Faculty of Science

Floer Homology for b-symplectic manifolds

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Summary

In this thesis we investigate various aspects of the dynamics of Hamiltonian vector fields in singular symplectic manifolds.

We concentrate on two questions: first, we investigate a generalization of the Arnold conjecture in the setting of singular symplectic geometry. Second, we explore constructions for integrable systems in this context.

In Chapter 2 we provide the background material required for this thesis. We start by delving into the theory of symplectic geometry. Then, we present the Arnold conjecture, which asserts that there is a lower bound on the number of 1-periodic orbits for a non-degenerate Hamiltonian system, and that this lower bound can be formulated strictly in topological terms. We also present a tool used in the investigation of this conjecture: Floer theory.

Then, we explain some notions of Poisson geometry before we explore a notion fundamental to this thesis: that of a *b^m-symplectic manifold*. These are manifolds with a structure that is symplectic almost everywhere but "blows up" at a hypersurface, which we call the singular hypersurface. We lay out some techniques used in the study of b^m-symplectic manifolds, with an emphasis on a procedure called *desingularization*.

Finally, we give a summary of the theory of integrable systems and the study of their singular points.

In Chapter 3 we investigate the dynamical behaviour of certain vector fields in b^m-symplectic geometry, coming from *b^m-Hamiltonians*. We focus on the study of their dynamics in a neighbourhood of the singular hypersurface, and find a family of b^m-Hamiltonians where a version of the Arnold conjecture can be formulated. Then, we explore new aspects of the desingularization procedure in relation to the b^m-Hamiltonian dynamics, and provide some techniques that allow us to relate these dynamics to those of classical symplectic geometry. We conclude with two results yielding partial versions of the Arnold conjecture for b^m-Hamiltonian vector fields.

In Chapter 4 we show the existence of a Floer homology for b^m-symplectic manifolds. This we manage through an investigation of the Floer equation for the family of b^m-Hamiltonians presented in Chapter 3.

In Chapter 5 we introduce the notion of the classes of *b-integrable* and *b-semitoric systems*. We study the features of b-semitoric systems using some interesting examples and the investigation of their singular points.

Samenvatting

In deze thesis bestuderen we verscheidene aspecten van de dynamica van Hamiltoniaanse vectorvelden op singuliere symplectische variëteiten.

We concentreren ons op twee vraagstukken: Als eerste bestuderen we een generalisatie van het vermoeden van Arnold in het gebied van de singuliere symplectische meetkunde. Ten tweede besturen we potentiële constructies voor integreerbare systemen binnen dit gebied.

In Hoofdstuk 2 geven we de basisdefinities die noodzakelijk zijn in dit proefschrift. We beginnen met het beschrijven van de theorie van de symplectische meetkunde, waarna we het vermoeden van Arnold presenteren. Het vermoeden van Arnold stelt dat er een ondergrens bestaat voor het aantal van 1-periodieke banen van een niet-ontaard Hamiltoniaans systeem, en dat deze ondergrens volledig in topologische termen gegeven kan worden. We presenteren tevens een hulpmiddel dat we gebruiken bij het bestuderen van dit vermoeden: de theorie van Floer.

Daarna leggen we enkele standaard noties uit de Poissonmeetkunde uit, alvorens een fundamentele notie in deze thesis te bestuderen: *b^m-symplectische variëteiten*. Dit zijn variëteiten met een structuur die bijna overal symplectisch is maar "opblaast" langs een hyperoppervlak, welke we het singuliere hyperopervlak noemen. We geven enkele technieken die gebruikt worden in de theorie van b^m-symplectische variëteiten, met de nadruk op een procedé genaamd *desingularisatie*.

Ten laatste geven we een samenvatting van de theorie van integreerbare systemen en de studie van hun singuliere punten.

In Hoofdstuk 3 bestuderen we het dynamisch gedrag van bepaalde vector velden in b^m-symplectische meetkunde, afkomstig van *b^m-Hamiltonianen.* We richten ons op de studie van hun dynamica in een omgeving van het singuliere hyperoppervlak. We zullen een familie van b^m-Hamiltonianen vinden waarvoor een versie van het vermoeden van Arnold geformuleerd

kan worden. Daarna onderzoeken we nieuwe aspecten van het desingularisatie procedé met betrekking tot b^m-Hamiltoniaanse dynamica, en we beschrijven enkele technieken die ons de mogelijkheid geven deze dynamica te relateren aan de dynamica in klassieke symplectische meetkunde. We concluderen met twee resultaten welke gedeeltelijke versies van het vermoeden van Arnold voor b^m-Hamiltoniaanse vector velden bewijzen.

In Hoofdstuk 4 tonen we aan dat er een Floer homologie voor b^m-symplectische variëteiten bestaat. Dit bereiken we door de Floer vergelijking te bestuderen voor de familie van b^m-Hamiltonianen die we in Hoofdstuk 3 hebben geïntroduceerd.

In Hoofdstuk 5 introduceren we de noties van b-integreerbare en b-semitorische systemen. We bestuderen de eigenschappen van b-semitorische systemen door middel van interessante voorbeelden en het bestuderen van hun singuliere punten.

Resum

En aquesta tesi investiguem diversos aspectes dinàmics sobre camps vectorials Hamiltonians en varietats simplèctiques singulars.

Ens centrem en dues facetes: primer investiguem una generalització de la conjectura d'Arnold en el context de la geometria simplèctica singular. En segon lloc, examinem construccions de sistemes integrables en aquest àmbit.

Al Capítol 2 oferim els coneixements preliminars necessaris per a aquesta tesi. Comencem fixant-nos en la teoria de la geometria simplèctica. Després presentem la conjectura d'Arnold, que proposa l'existència d'una fita inferior en el nombre d'òrbites 1-periòdiques en sistemes Hamiltonians no degenerats, la qual es pot formular en termes estrictament topològics. També presentem una eina emprada per investigar aquesta conjectura: la teoria de Floer.

Tot seguit exposem algunes nocions de la geometria de Poisson, abans d'explorar una noció fonamental d'aquesta tesi: la de *varietat b^m-simplèctica*. Les varietats b^m-simplèctiques tenen una estructura que és simplèctica gairebé arreu però que "explota" en una hipersuperfície, anomenada singular. També exposem algunes tècniques emprades en l'estudi de les varietats b^m-simplèctiques, posant èmfasi en un procés anomenat *desingularització*.

Concloem el capítol oferint un resum de la teoria de sistemes integrables i de l'estudi dels seus punts singulars.

Al Capítol 3 investiguem el comportament dinàmic d'uns camps vectorials particulars en geometria b^m-simplèctica, induïts per *b^m-Hamiltonians*. Ens centrem en estudiar la seva dinàmica en un entorn de la seva hipersuperfície singular, i trobem una família de b^m-Hamiltonians per a la qual es pot formular una versió de la conjectura d'Arnold. Després explorem alguns aspectes del procés de desingularització relacionats amb la dinàmica dels

camps b^m-Hamiltonians, i descrivim algunes tècniques que ens permeten connectar aquesta dinàmica amb la dinàmica que trobem en la geometria simplèctica clàssica. Per acabar, donem dos resultats que constitueixen versions parcials de la conjectura d'Arnold per a camps b^m-Hamiltonians.

Al Capítol 4 demostrem l'existència d'una homologia de Floer per a varietats b^m-simplèctiques. Per aconseguir-ho estudiem l'equació de Floer en el context dels b^m-Hamiltonians presentats al Capítol 3.

Al Capítol 5 introduïm les nocions de classe *b-integrable* i de *sistema b-semitòric*. Estudiem les característiques dels sistemes b-semitòrics a través d'alguns exemples i les propietats dels seus punts singulars.

Resumen

En esta tesis investigamos varios aspectos dinámicos de campos vectoriales Hamiltonianos en variedades simplécticas singulares.

Nos centramos en dos facetas: primero investigamos una generalización de la conjetura de Arnold en el contexto de la geometría simpléctica singular. En segundo lugar examinamos construcciones de sistemas integrables en este contexto.

En el Capítulo 2 proveemos los conocimientos previos necesarios para esta tesis. Empezamos fijándonos en la teoría de la geometría simpléctica. Después presentamos la conjetura de Arnold, que propone la existencia de una cota inferior en el número de órbitas 1-periódicas en sistemas Hamiltonianos no degenerados, la cual puede formularse en términos estrictamente topológicos. También presentamos una herramienta usada para investigar esta conjetura: la teoría de Floer.

A continuación exponemos algunas nociones de la geometría de Poisson, antes de explorar una noción fundamental en esta tesis: la de *variedad b^m-simpléctica*. Éstas son variedades dotadas de una estructura que es simpléctica casi en todas partes pero que "explota" en una hipersuperficie, la cual llamamos singular. También exponemos algunas técnicas usadas en el estudio de las variedades b^m-simplécticas, poniendo énfasis en un proceso llamado *desingularización*.

Cerramos el capítulo con un resumen de la teoría de sistemas integrables y del estudio de sus puntos singulares.

En el Capítulo 3 investigamos el comportamiento dinámico de unos campos vectoriales particulares en la geometría b^m-simpléctica, inducidos por *b^m-Hamiltonianos*. Nos centramos en estudiar su dinámica en un entorno de su hipersuperficie singular, y encontramos una familia de b^m-Hamiltonianos para la cual podemos formular una versión de la conjetura de Arnold. Después exploramos algunos aspectos del proceso de desingularización relacionados con la dinámica de los campos b^m-Hamiltonianos,

y describimos algunas técnicas que nos permiten conectar esta dinámica con la dinámica que encontramos en la geometría simpléctica clásica. Para acabar, damos dos resultados que constituyen versiones parciales de la conjetura de Arnold para campos b^m-Hamiltonianos.

En el Capítulo 4 demostramos la existencia de una homología de Floer para variedades b^m-simplécticas. Con este fin estudiamos la ecuación de Floer en el contexto de los b^m-Hamiltonianos presentados en el Capítulo 3.

En el Capítulo 5 introducimos las nociones de clase *b-integrable* y de *sistema b-semitórico*. Estudiamos las características de los sistemas b-semitóricos a través de algunos ejemplos y las propiedades de sus puntos singulares.

Chapter 1

Introduction

1.1 Preface

It is sometimes asserted that breakthroughs in science occur when one studies the intersections between different fields. This is certainly so in mathematics, where tools from one area can provide swift resolutions for problems that seemed insurmountable in a different one. The bridging between different fields can also inspire questions that propel our understanding of a field forward, and multiply its potential applications.

If there is a field in mathematics in which this pattern of collaboration has been fruitful, it is that of symplectic geometry. Symplectic geometry is the natural setting for the study of classical mechanics in full generality, particularly in their Lagrangian formulation. Symplectic manifolds, the focus of study in this field, encode the generalization of a phase space from classical mechanics. They are endowed with a structure that allows us to set up the equations of motion of a system given its preserved quantity, a function called the *Hamiltonian* of the system.

A connection grew between the world of symplectic geometry and that of topology, motivated by the hunt for periodic orbits: *If we have a Hamiltonian system on a symplectic manifold, is it possible to predict how many periodic orbits it will feature? Is it possible to locate them?* Arnold put forward in [Arn65] a conjecture on the lower bound of periodic orbits of period 1 for a family of Hamiltonian systems in terms of the topology of the underlying space. This conjecture triggered a series of investigations on the relationship between the topological properties of a symplectic manifold and both its geometric and dynamical features. Conley and Zehnder proved in [CZ83] a partial version of Arnold's conjecture using variational methods. In parallel, Gromov studied the behaviour of pseudoholomorphic curves in symplectic manifolds, which led to his famous non-squeezing theorem in [Gro85]. Floer took inspiration from both Conley and Zehnder, and Gromov to develop a construction called Floer homology, which relates the topology of a symplectic manifold with the dynamical behaviour of a Hamiltonian within it. Floer's work paved the way to a solution of Arnold's conjecture, and motivated a wide variety of questions in the intersection between symplectic geometry and topology.

Another interesting connection can be found between the fields of symplectic geometry and of foliation theory. Roughly speaking, foliation theory investigates the ways in which a space can be split in "well-behaved" subspaces, called *leaves*, possibly with different dimensions. For instance, a space can be partitioned into symplectic leaves, which provides a motivation to generalize the notion of symplectic manifold to what is known as a *Poisson manifold*. Poisson manifolds encode thus a generalization of symplectic structures. In fact, they can be thought of as a "symplectic structure" that is allowed to have some kind of degeneracy.

The category of Poisson manifolds is remarkably less restricted than that of symplectic manifolds, which renders the classification of Poisson manifolds much more challenging that that of symplectic manifolds. For this reason it is common to investigate families of manifolds within the Poisson category. A useful tool to define families within Poisson geometry are *Lie algebroids*, which intuitively encode limitations in the tangent space of a given manifold. Thus we can find the families of *b^m-symplectic* manifolds, *log-symplectic* manifolds, *c-symplectic* manifolds and so on.

b^m-symplectic manifolds represent a situation in which a symplectic structure "blows up" in a subspace of codimension 1. Their study is motivated by non-canonical changes of coordinates in the restricted three-body problem (see [BDM+19] or [MO21] for more information), which helps us to understand the dynamical behaviour of the third body towards infinity.

The field of b^m-symplectic geometry and the very much related field of b^m-contact geometry, by themselves, inspire many interesting questions because their behaviour can be compared easily to that of symplectic or contact geometry. However, b^m-symplectic and b^m-contact geometries can display behaviours distinctly different to that of symplectic or contact geometry.

The initial question motivating this thesis was the potential construction of a theory of Floer homology in the context of b^m-symplectic manifolds, and its use towards a potential analog to the Arnold conjecture on this setting. An inspiration for this project was the work of Frauenfelder and Schlenk [FS07] where periodic orbits are investigated in a symplectic manifold with contact boundary. The work of Pasquotto, Vandervorst and Wiśniewska [PW20, PVW22], where Rabinowitz Floer homology is studied for families of Hamiltonians (namely *tentacular Hamiltonians*) in non-compact hypersurfaces, also contributed to motivate our line of research. Many different avenues were explored trying to achieve an understanding of the dynamical behaviour relevant to setting up an Arnold conjecture. The final result has been a very promising collection of results providing definite lower bounds for certain families of b^m-symplectic manifolds. The proofs for these lower bounds suggest several directions in which they could be improved and perhaps unified in the future.

Another feature of symplectic geometry that can find its analog in b^m-symplectic geometry is that of an *integrable system*. Integrable systems arise when investigating the dynamics of Hamiltonian systems that have a certain number of symmetries that constraint the dynamics of the resulting system in interesting ways. In her famous theorem, Noether identified the relationship between these symmetries and conserved quantities in a system, inspiring the notion of moment map and its function. A particularly well-understood example of a family of integrable systems is that of *toric manifolds*. These are systems in which all the components of the flow of the momentum map are periodic. Toric manifolds were completely classified by Delzant in [Del88] using a single invariant, the polytope associated to the momentum map.

A generalization of toric manifolds within integrable systems are *semitoric systems*, which have a more complicated behaviour. In particular, semitoric systems have singular points which are called *focus-focus*, with have a more complicated behaviour. A complete classification was developed by Pelayo, Vũ Ngọc, Palmer and Tang in terms of a sophisticated invariant called the marked semitoric polygon (see [PVN12a, AH19, Pel21]).

In this thesis we introduce the notion of *b-semitoric system* and explore a particular example of such a system: the *b*-coupled spin-oscillator.

1.2 Publications

The contents of this thesis can be found in the following articles, written in collaboration with several coauthors.

- [BMO22] *The Arnold conjecture for singular symplectic manifolds*, joint with Eva Miranda and Cédric Oms, arXiv:2212.01344.
- [BHMM23] *Constructions of b-semitoric systems*, joint with Sonja Hohloch, Pau Mir and Eva Miranda. Journal of Mathematical Physics 64(7):072703 (2023). DOI:10.1063/5.0152551.

The preliminaries of the articles have been summarized and included in Chapter 2 of this thesis.

1.3 Structure and results

In **Chapter 2** we present all preliminary notions required to develop the results of this thesis. This comprises symplectic geometry, a presentation of the Arnold conjecture with an overview of its proof through Floer theory, Poisson structures, b^m-symplectic geometry and the techniques to desingularize them, and an overview of integrable systems and their generalization to the b^m-symplectic setting.

In **Chapter 3** we study the Arnold conjecture in the context of b^m-symplectic manifolds. We begin by identifying the conditions under which a generalization to the Arnold conjecture for a b^m-symplectic form can be formulated. We conclude with the definition of what an *admissible Hamiltonian* is and we study its basic dynamical properties. We explore the way in which our desingularization techniques can be adapted to these admissible Hamiltonians and their effects on the associated dynamics. Using these techniques we are then able to derive lower bounds on 1-periodic orbits for several families of manifolds within the b^m-symplectic family.

The first result we highlight is in the case of an acyclic b^{2k}-symplectic manifold:

Theorem A (Brugués, Miranda and Oms. Theorem 3.3.3) *Let* (M, Z, ω) *be a compact* b^{2k}*-symplectic manifold whose associated graph is acyclic. Let* H_t *be a time-dependent regular admissible* b^{2k}*-Hamiltonian function. Then*

$$\#\mathcal{P}(H) \geq \sum_i \dim HM_i(M; \mathbb{Z}_2).$$

We explore as well the more general case of b^{2k}-symplectic manifolds, and what are the limitations for a lower bound to be found in the b^{2k+1}-symplectic case.

The second theorem that we highlight can be found in the particular case of b^m-symplectic surfaces:

Theorem B (Brugués, Miranda and Oms. Theorem 3.3.5) *Let* (Σ, Z, ω) *be a closed* b^m*-symplectic orientable surface. Let* H_t *be a regular admissible* b^m*-Hamiltonian function. Then the number of 1-periodic orbits of* X_H *has the lower bound*

$$\#\mathcal{P}(H) \geq \sum_{v \in V} \left(2g_v + \left|\deg(v) - 2\right|\right).$$

Moreover, in Proposition 3.3.7 we prove that the lower bound in Theorem B is sharp.

In **Chapter 4** we explore the construction of a Floer complex from a regular admissible Hamiltonian as introduced in Chapter 3 through the study of the Floer equation. The study of the Floer equation yields the following important result:

Theorem C (Brugués, Miranda and Oms. Theorem 4.1.6) *Let* (M, Z, ω) *be a* b^m*-symplectic manifold, and let* $u : \Omega \subset \mathbb{C} \to \mathcal{N} \subset M$*, where* $\mathcal{N}$ *is a tubular neighbourhood of* Z *in* $M \setminus Z$*. Let also* $f : \mathcal{N} \to \mathbb{R}$ *be given by* $\log|z|$ *if* $m = 1$ *and* $-\frac{1}{(m-1)z^{m-1}}$ *if* $m > 1$*. Suppose also that* u *is a solution of the Floer equation for an admissible Hamiltonian* $H \in C^\infty(S^1 \times \mathcal{N})$*. If* $f \circ u$ *attains its maximum or minimum on* Ω*, then* $f \circ u$ *is constant.*

In **Chapter 5** we switch to the study of a particular subclass of b-integrable systems, *b-semitoric systems*. We concentrate on the study of a particular family of such systems, the *b-coupled spin-oscillators*, and classify all critical points in the system.

1.4 Open questions

As many projects of this kind, this thesis leaves a lot of questions open at its conclusion. We deem the following the most interesting questions raised by our work:

1. Is it possible to compute the homology defined in Chapter 4?

   In Chapter 4 we define a Floer complex in line with the classical definition, but there is no clear indication of how to compute its homology. In light of the results exposed in Chapter 3 we would expect this homology to be split in each of the connected components of $M \setminus Z$ and to depend on the relative topology of Z within each of the components. The second part of Theorem 4.1.6 should be helpful in this endeavour, in particular to build an isomorphism of the homology or at least to prove invariance with respect to the admissible Hamiltonian H.

   A reasonable direction in which to investigate would be to attempt to build a Morse theory that is compatible with the class of Hamiltonians introduced in Definition 3.1.11, taking into account the behaviour of these functions near the critical hypersurface Z.

2. Is it possible to generalize the desingularization arguments of Chapter 3 to b^{2k+1}-symplectic manifolds of dimension higher than 2?

   The desingularization arguments work for the case of b^{2k}-symplectic manifolds because the resulting structure is symplectic, and in the case of surfaces because the low dimension allows us to keep track of changes of a change of sign on certain components of the surface $\Sigma \setminus Z$. In higher dimensions we need to be able to glue smoothly the dynamics along the symplectic foliation of Z locally around each connected component, for which there is no general method yet.

3. Can the desingularization methods be considered for wider families of b^m-Hamiltonians?

   In this thesis we deal with the notion of admissible Hamiltonian (see Definition 3.1.11) within the wider family of b^m-Hamiltonians. The purpose of this definition is to work with induced dynamics that split along different connected components of $M \setminus Z$. The next step should be to investigate the possible roles that the dynamics in Z could play in different generalizations of the results exposed in this thesis. We deem the first natural generalization to be one in which finite periodic orbits are allowed within Z, and to develop an understanding of how the solutions to the Floer equation (2.5) could connect periodic orbits within Z to those outside.

4. This thesis contributes to providing toy models for the classification of b-semitoric system, a research project of Eva Miranda. The question has already been raised by Guillemin, Miranda and Pires [GMP14] for general b-integrable systems and by Guillemin, Miranda, Pires and Scott [GMPS17] for almost toric manifolds. We think that there should be good prospects for such a classification, taking into account the fact that in Proposition 5.1.9 we prove that fixed points of the system cannot belong to Z, and therefore the focus-focus critical points of the system will fall away from Z. Therefore, all invariants which are locally defined near singularities should be trivial to generalize to this context.

Chapter 2

Preliminaries

In this chapter we cover the mathematical background essential to this thesis. All the contents of this chapter are either widely known or available in the cited works whenever provided, with the sole exception of some of the examples in Subsection 2.4.3, which we will point out accordingly.

2.1 Symplectic geometry

In classical physics many systems can be formulated in a compact way by means of the *Hamiltonian formulation*. In these problems there is a conserved quantity, usually the sum of the kinetic and potential energies, called here the *Hamiltonian* of the system. If we take the position and momentum coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ in $\mathbb{R}^{2n}$ and let $H : \mathbb{R}^{2n} \to \mathbb{R}$ be the Hamiltonian, the trajectory of a particle under the forces derived from the potential energy is governed by *Hamilton's equations*,

$$\begin{cases} \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i}, \\ \frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}. \end{cases} \tag{2.1}$$

Symplectic geometry is a setting where one can generalize Hamilton's equations within the category of smooth manifolds.

Definition 2.1.1 Let M be a smooth manifold and $\omega \in \Omega^2(M)$ a differential form. We say that it is a *symplectic form* if it closed (i.e. $d\omega = 0$) and non-degenerate. In that case we call (M, ω) a *symplectic manifold*.

Remark 2.1.2 Symplectic manifolds are always even dimensional. This is a consequence of the non-degeneracy property in Definition 2.1.1. Also, as ω^n is a volume form, symplectic manifolds are oriented.

Example 2.1.3 In $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ the standard symplectic form is

$$\omega_{\mathrm{st}} = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

Example 2.1.4 If Σ is an orientable surface then any area form $\omega_\Sigma \in \Omega^2(\Sigma)$ has top degree and thus it is closed, and by definition it is non-degenerate. Hence, any orientable surface admits a symplectic structure.

In particular, let us consider the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, and let $\langle \cdot, \cdot \rangle$ be a Riemannian metric in $\mathbb{R}^3$. We say then that the differential form $\omega_{\mathbb{S}^2} \in \Omega^2(\mathbb{S}^2)$ given by $\omega_{\mathbb{S}^2,p}(u, v) = \langle p, u \times v \rangle$ is the *standard symplectic form* in $\mathbb{S}^2$. Let (θ, z) denote the cylindrical coordinates in $\mathbb{S}^2 \setminus \{N, S\}$, where N and S denote the north and south poles respectively. In these coordinates the standard symplectic form in $\mathbb{S}^2$ with respect to the Euclidean metric has the expression

$$\omega_{\mathbb{S}^2} = d\theta \wedge dz.$$

Example 2.1.5 Let M be a smooth manifold, and let $\pi : T^*M \to M$ denote the projection of the cotangent bundle onto the base space. The *tautological 1-form* $\lambda \in \Omega^1(T^*M)$ is the differential form given at every $\xi \in T^*M$ by $\lambda_\xi = \xi \circ d_\xi \pi$.

The canonical symplectic form on a cotangent bundle is then given by $\omega := -d\lambda$. If $U \subset M$ is a trivializing chart for the cotangent bundle with coordinates $(q_1, \ldots, q_n)$ in the base and coordinates $p_1, \ldots, p_n$ in the direction of the fibre, the canonical symplectic form has the local expression

$$\omega|_{T^*U} = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

Remark 2.1.6 Let M be a closed even dimensional manifold. If the second De Rham cohomology group $H^2_{DR}(M)$ is trivial then M does not admit a symplectic structure.

As an example, $\mathbb{S}^{2n}$ does not admit a symplectic structure for any $n > 1$.

Proof. Assume that the second De Rham cohomology group $H^2_{DR}(M)$ is trivial. Let $\omega \in \Omega^2(M)$ be a closed form. As the cohomology group is trivial, $\omega = d\eta$. By the Stokes theorem,

$$\int_M \omega^n = \int_M d\left(\eta \wedge \omega^{n-1}\right) = \int_{\partial M} \eta \wedge \omega^{n-1} = 0.$$

This implies that ω has to be degenerate at some point, and therefore it cannot be a symplectic form. □

A notable characteristic of symplectic forms is that they have no local invariants besides the dimension of the manifold. In particular, any symplectic form admits a local expression akin to that of Examples 2.1.3 and 2.1.5:

Theorem 2.1.7 (Darboux) *Let (M, ω) be a symplectic manifold and $p \in M$. Then there exists a chart centered on p, $(U; q_1, \dots, q_n, p_1, \dots, p_n)$, such that the expression of ω in given in these coordinates by*

$$\omega|_U = \sum_{i=1}^{n} dq_i \wedge dp_i.$$

As mentioned at the start of this section, symplectic geometry is tightly related to classical mechanics. In Definition 2.1.1 we can consider M to be describing the phase space of a physical system, and if $H : M \to \mathbb{R}$ is a smooth function it is possible to use the symplectic form ω to derive a system of equations locally equivalent to that of Equation 2.1.

Definition 2.1.8 A vector field $X \in \mathfrak{X}(M)$ is *symplectic* if it preserves the symplectic structure, this means, $\mathcal{L}_X \omega = 0$.

The *Hamiltonian vector field* or *symplectic gradient* of a smooth function H is the unique vector field that satisfies the equation

$$\omega(X_H, \cdot) = -dH.$$

Remark 2.1.9 The Hamiltonian vector field X_H is symplectic for any function $H \in C^\infty(M)$. This we can see by using Cartan's formula,

$$\mathcal{L}_{X_H}\omega = \iota_{X_H} d\omega + d\iota_{X_H}\omega = 0 - ddH = 0.$$

Example 2.1.10 Consider the standard symplectic structure in local Darboux coordinates $(q_1, \dots, q_n, p_1, \dots, p_n)$ as presented in Theorem 2.1.7. The Hamiltonian vector field of H is given in these coordinates by the expression

$$X_H = \sum_{i=1}^{n} \frac{\partial H}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i}\frac{\partial}{\partial p_i}.$$

Definition 2.1.11 Let (M_1, ω_1) and (M_2, ω_2) be symplectic manifolds, and consider $\psi : M_1 \to M_2$ a diffeomorphism. We say that it is a *symplectomorphism* if $\psi^*\omega_2 = \omega_1$.

We say (M_1, ω_1) and (M_2, ω_2) are *symplectomorphic* if there exists a symplectomorphism between them.

Remark 2.1.12 For any Hamiltonian H the flow of its Hamiltonian vector field, $\varphi^t_{X_H}$, is a symplectomorphism for every t when it is defined.

2.2 An introduction to Floer theory

In this section we will present the statement of the Arnold conjecture, a central question in the study of symplectic geometry and a focus of this thesis. This conjecture provides a lower bound on the number of fixed points for Hamiltonian isotopies on compact symplectic manifolds. This lower bound is set up in topological terms, which has implications on the broader study of symplectic topology and symplectic dynamics. Moreover, we will present a summary of one of the most useful techniques to tackle this conjecture: Floer theory.

2.2.1 Morse theory

We will first introduce a sketch of Morse theory, which provides both a model for Floer theory and a first inspiration for the Arnold conjecture. For a more detailed introduction to this subject we direct the reader to Milnor [Mil63] or to Part I of Audin and Damian [AD14].

Morse theory is centered on the relationship between the topology of a manifold and the critical points of certain smooth functions defined on the manifold, by means of a chain complex and the induced homology.

Definition 2.2.1 Let M be a closed smooth manifold and $f \in C^\infty(M)$, and let $p \in M$ be a critical point of f. The *Hessian of f at p* is the bilinear map $H_p[f]$ on T_pM such that

$$H_p[f](u, v) = v\left(\mathcal{L}_{X_u} H\right)$$

for all $u, v \in T_pM$. Here, $X_u \in \Gamma(U)$ is a (local) vector field extending u in a neighbourhood U of p, so $X_u|_p = u$.

Lemma 2.2.2 *The Hessian $H_p[f]$ is well defined in the sense that it does not depend on the choice of X_u, and it is a bilinear and symmetric map.*

Definition 2.2.3 Let M be a closed smooth manifold and $f \in C^\infty(M)$. A critical point p of f is *non-degenerate* if the Hessian $H_p[f]$ has maximal rank.

If p is a non-degenerate critical point, its *index* is the dimension of the maximal subspace of T_pM on which the Hessian $H_p[f]$ is negative definite. We denote the index of f at p by $\mu_f(p)$.

A function f is *Morse* if all its critical points are non-degenerate.

Definition 2.2.4 Let M be a closed smooth manifold endowed with a Riemannian metric g. Let $f : M \to \mathbb{R}$ be a Morse function, and let $\varphi^t_{\operatorname{grad} f}$ denote the flow of the gradient of f with respect to g. Let $p \in M$ be a critical point.

The *stable manifold* of f at p is the submanifold

$$W^s(p) := \left\{ q \in M \;\middle|\; \lim_{t\to+\infty} \varphi^t_{\operatorname{grad} f}(q) = p \right\}.$$

The *unstable manifold* of f at p is the submanifold

$$W^u(p) := \left\{ q \in M \;\middle|\; \lim_{t\to-\infty} \varphi^t_{\operatorname{grad} f}(q) = p \right\}.$$

It is possible to construct a cellular decomposition of a manifold using the stable manifolds of $-\operatorname{grad}(f)$ at each of the critical points of a Morse function defined on the manifold. This naturally provides a relationship between the topology of the manifold and the dynamics of the gradient vector field $-\operatorname{grad}(f)$.

We can formalize this concept by defining a chain complex.

Definition 2.2.5 Let M be a closed manifold, $f \in C^\infty(M)$ a Morse function and g a Riemannian metric on M.

We define the groups of the *Morse complex* of (M, f, g) as the group of degree k generated freely over $\mathbb{Z}_2$ by the critical points with index k, this means,

$$CM_k(M, f, g) := \left\{ \sum_{i=1}^{N} a_{p_i} p_i \,\middle|\, N \in \mathbb{N}, p_i \in \mathrm{Crit}(f), \mu_f(p_i) = k, a_{p_i} \in \mathbb{Z}_2 \right\}.$$

Remark 2.2.6 It is possible to define the Morse complex over $\mathbb{Z}$ as well. We choose as a coefficient ring $\mathbb{Z}_2$ because it simplifies considerably computations significantly later on and when working with the Floer complex (which we will model after the Morse complex), as the use of $\mathbb{Z}$ requires the consideration of orientations on the stable and unstable manifolds. Although the integral complex can eventually rely more information, $\mathbb{Z}_2$ suffices to prove the Morse inequalities.

Definition 2.2.7 The *boundary map* of the Morse complex $\partial_k : CM_k(M, f, g) \to CM_{k-1}(M, f, g)$ is defined on the generators by the expression

$$\partial_k(p) := \sum_{\mu_f(q)=k-1} n(p, q) q,$$

where $n(p, q)$ denotes the number (modulo 2) of solutions to the system of equations

$$\begin{cases} \gamma'(t) = -\operatorname{grad}(f)\big|_{\gamma(t)} \\ \lim\limits_{t\to-\infty} \gamma(t) = p \\ \lim\limits_{t\to+\infty} \gamma(t) = q. \end{cases}$$

This means, the number of flow lines of $-\operatorname{grad}(f)$ that connect the critical points p and q.

Lemma 2.2.8 *For any manifold M, Morse function f, and with the appropriate choice of g, we have that*

$$\partial_k \circ \partial_{k+1} = 0.$$

Definition 2.2.9 The *Morse homology* of (M, f, g) is the sequence of groups

$$HM_k(M, f, g) := \frac{\ker(\partial_k)}{\operatorname{im}(\partial_{k+1})}.$$

Theorem 2.2.10 *The Morse homology of a manifold M does not depend on the choice of Morse function f or the Riemannian metric g.*

In light of Theorem 2.2.10, it makes sense to denote the homology simply as $HM_\bullet(M)$.

Theorem 2.2.11 *The Morse homology of a manifold M is isomorphic to the singular homology of the manifold with coefficients on $\mathbb{Z}_2$,*

$$HM_\bullet(M) \cong H_\bullet(M;\mathbb{Z}_2).$$

Corollary 2.2.12 (Morse inequalities) *Let f be a Morse function over a closed manifold M. Then, there is a lower bound on the number of critical points of f,*

$$\#\mathrm{Crit}(f) \geq \sum_{k=0}^{n} \beta_k,$$

where β_k denotes the k-th Betti number of M, i.e., the rank of $H_k(M;\mathbb{Z}_2)$.

In particular, if we take $\mathrm{Crit}_k(f) = \left\{p \in \mathrm{Crit}(f) \mid \mu_f(p) = k\right\}$ *we have that*

$$\#\mathrm{Crit}_k(f) \geq \beta_k. \tag{2.2}$$

Moreover, for all $0 \leq l < n$ *we have*

$$\sum_{k=0}^{l}(-1)^k \#\mathrm{Crit}_k(f) \geq \sum_{k=0}^{l}(-1)^k \beta_k, \tag{2.3}$$

and

$$\sum_{k=0}^{n}(-1)^k \#\mathrm{Crit}_k(f) = \sum_{k=0}^{n}(-1)^k \beta_k. \tag{2.4}$$

Inequality 2.2 is often called the strong Morse inequality, *while inequalities 2.3 and 2.4 are called the* weak Morse inequalities.

Remark 2.2.13 The weak Morse inequalities reflect a general property of homology groups, namely that the Euler characteristic of the sequence of homology groups coincides with that of the chain complex from which it is computed.

2.2.2 The Arnold conjecture

The Arnold conjecture, stated by V. I. Arnold in [Arn65, Arn86], establishes a lower bound on the number of fixed points of certain symplectomorphisms on compact symplectic manifolds.

First, let us consider a direct consequence of the Morse inequalities in Corollary 2.2.12:

Corollary 2.2.14 *Let (M, ω) be a closed symplectic manifold and let $H \in C^\infty(M)$ be a Morse function. Let $\mathcal{P}(H)$ denote the set of 1-periodic orbits of the Hamiltonian vector field X_H. Then the number of (non-parametrized) 1-periodic orbits of the Hamiltonian vector field X_H is bounded below by the sum of the Betti numbers of M:*

$$\#\mathcal{P}(H) \geq \sum_{k=0}^{n} \beta_k.$$

The interesting idea by V. I. Arnold was to propose the same inequality for a wider family of Hamiltonians, as we will see.

Remark 2.2.15 Let $H : \mathbb{R} \times M \to \mathbb{R}$ be a smooth function. Its *Hamiltonian vector field* is defined in an analogous manner to Definition 2.1.8. Let us recall that the Hamiltonian vector field is defined as

$$\omega(X_{H_t}, \cdot) = -dH_t.$$

Here, dH_t denotes the differential of the function with respect to the coordinates in M, ignoring the variable t.

This is a section in the sense that it is a map $X_{H_t} : \mathbb{R} \times M \to TM$ such that, if $\pi : TM \to M$ denotes the vector bundle projection, $\pi \circ X_{H_t} : \mathbb{R} \times M \to M$ is the projection on the second component.

We use the same notation for a smooth function $H : \mathbb{S}^1 \times M \to \mathbb{R}$.

Definition 2.2.16 A periodic solution $x : \mathbb{S}^1 \to M$ to Hamilton's equations for a time-dependent Hamiltonian is said to be *non-degenerate* if

$$\det\left(\mathrm{Id} - d_{x(0)}\varphi^1_{X_{H_t}}\right) \neq 0.$$

A Hamiltonian $H : \mathbb{R} \times M \to \mathbb{R}$ is *non-degenerate* if all of its 1-periodic orbits are non-degenerate.

Remark 2.2.17 If a critical point of H is non-degenerate as a periodic orbit, then it is non-degenerate as a critical point.

With these definitions in mind, we can state the general formulation of the Arnold conjecture.

Theorem 2.2.18 (Arnold Conjecture) *Let (M, ω) be a compact symplectic manifold, and let $H : \mathbb{R} \times M \to \mathbb{R}$ be a non-degenerate smooth Hamiltonian.*

The number of 1-periodic orbits of X_H is bounded below by the sum of the Betti numbers of M:

$$\#\mathcal{P}(H) \geq \sum_{k=0}^{2n} \beta_k.$$

The Arnold conjecture was proved in full generality by the combined efforts of several researchers. A chronology of the proof can be found in Salamon [Sal99], which we will recall here:

A first proof for Riemann surfaces was found by Eliashberg [Eli79]. Then, Conley and Zehnder developed a proof for the $2n$-tori in [CZ83]. In [Gro85] Gromov proved the existence of at least one fixed point under the assumption that $\pi_2(M) = 0$. Floer introduced his homology in a series of papers [Flo88a, Flo88b, Flo88c, Flo89b, Flo89a] proving the Arnold conjecture, for aspherical manifolds first and for monotone manifolds later. Floer's proof was extended by Hofer and Salamon [HS95] and Ono [Ono95] to the weakly monotone case. Finally, Fukaya and Ono [FO99], Liu and Tian [LT98] and Ruan [Rua99] achieved the proof for general closed symplectic manifolds.

Further work has been devoted to iterate on and refine the proofs of the Arnold conjecture in the last decades. For example, Filippenko and Wehrheim [FW22] formulated a general proof using a perturbation scheme based on the polyfold theory developed by Hofer, Wysocki and Zehnder [HWZ21] following the scheme by Piunikhin, Salamon and Schwarz [PSS96]. Another approach for the full proof can be found in Pardon [Par16] using techniques on virtual fundamental cycles on moduli spaces of pseudo-holomorphic curves. Moreover, Abouzaid and Blumberg [AB21] have proved a more general version of the Arnold conjecture in terms of generalized homology with respect to Morava K-theory.

2.2.3 An overview of Floer theory

In this part we will give an overview of the proof of the Arnold Conjecture 2.2.18 in the particular case of aspherical manifolds. This will allow us to introduce an element of special interest in this thesis: Floer homology.

Definition 2.2.19 Let M be a smooth manifold. An *almost complex structure* is a section $J \in \mathrm{End}(TM)$ such that $J^2 = -\mathrm{Id}$.

If (M, ω) is a symplectic manifold and J is an almost complex structure, we say that J is *compatible with*, or *calibrated by* ω, if $\omega(JX, JY) = \omega(X, Y)$ for all $X, Y \in \mathfrak{X}(M)$ and the section $g_J := \omega(\cdot, J\cdot)$ is positive definite. In that case g_J defines a Riemannian metric on M. Let us denote by $\mathcal{J}(M, \omega)$ the space of almost complex structures compatible with ω.

Remark 2.2.20 (McDuff and Salamon [MS98]) The space $\mathcal{J}(M, \omega)$ of almost complex structures compatible with ω is contractible.

Let (M, ω) be a compact symplectic manifold and consider J an almost complex structure adapted to ω. Let us assume that the first Chern class of (M, J) vanishes on $\pi_2(M)$ and also that $[\omega]$ vanishes on the second homotopy group of M. These conditions can be written as $\langle \omega, \pi_2(M)\rangle = 0$ and $\langle c_1(TM, J), \pi_2(M)\rangle = 0$.

Let us introduce informally the domain on which we will work. For a formal introduction we refer the reader to [AD14, Section 6.8].

Definition 2.2.21 Let $p > 1$ and g a Riemannian metric on M. The *space of loops* of M, denoted by $\mathcal{L}^{1,p}M$, is a subset of $C^0(\mathbb{S}^1; M)$ given by the exponentials (with respect to g) of sections of the Banach bundle $W^{1,p}(x^*TM)$, where $x \in C^\infty(\mathbb{S}^1, M)$.

Theorem 2.2.22 (Schwarz [Sch93, Theorem 10]) *$\mathcal{L}^{1,p}M$ has a structure of smooth manifold for all $p > 1$ which does not depend on g. Moreover, $C^\infty(\mathbb{S}^1; M) \subset \mathcal{L}^{1,p}M \subset C^0(\mathbb{S}^1; M)$, where each inclusion is dense in the next one.*

From now on, we will denote by $\mathcal{L}M$ the subspace of $\mathcal{L}^{1,2}M$ of contractible loops.

Definition 2.2.23 Let $H : \mathbb{S}^1 \times M \to \mathbb{R}$ be a smooth 1-periodic Hamiltonian. The action functional $\mathcal{A}_H : \mathcal{L}M \to \mathbb{R}$ is defined as

$$\mathcal{A}_H(x) = \int_{\mathbb{S}^1} H_t(x(t))dt - \int_{D^2} v^*\omega,$$

where v is a filling of x within M, i.e., $v : D^2 \to M$ such that $v|_{\partial D^2} = x$.

Lemma 2.2.24 *Critical points of the action functional correspond to* 1*-periodic orbits of the Hamiltonian vector field.*

Proof. Let us consider $x \in \mathcal{L}M$. Let $\pi : TM \to M$ denote the projection of the tangent bundle. A tangent vector $Y \in T_x\mathcal{L}M$ is a section $Y : \mathbb{S}^1 \to TM$ such that $\pi \circ Y = x$. Let us compute the differential of $\mathcal{A}_H$ along Y.

First, let us consider a path through x, given by $z : (-\varepsilon, \varepsilon) \times \mathbb{S}^1 \to M$ such that $z(0, t) = x(t)$ for all t and $\frac{d}{ds}\big|_{s=0} z(s, t) = Y(t)$.

Let $v : D^2 \to M$ a filling of x. Let us consider $\widetilde{v} : (-\varepsilon, \varepsilon) \times D^2 \to M$ such that $\widetilde{v}\left(s, e^{it}\right) = z\left(s, e^{it}\right)$ and $\widetilde{v}(0, p) = v(p)$. In other words, $\widetilde{v}$ is a filling of z compatible with v.

Then, Y can be extended to the whole disk D^2 by taking $Y(p) = \frac{\partial \widetilde{v}}{\partial s}(0, p)$.

Then,

$$d\mathcal{A}_H(x) \cdot Y = \frac{d}{ds}\bigg|_{s=0} \mathcal{A}_H\left(z(s)\right) = \frac{d}{ds}\bigg|_{s=0} \left(\int_0^1 H_t\left(z(s,t)\right) dt - \int_{D^2} \widetilde{v}_s^* \omega\right).$$

If we look at the second component and apply Cartan's formula and Stoke's theorem, we get

$$\begin{aligned} -\int_{D^2} \left(\frac{d}{ds}\bigg|_{s=0} \widetilde{v}_s^* \omega\right) &= -\int_{D^2} v^*\left(\mathcal{L}_{Y(p)}\omega\right) = -\int_{D^2} v^*\left(d\iota_Y \omega\right) \\ &= -\int_{\mathbb{S}^1} x^*\left(\iota_Y \omega\right) = \int_{\mathbb{S}^1} \omega(x'(t), Y(t)) dt. \end{aligned}$$

On the other hand, for the first term we have

$$\int_0^1 \frac{\partial}{\partial s}\bigg|_{s=0} H_t\left(z(s,t)\right) dt = \int_0^1 dH_t\left(x(t)\right) \cdot Y(t) dt = -\int_{\mathbb{S}^1} \omega\left(X_{H_t}, Y\right) dt.$$

Therefore,

$$d\mathcal{A}_H(x) \cdot Y = \int_0^1 \omega\left(x'(t) - X_{H_t}\left(x(t)\right), Y(t)\right) dt.$$

A loop x is a critical point of $\mathcal{A}_H$ if and only if this last expression vanishes for all $Y \in T_x\mathcal{L}M$ and, as ω is non-degenerate, this happens if and only if $x'(t) = X_{H_t}(x(t))$ for all $t \in \mathbb{S}^1$, which means that x has to be an orbit of the Hamiltonian vector field. $\square$

Definition 2.2.25 Consider a Hamiltonian $H : \mathbb{S}^1 \times M \to \mathbb{R}$. We say that H is *regular* if all of the 1-periodic orbits of X_{H_t} are non-degenerate (see Definition 2.2.16).

Lemma 2.2.26 *Let H be an regular Hamiltonian defined on a compact manifold. Then, all* 1*-periodic orbits of* X_{H_t} *are isolated. In particular,* X_{H_t} *has a finite number of* 1*-periodic orbits.*

Definition 2.2.27 The *Floer complex* is the graded $\mathbb{Z}_2$-vector spaces generated freely by the critical points of $\mathcal{A}_H$:

$$CF_k(M,\omega,H) := \left\{ \sum_{i=1}^{N} a_{x_i} x_i \,\middle|\, N \in \mathbb{N}, x_i \in \mathrm{Crit}(\mathcal{A}_H),\ \mu_{CZ}(x_i) = k, a_{x_i} \in \mathbb{Z}_2 \right\}.$$

The grading on the Floer complex is given by a map μ_{CZ} called the *Conley-Zehnder index*, which we will introduce here. The reader is encouraged to look at Robbin and Salamon [RS93] or at Gutt [Gut12] for a definition beyond our brief sketch.

Definition 2.2.28 Let $\mathrm{SP}(n)$ be the set of continuous paths of matrices $\psi : [0,1] \to \mathrm{Sp}\left(\mathbb{R}^{2n}, \omega_{\mathrm{st}}\right)$ (the space of symplectic matrices) such that $\psi(0) = \mathrm{Id}$ and $\det(\mathrm{Id} - \psi(1)) \neq 0$.

The idea is to provide a map $\mathrm{SP}(n) \to \mathbb{Z}$ that keeps track of the "rotation" taken by a given path of matrices. To provide more context, take into consideration the topological properties of the space of symplectic matrices, $\mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$:

Lemma 2.2.29 *The first homotopy group of the space of symplectic matrices is* $\pi_1\left(\mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})\right) = \mathbb{Z}$. *Moreover, there exists a projection* $\rho : \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}}) \to \mathbb{S}^1$ *that induces an isomorphism of the homotopy groups.*

We will call the projection ρ from Lemma 2.2.29 the *rotation map*.

Informally, the Conley-Zehnder index can be constructed from a path $\psi \in \mathrm{SP}(n)$ by extending it into a particular path $\widetilde{\psi} : [0,2] \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ such that $\widetilde{\psi}(2) = \mathrm{Id}$. This means, $\widetilde{\psi} : \mathbb{S}^1 \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$. The index can then be defined to be the degree of the map $\rho \circ \widetilde{\psi} : \mathbb{S}^1 \to \mathbb{S}^1$.

We include this sketch to provide a topological intuition for this construction. However, the Conley-Zehnder can also be succintly defined axiomatically, as follows.

Proposition 2.2.30 ([Gut12, Proposition 37]) *There exists a unique map* $\mu_{CZ} : \mathrm{SP}(n) \to \mathbb{Z}$ *such that it satisfies the properties*

1. (Homotopy): *If* $\psi_0, \psi_1 \in \mathrm{SP}(n)$ *are homotopic paths, then* $\mu_{CZ}(\psi_0) = \mu_{CZ}(\psi_1)$.
2. (Loop): *Let* $\psi \in \mathrm{SP}(n)$ *and* $\phi : [0,1] \to \mathrm{Sp}(\mathbb{R}^{2n}, \omega_{\mathrm{st}})$ *such that* $\phi(0) = \phi(1) = \mathrm{Id}$. *Then,*
$$\mu_{CZ}(\phi\psi) = \mu_{CZ}(\psi) + 2\deg(\rho \circ \phi).$$
3. (Signature): *Let* S *a symmetric matrix such that* $0 < |\lambda_i| < 2\pi$ *for all its eigenvalues. Let* $\psi \in \mathrm{SP}(n)$ *given by* $\psi(t) = \exp(J_0 St)$, *where* $J_0 = \begin{pmatrix} 0 & \mathrm{Id} \\ -\mathrm{Id} & 0 \end{pmatrix}$. *Then,*
$$\mu_{CZ}(\psi) = \frac{1}{2}\mathrm{Sign}(S),$$
where Sign *denotes the signature of* S.

Definition 2.2.31 If $x \in \mathcal{P}(H)$ (a 1-periodic orbit of X_{H_t}), we say that the *Conley-Zehnder index* of x is the integer given by $\mu_{CZ}(x) := \mu_{CZ}\left(d\varphi^t_{X_{H_t}}\right)$.

Remark 2.2.32 The Conley-Zehnder index in Definition 2.2.31 is well defined in this context because of the condition of asphericality on the Chern class, which guarantees that the tangent bundle TM can be symplectically trivialized when restricted to any $\mathbb{S}^2$ embedded in M. As a consequence, two different choices of symplectic trivialization of TM over a disk filling a contractible orbit will be equivalent.

To define a boundary map for the Floer complex, we need a way to connect two periodic orbits, which we accomplish with the Floer equation.

Let (M, ω, J) be a symplectic manifold with a compatible almost-complex structure. Let $H \in C^\infty(\mathbb{R} \times M)$ a regular Hamiltonian, and let $u : \mathbb{R} \times \mathbb{S}^1 \to M$. The Floer equation can be found when computing the negative gradient flow of the action functional along u with respect to the metric induced by g_J on $\mathcal{L}M$:

Definition 2.2.33 Let $u : \mathbb{R} \to \mathcal{L}M$ smooth. The *Floer equation* is given by

$$\frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \mathrm{grad} H_t(u) = 0. \tag{2.5}$$

The *energy* of a solution u of Equation 2.5 is defined by

$$E(u) = \int_{\mathbb{S}^1 \times \mathbb{R}} u^* d\mathcal{A}_H,$$

and the *set of finite energy solutions* is defined by

$$\mathcal{M} = \left\{ u : \mathbb{R} \times \mathbb{S}^1 \to M \mid u \text{ is a solution to 2.5 and } E(u) < \infty \right\}. \tag{2.6}$$

The properties of $\mathcal{M}$ were studied by Floer using the techniques pioneered by Gromov in his analysis of pseudo-holomorphic curves. In the particular context of the Floer equation, very similar methods are used to show the following theorem:

Theorem 2.2.34 (Audin and Damian [AD14, Theorems 6.5.4 and 6.5.6]) *For a generic choice of the almost-complex structure J, the set $\mathcal{M}$ is compact in $C^\infty_{loc}(\mathbb{R} \times \mathbb{S}^1; M)$. Moreover, for any $u \in \mathcal{M}$, there exist two critical points of $\mathcal{A}_H$, x and y, such that*

$$\lim_{s \to -\infty} u(s, \cdot) = x, \ \lim_{s \to +\infty} u(s, \cdot) = y$$

in $C^\infty(\mathbb{S}^1, M)$, and

$$\lim_{s \to \pm\infty} \frac{\partial u}{\partial s}(s, \cdot) = 0.$$

Therefore, $\mathcal{M} = \bigcup_{x,y \in \mathrm{Crit}(\mathcal{A}_H)} \mathcal{M}(x, y)$.

In order the define a boundary map for the Floer complex one must study the sets $\mathcal{M}(x, y)$. It is possible to show that each $\mathcal{M}(x, y)$ is a finite dimensional smooth manifold. To do so, one studies the linear approximation of the Floer operator acting on the set of perturbations of elements of $\mathcal{M}$. The space of such perturbations is defined as

$$\mathcal{P}^{1,p}(x, y) = \left\{ P : (s, t) \mapsto \exp_{u(s,t)} Y(s, t) \mid u \in \mathcal{M}(x, y), Y \in W^{1,p}(u^*(TM)) \right\}.$$

Here $W^{1,p}(u^*(TM))$ denotes the set of maps $Y : \mathbb{R} \times \mathbb{S}^1 \to TM$ such that $\pi \circ Y = u$ and such that their local trivializations belong to the Sobolev space $W^{1,p}$. The exponential $\exp : T_pM \to M$ is the exponential induced by the Riemannian structure g_J.

Then, the linearised Floer operator is defined as

$$\begin{array}{rccc} \mathcal{F}: & \mathcal{P}^{1,p}(x,y) & \longrightarrow & L^p(\mathbb{R},\mathbb{S}^1) \\ & w & \longmapsto & \frac{\partial w}{\partial s} + J_w \frac{\partial w}{\partial t} + \operatorname{grad}_w H_t, \end{array}$$

and its linearisation has the expression

$$d_u\mathcal{F}(Y) = \frac{\partial Y}{\partial s} + J_u \frac{\partial Y}{\partial t} + (\mathcal{L}_Y J)_u \frac{\partial u}{\partial t} + \mathcal{L}_Y \left(\operatorname{grad}_u H_t\right).$$

One can conclude (see [AD14, Theorem 8.1.5] or [Sal99, Theorem 2.2]) that $d_u\mathcal{F}$ is a Fredholm map for any $u \in \mathcal{M}(x,y)$ and that its Fredholm index is $\operatorname{Ind}(d\mathcal{F}_u) = \mu_{CZ}(x) - \mu_{CZ}(y)$, the difference of Conley-Zehnder indices. Moreover, $d_u\mathcal{F}$ is a surjective map for any non-degenerate Hamiltonian H and any almost complex structure J compatible with ω (recall that in Remark 2.2.20 we mentioned that two almost complex structures compatible with the same symplectic form are always homotopic). From this, it is possible to determine the dimension of $\mathcal{M}(x,y)$ as follows:

Theorem 2.2.35 *For $p > 2$, $\mathcal{F}^{-1}(0)$ is a smooth manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y)$.*

Definition 2.2.36 Let x and y two 1-periodic Hamiltonian orbits. The set of *non-parametrized trajectories*, denoted by $\mathcal{T}(x,y)$, is the quotient of the manifold $\mathcal{M}(x,y)$ by the action of $\mathbb{R}$.

Remark 2.2.37 It is possible to conclude (see [AD14, Chapter 9]) that $\mathcal{T}(x,y)$ is Hausdorff. Moreover, for any pair (x,y) of orbits, $\mathcal{T}(x,y)$ can be compactified into a manifold of dimension $\mu_{CZ}(x) - \mu_{CZ}(y) - 1$, which we denote by $\overline{\mathcal{T}}(x,y)$.

Definition 2.2.38 Let (x,y) a pair of Hamiltonian orbits such that $\mu_{CZ}(x) = \mu_{CZ}(y) + 1$. Then, we denote by $n(x,y)$ the cardinality of the zero-dimensional and compact manifold $\overline{\mathcal{T}}(x,y)$, modulo 2.

Then, for each $k \in \mathbb{N}$ the *boundary map* of the Floer complex is the map $\partial_k : CF_{k+1}(M;H,J) \to CF_k(M;H,J)$ given by

$$\partial_k(x) = \sum_{y \in CF_k(M;H,J)} n(x,y)y.$$

Theorem 2.2.39 (Floer [Flo89a, Theorem 4])

$$\partial_k \circ \partial_{k+1} = 0.$$

Thus, the Floer complex $(CF_\bullet(M; H, J), \partial_\bullet)$ is well defined and it induces a homology. It is clear that we used both H and J to define this complex, so we would expect the complex (and therefore the homology) to depend on these choices. However, this is not the case for the homology:

Theorem 2.2.40 (Floer [Flo89a, Theorem 5], Audin and Damian [AD14, Chapter 11]) *The homology induced by the Floer complex does not depend on the choice of a pair (H, J).*

Indeed, this homology can be identified with the Morse homology:

Theorem 2.2.41 (Floer [Flo89a, Theorem 1], Audin and Damian [AD14, Theorem 10.1.1]) *The Floer homology is isomorphic to the Morse homology with a shift in the degree,*

$$HF_\bullet(M) \cong HM_{\bullet+n}(M),$$

where $\dim(M) = 2n$.

Theorem 2.2.41 proves that the Floer homology is indeed a topological invariant of an (aspherical) symplectic manifold. Moreover, it provides explicitly the dimensions of the homology groups, in relation to the groups of the Morse homology.

The power of this result resides in the fact that the groups of the Floer complex may be impossible to compute, rendering an explicit computation of the Floer homology impossible. However, the theorem allows the Morse inequalities to be translated to this setting without any further effort.

In particular, this proves the Arnold conjecture for the aspherical symplectic case.

2.3 Poisson geometry

In this section we generalize the notion of symplectic geometry to a broader setting. This generalization is particular in that we still can compute a Hamiltonian vector field for any given function $H \in C^\infty(M)$ while relaxing the non-degeneracy condition.

Definition 2.3.1 A *Poisson bracket* or *Poisson structure* on a manifold M is a map $\{\cdot,\cdot\} : C^\infty(M)\times C^\infty(M) \to C^\infty(M)$ such that for all $f, g, h \in C^\infty(M)$:

1. It is *skew-symmetric*, this means, $\{g, f\} = -\{f, g\}$.
2. It is *bilinear*, so $\{f, ag + bh\} = a\{f, g\} + b\{f, h\}$ for all $a, b \in \mathbb{R}$.
3. It satisfies the *Jacobi identity*: $\{f,\{g, h\}\}+\{g,\{h, f\}\}+\{h,\{f, g\}\} = 0$.
4. It satisfies the *Leibniz identity*: $\{f, gh\} = g\{f, h\} + h\{f, g\}$.

If $\{\cdot,\cdot\}$ is a Poisson bracket, we say that $(M,\{\cdot,\cdot\})$ is a *Poisson manifold*.

Example 2.3.2 Any symplectic manifold has a Poisson structure given by

$$\{f, g\}_\omega := \omega(X_f, X_g).$$

Thus, symplectic manifolds are particular cases of Poisson manifolds. However, there are meaningful differences between both classes. For instance, there exist no topological restrictions on the existence of a Poisson bracket on a given manifold, because the trivial bracket is always a possibility:

Example 2.3.3 Let M be a smooth manifold. Then, $(M,\{\cdot,\cdot\}_0)$ is a Poisson manifold, where $\{f, g\}_0 = 0$ for all $f, g \in C^\infty(M)$.

In Definition 2.1.8 we introduced the notion of symplectic and Hamiltonian vector fields. We have analogous notions of Poisson and Hamiltonian vector fields in this context:

Definition 2.3.4 Let $(M,\{\cdot,\cdot\})$ be a Poisson manifold. We say that $X \in \mathfrak{X}(M)$ is a *Poisson vector field* if $\forall f, g \in C^\infty(M)$ we have

$$\mathcal{L}_X\{f, g\} = \{\mathcal{L}_X f, g\} + \{f, \mathcal{L}_X g\}.$$

For a function $f \in C^\infty(M)$, its *Hamiltonian vector field* is given by the derivation

$$X_f(g) := \{f, g\}.$$

By the Jacobi identity in Definition 2.3.1 it is clear that a Hamiltonian vector field is always a Poisson vector field.

Remark 2.3.5 If $(M, \{\cdot,\cdot\})$ is induced by a symplectic structure ω, the Hamiltonian vector fields introduced in Definitions 2.1.8 and 2.3.4 coincide.

Poisson structures can be presented by means of a bivector field:

Definition 2.3.6 The *Poisson bivector field* associated to a Poisson bracket is the bivector field $\pi \in \Gamma\left(\bigwedge^2 TM\right)$ such that

$$\{f, g\} = \pi(df, dg)$$

for all $f, g \in C^\infty(M)$.

Conversely, a bivector field $\pi \in \Gamma\left(\bigwedge^2 TM\right)$ induces a Poisson structure on M if $[\pi, \pi] = 0$, where $[\cdot,\cdot]$ denotes the Schouten-Nijenhuis bracket.

We will not develop the notion of the Schouten-Nijenhuis bracket here. We just note that it is a generalization of the Lie bracket to multivector fields. A detailed introduction can be found, for example, in Laurent-Gengoux, Pichereau and Vanhaecke [LGPV13].

Example 2.3.7 For any symplectic manifold (M, ω), we will denote the Poisson bivector field associated to ω by ω^{-1}. In the case of the standard symplectic form in $\mathbb{R}^{2n}$ with coordinates $(x_1, \ldots, x_n, y_1, \ldots, y_n)$, the Poisson bivector field is

$$\omega^{-1} := \pi_{\mathrm{st}} = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}.$$

Lemma 2.3.8 *If f, g are smooth functions, then*

$$[X_f, X_g] = X_{\{f,g\}}.$$

Proof. Using the definition of a Hamiltonian vector field, we can compute the Lie derivative of a function h with respect to $[X_f, X_g]$:

$$\begin{aligned}\mathcal{L}_{[X_f,X_g]}h &= \mathcal{L}_{X_f}\mathcal{L}_{X_g}h - \mathcal{L}_{X_g}\mathcal{L}_{X_f}h = \{f,\{g,h\}\} - \{g,\{f,h\}\}\\ &= \{f,\{g,h\}\} + \{g,\{h,f\}\} = \{\{f,g\},h\} = \mathcal{L}_{X_{\{f,g\}}}h,\end{aligned}$$

where we use the Jacobi identity and the skew-symmetry of $\{\cdot,\cdot\}$. □

This fact allows us to define an involutive distribution for any Poisson manifold, induced by its set of Hamiltonian vector fields.

Definition 2.3.9 Let $(M, \{\cdot,\cdot\})$ be a Poisson manifold. Its *characteristic foliation*, also called *symplectic foliation* is the foliation induced by the distribution of Hamiltonian vector fields.

We say that the *rank* of a Poisson structure at a point $p \in M$ is the rank of the characteristic foliation at p.

Remark 2.3.10 The restriction of the Poisson structure to each leaf of the characteristic foliation is symplectic. In particular, the rank of the Poisson structure at a point is always even.

Poisson structures do not admit a uniform local expression as in Theorem 2.1.7, in part because the rank can vary depending on the point. However, Weinstein presented in [Wei83] a local form theorem:

Theorem 2.3.11 (Weinstein splitting theorem) *Let (M, π) be a Poisson manifold of dimension n and let $p \in M$. Let $2r$ denote the rank of the Poisson structure at p.*

Then, there exist a neighbourhood U of p and a local coordinate system in U centered at p with coordinates $(x_1, \dots, x_r, y_1, \dots, y_r, z_1, \dots, z_{n-2r})$, such that the Poisson structure has the local expression

$$\pi|_U = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{j<k} f_{j,k}(z) \frac{\partial}{\partial z_j} \wedge \frac{\partial}{\partial z_k},$$

where $f_{j,k}$ are smooth functions depending only on the variables $(z_1, \dots, z_{n-2r})$ and vanishing at the origin.

There is a particular vector field in the context of Poisson geometry that will be of special interest in our research: the modular vector field.

Definition 2.3.12 Let (M, π) be a Poisson manifold admitting a volume form Ω. The *modular vector field v_{mod}* associated to π and Ω is the unique vector field such that

$$\mathcal{L}_{X_f}\Omega = \left(\mathcal{L}_{v_{mod}} f\right)\Omega,$$

where X_f is the Hamiltonian vector field of f with respect to π, for all $f \in C^\infty(M)$.

Remark 2.3.13 Let (M, π) be a Poisson manifold and Ω and Ω' volume forms. Then there exists $g \in C^\infty(M)$ a strictly positive function, so that $\Omega' := g\Omega$. If v_{mod}^{Ω} and $v_{mod}^{\Omega'}$ denote the modular vector fields associated to the corresponding volume forms, then

$$v_{mod}^{\Omega'} = v_{mod}^{\Omega} + X_{\log g}.$$

Therefore, the difference between two modular vector fields is always a Hamiltonian vector field.

Remark 2.3.14 The modular vector field is always a Poisson vector field, this means, $\mathcal{L}_{v_{mod}}\pi = 0$.

We will introduce also a particular type of Poisson manifold: cosymplectic manifolds:

Definition 2.3.15 Let M be a smooth manifold of dimension $2n + 1$ and $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$ closed forms. We say that (α, β) is a *cosymplectic structure* if $\alpha \wedge \beta^n$ is a volume form. In that case we call (M, α, β) a *cosymplectic manifold*.

The *Hamiltonian vector field* associated to a function $f \in C^\infty(M)$ is the unique vector field X_f such that

$$\beta(X_f, \cdot) + \alpha(X_f)\alpha = -df,$$

and the Poisson structure induced by (α, β) is

$$\{f, g\} := \beta\left(X_f, X_g\right).$$

Remark 2.3.16 The symplectic foliation of a cosymplectic manifold (as introduced in Definition 2.3.9) is induced by the integrable distribution given by $\ker(\alpha)$.

Remark 2.3.17 A cosymplectic manifold (M, α, β) has the natural volume form $\alpha \wedge \beta^n$ by definition. Its modular vector field, also called *Reeb vector field* in this context, is given by

$$\alpha(v_{mod}) = 1, \ \beta(v_{mod}, \cdot) = 0.$$

Remark 2.3.18 There is an equivalent characterization of cosymplectic manifolds by means of the Poisson structure (see for instance Osorno-Torres [OT15]): Let (M, π) a Poisson manifold of dimension $2n + 1$ whose Poisson bivector field has constant rank $2n$. If there exists a Poisson vector field $X \in \mathfrak{X}(M)$ transverse to $\mathcal{F}$ everywhere, then the tuple (M, π, X) describes a cosymplectic manifold with modular vector field X.

Example 2.3.19 Consider $\mathbb{R}^{2n+1}$ with coordinates $(t, x_1, ..., x_n, y_1, ..., y_n)$. The standard cosymplectic structure is then given by

$$\alpha = dt,\ \beta = \sum_{i=1}^{n} dx_i \wedge dy_i.$$

In this case, $v_{mod} = \frac{\partial}{\partial t}$ and the Poisson structure has the form

$$\{f, g\} = \sum_{i=1}^{n} \left(\frac{\partial f}{\partial x_i} \frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial g}{\partial x_i} \right).$$

2.4 b^m-symplectic geometry

In this section we will provide a brief overview of the field of b^m-symplectic manifolds. The aim of this field is to reproduce techniques and results of symplectic geometry to a more general family of structures within Poisson geometry. Therefore, we will emphasize similarities and contrasts between b^m-symplectic and symplectic structures whenever possible.

A reader interested in furthering their understanding of this subject is advised to consult Guillemin, Miranda and Pires [GMP14] and Scott [Sco16].

The issue of terminology is relevant in this subject. The b in b^m-symplectic comes from *boundary*, and was introduced by Melrose in [Mel93] in the context of *b-calculus* while proving the Atiyah-Patodi-Singer theorem. The terminology was adopted by the aforementioned authors in their study of b^m-symplectic manifolds with a motivation in studying the dynamical aspects of the restricted 3-body problem under non-canonical changes of coordinates. For a complete development of this interpretation, see Kiesenhofer and Miranda [KM17] or Kiesenhofer, Miranda and Scott [KMS16].

However, the same structures and similar ones have been independently introduced and studied under the name of log-*symplectic structure* among others by Cavalcanti, Gualtieri, Li, Pelayo and Ratiu, see for instance [Cav17, GL14, GLPR17, CK19]. In their case the point of view is motivated by a study of generalized complex geometry and foliations of Poisson manifolds.

Further contributions may be found in [MP18, GMW18a, GMW18b, GMW19, KMS16, KM17, MOT14b, MOT14a].

2.4.1 *b*-manifolds

We begin by establishing the basic tools used to set up a theory of symplectic structures on singular manifolds.

Definition 2.4.1 A *b-manifold* is a pair (M, Z), where M is a smooth manifold and $Z \subset M$ is an embedded hypersurface. Z is often called the *singular hypersurface*, the *critical set* or the *divisor* of the b-manifold.

If (M_1, Z_1) and (M_2, Z_2) are b-manifolds and $f : M_1 \to M_2$ is a smooth map, we say that f is a *b-map* if $f^{-1}(Z_2) = Z_1$ and f is transversal to Z_2.

Definition 2.4.2 Let (M, Z) be a b-manifold, and let $\mathcal{N}(Z)$ denote a neighbourhood of Z, this means, an open set $\mathcal{N}(Z) \subset M$ containing Z. A *defining function* is a b-map $z : (\mathcal{N}(Z), Z) \to (\mathbb{R}, \{0\})$. In the case that a defining function z is defined in the whole manifold we say that it is a *global defining function* or *globally defined defining function*.

Moreover, if $U \subset M$ is an open set, we will say that a *local defining function* is a b-map $z : (U, U \cap Z) \to (\mathbb{R}, \{0\})$.

b-manifolds allow us to encode dynamical behaviours in which the hypersurface is left invariant by the system. A basic tool to understand this kind of systems is the b-tangent bundle.

Definition 2.4.3 Let (M, Z) be a b-manifold. A *b-vector field* is a vector field $X \in \mathfrak{X}(M)$ such that it is tangent to Z at all points of Z, this means, $X(z) \in TZ$ for all $z \in Z$. The subset of b-vector fields is denoted by ${}^b\mathfrak{X}(M, Z)$.

Remark 2.4.4 The set of b-vector fields is a submodule of $\mathfrak{X}(M)$. Moreover, it is closed under the Lie bracket.

Remark 2.4.5 If $p \in Z$ and we have a local chart around p given by $(U; z, x_2, \ldots, x_n)$, with $z : U \to \mathbb{R}$ a local defining function, it is possible to describe the set of b-vector fields in local coordinates as

$$ {}^b\mathfrak{X}(M, Z)\big|_U = \left\langle z\frac{\partial}{\partial z}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n} \right\rangle_{C^\infty(U)} $$

The set of b-vector fields can be related to a geometrical object, namely the b-tangent bundle, through the use of the Serre-Swan theorem.

Theorem 2.4.6 (Swan [Swa62]) *Let M be a smooth manifold and P a projective $C^\infty(M)$-module. Then, there exists a vector bundle $E \to M$ such that P is isomorphic to $\Gamma(E)$, the module of sections of E. Moreover, E is unique up to vector bundle isomorphisms.*

The module ${}^b\mathfrak{X}(M, Z)$ fulfils the requirements of Theorem 2.4.6, so we can define the vector bundle whose sections are precisely the b-vector fields.

Definition 2.4.7 Let (M, Z) be a b-manifold where $\dim(M) = n$. The *b-tangent bundle* is the unique vector bundle ${}^bTM \to M$ of rank n such that ${}^b\mathfrak{X}(M, Z) = \Gamma({}^bTM)$. The dual to this vector bundle, denoted by ${}^bT^*M$, is the *b-cotangent bundle*.

Definition 2.4.8 A *Lie algebroid* is a tuple $(\mathcal{A}, [\cdot, \cdot], \rho)$ where

- $\mathcal{A} \to M$ is a vector bundle,
- $\rho : \mathcal{A} \to TM$, called the *anchor map*, is a vector bundle morphism,
- $[\cdot, \cdot] : \Gamma(\mathcal{A}) \times \Gamma(\mathcal{A}) \to \Gamma(\mathcal{A})$ is a Lie bracket,

such that $[X, fY] = \left(\mathcal{L}_{\rho(X)} f\right) Y + f[X, Y]$ for all $X, Y \in \Gamma(\mathcal{A})$ and $f \in C^\infty(M)$.

Remark 2.4.9 The b-tangent bundle is a particular case of a Lie algebroid on M, where the anchor map ρ is induced by the inclusion of sections ${}^b\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$.

b-manifolds are particular cases of a more general family, that of b^m-manifolds:

Definition 2.4.10 (Scott [Sco16]) Let (M, Z) be a b-manifold and take an integer $m \geq 1$. A *b^m-vector field* is a vector field $X \in \mathfrak{X}(M)$ that is tangent to Z at order m with respect to some defining function $z : (\mathcal{N}(Z), Z) \to (\mathbb{R}, \{0\})$. The set of b^m-vector fields is denoted by ${}^{b^m}\mathfrak{X}(M)$.

The sheaf of b^m-vector fields constitutes a projective submodule of $\mathfrak{X}(M)$ for all $m \geq 1$, and therefore we can use Theorem 2.4.6 to define the b^m-tangent bundle and the as in Definition 2.4.7:

Definition 2.4.11 Let (M, Z) be a b-manifold and $m \geq 1$ an integer. The *b^m-tangent bundle* is the unique vector bundle ${}^{b^m}TM \to M$ whose sheaf of sections is isomorphic to ${}^{b^m}\mathfrak{X}(M)$. Its dual, denoted as ${}^{b^m}T^*M$, is the *b^m-cotangent bundle.*

By construction, when $m = 1$ we recover precisely the construction of Definition 2.4.7.

Remark 2.4.12 Let (M, Z) be a b-manifold with a defining function z and $m \geq 1$. Let $p \in Z$. Then, there is a chart centered on p, $(U; z, x_2, \ldots, x_n)$, such that

$$\Gamma\left({}^{b^m}T^*M\right)\Big|_U = \left\langle z^m\frac{\partial}{\partial z}, \frac{\partial}{\partial x_2}, \ldots, \frac{\partial}{\partial x_n}\right\rangle_{C^\infty(U)}$$

and

$$\Gamma\left({}^{b^m}T^*M\right)\Big|_U = \left\langle \frac{dz}{z^m}, dx_2, \ldots, dx_n\right\rangle_{C^\infty(U)}.$$

The notation $\frac{dz}{z^m}$ might seem misleading, as the expression cannot be evaluated at points where $z = 0$. However, what is meant by this expression is merely that the section $\frac{dz}{z^m}$ is such that its pairing with the non-vanishing section $z^m\frac{\partial}{\partial z} \in \Gamma\left({}^{b^m}TM\right)$ is precisely 1.

Definition 2.4.13 The set of *b^m-k-differential forms* on a b-manifold (M, Z) is the space of sections

$${}^{b^m}\Omega^k(M) := \Gamma\left(\Lambda^k\left({}^{b^m}T^*M\right)\right).$$

Proposition 2.4.14 (Guillemin-Miranda-Pires [GMP14]) *Let (M, Z) be a b-manifold with a fixed defining function z. Let $\omega \in {}^{b^m}\Omega^k(M)$. Then, there exist $\alpha \in \Omega^{k-1}(M)$ and $\beta \in \Omega^k(M)$ such that locally ω has the expression*

$$\omega = \alpha \wedge \frac{dz}{z^m} + \beta$$

around Z. The forms α and β are not necessarily unique, but $\alpha|_Z$ and $\beta|_Z$ are unique.

With this decomposition in mind we can define a structure completely analogous to that of the De Rham complex.

Definition 2.4.15 The *differential* is the graded map $d : {}^{b^m}\Omega^\bullet(M) \to {}^{b^m}\Omega^{\bullet+1}(M)$ such that, if $\omega = \alpha \wedge \frac{dz}{z^m} + \beta$, then

$$d\omega = d\alpha \wedge \frac{dz}{z^m} + d\beta.$$

The *b^m-De Rham complex* is then the chain complex

$$0 \to {}^{b^m}\Omega^0(M) \xrightarrow{d} {}^{b^m}\Omega^1(M) \xrightarrow{d} \cdots \xrightarrow{d} {}^{b^m}\Omega^n(M) \to 0.$$

The *b^m-De Rham cohomology* can be defined by

$${}^{b^m}H^k(M) := \frac{\ker\left(d^k\right)}{\operatorname{im}\left(d^{k-1}\right)}.$$

An interesting phenomenon in this case is that this homology can be very easily computed.

Theorem 2.4.16 (Mazzeo-Melrose) *Let (M, Z) be a b-manifold. The b^m-De Rham cohomology is*

$${}^{b^m}H^k(M) \cong \left(H^{k-1}(Z)\right)^m \oplus H^k(M).$$

There will be cases in which we will be interested in expanding the set $C^\infty(M)$ in such a way that the form $\frac{dz}{z^m}$ becomes exact. In such cases we will take b^m-functions into consideration.

Definition 2.4.17 The set of *b-functions* or *b-Hamiltonians* is the sheaf given locally by

$${}^bC^\infty(U) := \{a \log|z| + g \mid a \in \mathbb{R},\ g \in C^\infty(U)\}$$

when U intersects Z, where z denotes a local function $z : U \to \mathbb{R}$ defining $Z \cap U$. When U does not intersect Z we simply take ${}^bC^\infty(U) = C^\infty(U)$.

With the same notations, we define the set of *b^m-functions* or *b^m-Hamiltonians* as the sheaf constructed analogously as

$${}^{b^m}C^\infty(U) := \left(\bigoplus_{i=1}^{m-1} z^{-i}C^\infty(z)\right) \oplus {}^bC^\infty(U)$$

whenever $U \cap Z \neq \emptyset$, where $C^\infty(z)$ denotes smooth functions of one variable composed with z.

We will end this section by introducing a structure that will be useful in multiple places to encode topological information regarding the relative position of Z within M.

Definition 2.4.18 The *associated graph* to a b-manifold (M, Z) is the graph whose vertices are the connected components of $M \setminus Z$, often individually denoted by M_i. There is an edge between M_i and M_j if there is a connected component of Z that borders both M_i and M_j. We will usually refer to the edges (or connected components of Z) by Z_i.

In the case that $\dim(M) = 2$ we equip the graph with *weights*. The weight associated to the vertex M_i, denoted by g_{M_i}, is given by the genus of the surface $\overline{M_i}$ where each adjacent component of Z has been compactified into a point.

We say that a b-manifold is *cyclic* or *acyclic* if its associated graph contains (or, respectively, does not contain) a cycle.

An example of such a graph can be seen in Figure 2.1.

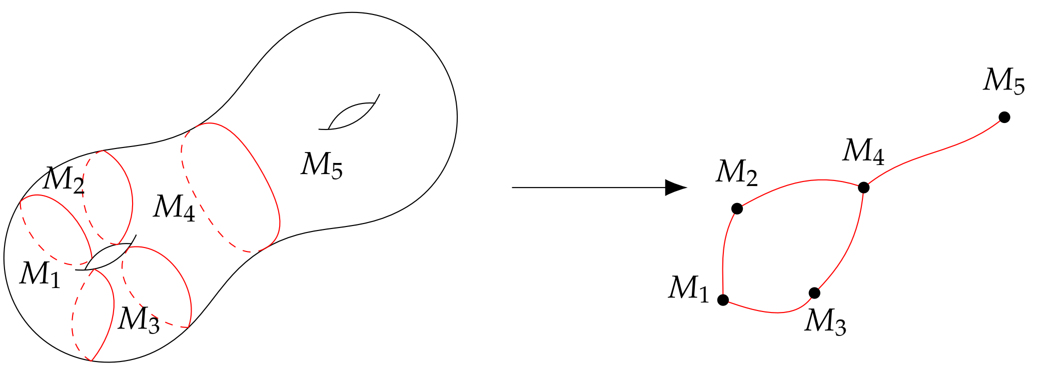

Figure 2.1: An example of the graph associated to a b-manifold (cyclic). All the weights are 0 except of g_{M_5}, which is 1.

A first use of the graph is the characterization of the b-manifolds that admit a global defining function (see Definition 2.4.2). The following result is essentially a reformulation from Miranda and Planas [MP18, Theorem 5.5].

Lemma 2.4.19 *A b-manifold admits a global defining function if and only if its associated graph is* 2*-colourable.*

Proof. Let $z : (M, Z) \to (\mathbb{R}, \{0\})$ a global defining function. By definition we know that $M \setminus Z = z^{-1}(\mathbb{R} \setminus \{0\})$. Therefore, we can choose a colouring of the graph by partitioning the set of connected components of $M \setminus Z$ between those that are subsets of $z^{-1}(\mathbb{R}_{>0})$ and those that are subsets of $z^{-1}(\mathbb{R}_{<0})$. Since z must vanish transversally in each connected component Z_i, the two connected components of $M \setminus Z$ adjacent to Z_i must belong to two different partitions, which in terms of the graph means that any two adjacent vertices must have different colours. In this way z induces a 2-colouring on the graph.

Conversely, let (M, Z) be a b-manifold whose graph admits a 2-colouring. Let $\mathcal{N}(Z)$ an open neighbourhood of Z small enough so that $M \setminus \mathcal{N}(Z)$ is homeomorphic to $M \setminus Z$. Consider now a partition $\{U_1, ..., U_k; V_1, ..., V_l\}$ of the connected components of $M \setminus Z$ given by the 2-colouring, so $M \setminus Z = U_1 \cup ... \cup U_k \cup V_1 \cup ... \cup V_l$, and let $\left\{\overline{U_i}, \overline{V_i}\right\}$ denote the associated connected components of $M \setminus \mathcal{N}(Z)$. Moreover, for each connected component $Z_i \subset Z$ consider a local defining function z_i such that $z_i^{-1}(\mathbb{R}_{>0})$ lies within one of the sets U_i and $z_i^{-1}(\mathbb{R}_{<0})$ lies within one of the V_i. Then, we can construct a global defining function $z : M \to \mathbb{R}$ by smoothly interpolating each of the local defining functions z_i with the constant 1 in $\cup_{i=1}^k \overline{U_i}$ and with the constant -1 in $\cup_{i=1}^l \overline{V_i}$. □

2.4.2 b^m-symplectic structures

As we mentioned at the beginning of this section, we are interested in generalizing the notion of symplectic structures to a setting where singularities are introduced. With the introduction of b^m-smooth forms we have the tools to do so.

Definition 2.4.20 Let (M, Z) be a b-manifold and $m \geq 1$. A b^m*-symplectic structure* on (M, Z) is a b^m-2-form $\omega \in {}^{b^m}\Omega^2(M)$ such that it is closed, i.e, $d\omega = 0$, and it is non-degenerate as a b^m-form.

Theorem 2.4.21 (b^m-Darboux) *Let ω be a b^m-symplectic form on (M^{2n}, Z). Let $p \in Z$. Then, there exists a coordinate chart $(U; z, t, x_2, y_2, \ldots, x_n, y_n)$ centered at p such that z is a local defining function for $U \cap Z$ and*

$$\omega|_U = \frac{dz}{z^m} \wedge dt + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Let us present some examples of b^m-symplectic structures on some b-manifolds.

Example 2.4.22 $(\mathbb{R}^{2n}, Z = \{x_1 = 0\})$ has a natural b^m-symplectic structure in the form of

$$\omega = \frac{dx_1}{x_1^m} \wedge dy_1 + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Example 2.4.23 Let $\mathbb{S}^2 \subset \mathbb{R}^3$ be the sphere and $Z := \{(x,y,z) \in \mathbb{S}^2 \mid z = 0\}$. The b-manifold $(\mathbb{S}^2, Z)$ admits a b^m-symplectic structure that has the expression

$$\omega = -\frac{dz}{z^m} \wedge d\theta$$

in cylindrical coordinates (z, θ) on $\mathbb{S}^2 \setminus \{N, S\}$, and the expression

$$\omega = -\frac{1}{1 - x^2 - y^2} dx \wedge dy$$

in Cartesian coordinates (x, y) in $U^+ := \left\{(x,y,z) \in \mathbb{S}^2 \mid z > 0\right\}$ and in $U^- := \left\{(x,y,z) \in \mathbb{S}^2 \mid z < 0\right\}$.

Example 2.4.24 Let $\mathbb{T}^{2n}$ denote the $2n$-torus with coordinates $(\theta_1, \ldots, \theta_{2n})$. Take $Z := \{\theta_1 = 0\} \sqcup \{\theta_1 = \pi\}$. The b-manifold then admits the b^m-symplectic structure

$$\omega = \frac{d\theta_1}{\sin^m \theta_1} \wedge d\theta_2 + \sum_{i=2}^{n} d\theta_{2i-1} \wedge d\theta_{2i}.$$

Remark 2.4.25 We note that there is a natural inclusion $\Omega^k(M) \hookrightarrow {}^{b^m}\Omega^k(M)$. In the particular case of $k = 1$, for any function $H \in C^\infty(M)$ we can consider that $dH \in {}^{b^m}\Omega^1(M)$ for any choice of Z and m. Therefore we can define the *Hamiltonian vector field* X_H of a Hamiltonian $H \in C^\infty(M)$ in the same way as in Definition 2.1.8 through the equation

$$\omega(X_H, \cdot) = -dH.$$

In the same way, following Example 2.3.2 we can define the Poisson structure associated to ω by

$$\{f, g\} = \omega\left(X_f, X_g\right).$$

The existence of a Poisson structure associated to a b^m-symplectic structure ω can be studied in more generality.

Definition 2.4.26 A Poisson structure π on M is *b-Poisson* if the multivector field π^n intersects the zero section of $\bigwedge^{2n} TM$ transversally along a hypersurface Z.

Let (M, Z) be a b-manifold and $m \geq 1$. A Poisson structure π on M is *b^m-Poisson* if its bracket satisfies that $\{\cdot,\cdot\}_\pi : I_Z \times C^\infty(M) \to I_Z^m$. In particular, b^1-Poisson structures are b-Poisson.

Lemma 2.4.27 *The Poisson bracket associated to a b^m-symplectic structure is a b^m-Poisson structure, and any b^m-Poisson structure can be represented by a b^m-symplectic structure.*

In particular, b-Poisson manifolds can be constructed from cosymplectic manifolds.

Example 2.4.28 Let (N, π, X) be a cosymplectic manifold induced by the Poisson structure π and the modular vector field X, and let $f : \mathbb{S}^1 \to \mathbb{R}$. Then, $(\mathbb{S}^1 \times N, f(\theta)\frac{\partial}{\partial\theta} \wedge X + \pi)$ is a b-Poisson manifold if and only if $df \neq$ at every point where f vanishes.

Proposition 2.4.29 (Guillemin, Miranda and Pires [GMP14]) *A form $\omega \in {}^{b^m}\Omega^2(M)$ is b^m-symplectic if and only if its associated bivector field π is b^m-Poisson.*

In Proposition 2.4.29 the Poisson structure associated to a b^m-symplectic form is defined in the same way as in Example 2.3.2, as $\{f, g\} = \omega\left(X_f, X_g\right)$.

Let us now introduce a particular vector field, the normal vector field, which will be useful to study the geometry of a b^m-symplectic manifold near Z.

Lemma 2.4.30 *There is a natural inclusion $\Gamma\left({}^{b^m}TM\big|_Z\right) \hookrightarrow \mathfrak{X}(Z)$. This induces a morphism of vector bundles $\varphi : {}^{b^m}TM\big|_Z \to TZ$. This morphism is surjective, and the kernel of φ is a trivial line bundle $L_Z \to Z$.*

The line bundle introduced in Lemma 2.4.30 can be extended to a tubular neighbourhood $\mathcal{N}(Z)$ of Z, which we will denote $L_{\mathcal{N}(Z)} \to \mathcal{N}(Z)$.

Definition 2.4.31 The *normal* or *normal symplectic b^m-vector field*, denoted X^σ is a trivializing vector field for $L_{\mathcal{N}(Z)}$. The normal vector field can moreover be chosen to be symplectic with respect to a given b^m-symplectic form, i.e., $\mathcal{L}_{X^\sigma}\omega = 0$.

Example 2.4.32 Let (M, ω, Z) be a b^m-symplectic manifold with a local defining function $z : \mathcal{N}(Z) \to \mathbb{R}$ such that the local expression of ω is $\omega = \frac{dz}{z^m} \wedge \alpha + \beta$, with $\alpha \in \Omega^1(M)$ and $\beta \in \Omega^2(M)$. The normal vector field can be then chosen as $X^\sigma = z^m \frac{\partial}{\partial z}$.

Remark 2.4.33 Let (M, ω, Z) be a b^m-symplectic manifold of dimension $2n$. The rank of the Poisson structure associated to ω is $2n$ for all points in $M \setminus Z$ and $2n-2$ for all points in Z.

In other words, the symplectic foliation associated to ω is composed of the connected components of $M \setminus Z$ and of a codimension 1 foliation within each of the connected components of Z.

It is possible to give an interpretation of b^m-symplectic manifolds as symplectic manifolds with boundary, where we have a symplectic vector field pointing towards the boundary. More precisely,

Lemma 2.4.34 (Frejlich, Martínez-Torres and Miranda [FMTM17]) *Let $(M, \partial M)$ be a manifold with boundary ∂M and let $\omega \in \Omega^2(M)$ be a symplectic form on $M \setminus \partial M$ such that there exists a symplectic vector field X^σ that points outwards or inwards at the boundary.*

Then, for each $m \in \mathbb{N}, m \geq 1$ there exists a b^m-symplectic structure on $(M, \partial M)$ with critical set given by ∂M that coincides with the symplectic structure outside of a tubular neighborhood of the boundary ∂M.

Proof. We start by showing that the boundary ∂M can be endowed with a cosymplectic structure.

We assume without loss of generality that X^σ points towards the interior of M at ∂M. Let $\varphi_{X^\sigma} : U \subset M \times \mathbb{R} \to M$ denote the flow of X^σ. As X^σ is transverse to ∂M, there exists a tubular neighbourhood $V_\varepsilon := \{(x, z) \mid 0 \leq z < \varepsilon(x)\} \subset \partial M \times \mathbb{R}$ for some function $\varepsilon : \partial M \to \mathbb{R}$ such that

$$\begin{array}{rccc} c: & V_\varepsilon & \longrightarrow & M \\ & (x,z) & \longmapsto & \varphi^z_{X^\sigma}(x) \end{array}$$

is an embedding.

Let $\theta := c^*(\iota_{X^\sigma}\omega)$ and $\eta := c^*\omega$. Both forms are invariant with respect to $\frac{\partial}{\partial z}$. We can show this for η by the observation that the vector fields $\frac{\partial}{\partial z}$ and X^σ are c-related, so $\mathcal{L}_{\frac{\partial}{\partial z}}\eta = c^*\left(\mathcal{L}_{X^\sigma}\omega\right) = 0$. Conversely,

$$\mathcal{L}_{\frac{\partial}{\partial z}}\theta = c^*\left(\mathcal{L}_{X^\sigma}\iota_{X^\sigma}\omega\right) = c^*\left(d\iota_{X^\sigma}\iota_{X^\sigma}\omega + \iota_{X^\sigma}d\iota_{X^\sigma}\omega\right) = c^*\left(\iota_{X^\sigma}\mathcal{L}_{X^\sigma}\omega\right) = 0,$$

where the first term vanishes because ω is skew-symmetric, and the second term vanishes because X^σ is a symplectic vector field.

Moreover, if $j_{\partial M} : \partial M \hookrightarrow V_\varepsilon$ denotes the inclusion by $j_{\partial M}(x) = (x, 0)$, $j^*_{\partial M}\left(\theta \wedge \eta^{n-1}\right)$ is a volume form for ∂M, because

$$\theta \wedge \eta^{n-1} = c^*\left(\iota_{X^\sigma}\omega \wedge \omega^{n-1}\right) = \frac{1}{n}c^*\left(\iota_{X^\sigma}\omega^n\right) = \frac{1}{n}\iota_{\frac{d}{dz}}c^*\omega^n,$$

whose pull-back by $j_{\partial M}$ is a well defined non-degenerate form due to the z-invariance.

Therefore, $(\partial M, \theta, \eta)$ is a cosymplectic manifold, and c is an embedding into $V_\varepsilon \subset M$. Moreover, by our definitions it is clear that

$$c^*\omega = dz \wedge \theta + \eta.$$

Let $\psi : V_\varepsilon \to \mathbb{R}$ be a smooth bump function such that $\psi(x, z) = 1$ when $0 \leq z < \frac{\varepsilon(x)}{3}$ and $\psi(x, z) = 0$ when $z > \frac{2\varepsilon(x)}{3}$. Then the b-form

$$\overline{\omega} = d(\psi \log z + (1-\psi)z) \wedge \theta + \eta$$

is clearly non-degenerate and therefore a b-symplectic form on V_ε. If we push it forward to M, it coincides with ω outside of a tubular neighborhood of ∂M.

In the same way, for $m > 1$ we can use the b^m-form

$$\overline{\omega} = d\left(-\frac{\psi}{m-1}\frac{1}{z^{m-1}} + (1-\psi)z\right) \wedge \theta + \eta,$$

which again is a b^m-symplectic form on V_ε whose push-forward to M coincides with ω outside a neighbourhood of ∂M. □

The tubular neighborhood and the defining function can be chosen to satisfy the following Proposition.

Proposition 2.4.35 (Guillemin, Miranda and Weitsman [GMW18a, Theorem 2]) *There exist a choice of defining function z for the critical set and a projection $\pi : \mathcal{N}(Z) \to Z$ such that there exists an expansion for ω of the form*

$$\omega|_{\mathcal{N}(Z)} = \sum_{i=1}^{m} \frac{dz}{z^i} \wedge \pi^*\alpha_i + \pi^*\beta$$

where $\alpha_i \in \Omega^1(Z)$ are closed and $\beta \in \Omega^2(Z)$ is symplectic on the foliation defined by α_m.

Remark 2.4.36 For convenience, sometimes we will denote the expansion from Proposition 2.4.35 by

$$\omega|_{\mathcal{N}(Z)} = \frac{dz}{z^m} \wedge \widetilde{\alpha} + \beta,$$

where $\widetilde{\alpha} = \sum_{i=1}^{m} z^{m-i}\pi^*\alpha_i$.

Recall from Remark 2.3.17 that the modular vector field can be characterized on $\mathcal{N}(Z)$ by the equations

$$\iota_{v_{mod}}\widetilde{\alpha} = 1 \ , \ \iota_{v_{mod}}\beta = 0.$$

Remark 2.4.37 Let (M, Z, π) be a b^m-symplectic manifold and let π be its Poisson structure. Then, $(\pi|_Z \, , v_{mod})$ is a cosymplectic structure.

If $\omega = \frac{dz}{z^m} \wedge \alpha + \beta$, the pull-back of the forms α and β are explicitly the forms defining a cosymplectic structure.

Theorem 2.4.38 (Guillemin, Miranda and Pires [GMP14]) *Let (M, Z, ω) be a b^m-symplectic manifold, let Z_i denote a connected component of Z and let $\mathcal{F}$ denote the symplectic foliation restricted to Z_i.*

If $\mathcal{F}$ contains a compact leaf $\mathcal{L}$, then every leaf of $\mathcal{F}$ is symplectomorphic to $\mathcal{L}$. Moreover, Z_i is the total space of a fibration over $\mathbb{S}^1$ given by the mapping torus of the symplectomorphism $\phi : \mathcal{L} \to \mathcal{L}$ induced by the flow of a modular vector field v_{mod}. This means, $\phi = \varphi^T_{v_{mod}}$ for some $T \in \mathbb{R}$, $T > 0$ and

$$Z_i \cong \frac{\mathcal{L} \times [0, T]}{(x, 0) \sim (\phi(x), T)}. \tag{2.7}$$

Definition 2.4.39 The *modular period* or the *modular weight* of a connected component $Z_i \subset Z$ is the period of the flow of v_{mod} on Z_i, this means, the number T in Theorem 2.4.38.

Remark 2.4.40 The situation exposed in Theorem 2.4.38 can be understood to be, in fact, generic. If the cosymplectic structure in Z induces a symplectic foliation with non-compact leaves, then there exists a family of cosymplectic structures that tend to it in the C^0-topology, all of which have a symplectic foliation with compact leaves (see Tischler [Tis70] and Frejlich, Martínez-Torres and Miranda [FMTM17]).

Let us recall from Definition 2.4.18 the notion of the graph of a b-manifold. Taking into account b^m-symplectic structures, we may have a restriction on the graph:

Remark 2.4.41 Let (M, Z, ω) be a b^{2k+1}-symplectic manifold of dimension $2n$, with M orientable. Then, the graph (V, E) of the b-manifold (M, Z) must admit a 2-colouring induced by the sign of ω^n with respect to an orientation on M.

To conclude this summary we highlight the fact that b-symplectic closed and oriented surfaces where first completely classified by Radko [Rad02].

Theorem 2.4.42 (Radko [Rad02]) *Let Σ be a closed and oriented surface. Two b-symplectic structures $\omega_1, \omega_2 \in {}^b\Omega(\Sigma)$ are equivalent if, and only if, the following invariants coincide:*

- *The* regularized Liouville volume *of the b-symplectic forms, which, if $z : \Sigma \to \mathbb{R}$ denotes a defining function for Z_i, is the well defined limit*

$$\mathrm{Vol} := \lim_{\varepsilon \to 0} \int_{|z| > \varepsilon} \omega_i.$$

- *The topology of Z.*
- *The modular weight of each connected component of Z.*

This classification was expanded to b^m-symplectic structures in closed surfaces by Scott [Sco16] and by Miranda and Planas [MP18] in the non-orientable case. The classification is expressed in terms of the b^m-de Rham cohomology classes:

Theorem 2.4.43 (Scott [Sco16], Miranda and Planas [MP18]) *Let (Σ, Z) a closed b^m-surface. Two b^m-symplectic structures ω_1, ω_2 on Σ are equivalent if, and only if, $[\omega_1] = [\omega_2]$ in ${}^{b^m}H^2(\Sigma)$.*

2.4.3 The topology of the b^m-tangent bundle

We end our first exploration of the properties of b^m-manifolds by giving some insights into the relationship between the tangent bundle of a manifold and its b^m-tangent bundle. In particular, we are interested on the possible topologies that the latter may exhibit.

Although most of the results in this subsection were already known, Examples 2.4.47 and 2.4.48 are original to this thesis.

We give first some examples of low dimensional b^m-manifolds and their associated bundles:

Example 2.4.44 Let $M = \mathbb{S}^1 \cong [0,1]/0 \sim 1$ and $Z = \{z_1, ..., z_N\}$ a finite number of points. Without loss of generality we can assume that $\{z_1, \ldots, z_N\}$ are equidistant, this means, $z_i = \left[\frac{i-1}{N}\right] \in \mathbb{S}^1$.

If m is even, the b^m-tangent bundle always admits a trivializing vector field, $X := \sin^m(Nt)\frac{\partial}{\partial t}$. The same family of b^m-vector fields works if m is odd but N is even. Since we know that all possible rank 1 vector bundles over $\mathbb{S}^1$ are either trivial or the Möbius strip, we conclude:

$$ {}^{b^m}T\mathbb{S}^1 \cong \begin{cases} T\mathbb{S}^1 & \text{if } N \text{ or } m \text{ are even,} \\ \text{the Möbius strip} & \text{if } N \text{ and } m \text{ are odd.} \end{cases} $$

Example 2.4.45 Consider the 2-torus $\mathbb{T}^2$ with Z being the disjoint union of N handles as in Figure 2.2. From Example 2.4.44 we can deduce that if N is even or m is even then we can parallelize ${}^{b^m}T\mathbb{T}^2$.

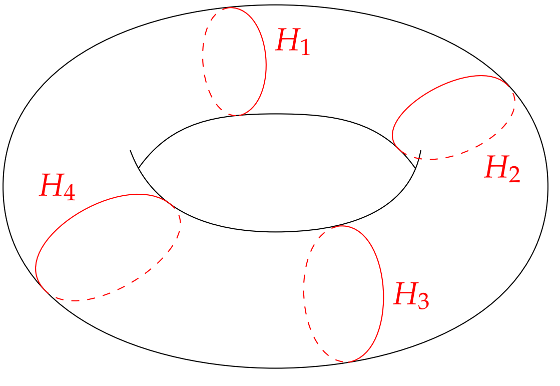

Figure 2.2: A torus with $N = 4$ handles

The parity of N in these examples matters because the trivializing sections allow us to associate a sign to each connected component of $M \setminus Z$.

Following this logic we can arrive at the following result.

Proposition 2.4.46 ([Kla17, Corollary 11.2.4]) *Let (M, Z) a b^{2k+1}-manifold such that both TM and ${}^bTM$ are orientable. Then, $M \backslash Z$ has two or more connected components.*

Proof. Let $n = \dim(M)$. If both TM and ${}^{b^{2k+1}}TM$ are orientable, this implies that both $\bigwedge^n T^*M$ and $\bigwedge^n {}^{b^{2k+1}}TM$ are trivializable. Let $\Omega \in \Gamma(\bigwedge^n T^*M)$ and $A \in \Gamma\left(\bigwedge^n {}^{b^{2k+1}}TM\right)$ trivializing sections.

The natural inclusion ${}^{b^{2k+1}}\mathfrak{X}(M) \hookrightarrow \mathfrak{X}(M)$ induces an inclusion $\Gamma\left(\bigwedge^n {}^{b^{2k+1}}TM\right) \hookrightarrow \bigwedge^n \mathfrak{X}(M)$. Abusing notation, let A denote the image of the trivializing section of $\Gamma\left(\bigwedge^n {}^{b^{2k+1}}TM\right)$ in $\bigwedge^n \mathfrak{X}(M)$.

Consider the function F resulting from the contraction of the trivializing sections, $F := \langle \Omega, A\rangle : M \to \mathbb{R}$. As a consequence of the local expression of the sections of the b^m-tangent bundle as seen in Remark 2.4.12, the function F must have the local expression $F(z, x_2, \dots, x_n) = z^m g$ around any point in Z, where g is a non-vanishing function. Moreover, F cannot vanish in $M \setminus Z$.

Consider the open sets $U_+ := F^{-1}(\mathbb{R}_{>0})$ and $U_- := F^{-1}(\mathbb{R}_{<0})$. By construction, $M \setminus Z = U_+ \sqcup U_-$. Also, if Z is non-empty, and as a consequence of the local expression of F that we have just presented, neither U_+ nor U_- can be empty.

Therefore, $M \setminus Z$ can be separated with two disjoint non-empty open sets. This means that $M \setminus Z$ is non-connected, so it has two or more connected components. □

The following example goes further into detail in a case where the b-tangent bundle is parallelizable but the tangent bundle is not:

Example 2.4.47 Consider the b-manifold $(\mathbb{S}^2, Z)$, where $Z \subset \mathbb{S}^2$ is the equator. Then, ${}^bT\mathbb{S}^2$ is parallelizable, so ${}^bT\mathbb{S}^2 \ncong T\mathbb{S}^2$

Proof. Consider $\mathbb{S}^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\} \subset \mathbb{R}^3$, and let the equator $Z = \{z = 0\} \subset \mathbb{S}^2$ be the singular hypersurface. Let $j : \mathbb{S}^2 \to \mathbb{R}^3$ the canonical inclusion, and let $g = j^* g_{\mathrm{st}}$, where g_{st} is the Euclidean metric in $\mathbb{R}^3$. We will study the vector field $\frac{\partial}{\partial x} := \nabla_g x$, where we denote by $x : \mathbb{S}^2 \to \mathbb{R}$ the x coordinate function composed with j. We will compute the local form of $\frac{\partial}{\partial x}$ in a coordinate chart, in order to prove that it does not vanish anywhere.

Consider the points $P = (1, 0, 0)$ and $Q = (-1, 0, 0)$, and the atlas given by the open sets $\mathbb{S}^2 \setminus \{P\}$ and $\mathbb{S}^2 \setminus \{Q\}$:

Chart 1: Given by the local diffeomorphism

$$\begin{array}{cccc} \varphi : & \mathbb{S}^2 \backslash \{P\} & \longleftrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \longmapsto & \left(\frac{y}{1-x}, \frac{z}{1-x}\right) \\ & \left(\frac{Y^2+Z^2-1}{1+Y^2+Z^2}, \frac{2Y}{1+Y^2+Z^2}, \frac{2Z}{1+Y^2+Z^2}\right) & \longleftarrow & (Y,Z)\,. \end{array}$$

In this chart, the local form of the function x is

$$x(Y,Z) = \frac{Y^2+Z^2-1}{1+Y^2+Z^2},$$

and the local expression of the metric g is

$$\varphi^* g = \left(\frac{2}{1+Y^2+Z^2}\right)^2 \left(dY^2 + dZ^2\right),$$

so the local form of $\frac{\partial}{\partial x}$ can be computed and it is

$$\frac{\partial}{\partial x} = Y\frac{\partial}{\partial Y} + Z\frac{\partial}{\partial Z},$$

where $Z\frac{\partial}{\partial Z}$ is a non-vanishing b-vector field, so $\frac{\partial}{\partial x}$ does not vanish anywhere in $\mathbb{S}^2\backslash\{P\}$.

Chart 2: Given the local diffeomorphism

$$\begin{array}{cccc} \psi : & \mathbb{S}^2 \backslash \{Q\} & \longleftrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \longmapsto & \left(\frac{y}{1+x}, \frac{z}{1+x}\right) \\ & \left(\frac{1-Y^2-Z^2}{1+Y^2+Z^2}, \frac{2Y}{1+Y^2+Z^2}, \frac{2Z}{1+Y^2+Z^2}\right) & \longleftarrow & (Y,Z)\,. \end{array}$$

In this chart, the local form of the function x is

$$x(Y,Z) = \frac{1-Y^2-Z^2}{1+Y^2+Z^2},$$

and the local expression of g is the same as before,

$$\varphi^* g = \left(\frac{2}{1+Y^2+Z^2}\right)^2 \left(dY^2 + dZ^2\right),$$

so the local form of $\frac{\partial}{\partial x}$ is

$$\frac{\partial}{\partial x} = -Y\frac{\partial}{\partial Y} - Z\frac{\partial}{\partial Z},$$

where $Z\frac{\partial}{\partial Z}$ is a non-vanishing b-vector field, so $\frac{\partial}{\partial x}$ is nonzero in $\mathbb{S}^2\backslash\{Q\}$.

This proves that $\frac{\partial}{\partial x}$ is a non-vanishing b-vector field for $(\mathbb{S}^2, Z)$.

This argument can be reproduced for $\frac{\partial}{\partial y} := \nabla_g y$, so $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ is a trivializing basis for ${}^bT\mathbb{S}^2$. $\square$

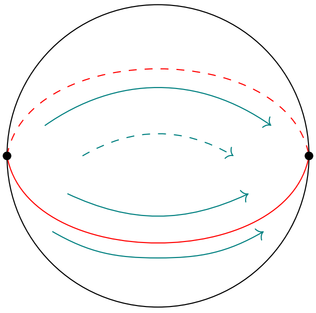

Figure 2.3: A sketch of one of the trivializing sections of ${}^bT\mathbb{S}^2$.

Example 2.4.48 Consider $Z \subset \mathbb{T}^2$ a contractible circle, as in Figure 2.4. Then, ${}^bT\mathbb{T}^2$ is parallelizable.

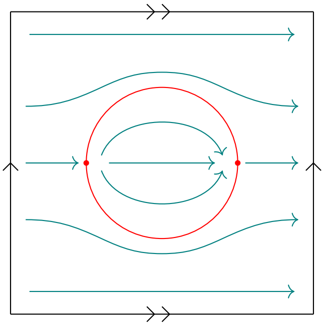

Figure 2.4: A representation of a non-vanishing b-vector field in $\mathbb{T}^2$

Proof. Let us find the pair of trivializing vector fields explicitly in $\mathbb{T}^2 \cong \frac{[0,1]\times[0,1]}{(0,y)\sim(1,y),\ (x,0)\sim(x,1)}$. We will consider $Z = \{(x,y) \in [0,1]\times[0,1] \,|\, 4x^2+4y^2 = 1\}$, the circle of radius $\frac{1}{2}$, and its image inside of $\mathbb{T}^2$. We will construct the two vector fields by patching together a collection of vector fields through a partition of unity.

Consider the function $f : \mathbb{T}^2 \to \mathbb{R}$ given in $[0,1] \times [0,1]$ by

$$f(x,y) = \begin{cases} \exp\left[36 + \frac{1}{x^2+y^2-\sqrt{x^2+y^2}+\frac{2}{9}}\right] & \text{if } \frac{1}{3} < \sqrt{x^2+y^2} < \frac{2}{3}, \\ 0 & \text{if } \sqrt{x^2+y^2} \leq \frac{1}{3},\ \sqrt{x^2+y^2} \geq \frac{2}{3}, \end{cases}$$

which is equal to 1 for all points in Z. It is clear by definition that f is well defined also in the quotient $\mathbb{T}^2$.

Consider also the two indicative functions $g_-, g_+ : \mathbb{T}^2 \to \mathbb{R}^2$ given in $[0,1] \times [0,1]$ by

$$g_-(x,y) = \begin{cases} \exp\left[1 + \frac{1}{144\left(x+\frac{1}{2}\right)^2+144y^2-1}\right] & \text{if } \sqrt{\left(x+\frac{1}{2}\right)^2+y^2} < \frac{1}{12}, \\ 0 & \text{if } \sqrt{\left(x+\frac{1}{2}\right)^2+y^2} \geq \frac{1}{12}, \end{cases}$$

$$g_+(x,y) = \begin{cases} \exp\left[1 + \frac{1}{144\left(x-\frac{1}{2}\right)^2+144y^2-1}\right] & \text{if } \sqrt{\left(x-\frac{1}{2}\right)^2+y^2} < \frac{1}{12}, \\ 0 & \text{if } \sqrt{\left(x-\frac{1}{2}\right)^2+y^2} \geq \frac{1}{12}, \end{cases}$$

with $g_-\left(-\frac{1}{2},0\right) = g_+\left(\frac{1}{2},0\right) = 1$.

Moreover, consider the vector fields defined on the appropriate open sets:

- $X' = \frac{\partial}{\partial x}$ in $\mathbb{T}^2$,
- $Z = y^2\frac{\partial}{\partial x} - xy\frac{\partial}{\partial y}$ in

$$U = \left\{(x,y) \in [0,1] \times [0,1] \mid \frac{1}{3} < \sqrt{x^2+y^2} < \frac{2}{3}\right\},$$

- $u_- = -x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$ and $u_+ = x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y}$ in

$$B_- = \left\{(x,y) \in [0,1] \times [0,1] \mid \sqrt{\left(x+\frac{1}{2}\right)^2 + y^2} < \frac{1}{12}\right\},$$

and

$$B_+ = \left\{(x,y) \in [0,1] \times [0,1] \mid \sqrt{\left(x-\frac{1}{2}\right)^2 + y^2} < \frac{1}{12}\right\},$$

respectively.

The first trivializing vector field is then defined as

$$X = f \cdot (g_- u_- + g_+ u_+ + (1 - g_- - g_+) Z) + (1 - f) X',$$

The second vector field Y can be defined by a 90° rotation on X.

It can be checked that X and Y are everywhere linearly independent on $\mathbb{T}^2$ as b-vector fields. □

An analogous construction can be carried out in higher dimensions if we consider $\mathbb{T}^n \cong [0,1] \times \cdots \times [0,1] /_{\sim}$ and $Z = \left\{(x_1, \ldots, x_n) \in [0,1] \times \cdots \times [0,1] \mid x_1^2 + \cdots + x_n^2 = \frac{1}{4}\right\}$. As in the last proof, we can construct a single non-vanishing b-vector field and rotate it to cover all the remaining $n-1$ dimensions in order to get a non-vanishing frame.

Further investigation of b^m-tangent bundles and their characteristic classes can be found in Klaasse's PhD thesis [Kla17].

2.5 Desingularization of b^m-symplectic manifolds

The desingularization of a b^m-symplectic manifold is a process that was first introduced in [GMW19] and allows us to approximate a particular b^m-symplectic form either by a family of symplectic forms or by a family of folded symplectic forms, depending on the parity of m. This construction will be crucial for our work with the Arnold conjecture in b^m-symplectic manifolds.

Let (M, Z, ω) be a compact b^{2k}-symplectic manifold with $Z \subset M$ compact.

As we saw in Proposition 2.4.35 there is a tubular neighbourhood $\mathcal{N}(Z)$ for each component of Z with coordinates such that

$$\omega|_{\mathcal{N}(Z)} = \frac{dz}{z^{2k}} \wedge \left(\sum_{i=0}^{2k-1} z^i \pi^* \alpha_i \right) + \pi^* \beta.$$

Let $f \in C^\infty(\mathbb{R})$ be an odd function such that $f'(x) > 0$ for all $x \in [-1, 1]$

and defined outside of the interval as

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for } x < -1, \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for } x > 1. \end{cases}$$

For instance, see the illustration on Figure 2.5.

Moreover, let

$$f_\varepsilon = \frac{1}{\varepsilon^{2k-1}} f\left(\frac{x}{\varepsilon}\right)$$

and $\mathcal{N}_\varepsilon(Z) = \{(z,p) \in \mathcal{N}(Z) \mid |z| < \varepsilon\}$.

Definition 2.5.1 An *f_ε-desingularization ω_ε of ω* is a form on M defined in a neighbourhood $\mathcal{N}_\varepsilon(Z)$ as

$$\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=0}^{2k-1} z^i \pi^* \alpha_i\right) + \pi^*\beta$$

and coinciding with ω outside of $\mathcal{N}_\varepsilon(Z)$.

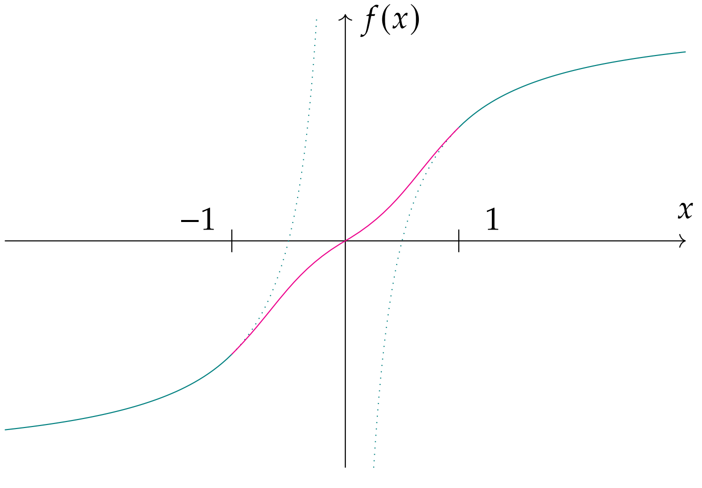

Figure 2.5: Construction of f: we interpolate two branches of the function $-\frac{1}{2k-1}x^{1-2k}$ (with a displacement) by a function whose derivative does not vanish.

This construction allows us to approximate ω^{-1}, seen as a b^{2k}-Poisson form, by the family of Poisson forms ω_ε^{-1}:

Theorem 2.5.2 (Guillemin, Miranda and Weitsman [GMW19, Theorem 3.1]) *If $\varepsilon > 0$ is small enough, the form ω_ε is symplectic. The family of bivector fields ω_ε^{-1} converges to the Poisson structure ω^{-1} in the C^{2k-1}-topology as $\varepsilon \longrightarrow 0$.*

As a consequence of this theorem, topological obstructions for the existence of b^{2k}-symplectic structures in M can be found without further work:

Corollary 2.5.3 *A manifold admitting a b^{2k}-symplectic structure must also admit a symplectic structure.*

For the odd case let us recall the definition of folded symplectic structures:

Definition 2.5.4 Let M be a $2n$-dimensional manifold. We say that a form $\omega \in \Omega^2(M)$ is *folded symplectic* if

- $d\omega = 0$ and ω^n intersects transversally the form $0 \in \Omega^{2n}(M)$.
- If $Z := \{p \in M \mid \omega_p^n = 0\}$ is the hypersurface where ω degenerates and $j : Z \rightarrow M$ denotes the natural inclusion, then $j^*\omega$ has rank $2n-2$ in Z.

This definition can be understood to be in some sense "dual" to that of b-Poisson structures, as we are allowing the rank of the symplectic form to decrease, whereas for b-Poisson structures it is the degree of the b-Poisson structure that decreases. However, we must note that a folded symplectic form does not induce a Poisson structure on M.

Let (M, Z, ω) be a compact b^{2k+1}-symplectic manifold with $Z \subset M$ compact, whose b^{2k+1}-symplectic form has a semilocal expression given by

$$\omega|_{\mathcal{N}(Z)} = \frac{dz}{z^{2k+1}} \wedge \left(\sum_{i=0}^{2k} z^i \pi^* \alpha_i \right) + \pi^* \beta,$$

and take $f \in C^\infty(\mathbb{R})$ an even function (this means, $f(-x) = f(x)$) such that

- $f > 0$,
- $f(x) = 2 - x^2$ for $x \in [-1, 1]$,

- for $x \in \mathbb{R} \setminus [-2, 2]$,

$$f(x) = \begin{cases} \log|x| & \text{if } k = 0, \\ \frac{-1}{(2k+2)x^{2k+2}} & \text{if } k > 0. \end{cases}$$

For an example, see the illustration in Figure 2.6.

Let

$$f_\varepsilon(x) = \frac{1}{\varepsilon^{2k}} f\left(\frac{x}{\varepsilon}\right)$$

and $\mathcal{N}_\varepsilon(Z) = \{(z, p) \in \mathcal{N}(Z) \mid |z| < \varepsilon\}$.

An f_ε-desingularization ω_ε of a b^{2k+1}-symplectic structure ω is defined as in Definition 2.5.1 with this choice of function f_ε.

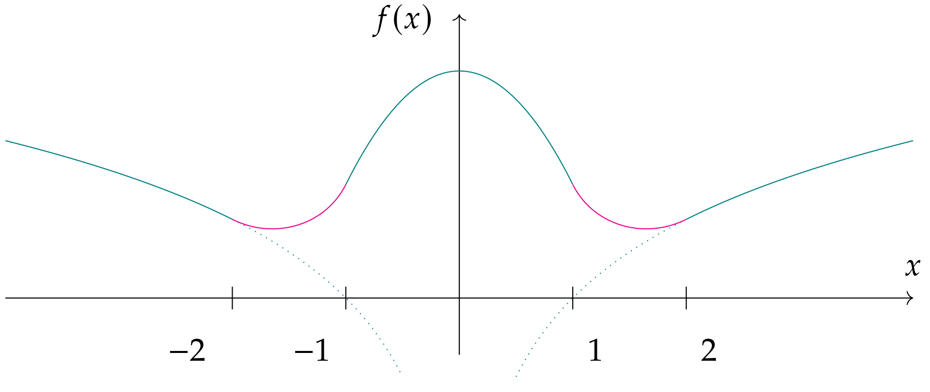

Figure 2.6: Construction of f: we interpolate two branches of the function $-\frac{1}{(2k+2)x^{2k+2}}$ (or $\log|x|$) with the function $2 - x^2$ by a function that does not vanish.

Theorem 2.5.5 (Guillemin, Miranda and Weitsman [GMW19, Theorem 5.1]) *If $\varepsilon > 0$ is small enough, the form ω_ε is folded symplectic.*

Corollary 2.5.6 *A manifold admitting a b^{2k+1}-symplectic structure must also admit a folded symplectic structure.*

2.6 Integrable and singular integrable systems

In this section we will introduce the notion of integrable systems, central among the applications of symplectic geometry to dynamical systems.

We will present the basic results for the best understood cases, namely toric and semitoric systems. We will also briefly introduce the equivalent notion in the context of b-symplectic manifolds, and examine the work done in analogy to toric systems.

2.6.1 Definition and critical points

In the setting of symplectic geometry there are systems induced by a Hamiltonian vector field that have certain symmetries that allow for simplifications and reductions of the system. As described by Noether's theorem, such symmetries can be encoded by preserved quantities, also called *first integrals* of a system. The best possible case, which we will focus on in this thesis, is that in which the number of first integrals is equal to half the dimension of the manifold. Such a system is called an integrable system.

Definition 2.6.1 Let (M, ω) be a symplectic manifold and $f, g \in C^\infty(M)$. We say that g *is preserved by* X_f if $\mathcal{L}_{X_f} g = 0$ or, equivalently, $\{f, g\} = 0$ in the associated Poisson structure (see Example 2.3.2).

With this in mind we can present the notion of a completely integrable system.

Definition 2.6.2 Let (M, ω) be a symplectic manifold of dimension $2n$ and let $F = (f_1, \dots, f_n) : M \to \mathbb{R}^n$ be a smooth function. The tuple (M, ω, F) is said to be an *integrable system* or a *completely integrable system* if dF has maximal rank almost everywhere (with respect to the symplectic volume ω^n) and $\{f_i, f_j\} = 0$ for all i, j.

The map $F = (f_1, \dots, f_n)$ is called the *momentum map* of the system, and the component maps $f_1, \dots, f_n$ are called the *momenta*.

A point $p \in M$ is called *regular* if $dF|_p$ has rank exactly equal to n and *singular* if the rank is strictly lower than n.

Definition 2.6.3 Let us assume that the flow of $X_{f_1}, \dots, X_{f_n}$ is defined for all times. The *flow of an integrable system* is the group action

$$\begin{array}{ccc} \mathbb{R}^n \times M & \longrightarrow & M \\ ((t_1, \dots, t_n), p) & \longmapsto & \left(\varphi^{t_1}_{X_{f_1}} \circ \cdots \circ \varphi^{t_n}_{X_{f_n}}\right)(p). \end{array}$$

We say that this map is a group action because the vector fields $X_{f_1}, ..., X_{f_n}$ are, by Lemma 2.3.8, in involution, and therefore their flows commute.

The distribution generated by the vector fields $X_{f_1}, ..., X_{f_n}$ is thus in involution and, therefore, induces a foliation of M. Moreover, the leaves of the foliation are the connected components of $F^{-1}(c)$ for values of $c \in \mathbb{R}^n$. When $c \in \mathbb{R}^n$ is a regular value of F the foliation has a very well understood behaviour:

Theorem 2.6.4 (Arnold-Liouville-Mineur) *Let (M, ω, F) be an integrable system, and let $c \in \mathbb{R}^n$ be a regular point of F, and let $L_c := F^{-1}(c)$. If L_c is compact and connected, then it is diffeomorphic to the n-dimensional torus $\mathbb{T}^n$, and there exist local coordinates $(\theta_1, \ldots, \theta_n, p_1, \ldots, p_n)$ around every such fiber with $\omega = \sum d\theta_i \wedge dp_i$, such that L_c are given locally as level sets of $(p_1, \ldots, p_n)$ and the flow of the system in the direction of θ_i is linear. The coordinates $(\theta_1, \ldots, \theta_n, p_1, \ldots, p_n)$ are called the* action-angle coordinates*.*

Let us shift our attention to the singular points of integrable systems.

Let (M, ω, F) be a completely integrable system and let $p \in M$ be a singular point with rank $r < n$. Then it is possible to choose coordinates around p in such a way that the action of $dF|_p$ induces a symplectic action of $\mathbb{R}^{n-r}$ on T_pM. This defines a Lie subgroup $G(p, F) \subset \mathrm{Sp}(2(n-r), \mathbb{R})$, which in turn defines by its linearization a Lie subalgebra $K(p, F) \subset \mathfrak{sp}(2(n-r), \mathbb{R})$.

Definition 2.6.5 A singular point $p \in M$ is *non-degenerate* if $K(p, F)$ is a Cartan subalgebra of $\mathfrak{sp}(2(n-r), \mathbb{R})$, this means, if it is nilpotent and self-normalizing.

Non-degenerate critical points have been studied and normal forms have been found for them in the works of Rüssman [Rüs64], Vey [Vey78], Colin de Verière and Vey [CdVV79], Eliasson [Eli90b, Eli90a], Dufour and Molino [DM88], Miranda [Mir03, Mir14], Miranda and Zung [MZ04], Miranda and Vũ Ngọc [MVN05], Vũ Ngọc and Wacheux [VNW13] and Chaperon [Cha13].

Theorem 2.6.6 (Local normal form for non-degenerate singularities) *Let (M, ω, F) be a completely integrable system and let $p \in M$ be a non-degenerate singular point. Then there exist a coordinate chart $(U; (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n))$ centered on p and smooth functions $q_1, \ldots, q_n : U \to \mathbb{R}$ such that*

$$\omega|_U = \sum_{i=1}^{n} dx_i \wedge d\xi_i,$$

we have that $\{q_i, f_j\} = 0$ *for all* $1 \leq i, j \leq n$, *and each of the components has one of the following forms:*

- $q_i = \left(x_i^2 + \xi_i^2\right)/2$, *called an* elliptic component.
- $q_i = x_i \xi_i$, *called a* hyperbolic component.
- $q_i = x_i \xi_{i+1} - x_{i+1}\xi_i$ *and* $q_{i+1} = x_i \xi_i + x_{i+1}\xi_{i+1}$, *called a* focus-focus component.
- $q_i = \xi_i$, *called a* regular component.

Moreover, if there are no hyperbolic components then the system of equations $\{q_i, f_j\} = 0$ *for* $1 \leq i, j \leq n$ *is equivalent to the existence of a local diffeomorphism* $g : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *such that*

$$g \circ f = (q_1, \ldots, q_n) \circ (x_1, \ldots, x_n, \xi_1, \ldots, \xi_n).$$

In what follows we will concentrate on integrable systems where the dimension of the manifold is 4, and therefore there are two integrals, which we usually call J and H.

In this particular case, the singular points of rank 1 must be either of elliptic-regular or hyperbolic-regular type. The singular points of rank 0 must be of type elliptic-elliptic, focus-focus, hyperbolic-hyperbolic or hyperbolic-elliptic type.

In order to classify the singular points of a 4-dimensional integrable system we present a criterion which can be found in Bolsinov and Fomenko in [BF04] in the line of Definition 2.6.5. In particular, we are interested in identifying the algebra $K(J, H) \subset \mathfrak{sp}(4, \mathbb{R})$, which is generated by the linearizations of X_J and X_H at the singular point p. These linearizations can be computed locally in Darboux coordinates centered at p as the matrices $A_J = \Omega_0^{-1} d^2 J$ and $A_H = \Omega_0^{-1} d^2 H$, where

$$\Omega_0 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$

Lemma 2.6.7 *The algebra* $K(J, H) := \{c_1 A_J + c_2 A_H \mid c_1, c_2 \in \mathbb{R}\}$ *is a Cartan subalgebra if and only if it is two-dimensional and it contains an element whose eigenvalues are all different.*

With this characterization it is possible to classify all Cartan subalgebras of $\mathfrak{sp}(4,\mathbb{R})$. This classification was found by Williamson [Wil36]. In particular, a Cartan subalgebra must be conjugate to one of the algebras induced by the following 2-parameter families:

$$\begin{pmatrix} 0 & 0 & -\alpha & 0 \\ 0 & 0 & 0 & -\beta \\ \alpha & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & -\beta \\ 0 & 0 & \alpha & 0 \\ 0 & \beta & 0 & 0 \end{pmatrix}, \begin{pmatrix} -\alpha & 0 & 0 & 0 \\ 0 & -\beta & 0 & 0 \\ 0 & 0 & \alpha & 0 \\ 0 & 0 & 0 & \beta \end{pmatrix}, \begin{pmatrix} -\alpha & -\beta & 0 & 0 \\ \beta & -\alpha & 0 & 0 \\ 0 & 0 & \alpha & -\beta \\ 0 & 0 & \beta & \alpha \end{pmatrix}, \tag{2.8}$$

where $\alpha, \beta \in \mathbb{R}$.

Each of the types of algebra is determined by the eigenvalues of the matrix, and they determine the type of the critical point:

- p is *elliptic-elliptic* if it has the four purely imaginary eigenvalues $\{\pm i\alpha, \pm i\beta\}$,
- p is *hyperbolic-hyperbolic* if it has four purely real eigenvalues $\{\pm\alpha, \pm\beta\}$,
- p is *elliptic-hyperbolic* if it has two purely imaginary and two purely real eigenvalues, $\{\pm i\alpha, \pm\beta\}$,
- p is *focus-focus* if it has complex eigenvalues of the form $\{\pm\alpha \pm i\beta\}$,

where $\alpha, \beta \neq 0$ and $\alpha \neq \beta$ in the elliptic-elliptic and hyperbolic-hyperbolic cases.

2.6.2 Toric manifolds

There is a particular family of systems notable because of their rigidity, known as toric systems. Toric systems arise as actions of a torus on a manifold, with subcategories of such systems defined by symplectic actions and Hamiltonian actions. Hamiltonian toric systems, thus, may only exhibit either regular or elliptic components in their systems.

Let us give a brief overview of their classification.

Definition 2.6.8 A completely integrable system (M, ω, F) is *toric* if the flow of its momentum map generates an effective action of $\mathbb{T}^n$ on M.

Hamiltonian toric systems on compact and connected manifolds can be completely classified by the image of the momentum map, $F(M) \subset \mathbb{R}^n$:

Definition 2.6.9 A convex polytope $\Delta \subset \mathbb{R}^n$ is a *Delzant polytope* if it is

- *simple*: each vertex lies at the intersection of exactly n edges,
- *rational*: all edges have a rational slope in the sense that for every vertex every edge has the parametrization $p + vt$ with $p \in \mathbb{R}^n$ and $v \in \mathbb{Z}^n$.
- *smooth*: at every vertex the n vectors parametrizing the edges meeting at the vertex form a basis of the module $\mathbb{Z}^n$.

Theorem 2.6.10 (Delzant [Del88]) *There is an equivalence of categories between the category of Hamiltonian toric systems* (M, ω, F) *(up to equivariant symplectorphisms) and the category of Delzant polytopes (up to translation). In particular, the image of the momentum map* $F(M)$ *determines* (M, ω, F) *up to symplectic equivariance, and for any Delzant polytope* $\Delta \subset \mathbb{R}^n$ *there exists a compact connected symplectic* $2n$*-dimensional manifold* (M, ω) *and a momentum map* $F : M \longrightarrow \mathbb{R}^n$ *such that* $F(M) = \Delta$.

Hamiltonian toric systems are thus very special. At the same time, its dynamical features are relatively simple, in the sense that all singular points of toric systems have exclusively a combination of regular and elliptic components.

2.6.3 b-Toric manifolds

In this section we will summarize a generalization of integrable systems into the setting of b-symplectic geometry, namely that of toric systems on b-symplectic manifolds. This subject was thoroughly studied by Guillemin, Miranda, Pires and Scott [GMPS15, GMPS17] and by Gualtieri, Li, Pelayo and Ratiu [GLPR17]. Here we present a summary of the formers' results.

We will denote by $\mathfrak{t}$ the Lie algebra of the torus $\mathbb{T}^n$ and by $X^\# \in \mathfrak{X}(M)$ the fundamental vector field associated to an element $X \in \mathfrak{t}$ through the infinitesimal action.

Definition 2.6.11 Let (M, Z, ω) be a b-symplectic manifold.

We say that a toric action $\mathbb{T}^n \curvearrowright M$ is *Hamiltonian* if for all $X, Y \in \mathfrak{t}$:

- $\iota_{X^\#}\omega = dH_X$ for some $H_X \in^b C^\infty(M)$.
- $\omega(X^\#, Y^\#) = 0$.

We say that the action is *toric* if it is effective and $\dim(\mathbb{T}^n) = \frac{1}{2}\dim(M)$.

Through an equivariant version of the b-Morse lemma it is possible to show a simple classification for toric Hamiltonian b-actions in the particular case of surfaces:

Theorem 2.6.12 (Guillemin, Miranda, Pires and Scott [GMPS15, Theorem 9]) *A b-symplectic surface with a toric $\mathbb{S}^1$-action is equivariantly b-symplectomorphic to either $(\mathbb{S}^2, Z)$ or $(\mathbb{T}^2, Z)$. Here, Z is a collection of latitude circles (in the $\mathbb{T}^2$ case, an even number of such circles), the action is the standard rotation, and the b-symplectic form is determined by the modular periods of the critical curves and the regularized Liouville volume.*

To study higher dimensional cases first we need to understand the behaviour of the $\mathbb{T}^n$-action semilocally near the hypersurface Z. To this end, an equivariant Darboux theorem is proved.

Definition 2.6.13 For each connected component $Z' \subseteq Z$ there is an element $v_{Z'} \in \mathfrak{t}^* = \mathrm{Hom}(\mathfrak{t}, \mathbb{R})$, the *toric modular weight* of Z', such that for every $X \in \mathfrak{t}$ the function H_X given by Definition 2.6.11 has the form $v_{Z'}(X)\log|z| + g$ in a tubular neighbourhood around Z', where z is a local defining function of Z' and $g \in C^\infty(M)$.

Remark 2.6.14 If the action is toric, then $v_{Z'} \neq 0$.

The following remark gives an understanding of the topology of Z in terms of the mapping torus description from Theorem 2.4.38.

Remark 2.6.15 Let (M, Z, ω) be a b-symplectic manifold with a toric action and assume that Z is connected. Let $\mathcal{L}$ be a leaf of Z. Then, $Z \cong \mathcal{L} \times \mathbb{S}^1$.

Using these tools it is possible to understand the global behaviour of a b-toric Hamiltonian action via an analogue to the Delzant polytope. In some sense, we want to understand the image of a momentum map, which as we see in Definition 2.6.13 is not a smooth function in a neighbourhood of Z.

Let us start by providing the notion of the codomain.

Remark 2.6.16 Let (M, Z, ω) be a b-symplectic manifold, and consider a toric Hamiltonian action $\mathbb{T}^n \curvearrowright M$. Let $G = (V, E)$ be the graph associated to (M, Z), and take $w : E \longrightarrow \mathfrak{t}^*$ the map that associates to each connected component Z its toric modular weight v_Z. When the action is effective, the graph G must either be a cycle with an even number of vertices or a line.

Definition 2.6.17 Consider a pair $\mathcal{G} = (G, w)$ of a graph satisfying the condition of Remark 2.6.16 and a function $w : E \longrightarrow \mathfrak{t}^*$ such that $w(e) = kw(e')$ for $k \in \mathbb{R}$ and $k < 0$ if e and e' meet at a vertex. The *b-momentum codomain* $(\mathcal{R}_\mathcal{G}, \mathcal{Z}_\mathcal{G}, \hat{x})$ is a b-manifold $(\mathcal{R}_\mathcal{G}, \mathcal{Z}_\mathcal{G})$ with a smooth map $\hat{x} : \mathcal{R}_\mathcal{G} \setminus \mathcal{Z}_\mathcal{G} \longrightarrow \mathfrak{t}^*$. A b-map $\mu : M \longrightarrow \mathcal{R}_\mathcal{G}$ is then a *momentum map* if it is $\mathbb{T}^n$ equivariant and $\mathfrak{t} \ni X \mapsto \mu^X \in C^\infty(M)$ with $\mu^X(p) = \langle \hat{x} \circ \mu(p), X \rangle$ is linear, and moreover

$$\iota_{X^\#}\omega = d\mu^X.$$

For the complete definition of the codomain see [GMPS15, Section 5]. See Figure 2.7 for an example.

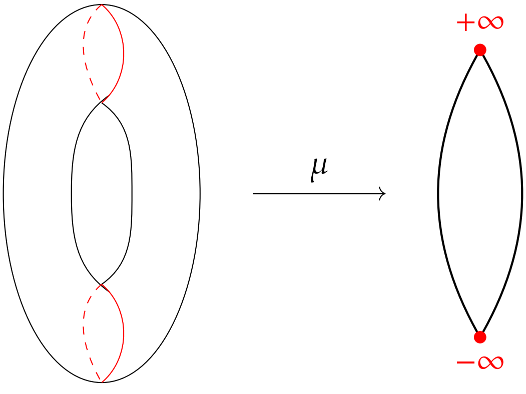

Figure 2.7: The momentum map $\mu : \mathbb{T}^2 \longrightarrow \mathcal{R}_\mathcal{G}$.

Definition 2.6.18 ([GMPS15, Definition 28]) A *b-symplectic toric manifold* is $(M^{2n}, Z, \omega, \mu : M \longrightarrow \mathcal{R}_\mathcal{G})$, where (M, Z, ω) is b-symplectic and μ is a momentum map for some b-toric action on (M, Z, ω).

Definition 2.6.19 A *b-polytope* in $\mathcal{R}_{\mathcal{G}}$ is a bounded subset P that intersects every component of $\mathcal{Z}_{\mathcal{G}}$ and can be expressed as a finite intersection of half-spaces.

Such a polytope is *Delzant* if

- In the case that G is a line, if for every vertex $v \in P$ there is a lattice basis $\{u_i\}$ of $\mathfrak{t}^*$ such that the edges incident to v can be written in a neighbourhood of v as $v + tu_i$ for $t \geq 0$.
- In the case that G is a cycle, $\Delta_Z \subseteq \mathfrak{t}^*_w$.

Now we have all the notions required to establish a classification:

Theorem 2.6.20 ([GMPS15, Theorem 35]) *The functor*

$$\left\{ \begin{array}{c} b\text{-symplectic toric manifolds} \\ (M, Z, \omega, \mu : M \to \mathcal{R}_{\mathcal{G}}) \end{array} \right\} \to \left\{ \begin{array}{c} \text{Delzant } b\text{-polytopes} \\ \text{in } \mathcal{R}_{\mathcal{G}} \end{array} \right\}$$

that sends a b-symplectic toric manifold to the image of its momentum map is a bijection, where b-symplectic toric manifolds are considered up to equivariant b-symplectomorphisms that preserve the momentum map.

Theorem 2.6.20 induces a particularly rigid classification of b-toric manifolds:

Corollary 2.6.21 *Every b-toric manifold is b-symplectomorphic to either*

- *A product of a b-symplectic* $\mathbb{T}^2$ *with a smooth toric manifold, or*
- *A manifold obtained from a product of a b-symplectic* $\mathbb{S}^2$ *with a smooth toric manifold by a sequence of symplectic cuts performed at the north and south "polar caps", away from the critical hypersurface* Z.

2.6.4 Semitoric manifolds

As we have seen, there are well understood generalizations of the theory of toric systems into the setting of b-symplectic manifolds. In this section we will present a type of integrable system more complicated than toric systems, whose generalization to the b-symplectic setting we will introduce in Chapter 5 and illustrate with examples.

There are many good surveys on semitoric systems and their classification, see for instance Pelayo and Vũ Ngọc [PVN12a], Alonso and Hohloch [AH19] and Pelayo [Pel21]. The research in the field of semitoric systems has been expanded recently by the inquiry into their natural generalization, that of *hypersemitoric systems*. A reader interested in these developments is encouraged to read Henriksen, Hohloch and Martynchuk [HHM23] and Gullentops and Hohloch [GH23].

Definition 2.6.22 (Pelayo and Vũ Ngọc [PVN09]) Let (M, ω) be a 4-dimensional symplectic manifold, and let $F = (L, H) : M \to \mathbb{R}^2$ be the momentum map for a completely integrable system on M. We say that the system is *semitoric* if

- L is a proper map.
- L generates an effective $\mathbb{S}^1$-action.
- All singularities are non-degenerate.
- There are no hyperbolic singularities.

We say that a semitoric system is *simple* if each fiber $L^{-1}(\theta)$ contains at most 1 focus-focus critical point.

Due to their definition semitoric systems can admit three types of singular points: elliptic-elliptic, elliptic-regular or focus-focus. Focus-focus fibers are of particular interest in the classification of semitoric systems. In the simple case they are always topologically equivalent to a *pinched torus* as in Figure 2.6.4. More generally, focus-focus fibers can have a slightly more complicated topology, adding more "pinches" to the torus.

Simple semitoric systems were completely classified by Pelayo and Vũ Ngọc [PVN09, PVN11]:

Theorem 2.6.23 ([PVN09, PVN11]) *Simple semitoric systems are completely classified in terms of five invariants:*

- *The* polygon invariant, *a generalization of the Delzant polytope.*
- *The* number of focus-focus points, $n_{FF} \in \mathbb{N}$.
- *The* height invariant, *a n_{FF}-tuple which denotes the "heights" of all the focus-focus points in the image of the momentum map.*

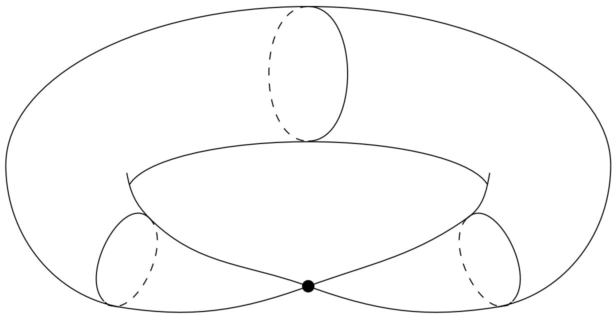

Figure 2.8: Focus-focus fiber of a simple semitoric system

- *The* twisting index invariant, *which roughly describes the "gluing" of the focus-focus points into the polygon.*
- *The* Taylor series invariant, *which further details the behaviour of the focus-focus point.*

The Taylor series invariant had been first constructed by Vũ Ngọc [VN03], and the number of focus-focus points, the polygon and the height invariant were developed in [VN07]. This classification was extended to non-simple systems by Palmer, Pelayo and Tang [PPT19], and in the work of Le Floch and Palmer [LFP18] they were condensed into a single invariant, the *marked semitoric polygon*. More recent developments towards the understanding of the twisting index invariant were carried out by Alonso, Hohloch and Palmer [AHP23].

We will describe two examples of systems that exhibit semitoric features, one of which we will generalize as a singular symplectic manifold in Chapter 5.

Example 2.6.24 Consider $M = \mathbb{S}^2 \times \mathbb{R}^2$ with Cartesian coordinates (x, y, z, u, v) and take as symplectic form $\omega = -\rho_1\omega_{\mathbb{S}^2} + \rho_2\omega_{\mathbb{R}^2}$, where $\omega_{\mathbb{S}^2}$ and $\omega_{\mathbb{R}^2}$ denote the standard symplectic forms on $\mathbb{S}^2$ and $\mathbb{R}^2$ respectively and $\rho_1, \rho_2 \in \mathbb{R}$ and $\rho_1, \rho_2 > 0$.

The *coupled spin oscillator* is the integrable system $(M, \omega, (L, H))$ given by

$$\begin{cases} L(x,y,z,u,v) := \rho_1 z + \frac{\rho_2}{2}(u^2+v^2), \\ H(x,y,z,u,v) := \frac{1}{2}(xu+yv). \end{cases}$$

This system is a simple semitoric system, and the image of its momentum map with the images of the critical points marked is as shown in Figure 2.9.

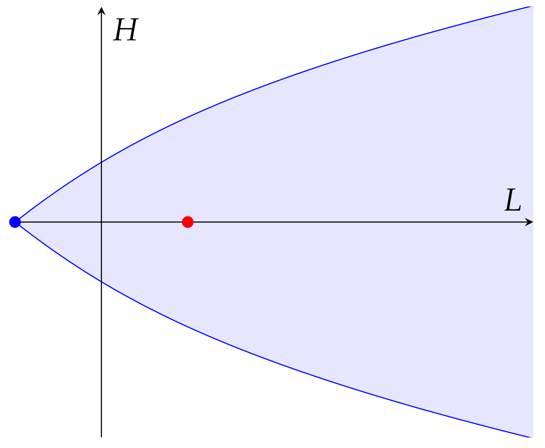

Figure 2.9: Momentum map of the coupled spin-oscillator. The blue dot marks the image of the elliptic-elliptic fiber and the red dot marks the image of the focus-focus fiber.

The system has two fixed points at the images of $(0,0,\pm 1,0,0)$. The singularity at the point $(0,0,-1,0,0)$ is of elliptic-elliptic type, whereas the singularity at the point $(0,0,1,0,0)$ is of focus-focus type.

Remark 2.6.25 The system in Example 2.6.24 is a simplification of the Jaynes-Cummings model [JC63] by Babelon, Cantini and Douçot [BCD09]).

The Jaynes-Cummings model is a theoretical model in quantum optics that describes the interaction between an atom and a quantized mode of an electromagnetic field.

In [BCD09], Babelon, Cantini and Douçot developed a generalization of the Jaynes-Cummings model, replacing the two-level atom by a single quantum spin s. The adapted model turns out to be an integrable system with two degrees of freedom: one spin in the sphere $\mathbb{S}^2$ and one harmonic oscillator in the plane $\mathbb{R}^2$. In particular, the model corresponds to a simple semitoric integrable system which exhibits one unstable fixed point of focus–focus type. The model has also appeared in other areas in physics, for instance in solid-state physics and quantum information. See Greentree, Koch and Larson and [GKL13] other articles in the special edition for an orientation on these applications.

This system has special interest from the perspective of semitoric systems because it has been the first such system that could be completely classified in terms of the invariants from Theorem 2.6.23. Specifically, Pelayo and Vũ Ngọc found in [PVN12b] the polygon and height invariants of the system, and computed the linear terms of the Taylor series invariant. Alonso, Dullin and Hohloch completed the classification in [ADH19] finding the twisted index invariant and all the higher order terms of the Taylor series invariant.

The following example, the *coupled angular momenta*, is a well known semitoric system in quantum physics. It was analysed notably by Sadovskii and Zhilinskii [SZ99]. The system models the coupling of two quantum angular momenta, by means of an interpolating parameter $t \in [0, 1]$.

Example 2.6.26 Consider $M = \mathbb{S}^2 \times \mathbb{S}^2$ with Cartesian coordinates $(x_1, y_1, z_1, x_2, y_2, z_2)$ and take as symplectic form $\omega = -R_1\omega_{\mathbb{S}^2} - R_2\omega_{\mathbb{S}^2}$, where $\omega_{\mathbb{S}^2}$ denotes the standard symplectic form on $\mathbb{S}^2$ and $R_1, R_2 \in \mathbb{R}$ with $0 < R_1 < R_2$.

The *coupled angular momenta* is the integrable system $(M, \omega, (L, H))$ given by

$$\begin{cases} L(x_1, y_1, z_1, x_2, y_2, z_2) := R_1 z_1 + R_2 z_2, \\ H(x_1, y_1, z_1, x_2, y_2, z_2) := (1-t)z_1 + t(x_1x_2 + y_1y_2 + z_1z_2). \end{cases}$$

Its interest from the mathematical point of view is that, despite its relative simplicity, the system can exhibit an interesting range of behaviours as t changes. The singularities of the system can change from degenerate and non-degenerate and viceversa, and their components may also change in type. Moreover, it is the first example of a compact semitoric system for which all invariants have been determined. Le Floch and Pelayo found in [LFP19] some of the invariants with a specific choice of the parameters of the system. Alonso, Dullin and Hohloch [ADH20] computed all the invariants for the whole family, and moreover analysed the dependence of the parameters with respect to the parameters of the system, particularly in the limit where the focus-focus singularity becomes degenerate.

The system is often used in the study of semitoric systems more broadly. For example, Hohloch and Palmer [HP18] derived a 4-parameter family of systems generalizing the coupled angular momenta, with which they were able to introduce the first example of a semitoric system with two focus-focus singularities. Alonso and Hohloch [AH21] analysed the system and computed the polygon and the height invariant for this family, and investigated the relationship between these two invariants and the symmetries of the system. Alonso, Hohloch and Palmer [AHP23] computed the twisting index invariant for a subfamily of the family, completing the classification of a collection of systems with two focus-focus singularities.

The coupled angular momenta is an interpolation through several integrable systems for different values of $t \in [0,1]$. It has 4 fixed points at $p_{\pm,\pm} = (0,0,\pm 1,0,0,\pm 1)$ independently of t. The points $p_{+,+}$, $p_{-,+}$ and $p_{-,-}$ are elliptic-elliptic fixed points regardless of t, but the type of the point $p_{+,-}$ depends on the interpolating parameter. In particular, it is non-degenerate of elliptic-elliptic type for $t < t^-$ and for $t > t^+$, of focus-focus type for $t^- < t < t^+$ and degenerate for $t \in \{t^-, t^+\}$, where

$$t^{\pm} = \frac{R_2}{2R_2 + R_1 \mp 2\sqrt{R_1 R_2}}.$$

In Figure 2.10 we can see how, as t changes, one of the elliptic-elliptic values enters the polygon, transforms into a focus-focus point, and it becomes an elliptic-elliptic point again as it reaches the opposite boundary.

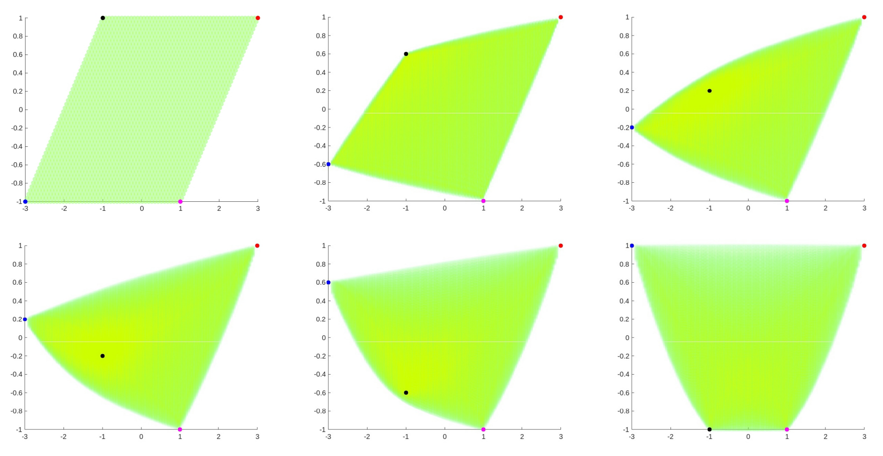

Figure 2.10: Image of the map of the coupled angular momenta for various values of t.

Chapter 3

The Arnold conjecture for b^m-symplectic manifolds

In this chapter we will investigate lower bounds on the number of 1-periodic orbits for certain b^m-Hamiltonians in the context of b^m-symplectic manifolds. We will employ desingularization techniques as introduced in Section 2.5 to achieve these lower bounds. We will be following closely the central sections of Brugués, Miranda and Oms [BMO22].

In Section 3.1 we will introduce the notion of admissible Hamiltonians, the setting around which the notions in this chapter and the following will revolve. In Section 3.2 we will expand on the desingularization techniques first visited on Section 2.5 to account for the dynamics of an admissible Hamiltonian and how desingularization affect them. Finally, in Section 3.3 we will see how to apply all of the desingularization techniques previously introduced to deduce several lower bounds for admissible Hamiltonians in b^m-symplectic manifolds.

3.1 Admissible Hamiltonians and semilocal dynamics

In this section we will investigate possible relationship between Hamiltonian vector fields of b^m-symplectic manifolds and the normal and modular vector fields. We will be particularly interested in the presence of periodic orbits in a neighbourhood of the singular hypersurface, and we will try to synthesize a subset of Hamiltonians, *admissible Hamiltonians*, on which the Arnold conjecture is reasonable given the previous discussion.

Remark 3.1.1 Throughout the following two chapters we will be considering *time dependent b^m-functions* or *Hamiltonians*, denoted as ${}^{b^m}C^\infty(\mathbb{R}\times M)$ or as ${}^{b^m}C^\infty(\mathbb{S}^1\times M)$. These denote simply elements $H_t:\mathbb{R}\to {}^{b^m}C^\infty(M)$ or $H_t:\mathbb{S}^1\to {}^{b^m}C^\infty(M)$ (see Definition 2.4.17) that are smooth in the time component.

Definition 3.1.2 Let (M,Z,ω) be a b^m-symplectic manifold and let X^σ be the normal symplectic vector field. We say that a (time dependent) Hamiltonian $H\in {}^{b^m}C^\infty(\mathbb{S}^1\times M)$ is *linear along X^σ* if $\mathcal{L}_{X^\sigma}H = K(t)$ on a tubular neighbourhood of Z, where $K\in C^\infty(\mathbb{S}^1)$ and therefore is a constant with respect to all variables on M.

The term "linear along X^σ" is chosen in analogy to the smooth case, as we are demanding that the growth of H along X^σ does not depend on any variable on M.

Example 3.1.3 Consider the b-symplectic manifold $\left(\mathbb{T}^2, Z=\{\sin\theta_1=0\},\ \omega=\frac{d\theta_1}{\sin\theta_1}\wedge\theta_2\right)$. In this case, the normal symplectic vector field can be found globally and expressed as $X^\sigma=\sin\theta_1\frac{\partial}{\partial\theta_1}$. Following the definition, the family of linear Hamiltonians is then given by

$$H(t,\theta_1,\theta_2)=K(t)\log\left|\frac{\sin\theta_1}{1+\cos\theta_1}\right|+h_t(\theta_2),$$

where $K\in C^\infty(\mathbb{S}^1)$ and $h_t\in C^\infty(\mathbb{S}^1\times\mathbb{S}^1)$. Notice that, although the family of Hamiltonians is not well defined as it is written locally in a neighbourhood of $\theta_1=\pi$, it can be rewritten as

$$H(t,\theta_1,\theta_2)=-K(t)\log\left|\frac{\sin\theta_1}{1-\cos\theta_1}\right|+h_t(\theta_2)$$

in such a neighbourhood. With this observation, it is clear that this family of functions satisfies the linearity condition in each tubular neighbourhood of Z. The Hamiltonian vector field is then given by

$$X_H=K(t)\frac{\partial}{\partial\theta_2}-\sin\theta_1\frac{\partial h_t}{\partial\theta_2}\frac{\partial}{\partial\theta_1}.$$

Remark 3.1.4 More generally, if we think of the local expression around Z with a defining function z, then linear Hamiltonians are given by the formula

$$H(t,z,x) = \begin{cases} K(t)\log|z| + h_t(x) & \text{if } m = 1, \\ -\frac{K(t)}{(m-1)z^{m-1}} + h_t(x) & \text{if } m > 1, \end{cases}$$

where K is a smooth function depending on time, and $\frac{\partial h_t}{\partial z} = 0$.

If we assume that the modular vector field is chosen such that $\omega(X^\sigma, v_{mod}) = 1$, then we have that

$$X_H = Kv_{mod} - \left(\mathcal{L}_{v_{mod}} h_t\right) X^\sigma + X_h,$$

where X_h denotes the Hamiltonian vector field of h restricted to a leaf of the symplectic foliation of ω on Z. The aspect that we want to highlight here is that K is the term that controls the weight of the modular vector field on the expression of the Hamiltonian vector field.

Remark 3.1.5 An important feature of Definition 3.1.2 is that it allows us to prescribe the form of our b^m-Hamiltonians in a coordinate-free way, which is expected in the context of differential geometry. But also the condition is strong in the sense that the prescription of the form of H happens in a neighbourhood of Z and not just the hypersurface. An advantage of this framing is that we can make sense of the notion of linear Hamiltonian also in the context of symplectic manifolds with cosymplectic boundary (or, alternatively, open symplectic manifolds that have cosymplectic behaviour at infinity), in line with Lemma 2.4.34. In the Lemma we see that X^σ has an important role to play in the geometry of such manifolds, and it is also the central piece in the notion of linear Hamiltonian.

This approach was inspired by the work in [FS07] and their conception of admissibility in the context of convex symplectic manifolds, characterized by a contact behaviour at infinity.

In light of Remark 3.1.4 it is worth asking why K must be a function instead of just being constant. We begin justifying this choice in the case of surfaces and for $m = 1$.

Proposition 3.1.6 *Let (Σ, Z, ω) be a compact b-symplectic surface with X^σ a normal symplectic vector field. Let H be a b-Hamiltonian function such that $\mathcal{L}_{X^\sigma} H = K \in \mathbb{R}$ locally constant and different from zero in each connected component of Z.*

Then, for any connected component $Z_i \subset Z$ with modular weight a any tubular neighbourhood $\mathcal{N}(Z_i)$ small enough contains infinite periodic orbits of period $\frac{a}{K}$. Further, if $K \in a\mathbb{Z}$ there exist infinite 1-periodic Hamiltonian orbit in $\mathcal{N}(Z)$.

The situation in Proposition 3.1.6 is undesirable in our case because we are looking for a situation in which the periodic orbits of the Hamiltonian vector are isolated, as a necessary prerequisite for them to be non-degenerate.

Proof. By compactness, Z must be a finite union of circles. For the purposes of this proof we can assume that $Z = \mathbb{S}^1 \cong [0,1] \big/ (0 \sim 1)$ and that $\mathcal{N}(Z)$ is diffeomorphic to $\mathbb{S}^1 \times]-\varepsilon, \varepsilon[$.

By Proposition 2.4.35, the b-symplectic form is given in this tubular neighbourhood by

$$\omega = a\frac{dz}{z} \wedge d\theta,$$

where $a \in \mathbb{R}$ is the modular weight of the particular component of Z.

In these coordinates the normal symplectic b-vector field is $X^\sigma = z\frac{\partial}{\partial z}$. As H is linear, we have that $H_t(z,\theta) = K \log|z| + h_t(\theta)$, so the Hamiltonian vector field is given by

$$aX_{H_t} = K\frac{\partial}{\partial \theta} - z\frac{\partial h_t}{\partial \theta}\frac{\partial}{\partial z}.$$

The flow of this vector field X_{H_t} can be computed explicitly in $\mathcal{N}(Z)$, taking initial conditions $(\theta_0, z_0) \in \mathbb{S}^1 \times]-\varepsilon, \varepsilon[$, as

$$\begin{aligned} \theta(t) &= \theta_0 + \tfrac{K}{a}t \\ z(t) &= z_0 \exp\left(-\int_0^t \frac{1}{a}\frac{\partial h_s}{\partial \theta}(\theta_0 + Ks)ds\right). \end{aligned}$$

Notice that, as the image of θ lies in $\mathbb{S}^1$, $\theta(t)$ is merely a periodic flow, and in particular $\theta(t + \frac{a}{K}) = \theta(t)$.

Let $F : \mathbb{S}^1 \to \mathbb{R}$ be given by

$$F(\theta) := \int_0^{\frac{a}{K}} \frac{\partial h_t}{\partial \theta}(\theta + Kt)dt.$$

The z-component of the flow has a 1-periodic orbit if and only if there is some $\theta_0 \in \mathbb{S}^1$ such that $F(\theta_0) = 0$. Indeed, integrating the function F over $\mathbb{S}^1$ and applying Fubini's theorem we obtain

$$\begin{aligned}\int_{\mathbb{S}^1} F(\theta)d\theta &= \int_{\mathbb{S}^1}\int_0^{\frac{a}{K}} \frac{\partial h_t}{\partial\theta}(\theta + Kt)dt\,d\theta = \int_0^{\frac{a}{K}}\int_{\mathbb{S}^1} \frac{\partial h_t}{\partial\theta}(\theta + Kt)d\theta\,dt\\ &= \int_0^{\frac{a}{K}} [h_t(\theta + Kt)]_{\theta=0}^{\theta=1}dt = 0,\end{aligned}$$

and therefore $F(\theta_0) = 0$ for some θ_0. This means that for any z_0 small enough there exists some θ_0 such that (z_0, θ_0) belongs to a periodic orbit of period $\frac{a}{K}$. In particular, if $K \in a\mathbb{Z}$, then there exists orbits of period $\frac{a}{K}$ in $\mathcal{N}(Z)$, and therefore also 1-periodic orbits. □

A similar statement holds for $m > 1$.

Proposition 3.1.7 *Let (Σ, Z, ω) be a compact b^m-symplectic surface with X^σ a normal symplectic vector field. Let H be a b^m-Hamiltonian function such that $\mathcal{L}_{X^\sigma}H = K \in \mathbb{R}$ is constant (different from zero) in a tubular neighbourhood $\mathcal{N}(Z)$ of Z. Then, any tubular neighbourhood small enough contained in $\mathcal{N}(Z)$ contains at least one periodic orbit.*

Proof. In a tubular neighbourhood, the b^m-symplectic form can be expressed by the expansion introduced in Proposition 2.4.35,

$$\omega = \left(\sum_{i=1}^m z^{m-i}a_i\right)\frac{dz}{z^m} \wedge d\theta.$$

As H is linear, we have that $H_t(z, \theta) = -\frac{1}{m-1}\frac{K}{z^{m-1}} + h_t(\theta)$, where K is constant. The Hamiltonian vector field is therefore given by

$$X_{H_t} = \left(\sum_{i=1}^m z^{m-i}a_i\right)^{-1}\left(K\frac{\partial}{\partial\theta} - z^m\frac{\partial h_t}{\partial\theta}\frac{\partial}{\partial z}\right).$$

The function $\left(\sum_{i=1}^m z^{m-i}a_i\right)^{-1}$ does not vanish, so we consider the time reparametrization of the vector field X_H given by

$$\widetilde{X}_{H_t} = K\frac{\partial}{\partial\theta} - z^m\frac{\partial h_t}{\partial\theta}\frac{\partial}{\partial z}.$$

As this is a reparametrization, both vector field are orbitally equivalent and thus periodic orbits of $\widetilde{X}_{H_t}$ correspond to periodic orbits of X_{H_t}.

We now analyse the flow of the vector field $\widetilde{X}_{H_t}$ and show that this vector field has always infinitely many periodic orbits around the critical set. Its flow can be computed explicitly as

$$\begin{aligned}\theta(t) &= \theta_0 + Kt \\ z(t) &= \left(\frac{1}{z_0^{m-1}} - (m-1)\int_0^t \frac{\partial h_t}{\partial \theta}(\theta_0 + Ks)ds\right)^{-\frac{1}{m-1}}\end{aligned}$$

The flow in the θ coordinate is periodic of period $\frac{1}{K}$ as in the proof of Proposition 3.1.6 and the same arguments apply. This means that for any z_0 there exists a θ_0 such that the integral curve of the reparametrization $\widetilde{X}_{H_t}$ with initial condition (z_0, θ_0) is periodic. The same thus holds for the Hamiltonian vector field X_{H_t}. However, due to the reparametrization the periods of the periodic orbits do not coincide. □

This result is also true in higher dimensions if H does not depend on time.

Lemma 3.1.8 *Assume H is a time-independent Hamiltonian that is a first integral of X^σ. Then there is a 1-parametric family of critical points approaching the critical set.*

Proof. As H is a first integral of X^σ, H can be viewed as a function on the level sets $z^{-1}(\varepsilon)$, where $Z = z^{-1}(0)$. As Z is compact, there exist critical points of H on each hypersurface. The critical points translate to trivial periodic orbits which appear as a 1-parametric family. □

The same result holds also in higher dimensions if $\mathcal{L}_{X^\sigma}H = 0$ and the geometry of Z is that of a trivial mapping torus as in Equation 2.7.

Proposition 3.1.9 *Let (M, Z, ω) be a b-symplectic manifold such that Z is compact and Z is a trivial mapping torus. Then all X^σ-invariant Hamiltonian b-functions have periodic orbits arbitrarily close to Z.*

Proof. Around the critical set we have the local expression $\omega = \frac{dz}{z} \wedge \alpha + \beta$. The tubular neighbourhood $\mathcal{N}(Z)$ admits the codimension 2 symplectic integrable distribution $\ker(dz) \cap \ker(\alpha)$. Let us denote this foliation around the critical set by $\mathcal{F}$ and its leaves by $\mathcal{L}$.

Let us consider Hamilton's equation $\iota_{X_H}\omega = -dH$. The Hamiltonian vector field can be computed using the b^m-Poisson structure, which is given by

$$\Pi = X^\sigma \wedge v_{mod} + \pi_{\mathcal{L}},$$

where $\pi_{\mathcal{L}} = \omega_{\mathcal{L}}^{-1}$ for the symplectic structure on the leaf, and $X^\sigma = z\frac{\partial}{\partial z}$. Thus, the Hamiltonian vector field is given by $\Pi(dH, \cdot)$.

As H does not depend on z, the Hamiltonian vector field has the expression

$$X_H = \frac{\partial H}{\partial \theta} z \frac{\partial}{\partial z} + (X_H)_{\mathcal{L}},$$

where $\frac{\partial H}{\partial \theta}$ is a function that does not depend on z and $(X_H)_{\mathcal{L}}$ is the Hamiltonian vector field along the leaf.

On one hand, by the Arnold conjecture applied to a compact symplectic leaf $\mathcal{L}$, there always exists a 1-periodic orbit of $(X_H)_{\mathcal{L}}$ in $\mathcal{L}$. On the other hand, we can apply Proposition 3.1.6 to the term $\frac{\partial H}{\partial \theta} z \frac{\partial}{\partial z}$ and therefore always find a 1-periodic orbits on the normal direction. Hence there are periodic orbits for X_H. Furthermore, they come in 1-parametric families. □

Remark 3.1.10 A sufficient condition so that (W, Z, ω) has a trivial mapping torus at Z is that the cohomology class $[\omega] \in {}^bH^2(M)$ is integral. See [GMW18b, Section 2] for more details.

In our context, we are interested in producing a category of Hamiltonians whose symplectic gradient has a finite number of periodic orbits, and therefore we look for cases in which we can avoid periodic orbits in $\mathcal{N}(Z)$. In other words, we are trying to pose a version of the Arnold conjecture for the open symplectic manifold $M \setminus Z$. With this interpretation in mind, we introduce the following concept:

Definition 3.1.11 A Hamiltonian $H_t \in {}^{b^m}C^\infty(\mathbb{S}^1 \times M)$ is *admissible* if there exists a tubular neighbourhood $\mathcal{N}(Z)$ of each component of Z such that

1. H_t is *linear* along the normal symplectic b^m-vector field X^σ: $\mathcal{L}_{X^\sigma} H_t = K(t)$.

2. H_t is invariant with respect to the modular vector field: $\mathcal{L}_{v_{mod}} H_t = 0$.

3. The neighbourhood $\mathcal{N}(Z)$ contains no 1-periodic orbits of X_{H_t}.

We denote the set of admissible Hamiltonians as ${}^{b^m}\mathrm{Adm}(M, X^\sigma, v_{mod})$.

Condition 3 in Definition 3.1.11 can seem a bit ad hoc given our interest particularly in periodic orbits of Hamiltonians. In Proposition 3.1.15 we will see a sufficient condition for the admissibility of a Hamiltonian in ${}^{b^m}C^\infty(\mathbb{S}^1 \times M)$ in terms of $K(t)$ and v_{mod}.

Remark 3.1.12 Definition 3.1.11 depends on the choice of modular vector field and normal symplectic vector field. However, as we will see later, the lower bounds on the 1-periodic Hamiltonian orbits, which we are interested in, will be shown to be independent of these choices.

Remark 3.1.13 Building on the notation from Remark 3.1.4, we can see that in local coordinates admissible Hamiltonians have the local expression

$$H(t, z, \theta, x) = \begin{cases} K(t) \log |z| + h_t(x) & \text{if } m = 1, \\ -\frac{K(t)}{(m-1)z^{m-1}} + h_t(x) & \text{if } m > 1, \end{cases}$$

where θ denotes precisely the coordinate in the modular direction, and therefore $\mathcal{L}_{X^\sigma} h_t = \mathcal{L}_{v_{mod}} h_t = 0$.

In the particular case of surfaces, admissible Hamiltonians are quite restricted in a neighbourhood of Z, as $h_t \equiv 0$.

Example 3.1.14 Let us consider the $2n$-torus $\mathbb{T}^{2n}$ with coordinates $(\theta_1, \dots, \theta_{2n})$ and the singular hypersurface $Z = \{\sin\theta_1 = 0\}$. Take the family of b-symplectic forms

$$\omega = \frac{d\theta_1}{\sin\theta_1} \wedge \alpha + \beta$$

for a pair of closed forms $\alpha \in \Omega^1\left(\mathbb{T}^{2n}\right)$ and $\beta \in \Omega^2\left(\mathbb{T}^{2n}\right)$ such that $\frac{d\theta_1}{\sin\theta_1} \wedge \alpha \wedge \beta^{n-1}$. We take also

$$H_t(\theta_1, \dots, \theta_{2n}) = K \log\left|\frac{\sin\theta_1}{1 + \cos\theta_1}\right|$$

for some $K \in \mathbb{R}$ constant, so $X_H = K v_{mod}$. Then, regardless of the expression of v_{mod}, K can always be chosen small enough so that X_H has no 1-periodic orbits, which implies that there is a family of admissible Hamiltonians that have no 1-periodic orbit in $\mathbb{T}^{2n}$.

We close this discussion by providing a sufficient condition to satisfy the last requirement of Definition 3.1.11 in the case that Z is given by the mapping torus of the modular vector field.

Proposition 3.1.15 *Let $H_t \in {}^{b^m}\mathrm{Adm}(M, X^\sigma, v_{mod})$ with $\mathcal{L}_{X^\sigma} H_t = K(t)$, and suppose that the singular hypersurface $Z \subset M$ is given by the mapping torus of the modular vector field,*

$$Z \cong \frac{\mathcal{L} \times [0,T]}{(x,0) \sim \left(\varphi^T_{v_{mod}}(x), T\right)}, \tag{3.1}$$

where T is the modular weight of Z.

Then, if

$$0 < \int_0^1 K(t)dt < T,$$

there exists a tubular neighbourhood of Z that contains no 1-periodic orbits of X_H.

Proof. First, let us observe that, since the forms $\widetilde{\alpha}$ and β defined in Remark 2.4.36 are invariant with respect to X^σ and $\widetilde{\alpha} \wedge \beta^{n-1} \neq 0$, then a tubular neighbourhood of Z small enough $\mathcal{N}(Z)$ will be foliated by symplectic tori, each of them of the form in Equation 3.1.

Taking the local expression of the b^m-Poisson structure $\Pi = X^\sigma \wedge v_{mod} + \pi_{\mathcal{L}}$ around Z, the Hamiltonian vector field admits a splitting in a tubular neighbourhood $\mathcal{N}(Z)$ of the form $X_H = K(t)v_{mod} + (X_H)_{\mathcal{L}}$.

If we identify Z with the expression of its mapping torus from Equation 3.1 we can choose coordinates so that we have $v_{mod} = \frac{\partial}{\partial\theta}$, where θ denotes the translation in the second coordinate. Therefore, by construction, the flow of X_H restricted to this coordinate cannot form a 1-periodic orbit because of the condition that $\int K(t) < T$, and therefore the complete flow cannot have a 1-periodic orbit in $\mathcal{N}(Z)$. □

3.2 New results on the desingularization of b^m-symplectic manifolds

In this section we will introduce versions of the desingularization process, which we first visited on Section 2.5, that account for the dynamical behaviour of a given admissible Hamiltonian on a particular compact b^m-symplectic manifold. Our aim will be to show that the flow of the Hamiltonian can be still Hamiltonian for the desingularized symplectic form in some cases.

We will see two families of results: on one hand, results for b^{2k}-symplectic manifolds, and on the other hand for b^m-symplectic surfaces. In both cases we will see that the result depends on whether the graph of the manifold (see Definition 2.4.18) is acyclic or not.

We begin with the even case.

Proposition 3.2.1 *Let (M, Z, ω) be a compact b^{2k}-symplectic manifold and take $H_t \in {}^{2k}C^\infty(\mathbb{S}^1 \times M)$ an admissible Hamiltonian. Then, there exists a symplectic structure $\widetilde{\omega}$ on M with respect to which X_H^ω is a symplectic vector field.*

Moreover, if the b-manifold (M, Z) is acyclic there exits a smooth Hamiltonian $\widetilde{H}_t \in C^\infty(\mathbb{S}^1 \times M)$ such that $X_{\widetilde{H}}^{\widetilde{\omega}}$ coincides with X_H^ω.

Proof. The idea of the proof will be to repeat the process of desingularization introduced in Definition 2.5.1 but taking also into account the b^{2k}-Hamiltonian H_t in the even case.

As we explained in Definition 2.5.1, there is a family of symplectic forms approximating ω with the semilocal expression

$$\omega_\varepsilon = df_\varepsilon \wedge \left(\sum_{i=0}^{2k-1} z^i \pi^* \alpha_i \right) + \pi^* \beta \tag{3.2}$$

on a tubular neighbourhood $\mathcal{N}(Z_i)$ around Z_i, and exactly equal to ω away from $\mathcal{N}(Z_i)$. As discussed in Remark 3.1.13 the admissible Hamiltonian will have the local expression in $\mathcal{N}(Z_i)$ given by

$$H_t = -K_i(t) \frac{1}{2k-1} \frac{1}{z^{2k-1}} + h_t. \tag{3.3}$$

We will use the function f_ε to build a smooth Hamiltonian $\widetilde{H}_\varepsilon$ semilocally in $\mathcal{N}(Z_i)$ to prove the first part of the Proposition, and then we will proceed to globalize the construction of $\widetilde{H}_\varepsilon$ in the acyclic case.

Let us take $\widetilde{H}_\varepsilon = K(t)f_\varepsilon + h_t$ in $\mathcal{N}(Z_i)$ and equal to H in $M \setminus \mathcal{N}(Z_i)$. A direct computation shows that

$$X_H^\omega\Big|_{\mathcal{N}(Z_i)} = X_{\widetilde{H}_\varepsilon}^{\omega_\varepsilon}\Big|_{\mathcal{N}(Z_i)}.$$

Moreover, as $d\widetilde{H}_\varepsilon$ coincides with dH_t outside of $\mathcal{N}(Z_i)$ for all connected components of Z, $X_{\widetilde{H}_\varepsilon}^{\omega_\varepsilon}$ can be extended to X_H^ω in $M \setminus \bigcup_i \mathcal{N}(Z_i)$. Thus, the vector field $X_{\widetilde{H}_\varepsilon}^{\omega_\varepsilon}$ is globally defined. Locally, $X_{\widetilde{H}_\varepsilon}^{\omega_\varepsilon}$ is generated by Hamiltonian functions, so it is a symplectic vector field with respect to ω_ε. We conclude therefore that X_H^ω is a symplectic vector field on $(M, \widetilde{\omega} = \omega_\varepsilon)$ for any $\varepsilon > 0$ small enough.

Let us assume from now on that the graph associated to (M, Z) is acyclic. As it is acyclic we can choose a colouring of the graph with two colours, which we will label with the signs $\{+, -\}$. Let M^+ denote the union of the connected components of $M \setminus Z$ labeled with the sign $+$ and M^- the union of the connected components labeled with the sign $-$, so that $M \setminus Z = M^+ \bigsqcup M^-$ and all the adjacent components to a connected component of M^+ belong to M^- and conversely.

We will define $\widetilde{H}_\varepsilon$ iteratively starting from a connected component of $M \setminus \mathcal{N}(Z)$. On this initial component we take $\widetilde{H}_\varepsilon = H$, the given b^{2k}-admissible Hamiltonian.

Let us now assume that we have defined $\widetilde{H}_\varepsilon$ on a connected component of $M \setminus Z$ whose vertex in the graph we denote by v. Consider a vertex adjacent to v on which we still have not constructed $\widetilde{H}_\varepsilon$, and denote by Z_i the connected component of Z separating it from the connected component associated to v.

In $\mathcal{N}(Z_i)$ we consider the local expression of H_t as given in Equation 3.3. In particular, we have a specific function $K_i(t)$. We define now $\widetilde{H}_\varepsilon$ in $\mathcal{N}(Z_i)$ depending on the label of the present vertex, v:

- If it has the $+$ label, this means, v represents a connected component in M^+, we define $\widetilde{H}_\varepsilon = f_\varepsilon(z)K_i(t) - \frac{2}{\varepsilon^{2k-1}}K_i(t) + h_t$.
- If it has the $-$ label, this means, v represents a connected component in M^-, we define $\widetilde{H}_\varepsilon = f_\varepsilon(z)K_i(t) + \frac{2}{\varepsilon^{2k-1}}K_i(t) + h_t$.

The "glued" function is then smooth. Indeed, if we look at the case when v is labeled with + and take $z > \varepsilon$,

$$\begin{aligned}\widetilde{H}_\varepsilon &= f_\varepsilon(z)K_i(t) - \frac{2}{\varepsilon^{2k-1}}K_i(t) + h_t \\ &= K_i(t)\left(\frac{-1}{(2k-1)z^{2k-1}} + \frac{2}{\varepsilon^{2k-1}}\right) - \frac{2}{\varepsilon^{2k-1}}K_i(t) + h_t \\ &= -K_i(t)\frac{1}{(2k-1)z^{2k-1}} + h_t.\end{aligned}$$

The same argument applies in the case when v is labeled with $-$ and we look at $z < -\varepsilon$.

Also, as $f'_\varepsilon > 0$ by construction, the function $\widetilde{H}_\varepsilon$ admits no critical points on $\mathcal{N}(Z_i)$.

By the same criterion, we define $\widetilde{H}_\varepsilon$ in the adjacent vertex as $\widetilde{H}_\varepsilon = H_t - \frac{4}{\varepsilon^{2k-1}}K_i(t)$ if v has the label +, or as $\widetilde{H}_\varepsilon = H_t + \frac{4}{\varepsilon^{2k-1}}K_i(t)$ if v has the label $-$. Following this process, as the graph is acyclic, we can construct consistently a smooth function $\widetilde{H}_\varepsilon$ on the whole manifold M.

A direct computation allows us to verify that the Hamiltonian vector field associated to $\widetilde{H}_\varepsilon$ by the symplectic form ω_ε constructed by a desingularization through the same functions f_ε coincides with X_H^ω, as we wanted to prove. Thus, X_H^ω is a Hamiltonian vector field for a choice of symplectic structure and smooth Hamiltonian. □

Proposition 3.2.1 has direct implications for the dynamics of X_H in the b^{2k}-symplectic setting, which we will get back to in Section 3.3

For now we will concentrate on a related result in a slightly different setting, that of b^m surfaces, where the parity of m will no longer play a role.

Proposition 3.2.2 *Let (Σ, Z, ω) be a b^m-symplectic surface such that Σ is orientable, and take H_t an admissible Hamiltonian. Then, there exists a symplectic structure $\widetilde{\omega}$ on Σ with respect to which X_H^ω is a symplectic vector field.*

Moreover, if (Σ, Z) is acyclic, then there exists a smooth Hamiltonian $\widetilde{H}$ such that $X_{\widetilde{H}}^{\widetilde{\omega}}$ coincides with X_H^ω.

Proof. It is clear that the case in which $m = 2k$ is already covered in Proposition 3.2.1, so we will assume in this proof that m is odd. The structure of the proof is also the same: we will first construct $X^{\widetilde{\omega}}_{\widetilde{H}}$ locally, and then we will construct its smooth Hamiltonian globally for the acyclic case.

As Σ is a compact surface and $Z \subset \Sigma$ is also compact it follows that Z is diffeomorphic to a disjoint union of circles, so each tubular neighbourhood is diffeomorphic to a cylinder, $\mathcal{N}(Z) \cong \left]-\varepsilon, \varepsilon\right[\times \mathbb{S}^1$, with coordinates (z, θ). In these local coordinates we can express the b^m-symplectic form as

$$\omega = \sum_{i=0}^{m-1} (z^i a_i) \frac{dz}{z^m} \wedge d\theta,$$

where $a_0 \neq 0$. As H_t is an admissible Hamiltonian, we have a local expression of H_t on $\mathcal{N}(Z)$ given by

$$H_t(z, \theta) = \begin{cases} K(t) \log |z| & \text{if } m = 1, \\ -K(t) \frac{1}{m-1} \frac{1}{z^{m-1}} & \text{if } m > 1. \end{cases}$$

As in the proof of Proposition 3.2.1 take Σ^+ and Σ^- disjoint components of $\Sigma \setminus Z$ corresponding to a two-colouring of the graph associated to (Σ, Z). Such a colouring exists even if the graph is not acyclic under the assumption that m is odd, as we commented in Remark 2.4.41.

We will perform now a desingularization of ω in a different manner to the one exposed in Definition 2.5.1, as we want to build a symplectic structure instead of a folded structure, with the caveat that we are no longer approximating ω as in the standard case.

Let $g_\varepsilon : \left]-\varepsilon, \varepsilon\right[\to \mathbb{R}$ be a smooth function such that $g_\varepsilon' > 0$ and

$$g_\varepsilon\big|_{\left]\frac{\varepsilon}{2}, \varepsilon\right[}(z) = \begin{cases} \log |z| & \text{if } m = 1, \\ -\frac{1}{m-1} \frac{1}{z^{m-1}} & \text{if } m > 1, \end{cases}$$

$$g_\varepsilon\big|_{\left]-\varepsilon, -\frac{\varepsilon}{2}\right[}(z) = \begin{cases} -\log |z| + 2 \log \left(\frac{\varepsilon}{2}\right) - \varepsilon & \text{if } m = 1, \\ \frac{1}{m-1} \frac{1}{|z|^{m-1}} - \frac{1}{m-1} \frac{2^{m+1}}{\varepsilon^m} & \text{if } m > 1. \end{cases}$$

This function is indeed well defined, because $g_\varepsilon\left(\frac{\varepsilon}{2}\right) - g_\varepsilon\left(-\frac{\varepsilon}{2}\right) > 0$ for all $m \in \mathbb{N}$, and therefore g_ε can be chosen in such a way that $g_\varepsilon' > 0$ everywhere.

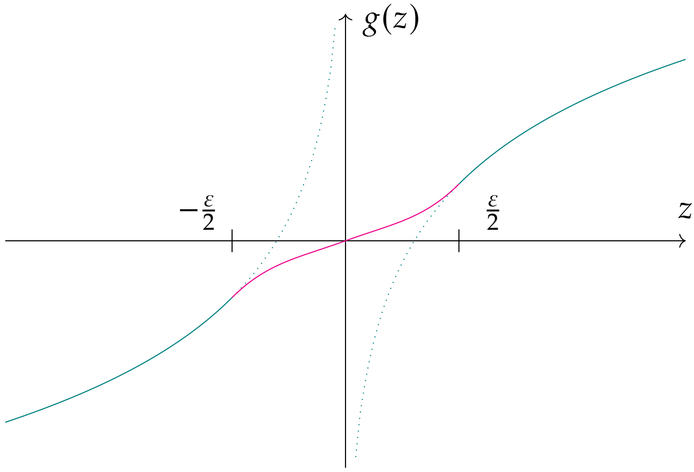

Figure 3.1: Construction of g_ε

See Figure 3.1 for an illustration of the construction of g_ε.

We define a smooth 2-form on Σ as

$$\widetilde{\omega}_\varepsilon = \begin{cases} \omega & \text{in } \Sigma^+ \setminus \mathcal{N}_\varepsilon(Z), \\ (-1)^m \omega & \text{in } \Sigma^- \setminus \mathcal{N}_\varepsilon(Z), \\ \sum_{i=0}^{m-1} (z^i a_i) dg_\varepsilon \wedge d\theta & \text{in } \mathcal{N}_\varepsilon(Z), \end{cases}$$

which can easily be shown to be a symplectic form on Σ coinciding with either ω or $(-1)^m\omega$ on each connected component of $\Sigma \setminus \mathcal{N}_\varepsilon(Z)$.

With this construction in mind we can now reproduce the arguments of the proof of Proposition 3.2.1: we can define the smooth Hamiltonian in the tubular neighbourhoods $\mathcal{N}_\varepsilon(Z)$ in such a way that $d\widetilde{H}_\varepsilon$ can be patched into a smooth 1-form, so $X^{\widetilde{\omega}}_{\widetilde{H}_\varepsilon}$ is a symplectic vector field for $\widetilde{\omega}_\varepsilon$ and it coincides with X^ω_H, proving the second part of the Proposition.

On the other hand, when the graph associated to Σ, Z is acyclic we can proceed as in Proposition 3.2.1 in order to construct $\widetilde{H}_\varepsilon$ globally, thus showing that $X^{\widetilde{\omega}}_{\widetilde{H}}$ can be recovered as a Hamiltonian vector field for the smooth symplectic form $\widetilde{\omega}$. $\square$

Remark 3.2.3 Both Propositions 3.2.1 and 3.2.2 can be rephrased as follows: there exists a symplectic (desingularized) structure $\widetilde{\omega}_\varepsilon$ for which the 1-form resulting from the contraction

$$\eta = \iota_{X_H^\omega}\widetilde{\omega}_\varepsilon$$

is closed. Moreover, if the graph corresponding to (M, Z) or to (Σ, Z) is acyclic, then η is exact.

Remark 3.2.4 The construction in Proposition 3.2.2 cannot be adapted to higher dimensions for m odd. This is because in the local expression $\omega = \frac{dz}{z^m} \wedge \widetilde{\alpha} + \beta$ we would be interpolating the first component from $\frac{dz}{z^m} \wedge \widetilde{\alpha}$ to $-\frac{dz}{z^m} \wedge \widetilde{\alpha}$, but there is not a process that can be adopted to interpolate β to $-\beta$ in general. Lacking such a process, we cannot generalize this construction to dimensions higher than 2 a priori.

We believe that a version of this construction should be possible in higher dimensions, at least in the case that Z has the structure of a mapping torus due to results that we will see in Section 3.3. However, this remains a mere conjecture.

While the argument may not be applicable to higher dimensions necessarily, it is trivial to use it for direct products of b^m-surfaces with symplectic manifolds.

Remark 3.2.5 In b^m-symplectic manifolds of the form $(\Sigma\times M, Z\times M, \omega_1 + \omega_2)$, where (Σ, Z, ω_1) is a b^m-symplectic surface and (M, ω_2) is a symplectic manifold, we can apply the same construction as in the proof of Proposition 3.2.2, as the admissibility condition implies that the Hamiltonian dynamics corresponds to the product dynamics. The same argument holds for c-symplectic manifolds (see Miranda and Scott [MS20]) arising from products of b^m-symplectic surfaces with an analogous notion of admissibility.

The desingularization process that we have studied so far can, in a certain way, be reversed in the particular case of surfaces. This process, which we call *singularization*, was already studied in full generality in Cavalcanti [Cav17, Section 5], but we diverge from his focus of attention slightly as we want to concentrate on the dynamical implications of such a singularization process. A similar trick was used to *singularize* contact structures along convex surfaces in [MO18]. This tool will be useful later to prove the sharpness of some of the lower bounds that we will introduce in Section 3.3.

Proposition 3.2.6 *Let $(\Sigma, \widetilde{\omega})$ be a symplectic surface, and let $Z = \bigcup_{i \in I} \gamma_i \subset \Sigma$ be a collection of embedded smooth curves such that the b-surface (Σ, Z) has a 2-colourable graph. Let Σ^+ and Σ^- denote the partition of $\Sigma \setminus Z$ given by the 2-colouring of the graph.*

Then, for every $m \in \mathbb{N}_{>0}$ there exists a b^m-symplectic structure on (Σ, Z) that agrees with $\widetilde{\omega}$ in $\Sigma^+ \setminus \mathcal{N}_\varepsilon(Z)$ (outside an ε-neighbourhood of Z) and with $(-1)^m \widetilde{\omega}$ in $\Sigma^- \setminus \mathcal{N}_\varepsilon(Z)$. Moreover, if $\widetilde{H} \in C^\infty(\Sigma)$ is a smooth function such that $\widetilde{H}(z, \theta) = z$ in a ε-neighbourhood of Z, then there exists an admissible b^m-function H such that $X_H^\omega = X_{\widetilde{H}}^{\widetilde{\omega}}$.

Proof. Let us start working in an ε-neighbourhood of a connected component γ_i, $\mathcal{N}_\varepsilon(\gamma_i) \cong \left]-\varepsilon, \varepsilon\right[\times \gamma_i$. We define the b^m-function $s_\varepsilon \in {}^{b^m}C^\infty\left(\mathcal{N}_\varepsilon(\gamma_i)\right)$ given by

$$s_\varepsilon(z, \theta) = \begin{cases} \log|z| & \text{if } z \in \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\text{ and } m = 1, \\ -\frac{1}{z^{m-1}} & \text{if } z \in \left]-\frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right[\text{ and } m > 1, \\ z & \text{if } z > \frac{\varepsilon}{2}, \\ (-1)^m z & \text{if } z < -\frac{\varepsilon}{2}, \end{cases}$$

which in particular satisfies that $\frac{\partial s_\varepsilon}{\partial z} \neq 0$. We repeat this process for each component γ_i so that we get a function defined in $\mathcal{N}_\varepsilon(Z)$. We can thus define $\omega = \frac{\partial s_\varepsilon}{\partial z} \widetilde{\omega} \in {}^{b^m}\Omega^2(\mathcal{N}_\varepsilon(Z))$. Moreover, ω can be trivially extended to the whole manifold by taking it to be equal to $\widetilde{\omega}$ in $\Sigma^+ \setminus \mathcal{N}_\varepsilon(Z)$ and to $(-1)^m \widetilde{\omega}$ in $\Sigma^- \setminus \mathcal{N}_\varepsilon(Z)$. The b^m-form ω can be seen to be a b^m-symplectic form.

By an analogous process, we can modify a smooth function $\widetilde{H} \in C^\infty(\Sigma)$ to a b^m-function by taking

$$H(p) = \begin{cases} \widetilde{H}(p) & \text{if } p \in \Sigma^+ \setminus \mathcal{N}_\varepsilon(Z), \\ (-1)^m \widetilde{H}(p) & \text{if } p \in \Sigma^- \setminus \mathcal{N}_\varepsilon(Z), \\ s_\varepsilon & \text{if } p \in \mathcal{N}_\varepsilon(Z). \end{cases}$$

A direct computation shows that indeed $X_H^\omega = X_{\widetilde{H}}^{\widetilde{\omega}}$. □

Next we will investigate particular constructions that can be performed in b^m-symplectic surfaces to isolate the dynamical behaviours in connected components of $\Sigma \setminus Z$.

Proposition 3.2.7 *Let (Σ, Z, ω) be a compact b^m-symplectic orientable surface and let H_t be an admissible Hamiltonian. Let Σ' denote a connected component of $\Sigma \setminus Z$ with genus g, and let $\{\gamma_1, \ldots, \gamma_d\}$ denote the connected components of Z adjacent to Σ'. For $\varepsilon > 0$, let $\Sigma'_\varepsilon = \Sigma' \setminus \mathcal{N}_\varepsilon(Z)$.*

Then, there exists a closed symplectic surface $\left(\overline{\Sigma'}, \overline{\omega}\right)$ of genus g such that

$$\left(\overline{\Sigma'} \setminus \bigsqcup_{1 \le i \le d} D_i,\, \overline{\omega}\right) \cong \left(\Sigma'_\varepsilon,\, \omega|_{\Sigma'_\varepsilon}\right)$$

symplectomorphically for certain contractible sets $D_1, \ldots, D_d \subset \overline{\Sigma'}$.

Moreover, there exists a smooth function defined on $\overline{\Sigma'}$ such that its Hamiltonian vector field coincides with X_H^ω on $\overline{\Sigma'} \setminus \bigsqcup_{1 \le i \le d} D_i$.

The idea of the proof is to attach a collection of 2-disks $D_1, \ldots, D_d$ to Σ' at the corresponding curves $\gamma_1, \ldots, \gamma_d$, as suggested by Figure 3.2, and then desingularize in order to recover the desired symplectic behaviour, and also to extend H to this new set in order to achieve the required dynamics.

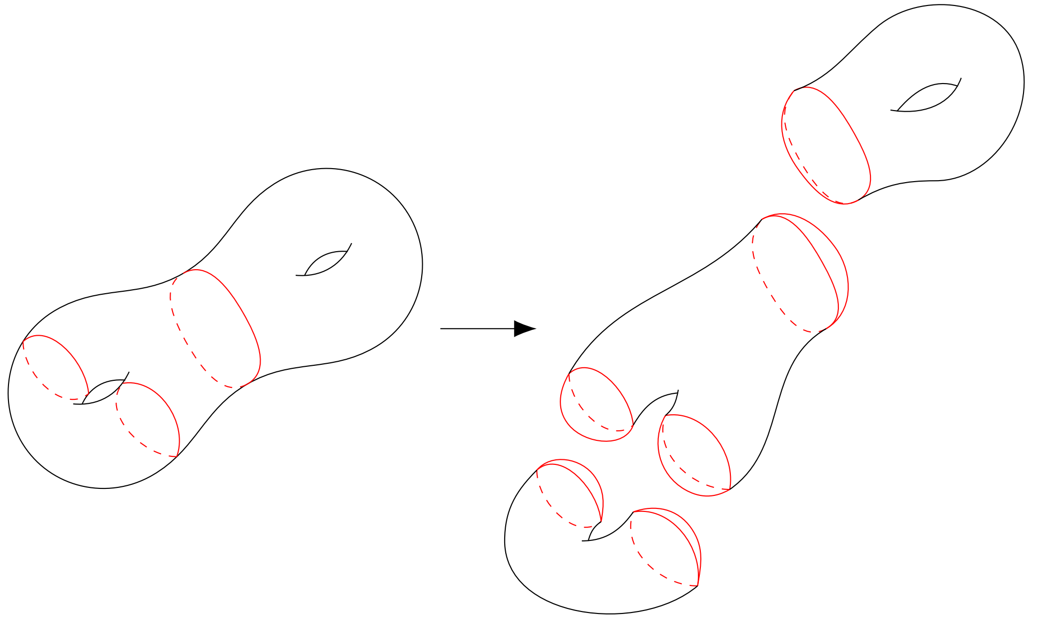

Figure 3.2: Completion of a b-symplectic surface with disks at the punctures

Consider the b-surface given by a disk and its boundary, denoted as $(D, \partial D)$. Any b^m-symplectic structure ω on $(D, \partial D)$ has the expansion given by Proposition 2.4.35 in a neighbourhood of Z,

$$\omega|_{\mathcal{N}(\partial D)} = \sum_{i=1}^{m} a_i \frac{dr}{(1-r)^i} \wedge d\theta, \tag{3.4}$$

where $a_i \in C^\infty(\mathbb{S}^1 \times D)$ are smooth functions and (r, θ) are polar coordinates.

Also, any admissible Hamiltonian must have the semilocal expression

$$H_t|_{\mathcal{N}(\partial D)}(r, \theta) = \begin{cases} K(t) \log|1-r| & \text{if } m = 1, \\ -\frac{K(t)}{(m-1)(1-r)^{m-1}} & \text{if } m > 1, \end{cases}$$

for some $K \in C^\infty(\mathbb{S}^1)$ such that $K(t) > 0$ and

$$\int_0^1 K(t)dt < \frac{2\pi}{a_m},$$

where $\frac{2\pi}{a_m}$ is the modular weight of ω.

Lemma 3.2.8 *Let $\mathcal{N}_\varepsilon(Z) = \{(r, \theta) \in D \mid r > 1-\varepsilon\}$ be an annulus around the boundary of the disk, and let ω and H_t be a b^m-symplectic form and an admissible Hamiltonian on $\mathcal{N}_\varepsilon(Z)$, respectively. Then there exist extensions of ω and H_t to the whole disk such that the Hamiltonian flow at time* 1 *has exactly one fixed point.*

Proof. The proof is a simple computation where we extend the expression given in Equation 3.4 in such a way that the Hamiltonian vector field is given globally by $X_H = -\frac{K(t)}{a_m}\frac{\partial}{\partial\theta}$, where $\frac{\partial}{\partial\theta}$ has exactly one fixed point in $(0, 0) \in D$ and the time 1 flow of X_H cannot have more fixed points because $0 < \int_0^1 K(t)dt < \frac{2\pi}{a_m}$. □

Proof of Proposition 3.2.7. Let us start with the assumption that $d = 1$, this means, Σ' is diffeomorphic to an orientable surface of genus g punctured at a single point, and the adjacent component of Z, which we can denote simply as γ, is diffeomorphic to $\mathbb{S}^1$. Equivalently, the vertex corresponding to Σ' in the graph of the b-manifold has degree 1. We will start by completing Σ' to a closed b^m-symplectic surface by attaching a 2-disk to the puncture.

The b^m-symplectic form ω in the neighbourhood near γ has the expression

$$\omega|_{\mathcal{N}_\varepsilon(Z)} = \sum_{i=1}^{m} a_i \frac{dr}{(1-r)^i} \wedge d\theta,$$

so we can apply Lemma 3.2.8 to attach a 2-disk D to γ in order to get a closed surface of genus g, $\overline{\Sigma'}$, with a b^m-symplectic form $\widetilde{\omega}$ on the extension. Lemma 3.2.8 can also be applied to a given Hamiltonian H_t to get a b^m-admissible Hamiltonian $\widetilde{H}_t$ in such a way that near γ the Hamiltonian vector field has the expression $X_{\widetilde{H}} = K\frac{\partial}{\partial\theta}$. Under these conditions, we also know that $X_{\widetilde{H}}\big|_D$ vanishes exactly at one point in the interior of the disk.

With these choices, $\left(\overline{\Sigma'}, \gamma\right)$ is a b-surface with a trivial graph, given by two vertices linked with a single edge. In particular, this graph is acyclic. Therefore, we can apply the desingularization procedure from Proposition 3.2.2 to the system $\left(\overline{\Sigma'}, \gamma, \widetilde{\omega}, \widetilde{H}\right)$ to get the symplectic desingularized surface $\left(\overline{\Sigma'}, \overline{\omega}\right)$ with a smooth Hamiltonian $\overline{H}_t$.

If the degree of the vertex of the open component under consideration is strictly greater than 1, the process is analogous. In this case, the singular hypersurface is $\bigcup_{1\le i\le d} \gamma_i \cong \mathbb{S}^1 \sqcup \cdots \sqcup \mathbb{S}^1$, and the application of Lemma 3.2.8 is done on each disjoint open neighbourhood $\mathcal{N}_\varepsilon(\gamma_i)$. The resulting closed b-surface after gluing the d 2-disks, $\left(\overline{\Sigma'}, \bigcup_{1\le i\le d}\gamma_i\right)$, has a star graph consistent of $d+1$ vertices, with a central vertex of degree d and all remaining vertices connected only to this central vertex. In particular, this graph in acyclic. Therefore we can apply Proposition 3.2.2 to $\left(\overline{\Sigma'}, \bigcup_{1\le i\le d}\gamma_i, \widetilde{\omega}, \widetilde{H}\right)$, where $\widetilde{\omega}$ and $\widetilde{H}$ denote the extended b^m-symplectic structure and the extended b^m-admissible Hamiltonian respectively, and we obtain a desingularized symplectic surface $\left(\overline{\Sigma'}, \overline{\omega}\right)$ of genus g with a smooth Hamiltonian $\overline{H}_t$. Notably, the Hamiltonian flow $X^{\overline{\omega}}_{\overline{H}}$ has a fixed point in the interior of each of the d attached disks and no other 1-periodic orbit in the neighbourhood of either of them. □

Remark 3.2.9 The process in Proposition 3.2.7 can be reproduced in an analogous way for symplectic manifolds in higher dimensions for which the singular hypersurface Z has the geometry of a mapping torus. A curious reader is encouraged to check [BMO22, Section 4.4] for more details about this subject.

3.3 The Arnold conjecture through desingularization

After the groundwork laid out in Section 3.2 we are now ready to present several results related to lower bounds on the number of 1-periodic orbits of the Hamiltonian flow of certain admissible b^m-Hamiltonians, this means, versions of the Arnold conjecture, for some categories of b^m-symplectic manifolds.

To start we have to narrow the family of admissible Hamiltonians to ones for which the Arnold conjecture is sensible, that of regular admissible Hamiltonians.

Definition 3.3.1 Let (M, Z, ω) be a b^m-symplectic manifold and take $H_t \in {}^{b^m}\mathrm{Adm}(M, X^\sigma, v_{mod})$. Let us denote by $\mathcal{P}(H)$ the set of 1-periodic orbits of the flow of X_H, which by construction are completely contained within $M \setminus Z$. We say that H_t is *regular* if for all periodic orbits $x \in \mathcal{P}(H)$ we have that

$$\det\left(\mathrm{Id} - d\varphi^1_{X_H}(x(0))\right) \neq 0.$$

This means, all 1-periodic orbits are *non-degenerate*

Remark 3.3.2 As is the case for smooth Hamiltonians, the set of regular admissible Hamiltonians is open and dense in ${}^{b^m}\mathrm{Adm}(M, X^\sigma, v_{mod})$ in the strong Whitney C^∞-topology.

Then, as a consequence of Proposition 3.2.1 we have that:

Theorem 3.3.3 (Brugués, Miranda and Oms [BMO22]) *Let (M, Z, ω) be a compact acyclic b^{2k}-symplectic manifold, and let H_t be a regular admissible Hamiltonian. Then,*

$$\#\mathcal{P}(H) \geq \sum_i \dim HM_i\,(M;\mathbb{Z}_2)\,,$$

where $HM_i\,(M;\mathbb{Z}_2)$ denotes the i-th group of the Morse homology on M with coefficients in $\mathbb{Z}_2$.

Proof. By Proposition 3.2.1 we can desingularize (M, Z, ω) into a symplectic manifold and H_t into a smooth Hamiltonian $\widetilde{H}$ whose Hamiltonian vector field coincides with X_H. Furthermore, we can assume that all the

1-periodic orbits of $X_{\widetilde{H}}$ are non-degenerate, so we can apply the standard Arnold conjecture for compact symplectic manifolds as presented in Theorem 2.2.18. □

Note that a crucial assumption in Theorem 3.3.3 is that the graph associated to (M, Z) is acyclic, which intrinsically is a topological property of the b-manifold. Proposition 3.2.1 only concludes that the vector field is symplectic when the graph is cyclic. Symplectic vector fields do not necessarily exhibit 1-periodic orbits at all. In that case, the best lower bound available for such flows can be found in the main theorem of [VO95], which induces the following lower bound:

Corollary 3.3.4 *Let (M, Z, ω) be a compact b^{2k}-symplectic manifold of dimension $2n$ which is aspherical, in the sense that $\pi_2(M) = 0$. Let H_t be a regular admissible b^{2k}-Hamiltonian, with Hamiltonian flow $\varphi^t_{X_H}$. Suppose that there exists an isotopy between $\varphi^1_{X_H}$ and the identity through b^{2k}-symplectomorphisms.*

Then, the number of fixed points of $\varphi^1_{X_H}$ is greater or equal to the sum of the Betti numbers of the Novikov homology over $\mathbb{Z}_2$ associated with the Calabi invariant of $\varphi^1_{X_H}$.

In the particular case introduced in Example 3.1.14 this lower bound turns out to be 0, as we can see in the Example for a particular Hamiltonian.

In what follows we will concentrate on a stronger version of this lower bounds for the particular case of b^m-symplectic surfaces. We will adopt the notation $G = (V, E)$ for the graph of a given b-surface (Σ, Z), where V denotes the set of vertices and E the set of edges of the graph. Moreover, if $v \in V$ is a vertex of the associated graph we will denote by $\Sigma_v \subset \Sigma \setminus Z$ the connected component associated to the vertex v, and the genus of Σ_v as a surface will be denoted by g_v. Finally, let us recall that the *degree* of a vertex $v \in V$, denoted by $\deg(v)$, is the number of vertices that are adjacent to v in the graph.

Theorem 3.3.5 (Brugués, Miranda and Oms [BMO22]) *Let (Σ, Z, ω) be a closed b^m-symplectic orientable surface. Let H_t be a regular admissible b^m-Hamiltonian. Then the number of 1-periodic orbits of X_H has the lower bound*

$$\#\mathcal{P}(H) \geq \sum_{v \in V} \left(2g_v + \left|\deg(v) - 2\right|\right).$$

Proof. We will separate this proof in two parts: first we will show that a connected component $\Sigma_v \subset \Sigma \setminus Z$ with $\deg(v) = 1$ contains at least $2g_v + 1$ periodic orbits, and then we will prove that in the case that $\deg(v) \geq 2$ then the lower bound is $2g_v + \deg(v) - 2$.

In the case that $\deg(v) = 1$, we can apply Proposition 3.2.7 to the open surface Σ_v, so we get a closed symplectic surface $\left(\overline{\Sigma}_v, \overline{\omega}_v\right)$ with a smooth Hamiltonian $\overline{H}_t$ in such a way that $X^{\overline{\omega}}_{\overline{H}}$ agrees with X^{ω}_{H} in $\Sigma_v \subset \overline{\Sigma}_v$. If we apply the standard Arnold conjecture to $\left(\overline{\Sigma}_v, \overline{\omega}, \overline{H}_t\right)$ we conclude that $X^{\overline{\omega}}_{\overline{H}}$ has at least $2g_v + 2$ one-periodic orbits. Moreover, we know that in the attached disk $D \subset \overline{\Sigma}_v$ the vector field $X^{\overline{\omega}}_{\overline{H}}$ has precisely one periodic orbit, so we can conclude that it has at least $2g_v + 1$ periodic orbits away from D, this means, in the interior of Σ_v.

Let us now consider the case when $\deg(v) \geq 2$. If we apply Proposition 3.2.7 to this case we get a closed symplectic surface $\left(\overline{\Sigma}_v, \overline{\omega}\right)$ and a smooth Hamiltonian $\overline{H}_t$, where $\overline{\Sigma}_v \setminus \Sigma_v$ is the disjoint union of $\deg(v)$ 2-disks. As remarked in Lemma 3.2.8, the flow of $X^{\overline{\omega}}_{\overline{H}}$ has $\deg(v)$ trivial periodic orbits, each of them located in the interior of each of these disks. As each of them has to be either a minimum or a maximum of $\overline{H}_t$, they have Conley-Zehnder index either $+1$ or -1.

Let us now consider the Floer complex induced by $\left(\overline{\Sigma}_v, \overline{\omega}, \overline{H}_t\right)$. We will use the notation $c_i := \#\{x \in \mathcal{P}(H) \mid \mu_{CZ}(x) = i\}$ for the dimension of the i-th Floer complex group, and b_i for the dimension of the i-th Floer homology group. As $\overline{\Sigma}_v$ is a closed surface of genus g_v, we know that $b_{-1} = b_1 = 1$ and $b_0 = 2g_v$. The dimensions c_i and b_i are related by the *weak Morse inequalities*, which in the case of the Floer complex yield

$$\sum (-1)^i c_i = \sum (-1)^i b_i. \tag{3.5}$$

By our previous observation that the points in the interior of the disks are maxima or minima we can deduce that $c_{-1} + c_1 \geq \deg(v)$. If we combine everything we know with Equation 3.5 we can arrive at the inequality

$$c_0 \geq 2g_v + \deg(v) - 2.$$

Therefore the number of periodic orbits of $X^{\overline{\omega}}_{\overline{H}}$ can be bounded below:

$$c_{-1} + c_0 + c_1 \geq 2g_v + 2\deg(v) - 2.$$

If we subtract the $\deg(v)$ periodic orbits that lie in the interior of the disks we will obtain the lower bound for the orbits of $X_{\overline{H}}^{\overline{\omega}} = X_H^{\omega}$ contained in the interior of Σ_v, which is precisely $2g_v + \deg(v) - 2$.

Repeating this process for each connected component of $\Sigma \setminus Z$ we arrive at the desired conclusion. □

Remark 3.3.6 The lower bound in Theorem 3.3.5 is always equal or better than the lower bound from Theorem 3.3.3 in the case when both results can be applied.

Moreover, the lower bound in Theorem 3.3.5 is sharp, in the sense that it is the best lower bound that can be obtained.

Proposition 3.3.7 (Brugués, Miranda and Oms [BMO22]) *Let (Σ, ω) be a compact orientable b^m-symplectic surface with critical set $Z = \bigsqcup \gamma_i$, where each γ_i is diffeomorphic to a circle. Then there exists an admissible b^m-function $F : (\Sigma, \bigsqcup \gamma_i) \to \mathbb{R} \cup \{\pm\infty\}$ such that X_F has exactly*

$$\sum_{v \in V} \left(2g_v + |\deg(v) - 2|\right)$$

1-*periodic orbits.*

The proposition follows from Proposition 3.2.6 combined with fairly basic Morse theory on surfaces. We will need the following result for the case that m is even:

Definition 3.3.8 Let $G = (V, E)$ an undirected graph. A *good orientation* of the graph is a choice of direction for each edge $e \in E$ in such a way that every vertex $v \in V$ with $\deg(v) > 1$ has at least one edge whose target is v and at least one edge whose source is v.

Lemma 3.3.9 *Let $G = (V, E)$ an undirected graph. Then, there exists a good orientation for G.*

Proof of Proposition 3.3.7. To prove this result we will split the surface at each connected component of Z and construct a Morse function for each of the compactified components, in a process similar to that of Proposition 3.2.7. We will then singularize the constructed Morse functions and glue them together in the initial surface. In order to perform this gluing operation, we will need to add more data to the graph of our surface, taking into account the parity of m:

- If m is odd, then we equip the graph with a colouring at the edges. This means that we associate to each connected component of Z a label, either $+$ or $-$. Such a colouring exists for any graph that has no odd cycles, which is true always for the graph of a b^m-symplectic manifold with m odd, as it accepts a vertex 2-colouring (see for instance [MP18]).

- If m is even, then we equip the graph with a direction. This means, for each edge of our graph we will select one of the two incident vertices as a source and the other one as a target. We select this directionality in such a way that each vertex of degree strictly greater than 1 is at least the source of an edge and the target of a different edge. Such a choice of direction always exists, by Lemma 3.3.9.

For each connected component of $\Sigma \setminus Z$ we construct a closed symplectic manifold $\left(\Sigma_j, \widetilde{\omega}_j\right)$ as in Proposition 3.2.7. Let v denote the vertex in the graph representing the connected component from which we construct Σ_j. Then Σ_j contains $\deg(v)$ connected components of Z, which we denote by $\gamma_1, \ldots, \gamma_{\deg(v)}$. If m is odd, each of these components is labeled with a sign $+$ or $-$ by the edge 2-colouring. Otherwise, if m is even, we can label each γ_i with the sign $-$ if v is the source of the associated edge, and the sign $+$ if v is the target. By construction, each of these connected components of Z bounds a 2-disk.

As Σ_j is orientable, it admits a perfect Morse function, this means, a Morse function that has the minimal number possible of critical points, in our case $2 + 2g_v$. Such a Morse function can be manipulated in Morse coordinates by adding non-degenerate critical points in such a way that we end up with a Morse function $F_j^1 : \Sigma_j \to \mathbb{R}$ such that each disk delimited by a critical curve γ_i contains exactly one critical point of F_j^1 in its interior: a maximum if γ_i has the $+$ label, or a minimum if γ_i has the $-$ label. Moreover, we can assume that in the tubular neighbourhood of any critical curve it is possible to choose cylindrical coordinates (z, θ) such that F_j^1 has the local expression $F_j^1(z, \theta) = z$ in the neighbourhood.

The constructed Morse function has then $2+2g_v$ critical points if $\deg(v) = 1$ or $2 + 2g_v + 2(\deg(v) - 2)$ critical points otherwise. In the first case, exactly $1 + 2g_v$ critical points are contained away from the disks delimited by the critical curve, and in the second case $2g_v + \deg(v) - 2$. If needed, we divide F_j^1 by a constant large enough so that $X^{\widetilde{\omega}}_{F_j^1}$ has no 1-periodic orbits besides its critical points.

We now apply Proposition 3.2.6 and obtain a b^m-function F_j^2 on the b^m-symplectic surface (Σ_j, ω) such that $X^{\omega}_{F_j^2}$ has exactly the same 1-periodic orbits as $X^{\widetilde{\omega}}_{F_j^1}$. In particular, there exists a tubular neighbourhood around each γ_i in such a way that F_j^2 has the local expression

$$F_j^2(z,\theta) = \begin{cases} \pm \log|z| & \text{if } m = 1, \\ \pm \frac{1}{z^{m-1}} & \text{if } m > 1, \end{cases} \tag{3.6}$$

with the sign coinciding with the label associated to γ_i.

Let D_i denote the disk delimited by the curve γ_i, and let us restrict to the interior of $\Sigma_j \setminus \bigcup_i D_i$. The vector field $X^{\omega}_{F_j^2}$ has then $1+2g_v$ 1-periodic orbits if $\deg(v) = 1$ and $2g_v + \deg(v) - 2$ 1-periodic orbits if $\deg(v) > 1$.

To conclude, we construct the b^m-function $F^3 \in {}^{b^m}C^\infty(\Sigma)$ by gluing together the components F_j^2. We can define F^3 by its restriction on each of the connected components of $\Sigma \setminus Z$ so that it coincides with F_j^2, and use the expression from Equation 3.6 in the tubular neighbourhood of each component of Z. Because of our choices of edge 2-colouring or directionality, the sign will be consistent along each of the tubular neighbourhoods of Z. Therefore, F^3 is a well defined b^m-function and, moreover, an admissible b^m-function.

□

Proof of Lemma 3.3.9. Consider the graph $G = (V, E)$. An orientation of a graph is good if and only if it is good for each connected component. Hence, we may assume that G is a connected graph.

Consider an ordering of the edges of G, $\{e_i\}_{1\le i\le n}$. We define inductively the graphs $G_0, \dots, G_n$ as follows: $G_0 = G$, and if $i > 0$ then $G_i = (V, E_i)$, where $E_i := E_{i-1} \setminus \{e_i\}$ if one of the two vertices to which e_i is incident has degree strictly greater than 2 in the graph G_{i-1}, and $E_i := E_{i-1}$ otherwise.

By this construction, all the vertices in the graph G_n have a degree lower or equal to 2. G_n can then be given a good orientation trivially, because each

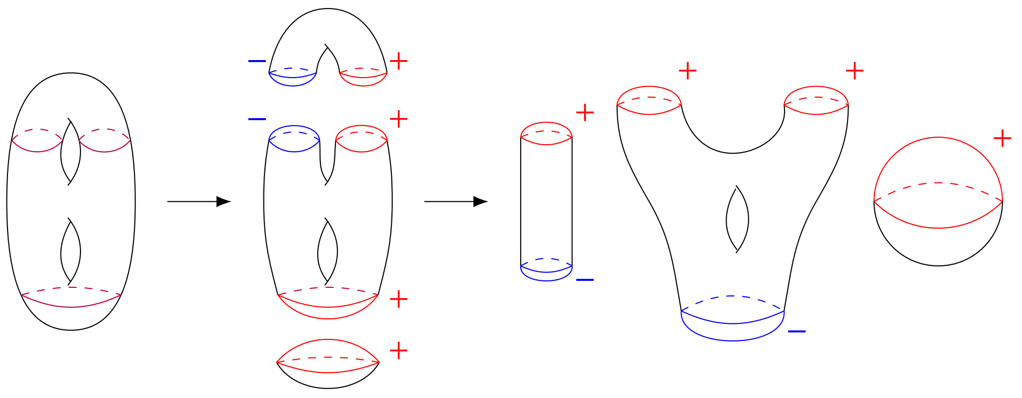

Figure 3.3: Cutting and filling a b^{2k+1}-symplectic surface with signs at the disks. Red disks contain maxima and blue disks contain minima

connected component of G_n will be either a cycle, a path, or an isolated vertex.

Now, let us construct a good orientation inductively on G_i for $i < n$.

Let us assume that G_{i+1} has a good orientation. We claim then that G_i admits a good orientation as well. Consider the edge e_i. If $G_i = G_{i+1}$, this means that we did not remove the edge e_i in the construction of $\{G_i\}_i$, and therefore G_i admits the same good orientation as G_{i+1}. Otherwise, e_i is the only edge of G_i for which a direction compatible with the good orientation (from G_{i+1}) must be determined. By our construction, one of vertices adjacent to e_i must have a degree greater than 2. We choose the direction for e_i depending on the degree of the other vertex in G_{i+1}:

- If the vertex has either degree 0 or greater or equal to 2, then either direction for e_i produces a good orientation for the graph G_i.
- If the vertex has degree 1, then it is the source (respectively the target) of an edge in E_{i+1}. Then, we pick the direction on e_i such that this vertex becomes also the target (respectively source) of e_i.

Following this process, a good orientation can be induced for each of the graphs G_i, and in particular for $G_0 = G$. □

As mentioned in Remark 3.2.5, the desingularization process can be reproduced if we take the product of a b^m-symplectic surface with a symplectic manifold, so Theorem 3.3.5 has the following immediate Corollary:

Corollary 3.3.10 (Arnold conjecture for product b^m-symplectic manifolds) *Let (Σ, Z, ω_1) be an orientable b^m-symplectic surface and (W, ω_2) a compact symplectic manifold, and consider the b^m-symplectic manifold obtained by their product, $(M = \Sigma \times W, Z \times W, \omega_1 + \omega_2)$. Let H_t be a regular admissible b^m-Hamiltonian on M.*

Then the number of periodic orbits of X_H has the lower bound

$$\#\mathcal{P}(H) \geq \left(\sum_{v\in V} \left(2g_v + \left|\deg(v) - 2\right|\right)\right) \cdot \left(\sum_{i} \dim HM_i\left(W; \mathbb{Z}_2\right)\right)$$

where $HM_i(W; \mathbb{Z}_2)$ is the i-th Morse homology group of W.

In Remark 3.2.9 we discussed the possibility to extend the desingularization techniques to the setting of b^m-symplectic manifolds of higher dimension where the singular hypersurface Z has the geometry of a mapping torus. This yields the following Proposition, in line with the results in this section.

Proposition 3.3.11 *Let (M, Z, ω) be a closed b^m-symplectic manifold of dimension $2n$ with trivial mapping tori Z_i. Let H_t be a regular admissible Hamiltonian. Then, the number of 1-periodic orbits of X_H has the lower bound*

$$\#\mathcal{P}(H) \geq \sum_{v\in V} \max\left\{\sum_{i=0}^{2n} \dim HM_i\left(\overline{M}_v; \mathbb{Z}_2\right) - \sum_{e\in E_v} \mathcal{P}\left(X_{h_t^e}\big|_{Z_e}\right), 0\right\}$$

where E_v denotes the set of edges adjacent to the vertex v, Z_e is the connected component of Z corresponding to the edge e, and h_t^e the smooth function such that

$$H_t|_{\mathcal{N}(Z_e)}(z, \theta, x) = \begin{cases} K(t)\log|z| + h_t^e(x) & \text{if } m = 1, \\ -K(t)\frac{1}{(m-1)z^{m-1}} + h_t^e(x) & \text{if } m > 1. \end{cases}$$

This result is not properly analogous to the Arnold conjecture, because the lower bound that we find is not topological in nature. Rather, it depends on the dynamics of the Hamiltonian restricted to Z. It merely suggests a way in which these can type of lower bounds could be found. Namely, an exploration of the dynamics of $X_{h_t^e}$ and potential topological restrictions thereof could be the key to find a strict lower bound for the family of b^m-symplectic compact manifolds with trivial mapping tori Z.

Chapter 4

Floer theory on b^m-symplectic manifolds

4.1 The Floer equation on b^m-symplectic manifolds

In this section we will derive the Floer equation as in Equation 2.5 for admissible Hamiltonians in b^m-symplectic manifolds and investigate some elementary properties of their solutions. From our study we will be able to conclude that the solutions of the Floer equation in this setting satisfy a minimum principle, which in the next section will be useful to set up the construction of a Floer complex.

Definition 4.1.1 Let (M, Z) be a b-manifold and $m \in \mathbb{N}_{>0}$. A *b^m-Riemannian metric* is a symmetric and strictly positive definite tensor $g \in \Gamma\left({}^{b^m}T^*M \otimes {}^{b^m}T^*M\right)$.

Definition 4.1.2 Let (M, Z, ω) be a b^m-symplectic manifold. An *almost complex structure* is a bundle endomorphism $J : {}^{b^m}TM \to {}^{b^m}TM$ such that $J^2 = -\mathrm{Id}$.

An almost complex structure can always be chosen so that it is *compatible* with the b^m-symplectic structure ω, this means, the tensor $g_J(\cdot,\cdot) := \omega(\cdot, J\cdot)$ defines a b^m-Riemannian metric.

Moreover, an almost complex structure can be chosen in such a way that it is compatible with the cosymplectic structure in a tubular neighbourhood $\mathcal{N}(Z)$ around the singular set, which means that

1. The restriction of J to the symplectic leaves is a smooth almost complex structure.

2. It leaves the distribution $\langle X^\sigma, v_{mod}\rangle$ invariant, and in particular $JX^\sigma = v_{mod}$.

3. J commutes with the flow of X^σ, which means that

$$d\varphi^t_{X^\sigma}(z,x)J_{(z,x)} = J_{\varphi^t_{X^\sigma}(z,x)}d\varphi^t_{X^\sigma}(z,x).$$

As is the case in Remark 2.2.20, the space of almost complex structures compatible with a b^m-symplectic structures (and also with the cosymplectic structure in $\mathcal{N}(Z)$) is contractible.

Remark 4.1.3 In this chapter we will use structures a bit more general than in Chapter 3. In particular, our Hamiltonian will depend on two real variables instead of one, and we will introduce the same dependencies for the almost complex structure. This will provide more general results about the Floer equation in b^m-symplectic manifolds, which we believe that could be used in the future to investigate the Floer homology of a b^m-symplectic manifold.

Now we can define the Floer equation in the context of admissible b^m-Hamiltonians:

Definition 4.1.4 Let (M, Z, ω) be a b^m-symplectic manifold, and let $\mathcal{N} \subset \mathcal{N}(Z)$ with $\mathcal{N} \cong (0,\varepsilon)\times Z$ be an open neighbourhood not including Z. Let X^σ be a normal symplectic vector field and v_{mod} a modular vector field, both defined in $\mathcal{N}$. Let $\Omega \subseteq \mathbb{C}$ with coordinates $\eta = s + it$, and take H an admissible Hamiltonian in ${}^{b^m}C^\infty(\Omega \times \mathcal{N})$, thus having the form

$$H(s,t,z,x) = \begin{cases} K(s,t)\log|z| + h(s,t,x) & \text{if } m = 1, \\ -K(s,t)\frac{1}{(m-1)z^{m-1}} + h(s,t,x) & \text{if } m > 1, \end{cases}$$

where $K \in C^\infty(\Omega)$ and $h \in C^\infty(\Omega \times Z)$ with $\mathcal{L}_{v_{mod}} h = 0$. Let $J \in \Gamma\left(\Omega \times \mathcal{N}, {}^{b^m}T^*M \otimes {}^{b^m}TM\right)$ be a compatible almost complex structure adapted to ω and to the cosymplectic structure.

The *Floer equation* for H is then

$$\frac{\partial u}{\partial s} + J_{(\eta,u(\eta))}\left(\frac{\partial u}{\partial t} - X_H(u)\right) = 0. \tag{4.1}$$

Due to the restrictions imposed on J using the vector fields X^σ and v_{mod}, the solutions of Equation 4.1 satisfy the following Proposition:

Proposition 4.1.5 (Brugués, Miranda and Oms [BMO22]) *Let $(\mathcal{N}, \omega, X^\sigma, v_{mod}, H, J)$ be as set up in Definition 4.1.4, and consider $u : \Omega \to \mathcal{N}$ a solution to Equation 4.1. Let $f : \mathcal{N} \to \mathbb{R}$ be given by*

$$f(z,p) = \begin{cases} \log|z| & \text{if } m = 1, \\ -\frac{1}{(m-1)z^{m-1}} & \text{if } m > 1. \end{cases}$$

Then

$$\Delta(f \circ u) = -\frac{\partial K}{\partial s}.$$

Proof. Let $d^c(v) := dv \circ i = \frac{\partial v}{\partial t}ds - \frac{\partial v}{\partial s}dt$. Then

$$-dd^c(v) = (\Delta v) ds \wedge dt.$$

Computing,

$$\begin{aligned} -d^c(f \circ u) =& \frac{\partial}{\partial s}(f \circ u)dt - \frac{\partial}{\partial t}(f \circ u)ds = \left(df(u)\frac{\partial u}{\partial s}\right)dt - \left(df(u)\frac{\partial u}{\partial t}\right)ds \\ =& \left(df(u)\left(\frac{\partial u}{\partial s} + J(\eta, u)\frac{\partial u}{\partial t}\right)\right)dt - \left(df(u)\left(J(\eta,u)\frac{\partial u}{\partial t}\right)\right)dt + \\ &+ \left(df(u)\left(J(\eta,u)\frac{\partial u}{\partial s} - \frac{\partial u}{\partial t}\right)\right)ds - \left(df(u)\left(J(\eta,u)\frac{\partial u}{\partial s}\right)\right)ds \\ =& -\omega(\nabla f(u), X_H(u))dt + \omega\left(\nabla f(u), \frac{\partial u}{\partial t}\right)dt - \\ &- \omega\left(\nabla f(u), J(\eta,u)X_H(u)\right)ds + \omega\left(\nabla f(u), \frac{\partial u}{\partial s}\right)ds \end{aligned} \tag{4.2}$$

Now, we apply the fact that with our choice of J we have $\nabla f = X^\sigma$, so for the first term

$$\omega(X^\sigma, X_H(u))dt = (\mathcal{L}_{X^\sigma}H)\,dt = K(s,t)dt.$$

For the second and fourth terms

$$\omega\left(X^\sigma, \frac{\partial u}{\partial t}\right)dt + \omega\left(X^\sigma, \frac{\partial u}{\partial s}\right)ds = u^* i_{X^\sigma}\omega.$$

Finally, for the third term

$$\omega(X^\sigma, J(\eta,u)X_H(u))ds = \omega(X_H(u), J(\eta,u)X^\sigma(u))ds = \\ \omega(X_H(u), v_{\mathrm{mod}}(u))ds = \left(\mathcal{L}_{v_{\mathrm{mod}}}H\right) ds = 0.$$

Collecting everything, Equation (4.2) yields that

$$-d^c(f\circ u) = -K(s,t)dt - u^* i_{X^\sigma}\omega.$$

If we apply the differential, it is clear that $d(K(s,t)dt) = \frac{\partial K}{\partial s} ds \wedge dt$, and

$$d\left(u^* i_{X^\sigma}\omega\right) = u^*(d i_{X^\sigma}\omega) = u^*(\mathcal{L}_{X^\sigma}\omega) = 0.$$

Therefore,

$$\left(\Delta\left(f\circ u\right)\right) ds \wedge dt = -dd^c(f\circ u) = -\frac{\partial K}{\partial s} ds \wedge dt.$$

□

As a corollary of Proposition 4.1.5, we obtain the following result.

Theorem 4.1.6 (Brugués, Miranda and Oms [BMO22]. Minimum principle) *Let $u \in C^\infty(\Omega, \mathcal{N})$ be a solution to the Floer equation 4.1. Then,*

1. *If u is a solution for an admissible b^m-Hamiltonian that only depends on t, $H \in {}^{b^m}C^\infty(\mathbb{S}^1 \times \mathcal{N})$ and if $f\circ u$ attains its maximum or minimum in Ω, then $f\circ u$ is constant.*
2. *If u is a general solution for an admissible b^m-Hamiltonian $H \in {}^{b^m}C^\infty(\Omega\times \mathcal{N})$ such that $\frac{\partial K}{\partial s}(s,t) \geq 0$ for all $(s,t)\in\Omega$ and if $f\circ u$ attains its minimum in Ω, then $f\circ u$ is constant.*

Proof. Both assertions are a direct application of the Maximum principle to $-(f\circ u)$, making use of the fact that $\Delta\left(-f\circ u\right) = \frac{\partial K}{\partial s} \geq 0$, with an equality in the case that H does not depend on s. □

4.2 A Floer complex

In this section we will use the results from Section 4.1 to construct a Floer complex in the setting of b^m-symplectic manifolds. In this Section we will consider (M, Z, ω) a b^m-symplectic manifold that is aspherical, in the sense that $[\omega]$ vanishes on $\pi_2(M)$.

Definition 4.2.1 Let H_t be a regular admissible b^m-Hamiltonian. Let $\mathcal{P}(H)$ denote the set of 1-periodic orbits of the Hamiltonian vector field X_H.

We define the *Floer chain complex* ${}^{b^m}CF(M, \omega, H)$ as the $\mathbb{Z}_2$-vector space generated over $\mathcal{P}(H)$, this means, the set of formal sums of the type

$$v = \sum_{x \in \mathcal{P}(H)} v_x x, \ v_x \in \mathbb{Z}_2.$$

Under the assumption that the first Chern class $c_1 = c_1(\omega) \in H^2(M, \mathbb{Z})$ of the bundle $({}^{b^m}TM, J)$ vanishes on $\pi_2(M)$, the Conley-Zehnder index μ_{CZ} of $x \in \mathcal{P}(H)$ is well-defined (see Definition 2.2.31). The index can be normalized in such a way that for any critical points of a C^2-small enough H it is satisfied that

$$\mu_{CZ}(x) = 2n - \mu_H(x),$$

where μ_H denotes the Morse index of H.

We can use the Conley-Zehnder index to turn ${}^{b^m}CF(M, H, \omega)$ into a graded vector space.

We denote by $\mathcal{M}$ the moduli space of Floer solutions with finite energy, this means

$$\begin{cases} \frac{\partial u}{\partial s} + J_u \frac{\partial u}{\partial t} + \operatorname{grad}_u H = 0, \\ E(u) < +\infty, \end{cases}$$

as in Equation 2.5.

As $(M \backslash Z, \omega)$ is aspherical, we cannot have bubbles of pseudo-holomorphic spheres (see, for instance, Section 6.6 on [AD14]). Moreover, as a consequence of Theorem 4.1.6 we cannot have solutions of the Floer equation approaching Z in any way. Thus, we can apply the same reasoning as in Theorem 2.2.34 to conclude that $\mathcal{M}$ is compact. As in standard Floer theory, we conclude from here that whenever $\mu_{CZ}(x) - \mu_{CZ}(y) = 1$, the quotient $\mathcal{M}(x, y) / \mathbb{R}$ by the action along the variable s is a finite set.

Let

$$n(x,y) := \#\left\{\mathcal{M}(x,y)\,/\mathbb{R}\right\} \mod 2.$$

Then, for each index k we can define the boundary operator of the Floer complex,

$$\partial_k : {}^{b^m}CF_k(M,\omega,H,J) \longrightarrow {}^{b^m}CF_{k-1}(M,\omega,H,J)$$

as defined in the generators of ${}^{b^m}CF_k(M,\omega,H,J)$ by

$$\partial_k x := \sum_{\substack{y\in\mathcal{P}(H)\\ \mu(y)=k-1}} n(x,y)y.$$

It follows, in the same way as in classical Floer theory, that $\partial_k \circ \partial_{k+1} = 0$ for all k. In other words, ${}^{b^m}CF_\bullet(M,\omega,H,J)$ forms a chain complex.

Definition 4.2.2 The Floer homology is the one given by

$${}^{b^m}HF_k(M,\omega,H,J) := \frac{\ker\partial_k}{\mathrm{im}\partial_{k+1}}.$$

Remark 4.2.3 The homology as it is constructed in Definition 4.2.2, as the notation implies, depends on the choice of H, J, and ω, besides depending on M, Z, and the relative topology between them. More precisely, the family of admissible Hamiltonian functions depends on X^σ and v_{mod}, but we do not include this dependence in the notation as they are accounted for in the choice of a Hamiltonian H.

We believe that this homology can actually be computed and shown to be invariant with respect to the aforementioned choices. In particular, in light of Proposition 4.1.5 it is clear that if we study only the dependence of the homology with respect to J we will also have that solutions to a parametrized Floer equation will satisfy the same principle as case 1 of Theorem 4.1.6. As $\mathcal{J}(M,\omega)$ is contractible also in the b^m-symplectic case, it seems very reasonable to conjecture that the homology may be invariant with respect to the choice of J.

We also conjecture that ${}^{b^m}HF_\bullet(M)$ is, in fact, a topological invariant of the b^m-manifold (M,Z). The identification of this invariant, falls outside the scope of this work, despite it being an intriguing question in the development of the theory of b^m-symplectic manifolds.

Remark 4.2.4 The construction of this complex (and homology) is related to the results exposed in Section 3.3 due to the conditions on admissible Hamiltonian functions, in particular to the fact that the dynamics of X_H is, in a sense, split between the connected components of $M\backslash Z$. Moreover, due to Theorem 4.1.6 we can deduce that solutions to the Floer equation with finite energy do not cross the singular hypersurface Z, this means, our Floer complex splits between the connected components of $M\backslash Z$,

$$ {}^{b^m}CF_{\bullet}(M,\omega,H,J) = \bigoplus_{M_i \in M\backslash Z} {}^{b^m}CF_{\bullet}(M_i,\omega,H,J). $$

Remark 4.2.5 In this section we have worked under the assumption that $M \setminus Z$ is aspherical. We would expect to be able to generalize the construction of this complex to more complicated b-manifolds following the same techniques used to generalize Floer homology to non-aspherical manifolds, due to the splitting phenomenon just mentioned in Remark 4.2.4.

Chapter 5

Features of b-semitoric systems

An aim that attracts special interest in the context of b-symplectic geometry is the generalization of integrable systems. As we explained in Section 2.6.3 this aim has been pursued in the particular case of b-toric manifolds. Further, integrable systems have been studied in the context of singular symplectic geometry among others by Guillemin, Miranda and Pires [GMP14], Guillemin, Miranda, Pires and Scott [GMPS15], Kiesenhofer, Miranda and Scott [KMS16] and by Gualtieri, Li, Pelayo and Ratiu [GLPR17].

In this Chapter we will lay out the results of Brugués, Hohloch, Mir and Miranda [BHMM23]. Our main objective is to define and provide some properties of b-semitoric systems which, as their name implies, generalize the notion of semitoric systems in the setting of singular symplectic geometry. Afterwards we will study a family of examples generalizing the coupled spin-oscillator (from Example 2.6.24), providing case studies that pave the way to a systematic study of these new type of systems. A similar family of examples from the coupled angular momenta (see Example 2.6.26) can be found in [BHMM23].

5.1 b-semitoric systems

Definition 5.1.1 Let (M, Z, ω) be a $2n$-dimensional b-symplectic manifold, and let $f_1, \ldots, f_n \in {}^bC^\infty(M)$. We say that $(M, Z, \omega, F := (f_1, \ldots, f_n))$ is a *b-integrable system* if $\{f_i, f_j\} = 0$ for all $1 \leq i, j \leq n$ and if the b-form $df_1 \wedge \cdots \wedge df_n \in {}^b\Omega^{2n}(M)$ does not vanish almost everywhere in M and also does not vanish almost everywhere in Z.

Remark 5.1.2 An important aspect of Definition 5.1.1 is that we require the b-1-forms $df_1, ..., df_n$ to be independent almost everywhere in Z. This is chosen to avoid a situation in which $(X_{f_1}, ..., X_{f_n})$ reduces to a distribution of rank $2n - 2$ on Z, which is too restrictive in order to prove normal form results. See Kiesenhofer, Miranda and Scott [KMS16, Section 3] for a more in-depth discussion.

Remark 5.1.3 The Hamiltonian vector fields $X_{f_1}, \ldots, X_{f_n}$ are tangent to Z everywhere and, therefore, are b-vector fields.

Remark 5.1.4 Throughout this Chapter we will consider only the notion of b-integrable system in contrast with previous parts of this work, where we have considered b^m-structures with arbitrary degree m. b^m-integrable systems can indeed be defined in an analogous way to Definition 5.1.1 by using b^m-symplectic forms and b^m-functions, but the inquiry therein falls beyond the scope of this thesis. The reader is invited to check Miranda and Planas [MP23] for a study of such systems more generally.

Near regular points in Z, a b-integrable system has a very constricted behaviour:

Lemma 5.1.5 (Kiesenhofer and Miranda [KMS16, Remark 18]) *Near a regular point of Z, a b-integrable system is equivalent to one of the form $F = (f_1, \ldots, f_n)$, where $f_2, \ldots, f_n$ are smooth functions and $f_1 = c \log |z|$, where $c \in \mathbb{R}$ and z is a defining function for Z.*

From now on we will restrict to the case of a b-integrable system of dimension 4.

Definition 5.1.6 A b-integrable system $(M, Z, \omega, F := (L, H))$ is b-*semitoric* if L is proper and the flow of the vector field X_L generates an effective $\mathbb{S}^1$ action on M and if all the critical points of F are non-degenerate and do not include any hyperbolic components.

This is the simplest class of systems that includes the possibility of focus-focus points in dimension 4. Notably, models for b-integrable systems of dimension 6 with focus-focus singularities and of dimension 4 with hyperbolic singularities were already stated in Kiesenhofer and Miranda [KM17].

Example 5.1.7 ([KM17]) Consider the group $G := \mathbb{S}^1 \times \mathbb{R}^+ \times \mathbb{S}^1$ acting on $M := \mathbb{S}^1 \times \mathbb{R}^2$ by $(\varphi, a, \alpha) \cdot (\theta, x_1, x_2) = (\theta + \varphi, aR_\alpha(x_1, x_2))$, where R_α denotes a rotation with angle α in the (x_1, x_2) plane.

Let us consider the twisted b-cotangent lift of the action to T^*M, by which we mean the b-symplectic structure $\omega := -d\lambda$, where

$$\lambda := \log|p|d\theta + y_1 dx_1 + y_2 dx_2,$$

and where the lifted action has the momentum map given by $f_1 = \log|p|$, $f_2 = x_1y_1 + x_2y_2$ and $f_3 = x_1y_2 - y_1x_2$.

In the set of points such that $x_1 = y_1 = x_2 = y_2 = 0$ the system has singularities with a focus-focus component and a regular component.

Example 5.1.8 ([KM17]) Consider the group $G := \mathbb{S}^1 \times \mathbb{R}^+$ acting on $M := \mathbb{S}^1 \times \mathbb{R}$ by $(\varphi, g) \cdot (\theta, x) := (\theta + \varphi, gx)$, and consider the twisted b-cotangent lift in an analogous way to Example 5.1.7. In this case the momentum map of the lifted action to T^*M has the expression

$$f_1 = \log|p|, \; f_2 = xy,$$

and therefore the family of points with $x = y = 0$ exhibit a hyperbolic-regular singularity.

Let us now present a restriction on the dynamics of the system on the singular hypersurface Z:

Proposition 5.1.9 (Brugués, Hohloch, Mir and Miranda [BHMM23]) *Let $(M, Z, \omega, F = (L, H))$ be a b-semitoric system. Then, the rank of the system on all points on Z is* 1 *or higher.*

Proof. This proof is based on the idea by Kiesenhofer and Miranda in [KM17, Remark 33]. We follow the notation from Bolsinov and Fomenko [BF04, Section 1.8].

Take $p \in Z$ a fixed point, this means, such that $dL|_p = dH|_p = 0$. Then, the linearization of the actions of the flows $\varphi^t_{X_L}$ and $\varphi^t_{X_H}$ generates an $\mathbb{R}^2$ action on T_pM which by construction preserves the b-symplectic form ω. This means that $\left(dL|_p \, , dH|_p\right)$ induces a dimension 2 commutative Lie subgroup $G(L, H) \subset \mathrm{Sp}(4, \mathbb{R})$ Let $K(L, H) \subset \mathfrak{sp}(4, \mathbb{R})$ denote the commutative Lie subalgebra induced by $G(L, H)$.

However, the vector fields X_L and X_H are tangent to Z at every point, and therefore $G(L, H)$ must preserve $T_pZ \subset T_pM$, a 3-dimensional linear subspace. Thus, the Lie algebra $K(L, H)$ must preserve T_pZ as well.

Since the point p is non-degenerate, we know that $K(L,H)$ must be a Cartan subalgebra, and therefore it must be conjugate to one of the matrix subalgebras from Equation 2.8. Of these algebras, only the ones that have hyperbolic components can leave a 3-dimensional subspace invariant, which cannot be present in a b-semitoric system. Thus, there cannot be such a point $p \in Z$. □

In the remainder of this Chapter we will devote ourselves to present a particular family of examples of b-semitoric systems and the properties of their singular points. The aim of this effort is to provide models from which more systematic theories on the study and classification of b-semitoric systems can be developed in the future.

Remark 5.1.10 The tools in Chapters 3 and 4 can be used to give some baseline information about b-toric and b-semitoric systems, in the same way that the Arnold conjecture and Floer theory find applications to the study of fixed points of toric manifolds (see for instance Givental [Giv95] and [GvS17]).

In Corollary 2.6.21 we saw that b-toric manifolds have a very restricted topology, and therefore we can sometimes predict a lower bound of the number of fixed points that any particular system may have, using the lower bound from Theorem 3.3.5.

It is worth noting that neither b-toric nor b-semitoric may be always aspherical. Indeed, Corollary 2.6.21 allows for the construction of plenty b-symplectic manifolds which do not satisfy this property. Moreover, the examples in [BHMM23, Section 5] show the construction of compact b-symplectic manifolds with a b-semitoric system that are, however, not aspherical. Thus the Floer homology that might be constructed from the ideas in Chapter 4 could not cover these cases, which provides further incentive to study their classification.

5.2 An example: The b-coupled spin oscillator

We give here the explicit computations related to an example of a b-semitoric system. All the results exposed in this section are original and can also be found in Brugués, Hohloch, Mir, Miranda [BHMM23].

Consider the (non-compact) b-manifold composed by $M = \mathbb{S}^2 \times \mathbb{R}^2$ with hypersurface $Z = \{(x,y,z) \in \mathbb{S}^2 \mid z = 0\} \times \mathbb{R}^2$. In this section we will introduce two possible generalizations of the semitoric system given by the coupled spin-oscillator in Example 2.6.24 as b-integrable systems. We will call them the b-coupled spin-oscillator and the reversed b-coupled spin-oscillator due to the difference being owed only to a change of sign on the b-symplectic form and in one of the b-functions. We will focus on deriving the dynamical behaviour of the two systems, this means, the number of fixed points and their classification.

Let us endow the b-manifold (M, Z) with either of the b-symplectic forms

$$\begin{aligned} \omega_1 &:= -\rho_1 {}^b\omega_{\mathbb{S}^2} + \rho_2 \omega_{\mathbb{R}^2}, \\ \omega_2 &:= \rho_1 {}^b\omega_{\mathbb{S}^2} + \rho_2 \omega_{\mathbb{R}^2}, \end{aligned}$$

where ${}^b\omega_{\mathbb{S}^2}$ is the standard b-symplectic form on $\left(\mathbb{S}^2, Z = \{z = 0\}\right)$ as in Example 2.4.23 for $m = 1$, $\omega_{\mathbb{R}^2}$ is the standard symplectic form in $\mathbb{R}^2$, and ρ_1, ρ_2 are strictly positive real numbers.

Definition 5.2.1 Consider the b-Hamiltonians $L_1, L_2, H \in {}^bC^\infty(M)$ given by

$$\begin{cases} L_1(x,y,z,u,v) = \rho_1 \log|z| + \frac{\rho_2}{2}\left(u^2+v^2\right), \\ L_2(x,y,z,u,v) = -\rho_1 \log|z| + \frac{\rho_2}{2}\left(u^2+v^2\right), \\ H(x,y,z,u,v) = \frac{1}{2}\left(xu+yv\right). \end{cases}$$

The *b-coupled spin-oscillator* is the tuple $(M, Z, \omega_1, F = (L_1, H))$, and the *reversed b-coupled spin-oscillator* is the tuple $(M, Z, \omega_2, F = (L_2, H))$.

For the sake of convenience we will now introduce a set of coordinate charts on M that we will use throughout the section.

Let $U^\pm \subset \mathbb{S}^2$ denote the two open sets given respectively by $U^+ = \{(x,y,z) \in \mathbb{S}^2 \mid z > 0\}$ and $U^- = \{(x,y,z) \in \mathbb{S}^2 \mid z < 0\}$, with the map

$$\begin{array}{rccc} \varphi : & \mathbb{S}^2 & \longrightarrow & \mathbb{R}^2 \\ & (x,y,z) & \longmapsto & (x,y). \end{array}$$

This induces the two coordinate charts $\left(M^\pm := U^\pm \times \mathbb{R}^2, \varphi|_{U^\pm} \times \mathrm{id}_{\mathbb{R}^2}\right)$.

The points not covered by the charts $U^\pm$ correspond precisely to Z. These points are included in a cylindrical chart covering the sphere minus the two poles: we take $U^0 = \{(x,y,z) \in \mathbb{S}^2 \mid |z| < 1\}$ and

$$\begin{array}{cccc} \psi : & U^0 & \longrightarrow & \mathbb{S}^1 \times \mathbb{R}^1 \\ & (x,y,z) & \longmapsto & (\theta, z), \end{array}$$

where (θ, z) denote the cylindrical coordinates. We complete our atlas with the pair $(M^0 := U^0 \times \mathbb{R}^2, \psi \times \mathrm{id})$. Even though M^0 is not a subset of $\mathbb{R}^4$ and thus the pair is not formally a coordinate chart for M, we will be able to use the coordinates (θ, z, u, v) to perform all computations.

In local coordinates we have that

$$\begin{aligned} \omega_1|_{M^+} = \omega_1|_{M^-} &= -\rho_1 \frac{1}{1-x^2-y^2} dx \wedge dy + \rho_2 du \wedge dv, \\ \omega_1|_{M^0} &= \rho_1 \frac{dz}{z} \wedge d\theta + \rho_2 du \wedge dv, \\ \omega_2|_{M^+} = \omega_2|_{M^-} &= \rho_1 \frac{1}{1-x^2-y^2} dx \wedge dy + \rho_2 du \wedge dv, \\ \omega_2|_{M^0} &= -\rho_1 \frac{dz}{z} \wedge d\theta + \rho_2 du \wedge dv. \end{aligned}$$

Proposition 5.2.2 *The b-coupled spin-oscillator and the reversed spin-oscillator are b-integrable systems.*

Proof. We will show that X_L and X_H are linearly independent almost everywhere in the proofs of Proposition 5.2.4 and of Proposition 5.2.7. For now, we will prove that $\{L_1, H\} = 0$ and $\{L_2, H\} = 0$.

The most convenient way to perform this computation is to take $\{L_{1,2}, H\} = -X_{L_{1,2}}(H)$. Computing the vector fields $X_{L_1}^{\omega_1}$ and $X_{L_2}^{\omega_2}$, we can see that both coincide. Thus set $X_L := X_{L_1}^{\omega_1} = X_{L_2}^{\omega_2}$. In cylindrical coordinates this vector field can be expressed as

$$X_L|_{M^0} = \frac{\partial}{\partial \theta} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v},$$

whereas in Cartesian coordinates we have

$$X_L|_{M^\pm} = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}.$$

Computing the Poisson bracket in each of the charts yields

$$X_L|_{M^0}(H) = \left(\frac{\partial}{\partial\theta} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)\left(\frac{\sqrt{1-z^2}}{2}(u\cos\theta + v\sin\theta)\right) = 0,$$
$$X_L|_{M^\pm}(H) = \left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}\right)\left(\frac{1}{2}(xu + yv)\right) = 0.$$

□

5.2.1 The b-coupled spin-oscillator

In this part we will classify the critical points of the b-coupled spin-oscillator from Definition 5.2.1. We start by summarizing the results that we are going to prove.

Proposition 5.2.3 *The b-coupled spin-oscillator has two fixed points:* $N := ((0,0,1),(0,0)) \in M$ *and* $S := ((0,0,-1),(0,0)) \in M$*, and both are focus-focus points.*

Proposition 5.2.4 *All the points on* $M \setminus \{N, S\}$ *are regular.*

Proposition 5.2.5 *Both leaves of the momentum map* $F = (L, H) : M \setminus Z \to \mathbb{R}^2$ *of the b-coupled spin-oscillator are surjective.*

Proof of Proposition 5.2.3. If we compute $dF = (dL_1, dH)$, we can see that $dL_1 = dH = 0$ if and only if $x = y = u = v = 0$, this means, precisely at the points N and S.

To prove that the points are non-degenerate and of focus-focus type we will use Lemma 2.6.5. In particular, we prove that the Hessians d^2L_1 and d^2H are linearly independent and that there exists a linear combination of the symplectic operators $\omega^{-1}d^2L_1$ and $\omega^{-1}d^2H$ with four different eigenvalues of the form $\pm a \pm ib$.

At N and S, in coordinates (x, y, u, v), the Hessians of L and H and the matrix form of ω have the following expressions:

$$d^2L_1 = \begin{pmatrix} -\rho_1 & 0 & 0 & 0 \\ 0 & -\rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_2 \end{pmatrix}, \quad d^2H = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\Omega = \begin{pmatrix} 0 & -\rho_1 & 0 & 0 \\ \rho_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_2 \\ 0 & 0 & -\rho_2 & 0 \end{pmatrix}.$$

The matrices d^2L_1 and d^2H are clearly independent and give rise to the following symplectic operators:

$$A_L := \Omega^{-1}d^2L_1 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad A_H := \Omega^{-1}d^2H = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \frac{1}{\rho_1} \\ 0 & 0 & \frac{-1}{\rho_1} & 0 \\ 0 & \frac{-1}{\rho_2} & 0 & 0 \\ \frac{1}{\rho_2} & 0 & 0 & 0 \end{pmatrix}.$$

The operator corresponding to the linear combination $A_L + 2A_H$ has the form

$$\begin{pmatrix} 0 & -1 & 0 & \frac{1}{\rho_1} \\ 1 & 0 & \frac{-1}{\rho_1} & 0 \\ 0 & \frac{-1}{\rho_2} & 0 & -1 \\ \frac{1}{\rho_2} & 0 & 1 & 0 \end{pmatrix},$$

and its four different complex eigenvalues are $\pm\frac{1}{\sqrt{\rho_1\rho_2}} \pm i$, proving that the poles are non-degenerate singularities of focus-focus type. □

Proof of Proposition 5.2.4. Let us consider first the set of points $P := \{(x, y, z, u, v) \in M \mid x = y = 0\}$. If we compute the differential of F in Cartesian coordinates, we get

$$dL_1(x, y, u, v) = -\frac{2\rho_1 x dx}{1 - x^2 - y^2} - \frac{2\rho_1 y dy}{1 - x^2 - y^2} + \rho_2 (udu + vdv),$$
$$dH(x, y, u, v) = \frac{1}{2}(udx + vdy + xdu + ydv).$$

Which, if we restrict to the set P by taking $x = y = 0$, are always linearly independent except for the points N, S, where both forms vanish.

The complement of P is precisely the domain of the polar coordinate chart, M^0, in which the differential has the expression

$$\begin{aligned} dL_1(z,\theta,u,v) &= \rho_1 \frac{dz}{z} + \rho_2 \left(udu + vdv\right), \\ dH(z,\theta,u,v) &= \frac{-z^2}{2\sqrt{1-z^2}} \left(u\cos\theta + v\sin\theta\right) \frac{dz}{z} \\ &\quad + \frac{\sqrt{1-z^2}}{2} \left(\left(-u\sin\theta + v\cos\theta\right) d\theta + \cos\theta du + \sin\theta dv\right). \end{aligned}$$

None of the differentials vanish in M^0, so if there exists some point in which the rank of the system is 1 there must exist some $\mu \neq 0$ such that $\mu dL_1 + dH = 0$ at that point. Looking at each of the components of this relationship, we get the system of equations

$$\begin{cases} \mu\rho_1 - \frac{z^2}{2\sqrt{1-z^2}} \left(u\cos\theta + v\sin\theta\right) = 0, \\ -u\sin\theta + v\cos\theta = 0, \\ \mu\rho_2 u + \frac{\sqrt{1-z^2}}{2} \cos\theta = 0, \\ \mu\rho_2 v + \frac{\sqrt{1-z^2}}{2} \sin\theta = 0. \end{cases}$$

However, if we combine the first, third and fourth equations we can deduce that

$$\mu\left(\rho_1 + \rho_2 \frac{z^2}{1-z^2}\left(u^2+v^2\right)\right) = 0,$$

of which the only solution is $\mu = 0$, in contradiction with the observation that μ cannot vanish. In conclusion, there are no points in M^0 such that the rank of dF is 1.

Therefore, the rank of dF is 2 for all points outside of $\{N, S\}$. □

Proof of Proposition 5.2.5. We first prove that $L_1 : M \setminus Z \to \mathbb{R}$ is surjective. Moreover, each point has two preimages, each of them in one of the connected components of $M \setminus Z$.

Indeed, it is clear that the equation $\rho_1 \log|z| + \frac{\rho_2}{2}(u^2+v^2) = \ell$ has two solutions for any choice of $\rho_1, \rho_2 > 0$ and $\ell \in \mathbb{R}$: if $\ell = 0$, then $z = \pm 1$,

$(u,v)=(0,0)$ are its two preimages; if $\ell>0$, then we can take $z=\pm 1$ and (u,v) such that $u^2+v^2=\frac{2\ell}{\rho_2}$; and if $\ell<0$ we can take $z=\pm\exp\left(\frac{\ell}{\rho_1}\right)$ and $(u,v)=(0,0)$.

Furthermore, we now show that H is surjective when restricted to any given fiber $\{L_1=\ell\}$.

For simplicity we restrict ourselves to the points $(x,y,z,u,v)\in M$ such that the vectors (x,y) and (u,v) are collinear in $\mathbb{R}^2$. In that case, H can be expressed as $H=\pm\frac{1}{2}\|(x,y)\|\|(u,v)\|$, where the sign depends on whether (x,y) and (u,v) point in the same or in opposite directions. Since (x,y,z) lies in the sphere, we know that $\|(x,y)\|=\sqrt{1-z^2}$. Let $r:=\|(u,v)\|$.

With these notations, the momentum map can be expressed as

$$\begin{cases} L_1(z,r)=\rho_1\log|z|+\frac{\rho_2}{2}r^2,\\ H(z,r)=\pm\frac{1}{2}\sqrt{1-z^2}r. \end{cases}$$

Let us now assume that $L_1(z,r)=\ell$ for some $\ell\in\mathbb{R}$. We will study separately the cases in which $\ell\geq 0$ and $\ell\leq 0$.

- If $\ell\geq 0$, then z may take any value within $[-1,1]$, and we can isolate r with respect to z,

$$r=\sqrt{\frac{2}{\rho_2}\left(\ell-\rho_1\log|z|\right)},$$

  which allows us to conclude that $r\geq\sqrt{\frac{2\ell}{\rho_2}}$. Moreover, the expression

$$H_+=\frac{1}{2}\sqrt{1-z^2}r=\frac{1}{2}\sqrt{1-z^2}\sqrt{\frac{2}{\rho_2}\left(\ell-\rho_1\log|z|\right)}$$

  may take any non-negative value (as $H_+(1)=0$, $\lim\limits_{z\to 0}H_+(z)=+\infty$, and H_+ is continuous), and thus H is surjective under the assumption that $L=\ell$.

- If $\ell\leq 0$, then r may take any non-negative value, and we can isolate $|z|$ with respect to r,

$$|z|=\exp\left(\frac{1}{\rho_1}\left(\ell-\frac{\rho_2}{2}r^2\right)\right),$$

which means that $|z| \leq \exp\left(\frac{\ell}{\rho_1}\right)$. We can conclude from this that the expression $H_+ = \frac{1}{2}\sqrt{1-z^2}r$ may take any non-negative value, and therefore H is surjective for the fiber $\{L_1 = \ell\}$.

□

5.2.2 The reversed b-coupled spin-oscillator

In this part we will study the reversed b-coupled spin-oscillator from Definition 5.2.1 and its critical points. Here is a summary of the results in this section:

Proposition 5.2.6 *The reversed b-coupled spin-oscillator has two fixed points, $N := ((0,0,1),(0,0)) \in M$ and $S := ((0,0,-1),(0,0)) \in M$, and both are elliptic-elliptic points.*

Proposition 5.2.7 *The reversed b-coupled spin-oscillator has four connected components of elliptic-regular points, emanating from N and S.*

Moreover, the image of the momentum map consists of a double covering of the region shown in Figure 5.1.

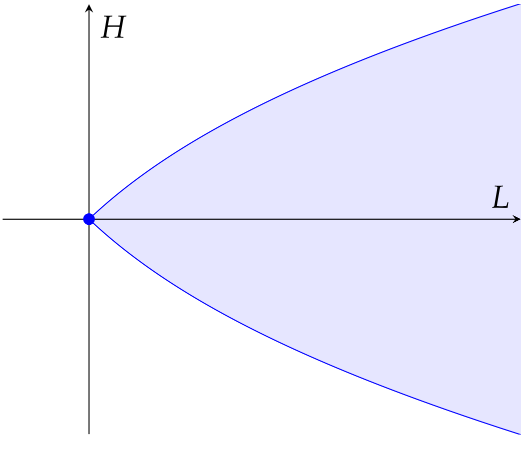

Figure 5.1: Image of the momentum map of the reversed b-coupled spin-oscillator. The blue dot is the image of the elliptic-elliptic singularities.

Proof of Proposition 5.2.6. Direct computation shows that the rank of F is 0 only at the points N and S. We follow again the scheme from Lemma 2.6.5

to prove that the poles are non-degenerate fixed points of elliptic-elliptic type.

At N and S, in coordinates (x, y, u, v), we have:

$$d^2L_2 = \begin{pmatrix} \rho_1 & 0 & 0 & 0 \\ 0 & \rho_1 & 0 & 0 \\ 0 & 0 & \rho_2 & 0 \\ 0 & 0 & 0 & \rho_2 \end{pmatrix}, \quad d^2H = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$
$$\Omega = \begin{pmatrix} 0 & \rho_1 & 0 & 0 \\ -\rho_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \rho_2 \\ 0 & 0 & -\rho_2 & 0 \end{pmatrix}.$$

The matrices d^2L_2 and d^2H are independent and give rise to:

$$A_L := \Omega^{-1}d^2L_2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad A_H := \Omega^{-1}d^2H = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & \frac{-1}{\rho_1} \\ 0 & 0 & \frac{1}{\rho_1} & 0 \\ 0 & \frac{-1}{\rho_2} & 0 & 0 \\ \frac{1}{\rho_2} & 0 & 0 & 0 \end{pmatrix}.$$

For any $\gamma > 0$, the linear combination $A_L + 2\gamma A_H$ has the form

$$\begin{pmatrix} 0 & -1 & 0 & \frac{-\gamma}{\rho_1} \\ 1 & 0 & \frac{\gamma}{\rho_1} & 0 \\ 0 & \frac{-\gamma}{\rho_2} & 0 & -1 \\ \frac{\gamma}{\rho_2} & 0 & 1 & 0 \end{pmatrix},$$

and has eigenvalues $\pm i\left(1 + \frac{\gamma}{\sqrt{\rho_1\rho_2}}\right), \pm i\left(1 - \frac{\gamma}{\sqrt{\rho_1\rho_2}}\right)$. Then, the linear combination $A_L + 2\gamma A_H$ has four different imaginary eigenvalues of the type $\pm ia, \pm ib$ (except when γ is exactly $\sqrt{\rho_1\rho_2}$, but we just need to show that there exists one linear combination of A_L and A_H with this property). This implies that the two points are non-degenerate singularities of elliptic-elliptic type. □

Proof of Proposition 5.2.7. As in the proof of Proposition 5.2.4, the rank of $F = (L_2, H)$ restricted to $P = \{(x, y, z, u, v) \in M \mid x = y = 0\}$ is lower than 2 only at the fixed points N and S.

On M^0, the complement of P in M, we have that

$$
\begin{aligned}
dL_2(z,\theta,u,v) &= -\rho_1\frac{dz}{z} + \rho_2\,(udu + vdv)\,, \\
dH(z,\theta,u,v) &= \frac{-z^2}{2\sqrt{1-z^2}}\,(u\cos\theta + v\sin\theta)\,\frac{dz}{z} \\
&\quad + \frac{\sqrt{1-z^2}}{2}\,((-u\sin\theta + v\cos\theta)\,d\theta + \cos\theta du + \sin\theta dv)\,.
\end{aligned}
$$

Neither dL_2 of dH vanish on M^0, so dF has rank 1 at a point only if there is some $\mu \neq 0$ such that $\mu dL_2 + dH = 0$. Examining this equation termwise,

$$
\begin{cases}
-\mu\rho_1 - \frac{z^2}{1-z^2}\,(u\cos\theta + v\sin\theta) = 0, \\
-u\sin\theta + v\cos\theta = 0, \\
\mu\rho_2 u + \cos\theta = 0, \\
\mu\rho_2 v + \sin\theta = 0,
\end{cases}
\quad\Rightarrow\quad
\begin{cases}
\frac{z^2}{1-z^2} = \mu^2\rho_1\rho_2, \\
\mu^2\rho_2^2\left(u^2+v^2\right) = 1, \\
u\sin\theta = v\cos\theta.
\end{cases}
$$

The last system can be solved for any value of $\mu \neq 0$, and has no solution with $z = 0$ or with $(u,v) = (0,0)$. The space of solutions can be parametrized using just 2 parameters. Explicitly, the set K_1 of singular points of rank 1 on M^0 is a 2-dimensional submanifold. K_1 can be parametrized by $\theta \in [0,2\pi[$ and $z \in]-1,0[\cup]0,1[$ as:

$$
(u(\theta,z), v(\theta,z)) = \pm\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\,(\cos\theta, \sin\theta)\,.
$$

Observe that, for any $(\theta,z) \in [0,2\pi[\times \left(]-1,0[\cup]0,1[\right) \subset \mathbb{S}^2$, there are two points $(u,v) \in \mathbb{R}^2$ that solve the system of equations. If we look at the northern hemisphere, where $\theta \in [0,2\pi[$ and $z \in]0,1[$, there are two of them, both emanating from the respective poles. The situation is analogous in the southern hemisphere, which means that the submanifold of singular points of rank 1 is the disjoint union of four connected components.

In the singular points of rank 1 the b-Hamiltonian vector fields X_{L_2} and X_H are colinear and their flows generate $\mathbb{S}^1$-orbits. In M^0 we have

$$
X_{L_2} = \frac{\partial}{\partial\theta} - v\frac{\partial}{\partial u} + u\frac{\partial}{\partial v}. \tag{5.1}
$$

Therefore, the $\mathbb{S}^1$-orbit of a singular point (z,θ,u,v) of rank 1 contains all the singular points of rank 1 that can be reached from (z,θ,u,v) by a simultaneous rotation of (z,θ) around the vertical axis of $\mathbb{S}^2$ and of (u,v) around the origin of $\mathbb{R}^2$. The four families of $\mathbb{S}^1$-orbits, in coordinates (z,θ,u,v), are the following:

$$\left(z,\theta,\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\cos\theta,\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\sin\theta\right), \qquad z\in]0,1[,\theta\in[0,2\pi[,$$
$$\left(z,\theta,-\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\cos\theta,-\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\sin\theta\right), \qquad z\in]0,1[,\theta\in[0,2\pi[,$$
$$\left(z,\theta,\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\cos\theta,\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\sin\theta\right), \qquad z\in]-1,0[,\theta\in[0,2\pi[,$$
$$\left(z,\theta,-\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\cos\theta,-\sqrt{\frac{\rho_1}{\rho_2}}\frac{\sqrt{1-z^2}}{z}\sin\theta\right), \quad z\in]-1,0[,\theta\in[0,2\pi[.$$

□

This concludes the analysis of the two b-symplectic variants of the coupled spin-oscillator. A similar analysis for b-variants of the coupled angular momenta (see Example 2.6.26) can be found in Brugués, Hohloch, Mir and Miranda [BHMM23].

The question that follows naturally is the one regarding the classification of such a system, which at this point is still open. In light of the classification explored in Subsection 2.6.3 (from Guillemin, Miranda, Pires and Scott [GMPS15]) and by Gualtieri, Li, Pelayo and Ratiu [GLPR17], it could be an interesting idea to attempt to reproduce the same invariants that can be found in classical semitoric systems to this context in the same sense that the aforementioned works generalize Delzant's theorem from toric into b- or log-toric systems. Such an exercise, unfortunately, lies beyond the scope of this thesis.

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Academic Resumé

Education

- (2012–2016) Bachelor in Mathematics at the Universitat Politècnica de Catalunya.
- (2016–2019) Master in Advanced Mathematics and Mathematical Engineering at the Universitat Politècnica de Catalunya.

Publications

- *The Arnold conjecture for singular symplectic manifolds*, joint with Eva Miranda and Cédric Oms, arXiv:2212.01344
- *Constructions of b-semitoric systems*, joint with Sonja Hohloch, Pau Mir and Eva Miranda. Journal of Mathematical Physics 64(7):072703 (2023). DOI:10.1063/5.0152551.

Teaching experience: exercise sessions for

- (2020) *Differential geometry* at the Master in Advanced Mathematics and Mathematical Engineering, at the Universitat Politècnica de Catalunya (problem sessions support).
- (2020) *Fundaments of Mathematics* at the Degree in Technical Architecture and Building Construction at the Universitat Politècnica de Catalunya.
- (2021–2023) *Multivariate Calculus* at the Degree in Mathematics at the University of Antwerp.
- (2023) *Calculus* at the Degree in Mathematics and *Mathematical methods for Physics I* at the Degree in Physics (joint course), at the University of Antwerp.

Talks in workshops and seminars

- (June 25, 2019) At the *Analysis and Geometry Seminar* at the University of Antwerp. Talk title: *An introduction to b-symplectic geometry*.
- (July 9, 2019) At the *13th International ICMAT Summer School on Geometry, Mechanics and Control* at ICMAT in Madrid. Talk title: *The construction of Floer Homology*.
- (January 29, 2020) At the *Seminari Informal de Matemàtiques de Barcelona* at the University of Barcelona. Talk title: *Morse theory: from dynamics to topology and back*.
- (February 12, 2020) At the *Seminari Informal de Matemàtiques de Barcelona* at the University of Barcelona. Talk title: *An introduction to Floer theory*.
- (March 17, 2021) At the *Analysis and Geometry Seminar* at the University of Antwerp. Talk title: *Singular Floer theory and singular Hamiltonian/Reeb Dynamics: First steps*.
- (March 24, 2022) At the *Poisson Seminars* at the KU Leuven. Talk title: *Towards a Poincaré-Hopf theorem for b-manifolds*.
- (April 27, 2022) At the *Analysis and Geometry Seminar* at the University of Antwerp. Talk title: *Towards a Poincaré-Hopf theorem for b-manifolds*.
- (July 15, 2022) At the *CRM Poisson Days 2022* at the Universitat Politècnica de Catalunya. Talk title: *A Floer complex for b-symplectic manifolds*.
- (August 16, 2022) At the workshop *Symplectic Dynamics Beyond Periodic Orbits 2022* at the Lorentz Center in Leiden. Talk title: *The Arnold conjecture for b-symplectic manifolds*.
- (November 23, 2022) At the *Arbeidsgemeinschaft Symplektische Topologie* at the University of Cologne. Talk title: *The Arnold conjecture for b-surfaces*.
- (January 31, 2023) At the *Geometry Seminar* at the University of Leuven. Talk title: *The Arnold conjecture for singular symplectic manifolds*.
- (February 6, 2023) At the *VI Congreso de Jóvenes Investigadores de la Real Sociedad Matemática Española* at the Universidad de León. Talk title: *The Arnold conjecture for b^m-symplectic manifolds*.

- (December 6, 2023) At the *Analysis and Geometry Seminar* at the University of Antwerp. Talk title: *The Arnold conjecture for b^m-symplectic manifolds*.

Poster presentation

- (October 6–11, 2019) Deformations and Rigidity in Algebra, Geometry and Analysis, University of Würzburg. Title: *Towards a Floer homology for singular symplectic manifolds*.
- (May 15, 2022) Belgian Mathematical Society's PhD Day at the University of Liège. Title: *Constructing a singular Floer homology*.
- (July 25–29, 2022) Poisson Conference in Madrid. Title: *A Hamiltonian Floer complex for b-manifolds*.

Organization of events

- (2021–2022) *b*–Lab Informal Seminar (co-organizer).
- (September 7 – October 5, 2021) Minicourse "Geometry and Dynamics of Singular Symplectic manifolds" at the University of Henan (co-organizer, problem sessions).
- (January 2023) EoS project "Beyond symplectic geometry" Minicourse at the University of Antwerp (local support).
- (December 18, 2023) Miniworkshop on Singular symplectic geometry (co-organizer).