\cos(\vec{k}\cdot\vec{x}). \tag{B.7}$$

To expand this expression we need to write the spherical harmonics in terms of the associated Legendre polynomials and expand them using the Rodrigues formula [1, Equation 22.11]

$$\chi_{\ell,m}(\theta,\varphi) = N_{\ell,m}P_\ell^m(\cos\theta)e^{im\varphi}, \tag{B.8}$$

$$N_{\ell,m} = (-1)^m\sqrt{\frac{(2\ell+1)(\ell-m)!}{4\pi(l+m)!}}, \tag{B.9}$$

$$P_\ell^m(x) = (-1)^m 2^\ell(1-x^2)^{m/2}\sum_{s=m}^{\ell}\frac{s!}{(s-m)!}x^{s-m}\binom{\ell}{s}\binom{\ell+s-1}{\ell} \tag{B.10}$$

$$= (-1)^m(1-x^2)^{m/2}\sum_{s=m}^{\ell}a_s(\ell,m)x^{s-m}, \tag{B.11}$$

where we have defined a function a_s to accommodate for the factorials and binomial terms of the sum. If we write now the expressions in momentum space using the spherical coordinates inverse transformation, we obtain

$$e^{im\varphi} = (k_1^2+k_2^2)^{-m/2}(k_1+ik_2)^m, \tag{B.12}$$

$$P_\ell^m(\vec{k}) = (-1)^m(k_1^2+k_2^2)^{m/2}\sum_{s=m}^{\ell}a_s(\ell,m)\omega^{-s}k_3^{s-m}. \tag{B.13}$$

A closer look at B.13 reveals that some terms in the sum performed to compute $\chi_{2\ell,m}$ are antisymmetric with respect to the change $\vec{k}\mapsto-\vec{k}$. This happens when s is an odd number, which results in the integral being zero. Therefore, only the terms with even values of s survive.

Finally, we observe that the terms depending on momentum variables can be obtained by derivation over the cosine function as

$$(k_1+ik_2)^{2m}k_3^{2s-2m}\cos(\vec{k}\cdot\vec{x}) = (-1)^s(\partial_{x_1}+i\partial_{x_2})^{2m}\partial_{x_3}^{2s-2m}\cos(\vec{k}\cdot\vec{x}). \tag{B.14}$$

Therefore, the expression in B.7 takes the following form, for $m\geq 0$,

$$g_{2\ell,2m}(\vec{x}) = N_{2\ell,2m}(\partial_{x_1}+i\partial_{x_2})^{2m}\sum_{s=m}^{\ell}a_{2s}\partial_{x_3}^{2s-2m}G_s(\vec{x}),\ m\leq\ell, \tag{B.15}$$

$$g_{2\ell,2m+1}(\vec{x}) = -N_{2\ell,2m+1}(\partial_{x_1}+i\partial_{x_2})^{2m+1}\sum_{s=m+1}^{\ell}a_{2s}\partial_{x_3}^{2s-2m-1}G_s(\vec{x}),\ m\leq\ell-1, \tag{B.16}$$

where we have introduced the 3-dimensional polyharmonic function

$$G_s(\vec{x}) = (-1)^s\frac{1}{(2\pi)^3}\int d^3k\,\omega^{-2s}\cos(\vec{k}\cdot\vec{x}), \tag{B.17}$$

that satisfies the equation $(\nabla^2)^sG_s(\vec{x}) = \delta^{(3)}(\vec{x})$. The expressions for $m<0$ can be obtained simply by taking the complex conjugate of the previous ones.

When compared to the $(2+1)$-dimensional polyharmonic functions, one has now a sum of partial derivatives with respect to x_3 that complicate the expressions without adding additional novelty to the analysis. For this reason, the analysis presented is conducted in the simpler $(2+1)$-dimensional case with the aim of understanding the non-local nature of the BMS transformations and their expression in position space.

Appendix C

Detailed computations for the non-linear realization of the BMS particle in Chapter 3

C.1 Computation of the term of the Maurer-Cartan form proportional to the P_0 generator

Using

$$e^X Y e^{-X} = e^{\mathrm{ad}_X} Y = Y + [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots \tag{C.1}$$

one has

$$\begin{aligned} \left[Y, e^{-X}\right] &= e^{-X}\left([X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots\right) \\ &\equiv e^{-X} K(X,Y), \end{aligned} \tag{C.2}$$

where

$$K(X,Y) = [X,Y] + \frac{1}{2!}[X,[X,Y]] + \frac{1}{3!}[X,[X,[X,Y]]] + \cdots, \tag{C.3}$$

which is linear in its second argument. We will call $K(X,Y)$ the K-action of X on Y.

By repeated use of (C.2) one arrives at

$$U^{-1} P_n U = P_n + K(-iL_1 u, P_n) + K(-iL_{-1} v, P_n) + K(-iL_1 u, K(-iL_{-1} v, P_n)). \tag{C.4}$$

The second and third terms in (C.4) are

$$\begin{aligned} K(-iL_1 u, P_n) &= \sum_{l=1}^{\infty} \frac{1}{l!}(-iu)^l [L_1, [L_1, \overset{l)}{\ldots}, [L_1, P_n] \ldots]] \\ &= \sum_{l=1}^{\infty} \frac{1}{l!}(-iu)^l (-i)^l (n-1)n \ldots (n+l-2) P_{n+l} \\ &= \sum_{l=1}^{\infty} \frac{1}{l!}(-1)^l u^l (n-1)n \ldots (n+l-2) P_{n+l}. \end{aligned} \tag{C.5}$$

$$\begin{aligned} K(-iL_{-1}v, P_n) &= \sum_{l=1}^{\infty} \frac{1}{l!}(-iv)^l [L_{-1}, [L_{-1}, \overset{l)}{\ldots}, [L_{-1}, P_n] \ldots]] \\ &= \sum_{l=1}^{\infty} \frac{1}{l!}(-iv)^l (-i)^l (n+1)n \ldots (n-l+2) P_{n-l} \\ &= \sum_{l=1}^{\infty} \frac{1}{l!}(-1)^l v^l (n+1)n \ldots (n-l+2) P_{n-l}. \end{aligned} \tag{C.6}$$

The fourth term in (C.4) is

$$\begin{aligned} K(-iL_1u, K(-iL_{-1}v, P_n)) &= K(-iL_1u, \sum_{l=1}^{\infty} \frac{1}{l!}(-1)^l v^l (n+1)n \ldots (n-l+2) P_{n-l}) \\ &= \sum_{l=1}^{\infty} \frac{1}{l!}(-1)^l v^l (n+1)n \ldots (n-l+2) K(-iL_1u, P_{n-l}), \end{aligned}$$

where we have used the linearity of K in its second argument. Finally

$$K(-iL_1u, K(-iL_{-1}v, P_n)) = \sum_{l=1}^{\infty} \frac{1}{l!}(-1)^l v^l (n+1)n \ldots (n-l+2)$$

$$\sum_{k=1}^{\infty} \frac{1}{k!}(-1)^k u^k (n-l-1)(n-l) \ldots (n-l+k-2) P_{n-l+k}. \tag{C.7}$$

As mentioned before, we are interested only in the P_0 terms in (C.4):

- P_n. Contributes only for $n = 0$, with P_0.
- $K(-iL_1u, P_n)$. Only the terms with $l = -n$ yield a P_0. Since $l \geq 1$, this means that there is no contribution for $n \geq 0$, while for $n = -m < 0$ one picks the term
$$\frac{1}{m!}(-1)^m u^m (-m-1)(-m) \ldots (-2) P_0 = (m+1) u^m P_0.$$
- $K(-iL_{-1}v, P_n)$. The P_0 contribution is obtained now for $l = n$ and since $l \geq 1$, there is only contribution if $n > 0$, which is
$$\frac{1}{n!}(-1)^n v^n (n+1)n \ldots 2\, P_0 = (-1)^n (n+1) v^n P_0.$$
- $K(-iL_1u, K(-iL_{-1}v, P_n))$. This term has multiple P_0 contributions, given by $l - k = n$, subjected to $l \geq 1, k \geq 1$. For given l one picks the $k = l - n$ term in the k series, but $k \geq 1$ implies that l must satisfy, besides $l \geq 1$, the constraint $l \geq 1 + n$. If $n \leq 0$ this just means $l \geq 1$, but, for $n > 0$, l is restricted by $l \geq 1 + n$.

  Selecting $k = l - n$ in (C.7) and restricting the series over l according to the above discussion one has, after re-arranging terms and cancelling some signs,
$$\sum_{l=n+1}^{\infty} \frac{l-n+1}{l!}(-1)^l (n+1)n \ldots (n-l+2) v^l u^{l-n} P_0 \tag{C.8}$$

for $n > 0$, and

$$\sum_{l=1}^{\infty} \frac{m+l+1}{l!}(m-1)m\dots(m+l-2)v^l u^{l+m} P_0 \tag{C.9}$$

for $n = -m \leq 0$.

Putting everything together, the coefficient of P_0 in (3.10) can be computed as follows, collecting the contributions proportional to the different $\mathrm{d}x^n$.

For $n = 0$, there is only contribution from the first and fourth terms in (C.4),

$$\mathrm{d}x^0(1 + \sum_{l=1}^{\infty} \frac{l+1}{l!}(0-1)(0)\dots(0+l-2)v^l u^l$$

Notice, however, that the above series finishes in fact after $l = 1$, so one gets

$$\mathrm{d}x^0(1-2uv). \tag{C.10}$$

For $n > 0$, only the third and fourth terms have a non-vanishing contribution, given by

$$\mathrm{d}x^n \left((-1)^n(n+1)v^n + \sum_{l=n+1}^{\infty} \frac{l-n+1}{l!}(-1)^l(n+1)n\dots(n-l+2)v^l u^{l-n} \right). \tag{C.11}$$

Actually, the product in the coefficients of the series always contains a zero except if $l = n+1$, and hence the above expression collapses to

$$\mathrm{d}x^n \left((-1)^n(n+1)v^n + 2(-1)^{n+1}uv^{n+1} \right) = \mathrm{d}x^n(-1)^n v^n(n+1-2uv). \tag{C.12}$$

Finally, for $n = -m < 0$, the contributions come from the second and fourth terms and are given by

$$\mathrm{d}x^{-m} \left((m+1)u^m + \sum_{l=1}^{\infty} \frac{m+l+1}{l!}(m-1)m\dots(m+l-2)v^l u^{l+m} \right). \tag{C.13}$$

The series is identically zero for $m = 1$, while for $m \geq 2$ it can be rewritten as

$$\mathrm{d}x^{-m} \left((m+1)u^m + \sum_{l=1}^{\infty} \frac{m+l+1}{l!} \frac{(l+m-2)!}{(m-2)!} v^l u^{l+m} \right). \tag{C.14}$$

This series can be summed (provided that $|uv| < 1$) and, after adding the $(m+1)u^m$ term, one gets

$$\mathrm{d}x^{-m} u^m \frac{m+1-2uv}{(1-uv)^m}. \tag{C.15}$$

Adding all the terms, the coefficient of P_0 in the Maurer-Cartan form is

$$\Omega_{P_0} = \mathrm{d}x^0(1-2uv) + \sum_{n=1}^{\infty} \mathrm{d}x^n(-1)^n v^n(n+1-2uv) + \mathrm{d}x^{-1}2u + \sum_{n=2}^{\infty} \mathrm{d}x^{-n}u^n \frac{n+1-2uv}{(1-uv)^n}. \tag{C.16}$$

The fact that the $\dot{x}^{-n}$ contribution is much more complex than that of $\dot{x}^n$, actually involving the series that has been mentioned, is due to the form of the last term in

(C.4), which in turn is a consequence of the ordering that we have selected for the two exponentials in U. For $n \geq 2$, the K-action of $-ivL_{-1}$ on P_n can only descend to P_{-1} (since the Poincaré part is BMS invariant), and then there is only one term in the K-action of $-iL_1u$ that returns to P_0. Instead, for $n \geq 2$, $K(-ivL_{-1}, P_{-n})$ produces terms P_k for any $k = -3, -4, \ldots$, and then, for each of them, there is a way to return to P_0 by the K-action of $-iL_1u$.

C.2 Quasi-invariance of the Lagrangian under gauge transformations

We consider first the full Lagrangian (3.13) and its variation under the full set of gauge transformations given by the first class constraints ϕ_m and $\bar{\phi}_m$, $m \geq 2$. In order to compute the transformation of the phase-space variables induced by $\phi_m, \bar{\phi}_m$ one needs

$$\begin{aligned}
\{x^1, \phi_m\} &= \{x^1, p_m + \mu p_{-1}^{-m} f_m^{\pm}(p_1 p_{-1})\} \\
&= \mu p_{-1}^{-m} (f_m^{\pm})'(p_1 p_{-1}) p_{-1} = \mp \mu p_{-1}^{-m+1} \frac{m+1}{2\sqrt{\mu^2 + p_1 p_{-1}}} f_{m-1}^{\pm}(p_1 p_{-1}) \\
&= \mu p_{-1}^{-m+1} \frac{m+1}{2P_0} f_{m-1}^{\pm}(p_1 p_{-1}) = -(m+1) \frac{P_{m-1}}{2P_0}, \qquad \text{(C.17)} \\
\{x^1, \bar{\phi}_m\} &= \{x^1, \bar{p}_m + \mu p_{-1}^{m} g_m^{\pm}(p_1 p_{-1})\} \\
&= \mu p_{-1}^{m} (g_m^{\pm})'(p_1 p_{-1}) p_{-1} = \pm \mu p_{-1}^{m+1} \frac{m-1}{2\sqrt{\mu^2 + p_1 p_{-1}}} g_{m+1}^{\pm}(p_1 p_{-1}) \\
&= -\mu p_{-1}^{m+1} \frac{m-1}{2P_0} g_{m+1}^{\pm}(p_1 p_{-1}) = (m-1) \frac{P_{-m-1}}{2P_0}. \qquad \text{(C.18)}
\end{aligned}$$

From these two it also follows that

$$\{x^{-1}, \phi_m\} = \{x^1, \bar{\phi}_m\}^* = (m-1) \frac{P_{m+1}}{2P_0}, \qquad \text{(C.19)}$$

$$\{x^{-1}, \bar{\phi}_m\} = \{x^1, \phi_m\}^* = -(m+1) \frac{P_{-m+1}}{2P_0}, \qquad \text{(C.20)}$$

which, together with $\{x^n, \phi_m\} = \delta_m^n$, $\{x^{-n}, \bar{\phi}_m\} = \delta_m^n$, $\{P_n, \phi_m\} = \{P_{-n}, \phi_m\} = \{P_n, \bar{\phi}_m\} = \{P_{-n}, \bar{\phi}_m\} = 0$, allow to compute the transformations of all the terms in the Lagrangian. For instance, if $G = \alpha_m \phi_m = \epsilon_m 2P_0 \phi_m$, one has

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\mathrm{d}}{\mathrm{d}\tau}(2P_0 \epsilon_m) P_m + \frac{\mathrm{d}}{\mathrm{d}\tau}(-(m+1)\epsilon_m P_{m-1}) p_1 + \frac{\mathrm{d}}{\mathrm{d}\tau}((m-1)\epsilon_m P_{m+1}) p_{-1} \\
&= \epsilon_m (2\dot{P}_0 p_m - (m+1)\dot{P}_{m-1} p_1 + (m-1)\dot{P}_{m+1} p_{-1}) \\
&\quad + \dot{\epsilon}_m (2P_0 P_m - (m+1) p_1 P_{m-1} + (m-1) P_{m+1} p_{-1}). \qquad \text{(C.21)}
\end{aligned}$$

This can be written as a total derivative, $\delta\mathcal{L} = \frac{\mathrm{d}}{\mathrm{d}\tau} F$, provided that

$$2P_0 \dot{P}_m - (m+1) P_{m-1} \dot{p}_1 + (m-1) P_{m+1} \dot{p}_{-1} = 0. \qquad \text{(C.22)}$$

Using[1]

$$\dot{P}_m = \frac{\mathrm{d}}{\mathrm{d}\tau}\left(-\frac{\mu}{p_{-1}^m} f_m^\pm\right) = m\frac{\mu}{p_{-1}^{m+1}}\dot{p}_{-1} f_m^\pm - \frac{\mu}{p_{-1}^m}(f_m^\pm)' \cdot (\dot{p}_1 p_{-1} + p_1 \dot{p}_{-1}) \tag{C.23}$$

$$= m\frac{\mu}{p_{-1}^{m+1}}\dot{p}_{-1} f_m^\pm - \frac{\mu}{p_{-1}^m}\left(\mp\frac{m+1}{2\sqrt{\mu^2 + p_1 p_{-1}}} f_{m-1}^\pm\right)(\dot{p}_1 p_{-1} + p_1 \dot{p}_{-1}), \tag{C.24}$$

the left-hand side of (C.22) is

$$\begin{aligned}
\mathrm{LHS(C.22)} &= \mp 2\sqrt{\mu^2 + p_1 p_{-1}}\, m\frac{\mu}{p_{-1}^{m+1}}\dot{p}_{-1} f_m^\pm - (m+1)\frac{\mu}{p_{-1}^m} f_{m-1}^\pm \cdot (\dot{p}_1 p_{-1} + p_1 \dot{p}_{-1}) \\
&\quad - (m+1) P_{m-1}\dot{p}_1 + (m-1) P_{m+1}\dot{p}_{-1} \\
&= \mp 2\sqrt{\mu^2 + p_1 p_{-1}}\, m\frac{\mu}{p_{-1}^{m+1}}\dot{p}_{-1} f_m^\pm - (m+1)\frac{\mu}{p_{-1}^m} f_{m-1}^\pm \cdot (\dot{p}_1 p_{-1} + p_1 \dot{p}_{-1}) \\
&\quad + (m+1)\frac{\mu}{p_{-1}^{m-1}} f_{m-1}^\pm \dot{p}_1 - (m-1)\frac{\mu}{p_{-1}^{m+1}} f_{m+1}^\pm \dot{p}_{-1}
\end{aligned} \tag{C.25}$$

The two terms containing $\dot{p}_1$ cancel each other, while the terms proportional to $\dot{p}_{-1}$ are

$$-\dot{p}_{-1}\frac{\mu}{p_{-1}^{m+1}}\left((m-1) f_{m+1}^\pm \pm 2m\sqrt{\mu^2 + p_1 p_{-1}}\, f_m^\pm + (m+1) p_1 p_{-1} f_{m-1}^\pm\right), \tag{C.26}$$

which is zero due to (3.56). This proves (C.22) and thus (3.79). Equation (3.81) is proved in a similar way.

We consider next the partially gauge fixed Lagrangian

$$\mathcal{L}_0 = \dot{x}^0 P_0 + \dot{x}^1 p_1 + \dot{x}^{-1} p_{-1}, \tag{C.27}$$

obtained from the full Lagrangian by setting $x^m = 0$ for $|m| \geq 2$, and which, as explained in the text, is just the standard Lagrangian for a massive Poincaré particle. This Lagrangian has the gauge symmetry transformation induced by the remaining first-class constraint ϕ_0, associated with reparametrization invariance, and the Poincaré invariance generated by p_0, p_1, p_{-1}, J, K_+ and K_-, but also the infinite set of symmetries given by the residual gauge transformations (3.113), (3.114), which we write in the condensed notation

$$\delta^m_{\mathrm{res}} x^1 = \epsilon^m (m+1)\frac{P_{m-1}}{2P_0} = A_m(p_1, p_{-1}), \tag{C.28}$$

$$\delta^m_{\mathrm{res}} x^{-1} = -\epsilon^m (m-1)\frac{P_{m+1}}{2P_0} = B_m(p_1, p_{-1}), \tag{C.29}$$

with all the other variables x^0, p_0, p_1 and p_{-1} invariant. One has

$$\delta_{\mathrm{res}}\mathcal{L}_0 = \delta_{\mathrm{res}}(\dot{x}^1 p_1 + \dot{x}^{-1} p_{-1}) = p_1\frac{\mathrm{d}}{\mathrm{d}\tau}A_m + p_{-1}\frac{\mathrm{d}}{\mathrm{d}\tau}B_m = C_m \dot{p}_1 + D_m \dot{p}_{-1}, \tag{C.30}$$

[1] We do not display the dependence of $f_m^\pm$ on $p_1 p_{-1}$.

where

$$C_m = p_1 \frac{\partial A_m}{\partial p_1} + p_{-1} \frac{\partial B_m}{\partial p_1}, \tag{C.31}$$

$$D_m = p_1 \frac{\partial A_m}{\partial p_{-1}} + p_{-1} \frac{\partial B_m}{\partial p_{-1}}. \tag{C.32}$$

The quasi-invariance of the Lagrangian, that is, the existence of a function F_0 such that $\delta_{\rm res}\mathcal{L}_0 = \frac{\mathrm{d}}{\mathrm{d}\tau} F_0$, is equivalent to

$$\frac{\partial C_m}{\partial p_{-1}} = \frac{\partial D_m}{\partial p_1}, \tag{C.33}$$

which boils down to

$$\frac{\partial A_m}{\partial p_{-1}} = \frac{\partial B_m}{\partial p_1}, \tag{C.34}$$

which in turn can be proved using the expressions of P_n, P_0 in terms of p_1 and p_{-1} and the properties of the functions $f_n^{\pm}$.

C.3 Invariance of the massless limit constraints under superrotations

Using $\{p_n, K_+^m\} = (n-m)p_{n+m}$ one has that the variation of φ_n under a superrotation induced by K_+^m is

$$\begin{aligned}
\{\varphi_n, K_+^m\} &= \{p_n \pm (\mp 1)^n p_{-1}^{-n} (\sqrt{p_1 p_{-1}})^{n+1}, K_+^m\} \\
&= (n-m)p_{n+m} \pm (\mp)^n (-n p_{-1}^{-n-1} (-1-m) p_{-1+m}) (\sqrt{p_1 p_{-1}})^{n+1} \\
&\pm (\mp)^n p_{-1}^{-n} (n+1) (\sqrt{p_1 p_{-1}})^n \frac{1}{2\sqrt{p_1 p_{-1}}} \left(p_1 (-1-m) p_{-1+m} + p_{-1} (1-m) p_{1+m} \right).
\end{aligned}$$

Since we only have to deal with the case $m \geq 2$, one has that $n+m$, $-1+m$ and $1+m$ are all positive. Using φ_{n+m}, φ_{-1+m} and φ_{1+m} one can express p_{n+m}, p_{-1+m} and p_{1+m} in terms of p_1 and p_{-1}, and one obtains, after re-arranging terms and extracting the common dependency of all the terms in p_1 and p_{-1},

$$\begin{aligned}
\{\varphi_n, K_+^m\} &\simeq (\mp)^{n+m+1} p_{-1}^{-n-m} (\sqrt{p_1 p_{-1}})^{n+m+1} \\
&\cdot \left(n - m - n(1+m) + (1+m)\frac{n+1}{2} - (1-m)\frac{n+1}{2} \right) = 0.
\end{aligned}$$

Similarly, from $\{p_n, K_-^m\} = -(m+n)p_{n-m}$,

$$\begin{aligned}
\{\varphi_n, K_-^m\} &= \{p_n \pm (\mp 1)^n p_{-1}^{-n} (\sqrt{p_1 p_{-1}})^{n+1}, K_-^m\} \\
&= -(n+m)p_{n-m} \pm (\mp)^n (-n p_{-1}^{-n-1} (-(m-1) p_{-1-m})) (\sqrt{p_1 p_{-1}})^{n+1} \\
&\pm (\mp)^n p_{-1}^{-n} (n+1) (\sqrt{p_1 p_{-1}})^n \frac{1}{2\sqrt{p_1 p_{-1}}} \left(p_1 (-(m-1) p_{-1-m}) + p_{-1} (-(m+1) p_{1-m}) \right).
\end{aligned}$$

Again, since we must only consider $m \geq 2$, both p_{-1-m} and p_{1-m} are BMS momenta with negative indexes, and can be expressed in terms of p_1 and p_{-1} using $\bar{\varphi}_{1+m}$ and $\bar{\varphi}_{m-1}$, respectively. For $n - m \geq 0$, one can use φ_{n-m} for p_{n-m}, while for $n - m < 0$ p_{n-m} has negative index and can be expressed in terms of p_1 and p_{-1} using $\bar{\varphi}_{m-n}$. It turns out that in both cases the term obtained from p_{n-m} is the same, and one has

$$\{\varphi_n, K_-^m\} \simeq (\mp)^{n+m+1} p_1^{-m} p_{-1}^{-n} (\sqrt{p_1 p_{-1}})^{n+m+1} \cdot \left(-(n+m) - n(m-1) + (m-1)\frac{n+1}{2} + (m+1)\frac{n+1}{2} \right) = 0.$$

It should be noticed that it is this case, the variation of a positive index constraint under a negative index superrotation, the one that breaks the invariance of the theory under superrotations in the massive case.

Due to the real character of the Poisson bracket, one will also have

$$\{\bar{\varphi}_n, K_-^m\}^* = \{\varphi_n, K_+^m\} \simeq 0,$$
$$\{\bar{\varphi}_n, K_+^m\}^* = \{\varphi_n, K_-^m\} \simeq 0,$$

and thus all the constraints are weakly invariant under all the superrotations.

C.4 Casimirs of the Lorentz and Poincaré groups in BMS space

The action of the Lorentz generators $K_\pm$, J on the p_n, $n \in \mathbb{Z}$, provided by Poisson brackets,

$$\delta_J p_n = \{p_n, J\} = -n p_n, \tag{C.35}$$
$$\delta_+ p_n = \{p_n, K_+\} = -(1-n) p_{n+1}, \tag{C.36}$$
$$\delta_- p_n = \{p_n, K_-\} = -(1+n) p_{n-1}, \tag{C.37}$$

leads to the definition of infinite dimensional matrices acting on vectors of the form

$$(\ldots, p_{-2}, p_{-1}, p_0, p_1, p_2, \ldots)$$

which implement this action, given by

$$J_{nm} = -n\delta_{nm}, \tag{C.38}$$
$$(K_+)_{nm} = -(1-m)\delta_{n,m+1}, \tag{C.39}$$
$$(K_-)_{nm} = -(1+m)\delta_{n,m-1}. \tag{C.40}$$

Using the structure constants of the $SO(2,1)$ algebra in the J, K_+, K_- basis one can construct the Killing form of the Lie algebra, and from that the quadratic Casimir, which is given by

$$C_2^L = \frac{1}{2}J^2 + \frac{1}{4}K_+K_- + \frac{1}{4}K_-K_+. \tag{C.41}$$

Using the above matrices one immediately obtains

$$\left(\frac{1}{2}J^2 + \frac{1}{4}K_+K_- + \frac{1}{4}K_-K_+\right)_{nm} = 1 \cdot \delta_{nm} \tag{C.42}$$

and hence this corresponds to an adjoint representation on the space of BMS momenta.

Similarly, one can consider the action on the space of BMS coordinates x^n, $n \in \mathbb{Z}$,

$$\delta_J x^n = \{x^n, J\} = n x^n, \tag{C.43}$$
$$\delta_+ x^n = \{x^n, K_+\} = (2-n) x^{n-1}, \tag{C.44}$$
$$\delta_- x^n = \{x^n, K_-\} = (2+n) x^{n+1}, \tag{C.45}$$

which leads to the matrices

$$\tilde{J}_{nm} = n\delta_{nm}, \tag{C.46}$$
$$(\tilde{K}_+)_{nm} = (2-m)\delta_{n,m-1}, \tag{C.47}$$
$$(\tilde{K}_-)_{nm} = (2+m)\delta_{n,m+1}. \tag{C.48}$$

Again

$$\left(\frac{1}{2}\tilde{J}^2 + \frac{1}{4}\tilde{K}_+\tilde{K}_- + \frac{1}{4}\tilde{K}_-\tilde{K}_+\right)_{nm} = 1 \cdot \delta_{nm}, \tag{C.49}$$

which shows that it also corresponds to an adjoint representation on the space of coordinates.

With respect to the Poincaré group, the quadratic Casimir in $2+1$ in our coordinates is

$$C_2^P = p_0^2 - p_1 p_{-1}, \tag{C.50}$$

which, for our system and taking into account the constraint ϕ_0, takes value $-\mu^2$. One may wonder if it is possible to obtain a quadratic Casimir involving the higher BMS momenta, of the form

$$C_2 = A_{mn} p_m p_n, \quad A_{mn} = A_{nm}. \tag{C.51}$$

Imposing the invariance under J one gets

$$\delta_J C_2 = -A_{mn} p_m p_n (m+n) = 0 \tag{C.52}$$

which implies that the only A_{mn} that can be different from zero are those corresponding to $m = -n$. Thus

$$A_{mn} = A_n \delta_{m,-n}, \tag{C.53}$$

with $A_n = A_{-n}$ due to the symmetry of A_{mn}. Computing now the variation under K_+ and using this form for A_{mn} one gets

$$\delta_+ C_2 = -p_{m+1} p_{-m}((1-m)A_m + (2+m)A_{m+1}) = 0. \tag{C.54}$$

In order to equal to zero the coefficients of this sum over m one must notice that the terms corresponding to $m = n$ and $m = -1-n$ yield the same product $p_{m+1}p_{-m}$. Taking this into account and using $A_m = A_{-m}$ one gets the first order recurrence relation

$$(1-m)A_m + (2+m)A_{m+1} = 0, \quad m = 0, 1, 2, \ldots. \tag{C.55}$$

The invariance under K_- does not add any new condition. For $m = 0$, (C.55) yields

$$A_0 + 2A_1 = 0,$$

from which $A_1 = -\frac{1}{2}A_0$ and hence also $A_{-1} = -\frac{1}{2}A_0$. For $m = 1$, however, the relation is

$$0 \cdot A_1 + 3A_2 = 0,$$

from which $A_2 = 0$ and thus $A_{-2} = 0$. From this point, using the recurrence for higher values of m leads to $A_m = A_{-m} = 0$ for $m = 2, 3, \ldots$. The final result is then that the only quadratic Casimir of the Poincaré group in BMS space is, up to a global constant, the standard one, given by (C.50).

The transformation of the coordinates x^m under J, K_+ and K_- is given by

$$\delta_J x^m = \{x^m, \sum_{n\in\mathbb{Z}} n x^n p_n\} = m x^m, \tag{C.56}$$

$$\delta_+ x^m = \{x^m, \sum_{n\in\mathbb{Z}} (1-n) x^n p_{n+1}\} = (2-m) x^{m-1}, \tag{C.57}$$

$$\delta_- x^m = \{x^m, \sum_{n\in\mathbb{Z}} (1+n) x^n p_{n-1}\} = (2+m) x^{m+1},. \tag{C.58}$$

The space of Poincaré coordinates $\{x^0, x^{\pm 1}\}$ is not invariant under $\delta_\pm$, since

$$\delta_+ x^{-1} = 3x^{-2}, \quad \delta_- x^1 = 3x^2$$

unless, as done in Section 3.6, we gauge away the BMS coordinates. In particular, one has that $-(x^0)^2 + x^1 x^{-1}$ is not invariant since, for instance

$$\delta_+(-(x^0)^2 + x^1 x^{-1}) = 12 x^1 x^{-2}.$$

We can repeat the construction above to try to obtain a quadratic function of the Poincaré and additional BMS coordinates which is invariant under Lorentz transformations. If we write

$$D_2 = B_{mn} x^m x^n \tag{C.59}$$

invariance under J imposes, as was the case for the quadratic Casimir of the Poincaré group, $B_{mn} = B_m \delta_{m,-n}$, with $B_m = B_{-m}$. Imposing now invariance under K_+ leads to the equations

$$(2-m) B_m + (1+m) B_{m-1} = 0, \quad m = 1, 2, \ldots \tag{C.60}$$

For $m = 2$ one obtains then $B_1 = 0$ (and hence $B_{-1} = 0$) and then $m = 1$ leads to $B_0 = 0$. This means that no quadratic invariant can be constructed out of the Poincaré coordinates. Solving the recurrence for $m \geq 3$ one gets

$$B_m = \frac{1}{6} m(m^2-1) B_2, \quad m \geq 2. \tag{C.61}$$

Choosing $B_2 = 3$ one arrives at the conclusion that the only Lorentz invariant quadratic function of the coordinates is, up to an arbitrary overall factor,

$$D_2 = \sum_{m\geq 2} m(m^2-1) x^m x^{-m}. \tag{C.62}$$

Appendix D

Computations associated with the symmetry generators for a scalar field

Let's consider a scalar field theory with Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{1}{2}m^2\phi^2, \tag{D.1}$$

where m is the mass, we are considering signature $\eta = (-, +, \dots, +)$ and d-dimensional spacetime.

D.1 Conserved charges for a scalar field

According to Noether's theorem, every continuous symmetry of the Lagrangian gives rise to a conserved current. In terms of infinitesimal transformations, $\delta\phi$ is a symmetry of the theory if the Lagrangian changes by a total derivative

$$\delta\mathcal{L} = \partial_\mu\mathcal{J}^\mu, \tag{D.2}$$

for some function $J^\mu(\phi)$.

If we transform the Lagrangian now under a general transformation $\delta\phi$,

$$\delta\mathcal{L} = \left[\frac{\partial\mathcal{L}}{\partial\phi} - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right)\right]\delta\phi + \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right), \tag{D.3}$$

and the first term in the RHS vanishes when the equations of motion are satisfied while the second term is a total derivative, thus

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi\right) = \partial_\mu\mathcal{J}^\mu. \tag{D.4}$$

Therefore, we get a conserved current by writing

$$J^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi - \mathcal{J}^\mu. \tag{D.5}$$

D.2 Derivation of global charges expressions in momentum space

Throughout this section, we will write $a(\vec{k},t)$ as a_k for brevity and clarity in the expressions.

For the case of the four-momentum P^μ we have

$$P^\mu = \int dx\, T_c^{0\mu} = \int dx \left(-\dot{\phi}\partial^\mu\phi - \frac{1}{2}\eta^{\mu 0}\partial^\rho\phi\partial_\rho\phi\right), \tag{D.6}$$

that can be divided into space and time components for simplicity. For $\mu = j$,

$$\begin{aligned} P^j &= -\int dx\, \dot{\phi}\partial^j\phi = \int dx\tilde{d}q\tilde{d}k\, \omega(k)q^j i^2 \left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right) \\ &= -\int dx\tilde{d}q\tilde{d}k\, \omega(k)q^j \left(a_k a_q e^{i(\vec{k}+\vec{q})\cdot\vec{x}} - a_k^* a_q e^{-i(\vec{k}-\vec{q})\cdot\vec{x}} - a_k a_q^* e^{i(\vec{k}-\vec{q})\cdot\vec{x}} + a_k^* a_q^* e^{-i(\vec{k}+\vec{q})\cdot\vec{x}}\right) \\ &= \int \tilde{d}k\, \frac{1}{2}k^j \left(a_k a_{-k} + a_k^* a_k + a_k a_k^* + a_k^* a_{-k}^*\right) = \int \tilde{d}k\, k^j a_k^* a_k \end{aligned} \tag{D.7}$$

where in the third line an integration over q was conducted and in the last step, antisymmetric properties from the first and fourth terms in k were used to drop them and obtain the final expression. For $\mu = 0$, following the same steps as before,

$$\begin{aligned} P^0 &= \int dx \left(\dot{\phi}^2 + \frac{1}{2}(-\dot{\phi}^2 + (\vec{\nabla}\phi)^2)\right) = \frac{1}{2}\int dx \left(\dot{\phi}^2 + (\vec{\nabla}\phi)^2\right) \\ &= \frac{1}{2}\int dx\tilde{d}k\tilde{d}q(-\omega(k)\omega(q) - k^j q_j) \left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right) \\ &= -\frac{1}{2}\int \tilde{k}\frac{1}{2\omega}\omega^2(-a_k^* a_k - a_k a_k^* - a_k^* a_k - a_k a_k^*) = \int \tilde{d}k\, \omega a_k^* a_k \end{aligned} \tag{D.8}$$

For the case of boosts $M^{\mu\nu}$, the expressions get larger

$$M^{\mu\nu} = \int dx\, (I^0)^{\mu\nu} = \int dx \left(x^\mu T_c^{0\nu} - x^\nu T_c^{0\mu}\right) \tag{D.9}$$

Again, we consider the two components separately

$$M^{0j} = \int dx\,(tT_c^{0j} - x^j T_c^{00}) = \int dx\,\left[t(-\dot\phi\partial^j\phi) - x^j\left(\dot\phi^2 + \frac{1}{2}(-\dot\phi^2 + (\vec\nabla\phi)^2)\right)\right] \tag{D.10}$$

$$= -t\int dx\tilde{d}k\tilde{d}q\omega(k)q^j\left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$+\int dx\tilde{d}k\tilde{d}q\frac{1}{2}x^j(\omega(k)\omega(q) + k^i q_i)\left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$= t\int \tilde{d}k\frac{1}{2}k^j(a_k a_{-k} + a_k^* a_k + a_k a_k^* + a_k^* a_{-k}^*)$$

$$-i\int dx\tilde{d}k\tilde{d}q\frac{1}{2}(\omega(k)\omega(q) + k^i q_i)\left(a_k\partial^j e^{i\vec{k}\cdot\vec{x}} + a_k^*\partial^j e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$= tP^j + i\int dx\tilde{d}k\tilde{d}q\frac{1}{2}(\omega(k)\omega(q) + k^i q_i)\left((\partial^j a_k)e^{i\vec{k}\cdot\vec{x}} + (\partial^j a_k^*)e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$+i\int dx\tilde{d}k\tilde{d}q\frac{1}{2}\left(\delta^{ij} - \frac{k^i k^j}{\omega^2}\right)q_i\left(a_k e^{i\vec{k}\cdot\vec{x}} + a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$= tP^j - i\int \tilde{d}k\frac{1}{2}\omega((\partial^j a_k)a_k^* - (\partial^j a_k^*)a_k) = tP^j - i\int \tilde{d}k\, a_k^*\omega\partial^j a_k$$

where after the third equal sign we have applied the change rule to get rid of the x^j term in the integrand and in the fourth equal sign we have integrated by parts to move the derivative around. The other component is

$$M^{ij} = \int dx\,\left(x^i T_c^{0j} - x^j T_c^{0i}\right) = \int dx\,\left(x^i(-\dot\phi\partial^j\phi) - x^j(-\dot\phi\partial^i\phi)\right) \tag{D.11}$$

$$= -\int dx\tilde{d}k\tilde{d}q\left[x^i\omega(k)q^j - x^j\omega(k)q^i\right]\left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

where the integral can be divided into two identical terms, but for the ordering in the indices. Thus, the expression for one of the terms can be computed and subtracted altering the order of appearance of i and j to obtain the boost final expression. The term is the following

$$A^{ij} = -\int dx\tilde{d}k\tilde{d}q\, x^i\omega(k)q^j\left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right) \tag{D.12}$$

$$= i\int dx\tilde{d}k\tilde{d}q\,\omega(k)q^j\left(a_k\partial^i e^{i\vec{k}\cdot\vec{x}} + a_k^*\partial^i e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$= -i\int dx\tilde{d}k\tilde{d}q\,\omega(k)q^j\left((\partial^i a_k)e^{i\vec{k}\cdot\vec{x}} + (\partial^i a_k^*)e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)$$

$$= i\int \tilde{d}k\,\frac{1}{2}k^j((\partial^i a_k)a_{-k} + (\partial^i a_k)a_k^* - (\partial^i a_k^*)a_k - (\partial^i a_k^*)a_{-k}^*)$$

$$= -\frac{i}{4}\int \tilde{d}k\,\left(\delta^{ij} - \frac{k^i k^j}{\omega^2}\right)a_k a_{-k} + i\int \tilde{d}k\, a_k^* k^j\partial^i a_k$$

$$+\frac{i}{2}\int \tilde{d}k\,\left(\delta^{ij} - \frac{k^i k^j}{\omega^2}\right)a_k^* a_k + \frac{i}{4}\int \tilde{d}k\,\left(\delta^{ij} - \frac{k^i k^j}{\omega^2}\right)a_k^* a_{-k}^*$$

Hence, the final boost expression can be obtained doing $A^{ij} - A^{ji}$, to keep only the non-symmetric term from the previous expression,

$$M^{ij} = -i \int \tilde{dk}\, a_k^* (k^i \partial^j - k^j \partial^i) a_k \tag{D.13}$$

For the dilatation, D, the expression is

$$\begin{aligned} D &= \int dx \left[x^\nu \left(-\dot\phi \partial_\nu \phi - \frac{1}{2}\eta^0_\nu \partial^\rho \phi \partial_\rho \phi \right) - \frac{d-2}{2}\dot\phi\phi \right] \\ &= \int dx \left[-t\dot\phi^2 - x^j \dot\phi \partial_j \phi - \frac{1}{2} t(-\dot\phi^2 + (\vec\nabla\phi)^2) - \frac{d-2}{2}\dot\phi\phi \right] \\ &= \int dx \left[-\frac{1}{2} t \left(\dot\phi^2 + (\nabla\phi)^2\right) - x^j \dot\phi \partial_j \phi - \frac{d-2}{2}\dot\phi\phi \right] \end{aligned} \tag{D.14}$$

The third term for the previous expression is the easiest to compute,

$$\begin{aligned} -\frac{d-2}{2}\int dx\, \dot\phi\phi &= i\frac{d-2}{2}\int dx\tilde{dk}\tilde{dq}\, \omega(k) \left(a_k e^{i\vec k\cdot\vec x} - a_k^* e^{-i\vec k\cdot\vec x}\right)\left(a_q e^{i\vec q\cdot\vec x} + a_q^* e^{-i\vec q\cdot\vec x}\right) \\ &= \frac{i}{4}(d-2)\int \tilde{dk}\, \left(a_k a_{-k} + a_k a_k^* - a_k^* a_k - a_k^* a_{-k}^*\right) \\ &= \frac{i}{4}(d-2)\int \tilde{dk}\, \left(a_k a_{-k} - a_k^* a_{-k}^*\right) \end{aligned} \tag{D.15}$$

The second term can be computed using the expressions found in (D.12) by contracting the indices with η_{ij}, that in a d-dimensional spacetime gives

$$\begin{aligned} -\int dx\, x^j \dot\phi \partial_j \phi = &-\frac{i}{4}(d-2)\int \tilde{dk}\, a_k a_{-k} + \int \tilde{dk}\, k_j a_k^* \partial^j a_k \\ &+ \frac{i}{2}(d-2)\int \tilde{dk}\, a_k^* a_- k + \frac{i}{4}(d-2)\int \tilde{dk}\, a_k^* a_{-k}^* \end{aligned} \tag{D.16}$$

Finally, the explicitly time-dependent term is just the expression for P^0 multiplied by time

$$-\int dx\, \frac{1}{2} t \left(\dot\phi^2 + (\nabla\phi)^2\right) = -tP^0 \tag{D.17}$$

Hence, the final expression for D, after canceling most of the terms, is

$$D = -tP^0 + i\int \tilde{dk}\, a_k^* \left(k_j \partial^j + \frac{i}{2}(d-2)\right) a_k \tag{D.18}$$

Lastly, for special conformal transformations K^μ, from equation (4.21), we have a more complicated expression

$$K^\mu = \int dx \left[(\underbrace{x^2\eta^{\mu\sigma}}_{(1a)} - \underbrace{2x^\mu x^\sigma}_{(1b)}) T^0_{c\,\sigma} + (d-2)\left(\underbrace{x^\mu \dot\phi\phi}_{(2a)} + \underbrace{\frac{1}{2}\eta^{\mu 0}\phi^2}_{(2b)} \right) \right] \tag{D.19}$$

The (2b) term is easily computed and gives

$$(d-2)\frac{1}{2}\int dx\, \eta^{\mu 0}\phi^2 = \frac{d-2}{4}\int \tilde{dk}\, \frac{1}{\omega}\eta^{\mu 0}\left(a_k a_{-k} + 2a_k^* a_k + a_k^* a_{-k}^*\right) \tag{D.20}$$

For the (2a) term, we must consider two possible values for the index μ

$$\int dx\,(d-2)x^\mu\phi\dot{\phi} = -i\int dx\tilde{dk}\tilde{dq}\,(d-2)x^\mu\omega(k)\left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} + a_q^* e^{-i\vec{q}\cdot\vec{x}}\right) \tag{D.21}$$

Thus, when $\mu = 0$ we don't need to integrate by parts to get rid of any integrable variable and the integrals can be done directly to obtain

$$-\frac{i}{2}(d-2)t\int \tilde{dk}\,\left(a_k a_{-k} - a_k^* a_{-k}^*\right) \tag{D.22}$$

While for $\mu = j$, we must get rid of x^j before integrating, and more steps must be made

$$\int dx\,(d-2)x^j\phi\dot{\phi} = -(d-2)\int dx\tilde{dk}\tilde{dq}\,\omega(k)\left(a_k\partial^j e^{i\vec{k}\cdot\vec{x}} + a_k^*\partial^j e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} + a_q^* e^{-i\vec{q}\cdot\vec{x}}\right) \tag{D.23}$$

$$\begin{aligned}
&= (d-2)\int dx\tilde{dk}\tilde{dq}\,\omega(k)\Big((\partial^j a_k)a_q e^{i(\vec{k}+\vec{q})\cdot\vec{x}} + (\partial^j a_k)a_q^* e^{i(\vec{k}-\vec{q})\cdot\vec{x}} \\
&\qquad + (\partial^j a_k^*)a_q^* e^{-i(\vec{k}-\vec{q})\cdot\vec{x}} + (\partial^j a_k^*)a_q e^{-i(\vec{k}+\vec{q})\cdot\vec{x}}\Big) \\
&= \frac{d-2}{2}\int \tilde{dk}\,\Big((\partial^j a_k)a_{-k} + (\partial^j a_k)a_k^* + (\partial^j a_k^*)a_k^* + (\partial^j a_k^*)a_{-k}\Big) \\
&= \frac{d-2}{2}\int \tilde{dk}\,\Big((\partial^j a_k)a_{-k} + (\partial^j a_k^*)a_{-k}^*\Big) + \frac{d-2}{2}\int \tilde{dk}\,\frac{k^j}{\omega^2}a_k^* a_k
\end{aligned}$$

where we have integrated by parts in the last step (taking into account the ω present in the integral measure $\tilde{dk}$). The (1a) term must be separated again in time and space components, so that for $\mu = 0$, one obtains

$$\int dx\,(-t^2 + x^j x_j)T_c^{00} = \int dx\,(-t^2 + x^j x_j)\frac{1}{2}(\dot{\phi}^2 + (\vec{\nabla}\phi)^2) \tag{D.24}$$

where the term in t^2 can be integrated using previous expression to get $-t^2 P^0$. The remaining part can be computed as follows

$$\frac{1}{2}\int dx\,x^j x_j(\dot{\phi}^2 + (\vec{\nabla}\phi)^2) \tag{D.25}$$

$$\begin{aligned}
&= -\frac{1}{2}\int dx\tilde{dk}\tilde{dq}\,x^j x_j(\omega(k)\omega(q) + k^i q_i)\left(a_k e^{i\vec{k}\cdot\vec{x}} - a_k^* e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right) \\
&= \frac{1}{2}\int dx\tilde{dk}\tilde{dq}\,(\omega(k)\omega(q) + k^i q_i)\left(a_k\partial^j\partial_j e^{i\vec{k}\cdot\vec{x}} - a_k^*\partial^j\partial_j e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_q e^{i\vec{q}\cdot\vec{x}} - a_q^* e^{-i\vec{q}\cdot\vec{x}}\right)
\end{aligned}$$

where the x^j components in the integrand have been expressed as derivatives with respect to k^j of the exponentials. Integrating by parts, these derivatives can be moved around to the other terms depending on k^j also. After the integration, the expression is

$$\frac{1}{2}\int dx\tilde{d}k\tilde{d}q\,\left(\omega(k)\omega(q)+k^iq_i\right)\left(\partial^j\partial_ja_ke^{i\vec{k}\cdot\vec{x}}-\partial^j\partial_ja_k^*e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_qe^{i\vec{q}\cdot\vec{x}}-a_q^*e^{-i\vec{q}\cdot\vec{x}}\right) \tag{D.26}$$
$$+\int dx\tilde{d}k\tilde{d}q\,\left(\delta^{ij}-\frac{k^ik^j}{\omega^2}\right)q_i\left(\partial_ja_ke^{i\vec{k}\cdot\vec{x}}-\partial_ja_k^*e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_qe^{i\vec{q}\cdot\vec{x}}-a_q^*e^{-i\vec{q}\cdot\vec{x}}\right)$$
$$-\frac{d-2}{2}\int dx\tilde{d}k\tilde{d}q\,\frac{k^iq_i}{\omega^2}\left(a_ke^{i\vec{k}\cdot\vec{x}}-a_k^*e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_qe^{i\vec{q}\cdot\vec{x}}-a_q^*e^{-i\vec{q}\cdot\vec{x}}\right)$$

that can be integrated over x and $\vec{q}$, noting that the expression in the second line cancels after integrating a Dirac delta and substituting all $\vec{q}$ by $\vec{k}$. What remains after the two integrations is

$$\frac{1}{2}\int\tilde{d}k\,\omega\left(-(\partial^j\partial_ja_k)a_k^*-(\partial^j\partial_ja_k^*)a_k\right)-\frac{d-2}{4}\int\tilde{d}k\,\frac{1}{\omega}\left(-a_ka_{-k}-2a_k^*a_k-a_k^*a_{-k}^*\right) \tag{D.27}$$
$$=-\int\tilde{d}k\,a_k^*\omega\partial^j\partial_ja_k+\frac{d-2}{2}\int\tilde{d}k\,\frac{1}{\omega}a_k^*a_k+\frac{d-2}{4}\int\tilde{d}k\,\frac{1}{\omega}\left(a_ka_{-k}+a_k^*a_{-k}^*\right)$$

For $\mu=j$, the integrand is the following

$$\int dx(-t^2+x^ix_i)T_c^{0j}=\int dx\,(-t^2+x^ix_i)(-\dot{\phi}\partial^j\phi) \tag{D.28}$$

where the time component can be integrated again using information from previous charges to obtain $-t^2P^j$, while the second one can be expressed in the momentum space as

$$\int dx\,x^ix_i(-\dot{\phi}\partial^j\phi) \tag{D.29}$$
$$=-\int dx\tilde{d}k\tilde{d}q\,x^ix_i\omega(k)q^j\left(a_ke^{i\vec{k}\cdot\vec{x}}-a_k^*e^{-i\vec{k}\cdot\vec{x}}\right)\left(a_qe^{i\vec{q}\cdot\vec{x}}-a_q^*e^{-i\vec{q}\cdot\vec{x}}\right)$$
$$=\int dx\tilde{d}k\tilde{d}q\,\omega(k)q^j\left(a_k\partial^i\partial_ie^{i\vec{k}\cdot\vec{x}}-a_k^*\partial^i\partial_ie^{-i\vec{k}\cdot\vec{x}}\right)\left(a_qe^{i\vec{q}\cdot\vec{x}}-a_q^*e^{-i\vec{q}\cdot\vec{x}}\right)$$

where we have used again the expression of x^i as derivatives of exponentials with respect to k^i. Following the same process as in the case of $\mu=0$, one obtains the following set of expressions

$$-\frac{1}{2}\int\tilde{d}k\,k^j\left((\partial^i\partial_ia_k)a_{-k}+(\partial^i\partial_ia_k)a_k^*+(\partial^i\partial_ia_k^*)a_k+(\partial^i\partial_ia_k^*)a_{-k}\right) \tag{D.30}$$
$$=-\int\tilde{d}k\,a_k^*k^j\partial^i\partial_ia_k+\frac{1}{2}\int\tilde{d}k\,\left(\delta^{ij}-\frac{k^ik^j}{\omega^2}\right)\left[a_{-k}\partial_ia_k+a_{-k}^*\partial_ia_k^*\right]$$
$$-\frac{d-2}{4}\int\tilde{d}k\,\frac{k^j}{\omega^2}\left[a_ka_{-k}+a_k^*a_{-k}^*\right]$$
$$-\int\tilde{d}k\,\left(\delta^{ij}-\frac{k^ik^j}{\omega^2}\right)\partial_ia_ka_k^*+\frac{d-2}{2}\int\tilde{d}k\,\frac{k^j}{\omega^2}a_k^*a_k$$

where the term in the second-to-last line cancels because it is antisymmetric in $\vec{k}$. Finally, the (1b) term can be computed using the results obtained for the Dilatation and the Boost. Thus, the temporal component of this term is the following

$$\int (-2tx^\sigma T^0_{c\,\sigma}) = -\int dx\, 2t \left[-t\frac{1}{2}\left(\dot\phi^2 + (\vec\nabla\phi)^2\right) + x^j(-\dot\phi\partial_j\phi)\right] \tag{D.31}$$

that can be computed using the expression for the Momentum charge and a term that appeared in equation (D.16) when computing the Dilatation. The result is

$$2t^2P^0 + \frac{d-2}{2}it\int \tilde{dk}\,(a_k a_{-k} - 2a_k^* a_k - a_k^* a_{-k}^*) - 2it\int \tilde{dk}\, k_j a_k^* \partial^j a_k \tag{D.32}$$

The term for $\mu = j$ gives the following expression

$$\int dx\, \left(-2x^j x^\sigma T^0_{c\,\sigma}\right) = -2\int dx\, x^j \left(-\frac{1}{2}t\left(\dot\phi^2 + (\vec\nabla\phi)^2\right) + x^i(-\dot\phi\partial_i\phi)\right) \tag{D.33}$$

where the first term was already computed for the boost in equation (D.10) and is

$$2ti\int \tilde{dk}\, a_k^*\omega\partial^j a_k \tag{D.34}$$

while the second must be computed explicitly. That is what will be done now

$$\begin{aligned}
2\int dx\, x^j x^i\dot\phi\partial_i\phi &= 2\int dx\tilde{dk}\tilde{dq}\, x^j x^i \omega(k) q_i \left(a_k e^{i\vec k\cdot\vec x} - a_k^* e^{-i\vec k\cdot\vec x}\right)\left(a_q e^{i\vec q\cdot\vec x} - a_q^* e^{-i\vec q\cdot\vec x}\right) \quad \text{(D.35)}\\
&= -2\int dx\tilde{dk}\tilde{dq}\,\omega(k)q_i\left(a_k\partial^i\partial^j e^{i\vec k\cdot\vec x} - a_k^*\partial^i\partial^j e^{-i\vec k\cdot\vec x}\right)\left(a_q e^{i\vec q\cdot\vec x} - a_q^* e^{-i\vec q\cdot\vec x}\right)\\
&= \int \tilde{dk}\,k_i\left((\partial^i\partial^j a_k)a_{-k} + (\partial^i\partial^j a_k)a_k^* + (\partial^i\partial^j a_k^*)a_k + (\partial^i\partial^j a_k^*)a_{-k}^*\right)\\
&= 2\int \tilde{dk}\, a_k^* k^i\partial_i\partial^j a_k + \int \tilde{dk}\, a_k^*\left[\left(\delta^{ij} - \frac{k^ik^j}{\omega^2}\right)\partial_i - \frac{d-2}{\omega^2}k^j + \frac{d-2}{\omega}\partial^j\right]a_k\\
&\quad -\frac{1}{2}(d-1)\int \tilde{dk}\, a_{-k}\partial^j a_k + \frac{1}{2}\int \tilde{dk}\,\frac{k^ik^j}{\omega^2}\partial_i a_k a_{-k}\\
&\quad -\frac{1}{2}(d-1)\int \tilde{dk}\, a_{-k}^*\partial^j a_k^* + \frac{1}{2}\int \tilde{dk}\,\frac{k^ik^j}{\omega^2}\partial_i a_k^* a_{-k}^*
\end{aligned}$$

where we have integrated by parts using the identity

$$\int \tilde{dk}\, k_i a_{-k}\partial^i\partial^j a_k = -\frac{1}{2}(d-1)\int \tilde{dk}\, a_{-k}\partial^j a_k + \frac{1}{2}(d-2)\int \tilde{dk}\,\frac{k^j}{\omega^2}a_k a_{-k} + \frac{1}{2}\int \tilde{dk}\,\frac{k^ik^j}{\omega^2}\partial_i a_k a_{-k} \tag{D.36}$$

and used that the second integral on the right-hand side vanishes because of anti-symmetry in k.

Adding all terms, the final expressions for K^μ are given by

$$K^0 = t^2P^0 + \int \tilde{dk}\, a_k^*\left[-\omega\partial^j\partial_j - 2itk^j\partial_j - it(d-2)\right]a_k \tag{D.37}$$

$$K^j = -t^2P^j + \int \tilde{dk}\, a_k^*\left[-k^j\partial_i\partial^i + 2k^i\partial^j\partial_i + 2it\omega\partial^j\right]a_k \tag{D.38}$$

D.2.1 Computation of D at $t = 0$

Since we know that the charges for which we are obtaining the momentum space expressions are conserved, we can instead compute all the expressions by setting $t = 0$ and then extracting the equivalent time-dependent expression by writing the time-independent Fourier modes in terms of the time-dependent ones. Let's see this for the case of the dilatation charge.

We have

$$\begin{aligned} D(t) = \int \mathrm{d}\vec{x}\, J_D^0 &= \int \mathrm{d}\vec{x} \left(x^\nu T^0_{c\,\nu} + \frac{d-2}{2}\phi\dot{\phi} \right) = \int \mathrm{d}\vec{x} \left(t T^0_{c\,0} + x^i T^0_{c\,i} + \frac{d-2}{2}\phi\dot{\phi} \right) \\ &= \int \mathrm{d}\vec{x} \left(-t T_c^{00} + x^i T^0_{c\,i} + \frac{d-2}{2}\phi\dot{\phi} \right). \end{aligned} \tag{D.39}$$

Then

$$\begin{aligned} D(0) &= \int \mathrm{d}\vec{x} \left(x^i T^0_{c\,i}\Big|_{t=0} + \frac{d-2}{2}\phi(0,\vec{x})\dot{\phi}(0,\vec{x}) \right) \\ &= \int \mathrm{d}\vec{x} \left(x^i \dot{\phi}(0,\vec{x})\partial_i\phi(0,\vec{x}) + \frac{d-2}{2}\phi(0,\vec{x})\dot{\phi}(0,\vec{x}) \right). \end{aligned} \tag{D.40}$$

We will now compute this using the Fourier expansion at $t = 0$ in the form (all the expressions follow from the time-dependent ones)

$$\phi(0,\vec{x}) = \int d\widetilde{k}\, (a(\vec{k})e^{i\vec{k}\cdot\vec{x}} + a^*(\vec{k})e^{-i\vec{k}\cdot\vec{x}}) = \int d\widetilde{k}\, (a(\vec{k}) + a^*(-\vec{k}))e^{i\vec{k}\cdot\vec{x}}, \tag{D.41}$$

$$\dot{\phi}(0,\vec{x}) = \int d\widetilde{k}\, (-i\omega(\vec{k}))(a(\vec{k}) - a^*(-\vec{k}))e^{i\vec{k}\cdot\vec{x}}, \tag{D.42}$$

$$\partial_i\phi(0,\vec{x}) = \int d\widetilde{k}\, (ik_i)(a(\vec{k}) + a^*(-\vec{k}))e^{i\vec{k}\cdot\vec{x}}. \tag{D.43}$$

The second term in (D.40) is

$$\frac{d-2}{2}\int \mathrm{d}\vec{x}\phi(0,\vec{x})\dot{\phi}(0,\vec{x}) = -i\frac{d-2}{4}\int d\widetilde{k}\, \left(a(\vec{k})a(-\vec{k}) - a(\vec{k})a^*(\vec{k}) + a^*(-\vec{k})a(-\vec{k}) - a^*(-\vec{k})a^*(\vec{k}) \right). \tag{D.44}$$

Under the change $\vec{k} \to -\vec{k}$, the third term in the integral cancels the second one, and one gets

$$\frac{d-2}{2}\int \mathrm{d}\vec{x}\phi(0,\vec{x})\dot{\phi}(0,\vec{x}) = -i\frac{d-2}{4}\int d\widetilde{k}\, \left(a(\vec{k})a(-\vec{k}) - a^*(-\vec{k})a^*(\vec{k}) \right). \tag{D.45}$$

The first term in (D.40) is

$$
\begin{aligned}
&\int \mathrm{d}\vec{x} x^i \dot{\phi}(0,\vec{x})\partial_i\phi(0,\vec{x}) \\
&= \int \mathrm{d}\vec{x} \widetilde{dk}\, \widetilde{dq}\, x^i(-i\omega(\vec{k}))(a(\vec{k})-a^*(-\vec{k}))e^{i\vec{k}\cdot\vec{x}}\,(iq_i)(a(\vec{q})+a^*(-\vec{q}))e^{i\vec{q}\cdot\vec{x}} \\
&= \int \mathrm{d}\vec{x} \frac{\mathrm{d}\vec{k}}{(2\pi)^{d-1}2\omega(\vec{k})}\frac{\mathrm{d}\vec{q}}{(2\pi)^{d-1}2\omega(\vec{q})}\omega(\vec{k})q_i(a(\vec{k})-a^*(-\vec{k})) \\
&\qquad (a(\vec{q})+a^*(-\vec{q}))e^{i\vec{q}\cdot\vec{x}}\left(-i\partial_{k_i}e^{i\vec{k}\cdot\vec{x}}\right) \\
&\text{(by parts in } k_i\text{, taking into account that there is no } \omega(\vec{k})\text{, and then } \vec{q}=-\vec{k}) \\
&= -\frac{i}{2}\int \widetilde{dk}\,(a(-\vec{k})+a^*(\vec{k}))k_i(\partial_i a(\vec{k})-\partial_i a^*(-\vec{k})) \\
&\text{(we integrate by parts 3 of the 4 resulting terms)} \\
&= -\frac{i}{2}\int \widetilde{dk}\, a^*(\vec{k})k_i\partial_i a(\vec{k}) + \frac{i}{2}\int \frac{\mathrm{d}\vec{k}}{(2\pi)^{d-1}2}\partial_i\left(\frac{k_i}{\omega(\vec{k})}a(-\vec{k})\right)a(\vec{k}) \\
&- \frac{i}{2}\int \frac{\mathrm{d}\vec{k}}{(2\pi)^{d-1}2}\partial_i\left(\frac{k_i}{\omega(\vec{k})}(a(-\vec{k})+a^*(\vec{k}))\right)a^*(-\vec{k}).
\end{aligned}
\tag{D.46}
$$

Using

$$
\partial_i\left(\frac{k_i}{\omega(\vec{k})}\right) = \frac{d-2}{\omega(\vec{k})}
\tag{D.47}
$$

and re-writing some of the terms with $\vec{k}\to-\vec{k}$ one gets finally

$$
\begin{aligned}
&\int \mathrm{d}\vec{x} x^i\dot{\phi}(0,\vec{x})\partial_i\phi(0,\vec{x}) = -i\int \widetilde{dk}\, a^*(\vec{k})k_i\partial_i a(\vec{k}) - \frac{i}{2}(d-2)\int \widetilde{dk}\, a^*(\vec{k})a(\vec{k}) \\
&+\frac{i}{2}(d-2)\int \widetilde{dk}\,(a(-\vec{k})a(\vec{k})-a^*(\vec{k})a^*(-\vec{k})) + \frac{i}{2}\int \widetilde{dk}\, k_i(\partial_i a(-\vec{k})a(\vec{k})-\partial_i a^*(\vec{k})a^*(-\vec{k})),
\end{aligned}
\tag{D.48}
$$

and

$$
\begin{aligned}
D(0) &= -i\int \widetilde{dk}\, a^*(\vec{k})k_i\partial_i a(\vec{k}) - \frac{i}{2}(d-2)\int \widetilde{dk}\, a^*(\vec{k})a(\vec{k}) \\
&+\frac{i}{4}(d-2)\int \widetilde{dk}\,(a(-\vec{k})a(\vec{k})-a^*(\vec{k})a^*(-\vec{k})) + \frac{i}{2}\int \widetilde{dk}\, k_i(\partial_i a(-\vec{k})a(\vec{k})-\partial_i a^*(\vec{k})a^*(-\vec{k})).
\end{aligned}
\tag{D.49}
$$

The last two integrals cancel each other because

$$
\begin{aligned}
\int \widetilde{dk}\, k_i\partial_i a(-\vec{k})a(\vec{k}) &= \int \widetilde{dk}\, k_i\partial_i a(\vec{k})a(-\vec{k}) \\
&\overset{\text{by parts}}{=} -(d-2)\int \widetilde{dk}\, a(\vec{k})a(-\vec{k}) - \int \widetilde{dk}\, k_i a(\vec{k})\partial_i a(-\vec{k}),
\end{aligned}
$$

from which

$$
\int \widetilde{dk}\, k_i\partial_i a(-\vec{k})a(\vec{k}) = -\frac{d-2}{2}\int \widetilde{dk}\, a(\vec{k})a(-\vec{k}),
\tag{D.50}
$$

and, similarly,

$$\int d\widetilde{k}\, k_i \partial_i a^*(\vec{k}) a^*(-\vec{k}) = \int d\widetilde{k}\, k_i \partial_i a^*(-\vec{k}) a^*(\vec{k}) = -\frac{d-2}{2} \int d\widetilde{k}\, a^*(\vec{k}) a^*(-\vec{k}). \quad \text{(D.51)}$$

Hence, the final form of the D charge at $t = 0$ is

$$D(0) = -i \int d\widetilde{k}\, a^*(\vec{k}) k_i \partial_i a(\vec{k}) - \frac{i}{2}(d-2) \int d\widetilde{k}\, a^*(\vec{k}) a(\vec{k}). \quad \text{(D.52)}$$

In order to obtain the time-dependent conserved quantity, we express the time-independent Fourier modes $a(\vec{k})$, $a^*(\vec{k})$ in terms of the time-dependent ones,

$$a(\vec{k}) = a(t, \vec{k}) e^{i\omega(\vec{k})t}, a^*(\vec{k}) = a^*(t, \vec{k}) e^{-i\omega(\vec{k})t}. \quad \text{(D.53)}$$

The derivatives with respect to k_i in (D.52) yield additional terms when acting on $\omega(\vec{k})$,

$$\begin{aligned} -ik_i \partial_i \left(a(t, \vec{k}) e^{i\omega(\vec{k})t} \right) &= -ik_i \partial_i a(t, \vec{k}) e^{i\omega(\vec{k})t} + k_i a(t, \vec{k}) \frac{k_i}{\omega(\vec{k})} t e^{i\omega(\vec{k})t} \\ &= -ik_i \partial_i a(t, \vec{k}) e^{i\omega(\vec{k})t} + a(t, \vec{k}) \omega(\vec{k}) t e^{i\omega(\vec{k})t}, \end{aligned}$$

and one gets

$$D(0) = -i \int d\widetilde{k}\, a^*(t, \vec{k}) k_i \partial_i a(t, \vec{k}) - \frac{i}{2}(d-2) \int d\widetilde{k}\, a^*(t, \vec{k}) a(t, \vec{k}) + t \int d\widetilde{k}\, a^*(t, \vec{k}) \omega(\vec{k}) a(t, \vec{k}). \quad \text{(D.54)}$$

The right-hand side of (D.54) is $D(t)$. It reduces to $D(0)$ for $t = 0$ and, by construction, is a conserved quantity. It can be expressed in terms of

$$P^0(t) = H(t) = \int d\widetilde{k}\, a^*(t, \vec{k}) \omega(\vec{k}) a(t, \vec{k}) = \int d\widetilde{k}\, a^*(\vec{k}) \omega(\vec{k}) a(\vec{k}) = H(0) \equiv H \quad \text{(D.55)}$$

as

$$D(t) = tH - i \int d\widetilde{k}\, a^*(t, \vec{k}) k_i \partial_i a(t, \vec{k}) - \frac{i}{2}(d-2) \int d\widetilde{k}\, a^*(t, \vec{k}) a(t, \vec{k}). \quad \text{(D.56)}$$

Appendix E

Further computations for BMS-extended conformal generators

This Section contains some extra relations and computations not presented in Chapter 4.[1] Although the idea of extending the conformal algebra with super-translations does not finally succeed in our approach, a new infinite-dimensional algebra, which does not contain the conformal one, is obtained, as discussed at the end of Chapter 4.

In this Thesis, we have introduced the generalization of the Lorentz group charges to infinite-dimensional ones $\mathcal{P}_\ell$ and $\mathcal{R}_m$ in $2+1$ dimensions as

$$\mathcal{P}_\ell = \int d\tilde{k}\, a^*(\vec{k},t)\widehat{\mathcal{P}_\ell} a(\vec{k},t), \quad \mathcal{R}_m = \int d\tilde{k}\, a^*(\vec{k},t)\widehat{\mathcal{R}_m} a(\vec{k},t), \tag{E.1}$$

where

$$\widehat{\mathcal{P}}_\ell = \omega_\ell = \omega^{1-\ell}(k^1+ik^2)^\ell, \tag{E.2}$$

$$\widehat{\mathcal{R}}_m = \frac{1}{\omega}\omega_m\left((k^2+imk^1)\frac{\partial}{\partial k^1} - (k^1-imk^2)\frac{\partial}{\partial k^2}\right) = \left(\xi^1_m\frac{\partial}{\partial k^1} - \xi^2_m\frac{\partial}{\partial k^2}\right). \tag{E.3}$$

For these generalized charges the algebra can be computed from (4.37)-(4.39) when substituting P^μ and $M^{\mu\nu}$ by the corresponding supercharges

$$[\widehat{\mathcal{R}}_m, \widehat{\mathcal{P}}_\ell] = i(m-\ell)\widehat{\mathcal{P}}_{m+\ell}, \qquad [\widehat{\mathcal{R}}_m, \widehat{\mathcal{R}}_n] = i(m-n)\widehat{\mathcal{R}}_{m+n}, \tag{E.4}$$

$$[\widehat{D}, \widehat{\mathcal{P}}_\ell] = i\widehat{\mathcal{P}}_\ell, \qquad [\widehat{D}, \widehat{\mathcal{R}}_m] = 0, \tag{E.5}$$

$$[\widehat{K}^0, \widehat{P}_\ell] = -2i\left[(1-\ell^2)\hat{D}_\ell + \ell\hat{\mathcal{R}}_\ell\right], \tag{E.6}$$

$$[\widehat{K}^1, \widehat{P}_\ell] = i\left[-(1-\ell)\ell\hat{D}_{\ell+1} + (1+\ell)\ell\hat{D}_{\ell-1} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} - (1+\ell)\hat{\mathcal{R}}_{\ell-1}\right], \tag{E.7}$$

$$[\widehat{K}^2, \widehat{P}_\ell] = \left[-(1-\ell)\ell\hat{D}_{\ell+1} - (1+\ell)\ell\hat{D}_{\ell-1} + (1-\ell)\hat{\mathcal{R}}_{\ell+1} + (1+\ell)\hat{\mathcal{R}}_{\ell-1}\right]. \tag{E.8}$$

where we have introduced a new superdilatation operator $D_\ell = \frac{1}{\omega}\omega_\ell D$. The last two commutators can be written using a combination of the operators K^1 and K^2 as

[1] Some relations are given in terms of the Poisson brackets of the conserved quantities and some in terms of commutators of the corresponding differential operators, see Appendix F, and we freely interchange the name of the conserved quantity and the name of the operator.

$K^{\pm} = \frac{1}{2}(K^1 \pm iK^2)$ to obtain

$$[\widehat{K}^+, \widehat{P}_\ell] = -i(1-\ell)\ell\hat{D}_{\ell+1} + i(1-\ell)\hat{\mathcal{R}}_{\ell+1}, \tag{E.9}$$
$$[\widehat{K}^-, \widehat{P}_\ell] = i(1+\ell)\ell\hat{D}_{\ell-1} - i(1+\ell)\hat{\mathcal{R}}_{\ell-1} \tag{E.10}$$

This superdilatation operator satisfies the following commutation relations

$$[\hat{D}_\ell, \hat{\mathcal{P}}_m] = i\hat{\mathcal{P}}_{\ell+m} \quad [\hat{D}_\ell, \hat{D}_m] = 0, \quad [\hat{D}_\ell, \hat{\mathcal{R}}_m] = i\ell\hat{D}_{\ell+m}. \tag{E.11}$$

For commutators with superrotations we should recover the original commutators for indices $\ell = 0, \pm 1$,

$$\{K^0, \mathcal{R}_0\} = 0, \qquad \{K^0, \mathcal{R}_1\} = -K^1 - iK^2, \qquad \{K^0, \mathcal{R}_{-1}\} = K^1 - iK^2, \tag{E.12}$$
$$\left\{K^1, \mathcal{R}_0\right\} = iK^2, \qquad \left\{K^1, \mathcal{R}_1\right\} = -K^0, \qquad \left\{K^1, \mathcal{R}_{-1}\right\} = K^0, \tag{E.13}$$
$$\{K^2, \mathcal{R}_0\} = -iK^1, \qquad \{K^2, \mathcal{R}_1\} = -iK^0, \qquad \{K^2, \mathcal{R}_{-1}\} = -iK^0, \tag{E.14}$$

Introducing again the operators $K^{\pm}$, we obtain

$$\{K^0, \mathcal{R}_0\} = 0, \qquad \{K^0, \mathcal{R}_1\} = -2K^+, \qquad \{K^0, \mathcal{R}_{-1}\} = 2K^-, \tag{E.15}$$
$$\{K^+, \mathcal{R}_0\} = K^+, \qquad \{K^+, \mathcal{R}_1\} = 0, \qquad \{K^+, \mathcal{R}_{-1}\} = K^0, \tag{E.16}$$
$$\{K^-, \mathcal{R}_0\} = -K^-, \qquad \{K^-, \mathcal{R}_1\} = -K^0, \qquad \{K^-, \mathcal{R}_{-1}\} = 0, \tag{E.17}$$

while for general super-rotations with index ℓ

$$\{K^+, \mathcal{R}_\ell\} = -\frac{1}{2}\ell(1-\ell)\omega_{\ell+1}\partial^i\partial_i + (1-\ell^2)\frac{\omega_\ell}{\omega}K^+ + \frac{1}{2}\ell(1-\ell^2)\frac{\omega_{\ell+1}}{\omega^2}k^i\partial_i \tag{E.18}$$
$$\{K^-, \mathcal{R}_\ell\} = -\frac{1}{2}\ell(1+\ell)\omega_{\ell-1}\partial^i\partial_i - (1-\ell^2)\frac{\omega_\ell}{\omega}K^- + \frac{1}{2}\ell(1-\ell^2)\frac{\omega_{\ell-1}}{\omega^2}k^i\partial_i \tag{E.19}$$
$$\{K^0, \mathcal{R}_\ell\} = -\ell(1+\ell)\frac{\omega_{\ell-1}}{\omega}K^+ - \ell(1-\ell)\frac{\omega_{\ell+1}}{\omega}K^- + \ell(1-\ell^2)\frac{\omega_\ell}{\omega^2}k^i\partial_i \tag{E.20}$$

More compact and clearer expressions can be obtained if we define the new operators $K^{\pm}_\ell = \frac{1}{\omega}\omega_\ell K^{\pm}$ and $K^0_\ell = \frac{1}{\omega}\omega_\ell K^0$,

$$\{K^+, \mathcal{R}_\ell\} = -\frac{1}{2}\ell(1-\ell)K^0_{\ell+1} + (1-\ell^2)K^+_\ell + \frac{1}{2}\ell(1-\ell^2)\frac{1}{\omega^2}\omega_{\ell+1}k^i\partial_i \tag{E.21}$$
$$\{K^-, \mathcal{R}_\ell\} = -\frac{1}{2}\ell(1+\ell)K^0_{\ell-1} - (1-\ell^2)K^-_\ell + \frac{1}{2}\ell(1-\ell^2)\frac{1}{\omega^2}\omega_{\ell-1}k^i\partial_i \tag{E.22}$$
$$\{K^0, \mathcal{R}_\ell\} = -\ell(1+\ell)K^+_{\ell-1} - \ell(1-\ell)K^-_{\ell+1} + \ell(1-\ell^2)\frac{1}{\omega^2}\omega_\ell k^i\partial_i \tag{E.23}$$

In any case, as explained in the main text, it seems that new sets of operators keep appearing and that it is not possible to include both super-rotations and special conformal transformations in a closed algebra.

Let us consider a generalization of the conformal operators for special conformal transformations of the form

$$\mathcal{O}_\ell = \frac{1}{\omega}\omega_\ell O, \tag{E.24}$$

where O is the original special conformal operator, in the same spirit as done for the dilatation operator. For $\ell = 0$ one recovers the base operator O. With the new operators obtained from K^0, $K^{\pm}$, one obtains the following commutators

$$[\hat{D}_\ell, \hat{\mathcal{P}}_m] = i\hat{\mathcal{P}}_{\ell+m} \quad [\hat{D}_\ell, \hat{D}_m] = 0, \quad [\hat{D}_\ell, \hat{\mathcal{R}}_m] = i\ell\hat{D}_{\ell+m} \tag{E.25}$$

$$\left[\hat{K}^0_m, \hat{\mathcal{P}}_\ell\right] = -2i\left(\ell\hat{\mathcal{R}}_{\ell+m} + (1-\ell(\ell+m))\hat{D}_{\ell+m}\right) + \ell m\frac{1}{\omega}\omega_{\ell+m} \tag{E.26}$$

$$\left[\hat{K}^+_m, \hat{\mathcal{P}}_\ell\right] = i(1-\ell)\left(\hat{\mathcal{R}}_{\ell+m+1} - (\ell+m)\hat{D}_{\ell+m+1}\right) - (1-\ell)m\frac{1}{2\omega}\omega_{\ell+m+1} \tag{E.27}$$

$$\left[\hat{K}^-_m, \hat{\mathcal{P}}_\ell\right] = -i(1+\ell)\left(\hat{\mathcal{R}}_{\ell+m-1} - (\ell+m)\hat{D}_{\ell+m-1}\right) + (1+\ell)m\frac{1}{2\omega}\omega_{\ell+m-1} \tag{E.28}$$

$$\left[\hat{K}^0_m, \hat{\mathcal{R}}_\ell\right] = im\hat{K}^0_{\ell+m} - i\ell(1-\ell)\hat{K}^-_{\ell+m+1} - i\ell(1+\ell)\hat{K}^+_{\ell+m-1} + i\ell(1-\ell^2)\frac{\omega_{\ell+m}}{\omega^2}k^i\partial_i \tag{E.29}$$

$$\left[\hat{K}^+_m, \hat{\mathcal{R}}_\ell\right] = -i\frac{1}{2}\ell(1-\ell)\hat{K}^0_{\ell+m+1} + i(1+m-\ell^2)\hat{K}^+_{\ell+m} + i\frac{1}{2}\ell(1-\ell^2)\frac{\omega_{\ell+m+1}}{\omega^2}k^i\partial_i \tag{E.30}$$

$$\left[\hat{K}^-_m, \hat{\mathcal{R}}_\ell\right] = -i\frac{1}{2}\ell(1+\ell)\hat{K}^0_{\ell+m-1} - i(1-m-\ell^2)\hat{K}^-_{\ell+m} + i\frac{1}{2}\ell(1-\ell^2)\frac{\omega_{\ell+m-1}}{\omega^2}k^i\partial_i \tag{E.31}$$

$$\begin{aligned}\left[\hat{K}^0_m, \hat{D}_\ell\right] =& i\hat{K}^0_{\ell+m} - i\ell\hat{K}^+_{\ell+m-1} + i\ell\hat{K}^-_{\ell+m+1} \\ & - i\ell\frac{1}{\omega^2}\omega_{\ell+m}\left(\frac{1}{2}\ell + \ell k^i\partial_i + 2i(k^2\partial_1 - k^1\partial_2)\right)\end{aligned} \tag{E.32}$$

$$\begin{aligned}\left[\hat{K}^+_m, \hat{D}_\ell\right] =& i\frac{1}{2}\ell\hat{K}^0_{\ell+m+1} + i(1-\ell)\hat{K}^+_{\ell+m} \\ & - i\frac{1}{2}\ell\frac{1}{\omega^2}\omega_{\ell+m+1}\left(\frac{1}{2}(1+\ell) - (1-\ell)k^i\partial_i + 2i(k^2\partial_1 - k^1\partial_2)\right)\end{aligned} \tag{E.33}$$

$$\begin{aligned}\left[\hat{K}^-_m, \hat{D}_\ell\right] =& -i\frac{1}{2}\ell\hat{K}^0_{\ell+m-1} + i(1+\ell)\hat{K}^-_{\ell+m} \\ & + i\frac{1}{2}\ell\frac{1}{\omega^2}\omega_{\ell+m-1}\left(\frac{1}{2}(1-\ell) - (1+\ell)k^i\partial_i - 2i(k^2\partial_1 - k^1\partial_2)\right)\end{aligned} \tag{E.34}$$

If we introduce now the new operators

$$\mathscr{K}^0_\ell := \frac{1}{\omega}\omega_\ell\hat{K}^0 - i\frac{1}{\omega^2}\ell\omega_\ell\hat{\mathcal{R}}_0 \tag{E.35}$$

$$\mathscr{K}^+_\ell := \frac{1}{\omega}\omega_\ell\hat{K}^+ - i\frac{1}{2}\frac{1}{\omega^2}\ell\omega_\ell\hat{\mathcal{R}}_1 \tag{E.36}$$

$$\mathscr{K}^-_\ell := \frac{1}{\omega}\omega_\ell\hat{K}^- - i\frac{1}{2}\frac{1}{\omega^2}\ell\omega_\ell\hat{\mathcal{R}}_{-1} \tag{E.37}$$

the previous commutators become

$$\left[\mathscr{K}^0_m, \hat{\mathcal{P}}_\ell\right] = -2i\left(\ell\hat{\mathcal{R}}_{\ell+m} + (1-\ell(\ell+m))\hat{D}_{\ell+m}\right) \tag{E.38}$$

$$\left[\mathscr{K}^+_m, \hat{\mathcal{P}}_\ell\right] = i(1-\ell)\left(\hat{\mathcal{R}}_{\ell+m+1} - (\ell+m)\hat{D}_{\ell+m+1}\right) \tag{E.39}$$

$$\left[\mathscr{K}^-_m, \hat{\mathcal{P}}_\ell\right] = -i(1+\ell)\left(\hat{\mathcal{R}}_{\ell+m-1} - (\ell+m)\hat{D}_{\ell+m-1}\right) \tag{E.40}$$

$$[\hat{\mathscr{K}}^0_m, \hat{\mathcal{R}}_\ell] = im\hat{\mathscr{K}}^0_{\ell+m} - i\ell(1-\ell)\hat{\mathscr{K}}^-_{\ell+m+1} - i\ell(1+\ell)\hat{\mathscr{K}}^+_{\ell+m-1} \tag{E.41}$$

$$[\hat{\mathscr{K}}^+_m, \hat{\mathcal{R}}_\ell] = -i\frac{1}{2}\ell(1-\ell)\hat{\mathscr{K}}^0_{\ell+m+1} + i(1+m-\ell^2)\hat{\mathscr{K}}^+_{\ell+m} \tag{E.42}$$

$$[\hat{\mathscr{K}}^-_m, \hat{\mathcal{R}}_\ell] = -i\frac{1}{2}\ell(1+\ell)\hat{\mathscr{K}}^0_{\ell+m-1} - i(1-m-\ell^2)\hat{\mathscr{K}}^-_{\ell+m} \tag{E.43}$$

$$[\hat{\mathscr{K}}^0_m, \hat{D}_\ell] = i\hat{\mathscr{K}}^0_{\ell+m} - i\ell\hat{\mathscr{K}}^+_{\ell+m-1} + i\ell\hat{\mathscr{K}}^-_{\ell+m+1} - i\frac{1}{2}\ell(\ell+m)\frac{1}{\omega^2}\omega_{\ell+m} \tag{E.44}$$

$$[\hat{\mathscr{K}}^+_m, \hat{D}_\ell] = i\frac{1}{2}\ell\hat{\mathscr{K}}^0_{\ell+m+1} + i(1-\ell)\hat{\mathscr{K}}^+_{\ell+m} - i\frac{1}{4}\ell(\ell+m+1)\frac{1}{\omega^2}\omega_{\ell+m+1} \tag{E.45}$$

$$[\hat{\mathscr{K}}^-_m, \hat{D}_\ell] = -i\frac{1}{2}\ell\hat{\mathscr{K}}^0_{\ell+m-1} + i(1+\ell)\hat{\mathscr{K}}^-_{\ell+m} - i\frac{1}{4}\ell(\ell+m-1)\frac{1}{\omega^2}\omega_{\ell+m-1} \tag{E.46}$$

Notice that $\ell = 0$, *i.e.* $\hat{\mathscr{K}}^a_0 = \hat{K}^a$ for $a = 0, \pm$.

One may try to get rid of the additional terms in the commutators with $\hat{D}_\ell$ by considering appropriate linear combinations of the $\hat{\mathscr{K}}^a_m$, $a = 0, \pm$. It turns out that such combinations exist, and are given by

$$\hat{\mathscr{K}}_m = \hat{\mathscr{K}}^0_m - \hat{\mathscr{K}}^+_{m-1} - \hat{\mathscr{K}}^-_{m+1}. \tag{E.47}$$

In terms of these, one obtains the following infinite-dimensional closed algebra, involving supertranslations, superrotations, superdilatations and the infinite set of new operators related to the special conformal transformations,

$$[\hat{\mathscr{K}}_m, \hat{\mathscr{K}}_\ell] = 0, \tag{E.48}$$

$$[\hat{\mathscr{K}}_m, \hat{\mathcal{P}}_\ell] = -4i\hat{D}_{\ell+m}, \tag{E.49}$$

$$[\hat{\mathscr{K}}_m, \hat{D}_\ell] = i\hat{\mathscr{K}}_{\ell+m}, \tag{E.50}$$

$$[\hat{\mathscr{K}}_m, \hat{\mathcal{R}}_\ell] = i(\ell+m)\hat{\mathscr{K}}_{\ell+m}, \tag{E.51}$$

and which is further discussed at the end of Chapter 4.

If one considers the same combinations for the operators $\hat{K}^0_m$, $\hat{K}^\pm_m$, that is, $\hat{K}_m = \hat{K}^0_m - \hat{K}^+_{m-1} - \hat{K}^-_{m+1}$, the commutators do not close

$$[\hat{K}_m, \hat{K}_\ell] = 0, \tag{E.52}$$

$$[\hat{K}_m, \hat{\mathcal{P}}_\ell] = -4i\hat{D}_{\ell+m} - \frac{1}{\omega}\omega_{\ell+m}, \tag{E.53}$$

$$[\hat{K}_m, \hat{D}_\ell] = i\hat{K}_{\ell+m}, \tag{E.54}$$

$$[\hat{K}_m, \hat{\mathcal{R}}_\ell] = i(\ell+m)\hat{K}_{\ell+m}. \tag{E.55}$$

Appendix F

Relation between Poisson brackets and operator commutators for a scalar field

Consider two quantities P and Q defined as

$$P(t) = \int \widetilde{dk}\, a^*(\vec{k})\hat{P}_k(t)a(\vec{k}), \tag{F.1}$$

$$Q(t) = \int \widetilde{dk}\, a^*(\vec{k})\hat{Q}_k(t)a(\vec{k}), \tag{F.2}$$

where $\hat{P}_k$ and $\hat{Q}_k$ are differential operators of finite order in the components of $\vec{k}$, that is, for instance

$$\hat{P}_k(t) = f^{(0)}(t,\vec{k}) + f^{(1)}_i(t,\vec{k})\partial_i + f^{(2)}_{ij}(t,\vec{k})\partial_i\partial_j + \ldots + f^{(m)}_{i_1 i_2 \ldots i_m}(t,\vec{k})\partial_{i_1}\partial_{i_2}\cdots\partial_{i_m}, \tag{F.3}$$

where m is the order of the differential operator, and can be different for different operators, with

$$\widetilde{dk} = \frac{\mathrm{d}^{d-1}k}{(2\pi)^{d-1}2\omega(\vec{k})}, \quad \omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}. \tag{F.4}$$

Using the equal-time Poisson brackets for the Fourier modes

$$\begin{aligned}\{a(\vec{k}), a^*(\vec{q})\} &= -i(2\pi)^{d-1}2\omega(\vec{k})\delta^{d-1}(\vec{k}-\vec{q}) \\ &= -i(2\pi)^{d-1}2\omega(\vec{q})\delta^{d-1}(\vec{k}-\vec{q}),\end{aligned} \tag{F.5}$$

one can prove that

$$\{P(t), Q(t)\} = -i\int \widetilde{dk}\, a^*(\vec{k})[\hat{P}_k(t), \hat{Q}_k(t)]a(\vec{k}). \tag{F.6}$$

Indeed, using the appropriate form of (F.5),

$$\begin{aligned}\{P(t), Q(t)\} = &\int \widetilde{dk}\,\mathrm{d}\widetilde{q}\,\big(a^*(\vec{k})\{\hat{P}_k(t)a(\vec{k}), a^*(\vec{q})\}\hat{Q}_q(t)a(\vec{q}) \\ &+ \{a^*(\vec{k}), \hat{Q}_q(t)a(\vec{q})\}\hat{P}_k(t)a(\vec{k})a^*(\vec{q})\big) \\ = &\int \widetilde{dk}\,\mathrm{d}\widetilde{q}\,(-2i(2\pi)^{d-1})\big(a^*(\vec{()})k)\hat{Q}_q(t)a(\vec{q})\hat{P}_k(t)(\omega(\vec{q})\delta^{d-1}(\vec{k}-\vec{q})) \\ &- \hat{P}_k(t)a(\vec{k})a^*(\vec{q})\hat{Q}_q(t)(\omega(\vec{k})\delta^{d-1}(\vec{k}-\vec{q}))\big)\end{aligned}$$

For both terms, we pull now the ω function, which is canceled by the corresponding factor in one of the measures, and shift the action of the differential operator from $\vec{k}$ to $\vec{q}$, or the other way around. When doing this, the derivatives pick a minus sign if the order of the term in the differential operator is odd, and remain equal if it is even. If, associated with each operator of the form (F.3), we define

$$\hat{P}^{\circ}_k(t) = f^{(0)}(t,\vec{k}) - f^{(1)}_i(t,\vec{k})\partial_i + f^{(2)}_{ij}(t,\vec{k})\partial_i\partial_j + \ldots + (-1)^m f^{(m)}_{i_1 i_2 \ldots i_m}(t,\vec{k})\partial_{i_1}\partial_{i_2}\cdots\partial_{i_m}, \tag{F.7}$$

we can write the result as

$$\begin{aligned}\{P(t),Q(t)\} = -i\int d\widetilde{k}\,\mathrm{d}^{d-1}q\; a^*(\vec{k})\hat{Q}_q(t)a(\vec{q})\hat{P}^{\circ}_q(t)\delta^{d-1}(\vec{k}-\vec{q})\\ i\int \mathrm{d}^{d-1}k\,\mathrm{d}\widetilde{q}\,\hat{P}_k(t)a(\vec{k})a^*(\vec{q})\hat{Q}^{\circ}_k(t)\delta^{d-1}(\vec{k}-\vec{q})\end{aligned}$$

We now integrate by parts the operators acting on the delta functions, which restores the original sign in each of the terms, and obtain

$$\begin{aligned}\{P(t),Q(t)\} = -i\int d\widetilde{k}\,\mathrm{d}^{d-1}q\; a^*(\vec{k})\hat{P}_q(t)\hat{Q}_q(t)a(\vec{q})\delta^{d-1}(\vec{k}-\vec{q})\\ i\int \mathrm{d}^{d-1}k\,\mathrm{d}\widetilde{q}\,\hat{Q}_k(t)\hat{P}_k(t)a(\vec{k})a^*(\vec{q})\delta^{d-1}(\vec{k}-\vec{q}),\end{aligned}$$

and using the delta functions one obtains indeed (F.6).

For the special case in which $\hat{Q}_k(t) = Q^{(0)}_k(t) = f(t,\vec{k})$ is just a function, without derivative terms, and $\hat{P}_k(t) = \hat{P}^{(1)}_k(t)$ is a first-order operator, the commutator becomes the function obtained by the action of $\hat{P}^{(1)}_k$ on f, and one gets

$$\{P^{(1)}(t),Q^{(0)}(t)\} = -i\int d\widetilde{k}\,a^*(\vec{k})a(\vec{k})\hat{P}^{(1)}_k(t)f(t,\vec{k}). \tag{F.8}$$

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