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Tube formulas for valuations

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Introduction

Problem statement

For a compact convex set $A \subset \mathbb{R}^m$, the *Steiner formula* expresses the volume of the set A_t consisting of points at distance smaller than t from A as the following polynomial on the radius t

$$\mathrm{vol}(A_t) = \sum_{i=0}^{m} \omega_{m-i}\mu_i(A)t^{m-i}. \tag{1}$$

Here the normalizing constant ω_k is the volume of the k-dimensional unit ball, and the functionals μ_i are the so-called *intrinsic volumes*. The intrinsic volumes are fundamental examples of the notion of *valuation*, which we introduce next. Denote by $\mathcal{K}(\mathbb{R}^m)$ the class of compact convex set of the euclidean space. A complex-valued functional φ on $\mathcal{K}(\mathbb{R}^m)$ such that

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B)$$

for $A, B, A \cup B \in \mathcal{K}(\mathbb{R}^m)$, is called a *valuation*. The space of continuous and translation invariant valuations is denoted by $\mathrm{Val}(\mathbb{R}^m)$. Hadwiger's characterisation theorem states that the subspace $\mathrm{Val}(\mathbb{R}^m)^{SO(m)} \subset \mathrm{Val}(\mathbb{R}^m)$ of $SO(m)$-invariant valuations is spanned by the intrinsic volumes $\mu_0 = \chi, \ldots, \mu_m = \mathrm{vol}_{\mathbb{R}^m}$, where χ denotes the Euler characteristic.

In his famous *tube formula*, H. Weyl ([65]) proved that (1) holds for $A \subset \mathbb{R}^m$ a smooth compact submanifold and $t \geq 0$ small enough, in which case A_t is called the *tube around A of radius t*. Additionally, he proved that the coefficients $\mu_i(A)$ depend only on the induced riemannian structure of A.

In [27], Federer showed (1) for the class of compact *sets of positive reach*. A further development due to Federer is the introduction of certain localizations of the intrinsic volumes, which he called *curvature measures*. From the modern viewpoint, a *curvature measure* is a functional Φ on $\mathcal{K}(\mathbb{R}^m)$ taking values in the space of Borel measures of $\mathbb{R}^m$, fulfilling, *at least*, the following

$$\Phi(A \cup B, U) = \Phi(A, U) + \Phi(B, U) - \Phi(A \cap B, U)$$

where $A, B, A \cup B \in \mathcal{K}(\mathbb{R}^m)$ and U is a Borel subset of $\mathbb{R}^m$. Taking $U = \mathbb{R}^m$ we obtain a map called *globalization* from curvature measures to valuations. A notable family of curvature measures, denoted by Δ_i, are the *Federer curvature measures*, whose globalizations correspond to the intrinsic volumes μ_i.

Tube formulas exist however in other spaces besides euclidean space. Indeed, already in Weyl's original work [65], the tube formula (1) was extended to the sphere and hyperbolic space. The perspective has shifted: our focus is now on altering the ambient space, not merely the class of sets where tubes are taken. The tube formula in spherical and hyperbolic spaces is expressed not as a polynomial in the radius t, but as

$$\mathrm{vol}(A_t) = \sum_{i=0}^{n-1}(n-i)\omega_{n-i}\left(\int_0^t \sin_\lambda^{n-i-1}(r)\cos_\lambda^i(r)dr\right)\nu_i(A) \tag{2}$$

where ν_i are certain functionals, λ is the curvature of the ambient,

$$\sin_\lambda(t) := \begin{cases} \dfrac{\sin(\sqrt{\lambda}t)}{\sqrt{\lambda}} & \lambda > 0, \\ t & \lambda = 0, \\ \dfrac{\sinh(\sqrt{|\lambda|}t)}{\sqrt{|\lambda|}} & \lambda < 0, \end{cases}$$

which is an analytic function in both λ and t, and $\cos_\lambda(t) := \frac{d}{dt}\sin_\lambda(t)$. The functionals ν_i in (2) are naturally seen as analogs of the μ_i in (1), which hints at a notion of valuation beyond the euclidean space. The current perspective is to view these ν_i as *smooth valuations* in the sense of Alesker's theory of valuations on manifolds (cf. [7]). To simplify matters, let M^n be a riemannian manifold and fix $\mathcal{R}(M)$ the class of sets of positive reach in M. Let $\pi\colon SM \to M$ be the sphere bundle of M. A *smooth valuation* on M is a complex-valued functional $\varphi = [\![\omega, \eta]\!]$ on $\mathcal{R}(M)$ of the form

$$\varphi(A) = \int_{N(A)}\omega + \int_A \eta, \qquad A \in \mathcal{R}(M),$$

where $\omega \in \Omega^{n-1}(SM)$ and $\eta \in \Omega^n(M)$, are complex-valued differential forms and $N(A) \subset SM$ is the so-called *normal cycle of* A (cf. e.g. [53]), consisting of outward normal vectors to A. We denote by $\mathcal{V}(M)$ the complex vector space of valuations on M.

It is natural to consider also the corresponding localized functional $\Phi = [\omega, \eta]$ given by

$$\Phi(A, U) = \int_{N(A)\cap\pi^{-1}(U)}\omega + \int_{A\cap U}\eta$$

where $A \in \mathcal{R}(M)$ and U is a Borel subset of M. Such a functional is called a *smooth curvature measure*, and the complex vector space they comprise is denoted by $\mathcal{C}(M)$. Once again, we have a *globalization* map $\mathcal{C}(M) \to \mathcal{V}(M)$ simply by taking $U = M$.

Federer's curvature measures Δ_i can be naturally extended from $\mathbb{R}^m$ to any riemannian manifold. The globalizations of these curvature measures Δ_i on M give us a family of valuations ν_i in M. Although there is no fixed terminology for these ν_i, it is natural to refer to them as *Federer valuations*.

Further results, in the vein of extending tube formulas to other ambient spaces, were formulated by Gray and others (cf. [38, 40, 39]). In particular, Gray and Vanvhecke [38] computed the volume of a tube around submanifolds of real, complex, quaternionic, and octonionic space forms. Here we continue their work in the real, complex, and quaternionic cases, leaving the octonionic one for a better occasion. For that purpose, let us fix some notation:

- *Real space forms*: we denote by $\mathbb{S}^m_\lambda$ the (unique up to isometry) complete m-dimensional simply connected riemannian manifold of constant sectional curvature λ. If $\lambda > 0$, it corresponds to the sphere of radius $r = \frac{1}{\sqrt{\lambda}}$. For $\lambda < 0$, it represents the real hyperbolic space, with the metric suitably rescaled. Finally, $\lambda = 0$ represents the standard euclidean space. We refer to the family $\mathbb{S}^m_\lambda$ as *real space forms*. We denote by $G_{\lambda,\mathbb{R}}$ the group $\mathrm{Isom}(\mathbb{S}^m_\lambda)$ of isometries of $\mathbb{S}^m_\lambda$.

- *Complex space forms*: we denote by $\mathbb{C}P^n_\lambda$ the (unique up to isometry) complete n-dimensional simply connected Kähler manifold of constant holomorphic sectional curvature 4λ. If $\lambda > 0$, it corresponds to the complex projective space endowed with the Fubini-Study metric. For $\lambda < 0$, it is the complex hyperbolic space equipped with the Bergman metric. Finally, $\lambda = 0$ represents the hermitian standard space $\mathbb{C}^n$. We refer to the family $\mathbb{C}P^n_\lambda$ as *complex space forms*. We denote by $G_{\lambda,\mathbb{C}}$ the group $\mathrm{Isom}(\mathbb{C}P^n_\lambda)$ of isometries on $\mathbb{C}P^n_\lambda$.

- *Quaternionic space forms*: we denote by $\mathbb{H}P^n_\lambda$ the (unique up to isometry) complete n-dimensional simply connected quaternionic Kähler manifold of constant quaternionic sectional curvature 4λ. If $\lambda > 0$, it corresponds to the quaternionic projective space. For $\lambda < 0$, it is called quaternionic hyperbolic space. Finally, $\lambda = 0$ represents the quaternionic standard space $\mathbb{H}^n$. We refer to the family $\mathbb{H}P^n_\lambda$ as *quaternionic space forms*. We denote by $G_{\lambda,\mathbb{H}}$ the group $\mathrm{Isom}(\mathbb{H}P^n_\lambda)$ of isometries of $\mathbb{H}P^n_\lambda$.

All previously mentioned works focused on computing the *volume* of tubes around geometric objects. Tube formulas, however, also exist for valuations beyond volume. For instance, by differentiating the Steiner formula one easily obtains

$$\mu_k(A_t) = \sum_{j=0}^{k} \binom{m-j}{m-k} \frac{\omega_{m-j}}{\omega_{m-k}} \mu_j(A) t^{k-j}, \qquad A \subset \mathbb{R}^m. \tag{3}$$

In real space forms, Santaló obtained similar tube formulas for all isometry invariant valuations (see [55]); explicitly, in terms of ν_i, the tube formula is

$$\nu_i(A_t) = \sum_{j=0}^{m} \phi^\lambda_{m,i,j}(t) \frac{(m-j)\omega_{m-j}}{(m-i)\omega_{m-i}} \nu_j(A), \tag{4}$$

where

$$\phi^\lambda_{m,i,j}(t) = \sum_{h\geq 0} (-\lambda)^{j-h} \binom{m-j}{i-h} \binom{j}{h} \sin_\lambda^{i+j-2h}(t) \cos_\lambda^{m-i-j+2h}(t).$$

We will refer to $\phi^{\lambda}_{m,i,j}$ as *Santaló's polynomials*. Thus, we have as in (2) a tube formula involving *trigonometric polynomials* instead of a polynomial on the radius like in (1) and (3). Gray and Vanhecke established tube formulas for ν_i in real, complex, and quaternionic space forms, albeit under stringent restrictions on the submanifold.

Main results

In this work, we establish the existence of tube formulas for any smooth valuation on a riemannian manifold. In real and complex space forms we will compute these tube formulas for every *invariant* valuation. In addition, we derive the tube formulas for a certain interesting family of valuations in $\mathbb{H}P^n_\lambda$ including the Federer valuations. Next, we describe these results in more detail.

Tubular operators

Given a riemannian manifold M^n, we construct the *tubular operator*, which consists in a family $\mathbf{T}_t$ of linear endomorphisms of $\mathcal{V}(M)$, such that

$$\mu(A_t) = \mathbf{T}_t\mu(A), \qquad \forall \mu \in \mathcal{V}(M)^G,$$

for $A \in \mathcal{R}(M)$ and $t \geq 0$ small enough. Differentiating $\mathbf{T}_t$ at $t = 0$ yields the *derivative operator* $\partial\colon \mathcal{V}(M) \to \mathcal{V}(M)$. Similarly, we define the *local tubular operator* $\mathbf{LT}_t$ in $\mathcal{C}(M)$ and its corresponding derivative operator also denoted by ∂.

Explicitly, these tubular operators are given in terms of differential forms as follows. Let $t \geq 0$ and $p_t\colon SM \times [0,t] \to SM$ be the projection on the first factor. Let us consider $\phi\colon SM \times \mathbb{R} \to SM$ the geodesic flow. Then, for $\mu = [\![\omega, \eta]\!]$ and $\Phi = [\omega, \eta]$; i.e.

$$\mathbf{T}_t(\mu) = [\![\phi_t^*\omega + (p_t)_*(\pi \circ \phi)^*\eta, \eta]\!], \tag{5}$$

$$\mathbf{LT}_t(\Phi) = [\phi_t^*\omega + (p_t)_*(\pi \circ \phi)^*\eta, \eta]. \tag{6}$$

Let now G be a subgroup of isometries of M and the subspaces $\mathcal{V}(M)^G$ and $\mathcal{C}(M)^G$ of invariant valuations and curvature measures. We will prove their invariance under $\mathbf{T}_t$ and $\mathbf{LT}_t$; i.e

$$\mathbf{T}_t\mu \in \mathcal{V}(M)^G, \quad \mathbf{LT}_t\Phi \in \mathcal{C}(M)^G,$$

for all $\mu \in \mathcal{V}(M)^G$ and $\Phi \in \mathcal{C}(M)^G$. Therefore, the same property holds for ∂ in $\mathcal{V}(M)^G$ and $\mathcal{C}(M)^G$.

Let us assume that (M, G) is an *isotropic space*, meaning that G acts transitively on SM. Then, $\mathcal{V}(M)^G$ and $\mathcal{C}(M)^G$ are finite-dimensional vector spaces and the tubular operator $\mathbf{T}_t|_{\mathcal{V}(M)^G}$ is given by the flow of $\partial|_{\mathcal{V}(M)^G}$

$$\mathbf{T}_t\mu = \exp(t\partial)\mu = \sum_{i\geq 0} \frac{t^i}{i!}\partial^i\mu, \qquad \forall \mu \in \mathcal{V}(M)^G. \tag{7}$$

The analyticity in t is ensured by the fact that $\dim \mathcal{V}(M)^G < \infty$, and may not hold without the finite-dimensionality restriction. We will apply (7) to real, complex, and quaternionic space forms, where the derivative operator will be explicitly displayed.

Complex space forms

Let us first focus on the space $\mathcal{V}^n_{\lambda,\mathbb{C}}$ of valuations invariant by holomorphic isometries, endowed with its corresponding derivative operator $\partial_{\lambda,\mathbb{C}}$. We begin by analyzing the null-curvature case $\lambda = 0$.

Let $\mathrm{Val}^{U(n)}$ be the vector space of translation and $U(n)$-invariant continuous valuations in $\mathbb{C}^n$. Bernig and Fu introduced an $\mathfrak{sl}_2$-module structure in $\mathrm{Val}^{U(n)}$, consisting of a triple $(\Lambda, L, H := [\Lambda, L])$, where Λ is a convenient normalization of the derivative operator on $\mathrm{Val}^{U(n)}$, and L is a normalization of the product with the first intrinsic volume. In particular, Λ is nilpotent, implying that ∂ is also nilpotent. Consequently, (7) reduces to a finite sum.

Among the many special bases for $\mathrm{Val}^{U(n)}$, Bernig and Fu introduced one that is compatible with the decomposition into $\mathfrak{sl}_2$-irreducible components; i.e., maximal and non-trivial subspaces that remain invariant under the action of Λ and L. In this basis, the successive powers of ∂ are straightforwardly given, and consequently, the tubular operator is a simple consequence of (7).

Remarkably, the $\mathfrak{sl}_2$-structure of $\mathrm{Val}^{U(n)}$ is somehow present also in the curved case $\lambda \neq 0$. Indeed, we will establish the existence of an isomorphism $\Phi_\lambda \colon \mathrm{Val}^{U(n)} \to \mathcal{V}^n_{\lambda,\mathbb{C}}$ such that

$$\partial_{\lambda,\mathbb{C}} = \Phi_\lambda \circ (\Lambda - \lambda L) \circ \Phi_\lambda^{-1}. \tag{8}$$

While it is natural to have a relation between $\partial_{\lambda,\mathcal{C}}$ and Λ, the appearance of L is quite unexpected.

The identity (8) turns the determination of tube formulas in $\mathcal{V}^n_{\lambda,\mathbb{C}}$ into the computation of $\exp t(\Lambda - \lambda L)$. By decomposing $\mathrm{Val}^{U(n)}$ into $\mathfrak{sl}_2$-irreducible components this computation can be easily performed in some abstract model space. In fact, the same abstract computation lies at the basis of most of the results of this thesis. The result we obtain for $\mathcal{V}^n_{\lambda,\mathbb{C}}$ is the following.

Theorem. *There exists a basis $\{\sigma^\lambda_{k,r}\}$ of $\mathcal{V}^n_{\lambda,\mathbb{C}}$ such that*

$$\mathbf{T}_t \sigma^\lambda_{k,r} = \sum_{j=0}^{2n-4r} \phi^\lambda_{2n-4r,k-2r,j}(t)\sigma^\lambda_{j+2r,r}, \tag{9}$$

where

$$\phi^\lambda_{m,k,j}(t) = \sum_{h\geq 0} (-\lambda)^{j-h} \binom{m-j}{k-h}\binom{j}{h} \sin_\lambda^{k+j-2h}(t) \cos_\lambda^{m-k-j+2h}(t).$$

We describe the basis $\sigma_{k,r}^{\lambda}$ explicitly in terms of the previously known valuations $\tau_{k,p}^{\lambda}$ of [21]. The tube formulas for the $\tau_{k,p}^{\lambda}$ can be directly obtained from the previous ones, as we also provide the expression of these valuations in terms of the $\sigma_{k,r}^{\lambda}$.

The expressions (9) closely resemble those obtained by Santaló in (4) within real space forms. To explain these similarities, we demonstrate the existence of a phenomenon that parallels (8). However, the volume tube formula exhibits significant differences.

We then switch to the local level by providing closed formulas for $\mathbf{LT}_t$ in $\mathbb{C}P_{\lambda}^{n}$, leveraging the complete classification of invariant curvature measures known for $\mathbb{C}P_{\lambda}^{n}$ ([51, 21]). The resulting formulas are substantially more involved than (9).

Federer valuations and curvature measures

Our study concludes in Chapter 4 with the determination of tube formulas for the Federer curvature measures Δ_i and the Federer valuations ν_i in quaternionic space forms. This election arises from the lack of a complete classification of invariant curvature measures in $\mathbb{H}P_{\lambda}^{n}$, unlike $\mathbb{S}_{\lambda}^{m}$ and $\mathbb{C}P_{\lambda}^{n}$. For this reason, we chose to begin with tube formulas for Federer valuations as a solid starting point for future developments. We will treat the cases of $\mathbb{C}P_{\lambda}^{n}$ and $\mathbb{H}P_{\lambda}^{n}$ in parallel. Although we know $\mathbf{T}_t$ and $\mathbf{LT}_t$ for all invariant valuations and curvature measures in complex space forms, it is interesting to establish particular formulas for Δ_i and ν_i in $\mathbb{C}P_{\lambda}^{n}$. Moreover, these formulas provide insights into approaching the case of $\mathbb{H}P_{\lambda}^{n}$.

The strategy in both $\mathbb{C}P_{\lambda}^{n}$ and $\mathbb{H}P_{\lambda}^{n}$ is to decompose the Federer valuations Δ_i as a sum of curvature measures that are conveniently adapted to the curvature tensor, and thus have a simple behavior under the geodesic flow. Explicitly, we put

$$\Delta_i = \sum_j [\psi_j \wedge \varphi_{i-j}, 0], \tag{10}$$

where ψ_j and φ_{i-j} are a family of invariant differential forms with a specially nice behavior to the geodesic flow. More precisely, we find that the action of $\mathcal{L}_T$ on the span of the ψ_j and the span of the φ_k follows the same abstract $\mathfrak{sl}_2$ model as $\partial_{\lambda,\mathbb{C}}$ on $\mathcal{V}_{\lambda,\mathbb{C}}^{n}$. By using the same abstract computations as in that case, we obtain $\phi_t^*\psi_j$ and $\phi_t^*\varphi_k$. These pull-backs along with formula (7) and representation (10), yield

Theorem. *The tube formulas for the Federer valuations and curvature measures in* $\mathbb{H}P_{\lambda}^{n}$ *are*

$$\mathbf{T}_t(\nu_i) = \frac{1}{(4n-i)\omega_{4n-i}} \sum_{j,k=0}^{3} \sum_{l=0}^{4n-4} \phi_{3,j,k}^{4\lambda}(t)\phi_{4n-4,i-j,l}^{\lambda}(t)[\![\psi_k \wedge \varphi_l, 0]\!],$$

$$\mathbf{LT}_t(\Delta_i) = \frac{1}{(4n-i)\omega_{4n-i}} \sum_{j,k=0}^{3} \sum_{l=0}^{4n-4} \phi_{3,i,k}^{4\lambda}(t)\phi_{4n-4,i-j,l}^{\lambda}(t)[\psi_k \wedge \varphi_l, 0],$$

for $0 \leq i \leq 4n-1$.

In fact, we have completely determined $\mathbf{LT}_t$ and $\mathbf{T}_t$ on the vector spaces spanned by

$$\{[\psi_j \wedge \varphi_k]\}, \quad \{[\![\psi_j \wedge \varphi_k]\!]\}$$

Therefore, both on $\mathbb{C}P^n_\lambda$ and $\mathbb{H}P^n_\lambda$ we have found tubular formulas in certain natural subspaces of isometry invariant curvature measures and valuations which contain Δ_i and ν_i respectively.

As application of our results, we can compute the Hopf push-forward of valuations through the Hopf fibration $p_{\lambda,\mathbb{H}}\colon S^{4n+3} \to \mathbb{H}P^n_\lambda$ following the approach of Georg Hofstätter and Thomas Wannerer ([37]). The essential component of this computation relies on the commutativity of $\mathbf{T}_t$ and the Hopf push-forward $(p_{\lambda,\mathbb{H}})_*$, a result they established. With these tools at hand, we can represent this push-forward in terms of differential forms and introduce new and interesting families of valuations in $\mathcal{V}^n_{\lambda,\mathbb{H}}$.

Chapter 1

Background

In this chapter we introduce the essential material about the riemannian manifolds we will be working with. We also review the basic notions of valuation theory, as well as some basic facts on $\mathfrak{sl}_2$-representations.

1.1 Riemannian manifolds

1.1.1 The sphere bundle

We review some fundamental notions on the sphere bundle of a riemannian manifold and its canonical contact structure. Everything is classical, but we include several proofs and details for self-containedness. For a more comprehensive treatment of this subject, see [25].

Let M^n be an oriented riemannian manifold, where $g = \langle \cdot, \cdot \rangle$ denotes its riemannian metric. Let ∇ be the *Levi-Civita connection* satisfying

$$\nabla g = 0, \quad [X, Y] = \nabla_X Y - \nabla_Y X,$$

and let R be the *Gauss curvature tensor* defined as

$$R(W, X)Y := \nabla_{[W,X]} Y - \nabla_W \nabla_X Y + \nabla_X \nabla_W Y.$$

The *Riemann curvature tensor* is given by $R(W, X, Y, Z) = g(R(W, X)Y, Z)$.

Definition 1.1.1. The *sphere bundle of* M, denoted by SM, is the $(2n-1)$-dimensional smooth manifold consisting of the set of unit tangent vectors with canonical projection $\pi \colon SM \to M$. The *canonical contact 1-form* of SM is

$$\alpha_\xi(v) := \langle \xi, d\pi(v) \rangle, \quad v \in T_\xi SM. \tag{1.1}$$

The 2-form $\Theta_s := -d\alpha$ is called *symplectic form* of SM. The top form $\alpha \wedge (d\alpha)^{n-1}$ is nowhere vanishing; i.e., it is a volume element (see [25]).

We systematically denote by $e_0\colon SM \to TM$ the smooth map given by $e_0(\xi) = \xi$ for all $\xi \in SM$, i.e, the inclusion map of SM in TM. Therefore, for any vector field in SM, we can describe the canonical contact 1-form introduced in (1.1) by

$$\alpha(X) = \langle e_0, d\pi X\rangle. \tag{1.2}$$

A diffeomorphism ϕ on SM is called a (oriented) *contactomorphism* if it preserves the oriented hyperplane field $\ker\alpha := \{X \in \mathfrak{X}(SM) : i_X\alpha = 0\}$, known as the *contact hyperplane*, with the orientation induced by $d\alpha^{n-1}$. This is equivalent to

$$\phi^*\alpha = \mathrm{e}^h\,\alpha \tag{1.3}$$

for some real-valued function h on SM. When h is identically zero ϕ is called *strict contactomorphism.* A vector field X on SM which satisfies $\mathcal{L}_X\alpha = f\cdot\alpha$ for some function $f\colon M \to \mathbb{R}$ is called a *contact vector field.*

1.1.2 Pull-back bundle and pull-back connection

Given the projections $\pi\colon SM \to M$ and $\pi'\colon TM \to M$, we consider the pull-back vector bundle

$$\pi^*TM := \{(\xi, X) \in SM \times TM : \pi\xi = \pi' X\}.$$

If Y is a section of TM (i.e., a vector field on M), we can pull back Y to obtain a section in π^*TM

$$\pi^*Y = (e_0, Y \circ \pi).$$

Locally, every section of π^*TM is a C^∞-linear combination of pull-back sections of TM (e.g. the pull-backs of $Y_i = \frac{\partial}{\partial x_i}$ where $(x_1, \ldots, x_n)$ are local coordinates on M). It follows that there exists a unique connection $\pi^*\nabla$ in the pullback bundle π^*TM characterized by the property that for every vector $X \in T_\xi SM$ and every vector field Y on M, the following holds:

$$(\pi^*\nabla)_X(\pi^*Y) = \pi^*\left(\nabla_{d\pi X}Y\right).$$

For each $\xi \in SM$, the pull-back connection $\pi^*\nabla$ induces a natural decomposition of $T_\xi SM$ given by

$$T_\xi SM = H_\xi \oplus V_\xi.$$

where $V_\xi := \ker(d\pi)_\xi$ is known as the *vertical* subspace, and $H_\xi := \ker(X \mapsto (\pi^*\nabla)_X e_0)_\xi$ is the *horizontal* subspace. The restriction $(d\pi)_\xi\colon H_\xi \to T_xM$ is an isomorphism, and likewise

$$(\pi^*\nabla)e_0\colon V_\xi \longrightarrow \langle\xi\rangle^\perp, \qquad X \longmapsto (\pi^*\nabla)_X e_0 \tag{1.4}$$

defines an isomorphism. Hence, we have an identification

$$\begin{array}{rcl} T_\xi SM = & H_\xi \oplus V_\xi & \cong T_xM \oplus \langle\xi\rangle^\perp \\ X & \longleftrightarrow & (X^H, X^V) \end{array}$$

where $x = \pi(\xi)$, $X^H = d\pi X$ and $X^V = (\pi^*\nabla)_X e_0$. Therefore, any vector field X on the sphere bundle SM is determined by $d\pi X$ and $(\pi^*\nabla)_X e_0$, i.e., through its horizontal and vertical parts, respectively. In this notation the canonical contact 1-form α previously defined in (1.1) can be expressed by

$$\alpha(X) = \langle X^H, e_0\rangle, \quad X \in \mathfrak{X}(SM). \tag{1.5}$$

The *bilinear connection form* ω is defined by

$$\omega(X,Y) := \langle (\pi^*\nabla)_X e_0, d\pi Y\rangle = \langle X^V, Y^H\rangle, \tag{1.6}$$

for $X, Y \in \mathfrak{X}(SM)$. Thus, by definition of the vertical subspace V, we have the following description

$$V = \{X \in \mathfrak{X}(SM) : i_X\omega = 0\}.$$

The connection form ω is bilinear but *not* skew-symmetric.

1.1.3 Moving frames and the Reeb vector field

Another key concept is *moving frames*, useful for computations with connections and curvature.

Definition 1.1.2. Let U be an open subset of SM. A *moving frame on U* is a collection $e_0, \dots, e_{n-1}$ of smooth maps $e_j : U \to TM$ such that $e_0(\xi) = \xi$, and $\{e_0(\xi), \dots, e_{n-1}(\xi)\}$ forms a basis of $T_{\pi\xi}M$ for all $\xi \in SM$. If the basis is orthonormal for each $\xi \in U$, we call it an *orthonormal moving frame.*

Consider an orthonormal moving frame $e_0, e_1, \dots, e_{n-1}$ defined on $U \subset SM$. Then each e_j can be seen as a section of π^*TM. The *coframe, connection* and *curvature forms*

$$\theta_i, \omega_{i,j} \in \Omega^1(U), \quad \Omega_{i,j} \in \Omega^2(U), \qquad 0 \le i, j \le n-1$$

are defined by

$$\theta_i(X) := \langle e_i, d\pi X\rangle, \tag{1.7}$$

$$\omega_{i,j}(X) := \langle e_i, (\pi^*\nabla)_X e_j\rangle, \tag{1.8}$$

$$\Omega_{i,j}(X,Y) := R(e_i, e_j; d\pi X, d\pi Y). \tag{1.9}$$

where R is the curvature tensor of M. Notice that $\theta_0 = \alpha$.

The corresponding *structure equations* are given by

$$d\theta_i = -\sum_{j=0}^{n-1} \omega_{i,j} \wedge \theta_j, \tag{1.10}$$

$$d\omega_{i,j} = -\sum_{k=0}^{n-1} \omega_{i,k} \wedge \omega_{k,j} + \Omega_{i,j}, \tag{1.11}$$

$$d\Omega_{i,j} = \sum_{k=0}^{n-1} (\Omega_{i,k} \wedge \omega_{k,j} - \omega_{i,k} \wedge \Omega_{k,j}). \tag{1.12}$$

Let us recall that any vector field on SM is completely determined by its vertical and horizontal components. Consider the vector field T on the sphere bundle SM such that

$$d\pi T = e_0, \quad (\pi^*\nabla)_T e_0 = 0.$$

This vector field on SM is called the *Reeb vector field*, and its flow ϕ is called the *Reeb flow*. In Proposition 1.1.7, we will see the classical description of the Reeb vector field within contact manifolds. First, we study the behavior of the Reeb vector field T concerning an orthonormal moving frame.

Proposition 1.1.3. *Let $e_0, \ldots, e_{n-1}$ be an orthonormal moving frame. Then*

$$i_T\theta_i = \delta_{i,0}, \quad i_T\omega_{i,0} = 0, \quad 0 \le i \le n-1,$$

where T is the Reeb vector field.

Proof. From (1.7) and (1.8)

$$i_T\theta_i = \langle e_i, d\pi T\rangle = \langle e_i, e_0\rangle = \delta_{i,0},$$
$$i_T\omega_{i,0} = \langle e_i, (\pi^*\nabla)_T e_0\rangle = 0.$$

□

Proposition 1.1.4. *Let $e_0, \ldots, e_{n-1}$ be an orthonormal moving frame. Then, for $0 \le i \le n-1$*

i)

$$\mathcal{L}_T\theta_i = \omega_{i,0} - \sum_{j=1}^{n-1} i_T(\omega_{i,j})\theta_j,$$

ii)

$$\mathcal{L}_T\omega_{i,0} = -\sum_{k=1}^{n-1} i_T(\omega_{i,k})\omega_{k,0} + i_T\Omega_{i,0}.$$

Proof. *i)* By the structure equation (1.10), for $0 \le i \le n-1$

$$d\theta_i = -\sum_{j=0}^{n-1} \omega_{i,j} \wedge \theta_j.$$

By Proposition 1.1.3

$$i_T\theta_i = \delta_{i,0}, \quad i_T\omega_{i,0} = 0.$$

Therefore

$$i_T d\theta_i = -\sum_{j=1}^{n-1} i_T(\omega_{i,j})\theta_j + \omega_{i,0}.$$

Since $i_T\theta_i = \delta_{i,0}$, then $(d \circ i_T)\theta_i = 0$ and subsequently

$$\mathcal{L}_T\theta_i = i_T d\theta_i = \omega_{i,0} - \sum_{j=1}^{n-1} i_T(\omega_{i,j})\theta_j.$$

ii) From the structure equation (1.11), for $0 \le i \le n-1$

$$d\omega_{i,0} = -\sum_{k=0}^{n-1} \omega_{i,k} \wedge \omega_{k,0} + \Omega_{i,0}.$$

Since $i_T\omega_{k,0} = 0$

$$i_T d\omega_{i,0} = -\sum_{k=1}^{n-1} i_T(\omega_{i,k})\omega_{k,0} + i_T\Omega_{i,0}.$$

Once again, $(d \circ i_T)\omega_{i,0} = 0$ and the conclusion follows. □

The expressions of Proposition 1.1.4 become simpler after choosing a suitable moving frame. The construction proceeds as follows: given $\xi \in SM$ and $\{e_0 = \xi, \dots, e_{n-1}\}$ an orthonormal basis of $T_{\pi\xi}M$, we *parallel transport* $e_0, \dots, e_{n-1}$ along the geodesic $\gamma(t) = \exp(t\xi)$. This defines a moving frame on the curve:

$$\Gamma := \{(\gamma(t), \gamma'(t)) \colon |t| < \varepsilon\} \subset SM.$$

Finally, we arbitrarily extend it to a moving frame $e_0, \dots, e_{n-1}$ defined on a neighborhood of Γ. We refer to $e_0, \dots, e_{n-1}$ as a *parallel orthonormal moving frame relative to* ξ. The property characterizing these moving frames, in addition to orthonormality, is that on Γ

$$(\pi^*\nabla)_T e_i = 0, \quad 0 \le i \le n-1. \tag{1.13}$$

Remark 1.1.5. It is important to note that the parallel moving frame is defined in an open neighborhood of ξ, but we only require parallelism (1.13) along the curve Γ.

Corollary 1.1.6. *Let $e_0, \dots, e_{n-1}$ be a parallel orthonormal moving frame relative to ξ. Then, for $\gamma(t) = \exp(t\xi)$ we have*

$$\mathcal{L}_T\theta_i = \omega_{i,0}, \quad \mathcal{L}_T\omega_{i,0} = i_T\Omega_{i,0}, \qquad 1 \le i \le n-1,$$

at every point $(\gamma(t), \gamma'(t)) \in SM$.

Proof. By Proposition 1.1.7 we know that $d\pi T = e_0$. Since $e_0, \dots, e_{n-1}$ is a parallel orthonormal moving frame relative to ξ, from equation (1.13)

$$i_T\omega_{i,j} = \langle e_i, (\pi^*\nabla)_T e_j \rangle = 0.$$

The result follows from Proposition 1.1.4. □

To conclude this section we recall two fundamental facts concerning the Reeb vector field.

Proposition 1.1.7. *The Reeb vector field T is the unique vector field on SM satisfying*

$$i_T\alpha = 1, \quad i_T d\alpha = 0. \tag{1.14}$$

Proof. Let $e_0, \dots, e_{n-1}$ be an orthonormal moving frame relative to $\xi \in SM$. From (1.10)

$$\alpha = \theta_0, \qquad d\alpha = \sum_{j=1}^{n-1} \omega_{j,0} \wedge \theta_j.$$

Let us prove that the Reeb vector field satisfies the identities (1.14). From Proposition 1.1.3

$$\begin{aligned} i_T\alpha &= i_T\theta_0 = 1, \\ i_T d\alpha &= \sum_{j=1}^{n-1} i_T\omega_{j,0} \wedge \theta_j - \omega_{j,0} \wedge i_T\theta_j = 0. \end{aligned}$$

Let us see the uniqueness. Assume that there exists $\tilde{T}$ fulfilling (1.14). Since $\alpha \wedge (d\alpha)^{n-1}$ is a volume element we have

$$\ker \alpha \cap \ker(d\alpha)^{n-1} = \{0\}.$$

The identity $i_X(d\alpha)^{n-1} = (n-1) i_X d\alpha \wedge d\alpha^{n-2}$ ensures that $\ker d\alpha \subset \ker(d\alpha)^{n-1}$. Therefore

$$\ker \alpha \cap \ker d\alpha = \{0\}.$$

Since $\ker \alpha$ is the contact hyperplane and intersects $\ker d\alpha$ trivially, $\ker d\alpha$ is one-dimensional. Thus, there exists $f \in C^\infty(SM)$ such that $\tilde{T} = f \cdot T$. This implies

$$1 = i_{\tilde{T}}\alpha = f i_T\alpha = f,$$

and yields the result. □

By Cartan's magic formula, the condition $i_T d\alpha = 0$ may be replaced by $\mathcal{L}_T\alpha = 0$. Therefore, T is a contact vector field whose flow is a family $\{\phi_t\}_{t\in\mathbb{R}}$ of *strict* contactomorphisms; i.e

$$\phi_t^*\alpha = \alpha, \qquad \forall t \in \mathbb{R}, \tag{1.15}$$

which corresponds to (1.3) with $h \equiv 0$.

Proposition 1.1.7 contains the classical notion of the Reeb field in contact manifolds. In our case, it has an additional key property. Recall that, for any riemannian manifold M, the *geodesic vector field* $G\colon SM \to TSM$ is defined as follows. For each $\xi \in SM$, let $\gamma\colon (-\varepsilon, \varepsilon) \to M$ be the unique geodesic in M such that $\gamma(0) = \pi\xi$ and $\gamma'(0) = \xi$, i.e, $\gamma(t) = \exp(t\xi)$. Then $G(\xi) = \widetilde{\gamma}'(0)$ where

$$\widetilde{\gamma}(t) := (\gamma(t), \gamma'(t)).$$

The following fact is well-known but remarkable. We include a proof for completeness.

Proposition 1.1.8. *The Reeb vector field and the geodesic vector field on SM coincide.*

Proof. Let $G\colon SM \to TSM$ be the geodesic field of SM; i.e, for each $\xi \in SM$,

$$\tilde{\gamma}(t) := \exp(tG(\xi)) = (\gamma(t), \gamma'(t)),$$

where $\gamma : (-\varepsilon, \varepsilon) \to M$ is the *unique* geodesic in M such that $\gamma(0) = \pi\xi$ and $\gamma'(0) = \xi$. Let us see that G is the Reeb field, i.e.,

$$d\pi G = e_0, \quad (\pi^*\nabla)_G e_0 = 0.$$

By definition,

$$(d\pi)_\xi(G(\xi)) = \left.\frac{d}{dt}\right|_{t=0} (\pi \circ \tilde{\gamma}) = \gamma'(0) = \xi.$$

Fix $\xi \in SM$ and let $Y \in \mathfrak{X}(M)$ such that $Y(\gamma(t)) = \gamma'(t)$ for all t. Then $e_0(\tilde{\gamma}(t)) = \pi^* Y(\tilde{\gamma}(t))$, and thus

$$(\pi^*\nabla)_{G(\xi)} e_0 = \pi^* \left(\nabla_{(d\pi)_\xi(G(\xi))} Y\right) = \pi^* \left(\nabla_{\gamma'(0)} Y\right) = 0. \qquad \square$$

The fact that the Reeb vector field coincides with the geodesic vector field of SM will play a crucial role throughout our study.

1.2 Valuations in linear spaces

In this section, we introduce the concept of *valuations*, which is the main focus of our study. We begin with the simplest case: valuations in linear spaces, which form the basis of the *classic valuation theory*. For more information on this topic, one can consult [15].

Let V be a m-dimensional euclidean vector space. We denote by $\mathcal{K} = \mathcal{K}(V)$ the set of convex compact subsets of V, *convex bodies of* V, and endow it with the *Hausdorff metric* d_H defined by

$$d_H(A, B) = \max \left\{ \sup_{a\in A} \inf_{b\in B} \{d(a, b)\}, \sup_{b\in B} \inf_{a\in A} \{d(a, b)\} \right\},$$

where d is the distance in the ambient space V. This metric d_H in $\mathcal{K}(V)$ induces a metric topological structure. We also denote by $\mathcal{K}^{sm}$ the dense subspace of convex subsets A with nonempty interior and smooth boundary, and such that all principal curvatures are strictly positive in ∂A.

The Lebesgue volume function $\mathrm{vol}_V\colon \mathcal{K} \to \mathbb{R}$ is continuous. Also the Minkowski sum $\mathcal{K} \times \mathcal{K} \to \mathcal{K}$ given by

$$A + B = \{a + b\colon a \in A, b \in B\}$$

is a continuous map.

Definition 1.2.1. A *valuation* on V is a $\mathbb{C}$-valued functional φ on $\mathcal{K}(V)$ such that

$$\varphi(A \cup B) = \varphi(A) + \varphi(B) - \varphi(A \cap B)$$

for $A, B, A \cup B \in \mathcal{K}(V)$.

Definition 1.2.2. A valuation $\varphi\colon \mathcal{K}(V) \to \mathbb{C}$ is

- *translation invariant* if
$$\varphi(A + x) = \varphi(A), \quad \forall A \in \mathcal{K}(V),\ \ x \in V;$$
- *G-invariant* with respect to a group acting linearly on V if
$$\varphi(gA) = \varphi(A), \quad A \in \mathcal{K}(V),\ \ g \in G;$$
- *k-homogeneous* if
$$\varphi(\lambda A) = \lambda^k \varphi(A), \quad \forall A \in \mathcal{K}(V),\ \ \forall \lambda \in (0, \infty);$$
- *even* (resp. odd) if $\varphi(-A) = \varphi(A)$ (resp. $\varphi(-A) = -\varphi(A)$) for all $A \in \mathcal{K}(V)$;
- *continuous* if $\varphi\colon (\mathcal{K}(V), d_H) \to (\mathbb{C}, |\cdot|)$ is a continuous map.

The space of continuous invariant translation valuations is denoted by $\mathrm{Val}(V)$, the subspace of $\mathrm{Val}(V)$ of the homogeneous valuations of degree k by $\mathrm{Val}_k(V)$ and the subspace of $\mathrm{Val}(V)$ of even valuations (resp. odd valuations) by $\mathrm{Val}^+(V)$ (resp. $\mathrm{Val}^-(V)$). If G acts linearly on V we denote by Val^G the space of continuous-translation and G-invariant valuations. We say that $\mu \in \mathrm{Val}(V)$ is *smooth* if the map $Gl(V) \to \mathrm{Val}(V)$ given by $g \mapsto \mu \circ g^{-1}$ is smooth. The space of smooth valuations on V forms a dense subspace $\mathrm{Val}^\infty(V) \subset \mathrm{Val}(V)$.

Let $A \in \mathcal{K}(V)$ be a convex body in V. For $r \geq 0$ consider the set
$$A_r := \{x \in V : d_A(A, x) \leq r\},$$
where d_A is the (minimum) distance to the set A. The set A_r is sometimes called the *tube around A of radius* r, and other times the *parallel set* at distance r.

A simple but important remark is that
$$(A_r)_s = (A + r\mathbb{B}^m) + s\mathbb{B}^m = A + (r+s)\mathbb{B}^m = A_{r+s}, \qquad r, s > 0,$$
where $\mathbb{B}^m \subset V$ is the euclidean m-dimensional unit ball. Note also that
$$d_H(A, B) = \min\{r \geq 0\colon A \subset B_r,\ \ B \subset A_r\}.$$
It follows that $A \mapsto A_r$ is an isometry of $(\mathcal{K}, d_H)$.

Theorem 1.2.3 (Steiner's formula)**.** *For every $A \in \mathcal{K}(V)$ and any $r \geq 0$*
$$\mathrm{vol}_V(A_r) = \sum_{i=0}^{m} r^{m-i} \omega_{m-i} \mu_i(A),$$
where $\omega_k = \mathrm{vol}_V(\mathbb{B}^m)$, and $\mu_0, \ldots, \mu_m \in \mathrm{Val}^\infty(V)$ are valuations, called intrinsic volumes.

The Steiner formula is also called *volume tube formula in linear spaces*. The valuations μ_i are, up to scale, to the *Quermassintegrale* introduced by Minkowski. The first and last ones are $\mu_0 = \chi$ and $\mu_m = \mathrm{vol}_V$ while $\mu_{m-1} = \frac{1}{2}$ perimeter. It follows from Steiner's formula that

$$\mu_i(\mathbb{B}^m) = \binom{m}{i}\frac{\omega_m}{\omega_{m-i}}.$$

The chosen normalization of the μ_i makes them independent of the ambient dimension. In other words, if $f\colon \mathbb{R}^n \to \mathbb{R}^N$ is a linear isometry, then $\mu_i(f(A)) = \mu_i(A)$. Moreover, the normalization is chosen such that $\mu_i(A) = \mathrm{vol}_i(A)$ whenever A is contained in a i-dimensional affine subspace of $\mathbb{R}^n$.

For $A \in \mathcal{K}^{sm}(V)$ (smooth convex body) the proof of Theorem 1.2.3 is simple using (cf. e.g. [15])

$$A_r = A \cup \exp_A(\partial A \times [0,r]), \quad \exp_A(x,t) := x + tn_A(x),$$

where $n_A\colon \partial A \to S^{m-1}$ is the Gauss map and $S^{m-1} \subset V$ is the unit standard euclidean sphere. Furthermore, in this case one has the following expression for the intrinsic volumes (cf. e.g. [15])

$$\mu_i(A) = \frac{1}{(m-i)\omega_{m-i}}\int_{\partial A}\sigma_{m-i-1}(k_1,\ldots,k_{m-1})$$

where σ_j is the j-th elementary symmetric polynomial and k_j are the principal curvatures.

For a general convex body A we have the *Crofton formula* (cf. e.g. [15])

$$\mu_k(A) = \binom{m}{k}\frac{\omega_m}{\omega_k\omega_{m-k}}\int_{\mathrm{Gr}_k}\mathrm{vol}_k(\pi_G(A))\,dG, \quad k = 0,\ldots,m-1$$

where Gr_k denotes the space of linear k-dimensional planes in V, i.e, the *grassmannian space*, and dG is *the Haar probability measure* in Gr_k.

Let $\overline{\mathrm{Gr}}_{n-k}$ denote the space of affine planes of dimension $n-k$ in $\mathbb{R}^n$. Then

$$\mu_k(A) = \int_{\overline{\mathrm{Gr}}_{n-k}}\chi(A \cap \overline{H})d\overline{H},$$

where χ is the Euler characteristic, and $d\overline{H}$ is a suitably normalized Haar measure.

Theorem 1.2.4 (Hadwiger's theorem)**.**

$$\mathrm{Val}^{SO(V)} = \langle \mu_0, \ldots, \mu_m \rangle.$$

The linear group $GL(V)$ of invertible linear transformations of V acts on $\mathrm{Val}(V)$ and preserves the homogeneity.

Theorem 1.2.5 (McMullen's theorem+Alesker's Irreducibility Theorem)**.**

$$\mathrm{Val}(V) = \bigoplus_{i=0}^{m} \bigoplus_{\epsilon=\pm} \mathrm{Val}_i^{\epsilon}$$

is the decomposition into $GL(V)$-irreducible representations, i.e, $\mathrm{Val}_i^{\epsilon}$ *has no proper closed $GL(V)$-invariant subspaces.*

There are however interesting $GL(V)$-invariant *dense* subspaces in $\mathrm{Val}(V)$. One of them is the so-called space of smooth valuations $\mathrm{Val}^{\infty}(V)$, consisting of those elements of $\mathrm{Val}(V)$ that behave smoothly under the action of $GL(V)$. Given $A \in \mathcal{K}(V)$, let

$$\mu_A(B) = \mathrm{vol}(A+B) = \int_{\mathbb{R}^m} \chi(A \cap (x-B))dx.$$

Then, assuming ∂A to be smooth and positively curved one has $\mu_A \in \mathrm{Val}^{\infty}(V)$.

Remarkably, the space $\mathrm{Val}^{\infty}(V)$ admits two product structures which turn it into a commutative algebra in two different ways: the Alesker product and the Bernig-Fu convolution. For $\varphi \in \mathrm{Val}^{\infty}(V)$, the Alesker product and the Bernig-Fu convolution of valuations are determined by

$$\mu_A \cdot \varphi\ (B) = \int_{\mathbb{R}^m} \varphi(A \cap (x-B))dx, \quad \mu_A * \varphi\ (B) = \varphi(A+B).$$

Furthermore, observe that $\chi \cdot \varphi = \varphi$ and $\mathrm{vol} * \varphi = \varphi$ for all $\varphi \in \mathrm{Val}^{\infty}(V)$.

1.3 Valuations in riemannian manifolds

The notion of valuation was extended to smooth manifolds by Alesker (cf. [5, 6, 14, 7]). To simplify matters, we focus on riemannian manifolds.

1.3.1 Basic notions

We consider the class of compact sets of positive reach in M, denoted by $\mathcal{R}(M)$. The definition and some basic properties of such sets are recalled in subsection 2.2.2 together with a general construction of the *normal cycle* for $\mathcal{R}(M)$. Let us describe only the normal cycle in the subclass $\mathcal{P}(M) \subset \mathcal{R}(M)$ of compact submanifolds with corners for now. For $A \in \mathcal{P}(M)$, the normal cycle is

$$N(A) = \{(p,v) \in SM : p \in A, \langle v, w\rangle \leq 0\ \forall w \in T_pA\},$$

where

$$T_pA = \left\{\gamma'(0) \in T_pM : \gamma \in C^1([0,1),A), \gamma(0) = p\right\}.$$

For each $A \in \mathcal{P}(M)$, the normal cycle $N(A)$ is a compact Lipschitz $(n-1)$-dimensional submanifold of the sphere bundle SM. Furthermore, $N(A)$ is *legendrian* and *lagrangian* with respect to the contact structure of SM, which means, respectively

$$\int_{N(A)} \rho \wedge \alpha = 0, \quad \forall \rho \in \Omega^{n-2}(SM),$$

and

$$\int_{N(A)} \omega \wedge d\alpha = 0, \quad \forall \omega \in \Omega^{n-3}(SM).$$

In Corollary 2.2.17, we establish this property in the general case of sets of positive reach.

Definition 1.3.1. A *smooth valuation* on M is a $\mathbb{C}$-valued functional $\varphi = [\![\omega, \eta]\!]$ on $\mathcal{R}(M)$ of the form

$$\varphi(A) = \int_{N(A)} \omega + \int_A \eta,$$

where $\omega \in \Omega^{n-1}(SM)$ and $\eta \in \Omega^n(M)$, are complex-valued differential forms. For any subgroup $G \leq \mathrm{Diff}(M)$, we will denote by $\mathcal{V}(M)^G$ the space of G-invariant valuations; i.e. $\mu \in \mathcal{V}(M)$ such that $\mu(gA) = \mu(A)$ for all $A \in \mathcal{R}(M)$ and $g \in G$.

We denote by $\mathcal{B}(M)$ the set of Borel subsets of M. Any smooth valuation may be localized, albeit non-uniquely.

Definition 1.3.2. A *smooth curvature measure* on M is a $\mathbb{C}$-valued functional $\Phi = [\omega, \eta]$ on $\mathcal{R}(M) \times \mathcal{B}(M)$ of the form

$$\Phi(A, U) = \int_{N(A)\cap\pi^{-1}(U)} \omega + \int_{A\cap U} \eta$$

where $\omega \in \Omega^{n-1}(SM)$ and $\eta \in \Omega^n(M)$, are complex-valued differential forms. For any subgroup $G \leq \mathrm{Diff}(M)$, we will denote by $\mathcal{C}(M)^G$ the space of G-invariant curvature measures; i.e. $\Psi \in \mathcal{C}(M)$ such that $\Psi(gA, gU) = \Psi(A, U)$ for all $A \in \mathcal{R}(M)$, $U \in \mathcal{B}(M)$ and $g \in G$.

The *globalization map* is given by

$$\mathrm{glob}\colon \mathcal{C}(M) \longrightarrow \mathcal{V}(M), \quad \Phi \longmapsto \Phi(\cdot, M).$$

Theorem 1.3.3 ([12, 14] and [21])**.** *The vector space $\mathcal{V}(M)$ has an algebraic structure with the so-called Alesker product, and the vector space $\mathcal{C}(M)$ has a module structure over $\mathcal{V}(M)$ such that*

$$\mathrm{glob}\,(\mu \cdot \Phi) = \mu \cdot \mathrm{glob}\,(\Phi)\,, \quad \mu \in \mathcal{V}(M), \Phi \in \mathcal{C}(M).$$

The differential forms defining a valuation are not unique. This difficulty was addressed by Bernig and Bröcker in [18] using the following operator introduced by Rumin.

Definition 1.3.4. Given $\omega \in \Omega^{n-1}(SM)$, there exists $\xi \in \Omega^{n-2}(SM)$ such that

$$D\omega := d(\omega + \alpha \wedge \xi), \tag{1.16}$$

is a multiple of α. The unique n-form $D\omega$ satisfying (1.16) is the *Rumin differential* of ω (cf. [54]).

Proposition 1.3.5 ([18]).

$$\ker[\![\cdot,\cdot]\!] = \{(\omega,\eta) : D\omega + \pi^*\eta = 0, \pi_*\omega = 0\},$$

$$\ker[\cdot,\cdot] = \{(\omega,0) : \omega \in (\alpha, d\alpha)\},$$

where $(\alpha, d\alpha)$ is the ideal generated by α and $d\alpha$ on $\Omega^\bullet(SM)$.

The following class of manifolds is central to our study.

Definition 1.3.6. Let M be a riemannian manifold and $G \leq \mathrm{Isom}(M)$. We say that the pair (M, G) is *isotropic* if G acts transitively over SM; i.e, given ξ and ξ' in the sphere bundle SM, there exist $g \in G$ such that $\xi' = g\xi$.

In an isotropic pair (M, G), given $\mu \in \mathcal{V}(M)^G$ and $\Phi \in \mathcal{C}(M)^G$, may be represented in terms of *non*-invariant differential forms. However, according to [21, Proposition 2.6], for isotropic pairs, we can always find G-invariant forms on SM representing them. Therefore the vector spaces $\mathcal{V}(M)^G$ and $\mathcal{C}(M)^G$ are *finite-dimensional*.

If $f\colon M_1 \to M_2$ is an immersion, then there exist pull-back maps

$$f^*\colon \mathcal{C}(M_2) \longrightarrow \mathcal{C}(M_1), \quad f^*\colon \mathcal{V}(M_2) \longrightarrow \mathcal{V}(M_1)$$

such that

$$(f^*\Phi)(A,B) = \Phi(f(A), f(B)),$$
$$(f^*\mu)(A) = \mu(f(A)).$$

Furthermore, f^* is an algebra and module homomorphism. These pulled-back objects can be represented, once again, by differential forms. This was explicitly proved in [34, Proposition 4.7]. From a broader perspective, the pull-back map was analyzed in [9].

We also introduce the standard notation $f_*\colon \mathcal{V}(M_1) \to \mathcal{V}(M_2)$ for the push-forward of valuations in the specific case where f is a *proper* submersion

$$(f_*\mu)(A) = \mu(f^{-1}(A)).$$

In [9], a representation of f_* in differential forms is provided. We refer to [11] for further developments about the push-forward of valuations.

1.3.2 Federer valuations

To conclude this section, we introduce the *Federer valuations*, which are globalizations of the *Federer curvature measures* (cf. [21]).

Definition 1.3.7. For any riemannian manifold M^n and $0 \leq i \leq n-1$, consider $\kappa_i \in \Omega^{n-1}(SM)$ such that for $X_1, \ldots, X_{n-1} \in \mathfrak{X}(SM)$

$$\kappa_i(X_1,\ldots,X_{n-1}) := \frac{1}{i!(n-i-1)!} \sum_{\sigma \in \mathcal{S}_{n-1}} |\sigma| \det\left(e_0, X^H_{\sigma(1)}, \ldots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \ldots, X^V_{\sigma(n-1)}\right)$$

$$:= \sum_{\sigma \in \mathcal{S}_{i,n-i-1}} |\sigma| \det\left(e_0, X^H_{\sigma(1)}, \ldots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \ldots, X^V_{\sigma(n-1)}\right),$$

where $\mathcal{S}_{n-1}$ represents the set of permutations of $\{1, \ldots, n-1\}$ and

$$\mathcal{S}_{i,n-i-1} := \{\varepsilon \in \mathcal{S}_{n-1} : \varepsilon(1) < \cdots < \varepsilon(i), \quad \varepsilon(i+1) < \cdots < \varepsilon(n-1)\}.$$

The i-th *Federer curvature measure* is

$$\Delta_i^M := \frac{1}{(n-i)\omega_{n-i}}[\kappa_i, 0],$$

and we call

$$\nu_i^M := \mathrm{glob}(\Delta_i^M, 0)$$

the i-th *Federer valuation.*

Remark 1.3.8. The Federer curvature measures and valuations are isometry invariant.

Proposition 1.3.9. *In an orthonormal moving frame* $e_0, e_1, \ldots, e_{n-1}$*, we have*

$$\begin{aligned}\kappa_i &= \frac{1}{i!(n-i-1)!} \sum_{\varepsilon \in \mathcal{S}_{n-1}} |\varepsilon| \theta_{\varepsilon(1)} \wedge \cdots \wedge \theta_{\varepsilon(i)} \wedge \omega_{\varepsilon(i+1),0} \wedge \cdots \wedge \omega_{\varepsilon(n-1),0} \\ &= \sum_{\varepsilon \in \mathcal{S}_{i,n-i-1}} |\varepsilon| \theta_{\varepsilon(1)} \wedge \cdots \wedge \theta_{\varepsilon(i)} \wedge \omega_{\varepsilon(i+1),0} \wedge \cdots \wedge \omega_{\varepsilon(n-1),0}.\end{aligned}$$

Proof. Since

$$\det = \theta_0 \wedge \theta_1 \wedge \cdots \wedge \theta_{n-1},$$

using the relation $i_{e_0}\theta_j = \delta_{j,0}$ and the identity

$$\theta_j\left((\pi^*\nabla)_{X_k} e_0\right) = \langle e_j, (\pi^*\nabla)_{X_k} e_0 \rangle = \omega_{j,0}(X_k),$$

we have

$$
\begin{aligned}
&\sum_\sigma |\sigma| \det \left(e_0, X^H_{\sigma(1)}, \dots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \dots, X^V_{\sigma(n-1)}\right)\\
&= \sum_\sigma |\sigma| \theta_0 \wedge \cdots \wedge \theta_{n-1} \left(\xi, X^H_{\sigma(1)}, \dots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \dots, X^V_{\sigma(n-1)}\right)\\
&= \sum_\sigma |\sigma| \theta_1 \wedge \cdots \wedge \theta_{n-1} \left(X^H_{\sigma(1)}, \dots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \dots, X^V_{\sigma(n-1)}\right)\\
&= \sum_\sigma |\sigma| \left(\sum_\varepsilon |\varepsilon| \theta_{\epsilon(1)} \otimes \cdots \otimes \theta_{\epsilon(n-1)} \left(X^H_{\sigma(1)}, \dots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \dots, X^V_{\sigma(n-1)}\right)\right)\\
&= \sum_\epsilon |\epsilon| \left(\sum_\sigma |\sigma| \theta_{\epsilon(1)} \otimes \cdots \otimes \theta_{\epsilon(n-1)} \left(X^H_{\sigma(1)}, \dots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \dots, X^V_{\sigma(n-1)}\right)\right)\\
&= \sum_\epsilon |\epsilon| \left(\sum_\sigma |\sigma| \theta_{\epsilon(1)}(X^H_{\sigma(1)}) \cdots \theta_{\epsilon(n-1)}(X^V_{\sigma(n-1)})\right)\\
&= \sum_\epsilon |\epsilon| \left(\sum_\sigma |\sigma| \theta_{\epsilon(1)}(X_{\sigma(1)}) \cdots \theta_{\epsilon(i)}(X_{\sigma(i)}) \omega_{\epsilon(i+1),0}(X_{\sigma(i+1)}) \cdots \omega_{\epsilon(n-1),0}(X_{\sigma(n-1)})\right)\\
&= \sum_\epsilon |\epsilon| \left(\sum_\sigma |\sigma| \theta_{\epsilon(1)} \otimes \cdots \otimes \theta_{\epsilon(i)} \otimes \omega_{\epsilon(i+1),0} \otimes \cdots \otimes \omega_{\epsilon(n-1),0}(X_{\sigma(1)}, \dots, X_{\sigma(n-1)})\right)\\
&= \sum_\epsilon |\epsilon| \theta_{\epsilon(1)} \wedge \cdots \wedge \theta_{\epsilon(i)} \wedge \omega_{\epsilon(i+1),0} \wedge \cdots \wedge \omega_{\epsilon(n-1),0}(X_1, \dots, X_{n-1}). \qquad \square
\end{aligned}
$$

Remark 1.3.10. If $A \subset M$ is a smooth domain, then $\Delta_i(A, \cdot)$ denotes a measure on ∂A. Its density relative to vol_{n-1} is $((n-i)\omega_{n-i})^{-1}$ times the $(n-i-1)$-th elementary symmetric function of the principal curvatures of ∂A. For $M = \mathbb{R}^n$, $\mathrm{glob}(\Delta_i) = \mu_i$.

1.4 Space forms

Our study takes place in the so-called *space forms.* More precisely we will work in the real, complex, and quaternionic space forms, which are the main examples of isotropic spaces. Further details on this topic can be found in [46, 47].

1.4.1 Real space forms

We will denote by $\mathbb{S}^m_\lambda$ the (unique up to isometry) m-dimensional simply connected riemannian manifold of constant sectional curvature λ. If $\lambda > 0$, it corresponds to the sphere of radius $r = \frac{1}{\sqrt{\lambda}}$. For $\lambda < 0$, it represents the real hyperbolic space. Finally, when $\lambda = 0$, it represents the standard euclidean space. These spaces of constant sectional curvature are also referred to as *real space forms.* We denote $G_{\lambda,\mathbb{R}} = \mathrm{Isom}(\mathbb{S}^m_\lambda)$ its full isometry group.

Proposition 1.4.1. *The curvature tensor of* $\mathbb{S}^m_\lambda$ *is given by* $R = \lambda R_1$ *where*

$$R_1(W,X;Y,Z) := g(W,Y)g(X,Z) - g(W,Z)g(X,Y).$$

The curvature tensor has a simple expression in a moving frame.

Proposition 1.4.2. *Let* $e_0,\ldots,e_{n-1}$ *be a moving frame on* $\mathbb{S}^m_\lambda$. *Then for* $0 \le i,j \le m-1$

$$\Omega_{i,j} = \lambda \theta_i \wedge \theta_j.$$

Proof. By definition of the curvature 2-forms $\Omega_{i,j}$ (see (1.9))

$$\Omega_{i,j}(X,Y) = R(e_i,e_j;d\pi X, d\pi Y).$$

The concrete description of the curvature tensor R for real space forms shown in Proposition 1.4.1 yields the result:

$$\begin{aligned}\Omega_{i,j}(X,Y) &= R(e_i,e_j;d\pi X,d\pi Y)\\ &= \lambda\left(g(e_i,d\pi X)g(e_j,d\pi Y) - g(e_i,d\pi Y)g(e_j,d\pi X)\right)\\ &= \lambda\left(\theta_i(X)\theta_j(Y) - \theta_i(Y)\theta_j(X)\right) = \lambda\theta_i\wedge\theta_j(X,Y).\end{aligned}$$ □

1.4.2 Complex space forms

Kähler manifolds have been the object of recent study in valuation theory and integral geometry (cf. [17, 20, 21, 22, 34]).

Definition 1.4.3. A *hermitian manifold* is a riemannian manifold M^{2n} endowed with an endomorphism $J\colon TM \to TM$, such that $J^2 = -\operatorname{id}$ and satisfying

$$g(JX,JY) = g(X,Y), \quad \forall X,Y \in \mathfrak{X}(M).$$

The endomorphism J is called the *almost complex structure* of M and g its *hermitian metric.* If the 2-form $F \in \Omega^2(M)$ defined by

$$F(X,Y) := g(JX,Y), \quad X,Y \in \mathfrak{X}(M)$$

is closed, i.e, $dF = 0$, we say that M is a *Kähler manifold* with *Kähler form* F.

The condition $dF = 0$ is equivalent to $\nabla J = 0$, where

$$(\nabla_X J)Y := \nabla_X(JY) - J\nabla_X Y, \quad X,Y \in \mathfrak{X}(M). \tag{1.17}$$

We will denote by $\mathbb{C}P^n_\lambda$ the (unique up to isometry) n-dimensional simply connected Kähler manifold of constant holomorphic curvature 4λ. If $\lambda > 0$, it corresponds to the complex projective space endowed with the Fubini-Study metric. For $\lambda < 0$, it represents the complex hyperbolic space equipped with the Bergman metric. Finally, $\lambda = 0$ represents the hermitian standard space $\mathbb{C}^n$. These spaces of constant holomorphic curvature are also referred to as *complex space forms.* We denote by $G_{\lambda,\mathbb{C}} = \operatorname{Isom}(\mathbb{C}P^n_\lambda)$ its group of holomorphic isometries which for $\lambda \neq 0$ coincides with the full isometry group.

Proposition 1.4.4 ([40, Lemma 6.9])**.** *The curvature tensor of* $\mathbb{C}P^n_\lambda$ *is given by* $R = \lambda G$, *where*

$$\begin{aligned} G(W, X; Y, Z) := & \, g(W; Y)g(X; Z) - g(W; Z)g(X; Y) \\ & + g(JW; Y)g(JX; Z) - g(JW; Z)g(JX; Y) \\ & + 2g(JW; X)g(JY; Z). \end{aligned}$$

Remark 1.4.5. The tensor G is called the *Gray double form* in [22, (42)].

Previously, we introduced the concept of an orthonormal moving frame for any riemannian manifold. Within the context of Kähler manifolds, we encounter the almost complex structure J. Consequently, we introduce a new type of orthonormal moving frames.

Definition 1.4.6. Let M^n be a Kähler manifold and $x \in M$. A *hermitian basis* of T_xM is an orthonormal basis $\{e_0, \ldots, e_{2n-1}\}$ of T_xM such that

$$e_{2i+1} = Je_{2i}, \quad 0 \leq i \leq n-1.$$

A moving frame $e_0, \ldots, e_{2n-1}$, locally defined on SM will be called *hermitian* if $\{e_0(\xi), \ldots, e_{2n-1}(\xi)\}$ is a hermitian basis for all ξ.

The following well-known result is a remarkable property intertwining the Levi-Civita connection and the almost complex structure.

Proposition 1.4.7. *Let* M *be a Kähler manifold,* $\xi \in SM$ *and* $\{e_0, \ldots, e_{2n-1}\}$ *a hermitian basis of* T_xM, $x = \pi\xi$. *Consider* $e_0(t), \ldots, e_{2n-1}(t)$ *the parallel transport of* $\{e_0, \ldots, e_{2n-1}\}$ *along the curve* $\gamma(t) = \exp(t\xi)$. *Then,* $\{e_0(t), \ldots, e_{2n-1}(t)\}$ *is a hermitian basis of* $T_{\gamma(t)}M$ *for each* t.

Proof. For all $0 \leq j \leq 2n-1$

$$\nabla_{e_0(t)} e_j(t) = 0, \quad \forall t.$$

From the compatibility condition (1.17)

$$\nabla_{e_0(t)} Je_{2i}(t) = J(\nabla_{e_0(t)} e_{2i}(t)) = 0.$$

Then, we have that $e_{2i+1}(t)$ and $Je_{2i}(t)$ are parallel and agree in $t = 0$, whence $e_{2i+1}(t) = Je_{2i}(t)$, for all t. □

Proposition 1.4.8. *Given* $\xi \in S\mathbb{C}P^n_\lambda$ *we have*

$$R(e_0, X)e_0 = 4\lambda X, \quad X \in \langle Je_0 \rangle, \tag{1.18}$$

$$R(e_0, X)e_0 = \lambda X, \quad X \in \langle e_0, Je_0 \rangle^\perp. \tag{1.19}$$

Proof. We prove only (1.18) because (1.19) is analogous. Let $X = aJe_0$ for some $a \in \mathbb{R}$. Since

$$\begin{aligned} G(e_0, X; e_0, Z) &= g(e_0, e_0)g(X, Z) - g(e_0, Z)g(X, e_0) \\ &\quad + g(Je_0, e_0)g(JX, Z) - g(Je_0, Z)g(JX, e_0) + 2g(Je_0, X)g(Je_0, Z) \\ &= g(X, Z) + g(Je_0, Z)g(X, Je_0) + 2g(Je_0, X)g(Je_0, Z) \\ &= g(X, Z) + 3ag(Je_0, Z) \\ &= 4g(X, Z) \\ &= 4(i_X g)(Z), \end{aligned}$$

equation (1.18) follows from Proposition 1.4.4. □

Corollary 1.4.9. *Let $e_0, \ldots, e_{2n-1}$ be a parallel hermitian moving frame in $\mathbb{C}P^n_\lambda$ relative to ξ. Then, for $\gamma(t) = \exp(t\xi)$ we have*

$$\begin{aligned} \mathcal{L}_T\theta_i &= \omega_{i,0}, & 1 \leq i \leq 2n-1 \\ \mathcal{L}_T\omega_{1,0} &= -4\lambda\theta_1, & \\ \mathcal{L}_T\omega_{j,0} &= -\lambda\theta_j, & 2 \leq j \leq 2n-1 \end{aligned}$$

at every point $(\gamma(t), \gamma'(t)) \in SM$.

Proof. This is a particular case of Corollary 1.1.6 using Proposition 1.4.8. □

1.4.3 Quaternionic space forms

We conclude this overview of the fundamental concepts of space forms with the quaternionic case. More comprehensive discussions about these manifolds can be found in [49]. For a more focused investigation on volume and area tube formulas, we refer to [38], an article we will reinterpret from our framework in the final chapter. Lastly, we particularly mention A. Bernig and G. Solanes' contributions in the articles [23, 24], concerning the quaternionic plane in the light of modern valuation theory.

Definition 1.4.10. An *almost quaternionic manifold* is a riemannian manifold M^{4n} such that every point $x \in M$ has an open neighborhood U endowed with three bundle endomorphisms $J_i \colon TU \to TU$, $1 \leq i \leq 3$, such that $J_i^2 = -I$, $J_1J_2 = -J_2J_1 = J_3$ and satisfying

$$g(J_iX, J_iY) = g(X, Y), \quad \forall X, Y \in \mathfrak{X}(U).$$

In this case, we say that g is a *quaternionic metric*. If there exist $a_{j,k} \in \Omega^1(U)$ such that

$$\nabla_X J_i = \sum_{j=1}^{3} a_{i,j}(X)J_j,$$

fulfilling $a_{j,i} = -a_{i,j}$, we will say that M is a *quaternionic Kähler manifold.*

We say that an almost quaternionic manifold M has constant quternionic sectional curvature a if every 2-plane $\langle X, Y\rangle$, with $X \in TM\backslash\{0\}$ arbitrary and $Y \in \langle X, J_1X, J_2X, J_3X\rangle$, has sectional curvature a. All such manifolds are locally isometric [42].

We will denote by $\mathbb{H}P^n_\lambda$ the (unique up to isometry) n-dimensional simply connected quaternionic Kähler manifold of constant quaternionic sectional curvature 4λ. If $\lambda > 0$, it corresponds to the quaternionic projective space. For $\lambda < 0$, it represents the quaternionic hyperbolic space. Finally, $\lambda = 0$ represents the quaternionic standard space $\mathbb{H}^n$. These spaces of constant curvature are also referred to as *quaternionic space forms.* We denote by $G_{\lambda,\mathbb{H}} = \mathrm{Isom}(\mathbb{H}P^n_\lambda)$ its full isometry group.

Proposition 1.4.11 ([49])**.** *The curvature tensor in $\mathbb{H}P^n_\lambda$ is given by $R = \lambda R_1$, where*

$$\begin{aligned}R_1(W,X;Y,Z) :=\, & g(W,Y)g(X,Z) - g(W,Z)g(X,Y)\\ & + g(J_1W,Y)g(J_1X,Z) - g(J_1W,Z)g(J_1X,Y) + 2g(J_1W,X)g(J_1Y,Z)\\ & + g(J_2W,Y)g(J_2X,Z) - g(J_2W,Z)g(J_2X,Y) + 2g(J_2W,X)g(J_2Y,Z)\\ & + g(J_3W,Y)g(J_3X,Z) - g(J_3W,Z)g(J_3X,Y) + 2g(J_3W,X)g(J_3Y,Z).\end{aligned}$$

The following notion will be useful for our purposes.

Definition 1.4.12. Let M^{4n} be a quaternionic Kähler manifold and $x \in M$. A *partially quaternionic basis* of T_xM is an orthonormal basis $\{e_0, \ldots, e_{4n-1}\}$ of T_xM such that

$$e_j \in \langle J_1e_0, J_2e_0, J_3e_0\rangle,$$

for $1 \leq j \leq 3$.

Note that *no* assumption is made on the vectors $e_i, i > 3$ beyond orthonormality in the quaternionic case. The following result naturally arises when comparing it with Proposition 1.4.7

Proposition 1.4.13. *Let M be a quaternionic manifold, $\xi \in SM$ and $\{e_0, \ldots, e_{4n-1}\}$ a partially quaternionic basis of T_xM, $x = \pi\xi$. Consider $e_0(t), \ldots, e_{4n-1}(t)$ the parallel transport of $\{e_0, \ldots, e_{4n-1}\}$ along the curve $\gamma(t) = \exp(t\xi)$. Then, $\{e_0(t), \ldots, e_{4n-1}(t)\}$ is a partially quaternionic basis of $T_{\gamma(t)}M$ for each t.*

Proof. Consider $e_0(t), e_1(t), e_2(t), e_3(t)$ the parallel transport along $\gamma(t) = \exp(t\xi)$ of $e_0 = \xi, e_1 = J_1(\xi), e_2 = J_2(\xi), e_3 = J_3(\xi)$. For $1 \leq i \leq 3$, let us look for functions $b_{i,j}(t)$ such that

$$\tilde{e}_i(t) := \sum_{j=1}^{3} b_{i,j}(t)J_je_0(t),$$

are parallel. Given that

$$\nabla_{e_0(t)}J_je_0(t) = \sum_{k=1}^{3} a_{j,k}(t)J_ke_0(t),$$

the $\tilde{e}_i(t)$ are parallel if and only if the $b_{i,j}(t)$ are a solution of

$$0 = \nabla_{e_0(t)}\tilde{e}_i(t) = \sum_{j,k=1}^{3} b'_{i,j}(t)J_j e_0(t) + b_{i,j}(t)a_{j,k}(t)J_k e_0(t),$$

Since this is a linear first-order differential system, such solutions exist with any given initial condition. Taking $b_{i,j}(0) = \delta_{i,j}$ we have $\tilde{e}_i(0) = e_i(0)$. Consequently, since both are parallel, we deduce $e_i(t) = \tilde{e}_i(t)$. □

Proposition 1.4.14. *Given $e_0 \in SM = S\mathbb{H}P^n_\lambda$, one has*

$$R(e_0, X)e_0 = 4\lambda X, \quad X \in \langle J_1e_0, J_2e_0, J_3e_0\rangle, \tag{1.20}$$

$$R(e_0, X)e_0 = \lambda X, \quad X \in \langle e_0, J_1e_0, J_2e_0, J_3e_0\rangle^\perp. \tag{1.21}$$

Proof. □

Proof. We only prove (1.20) because (1.21) is analogous. Let

$$X = a_1J_1e_0 + a_2J_2e_0 + a_3J_3e_0, \qquad a_1, a_2, a_3 \in \mathbb{R}.$$

From Proposition 1.4.11 $R = \lambda R_1$, and for any $Z \in T_x\mathbb{H}P^n_\lambda$, $x = \pi\xi$, $\xi \in SM$

$$\begin{aligned}
R_1(e_0, X; e_0, Z) = \,& g(e_0, e_0)g(X, Z) - g(e_0, Z)g(X, e_0)\\
&+ g(J_1e_0, e_0)g(J_1X, Z) - g(J_1e_0, Z)g(J_1X, e_0) + 2g(J_1e_0, X)g(J_1e_0, Z)\\
&+ g(J_2e_0, e_0)g(J_2X, Z) - g(J_2e_0, Z)g(J_2X, e_0) + 2g(J_2e_0, X)g(J_2e_0, Z)\\
&+ g(J_3e_0, e_0)g(J_3X, Z) - g(J_3e_0, Z)g(J_3X, e_0) + 2g(J_3e_0, X)g(J_3e_0, Z)\\
= \,& g(X, Z)\\
&- g(J_1e_0, Z)g(J_1X, e_0) + 2g(J_1e_0, X)g(J_1e_0, Z)\\
&- g(J_2e_0, Z)g(J_2X, e_0) + 2g(J_2e_0, X)g(J_2e_0, Z)\\
&- g(J_3e_0, Z)g(J_3X, e_0) + 2g(J_3e_0, X)g(J_3e_0, Z)\\
= \,& g(X, Z)\\
&+ g(J_1e_0, Z)g(X, J_1e_0) + 2g(J_1e_0, X)g(J_1e_0, Z)\\
&+ g(J_2e_0, Z)g(X, J_2e_0) + 2g(J_2e_0, X)g(J_2e_0, Z)\\
&+ g(J_3e_0, Z)g(X, J_3e_0) + 2g(J_3e_0, X)g(J_3e_0, Z)\\
= \,& g(X, Z) + 3g(J_1e_0, X)g(J_1e_0, Z) + 3g(J_2e_0, X)g(J_2e_0, Z) + 3g(J_3e_0, X)g(J_3e_0, Z)\\
= \,& g(X, Z)\\
&+ 3a_1g(J_1e_0, Z) + 3a_2g(J_2e_0, Z) + 3a_3g(J_3e_0, Z)\\
= \,& g(X, Z) + 3g(a_1J_1e_0 + a_2J_2e_0 + a_3J_3e_0, Z)\\
= \,& 4g(X, Z)\\
= \,& 4(i_Xg)(Z). \quad \square
\end{aligned}$$

Corollary 1.4.15. *Let* $e_0, \ldots, e_{4n-1}$ *be a parallel quaternionic moving frame in* $\mathbb{H}P^n_\lambda$ *relative to* ξ*. Then, for* $\gamma(t) = \exp(t\xi)$ *we have*

$$\begin{aligned} \mathcal{L}_T\theta_i &= \omega_{i,0}, && 0 \le i \le 4n-1 \\ \mathcal{L}_T\omega_{j,0} &= -4\lambda\theta_j, && 1 \le j \le 3 \\ \mathcal{L}_T\omega_{k,0} &= -\lambda\theta_j, && 4 \le k \le 4n-1, \end{aligned}$$

at every point $(\gamma(t), \gamma'(t)) \in SM$*.*

Proof. This is a particular case of Corollary 1.1.6 using Proposition 1.4.14. □

1.5 $\mathfrak{sl}_2$-representation theory

Here, we review some fundamental notions of representation theory and $\mathfrak{sl}_2$. We include some details for the sake of completeness and future reference but refer the reader to [57] and [35] for more information.

1.5.1 Representation theory

Definition 1.5.1. A *Lie algebra* $\mathfrak{g}$ is defined as a vector space over a field $\mathbb{K}$, endowed with a skew-symmetric bilinear map

$$[\,,\,]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g},$$

fulfilling the Jacobi identity

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0.$$

We say that a vector subspace $\mathfrak{h} \subset \mathfrak{g}$ is a *Lie subalgebra* if $[x, y] \in \mathfrak{h}, \forall x, y \in \mathfrak{h}$.

Example 1.5.2. If V is a vector space over $\mathbb{K}$, then the space $\mathfrak{g} = \mathrm{End}(V)$, consisting of all endomorphisms of V, is a Lie algebra with $[A, B] = A \circ B - B \circ A$. Similarly, The space $\mathfrak{gl}_n(\mathbb{K})$, which consists of all $n \times n$ matrices over $\mathbb{K}$, is a Lie algebra, and $\mathfrak{sl}_n(\mathbb{K})$, the subspace of $\mathfrak{gl}_n(\mathbb{K})$ consisting of matrices with trace zero, forms a Lie subalgebra of $\mathfrak{gl}_n(\mathbb{K})$.

Definition 1.5.3. A *representation* of a Lie algebra $\mathfrak{g}$ is a linear map

$$\rho\colon \mathfrak{g} \longrightarrow \mathrm{End}(V),$$

such that

$$[x, y] = [\rho(x), \rho(y)], \quad x, y \in \mathfrak{g}.$$

Clearly, the restriction of ρ to a subalgebra $\mathfrak{h}$ is also a representation.

Given a Lie algebra $\mathfrak{g}$, a $\mathfrak{g}$-module consists of a vector space V endowed with a bilinear map

$$\mathfrak{g} \times V \to V, \quad (x, v) \mapsto x \cdot v,$$

such that $[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v)$ for all $x, y \in \mathfrak{g}$, and $v \in V$.

If $\rho\colon \mathfrak{g} \to \operatorname{End}(V)$ is a representation, then V has a $\mathfrak{g}$-module structure defined by

$$\mathfrak{g} \times V \longrightarrow V, \quad (x, v) \longmapsto \rho(x)(v).$$

Conversely, every $\mathfrak{g}$-module V, has a corresponding representation given by

$$\rho\colon \mathfrak{g} \longrightarrow \operatorname{End}(V), \quad \rho(x)(v) = x \cdot v.$$

It is therefore customary to identify $\mathfrak{g}$-modules and representations of $\mathfrak{g}$.

Definition 1.5.4. A $\mathfrak{g}$-module V (or the corresponding representation) is called *irreducible* if the only $\mathfrak{g}$-invariant subspaces of V are 0 and V.

Theorem 1.5.5 (Schur's Lemma)**.** *If V is an irreducible $\mathfrak{g}$-module over $\mathbb{C}$, and $f \in \operatorname{End}(V)$ satisfies $f(xv) = xf(v)$ for all $x \in \mathfrak{g}, v \in V$, then $f = c \cdot \operatorname{Id}$ for some $c \in \mathbb{C}$.*

1.5.2 Irreducible $\mathfrak{sl}_2(\mathbb{C})$-representations

We construct a standard model for the finite-dimensional irreducible representations of $\mathfrak{sl}_2 \equiv \mathfrak{sl}_2(\mathbb{C})$, the space of two-dimensional matrices with null trace. The matrices X, Y, H given by

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{1.22}$$

form a basis of $\mathfrak{sl}_2$. The Lie bracket in this basis is simply

$$[X, Y] = H, \quad [H, X] = 2X, \quad [H, Y] = -2Y. \tag{1.23}$$

A triple of endomorhpsisms X, Y, H satisfying (1.23) is sometimes called an $\mathfrak{sl}_2$-*triple*.

We will denote

$$V^{(m)} := \langle x^k y^{m-k} : 0 \leq k \leq m \rangle.$$

This space of homogeneous polynomials plays a crucial role in this work and, therefore, deserves special attention. The distinguishing characteristic of $V^{(m)}$ is its structure as an $\mathfrak{sl}_2(\mathbb{C})$-module as we show next.

Let $X, Y, H \in \operatorname{End}(V^{(m)})$ be given by

$$X = x\frac{\partial}{\partial y}, \quad Y = y\frac{\partial}{\partial x}, \quad H = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}.$$

It is straighforward but tedious to check that these operators satisfy (1.23). Therefore, they induce an $\mathfrak{sl}_2$-module structure on $V^{(m)}$ and also on the whole $\mathbb{C}[x, y]$. We observe that $H = [X, Y]$ and is diagonalizable on $V^{(m)}$ with eigenvalues

$$\operatorname{spec}(H) = \{2k - m : 0 \leq k \leq m\},$$

because of

$$H(x^k y^{m-k}) = (2k-m)x^k y^{m-k}.$$

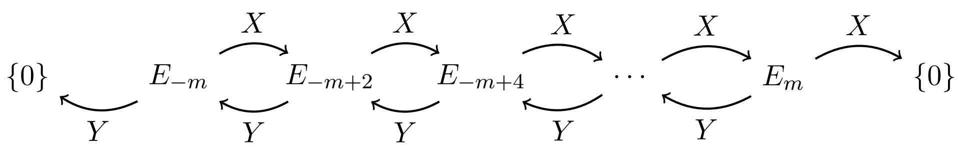
In particular, each eigenspace E_{2k-m} is one-dimensional for $0 \leq k \leq m$ and

$$E_{2k-m} = \mathbb{C} \cdot x^k y^{m-k}.$$

It is trivial to check

$$X(E_{2k-m}) \subset E_{2k-m+2}, \quad Y(E_{2k-m}) \subset E_{2k-m-2}, \quad 0 \leq k \leq m. \tag{1.24}$$

The following diagram represents this situation:



In particular, X and Y are nilpotent. Note also that for any $\pi \in E_{-m}$ the X-orbit $\{X^i(\pi) \colon 0 \leq i \leq m\}$ spans $V^{(m)}$. Such an element $\pi \in E_{-m}$ is called *Y-primitive.*

The proof of the following fundamental facts can be found for instance in [35, §11.1]

Proposition 1.5.6. *i) The $\mathfrak{sl}_2$-representations $V^{(m)}$ are irreducible, for all $m \in \mathbb{N}$.*

ii) If V is an m-dimensional irreducible representation of $\mathfrak{sl}_2$ we have $V \cong V^{(m-1)}$ as $\mathfrak{sl}_2$-modules.

If V is a $\mathfrak{sl}_2$-representation with $\dim V < \infty$, it can be decomposed as a finite sum

$$V = \bigoplus_{j=0}^{k} W_j$$

where each W_j is a subrepresentation isomorphic to some $V^{(m_j)}$. It follows that $v \in V$ can be expressed as

$$v = \sum_{j=0}^{k} \sum_{i} X^i(\pi_j)$$

where each π_j is Y-primitive in W_j. This is the so-called *Lefschetz decomposition* of v.

Chapter 2

Tube formulas

In this chapter, we delve into the study of *tube formulas*, the core of the present work.

We begin by examining tube formulas for smooth translation-invariant valuations in linear spaces, which are described in terms of the convolution of valuations. As an application we compute the tube formulas in $\mathrm{Val}^{U(n)}$ using the $\mathfrak{sl}_2$-module structure introduced in [20]. As a result of independent interest, we obtain the Lefschetz decomposition of the hermitian intrinsic volumes.

Then we move to curved spaces and prove that, for any smooth valuation μ on a riemannian manifold, the value of μ on a tube of radius t is determined by a smooth valuation. An analogous result is established for smooth curvature measures. To accomplish this, we first study certain operators on valuations and curvature measures spaces, and then we analyze the geometric properties of tubes surrounding sets of positive reach.

2.1 Tube formulas in linear spaces

The linear case is the paradigm for developing the tubular operator in riemannian manifolds.

Next, we introduce the *tubular operator*, which assigns the tube formula to each valuation. Additionally, we present the *derivative operator* to account for variations in terms of the radius.

Definition 2.1.1. Let V be an m-dimensional euclidean vector space. Given $t \geq 0$, let $\mathbf{T}_t \colon \mathrm{Val}^\infty(V) \to \mathrm{Val}^\infty(V)$ be given by

$$(\mathbf{T}_t\mu)(A) = \mu(A + t\mathbb{B}^m) = (\mu_{t\mathbb{B}^m} * \mu)(A) \qquad A \in \mathcal{K}(V), \tag{2.1}$$

where $\mathbb{B}^m$ is the unit ball and $\mu_{t\mathbb{B}^m}$ is the valuation given by

$$\mu_{t\mathbb{B}^m}(A) = \mathrm{vol}_V(A + t\mathbb{B}^m).$$

We will call $\mathbf{T}_t$ the *tubular operator*. Let also $\partial\colon \mathrm{Val}^\infty(V) \to \mathrm{Val}^\infty(V)$ be the *derivative* operator given by

$$\partial\mu := \left.\frac{d}{dt}\right|_{t=0} \mathbf{T}_t\mu. \tag{2.2}$$

The operator ∂ has sometimes been denoted by Λ in the literature, but following [20] we reserve the symbol Λ for a certain normalization of ∂ (see (2.8)).

Proposition 2.1.2. *The maps $\mathbf{T}_t$ and ∂ are well defined and*

$$\partial\mu = \mathrm{area}_V * \mu, \quad \mu \in \mathrm{Val}^\infty(V).$$

Proof. Let $\mu \in \mathrm{Val}^\infty(V)$. We need to verify that $\mathbf{T}_t\mu$ and $\partial\mu$ belong to $\mathrm{Val}^\infty(V)$. Since $\mathbf{T}_t\mu$ is, by (2.1), the convolution of two valuations, we have $\mathbf{T}_t\mu$ is another valuation. Now let us check that $\partial\mu$ exists; i.e. that $\mathbf{T}_t\mu$ is differentiable at $t = 0$. Steiner's formula tells us

$$\mu_{t\mathbb{B}^m} = \mathrm{vol}(\cdot + t\mathbb{B}^m) = \sum_{i=0}^{m} \omega_{m-i} t^{m-i} \mu_i.$$

Thus

$$\begin{aligned}\partial\mu &= \left.\frac{d}{dt}\right|_{t=0} \mathbf{T}_t\mu = \left.\frac{d}{dt}\right|_{t=0} (\mu_{t\mathbb{B}^m} * \mu) \\ &= \left(\left.\frac{d}{dt}\right|_{t=0} \mu_{t\mathbb{B}^m}\right) * \mu = \omega_1 \mu_{m-1} * \mu = 2\mu_{m-1} * \mu = \mathrm{area} * \mu.\end{aligned}$$ □

Proposition 2.1.3. *The tubular operator satisfies the following main properties.*

i)
$$\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s = \mathbf{T}_s \circ \mathbf{T}_t. \tag{2.3}$$

ii)
$$\frac{d}{dt}\mathbf{T}_t\mu = \mathbf{T}_t\partial\mu = \partial\mathbf{T}_t\mu. \tag{2.4}$$

iii)
$$\partial^i\mu = \left.\frac{d^i}{dt^i}\right|_{t=0} \mathbf{T}_t\mu. \tag{2.5}$$

iv)
$$\mathbf{T}_t\mu = \sum_{i=0}^{m} \frac{t^i}{i!}\partial^i\mu. \tag{2.6}$$

v) ∂ is $(m+1)$-nilpotent, that is $\partial^{m+1} = 0$ and $\partial^k \neq 0$ for $k \leq m$.

Proof. The properties of the Minkowski sum ensure that $\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s = \mathbf{T}_s \circ \mathbf{T}_t$. Differentiating with respect to s at zero yields

$$\frac{d}{dt}\mathbf{T}_t\mu = \mathbf{T}_t\partial\mu = \partial\mathbf{T}_t\mu. \tag{2.7}$$

It follows that

$$\partial^i\mu = \left.\frac{d^i}{dt^i}\right|_{t=0}\mathbf{T}_t\mu.$$

For each $\mu \in \mathrm{Val}^\infty(V)$, the map $t \mapsto \mathbf{T}_t\mu$ is a polynomial in t of degree m by (2.1) and the Steiner formula (or by [50]). Hence

$$\begin{aligned}\mathbf{T}_t\mu &= \sum_{i=0}^m \frac{t^i}{i!}\left.\frac{d^i}{dt^i}\right|_{t=0}\mathbf{T}_t\mu \\ &= \sum_{i=0}^m \frac{t^i}{i!}\partial^i\mu.\end{aligned}$$

Note also that, by (2.5) and (2.6), the derivative operator ∂ is $(m+1)$-nilpotent. □

Let us compute the tube formula for the intrinsic volumes $\mu_0, \ldots, \mu_m$ using (2.6). For that purpose, we first compute ∂ in those terms.

Lemma 2.1.4. *For* $0 \leq j \leq m$

$$\partial\mu_j = \frac{\omega_{m-j+1}}{\omega_{m-j}}(m-j+1)\mu_{j-1}.$$

Proof. Since $\mathbf{T}_{t+s} = \mathbf{T}_s \circ \mathbf{T}_t$ we have

$$\mathbf{T}_{t+s}\,\mathrm{vol} = \sum_{j=0}^m \omega_{m-j}t^{m-j}\mathbf{T}_s\mu_j.$$

On the other hand

$$\mathbf{T}_{t+s}\,\mathrm{vol} = \sum_{j=0}^m \omega_{m-j}(t+s)^{m-j}\mu_j,$$

Differentiating at $s = 0$ and comparing coefficients yields the desired formula. □

Theorem 2.1.5.

$$\mathbf{T}_t\mu_k = \sum_{j=0}^k \binom{m-j}{k-j}\frac{\omega_{m-j}}{\omega_{m-k}}t^{k-j}\mu_j,$$

Proof. Using (2.6), we get

$$\mathbf{T}_t\mu_k = \sum_{i=0}^{m} \frac{t^i}{i!}\partial^i\mu_k = \sum_{i=0}^{k} \frac{t^i}{i!}\frac{\omega_{m-k+i}}{\omega_{m-k}}\frac{(m-k+i)!}{(m-k)!}\mu_{k-i}$$
$$= \sum_{j=0}^{k} \binom{m-j}{k-j}\frac{\omega_{m-j}}{\omega_{m-k}}t^{k-j}\mu_j. \qquad \square$$

As an application of the previous considerations we compute the tube formulas for invariant valuations in $\mathbb{C}^n$. Recall that $\mathrm{Val}^{U(n)}$ is the space of traslation and $U(n)$-invariant valuations on $\mathbb{C}^n$. Bernig and Fu [20] introduced the so-called *hermitian intrinsic volumes* $\mu_{k,q} \in \mathrm{Val}^{U(n)}$ for $\max\{0, n-k\} \leq q \leq \frac{k}{2} \leq n$ which consitute a basis of $\mathrm{Val}^{U(n)}$. They also introduced a second basis consisting of the *Tasaki valuations* given by

$$\tau_{k,q} = \sum_{i=q}^{\lfloor k/2 \rfloor} \binom{i}{q}\mu_{k,i}.$$

In Chapter 3, which is devoted to real and complex space forms, we explore these bases in detail.

In order to compute the tube formulas of $\mathrm{Val}^{U(n)}$, it will be useful to recall the $\mathfrak{sl}_2$-module structure of found on this space by Bernig and Fu. For an euclidean vector space V, consider the linear maps

$$\Lambda, L, H\colon \mathrm{Val}^\infty(V) \to \mathrm{Val}^\infty(V),$$

defined as follows

$$\Lambda\nu := \frac{\omega_{m-k}}{\omega_{m-k+1}}\partial\nu, \qquad L\nu := \frac{2\omega_k}{\omega_{k+1}}\mu_1 \cdot \nu, \qquad H\nu = (2k-m)\nu, \tag{2.8}$$

where $\nu \in \mathrm{Val}_k^\infty(V)$ and $\cdot$ refers to the Alesker product.

Proposition 2.1.6. *On* $\mathrm{Val}^{O(m)}$ *the operators* Λ, L *are given by*

$$L\mu_k = (k+1)\mu_{k+1}, \tag{2.9}$$
$$\Lambda\mu_k = (m-k+1)\mu_{k-1}, \tag{2.10}$$

while on $\mathrm{Val}^{U(n)}$ *one has*

$$L\mu_{k,p} = (k-2q+1)\mu_{k+1,q} + 2(q+1)\mu_{k+1,q+1} \tag{2.11}$$
$$\Lambda\mu_{k,p} = (k-2q+1)\mu_{k-1,q-1} + 2(n-k+q+1)\mu_{k-1,q}, \tag{2.12}$$

which implies

$$L\tau_{k,q} = (k-2q+1)\tau_{k+1,q} \tag{2.13}$$
$$\Lambda\tau_{k,q} = (k-2q+1)\tau_{k-1,q-1} + (2n-2q-k+1)\tau_{k-1,q} \tag{2.14}$$

Proof. The first two equalities are [15, eqs. (2.3.12) and (2.3.13)]. The rest is [20, Lemma 5.2]. □

It follows that both $\mathrm{Val}^{O(n)}$ and $\mathrm{Val}^{U(n)}$ have a natural $\mathfrak{sl}_2$-module structure. We will see later that this structure also underlies the space of invariant valuations of $\mathbb{C}P^n_\lambda$, and also a certain subspace of valuations of $\mathbb{H}P^n_\lambda$.

Proposition 2.1.7 ([15, Prop. 2.3.10 (3)])**.** *The operators* Λ, L, H *define an* $\mathfrak{sl}_2$*-module structure on both* $\mathrm{Val}^{O(m)}$ *and* $\mathrm{Val}^{U(n)}$*; i.e.* $[L,\Lambda] = H,\ [H,L] = 2L, [H,\Lambda] = -2\Lambda$.

The decomposition into irreducible components is as follows

$$\mathrm{Val}^{O(m)} \cong V^{(m)}, \qquad \mathrm{Val}^{U(n)} \cong \bigoplus_{0\le 2r\le n} V^{(2n-4r)} \tag{2.15}$$

where $V^{(m)}$ is the $(m+1)-$dimensional irreducible $\mathfrak{sl}_2$-representation. In particular, for $0 \le 2r \le n$, there exists a unique, up to a multiplicative constant, primitive element (i.e. annihilated by Λ) in each irreducible component of $\mathrm{Val}^{U(n)}$. By the Lefschetz decomposition, the L-orbits of these primitive elements constitute a basis of $\mathrm{Val}^{U(n)}$. This basis was explicitly computed in [20] as follows.

Proposition 2.1.8 ([20, eq. (76)])**.** *The following valuations*

$$\pi_{2r,r} := (-1)^r(2n-4r+1)!!\sum_{i=0}^{r}(-1)^i\frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!}\tau_{2r,i}, \quad 0\le 2r\le n, \tag{2.16}$$

are Λ*-primitive; i.e.* $\Lambda\pi_{2r,r} = 0$*. The family*

$$\pi_{k,r} := L^{k-2r}\pi_{2r,r} \tag{2.17}$$

$$= (-1)^r(2n-4r+1)!!\sum_{i=0}^{r}(-1)^i\frac{(k-2i)!}{(2r-2i)!}\frac{(2r-2i-1)!!}{(2n-2r-2i+1)!!}\tau_{k,i}, \quad 2r\le k\le 2n-2r \tag{2.18}$$

forms a basis of $\mathrm{Val}^{U(n)}$*.*

In particular the irreducible components of $\mathrm{Val}^{U(n)}$ are the following subspaces

$$\mathcal{I}^{n,r}_{0,\mathbb{C}} := \{\pi_{k,r} : 2r \le k \le 2n-2r\}, \quad 0 \le 2r \le n. \tag{2.19}$$

We can now compute the tube formulas in the complex case using (2.6).

Theorem 2.1.9. *For* $0 \le 2r \le n$, $2r \le k \le 2n-2r$

$$\mathbf{T}_t\pi_{k,r} = \frac{(k-2r)!}{\omega_{2n-k}}\sum_{j=0}^{k-2r}\binom{2n-4r-j}{k-2r-j}t^{k-2r-j}\frac{\omega_{2n-2r-j}}{j!}\pi_{j+2r,r}. \tag{2.20}$$

Proof. By [20, Lemma 5.6],

$$\Lambda\pi_{k,r} = (k-2r)(2n-k-2r+1)\pi_{k-1,r},$$

and then

$$\Lambda^i\pi_{k,r} = \frac{(k-2r)!(2n-k-2r+i)!}{(k-2r-i)!(2n-k-2r)!}\pi_{k-i,r}.$$

Using (2.6), we obtain the tube formula

$$\begin{aligned}
\mathbf{T}_t\pi_{k,r} &= \sum_{i=0}^{2n}\frac{t^i}{i!}\frac{\omega_{2n-k+i}}{\omega_{2n-k}}\Lambda^i\pi_{k,r}\\
&= \frac{(k-2r)!}{\omega_{2n-k}}\sum_{i=0}^{k-2r}\frac{t^i}{i!}\omega_{2n-k+i}\frac{(2n-k-2r+i)!}{(k-2r-i)!(2n-k-2r)!}\pi_{k-i,r}\\
&= \frac{(k-2r)!}{\omega_{2n-k}}\sum_{j=0}^{k-2r}\binom{2n-4r-j}{k-2r-j}t^{k-2r-j}\frac{\omega_{2n-2r-j}}{j!}\pi_{j+2r,r}.
\end{aligned}$$

□

It might be convenient to express these tube formulas in terms of other bases of $\mathrm{Val}^{U(n)}$, like the Tasaki valuations $\tau_{k,q}$. To achieve this, we compute their *Lefschetz decomposition*, which amounts to expressing $\tau_{k,q}$ as a linear combination of the $\pi_{k,r}$.

Proposition 2.1.10. *The Lefschetz decomposition of* $\tau_{k,r}$ *is given by*

$$\tau_{k,r} = \frac{1}{(k-2r)!}\sum_{i=0}^{r}\binom{n-2i}{r-i}\frac{(2n-2i-2r)!}{(2n-4i)!}\pi_{k,i}. \tag{2.21}$$

Proof. Consider the linear map $\psi\colon \mathrm{Val}^{U(n)} \to \mathrm{Val}^{U(n)}$ mapping $\tau_{k,r}$ to the left hand side of (2.21). We need to show that $\psi = \mathrm{id}$. Let us check that this endomorphism commutes with both Λ and L. To check commutation with Λ, we only need to verify the following

$$\begin{aligned}
(k-2r)!\psi(\Lambda(\tau_{k,r})) =& \sum_{i=0}^{r-1}\frac{(n-2i)!(2n-2i-2r+2)!}{(r-i-1)!(n-r-i+1)!(2n-4i)!}\pi_{k-1,i}\\
&+(k-2r)(2n-k-2r+1)\sum_{i=0}^{r}\frac{(n-2i)!(2n-2i-2r)!}{(r-i)!(n-i-r)!(2n-4i)!}\pi_{k-1,i}\\
=&\sum_{i=0}^{r}\frac{(n-2i)!(2n-2i-2r)!}{(r-i)!(n-i-r)!(2n-4i)!}(k-2i)(2n-k-2i+1)\pi_{k-1,i}\\
=&(k-2r)!\Lambda\psi(\tau_{k,r}).
\end{aligned}$$

Comparing term by term, the previous identities boil down to

$$2(r-i)(2n-2i-2r+1)+(k-2r)(2n-k-2r+1) = (k-2i)(2n-k-2i+1)$$

which is trivial.

Commutation with L is straightforward using $L\pi_{k,i} = \pi_{k+1,i}$.

Given that ψ commutes with the operators Λ and L and $\mathrm{Val}^{U(n)}$ is multiplicity-free, Schur's lemma implies that for each $0 \leq 2r \leq n$, there exists a constant c_r such that $\psi|_{\mathcal{I}^{n,r}_{0,\mathbb{C}}} = c_r \,\mathrm{id}$.

Let $a_{2r,j}$ and $b_{2r,i}$ be the coeficients of $\pi_{2r,j}$ and $\tau_{2r,i}$ in (2.21) and (2.16) respectively, so that $\psi(\tau_{2r,i}) = \sum_{j=0}^{i} a_{2r,j}\pi_{2r,j}$ and $\pi_{2r,r} = \sum_{i=0}^{r} b_{2r,i}\tau_{2r,i}$. Then

$$c_r\pi_{2r,r} = \psi(\pi_{2r,r}) = \sum_{i=0}^{r} b_{2r,i}\left(\sum_{j=0}^{i} a_{2r,j}\pi_{2r,j}\right) = \sum_{j=0}^{r}\sum_{i=j}^{r} b_{2r,i}a_{2r,j}\pi_{2r,j}.$$

Comparing the coefficient of $\pi_{2r,r}$ on both sides we get $c_r = b_{2r,r}a_{2r,r} = 1$ for each $0 \leq 2r \leq n$. To obtain $\pi_{2r,r}$ on the right-hand side, we must set $j = r$. However, this value of j can only be obtained by setting $i = r$. Therefore, we have

$$c_r = a_{2r,r}b_{2r,r} = \frac{1}{0!}\binom{n-2r}{0}\frac{(2n-4r)!}{(2n-4r)!}\cdot(-1)^{2r}(2n-4r+1)!!\frac{(-1)!!}{(2n-4r+1)!!} = 1.$$

Hence $\psi = \mathrm{id}$, which proves (2.21). □

By plugging (2.18) and (2.21) in (2.20) one gets the tube formulas $\mathbf{T}_t\tau_{k,p}$ in terms of the $\tau_{i,j}$.

2.2 Tube formulas in riemannian manifolds

We introduce the *tubular and derivative operators* in riemannian manifolds, extending the definition from linear spaces. We also review the construction of the normal cycle for the class of *sets of positive reach.*

2.2.1 Tubular and derivative operators

Let M^n be a complete riemannian manifold. Recall that $\pi : SM \to M$ denotes the canonical projection and $\phi : SM \times \mathbb{R} \to SM$ denotes the Reeb flow, which is the flow associated with the Reeb vector field T on SM.

Definition 2.2.1 (Tubular and derivative operators)**.** Given $t \geq 0$, we define the *tubular operator* $\mathbf{T}_t$ by

$$\mathbf{T}_t \colon \mathcal{V}(M) \longrightarrow \mathcal{V}(M), \quad [\![\omega,\eta]\!] \longmapsto [\![\phi_t^*\omega + (p_t)_*(\pi\circ\phi)^*\eta, \eta]\!],$$

where $p_t \colon SM \times [0,t] \to SM$ is the projection on the first factor, and $\phi_t = \phi(\cdot,t)$. We define the *derivative operator* $\partial = \partial_M$ by

$$\partial_M \colon \mathcal{V}(M) \longrightarrow \mathcal{V}(M), \quad \mu \longmapsto \left.\frac{d}{dt}\right|_{t=0} \mathbf{T}_t\mu.$$

Definition 2.2.2 (Local tubular and derivative operators)**.** Given $t \geq 0$, we define the *local tubular operator* $\mathbf{LT}_t$ by

$$\mathbf{LT}_t\colon \mathcal{C}(M) \longrightarrow \mathcal{C}(M), \quad [\omega, \eta] \longmapsto [\phi_t^*\omega + (p_t)_*(\pi \circ \phi)^*\eta, \eta],$$

where $p_t\colon SM \times [0,t] \to SM$ is the projection on the first factor, and $\phi_t = \phi(\cdot, t)$. We define the *derivative operator* $\partial = \partial_M$ for curvature measures by

$$\partial_M\colon \mathcal{C}(M) \longrightarrow \mathcal{C}(M), \quad \Phi \longmapsto \left.\frac{d}{dt}\right|_{t=0} \mathbf{LT}_t\Phi.$$

Proposition 2.2.3. *The definitions of* $\mathbf{T}_t$ *and* $\mathbf{LT_t}$ *are consistent, i.e, if* $\mu = [\![\omega, \eta]\!] = 0$ *then* $\mathbf{T}_t\mu = 0$*, and if* $\Phi = [\omega, \eta] = 0$ *then* $\mathbf{LT}_t\Phi = 0$.

Proof. Suppose $\mu = [\![\omega, \eta]\!] = 0$, and let us check that $\mathbf{T}_t\mu = 0$ for all $t \geq 0$, i.e.

$$\int_{N(A)} \phi_t^*\omega + \int_{N(A)} (p_t)_*(\pi \circ \phi)^*\eta + \int_A \eta = 0,$$

for all $A \in \mathcal{R}(M)$. By Proposition 1.3.5 we have $\pi^*\eta = -D\omega = -d(\omega + \xi \wedge \alpha)$. Hence

$$\begin{aligned}\int_{N(A)} (p_t)_*(\pi \circ \phi)^*\eta &= -\int_{N(A)} (p_t)_* \circ \phi^* D\omega = -\int_{N(A)\times[0,t]} \phi^* d(\omega + \xi \wedge \alpha)\\ &= -\int_{N(A)\times[0,t]} d\phi^*(\omega + \xi \wedge \alpha) = -\int_{N(A)\times\{0,t\}} \phi^*\omega + \phi^*\xi \wedge \alpha\\ &= \int_{N(A)} \phi_0^*\omega - \int_{N(A)} \phi_t^*\omega = \int_{N(A)} \omega - \int_{N(A)} \phi_t^*\omega,\end{aligned}$$

as α vanishes on $N(A)$. Since $[\![\omega, \eta]\!] = 0$, we have $\int_{N(A)} \omega = -\int_A \eta$. Therefore $\mathbf{T}_t\mu = 0$.

Assume $\Phi = [\omega, \eta] = 0$. From Proposition 1.3.5, we know $\omega \in (\alpha, d\alpha)$ and $\eta = 0$. Since ϕ_t is a strict contactomorphism (cf. (1.15)), $\phi_t^*\omega \in (\alpha, d\alpha)$, implying $\mathbf{LT}_t\Phi = 0$. □

Let us next establish some basic properties of these operators.

Lemma 2.2.4.

$$\frac{d}{dt}(p_t)_*\phi^*\rho = i_T\phi_t^*\rho, \qquad \rho \in \Omega^*(SM)$$

Proof. Given a compact smooth submanifold $N \subset SM$,

$$\begin{aligned}\int_N (p_t)_*\phi^*\rho &= \int_{N\times[0,t]} \phi^*\rho\\ &= \int_{N\times[0,t]} i_{\frac{\partial}{\partial t}}\phi^*\rho \wedge dt\\ &= \int_0^t \left(\int_N \phi_t^* i_{\frac{\partial\phi}{\partial t}}\rho\right) dt,\end{aligned}$$

Since i_T and ϕ_t^* commute, the result follows.

□

Proposition 2.2.5. *For* $\mu = [\![\omega, \eta]\!]$ *and* $\Phi = [\omega, \eta]$

$$\begin{aligned} \partial\mu &= [\![\mathcal{L}_T\omega + i_T(\pi^*\eta), 0]\!] = [\![i_T\left(d\omega + \pi^*\eta\right), 0]\!], \\ \partial\Phi &= [\mathcal{L}_T\omega + i_T(\pi^*\eta), 0]. \end{aligned}$$

In particular, if $\eta = 0$

$$\begin{aligned} \partial\mu &= [\![\mathcal{L}_T\omega, 0]\!], \\ \partial\Phi &= [\mathcal{L}_T\omega, 0]. \end{aligned}$$

Proof. Modulo exact forms we have

$$\frac{d}{dt}\phi_t^*\omega = \left.\frac{d}{ds}\right|_{s=0} \phi_{t+s}^*\omega = \mathcal{L}_T\phi_t^*\omega \equiv i_T\phi_t^* d\omega. \tag{2.22}$$

Together with Lemma 2.2.4, taking $t = 0$ yields the result. □

We generalize Proposition 2.1.3 within the riemannian context.

Proposition 2.2.6. *The tubular operator satisfies the following main properties.*

i)

$$\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s = \mathbf{T}_s \circ \mathbf{T}_t. \tag{2.23}$$

ii)

$$\frac{d}{dt}\mathbf{T}_t\mu = \mathbf{T}_t\partial\mu = \partial\mathbf{T}_t\mu. \tag{2.24}$$

iii)

$$\partial^i\mu = \left.\frac{d^i}{dt^i}\right|_{t=0} \mathbf{T}_t\mu. \tag{2.25}$$

iv) The Taylor series expansion of $\mathbf{T}$ *is*

$$\mathbf{T}_t\mu \sim \sum_{i\geq 0} \frac{t^i}{i!}\partial^i\mu. \tag{2.26}$$

Proof. *i)* It is enough to check that both sides have the same derivative with respect to s, as they agree for $s = 0$. By (2.22), we have

$$\begin{aligned} \frac{d}{ds}\mathbf{T}_t \circ \mathbf{T}_s(\mu) &= \mathbf{T}_t \circ \frac{d}{ds}\mathbf{T}_s(\mu) \\ &= \mathbf{T}_t[\![\mathcal{L}_T\phi_s^*\omega + i_T\phi_s^*\pi^*\eta, 0]\!] \\ &= [\![\phi_t^*(\mathcal{L}_T\phi_s^*\omega + i_T\phi_s^*\pi^*\eta), 0]\!] \\ &= [\![\mathcal{L}_T\phi_{t+s}^*\omega + i_T\phi_{t+s}^*\pi^*\eta), 0]\!], \end{aligned}$$

since ϕ_t^* commutes with $\mathcal{L}_T$ and i_T. It follows from (2.22) that $\frac{d}{ds}\mathbf{T}_{t+s} = \frac{d}{ds}\mathbf{T}_t \circ \mathbf{T}_s$.

ii) Since $\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s = \mathbf{T}_s \circ \mathbf{T}_t$, differentiating with respect to s at zero yields

$$\frac{d}{dt}\mathbf{T}_t\mu = \mathbf{T}_t\partial\mu = \partial\mathbf{T}_t\mu. \tag{2.27}$$

iii) It is a direct consequence of (2.27).

iv) The function $f : \mathbb{R} \to \mathcal{V}(M)$ defined by

$$f(t) := \mathbf{T}_t\mu$$

is C^∞. To establish this, we first observe that by *ii)*, the first derivative of $f(t)$ with respect to t exists and the derivative can be expressed as

$$\frac{d}{dt}\mathbf{T}_t\mu = \mathbf{T}_t\partial\mu.$$

In the same way, we can take the second derivative of $f(t)$ with respect to t, resulting in

$$\frac{d^2}{dt^2}\mathbf{T}_t\mu = \frac{d}{dt}\left(\mathbf{T}_t\partial\mu\right) = \mathbf{T}_t\partial^2\mu.$$

The pattern observed here can be extended by induction to higher-order derivatives

$$\frac{d^k}{dt^k}\mathbf{T}_t\mu = \mathbf{T}_t\partial^k\mu.$$

Thus, we conclude that the function $f(t)$ has derivatives of all orders. By definition, the Taylor expansion around zero is

$$\sum_{i\geq 0}\frac{t^i}{i!}\left.\frac{d^i}{dt^i}\right|_{t=0} f(t) = \sum_{i\geq 0}\frac{t^i}{i!}\left.\frac{d^i}{dt^i}\right|_{t=0}\mathbf{T}_t\mu.$$

From (2.27) the result follows. □

At the level of curvature measures, we have the same properties.

Proposition 2.2.7. *The* local *tubular operator satisfies the following main properties.*

i)

$$\mathbf{LT}_{t+s} = \mathbf{LT}_t \circ \mathbf{LT}_s = \mathbf{LT}_s \circ \mathbf{LT}_t. \tag{2.28}$$

ii)

$$\frac{d}{dt}\mathbf{LT}_t\Phi = \mathbf{LT}_t\partial\Phi = \partial\mathbf{LT}_t\Phi. \tag{2.29}$$

iii)

$$\partial^i\Phi = \left.\frac{d^i}{dt^i}\right|_{t=0}\mathbf{LT}_t\Phi. \tag{2.30}$$

iv) The Taylor series expansion of **LT** *is*

$$\mathbf{LT}_t\Phi \sim \sum_{i\geq 0}\frac{t^i}{i!}\partial^i\Phi. \tag{2.31}$$

Proof. The proof is similar to the preceding proposition. □

Remark 2.2.8. Note that (2.26) coincides with (2.6) except that the derivative operator does not need to be nilpotent on a riemannian manifold.

Remark 2.2.9. The functions $\mathbf{T}: t\mapsto \mathbf{T}_t$ and $\mathbf{LT}: t\mapsto \mathbf{LT}_t$ are not analytic in general. However, if we consider a finite-dimensional and ∂_M-invariant subspace $E\subset \mathcal{V}(M)$, then $\mathbf{T}|_E$ and $\mathbf{LT}|_E$ are analytic and agree with their Taylor series expansions (2.26) and (2.31). This is the case for $E=\mathcal{V}(M)^G$ where (M,G) is an isotropic pair.

2.2.2 Tubes in riemannnian manifolds

We review the basic properties of tubes around sets of positive reach and establish their relation with the tubular operator from the previous section.

Let M^n be a complete riemannian manifold and let $d\colon M\times M\to[0,\infty)$ be its riemannian distance.

Definition 2.2.10. For $t\geq 0$, the *tube* of radius t around a subset $A\subset M$ is defined as

$$A_t := \{p\in M : d_A(p)\leq t\}\,,$$

where

$$d_A(p) := \inf\{d(p,q) : q\in A\}\,.$$

We review some basic facts about tubes around sets of positive reach (introduced by Federer in euclidean spaces and by Kleinjohann in riemannian manifolds). For such sets A we will prove that $\mathbf{T}_t\mu(A)=\mu(A_t)$ for any $\mu\in\mathcal{V}(M)$ and sufficiently small t.

Definition 2.2.11 (Sets of positive reach)**.** A *set of positive reach* in M is a closed subset $A\subset M$ for which there exists an open neighborhood $U_A\supset A$ such that for every $p\in U_A\setminus A$ there exists a unique point $f_A(p)\in A$, called the *footpoint associated to the point* p, such that $d(p,f_A(p))=d_A(p)$, and an *unique* minimizing geodesic joining p with $f_A(p)$. We denote by $\mathcal{R}(M)$ the class of compact sets of positive reach in M.

By the previous definition, there is a well-defined map

$$F_A : U_A\setminus A \longrightarrow SM, \qquad F_A(p)=\big(\gamma(0),\gamma'(0)\big) \tag{2.32}$$

where γ is the *unique* minimizing geodesic such that $\gamma(0)=f_A(p)$ and $\gamma(d_A(p))=p$.

Proposition 2.2.12 ([45, Satz 3.3])**.** *For $A\in\mathcal{R}(M)$, $N(A):=F_A(U_A\setminus A)$ is a $(n-1)$-dimensional oriented compact Lipschitz submanifold of SM called the* normal cycle *of A.*

Proposition 2.2.13 ([45, Satz 3.3, Korollar 2.7]). *Given a set of positive reach A in M there exists $r = r_A > 0$ such that $A_r \subset U_A$ and*

i) for $0 < t < r$ the restriction $F_A|_{\partial A_t}$ gives a bilipschitz homeomorphism between ∂A_t and $N(A)$, preserving the natural orientations,

ii) the distance function d_A is of class C^1 in $A_r \setminus A$ and

$$\phi_{d_A(p)}(F_A(p)) = (p, \nabla d_A(p)), \qquad \partial A_t = d_A^{-1}(\{t\})$$

for $0 < t < r$. In particular, each level set ∂A_t with $0 < t < r$ is a C^1-regular hypersurface with unit normal vector field ∇d_A.

The following propositions are certainly well-known.

Proposition 2.2.14. *For $0 < s < r = r_A$ the set A_s has positive reach and on $A_r \setminus A_s$ we have*

$$d_{A_s} = d_A - s, \qquad F_{A_s} = \phi_s \circ F_A. \tag{2.33}$$

In particular $(A_s)_t = A_{t+s}$ for $t + s < r$.

Proof. Let $p \in A_r \setminus A_s$, and put $d = d_A(p)$. Let $\gamma \colon [0, d] \to A_r$ be the unique minimizing geodesic with $\gamma(0) = f_A(p)$ and $\gamma(d) = p$. In particular $|\gamma'| = 1$ and thus $\gamma(s) \in A_s$.

Assume that $\gamma|_{[s,d]}$ does not minimize the distance between p and A_s, i.e., there exists a smooth curve $\alpha : [0, 1] \to M$ with $q := \alpha(0) \in A_s$, $\alpha(1) = p$ and length $\ell(\alpha) < d - s$. It follows that

$$d_A(p) \leq \ell(\alpha) + d_A(q) \leq \ell(\alpha) + s < d_A(p),$$

a contradiction. We conclude that $\gamma|_{[s,d]}$ realizes the distance $d_{A_s}(p)$. Hence $d_{A_s}(p) = d_A(p) - s$ and

$$F_{A_s}(p) = (\gamma(s), \gamma'(s)) = \phi_s(\gamma(0), \gamma'(0)) = \phi_s(F_A(p)). \qquad \square$$

Proposition 2.2.15. *For $0 < s < r_A$, the restriction $\phi_s|_{N(A)}$ is a bilipschitz homeomorphism between $N(A)$ and $N(A_s)$.*

Proof. Take t with $s < t < \min(r_A, s + r_{A_s})$. By Proposition 2.2.13, both $F_A|_{\partial A_t} : \partial A_t \to N(A)$ and $F_{A_s}|_{\partial A_t} : \partial A_t \to N(A_s)$ are bilipschitz homeomorphisms. By (2.33) we have

$$\phi_s|_{N(A)} = F_{A_s}|_{\partial A_t} \circ (F_A|_{\partial A_t})^{-1}.$$

The statement follows. $\square$

Proposition 2.2.16. *For $0 < t < r_A$ the composition $\pi \circ \phi$ gives a bijective Lipschitz map between $N(A) \times (0, t]$ and $A_t \setminus A$.*

Proof. Since π, ϕ are smooth, the restriction of $\pi\circ\phi$ to the Lipschitz manifold $N(A)\times(0,t]$ is Lipschitz. Given that π and ϕ are smooth in $SM\times\mathbb{R}$, it follows that $\pi\circ\phi$ is also smooth in $SM\times\mathbb{R}$. Furthermore, since A is compact, then $N(A)$ is compact. Consequently, as $\pi\circ\phi$ is smooth and $N(A)\times[-t,t]$ is compact, we deduce that $\pi\circ\phi$ is Lipschitz on $N(A)\times[-t,t]$. Thus, $\pi\circ\phi$ is Lipschitz in $N(A)\times(0,t]$.

Given $(\xi,s)\in N(A)\times(0,t]$, we know by the previous proposition that $\phi(\xi,s)\in N(A_s)$ and thus $\pi\circ\phi(x,s)\in\partial A_s\subset A_t\setminus A$.

To check surjectivity, given $p\in A_t\setminus A$ take $\xi=F_A(p), s=d_A(p)$ and note that $\pi\circ\phi(\xi,s)=p$.

As for injectivity, suppose $\pi\circ\phi(\xi_1,t_1)=\pi\circ\phi(\xi_2,t_2)=: p$ for some $(\xi_1,t_1),(\xi_2,t_2)\in N(A)\times(0,t]$. By the previous proposition p belongs to both $\partial A_{t_1},\partial A_{t_2}$, so $t_1=t_2$. For $s\in[0,t_1]$, the geodesics $\gamma_1(s)=\pi\circ\phi(\xi_1,s),\gamma_2(s)=\pi\circ\phi(\xi_2,s)$ realize the distance between p and A. Since $A_s\subset A_{r_A}\subset U_A$, we have $\gamma_1=\gamma_2$ and thus $\xi_1=\xi_2$. □

Corollary 2.2.17. *For $A\in\mathcal{R}(M)$, the normal cycle $N(A)$ is* legendrian *and* lagrangian *i.e,*

$$\int_{N(A)}\rho\wedge\alpha=0,\qquad\int_{N(A)}\eta\wedge d\alpha=0,$$

for all $\rho\in\Omega^{n-2}(SM)$, $\eta\in\Omega^{n-3}(SM)$.

Proof. The second identity follows from the first one by integration by parts. Since $d_{A_s}(p)=\varepsilon$ for all $p\in\partial A_{s+\varepsilon}$, by Proposition 2.2.13, the map

$$F_{A_s}|_{\partial A_{s+\varepsilon}}:\partial A_{s+\varepsilon}\to N(A_s)$$

is a homeomorphism of class C^1. Then

$$\int_{N(A_s)}\rho\wedge\alpha=\int_{\partial A_{s+\varepsilon}}F_{A_s}^*\rho\wedge F_{A_s}^*\alpha.$$

Now, let us evaluate $F_{A_s}^*\alpha$. Let $p\in\partial A_{s+\varepsilon}$ and let $V\in T_p(\partial A_{s+\varepsilon})$. Then,

$$\begin{aligned}(F_{A_s}^*\alpha)_p(V)&=\alpha_{F_{A_s}(p)}(dF_{A_s}(V))\\&=\langle F_{A_s}(p),(d\pi)_{F_{A_s}(p)}((dF_{A_s})_p(V))\rangle.\end{aligned}$$

Since $p\in\partial A_{s+\varepsilon}\subset A_{s+\varepsilon}\setminus A$,

$$F_{A_s}(p)=\phi_s\circ F_A(p)=(p,\nabla d_A(p)).$$

Since $\pi\circ F_{A_s}|_{\partial A_{s+\varepsilon}}=\mathrm{id}|_{\partial A_{s+\varepsilon}}$, we have

$$(d\pi)_{F_{A_s}(p)}((dF_{A_s})_p(V))=V.$$

Therefore, since $\nabla d_A(p)$ is orthogonal to $\partial A_{s+\varepsilon}$ and $V\in T_p(\partial A_{s+\varepsilon})$,

$$(F_{A_s}^*\alpha)_p(V)=\langle\nabla d_A(p),V\rangle=0.$$

Finally, since $N(A_s)=\phi_s(N(A))$ with ϕ_s bi-Lipschitz

$$\int_{N(A)}\rho\wedge\alpha=\lim_{s\to0^+}\int_{N(A_s)}\rho\wedge\alpha=0.\quad\square$$

Corollary 2.2.18. *For every $A \in \mathcal{R}(M)$ and $\mu \in \mathcal{V}(M)$ we have $\mu(A_t) = \mathbf{T}_t\mu(A)$ for $0 \leq t \leq r_A$.*

Proof. Let $\mu = [\![\omega, \eta]\!]$. By Propositions 2.2.15 and 2.2.16 and the coarea formula,

$$\begin{aligned}
\mu(A_t) &= \int_{N(A_t)} \omega + \int_{A_t} \eta \\
&= \int_{\phi_t(N(A))} \omega + \int_{(\pi\circ\phi)(N(A)\times(0,t])} \eta + \int_A \eta \\
&= \int_{N(A)} \phi_t^*\omega + \int_{N(A)} (p_t)_*\eta + \int_A \eta \\
&= \mathbf{T}_t\mu(A).
\end{aligned}$$ □

Proposition 2.2.19. *Given $A \in \mathcal{R}(M)$ and a Borel set U, let $U_t = \{p \in A_t \colon f_A(p) \in U\}$. Then, for every $\Phi \in \mathcal{C}(M)$ we have $\Phi(A_t, U_t) = \mathbf{LT}_t\Phi(A, U)$ for $0 \leq t \leq r_A$.*

Proof. Let $A \in \mathcal{R}(M)$ and U a Borel set of M. For each $t \in \mathbb{R}$

$$\begin{aligned}
U_t &= (\pi \circ \phi)\left(\left(N(A) \cap \pi^{-1}(U)\right) \times (0, t]\right), \\
N(A_t) \cap \pi^{-1}(U_t) &= \phi_t\left(N(A) \cap \pi^{-1}(U)\right).
\end{aligned}$$

Thus

$$\begin{aligned}
\Phi(A_t, U_t) &= \int_{N(A_t)\cap\pi^{-1}(U_t)} \omega + \int_{U_t} \eta \\
&= \int_{\phi_t(N(A)\cap\pi^{-1}(U))} \omega + \int_{(\pi\circ\phi)((N(A)\cap\pi^{-1}(U))\times(0,t])} \eta + \int_U \eta \\
&= \int_{N(A)\cap\pi^{-1}(U)} \phi_t^*\omega + \int_{(N(A)\cap\pi^{-1}(U)} (p_t)_*\eta + \int_U \eta \\
&= \mathbf{LT}_t\Phi(A, U).
\end{aligned}$$ □

Remark 2.2.20. In the subclass $\mathcal{P}(M) \subset \mathcal{R}(M)$ of compact submanifolds with corners, the normal cycle is more naturally defined as follows. For $A \in \mathcal{P}(M)$ and $p \in A$, let

$$\begin{aligned}
T_pA &= \left\{\gamma'(0) \in T_pM : \gamma \in C^1([0,1), A), \gamma(0) = p\right\}, \\
N'(A) &= \{(p, v) \in SM : p \in A, \langle v, w\rangle \leq 0 \ \forall w \in T_pA\}.
\end{aligned}$$

Proposition 2.2.21. *If $A \in \mathcal{P}(M)$ then $N(A) = N'(A)$.*

Proof. Let us check that indeed $N'(A)$ equals $N(A) = F_A(U_A)$. Covering A by local charts (locally modeled on $\mathbb{R}^k \times [0,\infty)^l \subset \mathbb{R}^m$), and considering the copy of $N'(A)$ in the cosphere bundle of M, one sees that $N'(A)$ is a compact topological manifold. It is also easy to show that $N(A) \subset N'(A)$. It follows by the invariance of the domain theorem that $N(A)$ is an open subset of $N'(A)$. Since $N'(A)$ is a Hausdorff space and $N(A)$ is compact, we also have that $N(A)$ is a closed subset of $N'(A)$. Since the number of connected components of both $N(A)$ and $N'(A)$ equals the number of connected components of A, we necessarily have $N(A) = N'(A)$. □

2.2.3 Tubular and derivative operators for invariant valuations

Let M^n be a complete riemannian manifold, $G \leq \mathrm{Isom}(M)$ and $g \in G$. Denote by $\tilde{g}\colon SM \to SM$ the induced strict contactomorphism on the sphere bundle, i.e. the only diffeomorphism satisfying

$$\pi \circ \tilde{g} = g \circ \pi, \qquad \tilde{g}^*\alpha = \alpha.$$

Since g is an isometry, $gA \in \mathcal{R}(M)$ for all $A \in \mathcal{R}(M)$, and

$$N(gA) = \tilde{g}N(A), \quad \forall A \in \mathcal{R}(M).$$

Using the above equations, if $\mu = [\![\omega, \eta]\!]$, where $\omega \in \Omega^{2n-1}(SM)$ and $\eta \in \Omega^n(M)$, then

$$(g^*\mu)(A) = \mu(gA) = [\![\tilde{g}^*\omega, g^*\eta]\!](A), \quad \forall A \in \mathcal{R}(M). \tag{2.34}$$

The same holds for $\Phi \in \mathcal{C}(M)$.

Proposition 2.2.22. *If $\mu \in \mathcal{V}(M)$ and $\Phi \in \mathcal{C}(M)$*

$$g^*\mathbf{T}_t\mu = \mathbf{T}_t g^*\mu, \quad g^*\mathbf{LT}_t\Phi = \mathbf{LT}_t g^*\Phi, \quad \forall g \in \mathrm{Isom}(M).$$

In particular, if μ and Φ are G-invariant then $\mathbf{T}_t\mu, \partial_M\mu, \mathbf{LT}_t\Phi$ and $\partial_M\Phi$ are also G-invariant.

Proof. Since $\tilde{g}^*$ commutes with ϕ_t^*, π^* and $(p_t)_*$, where $p_t\colon SM \times \mathbb{R} \to SM$ is the projection onto the first factor, from (2.34) we have for $\mu = [\![\omega, \eta]\!]$

$$\begin{aligned}\mathbf{T}_t g^*\mu &= [\![\phi_t^*\tilde{g}^*\omega + (p_t)_*(\pi \circ \phi)^* g^*\eta, g^*\eta]\!] \\ &= [\![\tilde{g}^*\phi_t^*\omega + \tilde{g}^*(p_t)_*(\pi \circ \phi)^*\eta, g^*\eta]\!] \\ &= g^*\mathbf{T}_t\mu.\end{aligned}$$

Differentiating with respect to t at $t = 0$ yields

$$\partial_M g^*\mu = g^*\partial_M\mu, \quad \forall \mu \in \mathcal{V}(M).$$

In particular, if $\mu \in \mathcal{V}(M)^G$, then $\mathbf{T}_t\mu$ and $\partial_M\mu$ are also G-invariant. The same argument applies to curvature measures. □

Fix $\mu \in \mathcal{V}(M)^G$. It follows from (2.24) in Proposition 2.2.6 that

$$\frac{d}{dt}\mathbf{T}_t\mu = \partial\mathbf{T}_t\mu. \tag{2.35}$$

Since (M, G) is isotropic, $\mathcal{V}(M)^G$ is finite-dimensional, and hence computing $\mathbf{T}_t\mu$ boilws down to solving the first order linear Cauchy problem (2.35) with initial condition $\mathbf{T}_0\mu = \mu$; i.e.

$$\mathbf{T}_t\mu = \exp(t\partial)\mu = \sum_{i\geq 0}\frac{t^i}{i!}\partial^i\mu. \tag{2.36}$$

This is the approach we will follow to obtain the tube formulas for invariant valuations in $\mathbb{S}^m_\lambda, \mathbb{C}P^n_\lambda$ and $\mathbb{H}P^n_\lambda$.

Chapter 3

Tube formulas in real and complex space forms

Here we will obtain our main result: the tube formulas for invariant valuations of $\mathbb{C}P^n_\lambda$. We will also recover Santaló's tube formulas for invariant valuations of real space forms (cf. [55]) in a way that explains the similarities between the real and the complex space forms.

For $\lambda \in \mathbb{R}$, let $\mathcal{V}^m_{\lambda,\mathbb{R}}$ and $\mathcal{V}^n_{\lambda,\mathbb{C}}$ denote the vector spaces of smooth valuations in $\mathbb{S}^m_\lambda$ and $\mathbb{C}P^n_\lambda$ invariant under $G_{\lambda,\mathbb{R}}$ and $G_{\lambda,\mathbb{C}}$, respectively, and define $\partial_{\lambda,\mathbb{R}}$ and $\partial_{\lambda,\mathbb{C}}$ as the restrictions of $\partial_{\mathbb{S}^m_\lambda}$ to $\mathcal{V}^m_{\lambda,\mathbb{R}}$ and $\partial_{\mathbb{C}P^n_\lambda}$ to $\mathcal{V}^n_{\lambda,\mathbb{C}}$. These restrictions are based on Proposition 2.2.22.

3.1 Derivative operator in $\mathbb{S}^m_\lambda$

The computation of $\partial_{\lambda,\mathbb{R}}$ is the goal of this section. Recall that we denote by Δ_i^M and ν_i^M the Federer curvature measures and valuations of a riemannian manifold M.

For $M = \mathbb{S}^m_\lambda$ it will be convenient to renormalize the Federer valuations as follows. For $0 \leq i \leq m-1$ we define

$$\sigma_i^\lambda := [\![\kappa_i, 0]\!] = (m-i)\omega_{m-i}\nu_i^{\mathbb{S}^m_\lambda},$$

and $\sigma_m^\lambda = \mathrm{vol}_{\mathbb{S}^m_\lambda}$. Another common basis in the literature (cf. [15, 34]) is $\tau_0^\lambda, \ldots, \tau_m^\lambda \in \mathcal{V}^m_{\lambda,\mathbb{R}}$. These are proportional to σ_i^λ as follows

$$\tau_i^\lambda = \frac{i!\omega_i}{\pi^i(m-i)\omega_{m-i}}\sigma_i^\lambda, \quad 0 \leq i \leq m-1,$$
$$\tau_m^\lambda = \frac{m!\omega_m}{\pi^m}\sigma_m^\lambda.$$

To compute $\partial_{\lambda,\mathbb{R}}\sigma_i^\lambda$, we only need to determine $\mathcal{L}_T\kappa_i$ using Proposition 2.2.5. In Chapter 4, dedicated to Federer valuations, we establish a general formula for $\mathcal{L}_T\kappa_i$ in real, complex, and quaternionic space forms, generalizing the following result.

Lemma 3.1.1 ([31]). *For $0 \leq i \leq m-1$ and $\kappa_{-1} \equiv \kappa_m \equiv 0$*

$$\mathcal{L}_T \kappa_i = (m-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1}, \quad 0 \leq i \leq m-1.$$

Proposition 3.1.2.

$$\partial_{\lambda,\mathbb{R}} \sigma_i^\lambda = (m-i)\sigma_{i-1}^\lambda - \lambda(i+1)\sigma_{i+1}^\lambda, \qquad 0 \leq i \leq m-2, \tag{3.1}$$

$$\partial_{\lambda,\mathbb{R}} \sigma_{m-1}^\lambda = \sigma_{m-2}^\lambda, \tag{3.2}$$

$$\partial_{\lambda,\mathbb{R}} \sigma_m^\lambda = \sigma_{m-1}^\lambda, \tag{3.3}$$

where it is understood that $\sigma_{-1}^\lambda = 0$.

Let us emphasize that (3.1) would make formal sense but does not hold for $i = m-1$.

Proof. The derivative of the volume can be straightforwardly computed as follows

$$\begin{aligned} \partial_{\lambda,\mathbb{R}} \sigma_m^\lambda &= \partial_{\lambda,\mathbb{R}} \mathrm{vol}_{\mathbb{R}^m} = \mathrm{area}_{\mathbb{R}^m} \\ &= [\![\kappa_{m-1}, 0]\!] = \sigma_{m-1}^\lambda. \end{aligned}$$

Finally, by using the relation $\sigma_i^\lambda = [\![\kappa_i, 0]\!]$ for $0 \leq i \leq m-1$, along with Lemma 3.1.1 and Proposition 2.2.5, we obtain the desired formula. □

3.2 Derivative operator in $\mathbb{C}P_\lambda^n$

Here we compute the derivative operator on $\mathcal{V}_{\lambda,\mathbb{C}}^n$. Remarkably we will encounter an isomorphism with $\mathrm{Val}^{U(n)}$ that intertwines $\partial_{\lambda,\mathbb{C}}$ with $\Lambda - \lambda L$.

3.2.1 Preliminar computations

We start introducing general notions on Kähler manifolds.

Definition 3.2.1. For M a Kähler manifold and $\pi\colon SM \to M$ its sphere bundle we define the *canonical 1-forms* $\alpha, \beta, \gamma \in \Omega^1(SM)$

$$\begin{aligned} \alpha_\xi(X) &:= \langle \xi, d\pi X \rangle, \\ \beta_\xi(X) &:= \langle J\xi, d\pi X \rangle, \\ \gamma_\xi(X) &:= \langle J\xi, (\pi^*\nabla)_X \xi \rangle, \end{aligned}$$

for $\xi \in SM$ and X a vector field of SM.

Remark 3.2.2. If $M = \mathbb{C}P_\lambda^n$, then these forms agree with the like-named forms defined in [20] and [2]. The differential form β was also crucial in [22].

Definition 3.2.3. Let M be a Kähler manifold. Let $\Theta_0, \Theta_1, \Theta_2, \Theta_s \in \Omega^2(SM)$ be given by

$$\begin{aligned}
\Theta_0(X,Y) &:= \langle J(\pi^*\nabla)_X e_0, (\pi^*\nabla)_Y e_0\rangle,\\
\Theta_1(X,Y) &:= \langle Jd\pi X, (\pi^*\nabla)_Y e_0\rangle - \langle Jd\pi Y, (\pi^*\nabla)_X e_0\rangle\\
\Theta_2(X,Y) &:= \langle Jd\pi X, d\pi Y\rangle,\\
\Theta_s &:= -d\alpha.
\end{aligned}$$

for $\xi \in SM$, and $X, Y \in T_\xi(SM)$.

Proposition 3.2.4. *Let M^{2n} be a Kähler manifold and $e_0, \dots, e_{2n-1}$ an hermitian moving frame. Then*

$$\begin{aligned}
&(1)\quad \alpha = \theta_0, && (4)\quad \Theta_0 = \sum_{i=0}^{n-1} \omega_{2i,0} \wedge \omega_{2i+1,0},\\
&(2)\quad \beta = \theta_1, && (5)\quad \Theta_1 = \sum_{i=0}^{n-1} \left(\theta_{2i} \wedge \omega_{2i+1,0} - \theta_{2i+1} \wedge \omega_{2i,0}\right),\\
&(3)\quad \gamma = \omega_{1,0}, && (6)\quad \Theta_2 = \sum_{i=0}^{n-1} \theta_{2i} \wedge \theta_{2i+1}.
\end{aligned}$$

Proof. The identities for α, β, and γ follow directly from the definition:

(1)

$$\alpha(X) = \langle e_0, d\pi X\rangle = \theta_0(X).$$

(2) Since $e_0, \dots, e_{2n-1}$ is hermitian $e_1 = Je_0$. Thus

$$\beta(X) = \langle Je_0, d\pi X\rangle = \langle e_1, d\pi X\rangle = \theta_1(X).$$

(3)

$$\gamma(X) = \langle Je_0, (\pi^*\nabla)_X e_0\rangle = \langle e_1, (\pi^*\nabla)_X e_0\rangle = \omega_{1,0}(X).$$

(4) Since

$$\theta_{2i} \circ J = -\theta_{2i+1}, \quad \theta_{2i+1} \circ J = \theta_{2i}, \quad 0 \le i \le n-1.$$

we have

$$\begin{aligned}
\Theta_0(X,Y) &= \sum_{i=0}^{2n-1} \theta_i \otimes \theta_i(J(\pi^*\nabla)_X e_0, (\pi^*\nabla)_Y e_0) \\
&= \sum_{i=0}^{n-1} \theta_{2i}(J(\pi^*\nabla)_X e_0)\theta_{2i}((\pi^*\nabla)_Y e_0) + \sum_{i=0}^{n-1} \theta_{2i+1}(J(\pi^*\nabla)_X e_0)\theta_{2i+1}((\pi^*\nabla)_Y e_0) \\
&= -\sum_{i=0}^{n-1} \theta_{2i+1}((\pi^*\nabla)_X e_0)\theta_{2i}((\pi^*\nabla)_Y e_0) + \sum_{i=0}^{n-1} \theta_{2i}((\pi^*\nabla)_X e_0)\theta_{2i+1}((\pi^*\nabla)_Y e_0) \\
&= -\omega_{2i+1,0}(X)\omega_{2i,0}(Y) + \sum_{i=0}^{n-1} \omega_{2i,0}(X)\omega_{2i+1,0}(Y) \\
&= \sum_{i=0}^{n-1} \left(\omega_{2i,0} \otimes \omega_{2i+1,0} - \omega_{2i+1,0} \otimes \omega_{2i,0}\right)(X,Y) = \sum_{i=0}^{n-1} \left(\omega_{2i,0} \wedge \omega_{2i+1,0}\right)(X,Y).
\end{aligned}$$

(5) Consider

$$\eta(X,Y) := \langle Jd\pi X, (\pi^*\nabla)_Y e_0\rangle$$

Then

$$\Theta_1(X,Y) = \eta(X,Y) - \eta(Y,X).$$

Since

$$\begin{aligned}
\eta_\xi(X,Y) &= \sum_{i=0}^{2n-1} \theta_i \otimes \theta_i(Jd\pi X, (\pi^*\nabla)_Y e_0) \\
&= \sum_{i=0}^{n-1} \theta_{2i} \otimes \theta_{2i}(Jd\pi X, (\pi^*\nabla)_Y e_0) + \sum_{i=0}^{n-1} \theta_{2i+1} \otimes \theta_{2i+1}(Jd\pi X, (\pi^*\nabla)_Y e_0) \\
&= -\sum_{i=0}^{n-1} \theta_{2i+1} \otimes \theta_{2i}(d\pi X, (\pi^*\nabla)_Y e_0) + \sum_{i=0}^{n-1} \theta_{2i} \otimes \theta_{2i+1}(d\pi X, (\pi^*\nabla)_Y e_0) \\
&= \sum_{i=0}^{n-1} \theta_{2i} \otimes \omega_{2i+1,0}(X,Y) - \sum_{i=0}^{n-1} \theta_{2i+1} \otimes \omega_{2i,0}(X,Y),
\end{aligned}$$

we have

$$\begin{aligned}
\Theta_1 &= \sum_{i=0}^{n-1} \theta_{2i} \otimes \omega_{2i+1,0} - \theta_{2i+1} \otimes \omega_{2i,0} - (\omega_{2i+1,0} \otimes \theta_{2i} - \omega_{2i,0} \otimes \theta_{2i+1}) \\
&= \sum_{i=0} \theta_{2i} \wedge \omega_{2i+1,0} - \theta_{2i+1} \wedge \omega_{2i,0}.
\end{aligned}$$

(6) The given expression for Θ_2 follows similarly. □

To describe $\mathcal{V}^n_{\lambda,\mathbb{C}}$, we introduce the classification of $\Omega^{2n-1}(S\mathbb{C}P^n_\lambda)^{G_{\lambda,\mathbb{C}}}$ (cf. [51, 21]).

Theorem 3.2.5 ([51])**.** *For* $\max\{0, k-n\} \leq q < \frac{k}{2} < n$ *define*

$$\begin{aligned} \beta_{k,q} &:= c_{n,k,q}\beta \wedge \Theta_{n-k+q,k-2q-1,q}, \quad k > 2q, \\ \gamma_{k,q} &:= \frac{c_{n,k,q}}{2}\gamma \wedge \Theta_{n-k+q-1,k-2q,q}, \quad n > k-q. \end{aligned}$$

where

$$\begin{aligned} \Theta_{a,b,c} &:= \Theta_0^a \wedge \Theta_1^b \wedge \Theta_2^c, \\ c_{n,k,q} &:= \frac{1}{q!(n-k+q)!(k-2q)!\omega_{2n-k}}. \end{aligned}$$

Then

$$\Omega^{2n-1}(S\mathbb{C}P^n_\lambda)^{G_{\lambda,\mathbb{C}}}/(\alpha, d\alpha) \equiv \langle \beta_{k,q} \colon k > 2q\rangle \oplus \langle \gamma_{k,q} \colon n > k-q\rangle.$$

Theorem 3.2.6 ([21])**.** *For* $\max\{0, k-n\} \leq q \leq \frac{k}{2} \leq n \leq \infty$*, we set*

$$\mu^\lambda_{k,q} := [\![\beta_{k,q}, 0]\!] \in \mathcal{V}^n_{\lambda,\mathbb{C}}, \quad k > 2q,$$

$$\mu^\lambda_{2q,q} := \sum_{i\geq 0} \left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!}[\![\gamma_{2q+2i,q+i}, 0]\!] \in \mathcal{V}^n_{\lambda,\mathbb{C}},$$

$$\tau^\lambda_{k,q} := \sum_{i=q}^{\lfloor k/2 \rfloor} \binom{i}{q} \mu^\lambda_{k,i} \in \mathcal{V}^n_{\lambda,\mathbb{C}}.$$

Then $\{\mu^\lambda_{k,q}\}_{k,q}$ *and* $\{\tau^\lambda_{k,q}\}_{k,q}$ *are two bases for* $\mathcal{V}^n_{\lambda,\mathbb{C}}$ *and* $\mathcal{V}^n_{\lambda,\mathbb{C}} \cong \mathcal{V}^n_{0,\mathbb{C}} \equiv \mathrm{Val}^{U(n)}$ *as vector spaces.*

The valuations $\mu^0_{kq} \in \mathcal{V}^n_{0,\mathbb{C}} \equiv \mathrm{Val}^{U(n)}$ coincide with those previously studied as $\mu_{k,q}$ in [20]. For $\lambda \neq 0$ they agree with the basis elements examined in [2] only if $k > 2q$.

In light of Proposition 2.2.5, we aim to compute the Lie derivatives of the differential forms $\beta_{k,q}$ and $\gamma_{k,q}$, as their globalizations generate the entire space of valuations $\mathcal{V}^n_{\lambda,\mathbb{C}}$. To achieve this, we present several technical lemmas.

Proposition 3.2.7. *In* $\mathbb{C}P^n_\lambda$

(1) $d\alpha = -\Theta_s$,

(2) $d\beta = \Theta_1$,

(3) $d\gamma = 2\Theta_0 - 2\lambda\Theta_2 - 2\lambda\alpha \wedge \beta$,

(4) $d\Theta_0 = -\lambda\left(\alpha \wedge \Theta_1 + \beta \wedge \Theta_s\right)$,

(5) $d\Theta_1 = 0$,

(6) $d\Theta_2 = 0$

Proof. (1) It is the definition.

(2) By Proposition 3.2.4 we have $\beta = \theta_1$, in a hermitian moving frame. The structure equation (1.10) ensures us

$$d\theta_1 = -\sum_{j=0}^{n-1} \omega_{1,j} \wedge \theta_j. \tag{3.4}$$

On the other hand, since $e_1 = Je_0$ and $\pi^*\nabla$ commutes with the complex structure J, we have

$$\omega_{1,2j}(X) = \langle Je_0, (\pi^*\nabla)_X e_{2j}\rangle = -\langle e_0, (\pi^*\nabla)_X Je_{2j}\rangle = -\omega_{0,2j+1}.$$

In the same way $\omega_{1,2j+1} = \omega_{0,2j}$. Finally, by (3.4) the result follows

(3) In a hermitian moving frame, from Proposition 3.2.4 we have $\gamma = \omega_{1,0}$. By equation (1.11) and $\omega_{1,2j} = -\omega_{0,2j+1}$, $\omega_{1,2j+1} = \omega_{0,2j}$, we have

$$d\omega_{1,0} = -\sum_{k=0}^{n-1} \omega_{1,k} \wedge \omega_{k,0} + \Omega_{1,0} = 2\Theta_0 + \Omega_{1,0}.$$

The curvature tensor of $\mathbb{C}P^n_\lambda$ is given by $R = \lambda G$. Then, since $\Omega_{1,0} = R(\cdot,\cdot;e_0,Je_0)$ it follows

$$\begin{aligned} G(X,Y;e_0,Je_0) =& g(X;e_0)g(Y;Je_0) - g(X;Je_0)g(Y;e_0) \\ &+ g(JX;e_0)g(JY;Je_0) - g(JX;Je_0)g(JY;e_0) \\ &+ 2g(JX;Y)g(Je_0;Je_0) \\ =&\alpha(X)\beta(Y) - \beta(X)\alpha(Y) \\ &+ \alpha(JX)\beta(JY) - \beta(JX)\alpha(JY) \\ &+ 2\Theta_2(X,Y) \\ &= \alpha(X)\beta(Y) - \beta(X)\alpha(Y) \\ &- \beta(X)\alpha(Y) + \alpha(X)\beta(Y) \\ &+ 2\Theta_2(X,Y) = (2\alpha\wedge\beta + 2\Theta_2)(X,Y). \end{aligned}$$

(6) Since Θ_2 is the Kähler form $d\Theta_2 = 0$.

(4) Since $d^2\gamma = d\Theta_2 = 0$,

$$0 = d^2\gamma = 2d\Theta_0 - 2\lambda d\Theta_2 - 2\lambda d(\alpha\wedge\beta) = 2d\Theta_0 + 2\lambda\Theta_s \wedge \beta + 2\lambda\alpha\wedge\Theta_1,$$

which yields the result.

(5) Since $\Theta_1 = d\beta$ then $d\Theta_1 = d^2\beta = 0$.

□

Lemma 3.2.8. *For any Kähler manifold, it is satisfied*

$$\begin{aligned} &i_T\alpha = 1, \quad i_T\Theta_1 = \gamma \\ &i_T\Theta_2 = \beta, \quad i_T\beta = i_T\gamma = i_T\Theta_0 = i_T\Theta_s = 0. \end{aligned}$$

Proof. Since $d\pi(e_0) = e_0$ and $(\pi^*\nabla)_T e_0 = 0$

$$
\begin{aligned}
i_T\alpha &= \langle e_0, d\pi(e_0)\rangle = \langle e_0, e_0\rangle = 1 \\
i_T\beta &= \langle Je_0, d\pi(e_0)\rangle = \langle Je_0, e_0\rangle = 0 \\
i_T\gamma &= \langle Je_0, (\pi^*\nabla)_T e_0\rangle = \langle Je_0, 0\rangle = 0 \\
(i_T\Theta_0)(Y) &= \langle J(\pi^*\nabla)_T e_0, (\pi^*\nabla)_Y e_0\rangle = \langle 0, (\pi^*\nabla)_Y e_0\rangle = 0 \\
(i_T\Theta_1)(Y) &= \langle Je_0, (\pi^*\nabla)_Y e_0\rangle - \langle JY, (\pi^*\nabla)_T e_0\rangle = \gamma - \langle JY, 0\rangle = \gamma(Y) \\
(i_T\Theta_2)(Y) &= \langle Je_0, d\pi Y\rangle = \beta(Y)
\end{aligned}
$$

Finally, form Proposition 1.1.7 we know $i_T\Theta_s = -i_T d\alpha = 0$. □

Corollary 3.2.9. *In* $\mathbb{C}P^n_\lambda$

$$
\begin{aligned}
&(1)\quad \mathcal{L}_T\alpha = 0, && (4)\quad \mathcal{L}_T\Theta_0 = -\lambda\Theta_1 + \lambda\alpha\wedge\gamma, \\
&(2)\quad \mathcal{L}_T\beta = \gamma, && (5)\quad \mathcal{L}_T\Theta_1 = 2\Theta_0 - 2\lambda\Theta_2 - 2\lambda\alpha\wedge\beta, \\
&(3)\quad \mathcal{L}_T\gamma = -4\lambda\beta, && (6)\quad \mathcal{L}_T\Theta_2 = \Theta_1.
\end{aligned}
$$

Proof. This straightforward computation uses Cartan's magic formula, Proposition 3.2.7 and Lemma 3.2.8. □

Proposition 3.2.10. *In* $\mathbb{C}P^n_\lambda$

$$
\begin{aligned}
\mathcal{L}_T(\Theta_{a,b,c}) =& -a\lambda\Theta_{a-1,b+1,c} + a\lambda\alpha\wedge\gamma\wedge\Theta_{a-1,b,c} \\
& + 2b\Theta_{a+1,b-1,c} - 2b\lambda\Theta_{a,b-1,c+1} - 2b\lambda\alpha\wedge\beta\wedge\Theta_{a,b-1,c} + c\Theta_{a,b+1,c-1} \\
\mathcal{L}_T(\beta\wedge\Theta_{a,b,c}) =& \gamma\wedge\Theta_{a,b,c} - a\lambda\beta\wedge\Theta_{a-1,b+1,c} \\
& - a\lambda\alpha\wedge\beta\wedge\gamma\wedge\Theta_{a-1,b,c} + 2b\beta\wedge\Theta_{a+1,b-1,c} \\
& - 2b\lambda\beta\wedge\Theta_{a,b-1,c+1} + c\beta\wedge\Theta_{a,b+1,c-1} \\
\mathcal{L}_T(\gamma\wedge\Theta_{a,b,c}) =& -4\lambda\beta\wedge\Theta_{a,b,c} - a\lambda\gamma\wedge\Theta_{a-1,b+1,c} \\
& + 2b\gamma\wedge\Theta_{a+1,b-1,c} - 2b\lambda\gamma\wedge\Theta_{a,b-1,c+1} \\
& - 2b\lambda\alpha\wedge\beta\wedge\gamma\wedge\Theta_{a,b-1,c} + c\gamma\wedge\Theta_{a,b+1,c-1}.
\end{aligned}
$$

Proof. Given that $\mathcal{L}_T$ is a derivation

$$
\begin{aligned}
\mathcal{L}_T(\Theta_{a,b,c}) &= a\left(-\lambda\Theta_1 + \lambda\alpha\wedge\gamma\right)\Theta_{a-1,b,c} \\
&\quad + b\left(2\Theta_0 - 2\lambda\Theta_2 - 2\lambda\alpha\wedge\beta\right)\Theta_{a,b-1,c} + c\Theta_1\Theta_{a,b,c-1} \\
&= -a\lambda\Theta_{a-1,b+1,c} + a\lambda\alpha\wedge\gamma\wedge\Theta_{a-1,b,c} \\
&\quad + 2b\Theta_{a+1,b-1,c} - 2b\lambda\Theta_{a,b-1,c+1} - 2b\lambda\alpha\wedge\beta\wedge\Theta_{a,b-1,c} + c\Theta_{a,b+1,c-1}.
\end{aligned}
$$

Corollary 3.2.9 for β and γ yields the result. □

Proposition 3.2.11.

$$\begin{aligned}\frac{\omega_{2n-k}}{\omega_{2n-k+1}}\mathcal{L}_T\beta_{k,q} =& 2(n-k+q+1)\gamma_{k-1,q} + (k-2q+1)\beta_{k-1,q-1}\\ &+ 2\frac{(n-k+q+1)(k-2q-1)}{(k-2q)}\beta_{k-1,q}\\ &- \frac{\lambda}{2\pi}(k-2q+1)(2n-k+1)\beta_{k+1,q}\\ &- 2\lambda(k-2q-1)\frac{(2n-k+1)(q+1)}{2\pi(k-2q)}\beta_{k+1,q+1}\\ &- (n-k+q)c_{n,k,q}\lambda\alpha\wedge\beta\wedge\gamma\wedge\Theta_{n-k+q-1,k-2q-1,q}\\ \frac{\omega_{2n-k}}{\omega_{2n-k+1}}\mathcal{L}_T\gamma_{k,q} =& -2\lambda\frac{(2n-k+1)(k-2q+1)}{2\pi(n-k+q)}\beta_{k+1,q}\\ &+ (k-2q+1)\gamma_{k-1,q-1}\\ &+ 2(n-k+q+1)\gamma_{k-1,q}\\ &- \lambda(n-k+q-1)\frac{(2n-k+1)(k-2q+1)}{2\pi(n-k+q)}\gamma_{k+1,q}\\ &- \lambda(k-2q)\frac{(2n-k+1)(q+1)}{\pi(k-2q)}\gamma_{k+1,q+1}\\ &- (k-2q)c_{n,k,q}\lambda\alpha\wedge\beta\wedge\gamma\wedge\Theta_{n-k+q-1,k-2q-1,q}\end{aligned}$$

Proof. For $a = n-k+q, b = k-2q-1$ and $c = q$, in Proposition 3.2.10

$$\begin{aligned}\mathcal{L}_T\beta_{k,q} =& 2\frac{c_{n,k,q}}{c_{n,k-1,q}}\gamma_{k-1,q} + q\frac{c_{n,k,q}}{c_{n,k-1,q-1}}\beta_{k-1,q-1}\\ &+ 2(k-2q-1)\frac{c_{n,k,q}}{c_{n,k-1,q}}\beta_{k-1,q} - \lambda(n-k+q)\frac{c_{n,k,q}}{c_{n,k+1,q}}\beta_{k+1,q}\\ &- 2\lambda(k-2q-1)\frac{c_{n,k,q}}{c_{n,k+1,q+1}}\beta_{k+1,q+1}\\ &- (n-k+q)c_{n,k,q}\lambda\alpha\wedge\beta\wedge\gamma\wedge\Theta_{n-k+q-1,k-2q-1,q}.\end{aligned}$$

For $a = n-k+q-1, b = k-2q$ and $c = q$, in Proposition 3.2.10

$$\begin{aligned}\mathcal{L}_T\gamma_{k,q} =& -2\lambda\frac{c_{n,k,q}}{c_{n,k+1,q}}\beta_{k+1,q} + q\frac{c_{n,k,q}}{c_{n,k-1,q-1}}\gamma_{k-1,q-1}\\ &+ 2(k-2q)\frac{c_{n,k,q}}{c_{n,k-1,q}}\gamma_{k-1,q} - \lambda(n-k+q-1)\frac{c_{n,k,q}}{c_{n,k+1,q}}\gamma_{k+1,q}\\ &- 2\lambda(k-2q)\frac{c_{n,k,q}}{c_{n,k+1,q+1}}\gamma_{k+1,q+1}\\ &- (k-2q)c_{n,k,q}\lambda\alpha\wedge\beta\wedge\gamma\wedge\Theta_{n-k+q-1,k-2q-1,q}.\end{aligned}$$

Lastly, we just need to apply the identity $\frac{\omega_n}{\omega_{n-2}} = \frac{2\pi}{n}$. □

Proposition 3.2.11, which we have just established, provides a general formula for the Lie derivative of a basis of the vector space $\Omega^{2n-1}(S\mathbb{C}P^n_\lambda)/(\alpha, d\alpha)$. Therefore, by globalizing these Lie derivatives, given Proposition 2.2.5, we will obtain a closed expression for $\partial_{\lambda,\mathbb{C}}\mu^\lambda_{k,q}$. Recall that $[\![\beta_{k,q}, 0]\!] = \mu^\lambda_{k,q}$ (for $k > 2q$), while $[\![\gamma_{k,q}, 0]\!]$ is given next.

Lemma 3.2.12 ([2]).

$$[\![\gamma_{k,q}, 0]\!] = \mu^\lambda_{k,q} - \lambda\frac{(2n-k)(q+1)}{2\pi(n-k+q)}\mu^\lambda_{k+2,q+1}, \quad n-k+q > 0. \tag{3.5}$$

Proof. Consider $\eta = (\Theta_s - \beta\wedge\gamma)\wedge\Theta_{n-k+q-1,k-2q-1,q}$. Then

$$\begin{aligned} d\eta \equiv & -\gamma\wedge\Theta_{n-k+q-1,k-2q,q} \\ & + 2\beta\wedge\Theta_{n-k+q,k-q-1,q} \\ & - 2\lambda\beta\wedge\Theta_{n-k+q-1,k-2q-1,q+1}, \qquad \text{mod } (\alpha, d\alpha). \end{aligned}$$

Since $[\![d\eta, 0]\!] = 0$, globalizing both members the result follows. □

Proposition 3.2.13. *For $k > 2q$*

$$\begin{aligned} \frac{\omega_{2n-k}}{\omega_{2n-k+1}}\partial_{\lambda,\mathbb{C}}\mu^\lambda_{k,q} &= (k-2q+1)\mu^\lambda_{k-1,q-1} + 2(n-k+q+1)\mu^\lambda_{k-1,q} \\ &\quad - \frac{\lambda}{2\pi}(2n-k+1)\left((k-2q+1)\mu^\lambda_{k+1,q} + 2(q+1)\mu^\lambda_{k+1,q+1}\right) \end{aligned} \tag{3.6}$$

and

$$\frac{\omega_{2n-2q}}{\omega_{2n-2q+1}}\partial_{\lambda,\mathbb{C}}\mu^\lambda_{2q,q} = \mu_{2q-1,q-1} - (2n-2q+1)\frac{\lambda}{2\pi}\mu_{2q+1,q}. \tag{3.7}$$

Proof. Equality (3.6) follows from Proposition 2.2.5 and Proposition 3.2.11 together with Lemma 3.2.12.

Let us now prove (3.7). Note first that from Proposition 3.2.11 and Lemma 3.2.12 we get

$$\begin{aligned} [\![\mathcal{L}_T\gamma_{2j,j}, 0]\!] &= \frac{\omega_{2n-2j+1}}{\omega_{2n-2j}}\mu^\lambda_{2j-1,j-1} \\ &\quad - \frac{\omega_{2n-2j+1}}{\omega_{2n-2j}}\frac{(2n-2j+1)(n+1)}{n-j}\frac{\lambda}{2\pi}\mu^\lambda_{2j+1,j} \\ &\quad + \frac{\omega_{2n-2j+1}}{\omega_{2n-2j}}\frac{(2n-2j+1)(2n-2j-1)(j+1)}{n-j}\frac{\lambda^2}{4\pi^2}\mu^\lambda_{2j+3,j+1} \\ &=: a_j\mu^\lambda_{2j-1,j-1} + b_j\frac{\lambda}{\pi}\mu^\lambda_{2j+1,j} + c_j\frac{\lambda^2}{\pi^2}\mu^\lambda_{2j+3,j+1} \end{aligned} \tag{3.8}$$

Then, by Proposition 2.2.5 and observing that $a_n = 2$

$$
\begin{aligned}
\partial_{\lambda,\mathbb{C}}\mu^\lambda_{2q,q} &= \sum_{i=0}^{n-q-1}\left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!}[\![\mathcal{L}_T\gamma_{2q+2i,q+i},0]\!] + 2\left(\frac{\lambda}{\pi}\right)^{n-q}\frac{n!}{q!}\mu^\lambda_{2n-1,n-1}\\
&= \sum_{i=0}^{n-q}\left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!}a_{q+i}\mu^\lambda_{2q+2i-1,q+i-1}\\
&+ \sum_{i=0}^{n-q-1}\left(\frac{\lambda}{\pi}\right)^{i+1} \frac{(q+i)!}{q!}b_{q+i}\mu^\lambda_{2q+2i+1,q+i}\\
&+ \sum_{i=0}^{n-q-2}\left(\frac{\lambda}{\pi}\right)^{i+2} \frac{(q+i)!}{q!}c_{q+i}\mu^\lambda_{2q+2i+3,q+i+1}\\
&= a_q\mu_{2q-1,q-1} + \frac{\lambda}{\pi}((q+1)a_{q+1}+b_q)\mu^\lambda_{2q+1,q}\\
&+ \sum_{j=2}^{n-q}\left(\frac{\lambda}{\pi}\right)^j\left(\frac{(q+j)!}{q!}a_{q+j} + \frac{(q+j-1)!}{q!}b_{q+j-1} + \frac{(q+j-2)!}{q!}c_{q+j-2}\right)\mu^\lambda_{2q+2j-1,q+j-1}
\end{aligned}
$$

A straightforward computation using $k\omega_k = 2\pi\omega_{k-2}$ shows

$$j(j-1)a_j + (j-1)b_{j-1} + c_{j-2} = 0$$

and the result follows. □

3.2.2 A remarkable isomorphism

Note that by (2.8) the linear map $\Phi_0\colon \operatorname{Val}^{U(n)} \to \operatorname{Val}^{U(n)}$ given by $\Phi_0|_{\operatorname{Val}_k^{U(n)}} = \omega_{2n-k}\,\mathrm{id}$ satisfies

$$\partial_{0,\mathbb{C}} = \Phi_0 \circ \Lambda \circ \Phi_0^{-1}.$$

A similar identity holds for all λ, which will be crucial for our determination of tube formulas in $\mathbb{C}P^n_\lambda$. Let us consider the linear isomorphism

$$\mathcal{F}_{\lambda,\mathbb{C}}\colon \operatorname{Val}^{U(n)} \longrightarrow \mathcal{V}^n_{\lambda,\mathbb{C}}, \quad \mu_{k,q} \longmapsto \mu^\lambda_{k,q}.$$

Theorem 3.2.14. *The linear isomorphism*

$$\Phi_\lambda := \mathcal{F}_{\lambda,\mathbb{C}} \circ \Phi_0\colon \operatorname{Val}^{U(n)} \longrightarrow \mathcal{V}^n_{\lambda,\mathbb{C}}, \qquad \mu_{k,q} \longmapsto \omega_{2n-k}\mu^\lambda_{k,q}.$$

fulfills

$$\partial_{\lambda,\mathbb{C}} = \Phi_\lambda \circ (\Lambda - \lambda L) \circ \Phi_\lambda^{-1}.$$

Proof. By combining Proposition 3.2.13, Proposition 2.1.6 and the fact $\frac{\omega_n}{\omega_{n-2}} = \frac{2\pi}{n}$, this is straightforward to check:

$$\begin{aligned}
\Phi_\lambda \circ (\Lambda - \lambda L)(\mu_{k,q}) &= (k-2q+1)\omega_{2n-k+1}\mu^\lambda_{k-1,q-1} + 2(n-k+q+1)\omega_{2n-k+1}\mu^\lambda_{k-1,q} \\
&\quad - \lambda(k-2q+1)\omega_{2n-k-1}\mu^\lambda_{k+1,q} - 2\lambda(q+1)\omega_{2n-k-1}\mu^\lambda_{k+1,q+1} \\
&= \omega_{2n-k+1}\Big((k-2q+1)\mu^\lambda_{k-1,q-1} + 2(n-k+q+1)\mu^\lambda_{k-1,q} \\
&\quad - \frac{\lambda}{2\pi}(2n-k+1)\Big((k-2q+1)\mu^\lambda_{k+1,q} + 2(q+1)\mu^\lambda_{k+1,q+1}\Big) \\
&= \omega_{2n-k}\partial_{\lambda,\mathbb{C}}\mu^\lambda_{k,q} = \partial_{\lambda,\mathbb{C}} \circ \Phi_\lambda(\mu_{k,q}).
\end{aligned}$$
□

Remark 3.2.15. The linear isomorphism we have constructed serves as a valuable tool for expressing the derivative operator simply in terms of the $\mathfrak{sl}_2$-representation within $\mathrm{Val}^{U(n)}$. However, it is important to note that this isomorphism does *not* constitute an algebra isomorphism.

In real space forms we have a result similar to Theorem 3.2.14 but only in a hyperplane of $\mathcal{V}^m_{\lambda,\mathbb{R}}$.

Theorem 3.2.16. *The linear monomorphism*

$$\Psi_\lambda\colon \mathrm{Val}^{O(m)} \longrightarrow \mathcal{V}^{m+1}_{\lambda,\mathbb{R}}, \qquad \mu_k \longmapsto \sigma^\lambda_k$$

fulfills

$$\partial_{\lambda,\mathbb{R}} \circ \Psi_\lambda = \Psi_\lambda \circ (\Lambda - \lambda L).$$

Proof. By Proposition 3.1.2 and Theorem 2.1.9

$$\begin{aligned}
\partial_{\lambda,\mathbb{R}} \circ \Psi_\lambda(\mu_k) &= \partial_{\lambda,\mathbb{R}}\sigma^\lambda_k = (m-k+1)\sigma^\lambda_{k-1} - \lambda(k+1)\sigma^\lambda_{k+1} \\
&= \Psi_\lambda((m-k+1)\mu_{k-1} - \lambda(k+1)\mu_{k+1}) \\
&= \Psi_\lambda(\Lambda\mu_k - \lambda L\mu_k). \quad \square
\end{aligned}$$

Note the dimension difference between the source and the target of Ψ_λ. We will show that there is *no* isomorphism between $\mathrm{Val}^{O(m)}$ and $\mathcal{V}^m_{\lambda,\mathbb{R}}$ intertwining ∂ and $\Lambda - \lambda L$. This is essentially because (3.2) and (3.3) differ from (3.1).

3.3 A model space for tube formulas

We next perform some abstract computations that will easily lead to the tube formulas in both complex and real space forms via (3.24) and (3.29). The same approach will allow us to determine the kernel, the image, and the spectrum of the derivative operator on $\mathcal{V}^m_{\lambda,\mathbb{R}}$ and $\mathcal{V}^n_{\lambda,\mathbb{C}}$.

3.3.1 A system of differential equations

Consider the decomposition

$$\mathbb{C}[x,y] = \bigoplus_{m\geq 0} V^{(m)} \tag{3.9}$$

where $V^{(m)}$ consists of m-homogeneous polynomials in two variables

$$V^{(m)} := \langle x^k y^{m-k} : 0 \leq k \leq m \rangle,$$

and recall that $X = x\frac{\partial}{\partial y}$, $Y = y\frac{\partial}{\partial x}$, and $H = [X,Y]$ induce an $\mathfrak{sl}_2$-structure on each $V^{(m)}$. It follows that X, Y, H induce an $\mathfrak{sl}_2$-structure on $\mathbb{C}[x,y]$, whose decomposition into irreducible components is (3.9).

In Proposition 1.5.6, we established that any finite-dimensional irreducible representation of $\mathfrak{sl}_2$ is isomorphic to some $V^{(m)}$.

Motivated by Theorem 3.2.14, we consider

$$Y_\lambda = Y - \lambda X = y\frac{\partial}{\partial x} - \lambda x \frac{\partial}{\partial y},$$

which is a derivation on $\mathbb{C}[x,y]$. Observe that $V^{(m)}$ is Y_λ-invariant for all $m \geq 0$. Therefore, we can consider $Y_\lambda|_{V^{(m)}}$. It will be sometimes convenient to consider the monomials $p_{m,k}(x,y) := \binom{m}{k} x^k y^{m-k}$. In these terms

$$Y_\lambda p_{m,k} = (m-k+1)p_{m,k-1} - \lambda(k+1)p_{m,k+1}. \tag{3.10}$$

Our goal here is to solve the following Cauchy problem: find $p_{m,k} \colon \mathbb{R} \to V^{(m)}$ such that

$$\begin{cases} \dfrac{d}{dt} p_{m,k}(t) = Y_\lambda p_{m,k}(t), \\ p_{m,k}(0) = \dbinom{m}{k} x^k y^{m-k}, \quad 0 \leq k \leq m. \end{cases} \tag{3.11}$$

Since (3.11) is a linear system of differential equations, we determine its solution by

$$\begin{aligned} p_{m,k}(t) &= \exp(t\, Y_\lambda|_{V^{(m)}}) p_{m,k} \\ &= \binom{m}{k} \exp(t\, Y_\lambda|_{V^{(m)}})(x^k y^{m-k}), \quad 0 \leq k \leq m. \end{aligned} \tag{3.12}$$

Given that $\dim V^{(m)} < \infty$ and $Y_\lambda|_{V^{(m)}} \in \operatorname{End}(V^{(m)})$, the function $\exp(t\, Y_\lambda|_{V^{(m)}})$ is always defined and analytic in t and λ. We simply write $\exp(tY_\lambda)$.

We will use the standard notation

$$\sin_\lambda(t) := \begin{cases} \dfrac{\sin(\sqrt{\lambda} t)}{\sqrt{\lambda}} & \lambda > 0, \\ t & \lambda = 0, \\ \dfrac{\sinh(\sqrt{|\lambda|} t)}{\sqrt{|\lambda|}} & \lambda < 0, \end{cases}$$

which is an analytic function in both λ and t, and $\cos_\lambda(t) := \frac{d}{dt}\sin_\lambda(t)$.

Proposition 3.3.1. *For any $\lambda, t \in \mathbb{R}$, we have*

$$\exp(tY_\lambda)x = x\cos_\lambda(t) + y\sin_\lambda(t) =: u, \quad \exp(tY_\lambda)y = y\cos_\lambda(t) - \lambda x\sin_\lambda(t) =: v.$$

Proof. Since clearly

$$Y_\lambda^{2k}x = (-\lambda)^k x, \quad Y_\lambda^{2k+1}x = (-\lambda)^k y,$$

we have

$$\begin{aligned}\exp(tY_\lambda)x &= \sum_{k\geq 0}\frac{t^k}{k!}Y_\lambda^k x \\ &= \sum_{k\geq 0}\frac{t^{2k}}{(2k)!}(-\lambda)^k x + \sum_{k\geq 0}\frac{t^{2k+1}}{(2k+1)!}(-\lambda)^k y \\ &= x\cos_\lambda(t) + y\sin_\lambda(t).\end{aligned}$$

In the same way, we can compute $\exp(tY_\lambda)y$. Since

$$Y_\lambda^{2k}y = (-\lambda)^k y, \quad Y_\lambda^{2k+1}y = (-\lambda)^{k+1} x,$$

we have

$$\begin{aligned}\exp(tY_\lambda)y &= \sum_{k\geq 0}\frac{t^k}{k!}Y_\lambda^k y \\ &= \sum_{k\geq 0}\frac{t^{2k}}{(2k)!}(-\lambda)^k y + \sum_{k\geq 0}\frac{t^{2k+1}}{(2k+1)!}(-\lambda)^{k+1} x \\ &= y\sum_{k\geq 0}\frac{t^{2k}}{(2k)!}(-\lambda)^k - \lambda x\sum_{k\geq 0}\frac{t^{2k+1}}{(2k+1)!}(-\lambda)^k \\ &= y\cos_\lambda(t) - \lambda x\sin_\lambda(t). \quad \square\end{aligned}$$

The following standard and elementary fact will be useful.

Lemma 3.3.2. *Let* **A** *be a finite-dimensional algebra. A vector field on the underlying vector space of* **A** *is a derivation if and only if its flow ϕ_t satisfies*

$$\phi_t(p \cdot q) = \phi_t(p) \cdot \phi_t(q), \quad \forall p, q \in \mathbf{A}, \forall t \in \mathbb{R}.$$

In other words, each ϕ_t is an **A***-morphism.*

Proof. Assume that $X = \phi_t'$ is a derivation. Let $p, q \in \mathbf{A}$ be and consider $g(t) := \phi_t(p)\phi_t(q)$. Since X is a derivation

$$\begin{aligned}X(g(t)) &= X(\phi_t(p))\phi_t(q) + \phi_t(p)X(\phi_t(q)) \\ &= \phi_t'(p)\phi_t(q) + \phi_t(p)\phi_t'(q) \\ &= \frac{d}{dt}\phi_t(p)\phi_t(q) = g'(t).\end{aligned}$$

Therefore, $g(t)$ is an integral curve of X. Since $g(0) = \phi_0(pq)$ it follows that $g(t) = \phi_t(pq)$.

Conversely, assume that ϕ_t is an **A**-morphism for all $t \in \mathbb{R}$. Then, for all $p, q \in \mathbf{A}$ and $t \in \mathbb{R}$, we have

$$\phi_t(p \cdot q) = \phi_t(p) \cdot \phi_t(q).$$

Differentiating both sides with respect to t at $t = 0$, we obtain

$$\left.\frac{d}{dt}\right|_{t=0} \phi_t(p \cdot q) = \left.\frac{d}{dt}\right|_{t=0} (\phi_t(p) \cdot \phi_t(q)).$$

Using the fact that $\left.\frac{d}{dt}\right|_{t=0}\phi_t(p) = X(p)$, we get

$$X(p \cdot q) = X(p) \cdot q + p \cdot X(q).$$

Therefore, X is a derivation. □

Theorem 3.3.3. *The solution of the Cauchy problem* (3.11) *is*

$$p_{m,k}(t) = \binom{m}{k} u^k v^{m-k} \tag{3.13}$$

$$= \binom{m}{k} (x \cos_\lambda(t) + y \sin_\lambda(t))^k (y \cos_\lambda(t) - \lambda x \sin_\lambda(t))^{m-k} \tag{3.14}$$

$$= \sum_{j=0}^{m} \phi^\lambda_{m,k,j}(t) \binom{m}{j} x^j y^{m-j}, \tag{3.15}$$

where

$$\phi^\lambda_{m,k,j}(t) = \sum_{h \geq 0} (-\lambda)^{j-h} \binom{m-j}{k-h} \binom{j}{h} \sin_\lambda^{k+j-2h}(t) \cos_\lambda^{m-k-j+2h}(t). \tag{3.16}$$

Proof. Since Y_λ is a derivation, $\exp(tY_\lambda)$ is a $\mathbb{C}[x, y]$-morphism by Lemma 3.3.2. Hence

$$\exp(tY_\lambda) x^k y^{m-k} = (\exp(tY_\lambda)x)^k (\exp(tY_\lambda)y)^{m-k} = u^k v^{m-k}.$$

Comparing with (3.12) yields (3.14).

It remains to prove (3.15). Putting $s = \sin_\lambda(t), c = \cos_\lambda(t)$ we have

$$\begin{aligned}
\binom{m}{k}(xc + ys)^k (yc - \lambda xs)^{m-k} &= \binom{m}{k} \sum_{a,b} \binom{k}{a} (ys)^a (xc)^{k-a} \binom{m-k}{b} (-\lambda xs)^b (yc)^{m-k-b} \\
&= \binom{m}{k} \sum_{a,b} \binom{k}{a} \binom{m-k}{b} (-\lambda)^b s^{a+b} c^{m-a-b} x^{k-a+b} y^{m-k+a-b} \\
&= \binom{m}{k} \sum_{j,h} \binom{k}{h} \binom{m-k}{j-h} (-\lambda)^{j-h} s^{j+k-2h} c^{m-j-k+2h} x^j y^{m-j}
\end{aligned}$$

where we changed $a = k - h, b = j - h$. Using

$$\binom{m}{k}\binom{k}{h}\binom{m-k}{j-h} = \binom{m-j}{k-h}\binom{j}{h}\binom{m}{j}, \tag{3.17}$$

which is elementary, yields (3.15). □

3.3.2 Eigenvalues and eigenvectors of Y_λ

Given $f\colon V \to V$ an endomorphism of vector spaces, we denote by $\mathrm{spec}(f)$ the set of eigenvalues of f and by $E_\alpha(f)$ the eigenspace associated with each $\alpha \in \mathrm{spec}(f)$.

Lemma 3.3.4. *The endomorphism $Y_\lambda|_{V^{(m)}}$ is diagonalizable with simple multiplicities and*

$$\mathrm{spec}(Y_\lambda|_{V^{(m)}}) = \left\{(2k-m)\sqrt{-\lambda} : 0 \le k \le m\right\},$$
$$E_{(2k-m)\sqrt{-\lambda}}(Y_\lambda|_{V^{(m)}}) = \mathbb{C}\cdot e_1^k e_2^{m-k},$$

where $e_1 := y + \sqrt{-\lambda}x$ and $e_2 := y - \sqrt{-\lambda}x$.

Proof. The result is trivial to check for $m = 1$. Since

$$Y_\lambda(x) = y,$$
$$Y_\lambda(y) = -\lambda x,$$

we have

$$Y_\lambda(e_1) = \sqrt{-\lambda}y - \lambda x = \sqrt{-\lambda}e_1,$$
$$Y_\lambda(e_2) = -\sqrt{-\lambda}y - \lambda x = -\sqrt{-\lambda}e_2.$$

Given that Y_λ is a derivation

$$Y_\lambda e_1^k = k e_1^{k-1} Y_\lambda e_1 = k\sqrt{-\lambda}e_1^k,$$
$$Y_\lambda e_2^{m-k} = (m-k)e_2^{m-k-1}Y_\lambda e_2 = -\sqrt{-\lambda}(m-k)e_2^{m-k}.$$

Hence

$$Y_\lambda(e_1^k e_2^{m-k}) = (2k-m)\sqrt{-\lambda}e_1^k e_2^{m-k},$$

as stated. ∎

Remark 3.3.5. It is interesting to notice that the spectra of Y_λ and $\sqrt{-\lambda}H$ when restricted to each $V^{(m)}$, are identical. These two operators are thus intertwined by the linear isomorphism $x^k y^{m-k} \mapsto e_1^k e_2^{m-k}$.

Lemma 3.3.6. *For $0 \le k \le m$, the eigenvector asociated to $(2k-m)\sqrt{-\lambda}$ is given by*

$$\binom{m}{k}(y+\sqrt{-\lambda}x)^k(y-\sqrt{-\lambda}x)^{m-k} = \sum_{a,b}\binom{a+b}{a}\binom{m-a-b}{k-a}(-1)^b(\sqrt{-\lambda})^{a+b}p_{m,a+b}$$

Proof. Using the binomial theorem, we have

$$(y+\sqrt{-\lambda}x)^k = \sum_{a=0}^{k}(\sqrt{-\lambda})^a\binom{k}{a}x^a y^{k-a}.$$

Similarly,

$$(y-\sqrt{-\lambda}x)^{m-k} = \sum_{b=0}^{m-k}(-1)^b(\sqrt{-\lambda})^b\binom{m-k}{b}x^b y^{m-k-b}.$$

Therefore

$$\binom{m}{k}(y+\sqrt{-\lambda}x)^k(y-\sqrt{-\lambda}x)^{m-k} = \sum_{a,b}\binom{m}{k}\binom{k}{a}\binom{m-k}{b}(-1)^b(\sqrt{-\lambda})^{a+b}x^{a+b}y^{m-a-b}.$$

Finally, we only need

$$\binom{m}{k}\binom{k}{a}\binom{m-k}{b} = \binom{a+b}{a}\binom{m-a-b}{k-a}\binom{m}{a+b}.$$

which is straightforward. □

3.3.3 Image of Y_λ

Using Lemma 3.3.4, we can conclude that $Y_\lambda|_{V(m)}$ is bijective if and only if m is odd. If m is even, then the kernel is one-dimensional. An explicit description is the following.

Proposition 3.3.7. *If m is even, then*

$$\operatorname{im}(Y_\lambda|_{V(m)}) = \ker Z_{m,\lambda}, \tag{3.18}$$

where

$$Z_{m,\lambda} := \left(\frac{\partial^2}{\partial x^2} + \lambda\frac{\partial^2}{\partial y^2}\right)^{m/2}.$$

Proof. By the binomial formula

$$\begin{aligned} Z_{m,\lambda}(x^k y^{m-k}) &= \sum_{i=0}^{m/2}\lambda^{m/2-i}\binom{m/2}{i}\frac{\partial^m}{\partial x^{2i}\partial y^{m-2i}}x^k y^{m-k}\delta_{k,2i} \\ &= \lambda^{\frac{m-k}{2}}\binom{m/2}{k/2}k!(m-k)! \end{aligned} \tag{3.19}$$

if k is even, and $Z_{m,\lambda}(x^k y^{m-k}) = 0$ if k is odd. Therefore

$$\begin{aligned} Z_{m,\lambda}\circ Y_\lambda(x^{2l+1}y^{m-2l-1}) &= Z_{m,\lambda}((2l+1)x^{2l}y^{m-2l} - \lambda(m-2l-1)x^{2l+2}y^{m-2l-2}) \\ &= \lambda^{\frac{m}{2}-l}\binom{m/2}{l}(2l+1)!(m-2l)! \\ &\quad - \lambda^{\frac{m}{2}-l}\binom{m/2}{l+1}(2l+2)!(m-2l-1)! = 0 \\ Z_{m,\lambda}\circ Y_\lambda(x^{2l}y^{m-2l}) &= 0. \end{aligned}$$

This shows that $\operatorname{im}(Y_\lambda)$ is a subspace of $\ker Z_{m,\lambda}$. Given that $Z_{m,\lambda}$ is not zero, we have $\dim\ker Z_{m,\lambda} = m$, and by Lemma 3.3.4, we know that the image of $Y_\lambda|_{V(m)}$ has the same dimension. This yields (3.18) □

Next we compute, for even m and given φ in the image of $Y_\lambda|_{V(m)}$, the preimage $Y_\lambda^{-1}(\{\varphi\})$.

Proposition 3.3.8. *Consider*

$$P_{m,k} := \sum_{j\geq 0} \lambda^j \frac{(k+2j-1)!!(m-k-2j-1)!!}{(k-1)!!(m-k+1)!!} p_{m,k+2j} \in V^{(m)}. \tag{3.20}$$

Then

$$Y_\lambda P_{m,k} = p_{m,k-1} - c_{m,k} x^m \tag{3.21}$$

where $c_{m,k} = 0$ *if* $m-k$ *is even, and otherwise*

$$c_{m,k} = \lambda^{\frac{m-k+1}{2}} \frac{m!!}{(k-1)!!(m-k+1)!!}. \tag{3.22}$$

Proof. Using (3.10)

$$Y_\lambda p_{m,k+2j} = (m-k-2j+1)p_{m,k+2j-1} - \lambda(k+2j+1)p_{m,k+2j+1}.$$

$$\begin{aligned}
&(k-1)!!(m-k+1)!! Y_\lambda P_{m,k} \\
=&\sum_{j\geq 0} \lambda^j (k+2j-1)!!(m-k-2j-1)!! \left\{(m-k-2j+1)p_{m,k+2j-1} - \lambda(k+2j+1)p_{m,k+2j+1}\right\} \\
=&\sum_{l\geq 0} \lambda^{\lfloor (l+1)/2 \rfloor}(k+l)!!(m-k-l-2)!!(m-k-l) - \sum_{l\geq 0} \lambda^{\lfloor (l-1)/2\rfloor+1}(k+l-2)!!(m-k-l)!!(k-l)p_{m,k+l} \\
=&p_{m,k-1} - c_{m,k}x^m.
\end{aligned}$$

□

With these ingredients at hand, for even m, we can now compute a preimage by Y_λ of any element in $\operatorname{im} Y_\lambda$ as follows.

Proposition 3.3.9. *Let* $\Pi\colon V^{(m)} \to V^{(m)}$ *be given by* $p_{m,k} \mapsto P_{m,k+1}$. *If* m *is even then*

$$Y_\lambda \circ \Pi(\varphi) = \varphi, \qquad \forall \varphi \in \operatorname{im} Y_\lambda|_{V^{(m)}} \tag{3.23}$$

Proof. Let $0 < k < m$. Since $(m-k+1)c_{m,k} - \lambda(k+1)c_{m,k+2} = 0$, using (3.10) and (3.20) we get

$$\begin{aligned}
(Y_\lambda \circ \Pi \circ Y_\lambda)p_{m,k} =& Y_\lambda \circ \Pi\left((m-k+1)p_{m,k-1} - \lambda(k+1)p_{m,k+1}\right) \\
=&(m-k+1)Y_\lambda P_{m,k} - \lambda(k+1)Y_\lambda P_{m,k+2} \\
=&(m-k+1)p_{m,k-1} - \lambda(k+1)p_{m,k+1} \\
&- ((m-k+1)c_{m,k} - \lambda(k+1)c_{m,k+2})x^m = Y_\lambda p_{m,k}.
\end{aligned}$$

For $k=0$ and $k=m$,

$$\begin{aligned}
Y_\lambda \circ \Pi \circ Y_\lambda(y^m) &= -\lambda Y_\lambda(P_{m,2}) = -\lambda(mxy^{m-1} - c_{m,2}x^m) = Y_\lambda(y^m) + \lambda c_{m,2}x^m, \\
Y_\lambda \circ \Pi \circ Y_\lambda(x^m) &= Y_\lambda(P_{m,m}) = mx^{m-1}y - c_{m,m}x^m = Y_\lambda(x^m) + c_{m,m}x^m.
\end{aligned}$$

Since $c_{m,m} = 0$, and $c_{m,2} = 0$ if m is even, the result follows. □

The following consequence of Proposition 3.3.8 will be useful later.

Proposition 3.3.10.

$$p_{m,k} = \sum_{j\geq 0} \lambda^j \frac{(k+2j)!!(m-k-2j-2)!!}{k!!(m-k)!!} Y_\lambda p_{m,k+2j+1} + c_{m,k+1}x^m$$

Proof. Applying Y_λ on both sides of (3.20), and using (3.21) yields the result. □

3.4 Tube formulas in complex space forms

Recalling (2.15) and Proposition 2.1.8, we get an isomorphism $I_n \colon W_n \to \mathrm{Val}^{U(n)}$ of $\mathfrak{sl}_2$-modules from

$$W_n := \bigoplus_{0\leq 2r\leq n} V^{(2n-4r)}$$

to $\mathrm{Val}^{U(n)}$ by putting $I_n(y^{2n-4r}) = \pi_{2r,r}$ (i.e. mapping Y-primitive elements to Λ-primitive elements) and

$$\begin{aligned} I_n(p_{2n-4r,k-2r}) &= \frac{1}{(k-2r)!} I_n(X^{k-2r}(y^{2n-4r})) \\ &= \frac{1}{(k-2r)!} L^{k-2r} I_n(y^{2n-4r}) = \frac{1}{(k-2r)!}\pi_{k,r}. \end{aligned}$$

By Theorem 3.2.14, the map $J_{\lambda,\mathbb{C}} := \Phi_\lambda \circ I_n \colon W_n \to \mathcal{V}^n_{\lambda,\mathbb{C}}$ fulfills

$$\partial_{\lambda,\mathbb{C}} \circ J_{\lambda,\mathbb{C}} = J_{\lambda,\mathbb{C}} \circ Y_\lambda. \tag{3.24}$$

We define

$$\sigma^\lambda_{k,r} := J_\lambda(p_{2n-4r,k-2r}) = \frac{\omega_{2n-k}}{(k-2r)!}\pi^\lambda_{k,r}, \tag{3.25}$$

which satisfies

$$\partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k,r} = (2n-k-2r+1)\sigma^\lambda_{k-1,r} - \lambda(k-2r+1)\sigma^\lambda_{k+1,r}. \tag{3.26}$$

We arrive at our main theorem.

Theorem 3.4.1. *The tubular operator* $\mathbf{T}_t$ *in* $\mathcal{V}^n_{\lambda,\mathbb{C}}$ *is given by*

$$\mathbf{T}_t(\sigma^\lambda_{k,r}) = \sum_{j=0}^{2n-4r} \phi^\lambda_{2n-4r,k-2r,j}(t)\sigma^\lambda_{j+2r,r},$$

where

$$\phi^\lambda_{m,i,j}(t) = \sum_{h\geq 0}(-\lambda)^{j-h}\binom{m-j}{i-h}\binom{j}{h}\sin_\lambda^{i+j-2h}(t)\cos_\lambda^{m-i-j+2h}(t).$$

Proof. By (2.36), using (3.24) and (3.25), and putting $m = 2n - 4r$, we get

$$\begin{aligned}\mathbf{T}_t\sigma_{k,r}^{\lambda} &= \exp(t\partial_{\lambda,\mathbb{C}})(\sigma_{k,r}^{\lambda})\\ &= \exp(t\partial_{\lambda,\mathbb{C}}) \circ J_{\lambda,\mathbb{C}}(p_{m,k-2r})\\ &= J_{\lambda,\mathbb{C}} \circ \exp(tY_\lambda)(p_{m,k-2r})\\ &= J_{\lambda,\mathbb{C}}(p_{m,k-2r}(t)).\end{aligned}$$

Using (3.15) the result follows. □

Remark 3.4.2. It is also worth noting that $\mathcal{V}_{\lambda,\mathbb{C}}^n = \bigoplus_{0\le 2r\le n} \mathcal{I}_{\lambda,\mathbb{C}}^{n,r}$ where

$$\mathcal{I}_{\lambda,\mathbb{C}}^{n,r} := J_{\lambda,\mathbb{C}}(V^{(2n-4r)}) = \left\{\sigma_{k,r}^{\lambda} : 2r \le k \le 2n - 2r\right\}, \tag{3.27}$$

and that these subspaces are $\partial_{\lambda,\mathbb{C}}$-invariant. In particular, given $\varphi \in \mathcal{I}_{\lambda,\mathbb{C}}^{n,r}$ one has $\mathbf{T}_t(\varphi) \in \mathcal{I}_{\lambda,\mathbb{C}}^{n,r}$. This definition is an extension of (2.19) within $\mathrm{Val}^{U(n)}$.

The tube formulas in terms of the $\tau_{k,i}^{\lambda}$ can be obtained from Theorem 3.4.1 using (2.18) and (2.21) which hold verbatim replacing $\pi_{k,r}^{\lambda}, \tau_{k,r}^{\lambda}$ for $\pi_{k,r}, \tau_{k,r}$.

Remark 3.4.3. The tube formula for the volume $\sigma_{2n,0}^{\lambda} = \mathrm{vol}_{\mathbb{C}P_\lambda^n}$ is given by the following simple expression

$$\mathrm{vol}_{\mathbb{C}P_\lambda^n}(A_t) = \sum_{j=0}^{2n} \sin_\lambda^{2n-j}(t)\cos_\lambda^j(t)\sigma_{j,0}^{\lambda}(A), \tag{3.28}$$

which is Theorem 4.3 of [21], since $\sigma_{j,0}^{\lambda} = \omega_{2n-j}\tau_{j,0}^{\lambda} = \Phi_\lambda(\mu_j)$. The tube formulas $\mathbf{T}_t\sigma_{2n-2r,r}^{\lambda}$ are equally simple

$$\mathbf{T}_t\sigma_{2n-2r,r}^{\lambda} = \sum_{j=0}^{2n-4r} \sin_\lambda^{2n-4r-j}(t)\cos_\lambda^j(t)\sigma_{j+2r,r}^{\lambda}.$$

The tube formula for $\sigma_{2r,r}^{\lambda}$ is also simple

$$\mathbf{T}_t\sigma_{2r,r}^{\lambda} = \sum_{j=0}^{2n-4r} (-\lambda)^j \sin_\lambda^j(t)\cos_\lambda^{2n-4r-j}(t)\sigma_{j+2r,r}^{\lambda}.$$

We have a clear explanation of this phenomenon. By Theorem 3.3.3

$$\begin{aligned}p_{2n-4r,2n-4r}(t) &= \exp(tY_\lambda)x^{2n-4r} = (x\cos_\lambda(t) + y\sin_\lambda(t))^{2n-4r}\\ p_{2n-4r,0}(t) &= \exp(tY_\lambda)y^{2n-4r} = (y\cos_\lambda(t) - \lambda x\sin_\lambda(t))^{2n-4r}.\end{aligned}$$

Finally, since

$$\mathbf{T}_t\sigma_{2r,r} = J_\lambda(p_{2n-4r,0}(t)), \quad \mathbf{T}_t\sigma_{2n-4r,r} = J_\lambda(p_{2n-4r,2n-4r}(t)),$$

we have that the simplicity of the formulas is due to the lack of x or y in each polynomic representation of $\sigma_{2r,r}^{\lambda}$ and $\sigma_{2n-4r,r}^{\lambda}$.

Remark 3.4.4. An interesting feature of the previous tube formulas is the following *self-similarity property*, which is explained by (3.24). Let

$$\mathbf{G}^{\lambda}_{n,j}\colon \mathcal{V}^{n}_{\lambda,\mathbb{C}} \longrightarrow \mathcal{V}^{n+2j}_{\lambda,\mathbb{C}}, \qquad \mathbf{G}^{\lambda}_{n,j}(\sigma^{\lambda}_{k,r}) = \sigma^{\lambda}_{k+2j,r+j}.$$

Then one has $\mathbf{T}_t \circ \mathbf{G}^{\lambda}_{n,j} = \mathbf{G}^{\lambda}_{n,j} \circ \mathbf{T}_t$.

Example 3.4.5. To illustrate the *self-similarity property*, we present an example in lower dimensions. It is essential to note that the maps $\mathbf{G}^{\lambda}_{n,j}$ connect spaces with the same dimensional parity, dividing $\{\mathcal{V}^{n}_{\lambda,\mathbb{C}}\}_{n\in\mathbb{N}}$ into two equivalence classes. Let us consider $n = 2$ and $j = 1$. Thus, we have the map $\mathbf{G}^{\lambda}_{2,1}\colon \mathcal{V}^{2}_{\lambda,\mathbb{C}} \to \mathcal{V}^{4}_{\lambda,\mathbb{C}}$. The space $\mathcal{V}^{2}_{\lambda,\mathbb{C}}$ has two invariant subspaces

$$\mathcal{V}^{2}_{\lambda,\mathbb{C}} = \mathcal{I}^{2,0}_{\lambda,\mathbb{C}} \oplus \mathcal{I}^{2,1}_{\lambda,\mathbb{C}}$$

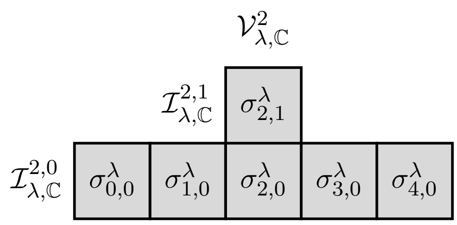
and $\mathcal{V}^{4}_{\lambda,\mathbb{C}}$ has three of them

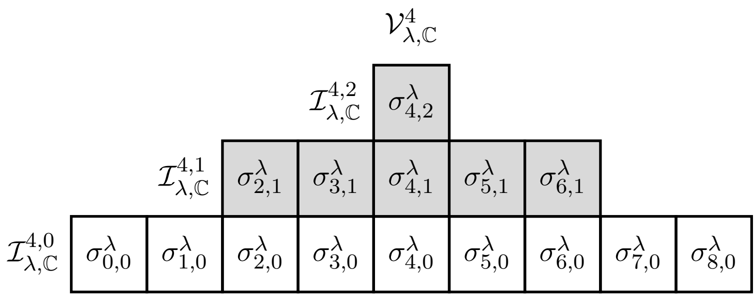
$$\mathcal{V}^{4}_{\lambda,\mathbb{C}} = \mathcal{I}^{4,0}_{\lambda,\mathbb{C}} \oplus \mathcal{I}^{4,1}_{\lambda,\mathbb{C}} \oplus \mathcal{I}^{4,2}_{\lambda,\mathbb{C}}.$$

According to the definition of $\mathbf{G}^{\lambda}_{2,1}$, we have

$$\mathbf{G}^{\lambda}_{2,1}\left(\mathcal{V}^{2}_{\lambda,\mathbb{C}}\right) = \mathcal{I}^{4,1}_{\lambda,\mathbb{C}} \oplus \mathcal{I}^{4,2}_{\lambda,\mathbb{C}},$$

This scenario can be represented using two diagrams, constructed as follows. Each invariant subspace is represented as a row of squares and each square as a basic element $\sigma^{\lambda}_{k,r}$. The subspace $\mathcal{I}^{n,0}_{\lambda,\mathbb{C}}$ forms the base of a pyramid. By stacking the subsequent rows, we construct the corresponding diagram. In this example, we have two diagrams: one associated with $\mathcal{V}^{2}_{\lambda,\mathbb{C}}$ and another with $\mathcal{V}^{4}_{\lambda,\mathbb{C}}$. The domain and range of the map $\mathbf{G}^{\lambda}_{2,1}$ are shaded accordingly.





Example 3.4.6. We draw a diagram for an odd dimension to observe the structural difference. Let us consider $n = 5$.

$\mathcal{V}^5_{\lambda,\mathbb{C}}$

$\mathcal{I}^{5,2}_{\lambda,\mathbb{C}}$					$\sigma^\lambda_{4,2}$	$\sigma^\lambda_{5,2}$					
$\mathcal{I}^{5,1}_{\lambda,\mathbb{C}}$			$\sigma^\lambda_{2,1}$	$\sigma^\lambda_{3,1}$	$\sigma^\lambda_{4,1}$	$\sigma^\lambda_{5,1}$	$\sigma^\lambda_{6,1}$	$\sigma^\lambda_{7,1}$	$\sigma^\lambda_{8,1}$		
$\mathcal{I}^{5,0}_{\lambda,\mathbb{C}}$	$\sigma^\lambda_{0,0}$	$\sigma^\lambda_{1,0}$	$\sigma^\lambda_{2,0}$	$\sigma^\lambda_{3,0}$	$\sigma^\lambda_{4,0}$	$\sigma^\lambda_{5,0}$	$\sigma^\lambda_{6,0}$	$\sigma^\lambda_{7,0}$	$\sigma^\lambda_{8,0}$	$\sigma^\lambda_{9,0}$	$\sigma^\lambda_{10,0}$

The invariant subspaces possess different dimensions, which discards the possibility of finding an isomorphism between them.

3.5 Tube formulas in real space forms

Let $I_m\colon V^{(m)} \to \mathrm{Val}^{O(m)}$ be the isomorphism of irreducible $\mathfrak{sl}_2$-representations determined by $I_m(y^m) = \chi$; i.e.

$$\binom{m}{i} I_m(x^i y^{m-i}) = \frac{1}{i!} I_m(X^i(y^m)) = \frac{1}{i!} L^i(I(y^m))$$
$$= \frac{1}{i!} L^i(\mu_0) = \mu_i$$

where we used (2.9). By Theorem 3.2.16, the map $J_{\lambda,\mathbb{R}} = \Psi_\lambda \circ I_m$ satisfies

$$\partial_{\lambda,\mathbb{R}} \circ J_{\lambda,\mathbb{R}} = J_{\lambda,\mathbb{R}} \circ Y_\lambda. \tag{3.29}$$

The map $J_{\lambda,\mathbb{R}}$ is explicitly given by

$$J_{\lambda,\mathbb{R}}\colon V^{(m)} \longrightarrow \mathcal{V}^{m+1}_{\lambda,\mathbb{R}}, \qquad p_{m,i} \longmapsto \sigma^\lambda_i. \tag{3.30}$$

The image of $J_{\lambda,\mathbb{R}}$ is the hyperplane $\mathcal{H}^{m+1}_\lambda := \mathrm{im}\, J_{\lambda,\mathbb{R}} = \langle \sigma^\lambda_0, \dots, \sigma^\lambda_m \rangle$.

Theorem 3.5.1. *The tubular operator on $\mathcal{V}^{m+1}_{\lambda,\mathbb{R}}$ is given as follows. For $0 \le i \le m$,*

$$\mathbf{T}_t \sigma^\lambda_i = \sum_{j=0}^m \phi^\lambda_{m,i,j}(t) \sigma^\lambda_j. \tag{3.31}$$

In particular

$$\mathbf{T}_t \sigma^\lambda_m = \sum_{j=0}^m \sin^{m-j}_\lambda(t) \cos^j_\lambda(t) \sigma^\lambda_j, \tag{3.32}$$

and thus

$$\mathbf{T}_t \sigma^\lambda_{m+1} = \sum_{j=0}^m \left(\int_0^t \sin^{m-j}_\lambda(s) \cos^j_\lambda(s) ds \right) \sigma^\lambda_j + \sigma^\lambda_{m+1}. \tag{3.33}$$

These formulas where first obtained by Santaló [55].

Proof. By (2.36), (3.29) and (3.30), we have for $0 \leq i \leq m$,

$$\begin{aligned} \mathbf{T}_t \sigma_i^\lambda &= \exp(t\partial_{\lambda,\mathbb{C}})(\sigma_i^\lambda) \\ &= \exp(t\partial_{\lambda,\mathbb{C}}) \circ J_{\lambda,\mathbb{C}}(p_{m,i}) \\ &= J_{\lambda,\mathbb{C}} \circ \exp(tY_\lambda)(p_{m,i}) \\ &= J_{\lambda,\mathbb{C}}(p_{m,i}(t)). \end{aligned}$$

This proves (3.31) of which (3.32) is a particular case. Integrating with respect to t yields (3.33). □

Remark 3.5.2. It is worth pointing out the similarity between tube formulas in real and complex space forms. More precisely, note that the isomorphism

$$\mathbf{F}_{n,r}^\lambda \colon \left(\mathcal{H}_\lambda^{2n-4r+1}, \partial_{\lambda,\mathbb{R}}\right) \longrightarrow \left(\mathcal{I}_{\lambda,\mathbb{C}}^{n,r}, \partial_{\lambda,\mathbb{C}}\right), \qquad \sigma_j^\lambda \longmapsto \sigma_{j+2r,j+r}^\lambda \tag{3.34}$$

between the linear subspaces $\mathcal{H}_\lambda^{2n-4r+1} \subset \mathcal{V}_{\lambda,\mathbb{R}}^{2n-4r+1}$ and $\mathcal{I}_{\lambda,\mathbb{C}}^{n,r} \subset \mathcal{V}_{\lambda,\mathbb{C}}^{n}$ such that

$$\mathbf{F}_{n,r}^\lambda \circ \partial_{\lambda,\mathbb{R}} = \partial_{\lambda,\mathbb{C}} \circ \mathbf{F}_{n,r}^\lambda;$$

i.e. $\mathbf{F}_{n,r}^\lambda$ commutes with the tubular operator $\mathbf{T}_t$. This is explained by (3.24) and (3.29), because of

$$\begin{aligned} \partial_{\lambda,\mathbb{R}} \sigma_i^\lambda &= (m-i+1)\sigma_{i-1}^\lambda - \lambda(i+1)\sigma_{i+1}^\lambda, \\ \partial_{\lambda,\mathbb{C}} \sigma_{k,r}^\lambda &= (2n-k-2r+1)\sigma_{k-1,r}^\lambda - \lambda(k-2r+1)\sigma_{k+1}^\lambda. \end{aligned}$$

3.6 Local tube formulas in complex space forms

We determine the local tube formulas in $\mathbb{C}P_\lambda^n$.

From Theorem 3.2.5 we know

$$\mathcal{C}(\mathbb{C}P_\lambda^n)^{G_{\lambda,\mathbb{C}}} = \langle B_{k,q} \colon k > 2q\rangle \oplus \langle \Gamma_{k,q} \colon n > k-q\rangle \oplus \langle d\mathrm{vol}\rangle,$$

where $B_{k,q} = [\beta_{k,q}, 0]$ and $\Gamma_{k,q} = [\gamma_{k,q}, 0]$. Recall that these differential forms $\beta_{k,q}, \gamma_{k,q}$ are defined by

$$\begin{aligned} \beta_{k,q} &:= c_{n,k,q} \beta \wedge \Theta_{n-k+q,k-2q-1,q}, \quad k > 2q, \\ \gamma_{k,q} &:= \frac{c_{n,k,q}}{2} \gamma \wedge \Theta_{n-k+q-1,k-2q,q}, \quad n > k-q. \end{aligned}$$

We directly compute $\phi_t^* \beta_{k,q}$ and $\phi_t^* \gamma_{k,q}$ to compute the local tube formulas.

Consider

$$W_1^{\mathbb{C}} := \langle \gamma, \beta\rangle, \quad U_2^{\mathbb{C}} = \langle \Theta_1, \Theta_2, \Theta_3\rangle/(\alpha, d\alpha).$$

From Corollary 3.2.9 the subspaces $W_1^{\mathbb{C}} \subset \Omega^1(S\mathbb{C}P_\lambda^n)$ and $U_2^{\mathbb{C}} \subset \Omega^2(S\mathbb{C}P_\lambda^n)/(\alpha, d\alpha)$ are $\mathcal{L}_T$-invariant.

Theorem 3.6.1. *The maps*

$$F_{\beta,\gamma}\colon (V^{(1)}, Y_{4\lambda}) \longrightarrow \left(W_1^{\mathbb{C}}, \mathcal{L}_T\right), \qquad x \longmapsto \beta, \quad y \longmapsto \psi_0$$
$$F_{\Theta}\colon (V^{(2)}, Y_{\lambda}) \longrightarrow \left(U_2^{\mathbb{C}}, \mathcal{L}_T\right), \qquad p_{2,i} \longmapsto \Theta_i$$

are linear isomorphisms which satisfy

$$\mathcal{L}_T \circ F_{\beta,\gamma} = F_{\beta,\gamma} \circ Y_{4\lambda}, \quad \mathcal{L}_T \circ F_{\Theta} = F_{\Theta} \circ Y_{\lambda}.$$

Proof. It suffices to compare Corollary 3.2.9 with

$$\begin{aligned} &Y_{4\lambda}(x) = y \\ &Y_{4\lambda}(y) = -4\lambda x \\ &Y_{\lambda}(p_{2,i}) = (3-i)p_{2,i-1} - \lambda(i+1)p_{2,i+1}. \quad \square \end{aligned}$$

Proposition 3.6.2. *i)*

$$\begin{aligned} \phi_t^*\beta &= \sin_\lambda(t)\cos_\lambda(t)\gamma + (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\beta \\ \phi_t^*\gamma &= (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\gamma - \lambda\sin_\lambda(t)\cos_\lambda(t)\beta \end{aligned}$$

ii)

$$\phi_t^*\Theta_i = \sum_{j=0}^{2} \phi_{2,i,j}^{\lambda}(t)\Theta_j$$

Explicitly,

$$\begin{aligned} \phi_t^*\Theta_0 =& \cos_\lambda^2(t)\Theta_0 - \lambda\sin_\lambda(t)\cos_\lambda(t)\Theta_1 + \lambda^2\sin_\lambda^2(t)\Theta_2 \\ \phi_t^*\Theta_1 =& 2\sin_\lambda(t)\cos_\lambda(t)\Theta_0 + (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\Theta_1 - 2\lambda\sin_\lambda(t)\cos_\lambda(t)\Theta_2 \\ \phi_t^*\Theta_2 =& \sin_\lambda^2(t)\Theta_0 + \sin_\lambda(t)\cos_\lambda(t)\Theta_1 + \cos_\lambda^2(t)\Theta_2, \end{aligned}$$

Proof. *i)* From Theorem 3.3.3 and applying

$$\begin{aligned} \sin_{4\lambda}(t) &= \sin_\lambda(t)\cos_\lambda(t), \\ \cos_{4\lambda}(t) &= \cos_\lambda^2(t) - \lambda\sin_\lambda^2(t), \end{aligned}$$

it follows

$$\begin{aligned} \phi_t^*\beta &= \sin_\lambda(t)\cos_\lambda(t)\gamma + (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\beta \\ \phi_t^*\gamma &= (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\gamma - \lambda\sin_\lambda(t)\cos_\lambda(t)\beta \end{aligned}$$

ii) This formula is completely analogous using once again Theorem 3.3.3. $\square$

Corollary 3.6.3.

$$\mathbf{LT}_t B_{k,q} = c_{n,k,q}\Big[(\sin_\lambda(t)\cos_\lambda(t)\gamma + (\cos^2_\lambda(t) - \lambda\sin^2_\lambda(t))\beta) \wedge \left(\sum_{a=0}^{2}\phi^\lambda_{2,n-k+q,a}(t)\Theta_0\right)^{n-k+q}\left(\sum_{b=0}^{2}\phi^\lambda_{2,k-2q-1,b}(t)\Theta_1\right)^{k-2q-1}\left(\sum_{c=0}^{2}\phi^\lambda_{2,q,c}(t)\Theta_2\right)^{q}, 0\Big]$$

$$\mathbf{LT}_t\Gamma_{k,q} = \frac{c_{n,k,q}}{2}\Big[((\cos^2_\lambda(t) - \lambda\sin^2_\lambda(t))\gamma - \lambda\sin_\lambda(t)\cos_\lambda(t)\beta)\wedge \wedge\left(\sum_{a=0}^{2}\phi^\lambda_{2,n-k+q+1,a}(t)\Theta_0\right)^{n-k+q+1}\left(\sum_{b=0}^{2}\phi^\lambda_{2,k-2q,b}(t)\Theta_1\right)^{k-2q}\left(\sum_{c=0}^{2}\phi^\lambda_{2,q,c}(t)\Theta_2\right)^{q}, 0\Big]$$

Remark 3.6.4. In Corollary 4.3.4 we will obtain simpler local tube formulas for some particular elements of $\mathcal{C}(\mathbb{C}P^n_\lambda)^{G_{\lambda,\mathbb{C}}}$.

3.7 Spectral analysis of the derivative operator

Here we compute the eigenvalues and eigenvectors of $\partial_{\lambda,\mathbb{R}}$ and $\partial_{\lambda,\mathbb{C}}$. Note that the tube formulas for such valuations are extremely simple: if $\partial\mu = a\mu$ with $a \in \mathbb{C}$, then $\mathbf{T}_t\mu = e^{at}\mu$.

Proposition 3.7.1. *For* $0 \leq 2r \leq n$, $0 \leq k \leq 2n-4r$, *set*

$$\upsilon^\lambda_{k,r} := \binom{2n-4r}{k} J_{\lambda,\mathbb{C}}(e_1^k e_2^{2n-4r-k})$$

Then

i) The restriction of $\partial_{\lambda,\mathbb{C}}$ *to* $\mathcal{I}^{n,r}_{\lambda,\mathbb{C}}$ *has the following (simple) eigenvalues and eigenspaces:*

$$\operatorname{spec}\left(\partial_{\lambda,\mathbb{C}}|_{\mathcal{I}^{n,r}_{\lambda,\mathbb{C}}}\right) = \left\{0, \pm2\sqrt{-\lambda}, \pm4\sqrt{-\lambda}, \ldots, \pm2(n-2r)\sqrt{-\lambda}, \right\},$$
$$E_{(2k-2n+4r)\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{C}}) = \mathbb{C}\cdot\upsilon^\lambda_{k,r}, \qquad 0 \leq k \leq 2n-4r.$$

ii) The endomorphism $\partial_{\lambda,\mathbb{C}}$ *diagonalizes on* $\mathcal{V}^n_{\lambda,\mathbb{C}}$ *with the following eigenspaces:*

$$E_{2j\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{C}}) = \langle \upsilon^\lambda_{k,r} : 0 \leq 2r \leq \min\{n-j, n+j\}\rangle,$$

for $-n \leq j \leq n$.

iii) Each eigenvector $\upsilon^\lambda_{k,r}$ *has the next expansion*

$$\upsilon^\lambda_{k,r} = \sum_{a,b}\binom{a+b}{a}\binom{2n-4r-a-b}{k-a}(-1)^b(\sqrt{-\lambda})^{a+b}\sigma^\lambda_{a+b+2r,r}.$$

Proof. *i)* By (3.24) we now that $\partial_{\lambda,\mathbb{C}} \circ J_{\lambda,\mathbb{C}} = J_{\lambda,\mathbb{C}} \circ Y_\lambda$, it follows that $J_{\lambda,\mathbb{C}}$ preserves the eigenvectors. Specifically, if $v \in E_\alpha(Y_\lambda)$, then $J_{\lambda,\mathbb{C}}(v) \in E_\alpha(\partial_{\lambda,\mathbb{C}})$. The eigenvectors and their corresponding eigenvalues in $(V^{(m)}, Y_\lambda)$ were classified in Proposition 3.3.4. Consequently,

$$\binom{2n-4r}{k} e_1^k e_2^{2n-4r-k} \in E_{(2k-2n+4r)\sqrt{-\lambda}}(Y_\lambda).$$

Since $J_{\lambda,\mathbb{C}}$ preserves the eigenvectors, we have that

$$\upsilon_{k,r}^\lambda = \binom{2n-4r}{k} J_{\lambda,\mathbb{C}}(e_1^k e_2^{2n-4r-k})$$

is an eigenvector of $\partial_{\lambda,\mathbb{C}}$ associated with the eigenvalue $(2k-2n+4r)\sqrt{-\lambda}$.

ii) The endomorphism $\partial_{\lambda,\mathbb{C}}$ diagonalizes because it has simple eigenvalues on each invariant subspace.

iii) The $\upsilon_{k,r}^\lambda$ expansion formula yields from Lemma 3.3.6. □

Proposition 3.7.2. *For* $0 \le k \le m-1$ *set*

$$\upsilon_k^\lambda = \binom{m-1}{k} J_{\lambda,\mathbb{R}}(e_1^k e_2^{m-k-1}).$$

Then

i) In $\mathbb{S}_\lambda^{2n}$ *the derivative operator is diagonalizable with*

$$\operatorname{spec}(\partial_{\lambda,\mathbb{R}}) = \left\{0, \pm\sqrt{-\lambda}, \pm 3\sqrt{-\lambda}, \dots, \pm(2n-1)\sqrt{-\lambda}\right\}, \tag{3.35}$$

$$E_0(\partial_{\lambda,\mathbb{R}}) = \mathbb{C}\cdot\chi \tag{3.36}$$

$$E_{(2k-2n+1)\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{R}}) = \mathbb{C}\cdot\upsilon_k^\lambda, \quad 0 \le k \le 2n-1 \tag{3.37}$$

ii) In $\mathbb{S}_\lambda^{2n+1}$ *the derivative operator is not diagonalizable since*

$$\operatorname{spec}(\partial_{\lambda,\mathbb{R}}) = \left\{0, 0, \pm 2\sqrt{-\lambda}, \pm 4\sqrt{-\lambda}, \dots, \pm 2n\sqrt{-\lambda}\right\}, \tag{3.38}$$

$$E_0(\partial_{\lambda,\mathbb{R}}) = \mathbb{C}\cdot\chi, \tag{3.39}$$

$$E_{(2k-2n)\sqrt{-\lambda}}(\partial_{\lambda,\mathbb{R}}) = \mathbb{C}\cdot\upsilon_k^\lambda, \quad 0 \le k \le 2n. \tag{3.40}$$

iii) Each eigenvector υ_k^λ *has the next expansion*

$$\upsilon_k^\lambda = \sum_{a,b} \binom{a+b}{a}\binom{m-a-b-1}{k-a}(-1)^b(\sqrt{-\lambda})^{a+b}\sigma_{a+b}^\lambda.$$

Proof. *i)* By Lemma 3.3.4 and (3.29) we have that $(2k-2n+1)\sqrt{-\lambda}$, $0 \leq k \leq 2n-1$, is an eigenvalue of $\partial_{\lambda,\mathbb{R}}$ with eigenspace given by (3.37). The Euler characteristic is an eigenvector with zero eigenvalue. We thus have at least $2n+1$ eigenvalues. Since this is precisely the dimension of $\mathcal{V}^{2n}_{\lambda,\mathbb{R}}$, the statement follows.

ii) In light of Lemma 3.3.4 and (3.29), we ascertain that $(2k-2n)\sqrt{-\lambda}$, $0 \leq k \leq 2n$, is an eigenvalue of $\partial_{\lambda,\mathbb{R}}$ and the corresponding eigenspace is described by (3.40).

Our next objective is to prove that while the algebraic multiplicity of the zero eigenvalue is two, its geometric multiplicity is one. This will entail finding a valuation μ that satisfies $\partial^2_{\lambda,\mathbb{R}}\mu = 0$, while also ensuring that $\partial_{\lambda,\mathbb{R}}\mu \neq 0$. Consider $\sigma^\lambda_{2n} = J_{\lambda,\mathbb{R}}(x^{2n}) \in \mathcal{V}^{2n+1}_{\lambda,\mathbb{R}}$. In the notation of Lemma 3.3.4,

$$x = \frac{1}{2\sqrt{-\lambda}}(e_1 - e_2), \qquad x^{2n} = (-4\lambda)^{-n}\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}e_1^i e_2^{2n-i}.$$

Hence

$$\partial_{\lambda,\mathbb{R}}\sigma^\lambda_{2n+1} = \sigma^\lambda_{2n} = (-4\lambda)^{-n}\sum_{i=0}^{2n}(-1)^i\binom{2n}{i}J_{\lambda,\mathbb{R}}(e_1^i e_2^{2n-i}).$$

Consider

$$\nu := (-4\lambda)^{-n}\sum_{\substack{i=0\\ i\neq n}}^{2n}\binom{2n}{i}\frac{(-1)^i}{(2i-2n)\sqrt{-\lambda}}J_{\lambda,\mathbb{R}}(e_1^i e_2^{2n-i}),$$

and note that, by Lemma 3.3.4,

$$\partial_{\lambda,\mathbb{R}}\nu := (-4\lambda)^{-n}\sum_{\substack{i=0\\ i\neq n}}^{2n}\binom{2n}{i}(-1)^i J_{\lambda,\mathbb{R}}(e_1^i e_2^{2n-i}),$$

since $e_1^n e_2^n \in \ker Y_\lambda$. Finally, we define $\mu = \sigma^\lambda_{2n+1} - \nu$. Then

$$\partial_{\lambda,\mathbb{R}}\mu = (-4\lambda)^{-n}\binom{2n}{n}(-1)^n J_{\lambda,\mathbb{R}}(e_1^n e_2^n) \neq 0,$$

while

$$\begin{aligned}\partial^2_{\lambda,\mathbb{R}}\mu &= (-4\lambda)^{-n}\binom{2n}{n}(-1)^n\partial_{\lambda,\mathbb{R}}J_{\lambda,\mathbb{R}}(e_1^n e_2^n)\\ &= (-4\lambda)^{-n}\binom{2n}{n}(-1)^n J_{\lambda,\mathbb{R}}(Y_\lambda(e_1^n e_2^n)) = 0.\end{aligned}$$

It follows that $\dim\ker\partial_{\lambda,\mathbb{R}} < \dim\ker\partial^2_{\lambda,\mathbb{R}}$. Noting that $\chi \in \ker\partial_{\lambda,\mathbb{R}}$ this implies the statement.

iii) The v_k^λ expansion formula follows from Lemma 3.3.6. ☐

Remark 3.7.3. We conclude from Proposition 3.7.2 and Lemma 3.3.4 that there is no isomorphism between $\mathrm{Val}^{O(m)}$ and $\mathcal{V}^m_{\lambda,\mathbb{R}}$ intertwining $\Lambda - \lambda L$ and $\partial_{\lambda,\mathbb{R}}$. Indeed, these two operators have different spectra no matter the parity of m.

Remark 3.7.4. There is no $\mathfrak{sl}_2$-module structure in $\mathcal{V}^n_{\lambda,\mathbb{C}}$ nor in $\mathcal{V}^m_{\lambda,\mathbb{R}}$, where $\partial_{\lambda,\mathbb{C}}$ and $\partial_{\lambda,\mathbb{R}}$ play the role of X or Y due to their lack of nilpotency.

3.7.1 Stable valuations in complex space forms

Definition 3.7.5. We say that a valuation $\varphi \in \mathcal{V}(M)$ on a riemannian manifold M is *stable* if $\partial\mu = 0$, or equivalently, if $\mathbf{T}_t\mu = \mu$ for all t. The space of stable valuations is $\ker\partial$.

Proposition 3.7.6. *The Euler characteristic is the unique isometry-invariant stable valuation in $\mathbb{S}^m_\lambda$ up to multiplicative constants.*

Proof. By Proposition 3.7.2

$$E_0(\partial_{\lambda,\mathbb{R}}) = \langle\chi\rangle = \ker\partial_{\lambda,\mathbb{R}}$$

with independence of the dimension. □

The complex case is more interesting. Previously, in Proposition 3.7.1, we provided a detailed description of each eigenvalue and eigenvector of $\partial_{\lambda,\mathbb{C}}$. In particular, we derived an expansion for the elements of the kernel of $\partial_{\lambda,\mathbb{C}}$. We now refine these formulas for this kernel, obtaining more simplified expressions.

Proposition 3.7.7. *The unique (up to multiplicative constants) stable valuation on $\mathcal{I}^{n,r}_{\lambda,\mathbb{C}}$ is given by*

$$\psi^\lambda_{2r} = \sum_{i=r}^{n-r}\binom{n-2r}{i-r}\binom{2n-4r}{2i-2r}^{-1}\lambda^{i-r}\sigma^\lambda_{2i,r}.$$

In particular

$$\ker\partial_{\lambda,\mathbb{C}} = \langle\psi^\lambda_{2r} : 0 \le 2r \le n\rangle.$$

Proof. By Lemma 3.3.4 the kernel of Y_λ on the space $V^{(m)}$ of homogeneous polynomials of degree $m = 2n-4r$ is spanned by

$$\begin{aligned} e_1^{n-2r}e_2^{n-2r} &= (y+\sqrt{-\lambda}x)^{n-2r}(y-\sqrt{-\lambda}x)^{n-2r} \\ &= (y^2+\lambda x^2)^{n-2r} = \sum_{j=0}^{n-2r}\binom{n-2r}{j}\lambda^j x^{2j}y^{m-2j} \\ &= \sum_{i=r}^{n-r}\binom{n-2r}{i-r}\binom{2n-4r}{2i-2r}^{-1}\lambda^{i-r}\binom{2n-4r}{2i-2r}x^{2i-2r}y^{2n-2i-2r} \end{aligned}$$

Therefore the kernel of $\partial_{\lambda,\mathbb{C}}$ in $\mathcal{I}^{n,r}_{\lambda,\mathbb{C}}$ is spanned by $\psi_{2r} = J_\lambda(e_1^{n-2r}e_2^{n-2r})$, for each $0 \le 2r \le n$. □

Next, we express the Euler characteristic as a combination of the stable valuations ψ_{2r}^{λ}. Note in particular that χ is *not* confined to any $\partial_{\lambda,\mathbb{C}}$-invariant subspace $\mathcal{I}_{\lambda,\mathbb{C}}^{n,r}$. More precisely:

Proposition 3.7.8.

$$\chi = \sum_{0\leq 2r\leq n} \left(\frac{\lambda}{4\pi}\right)^r \binom{2r}{r} \frac{r!}{\omega_{2n-2r}} \psi_{2r}^{\lambda}.$$

Proof. Since χ is stable, it can be expressed as $\chi = \sum_j a_j \psi_{2j}^{\lambda}$. By [21, Theorem 3.11]

$$\chi = \sum_{k,p\geq 0} \left(\frac{\lambda}{\pi}\right)^{k+p} \frac{\partial^{k+p}}{\partial\xi^k \partial\eta^p} \frac{1}{\sqrt{1-\xi}\sqrt{1-\eta}}\Bigg|_{(0,0)} \tau_{2k+2p,p}^{\lambda}.$$

The coefficient of $\tau_{2r,r}^{\lambda}$ in this expansion is

$$\begin{aligned}[\tau_{2r,r}^{\lambda}](\chi) &= \left(\frac{\lambda}{\pi}\right)^r \frac{\partial^r}{\partial\eta^r} \frac{1}{\sqrt{1-\eta}}\Bigg|_0 \\ &= \left(\frac{\lambda}{\pi}\right)^r \binom{2r}{r} r! 4^{-r}.\end{aligned}$$

By Proposition 2.1.8, we have

$$[\tau_{2r,r}^{\lambda}](\sigma_{k,r}^{\lambda}) = \frac{\omega_{2n-k}}{(k-2r)!}[\tau_{2r,r}^{\lambda}](\pi_{k,r}^{\lambda}) = \frac{\omega_{2n-k}}{(k-2r)!}\delta_{k,2r},$$

whence

$$\begin{aligned}\left[\tau_{2r,r}^{\lambda}\right]\left(\sum_j a_j \psi_{2j}^{\lambda}\right) &= a_r [\tau_{2r,r}^{\lambda}](\sigma_{2r,r}^{\lambda}) \\ &= a_r \omega_{2n-2r}.\end{aligned}$$

Hence

$$a_r = \left(\frac{\lambda}{\pi}\right)^r \binom{2r}{r} \frac{r!}{4^r \omega_{2n-2r}}$$

and the result follows. □

3.7.2 Image of $\partial_{\lambda,\mathbb{C}}$ and $\partial_{\lambda,\mathbb{R}}$

Next, we describe the image of the operators $\partial_{\lambda,\mathbb{C}}$ and $\partial_{\lambda,\mathbb{R}}$, and we compute the preimage of any element belonging to them.

Proposition 3.7.9. *Given any* $\varphi = \sum_{k,r} a_{k,r}\sigma_{k,r}^{\lambda} \in \mathcal{V}_{\lambda,\mathbb{C}}^{n}$*, we have* $\varphi \in \operatorname{im} \partial_{\lambda,\mathbb{C}}$ *if and only if*

$$\sum_{l=r}^{n-2r} a_{2l,r} \binom{n-2r}{l-r} \lambda^{n-l-r} = 0, \qquad \textit{for} \quad 0 \leq 2r \leq n. \tag{3.41}$$

Proof. Note that $\varphi = \sum_r \varphi_r$ with $\varphi_r = \sum_k a_{k,r}\sigma^\lambda_{k,r}$ is the decomposition of φ corresponding to

$$\mathcal{V}^n_{\lambda,\mathbb{C}} = \bigoplus_{r=0}^{\lfloor n/2\rfloor} \mathcal{I}^{n,r}_{\lambda,\mathbb{C}},$$

where $\mathcal{I}^{n,r}_{\lambda,\mathbb{C}}$ are the invariant subspaces introduced in (3.27). By (3.24) and Proposition 3.3.7 we have $\varphi \in \operatorname{im}\partial_{\lambda,\mathbb{C}}$ if and only if for every r

$$\begin{aligned}
0 = Z_{2n-4r,\lambda}(\varphi_r) &= \sum_{k=2r}^{2n-4r} a_{k,r}\binom{2n-4r}{k-2r} Z_{2n-4r,\lambda}(x^{k-2r}y^{2n-k-2r})\\
&= \sum_{l=r}^{n-2r} a_{2l,r}\binom{2n-4r}{2l-2r}\binom{n-2r}{l-r}\lambda^{n-l-r}(2l-2r)!(2n-2l-2r)!\\
&= (2n-4r)!\sum_{l=r}^{n-2r} a_{2l,r}\binom{n-2r}{l-r}\lambda^{n-l-r}
\end{aligned}$$

where we used (3.19). □

Proposition 3.7.10. *Given $\varphi = \sum_{k,r} a_{k,r}\sigma^\lambda_{k,r} \in \mathcal{V}^n_{\lambda,\mathbb{C}}$ satisfying* (3.41) *we have*

$$\partial^{-1}_{\lambda,\mathbb{C}}(\{\varphi\}) = \sum_{k,r} a_{k,r} J_{\lambda,\mathbb{C}}(P_{2n-4r,k-2r+1}) + \langle \psi_{2r} \colon 0 \le 2r \le n\rangle$$

where $P_{m,l}$ is given by (3.20).

Proof. This follows at once from Proposition 3.3.9 after decomposing $\varphi = \sum_r \varphi_r$ as in the previous proof. □

The following proposition will be useful later.

Proposition 3.7.11. *In $\mathbb{C}P^n_\lambda$ for $0 \le 2r \le n$ and $2r \le k \le 2n-2r$*

$$\sigma^\lambda_{k,r} = \sum_{j\ge 0}\lambda^j \frac{(k+2j-2r)!!(2n-k-2j-2r-2)!!}{(k-2r)!!(2n-4r-k+2)!!}\partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k+2j+1,r} + c_{2n-4r,k-2r+1}\sigma^\lambda_{2n-2r,r},$$

where $c_{2n-4r,k-2r+1}$ are the constants introduced in (3.22).

Proof. By Proposition 3.3.10

$$p_{2n-4r,k-2r} = \sum_{j\ge 0}\lambda^j \frac{(k+2j-2r)!!(2n-k-2j-2r-2)!!}{(k-2r)!!(2n-4r-k+2)!!} Y_\lambda p_{2n-4r,k+2j-2r+1} + c_{2n-4r,k-2r+1}x^{2n-4r}.$$

Then, transfering the formula via

$$J_{\lambda,\mathbb{C}}\colon \bigoplus_{0\le 2r\le n} V^{(2n-4r)} \longrightarrow \mathcal{V}^n_{\lambda,\mathbb{C}}$$

given by (3.25), we have

$$\sigma^\lambda_{k,r} = \sum_{j\geq 0} \lambda^j \frac{(k+2j-2r)!!(2n-k-2j-2r-2)!!}{(k-2r)!!(2n-4r-k+2)!!} \partial_{\lambda,\mathbb{C}} \sigma^\lambda_{k+2j+1,r} + c_{2n-4r,k-2r+1}\sigma^\lambda_{2n-2r,r}.$$

□

Next, we characterize the image of $\partial_{\lambda,\mathbb{R}}$. To this end, let us recall one of the several natural bases of $\mathcal{V}^m_{\lambda,\mathbb{R}}$. According to [15, eq. (118)], we define

$$\phi^k = \int_{G_{\lambda,\mathbb{C}}} \chi(\cdot \cap g\mathbb{S}^{m-k}_\lambda) dg \tag{3.42}$$

where dg denotes a properly normalized Haar measure on $G_{\lambda,\mathbb{C}}$, and $\mathbb{S}^{m-k}_\lambda$ represents an $(m-k)$-dimensional totally geodesic submanifold in $\mathbb{S}^m_\lambda$. The family $\phi^0,\dots,\phi^m$ forms a basis of $\mathcal{V}^m_{\lambda,\mathbb{R}}$ and is further expressed as

$$\phi^k = \sum_{j\geq 0} \left(\frac{\lambda}{4}\right)^j \tau^\lambda_{k+2j}, \tag{3.43}$$

where, we recall $\tau^\lambda_0,\dots,\tau^\lambda_m \in \mathcal{V}^m_{\lambda,\mathbb{R}}$ are the valuations proportional to $\sigma^\lambda_0,\dots,\sigma^\lambda_m$

$$\tau^\lambda_i = \frac{i!\omega_i}{\pi^i(m-i)\omega_{m-i}}\sigma^\lambda_i, \quad 0\leq i\leq m-1,$$
$$\tau^\lambda_m = \frac{m!\omega_m}{\pi^m}\sigma^\lambda_m$$

Proposition 3.7.12. *The image of $\partial_{\lambda,\mathbb{R}}$ in $\mathcal{V}^m_{\lambda,\mathbb{R}}$ is the hyperplane $\mathcal{H}^m_\lambda$ generated by $\sigma^\lambda_0,\dots,\sigma^\lambda_{m-1}$. Moreover*

$$\partial_{\lambda,\mathbb{R}}\phi^k = \frac{k!\omega_k}{\pi^k\omega_{m-k}}\sigma^\lambda_{k-1}, \qquad 1\leq k\leq m. \tag{3.44}$$

In particular

$$\partial^{-1}_{\lambda,\mathbb{R}}(\{\sigma^\lambda_{k-1}\}) = \frac{\pi^k\omega_{m-k}}{k!\omega_k}\phi^k + \mathbb{C}\cdot\chi.$$

Proof. Since

$$\sigma^\lambda_i = \frac{\pi^i(m-i)\omega_{m-i}}{i!\omega_i}\tau^\lambda_i, \quad 0\leq i\leq m-1,$$
$$\sigma^\lambda_m = \frac{\pi^m}{m!\omega_m}\tau^\lambda_m.$$

we have

$$\phi^k = \sum_{j=0}^{\lfloor\frac{m-k-1}{2}\rfloor} \left(\frac{\lambda}{4\pi^2}\right)^j \frac{(k+2j)!\omega_{k+2j}}{\pi^k(m-k-2j)\omega_{m-k-2j}}\sigma^\lambda_{k+2j} + \left(\left(\frac{\lambda}{4}\right)^{\frac{m-k}{2}} \frac{m!\omega_m}{\pi^m}\sigma^\lambda_m\right),$$

where the term between brackets appears only if $m-k$ is even. Using Proposition 3.1.2, this yields (3.44). The rest of the statement follows. □

Remark 3.7.13. Equation (3.44) also follows from Theorem 4 in [59].

Chapter 4

Tube formulas for the Federer valuations

In this chapter, we determine the tube formulas for the Federer curvature measures and valuations in $\mathbb{C}P^n_\lambda$ and $\mathbb{H}P^n_\lambda$. In fact, we obtain such formulas for an interesting space of curvature measures that includes Federer's ones. Additionally, we identify bases in which these formulas are particularly simple. As an application, we compute the push-forward of invariant valuations of the sphere through the quaternionic Hopf fibration, thus extending a result of Hofstätter-Wannerer.

4.1 Preliminary constructions

Definition 4.1.1. Let M^n be a riemannian manifold. For $1 \leq r \leq n$, let $\mathrm{Gr}_r(M)$ be the bundle over M whose fiber over $x \in M$ is the grassmannian of oriented linear r-planes of T_xM. A *distribution along geodesics* is a smooth map $f\colon SM \to \mathrm{Gr}_r(M)$ such that $\xi \in f(\xi)$ for all $\xi \in SM$.

Given any $\xi \in SM$ and any basis $e_0, \ldots, e_{r-1}$ of $f(\xi)$, we can consider the parallel transport $e_0(t), \ldots, e_{r-1}(t)$ along the geodesic $\gamma(t) = \exp(t\xi)$. We say that f is a *parallel distribution along geodesics* if

$$e_i(t) \in f(e_0(t)), \quad \forall t.$$

Example 4.1.2. The map $f_2\colon S\mathbb{C}P^n_\lambda \to \mathrm{Gr}_2(\mathbb{C}P^n_\lambda)$ given by

$$f_2\colon \xi \longmapsto \langle e_0(\xi), Je_0(\xi)\rangle.$$

is a distribution along geodesics.

Recall our convention $e_0(\xi) = \xi$. More precisely, given $\xi = (x, v)$ with $x \in M$ and $v \in T_xM$, we take $e_0(\xi) = v$.

Example 4.1.3. The global map $f_4\colon S\mathbb{H}P^n_\lambda \to \mathrm{Gr}_4(\mathbb{H}P^n_\lambda)$ locally given by

$$f_4\colon \xi \longmapsto \langle e_0(\xi), J_1e_0(\xi), J_2e_0(\xi), J_3e_0(\xi)\rangle.$$

is a distribution along geodesics.

Proposition 4.1.4. *The maps*

i) $f_2\colon S\mathbb{C}P^n_\lambda \longrightarrow \mathrm{Gr}_2(\mathbb{C}P^n_\lambda), \quad f_2\colon \xi \longmapsto \langle e_0(\xi), Je_0(\xi)\rangle,$

ii) $f_4\colon S\mathbb{H}P^n_\lambda \longrightarrow \mathrm{Gr}_4(\mathbb{H}P^n_\lambda), \quad f_4\colon \xi \longmapsto \langle e_0(\xi), J_1e_0(\xi), J_2e_0(\xi), J_3e_0(\xi)\rangle,$

are parallel distributions along geodesics.

Proof. Both points are direct consequences of Proposition 1.4.7 and 1.4.13. □

Definition 4.1.5. Let $f\colon SM \to \mathrm{Gr}_r(M)$ be a distribution along geodesics. A positively oriented orthonormal moving frame $e_0, \dots, e_{n-1}$ on SM such that $e_0(\xi), \dots, e_{r-1}(\xi)$ is a positive basis of $f(\xi)$ will be called a *frame adapted to* f.

Let us define, for $0 \leq i \leq r-1$, the differential forms $\mathcal{T}_i(f) \in \Omega^{r-1}(SM)$ given by

$$\mathcal{T}_i\left(f\right)\left(X_1, \dots, X_{r-1}\right) := \sum_{\sigma \in \mathcal{S}_{i,r-i-1}} |\sigma| \det(e_0, X^H_{\sigma(1)}, \dots, X^H_{\sigma(i)}, X^V_{\sigma(i+1)}, \dots, X^V_{\sigma(r-1)}, e_r, \dots, e_{n-1}),$$

where $e_0, \dots, e_{n-1}$ is a frame adapted to f.

It is easy to see that the previous definition is independent of the chosen adapted frame.

Example 4.1.6. Observe that for $r = n$ and $f_n\colon SM \to \mathrm{Gr}_n(M)$ given by

$$f_n\colon \xi \longmapsto \langle e_0(\xi), e_1(\xi), \dots, e_{n-1}(\xi)\rangle,$$

we obtain the differential forms κ_i, i.e

$$\mathcal{T}_i(f_n) = \kappa_i, \quad 0 \leq i \leq n-1.$$

From now on, for simplicity in the computations, we will omit the $\wedge$ symbol.

Proposition 4.1.7. *Let $f\colon SM \to \mathrm{Gr}_r(M)$ be a distribution along geodesics. Then, for $0 \leq i \leq r-1$ and with respect to an adapted frame*

$$\mathcal{T}_i\left(f\right) = \sum_{\sigma \in \mathcal{S}_{i,r-i-1}} |\sigma| \theta_{\sigma(1)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1)} \cdots \omega_{\sigma(r-1)}$$

Proof. Since $\det = \theta_0 \wedge \cdots \wedge \theta_{n-1}$ and

$$\omega_{j,0}(X) = \langle e_j, (\pi^*\nabla)_X e_0\rangle = \theta_j((\pi^*\nabla)_X e_0) = \theta_j(X^V)$$

The same argument of Proposition 1.3.9 gives us the result. □

Example 4.1.8. Let $f_2\colon S\mathbb{C}P^n_\lambda \to \mathrm{Gr}_2(\mathbb{C}P^n_\lambda)$ be given by

$$f_2\colon \xi \longmapsto \langle e_0(\xi), Je_0(\xi)\rangle.$$

We denote

$$\psi_i = \mathcal{T}_i(f_2), \quad \varphi_j = \mathcal{T}_j(e_0 \oplus f_2^\perp),$$

for $0 \le i \le 1$, $0 \le j \le 2n-2$. Let us show a description of this family of differential forms. Let $e_0, \dots, e_{2n-1}$ be a hermitian moving frame and note that $\{e_0, e_1\}$ and $\{e_2, \dots, e_{2n-1}\}$ are positively oriented bases of $f_2(e_0)$ and $f_2(e_0)^\perp$ respectively. Then

$$\begin{aligned}
\psi_0 &= \omega_{1,0} = \gamma,\\
\psi_1 &= \theta_1 = \beta,\\
\varphi_j &= \sum_\sigma |\sigma| \theta_{\sigma(2)} \cdots \theta_{\sigma(j+1)} \omega_{\sigma(j+2),0} \cdots \omega_{\sigma(2n-1),0},
\end{aligned}$$

where σ is in the range of permutations of $\{2, \dots, 2n-1\}$ such that

$$\sigma(2) < \cdots < \sigma(j+1); \quad \sigma(j+2) < \cdots < \sigma(2n-1).$$

Example 4.1.9. Let $f_4\colon S\mathbb{H}P^n_\lambda \to \mathrm{Gr}_4(\mathbb{H}P^n_\lambda)$ be given by

$$f_4\colon \xi \longmapsto \langle e_0(\xi), J_1e_0(\xi), J_2e_0(\xi), J_3e_0(\xi)\rangle,$$

We consider, as in $\mathbb{C}P^n_\lambda$, the family of differential forms

$$\psi_i = \mathcal{T}_i(f_4), \quad \varphi_j = \mathcal{T}_j(e_0 \oplus f_4^\perp),$$

for $0 \le i \le 3$, $0 \le j \le 4n-4$. Let $e_0, \dots, e_{2n-1}$ a quaternionic moving frame such that $\{e_0, e_1, e_2, e_3\}$ and $\{e_4, \dots, e_{4n-1}\}$ are positively oriented bases of $f_4(e_0)$ and $f_4^\perp(e_0)$ respectively. Then

$$\begin{aligned}
\psi_0 &= \omega_{1,0}\omega_{2,0}\omega_{3,0},\\
\psi_1 &= \theta_1\omega_{2,0}\omega_{3,0} + \omega_{1,0}\theta_2\omega_{3,0} + \omega_{1,0}\omega_{2,0}\theta_3,\\
\psi_2 &= \theta_1\theta_2\omega_{3,0} + \theta_1\omega_{2,0}\theta_3 + \omega_{1,0}\theta_2\theta_3,\\
\psi_3 &= \theta_1\theta_2\theta_3,\\
\varphi_j &= \sum_\sigma |\sigma| \theta_{\sigma(4)} \cdots \theta_{\sigma(j+3)} \omega_{\sigma(j+4),0} \cdots \omega_{\sigma(4n-1),0},
\end{aligned}$$

where σ is in the range of permutations of $\{4, \dots, 4n-1\}$ such that

$$\sigma(4) < \cdots < \sigma(j+3); \quad \sigma(j+4) < \cdots < \sigma(4n-1).$$

Proposition 4.1.10. *Let $0 \le i \le n-1$, $1 \le r \le n$ and $f\colon SM \to \mathrm{Gr}_r(M)$ be a distribution along geodesics. Then*

$$\kappa_i = \sum_{j=0}^{r-1} \mathcal{T}_j(f) \wedge \mathcal{T}_{i-j}(e_0 \oplus f^\perp).$$

Proof. Let us write

$$\psi_i^{(r)} := \mathcal{T}_i(f_r), \quad \varphi_j^{(r)} := \mathcal{T}_j(e_0 \oplus f_r^{\perp}).$$

We choose a common orthonormal moving frame, denoted by $e_0, \ldots, e_{n-1}$, such that

$$f_r(\xi) = \langle e_0(\xi), e_1(\xi), \ldots, e_{r-1}(\xi)\rangle,$$
$$e_0 \oplus f_r^{\perp}(\xi) = \langle e_0(\xi), e_r(\xi), \ldots, e_{n-1}(\xi)\rangle,$$

and such that $\{e_0(\xi), e_1(\xi), \ldots, e_{r-1}(\xi)\}$ and $\{e_0(\xi), e_r(\xi), \ldots, e_{n-1}(\xi)\}$ are positive basis. Then

$$\psi_i^{(r)} = \sum_{\epsilon \in \mathcal{S}_{i,r-i-1}} |\epsilon| \theta_{\epsilon(1)} \cdots \theta_{\epsilon(i)} \omega_{\epsilon(i+1),0} \cdots \omega_{\epsilon(r-1),0}, \tag{4.1}$$

$$\varphi_i^{(r)} = \sum_{\sigma} |\sigma| \theta_{\sigma(r)} \cdots \theta_{\sigma(i+r-1)} \omega_{\sigma(i+r),0} \cdots \omega_{\sigma(n-1),0}, \tag{4.2}$$

where σ is in the range of the set of permutations of $\{r, \ldots, n-1\}$ such that

$$\sigma(r) < \cdots < \sigma(i+r-1), \quad \sigma(i+r) < \cdots < \sigma(n-1).$$

We give an inductive proof over r.

i) For $r = 1$, $\kappa_i = \mathcal{T}_i(e_0 \oplus f_r^{\perp})$.

ii) Assume $\kappa_i = \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \varphi_{i-j}^{(r-1)}$, where

$$f_{r-1} \colon \xi \longmapsto \langle e_0(\xi), e_1(\xi), \ldots, e_{r-2}(\xi)\rangle,$$

$$\psi_i^{(r-1)} := \mathcal{T}_i(f_{r-1}), \quad \varphi_j^{(r-1)} := \mathcal{T}_j(e_0 \oplus f_{r-1}^{\perp}).$$

We have to show $\kappa_i = \sum_{j=0}^{r-1} \psi_j^{(r)} \wedge \varphi_{i-j}^{(r)}$. Let us decompose (4.1) as

$$\begin{aligned}\psi_i^{(r)} = &\sum_{\epsilon(i)=r-1} |\epsilon| \theta_{\epsilon(1)} \cdots \theta_{\epsilon(i)} \omega_{\epsilon(i+1),0} \cdots \omega_{\epsilon(r-1),0} \quad (I)\\ &+ \sum_{\epsilon(r-1)=r-1} |\epsilon| \theta_{\epsilon(1)} \cdots \theta_{\epsilon(i)} \omega_{\epsilon(i+1),0} \cdots \omega_{\epsilon(r-1),0} \quad (II)\end{aligned}$$

The first sum is

$$\begin{aligned}(I) &= |(r-1,i)| \sum_{\epsilon(r-1)=r-1} |\epsilon| \theta_{\epsilon(1)} \cdots \theta_{\epsilon(i-1)} \theta_{r-1} \omega_{\epsilon(i+1),0} \cdots \omega_{\epsilon(r-2),0} \omega_{\epsilon(i),0}\\ &= |(r-1,i)|(-1)^{i+1} \theta_{r-1} \wedge \sum_{\epsilon(r-1)=r-1} |\epsilon| \theta_{\epsilon(1)} \cdots \theta_{\epsilon(i-1)} \omega_{\epsilon(i+1),0} \cdots \omega_{\epsilon(r-2),0} \omega_{\epsilon(i),0}\\ &= |(r-1,i)|(-1)^{i+1}(-1)^{r-i} \theta_{r-1} \wedge \sum_{\epsilon(r-1)=r-1} |\epsilon| \theta_{\epsilon(1)} \cdots \theta_{\epsilon(i-1)} \omega_{\epsilon(i),0} \omega_{\epsilon(i+1),0} \cdots \omega_{\epsilon(r-2),0}\\ &= (-1)^r \theta_{r-1} \wedge \psi_{i-1}^{r-1}\end{aligned}$$

Arguing similarly with (II) we get

$$\psi_i^{(r)} = (-1)^r \theta_{r-1} \wedge \psi_{i-1}^{(r-1)} + (-1)^r \omega_{r-1,0} \wedge \psi_i^{(r-1)},$$

and an analogous argument with (4.2) gives

$$\varphi_i^{(r-1)} = \theta_{r-1} \wedge \varphi_{i-1}^{(r)} + \omega_{r-1,0} \wedge \varphi_i^{(r)}.$$

Here and in the following $\psi_j^{(k)}, \varphi_j^{(k)}$ are taken to be 0 if $j = -1$ or $j = k$. Thus

$$\begin{aligned}
\sum_{j=0}^{r-1} \psi_j^{(r)} \wedge \varphi_{i-j}^{(r)} &= \sum_{j=0}^{r-1} (-1)^r \theta_{r-1} \wedge \psi_{j-1}^{(r-1)} \wedge \varphi_{i-j}^{(r)} + (-1)^r \omega_{r-1,0} \wedge \psi_j^{(r-1)} \wedge \varphi_{i-j}^{(r)} \\
&= \sum_{j=0}^{r-1} \psi_{j-1}^{(r-1)} \wedge \theta_{r-1} \wedge \varphi_{i-j}^{(r)} + \sum_{j=0}^{r-1} \psi_j^{(r-1)} \wedge \omega_{r-1,0} \wedge \varphi_{i-j}^{(r)} \\
&= \sum_{j=0}^{r-1} \psi_{j-1}^{(r-1)} \wedge (\varphi_{i-j+1}^{(r-1)} - \omega_{r-1,0} \wedge \varphi_{i-j+1}^{(r)}) \\
&\quad + \sum_{j=0}^{r-1} \psi_j^{(r-1)} \wedge (\varphi_{i-j}^{(r-1)} - \theta_{r-1} \wedge \varphi_{i-j-1}^{(r)}) \\
&= \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \varphi_{i-j}^{(r-1)} + \sum_{j=0}^{r-1} \psi_j^{(r-1)} \wedge \varphi_{i-j}^{(r-1)} \\
&\quad - \sum_{j=0}^{r-1} \psi_{j-1}^{(r-1)} \wedge \omega_{r-1,0} \wedge \varphi_{i-j+1}^{(r)} + \psi_j^{(r-1)} \wedge \theta_{r-1} \wedge \varphi_{i-j-1}^{(r)} \\
&\overset{(*)}{=} \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \varphi_{i-j}^{(r-1)} \overset{(**)}{=} \kappa_i,
\end{aligned}$$

where in $(*)$ we have used that

$$\begin{aligned}
&\sum_{j=0}^{r-1} \psi_{j-1}^{(r-1)} \wedge \omega_{r-1,0} \wedge \varphi_{i-j+1}^{(r)} + \sum_{j=0}^{r-1} \psi_j^{(r-1)} \wedge \theta_{r-1} \wedge \varphi_{i-j-1}^{(r)} \\
&= \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \omega_{r-1,0} \wedge \varphi_{i-j}^{(r)} + \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \theta_{r-1} \wedge \varphi_{i-j-1}^{(r)} \\
&= \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \left(\omega_{r-1,0} \wedge \varphi_{i-j}^{(r)} + \theta_{r-1} \wedge \varphi_{i-j-1}^{(r)}\right) \\
&= \sum_{j=0}^{r-2} \psi_j^{(r-1)} \wedge \varphi_{i-j}^{(r-1)},
\end{aligned}$$

and $(**)$ is the induction hypothesis. □

Corollary 4.1.11. *i) In* $\mathbb{C}P^n_\lambda$

$$\begin{aligned}\kappa_i &= \psi_0 \wedge \varphi_i + \psi_1 \wedge \varphi_{i-1}\\ &= \gamma \wedge \varphi_i + \beta \wedge \varphi_{i-1}, \quad 0 \leq i \leq 2n-1.\end{aligned}$$

ii) In $\mathbb{H}P^n_\lambda$

$$\kappa_i = \psi_0 \wedge \varphi_i + \psi_1 \wedge \varphi_{i-1} + \psi_2 \wedge \varphi_{i-2} + \psi_3 \wedge \varphi_{i-3}, \quad 0 \leq i \leq 4n-1.$$

The following fact will be key for determining the Lie derivative $\mathcal{L}_T\kappa_i$ in complex and quaternionic space forms.

Proposition 4.1.12. *Suppose* $f\colon SM \to \mathrm{Gr}_r(M)$ *is a parallel distribution along geodesics and for certain* $a \in \mathbb{R}$ *it is satisfied*

$$R(X, e_0)e_0 = a \cdot i_X g, \quad \forall X \in f(e_0).$$

Then, for $0 \leq i \leq r-1$

$$\mathcal{L}_T \mathcal{T}_i(f) = (r-i)\mathcal{T}_{i-1}(f) + a(i+1)\mathcal{T}_{i+1}(f).$$

Proof. Given $\xi \in SM$ let $e_0, \ldots, e_n$ be a parallel moving frame relative to ξ such that $e_0(\xi), \ldots, e_{r-1}(\xi)$ is a positive basis of $f(\xi)$. Since f is parallel, the same happens for all t at $\xi(t) = (\gamma(t), \gamma'(t))$ where $\gamma(t) = \exp(t\xi)$. Consequently, by Corollary 1.1.6, the following holds:

$$\mathcal{L}_T\theta_i = \omega_{i,0}, \quad \mathcal{L}_T\omega_{j,0} = i_T\Omega_{j,0} = a\theta_j$$

for all $0 \leq i \leq r-1$ and $1 \leq j \leq r-1$. By Proposition 4.1.7

$$\begin{aligned}\mathcal{T}_i(f) &= \sum_{\sigma \in \mathcal{S}_{i,r-i-1}} |\sigma| \theta_{\sigma(1)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \omega_{\sigma(r-1),0}\\ &= \frac{1}{i!(r-i-1)!} \sum_{\sigma \in \mathcal{S}_{r-1}} |\sigma| \theta_{\sigma(1)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \omega_{\sigma(r-1),0}.\end{aligned}$$

Since $\mathcal{L}_T$ is a derivation, we have

$$\begin{aligned}i!(r-i-1)!\mathcal{L}_T\mathcal{T}_i(f) =& \sum_{j=1}^{i} \sum_{\sigma \in \mathcal{S}_{r-1}} |\sigma| \theta_{\sigma(1)} \cdots \mathcal{L}_T\theta_{\sigma(j)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \omega_{\sigma(r-1),0}\\ &+ \sum_{j=i+1}^{r-1} \sum_{\sigma \in \mathcal{S}_{r-1}} |\sigma| \theta_{\sigma(1)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \mathcal{L}_T\omega_{\sigma(j),0} \cdots \omega_{\sigma(r-1),0}\\ =& \sum_{j=1}^{i} \sum_{\sigma \in \mathcal{S}_{r-1}} |\sigma| \theta_{\sigma(1)} \cdots \omega_{\sigma(j),0} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \omega_{\sigma(r-1),0}\\ &+ a \sum_{j=i+1}^{r-1} \sum_{\sigma \in \mathcal{S}_{r-1}} |\sigma| \theta_{\sigma(1)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \theta_{\sigma(j)} \cdots \omega_{\sigma(r-1),0}\end{aligned}$$

Consider $\tau_{i,j} = (j, i)$. Then

$$\theta_{\sigma(1)} \cdots \omega_{\sigma(j),0} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \omega_{\sigma(r-1),0} =$$
$$\theta_{\sigma\circ\tau_{i,j}(1)} \cdots \theta_{\sigma\circ\tau_{i,j}(i-1)} \omega_{\sigma\circ\tau_{i,j}(i),0} \cdots \omega_{\sigma\circ\tau_{i,j}(r-1),0}$$

In the same way,

$$\theta_{\sigma(r)} \cdots \theta_{\sigma(r+i-1)} \omega_{\sigma(r+i),0} \cdots \theta_{\sigma(j)} \cdots \omega_{\sigma(r-1),0} =$$
$$\theta_{\sigma\circ\tau_{i,j}(1)} \cdots \theta_{\sigma\circ\tau_{i,j}(i+1)} \omega_{\sigma\circ\tau_{i,j}(i+2)} \cdots \omega_{\sigma\circ\tau_{i,j}(r-1)},$$

Therefore, the first sum is given by

$$\sum_{j=1}^{i} \sum_{\sigma\in\mathcal{S}_{r-1}} |\sigma \circ \tau_{i,j}| \theta_{\sigma\circ\tau_{i,j}(1)} \cdots \theta_{\sigma\circ\tau_{i,j}(i-1)} \omega_{\sigma\circ\tau_{i,j}(i)} \cdots \omega_{\sigma\circ\tau_{i,j}(r-1)} =$$
$$= \sum_{j=1}^{i} \sum_{\pi\in\mathcal{S}_{r-1}} |\pi| \theta_{\pi(1)} \cdots \theta_{\pi(i-1)} \omega_{\pi(i)} \cdots \omega_{\pi(r-1)} =$$
$$= i \sum_{\pi\in\mathcal{S}_{r-1}} |\pi| \theta_{\pi(1)} \cdots \theta_{\pi(i-1)} \omega_{\pi(i)} \cdots \omega_{\pi(r-1)} = i!(r-i)!\mathcal{T}_{i-1}(f).$$

and similarly

$$\sum_{j=i+1}^{r-1} \sum_{\sigma\in\mathcal{S}_{r-1}} |\sigma| \theta_{\sigma(1)} \cdots \theta_{\sigma(i)} \omega_{\sigma(i+1),0} \cdots \theta_{\sigma(j)} \cdots \omega_{\sigma(r-1),0} = (r-i-1)!(i+1)!\mathcal{T}_{i+1}(f).$$

Thus the result follows. □

Remark 4.1.13. The previous proposition generalizes Lemma 3.1.1, considering that, as in Example 4.1.6,

$$f_n : S\mathbb{S}^n_\lambda \to \mathrm{Gr}_n(\mathbb{S}^n_\lambda), \quad \xi \longmapsto \langle e_0(\xi), \ldots, e_{n-1}(\xi) \rangle$$

satisfies

$$\mathcal{T}_i(f_n) = \kappa_i, \qquad 0 \le i \le n-1.$$

According to Proposition 4.1.12, for $r = n$,

$$\mathcal{L}_T \kappa_i = (n-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1},$$

which coincides with Lemma 3.1.1.

Proposition 4.1.14. *In* $\mathbb{C}P^n_\lambda$

$$\mathcal{L}_T \psi_0 = -4\lambda\psi_1, \quad \mathcal{L}_T \psi_1 = \psi_0,$$

and for $0 \le i \le 2n-2$,

$$\mathcal{L}_T \varphi_i = (2n-i-1)\varphi_{i-1} - \lambda(i+1)\varphi_{i+1}.$$

Proof. By definition

$$\psi_i = \mathcal{T}_i(f_2), \quad \varphi_j = \mathcal{T}_j(e_0 \oplus f_2^\perp)$$

where $f_2 \colon S\mathbb{C}P^n_\lambda \to \mathrm{Gr}_2(\mathbb{C}P^n_\lambda)$ and $e_0 \oplus f_2^\perp \colon S\mathbb{C}P^n_\lambda \to \mathrm{Gr}_{2n-2}(\mathbb{C}P^n_\lambda)$. By Propositions 1.4.8 and 4.1.4, Proposition 4.1.12 applies and yields the result. □

Corollary 4.1.15. *In* $\mathbb{C}P^n_\lambda$*, for* $0 \leq i \leq 2n-1$

$$\mathcal{L}_T\kappa_i = (2n-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1} - 3\lambda\beta \wedge \varphi_i.$$

Proof.

$$\begin{aligned}\kappa_i =& \mathcal{L}_T\gamma \wedge \varphi_i + \gamma \wedge \mathcal{L}_T\varphi_i + \mathcal{L}_T\beta \wedge \varphi_{i-1} + \beta \wedge \mathcal{L}_T\varphi_{i-1}\\ =& -4\lambda\beta \wedge \varphi_i + (2n-i-1)\gamma \wedge \varphi_{i-1} - \lambda(i+1)\gamma \wedge \varphi_{i+1}\\ &+ \gamma \wedge \varphi_{i-1} + (2n-i)\beta \wedge \varphi_{i-2} - \lambda i\beta \wedge \varphi_i\\ =&(2n-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1} - 3\lambda\beta \wedge \wedge\varphi_i.\end{aligned}$$

□

Proposition 4.1.16. *In* $\mathbb{H}P^n_\lambda$

$$\begin{aligned}\mathcal{L}_T\psi_i =&(4-i)\psi_{i-1} - 4\lambda(i+1)\psi_{i+1}, && 0 \leq i \leq 3,\\ \mathcal{L}_T\varphi_i =&(4n-i-3)\varphi_{i-1} - \lambda(i+1)\varphi_{i+1}, && 0 \leq i \leq 4n-4.\end{aligned}$$

Proof. By definition

$$\psi_i = \mathcal{T}_i(f_4), \quad \varphi_j = \mathcal{T}_j(e_0 \oplus f_4^\perp)$$

where $f_4 \colon S\mathbb{H}P^n_\lambda \to \mathrm{Gr}_4(\mathbb{H}P^n_\lambda)$ and $e_0 \oplus f_4^\perp \colon S\mathbb{H}P^n_\lambda \to \mathrm{Gr}_{4n-4}(\mathbb{C}P^n_\lambda)$. By Propositions 1.4.14 and 4.1.4, Proposition 4.1.12 applies and yields the result. □

Corollary 4.1.17. *In* $\mathbb{H}P^n_\lambda$*, for* $0 \leq i \leq 4n-1$

$$\begin{aligned}\mathcal{L}_T\kappa_i =&(4n-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1}\\ &- 3\lambda\psi_1 \wedge \varphi_i - 6\lambda\psi_2 \wedge \varphi_{i-1} - 9\lambda\psi_3 \wedge \varphi_{i-2}\end{aligned}$$

Proof. By Corollary 4.1.11

$$\kappa_i = \psi_0 \wedge \varphi_i + \psi_1 \wedge \varphi_{i-1} + \psi_2 \wedge \varphi_{i-2} + \psi_3 \wedge \varphi_{i-3}, \quad 0 \leq i \leq 4n-1.$$

Therefore

$$
\begin{aligned}
\mathcal{L}_T\kappa_i =& \mathcal{L}_T\psi_0 \wedge \varphi_i + \psi_0 \wedge \mathcal{L}_T\varphi_i \\
& + \mathcal{L}_T\psi_1 \wedge \varphi_{i-1} + \psi_1 \wedge \mathcal{L}_T\varphi_{i-1} \\
& + \mathcal{L}_T\psi_2 \wedge \varphi_{i-2} + \psi_2 \wedge \mathcal{L}_T\varphi_{i-2} \\
& + \mathcal{L}_T\psi_3 \wedge \varphi_{i-3} + \psi_3 \wedge \mathcal{L}_T\varphi_{i-3} \\
=& -4\lambda\psi_1 \wedge \varphi_i + (4n-i-2)\psi_0 \wedge \varphi_{i-1} - \lambda(i+1)\psi_0 \wedge \varphi_{i+1} \\
& + 3\psi_0 \wedge \varphi_{i-1} - 8\lambda\psi_2 \wedge \varphi_{i-1} + (4n-i-1)\psi_1 \wedge \varphi_{i-2} - \lambda i\psi_1 \wedge \varphi_i \\
& + 2\psi_1 \wedge \varphi_{i-2} - 12\lambda\psi_3 \wedge \varphi_{i-2} + (4n-i)\psi_2 \wedge \varphi_{i-3} - \lambda(i-1)\psi_2 \wedge \varphi_{i-1} \\
& + \psi_2 \wedge \varphi_{i-3} + (4n-i+1)\psi_3 \wedge \varphi_{i-4} - \lambda(i-2)\psi_3 \wedge \varphi_{i-2} \\
=& (4n-i-3)\psi_0 \wedge \varphi_{i-1} + 3\psi_0 \wedge \varphi_{i-1} \\
& + (4n-i-2)\psi_1 \wedge \varphi_{i-2} + 2\psi_1 \wedge \varphi_{i-2} \\
& + (4n-i-1)\psi_2 \wedge \varphi_{i-3} + \psi_2 \wedge \varphi_{i-3} \\
& + (4n-i)\psi_3 \wedge \varphi_{i-4} \\
& - \lambda(i+1)\psi_0 \wedge \varphi_{i+1} \\
& - 4\lambda\psi_1 \wedge \varphi_i - \lambda i\psi_1 \wedge \varphi_i \\
& - 8\lambda\psi_2 \wedge \varphi_{i-1} - \lambda(i-1)\psi_2 \wedge \varphi_{i-1} \\
& - 12\lambda\psi_3 \wedge \varphi_{i-2} - \lambda(i-2)\psi_3 \wedge \varphi_{i-2} \\
& = (4n-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1} \\
& - 3\lambda\psi_1 \wedge \varphi_i - 6\lambda\psi_2 \wedge \varphi_{i-1} - 9\lambda\psi_3 \wedge \varphi_{i-2}.
\end{aligned}
$$

□

4.2 Globalization of $\psi_i \wedge \varphi_j$ in $\mathbb{C}P^n_\lambda$

After determining the Lie derivative of each of $\varphi_i \wedge \varphi_j$, we establish their globalizations in $\mathbb{C}P^n_\lambda$ in terms of the basis $\sigma^\lambda_{k,r}$ of $\mathcal{V}^n_{\lambda,\mathbb{C}}$. Let us recall the Definition 1.3.7 adapted to $M = \mathbb{C}P^n_\lambda$

$$[\![\kappa_i, 0]\!] = (2n-i)\omega_{2n-i}\nu_i = (2n-i)\omega_{2n-i}\,\mathrm{glob}(\Delta_i), \quad 0 \le i \le 2n-1.$$

Lemma 4.2.1 ([21, Lemma 3.9]). *For* $\max\{0, k-n\} \le q \le k/2 < n$ *we consider*

$$\delta_{k,q} = \frac{1}{2n-k}\left(2(n-k+q)\gamma_{k,q} + (k-2q)\beta_{k,q}\right).$$

Then, for $\Delta_{i,q} := [\delta_{i,q}, 0]$

$$\Delta_i = \sum_{q=0}^{\lfloor i/2 \rfloor} \Delta_{i,q}$$

and

$$[\![\delta_{k,q}, 0]\!] = \mu^\lambda_{k,q} - \lambda\frac{q+1}{\pi}\mu^\lambda_{k+2,q+1}, \quad \mu^\lambda_{k,q} = \sum_{i\ge 0}\left(\frac{\lambda}{\pi}\right)^i \frac{(q+i)!}{q!}[\![\delta_{k+2i,q+i}, 0]\!].$$

Proposition 4.2.2. *In* $\mathbb{C}P^n_\lambda$, *for* $0 \leq i \leq 2n-1$

$$\llbracket \beta \wedge \varphi_{i-1}, 0 \rrbracket_\lambda = \frac{i(2n-i)}{2n-1}\sigma^\lambda_{i,0} - 2\sigma^\lambda_{i,1},$$

$$\llbracket \gamma \wedge \varphi_i, 0 \rrbracket_\lambda = \frac{(2n-i)(2n-i-1)}{2n-1}\sigma^\lambda_{i,0} - \lambda\frac{(i+1)(i+2)}{(2n-1)}\sigma^\lambda_{i+2,0} + 2\sigma^\lambda_{i,1} - 2\lambda\sigma^\lambda_{i+2,1},$$

$$\llbracket \kappa_i, 0 \rrbracket = (2n-i)\sigma^\lambda_{i,0} - \lambda\frac{(i+1)(i+2)}{(2n-1)}\sigma^\lambda_{i+2,0} - 2\lambda\sigma^\lambda_{i+2,1}.$$

Proof. By Lemma 4.2.1

$$\begin{aligned}\llbracket \kappa_i, 0 \rrbracket &= (2n-i)\omega_{2n-i}\sum_{q=0}^{\lfloor i/2 \rfloor}\llbracket \delta_{i,q}, 0 \rrbracket \\ &= (2n-i)\omega_{2n-i}\sum_{q=0}^{\lfloor i/2 \rfloor}\left(\mu^\lambda_{i,q} - \lambda\frac{q+1}{\pi}\mu^\lambda_{i+2,q+1}\right) \\ &= (2n-i)\omega_{2n-i}\left(\tau_{i,0} - \frac{\lambda}{\pi}\tau_{i+1,1}\right).\end{aligned}$$

By the Lefschetz decomposition given in Proposition 2.1.10

$$\begin{aligned}\llbracket \kappa_i, 0 \rrbracket &= (2n-i)\omega_{2n-i}\left(\tau^\lambda_{i,0} - \frac{\lambda}{\pi}\tau^\lambda_{i+2,1}\right) \\ &= (2n-i)\omega_{2n-i}\left(\frac{1}{\omega_{2n-i}}\sigma^\lambda_{i,0} - \frac{\lambda}{\pi}\left(\frac{(i+1)(i+2)}{2(2n-1)\omega_{2n-i-2}}\sigma^\lambda_{i+2,0} + \frac{1}{\omega_{2n-i-2}}\sigma_{i+2,1}\right)\right) \\ &= (2n-i)\sigma^\lambda_{i,0} - \lambda\frac{(i+1)(i+2)}{(2n-1)}\sigma^\lambda_{i+2,0} - 2\lambda\sigma^\lambda_{i+2,1}.\end{aligned}$$

From Corollary 4.1.15

$$\beta \wedge \varphi_i = \frac{1}{3\lambda}\left((2n-i)\kappa_{i-1} - \lambda(i+1)\kappa_{i+1} - \mathcal{L}_T\kappa_i\right).$$

Then, globalizing

$$\llbracket \beta \wedge \varphi_i, 0 \rrbracket_\lambda = \frac{1}{3\lambda}\left((2n-i)\llbracket \kappa_{i-1}, 0 \rrbracket_\lambda - \lambda(i+1)\llbracket \kappa_{i+1}, 0 \rrbracket_\lambda - \llbracket \mathcal{L}_T\kappa_i, 0 \rrbracket_\lambda\right). \tag{4.3}$$

On the other hand, by Proposition 2.2.5

$$\llbracket \mathcal{L}_T\kappa_i, 0 \rrbracket_\lambda = \partial_{\lambda,\mathbb{C}}\llbracket \kappa_i, 0 \rrbracket = \partial_{\lambda,\mathbb{C}}\left((2n-i)\sigma^\lambda_{i,0} - \lambda\frac{(i+1)(i+2)}{(2n-1)}\sigma^\lambda_{i+2,0} - 2\lambda\sigma^\lambda_{i+2,1}\right).$$

Using

$$\partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k,r} = (2n-k-2r+1)\sigma^\lambda_{k-1,r} - \lambda(k-2r+1)\sigma^\lambda_{k+1,r}.$$

and equation (4.3) we get

$$[\![\beta \wedge \varphi_{i-1}, 0]\!]_\lambda = \frac{i(2n-i)}{2n-1}\sigma^\lambda_{i,0} - 2\sigma^\lambda_{i,1}.$$

From Corollary 4.1.11

$$[\![\gamma \wedge \varphi_i, 0]\!] = [\![\kappa_i, 0]\!] - [\![\beta \wedge \varphi_{i-1}, 0]\!].$$

The conclusion follows. □

Corollary 4.2.3. *In* $\mathbb{C}P^n_\lambda$*, for* $0 \leq i \leq 2n-1$

$$\nu_i = \frac{1}{(2n-i)\omega_{2n-i}}\left\{(2n-i)\sigma^\lambda_{i,0} - \lambda\frac{(i+1)(i+2)}{(2n-1)}\sigma^\lambda_{i+2,0} - 2\lambda\sigma^\lambda_{i+2,1}\right\}.$$

4.3 Tube formulas for the Federer valuations in complex and quaternionic space forms

Although we know $\mathbf{T}_t$ and $\mathbf{LT}_t$ in complex space forms, it is interesting to establish particular formulas for Δ_i and ν_i also in $\mathbb{C}P^n_\lambda$. Moreover, the same approach will apply also in $\mathbb{H}P^n_\lambda$.

4.3.1 Complex space forms

Let us recall that in $\mathbb{C}P^n_\lambda$ we have a particular construction of families of differential forms. We stated in Proposition 4.1.4 that $f_2 \colon S\mathbb{C}P^n_\lambda \to \mathrm{Gr}_2(\mathbb{C}P^n_\lambda)$

$$f_2 \colon \xi \longmapsto \langle e_0(\xi), Je_0(\xi)\rangle,$$

is a parallel distribution along geodesics. We consider the differential forms introduced in the Example 4.1.8

$$\psi_i = \mathcal{T}_i(f_2), \quad \varphi_j = \mathcal{T}_j(e_0 \oplus f_2^\perp),$$

for $0 \leq i \leq 1$ and $0 \leq j \leq 2n-2$. We denote by $W^{\mathbb{C}}_1$ and $W^{\mathbb{C}}_{2n-2}$ the vector subspaces

$$W^{\mathbb{C}}_1 := \langle \psi_0, \psi_1\rangle = \langle \gamma, \beta\rangle, \quad W^{\mathbb{C}}_{2n-2} := \langle \varphi_i : 0 \leq i \leq 2n-2\rangle \tag{4.4}$$

and $W^{\mathbb{C}} := W^{\mathbb{C}}_1 \otimes W^{\mathbb{C}}_{2n-2}$ its tensor product. Recall also that $V^{(m)}$ denotes the space of m-homogeneous polynomials on x, y, on which we considered the operators

$$X = x\frac{\partial}{\partial y}, \qquad Y = y\frac{\partial}{\partial x}, \qquad Y_\lambda = Y - \lambda X.$$

We can rephrase Proposition 4.1.14 as follows.

Theorem 4.3.1. *The linear isomorphisms*

$$F_\psi\colon (V^{(1)}, Y_{4\lambda}) \longrightarrow \left(W_1^{\mathbb{C}}, \mathcal{L}_T\right), \qquad x \longmapsto \psi_1, \quad y \longmapsto \psi_0$$
$$F_\varphi\colon (V^{(2n-2)}, Y_\lambda) \longrightarrow \left(W_{2n-2}^{\mathbb{C}}, \mathcal{L}_T\right), \qquad p_{2n-2,i} \longmapsto \varphi_i,$$

satisfy

$$\mathcal{L}_T \circ F_\psi = F_\psi \circ Y_{4\lambda}, \qquad \mathcal{L}_T \circ F_\varphi = F_\varphi \circ Y_\lambda.$$

Proof. It suffices to compare Proposition 4.1.14 with

$$\begin{aligned} &Y_{4\lambda}(x) = y, \\ &Y_{4\lambda}(y) = -4\lambda x, \\ &Y_\lambda(p_{2n-2,i}) = (2n-i+1)p_{2n-2,i-1} - \lambda(i+1)p_{2n-2,i+1}. \quad \square \end{aligned}$$

Remark 4.3.2. The map F_ψ was previously introduced in Theorem 3.6.1 under the notation $F_{\beta,\gamma}$.

Proposition 4.3.3.

$$\begin{aligned} \phi_t^*\gamma &= (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\gamma - \lambda\sin_\lambda(t)\cos_\lambda(t)\beta \\ \phi_t^*\beta &= \sin_\lambda(t)\cos_\lambda(t)\gamma + (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\beta \\ \phi_t^*\varphi_i &= \sum_{j=0}^{2n-2} \phi_{2n-2,i,j}^\lambda(t)\varphi_j. \end{aligned}$$

Proof. Since F_ψ intertwines $\mathcal{L}_T$ and $Y_{4\lambda}$, according to Theorem 3.3.3, we obtain the following expressions

$$\begin{aligned} \phi_t^*\gamma &= \exp(t\mathcal{L}_T)\gamma = \cos_{4\lambda}(t)\gamma - \lambda\sin_{4\lambda}(t)\beta, \\ \phi_t^*\beta &= \exp(t\mathcal{L}_T)\beta = \cos_{4\lambda}(t)\beta + \sin_{4\lambda}(t)\gamma. \end{aligned}$$

The identities

$$\begin{aligned} \sin_{4\lambda}(t) &= \sin_\lambda(t)\cos_\lambda(t), \\ \cos_{4\lambda}(t) &= \cos_\lambda^2(t) - \lambda\sin_\lambda^2(t), \end{aligned}$$

allow us to derive the first two formulas. The third formula can be established by using Theorem 3.3.3. $\square$

Corollary 4.3.4.

$$\phi_t^*(\gamma\wedge\varphi_j) = \left((\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\gamma - \lambda\sin_\lambda(t)\cos_\lambda(t)\beta\right)\sum_{k=0}^{2n-2}\phi_{2n-2,j,k}^\lambda(t)\varphi_k,$$
$$\phi_t^*(\beta\wedge\varphi_j) = (\sin_\lambda(t)\cos_\lambda(t)\gamma + (\cos_\lambda^2(t) - \lambda\sin_\lambda^2(t))\beta)\sum_{k=0}^{2n-2}\phi_{2n-2,j,k}^\lambda(t)\varphi_k.$$

Globalizing we get the following results.

Theorem 4.3.5. *In* $\mathbb{C}P^n_\lambda$

$$\mathbf{LT}_t[\beta\wedge\varphi_i,0] = (\cos^2_\lambda(t)-\lambda\sin^2_\lambda(t))\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\beta\wedge\varphi_j,0]+\sin_\lambda(t)\cos_\lambda(t)\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\gamma\wedge\varphi_j,0],$$

$$\mathbf{LT}_t[\gamma\wedge\varphi_i,0] = -\lambda\sin_\lambda(t)\cos_\lambda(t)\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\beta\wedge\varphi_j,0]+(\cos^2_\lambda(t)-\lambda\sin^2_\lambda(t))\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\gamma\wedge\varphi_j,0],$$

$$\mathbf{T}_t[\![\beta\wedge\varphi_i,0]\!] = (\cos^2_\lambda(t)-\lambda\sin^2_\lambda(t))\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\![\beta\wedge\varphi_j,0]\!]+\sin_\lambda(t)\cos_\lambda(t)\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\![\gamma\wedge\varphi_j,0]\!]$$

$$\mathbf{T}_t[\![\gamma\wedge\varphi_i,0]\!] = -\lambda\sin_\lambda(t)\cos_\lambda(t)\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\![\beta\wedge\varphi_j,0]\!]+(\cos^2_\lambda(t)-\lambda\sin^2_\lambda(t))\sum_{j=0}^{2n-2}\phi^\lambda_{2n-2,i,j}(t)[\![\gamma\wedge\varphi_j,0]\!]$$

Corollary 4.3.6. *The tube formulas for the Federer valuations and curvature measures in* $\mathbb{C}P^n_\lambda$ *are*

$$\mathbf{T}_t(\nu_i) = \frac{\cos^2_\lambda(t)}{(2n-i)\omega_{2n-i}}\sum_{k=0,1}\sum_{j=0}^{2n-2}\left((1-\lambda\tan^2_\lambda(t))\phi_{2n-2,i-k,j}(t)+(-\lambda)^k\tan_\lambda(t)\phi_{2n-2,i+k-1,j}(t)\right)[\![\psi_k\wedge\varphi_j,0]\!],$$

$$\mathbf{LT}_t(\Delta_i) = \frac{\cos^2_\lambda(t)}{(2n-i)\omega_{2n-i}}\sum_{k=0,1}\sum_{j=0}^{2n-2}\left((1-\lambda\tan^2_\lambda(t))\phi_{2n-2,i-k,j}(t)+(-\lambda)^k\tan_\lambda(t)\phi_{2n-2,i+k-1,j}(t)\right)[\psi_k\wedge\varphi_j,0],$$

for $0\le i\le 2n-1$.

4.3.2 Quaternionic space forms

The procedure used in the previous subsection can be applied in $\mathbb{H}P^n_\lambda$. We stated in Proposition 4.1.4 that $f_4\colon S\mathbb{H}P^n_\lambda\to \mathrm{Gr}_4(\mathbb{H}P^n_\lambda)$ given by

$$f_4\colon \xi\longmapsto \langle e_0(\xi),J_1e_0(\xi),J_2e_0(\xi),J_3e_0(\xi)\rangle$$

is a parallel distribution along geodesics. Thus, we have a family of differential forms

$$\psi_i=\mathcal{T}_i(f_4),\quad \varphi_j=\mathcal{T}_j(e_0\oplus f_4^\perp),$$

for $0\le i\le 3$ and $0\le j\le 4n-4$, as in the Example 4.1.9. We consider the vector subspaces of $\Omega^\bullet(S\mathbb{H}P^n_\lambda)$

$$W_3^{\mathbb{H}}:=\langle\psi_0,\psi_1,\psi_2,\psi_3\rangle,\quad W_{4n-4}^{\mathbb{H}}:=\langle\varphi_i : 0\le i\le 4n-4\rangle, \tag{4.5}$$

and $W^{\mathbb{H}}:=W_3^{\mathbb{H}}\otimes W_{4n-4}^{\mathbb{H}}$ its tensor product. Proposition 4.1.16 is equivalent to the following.

Theorem 4.3.7. *The linear isomorphisms*

$$F_\psi\colon (V^{(3)}, Y_{4\lambda}) \longrightarrow \left(W_3^{\mathbb{H}}, \mathcal{L}_T\right) \qquad p_{3,i} \longmapsto \psi_i,$$
$$F_\varphi\colon (V^{(4n-4)}, Y_\lambda) \longrightarrow \left(W_{4n-4}^{\mathbb{H}}, \mathcal{L}_T\right), \qquad p_{4n-4,i} \longmapsto \varphi_i,$$

satisfy

$$\mathcal{L}_T \circ F_\psi = F_\psi \circ Y_{4\lambda}, \qquad \mathcal{L}_T \circ F_\varphi = F_\varphi \circ Y_\lambda.$$

Proof. It suffices to compare Proposition 4.1.16 with

$$Y_{4\lambda}(p_{3,i}) = (4-i)p_{3,i-1} - 4\lambda(i+1)p_{3,i+1},$$
$$Y_\lambda(p_{4n-4,i}) = (4n-i-3)p_{4n-4,i-1} - \lambda(i+1)p_{4n-4,i+1}. \quad \square$$

Proposition 4.3.8. *For* $0 \le i \le 3$ *and* $0 \le j \le 4n-4$,

$$\phi_t^*\psi_i = \sum_{k=0}^{3} \phi_{3,i,k}^{4\lambda}(t)\psi_k,$$
$$\phi_t^*\varphi_j = \sum_{l=0}^{4n-4} \phi_{4n-4,j,l}^{\lambda}(t)\varphi_l.$$

Proof. This is a direct application of Theorem 3.3.3. $\square$

Corollary 4.3.9. *In* $\mathbb{H}P_\lambda^n$

$$\phi_t^*(\psi_i \wedge \varphi_j) = \sum_{k=0}^{3}\sum_{l=0}^{4n-4} \phi_{3,i,k}^{4\lambda}(t)\phi_{4n-4,j,l}^{\lambda}(t)\psi_k \wedge \varphi_l.$$

Proof. By Corollary 4.1.11

$$\kappa_i = \psi_0 \wedge \varphi_i + \psi_1 \wedge \varphi_{i-1} + \psi_2 \wedge \varphi_{i-2} + \psi_3 \wedge \varphi_{i-3}, \quad 0 \le i \le 4n-1.$$

Then, applying Corollary 4.3.9, the result follows. $\square$

Globalizing we get

Theorem 4.3.10. *In* $\mathbb{H}P_\lambda^n$

$$\mathbf{LT}_t[\psi_i \wedge \varphi_j, 0] = \sum_{k=0}^{3}\sum_{l=0}^{4n-4} \phi_{3,i,k}^{4\lambda}(t)\phi_{4n-4,j,l}^{\lambda}(t)[\psi_k \wedge \varphi_l, 0]$$
$$\mathbf{T}_t[\![\psi_i \wedge \varphi_j, 0]\!] = \sum_{k=0}^{3}\sum_{l=0}^{4n-4} \phi_{3,i,k}^{4\lambda}(t)\phi_{4n-4,j,l}^{\lambda}(t)[\![\psi_k \wedge \varphi_l, 0]\!]$$

Corollary 4.3.11. *The tube formulas for the Federer valuations and curvature measures in* $\mathbb{H}P^n_\lambda$ *are*

$$\mathbf{T}_t(\nu_i) = \frac{1}{(4n-i)\omega_{4n-i}} \sum_{j,k=0}^{3} \sum_{l=0}^{4n-4} \phi^{4\lambda}_{3,j,k}(t)\phi^{\lambda}_{4n-4,i-j,l}(t)[\![\psi_k \wedge \varphi_l, 0]\!],$$

$$\mathbf{LT}_t(\Delta_i) = \frac{1}{(4n-i)\omega_{4n-i}} \sum_{j,k=0}^{3} \sum_{l=0}^{4n-4} \phi^{4\lambda}_{3,j,k}(t)\phi^{\lambda}_{4n-4,i-j,l}(t)[\psi_k \wedge \varphi_l, 0],$$

for $0 \leq i \leq 4n-1$.

4.4 Area tube formula in complex and quaternionic space forms

The rest of this chapter is essentially devoted to the determination of bases where the previous tube formulas have a simpler form. We begin by considering the tube formulas for the area using two alterniavtive approaches.

Proposition 4.4.1.

$$\phi_t^* d\,\mathrm{area}_{\mathbb{C}P^n_\lambda} = \sum_{i=0}^{2n} \sin_\lambda^{2n-i}(t)\cos_\lambda^i(t)\rho_{i,0},$$

where

$$\rho_{i,0} := \kappa_{i-1} - \lambda\beta \wedge \varphi_i, \quad 0 \leq i \leq 2n.$$

As usual, it is understood above that $\kappa_{i-1} = 0$ for $i = 0$, and $\varphi_{2n-1} = \varphi_{2n} = 0$.

Proof. Since

$$\kappa_{2n-1} = \beta \wedge \varphi_{2n-2}.$$

and $\phi_{m,m,j}(t) = \sin_\lambda^{m-j}(t)\cos_\lambda^j(t)$, by Corollary 4.3.4 we have

$$\begin{aligned}
\phi_t^*(\beta \wedge \varphi_{2n-2}) &= \sum_{j=0}^{2n-2} \sin_\lambda^{2n-j-2}\cos_\lambda^{j+2}\beta \wedge \varphi_j \\
&\quad + \sum_{j=0}^{2n-2} \sin_\lambda^{2n-j-1}\cos_\lambda^{j+1}\gamma \wedge \varphi_j \\
&\quad - \lambda \sum_{j=0}^{2n-2} \sin_\lambda^{2n-j}\cos_\lambda^{j}\beta \wedge \varphi_j \\
&= \sum_{j=0}^{2n-2} \sin_\lambda^{2n-1-j}(t)\cos_\lambda^{j+1}(t)(\beta \wedge \varphi_{j-1} + \gamma \wedge \varphi_j) \\
&\quad - \lambda \sum_{j=0}^{2n-2} \sin_\lambda^{2n-j}(t)\cos_\lambda^{j}(t)\beta \wedge \varphi_j.
\end{aligned}$$

Since $\kappa_i = \gamma \wedge \varphi_i + \beta \wedge \varphi_{i-1}$, the result follows. $\square$

Corollary 4.4.2.

$$\begin{aligned}
\mathbf{LT}_t \operatorname{area}_{\mathbb{C}P^n_\lambda} &= \sum_{i=0}^{2n} \sin_\lambda^{2n-i}(t) \cos_\lambda^i(t) [\rho_{i,0}, 0], \\
\mathbf{LT}_t \operatorname{vol}_{\mathbb{C}P^n_\lambda} &= \sum_{i=0}^{2n} \left(\int_0^t \sin_\lambda^{2n-i}(s) \cos_\lambda^i(s)\, ds \right) [\rho_{i,0}, 0] + [0, d \operatorname{vol}_{\mathbb{C}P^n_\lambda}].
\end{aligned}$$

Proposition 4.4.3.

$$\begin{aligned}
\phi_t^* d \operatorname{area}_{\mathbb{H}P^n_\lambda} &= \sum_{i=0}^{3} \sum_{j=0}^{4n-4} \sum_{k=0}^{i} (-\lambda)^k \binom{i}{k} \sin_\lambda^{4n-i-j+2k-1}(t) \cos_\lambda^{i+j-2k+3}(t) \psi_i \wedge \varphi_j \\
&= \sum_{l=0}^{4n+2} \sin_\lambda^{4n-l+2}(t) \cos_\lambda^l(t) \zeta_{l,0},
\end{aligned}$$

where

$$\begin{aligned}
\zeta_{l,0} := &\sum_{0 \le k \le i \le 3} (-\lambda)^k \binom{i}{k} \psi_i \wedge \varphi_{l-i+2k-3} \\
= &\,\psi_0 \wedge \varphi_{l-3} + \psi_1 \wedge \varphi_{l-4} + \psi_2 \wedge \varphi_{l-5} + \psi_3 \wedge \varphi_{l-6} \\
&- \lambda \psi_1 \wedge \varphi_{l-2} - 2\lambda \psi_2 \wedge \varphi_{l-3} - 3\lambda \psi_3 \wedge \varphi_{l-4} \\
&+ \lambda^2 \psi_2 \wedge \varphi_{l-1} + 3\lambda^2 \psi_3 \wedge \varphi_{l-2} \\
&- \lambda^3 \psi_3 \wedge \varphi_l.
\end{aligned}$$

Proof. Since $d \operatorname{area}_{\mathbb{H}P^n_\lambda} = \kappa_{4n-1} = \psi_3 \wedge \varphi_{4n-4}$ and $\phi^\lambda_{m,m,j}(t) = \sin_\lambda^{m-j}(t) \cos_\lambda^j(t)$ we only have to use Corollary 4.3.9

$$\begin{aligned}
\phi_t^*(\psi_3 \wedge \varphi_{4n-4}) &= \sum_{i=0}^{3} \sum_{j=0}^{4n-4} \phi^{4\lambda}_{3,3,i}(t) \phi^\lambda_{4n-4,4n-4,j}(t) \psi_i \wedge \varphi_j \\
&= \sum_{i=0}^{3} \sum_{j=0}^{4n-4} \sin_{4\lambda}^{3-i}(t) \cos_{4\lambda}^i(t) \sin_\lambda^{4n-j-4}(t) \cos_\lambda^j(t) \psi_i \wedge \varphi_j
\end{aligned}$$

Using the identities

$$\begin{aligned}
\sin_{4\lambda}(t) &= \sin_\lambda(t) \cos_\lambda(t), \\
\cos_{4\lambda}(t) &= \cos_\lambda^2(t) - \lambda \sin_\lambda^2(t),
\end{aligned}$$

we get

$$\begin{aligned}
\sin_{4\lambda}^{3-i} \cos_{4\lambda}^i(t) &= \sin_\lambda^{3-i}(t) \cos_\lambda^{3-i}(t) (\cos_\lambda^2(t) - \lambda \sin_\lambda^2(t))^i \\
&= \sum_{k=0}^{i} (-\lambda)^k \binom{i}{k} \sin_\lambda^{3-i+2k}(t) \cos_\lambda^{3+i-2k}(t).
\end{aligned}$$

Then,

$$\begin{aligned}
\phi_t^*(\psi_3 \wedge \varphi_{4n-4}) &= \sum_{i=0}^{3} \sum_{j=0}^{4n-4} \sin_{4\lambda}^{3-i}(t) \cos_{4\lambda}^{i}(t) \sin_{\lambda}^{4n-j-4}(t) \cos_{\lambda}^{j}(t) \psi_i \wedge \varphi_j \\
&= \sum_{i=0}^{3} \sum_{j=0}^{4n-4} \sum_{k=0}^{i} (-\lambda)^k \binom{i}{k} \sin_{\lambda}^{4n-i-j+2k-1}(t) \cos_{\lambda}^{i+j-2k+3}(t) \psi_i \wedge \varphi_j .
\end{aligned}$$

Setting $l = i + j - 2k + 3$ yields the result. □

Corollary 4.4.4.

$$\begin{aligned}
\mathbf{LT}_t \operatorname{area}_{\mathbb{H}P_\lambda^n} &= \sum_{i=0}^{4n+2} \sin_{\lambda}^{4n-i+2}(t) \cos_{\lambda}^{i}(t) [\zeta_{i,0}, 0], \\
\mathbf{LT}_t \operatorname{vol}_{\mathbb{H}P_\lambda^n} &= \sum_{i=0}^{4n+2} \left(\int_0^t \sin_{\lambda}^{4n-i+2}(s) \cos_{\lambda}^{i}(s) \right) [\zeta_{i,0}, 0] + [0, d \operatorname{vol}_{\mathbb{H}P_\lambda^n}].
\end{aligned}$$

Next, we will provide an alternative computation of the local tube area formulas in both $\mathbb{C}P_\lambda^n$ and $\mathbb{H}P_\lambda^n$, employing the tools we have developed so far in a different way. The expression we obtain coincides with the formulas derived by Gray and Vanhecke through different methods in [38].

Proposition 4.4.5 ([38])**.** *With respect to any hermitian moving frame of* $\mathbb{C}P_\lambda^n$

$$\phi_t^* d \operatorname{area}_{\mathbb{C}P_\lambda^n} = \cos_{\lambda}^{2n}(t)(\theta_1 + \tan_\lambda(t)\omega_{1,0} - \lambda \tan_\lambda^2(t)\theta_1)(\theta_2 - \tan_\lambda(t)\omega_{2,0}) \cdots (\theta_{2n-1} - \tan_\lambda(t)\omega_{2n-1,0}).$$

Proof. The expression on the right-hand side is independent of the chosen hermitian moving frame $e_0, \ldots, e_{2n-1}$. We can therefore assume that $e_0, \ldots, e_{2n-1}$ is a parallel hermitian moving frame. Then, by Corollary 1.4.9

$$\mathcal{L}_T \theta_j = \omega_{j,0}, \quad \mathcal{L}_T \omega_{1,0} = -4\lambda\theta_1, \quad \mathcal{L}_T \omega_{i,0} = -\lambda\theta_i,$$

where $1 \le j < i \le 2n-1$. Hence

$$\begin{aligned}
&F_1 \colon \langle \theta_1, \omega_{1,0} \rangle, \mathcal{L}_T) \longrightarrow (V^{(1)}, Y_{4\lambda}), && \theta_1 \longmapsto x, \quad \omega_{1,0} \longmapsto y, \\
&F_i \colon (\langle \theta_i, \omega_{i,0} \colon 2 \le i \le 2n-1 \rangle, \mathcal{L}_T) \longrightarrow (V^{(1)}, Y_\lambda), && \theta_i \longmapsto x, \quad \omega_{i,0} \longmapsto y,
\end{aligned}$$

are linear isomorphisms intertwining the indicated derivations; i.e. satisfying

$$\begin{aligned}
&Y_{4\lambda} \circ F_1 = F_1 \circ \mathcal{L}_T, \\
&Y_\lambda \circ F_i = F_i \circ \mathcal{L}_T, \quad 2 \le i \le 2n-1.
\end{aligned}$$

By Proposition 3.3.1

$$\begin{aligned}
\phi_t^* \theta_1 &= \exp(t\mathcal{L}_T)\theta_1 = \cos_{4\lambda}(t)\theta_1 + \sin_{4\lambda}(t)\omega_{1,0}, \\
\phi_t^* \theta_i &= \exp(t\mathcal{L}_T)\theta_i = \cos_{\lambda}(t)\theta_i + \sin_{\lambda}(t)\omega_{i,0}, \quad i > 1.
\end{aligned}$$

Since

$$\sin_{4\lambda}(t) = \sin_\lambda(t)\cos_\lambda(t),$$
$$\cos_{4\lambda}(t) = \cos^2_\lambda(t) - \lambda\sin^2_\lambda(t),$$

we have

$$\phi_t^*\theta_1 = \cos^2_\lambda(t)(\theta_1 + \tan_\lambda(t)\omega_{1,0} - \lambda\tan^2_\lambda(t)\theta_1).$$

For $2 \leq i \leq 2n-1$

$$\phi_t^*\theta_i = \cos_\lambda(t)(\theta_i - \tan_\lambda(t)\omega_{i,0}).$$

Therefore

$$\begin{aligned}\phi_t^* d\,\mathrm{area}_{\mathbb{C}P^n_\lambda} &= \phi_t^*(\theta_1 \wedge \cdots \wedge \theta_{2n-1})\\ &= \cos^{2n}_\lambda(t)(\theta_1 + \tan_\lambda(t)\omega_{1,0} - \lambda\tan^2_\lambda(t)\theta_1)(\theta_2 - \tan_\lambda(t)\omega_{2,0})\cdots(\theta_{2n-1} - \tan_\lambda(t)\omega_{2n-1,0}).\end{aligned}$$

□

Proposition 4.4.6 ([38]). *With respect to a partially quaternionic moving frame of* $\mathbb{H}P^n_\lambda$,

$$\begin{aligned}\phi_t^* d\,\mathrm{area}_{\mathbb{H}P^n_\lambda} = \cos^{4n+2}_\lambda(t)&(\theta_1 + \tan_\lambda(t)\omega_{1,0} - \lambda\tan^2_\lambda(t)\theta_1)(\theta_2 + \tan_\lambda(t)\omega_{2,0} - \lambda\tan^2_\lambda(t)\theta_2)\\ &(\theta_3 + \tan_\lambda(t)\omega_{3,0} - \lambda\tan^2_\lambda(t)\theta_3)(\theta_4 + \tan_\lambda(t)\omega_{4,0})\cdots(\theta_{4n-1} + \tan_\lambda(t)\omega_{4n-1,0})\end{aligned}$$

Proof. Let $e_0, \ldots, e_{4n-1}$ be a parallel and partially quaternionic moving frame. By Corollary 1.4.15 we have

$$\mathcal{L}_T\theta_k = \omega_{k,0}, \quad \mathcal{L}_T\omega_{i,0} = -4\lambda\theta_i, \quad \mathcal{L}_T\omega_{j,0} = -\lambda\theta_j,$$

for $0 \leq k \leq 4n-1$, $1 \leq i \leq 3$ and $4 \leq j \leq 4n-1$. Therefore, the following linear isomorphisms intertwine the indicated derivations

$$\begin{aligned}(\langle\theta_i, \omega_{i,0} \colon 1 \leq i \leq 3\rangle, \mathcal{L}_T) &\longrightarrow (V^{(1)}, Y_{4\lambda}), \qquad \theta_i \longmapsto x, \quad \omega_{i,0} \longmapsto y,\\ (\langle\theta_j, \omega_{j,0} \colon 4 \leq j \leq 4n-1\rangle, \mathcal{L}_T) &\longrightarrow (V^{(1)}, Y_\lambda), \qquad \theta_j \longmapsto x, \quad \omega_{j,0} \longmapsto y.\end{aligned}$$

Then, arguing as in Proposition 4.4.5, the result follows. □

Let us check that the formulas obtained in Proposition 4.4.1 and Proposition 4.4.5 are equivalent. To this end, we just need to use the following expansion

$$\bigwedge_{i=a}^{b}(x_i + y_i) = \sum_{i=a-1}^{b}\sum_\sigma |\sigma| x_{\sigma(a)}\cdots x_{\sigma(i)}y_{\sigma(i+1)}\cdots y_{\sigma(b)}. \tag{4.6}$$

where σ is in range of permutation of the set $\{a, \ldots, b\}$ such that

$$\sigma(a) < \cdots < \sigma(i); \quad \sigma(i+1) < \cdots < \sigma(b).$$

In this way,

$$\bigwedge_{i=2}^{2n-1}(\theta_i+\tan_\lambda\omega_{i,0})=\sum_{i=1}^{2n-1}\tan_\lambda^{2n-i-1}\sum_\sigma|\sigma|\theta_{\sigma(2)}\cdots\theta_{\sigma(i)}\omega_{\sigma(i+1)}\cdots\omega_{\sigma(2n-1)}$$
$$=\sum_{i=1}^{2n-1}\tan_\lambda^{2n-i-1}\varphi_{i-1},$$

Therefore the area tube formula given in Proposition 4.4.5 becomes

$$\begin{aligned}\phi_t^* d\,\mathrm{area}_{\mathbb{C}P^n_\lambda} &= \cos_\lambda^{2n}(t)(\beta+\tan_\lambda(t)\gamma-\lambda\tan_\lambda^2(t)\beta)\sum_{i=0}^{2n-2}\tan_\lambda^{2n-i-2}(t)\varphi_i\\
&=\cos_\lambda^{2n}(t)\sum_{i=0}^{2n-2}\tan_\lambda^{2n-i-2}(t)\beta\wedge\varphi_i\\
&+\cos_\lambda^{2n}(t)\sum_{i=0}^{2n-2}\tan_\lambda^{2n-i-1}(t)\gamma\wedge\varphi_i\\
&-\lambda\cos_\lambda^{2n}(t)\sum_{i=0}^{2n-2}\tan_\lambda^{2n-i}(t)\beta\wedge\varphi_i\\
&=\cos_\lambda^{2n}(t)\sum_{i=0}^{2n-1}(\kappa_{i-1}-\lambda\beta\wedge\varphi_i)\tan_\lambda^{2n-i}(t)\\
&=\sum_{i=0}^{2n-1}(\kappa_{i-1}-\lambda\beta\wedge\varphi_i)\sin_\lambda^{2n-i}(t)\cos_\lambda^i(t),\end{aligned}$$

which agrees with Proposition 4.4.1.

Likewise, the formulas in Proposition 4.4.3 and Proposition 4.4.6 are equivalent, since

$$\bigwedge_{i=4}^{4n-1}(\theta_i+\tan_\lambda\omega_{i,0})=\sum_{i=3}^{4n-1}\tan_\lambda^{4n-i-1}\sum_\sigma|\sigma|\theta_{\sigma(4)}\cdots\theta_{\sigma(i)}\omega_{\sigma(i+1),0}\cdots\omega_{\sigma(4n-1),0}$$
$$=\sum_{i=3}^{4n-1}\tan_\lambda^{4n-i-1}\varphi_{i-3}=\sum_{j=0}^{4n-4}\tan_\lambda^{4n-j-4}\varphi_j.$$

In the same manner

$$\begin{aligned}
&\bigwedge_{i=1}^{3}((1-\lambda\tan_\lambda^2(t))\theta_i+\tan_\lambda(t)\omega_{i,0})\\
&=\sum_{i=0}^{3}(1-\lambda\tan_\lambda^2(t))^i\tan_\lambda^{3-i}(t)\sum_\sigma\theta_{\sigma(1)}\cdots\theta_{\sigma(i)}\omega_{\sigma(i+1),0}\cdots\omega_{\sigma(3),0}\\
&=\sum_{i=0}^{3}(1-\lambda\tan_\lambda^2(t))^i\tan_\lambda^{3-i}(t)\psi_i\\
&=\sum_{i=0}^{3}\sum_{k=0}^{i}(-\lambda)^k\binom{i}{k}\tan_\lambda^{3-i+2k}(t)\psi_i.
\end{aligned}$$

Finally,

$$\begin{aligned}
\phi_t^*d\,\mathrm{area}_{\mathbb{H}P_\lambda^n}&=\cos_\lambda^{4n+2}(t)\bigwedge_{i=1}^{3}((1-\lambda\tan_\lambda^2(t))\theta_i+\tan_\lambda(t)\omega_{i,0})\bigwedge_{i=4}^{4n-1}(\theta_i+\tan_\lambda\omega_{i,0})\\
&=\cos_\lambda^{4n+2}(t)\sum_{i=0}^{3}\sum_{j=0}^{4n-4}\sum_{k=0}^{i}(-\lambda)^k\binom{i}{k}\tan_\lambda^{4n-i-j+2k-1}(t)\psi_i\wedge\varphi_j\\
&=\sum_{i=0}^{3}\sum_{j=0}^{4n-4}\sum_{k=0}^{i}(-\lambda)^k\binom{i}{k}\sin_\lambda^{4n-i-j+2k-1}(t)\cos_\lambda^{i+j-2k+3}(t)\psi_i\wedge\varphi_j,
\end{aligned}$$

which agrees with Proposition 4.4.3.

4.5 Simpler local tube formulas

Here we endow certain subspaces of differential forms with a $\mathfrak{sl}_2$-module structure compatible with $\mathcal{L}_T$. As a result we obtain specially simple tube formulas on some spaces of curvature measures.

Proposition 4.5.1. *In $\mathbb{C}P_\lambda^n$, for $0\leq k\leq 2n$*

$$\mathcal{L}_T\rho_{k,0}=(2n-k+1)\rho_{k-1,0}-\lambda(k+1)\rho_{k+1,0}.$$

Proof. We write for simplicity $h_l(t)=\sin_\lambda^{2n-l}(t)\cos_\lambda^l(t)$, which is a famliy of linearly independent functions. By Proposition 4.4.1

$$\phi_t^*d\,\mathrm{area}_{\mathbb{C}P_\lambda^n}=\sum_{l=0}^{2n}h_l(t)\rho_{l,0},\tag{4.7}$$

and $h_l'(t) = (2n-l)h_{l+1}(t) - \lambda l h_{l-1}(t)$,we have

$$\begin{aligned}
\frac{d}{dt}\phi_t^* d\mathrm{area}_{\mathbb{C}P^n_\lambda} &= \sum_{l=0}^{2n} h_l'(t)\rho_{l,0} \\
&= \sum_{l=0}^{2n} \left((2n-l)h_{l+1}(t) - \lambda l h_{l-1}(t)\right)\rho_{l,0} \\
&= \sum_{l=0}^{2n} \left((2n-l+1)\rho_{l-1,0} - \lambda(l+1)\rho_{l+1,0}\right) h_l(t).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{d}{dt}\phi_t^* d\mathrm{area}_{\mathbb{C}P^n_\lambda} &= \left.\frac{d}{ds}\right|_0 \phi_{s+t}^* d\,\mathrm{area}_{\mathbb{C}P^n_\lambda} = \left.\frac{d}{ds}\right|_0 \phi_s^*(\phi_t^* d\,\mathrm{area}_{\mathbb{C}P^n_\lambda}) \\
&= \mathcal{L}_T \phi_t^* d\,\mathrm{area}_{\mathbb{C}P^n_\lambda} = \sum_{l=0}^{2n} h_l(t)\mathcal{L}_T\rho_{l,0}.
\end{aligned}$$

The result follows by comparing coefficients of $h_l(t)$. □

Proposition 4.5.2. *In* $\mathbb{H}P^n_\lambda$*, for* $0 \le l \le 4n+2$

$$\mathcal{L}_T\zeta_{l,0} = (4n-l+3)\zeta_{l-1,0} - \lambda(l+1)\zeta_{l+1,0}, \quad 0 \le l \le 4n+2.$$

Proof. Put $h_l(t) = \sin_\lambda^{4n-l+2}(t)\cos_\lambda^l(t)$. By Proposition 4.4.3

$$\phi_t^* d\,\mathrm{area}_{\mathbb{H}P^n_\lambda} = \sum_{l=0}^{4n+2} h_l(t)\zeta_{l,0}.$$

Since $h_l'(t) = (4n-l+2)h_{l+1}(t) - \lambda l h_{l-1}(t)$, we have

$$\begin{aligned}
\frac{d}{dt}\phi_t^* d\mathrm{area}_{\mathbb{H}P^n_\lambda} &= \sum_{l=0}^{4n+2} h_l'(t)\zeta_{l,0} \\
&= \sum_{l=0}^{4n+2} \left((4n-l+2)h_{l+1}(t) - \lambda l h_{l-1}(t)\right)\zeta_{l,0} \\
&= \sum_{l=0}^{4n+2} \left((4n-l+3)\zeta_{l-1,0} - \lambda(l+1)\zeta_{l+1,0}\right) h_l(t).
\end{aligned}$$

On the other hand

$$\begin{aligned}
\frac{d}{dt}\phi_t^* d\mathrm{area}_{\mathbb{H}P^n_\lambda} &= \left.\frac{d}{ds}\right|_0 \phi_{s+t}^* d\,\mathrm{area}_{\mathbb{H}P^n_\lambda} = \left.\frac{d}{ds}\right|_0 \phi_s^*(\phi_t^* d\,\mathrm{area}_{\mathbb{H}P^n_\lambda}) \\
&= \mathcal{L}_T \phi_t^* d\,\mathrm{area}_{\mathbb{H}P^n_\lambda} = \sum_{l=0}^{4n+2} h_l(t)\mathcal{L}_T\zeta_{l,0}.
\end{aligned}$$

The result follows by comparing the coefficients of $h_l(t)$. □

In particular, the following subspaces of $W^{\mathbb{C}}$ and $W^{\mathbb{H}}$, are $\mathcal{L}_T$-invariant

$$\mathcal{R}^{n,0}_{\lambda,\mathbb{C}} := \{\rho_{k,0} : 0 \leq k \leq 2n\} \subset W^{\mathbb{C}},$$
$$\mathcal{R}^{n,0}_{\lambda,\mathbb{H}} := \{\zeta_{k,0} : 0 \leq k \leq 4n+2\} \subset W^{\mathbb{H}}.$$

We will find an additional invariant subspace of $W^{\mathbb{C}}$. It is suggested by the following proposition.

Proposition 4.5.3. *In $\mathbb{C}P^n_\lambda$, for $0 \leq k \leq 2n$*

$$[\![\rho_{k,0}, 0]\!] = \partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k,0}.$$

Proof. The volume tube formula in $\mathbb{C}P^n_\lambda$ is

$$\mathbf{T}_t \operatorname{vol}_{\mathbb{C}P^n_\lambda} = \sum_{k=0}^{2n} \sin^{2n-k}_\lambda(t) \cos^k_\lambda(t) \sigma^\lambda_{k,0}.$$

Hence

$$\begin{aligned}
\mathbf{T}_t \operatorname{area}_{\mathbb{C}P^n_\lambda} = \mathbf{T}_t\partial_{\lambda,\mathbb{C}} \operatorname{vol}_{\mathbb{C}P^n_\lambda} &= \left.\frac{d}{ds}\right|_{s=0} \mathbf{T}_{t+s} \operatorname{vol}_{\mathbb{C}P^n_\lambda} \\
&= \sum_{k=0}^{2n} \sin^{2n-k}_\lambda(t) \cos^k_\lambda(t) \left.\frac{d}{ds}\right|_{s=0} \mathbf{T}_s\sigma^\lambda_{k,0} \\
&= \sum_{k=0}^{2n} \sin^{2n-k}_\lambda(t) \cos^k_\lambda(t) \partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k,0}.
\end{aligned}$$

Globalizing (4.7), yields

$$\mathbf{T}_t \operatorname{area}_{\mathbb{C}P^n_\lambda} = \sum_{j=0}^{2n} \sin^{2n-j}_\lambda(t) \cos^j_\lambda(t) [\![\rho_{j,0}, 0]\!]. \tag{4.8}$$

By comparing the coefficients of the basis $\{\sin^{2n-j}_\lambda(t)\cos^j_\lambda(t)\}_{j=0}^{2n}$ the result follows. $\square$

Proposition 4.5.4. *For $2 \leq j \leq 2n-2$, set*

$$\rho_{j,1} := \frac{1}{2}\left\{-\frac{(2n-j)(2n-j-1)}{2n-1}\beta\wedge\varphi_{j-2} + \frac{(2n-j-1)(j-1)}{2n-1}\gamma\wedge\varphi_{j-1} + \lambda\frac{j(j-1)}{2n-1}\beta\wedge\varphi_j\right\}.$$

Then, for $r \in \{0,1\}$ and $2r \leq k \leq 2n-2r$

i)

$$[\![\rho_{k,r}, 0]\!] = \partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k,r}.$$

ii)

$$\mathcal{L}_T\rho_{k,r} = (2n-k-2r+1)\rho_{k-1,r} - \lambda(k-2r+1)\rho_{k+1,r}.$$

Proof. It remains only to prove the case $r = 1$.

i) By Proposition 4.2.2

$$\begin{aligned}2\partial_{\lambda,\mathbb{C}}\sigma_{i,1}^{\lambda} =&\frac{i(2n-i)}{2n-1}[\![\kappa_{i-1}-\lambda\beta\wedge\varphi_i,0]\!]\\&-[\![\gamma\wedge\varphi_{i-1},0]\!]-(2n-i)[\![\beta\wedge\varphi_{i-2},0]\!]+\lambda i[\![\beta\wedge\varphi_i,0]\!]\\=&-\frac{(2n-i)(2n-i-1)}{2n-1}[\![\beta\wedge\varphi_{i-2},0]\!]\\&+\frac{(2n-i-1)(i-1)}{2n-1}[\![\gamma\wedge\varphi_{i-1},0]\!]\\&+\lambda\frac{i(i-1)}{2n-1}[\![\beta\wedge\varphi_i,0]\!].\end{aligned}$$

Thus

$$[\![\rho_{i,1},0]\!]=\partial_{\lambda,\mathbb{C}}\sigma_{i,1},\quad 2\le i\le 2n-2.$$

ii) We need to prove

$$\mathcal{L}_T\rho_{i,1}=(2n-i-1)\rho_{i-1,1}-\lambda(i-1)\rho_{i+1,1}.$$

Since

$$\begin{aligned}\mathcal{L}_T(\beta\wedge\varphi_{j-2})&=\gamma\wedge\varphi_{j-2}+(2n-j+1)\beta\wedge\varphi_{j-3}-\lambda(j-1)\varphi_{j-1}\\\mathcal{L}_T(\gamma\wedge\varphi_{j-1})&=-4\lambda\beta\wedge\varphi_{j-1}+(2n-j)\varphi_{j-2}-\lambda j\varphi_j\\\mathcal{L}_T(\beta\wedge\varphi_j)&=\gamma\wedge\varphi_j+(2n-j-1)\beta\wedge\varphi_{j-1}-\lambda(j+1)\varphi_{j+1},\end{aligned}$$

we have

$$\begin{aligned}\mathcal{L}_T\rho_{j,1}=&-\frac{(2n-j)(2n-j-1)}{2(2n-1)}\left(\gamma\wedge\varphi_{j-2}+(2n-j+1)\beta\wedge\varphi_{j-3}-\lambda(j-1)\varphi_{j-1}\right)\\&+\frac{(2n-j-1)(j-1)}{2(2n-1)}\left(-4\lambda\beta\wedge\varphi_{j-1}+(2n-j)\varphi_{j-2}-\lambda j\varphi_j\right)\\&+\lambda\frac{j(j-1)}{2(2n-1)}\left(\gamma\wedge\varphi_j+(2n-j-1)\beta\wedge\varphi_{j-1}-\lambda(j+1)\varphi_{j+1}\right)\\=&(2n-j-1)\frac{1}{2}\left(-\frac{(2n-j+1)(2n-j)}{2n-1}\beta\wedge\varphi_{j-3}\right.\\&\left.+\frac{(2n-j)(j-2)}{2n-1}\gamma\wedge\varphi_{j-2}+\lambda\frac{(j-1)(j-2)}{2n-1}\beta\wedge\varphi_{j-1}\right)\\&-\lambda(j-1)\frac{1}{2}\left(-\frac{(2n-j-1)(2n-j-2)}{2n-1}\beta\wedge\varphi_{j-1}\right.\\&\left.+\frac{(2n-j-2)j}{2n-1}\gamma\wedge\varphi_j+\lambda\frac{(j+1)j}{2n-1}\beta\wedge\varphi_{j+1}\right).\end{aligned}$$

□

We have thus a decomposition $W^{\mathbb{C}} = \mathcal{R}^{n,0}_{\lambda,\mathbb{C}} \oplus \mathcal{R}^{n,1}_{\lambda,\mathbb{C}}$ into $\mathcal{L}_T$-invariant subspaces given by

$$\mathcal{R}^{n,r}_{\lambda,\mathbb{C}} := \langle \rho_{k,r} : 2r \leq k \leq 2n-2r \rangle \subset W^{\mathbb{C}}, \qquad r = 0, 1.$$

The operator $\mathcal{L}_T$ acts familiarly on each invariant subspace.

Theorem 4.5.5. *i) In $\mathbb{C}P^n_\lambda$, for $r \in \{0,1\}$, the linear isomorphisms given by*

$$F_\rho \colon (V^{(2n-4r)}, Y_\lambda) \longrightarrow \left(\mathcal{R}^{n,r}_{\lambda,\mathbb{C}}, \mathcal{L}_T\right), \quad p_{2n-4r,k-2r} \longmapsto \rho_{k,r},$$

satisfy

$$\mathcal{L}_T \circ F_\rho = F_\rho \circ Y_\lambda.$$

ii) In $\mathbb{H}P^n_\lambda$, the linear isomorphism given by

$$F_\rho \colon (V^{(4n)}, Y_\lambda) \longrightarrow \left(\mathcal{R}^{n,0}_{\lambda,\mathbb{H}}, \mathcal{L}_T\right), \quad p_{4n,k} \longmapsto \zeta_{k,0},$$

satisfies

$$\mathcal{L}_T \circ F_\rho = F_\rho \circ Y_\lambda.$$

Corollary 4.5.6. *i) In $\mathbb{C}P^n_\lambda$, for $r \in \{0,1\}$ and $2r \leq k \leq 2n-2r$*

$$\mathbf{LT}_t[\rho_{k,r}, 0] = \sum_{j=0}^{2n-4r} \phi^\lambda_{2n-4r,k-2r,j}(t)[\rho_{j+2r,r}, 0].$$

ii) In $\mathbb{H}P^n_\lambda$

$$\mathbf{LT}_t[\zeta_{k,0}, 0] = \sum_{j=0}^{4n} \phi^\lambda_{4n,k,j}(t)[\zeta_{j,0}, 0]$$

Remark 4.5.7. To derive the tubular formulas for the Federer valuations in $\mathbb{C}P^n_\lambda$ and $\mathbb{H}P^n_\lambda$, we considered the spaces of differential forms $W^{\mathbb{C}}$ and $W^{\mathbb{H}}$. The results involve products of Santaló polynomials. However, by identifying the differential forms $\rho_{k,r}$, we have been able to linearize the expression and represent the Lie derivative on $W^{\mathbb{C}}$ in the form $Y - \lambda X$, where $(X, Y, [X,Y])$ forms an $\mathfrak{sl}_2$-triple. In Section 4.7.2 we will achieve something similar for the whole $W^{\mathbb{H}}$.

At this stage, having obtained a representation in terms of differential forms for the valuations $\partial_{\lambda,\mathbb{C}}\sigma^\lambda_{k,r}$ for $r \in \{0,1\}$, it is natural to ask whether we can also find a representation for $\sigma^\lambda_{k,r}$. The answer is affirmative, and the proof relies on our results on the derivative operator $\partial_{\lambda,\mathbb{C}}$ obtained in section 3.7.2.

Proposition 4.5.8. *For $0 \leq i \leq 2n-1$ consider the $(2n-1)$-differential forms given by*

$$\begin{aligned}\eta_{i,0} :=& \sum_{j\geq 0} \lambda^j \frac{(i+2j)!!(2n-i-2j-2)!!}{i!!(2n-i+2)!!}\left(\kappa_{i+2j} - \lambda\beta\wedge\varphi_{i+2j+1}\right),\\ =& \sum_{j\geq 0} \lambda^j \frac{(i+2j)!!(2n-i-2j-2)!!}{i!!(2n-i+2)!!}\rho_{i+2j+1,0},\\ \eta_{i,1} :=& \frac{i(2n-i)}{2(2n-1)}\eta_{i,0} - \frac{1}{2}\beta\wedge\varphi_{i-1}.\end{aligned}$$

and $\eta_{2n,0}=0$. Then, for $0 \leq i \leq 2n$ and $2 \leq j \leq 2n-2$, we have

$$\begin{aligned}\sigma^\lambda_{i,0} &= [\![\eta_{i,0}, c_{2n,i+1} d\operatorname{vol}]\!],\\ \sigma^\lambda_{j,1} &= [\![\eta_{j,1}, \frac{j(2n-j)}{2(2n-1)} c_{2n,j+1} d\operatorname{vol}]\!],\end{aligned}$$

where $c_{m,k}=0$ if $k-m$ is even and otherwise is given by (3.22)*.*

Proof. From Proposition 4.5.3

$$[\![\kappa_{i-1} - \lambda\beta\wedge\varphi_i, 0]\!] = (2n-i-1)\sigma^\lambda_{i-1,0} - \lambda(i+1)\sigma^\lambda_{i+1,0} = \partial_{\lambda,\mathbb{C}}\sigma^\lambda_{i,0}.$$

Then by Proposition 3.7.11

$$\begin{aligned}\sigma^\lambda_{i,0} &= \sum_{j\geq 0} \lambda^j \frac{(i+2j)!!(2n-i-2j-2)!!}{i!!(2n-i+2)!!}\partial_{\lambda,\mathbb{C}}\sigma^\lambda_{i+2j+1,0} + c_{2n,i+1}\operatorname{vol}_{\mathbb{C}P^n_\lambda}\\ &= \sum_{j\geq 0} \lambda^j \frac{(i+2j)!!(2n-i-2j-2)!!}{i!!(2n-i+2)!!}\left([\![\kappa_{i+2j} - \lambda\beta\wedge\varphi_{i+2j+1}, 0]\!]\right) + c_{2n,i+1}\operatorname{vol}_{\mathbb{C}P^n_\lambda}.\end{aligned}$$

By Proposition 4.2.2

$$\begin{aligned}\sigma^\lambda_{j,1} &= \frac{j(2n-j)}{2(2n-1)}\sigma^\lambda_{j,0} - \frac{1}{2}[\![\beta\wedge\varphi_{j-1}, 0]\!]\\ &= \frac{j(2n-j)}{2(2n-1)}[\![\eta_{j,0}, c_{2n,j+1} d\operatorname{vol}]\!] - \frac{1}{2}[\![\beta\wedge\varphi_{j-1}, 0]\!]\\ &= [\![\eta_{j,1}, \frac{j(2n-j)}{2(2n-1)} c_{2n,j+1} d\operatorname{vol}]\!]. \quad \square\end{aligned}$$

4.6 Push-forward through the Hopf fibration

Wannerer and Hofstätter have recently computed the push-forward through the complex Hopf fibration of the invariant valuations of the sphere. We will extend their result to the quaternionic case.

Let us first recall how these fibrations are constructed.

Definition 4.6.1. For $\lambda > 0$, $(z_1, \ldots, z_{n+1}) \in \mathbb{S}_\lambda^{2n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$, we define the *complex Hopf fibration* by

$$p_{\lambda,\mathbb{C}} \colon \mathbb{S}_\lambda^{2n+1} \longrightarrow \mathbb{C}P_\lambda^n, \quad (z_1, \ldots, z_{n+1}) \longmapsto [z_1, \ldots, z_{n+1}] \in \mathbb{C}P_\lambda^n \equiv \mathbb{C}^{n+1} \setminus \{0\}/\mathbb{C}^*,$$

with fiber S^1.

For $\lambda > 0$, $(q_1, \ldots, q_{n+1}) \in \mathbb{S}_\lambda^{4n+3} \subset \mathbb{H}^{n+1} \setminus \{0\}$, we define the *quaternionic Hopf fibration* by

$$p_{\lambda,\mathbb{H}} \colon \mathbb{S}_\lambda^{4n+3} \longrightarrow \mathbb{H}P_\lambda^n, \quad (q_1, \ldots, q_{n+1}) \longmapsto [q_1, \ldots, q_{n+1}] \in \mathbb{H}P_\lambda^n \equiv \mathbb{H}^{n+1} \setminus \{0\}/\mathbb{H}^*,$$

with fiber S^3.

Both Hopf fibrations are riemannian submersions. The following results are thus relevant to us.

Proposition 4.6.2 ([37])**.** *Let $f \colon M_1 \to M_2$ be a proper riemannian submersion and assume that M_2 is connected. Denote by F the fiber of f. Then*

i)

$$f_*\chi = c \cdot \chi,$$

where $c = \chi(F)$.

ii) For every $B \in \mathcal{P}(M_2)$,

$$(f_*\mathrm{vol}_{M_1})(B) = \left(\int_B \mathrm{vol}_{M_1}(f^{-1}(\cdot))\right) \mathrm{vol}_{M_2}.$$

In particular, if the fiber of f has constant volume $a \in \mathbb{R}$, then

$$f_*\mathrm{vol}_{M_1} = a \cdot \mathrm{vol}_{M_2}.$$

The next result by Hofstätter and Wannerer is crucial for determining the push-forward of valuations under the complex fibration.

Theorem 4.6.3 ([37])**.** *The push-forward map f_* commutes with ∂, and therefore with $\mathbf{T}_t$.*

4.6.1 Push-forward through the complex Hopf fibration

Let us recall Wannerer-Hofstätter's result. We include the proof for completeness and also because our approach in the quaternionic case is analogous.

For $\lambda > 0$, let $p_{\lambda,\mathbb{C}} \colon \mathbb{S}_\lambda^{2n+1} \to \mathbb{C}P_\lambda^n$ be the *Hopf fibration.* Since the fiber is S^1, Proposition 4.6.2 provides

$$(p_{\lambda,\mathbb{C}})_* \mathrm{vol}_{\mathbb{S}_\lambda^{2n+1}} = 2\pi \, \mathrm{vol}_{\mathbb{C}P_\lambda^n} . \tag{4.9}$$

Theorem 4.6.4 ([37])**.**

$$\frac{1}{2\pi}(p_{\lambda,\mathbb{C}})_*\sigma_k^\lambda = (2n-k+1)\sigma_{k-1,0}^\lambda - \lambda(k+1)\sigma_{k+1,0}^\lambda = \partial_{\lambda,\mathbb{C}}\sigma_{k,0}^\lambda = [\![\rho_{k,0}, 0]\!].$$

In particular, the image of $(p_{\lambda,\mathbb{C}})_*$ *is contained in* $\mathcal{I}_{\lambda,\mathbb{C}}^{n,0}$.

Proof. From Theorem 3.2.14, we have

$$\mathbf{T}_t \mathrm{vol}_{\mathbb{S}_\lambda^{2n+1}} = \sum_{j=0}^{2n}\left(\int_0^t \sin_\lambda^{2n-j}(s)\cos_\lambda^j(s)\,ds\right)\sigma_j^\lambda + \mathrm{vol}_{\mathbb{S}_\lambda^{2n+1}}.$$

By equation (4.9) and the property $\mathbf{T}_t \circ (p_{\lambda,\mathbb{C}})_* = (p_{\lambda,\mathbb{C}})_* \circ \mathbf{T}_t$, as stated in Theorem 4.6.3, it follows that

$$\begin{aligned} 2\pi\mathbf{T}_t\mathrm{vol}_{\mathbb{C}P_\lambda^n} &= \mathbf{T}_t(p_{\lambda,\mathbb{C}})_*\,\mathrm{vol}_{\mathbb{S}_\lambda^{2n+1}} = (p_{\lambda,\mathbb{C}})_*\mathbf{T}_t\mathrm{vol}_{\mathbb{S}_\lambda^{2n+1}} \\ &= \sum_{j=0}^{2n}\left(\int_0^t \sin_\lambda^{2n-j}(s)\cos_\lambda^j(s)\,ds\right)(p_{\lambda,\mathbb{C}})_*\sigma_j^\lambda + (p_{\lambda,\mathbb{C}})_*\mathrm{vol}_{\mathbb{S}_\lambda^{2n+1}}. \end{aligned}$$

Differentiating with respect to t, we obtain

$$2\pi\frac{d}{dt}\mathbf{T}_t\mathrm{vol}_{\mathbb{C}P_\lambda^n} = \sum_{j=0}^{2n}\sin_\lambda^{2n-j}(t)\cos_\lambda^j(t)(p_{\lambda,\mathbb{C}})_*\sigma_j^\lambda.$$

Moreover, since $\mathbf{T}_{t+s} = \mathbf{T}_t \circ \mathbf{T}_s$, using (3.28) we have

$$\begin{aligned} \frac{d}{dt}\mathbf{T}_t\mathrm{vol}_{\mathbb{C}P_\lambda^n} &= \left.\frac{d}{ds}\right|_{s=0}\mathbf{T}_{t+s}\mathrm{vol}_{\mathbb{C}P_\lambda^n} = \left.\frac{d}{ds}\right|_{s=0}\mathbf{T}_s\circ\mathbf{T}_t\mathrm{vol}_{\mathbb{C}P_\lambda^n} \\ &= \sum_{j=0}^{2n}\sin_\lambda^{2n-j}(t)\cos_\lambda^j(t)\left.\frac{d}{ds}\right|_{s=0}\mathbf{T}_s\sigma_{j,0}^\lambda = \sum_{j=0}^{2n}\sin_\lambda^{2n-j}(t)\cos_\lambda^j(t)\partial_{\lambda,\mathbb{C}}\sigma_{j,0}^\lambda. \end{aligned}$$

Since $\{\sin_\lambda^{2n-j}(t)\cos_\lambda^j(t)\}_{j=0}^{2n}$ are linearly independent functions, equating coefficients in the previous two identities yields the result. □

A precise description of the image and the kernel of $(p_{\lambda,\mathbb{C}})_*$ will be provided below in Proposition 4.6.6. First, we relate $(p_{\lambda,\mathbb{C}})_*$ to some maps we constructed before.

Let us recall the linear isomorphism (3.34)

$$\mathbf{F}_{n,r}^\lambda : \mathcal{H}_\lambda^{2n-4r+1} \longrightarrow \mathcal{I}_{\lambda,\mathbb{C}}^{n,r}, \qquad \sigma_j^\lambda \longmapsto \sigma_{j+2r,j+r}^\lambda$$

between the linear subspaces $\mathcal{H}_\lambda^{2n-4r+1} \subset \mathcal{V}_{\lambda,\mathbb{R}}^{2n-4r+1}$ and $\mathcal{I}_{\lambda,\mathbb{C}}^{n,r} \subset \mathcal{V}_{\lambda,\mathbb{C}}^n$. This map commutes with the tubular operator $\mathbf{T}_t$, i.e, both $\mathbf{F}_{n,0}^\lambda$ and $p_{\lambda,\mathbb{C}}$ preserve the tube formulas. The two maps are closely related.

Proposition 4.6.5.

$$\frac{1}{2\pi}(p_{\lambda,\mathbb{C}})_* = \mathbf{F}^\lambda_{n,0} \circ \partial_{\lambda,\mathbb{R}}.$$

Proof. By Proposition 3.7.12 the image of $\partial_{\lambda,\mathbb{R}}$ is contained in the hyperplane $\mathcal{H}^{2n+1}_\lambda$ and therefore the expression is well defined. The equality follows from (3.1). □

Proposition 4.6.6. *For $\lambda > 0$ the complex Hopf fibration satisfies*

i) $\ker(p_{\lambda,\mathbb{C}})_* = \langle\chi\rangle$.

ii) $\operatorname{Im}(p_{\lambda,\mathbb{C}})_* = \mathcal{I}^{n,0}_{\lambda,\mathbb{C}} = \langle \mu^\lambda_0, \dots, \mu^\lambda_{2n}\rangle$.

Proof. *i)* Since the fiber is isomorphic to S^1 and $\chi(S^1) = 0$, by Proposition 4.6.2 $(p_{\lambda,\mathbb{C}})_*\chi = 0$. Let $\mu \in \mathcal{V}^{2n+1}_{\lambda,\mathbb{R}}$ be such that $(p_{\lambda,\mathbb{C}})_*\mu = 0$. By Proposition 4.6.5, $\mathbf{F}^\lambda_{n,0}(\partial_{\lambda,\mathbb{R}}\mu) = 0$. Thus, $\partial_{\lambda,\mathbb{R}}\mu \in \ker \mathbf{F}^\lambda_{n,0} = \{0\}$ and therefore $\partial_{\lambda,\mathbb{R}}\mu = 0$. Since $\ker \partial_{\lambda,\mathbb{R}} = \langle\chi\rangle$ by Proposition 3.7.6, the result follows.

ii) By Proposition 3.7.12, the image of $\partial_{\lambda,\mathbb{R}}$ is $\mathcal{H}^{2n+1}_\lambda$. Since $\mathbf{F}^\lambda_{n,0} \colon \mathcal{H}^{2n+1}_\lambda \to \mathcal{I}^{n,0}_{\lambda,\mathbb{C}}$ is a linear isomorphism, the result follows. □

Let us recall the basis $\phi^0, \dots, \phi^{2n+1} \in \mathcal{V}^{2n+1}_{\lambda,\mathbb{R}}$ introduced in (3.42).

Proposition 4.6.7. *For $k \geq 1$*

$$(p_{\lambda,\mathbb{C}})_*\phi^k = 2\frac{k!\omega_k}{\pi^{k-1}\omega_{2n-k+1}}\sigma^\lambda_{k-1,0} = 2\frac{k!\omega_k}{\pi^{k-1}}\tau^\lambda_{k-1,0}.$$

Proof. By Proposition 4.6.5 and Proposition 3.7.12

$$\frac{1}{2\pi}(p_{\lambda,\mathbb{C}})_*\phi^k = \mathbf{F}^\lambda_{n,0} \circ \partial_{\lambda,\mathbb{R}}\phi^k = \frac{k!\omega_k}{\pi^k\omega_{m-k}}\sigma^\lambda_{k-1}.$$

Since $\pi^\lambda_{k,0} = k!\tau^\lambda_{k,0}$ we have

$$\sigma^\lambda_{k,0} = \frac{\omega_{2n-k}}{k!}\pi^\lambda_{k,0} = \omega_{2n-k}\tau^\lambda_{k,0}.$$

The stated formula follows. □

Another basis of $\mathcal{V}^m_{\lambda,\mathbb{R}}$ is given by the so-called Lipschitz-Killing valuations $t^0 = \chi, t, \dots, t^{2n+1}$ where $t = \frac{2}{\pi}\mu_1$ and μ_1 is the restriction to $\mathbb{S}^{2n+1}$ of the intrinsic volume $\mu_1 \in \operatorname{Val}(\mathbb{R}^{2n+2})$.

Proposition 4.6.8.

$$(p_{\lambda,\mathbb{C}})_*t^k = \sum_{j\geq 0}\binom{k/2+j-1}{j}\left(\frac{\lambda}{4}\right)^j\left\{2\frac{(k+2j)!\omega_{k+2j}}{\pi^{k+2j-1}}\tau^\lambda_{k+2j-1,0}\right\}.$$

Proof. By [34, Lemma 3.4]

$$t^k = \sum_{j\geq 0}\binom{k/2+j-1}{j}\left(\frac{\lambda}{4}\right)^j\phi^{k+2j}.$$

Proposition 4.6.7 yields the result. □

4.6.2 Push-forward through the quaternionic Hopf fibration

For $\lambda > 0$, let $p_{\lambda,\mathbb{H}}\colon \mathbb{S}^{4n+3}_\lambda \to \mathbb{H}P^n_\lambda$ be the *Hopf fibration.* Since the fiber is S^3, Proposition 4.6.2 provides

$$(p_{\lambda,\mathbb{H}})_* \operatorname{vol}_{\mathbb{S}^{4n+3}_\lambda} = 2\pi^2 \operatorname{vol}_{\mathbb{H}P^n_\lambda} . \tag{4.10}$$

Let us recall the forms

$$\zeta_{l,0} := \sum_{0\le k\le i\le 3} (-\lambda)^k \binom{i}{k} \psi_i \wedge \varphi_{l-i+2k-3}.$$

Proposition 4.6.9.

$$\frac{1}{2\pi^2}(p_{\lambda,\mathbb{H}})_*\sigma_i^\lambda = [\![\zeta_{i,0}, 0]\!], \quad 0 \le i \le 4n+2.$$

Proof. By Corollary 4.4.4 and equation (3.32)

$$\begin{aligned}\mathbf{T}_t \operatorname{area}_{\mathbb{H}P^n_\lambda} &= \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t)[\![\zeta_{i,0}, 0]\!]\\ \mathbf{T}_t \operatorname{area}_{\mathbb{S}^{4n+3}_\lambda} &= \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t)\sigma_i^\lambda.\end{aligned}$$

Since $(p_{\lambda,\mathbb{H}})_*$ commutes with the derivative map, and $\partial \operatorname{vol} = \operatorname{area}$, by Proposition 4.6.2 we have

$$\begin{aligned}(p_{\lambda,\mathbb{H}})_* \operatorname{area}_{\mathbb{S}^{4n+3}_\lambda} &= (p_{\lambda,\mathbb{H}})_* \partial \operatorname{vol}_{\mathbb{S}^{4n+3}_\lambda}\\ &= \partial (p_{\lambda,\mathbb{H}})_* \operatorname{vol}_{\mathbb{S}^{4n+3}_\lambda}\\ &= 2\pi^2 \partial \operatorname{vol}_{\mathbb{H}P^n_\lambda}\\ &= 2\pi^2 \operatorname{area}_{\mathbb{H}P^n_\lambda} .\end{aligned}$$

Therefore

$$\begin{aligned}(p_{\lambda,\mathbb{H}})_* \mathbf{T}_t \operatorname{area}_{\mathbb{S}^{4n+3}_\lambda} &= \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t)(p_{\lambda,\mathbb{H}})_*\sigma_i^\lambda\\ &= 2\pi^2 \mathbf{T}_t \operatorname{area}_{\mathbb{H}P^n_\lambda} = 2\pi^2 \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t)[\![\zeta_{i,0}, 0]\!].\end{aligned}$$

Comparing coefficients in the last two lines yields the result. □

Proposition 4.6.10.

$$(p_{\lambda,\mathbb{H}})_*\chi = 0.$$

Proof. Since $\chi(S^3) = 0$, by Proposition 4.6.2, the result follows. □

Since we do not know much about the globalizations $[\![\zeta_{i,0},0]\!]$ we can not determine the kernel and the image of $(p_{\lambda,\mathbb{H}})_*$.

Next we will look for valuations $S^\lambda_{i,0}$ in $\mathbb{H}P^n_\lambda$ playing the same role as the $\sigma^\lambda_{i,0}$ in $\mathbb{C}P^n_\lambda$; i.e, such that

$$[\![\zeta_{i,0},0]\!] = \frac{1}{2\pi^2}(p_{\lambda,\mathbb{H}})_*\sigma_i^\lambda = \partial_{\lambda,\mathbb{H}} S^\lambda_{i,0}.$$

Our approach consists of integrating the valuations $\sigma_i^\lambda \in \mathcal{V}^{4n+3}_{\lambda,\mathbb{R}}$ with respect to $\partial_{\lambda,\mathbb{R}}$ and then transferring them through the push-forward. Let us consider the valuations given by

$$\Sigma_i := \frac{\pi^{i+1}\omega_{4n-i+2}}{(i+1)!\omega_{i+1}}\phi^{i+1} \in \mathcal{V}^{4n+3}_{\lambda,\mathbb{R}}, \qquad 0 \leq i \leq 4n+2,$$

where $\phi^0,\dots,\phi^{4n+3} \in \mathcal{V}^{4n+3}_{\lambda,\mathbb{R}}$ are the valuations introduced in (3.42). By Proposition 3.7.12 it is satisfied

$$\partial_{\lambda,\mathbb{R}}\Sigma_i = \sigma_i^\lambda, \quad 0 \leq i \leq 4n+2.$$

In particular, since $\phi^m = t^m = \frac{m!\omega_m}{\pi^m}\mu_m$ on $\mathbb{S}^m_\lambda$,

$$\partial_{\lambda,\mathbb{R}}\Sigma_{4n+2} = \partial_{\lambda,\mathbb{R}} \operatorname{vol}_{\mathbb{S}^{4n+3}_\lambda} = \operatorname{area}_{\mathbb{S}^{4n+3}_\lambda}.$$

Proposition 4.6.11. *Define*

$$S^\lambda_{i,0} := \frac{1}{2\pi^2}(p_{\lambda,\mathbb{H}})_*\Sigma_i, \quad 0 \leq i \leq 4n+2.$$

Then

$$\partial_{\lambda,\mathbb{H}} S^\lambda_{i,0} = \frac{1}{2\pi^2}(p_{\lambda,\mathbb{H}})_*\sigma_i^\lambda,$$

and

$$\mathbf{T}_t \operatorname{vol}_{\mathbb{H}P^n_\lambda} = \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t) S^\lambda_{i,0}.$$

In particular

$$S^\lambda_{4n+2,0} = \frac{1}{2\pi^2}(p_{\lambda,\mathbb{H}})_* \operatorname{vol}_{\mathbb{S}^{4n+3}_\lambda} = \operatorname{vol}_{\mathbb{H}P^n_\lambda}.$$

Proof. Given that

$$\partial_{\lambda,\mathbb{R}}\Sigma_i = \sigma_i^\lambda, \quad 0 \leq i \leq 4n+2,$$

the first equality holds. Theorem 3.5.1 reads

$$\begin{aligned}\mathbf{T}_t\partial_{\lambda,\mathbb{R}}\Sigma_{4n+2} =& \mathbf{T}_t \operatorname{area}_{\mathbb{S}^{4n+3}_\lambda} = \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t)\sigma_i^\lambda \\ &= \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t)\cos_\lambda^i(t)\partial_{\lambda,\mathbb{R}}\Sigma_i.\end{aligned}$$

Since $\partial_{\lambda,\mathbb{R}} \circ \mathbf{T}_t = \mathbf{T}_t \circ \partial_{\lambda,\mathbb{R}}$, it follows that

$$\partial_{\lambda,\mathbb{R}} \left(\mathbf{T}_t \Sigma_{4n+2}\right) = \partial_{\lambda,\mathbb{R}} \left(\sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t) \cos_\lambda^i(t) \Sigma_i \right).$$

From Proposition 3.7.6, we know that $\ker \partial_{\lambda,\mathbb{R}} = \langle \chi \rangle$. Therefore, there exists a constant $c \in \mathbb{R}$ such that:

$$\mathbf{T}_t \Sigma_{4n+2} = \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t) \cos_\lambda^i(t) \Sigma_i + c\chi.$$

Given that $(p_{\lambda,\mathbb{H}})_* \chi = 0$ applying $(p_{\lambda,\mathbb{H}})_*$ on both sides of the previous formula yields

$$\mathbf{T}_t S^\lambda_{4n+2,0} = \sum_{i=0}^{4n+2} \sin_\lambda^{4n-i+2}(t) \cos_\lambda^i(t) S^\lambda_{i,0}. \qquad \square$$

Corollary 4.6.12.

$$\partial_{\lambda,\mathbb{H}} S^\lambda_{i,0} = (4n-i+3) S^\lambda_{i-1,0} - \lambda(i+1) S^\lambda_{i+1,0}, \quad 0 \le i \le 4n+2,$$

$$\mathbf{T}_t S^\lambda_{i,0} = \sum_{j=0}^{4n+2} \phi_{4n+2,i,j}(t) S^\lambda_{j,0}.$$

In analogy with the construction carried out in the complex case and the notation adopted in (3.27), we write

$$\mathcal{I}^{n,0}_{\lambda,\mathbb{H}} := \langle S^\lambda_{i,0} : 0 \le i \le 4n+2 \rangle. \tag{4.11}$$

Proposition 4.6.13. *The linear isomorphism*

$$J_{\lambda,\mathbb{H}} \colon (V^{(4n+2)}, Y_\lambda) \longrightarrow \left(\mathcal{I}^{n,0}_{\lambda,\mathbb{H}}, \partial_{\lambda,\mathbb{H}} \right), \quad p_{4n+2,i} \longmapsto S^\lambda_{i,0}$$

satisfies

$$J_{\lambda,\mathbb{H}} \circ Y_\lambda = \partial_{\lambda,\mathbb{H}} \circ J_{\lambda,\mathbb{H}}.$$

We shall now express $S^\lambda_{i,0}$ in terms of differential forms for $0 \le i \le 4n+1$.

Proposition 4.6.14. *For* $0 \le i \le 4n+2$

$$S^\lambda_{i,0} = \sum_{s \ge 0} \lambda^s \frac{(i+2s)!!(4n-i-2s)!!}{i!!(4n-i+2)!!} [\![\zeta_{i+2s+1,0}, 0]\!] + c_{4n+2,i+1} \operatorname{vol}_{\mathbb{H}P^n_\lambda}.$$

Proof. By Proposition 4.6.13, we have an $\mathfrak{sl}_2$-isomorphism $S^\lambda_{i,0} \mapsto p_{4n+2,i}$. From Proposition 3.3.10

$$p_{4n+2,i} = \sum_{s \ge 0} \lambda^s \frac{(i+2s)!!(4n-i-2s)!!}{i!!(4n-i+2)!!} Y_\lambda p_{4n+2,i+2s+1} + c_{4n+2,i+1} x^{4n+2}.$$

Then, transferring this expresion and using $\partial_{\lambda,\mathbb{H}} S^\lambda_{i,0} = [\![\zeta_{i,0}, 0]\!]$

$$\begin{aligned} S^\lambda_{i,0} &= \sum_{s\geq 0} \lambda^s \frac{(i+2s)!!(4n-i-2s)!!}{i!!(4n-i+2)!!} \partial_{\lambda,\mathbb{H}} S^\lambda_{i+2s+1} + c_{4n+2,i+1} \operatorname{vol}_{\mathbb{H}P^n_\lambda} \\ &= \sum_{s\geq 0} \lambda^s \frac{(i+2s)!!(4n-i-2s)!!}{i!!(4n-i+2)!!} [\![\zeta_{i+2s+1,0}, 0]\!] + c_{4n+2,i+1} \operatorname{vol}_{\mathbb{H}P^n_\lambda}. \quad \square \end{aligned}$$

4.7 Diagonalization of $\mathcal{L}_T$ on $W^{\mathbb{C}}$ and $W^{\mathbb{H}}$

Here we diagonalize the operator $\mathcal{L}_T$ in $W^{\mathbb{C}}$ and $W^{\mathbb{H}}$.

Note that we can diagonalize $\mathcal{L}_T$ on each component of the tensor product using Lemma 3.3.4. Hence, we just need to use the following observation. Given two vector spaces U_1, U_2 and two linear endomorphisms $f_i\colon U_i \to U_i$, $i = 1, 2$, consider $f\colon U_1 \otimes U_2 \to U_1 \otimes U_2$ given by

$$f(v \otimes w) = f_1(v) \otimes w + v \otimes f_2(w), \quad v \in U_1,\ w \in U_2.$$

Let $u_1, u_2 \in U_1, U_2$ be such that

$$f_1(u_1) = \alpha_1 u_1, \quad f_2(u_2) = \alpha_2 u_2.$$

Then

$$f(u_1 \otimes u_2) = (\alpha_1 + \alpha_2) u_1 \otimes u_2.$$

4.7.1 Complex space forms

Let us recall the space $W^{\mathbb{C}} = W^{\mathbb{C}}_1 \otimes W^{\mathbb{C}}_{2n-2}$, where

$$W^{\mathbb{C}}_1 = \langle \gamma, \beta \rangle, \quad W^{\mathbb{C}}_{2n-2} = \langle \varphi_i : 0 \leq i \leq 2n-2 \rangle.$$

By the previous considerations, if $\psi \in W^{\mathbb{C}}_1$ and $\varphi \in W^{\mathbb{C}}_{2n-2}$ satisfy

$$\mathcal{L}_T \psi = a\psi, \quad \mathcal{L}_T \varphi = b\varphi$$

for $a, b \in \mathbb{C}$, then

$$\mathcal{L}_T(\psi \otimes \varphi) = (a+b)\psi \otimes \varphi.$$

Proposition 4.7.1. *i) $\mathcal{L}_T$ is diagonalizable in $W^{\mathbb{C}}_1$ and*

$$\operatorname{spec}(\mathcal{L}_T|_{W^{\mathbb{C}}_1}) = \left\{-2\sqrt{-\lambda}, 2\sqrt{-\lambda}\right\},$$

with eigenspaces

$$\begin{aligned} E_{-2\sqrt{-\lambda}}(\mathcal{L}_T|_{W^{\mathbb{C}}_1}) &= \langle \upsilon_{1,0} \rangle, \\ E_{2\sqrt{-\lambda}}(\mathcal{L}_T|_{W^{\mathbb{C}}_1}) &= \langle \upsilon_{1,1} \rangle. \end{aligned}$$

where

$$\begin{aligned} \upsilon_{1,0} &= \gamma - 2\sqrt{-\lambda}\beta, \\ \upsilon_{1,1} &= \gamma + 2\sqrt{-\lambda}\beta. \end{aligned}$$

ii) $\mathcal{L}_T$ *is diagonalizable in* $W_{2n-2}^{\mathbb{C}}$ *and*

$$\operatorname{spec}(\mathcal{L}_T|_{W_{2n-2}^{\mathbb{C}}}) = \left\{(2k-2n+2)\sqrt{-\lambda}\colon 0 \leq k \leq 2n-2\right\},$$

$$E_{(2k-2n+2)\sqrt{-\lambda}}(\mathcal{L}_T|_{W_{2n-2}^{\mathbb{C}}}) = \langle \upsilon_{2n-2,k} \rangle, \quad 0 \leq k \leq 2n$$

where

$$\upsilon_{2n-2,k} = \sum_{a,b} \binom{a+b}{a} \binom{2n-a-b-2}{k-a} (-1)^b (\sqrt{-\lambda})^{a+b} \varphi_{a+b}.$$

Proof. *i)* By Lemma 3.3.4, the eigenvalues of $Y_{4\lambda}$ in $V^{(1)}$ are

$$\alpha_0 = -\sqrt{-4\lambda} = -2\sqrt{-\lambda}, \qquad \alpha_1 = \sqrt{-4\lambda} = 2\sqrt{-\lambda},$$

with the corresponding eigenvectors given by

$$u_0 = y - 2\sqrt{-\lambda}x, \quad u_1 = y + 2\sqrt{-\lambda}x.$$

By Theorem 4.3.1, the linear isomorphism F_ψ preserves both the eigenvalues and the eigenvectors of $\mathcal{L}_T$. Thus, $\upsilon_{1,i} = F_\psi(u_i)$, explicitly given by

$$\upsilon_{1,0} = \gamma - 2\sqrt{-\lambda}\beta, \quad \upsilon_{1,1} = \gamma + 2\sqrt{-\lambda}\beta,$$

are the eigenvectors associated with the eigenvalues $-2\sqrt{-\lambda}$ and $2\sqrt{-\lambda}$, respectively.

ii) By Lemma 3.3.4, the eigenvalues of Y_λ in $V^{(2n-2)}$ are $\beta_k = (2k-2n+2)\sqrt{-\lambda}$ for $0 \leq k \leq 2n-2$, with the corresponding eigenvectors given by

$$v_k = \binom{2n-2}{k} (y + \sqrt{-\lambda}x)^k (y - \sqrt{-\lambda}x)^{2n-k-2}.$$

According to Lemma (3.3.6), we can expand this expression as follows:

$$v_k = \sum_{a,b} \binom{a+b}{a} \binom{2n-a-b-2}{k-a} (-1)^b (\sqrt{-\lambda})^{a+b} p_{2n-2,a+b}.$$

By Theorem 4.3.1, the linear isomorphism F_φ preserves the eigenvectors and eigenvalues of $\mathcal{L}_T$. Then $\upsilon_{2n-2,k} = F_\varphi(v_k)$ are $\mathcal{L}_T$-eigenvectors with the associated eigenvalue β_k, explicitly given by

$$\upsilon_{2n-2,k} = \sum_{a,b} \binom{a+b}{a} \binom{2n-a-b-2}{k-a} (-1)^b (\sqrt{-\lambda})^{a+b} \varphi_{a+b},$$

for $0 \leq k \leq 2n-2$. □

Proposition 4.7.2. *$\mathcal{L}_T$ is diagonalizable in $W^{\mathbb{C}} = W_1^{\mathbb{C}} \otimes W_{2n-2}^{\mathbb{C}}$ and*

$$\operatorname{spec}(\mathcal{L}_T|_{W^{\mathbb{C}}}) = \left\{(4i+2j-2n)\sqrt{-\lambda} : 0 \le i \le 1, 0 \le j \le 2n-2\right\}$$

$$E_{(4i+2j-2n)\sqrt{-\lambda}}(\mathcal{L}_T|_{W^{\mathbb{C}}}) = \langle \upsilon_{1,i} \otimes \upsilon_{2n-2,j}\rangle, \quad 0 \le i \le 1, 0 \le j \le 2n-2.$$

Proof. By definition, for $0 \le i \le 1$ and $0 \le j \le 2n$

$$\mathcal{L}_T \upsilon_{1,i} = (4i-2)\sqrt{-\lambda}\upsilon_{1,i},$$

$$\mathcal{L}_T \upsilon_{2n-2,j} = (2j-2n+2)\sqrt{-\lambda}\upsilon_{2n-2,j},$$

therefore for $0 \le i \le 1$ and $0 \le j \le 2n-2$, since $\mathcal{L}_T$ is a derivation

$$\mathcal{L}_T(\upsilon_{1,i} \otimes \upsilon_{2n-2,j}) = (4i+2j-2n)\sqrt{-\lambda}\upsilon_{1,i} \otimes \upsilon_{2n-2,j}.$$

From a dimensionality argument, it follows that $\mathcal{L}_T$ is diagonalizable. □

4.7.2 Quaternionic space forms

Proposition 4.7.3. *i) $\mathcal{L}_T$ is diagonalizable in $W_3^{\mathbb{H}}$ and*

$$\operatorname{spec}(\mathcal{L}_T|_{W_3^{\mathbb{H}}}) = \left\{(4i-6)\sqrt{-\lambda} : 0 \le i \le 3\right\},$$

$$E_{(4i-6)\sqrt{-\lambda}} = \langle \upsilon_{3,i} : 0 \le i \le 3\rangle,$$

where

$$\upsilon_{3,i} := \sum_{a,b} \binom{a+b}{a}\binom{3-a-b}{k-a}(-1)^b(2\sqrt{-\lambda})^{a+b}\psi_{a+b}.$$

ii) $\mathcal{L}_T$ is diagonalizable in $W_{4n-4}^{\mathbb{H}}$ and

$$\operatorname{spec}(\mathcal{L}_T|_{W_{4n-4}^{\mathbb{H}}}) = \left\{(2j-4n+4)\sqrt{-\lambda} : 0 \le j \le 4n-4\right\},$$

$$E_{(2j-4n+4)\sqrt{-\lambda}} = \langle \upsilon_{4n-4,j} : 0 \le j \le 4n-4\rangle,$$

where

$$\sum_{a,b} \binom{a+b}{a}\binom{4n-a-b-4}{k-a}(-1)^b(\sqrt{-\lambda})^{a+b}\varphi_{a+b}.$$

Proof. *i)* The eigenvalues of $Y_{4\lambda}$ in $V^{(3)}$ are $(4k-6)\sqrt{-\lambda}$ for $0 \le k \le 3$, with the associated eigenvectors given by

$$u_k := \binom{3}{k}(y+2\sqrt{-\lambda}x)^k(y-2\sqrt{-\lambda}x)^{3-k}, \quad 0 \le k \le 3.$$

Specifically, using Lemma 3.3.6, this can be expanded as:

$$u_k = \sum_{a,b} \binom{a+b}{a}\binom{3-a-b}{k-a}(-1)^b(2\sqrt{-\lambda})^{a+b}\binom{3}{a+b}x^{a+b}y^{3-a-b}.$$

By Theorem 4.3.7 the differential forms $\upsilon_{3,k} := F_\psi(u_k)$ given by

$$\upsilon_{3,k} = \sum_{a,b} \binom{a+b}{a}\binom{3-a-b}{k-a}(-1)^b(2\sqrt{-\lambda})^{a+b}\psi_{a+b}$$

are $\mathcal{L}_T$-eigenvectors with associanted eigenvalue $(4k-6)\sqrt{-\lambda}$.

ii) The eigenvalues of Y_λ in $V^{(4n-4)}$ are $\beta_k = (2k-4n+4)\sqrt{-\lambda}$ for $0 \leq k \leq 4n-4$, with the associated eigenvectors given by

$$u_k := \binom{4n-4}{k}(y+\sqrt{-\lambda}x)^k(y-\sqrt{-\lambda}x)^{4n-k-4}.$$

Specifically, using Lemma 3.3.6, this can be expanded as:

$$u_k = \sum_{a,b} \binom{a+b}{a}\binom{4n-a-b-4}{k-a}(-1)^b(\sqrt{-\lambda})^{a+b}\binom{4n-4}{a+b}x^{a+b}y^{4n-a-b-4}.$$

By Theorem 4.3.7, the differential forms $\upsilon_{4n-4,k} := F_\varphi(u_k)$ given by

$$\upsilon_{4n-4,k} = \sum_{a,b} \binom{a+b}{a}\binom{4n-a-b-4}{k-a}(-1)^b(\sqrt{-\lambda})^{a+b}\varphi_{a+b}$$

are $\mathcal{L}_T$-eigenvectors with associanted eigenvalue $(2k-4n+4)\sqrt{-\lambda}$. □

Proposition 4.7.4. *$\mathcal{L}_T$ is diagonalizable in $W^{\mathbb{H}} = W_3^{\mathbb{H}} \otimes W_{4n-4}^{\mathbb{H}}$ and*

$$\operatorname{spec}(\mathcal{L}_T|_{W^{\mathbb{H}}}) = \{(2k-m)\sqrt{-\lambda} \colon 0 \leq k \leq m\},$$

where $m = 4n+2$, and

$$E_{(2k-m)\sqrt{-\lambda}}(\mathcal{L}_T|_{W^{\mathbb{H}}}) = \langle \upsilon_{3,i} \otimes \upsilon_{4n-4,j} : 2i+j=k, 0 \leq i \leq 3, 0 \leq j \leq 4n-4\rangle.$$

Proof. By the previous proposition $\upsilon_{3,i} \otimes \upsilon_{4n-4,j}$, with $0 \leq i \leq 3$ and $0 \leq j \leq 4n-4$, is an eigenvector with eigenvalue

$$(4i+2j-4n-2)\sqrt{-\lambda}.$$

Such vectors form a basis, and thus the proposition follows. □

A straightforwrad but tedious analysis shows (for $n \geq 3$)

$$\dim E_{(2k-m)\sqrt{-\lambda}} = \begin{cases} \lfloor \frac{k}{2} \rfloor + 1, & \text{if } 0 \leq k \leq 6, \\ 4, & \text{if } 7 \leq k \leq 4n-5, \\ \lfloor 2n+1-\frac{k}{2} \rfloor + 1, & \text{if } 4n-4 \leq k \leq 4n+2. \end{cases}$$

Using Lemma 3.3.4 one checks that Y_λ restricted to

$$V^{(4n-10)} \oplus V^{(4n-6)} \oplus V^{(4n-2)} \oplus V^{(4n+2)}$$

has the same spectrum and the same multiplicities as $\mathcal{L}_T$ on $W^{\mathbb{H}}$. It follows that there exists an isomorphism

$$\Phi \colon V^{(4n-10)} \oplus V^{(4n-6)} \oplus V^{(4n-2)} \oplus V^{(4n+2)} \longrightarrow W^{\mathbb{H}}$$

such that $\Phi \circ Y_\lambda = \mathcal{L}_T \circ \Phi$.

Corollary 4.7.5. *There exists a basis* $\{\zeta_{k,r} \colon 0 \leq r \leq 3, 2r \leq k \leq m-2r\}$ *of* $W^{\mathbb{H}}$ *such that*

$$\mathbf{LT}_t[\zeta_{k,r}, 0] = \sum_{j=0}^{m-4r} \phi^\lambda_{m-4r,k-2r,j}(t)[\zeta_{j+2r,r}, 0].$$

and the following subspaces of $W^{\mathbb{H}}$*, are* $\mathcal{L}_T$*-invariant*

$$\mathcal{R}^{n,r}_{\lambda,\mathbb{H}} := \{\zeta_{k,r} : 2r \leq k \leq m-2r\}.$$

Proof. It suffices to take $\zeta_{k,r} = \Phi(p_{m-4r,k-2r})$ and use

$$\begin{aligned} \exp(t\mathcal{L}_T)(\zeta_{k,r}) &= \exp(t\mathcal{L}_T)(\Phi p_{m-4r,k-2r}) \\ &= \Phi \circ \exp(tY_\lambda)(p_{m-4r,k-2r}) \\ &= \Phi(p_{m-4r,k-2r}(t)) \\ &= \sum_j \phi^\lambda_{m-4r,k-2r,j}(t)\Phi(p_{m-4r,j}) \\ &= \sum_j \phi^\lambda_{m-4r,k-2r,j}(t)\zeta_{j+2r,r}. \end{aligned}$$

□

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