Universitat Pompeu Fabra

Department of Economics and Business

Doctoral Thesis

On Two-Sided Network Markets

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by

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Chapter 1

Introduction

This thesis deals with a particular problem, and analyzes and draws conclusions looking at it from three different perspectives, always using game theoretic tools. All chapters are deeply inter-related among themselves.

The center of all discussions are situations in which we have two types of agents, identical among each type, each of them willing to trade with an agent of the other type. The communication structure is given by a network linking the agents, with only connected agents being able to reach an agreement. We will refer to these situations as *two-sided network markets*.

1.1 Motivation

Consider the following example: some people want to sell a car and some people want to buy it. The car is exactly the same and so are the utilities of sellers and buyers. Somehow, though, not all sellers and buyers are "connected", meaning that agents face some sort of communication constraints. The following network tells us which is the precise communication structure.

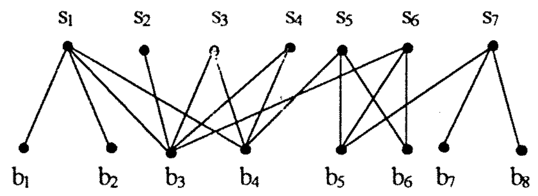


Figure 1

Agents on the top will be referred as sellers and agents on the bottom, as buyers. Thus, the network means that, say, b_1 can possibly trade only with s_1, but buyer b_3 can trade with either s_1, s_2, s_3, s_4 or s_6. Now, suppose that we had to say what is a good position in this network and what is not. Or, alternatively, suppose that each link symbolizes a surplus of 1 that can be shared, and that we want to select a value that will allocate the total surplus among nodes. How should we do it? Recall that all sellers are alike, and that all buyers are alike. The only difference among them is given by their communication structure. But how can we gain insight on communication structure? These are the central questions we address in this thesis:

1. What is the power of each agent in a given general network? What does it mean being "well-connected"? Which connections and valuable and which are irrelevant?

2. As a consequence, were I allowed to pay to form or sever links, what should I do?

To gain some intuition on the solution we propose, look at the most easy situation we can think about:

Figure 2

One seller and one buyer, connected among themselves: this is a simple market and a simple network. If we assume that the seller and the buyer engage in a bargaining process, then we can think that the "power" each of them has is sort of similar. If the surplus they can share equals 1, we would give to each of them 1/2 of it.

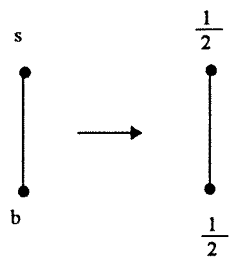


Figure 3

The second easiest network concerns two sellers and one buyer.

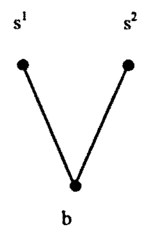


Figure 4

The buyer in this market is lucky; he has two sellers both willing to sell the car to him. Both sellers have no other option but selling the car to him, thus, competition will drive the price to be very cheap, so cheap that in equilibrium the buyer will get all the surplus.

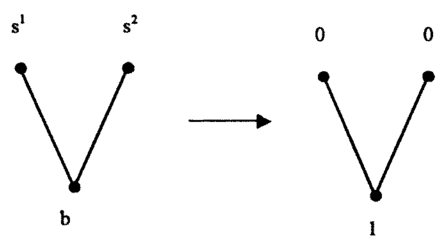


Figure 5

Let us complicate a little bit the setup. What would happen if the market and the network would be given by the following figure?

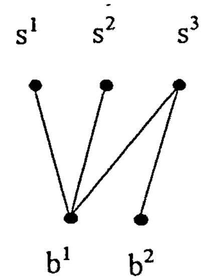
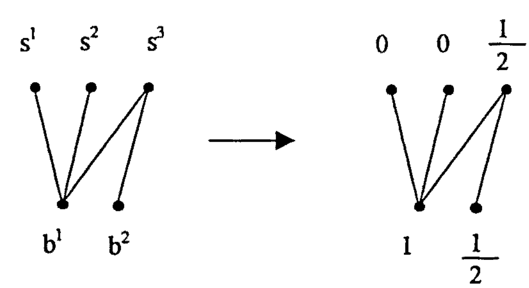


Figure 6

After looking at it for a while we can draw some conclusions. Sellers s_1 and s_2 are again unlucky, since they are both connected to only one buyer, b_1. Again, competition will force them to give away all the surplus. Now, let's look at seller s_3. He knows he has a connection with b_1, but he also knows that b_1 can get 1 from sellers s_1 and s_2. There is no way s_3 can offer something better, and therefore neither s_3 or b_1 are interested in each other. Actually, s_3 will settle an agreement with buyer b_2, and they will simply split the pie and get 1/2 each. This situation would be equivalent to one in which s_3 and b_1 were not connected.



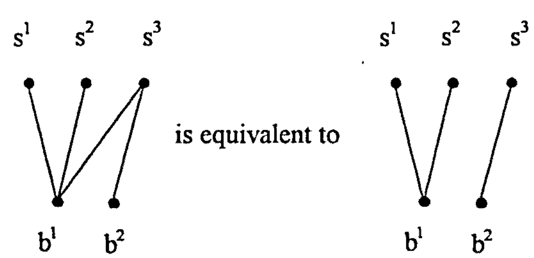


Figure 7

Now, let us go back to the initial network in figure 1. By the same intuition gained before, the network can be decomposed in smaller, easier networks, some of them "competitive" (as the one in figure 4, with the short part getting all the surplus) and some of them "even" (as the one in figure 2, with the same number of buyers and sellers and splitting the surplus evenly).

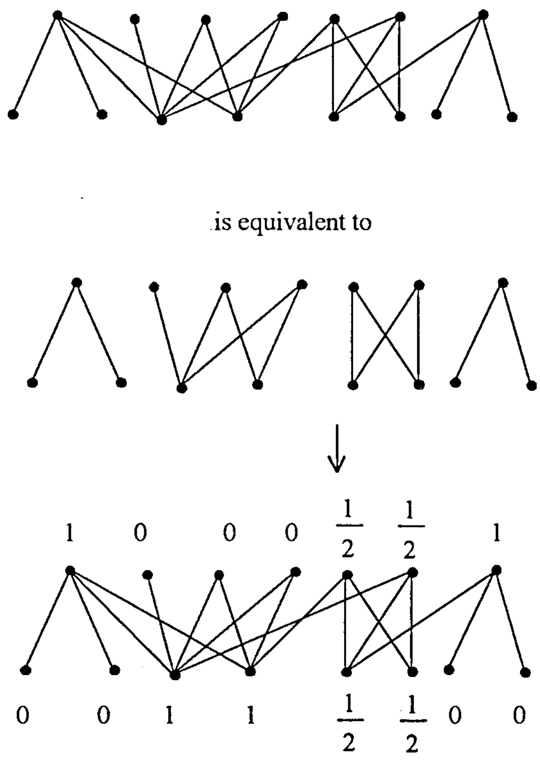


Figure 8

By analizing both a non-cooperative bargaining game played in a two-sided network market and some of its cooperative solutions, this thesis will introduce a new solution that we believe is an appropriate value for these type of situations.

1.2 Map of the thesis

Networks can be elegantly analyzed using Graph Theory. Chapter 2 is the technical chapter that gives us the background we need for all the further analysis. The focus there is on using some known results and developing some new ones to be used in our bipartite networks. The central result shown in Chapter 2 (theorem 2) concerns a decomposition that can be implemented in any bipartite network. Using this decomposition we can split any bipartite graph, no matter how complicated it is, into smaller and easier subgraphs which are of one of three possible types. We also study carefully the properties of this decomposition. Mainly, we are interested in knowing the changes it suffers when one new link is added or deleted. The results are located in the literature known as matching theory in graph theory.

Once we have the tools, we move to the first question: what is a good position and what is not. This will be addressed from a non-cooperative viewpoint and from a cooperative viewpoint , and the comparison among both will turn out to be very fruitful.

Chapter 3 deals with a non-cooperative game in which the two sides of the network engage in a simultaneous bilateral bargaining process. Specifically, sellers and buyers are connected through a network and make repeated alternating public offers that can be possibly accepted by any of the responders linked to each specific proposer. The structure there is inherited from the bargaining literature in decentralized markets, which tries to explain what is going on in a market without relying on an auctioneer, but only on the agreements agents fulfill strategically through bargaining. Our purpose is finding the conditions of the network driving the price distribution in equilibrium.

We quickly realize that the decomposition of chapter 2 plays a role when characterizing the subgame perfect equilibria of the game. Indeed, there exists an equilibrium in which the payoff is directly determined by the decomposition (proposition 8), and this equilibrium is unique in many cases. The agents belonging to a "competitive" subgraph (with more agents on one of the sides and with the short part being "well-connected" enough) turn out to give all the surplus to the short part, while agents belonging to an "even" subgraph split the surplus equally. In a sense, the connections which link these smaller subgraphs are irrelevant since the same outcome would be obtained in a network in which these links were ruled out.

The central result of Chapter 3 is given by theorem 4. There, we pose a question which relates our game to a concrete value given by competitive equilibrium. This solution would tell us that whenever there are more agents on one of the sides, the short part gets all the surplus, and whenever the number of sellers and buyers is equal, the surplus is split evenly. Interestingly, the only networks which implement this solution are the ones which are a union of subgraphs all of the same type. Thus, the structure which is equivalent both to the competitive equilibrium or to complete communication is characterized.

In Chapter 4 we move to cooperative game theory. We study two-sided network markets by relating them to assignment games in which the cost sellers face is always zero and the valuation of the buyers can be either zero or one. For these games, we first show that using the decomposition of Chapter 2, we can derive properties on the structure of its core and we can identify the cases in which the core is a singleton. Indeed, the core of any graph is equivalent to the union of the core of each of the subgraphs of the decomposition (theorem 5). Moreover, for the "competitive" networks the core is a singleton and gives all the surplus to the short side of the market, as one would expect. We also remark that a particular selection of the core, the so called fair solution, coincides with the limit of the equilibrium of the non-cooperative game of Chapter 3.

We then move to Chapter 5, where we try to find which are the networks we expect to arise in an endogenous process of link formation. Starting from any initial network, we show that *i)* every new link is weakly Pareto improving for the two newly linked agents, *ii)* if the network is not payoff equivalent to full communication, there exists always an agent that can strictly increase his payoff by creating a new link (corollaries 5 and 6). Building on these results we analyze stable and efficient networks, and we then build up a model of link formation, and characterize the set Nash equilibria. We find that for small costs networks which arise as Nash equilibria are characterized as those networks which are minimal (in the sense that no connection is redundant) and equivalent, in terms of payoff, to competitive equilibrium (see theorem 6). Interestingly, these networks are both stable (in the sense that no coalition wants to form or sever a link) and efficient (no other network, with the same players, would have a larger total sum of payoffs). We conclude that in our setup the only networks that can be expected to arise in the limit if the cost of building a new connection is small are the ones that give rise to competitive equilibrium, which are also equivalent to what would happen if every agent can communicate with every other agent.

Chapter 2

Graph Theory

2.1 Introduction

This thesis deals with two-sided markets in which the communication structure is given by a network. It will be useful to relate these networks to graphs, and to use some results in graph theory. In this chapter we will introduce the notation, review the related literature, and derive some new results to be applied subsequently in the rest of the thesis.

We will start by introducing the notation which concerns bipartite graphs, followed by the literature on matching theory and the marriage theorem shown by Hall (35). We will then turn to showing the existence of a decomposition which splits any bipartite graph into a union of three types of subgraphs. It will be shown that the type of subgraph a node belongs to by the decomposition is uniquely determined. This decomposition is a refinement of the Dumange and Mendelsohn (58), (59), (67) decomposition, later generalized by Gallai (63), (64) and Edmonds (65) for any (not necessarily bipartite) graph. The relation to these algorithms will be studied. We will also be concerned about the changes in the decomposition when a new link is added or deleted, and we will cover all the possible new connections and their consequences.

Basart (94), Bollobas (78) and Gould (88) are good graph theory manuals which contain chapters relating to matching theory, while Lovász and Plummer (86) offers a comprehensive review of the literature concerning matching theory.

The structure of this chapter is as follows. In section 2.2 we introduce the notation and definitions, and in section 2.3 we review the results in matching theory we will use. Section 2.4 introduces three types of subgraphs in which we will be particularly interested. The core of this chapter is in sections 2.5 and 2.6. In 2.5 we describe our decomposition and relate it to Gallai-Edmonds decomposition. This decomposition will play a crucial role in all chapters of this thesis. Section 2.6 studies the changes in the decomposition when a new link is added or deleted (section 2.6 will be used specifically in ch. 5).

2.2 The notation and definitions

We now introduce some notation and concepts of graph theory to be used in our analysis.

• A non-directed *bipartite graph* $G = \langle S \cup B, L \rangle$ consists of a set of *nodes* formed by sellers $S = \{s_1, ..., s_n\}$ and buyers $B = \{b_1, ..., b_m\}$, and a set of *links* L, each link joining a seller with a buyer. An element of L, say a link from s_i to b_j will be denoted as $s_i : b_j$.

We will say that a node *a belongs to a graph* $G = \langle S \cup B, L \rangle$ if $a \in S \cup B$.

We say that a node s_i is *adjacent* or *linked* to another node b_j if there is a link joining the two.

• A bipartite graph G is *connected* if there exists a path linking any two nodes of the graph. Formally, a *path* linking nodes s_j and b_i will be a collection of t buyers and t sellers, $t \geq 0$, $s_1, ..., s_t, b_1, ..., b_t$ among $S \cup B$ such that

$$\{s_j : b_1, b_1 : s_1, s_1 : b_2, ..., s_{t-1} : b_t, b_t : s_t, s_t : b_i\} \in L.$$

Some of the nodes in $s_1, ..., s_t, b_1, ..., b_t$ may coincide, that is, the path can repeat some nodes and some links.

• A *subgraph* $G_0 = \langle S_0 \cup B_0, L_0 \rangle$ of G is a graph such that $S_0 \subseteq S$, $B_0 \subseteq B$, $L_0 \subseteq L$ and such that each link in L_0 connects a seller of S_0 with a buyer in B_0.

When we speak of the *subgraph G_0 induced by the set of nodes $S_0 \cup B_0$ in G* we mean the subgraph formed by the nodes $S_0 \cup B_0$ and all the links that connect a seller in S_0 and a buyer in B_0 in G.

We will also speak about the *subgraphs $G_1, ..., G_t$ that result when we remove the set of nodes $S_0 \cup B_0$ from $G = \langle S \cup B, L \rangle$*. They will be defined as the maximal connected parts of the subgraph induced by the set of nodes $(S - S_0) \cup (B - B_0)$ in G. Similarly, for a graph $G = \langle S \cup B, L \rangle$ and a subgraph of G denoted $G_0 = \langle S_0 \cup B_0, L_0 \rangle$ we will sometimes write $G - G_0$ meaning the subgraph that results when we remove the set of nodes $S_0 \cup B_0$ from G.

Whenever we will speak about a graph G being equal to a *union of subgraphs* of G, written as: $G = G_1 \cup G_2 \cup ... \cup G_t$, we will mean that all nodes in G can be found in one of the subgraphs $G_1, ..., G_t$. We will say that the union is *disjoint* if each node of G can be found exactly in only one of the subgraphs $G_1, ..., G_t$.

• $N_G(s_j)$ will denote *the set of buyers linked with s_j* in $G = \langle S \cup B, L \rangle$; more formally:

$$N_G(s_j) = \{b_i \in B \text{ such that } b_i : s_j \in L\}$$

and similarly $N_G(b_i)$ stands for the set of sellers linked with b_i.

Similarly, for a subset of sellers $S_0 = \{s_1, ..., s_t\} \subseteq S$:

$$N_G(S_0) = \bigcup_{j=1}^{t} N_G(s_j)$$

We will read $N_G(S_0)$ as the *set of buyers collectively linked to S_0* in G. Similarly, for a set of buyers $B_0 \subseteq B$, $N_G(B_0)$ will be the set of sellers collectively linked to B_0 in G.

2.3 Matching theory

2.3.1 The marriage theorem

A natural question one may ask concerning bipartite graphs is when does it exist a way to match the nodes of a bipartite graphs in pairs. This was successfully solved by Hall (35) and stands as a very elegant and crucial result in matching theory, which graph theorists often refer to as "the marriage theorem"[1]. We now review the concept of a non-deficient set of nodes and its relation to Hall's theorem.

In the graph $G = \langle S \cup B, L \rangle$, consider a set of nodes $V \subseteq S$ or $V \subseteq B$ (either a subset of sellers or a subset of buyers). We will say that a set of nodes is non-deficient if all its subsets of nodes are collectively linked to a set of at least the same number of members. We will also talk about a matching in a bipartite graph, which is simply a collection of different linked pairs. Formally,

- A set of nodes V is *non-deficient* in $G = \langle S \cup B, L \rangle$ if:

  $|N_G(V_0)| \geq |V_0|$ for all $V_0 \subseteq V$.

  A set of nodes V is *strictly non-deficient* in $G = \langle S \cup B, L \rangle$ if:

  $|N_G(V_0)| > |V_0|$ for all $V_0 \subseteq V$

- A *matching* M in a bipartite graph $G = \langle S \cup B, L \rangle$ is a collection of linked pairs of B and S such that each agent in $S \cup B$ belongs to at most one pair. Formally, $M = \{\{s_{i_1}, b_{j_1}\}, ..., \{s_{i_t}, b_{j_t}\}\}$ with $s_{i_1} : b_{j_1}, ..., s_{i_t} : b_{j_t} \in L$ and such that $s_{i_k} \neq s_{i_{k'}}$ and $b_{j_k} \neq b_{j_{k'}}$ for $k \neq k'$.

  A matching M *saturates* all the nodes in V (or saturates the set V) if the set of pairs in M contains all members of V.

  If $G = \langle S \cup B, L \rangle$ is such that $|S| = |B|$, then a matching which saturates S (eq., saturates B) is called a *perfect matching.*

[1] Not to be confused with the results shown by Gale and Shapley (62), which game theorists also refer to as "the marriage theorem", but which are not directly related.

Consider a graph $G = \langle S \cup B, L \rangle$. A matching M will be called a *maximum matching* in G if there exists no other matching M' in G such that M' saturates strictly more nodes than M.

We now state the marriage theorem which finds the necessary and sufficient conditions for the existence of a matching saturating a given set of nodes. The first version of the marriage theorem was shown by G. Frobenius in 1917, the version we provide here is a generalization an was shown by P. Hall (35). The marriage theorem and its equivalent results is probably the single most important result to date in all of matching theory (see Lovász and Plummer (86), ch1, for details).

Theorem 1 *(*The marriage theorem, *Hall 35) There exists a matching in G that saturates all the nodes in $V \iff V$ is non-deficient in G.*

In the following figure we see a graph with 4 sellers and 3 buyers. The dotted lines represent the matching $M = \{s^1 : b^1, s^2 : b^2, s^4 : b^3\}$ formed by three pairs. The matching M saturates the set $V_1 = \{s^1, s^2, s^4\}$ and also saturates the set $V_2 = \{b^1, b^2, b^3\}$. Note though that there exists no matching saturating the set $V_3 = \{s^1, s^2, s^3\}$, since V_3 it is not a non-deficient set in G, i.e., $|N_G(V_3)| = |\{b^1, b^2\}| = 2 < 3 = |V_3|$.

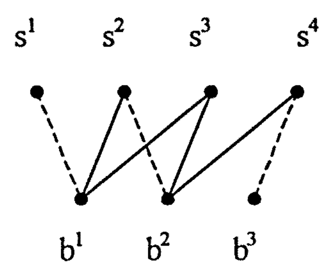


Figure 1

2.3.2 Almost non-deficiency

In this section we define a new concept very related to non-deficiency which we will call almost non-deficiency. To motivate almost non-deficiency, suppose that we are in a graph $G = \langle V_1 \cup V_2, L \rangle$ of sizes $|V_1| = n, |V_2| = m$ with $n > m$. The condition of a subset of nodes being non-deficient may hold for the set V_2, but never for the set V_1, since $|N_G(V_1)| = |V_2| = m < |V_1| = n$. That is, we may be able to find a matching that saturates all the nodes of the short part of the market, but we will never saturate the side with an excess of nodes. The set V_1 will be called almost non-deficient in G if the condition of being non-deficient is consistent with what is feasible.

Definition 1 *In a graph $G = \langle V_1 \cup V_2, L \rangle$ of sizes $|V_1| = n, |V_2| = m$ with $n \geq m$ we will say that the set of nodes V_1 is* almost non-deficient *in G if:*

For any subset $V_{11} \subset V_1$ of size $|V_{11}| \leq m$ we have that $|N(V_{11})| \geq |V_{11}|$

(note that this implies that for any subset $V_{11} \subset V_1$ of size $|V_{11}| > m$ we have that $|N(V_{11})| = m$).

We will now prove a lemma that will be useful for our results.

Lemma 1 *In a graph $G = \langle V_1 \cup V_2, L \rangle$ of sizes $|V_1| = n, |V_2| = m$ with $n > m$ we have that V_1 is* almost non-deficient *in G $\Rightarrow$ V_2 is* strictly non-deficient *in G.*

Proof. Suppose not. This implies that there exists a subset in V_2. Call this subset V_{21}, of size $|V_{21}| \leq m$ such that $|N(V_{21})| \leq |V_{21}|$, that is, it is collectively linked to at most the same number of partners. This implies that the rest of agents in V_1, that is, the set $V_1 - N(V_{21})$ that has size $n - |N(V_{21})|$, is collectively linked to at most the remaining agents in V_2, the set $V_2 - V_{21}$, a set of size $m - |V_{21}|$. This is so since, if not, one of the agents in V_{21} would be linked to an agent not belonging to $N(V_{21})$. This is the same as saying that:

$$N(V_1 - N(V_{21})) \leq m - |V_{21}| < n - |N(V_{21})| = |V_1 - N(V_{21})|,$$

contradicting the assumption. ∎

Note that the reverse implication is not true, since there are graphs in which the short side is strictly non-deficient but the large part is not almost non-deficient. For an example, see Figure 2, in which the set of all sellers (top side) is not almost non-deficient (indeed, we can find three sellers linked to only two buyers) but the set of buyers (bottom side) is strictly non-deficient.

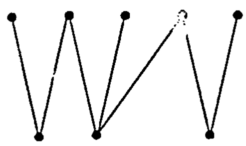


Figure 2

Finally, note that in a graph $G = \langle S \cup B, L \rangle$ with $|S| = |B| = n$, the set S is *non-deficient* in G iff the set B is *non-deficient* in G.

We are now going to use these tools to decompose any general graph into subgraphs.

2.4 Description of three types of subgraphs

We will now define three different types of subgraphs in a given graph. We start giving the definition for the easiest cases: those in which the subgraph is actually the whole graph. Figure 3 below has three graphs, each of them being of one of the types.

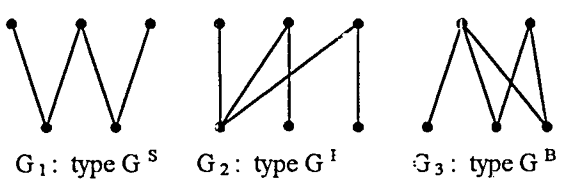


Figure 3

Note that graph G_1 (will be called of the G^S type) has more sellers than buyers and is such that the set of all sellers is almost non-deficient. Graph

G_2 (denoted of the G^E type) has an equal number of buyers and sellers and is such that a perfect matching exists (equivalent to the set of all sellers being non-deficient or to the set of all buyers being non-deficient). On the other hand graph G_3 (will be called of the G^B type) has too many buyers and the set of all buyers is also almost non-deficient. We now write the three definitions:

Definition 2 *We will say that a graph G with n sellers and m buyers is:*

- *of type G^S if $n > m$ and the set of all sellers in G is almost non-deficient.*
- *of type G^B if $n < m$ and the set of all buyers in G is almost non-deficient.*
- *of type G^E if $n = m$ and there exists a matching with n pairs.*

Besides discussing graphs of a certain type, we will also discuss subgraphs (included in a given graph) of a certain type. As an illustration, in Figure 4 we have underlined a subgraph of type G^S, one of type G^E and one of type G^B.

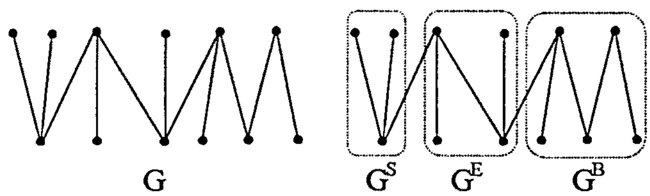


Figure 4

Subgraph G^S in the picture is such that there is an excess of sellers, all sellers in G^S are (in G), collectively linked to the buyers in G^S only, with the set of sellers being almost non-deficient. The symmetric idea applies to the subgraph of the type G^B: the buyers in G^B are collectively linked in G to the sellers in G^B only, there are more buyers than sellers, and the set of buyers is almost non-deficient. On the other hand, graphs of the type G^E always have an equal number of buyers and sellers, are such that a perfect matching exists in G^E, and their sellers are linked to buyers in G^E and possibly also to buyers in G^S; and similarly, their buyers are linked to sellers in G^E and possibly also to sellers in G^B.

Moreover, we also want to include other types of subgraphs in the same definition . Suppose that we remove the subgraphs of type G^S from the graph G . If in the remaining graph we find a subgraph of the G^S type, we will also include this subgraph in the definition of G^S-type. The sellers in G^S will therefore be collectively linked to buyers in G^S and also to buyers that belong to another subgraph, also of the G^S-type, that has previously been removed. Look at Figure 5 for an example:

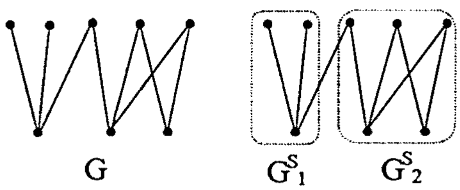


Figure 5

In Figure 5 we see that G_1^S is such that its sellers are collectively linked in G to only buyers in G_1^S. On the other hand, sellers in G_2^S are collectively linked not only to buyers in G_2^S but also to buyers in G_1^S. This subgraph G_2^S will also be called of the G^S type. The definition is symmetric for the G^B type.

We now formally state the recursive definitions for the three types.

Definition 3 *Given a graph $g = \langle S \cup B, L \rangle$, we will say that a connected strict subgraph $G_1 = \langle S_1 \cup B_1, L_1 \rangle$, with $|S_1| = n_1$ sellers and $|B_1| = m_1$ buyers is:*

- *of type G^S if conditions* a) *and* bi) *(for one $i \in \{1, 2, ...\}$) hold:*

  a) *considered as a graph, G_1 is of type G^S (see definition 2)*

  b1) $N_g(S_1) = B_1$.

  bi) $N_g(S_1) = B_1 \cup B_2$, *where $B_2 \subseteq B$ is such that if $b_2 \in B_2$, then b_2 belongs to a subgraph satisfying* a) *and* bj) *with $j \leq i - 1$.*

- *of type G^B if conditions* c) *and* di) *(for one $i \in \{1, 2, ...\}$) hold:*

  c) *considered as a graph, G_1 is of type G^B (see definition 2)*

  d1) $N_g(B_1) = S_1$.

  di) $N_g(B_1) = S_1 \cup S_2$, *where $S_2 \subseteq S$ is such that if $s_2 \in S_2$, then s_2 belongs to a subgraph satisfying* c) *and* dj) *with $j \leq i - 1$.*

- *of type G^E if conditions* e) *and* f) *hold:*

  e) *considered as a graph, G_1 is of type G^E (see definition 2)*

  f) $N_g(S_1) = B_1 \cup B_2$ *and* $N_g(B_1) = S_1 \cup S_2$, *with condition f1) and/or f2) being fulfilled:*

  f1) either $B_2 = \phi$ or for any $b_2 \in B_2$, buyer b_2 belongs to a subgraph of type G^S.

  f2) either $S_2 = \phi$ or for any $s_2 \in S_2$, seller s_2 belongs to a subgraph of type G^B.

2.5 The graph decomposition

After defining of the three types of subgraphs, we will be able to show the existence of a decomposition that will prove to be crucial for our results: each bipartite graph can be decomposed into a disjoint union of subgraphs, with each subgraph being of one of the three types. Moreover we also show that the property of belonging to a subgraph of a certain type is exclusive: if a node belongs, say, to a subgraph of type G^S, then it cannot belong for another decomposition to a subgraph of type G^B or G^E, and similarly for other types of subgraphs.

Theorem 2 *1) Every graph G can be decomposed into a number of connected subgraphs $G_1^S, ..., G_{n_S}^S$ (of the G^S type), $G_1^B, ..., G_{n_B}^B$ (of the G^B type), $G_1^E, ..., G_{n_E}^E$ (of the G^E type) in such a way that each node of G belongs to exactly one of the subgraphs. We will write $G = G_1^S \cup ... \cup G_{n_S}^S \cup G_1^B \cup ... \cup G_{n_B}^B \cup G_1^E, ..., G_{n_E}^E$.*

2) Moreover, a given node always belongs to the same type of subgraph for any such decomposition.

In the following subsection we describe an algorithm which implements the decomposition, and thus show part 1) if theorem 2. We later relate our decomposition to the Gallai-Edmonds decomposition, and as a consequence we show part 2) of theorem 2.

2.5.1 Algorithm implementing the decomposition

We now provide an algorithm which implements the decomposition of theorem 2.

An outline of the algorithm would be the following. In the first part we remove the subgraphs that have a set of sellers of size t collectively linked to less than t buyers. We do it starting from the subgraphs in which several sellers are collectively linked to only one buyer. Then we remove the subgraphs in which more than 2 sellers are collectively linked to only 2 buyers. We proceed in this way and when we have exhausted all the possibilities we start the second part in which we remove the subgraphs that have a set of buyers of size t collectively linked to less than t sellers. When this process is finished, the subgraphs removed in part one will be type G^S, the ones removed in part two will be type G^B and the ones that remain after the algorithm is finished, of type G^E.

==============ALGORITHM==============

Starting from a graph G_t, with the initial graph being $G_1 = G$.

$\longrightarrow$ **Part 1**

• Step s1)

step s1.1) We start from $G_t = \langle S_t \cup B_t, L_t \rangle$, with the initial graph being $G_1 = G$. Label all agents with a subindex. Look at every subset $\widetilde{S}$ of S_t such that $\left|\widetilde{S}\right| = 2$, starting from the subsets that contain s_1 in the order $\{s_1, s_2\}$, $\{s_1, s_3\}$, ..., $\{s_1, s_n\}$,then with the ones that contain s_2 in the order $\{s_2, s_3\}$, $\{s_2, s_4\}$, ..., $\{s_2, s_n\}$ and so on. That is, the order for looking at the subsets is $\{s_k, s_t\}$, $t = k+1, k+2, ..., n$, starting from $k = 1, k = 2$ up to $k = n$ (in short, lexicographic ordering).

Once you find one $\widetilde{S} \subseteq S_t$ with $\left|\widetilde{S}\right| = 2$ such that $\left|N(\widetilde{S})\right| = 1$, stop. For every seller $s^i \notin \widetilde{S}$ (again here we will follow the ordering given by their subindexes), if it is true that $N(\widetilde{S} \cup s^i) = N(\widetilde{S})$, then relabel $\widetilde{S} := \widetilde{S} \cup s^i$.

Call G_t^1 (superindex 1 stands for "Part 1") the subgraph in G_t induced by the set of sellers $\widetilde{S}$ and the set of buyers $N(\widetilde{S})$.

step s1.2) If we run step s1.1 and we found a G_t^1, then call $\bigcup_{j=t+1}^{k_t} G_j := G_t - G_t^1$, i.e., the connected subgraphs that we get when we remove G_t^1 from G_t, and run again step s1 with each G_j with $j > t$.

If we run step s1.1 without finding any G_t^1, then go to step s2.

...

- step sk)

step sk.1) We start from G_t. Look at every subset $\widetilde{S}$ of S_t such that $\left|\widetilde{S}\right| = k+1$, following the lexicographic ordering.

Once you find one $\widetilde{S} \subseteq S_t$ with $\left|\widetilde{S}\right| = k+1$ such that $\left|N(\widetilde{S})\right| = k$, stop. For every seller $s^i \notin \widetilde{S}$ (again here we will follow the ordering given by their subindexes), if it is true that $N(\widetilde{S} \cup s^i) = N(\widetilde{S})$, then relabel $\widetilde{S} := \widetilde{S} \cup s^i$.

Call G_t^1 the subgraph in G_t induced by the set of sellers $\widetilde{S}$ and the set of buyers $N(\widetilde{S})$.

step sk.2) If we run step sk.1 and we found a G_t^1, then call $\bigcup_{j=t+1}^{k_t} G_j := G_t - G_t^1$, i.e., the connected subgraphs that we get when we remove G_t^1 from G_t, and go again to step s1 with each G_j with $j > t$.

If we run step sk.1 without finding any G_t^1, then go to step sk+1.

.....

- step sm)

step sm.1) We start from G_t. Look at every subset $\widetilde{S}$ of S_t such that $\left|\widetilde{S}\right| = m+1$, following the same ordering as before.

Once you find one $\widetilde{S} \subseteq S_t$ with $\left|\widetilde{S}\right| = m+1$ such that $\left|N(\widetilde{S})\right| = m$, stop. For every seller $s^i \notin \widetilde{S}$ (again here we will follow the ordering given by their subindexes), if it is true that $N(\widetilde{S} \cup s^i) = N(\widetilde{S})$, then relabel $\widetilde{S} := \widetilde{S} \cup s^i$.

Call G_t^1 the subgraph in G_t induced by the set of sellers $\widetilde{S}$ and the set of buyers $N(\widetilde{S})$.

step sm.2) If we run step sm.1 and we found a G_t^1, then call $\bigcup_{j=t+1}^{k_t} G_j := G_t - G_t^1$, i.e., the connected subgraphs that we get when we remove G_t^1 from G_t, and go again to step s1 with each G_j with $j > t$.

If we run step sm.1 without finding any G_1^1, then end Part 1

• Once we are finished with Part 1, we go to Part 2.

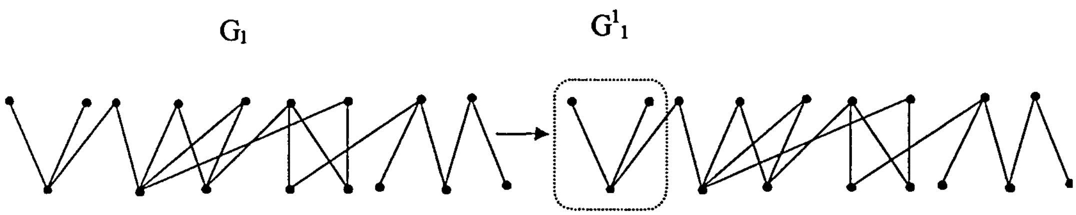
$\longrightarrow$ **Part 2**

Part 2 is completely symmetric to Part 1, with the roles of buyers and sellers get reversed. The steps go from step b1) to step bn). We start Part 2 with the G_t that come from the last iteration in Part 1. The subgraphs that we remove will now be called G_t^2 (superindex is 2 for Part 2).

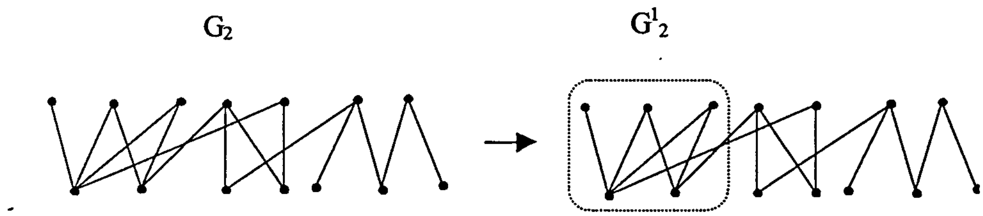
$\longrightarrow$ End of the algorithm.

==============================

Look at Figure 6 to see an example of how does the algorithm work.

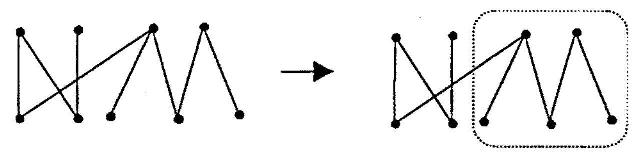


G^1_1 is removed



G^1_2 is removed

G_3 G^2_3



G^2_3 is removed

Therefore, the decomposition is:

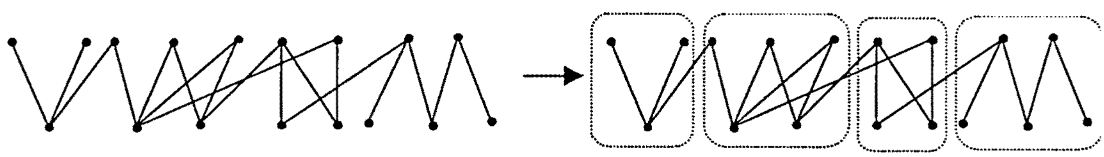


Figure 6

The two following lemmas show that the algorithm indeed finds subgraphs of the three defined types. Therefore, the two lemmas below show part 1) of theorem 2

Lemma 2 *In the initial graph G, subgraphs denoted by G^1_j for $j = 1, ..., t_1$ (removed in part 1 of the algorithm) are of the G^S type, while subgraphs denoted by G^2_j for $j = 1, ..., t_2$ (removed in part 2 of the algorithm) are of the G^B type.*

Proof. What we have to show is the following:

a1) In the initial graph G, sellers in G^1_j for $j = 1, ..., t_1$ are linked to buyers in G^1_j and maybe to other buyers that belong to a G^1_k for a $k < j$ (but not to buyers other than that). Moreover,

a2) the set of all sellers in G^1_j is *almost non-deficient* in G^1_j.

b1) In the initial graph G, buyers in G^2_j for $j = 1, ..., t_2$ are collectively linked to sellers in G^2_j and maybe to other sellers that belong to a G^2_k for a $k < j$. (but not to sellers other than that). Moreover, b2) the set of all buyers in G_j^2 is *almost non-deficient* in G_j^2.

showing a1) Immediate given that previously to G_j^1 the only subgraphs that have been removed from G are G_k^1 with $k < j$.

showing a2) Now, suppose that the subgraph $G_j^1 = \left\langle S_j^1 \cup B_j^1, L_j^1 \right\rangle$ does not fulfill a2) in G_j^1. This means that there exists a subset $S_0 \subseteq S_j$ of size $|S_0| \leq |B_j|$ for which $\left|N_{G_j^1}(S_0)\right| < |S_0|$. This is a contradiction since by construction of the algorithm, the subgraph of G_j^1 induced by the sellers S_0 and the buyers $N_{G_j^1}(S_0)$ should have been ruled out in a previous step.

showing b1 and b2) Symmetric to a1) and a2). What remains to be shown here is only that the buyers in G_j^2, when in G, are not linked to any of the sellers already ruled out in Part 1 of the algorithm, that is, that they are not linked to any seller belonging to G_j^1 for $j = 1, ..., t_1$. But this is immediate given a1 and a2. ■

When the algorithm is finished, the remaining subgraphs will be of the G^E type.

Lemma 3 *After running the algorithm, we are left with a number of disconnected subgraphs, $G_1^3, ..., G_t^3$. All of them are of the G^E type.*

Proof. What we must show is that all of them are such that G_i^3 is $n_i = m_i$ and such that the set of all sellers in G_i^3 is non-deficient in G_i^3 (and equivalently the set of all buyers in G_i^3 is non-deficient in G_i^3). Moreover, we must show that sellers in a G_i^3 are linked to buyers in a G^S and buyers in a G_i^3 are linked to a G^B.

Suppose subgraph G_i^3 was not $n_i = m_i$, but it was $n_i < m_i$. This is a contradiction since this subgraph should have been ruled out at some step in Part 2.

Suppose subgraph G_i^3 was $n_i > m_i$. The only possibility is that after running Part 1) we had no subgraph of this type (G^1), but that it appeared after running Part 2). Say that after running Part 1) we had a graph G. It must be that this graph G is $n \leq m$. By running Part 2), we are left with a subgraph of G , G_i^1, that has $n_i > m_i$. This means that while running Part

2), some buyer (say, b^i) that belonged to G (and therefore was linked to at least one seller s^i) dropped from G, while s^i did not. But recall that in Part 2), all the buyers that we remove who belong to a G_j^2 are such that its buyers are collectively linked to only the sellers in G_j^2 and sellers that belonged to another G_k^2 with $k < j$. Therefore, all the sellers linked to b_i in G should have been removed by the end of the algorithm.

Finally, the sellers in subgraph G_i^3 have to be non-deficient in G_i^3 since if not it would contain a subgraph in which t sellers would be collectively linked to t' buyers, with $t > t'$ (and this can't be since this should have been ruled out in Part 1) or it would contain a subgraph in which t buyers would be collectively linked to t' sellers, $t > t'$ (that should have been ruled out in Part 2). ■

Therefore, in the graph decomposed in Figure 6, we know which is the type of each subgraph:

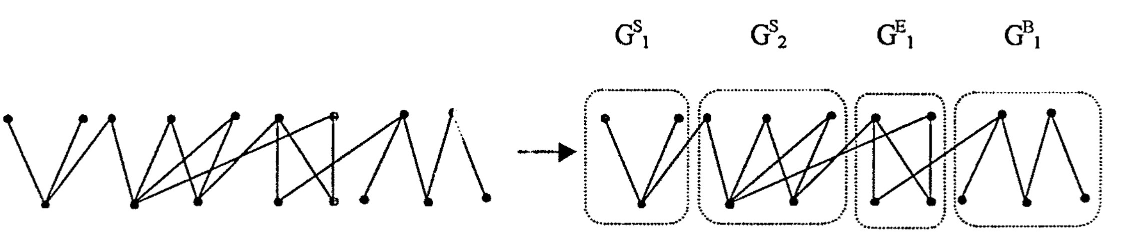


Figure 7

Finally, we would like to mention that we have developed a program in C++ which implements our decomposition. The user simply has to enter the number of buyers and sellers and its links, and the program gives a list of the subgraphs the graph decomposes into. If interested, we will be happy to send it to the reader.

2.5.2 Relation to the Gallai-Edmonds decomposition

The decomposition we have described above has close links with the canonical structure theorem due independently to T. Gallai (63, 64) and J. Edmonds

(65) (see Lovász and Plummer, ch. 3). This is a unique decomposition of any graph into three subgraphs, which is notably simpler in the case of a bipartite graph, and which was in this set up first worked out by Dulmage and Mendelsohn (58, 59 and 67).

We now state some of the results of Gallai-Edmonds, (subsequently, the G-E decomposition) adapted to our context and to our notation.

Let us first construct, for a bipartite graph $G = \langle S \cup B, L \rangle$, three sets of nodes belonging to $S \cup B$.

$D =$ Nodes in $S \cup B$ not covered by at least one maximum matching of G.

$A =$ Nodes in G adjacent to at least one node in D.

$C = (S \cup B) - (A \cup D)$.

We now define three particular subgraphs of G.

- G^{SS} is defined as the subgraph induced by the set of nodes $(D \cap S) \cup (A \cap B)$ in G.
- G^{EE} is defined as the subgraph induced by the set of nodes $(C \cap S) \cup (C \cap B)$ in G.
- G^{BB} is defined as the subgraph induced by the set of nodes $(A \cap S) \cup (D \cap B)$ in G.

Note that $G = G^{SS} \cup G^{EE} \cup G^{BB}$ and that the union is disjoint (each node in G belongs to exactly one of the subgraphs). Then:

Theorem 3 *(Gallai-Edmonds) For a given graph G, construct three subgraphs G^{SS}, G^{EE}, G^{BB} as above. Then:*

(1) A seller in G^{SS} is only linked in G to buyers in G^{SS}. Similarly, a buyer in G^{BB} is only linked to sellers in G^{BB}.

(2) The subgraph G^{EE} has a perfect matching.

(3) In G^{SS}, $|D \cap S| > |A \cap B|$ and the set $A \cap B$ is non deficient. Therefore, the maximum matching in G^{SS} involves $|A \cap B|$ pairs.

Similarly, in G^{BB}, $|D \cap B| > |A \cap S|$ and the set $A \cap S$ is non deficient. Therefore, the maximum matching in G^{EE} involves $|A \cap S|$ pairs.

(4) A maximum matching in G consists of a perfect matching of G^{EE}, a matching of $A \cap B$ into $D \cap S$ and a matching of $A \cap S$ into $D \cap B$.

Therefore, the property of a node always belonging to the same type of subgraphs by our decomposition is directly implied by the uniqueness of the G-E decomposition, as we state in the following lemma.

Lemma 4 *If a node belongs to a subgraph of a certain type for a decomposition following theorem 2, then it never belongs for a decomposition following theorem 2 to a subgraph of a different type.*

Proof. Immediate by the uniqueness of the G-E decomposition. ■

This lemma shows part 2) or Theorem 2.

Note that the decomposition is not unique only up to the following degree: it can happen that a node belongs to a subgraph G of a certain type for a decomposition and to a subgraph G' also of the same type for another decomposition. This will not be a problem for the results found in the subsequent chapters, since we will only be interested in the type of each node. Look at Figure 8 for an example.

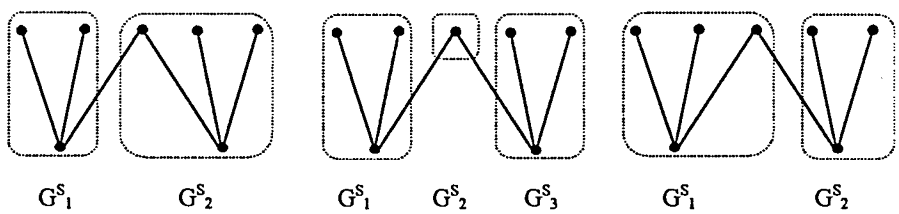


Figure 8

The relation between the two decompositions should now be clear. The union of subgraphs of the same type results in three different subgraphs which coincide with those of the G-E decomposition, as stated in the following proposition.

Proposition 1 *Take G and decompose it as $G_1^S, ..., G_{n_S}^S$ (of the G^S type), $G_1^B, ..., G_{n_B}^B$ (of the G^B type), $G_1^E, ..., G_{n_E}^E$ (of the G^E type), according to the algorithm. Then, according to the Gallai-Edmonds decomposition:*

G^{SS} is the subgraph induced in G by the nodes in $G_1^S \cup ... \cup G_{n_S}^S$, G^{EE} is the subgraph induced in G by the nodes in $G_1^E \cup ... \cup G_{n_E}^E$ and G^{BB} is the subgraph induced in G by the nodes in $G_1^B \cup ... \cup G_{n_B}^B$.

Proof. Straightforward and therefore omitted. ■

See Figure 9 for a comparision among our decomposition and G-E decomposition.

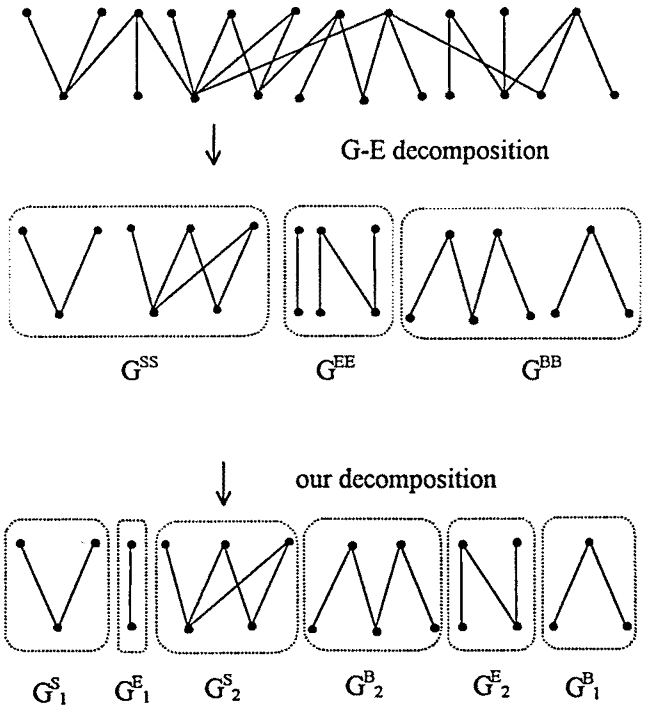


Figure 9

By the Gallai-Edmonds decomposition we know that a graph G is the union of three particular subgraphs plus some extra connections among them (in particular, either from a seller $s_i \in G^{BB}$ to a buyer $b_j \in \{G^{SS} \cup G^{EE}\}$ or from a buyer $b_j \in G^{SS}$ to a seller $s_i \in \{G^{BB} \cup G^{EE}\}$). Note that G^{SS},

G^{EE} and G^{BB} need not be connected. A very important consequence is number (4) in theorem 3, which says that the number of pairs involved in a maximum matching is immediate once we calculate the three subgraphs. Moreover, an immediate consequence of (4) tells us that a maximum matching never involves the "extra" links that join a seller $s_i \in G^{BB}$ to a buyer $b_j \in \left\{G^{SS} \cup G^{EE}\right\}$ or links that join a buyer $b_j \in G^{SS}$ to a seller $s_i \in \left\{G^{BB} \cup G^{EE}\right\}$. Similarly, a maximum matching always covers all buyers in G^{SS}, all sellers in G^{BB}, and all agents in G^{EE}.

Our decomposition adds to a further understanding of the interior structure of the sets G^{SS}, G^{EE} and G^{BB}. Indeed, we find that the set G^{SS} can be decomposed into a union of connected subgraphs $G_1^S, ..., G_{n_S}^S$ (plus some extra connections linking them), in such a way that the set of its sellers is always almost non deficient.

For some of the results we will derive in this thesis it will be sufficient to know the "type" of each node, that is, whether it belongs to a G_i^S, to a G_i^E or to a G_i^B (i.e., the G-E decomposition), while for others, specially to show some of the proofs in ch. 3 and ch. 5 and for the analysis of the effect of a new link (section 2.6 here) it will be necessary to know the precise subgraph they belong to.

Note also that a needed first step in order to implement the G-E decomposition is to find a maximum matching. The first algorithm which could find a maximum bipartite matching is the so called Hungarian Method, which was stated in terms of dual linear program and assignment problems by Kuhn (55). The most powerful algorithm for finding maximum matchings is considered to be Edmonds (65) algorithm.

As a last remark, note that if we have already implemented the G-E decomposition, finding our decomposition can be much simplified. We should only follow these steps:

1- Find the maximal connected subgraphs in G^{EE}. They constitute $G_1^E, ..., G_{n_E}^E$.

2- Use our algorithm to decompose G^{SS} into $G_1^S, ..., G_{n_S}^S$.

3- Use our algorithm to decompose G^{BB} into $G_1^B, ..., G_{n_B}^B$.

2.6 The effect of a new link.

We are now interested in studying the properties of the graph decomposition when a new link is added. The results explained here will be used when dealing with endogenous creation of links in ch. 5.

In this section we will be comparing two graphs, g and g', which will differ only on one link. We will speak about the g-decomposition when applying the decomposition to graph g, and about the g'-decomposition when applying it to graph g'. Similarly, we will speak about a graph or subgraph of type gG^S when it is of type G^S by the g-decomposition, and similar with $g'G^S$ for graph g' (similarly for $gG^E, g'G^E, gG^B, g'G^B$). Finally, we will also speak about the type of a node meaning the type of the subgraph the node belongs to by the decomposition (recall that by theorem 2, the type of subgraph a node belongs to by the decomposition is uniquely determined).

2.6.1 Notation and definitions.

We now introduce a new notation which will be needed in order to show the forthcoming results. We will differentiate two types of subgraphs of type G^S and G^B, the recursive ones and the non-recursive.

Definition 4 *We will say that a subgraph $G_1 = \langle S_1 \cup B_1, L_1 \rangle$ of a graph g which is of type G^S is* non-recursive *if it satisfies* a) *and* b1) *in the definition. Otherwise, we will say that G_1 is* recursive.

Similarly, a subgraph $G_1 = \langle S_1 \cup B_1, L_1 \rangle$ of a graph g which is of type G^B is non-recursive *if it satisfies* c) *and* d1) *in the definition. Otherwise, we will say that G_1 is* recursive.

Now, consider all the subgraphs of graph g that are of type G^S. We can organize all these subgraphs in levels, in the following sense. By the definition some of them are non-recursive, call them $G_{11}^S, ..., G_{1i_1}^S$. Those would be initial subgraphs. Some others may be recursive subgraphs of type G^S which are non-recursive in the graph $g \setminus \left\{G_1^S \cup ... \cup G_{i_1}^S\right\}$, and we will call them level 2 subgraphs, call them $G_{21}^S, ..., G_{2i_2}^S$. Next level would be formed by the recursive subgraphs of type G^S which are non-recursive in the graph $g \setminus \left\{G_1^S \cup ... \cup G_{i_1}^S \cup G_{21}^S \cup ... \cup G_{2i_2}^S\right\}$. Let us now define this notation formally.

Definition 5 *Take the set of all subgraphs of type G^S by the g-decomposition. We classify these subgraphs in levels, in the following way.*

level 0: *Non recursive subgraphs $G_{01}^S, ..., G_{0i_0}^S$ such that buyers in each subgraph are not linked to sellers in another G^S subgraph.*

level 1: *Non recursive subgraphs $G_{11}^S, ..., G_{1i_1}^S$ such that buyers in each subgraph are linked to sellers in another G^S subgraph.*

level 2: *Recursive subgraphs $G_{21}^S, ..., G_{2i_j}^S$ such that each of these subgraphs is non-recursive in $g \setminus \left\{G_{11}^S, ..., G_{1i_1}^S\right\}$*

...

level j: *Recursive subgraphs $G_{j1}^S, ..., G_{ji_j}^S$ such that each of these subgraphs is non-recursive in $g \setminus \left\{G_{11}^S, ..., G_{1i_1}^S, ..., G_{(j-1)1}^S, ..., G_{(j-1)i_{(j-1)}}^S\right\}$*

Call the last level according to the g-decomposition, level l. Besides being organized in levels, subgraphs of type gG^S can also be classified according to the relation among them.

Definition 6 *Take a subgraph of type G^S by the g-decomposition which is of level j, with $j \in \{2, 3, ..., l\}$, call it G_j^S. We will say that subgraphs $\left(G_j^S\right)^L = \left\{G_{Lj_1}^S, ..., G_{Lj_j}^S\right\}$ are the direct lower associated subgraphs of subgraph G_j^S if:*

1) Every subgraph in $\left(G_j^S\right)^L$ is of level strictly smaller than j.

2) Sellers in G_j^S are linked either to buyers in G_j^S or to buyers belonging to $\left(G_j^S\right)^L$. Moreover, every subgraph in $\left(G_j^S\right)^L$ has at least one buyer being linked to a seller in G_j^S.

Consider the direct lower associates of G_j^S, denoted by $\left(G_j^S\right)^L$, and also the direct lower associates of each of the subgraphs in $G_{Lj_1}^S, ..., G_{Lj_j}^S$, call it

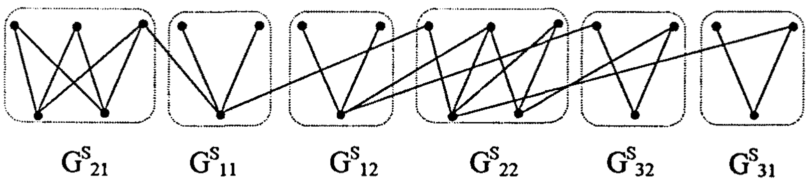
$\left(\left(G_j^S\right)^L\right)^L$ and iterate until we reach level 1 subgraphs. Then, those would be the minimal number of subgraphs such that when removing them from g, them, G_j^S turns out to be a non-recursive subgraph.

Note also that we can parallel the above definition and speak about direct upper associates of a subgraph.

See the Figure 10 for an example of the classification.

Classification figure

Take the following graph, which is a graph with all subgraphs of type G^S.



We can draw these subgraphs of type G^S as follows:

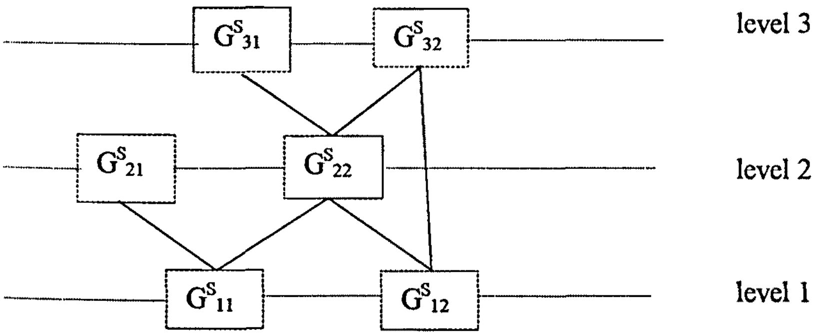


Figure 10

We would have that:

$$
\begin{array}{rclr}
\left(G_{31}^S\right)^L & = & \left\{G_{22}^S\right\} & \left(G_{31}^S\right)^U = \emptyset \\
\left(G_{32}^S\right)^L & = & \left\{G_{22}^S, G_{12}^S\right\} & \left(G_{32}^S\right)^U = \emptyset \\
\left(G_{21}^S\right)^L & = & \left\{G_{11}^S\right\} & \left(G_{21}^S\right)^U = \emptyset \\
\left(G_{22}^S\right)^L & = & \left\{G_{11}^S, G_{12}^S\right\} & \left(G_{22}^S\right)^U = \left\{G_{31}^S, G_{32}^S\right\}
\end{array}
$$

$$\left(G_{11}^S\right)^L = \emptyset \qquad \left(G_{11}^S\right)^U = \left\{G_{21}^S, G_{22}^S\right\}$$
$$\left(G_{12}^S\right)^L = \emptyset \qquad \left(G_{12}^S\right)^U = \left\{G_{22}^S, G_{32}^S\right\}$$

All this notation can be paralleled to G^B subgraphs.

2.6.2 The results

Propositions 2, 3 and 4 in this section tell us what happens when a new link is created, depending on the type of the newly linked agents. The results will be shown for a seller in a G^S, in a G^E or in a G^B (the other cases can be shown symmetrically).

To start with, note that whenever a new link is created among members of the same subgraph, nothing changes. This is immediate by the definition. Therefore we only need to see what happens for new links among two agents belonging to initially different subgraphs.

Call $g = \langle S \cup B, L \rangle$ the initial graph. Take any seller s and buyer b that are not connected in g. Now, call g' the graph that arises when we add this new link to g, *i.e.*, $g' = \langle S \cup B, L \cup \{s : b\} \rangle$.

Say that by the decomposition of g, seller s belongs to a subgraph G_1 and buyer b belongs to a subgraph G_2. Call n_1^S, m_1^S the number of sellers and buyers respectively of G_1. Similarly, call n_2^B, m_2^B the number of sellers and buyers respectively of G_2.

If G_1 is a subgraph of type G^S or G^B, then it is of a certain level, call its level l_1. Similarly, if G_2 is a subgraph of type G^S or G^B, then it is of a certain level, call its level l_2. Now define l_S as the minimum level of the subgraphs of type gG^S among G_1 and G_2 (similar for l_B).

We start by showing the following lemma:

Lemma 5 a) *If neither G_1 nor G_2 are of gG^S (gG^B) type, then all nodes that are of type gG^S (gG^B) are also of type $g'G^S$ ($g'G^B$).*

b) *The nodes $g \setminus \{G_1 \cup G_2\}$ which belong to subgraphs of type gG^S of a level $< l_S$ or subgraphs of type gG^B of a level $< l_B$ keep being of the same type by the g'-decomposition.*

Proof. Straightforward and therefore omitted. ■

The three following propositions completely characterize the effect of a new link in a graph. They tell us what is the effect of the new link on the two newly linked agents, but also on all the other agents. We leave their proofs for the appendix.

Proposition 2 *Suppose that seller s is of type gG^S. Then, all members of $g \setminus \{G_1 \cup G_2\}$ belong to the same type either by the g-decomposition or by the g'-decomposition. Moreover,*

a) *if buyer b is of type gG^S, all members of g belong to the same type by the g-decomposition or by the g'-decomposition (no changes).*

b) *if buyer b is of type gG^E, then both s and b are of type $g'G^S$. Nodes in G_1 are all of type $g'G^S$. Nodes in $G_2 \setminus b$ are either of type $g'G^S$ or $g'G^E$.*

c) *if buyer b is of type gG^B, then both s and b are of type $g'G^E$. Then, regarding members of G_1 and G_2 :*

ca) *If $n_1^S - m_1^S = 1$ and $m_2^B - n_2^B = 1$, all members of $G_1 \cup G_2$ are of type $g'G^E$.*

cb) *If $n_1^S - m_1^S = 1$ and $m_2^B - n_2^B > 1$, members of $G_1 \cup b$ are of type $g'G^E$, while members of $G_2 \setminus b$ are of type $g'G^B$.*

cc) *If $n_1^S - m_1^S > 1$ and $m_2^B - n_2^B = 1$, all members of $G_2 \cup s$ are of type $g'G^E$, while members of $G_1 \setminus s$ are of type $g'G^S$.*

cd) *If $n_1^S - m_1^S > 1$ and $m_2^B - n_2^B > 1$, $s \cup b$ are of type $g'G^E$, while members of $G_1 \setminus s$ are of type $g'G^S$ and members of $G_2 \setminus b$ are of type $g'G^B$.*

Proof. See the appendix. ■

Proposition 3 *Suppose seller s is of type gG^E. Then, all members of $g \setminus \{G_1 \cup G_2\}$ belong to the same type either by the g-decomposition or by the g'-decomposition. Moreover,*

a) *if buyer* b *is of type* gG^S*, then all members of* g *belong to the same type by the* g*-decomposition or by the* g'*-decomposition.*

b) *if buyer* b *is of type* gG^E*, then all members of* g *belong to the same type by the* g*-decomposition or by the* g'*-decomposition.*

c) *if buyer* b *is of type* gG^B*, both* s *and* b *are of type* $g'G^B$*. Nodes in* G_2 *are all of type* $g'G^B$*. Nodes in* $G_1 \setminus s$ *are either of type* $g'G^B$ *or* $g'G^E$*.*

Proof. See the appendix. ■

Proposition 4 *Suppose seller* s *is of type* gG^B*. Then, all members of* g *belong to the same type by the* g*-decomposition or by the* g'*-decomposition.*

Proof. See the appendix. ■

Note that in proposition 2 we say that when s is of type gG^S and b is of type gG^E, then both s and b are of type $g'G^S$, nodes in G_1 are all of type $g'G^S$ while nodes of type $G_2 \setminus b$ can be either of type $g'G^S$ or $g'G^E$. We now want to be a more precise about the type of the nodes belonging to $G_2 \setminus b$. This can not be concluded directly from the size of G_1 and G_2. We want to remark now that it is not necessary to run the algorithm on the whole graph g' to know the g'-type of $G_2 \setminus b$, though. It is enough to look deeper into the structure of G_2, in the following way:

Consider subgraph G_2. Then, G_2 can be split into two subgraphs, G_{21} and G_{22}. Call S_{21} and B_{21} the set of all sellers and buyers respectively in G_{21}. Then, G_{21} is characterized by the following properties:

i) $b \in B_{21}$.

ii) $|S_{21}| = |B_{21}|$

iii) for any subset S' of S_{21} of size $|S'| < S_{21}$, it holds that $|N_{G_{21}}(S')| \geq |S'| + 1$, while $N_{G_{21}}(S_{21}) = B_{21}$.

iv) Moreover, G_{21} is maximal in the sense that no other larger subgraph has the same properties.

Remark 1 *Construct G_{21} as explained above and define G_{22} as $G_2 \setminus G_{21}$. Then, nodes belonging to G_{21} will be of type $g'G^S$ while nodes of G_{22} will be of type $g'G^E$.*

Proof. a) If the size of G_1 is such that $n_1 - m_1 > 1$, then define $G' = \langle S_{21} \cup s \cup B_{21}, L_{21} \cup \{s : b\}\rangle$. Then, in the g'-decomposition, $G_1 \setminus s$ is a subgraph $g'G^S$, G' is also a subgraph $g'G^S$, and G_{22} is a subgraph $g'G^E$. It can be shown that this is true since the subgraphs fulfill definition 3. Clearly this is true for subgraph $G_1 \setminus s$. Subgraph G' is by definition such that it has more sellers than buyers, the set of its sellers is almost non-deficient, and its sellers are only linked either to sellers in $G_1 \setminus s$ or to buyers in G'. On the other hand, G_{22} has as many buyers as sellers, there exists a perfect matching among its members (since by construction no buyer of G_{22} is linked to a seller in G') and is such that its sellers may be linked to buyers in G' or $G_1 \setminus s$.

b) Otherwise, define $G' = G_1 \cup G_{21}$. Then, similarly as before it will be the case that in the g'-decomposition G' is a subgraph $g'G^S$, and G_{22} is a subgraph $g'G^E$. ■

For convenience the results of propositions 2, 3 and 4 are summarized in the two following tables. The first table tells us the effect of the new link on the two newly linked agents only.

s and b newly linked in g'	b is gG^S	b is gG^E	b is gG^B
s is gG^S	=	s is $g'G^S$ (=), b is $g'G^S$	s is $g'G^E$, b is $g'G^E$
s is gG^E	=	=	s is $g'G^B$, b is $g'G^B$ (=)
s is gG^B	=	=	=

Table 1: Entries tell us the types of seller s and buyer b for the g'-decomposition. If they are the same as in the g-decomposition, we write "=".

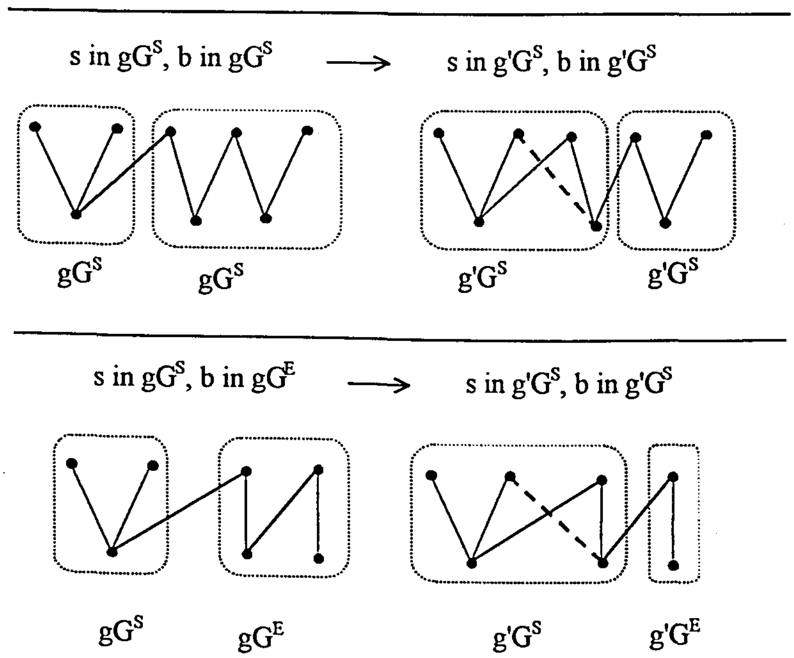
The second table tells us about the effect of the new link on $G_1 \setminus s$ and $G_2 \setminus b$.

s and b newly linked in g'	b is gG^S	b is gG^E	b is gG^B
s is gG^S	=	$G_1 \setminus s$ nodes are $g'G^S$ (=) $G_2 \setminus b$ nodes are $g'G^S$ or $g'G^E$	$G_1 \setminus s$ nodes are $g'G^S$or $g'G^E$ $G_2 \setminus b$ nodes are $g'G^B$or $g'G^E$
s is gG^E	=	=	$G_1 \setminus s$ nodes are $g'G^B$or $g'G^E$ $G_2 \setminus b$ nodes are $g'G^B$(=)
s is gG^B	=	=	=

Table 2: Entries tell us the types of nodes in subgraphs $G_1 \setminus s$ and $G_2 \setminus b$ for the g'-decomposition. If they are the same as in the g-decomposition, we write "=".

See Figure 11 for some examples on the effect of a new link on the two newly linked agents.

EXAMPLES



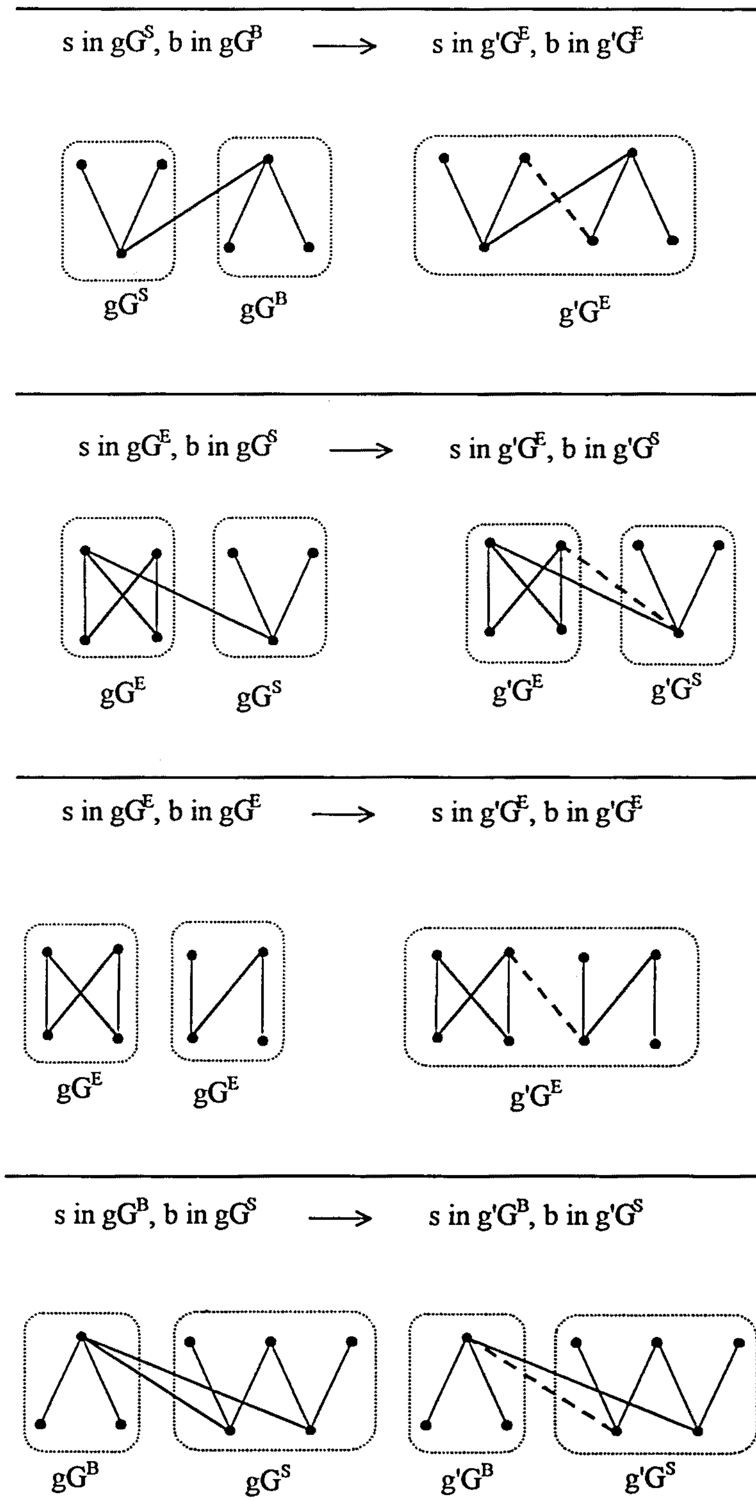

s in gG^S, b in gG^B ⟶ s in $g'G^E$, b in $g'G^E$
gG^S
gG^B
$g'G^E$
s in gG^E, b in gG^S ⟶ s in $g'G^E$, b in $g'G^S$
gG^E
gG^S
$g'G^E$
$g'G^S$
s in gG^E, b in gG^E ⟶ s in $g'G^E$, b in $g'G^E$
gG^E
gG^E
$g'G^E$
s in gG^B, b in gG^S ⟶ s in $g'G^B$, b in $g'G^S$
gG^B
gG^S
$g'G^B$
$g'G^S$

Figure 11

See Figure 12 for an example of the several consequences that a new link among a seller of type gG^S and a buyer of type gG^B can cause on the nodes of subgraphs G_1 and G_2.

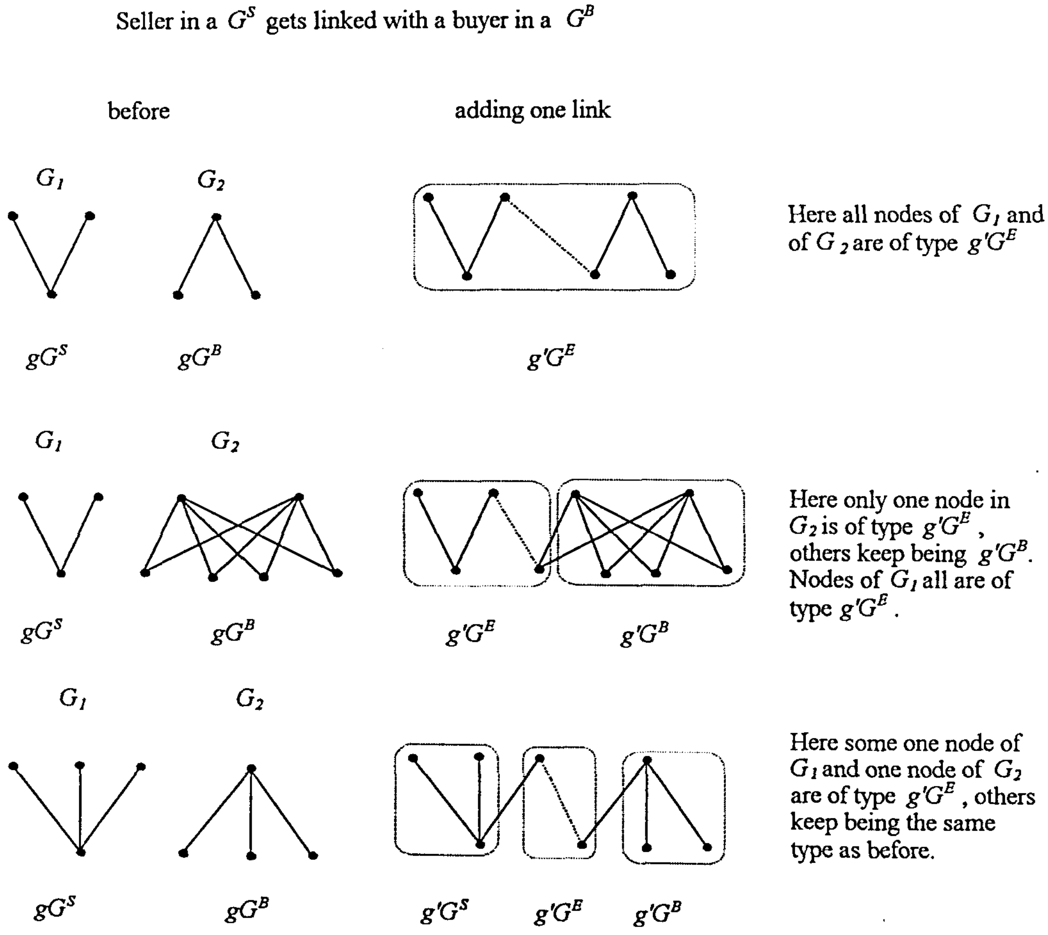


Figure 12

See Figure 13 for the effect a new link among a seller of type gG^S and a buyer of type gG^E may cause on the nodes of subgraphs G_1 and G_2.

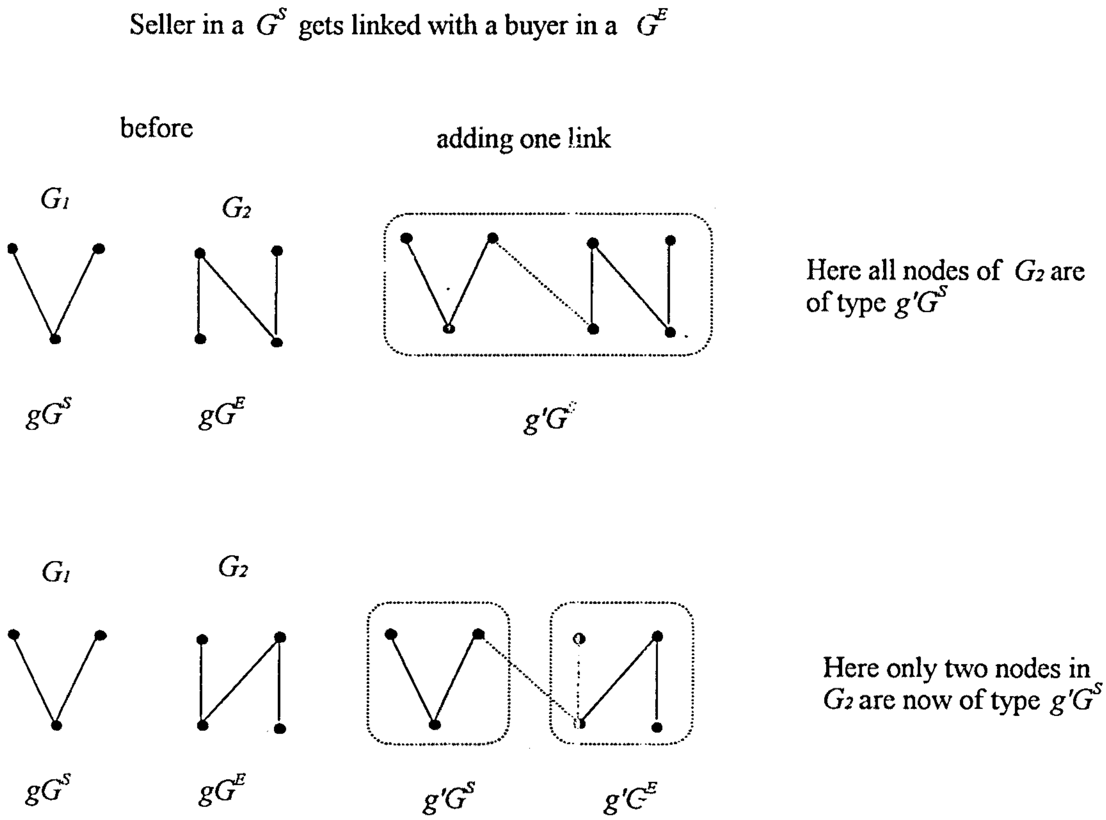


Figure 13

Corollary 1 below is a direct implication of the results in propositions 2, 3 and 4. This corollary states that the effect of a new link among s and b can only change the type of the agents that belonged to the same subgraph as s or b in the $g-$decomposition, but not to others. In a way this says that the creation of a new link has a "local" effect. The changes in the graphs can not be too fast.

Corollary 1 *A new link among seller s and buyer b only may affect agents belonging to the two subgraphs s and b belonged to by the g-decomposition.*

Proof. Included in propositions 2, 3 and 4. ■

2.7 Appendix

We devote the appendix to the proof of propositions 2, 3 and 4.

2.7.1 Proof of proposition 2

We first start by showing a lemma that applies in all cases. Call G^{SS} the subgraph induced by all nodes that are of type gG^S. Similarly, call G^{BB} the subgraph induced by all nodes that are of type gG^B.

Lemma 6 *No node of G^{SS} is of type $g'G^B$. (Similarly, no node of G^{BB} is of type $g'G^S$).*

Proof. Recall that a1) any buyer of type gG^B is linked to only sellers of type gG^B, and that a2) any set of sellers of type gG^B is strictly non-deficient in its subgraph (similarly, b1) any seller of type gG^S is linked to only buyers of type gG^S, and b2) any set of buyers of type gG^S is strictly non-deficient in its subgraph).

Suppose to the contrary that there exists one buyer in G^{SS}, call him b_1, which is of type $g'G^B$. Since b_1 belongs to G^{SS}, it is of type gG^S. By b2), there exist two sellers, call them s_1 and s_2, such that $N_G(b_1) \supseteq \{s_1, s_2\}$ with both s_1 and s_2 being of type gG^S. By a1), both s_1 and s_2 are of type $g'G^B$.

Now, by a2) these two sellers must be linked at least to three buyers (one of them is b_1), call them b_1,b_2,b_3, i.e. $N_G(\{s_1, s_2\}) \supseteq \{b_1, b_2, b_3\}$, all of them of type $g'G^B$. Recall that in g', sellers in G^{SS} are linked only to buyers that are of type gG^S and one of them, seller s, is linked to buyer b. This implies by b1) that at least two of the buyers in $\{b_1, b_2, b_3\}$, relabel them as $\{b_2, b_3\}$, must be of type gG^S.

We can iterate on the above reasoning, and eventually we will see that if a buyer in G^{SS} is of type $g'G^B$, this implies that a number of $b_1, b_2, ..., b_t$ buyers are of type $g'G^B$, with t greater than the number of buyers in g, which is a contradiction.

Alternatively, suppose that there exists one seller in G^{SS}, call him s_1, which is of type $g'G^B$. By a2) seller s_1 is linked to two buyers, b_1 and b_2, i.e. $N_G(s_1) \supset \{b_1, b_2\}$ with b_1, b_2 of type $g'G^B$. We also know by b1) that seller s_1, which by belonging to G^{SS} is of type gG^S, is linked to only buyers of type gG^S and maybe to buyer b. In any case, there exists one buyer among $\{b_1, b_2\}$, relabel him as b_2, which is of type gG^S and also of type $g'G^B$. But this is a contradiction since we showed before that no buyer of type gG^S can be of type $g'G^B$. ■

Now we show each of the cases separately.

CASE a: b belongs to a subgraph of type G^S*)*

First step: nodes in G^{SS} *are of type* $g'G^S$.

Using the lemma, we just have to show that they are not of type $g'G^E$.

Suppose that some of the nodes in G^{SS} are of type $g'G^E$. Denote by B^E_{SS} the buyers in G^{SS} that are of type $g'G^E$. By b2), it is the case that $N_{G^{SS}}\left(B^E_{SS}\right) > \left|B^E_{SS}\right|$. But recall that buyers in a $g'G^E$ are linked either only to sellers of type $g'G^E$ or to sellers of type $g'G^B$. By the above lemma, sellers in $N_{G^{SS}}\left(B^E_{SS}\right)$ must be of type $g'G^E$. This implies that sellers in $N_{G^{SS}}\left(B^E_{SS}\right)$ must be collectively linked to buyers (of type $g'G^E$) which belong to $g' \setminus G^{SS}$.

Now, recall that sellers in G^{SS} are by definition only linked to buyers inside G^{SS}. When going from graph g to graph g', we only add a link among seller s (belonging to G^{SS}) and buyer b (also belonging to G^{SS}), so again in g' sellers in G^{SS} are only linked to buyers inside G^{SS}. Therefore we reach a contradiction.

Now, suppose that some sellers in G^{SS} are of type $g'G^E$. Since sellers (in g') in G^{SS} are only linked to buyers inside G^{SS}, it has to be the case that there exist some buyers in G^{SS} of type $g'G^E$. This is a contradiction as shown above.

Second step: All nodes in g *keep being of the same type in the* g'*-decomposition.*

We previously showed that all nodes in G^{SS} keep being of the same type. By the lemma, it is clear that all nodes in G^{BB} also keep being of the same type. This implies in turn that any node of type gG^E will now be of type $g'G^E$ as well.

CASE b: b belongs to a subgraph of type G^E*)*

First step: Both s *and* b *are of type* $g'G^S$*. Members of* G_1 are of type $g'G^S$, and members of $G_2 \setminus b$ *are either of type* $g'G^E$ *or of type* $g'G^S$*.*

Call G_1^S the subgraph, of type G^S, that seller s belongs to by the decomposition after running the algorithm on graph g. Similarly, call G_2^E the subgraph, of type G^E, that buyer b belongs to by the decomposition of graph g. Similarly, call S_1^S, B_1^S to the sellers and buyers in G_1^S and similarly, S_2^E and B_2^E to sellers and buyers in G_2^E. Finally, call $\overline{G}$ to the subgraph induced by agents $S_1^S \cup B_1^S \cup S_2^E \cup B_2^E$ in g'.

Subgraph G_1^S is either recursive or non-recursive in the g-decomposition. By *b)* in lemma 5 lower associated subgraphs are of type $g'G^S$.

Now, take the set of sellers $S_1^S \setminus s$ and see whether they are collectively linked in G_1^S to a set of strictly less agents or not. Since there are more sellers than buyers in G_1^S, it must be the case that $\left|N_{G_1^S}(S_1^S \setminus s)\right| \leq \left|S_1^S \setminus s\right|$.

(subcase b.1) $\left|N_{G_1^S}(S_1^S \setminus s)\right| < \left|S_1^S \setminus s\right|$. Note that the subgraph induced in g' by sellers $S_1^S \setminus s$ and buyers $N_{G_1^S}(S_1^S \setminus s)$ has more sellers than buyers and is such that the set of all its sellers is almost non-deficient (given that G_1^S is of type gG^S). Moreover, sellers in $S_1^S \setminus s$ may be linked in g only to buyers in $N_{G_1^S}(S_1^S \setminus s)$ (this will happen if G_1^S is non-recursive) or they are linked to buyers that belonged to a lower associated subgraph by the g-decomposition. By *b)* in lemma 5, this implies that sellers in $S_1^S \setminus s$ are either linked to buyers in $N_{G_1^S}(S_1^S \setminus s)$ or to buyers of type $g'G^S$. This implies that sellers $S_1^S \setminus s$ and buyers $N_{G_1^S}(S_1^S \setminus s)$ constitute a subgraph of type $g'G^S$.

Now, take subgraph G_2^E. Relabel buyer b which belongs to G_2^E as b_1. There exists a matching involving all agents in G_2^E, relabel sellers and buyers in G_2^E as $s_1, s_2, ..., s_t, b_1, b_2, ..., b_t$ where s_i and b_i are linked by the matching. Now, take the subsets of sellers $S_{2i}^E = \{s_1, .., s_i\}$ for $i = 1, 2, ..., t$ and consider the set of sellers $S_{2i}^E \cup s$. We know that $N_{G_2^E \cup s}\left(S_{2t}^E \cup s\right) = \{b_1, b_2, ..., b_t\}$. Then, choose the smallest i for which $N_{G_2^E \cup s}\left(S_{2i}^E \cup s\right) = \{b_1, b_2, ..., b_i\}$. Then, the subgraph induced by sellers $S_{2i}^E \cup s$ and buyers $\{b_1, b_2, ..., b_i\}$ will be a subgraph of type $g'G^S$. To see why, note that it has more sellers than buyers and the set of its sellers is almost non-deficient by construction. It remains to be shown that its sellers are linked either only to buyers $\{b_1, b_2, ..., b_i\}$ or to buyers that we know are of type $g'G^S$. Note that s may only be linked, outside $b_1, b_2, ..., b_i$, to buyers of type $g'G^S$. Also note that sellers in G_2^E could only be linked, outside from buyers in G_2^E, to buyers of type gG^S. Given that buyers in G_1^S have been shown to be of type $g'G^S$, we can assert that sellers in G_2^E may only be linked to buyers of type $g'G^S$.

We now have to show that the remaining agents in G_2^E are of type $g'G^E$. Note that $\{b_i, b_{i+1}, ..., b_t\}$ and $S_2^E \setminus S_{2i}^E \cup s$ have the same number of elements, and that a perfect matching exists by construction. Moreover, buyers in G_2^E are only linked, apart from $S_2^E \setminus S_{2i}^E \cup s$, to sellers of type gG^B. By *a)* in lemma 5 sellers of type gG^B are also of type $g'G^B$. Sellers in G_2^E are only linked, apart from $\{b_i, b_{i+1}, ..., b_t\}$, to buyers of type gG^S. Since all buyers in G_1^S have been shown to be of type $g'G^S$, and also buyers belonging to $g' \setminus \left\{G_1^S \cup G_2^E\right\}$ are of type $g'G^S$, we conclude that the remaining agents in G_2^E are of type $g'G^E$.

(subcase b.2) Otherwise it will be the case that $\left|N_{G_1^S}(S_1^S \setminus s)\right| = \left|S_1^S \setminus s\right|$. With the same notation as in the above paragraph, find the smallest i for which $N_{G_2^E \cup s}\left(S_{2i}^E \cup s\right) = \{b_1, b_2, ..., b_i\}$. Then, the subgraph induced in g' by sellers $S_1^S \cup S_{2i}^E$ and buyers $B_1^S \cup \{b_1, b_2, ..., b_i\}$ is of type $g'G^S$. Again, it has more sellers than buyers, the set of its sellers is almost non-deficient by construction, and sellers in $S_1^S \cup S_{2i}^E$ may only be linked to $B_1^S \cup \{b_1, b_2, ..., b_i\}$ or to buyers that we know are of type $g'G^S$. Similarly as before, it can be shown that the rest of agents in G_2^E are of type $g'G^E$.

Second step: All members of $G \setminus \{G_1 \cup G_2\}$ belong to the same type either by the G-decomposition or by the g'-decomposition.

By the lemma any node that was of type gG^B is of type $g'G^B$. Let's look at nodes in $G \setminus \{G_1 \cup G_2\}$ that were of type gG^S. If non recursive, by *b)* in lemma 5 they are of type $g'G^S$. If non recursive with lower subgraphs in $G \setminus \{G_1 \cup G_2\}$, then they also are of type $g'G^S$. If recursive with a lower subgraph in $\{G_1 \cup G_2\}$, it means that these nodes belonged to a G^S by the g-decomposition that had as a lower associated subgraph G_1^S. Since all buyers in G_1^S keep being of type $g'G^S$, again these nodes are of type $g'G^S$.

Finally, let's look at nodes in $G \setminus \{G_1 \cup G_2\}$ that were of type gG^E. The only case that needs to be checked is those nodes of type gG^E that belonged to a subgraph by the g-decomposition such that its sellers were linked outside its subgraph to buyers in G_1^S. Again, since those buyers are of type $g'G^S$, the result holds.

CASE c: b belongs to a subgraph of type G^B)

Call G_1^S the subgraph, of type G^S, that seller s belongs to by the decomposition after running the algorithm on graph G. Similarly, call G_2^B the subgraph, of type G^B, that buyer b belongs to by the G-decomposition.

First step: Both s and b are of type $g'G^E$.

By the lemma s can only be of type $g'G^E$ or $g'G^S$. Similarly, b can only be of type $g'G^E$ or $g'G^B$. Now, suppose that s is of type $g'G^S$. Then, given that s is linked to b in graph g', we have that b must also be of type $g'G^S$ (see remark b1) above), which is a contradiction.

Now, define $\overline{G}_1^S$ as the subgraph that includes subgraph G_1^S and all its directly upper associates subgraphs of type G^S. Similarly, define $\overline{G}_2^B$ as the subgraph that includes subgraph G_2^B and all its directly upper associates subgraphs of type G^B.

Second step: Members of $\overline{G}_1^S \setminus G_1^S$ are of type $g'G^S$, members of $\overline{G}_2^B \setminus G_2^B$ are of type $g'G^B$.

Take the members of $\overline{G}_1^S \setminus G_1^S$. Suppose that $b_1 \in \overline{G}_1^S \setminus G_1^S$ is of type $g'G^E$. We know that b_1 is of type gG^S. Then, by b2) there exist two sellers s_1 and s_2 with $N_G(b_1) \supseteq \{s_1, s_2\}$ with both s_1 and s_2 of gG^S type (and therefore belonging to G^{SS}). Now, note that buyers of type $g'G^E$ can be linked either to sellers of type $g'G^E$ or to sellers of type $g'G^B$. By the above lemma, it must be the case that both s_1 and s_2 are of type $g'G^E$. By b1) all the buyers collectively linked to s_1 and s_2 are of type gG^S. Since any set of sellers of type $g'G^E$ is non-deficient in its subgraph, there exist two buyers b_1 and b_2 such that $N_G(\{s_1, s_2\}) \supseteq \{b_1, b_2\}$ with b_1, b_2 of type $g'G^E$.

Now, similarly as in the lemma above, iterating on this procedure we will reach a contradiction.

Similarly we can show that members of $\overline{G}_2^B \setminus G_2^B$ are of type $g'G^B$.

Third step: Members of $g \setminus G_1^S$ and members of $g \setminus G_2^B$ belong to the same type of subgraphs by the two decompositions.

We already saw that nodes belonging to $G^{SS} \setminus G_1^S$ keep being of type $g'G^S$, and similarly nodes belonging to $G^{BB} \setminus G_1^B$ keep being of type $g'G^B$.

It remains to show that nodes that were of type gG^E keep on being of type $g'G^E$. The only case that needs to be considered is those nodes of type gG^E that belonged to a subgraph in the g-decomposition, call it G_2^E such that sellers in G_2^E were linked to buyers in G_1^S and/or buyers in G_2^E were linked to sellers in G_2^B.

Recall that buyers in G_1^S are of type $g'G^S$ or of type $g'G^E$ and similarly sellers in G_2^B are of type $g'G^B$ or of type $g'G^E$. If some sellers in G_2^E are linked to buyers in G_1^S of type $g'G^S$ or some buyers in G_2^E are linked to sellers in G_2^B of type $g'G^B$, then it is immediate that nodes in G_2^E are of type $g'G^E$. If alternatively some sellers in G_2^E are linked to buyers in G_1^S of type $g'G^E$ or some buyers in G_2^E are linked to sellers in G_2^B of type $g'G^E$, then it must be the case that in the g'-decomposition, all these nodes appear together in the same $g'G^E$.

Fourth step: characterization of members of $G_1 \setminus s$ and $G_2 \setminus b$ depending on the size of G_1 and G_2.

Here we use a result previously shown, which is that members of $G_1 \setminus s$ are of type $g'G^S$ or $g'G^E$ and members of $G_2 \setminus b$ are of type $g'G^B$ or $g'G^E$.

ca) Note that there exists a matching involving m_1^S pairs in G_1. More precisely, any set of m_1^S sellers can be matched with the m_1^S buyers. Then, in particular, we can take out s and construct a matching with the remaining m_1^S sellers and m_1^S buyers in G_1. Call this matching M_1. Do the symmetric construction with G_2, i.e., find n_2^B sellers and n_2^B buyers in $G_2 \setminus b$ in such a way that they constitute a matching M_2. Then, we can construct a matching among the $n_1^S + n_2^B$ sellers and the $m_1^S + m_2^B$ buyers in $G_1 \cup G_2$ which corresponds to $M_1 \cup \{s, b\} \cup M_2$. (recall that a matching is a set of pairs). Therefore, members of $G_1 \cup G_2$ are of type $g'G^E$.

cb) Clearly there exists a matching in $G_1 \cup b$ involving the n_1^S sellers and n_1^S buyers. Moreover, note that the set $G_2 \setminus b$ has a number of buyers equal to $m_2^B - 1$ which is strictly greater than n_2^B, the number of sellers in $G_2 \setminus b$. Moreover, the set of all buyers in $m_2^B - 1$ is almost non deficient in $G_2 \setminus b$. This implies that members of $G_1 \cup b$ are of type $g'G^E$ while member of $G_2 \setminus b$ are of type $g'G^B$.

cc) Symmetric to cb)

cd) There are strictly more sellers than buyers in $G_1 \setminus s$ and the set of all its sellers is almost non deficient. Similarly, there are strictly more buyers than sellers in $G_2 \setminus b$ and the set of all its buyers is almost non deficient. This implies that members of $G_1 \setminus s$ are of type $g'G^S$, members of $G_2 \setminus b$ are of type $g'G^B$ and $s \cup b$ are of type G^E. ■

2.7.2 Proof of proposition 3

a) Call G_1^E the subgraph of the G^E type that s belongs to by the G-decomposition, and G_2^S the subgraph of the G^S type that b belongs to by the G-decomposition.

Note that all nodes of type gG^B are now $g'G^B$, by *a)* in lemma 5. Also by *b)* in lemma 5 we know that any node that belonged to a G^S of a lower level than G_2^S in the g-decomposition is now also of type $g'G^S$. Therefore, note that subgraph G_2^S keeps being of type $g'G^S$. This is true since G_2^S is itself a G^S graph and since sellers in G_2^S are collectively linked either only to buyers in G_2^S or to buyers that belong to a subgraph that is a directly lower associate of G_2^S by the g-decomposition (and, therefore, subgraphs that are of type $g'G^S$).

Now, consider a node of type gG^S which belongs to $g \setminus G_2^S$. If it belonged by the g-decomposition to a subgraph of level zero, or of a lower level than

G_2^S, then clearly these nodes are $g'G^S$. If, alternatively, they belonged to an upper level, then given that G_2^S is itself of type $g'G^S$, then its nodes are also of type $g'G^S$.

This implies that all subgraphs of type gG^E (including G_1^E) are also of type $g'G^E$.

b) Call G_1^E the subgraph of the G^E type that s belongs to by the g-decomposition, and G_2^E the subgraph of the G^E type that b belongs to by the g-decomposition. Call $\overline{G}$ the subgraph induced in graph g' by the nodes in the subgraph $G_1^E \cup G_2^E$. We are now going to show that subgraph $\overline{G}$ is of type $g'G^E$.

To see why this happens, think that clearly $\overline{G}$ is such that a matching exists among all its members (since a matching with all members of G_1^E and a matching with all members of G_2^E both exist). Moreover, by *a)* in the lemma we know that all subgraphs that were of types gG^S or gG^B also of the same type by the g'-decomposition. Therefore, sellers (buyers) in $\overline{G}$ are linked either only to buyers (sellers) in $\overline{G}$ or also to buyers (sellers) belonging to subgraphs of type $g'G^S$ ($g'G^B$). That is, the subgraph $\overline{G}$ is of type $g'G^E$.

c) Symmetric to *b)* in previous proposition, with the roles of buyers and sellers reversed. ■

2.7.3 Proof of proposition 4

If b is of type gG^E, the proof is symmetric to proposition 3a). If b is of type gG^B, the proof is symmetric to proposition 2a).

Now, suppose that b is of type gG^S. Call G_1^B the subgraph of the G^B type that s belongs to by the g-decomposition, and G_2^S the subgraph of the G^S type that b belongs to by the g-decomposition. By *b)* in lemma 5, all nodes that belong to a subgraph G^B of smaller level than G_1^B keep being of the same type in the g'-decomposition, and nodes that belong to a subgraph G^S of smaller level than G_2^S keep being of the same type in the g'-decomposition. Then, it is immediate that G_1^B keeps being of type $g'G^B$ and that G_2^S keeps being of type $g'G^S$. This implies in turn that any node of type gG^E is of type $g'G^E$. ■

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Chapter 3

Bargaining in a Network of Buyers and Sellers.

3.1 Introduction

Network structures appear in many social and economic relationships. Commonly referred literature documenting the importance of networks contains Boorman (75) relating it to the internal organization of firms, Wellman and Berkowitz (88) for social networks, Goyal (93) for information transmission and Hendricks et al.(95) for the structure of airline routes influencing competition (see Jackson and Wolinsky (96) and Jackson and Watts (98) for a rich survey of related literature).

In particular, much of the communication that is important in economic and social contexts does not take place via centralized institutions but rather through networks of decentralized bilateral relationships. Specifically, we are interested in two-sided markets which are organized through a network that communicates potential traders. Agents put effort in building the connections that form the network since, once the network is fixed, the position of each agent in the network will be crucial in determining the payoff. It is not clear in these sort of markets what it means to be "well-connected", since it is not only the number of connections that an agent has what matters but also how "good" these connections are. The position of an agent in the network determines his bargaining power relative to others. It is intuitive to think that the whole context matters, and that connections are better if they link you with a poorly connected partner that would be very eager to reach agreement.

With this motivation in mind we construct a model in which sellers and buyers are connected through a fixed network. Sellers each own an indivisible good, and buyers own money. A trade for price p gives a payoff of $\delta^t p$ to the seller and $\delta^t(1-p)$ to the buyer, and agents trade at most once. A buyer and a seller can only possibly trade if there is a connection that links the two agents. Sellers post prices and each responder can accept one among the posted prices to which he has access. Some pairs trade and leave the market, while the rest keep bargaining with alternating offers. The game is played repeatedly among the players that did not trade in previous periods. We wish to analyze results for a market imbedded in a fixed network where players have the possibility of trading until the market clears.

In our analysis all sellers have the same utility function and all buyers are also alike, since we want the differences in prices to be driven only by the network structure. This is precisely our purpose: finding the conditions on the network that will determine which are the realized prices in equilibrium. The results turn out to depend on the properties of the graph induced by the network. More precisely, we extensively use a modified version of the concept of a set of nodes being non-deficient. The concept of non-deficiency appears in the necessary and sufficient conditions for a perfect matching to exist in a bipartite graph. This result is crucial in the matching literature in graph theory and is known as Hall's theorem (see ch.2). Thus we combine tools of graph theory and game theory in order to solve our model.

This paper has common features with two streams of literature. One is the literature on markets with decentralized trade. Trying to microfound competitive equilibria without an auctioneer, much of this literature constructs markets in which the agents get matched randomly and conduct a bilateral bargaining process (see Osborne and Rubinstein (90) for an overview of the existing literature). We depart from these models by examining environments in which the communication structure is rigid (the matching is not random) and the bargaining process is not bilateral in the sense that it is conducted among all players collectively linked through the network.

There is a growing literature on networks, much developed in recent years, mainly relating to endogenous link formation. Questions like how networks form among agents, and which networks fulfill stability or efficiency properties have been addressed by Jackson and Wolinsky (96) or Dutta, van den Nouweland and Tijs (98), mainly from a cooperative point of view. Kranton and Minehart (98) study a two-sided market using a two stage game that is completely noncooperative, in which in the second stage the partition is decided using an ascending bid auction. In a more recent paper, Kranton and Minehart (99) compare three alternative supply structures: vertical integration, networks, and markets. Also related to industrial organization issues, Goyal and Joshi (99) is a contribution to the study of group formation and cooperation in oligopolies. Finally, the papers of Jackson and Watts (98) and Bala and Goyal (98) address the dynamics of non-cooperative network formation.

Calvó-Armengol (98) combines bargaining and network techniques, in a non-cooperative setup, with the focus on partner selection. Less related to our work, there are other papers which combine bargaining and two-sided markets using mainly cooperative game theory: they focus on general assignment games (for a discussion on assignment games and their relation to our model, see ch. 3). Among them we find Crawford and Rochford (86), Rochford (84) and Bennett (88).

The chapter proceeds as follows. Section 3.2 defines the model and introduces the notation. Section 3.3 deals with small markets. Section 3.4 defines what we will call the reference solution, which we will compare to the equilibria of our game. Section 3.5 and 3.6 explain how to implement the reference solution. Section 3.7 deals with the multiplicity of equilibria.

Section 3.8 discusses some of the modelling choices. Finally, the last section concludes and suggests future research.

3.2 The model

The market consists of n sellers $s_1, s_2, ..., s_n$ and m buyers $b_1, b_2, ..., b_m$. Each seller owns an indivisible good and each buyer owns money. If a seller and a buyer trade at price p at period t, the seller receives a utility of $\delta^t p$ and the buyer receives $\delta^t(1-p)$.

Agents are imbedded in a network that links sellers and buyers. A link is interpreted as meaning that trade is possible among linked agents.

Formally, it will be useful to represent the network as a bipartite graph $G = \langle S \cup B, L \rangle$ which will consist of a set of *nodes*, formed by sellers $S = \{s_1, ..., s_n\}$ and buyers $B = \{b_1, ..., b_m\}$, and a set of *links* L, each link joining a seller with a buyer. The graph theory notation used subsequently has been defined in the previous chapter.

3.2.1 Description of the game

The game is now described informally. There is a bargaining process in which players alternate offers and responses in the following way: all proposers simultaneously announce a price at which they are willing to trade the good, then responders accept or reject a price simultaneously. Accepting a price means that buyers are willing to trade the good at that price, and they do not care with which specific seller they trade (that is, proposers accept only prices, rather than prices and proposers). See section 3.8 for a discussion on this issue and on other modeling choices.

If the prices proposed are different and responders accept different prices, the pairs that form are immediately determined. But note that we must be explicit as to what happens when t responders accept an identical price proposed by a number t' of their linked sellers. We endow sellers and buyers with a subindex for the duration of the game; this will be used as a priority ranking in order to resolve ties. A mechanism (formally described later) will then decide which specific agents receive the good in case of coincidence of responses or proposals. This mechanism will be such that it is always guaranteed that the maximum possible number of pairs trade.

After some agents trade, they leave the market. Remaining agents continue making alternating offers among sellers and buyers, always preserving the network. The game is repeated until either all agents trade or have no remaining connections.

We now formally describe the game for a given graph G :

Call s-game (b-game), the subgames that start when sellers (buyers) propose and buyers (sellers) respond. They propose simultaneously and they respond also simultaneously.

- *period* t (starting from $t = 1, t = 2$, to infinity):

If t is odd, we are in a s-game, if t is even, we are in a b-game. In the first period ($t = 1$) the market is denoted by $G^t = G^1 = G$.

All proposers in G^t simultaneously propose a price $\in [0, 1]$. Next, each responder in G^t can either accept one of the prices proposed by his linked agents or reject all. If some responders accept, then pairs form and trade according to the mechanism below.

Say that there are k^t prices accepted, relabel them as $p^1, ..., p^{k^t}$, so that, for each p^i with $i = 1, .., k^t$, a number l_{p^i} of responders accept the same price p^i proposed by a number l'_{p^i} of proposers. Define the subgraph $G^t_{p^i}$ as the subgraph in G^t induced by the set of l_{p^i} responders and l'_{p^i} proposers. Call $M^t_{p^i}$ the maximum number of possible pairs that can form in the subgraph $G^t_{p^i}$. The mechanism will guarantee that $M^t_{p^i}$ pairs trade in each subgraph. There may be several possible matchings with $M^t_{p^i}$ pairs. To choose which agents will match in equilibrium, the mechanism uses the subindex of the agents. The mechanism will select the matching involving the responder with the smallest possible subindex. If several matchings fulfill this condition, then it will select the one involving the responder with the second smallest index, and so on. If this still does not uniquely determine the matching, then it selects the one involving the responder with the second smallest index, and so on. If this still does not determine the matching, then it chooses the one that involves the proposer with smallest index and so on. Once all the agents involved in the matching are chosen, it still may happen that the actual matching is not determined. If this is the case, the mechanism chooses the matching randomly (this only decides the specific agent with whom you are going to trade; the payoff is already determined once the agents are chosen). Then in each subgraph $G_{p^i}^t$ a number of $M_{p^i}^t$ sellers and buyers trade and leave the market, and the others stay in the market.

Call $G_{p^i}^{tT}$ the subgraph of $G_{p^i}^t$ formed by the $M_{p^i}^t$ sellers and buyers that trade and leave the market.

• *period t+1*: Now we take G^t and we remove all the subgraphs $G_{p^1}^{tT}, ..., G_{p^{k^t}}^{tT}$ from G^t. We are left with a set of several connected subgraphs. That is, $\left(G^t - \left\{G_{p^1}^{tT}, ..., G_{p^{k^t}}^{tT}\right\}\right) = \bigcup_{j=1}^{m_{t+1}} G_j^{t+1}$ with G_j^{t+1} being connected. If some of these subgraphs G_j^{t+1} consist of only one node, then these automatically get zero and leave the market. With the rest of subgraphs we repeat the game exactly as before.

The game repeats itself until the market clears or there is no remaining link among the agents.

The recursive structure of the game implies that in order to solve the subgame perfect equilibrium payoffs (denoted by PEP) of a graph G, we have to solve for the PEP of any subgraph G_0 of G. Note that if G consists of only one connected pair, then our game coincides exactly with the 2-players alternating offers bargaining game in Rubinstein (82).

3.3 Results for small markets (n≤2, m≤2)

We start by analyzing the small markets with less than 2 players in each side. One possible type of networks would have only one player on one side and two players on the other one. The other possibilities have two agents on each side. We analyze each case separately.

3.3.1 Case $n = 2, m = 1$ and $n = 1, m = 2$

Here we have a market with one of the sides shorter than the other. Figure 1 displays the two possible cases. In all figures sellers will be drawn as the nodes in the upper part of the graph, while buyers will be in the bottom part.

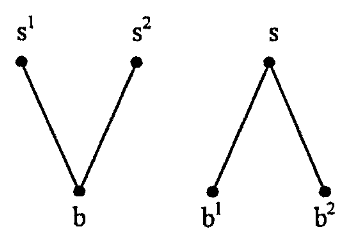

Figure 1

We will now show that we have a unique subgame-perfect equilibrium payoff (henceforth PEP) in both cases. Not surprisingly, the competitive result holds: the short side of the market gets all the surplus. Note for instance that when $n = 2$ and $m = 1$, both sellers compete to trade with the same buyer, this competition drives the price down to zero, and the buyer gets all the surplus. Similarly, if $n = 1, m = 2$ the only PEP gives the seller a payoff of 1.

These results are in the same spirit as those found for similar games in Binmore (1985) or Wilson (1984) (see Osborne and Rubinstein (1990), section 9.3, for a review of models with public price announcements with only 3 agents).

Proposition 5 *In the case $n = 2, m = 1$ or $n = 1, m = 2$ there exists a unique PEP. In this PEP the short side gets 1 and the long side gets 0.*

Proof. $\rightarrow$ Case $n = 2, m = 1$:

Existence : We can support the PEP with the following strategies

• s_i (when proposing) propose always $p = 0$, (when responding) accept any price.

• b (when proposing) proposes always $p = 0$, (when responding) accepts the minimum of the two offered prices if it is smaller or equal than $p = 1 - \delta$.

One can verify that this constitutes a PEP.

Uniqueness : It is immediate that at least one of the sellers will get a payoff of zero, since a positive payoff can only be reached by trading. Suppose that s_1 gets a payoff of zero. Then, in the first period (in which sellers propose) s_1 would do anything to get an agreement. The deviation in which s_1 proposes a price of ε should not be profitable. Note that a price of ε gives the buyer a payoff of $1 - \varepsilon$, which for a sufficiently small ε will be strictly better than δ, and δ is the maximum one can get in the second period. Therefore, this is a very good offer for the buyer. The only reason why the buyer would not accept a price of ε is if the other seller, s_2 would already be proposing a price of zero that the buyer could accept. Therefore in any PEP sellers will both propose a price of zero. This implies that in equilibrium both sellers get a payoff of 0 and the buyer gets a payoff of 1

$\rightarrow$ Case $n = 1, m = 2$:

Existence : We can support the PEP with the following strategies

• s (when proposing) always proposes $p = 1$, (when responding) accepts the maximum of the two offered prices provided it is greater than or equal to δ.

• b_i (when proposing) propose always $p = 1$, (when responding) accept any price.

Uniqueness : As before, one of the buyers must get a payoff of zero. Now, suppose that one of the buyers gets a positive payoff. If trade occurs for a positive price in a $b - game$, then the buyer that gets a payoff of zero would undercut the offer by posting a price of $1 - \varepsilon$. As before, that would constitute a profitable deviation. If trade occurs in a $s - game$, then suppose that the seller deviates and offers a slightly higher price than the one proposed in equilibrium. If both buyers reject, they enter a b-game with 2 buyers and one seller. That would be symmetric to the case studied above ($n = 2, m = 1$) and therefore we know that both buyers would get zero. Therefore, at least one buyer will accept and the deviation will be profitable. ■

3.3.2 Case $n = m = 2$

Now we analyze the case in which there are two pairs of agents in the market. Here there are the two possible connected networks, denoted by graph G_1 and graph G_2, as shown in Figure 2.

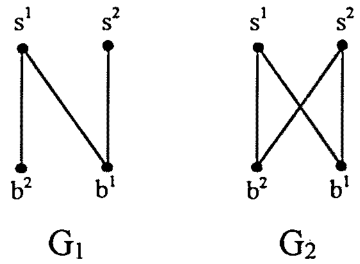

Figure 2

G_1 is the network in which one seller (s^1) is linked to all buyers and the other one (s^2) is linked to only one buyer (b^1). The other network, G_2, is the one denoted by the complete graph, in which all sellers are linked to all buyers.

First case: G_1

A first look to the network tells us that s_1 is in a better situation since he has access to both buyers, while s^2 has only access to one. Since he risks not being able to get anything at all, s^2 is competing with s^1 to be matched with b^1. Similarly, b^2 is competing with b^1 to get matched with s^1. Then one could think that maybe two different prices would coexist in equilibria: s^1 could maybe get a better deal than s^2. As we will shortly see, though, both sellers trade at the same price and no seller benefits from the situation[1].

[1] The fact that agents have the same utility function is of course crucial for this result to hold. Indeed, the situation has somewhat the flavour of an outside opportunity that is not better than the already existing partner.

The intuition behind the result is the following. It will never happen that s^1 trades with b^1, since if that would be the case, s^2 would get a payoff of 0 and would therefore have incentives to deviate and undercut the price of s^1. Knowing that such a pair will never trade in equilibrium, the situation is as if two separate pairs would be playing, which implies that all sellers will trade at the same price. Besides, the price corresponds to $z = \frac{1}{1+\delta}$, which is the price that a pair of players playing the usual alternating offers bargaining game would realize in equilibrium.

Proposition 6 *There exists a unique PEP in G_1. In this PEP all sellers get a payoff of z and all buyers get a payoff of $1 - z$.*

Proof. Note that strategies must specify which is the distribution of proposed prices and which are the responses of the agents not only when the market is given by the graph G_1, but also when the market is given by any subgraph G_{11} that results from G_1 when a pair trades and leaves the market. In this case two possibilities can happen: either one pair trades and two agents get isolated or one pair trades and the remaining two agents are still connected and can keep playing. The strategies followed in any of these subgraphs are simple. If agents get isolated, they automatically get zero and have no actions to chose. If a pair remains in the market, then the strategies are as in the 2-players alternating offers bargaining game. We now formally write the strategies. Call p_{b^i}, p_{s^i} the offered prices by each agent.

Existence: The strategies that would support such a PEP would be:

$\rightarrow$ If all agents are still in the initial graph G :

Proposals: both s^1 and s^2 propose $p_{s^1} = p_{s^2} = z$, both b^1 and b^2 propose $p_{b^1} = p_{b^2} = \delta z$

Acceptances: s^2 accepts p_{b^1} if $p_{b^1} \geq \delta z$, b^2 accepts p_{s^1} if $p_{s^1} \leq z$

The strategies followed by s^1 and b^1 when responding depend on the subindex they have (recall that the subindex is used only when ties have to be solved, see section 3.2.1).

- About seller s^1 :

case a) $s^1 = s_1^1$, $s^2 = s_2^2$ (case in which the priority of s^1 is higher than that of s^2, so that in case of ties among them s^1 will have preference over s^2).

s^1 accepts the maximum of the offered prices provided $max_i \{p_{b^i}\} \geq \delta z$.

case b) $s^1 = s_2^1$, $s^2 = s_1^2$ (case in which the priority of s^1 is smaller than that of s^2, so that in case of ties among them s^2 will have preference over s^1)

case b1) if $p_{b^1} \leq \delta z$, then s^1 accepts the maximum of the offered prices provided $max_i \{p_{b^i}\} \geq \delta z$

case b2) if $p_{b^1} > \delta z$, then if $p_{b^2} < \delta z$, s^1 accepts p_{b^1}, but if $p_{b^2} \geq \delta z$, then s^1 accepts $\min_i \{p_{b^i}\}$

• About buyer b^1 :

case a) $b^1 = b_1^1$, $b^2 = b_2^2$ (case in which the priority of b^1 is higher than that of b^2, so that in case of ties among them b^1 will have preference over b^2).

b^1 accepts the minimum of the offered prices provided $\min_i \{p_{s^i}\} \leq z$.

case b) $b^1 = b_2^1$, $b^2 = b_1^2$ (case in which the priority of b^1 is smaller than that of b^2, so that in case of ties among them b^2 will have preference over b^1).

case b1) if $p_{s^1} \geq z$, b^1 accepts the minimum of the offered prices provided $\min_i \{p_{s^i}\} \leq z$

case b2) if $p_{s^1} < z$, then if $p_{s^2} > z$, b^1 accepts p_{s^1} , but if $p_{s^2} \leq z$,then b^1 accepts $max_i \{p_{s^i}\}$

→ If there is only a pair of agents s^i and b^j (the possibilities can be $s^1 : b^1$ *or* $s^2 : b^1$ *or* $s^1 : b^2$) in the market, then:

Proposals: s^i proposes $p_{s^i} = z$, b^j proposes $p_{b^j} = \delta z$

Acceptances: s^i accepts p_{b^j} if $p_{b^j} \geq \delta z$, b^j accepts p_{s^i} if $p_{s^i} \leq z$

It can be checked that these strategies form a PEP.

Uniqueness: Call M_{s^i}, m_{s^i} the supremum and infimum of PEP for sellers in a $s-game$ (when it is the sellers' turn to propose; respectively M_{b^i}, m_{b^i} for buyers in a $b-game$), when all four agents are still in the market, that is, when the market is imbedded in graph G_1. We will find inequalities in order to show that $M_{s^i} = m_{s^i} = M_{b^i} = m_{b^i} = z$. Note that we already know by existence that $M_{s^i} \geq z$, $M_{b^i} \geq z$ and $m_{s^i} \leq z$, $m_{b^i} \leq z$. We can now show that:

$$m_{s^1} \geq 1 - \delta \max\{z, M_{b^2}\} = 1 - \delta M_{b^2}$$

That is, whenever seller s^1 will offer a price strictly smaller than $1 - \delta \max\{z, M_{b^2}\}$, he is sure to be accepted by buyer b^2. This is so since if b^2 rejects, he may get δz, δM_{b^2}, or even zero, depending on what the other agent does and on their priority. Note that this implies in particular that seller s^1 will never propose a price $p^{s^1} < 1 - \delta \max\{z, M_{b^2}\} = 1 - \delta M_{b^2}$ in equilibrium, since it would be accepted for sure, while a price $p^{s^1} + \varepsilon < 1 - \delta M_{b^2}$ would also be accepted for sure. Knowing this we can show that:

$$m_{s^2} \geq 1 - \delta \max\{z, M_{b^1}, M_{b^2}\} \geq 1 - \delta \max_i \{M_{b^i}\}$$

Suppose to the contrary that in equilibrium seller s^2 gets a payoff less than $1 - \delta \max_i \{M_{b^i}\}$. Then, s^2 could deviate and ask for a price slightly smaller than $1 - \delta \max_i \{M_{b^i}\}$ and he would be sure to be accepted by b^1. This is true since by rejecting, b^1 can get either δz or at most δM_{b^1}, and by trading with seller s^1, buyer b^1 can get at most a payoff of δM_{b^2} since we said above that never in equilibrium seller s^1 will propose a price smaller than $1 - \delta M_{b^2}$.

On the other hand, we can show that:

$$M_{b^2} \leq 1 - \delta \min\{z, m_{s^1}\} \leq 1 - \delta m_{s^1}$$

To see why this happens, note that if b^2 proposes a price of $\delta \min\{z, m_{s^1}\}$, s^1 will not accept since by rejecting s^1 can get δz or δm_{s^1}. Moreover, b^2 cannot get the amount $1 - \delta \min\{z, m_{s^1}\}$ in equilibrium trading with delay in the next period, since the most that b^2 can get in the next period is only $\delta(1 - m_{s^1})$. Therefore the above inequality holds. We can also show that:

$$M_{b^1} \leq 1 - \delta \min\{z, m_{s^1}, m_{s^2}\} \leq 1 - \delta \min_i \{m_{s^i}\}$$

Again, if b^1 offers a price strictly smaller than $\delta \min\{z, m_{s^1}, m_{s^2}\}$, none of the sellers will accept. As above, seller s^1 will not accept. Knowing this, s^2

can get δz or at least δm_{s^2} by rejecting; therefore seller s^2 would not accept this price either. Moreover, b^1 cannot get a payoff as high as $1-\delta \min_i \{m_{s^i}\}$ by trading next period. Note that these inequalities imply:

$$m_{s^i} \geq 1 - \delta \max_i \{M_{b^i}\}\ ,\ M_{b^i} \leq 1 - \delta \min_i \{m_{s^i}\}$$

The inequalities together with $M_{s^i} \geq z$, $M_{b^i} \geq z$ and $m_{s^i} \leq z$, $m_{b^i} \leq z$ imply that $m_{s^i} = M_{b^i} = z$. In a similar way we can show $m_{b^i} = M_{s^i} = z$. ■

Second case: G_2

We now examine the completely symmetric case in which all sellers are linked to all buyers. Not surprisingly, this case yields the same result as before: trade always takes place at the same price, which is again equal to z.[2]

Proposition 7 *There exists a unique PEP. In this PEP all sellers get a payoff of* z *and all buyers get a payoff of* $1 - z$

Proof. We have to specify which are the strategies followed when we are in G_2 and when we are in each possible subgraph that can result from G_2 when a pair trades. The only possibility is that one pair trades and the other one remains in the market, in which case the situation is like the 2-players usual game. We now formally write the strategies.

Existence: The strategies that would support such a PEP would be:

→ If they are in the initial graph G :

Proposals: s^i propose $p_{s^i} = z$, b^i propose $p_{b^i} = \delta z$

Acceptances: The strategies followed when responding depend on the subindex agents have.

• About sellers:

If one of the two proposed prices is smaller than or equal to δz, then sellers accept both the largest price provided it is larger or equal than δz.

[2]See Chaterjee and Dutta (1998), section 3, for a related result.

If both of the proposed prices are greater than δz, then the seller with highest priority accepts the largest one and the seller with lowest priority accepts the other.

• About buyers:

If one of the two proposed prices is greater than or equal to z, then sellers accept both the smallest price provided it is smaller than or equal to z.

If both of the proposed prices are smaller than z, then the buyer with highest priority accepts the smallest one and the buyer with lowest priority accepts the other.

It can be checked that these strategies form a PEP.

Uniqueness: We use the same notation as in the previous proposition. As before, we will be able to show that:

$$m_{s^i} \geq 1 - \delta \max \{z, M_{b^1}, M_{b^2}\}, \; M_{b^i} \leq 1 - \delta \min \{z, m_{s^1}, m_{s^2}\}$$

For the first inequality, note that in G_2 whenever a buyer rejects in a s-game, he is sure to get either δz or at most δM_{b^i}. Then, whenever a seller will propose a price that gives a payoff higher than $\delta \max \{z, M_{b^i}\}$ to the buyer, it will be accepted by one of the buyers. It cannot happen that both buyers are able to accept a better offer. Therefore either only one of them accepts (the other buyer getting δz) or they both reject and get at most δM_{b^i}.

The second inequality holds since any seller would prefer to reject than accepting $\delta \min \{z, m_{s^i}\}$. By rejecting, any player will get at least the payoff $\delta \min \{z, m_{s^i}\}$. These inequalities imply the result. ■

3.4 Competitive solution versus PEP

To explain our results, we choose to compare the equilibria of our game to the competitive equilibrium of a market with the same agents. The market consists of n sellers that each own a unit of an indivisible good and m buyers that each own one unit of money. If a seller trades for price p, he gets utility p, and if a buyer trades for price p, he gets utility $1 - p$. The competitive equilibrium of such a market is easy to calculate. If there are more sellers than buyers, the equilibrium price is 0, so that buyers get all the surplus. Similarly, if there are more buyers than sellers, the equilibrium price equals 1. When the number of sellers and buyers is equal, though, the prediction given by the competitive equilibrium is not sharp. We get that any price between zero and one can be supported as the competitive equilibrium price.

The idea of competitive equilibrium has an implicit free communication assumption among agents. Our model though is one of rigid communication, a market in which agents are connected through a given exogenous network. Therefore the differences among the competitive equilibrium and the equilibrium of our game will help us understand the role of the network.

We choose to speak about a "*reference*" solution, which will be a selection of the competitive solution in each possible case. If the number of buyers and sellers is not equal, the reference solution will coincide with the competitive equilibrium, that is, the one that allocates all the surplus to the short part of the market. If the number of buyers and sellers coincides, the reference solution will be a point in the set of competitive equilibrium: it will denote the one in which all proposers get payoff $z = \frac{1}{1-\delta}$ and all responders get $1 - z$. This is the most natural selection since price z is the same price that two agents playing the usual bargaining game alone would realize; moreover, when the discount tends to one this price turns out to be $\frac{1}{2}$, which corresponds to players splitting evenly the surplus.

There is another good reason to choose this outcome as our reference solution. When the bipartite graph is complete (that is, when all buyers are linked to all sellers) our game will yield a unique PEP in all cases (see section 3.7.1). If the number of buyers and sellers is not equal, the short part extracts all the surplus, otherwise all pairs trade for price z. That is, when communication is completely free the unique PEP coincides with the reference solution.

Moreover, the reference solution also coincides exactly with the unique equilibrium of our game for small markets. We have seen in the previous section that cases $n = 2, m = 1$ or $n = 1, m = 2$ have a unique PEP in which all the surplus is allocated to the short part of the market. The cases with $n = m = 2$ yield a unique equilibrium in which the realized price is z.

We now wish to answer the following questions:

1. Which types of networks will support the reference solution? Or, in other words, when will the communication structure be "free" enough for the competitive assumption to be reasonable?

2. Which of those networks will only support the reference solution?

We have seen that if $n, m \leq 2$, the reference solution is the unique PEP of our game in each possible type of network. Markets with a larger number or people are more challenging, since they allow for a richer structure, with many types of graphs for each number or agents. It is in this context that determining which structure will support the reference solution has more meaning.

In the two following sections we address the first question: which networks will support the reference solution?

3.5 Implementing the reference solution: examples

In order to see when the reference solution can be implemented we will use the three types of graphs defined in ch. 2. Recall that one type will have more sellers than buyers, another type will have more buyers than sellers, and the last type will have an equal number of sellers and buyers. Moreover, each graph will fulfill a condition that says that each set of nodes in the long side of the market is collectively connected to at least the same number of agents on the other side. We will then identify the role of this condition with respect to the implementation of the reference condition. In particular, if the number of agents in each side is not equal, then the condition will mean that the short side of the market is well connected enough to extract all the surplus. More interestingly, we will see in the next section that the allocation which gives the reference solution with respect to each of the subgraphs in the decomposition will be supported in equilibrium.

In this section we will start by showing some examples to explain the intuition behind the result. We leave the main results to be stated and shown in the next section.

We will now discuss the result of our game in two examples in which there is a surplus of sellers. The reference solution gives a payoff of 1 to all buyers and 0 to all sellers. One of the graphs will support this reference solution, but the other will not. The crucial difference will be that in the graph that supports the reference solution, a set of sellers is always collectively linked to at least the same number of buyers (non-deficiency, defined in ch. 2). Let us now motivate this idea.

Both of the examples exhibited in Figure 3 have 3 sellers and 2 buyers; i.e.e, a surplus of one seller.

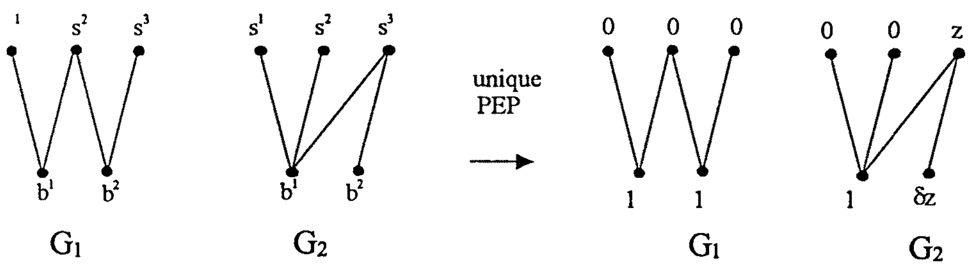

Figure 3

Also note that both of them have six connections. Still, they exhibit important differences. The unique PEP (drawn in the figure) will not coincide in both graphs. This result is an implication of the results which will be found later in section 3.6 and 3.7.

Let us start by analyzing graph G_1, trying to see why the reference solution is the unique PEP. Note first of all that in both examples at least one seller will get a payoff of zero, since at most 2 pairs can form. Suppose that this seller is s^1. To try to attract a buyer to trade with, s^1 could deviate by proposing a price as small as ε, with $\varepsilon > 0$. However, the fact that he cannot deviate implies that his linked buyer b^1 can accept a better price from s^2. Then, s^2 is proposing a price of zero in equilibrium. Again, why can't s^2 deviate, propose a price of ε and be accepted? None of his linked buyers, b^1 and b^2 would accept this price. Since b^1 and b^2 are collectively linked to $\{s^1, s^2, s^3\}$, we conclude that s^1 and s^3 are proposing a price of zero in equilibrium. Once we conclude that all sellers must propose zero we understand why the reference solution will indeed be an equilibrium: deviating will not be worthwhile for any seller. If it is the case that one seller deviates, all buyers will continue to accept zero. Given that in the network there exists a way to match the 2 buyers with the 2 sellers that still propose zero, by the definition of the game these 2 pairs will trade. Thus the seller that tried to deviate will be left isolated.

A close look to G_2 will quickly tell us that both s^1 and s^2 will get a payoff of zero, since they are collectively linked to only one buyer, so that if one of them would get a positive payoff, the other would be able to undercut the accepted price. But unlike the previous case, this fact does not imply that s^3 also gets a zero payoff. Indeed, what will happen in G_2 is that the subgraphs induced by the nodes $\{s^1, s^2\} \cup \{b^1\}$ and by $\{s^3\} \cup \{b^2\}$ will behave as if the are completely separated in the market. Sellers s^1 and s^2 have no way to communicate with b^2, and b^1 is already extracting all the surplus. Knowing this, s^3 will prefer to trade with b^2, and they end up doing it for price z in equilibrium. Note from figure 4 that this "decomposition" of G_2 coincides with the decomposition defined and studied in ch. 2.

The relevant difference among the two graphs is that the structure in G_1 is rich enough for the zero payoff to "propagate" from seller to seller, which implies that the buyers will be able to extract all the surplus. On the other hand in G_2 one of the buyers (b^2) is not "well-connected" enough to get all the surplus. This "well-connectedness", as informally stated here, does not have to do with the number of links, but the property called non-deficiency, studied in ch. 2.

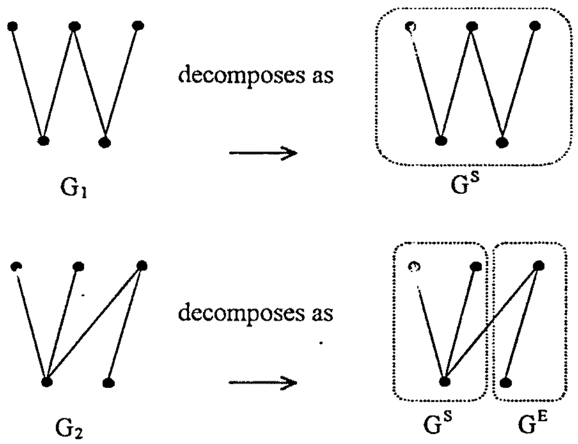

Figure 4

What we can see from these examples is the following: belonging to a set that is collectively linked to a set of partners of a larger size will be good for the short part. But for the agents in the short part to be able to completely extract all the surplus, the structure has to be rich enough so that if one proposer tries to propose a positive payoff, responders have sufficiently many links to accept the price of zero from other proposers.

3.6 Implementing the reference solution: results

The starting point is a proposition that tells us about the existence of a particular PEP for a general graph. This is a PEP which is the union of the reference solutions with respect to each of the subgraphs of the decomposition described in ch. 2. See Figure 5 for an example.

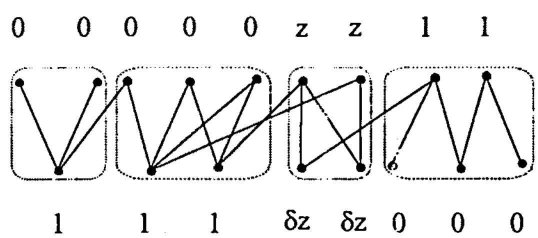

Figure 5

Proposition 8 *There exists a PEP in which:*

- *sellers in subgraphs G^S get 0, buyers in subgames G^S get 1.*
- *sellers in subgraphs G^B get 1, buyers in subgames G^B get 0.*
- *sellers in subgraphs G^E get z and buyers in subgames G^E get δz.*

Proof. We will show the result using induction. Let's first show the result for a number of agents $n \leq t, m \leq t$ with $t = 2$.

step 0) $(t = 2)$

Only four different graphs are possible. These are the graphs analyzed in section 3, in which the statement above is true.

step 1) $(t = k)$

Now, let's suppose that the result is true for graphs of sizes $n \leq k-1, m \leq k-1$. The strategies must specify (for the graph G and for any subgraph G_i of G that results from removing a set of pairs of nodes from G) for proposers, what price to propose in any given subgraph G_i, and for responders, what to do for any given distribution of prices, in any given graph G_i.

We now write the strategies that constitute a PEP. The strategy of the proposers depends on which subgraph they are in according to the algorithm. The strategy of the responders depends on the subgraph and each distribution of prices. Strategies do not depend on past history.

- *Strategies whenever we are in G_i, strict subgraph of G (somebody has traded)*

If this is the case, at least one pair of agents has traded and has left the market; therefore we are in a subgraph, call it G_i, such that the number of agents is strictly smaller than k. By the induction step we know of the existence of a PEP in this subgame. This is the PEP strategies will prescribe agents to follow.

- *Strategies whenever we are in G (nobody has traded)*

Apply the algorithm. The graph G is split in subgraphs of three types, G^S, G^B and G^E.

The price proposal in equilibrium will be the following:

Proposals: all proposed prices in G^S are 0 (both in a s-game or in a b-game), all proposed prices in G^B are 1 (both in a s-game or in a b-game), all prices in G^E are z in a s-game and δz in a b-game.

For future reference we call this price proposal price proposal P.

Acceptances: If the price proposal has been equal to P, then all responders accept (both s-game or b-game).

Now we will write what agents do facing some of the possible unilateral deviations:

$\longrightarrow$ Members of G_i^S (s-game) *Buyers*: If the distribution differs from P in one price only, then buyers all accept zero.

$\longrightarrow$ Members of G_i^B (b-game) *Sellers*: If the distribution differs from the one above in one price only, then sellers all accept one.

$\longrightarrow$ Members of G_i^E. We know that if the distribution is P and all responders in G_i^E accept the price, then there will exist a matching (the one that the mechanism will determine) that will saturate all buyers and sellers. Take this matching and relabel the agents $s^1, ..., s^{n_i}$ and $b^1, ..., b^{n_i}$, (n_i being the number of sellers in G_i^E) meaning that in the matching agent s^i trades with b^i. Now, the strategies will be:

(s-game) *Buyers*: If the distribution differs of P in one price, then:

If the difference is in G_i^E, with the price proposed by s^j being higher, then the $n-1$ buyers $b^1, ..., b^{j-1}, b^{j+1}, ..., b^n$ accept the price z and b^j rejects.

(b-game) *Sellers*: If the distribution differs of P in one price, then:

If the difference is in G_i^E, with the price proposed by b^j being lower, then the $n-1$ sellers $s^1, ..., s^{j-1}, s^{j+1}, ..., s^n$ accept the price δz and s^j rejects.

Note that in the strategies above we have only specified what responders should propose, what they should do when they face the distribution of prices

P, and what they should do when they face some of the possible unilateral deviations from P. Strategies should also specify what would responders do when facing other situations. This is what we specify below.

For any given distribution of prices, agents have a finite set of actions, that consist of either accepting one of the proposed prices by their linked proposers or rejecting all. Note also that if not all the possible number of pairs forms, then by the induction step we know that there exists a PEP in the resulting subgraph (since this will be a subgraph that has a number of agents strictly smaller than k). We define strategies so that if not all the possible number of pairs forms, then strategies follow the PEP of the resulting subgraph (which we know exists by the induction step). If all agents reject, then the strategies will prescribe for proposers to propose price distribution P and for responders to accept. Therefore we can conclude that given an action for all responders, the payoffs are immediately determined. For a given distribution of prices, the game is as a one-shot game with a finite set of actions. This must have at least one NE (in mixed strategies). We will define the strategies as follows: for a given distribution of prices, strategies will tell responders to play according to this NE. Note though that we may have a multiplicity of NE. If this is the case, strategies must specify which of the several NE will be played. Any specification would do the job. We will define strategies that specify agents to play the NE that involves the seller with subindex 1 trading with the buyer with the smallest possible subindex with highest probability (if not seller 2, or the one with smallest index, and like that up to seller n).

To check that what we have above is a PEP, see Appendix 1. ■

We will now answer the first question stated in section 4: which are the networks that are able to support the reference solution as a PEP of our game. The answer is easy in all cases: it must be the case that we have a graph that decomposes into a union of subgraphs all of the appropriate type. For instance, if we have a graph with more sellers than buyers, there will exist a PEP that coincides with the reference solution (giving all surplus to buyers) if and only if the graph decomposes into a union of subgraphs all of the G^S type.

We now state and prove the main result of this chapter.

Theorem 4 *Say that we have $G = \langle S \cup B, L\rangle$ with $|S| = n$, $|B| = m$. Then,*

G will support the reference solution $\Longleftrightarrow$ G decomposes as a union of subgraphs which are all of the same type.

Equivalently, if $n > m$, G supports the reference iff it decomposes as a union of subgraphs G^S, if $n = m$, G supports the reference iff it decomposes as a G^E and if $n < m$, G supports the reference iff it decomposes as a union of subgraphs G^B.

Proof. We know by the previous proposition that these type of graphs indeed support the reference solution. We now have to show that other types of graphs do not. For this purpose, we will show first a preliminary result.

Step 0): Call C^S the set of sellers and buyers that belong to subgraphs of type G^S in the graph. By definition, the set C^S consists of S^S sellers and B^S buyers with $\left|S^S\right| = n^S > m^S = \left|B^S\right|$. Then, in any PEP, the sum of the payoffs of all agents in C^S has to be at most m^S. To see why, note that by the definition of a G^S subgraph, we will have that $N_G(S^S) = B^S$. That is, sellers belonging to subgraphs G^S are linked to buyers in subgraphs G^S, but never to anything else. This implies that in any PEP at most a number m^S of the sellers in S^S will be able to trade. It is immediate then than the sum of the payoffs of all agents in C^S can be at most m^S.

Similarly, the set C^B of sellers and buyers that belong to subgraphs of type G^B cannot have in any PEP a total sum of payoffs exceeding the number of sellers, n^B.

Step 1: case a) Suppose to the contrary that we have a graph G that decomposes into a union of subgraphs, not all of type G^S., in which all sellers get zero and all buyers get one.

Note that it cannot happen that in the decomposition we have some subgraphs of type G^B. If that would be the case, we would have a number of n_B sellers and m_B buyers in subgraphs G^B with $n_B < m_B$ but in which all sellers get zero and all buyers get one, i.e, the sum of the payoffs of all agents in subgraphs G^B would be equal to m_B. This contradicts step 0).

Then the only possibility is that G decomposes into a number of subgraphs of type G^S and some of type G^E. If this is the case, then all buyers in a subgraph G^E are only linked in G to sellers in G^E. Then, if all sellers get zero and all buyers get one in all G^E, it must be the case that in all G^E, all sellers propose a price of zero, all buyers accept it, and the maximum number of pairs forms and trades. We now show that this is a contradiction. Take one particular G^E. Call S^E the set of sellers in G^E and B^E the set of buyers in G^E. Among all the sellers in S^E, take one of them (call him s) who is linked to the buyer with the lowest priority among the ones in B^E (call this buyer b). Then s could deviate and ask in the first period for a price of ε. Then it must be the case that somebody accepts this price of ε. To see why this happens, note that if a seller proposes zero, he is certain to get zero. Therefore, after the deviation at most $\left|S^E\right| - 1$ sellers are still proposing zero and $\left|B^E\right| = \left|S^E\right|$ buyers are willing to accept this price. Clearly b cannot accept price zero, since somebody else in B^E with highest priority will prevent him from trading for this price, and therefore b will be happy to accept price ε. Thus the deviation would be profitable.

case b) If the graph does not decompose in G^E only, then it must have G^S and G^B, with all sellers getting z and all buyers getting δz in the PEP. But then, the sum of the payoff of agents in all G^S equals $zn^S + \delta zm^S = z(n^S - m^S) + zm^S + \delta zm^S = zn^S + m^S > m^S$ therefore contradicting step 0).

case c) Suppose that the graph does not decompose in G^B only, but there is a PEP in which all sellers get one and all buyers get zero.

Note that it can not happen that in the decomposition we have some subgraphs of type G^S. If that would be the case, we would have a number of, n_S sellers and m_S buyers in subgraphs G^S with $n_S > m_S$ but in which all sellers get one and all buyers get zero, i.e, the sum of the payoffs of all agents in subgraphs G^S would be equal to n_S. This contradicts step 0).

Then the only possibility is that G decomposes into a number of subgraphs of type G^B and some of type G^E. If this is the case, then all sellers in a subgraph G^E are only linked in G to buyers in G^E. Then, if all sellers get one and all buyers get zero in all G^E, it must be the case that in all G^E, all sellers propose a price of one, all buyers accept it, and the maximum number of pairs forms and trades. This is a contradiction since any buyer in a G^E would do better by rejecting. If a buyer would reject, he would be alone next period with a seller, getting a payoff of δz. ■

Thus, for a network to support the reference solution, it must be the case that all its subgraphs are of the same type. This is a strong result that tells us precisely which properties must the network have to get the same result that we would obtain in a complete communication framework. This property formalizes the idea of the graph having sufficiently many connections.

Networks which support the reference solution:

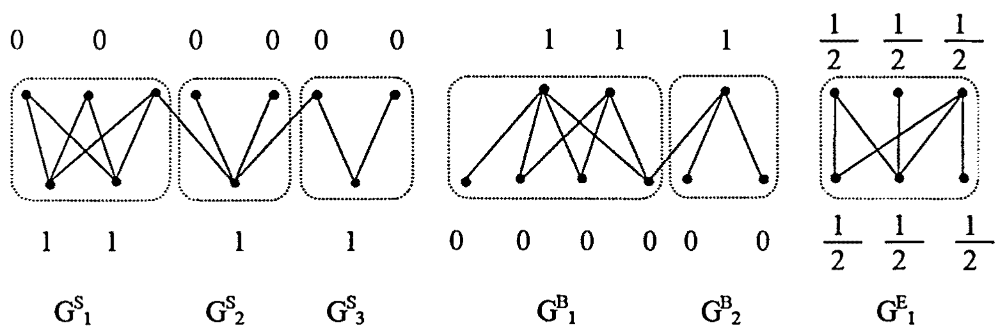

Figure 6

3.7 Is the reference solution the unique PEP?

In the previous section we have seen an implementation of the reference solution, but we do not know whether this is the only PEP that can be supported. We now address this issue starting from an extreme case: that of complete communication.

3.7.1 Complete graphs

In this case all agents of opposite types are connected, as in Figure 7.

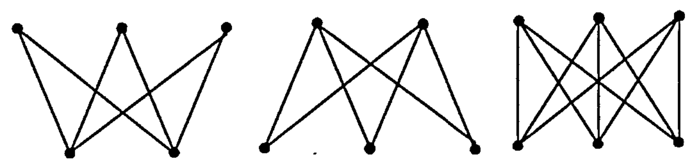

Figure 7

Our game will have in this case a unique PEP that coincides exactly with the reference solution, as we now show.

Proposition 9 *If the graph G is complete, there exists a unique PEP. In this PEP, if $n \neq m$ the short part extracts all the surplus. If $n = m$ all proposers get z and responders get $1 - z$.*

Proof. It is immediate that a complete graph decomposes as only one subgraph. A complete graph with $n > m$ is a graph of type G^S, while a complete graph with $n < m$ is of type G^B and a complete graph with $n = m$ is of type G^E. Therefore, existence of the equilibrium above is shown in proposition 8. We now prove uniqueness.

Case $n > m$) Clearly at least one seller (call him s^1), gets a payoff of zero in equilibrium. If he cannot deviate and propose a price of ε, this implies that all his linked buyers are able to accept zero from somebody else. Therefore, all sellers other than s^1 propose a price of zero. This implies that a number of $m + 1$ sellers are proposing zero in equilibrium. They will be accepted, implying that all sellers get a payoff of zero in equilibrium and all buyers get a payoff of one.

Case $n < m$) For any proposed distribution of prices, if some or all buyers reject, the ones that reject all receive zero. This is so since the graph next period would be complete and would have more buyers than sellers and buyers proposing. This is the symmetric situation of the one in the previous paragraph; therefore we know that in this case buyers would get zero.

This implies that if the prices proposed are positive, all buyers will immediately accept. Therefore, prices can be very high for the sellers. Then in equilibrium all sellers will propose a price of 1 and buyers will accept.

Case $n = m$) We use induction. The case $n = m \leq 2$ has been shown in section 3.3.2. Suppose the result is true for a number of buyers and sellers smaller than t, let us now show the result for a number of sellers n and of buyers m with $n = m \leq t + 1$.

We use the same notation as in proposition 6. Then the following inequalities will hold:

$$m_{s^i} \geq 1 - \delta \max \left\{ z, \max_i \{M_{b^i}\} \right\}, \quad M_{b^i} \leq 1 - \delta \min \left\{ z, \min_i \{m_{s^i}\} \right\}$$

For the first inequality, note that whenever buyer b^i rejects in a s-game, he is sure to get either δz (by the induction hypothesis, since he will face a complete graph with same number of buyers and sellers, this number being smaller or equal than t), or at most δM_{b^i}. Then, whenever a seller s^i will propose a price that gives a payoff higher than $\delta \max \{z, \max_i \{M_{b^i}\}\}$ to the buyer, it will be accepted by one of the buyers. This is so since it cannot happen that all buyers are able to accept a better offer, given that the number of sellers and buyers is equal.

The second inequality holds since any seller would prefer to reject than accepting $\delta \min \{z, \min_i \{m_{s^i}\}\}$. By rejecting, any player will get either δz or at least the payoff δm_{s^i}.

The inequalities imply the result. ■

It is worth noting that there is another case that is easy to solve. This is the case in which after decomposing the graph, all the bipartite subgraphs we get are complete. The equilibrium in this case is unique and coincides with the reference solution with respect to each subgraph. As an example see Figure 8.

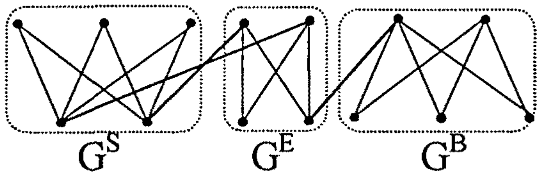

Figure 8

Proposition 10 *If the graph G is decomposed into complete subgraphs, there exists a unique PEP. In this PEP sellers in G^S get zero, buyers in G^S get one, buyers in G^B get zero, sellers in G^B get one, sellers in G^E get z, buyers in G^E get δz.*

Proof. It can be adapted from the previous proposition. ■

Note that there are several interesting implications one can get from these results. In particular, suppose that we have several independent markets, each market being represented by a complete graph. Now, suppose that one or several links are created, and that these links connect the previously independent markets. Depending on these links, the equilibrium of our game can differ completely.

For an example, think of a complete market (call it G_1) with 3 sellers and 2 buyers and a second complete market (G_2) with 2 sellers and 3 buyers (see Figure 9). Considered independently, in G_1 all sellers get a payoff of 0 and all buyers get 1. In G_2, however, all sellers get 1 and all buyers get 0. Now, if one buyer from G_1 manages to get connected with one seller from G_2, nothing changes at all. This is so since the new connection links two agents that are already extracting all the surplus, and therefore they gain nothing by trading with each other. In a way, we can think of this new connection as irrelevant. However, if the new connection is linking a seller from G_1 with a buyer from G_2, we suddenly get that the graph has 6 sellers and 6 buyers and that a perfect matching exists. Therefore the reference solution in which sellers will get z and buyers will get δz can be implemented. In this new situation, two agents that previously received zero are able to trade with each other. Clearly, they have much to gain from reaching an agreement. This fact influences the whole market in such a way that all prices change.

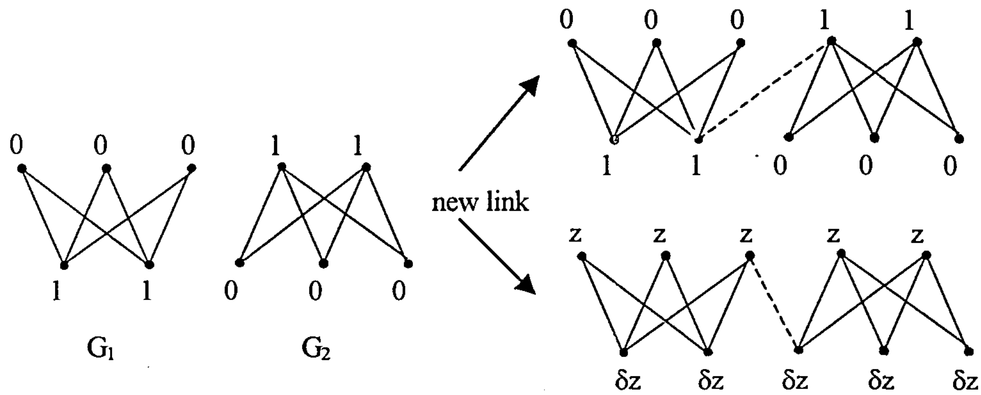

Figure 9

This particular example motivated the exploration of coalitionally stable networks which takes place in ch. 5.

3.7.2 Graphs with the long side starting to propose

The conclusions for the cases in which there are more sellers than buyers will equally be very sharp. We will be able to conclude that all graphs of the G^S type support a unique PEP, that of the reference solution. Moreover, uniqueness is shown in a broader sense: whenever we find a subgraph of type G^S imbedded in a graph G, any PEP in G will give zero for sellers in G^S and one for buyers in G^S, independently of the rest of the graph.

The intuition of the result is the following. The fact that there are more sellers than buyers immediately implies that at least one seller gets a payoff of zero. It must be the case that this seller will not deviate and ask for a very small price, which implies that other sellers are proposing zero. But then we can separate the sellers into 2 groups: the ones that trade for price zero and the ones that trade for a positive price. The property that the set of buyers is strictly non-deficient in G^S will imply that one of the buyers that trades for a positive price has access to one of the sellers that trades for price zero. Therefore the deviation in which this seller offers a price of ε must be accepted.

Proposition 11 *Suppose that we are in a network G that contains a subgraph G^S. Then in any PEP in G, all sellers in G^S will get 0 and buyers in G^S will get 1.*

Proof. Say that G has n sellers and m buyers, while G^S has n^S sellers and m^S buyers. Fix one PEP of the graph G. Call P the distribution of prices that is proposed in the first period (when sellers propose) of the PEP.

Case a) Suppose that the graph G^S is such that its sellers are collectively linked in G only to buyers in G^S.

- *step 1): We can separate the sellers in $G^S = \left\langle S^S \cup B^S, L^S \right\rangle$ into two sets: set S^+, sellers that get an strictly positive payoff in the PEP, and set S^0, sellers that get a payoff of zero in the PEP, with $S^+ \cup S^0 = S^S$, and with $|S^0| \geq n^S - m^S$.*

It is immediate that $n^S - m^S$ or the sellers belonging to G^S get a payoff of 0 in any PEP of G, since a positive payoff can only happen through trading for a positive price. Aiming for a contradiction, suppose now that $|S^0| < n^S$ (not all sellers in G^S get a payoff of zero in the PEP).

- *step 2): There exists a seller belonging to $S^0 \subseteq S^S$ (call him s^0), that is linked to a buyer $b^1 \in B^S$ such that in the PEP b^1 gets a payoff strictly less than 1.*

Let us now show *step 2)*. First, note that sellers in S^+ must get an strictly positive payoff through eventually trading for a strictly positive price. Call B^+ the set of $|B^+| = |S^+|$ buyers that trade with S^+ in the PEP. It must be the case that $B^+ \subseteq B^S$. Since by hypothesis $N_{G^S}(B^+) > |B^+|$, there must exist a seller $s^0 \in S^0$ linked to a buyer $b^1 \in B^+$.

- *step 3): There exists a profitable deviation in the PEP for seller s^0. This is the one that consists in s^0 proposing a price of $\varepsilon > 0$ at period 0.*

Call $\widetilde{P}$ the distribution of prices that coincides with P, except for the price proposed by s^0 (which is now equal to $\varepsilon > 0$), and suppose that the deviation is not profitable. This implies, in particular, that when facing distribution of prices $\widetilde{P}$, b^1 does not accept the price ε. Then, it must be the case that b^1 is able to accept and trade for price zero with a seller linked to him (i.e., belonging to $N(b^1)$. Call this seller $s^1 \in S$, with $s^1 \neq s^0$.

Say that there are a number t, with $1 \leq t \leq n$ of sellers in G that propose zero in the distribution P (s^1 is one of them). Call this set $S^{p0} \subseteq S$, with $|S^{p0}| = t$.

Now let's return to the distribution P. We know that when facing P buyer b^1 cannot trade for price zero. Then, it must be the case that another buyer $b^2 \in N(s^1)$ with a higher priority than b^1 accepts zero and trades with s^1.

When facing $\widetilde{P}$ it must be the case that b^2 can trade for price zero with a seller $s^2 \in N(b^2)$ with $s^2 \neq s^1$. This is so since b^1 can trade for price zero facing $\widetilde{P}$, even if it has a lower priority than b^2.

But now we can repeat the reasoning above and say that when facing P, it must be the case that there exists a buyer $b^3 \in N(s^2)$ with a higher priority than b^1 that accepts zero and trades with s^2 (if not, both b^1 and b^2 could accept zero and trade for price zero). This implies in turn that when facing $\widetilde{P}$ it will be the case that b^3 can trade for price zero with a seller $s^3 \in N(b^3)$ with $s^3 \neq s^2 \neq s^1$.

We can iterate this procedure until we conclude that there must exist $t+1$ different sellers $s^{t+1}, s^t, ..., s^2, s^1$ that all propose zero (and hence belong to S^{p0}). This is a contradiction since $|S^{p0}| = t$.

Case b) Suppose we are not in *case a)*. Then it must be that the graph G^s has been found in a recursive way. That is, sellers in G^S are collectively linked in G to buyers in G^S but also to buyers belonging to a subgraph (call it G_1^S) that is also of the G^S type. Using the first step and iterating, we can say that sellers in G^S are actually collectively linked in G to buyers in G^S and also to other buyers that we know get a payoff of 1 in equilibrium.

If this is the case, again we can conclude that $n_s - m_s$ or the sellers belonging to G^S get a payoff of 0 in any PEP of G (since by trading with buyers outside G^S sellers in G^S would get zero anyway). The same reasoning as before can now be adapted. ■

3.7.3 Graphs with the short side starting to propose

We now move to the cases in which the side that proposes is the short one. We want to know whether this is the unique PEP of this type of graphs. The answer is: not for all of them. The ranking we have imposed on the agents will imply that in some particular cases other price distributions can also be supported as a PEP. We offer an example in Figure 10.

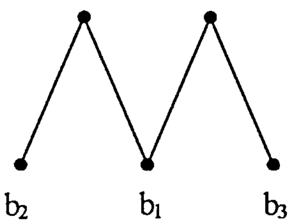

Figure 10

Suppose that the priority for the buyers is given by their subindexes.. Then the following would be an equilibrium: both sellers propose p, and if prices proposed have both been p, all buyers immediately accept, with $z \leq p \leq 1$. This can be supported as a PEP. If one seller proposes $p+\varepsilon$ and the other proposes p, then buyer b_1 will accept the smallest price and the agent having access to $p+\varepsilon$ will reject in order to get δz in the next period. Therefore no seller can deviate and ask for a larger price.

3.7.4 Graphs with an equal number of sellers and buyers

As in the previous case, graphs with an equal number of sellers and buyers will not yield a unique PEP in all cases. Indeed, we can have cases in which some agents trade immediately and some trade with delay. See Appendix 2 to see an example in which some proposers get z, some get $\delta^2 z$. The priority ranking plays again a crucial role. This is an example in which, after a deviation from the sellers, some buyers can get 1 in the next period. For the example to work, we need $\delta > \sqrt{\frac{1}{2}}$.

On the other hand, we can prove uniqueness if the discount is small enough, since in this case everybody will immediately trade.

Proposition 12 *Suppose that we are in a network G that contains a subgraph G^E and $\delta < \sqrt{\frac{1}{2}}$ (agents are sufficiently impatient). Then in any PEP in G, all sellers in G^E will get z and buyers in G^E will get δz.*

Proof. See Appendix 3 ■

3.8 Discussion on modelling assumptions

In our model the bargaining is conducted simultaneously among all linked players. One of the decisions we were confronted with when designing the model is how should responders accept prices, and how should ties be solved when they happen. In this chapter we have chosen to assume that responders will only be concerned with price (not identity) and that the precise matching will be decided by a mechanism that will maximize the number of pairs in the matching. We have also assumed that when ties happen, they will be resolved by the priority that all the agents are endowed with. We now explore the consequences of changing these two assumptions, respectively, by agents choosing prices and proposers, and by solving ties randomly.

About the first assumption, we conclude that the mechanism we have imposed plays an important role, since the results are weaker if we do not impose it. Therefore, the introduction of the mechanism improves the scope of the result even if it weakens the appeal of a totally decentralized market model. Random tie-solving does not play such a critical role and most of the results keep holding with this modification, while the multiplicity we get in our first model reproduces here as well.

3.8.1 On choosing prices only and not prices and proposers.

Assume now that, when facing the proposals made by one of the sides, responses have to specify whether they accept a proposal and from which proposer do they accept or whether they chose to reject all proposals.

The results of sections 3.3 and 3.7.1 keep holding (i.e, the reference solution is the unique PEP for small markets (up to 2 sellers and 2 buyers) and for complete or almost complete markets[3]. The results of section 3.6, though, do not reproduce here. In particular, though it is in general true that graphs of type G^B will support the reference solution (which gives 1 for all sellers and 0 for all buyers), we cannot say that this is true for all or them. The reason is that with this new model, responders have more strategic power than before.

Example 1 (not true all G^B support the reference solution)

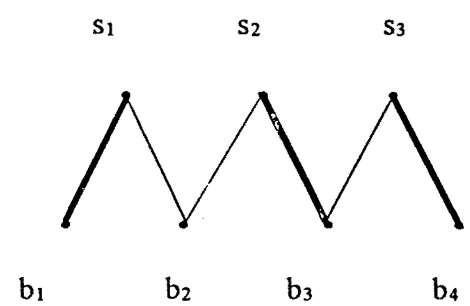

Figure 11

Here we have a graph which is of type G^B and in which we cannot support a PEP with all sellers getting 1 and all buyers getting 0. To see why, suppose it is true. Then all sellers must propose a price of 1, and at least 3 buyers have to accept the price of 1, from the 3 different sellers. But, whatever

[3] We chose not to include here all the complete proofs. Please contact the author for full details on subsections 3.8.1 and 3.8.2

the priorities, there will always exist one responder that would do better by deviating. In figure 11 , b_4 would do better by deviating since he would be able to get δz in the next period.

Recall that in the previous model the reference solution was supported by all sellers proposing a price of 1 and all buyers accepting. In that case, if one buyer did not accept, he got zero (since always the mechanism was allocating the three remaining buyers who did accept to the 3 sellers). This is no longer the case here, since agents have specified from whom they want to accept.

We now exhibit another case in which the results of section 3.6 do not hold. In the following example, we see a graph of type G^E in which we can support a PEP in which sellers get zero and buyers get 1 (again, note that this new model gives more power to the responders). This equilibrium can be supported for the particular case of the star graph.

Example 2 (when $n > m$, the reference solution is supported even if G is not the union of G^S)

This is an example of a graph which is a G^E and in which there is an equilibrium in which sellers get zero and buyers get 1.

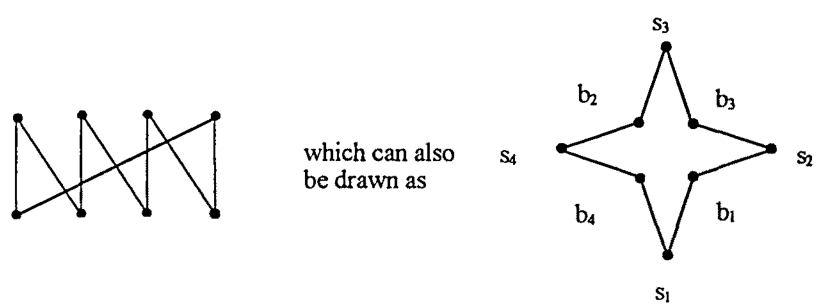

Figure 12

In equilibrium, all sellers propose zero, b1 accepts 0 from s1, b3 accepts 0 from s2, b2 accepts 0 from s3, b4 accepts 0 from s4, as shown in the picture.

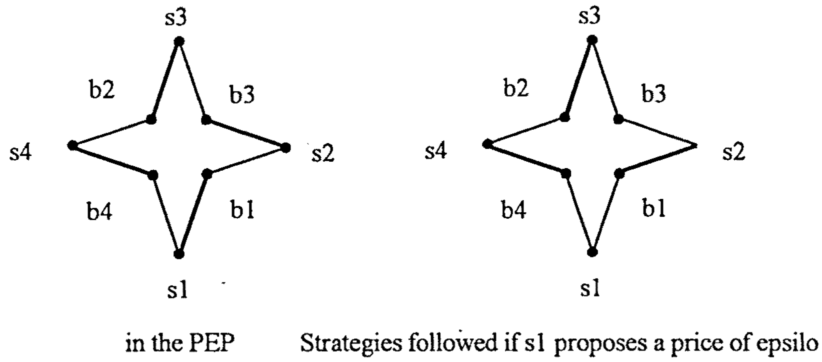

Figure 13

Now, if one seller, say, s1, tries to propose a small positive price, b1 accepts zero from s2, b2 accepts zero from s3, b4 accepts zero from s4. Buyer b3 can do nothing but get a payoff of zero. What happens is that if one seller proposes an ε, then the other sellers will respond in such a way that the ε will never be accepted (and one of the responders will get zero, since he won't be able to accept anything, he will be "blocked"). The PEP would be similar for other deviations.

In a similar way, we can find an example in which a graph with $n > m$ gives the references solution even if not being a G^S. For instance, take a star plus a G^S, and it will do the job. A PEP in which all sellers get zero and all buyers get 1 can be supported.

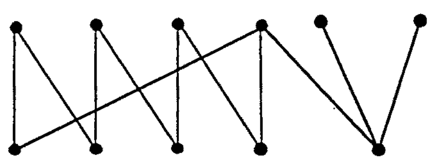

Figure 14

The examples we have provided do not depend on the model of how to solve ties, and would reproduce if ties are solved randomly.

3.8.2 On solving ties via a priority ordering and not randomly.

In order to solve ties, in this paper we endow agents with a priority ranking that the mechanism uses in order to decide which pairs trade. But, by doing this, we introduce differences among agents and the results in section 3.7 get influenced by this priority ranking. Along these lines, we have also tried to solve the same model but with ties resolved randomly.

Assume therefore that now, if two or more responders accept the same price, then the tie is resolved randomly, with the probability of each responder being matched being proportional to the number of links he has restricted to the tie.

It is worth stressing that all the results from 3.3 up to 3.7.2 keep holding if we allow ties to be resolved randomly. Nevertheless, we do not get a shrink in the set of equilibria. That is, when exploring the uniqueness of the equilibrium, we find that under some conditions we can find a G^S graph in which equilibria other than the reference solution can be supported, as shown below.

Recall that under priority ranking, in any G^S the only PEP gives 0 to sellers and 1 to buyers. This is so since the fact that always at least one seller will get a payoff of zero (since he will not be able to get matched) strengthens the competition that sellers face, pushing the prices down to zero. If ties are solved randomly, though, all sellers will get, in all PEP, some positive payoff in expected value. This reduces competition drastically, and sellers do not have such a high incentive to undercut prices.

Example 3 (non-uniqueness of equilibria for G^S)

The PEP in which all sellers get zero and all buyers get one is not the unique PEP for all graphs. We will now explain how to support an equilibrium in which all sellers propose a price p and all buyers accept it, with $\delta \geq \sqrt{\frac{2}{3}}$, $2(1-\delta) \leq p \leq \frac{2-\delta^2}{2+2\delta}$.

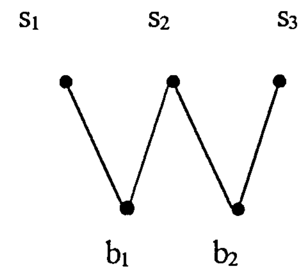

Figure 15

On the equilibrium path

Period 0: All sellers propose the same price p. All buyers accept price p.

Period 1: All buyers propose the same price $p = 1$. All sellers accept price $p = 1$.

Period 2: repeated.

After a deviation from the sellers:

If seller s_1 proposes a price $\widetilde{p}$ lower than p, if $\widetilde{p} < 1 - \delta$, then b_1 accepts price $\widetilde{p}$ and b_2 rejects. Otherwise, if $\widetilde{p} \geq 1 - \delta$, then all buyers reject and play as in the equilibrium path.

If seller s_3 proposes a price $\widetilde{p}$ lower than p, if $\widetilde{p} < 1 - \delta$, then b_2 accepts price $\widetilde{p}$ and b_1 rejects. Otherwise, if $\widetilde{p} \geq 1 - \delta$, then all buyers reject and play as in the equilibrium path.

If seller s_2 proposes a price $\widetilde{p}$ lower than p, if $\widetilde{p}$ is such that $\widetilde{p} \leq 1 - \delta$ and $\frac{1-\widetilde{p}+\delta z}{2} > 1 - p$, then both b_1 and b_2 accept price $\widetilde{p}$. If $\widetilde{p} \leq 1 - \delta$ but $\frac{1-\widetilde{p}+\delta z}{2} \leq 1 - p$ then b_1 accepts $\widetilde{p}$ and b_2 accepts p. Otherwise, both buyers reject and play as in the equilibrium path.

If any seller tries to propose a price $\widetilde{p}$ larger than p, then both buyers reject and play as in the equilibrium path.

After a deviation from the buyers:

If buyer b_1 tries to propose a price greater than $p = 0$, then both s_1 and s_2 accept this price. Next period follow the equilibrium path.

Then the above strategies are a PEP provided that $\delta \geq \sqrt{\frac{2}{3}}$ and that p

satisfies:

$$2(1-\delta) \leq p \leq 1 - \frac{\delta(1+z)}{2}$$

which simplifying are equivalent to:

$$\delta \geq \sqrt{\frac{2}{3}},\ 2(1-\delta) \leq p \leq \frac{2-\delta^2}{2+2\delta}$$

3.9 Conclusion and possible extensions

The purpose of this chapter is to analyze how the structure of a network connecting buyers and sellers will affect the prices realized in equilibrium, when players conduct a repeated alternating bargaining process. In order to see the role the network plays, we compare the equilibria of the game with a reference solution that is the most natural competitive equilibrium: if the number of buyers and sellers differs, the reference solution gives all the surplus to the short part; otherwise the surplus gets split evenly. We want to know how the network must be in order to support the reference solution, which corresponds to the free communication case.

We use a graph decomposition which splits any bipartite graph into a union of graphs which are of 3 possible types. Each of these subgraphs fulfills a property that plays a crucial role, that of the largest set of nodes being almost non-deficient. This property is related to graph theory techniques which are used to establish the properties that guarantee a perfect matching to exist. Our strongest result tells us that the equilibria of our game are strongly related to the structure of the graph according to this decomposition: we show that the reference solution is supported by an equilibrium in our game if and only if the graph can be split into subgraphs which are all of the same type.

The conclusion we reach is that agents will be able to extract all the surplus if they belong to a set of nodes that is collectively linked to a set of larger size, and also have enough many links to the other side, in the sense of the large part being almost non deficient. In other situations, agents will split the payoff evenly. In particular, this also means that being collectively linked to strictly more agents does not always guarantee a good payoff, since it depends on the interior structure of the network. We believe that this results are able to capture what we mean by being well-connected in a global market in which the entire context matters.

The equilibrium of our game is not unique in all cases, though. Indeed, the priority with which we endow agents in order to solve ties implies multiplicity for some specific graphs.

In our model the agents have complete information since our model is one of global interaction, in which agents internalize the structure of the whole network and understand the power they have given their context. It would be interesting to model a game in which agents only have a local knowledge of the network structure. On the other hand, it is also natural to ask for an introduction of a little bit of heterogeneity in the model. We believe that this line of research is unlikely to lead to fruitful results. Indeed, Chatterjee and Dutta (98) analyze the setup of 2 buyers and 2 sellers, fully linked with each other (i.e, network G_2 in figure 2). when buyers have different valuations and sellers are identical. One of the bargaining games they analyze (referred to as "the public offers model") is very similar to the one used here, and the results they get are negative both in the sense of inefficiencies of the equilibria and of lack of robustness. Their results seem to suggest that the model will turn out too complicated if heterogeneity is introduced.

Two other natural lines of research are explored in this thesis. In ch. 4 we look at the cooperative values of the graph exhibited here and in ch. 5, we move to a model of endogenous link formation. Finally, in Charness and Corominas (99) we test this model in a lab. Indeed, the simplicity of our model makes it suitable for being implemented in an experiment if applied to relatively small markets. The purpose of the experimental paper is to explore whether after a few rounds players "understand" which are the irrelevant connections and play according to theory.

3.10 Appendix

3.10.1 Appendix 1: checking strategies in proposition 8.

First of all, recall that by definition we know that: sellers in subgames G^S may be linked to buyers in another G^S only, while buyers in G^S may be linked to many other sellers in G (no restriction here).

Sellers in subgames G^B may be linked to many other buyers in G (no restriction here), while buyers in G^B may be linked to sellers in another G^B only.

Sellers in subgames G^E may be linked to buyers in a G^S , while buyers in G^E may be linked to sellers in a G^B

Let's check for unilateral deviations. We do not need to check for deviations in the strict subgraphs G_i of G, since we know by construction that what the strategies prescribe is a PEP. We have to check only for the graph G. We will check only for the sellers, the strategies for the buyers can be checked in a similar way.

(s-game) (in a PEP) Seller in a G_i^S, call him s^S, proposes 0 and is accepted. Suppose that s^S would propose something greater than zero. The equilibrium tells all buyers in a G^S, after the deviation, to continue accepting zero. By accepting zero they will all be able to trade for this price, since buyers belonging to G_i^S (call them B_i^S) know that they are collectively linked after the deviation to at least $\left|B_i^S\right|$ sellers proposing 0, in such a way that a perfect matching with B_i^S pairs exists, in equilibrium. Therefore s^S will be left isolated after the deviation and the deviation will not be profitable.

Any seller in G_i^B, by proposing a price of 1, is accepted and always gets a payoff of 1. No deviation can ever improve that.

Any seller in a G_i^E (call him s^E), can be accepted by proposing a price of z and always gets a payoff of z. He may be linked to buyers in a G^S. Suppose that he tries to deviate and offers a price strictly higher than z. It will happen that $n_i - 1$ buyers in his G_i^E will accept the price z and the other will reject, in such a way that all the $n_i - 1$ buyers that have accepted will get the good and in the next period seller s^E will be alone with one buyer. His payoff will therefore be $\delta^2 z$, and then the deviation is not profitable.

Now suppose that s^E tries to deviate by proposing a price smaller than z. If he is accepted or if all buyers reject, that would not be a profitable deviation. The only case in which this could be a profitable deviation is if s^E is rejected, but some of the other sellers are accepted, in such a way that next period s^E is left out in a G_i^B. But this would imply that buyers linked to him would have rejected all prices initially proposed. Therefore this cannot constitute a profitable deviation either.

(b-game) (in a PEP) Seller in a G_i^S (call him s^S) cannot accept anything other than zero. By rejecting he will be left isolated, since other sellers in a G^S will accept. Any seller in a G_i^E accepts a price of δz. By rejecting he will be left alone with only one buyer and will get exactly the same payoff.

3.10.2 Appendix 2: an example of non uniqueness in G^E.

This is an example in which delay in reaching agreement occurs. Look at the graph in Figure 16. This is a graph that is of the G^E type. The subindex of the buyers and sellers corresponds to their ranking.

Any seller in a G_i^B accepts a price of 1 and sells the good for this price. Nothing is better than this.

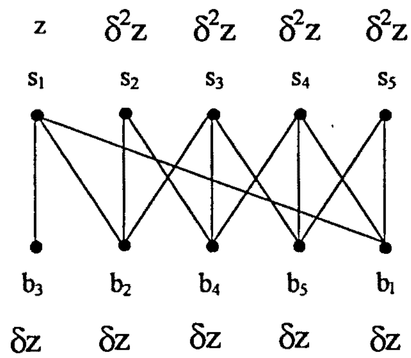

Figure 16

Now, a PEP would be the following:

$\rightarrow$ If we are in G, sellers proposing:

P^* : s_1 proposes z, s_2 , s_3, s_4, s_5 propose prices p_2, p_3, p_4, p_5, all of them strictly greater than z. Facing the distribution P^*, b_3 accepts z, all other buyers reject.

If s_1 tries to propose a higher price: all buyers reject.

If s_2 or s_3 try to propose a different price: b_1 accepts z, all other buyers reject.

If s_4 or s_5 try to propose a different price: b_2 accepts z, all other buyers reject.

$\rightarrow$ If all buyers reject, play as in proposition 8 (all buyers get z, all sellers get δz).

$\rightarrow$ If we are in a strict subgraph of G, play as the strategies in proposition 8

This is a PEP in which some sellers get a payoff of z, some get a payoff of $\delta^2 z$ and all buyers get δz.

3.10.3 Appendix 3: proof of proposition 12

We have shown before that in a G^E a buyer or a seller never get a payoff of zero (equivalently, no buyer or seller get a payoff of one). This implies that in a PEP in G, all sellers in G^E trade eventually with all buyers in G^E for a positive price.

Since no seller belonging to G^E trades with a buyer belonging to $G - G^E$ the problem is now reduced in finding the PEP of the graph G^E.

Fix a PEP. We start with a s-game, that is, with sellers proposing.

By the corollary we know that all agents in G^E eventually trade among them. Now, relabel sellers in G^E as $s^1, ..., s^t$ and buyers in G^E as $b^1, b^2, ..., b^t$ in such a way that in the PEP seller s^i trades with buyer b^i.

step 1) If some sellers trade immediately and some with delay, then the sellers that trade with delay get a payoff of $\delta^2 z$ and the buyers that trade with delay get δz.

Suppose to the contrary that some agreements are achieved with delay, some agreements are achieved immediately. Say that the set of sellers $S^{nd} = s^1, .., s^{t'}$ trade immediately (no delay) and $S^d = s^{t'}, ..., s^t$ trade with delay, for a $t' < t$. Equivalently, define $B^{nd} = b^1, .., b^{t'}$ trade immediately and $B^d = b^{t'}, ..., b^t$. Note that after sellers $s^1, .., s^{t'}$ trade in the first period, the remaining market is split into a number of subgraphs. We know that all of them are $n_i = m_i$ and allow for a perfect matching (this is so since we know that the pairs $s^{t'} : b^{t'}, ..., s^t : b^t$ will form). That is, the subgraphs are $n_i = m_i$ and the set of sellers in each subgraph is non deficient in the subgraph. Then the induction step tells us that in the next period, proposers (buyers) will get z and responders (sellers) will get δz. Therefore *step 1)* holds.

step 2) It never happens that some sellers trade immediately and some trade with delay.

The distribution of prices P that occurs in the first period in the PEP must be: sellers in S^{nd} propose a price $\leq z$ (since buyers in B^{nd} would be sure by rejecting to get z next period), sellers in S^d propose a price $\geq z$ (by the same argument).

Now we take one of the sellers belonging to S^d, call him s^d. Suppose that he deviates and proposes a price $p^* = \delta^2 z + \varepsilon$. We will now show that this will be a profitable deviation since it will be accepted.

Suppose to the contrary that facing the distribution of prices P^* (which is equal to P excepting for the price proposed by s^d that is now p^*), s^d is not accepted by any of his linked buyers in $N(s^d)$. It cannot be that all buyers in $N(s^d)$ decide to reject, since by rejecting the most they can get next period is 1 discounted, and we have that $\delta < 1 - \delta^2 z - \varepsilon$ since $\delta < \sqrt{\frac{1}{2}}$. That is, the price p^* is really good since it is more than anything responders can get in the next period. Therefore, it must be the case that when facing distribution P^* , buyers in $N(s^d)$ are able to accept a price and trade for that price, with this price being smaller or equal than p^*. That is, there exist a number u with $u \geq \left|N(s^d)\right|$ of prices being proposed by sellers belonging to S^{nd} in P that are smaller or equal than p^*. Facing distribution P buyers in $N(s^d)$ could not accept and trade for any of these prices, since a number of u other buyers, call them B^*, with $|B^*| = u$ had higher priority and traded for these prices, but facing distribution P^*, they can. Then it should be the case that facing distribution P^*, also buyers in B^* trade for prices smaller or equal than p^*. But this is a contradiction since there are only u of such prices.

step 3) If all agents trade immediately, they do it for price = z.

Note first of all that if all agents immediately trade, then they do it for a price $\leq z$ (since by rejecting responders know by the induction hypothesis that they will get δz).

Take the smallest price and call it p^1. Say that a number of pairs trade for this price. Then, select the seller s^1 that trades for price p^1 in equilibrium with buyer b^1, where b^1 is the buyer that has the lowest priority among all buyers that trade for price p^1 in equilibrium. Suppose (aiming for a contradiction) that $p^1 < z$. It must be then that if s^1deviates and asks for a price slightly larger, but still smaller than z, he will not be accepted. We call this distribution of prices $\widehat{P}$ (the one that coincides with P excepting for the price proposed by seller s^1 that is now $\widehat{p} = p^1 + \varepsilon < z$). It should happen that after this deviation, none of the buyers in $B^1 = N(s^1)$ accepts and trades with s^1.

case a) Facing distribution $\widehat{P}$, all buyers in B^1 accept a price and trade (not with s^1). It can only happen that they accept another price if this other price is p^1 (given that p^1 was the smallest price). Now, say that there are a number $v = \left|S^{p^1}\right|$ of sellers proposing price p^1 in distribution P. All of them are being accepted and trade in equilibrium. Facing distribution $\widehat{P}$, b^1 accepts and trades for price p^1. But this is a contradiction since at least one of the buyers in B^{p^1} cannot accept and trade facing distribution $\widehat{P}$ the price p^1, and we selected b^1 to be the one with lowest priority.

case b) Facing distribution $\widehat{P}$, all buyers in B^1 reject. If they all reject, after the deviation B^1 will be left out in a subgraph. Applying the algorithm to the subgraph, we will know that if after the separation B^1 is left out in a subgraph of the type G^E, then buyers would get (by the induction step) a payoff of δz and therefore should have accepted, or they are left out in a subgraph of the type G^B, which would imply that at least one buyer in B^1 would get zero and therefore should have accepted again, or they are left out in a subgraph of the type G^S. We conclude that the only possibility is that all members of B^1, after the deviation, and after the separation with the algorithm, belong to a subgraph G^S (doesn't need to be the same one for all buyers in B^1).

This implies that among the rest of buyers in G^E, $B - B^1$, at least one buyer got zero. This implies that this buyer was blocked and could not accept any price (since the mechanism was giving preference to the other acceptors). But recall that in equilibrium all prices were accepted. This implies that after the deviation at least one buyer decides to accept a different price than before, has a high priority and gets it. This is a contradiction since he could have done it before (if, say, before the buyers $b_1, .., b_n$ where accepting prices $p_1, ..., p_n$ now they will accept either the same price or a price strictly better for them. That is, buyer b_i will not accept p_j for $i < j$ even if he would have access to it, since if that was a better option he could have accepted it before).

step 4) It never happens that all agents trade with delay.

Suppose that all agents trade with delay. Then, either all agents immediately trade at period 2, or at period 3, or later. By the previous step we know that they reach agreement for a price z (discounted) in the s-games (odd periods) and for price δz (discounted) in the b-games (even periods).

This implies that sellers at period 1 get at most a payoff of $\delta^2 z$. Then one of the sellers could deviate at period 1 and propose $p^* = \delta^2 z + \varepsilon$. See *step 2)* for details. ■

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Chapter 4

On the core of two-sided network markets.

4.1 Introduction

The present chapter is devoted to the study of two-sided network markets from the point of view of cooperative game theory, relating them to assignment games and studying the structure of its core.

Assignment games have been long studied from the seminal paper of Shapley and Shubik (72). Indeed, assignment games appear in many relevant situations. Classical examples are the placement of medical interns and residents at hospitals, the assignment of workers to jobs, or the pairing of men and women in marriage. A particular attention has been devoted to studying the structure of the core of assignment games (Shapley and Shubik (72), Balinski and Gale (90), Thompson (81)) as well as its price equilibrium (Shapley and Shubik (72)), and the kernel and the nucleolus (Solymosi and Raghavanh (94), Driessen (98) and Granot (95)).

This papers links the assignment games literature and the networks literature (see ch. 3 and ch. 5) by studying a particular type of two-sided network market which is in turn an assignment game.

In the assignment game we study, n sellers all produce a different good.

There are m buyers which value differently each of the n goods, but these valuations are always either one or zero. Sellers all value their good at zero, and therefore the surplus any pair of a seller and a buyer may split is either one or zero. This allows us to define an induced network of this market, which will consist of the buyers and the sellers as the set of nodes, with buyer b_i being connected to seller s_j if its valuation is not zero. We will derive interesting properties for the core of such a market using properties on the induced network, more precisely, applying the graph theory decomposition studied in ch. 2. We will point to special situations in which the core is a singleton, and more generally we will be able to precise which nodes always receive the same payoff in any core allocation. Moreover, we will show that the task of computing the core of any graph can be reduced to computing the core for each of its subgraphs by the decomposition.

We will also study a particular selection of the core. This selection will correspond to the limiting value, as the discount value of the agents goes to one, of the non-cooperative bargaining game studied in ch. 3. Interestingly, this limiting value also corresponds to the "fair" solution described by Thompson (81), which is an interior point of the core.

Also linking cooperative game theory and network structures (but not related to our work) there is a large literature started by Myerson (77) and Owen (86) and summarized in van den Nouweland (93) and Borm et al. (94). The focus there is on introducing graphs to model communication channels between players. A cooperative game together with a communication graph on its player set is called a communication situation. Curiel (97) and van den Nouweland (93) are also a good review of the literature which uses both combinatorial techniques and cooperative game theory. In this chapter graphs are used only as a useful way of representing a given cooperative game, but, unlike their setup, the graph does not add new structure into the game.

The chapter is structured as follows. In section 4.2, we provide an example of the type of markets we study and introduce the model. Section 4.3 studies the core of our markets and section 4.4 studies other solution concepts like the kernel and the fair solution. Finally, section 4.5 concludes.

4.2 The Model

4.2.1 An illustrative example

Imagine a market with a number of translators, each knowing a set of different languages. On the other hand, there are a number of potential clients that need a translator that speaks two given languages. A translator is useful to a given client only if he speaks its two required languages. All clients want to pay at most 1 unit of money for the service, while translators are willing to accept any positive price.

More specifically, suppose that the set of translators is given by $T = \{T^1, T^2, T^3, T^4\}$ and the set of clients is given by $C = \{C^1, C^2\}$, and the languages translators speak and the languages clients require are given in the graph in Figure 1[1]. Note that linked agents correspond to those pairs in which trade is possible.

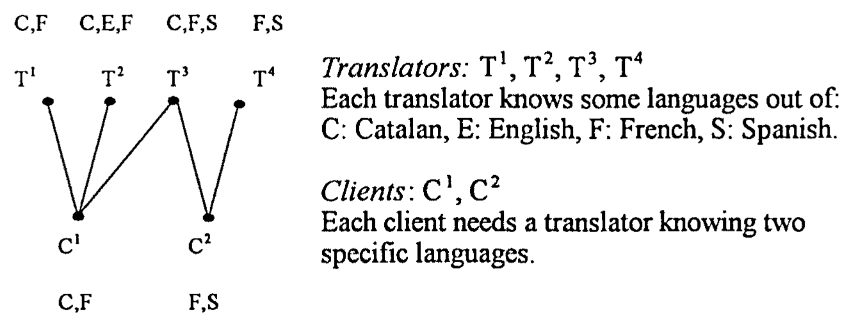


Figure 1

On the other hand, this is also a particular type of assignment market (in the sense of Shapley and Shubik (72)) with the following characteristics: translators value their good at zero and each client has different valuations for each translator, which can be either zero or one. For instance, client C^1

[1] I would like to thank Nir Dagan for suggesting this example and for discussions on this chapter.

values 1 translators T^1,T^2 and T^3 but values zero translator T^4. A valuation of one means the translator is useful for the client since it does know the two required languages, otherwise the valuation is zero.

This bijection among an assignment game of this type and a bipartite graph will prove to be crucial for our results.

We go on to formally define the particular assignment market we will study.

4.2.2 Our assignment market

There are n sellers, $s_1, ..., s_n$ and m buyers, $b_1, ..., b_m$. We call the set of all sellers $S = \{s_1, ..., s_n\}$ and the set of all buyers $B = \{b_1, ..., b_m\}$. The grand coalition N is given by all buyers and sellers, $N = S \cup B$. Each seller has a different good which he values at zero. Each buyer i has a different valuation for seller j, which is given by a_{ij}. Thus, the surplus a buyer i and a seller j may split is given by a_{ij}. In our case $a_{ij} \in \{0, 1\}$. The cooperative game we have is given by the assignment game of Shapley and Shubik (72), which means that for a given coalition of agents C, the valuation of the coalition is given by:

$$v(C) = \max\{a_{i_1 j_1} + a_{i_2 j_2} + ... + a_{i_k j_k}\}$$

where the maximum is taken over all arrangements of $2k$ distinct players, k buyers $b_{i_1}, ..., b_{i_k}$ in $C \cap B$ and k sellers $s_{j_1}, ..., s_{j_k}$ in $C \cap S$, where $k = \min\{|C \cap B|, |C \cap S|\}$. The cooperative game will be denoted by (N, v).

Allocations of this game will be denoted by vectors $x \in \Re^{n \times m}$. The notation we will use is $x = (x_{s_1}, ..., x_{s_n}; x_{b_1}, ..., x_{b_m})$ to indicate the payoff for each specific player. Also, we will sometimes write $x = (x_S, x_B)$ where x_S is the $\Re^n$ vector $x_S = (x_{s_1}, ..., x_{s_n})$ and x_B is the $\Re^m$ vector $x_B = (x_{b_1}, ..., x_{b_m})$.

We now define a bipartite graph associated to the cooperative game. This association will enable us to deduce powerful results about the core of this game.

4.2.3 The bijection among our assignment game and a bipartite graph

Recall that any graph is defined by its set of nodes and the set of links connecting the nodes. The associated graph to the game (N, v) will have as nodes all sellers and buyers in the market, and a seller s_i and a buyer b_j will be connected iff the valuation of buyer b_j for the good provided by s_i is equal to $a_{ij} = 1$. No buyer will be connected to another buyer, and no seller will be connected to another seller (i.e., the graph will be bipartite). Then, a connection implies that there are gains from trading among that pair of agents. Similarly, for a given bipartite graph we can speak about its associated cooperative game. More formally:

Definition 7 • *Take an assignment game (N, v) as defined above. Then, the unique bipartite graph associated to (N, v), denoted by $G_v = \langle N, L\rangle$ will be such that:*

1) The set of nodes is given by $N = S \cup B$, with $S = \{s_1, ..., s_n\}$ and $B = \{b_1, ..., b_m\}$

2) The set of links is given by $L = \{b_i : s_j \mid \text{ such that } a_{ij} = 1\}$.

- *Similarly, for a given bipartite graph $G = \langle S \cup B, L\rangle$, with $S = \{s_1, ..., s_n\}$ and $B = \{b_1, ..., b_m\}$ we will define the unique game associated to G, denoted by v_G, as an assignment game $(S \cup B, v_G)$ such that $a_{ij} = 1$ if $b_i : s_j \in L$ and $a_{ij} = 0$ otherwise.*

4.3 Results about the core of the game

We now devote this section to the study of the core of the game. We start with its definition and we then review the related literature. The last subsection concentrates on our results, after some introductory examples.

4.3.1 The definition of the core

The core consists of the imputations (allocations which add up to the value of the grand coalition) that cannot be improved upon by any subset of players. We now write its formal definition.

Definition 8 *A $n+m$ dimensional vector $x = (x_{s_1}, \dots, x_{s_n}, x_{b_1}, \dots, x_{b_m})$ will be in the core of the game (N,v), denoted by $C(N,v)$, if it fulfils the following inequalities:*

$$\begin{aligned} \sum_{a \in C} x_a &\geq v(C) \text{ for any coalition } C \subset N \\ \sum_{a \in N} x_a &= v(N) \end{aligned} \tag{4.1}$$

Given that we are in an assignment game, and given that valuations are only zero or one, the set of inequalities in (4.1) is equivalent to:

$$\begin{aligned} x_a &\geq 0 \text{ for all } a \in N \\ x_{s_i} + x_{b_j} &\geq 1 \text{ for all } i,\ j \text{ s.t. } a_{ji} = 1 \\ \sum_{i=1}^{n} x_{s_i} + \sum_{i=1}^{m} x_{b_i} &= v(N) \end{aligned} \tag{4.2}$$

4.3.2 Literature on the core of an assignment game

We know enumerate some properties for the core of assignment games which are relevant for our results. Roth and Sotomayor (90) (ch. 8), and Curiel (97) (ch. 3), as well as Balinsky and Gale (90), are good summaries of known results for the core of an assignment game.

- **Existence, computation:** From Shapley and Shubik (72), we know that any assignment game has a non empty core. We also know that the allocations in the core can be computed from a relatively small number of inequalities, given that only pairwise coalitions need to be considered.

The evaluation of the maximization problem to determine $v(C)$ is called an *optimal assignment problem*. Here $v(C)$ is thought as the maximum output that the society in C can produce. Recall that

$$v(C) = \max \{a_{i_1 j_1} + a_{i_2 j_2} + ... + a_{i_k j_k}\}$$

where the maximum is taken over all arrangements of $2k$ distinct players, k buyers $b_{i_1}, ..., b_{i_k}$ in $C \cap B$ and k sellers $s_{j_1}, ..., s_{j_k}$ in $C \cap S$, where $k = \min \{|C \cap B|, |C \cap S|\}$. The arrangement of $2k$ distinct players that reaches the maximum is called an *optimal assignment*. Thus, if the maximum is achieved for $a_{i_1 j_1} + a_{i_2 j_2} + ... + a_{i_k j_k}$, the corresponding optimal assignment is given by the set of distinct pairs $\{s_{i_1} : b_{j_1}, s_{i_2} : b_{j_2}, ..., s_{i_k} : b_{j_k}\}$. Equivalently, an optimal assignment can also be described as a matrix $m = (m_{ij})$ such that $m_{ij} = 1$ if a_{ij} belongs to the optimal assignment, and zero otherwise.

Now, consider the following linear programming:

$$\begin{aligned} & \max_{i,j} \sum_i a_{ij} m_{ij} \\ \sum_i m_{ij} &\leq 1, \sum_j m_{ij} \leq 1 \\ m_{ij} &\geq 0 \end{aligned} \tag{4.3}$$

This linear programming finds the optimal assignment of an assignment game, as shown in Dantzig (63). The dual of equations 4.3 is given by

$$\begin{aligned} & \min_{x_{s_i}, x_{b_j}} \sum_i x_{s_i} + \sum_j x_{b_j} \\ x_{s_i} &\geq 0, x_{b_j} \geq 0 \\ x_{s_i} + x_{b_j} &\geq a_{ij} \end{aligned} \tag{4.4}$$

The objective functions of equations 4.3 and 4.4 must attain the same value. Shapley and Shubik (72) showed that the solution of 4.4 gives core allocations.

Two consequences of this result are the following:

(1) An optimal assignment $m = (m_{ij})$ is such that:

$\sum_i x_{s_i} + \sum_j x_{b_j} = \sum_i a_{ij} m_{ij}$ for all allocations (x_S, x_B) in the core.

(2) For a core allocation (x_S, x_B), if $\sum_i x_{s_i} + \sum_j x_{b_j} = \sum_i a_{ij} m_{ij}$ then $m = (m_{ij})$ is an optimal assignment.

As a consequence, in any core allocation $(x_{s_1}, ..., x_{s_n}, x_{b_2}, ..., x_{b_m})$, it is the case that $x_{s_i} + x_{b_j} = a_{ij}$ for all pairs that belong to the optimal assignment, while $x_a = 0$ if a is a player which does not belong to the optimal assignment.

- **Geometric properties:** The core of an assignment game is a compact, convex polyhedron whose dimension is equal to at most the minimum of the number of members in one group or in the other.

  Now, for and allocation $x = (x_S, x_B)$ and $x' = (x'_S, x'_B)$ in the core of an assignment game, define the following partial order: $x >_S x'$ if $x_{s_i} \geq x'_{s_i}$ for all s_i in S and $x_{s_i} > x'_{s_i}$ for at least one s_i in S . Then, the core of the assignment game endowed with the partial order $\geq_S$ forms a complete lattice (dual to the lattice with ordering $\geq_P$) (see Roth and Sotomayor (90) for details). A consequence of this result is the following:

  There exists an S-optimal core outcome $(\overline{x_S}, \underline{x_B})$ with the property that for any core payoff (x_S, x_B), $\overline{x_S} \geq x_S$ and $\underline{x_B} \leq x_B$. Similarly, there exists a B-optimal core outcome $(\underline{x_S}, \overline{x_B})$ with the property that for any core payoff (x_S, x_B), $\underline{x_S} \leq x_S$ and $\overline{x_B} \geq x_B$. In other words, there is a vertex in the core at which every player from one side gets the maximum payoff and every agent from the other side gets the minimum payoff. There is another vertex with symmetric properties. It is in this sense that the core of an assignment game is sometimes said to be "elongated", with its "long axis" being the line that joins its two optimal core vertices.

4.3.3 Some examples for easy games

We start now by characterizing the core for some easy examples. For simplicity, instead of defining the cooperative game each time, we chose to display the associated graph with the equations describing their core allocations.

CORE ALLOCATIONS

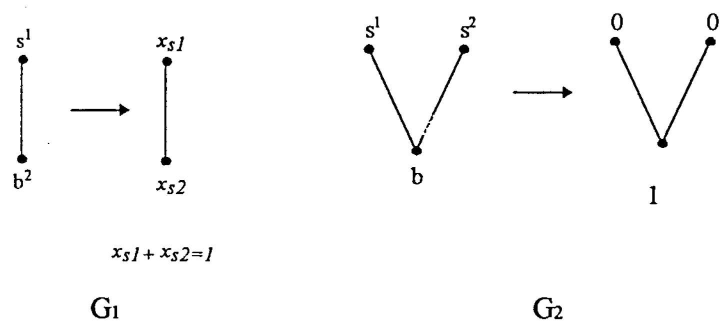


Figure 2

For the easiest case, G_1, which consists of only a pair of linked agents, any division of 1 is in the core. Note though that in G_2, which involves three linked agents, the core is a singleton and gives all the surplus to the short part of the market.

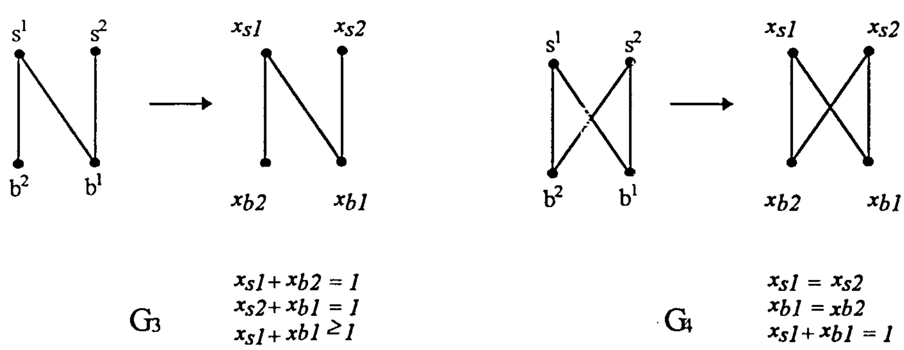


Figure 3

Graphs G_3 and G_4 are examples of situations with 2 sellers and 2 buyers. Note that in G_3 we have in particular that $x_{s_1} \geq x_{s_2}$ and also that $x_{b_1} \geq x_{b_2}$. That is, indeed the agents that have more links (i.e., s_1 and b_1) get a higher payoff. On the other hand in G_4 the situation is very symmetric.

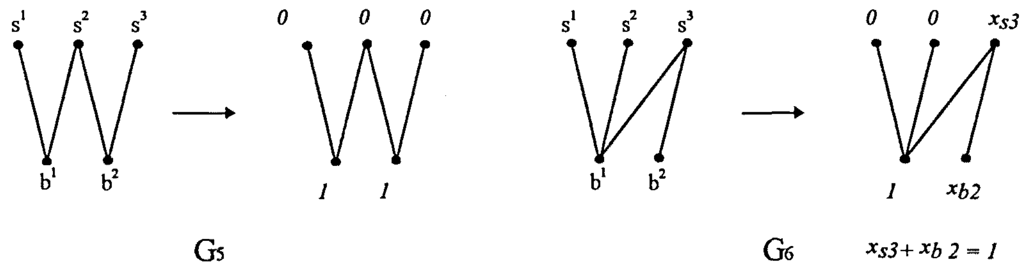
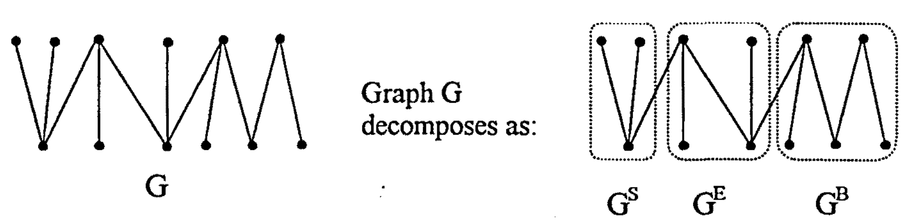


Figure 4

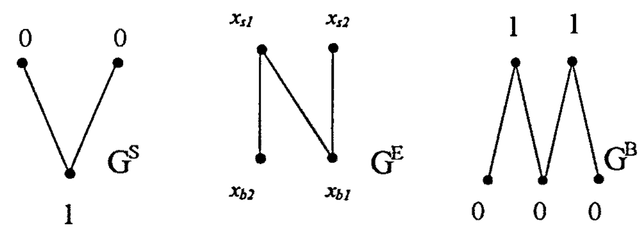
In G_5 we get, as in G_2, that the unique allocation in the core gives zero to sellers and 1 to buyers. Indeed, the sum of the payoff of any pair has to be at least 1, so starting from any seller having a positive payoff easily implies a contradiction. For the same reason, in G_6 any core allocation gives zero to s_1 and s_2 and gives one to b_1. Moreover, it has to be the case that $x_{s_3} + x_{b_2} = 1$.

These examples already give us the intuition of the results we will derive. Indeed, we will be able to use the decomposition in ch. 2 and say that in two out of the three types of subgraphs the core allocation is uniquely determined and gives agents zero or one, while in the other type of subgraphs, core allocations are non uniquely determined. Moreover, the core allocations of the graph can be found by computing the core allocations of each of the subgraphs.

As an example:

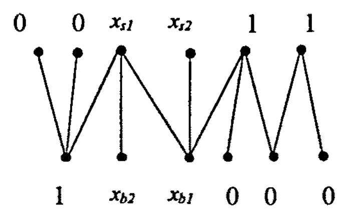


The core for each of the subgraphs is given by:



$x_{s1} + x_{b2} = 1$
$x_{s2} + x_{b1} = 1$
$x_{s1} + x_{b1} \geq 1$

Then the core of G will be given by:



$x_{s1} + x_{b2} = 1$
$x_{s2} + x_{b1} = 1$
$x_{s1} + x_{b1} \geq 1$

Figure 5

4.3.4 Our results relating to the core

First of all, let us now show a way to compute $v(C)$ for any coalition C in N by using the associated graph G_v. In our set up the optimal assignment will correspond to the matching in the graph G_v that involves the maximum number of connected pairs. We now show this in detail in the following lemma.

Lemma 7 *Take a cooperative game (N, v) of the type described above and construct its associated bipartite graph, $G_v = \langle S \cup B, L \rangle$. For a set of nodes $C \subseteq S \cup B$, define t_{GC} as the number of pairs involved in the maximum matching with respect to coalition C in G. Formally,*

$$t_{GC} = \max \left\{ \begin{array}{c} |T| : \textit{ there exists a matching in } G_v \textit{ that saturates } T, \\ \textit{with } T \subseteq S \cap C \textit{ or } T \subseteq B \cap C \end{array} \right\}$$

Then, we have that $v(C) = t_{GC}$.

Proof. Recall that for a given game (N, v):

$$v(C) = \max\{a_{i_1 j_1} + a_{i_2 j_2} + \dots + a_{i_k j_k}\}$$

where the maximum is taken over all arrangements of $2k$ distinct players, k buyers $b_{i_1}, \dots, b_{i_k}$ in $C \cap B$ and k sellers $s_{j_1}, \dots, s_{j_k}$ in $C \cap S$, where $k = \min\{|C \cap B|, |C \cap S|\}$. Since by construction a_{ij} is always either 0 or 1, we are actually choosing the set of pairs such that the sum of its $a_{i_k j_l}$ involves the maximum number of 1's. Therefore, an optimal assignment must contain a maximum matching in G_v involving connected pairs among buyers $b_{i_1}, \dots, b_{i_k}$ and sellers $s_{j_1}, \dots, s_{j_k}$. This coincides with the definition of t_{GC}. ■

We now use some of the results shown in ch. 2. Recall that any graph G can be decomposed into a union of subgraphs:

$$G = G_1^S \cup G_2^S \cup \dots \cup G_{i_S}^S \cup G_1^E \cup G_2^E \cup \dots \cup G_{i_E}^E \cup G_1^B \cup G_2^B \cup \dots \cup G_{i_B}^B$$

where G_i^S for $i = 1, \dots, i_S$ are of the G^S type, G_i^E for $i = 1, \dots, i_E$ are of the G^E type and G_i^B for $i = 1, \dots, i_B$ are of the G^B type. Then call C_i^S for $i \in \{1, \dots, i_S\}$ the nodes in subgraph G_i^S, similarly call C_i^E for $i \in \{1, \dots, i_E\}$ the nodes in subgraph G_i^E and C_i^B for $i \in \{1, \dots, i_B\}$ the nodes in subgraph G_i^B. Also call n_i^S, n_i^E, n_i^B the number of sellers of C_i^S, C_i^E, C_i^B respectively (similarly for m regarding buyers). Recall that the decomposition gives us immediately the number of pairs involved in any maximum matching (see ch. 2). This is what we state below.

Lemma 8 *If a graph G decomposes into a union of subgraphs*

$$G = G_1^S \cup G_2^S \cup \dots \cup G_{i_S}^S \cup G_1^E \cup G_2^E \cup \dots \cup G_{i_E}^E \cup G_1^B \cup G_2^B \cup \dots \cup G_{i_B}^B$$

then:

$$v_G(N) = m_{C_1^S} + \dots + m_{C_{i_S}^S} + m_{C_1^E} + \dots + m_{C_{i_E}^E} + n_{C_1^B} + \dots + n_{C_{i_B}^B}$$

Proof. By the previous lemma, we simply have to count the number of pairs involved in any maximum matching. This result was already shown in ch. 2. ■

In the next proposition we characterize certain types of graphs for which the core is a singleton.

Proposition 13 *If a graph G is of type G^S, then the unique core allocation of v_G gives zero to all sellers and one to all buyers. (symmetrically, if a graph is of type G^B, the unique core allocation of v_G gives one to all sellers and zero to all buyers).*

Proof. We have a graph $G = \langle N, L \rangle$ that is of type G^S, with $N = S \cup B$ having $|S| = n$ sellers and $|B| = m$ buyers, $n > m$.

Given that the set of all buyers is non-deficient, we can assure that any set of m sellers is almost non deficient, and therefore that there exist as many as $\binom{n}{m}$ possible maximum matchings (all of them involving m different sellers and the m buyers). Moreover, this is clearly the maximum number of pairs involved in a matching in G, that is, $t_{GN} = v_G(N) = m$.

Now, fix one seller, call it s^*, and select one of the $\binom{n}{m}$ maximum matchings in which he is not present. In the optimal matching corresponding to core allocations, only matched pairs distribute their payoff among them, while unmatched agents get a payoff of zero (see subsection 4.3.2). Therefore we conclude that $x_{s^*} = 0$ in any core allocation. Given that this procedure can be repeated for any seller, we conclude that $x_{s_i} = 0$ for all sellers s_i.

Now take one of the maximum matchings and relabel $s_1, ..., s_m$ and $b_1, ..., b_m$ the sellers and the buyers involved in the matching, in such a way that s_i is linked to b_i in the matching, for $i = 1, ..., m$. Finally, given that in the core allocations only matched pairs distribute their payoff among them,

$$x_{s_i} + x_{b_i} = 1 \text{ for all } i$$

We easily conclude that $x_{s_i} = 0$ for all $i = 1, ..., n$ and that $x_{b_i} = 1$ for all $i = 1, ...m$. ∎

Let us now move to a general market given by graph G. We will show below the main result of this chapter, namely that an allocation is in the core of v_G if and only if it is also in the core of each of the associated games of the subgraphs in the decomposition. In particular this will imply that in any allocation in the core, sellers in a G^S or buyers in a G^B get 0, and buyers in a G^S and sellers in a G^B get 1. Moreover, this result also allows us to find a simplified method of characterizing the core of this type of assignment markets.

Theorem 5 *If graph G decomposes as:*

$G = G_1^S \cup G_2^S \cup ... \cup G_{i_S}^S \cup G_1^E \cup G_2^E \cup ... \cup G_{i_E}^E \cup G_1^B \cup G_2^B \cup ... \cup G_{i_B}^B$. *Then:*

An allocation x belongs to the core of $v_G \iff$ the allocation $x_{|C_i^S}$ belongs to the core of $v_{G_i^S}$, $x_{|C_i^E}$ belongs to the core of $v_{G_i^E}$ and $x_{|C_i^B}$ belongs to the core of $v_{G_i^B}$.

Proof. We will now translate the statement in terms of the equations that characterize the core. in (4.2). The left side in the implication tells us that $x = (x_a)_{a\in N}$ belongs to the core of v_G, that is:

$$\begin{aligned} x_a &\geq 0 \text{ for all } a \in N \\ x_{s_i} + x_{b_j} &\geq 1 \text{ for all } i,\ j \text{ s.t. } a_{ji} = 1 \\ \sum_{i=1}^{n} x_{s_i} + \sum_{j=1}^{m} x_{b_j} &= v_G(N) \end{aligned} \tag{4.5}$$

where by lemma 8 we know that $v_G(N) = m_{C_1^S} + ... + m_{C_{i_S}^S} + m_{C_1^E} + ... + m_{C_{i_S}^E} + n_{C_1^B} + ... + n_{C_{i_S}^B}$.

The right side in the implication tells us that $x_{|C_i^S}$ belongs to the core of $v_{G_i^S}$, $x_{|C_i^E}$ belongs to the core of $v_{G_i^E}$ and $x_{|C_i^B}$ belongs to the core of $v_{G_i^B}$, that is:

$$\begin{aligned} x_a &\geq 0 \text{ for all } a \in C_i^S,\ a \in C_j^E,\ a \in C_k^B \\ x_{s_l} + x_{b_m} &\geq 1 \text{ for all } s_l,\ b_m \text{ s.t. } s_l \text{ is connected to } b_m \text{ in } G_i^S \\ x_{s_l} + x_{b_m} &\geq 1 \text{ for all } s_l,\ b_m \text{ s.t. } s_l \text{ is connected to } b_m \text{ in } G_j^E \\ x_{s_l} + x_{b_m} &\geq 1 \text{ for all } s_l,\ b_m \text{ s.t. } s_l \text{ is connected to } b_m \text{ in } G_k^B \\ \sum_{s_l\in C_i^S} x_{s_l} + \sum_{b_m\in C_i^S} x_{b_m} &= v_{G_i^S}(C_i^S),\ \sum_{s_l\in C_j^E} x_{s_l} + \sum_{b_m\in C_j^E} x_{b_m} = v_{G_j^E}(C_j^E) \\ \sum_{s_l\in C_k^B} x_{s_l} + \sum_{b_m\in C_k^B} x_{b_m} &= v_{G_k^B}(C_k^B) \end{aligned} \tag{4.6}$$

for $i = 1, ..., i_s$, $j = 1, ..., i_E$, $k = 1, ..., i_B$.

Step 1 $\Longrightarrow$) We have to show that equations in (4.5) imply equations in (4.6). Recall that by lemma 8 we know that $v_{G_i^S}(C_i^S) = m_i^S$, $v_{G_j^E}(C_j^E) = m_j^E$, $v_{G_k^B}(C_k^B) = n_k^B$ for $i = 1, ..., i_s$, $j = 1, ..., i_E$, $k = 1, ..., i_B$. The only inequalities that does not follow directly are the three last ones in (4.6). But recall that by the core definition in (4.1) v_G it has to be the case that:

$$\begin{aligned} \sum_{s_l \in C_i^S} x_{s_l} + \sum_{b_m \in C_i^S} x_{b_m} &\geq v_G(C_i^S), \ \sum_{s_l \in C_j^E} x_{s_l} + \sum_{b_m \in C_j^E} x_{b_m} \geq v_G(C_j^E) \quad (4.7) \\ \sum_{s_l \in C_k^B} x_{s_l} + \sum_{b_m \in C_k^B} x_{b_m} &\geq v_G(C_k^B) \end{aligned}$$

Now, by lemma 8 we have that $v_G(C_i^S) = m_i^S$, $v_G(C_j^E) = m_j^E$ and $v_G(C_k^B) = n_k^B$. Substituting above and adding up for all agents, we have that:

$$\sum_{i=1}^{n} x_{s_i} + \sum_{j=1}^{m} x_{b_j} \geq m_{C_1^S} + ... + m_{C_{i_S}^S} + m_{C_1^E} + ... + m_{C_{i_S}^E} + n_{C_1^B} + ... + n_{C_{i_S}^B} = v_G(N) \quad (4.8)$$

By the last equation in (4.5) the inequality in (4.8) is actually an equality, and all the equations in (4.7) are also equalities.

Step 2 $\Longleftarrow$) Take vectors $(x_a)_{a \in C_i^S}$, $(x_a)_{a \in C_j^E}$, $(x_a)_{a \in C_k^B}$ which fulfill (4.6) for $i = 1, ..., i_s$, $j = 1, ..., i_E$, $k = 1, ..., i_B$. Consider then the juxtaposition of all this vectors into a single vector $(x_a)_{a \in N}$. We should show that $(x_a)_{a \in N}$ fulfills (4.5).

The first and the last inequalities in (4.5) are immediately implied.

The second inequality in (4.5) tells us that each pair of connected agents gets no less than 1. We know that this is the case for connected agents inside subgraphs G_i^S, G_j^E, G_k^B. Therefore, it remains to be shown that a pair of connected agents belonging to different subgraphs in G also get a payoff that adds up to at least 1.

Recall that by construction of the decomposition (see ch. 2), the links among agents of different subgraphs may be only of one of these types:

1- A seller in a $G_{i_1}^S$ with a buyer in a $G_{i_2}^S$ (eq. a buyer in a $G_{i_1}^B$ with a seller in a $G_{i_2}^B$)

2- A seller in a G_j^E with a buyer in a G_i^S (eq. a buyer in a G_j^E with a seller in a G_k^B)

3- A seller in a G_k^B with a buyer in a G_j^E (eq. a buyer in a G_i^S with a seller in a G_j^E)

Also, recall that we know that core allocations in G^S and G^B are given by the proposition above (only zeros and ones). That is, sellers in a G^B or buyers in a G^S get one. Since any link of type 1, 2 or 3 involves one agent that we know gets a payoff of one, it is immediate that the sum of the payoffs of two agents of types 1,2 or 3 will add up to at least one. ■

We then get the immediate corollary which characterizes the core allocations for nodes in subgraphs G^S and G^B.

Corollary 2 *Take any graph G. Then its core allocations must give 0 to sellers in a G^S or buyers in a G^B, and 1 to buyers in a G^S or sellers in a G^B.*

We now show another result that will be useful to characterize the nodes in G which get a unique payoff in any core allocation. The following proposition will tell us that nodes belonging to a subgraph of type G^E will never get a unique payoff by any core allocation, since the allocation which gives 1 to all sellers, 0 to all buyers and the allocation which gives 0 to all sellers, 1 to all buyers are always in the core.

Proposition 14 *Take a graph G which is of type G^E, with $n = m$ buyers and sellers. Define allocation x_1 as $x_{s_i} = 1$ for all s_i in g and $x_{b_j} = 0$ for all b_j in G. Symmetrically, define allocation x_2 as $x_{s_i} = 0$ for all s_i in G and $x_{b_j} = 1$ for all b_j in G. Then, both x_1 and x_2 are in the core of v_G.*

Proof. We just have to check that the equations:

$$\begin{aligned} x_a &\geq 0 \text{ for all } a \in N \\ x_{s_i} + x_{b_j} &\geq 1 \text{ for all } i,\ j \text{ s.t. } a_{ji} = 1 \\ \sum_{i=1}^{n} x_{s_i} + \sum_{i=1}^{m} x_{b_i} &= v(N) \end{aligned}$$

are fulfilled. In any G^E we have that $n = m$ and that $v(N) = n$. Therefore the equations simplify to:

$$\begin{aligned} x_a &\geq 0 \text{ for all } a \in N \\ x_{s_i} + x_{b_j} &\geq 1 \text{ for all } i,\ j \text{ s.t. } a_{ji} = 1 \\ \sum_{i=1}^{n} x_{s_i} + \sum_{i=1}^{n} x_{b_i} &= n \end{aligned}$$

which are trivially satisfied both for allocation x_1 and x_2. ∎

As a corollary we characterize the graphs for which the core is a singleton.

Corollary 3 *Take any graph G_v. Then:*

1. *The core of G_v is a singleton $\Leftrightarrow$ G_v decomposes as a union of subgraphs of types G^S or G^B.*

2. *A node in G_v gets always the same payoff by any core allocation $\Leftrightarrow$ this node belongs by the decomposition to a G^S or to a G^B.*

4.4 Selections of the core: kernel and fair solution.

In the previous section we studied some properties of the core of our game. We now wish to further analyze two related solutions. First, we will review the literature that deals with the kernel of assignment games, which is equivalent to the symmetrically pairwise bargained solution studied by Rochford (84) (for definitions and discussions on the kernel, see Osborne and Rubinstein (94) or Myerson (91)). The kernel turns out to have nice properties which have highlighted this solution as a very natural value for assignment games. Second, we will study the solution which comes as the limit of the most natural PEP in the non-cooperative game of ch. 3. This second solution will turn out to coincide with the fair solution defined by Thompson (81), which is considered to be appealing for its simplicity.

4.4.1 The kernel or the symmetrically pairwise bargained solution

Relating assignment games to bargaining, Rochford (84) characterized certain points in the interior of the core of an assignment game as fixed points of a "rebargaining" process, in which matched pairs are thought of as bargaining over their transfer payments. Rochford called the corresponding solution the symmetrically pairwise bargained allocations (SPB) for assignment games, and she saw it as a way of arriving at a distribution of the profit generated by a matched pair between the two players who constitute the pair. Moreover, she showed that the set of SPB allocations is equal to the intersection of the kernel and the core. Roth and Sotomayor (88) further studied the structure of the SPB allocations (mainly, that its structure shares the lattice property of the core), while Benett (88) formulated a similar bargaining process which led again to a solution lying in the core.

A recent result shown independently by Driessen (98) and Granot (95) clarifies the role of the SPB solution. These authors have shown that in an assignment game the kernel is totally contained in the core. As an implication, we have now that the SPB set is actually equivalent to the kernel. This reinforces the kernel as a natural and appropriate concept for assignment games.

4.4.2 The fair solution or the limit of the PEP of ch. 3

We now define a second solution concept that will be very related to the non-cooperative game studied in ch. 3. We choose to concentrate on the most natural equilibrium of the non-cooperative game, which is characterized directly by the Gallai-Edmonds decomposition (see ch. 2 and ch. 3). Indeed, recall that one of the equilibria of the non-cooperative bargaining game is the following:

- $x_{s_i} = 0$ if $s_i \in G_j^S$ for some $j \in \{1, .., i_S\}$, $x_{s_i} = 1$ if $s_i \in G_j^B$ for some $j \in \{1, .., i_B\}$, $x_{s_i} = \frac{1}{1+\delta}$ if $s_i \in G_j^E$ for some $j \in \{1, .., i_E\}$

- $x_{b_i} = 0$ if $b_i \in G_j^B$ for some $j \in \{1, .., i_B\}$, $x_{b_i} = 1$ if $b_i \in G_j^S$ for some $j \in \{1, .., i_S\}$, $x_{b_i} = \frac{\delta}{1+\delta}$ if $b_i \in G_j^E$ for some $j \in \{1, .., i_E\}$

Even if this is not the unique equilibrium in all cases, it is the unique equilibrium for most of the graphs. Moreover, this equilibrium is immediately characterized by the decomposition (i.e., by knowing the "type" of each node, see ch. 2).

As the discount δ tends to one (that is, when the friction given by the impatience of agents vanish), the above solution will converge to:

- $x_{s_i} = 0$ if $s_i \in G_j^S$ for some $j \in \{1, .., i_S\}$, $x_{s_i} = 1$ if $s_i \in G_j^B$ for some $j \in \{1, .., i_B\}$, $x_{s_i} = \frac{1}{2}$ if $s_i \in G_j^E$ for some $j \in \{1, .., i_E\}$

- $x_{b_i} = 0$ if $b_i \in G_j^B$ for some $j \in \{1, .., i_B\}$, $x_{b_i} = 1$ if $b_i \in G_j^S$ for some $j \in \{1, .., i_S\}$, $x_{b_i} = \frac{1}{2}$ if $b_i \in G_j^E$ for some $j \in \{1, .., i_E\}$

This solution coincides with the "fair solution" described in Thompson (81). He defined his fair division point of an assignment game as the midpoint of the "long axis" of the core. Recall that when attention is confined to the core outcomes, there exist seller-optimal and buyer-optimal allocations that are the endpoints of the so called "long axis" (see subsection 4.3.2). Thompson's solution is simply the midpoint of this axis: at this pair division point each individual receives the average of the most and the least that he can receive under any core allocation. Thompson's fair division point reconciles the conflicting interests of the sellers and the buyers over the set of core allocations by treating the two sides of the market as two agents in a constant-sum game, represented by the long axis of the core, and, invoking symmetry, choosing its midpoint. Figure 6 below represents the core of a particular assignment game studied in Balinsky and Gale (91). In the figure we can see the buyer-optimal, the seller-optimal, and the fair solution points.

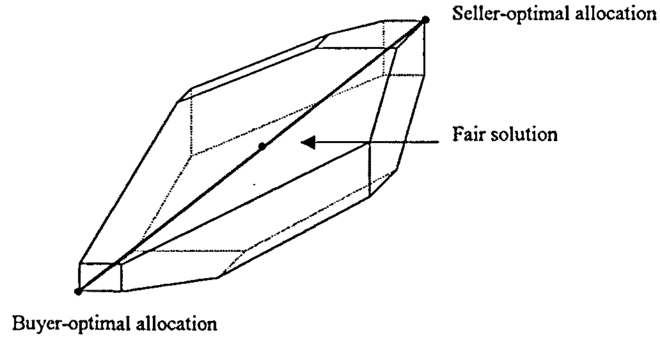


Figure 6

Let us now check that the fair solution coincides with the limit of the non-cooperative game of ch. 3.

Proposition 15 *The fair outcome $x^F = (x^F_{s_1}, ..., x^F_{s_n}, x^F_{b_1}, ..., x^F_{b_m})$ of a bipartite graph G (equivalently, to an assignment game v of our class) is given by:*

- $x^F_{s_i} = 0$ *if* $s_i \in G^S_j$ *for some* $j \in \{1, .., i_S\}$, $x^F_{s_i} = 1$ *if* $s_i \in G^B_j$ *for some* $j \in \{1, .., i_B\}$, $x^F_{s_i} = \frac{1}{2}$ *if* $s_i \in G^E_j$ *for some* $j \in \{1, .., i_E\}$
- $x^F_{b_i} = 0$ *if* $b_i \in G^B_j$ *for some* $j \in \{1, .., i_B\}$, $x^F_{b_i} = 1$ *if* $b_i \in G^S_j$ *for some* $j \in \{1, .., i_S\}$, $x^F_{b_i} = \frac{1}{2}$ *if* $b_i \in G^E_j$ *for some* $j \in \{1, .., i_E\}$

Proof. By our previous results we know that sellers in G^S or buyers in G^B always get zero in a core allocation, and similarly buyers in G^S or sellers in G^B always get one. Moreover, by proposition 14, we know that in a G^E the allocations which give one to sellers, zero to buyers and the one that gives zero to sellers, one to buyers, are also in the core. Clearly, these two allocations are the best for sellers and the best for buyers respectively. Therefore, the midpoint of the two extreme points in the core will give a payoff of $\frac{1}{2}$ to all members of a subgraph of type G^E. ■

Note that the fair division point is a kernel allocation only in special circumstances. For instance, both graphs G_3 and G_4 in Figure 7 have the allocation $x^* = (x_{s_1}, x_{s_2}, x_{b_1}, x_{b_2}) = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ as its fair solution. It is easy to check though that while the allocation x^* belongs to the kernel of graph G_4, it does not belong to the kernel of G_3. Indeed, the kernel of G_3 takes into account the asymmetry among seller s_1 and s_2 and among b_1 and b_2, and gives a payoff higher to s_1 (b_1) than to s_2 (b_2).

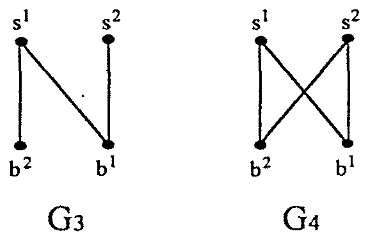


Figure 7

Finally, note also that we believe that the coincidence in our setup of the most natural PEP of the game in ch. 3 and the fair solution is not robust. Even if we have not explored heterogeneity in ch. 3 we believe that it will easily yield to a large set of PEP. Given that it is not obvious how should one select from the set of equilibria, and that the fair solution is always a singleton, the correspondence will no longer hold.

4.4.3 Algorithms related to the fair solution

Several authors have developed algorithms to find the extremal points of the core and more precisely to find the S-optimal core payoff and the B-optimal core payoff (which determine the fair solution). Recall that in ch. 2 we reviewed the algorithm which is used to find the Gallai-Edmonds decomposition. Note that given the equivalence in our context of the fair solution and the Gallai-Edmonds decomposition, these algorithms can now be seen as equivalent. Among them, there is the auction mechanism described by Demange and Gale (85) and Demange, Gale and Sotomayor (86), which computes the minimum price equilibrium and the maximum price equilibrium. From this price equilibrium it is easy to compute the S-optimal and the B-optimal core payoffs. The algorithm in Solymosi, T. and Raghavan, T.

E. S. (1994), constructed to compute the nucleolus of an assignment game, can also be used for the same purpose.

4.5 Conclusion

In this chapter we have looked at the two-sided network markets studied in this thesis from a cooperative point of view.

As a first step we realize that ours is a special type of assignment game, which already gives us a lot of information about the core of our cooperative game. We then move to a further analysis of the core. Using an equivalence among the cooperative game and a bipartite graph, we are able use the decomposition characterized in ch. 2 and give several results about its core. We characterize which are the nodes for which any core allocation always gives the same payoff and, more importantly, we find that the core follows the same structure as the decomposition of ch. 2. Indeed, the core of the associated graph can be found as the union of the cores of each of the subgraphs, and viceversa. This simplifies considerably the task of characterizing the core in general.

We then point to a particular selection of the core, the one that arises when taking the limit, as the discount value of the agents tends to 1, of the most natural equilibrium solution found in the non-cooperative game of ch. 3. This selection turns out to coincide with the fair solution described by Thompson (81). Moreover, the fair solution can in this context be characterized immediately in terms of the Gallai-Edmonds decomposition (see ch. 2).

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Chapter 5

Endogenous link formation on a two-sided market

5.1 Introduction

There is now an emerging literature of theoretical models explaining endogenous formation of networks. This literature addresses questions as how do networks among agents get created and how do they evolve and whether the resulting communication is efficient or stable.

Aumann and Myerson (88) were among the first in studying the issue of graph endogeneity, in the specific context of cooperative games under the Shapley value. They construct a linking game in which agents are offered to form links according to a rule of order, and once the cooperation graph g has been determined, the payoff to each player is defined as the Myerson value (Myerson (77) value is an extension of the Shapley value of a coalitional game v to the case of an arbitrary cooperation structure g). Dutta et al. (98) model cooperation structure formation as a game in normal form, and under a superadditive game, show that several equilibrium refinements predict the formation of the complete cooperation structure or a payoff-equivalent structure provided the solution concept satisfies reasonable properties. As a by-product, they characterize the class of weighted Myerson values. Kranton and Minehart (98) also model link formation through a two-stages game, and in their setup the payoff is realized according to an ascending bid auction.

Jackson and Wolinsky (96) analyze the stability and efficiency of social and economic networks, when self-interested individuals can form or sever links, in the context of non-directed networks (where we need the agreement of both sides to form a link). They allow the graph, the value of the graph and the allocation rule to be general. They show that there is no allocation rule satisfying anonymity and component balance such that for each value of the graph we can find a graph which is both efficient and stable. Dutta and Mutuswami (97) look at the relationship between stable and efficient networks in further detail using an implementation approach and show that certain efficient networks can be supported as being individually stable by weakening anonymity. Jackson (99) revisits these questions under directed networks (where we do not need the agreement of both sides to form a link).

We construct here a model of endogenous link formation in the particular setup of a two-sided network of identical buyers and sellers. We build on the results of ch.3 and ch.4, which justify the use of the fair solution as a natural value in this context. We assume that after the creation of links, agents will split the payoffs according to this value. Our paper differs in the ones mentioned above in that our setup is not general: we apply an specific allocation rule for which the graph completely determines both its value and the allocation.

There is also literature concerning creation of links modelled as a dynamic process. Watts (97) extends the Jackson and Wolinsky (96) model to a dynamic game, limiting attention to the specific context of the "connections model" discussed by Jackson and Wolinsky (96) and a particular deterministic dynamic. Jackson and Watts (98) analyze the dynamic formation and stochastic evolution of networks. Over time individuals form and sever links, creating a sequence of networks, which can either cycle or lead to a stable network. The evolutionary process is used to select from those. Their focus is on non-directed networks . On the other hand, Bala and Goyal (96) examine directed communication networks in a repeated game with a focus on learning, and find that learning leads fairly quickly to the emergence of stable social networks, which in many instances are socially efficient. Finally, also related to learning and evolution, Quin discusses how cooperation evolves under learning processes, Goyal and Janssen (97) show that network structure affects whether or not coordination occurs in evolutionary game theory, and Young (99) and Bala and Goyal (98) study issues relating networks and social learning.

The chapter starts by carefully studying the effect of one new link on the payoffs that agents get. Once we know which are the consequences of the addition or the removal of one link (and, as a consequence, of several links), we move to the study of stable and efficient networks. Stable networks are those in which no pair of agents has incentives to build or sever a new link among them, and efficient networks are those networks such that there exists no other network in which the sum of the total payoffs is larger. We address both the issue of pairwise stability and coalitional stability. We then model an static one-stage game in which links are produced at a cost, and payoffs are determined according to the fair value.

The chapter proceeds as follows. After defining the model in section 5.2, we move in section 5.3 to the study of the consequences of adding a new link. Section 5.4 deals with stability and efficiency properties, and section 5.5 analyzes the one-shot game of link formation. Finally, sections 5.6 and 5.7 discuss some extensions and conclude.

5.2 The Model

The situations we are interested are those in which a set of n sellers $\{s_1, ..., s_n\}$ and m buyers $\{b_1, ..., b_m\}$ are connected through a bipartite graph $g = \langle S \cup B, L \rangle$, where $S \cup B$ are the set of *nodes*, and L are a set of *links*, each link joining a seller with a buyer. A connection means that trade is possible among linked agents. In ch. 3 we constructed a non-cooperative game of alternating offers played on this market and analyzed its equilibria. Moreover, in ch. 4 a particular selection of the core (the fair value) is highlighted as an appropriate value, as it arises when taking the limit as the discount tends to 1 of the most natural equilibrium of the non-cooperative game. In this paper we take as given that, once the network is formed, the payoff agents will get is given by the fair value. The purpose here is therefore to concentrate in the process by which a network evolves.

The notation $x_a(g)$ denotes the payoff agent a gets if the current network is given by g. The payoff is given by the fair solution, which gives:

- $x_{s_i}(g) = 0$ if $s_i \in G_j^S$ for some $j \in \{1, .., i_S\}$, $x_{s_i}(g) = 1$ if $s_i \in G_j^B$ for some $j \in \{1, .., i_B\}$, $x_{s_i}(g) = \frac{1}{2}$ if $s_i \in G_j^E$ for some $j \in \{1, .., i_E\}$
- $x_{b_i}(g) = 0$ if $b_i \in G_j^B$ for some $j \in \{1, .., i_B\}$, $x_{b_i}(g) = 1$ if $b_i \in G_j^S$ for some $j \in \{1, .., i_S\}$, $x_{b_i}(g) = \frac{1}{2}$ if $b_i \in G_j^E$ for some $j \in \{1, .., i_E\}$

where G_j^S, G_j^E, G_j^B are the subgraphs of graph g by the decomposition studied in ch. 2.

Therefore, note that in our analysis there is a bijection among the type of subgraph a node belongs to by the decomposition and its payoff.

We now devote the next section to the study of the change in the payoffs when one new connection is added or deleted. For all our subsequent results the information we will derive now will be very useful.

5.3 Adding one link

We now devote this section to the study of the consequences of the addition or the removal of one link. For a given graph g, the decomposition is telling us precisely the payoff that each agent gets. Therefore, all we need to study is how does the decomposition change when one new link is added or when one link is deleted, which has already been studied in ch. 2., subsection 2.6.2.

5.3.1 What happens when a new link is created?

Call $g = \langle S \cup B, L \rangle$ the initial graph. Take any seller s and buyer b that are not connected in g. Now, call g' the graph that arises when we add this new link to g, *i.e.*, $g' = \langle S \cup B, L \cup \{s : b\} \rangle$. Say that by the decomposition of g, seller s belongs to a subgraph G_1 and buyer b belongs to a subgraph G_2.

We will now write in the propositions below the effect of a new link, speaking in terms of the payoff agents get. All we need to do is translating the results of subsection 2.6.2 in terms of payoffs.

For a better exposition, we state first the results in terms of the effect of a new link on the two newly linked agents, and then the effect on the whole graph.

Effect on the two newly linked agents

The results are stated for a seller in a G^S, in a G^E or in a G^B (the other cases can be shown symmetrically).

Proposition 16 • *Suppose that seller s has $x_s(g) = 0$. Then,*

Remark 2 a) *if buyer b has $x_b(g) = 1$, then $x_s(g') = 0$, $x_b(g') = 1$ (no change).*

b) *if buyer b has $x_b(g) = \frac{1}{2}$, then $x_s(g') = 0$, $x_b(g') = 1$ (no change for the seller, but the buyer increases his payoff from $\frac{1}{2}$ to 1).*

c) *if buyer b has $x_b(g) = 0$, then $x_s(g') = \frac{1}{2}$, $x_b(g') = \frac{1}{2}$ (both agents increase from 0 to $\frac{1}{2}$).*

• *Suppose seller s has $x_s(g) = \frac{1}{2}$. Then,*

a) *if buyer b has $x_b(g) = 1$, then $x_s(g') = \frac{1}{2}$, $x_b(g') = 1$ (no change).*

b) *if buyer b has $x_b(g) = \frac{1}{2}$, then $x_s(g') = \frac{1}{2}$, $x_b(g') = \frac{1}{2}$ (no change).*

c) *if buyer b has $x_b(g) = 0$, then $x_s(g') = 1$, $x_b(g') = 0$ (no change for the buyer, but the seller increases his payoff from $\frac{1}{2}$ to 1).*

- *Suppose that seller s has $x_s(g) = 1$. Then, there is never a change in the payoffs.*

For convenience the results of the last propositions are summarized in the table below. The entries tell us which is the effect of the newly linked agents in the new graph.

sand bare newly linked in g'	$x_b(g) = 1$	$x_b(g) = \frac{1}{2}$	$x_b(g) = 0$
$x_s(g) = 0$	$=$	$x_s(g') = 0$ (=), $x_b(g') = 1$	$x_s(g') = \frac{1}{2}, x_b(g') = \frac{1}{2}$
$x_s(g) = \frac{1}{2}$	$=$	$=$	$x_s(g') = 1, x_b(g') = 0$ (=)
$x_s(g) = 1$	$=$	$=$	$=$

Table 1: Entries tell us the payoffs for seller s and buyer b given by the fair value on the g'-decomposition. If they are the same as in the g-decomposition, we write "=".

The following table displays the change in the payoffs for seller s and buyer b respectively.

sand b are newly linked in g'	$x_b(g) = 1$	$x_b(g) = \frac{1}{2}$	$x_b(g) = 0$
$x_s(g) = 0$	$=,=$	$=,\uparrow$	$\uparrow,\uparrow$
$x_s(g) = \frac{1}{2}$	$=,=$	$=,=$	$\uparrow,=$
$x_s(g) = 1$	$=,=$	$=,=$	$=,=$

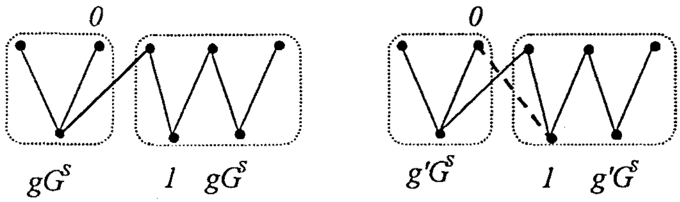
Table 2: Entries tell us the change in the payoffs for seller s and buyer b.

Note that if an agent has in network g the maximum payoff he can get (which is equal to 1), he can not lose and have a lower payoff by creating a new link. In other situations, agents may increase their payoff if linked with the appropriate partner. Note also that no player can increase from a payoff of 0 to a payoff of 1 (we can not jump two steps in one shot).

See Figure 1 for some examples. In the figures, we display the change in payoffs from the addition of a new link in several different contexts.

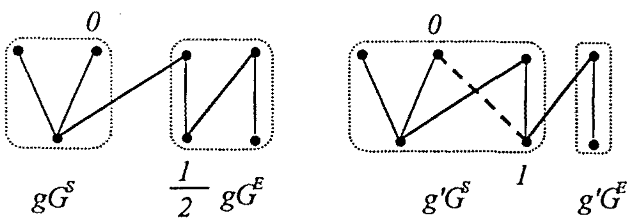
EXAMPLES

→ a seller who gets 0 is linked with a buyer who gets 1



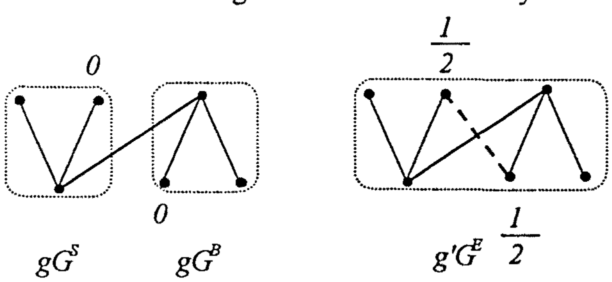
no change

→ a seller who gets 0 is linked with a buyer who gets 1/2



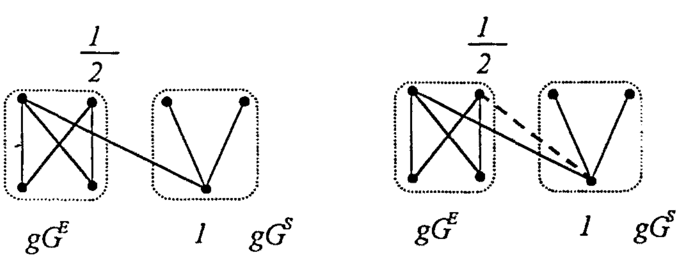
seller equal,
buyer increases

→ a seller who gets 0 is linked with a buyer who gets 0



seller increases,
buyer increases

→ a seller who gets 1/2 is linked with a buyer who gets 1



no change

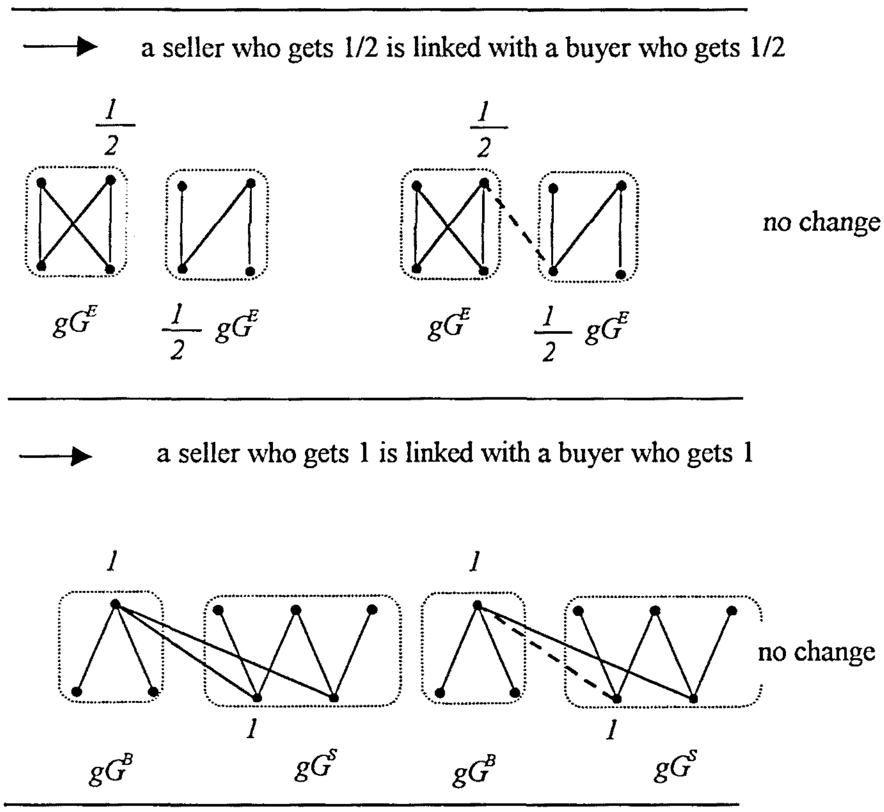


Figure 1

Other examples are also displayed later in Figure 2 and 3.

Effect on the nodes not related to the new link

We now summarize the effect of the new link on the agents other than s and b. From corollary 1 in ch. 2 we know that the effect of a new link is "local", in the sense that only subgraphs G_1 and G_2 can be affected by a link among s and b. This feature makes the analysis much easier.

Corollary 4 *A new link among seller s and buyer b only may change the payoffs of agents belonging to the two subgraphs s and b belonged to by the g-decomposition.*

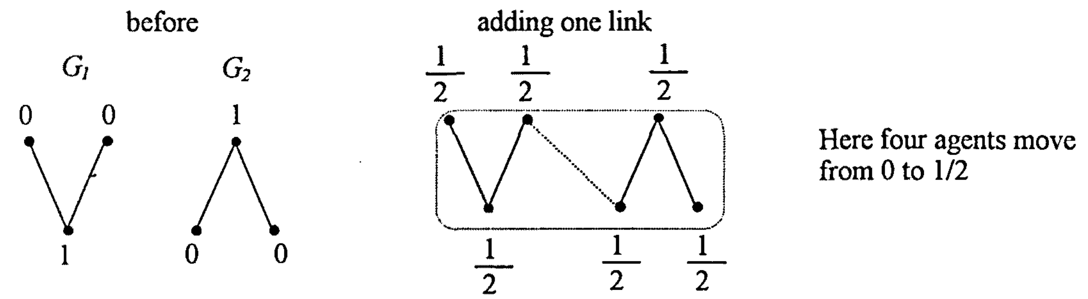
Let us concentrate then on the effect of a new link among seller s and b on nodes belonging to subgraphs G_1 and G_2. Again we just need to translate the results of 2.6.2 in chapter 2.

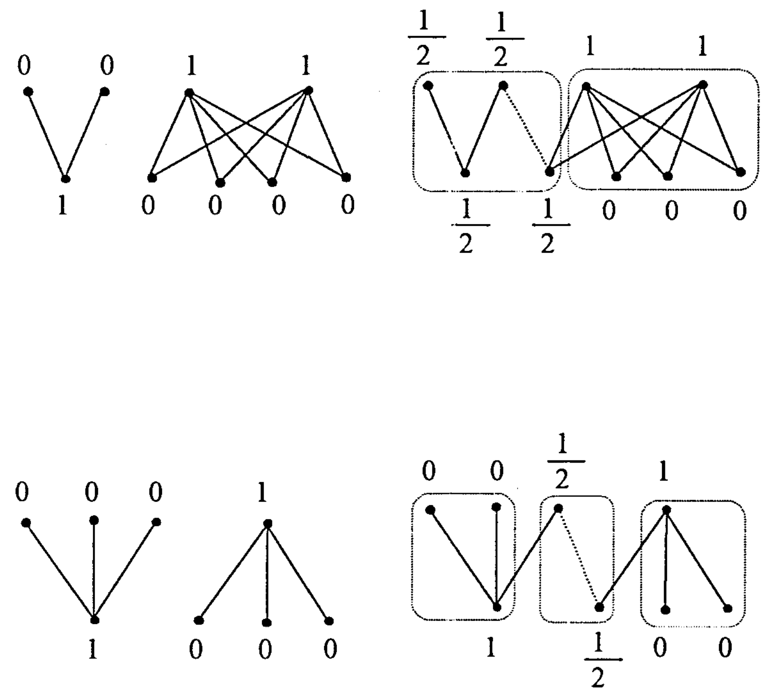
s and b newly linked in g'	$x_b(g) = 1$	$x_b(g) = \frac{1}{2}$	$x_b(g) = 0$
$x_s(g) = 0$	=	in $G_1 \setminus s$ no change (=) sellers in $G_2 \setminus b$ get 0 or $\frac{1}{2}$ buyers in $G_2 \setminus b$ get $\frac{1}{2}$ or 1	sellers in $G_1 \setminus s$ get 0 or $\frac{1}{2}$ buyers in $G_1 \setminus s$ get $\frac{1}{2}$or 1 sellers in $G_2 \setminus b$ get $\frac{1}{2}$or 1 buyers in $G_2 \setminus b$ get 0 or $\frac{1}{2}$
$x_s(g) = \frac{1}{2}$	=	=	sellers in $G_1 \setminus s$ get $\frac{1}{2}$ or 1. buyers in $G_1 \setminus s$ get 0or $\frac{1}{2}$ in $G_2 \setminus b$ no change (=)
$x_s(g) = 1$	=	=	=

Table 3: Inside entries tell us the payoffs for nodes in subgraphs $G_1 \setminus s$ and $G_2 \setminus b$ for the g'-decomposition.

See Figures 2 and 3 for some examples on the effect of a new link on all agents in the graph.

Seller that gets 0 newly linked with buyer that gets 0





Here three agents move from 0 to 1/2

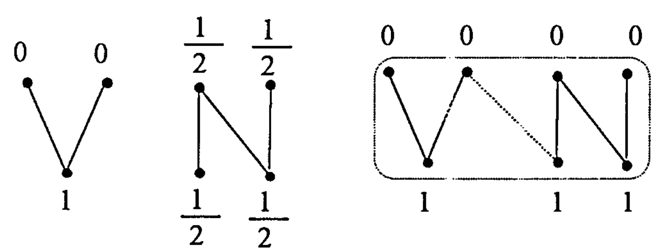
Here two agents move from 0 to 1/2

Figure 2

Seller that gets 0 newly linked with a buyer that gets 1/2

before

G_1 G_2 adding one link



Here two agents move from 1/2 to 1

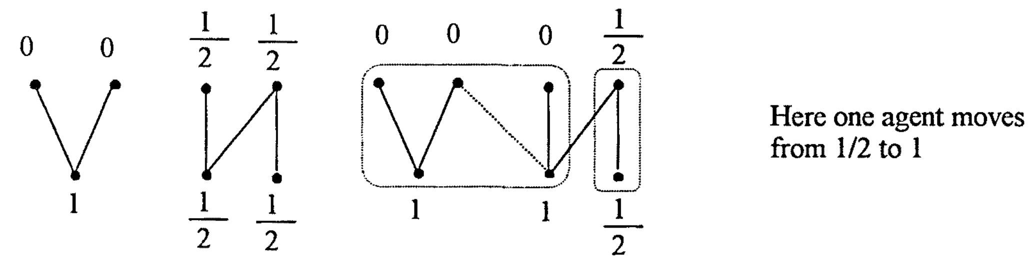


Figure 3

5.3.2 Main results

Interesting results come now as easy corollaries of the above results.

The first corollary below tells us that having more connections is never harmful for the two newly linked agents. Indeed, any link is weakly Pareto improving for the agents that build the new link. That is, when two agents create a new link among themselves, it is never the case that at least one of them gets a strictly worse payoff. As we can easily see from table 2, either none of the agents improves his payoff or at least one of them strictly improves his payoff. A new link is not always strictly Pareto improving for the two agents since some links leave the agents with the same payoff. Note also that a new link may worsen off some agents other than the two involved in the new link. See figure 4 for an example in which a new link affects the buyer in G_1 and the seller in G_2 (agents not involved in the new link) in a negative way (their payoff decreases from 1 to $\frac{1}{2}$).

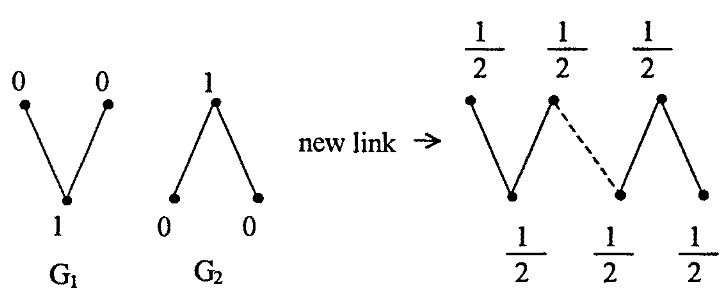


Figure 4

For simplicity we abuse notation and denote $g' = g + \{s_i : b_j\}$ the graph $g' = \langle S \cup B, L \cup \{s_i : b_j\}\rangle$

Corollary 5 *Take any graph g. Then any new link that connects two nodes is weakly Pareto improving for the two newly linked nodes, i.e, for any s_i and b_j, $x_{s_i}(g) \leq x_{s_i}(g + \{s_i : b_j\})$ and $x_{b_j}(g) \leq x_{b_j}(g + \{s_i : b_j\})$*

Proof. Immediate from table 2. ■

The second corollary refers to the structure of the initial graph g. Suppose that the graph g does not decompose as a union of subgraphs which are all of the same type. This is the same as saying that there exist two buyers or two sellers getting a different payoff in g. Then, there will exist at least one agent that, by building a new connection, would be strictly better off.

Corollary 6 *Take any graph g. If $x_{s_i}(g) \neq x_{s_j}(g)$ for two sellers $s_i \neq s_j$ or $x_{b_i}(g) \neq x_{b_j}(g)$ for two buyers $b_i \neq b_j$, then there exists at least one agent $a \in S \cup B$ such that $x_a(g') > x_a(g)$ for a graph g' which results of the addition to g of a new link involving agent a.*

Proof. Case $n > m$) Either the decomposition involves subgraphs of types G^S, G^E and G^B , or it involves G^S and G^B, or it involves subgraphs of types G^S and G^E. In the first case, any seller in G^S and buyer in G^B would strictly increase their payoff from zero to $\frac{1}{2}$ (seller) and $\frac{1}{2}$ (buyer). In the second case, any buyer in G^E, if newly linked to a seller in G^S, would increase his payoff from $\frac{1}{2}$ to 1.

Case $n = m$) The decomposition either involves G^S and G^B or it involves subgraphs of types G^S, G^E and G^B. Then any seller in a G^S and buyer in a G^B would strictly increase their payoff by forming a link among them.

Case $n < m$) Symmetric to cased $n > m$, with roles of buyers and sellers reversed. ■

We build on the above results for the rest of the chapter. We now move to studying the stable networks, those in which no agent has incentives to build new links.

5.4 Stability and efficiency properties of a graph

We will start here by studying two cooperative properties of a network, already studied in Jackson and Wolinsky (96), and Dutta and Mutuswami (97): stability and efficiency. We will then study which networks fullfil these properties.

As before, call $x_a(g) = (x_{s_1}(g), ..., x_{s_n}(g), x_{b_1}(g), ..., x_{b_m}(g))$ the payoff given by the fair value for a given graph g.

5.4.1 Definitions

We first define the concepts of pairwise stability and coalitional stability, and we then move to efficiency.

Pairwise Stability

Following the definition of Jackson-Wollinsky (96), a network will be *stable* if there exists no pair of agents which has incentives to create a new link among them by facing a cost[1]. We examine two possible definitions, when transfers are allowed and when they are not.

If transfers are not allowed, we say that a network will be pairwise stable if for all agents not linked in g, if a new link would benefit one of them (so that he would be willing to pay a cost equal to c), then it makes the other worse off. In other words, there exists no new link which strictly benefits at least one of them in an amount greater than c and still makes the other agent not worse off. In this definition we assume that agents in a pair can not share the cost or the benefits of a new connection.

Our initial graph is given by $g = \langle S \cup B, L \rangle$.

Definition 9 *A network g is pairwise stable without transfers iff:*
for all $s_i : b_j \notin g$, if $x_{s_i}(g) < x_{s_i}(g + \{s_i : b_j\}) - c$ then $x_{b_j}(g) > x_{b_j}(g - \{s_i : b_j\})$ and:

[1] For simplicity we assume that agents can not sever links. See section 5.6 for a discussion on this issue.

for all $s_i : b_j \notin g$, *if* $x_{b_j}(g) < x_{b_j}(g + \{s_i : b_j\}) - c$ *then* $x_{s_i}(g) > x_{s_i}(g - \{s_i : b_j\})$

An alternative concept would be that of pairwise stability when transfers are allowed. In this context a network will be pairwise stable if for all agents not linked in g, the amount they would jointly get by making a new link does not compensate for the cost of the new link. In this definition we assume that agents in a pair can share the cost and the benefits of a new connection.

Definition 10 *A network g is pairwise stable with transfers iff:*
for all $s_i : b_j \notin g$, $x_{s_i}(g) + x_{b_j}(g) > x_{s_i}(g + \{s_i : b_j\}) + x_{b_j}(g + \{s_i : b_j\}) - c$

Coalitional Stability (with transfers)

We will now address a different issue. Which are the networks that are coalitionally-stable, in the sense that there exists no coalition which has incentives to build new links, sharing the costs among them? For simplicity we assume that the coalition can only form 1 new link. (further research could try to solve the general case in which a coalition can form a number of links among them).

As an example, look at the following figure.

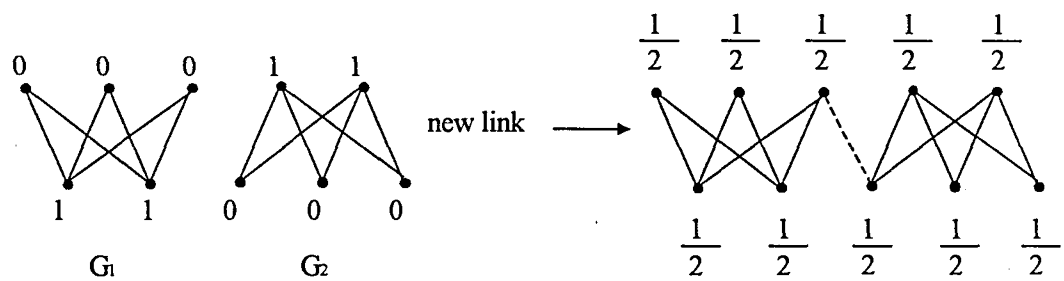


Figure 5

Clearly, if the cost of creating a new link would be very large (say, 2), a seller and a buyer could not afford building a new link. Note, though, that if the six nodes that used to get zero form a coalition and jointly pay for a new link, now they would all six get $\frac{1}{2}$. That is, they would jointly gain 3, which compensates for the cost.

Definition 11 *A network g is coalitionally pairwise stable iff, for every coalition of players $C \subseteq N$ we have that:*

$$\sum_{a \in C} x_a(g) > \max_{for\ each\ pair\ s_i\ ,b_j \in C} \left\{ \sum_{a \in C} x_a(g + \{s_i : b_j\}) \right\} - c$$

(it doesn't pay for any coalition C to connect two of its members and share the cost of the new connection).

Note that coalitional stability is equivalent to pairwise stability with transfers for the case of coalitions of size 2.

Efficiency

We will define now the efficiency of a network, relating it to the total welfare of a network. We define the *total welfare* of a network g as the sum of the payoffs for each agent, that is, $W(g) = \sum_{i=1}^{n} x_{s_i}(g) + \sum_{j=1}^{m} x_{b_j}(g)$. Recall (see ch. 4) that in our case the welfare of a graph can be directly computed from its decomposition: it equals the number of pairs involved in any maximum matching. That is, if g decomposes as:

$$g = G_1^S \cup G_2^S \cup ... \cup G_{i_S}^S \cup G_1^E \cup G_2^E \cup ... \cup G_{i_E}^E \cup G_1^B \cup G_2^B \cup ... \cup G_{i_B}^B$$

Call n_i^S, n_i^E, n_i^B the number of sellers of G_i^S, G_i^E, G_i^B respectively (similarly for m regarding buyers). Then:

$$W(g) = m_{C_1^S} + ... + m_{C_{i_S}^S} + m_{C_1^E} + ... + m_{C_{i_S}^E} + n_{C_1^B} + ... + n_{C_{i_S}^B}$$

We will say that a network is efficient if it maximizes the total welfare, i.e, if there exists no other network involving the same agents in which the total welfare is greater.

Definition 12 *For a given set of agents $S \cup B$, a network $g = \langle S \cup B, L \rangle$ is efficient if there exists no $g' = \langle S \cup B, L' \rangle$ such that $W(g') > W(g)$.*

(i.e, there is no other network involving the same agents in which the welfare is greater).

5.4.2 The results

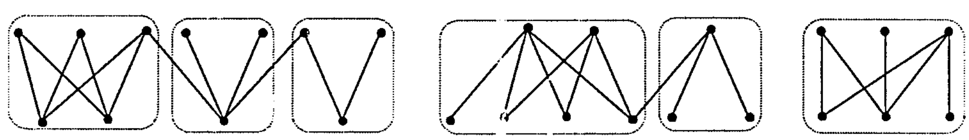
We now wish to characterize pairwise stable, coalitionally stable and efficient networks. In order to do so we now introduce a special type of graphs that will turn out to play an important role.

Reference Networks

Definition 13 *We will say that a graph g is a* reference graph *if g decomposes into subgraphs which are all of the same type.*

See the figure for some examples.

REFERENCE NETWORKS



G^S_1 G^S_2 G^S_3 G^B_1 G^B_2 G^E_1

Figure 6

Reference networks have three useful properties that we now enumerate..

- *A graph g is a reference network iff all sellers get the same payoff and all buyers get the same payoff.*

  In terms of payoffs, reference networks are easy to characterize: they are exactly those in which all sellers get the same payoff and all buyers also get the same payoff (recall that the type of subgraph a node belongs to determines its payoff and viceversa).

- *A graph g is a reference network iff it supports the reference solution as a PEP of the non-cooperative game of ch. 3.*

  We also want to remark that reference graphs have close links with the research relating to the non-cooperative game studied in chapter 3. The main result in ch. 3 relates reference networks and the competitive equilibria. Recall that in ch.3 we define the reference solution of a network as the most natural competitive equilibrium we would get in the case of complete communication. That is, if there are more agents on one side than on the other, the reference solution gives 0 to the long side and 1 to the short side, while if the number of buyers and sellers is equal, it gives a payoff of $\frac{1}{2}$ to all agents. Theorem 4 shows that a network supports the reference solution as a PEP of the non-cooperative game iff it is a reference network.

- *A reference graph is always efficient.*

  Another important property of reference graphs is that they are always efficient. Indeed, in any reference graph the total welfare adds up to $\min\{n, m\}$, which is the maximum welfare any network with n sellers and m buyers can achieve. Note that the reverse is not true, since a graph g involving one G^E and one G^B is efficient but it is not a reference network.

Results relating to pairwise stability

We now show that stable and reference networks are equivalent when the cost is sufficiently small, both for pairwise stability with or without transfers.

Proposition 17 *(Pairwise stability without transfers)*

If $c < \frac{1}{2}$, then a network g is pairwise stable without transfers $\Longleftrightarrow$ g is a reference network

If $c \geq \frac{1}{2}$, then all networks are pairwise stable without transfers.

Proof. It is clear that if the cost is $\geq \frac{1}{2}$, no agent can afford to pay the cost, since the maximum gain a new connection can imply is never larger than $\frac{1}{2}$.

Suppose then that $c < \frac{1}{2}$

$\Longleftarrow$) It is easy to show that reference networks are stable. Suppose that g is a reference network. The creation of a new link does not change the type of any of its nodes, while it involves a cost.

$\Longrightarrow$) Suppose to the contrary that graph g is not a reference network. By Corollary 6, there exists a pair of agents such that by creating a new link among them, one of them strictly increases his payoff by $\frac{1}{2}$ (and by corollary 5, the other one is not worse off). Then, such a network can not be stable. ■

Proposition 18 *(Pairwise stability with transfers)*

If $c \leq \frac{1}{2}$, then a network g is pairwise stable with transfers $\Longleftrightarrow$ g is a reference network

If $\frac{1}{2} < c < 1$, then a network g is pairwise stable with transfers $\Longleftrightarrow$ g does not contain both G^S and G^B in its decomposition. (equivalently, g is either a reference network or is such that in its decomposition all subgraphs are of type G^E and G^B or are of type G^E and G^S only).

If $c \geq 1$, then all networks g are pairwise stable with transfers.

Proof. As before it is clear that if $c \leq \frac{1}{2}$, then pairwise stable w.t. networks are equivalent to reference networks, since in any non reference network there is the possibility of creating a new link, while the common gain is at least $\frac{1}{2}$.

Clearly, if the cost exceeds 1, no new link is worth its cost.

Suppose now that the cost is $\frac{1}{2} < c < 1$.

$\Longleftarrow$) If g does not contain both G^S and G^B, then no pair can afford to pay more than $\frac{1}{2}$ for a new connection, since the only profitable new link would be that of a seller in G^E with a buyer in G^B (or of a buyer in G^E with a seller in G^S), with the common profit $\frac{1}{2}$ only.

$\Longrightarrow$) Suppose to the contrary that g contains both G^S and G^B. Then a seller in G^S and a buyer in G^B could get link, pay for the cost, and get a joint profit of 1. Therefore such a g is not pairwise stable with transfers. ■

Results relating to coalitional stability

We now wish to characterize the coalitionally stable networks. Recall that in pairwise stable networks with transfers the cost can be shared by any pair of players. Here, the cost of building a new connections can be shared among any coalition. As before, we will be able to conclude that for small costs, coalitionally stable networks are reference networks, while for very large costs, any network is stable.

To go on with the analysis we need to use the results on the effect of a new link not only on the two newly linked agents but on the other members of the network. Precisely, we will be able to construct an algorithm which computes a quota for a given graph g which we will call v_g. This value v_g will tells us which is the maximum benefit that a coalition in the network would get from a new link among them. Clearly if the cost is greater than this value, the network will be coalitionally stable.

As before, call $g = \langle S \cup B, L \rangle$ the initial graph. Take any seller s and buyer b that are not connected in g. Now, call g' the graph that arises when we add this new link to g, *i.e.*, $g' = \langle S \cup B, L \cup \{s : b\} \rangle$. Say that by the decomposition of g, seller s belongs to a subgraph G_1 and buyer b belongs to a subgraph G_2.

Construction of the quota v_g

We will now explain how to compute v_g, which will give us the maximum benefit a coalition can gain from a new link.

• *Computation of maximum benefit a coalition can have through a new link among a seller in gG^S and a buyer in gG^B.*

If there are no subgraphs of type G^S and G^B in g, call $v_{SB} = 0$. Otherwise, take all subgraphs of g which are of type gG^S. Call them $G_1^S, ..., G_{is}^S$. Now, suppose that some of these subgraphs, relabel them as $G_1^S, ..., G_j^S$ are such that $n_i^S = m_i^S + 1$ for $i = 1, .., j$. Then, call:

$\overline{G^S} = \left\{ G_k^S : \text{ such that } n_k^S = \max \left\{ n_i^S : \text{ for } i = 1, ..., j \right\} \right\}$.

Otherwise, if all subgraphs in $G_1^S, ..., G_{is}^S$ are such that $n_i^S > m_i^S + 1$ for $i = 1, .., i_s$ then select any of them an relabel it as $\overline{G^S}$.

Similarly, take all subgraphs of g which are of type gG^B. Call them $G_1^B, ..., G_{i_B}^B$. Now, suppose that some of these subgraphs, relabel them as $G_1^B, ..., G_j^B$ are such that $m_i^B = n_i^B + 1$ for $i = 1, .., j$. Then, call $\overline{G^B} = \left\{G_k^B : \text{ such that } m_k^B = \max\left\{m_i^B : \text{ for } i = 1, ..., j\right\}\right\}$.

Otherwise, if all subgraphs in $G_1^B, ..., G_{i_S}^B$ are such that $m_i^B > n_i^B + 1$ for $i = 1, .., i_s$ then select any of them an relabel it as $\overline{G^B}$. Call $\overline{n^S}$ and $\overline{m^S}$ the number of sellers and buyers in $\overline{G^S}$ and $\overline{n^B}$ and $\overline{m^B}$ the number of sellers and buyers in $\overline{G^B}$.

If $\overline{G^S}$ is such that $\overline{n^S} = \overline{m^S} + 1$ and $\overline{G^B}$ is such that $\overline{m^B} = \overline{n^B} + 1$ then call $v_{SB}^1 = \overline{n^S} + \overline{m^B}$

If $\overline{G^S}$ is such that $\overline{n^S} = \overline{m^S} + 1$ and $\overline{G^B}$ is such that $\overline{m^B} > \overline{n^B} + 1$ then call $v_{SB}^2 = \overline{n^S} + 1$

If $\overline{G^S}$ is such that $\overline{n^S} > \overline{m^S} + 1$ and $\overline{G^B}$ is such that $\overline{m^B} = \overline{n^B} + 1$ then call $v_{SB}^3 = 1 + \overline{m^B}$

If $\overline{G^S}$ is such that $\overline{n^S} > \overline{m^S} + 1$ and $\overline{G^B}$ is such that $\overline{m^B} > \overline{n^B} + 1$ then call $v_{SB}^4 = 1 + 1 = 2$

Call $v_{SB} = \frac{1}{2}\max\left\{v_{SB}^1, v_{SB}^2, v_{SB}^3, v_{SB}^4\right\}$. This is the maximum benefits that a coalition formed by sellers and buyers that used to get zero and now get $\frac{1}{2}$ would get. (see figure 2 and recall ch. 2, section 2.6.2).

- *Computation of maximum benefit a coalition can have through a new link among a seller in gG^S and a buyer in gG^E.*

Consider all subgraphs of g which are of type gG^E and relabel them as $G_1^E, ..., G_{i_E}^E$. Recall that in remark 1, ch. 2, section 2.6.2, we explained a decomposition of a subgraph of type G^E into two different subgraphs. Do the decomposition for all G_i^E, spliting every subgraph into G_{j1}^E and G_{j2}^E for $j \in \left\{G_1^E, ..., G_{i_E}^E\right\}$. Now, consider the number of buyers for each subgraph G_{j1}^E and denote by $\overline{G^E}$ the subgraph among $\left\{G_{11}^E, ..., G_{i_E 1}^E\right\}$ with the maximum number of buyers. Call $\overline{m^E}$ =number of buyers in $\overline{G^E}$. Then, define v_{SE} as 0 if there are no subgraphs of type G^S and G^E in g and as $v_{SE} = \frac{1}{2}\overline{m^E}$ otherwise. This is the maximum benefit that a coalition can get if forming a new link among a seller in a G^S and a buyer in a G^E.

• *Computation of maximum benefit a coalition can have through a new link among a buyer in* gG^B *and a seller in* gG^E.

Symmetrically as before, construct v_{BE}. This is the maximum benefit that a coalition can get if forming a new link among a buyer in G^B and a seller in G^E.

• *Computation of maximum benefit any coalition can have through a new link.*

Finally, consider $v_g = \max\{v_{SB}, v_{SE}, v_{BE}\}$.

We are now ready to characterize coalitionally stable networks depending on their quota v_g.

Proposition 19 *For graph* g*, construct the quota* v_g*. Then,*

If $c \leq \frac{1}{2}$*, then* g *is coalitionally stable* $\Leftrightarrow$ g *is a reference network*

If $c > \frac{1}{2}$*, then* g *is coalitionally stable* $\Leftrightarrow$ *the cost* $c > v_g$.

Proof. First of all, note that as before all reference networks will be coalitionally stable, since the addition of a new link changes nothing. On the other hand, similarly as before, coalitionally stable networks if $c \leq \frac{1}{2}$ will be reference networks. Therefore, let us now assume that g is not a reference network, so that in its decomposition there are subgraphs of different types. Then, clearly if $c \leq v_g$, graph g is not coalitionally stable. If $c > v_g$, then graph g is coalitionally stable. ■

Results relating to efficiency

We now characterize which are the efficient networks.

Proposition 20 *A network* g *is efficient* $\Longleftrightarrow$

$(n > m)$ g *is a reference network or* g *decomposes as a union of* G^E *and* G^S.

$(n = m)$ g *is a reference network.*

$(n < m)$ g *is a reference network or* g *decomposes as a union of* G^E *and* G^B.

(equivalently, g is efficient $\Longleftrightarrow$ the decomposition of g does not contain both G^B and G^S).

Proof. $\Longleftarrow$) It can be checked that all the networks above satisfy that $W(g) = \max\{n, m\}$, with n being the number of sellers in g and m the number of buyers in g. It is immediate that this is the maximum welfare of any network involving these agents.

$\Longrightarrow$) Suppose to the contrary that g decomposes as a union of subgraphs, one of them being of type G^B and another one of type G^S. Formally, $g = G_1^S \cup G_2^B \cup G_3 \cup ... \cup G_t$ with G_1^S of type G^S and G_2^B of type G^B. Call C_1^S and C_2^B respectively the agents belonging to G_1^S and G_2^B. Also, call n_1, n_2 and m_1, m_2 the number of sellers and buyers respectively of subgraphs G_1^S, G_2^B. By definition we have that $n_1 > m_1$ and that $n_2 < m_2$. Then :

$$W(g) = m_1 + n_2 + W(G_3 \cup ... \cup G_t).$$

Now, we will construct a network g' different than g, involving the same agents, which has larger total welfare. Indeed, take agents belonging to $C_1^S \cup C_2^B$ and link each seller with all buyers. Call this new graph G_{12}. The graph G_{12} is complete. Complete graphs always decompose as a unique subgraph (see ch. 3) and are reference networks. They are therefore efficient, and satisfy that $W(G_{12}) = \max\{\text{sellers in } G_{12}, \text{buyers in } G_{12}\} = \max\{n_1 + n_2, m_1 + m_2\}$. Now, consider graph g' which decomposes as $g' = G_{12} \cup G_3 \cup ... \cup G_t$. Graph g' involves the same agents as graph g. Moreover,

$$W(g') = \max\{n_1 + n_2,\ m_1 + m_2\} + W(G_3 \cup ... \cup G_t)$$

Given that $n_1 > m_1$ and that $n_2 < m_2$, $m_1 + n_2 < n_1 + n_2 \leq \max\{n_1 + n_2, m_1 + m_2\}$, which implies that $W(g') > W(g)$. ■

Note then that all stable networks are efficient.

5.5 Static game of link formation

We now move to a simple one-shot game of link formation, to study which networks will arise when agents are allowed to create links.

5.5.1 The game

The one-period game agents play is the following:

Each seller s_i announces a vector $g_{s_i} = (g_{s_i,b_1}, ..., g_{s_i,b_m})$, where each $g_{s_i,b_j} \in \{0,1\}$.

Each buyer b_i announces a vector $g_{b_i} = (g_{b_i,s_1}, ..., g_{b_i,s_n})$, where each $g_{b_i,s_j} \in \{0,1\}$.

(this is interpreted as: "$g_{b_i,s_j} = 1$" means buyer b_i wants to be linked to seller s_j, while "$g_{b_i,s_j} = 0$" means buyer b_i does not want to be linked to seller s_j).

The union of all actions defines a network. We consider that it is enough that one person wants a link for this link to be created, or if you want, an agent does not need the permission of the other agent to build a road connecting the two (see section 5.6 for a different model). Formally, the result of the actions (the $n + m$ vectors) is graph g, where:

$$g = \langle S \cup B, L \rangle \text{ with } L = \left\{ s_i : b_j \text{ such that } g_{s_i,b_j} = 1 \text{ or } g_{b_j,s_i} = 1 \right\}$$

Then, the final payoff of the game is the one that comes as a result of the fair value on the network (that, recall, can be either 0, 1, or $\frac{1}{2}$) minus tc where t is the number of connections that the agent decided to build.

5.5.2 Minimal networks

Let us now define a special type of graphs that we will call minimal graphs. Those will be the graphs that contain the minimal number of links to preserve the structure.

Definition 14 *Take graph $g = \langle S \cup B, L \rangle$, and consider the payoff of each of its nodes. This graph will be called minimal*[2] *if there exists no other graph*

[2]Equivalently, we can say that a network g is minimal if by removing one connection, some of its agents change its type. Indeed, if by removing (or adding) a connection, the two involved agents do not change its type, then neither do the other agents change their type (see tables 1 and 3).

$g' = \langle S \cup B, L' \rangle$ (involving the same agents) such that all the nodes get the same payoff in g or in g', and g' is the result of removing one link from g, i.e, there exist a seller $s \in S$ and buyer $b \in B$ such that $L' = L \setminus \{s : b\}$.

We now state and show a result that completely characterizes the structure of minimal networks. Indeed, minimal networks are very simple. They will always be a union of disconnected subgraphs of the type shown in Figure 7. Either they are linked pairs (subgraphs of type G^E), isolated nodes, or a subgraph G^S with one more seller than buyer such that every buyer is linked to exactly two sellers. (symmetrically, subgraphs G^B with one more buyer than seller such that every seller is linked to exactly two buyers).

MINIMAL NETWORKS

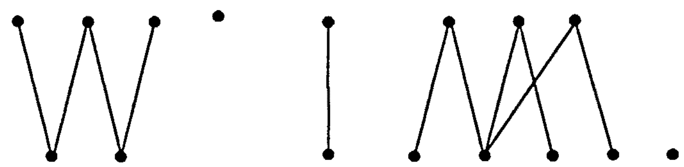


Figure 7

We now state the proposition and leave the proof for the appendix.

Proposition 21 *Take a graph g and its decomposition, $g = G_1^S \cup ... \cup G_{i_S}^S \cup G_1^E \cup ... \cup G_{i_E}^E \cup G_1^B \cup ... \cup G_{i_B}^B$.*

Then, g is a minimal graph $\Longleftrightarrow$

- *There are no connections linking nodes belonging to different subgraphs among $\left\{G_1^S, ..., G_{i_S}^S, G_1^E, ..., G_{i_E}^E, G_1^B, ..., G_{i_B}^B\right\}$.*
- *subgraphs G_i^S for $i = 1, .., i_S$ are of one of these two types:*

  $G_i^S = \langle \{s_{i_1}\}, \phi \rangle$ for a seller $s_{i_1} \in S$. That is, G_i^S is one isolated seller.

  $G_i^S = \langle S_i \cup B_i, L_i \rangle$, connected, with $|S_i| = |B_i| + 1$, and with $\left|N_{G_i^S}(b_j)\right| = 2$ for each buyer b_j belonging to G_i^S.

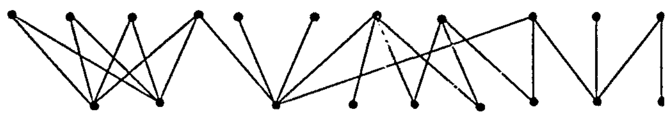
- *subgraphs G_i^B for $i = 1, ..., i_B$ symmetric to G_i^S, with the roles of buyers and sellers reversed.*

- *subgraphs G_i^E for $i = 1, ..., i_E$ are equal to $G_i^E = \langle \{s_i, b_j\}, \quad \{s_i : b_j\} \rangle$, that is, a pair of linked agents.*

Proof. See the Appendix ■

We display below two examples of a minimal and a not minimal network. Moreover, note that by definition any not minimal network has some "spare" connections that we can rule out, obtaining a minimal network with exactly the same structure.

The following network is not minimal:



The following network is minimal:

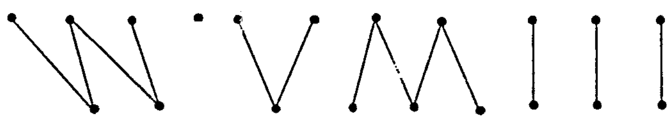


Figure 8

Note that once we have characterized minimal networks, we realize that they have an interesting property. The number of links a minimal network has is again minimal, in the sense that no other network would have the same structure with a smaller number of links. In other words, minimal networks are the "cheapest" when we consider the total cost of building them, compared with other networks which give the same payoff to all agents.

Corollary 7 *If a network $g = \langle S \cup B, L \rangle$ is minimal, then there exists no other network $g' = \langle S \cup B, L' \rangle$ with the same structure (i.e, with all agents getting the same payoffs in g or in g') and with $|L'| < |L|$.*

Proof. Simply note that in any G^E, by the fact that any G^E has at least 2 nodes, any node must have at least one connection. Then, the minimum number of connections in a G^E with n sellers and n buyers is exactly n, i.e, the number of connections of a minimal network. Similarly, in any G^S any buyer must have at least two connections, which is also the case in all minimal networks. ■

5.5.3 Nash networks

We will call Nash network the ones that arise as a Nash equilibria of the one-shot game. We proceed now to characterize the set of Nash networks.

As a starting point we get two immediate lemma. First of all we deduce that a Nash network will be minimal, since if not there would be an agent who would be able to pay for one connection less and still get the same payoff.

Lemma 9 *If graph g is a Nash Network, then it is a minimal network.*

Proof. Suppose to the contrary that $g = \langle S \cup B, L \rangle$ is not minimal. Then, there exists $g' = \langle S \cup B, L' \rangle$ with $L' = L \setminus \{s : b\}$ and such that the payoff for all agents is the same in g or in g'. Then, consider the actions taken in equilibrium by agents s and b. Given that g contains the link $s : b$, it must be the case that either s or b (or both) in equilibrium asked for the connection $s : b$. But a profitable deviation would be asking for exactly the same connections excepting connection $s : b$. We know that the payoff will be the same, hence saving the cost of one connection. ■

Similarly as when we were studying stability, we see now that Nash networks, if the cost is small, will have to necessarily be reference networks, since otherwise there would be one agent who should have deviated in the game and should have asked for one more connection.

Lemma 10 *If the cost $c \leq \frac{1}{2}$, then if graph g is a Nash Network, it is a reference network.*

Proof. Note that it is always worthwhile for a seller in a G_i^S (buyer in a G_j^B) to pay for a new link with a buyer in G_j^B (seller in a G_i^S). Similarly, it is always worthwhile for a seller in G_k^E (buyer in a G_k^E) to pay for a new link with a buyer in a G_j^B (seller in a G_i^S). This is true since in all the cases mentioned above, the agent that creates the new link faces a cost of c and improves his benefit in the amount of $\frac{1}{2}$. Then, clearly if we are not in a reference network, then, if $n > m$ (symmetric to $n < m$), either we have a graph g that decomposes as G^S, G^E and G^B or a graph g that decomposes as a G^S and a G^E, but in all cases one agent will want to create a new link. Similarly, if $n = m$, and g is not a reference network, g decomposes as a G^S, G^E and G^B, and again there is one agent who would get a better payoff by paying for one more link. ■

We are now ready to completely characterize the set of Nash equilibria of our game. For small costs, they will be equivalent to minimal and reference networks, and for large costs the only equilibria will be the one in which nobody pays for any link.

See Figure 9 for example of minimal and reference networks.

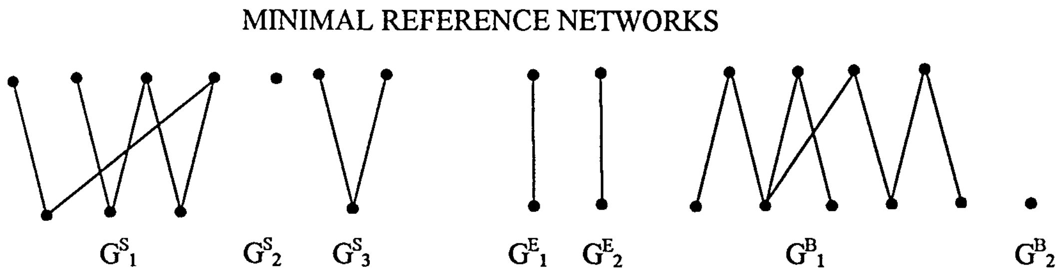


Figure 9

Theorem 6 • *If $c \leq \frac{1}{2}$ then g is a Nash network $\Longleftrightarrow$ g is a minimal reference network*

• *Otherwise, if $c > \frac{1}{2}$, then the only Nash network is the one in which all nodes are isolated.*

Proof. case $c \leq \frac{1}{2}$) Then $\Longrightarrow$) is immediate by lemmas 9 and 10.

$\Longleftarrow$) We separate two different cases.

Case $n = m$) Here we are able to show that if g is a minimal reference network, then it can be supported as a Nash equilibrium.

Take g which is a minimal reference network. Then, g is simply a union of n linked pairs. Relabel sellers as $S = \{s_1, .., s_n\}$ and buyers as $B = \{b_1, .., b_n\}$ where $g = \langle S \cup B, \{s_1 : b_1, ..., s_n : b_n\}\rangle$. Then the strategies supporting g as a Nash network would be:

$\rightarrow$ Seller s_i pays to be linked with b_i.

$\rightarrow$ Any buyer does not pay for any connection.

To see that this is a Nash Equilibrium, let us check possible deviations. Regarding sellers, note that each seller gets a payoff of $\frac{1}{2}$ and pays for 1 links, and therefore each seller gets a final payoff of $\frac{1}{2} - c$. Now, let us use the results from proposition 16 and table 1. If seller s_i is isolated (that would be the result taking the actions of all others as given, and by seller s_i taking the action of asking for zero connections), he gets a payoff of 0, which is worse than $\frac{1}{2} - c$. By adding one connection, he can at most get a payoff of $\frac{1}{2} - c$, which is what he gets now. By adding two connections, he can at most get $1 - 2c$, and $1 - 2c = 2\left(\frac{1}{2} - c\right) > \frac{1}{2} - c$. By adding more than two connections (say, k), he can at most get still 1, and pay for kc, getting $1 - kc$, which may be larger than $\frac{1}{2} - c$ depending on the cost. Then we should check that there is no way that, fixing the actions of other agents, seller s_i can pay for k connections and get more than $\frac{1}{2} - c$.

Recall that in g seller s_i belongs to a G^E and gets a payoff (without cost) of $\frac{1}{2}$, and so do all other agents. Then, if seller s_i pays for a new extra link, he will simply pay for a connection that links him with another agent who also gets $\frac{1}{2}$. Such a new connection will not change the payoff of any of the agents (see again proposition 16 or table 1). Similarly, k links, including $\{s_i : b_i\}$, with $k > 1$ will imply a resulting graph g' in which again all agents get $\frac{1}{2}$, but he would have to pay a higher cost.

Suppose alternatively that seller s_i decides to pay for k links, with $1 \leq k \leq n - 1$, with none of them connecting s_i and b_i. To start with, we see that if s_i decides not to pay for any link, he is isolated and gets zero, buyer b_i is also isolated and gets zero, and other agents still get $\frac{1}{2}$ each. Then, by paying a new link with an agent who gets $\frac{1}{2}$, he will for sure stay getting a payoff of zero. That is, seller s_1 gets a lower payoff, 0, and has to pay a cost of at least c. Clearly this is not a profitable deviation.

Similarly, buyers gain nothing by paying for some links, since in g they get $\frac{1}{2}$, and by creating a new link with a seller that has $\frac{1}{2}$, the structure doesn't change and they keep having the same payoff.

Case $n > m$) Take g which is a minimal reference network. By proposition 21 g decomposes as a union of isolated nodes plus subgraphs G_i^S which fulfill $\left|N_{G_i^S}(b_j)\right| = 2$ for each buyer b_j belonging to G_i^S with $i \in \{t+1, ..., i_S\}$, and moreover with G_i^S having one more seller than buyers (similarly for G_j^B).

Then, let us write $g = G_1^S \cup ... \cup G_t^S \cup G_{t+1}^S \cup ... \cup G_{i_S}^S$, with $G_1^S, ..., G_t^S$ being t isolated sellers (call them $s_1, ..., s_t$) and $G_{t+1}^S, ..., G_{i_S}^S$ being all subgraphs with a number of sellers equal to the number of buyers plus one, and moreover, with $\left|N_{G_i^S}(b_j)\right| = 2$ for each buyer b_j belonging to G_i^S with $i \in \{t+1, ..., i_S\}$. Then the strategies supporting such a network as a Nash equilibria would be:

→ Each buyer in G_i^S pays for its two links in g, for $i \in \{t+1, ..., i_S\}$.

→ All sellers do not pay for any link.

Let us now check for possible deviations. Regarding buyers, note that each buyer gets a payoff of 1 and pays for 2 links, and therefore each buyer gets a final payoff of $1 - 2c$. Now, let us use the results from proposition 16 and table 1. If buyer b_i is isolated, he gets a payoff of 0. By adding one connection, he can at most get a payoff of $\frac{1}{2} - c$. By adding two connections, he can at most get $1 - 2c$, and $1 - 2c = 2\left(\frac{1}{2} - c\right) > \frac{1}{2} - c$. By adding more than two connections, he can at most get still 1, and pay more than $2c$. Therefore, clearly there will exist no better option for buyer b_i than the one taken in the equilibrium.

Now, note that sellers do not pay for any link. Consider sellers which end up isolated, $s_1, ..., s_t$ in g, thus getting a payoff of 0. On the other hand, buyers in g get a payoff of 1. Therefore, would a seller pay for one link with a buyer in $b_1, ..., b_m$, all agents would still get the same payoff (see proposition 16 and table 1), while they would have to face a cost. Paying for 2 or more links would do no better: a new link from a seller who gets 0 with a buyer who gets 1 changes nothing. Therefore, there is no profitable deviation for the sellers either.

Case $n < m$) Symmetric to $n > m$.

Note that in this equilibrium all buyers in G_i^S and all sellers in G_j^B get a payoff of $1 - 2c$, all sellers in G_i^S and all buyers in G_j^B get a payoff of 0, all sellers in G_k^E get $\frac{1}{2} - c$ and all sellers in G_k^E get $\frac{1}{2}$.

case $c > \frac{1}{2}$) Suppose that there exists a Nash network g in which some nodes are connected. Now, look at the decomposition of g. Recall that in any subgraph G_i^S of type G^S any set of buyers is strictly non-deficient. This implies in particular that every single buyer in G_i^S is linked to at least 2 sellers in G_i^S. Therefore, if the g−decomposition involves a subgraph G^S, this implies that each buyer payed for two links, since clearly sellers would not pay a cost to get linked and then get a zero payoff. But, if $c > \frac{1}{2}$, then it doesn't compensate for the buyers to create the links either, since the payoff they will get is 1 but they will have to pay $2c > 1$. Similarly if it involves a G^B. Finally, if it involves a G^E, then one of the two agents in the pair payed for the link. Again, this is impossible since he would get a payoff of $\frac{1}{2}$ and would have to pay a cost $c > \frac{1}{2}$. ∎

5.6 Extensions

We first address two small modifications of the model we have explored.

On allowing to sever a link in the definition of stability.

In our definition of stability, agents can not sever a link. Indeed, in our setup severing a link that connected s_i and b_j can never improve the payoff agents s_i and b_j get. Therefore, if breaking a link would be costly, no agent would ever break a link. If, to the contrary, by breaking a link agents would somhow recover the cost of mantaining it, then the results would keep holding, but now stable networks for small costs would correspond to reference networks which are also minimal (in the sense of 5.5.2).

An alternative model for the static game.

An alternative way to construct the creation of the network would be considering that it is necessary that both agents agree in building a new connection for this connection to be active. That is, for given actions by sellers and buyers, the constructed graph g equals:

$g = \langle S \cup B, L \rangle$ with $L = \left\{ s_i : b_j \text{ such that } g_{s_i,b_j} = 1 \text{ and } g_{b_j,s_i} = 1 \right\}$

We now reproduce the analysis carried out in the last subsection.

Proposition 22 *If $c \leq \frac{1}{2}$, then a graph g is a Nash network $\Longleftrightarrow$ it is the minimal reference network of type G^E.*

Otherwise, if $c > \frac{1}{2}$, the only Nash network is the one in which all nodes are isolated.

Proof. To start with, note that we will never get as a Nash Network a graph g which has in its decomposition subgraphs of type G^S or G^B. This is so since, for instance in G^S sellers get a payoff of 0, while they have to pay for one connection, facing a cost of c. It would be better not to pay for any connection at all and get zero as well. Therefore, if graph g is a Nash Network, then it is a minimal reference network of type G^E.

case $c > \frac{1}{2}$) Here we have that the payoff that connected agents would get, $\frac{1}{2}$, does not compensate for the cost they face. Therefore the only Nash Network will be the one in which no agent pays for any link, and all agents get zero.

case $c \leq \frac{1}{2}$) We have to show that any minimal reference network of type G^E can be supported as a Nash Network. Take a minimal reference network of type G^E which involves a number p of connected agents. Relabel these connected agents as $s_1, ..., s_p$ and $b_1, ..., b_p$, where s_i is connected to b_i. Then the strategies would be:

Each s_i pays to be connected to b_i.

Each b_i pays to be connected to s_i.

Other agents do not pay for any link. ■

Note in particular that, unlike in the first model, many Nash Networks are inefficient. Even if $c \leq \frac{1}{2}$, the Nash Network may involve a number of connected pairs that may go from only 1 pair to the maximum number of pairs $t = \min\{n, m\}$.

An static game starting from a given network.

Regarding possible extensions, we are currently working on building an static game starting from a given network. Suppose now that there is a fixed network g_0 which we interpret as a status quo. Nobody payed any cost to be in g_0, agents were just born in g_0. But now agents are given the opportunity of building new links. This is an interesting extension (also, because it can be used to further investigate dynamic models of link formation) that we have not developed yet. Indeed, we believe that we will get similar results to before, in the sense that all Nash networks will be minimal with respect to g_0 (that is, by removing any one link in $g \setminus g_0$ some agents in g change its type) and will also be reference networks.

5.7 Conclusions

To analyze link formation in the context of a two-sided network market, we start by carefully analyzing which is the effect of a new link on the payoffs that agents get. We reach two interesting and simple results, namely, (i) that a new link is always weakly Pareto improving for the two newly linked agents, and (ii) that if the payoff that all agents of the same type (either sellers or buyers) get is not the same, then there exists at least one agent with incentives to create a new link.

Using this results we move to analyzing stable networks, which if the cost is sufficiently small, are characterized by a special type of networks that we call "reference" networks. These are the networks which are equivalent, in payoff terms, to competitive equilibrium (more specifically, to the reference solution defined in ch. 3) or to complete communication. Efficient networks can also be simply characterized, and all stable networks are efficient.

We then move to the one-shot game of link formation. The result we get is that Nash equilibria of this game must yield again stable networks, and moreover they must be minimal in the sense that no connection is irrelevant. Moreover, every minimal stable network can be supported as a Nash equilibrium. Thus, the set of Nash equilibria is characterized, and turns out to be simple and relatively small.

As a consequence of properties (i) and (ii), the analysis highlights a special type of networks, namely, minimal and reference networks, as the ones one would expect to arise if the cost is sufficiently small. These networks fulfill two interesting properties they are equivalent, in terms of structure or of payoff, to the complete network in which everybody is linked to everybody. Moreover, the payoff agents get in this networks is very simple: all sellers get the same payoff, all buyers get the same payoff, and it corresponds to the "reference" solution defined in ch. 3, which is the most natural competitive equilibrium. In other words: no coordination failure arises because of hte network effect.

The context we have studied is one of a two-sided market with complete information, with only one type of good being traded, and with agents of each side being complete homogeneous. Then, if the cost of a new link is small, one should not expect to see in our context networks that yield results other than competitive equilibrium. Or, in other words, if we do not see the competitive equilibrium to happen even if agents could in principle build new links, it must be the case either than agents are not aware of their full context (and, say, have only a local knowledge of the structure of the network) or because the agents are heterogeneous. It may be true after all that when we see networks in reality, the network is driven by a willingness to be connected to somebody which is different or special.

5.8 Appendix

5.8.1 Proof of proposition 21

$\Longleftarrow$[3] It can be checked that graphs of the type described above are minimal. Indeed, the removal of a connection implies that at least one agent gets a different payoff.

For a linked pair, if we rule out the connection we get two disconnected nodes who will get zero. Thus a linked pair is a minimal network. Call g a connected G^S graph with $\left|N_{G_i^S}(b_j)\right| = 2$. We will show that g is indeed minimal. To see why, recall that in any subgraph of type G^S, any set of buyers is strictly non-deficient (see lemma 1). In particular, this means that every buyer in a subgraph G^S must be linked to at least two sellers. Suppose now that we remove one link from g, say the link $\dot{s}_i : b_j$ with $s_i, b_j \in g$, so that we get a new graph $\overline{g} = g - \{\dot{s}_i : b_j\}$. Clearly in $\overline{g}$ buyer b_j is no longer linked to 2 sellers but only to 1 seller, that is, $|N_{\overline{g}}(b_j)| = 1$. Therefore, b_j is not of type $\overline{g}G^S$. This implies that g is minimal.

$\Longrightarrow$ Take any graph g. By the decomposition we know that $g = G_1^S \cup \ldots \cup G_{i_S}^S \cup G_1^E \cup \ldots \cup G_{i_E}^E \cup G_1^B \cup \ldots \cup G_{i_B}^B$.

As a first step we will show that there are no connections among nodes belonging to different subgraphs among $\left\{G_1^S, \ldots, G_{i_S}^S, G_1^E, \ldots, G_{i_E}^E, G_1^B, \ldots, G_{i_B}^B\right\}$. Suppose that one agent in a subgraph is linked to another agent in another subgraph. W.l.o.g, suppose that a buyer b in G_i^S is linked to a seller s in G_j^E. Then, consider the graph that we get when removing the connection $s : b$ from g. It is immediate that such a subgraph will decompose exactly in the same union as g, and therefore, that all the agents will be of the same type. Therefore, g would not be a minimal subgraph.

Suppose now that at least one of the subgraphs is not of the type described above. We will show that this subgraph must have some connections which can be removed without changing the payoff of any agent, so that g is not minimal.

[3] I would like to thank Joan Vidal for his ideas on how to solve this proof.

a) Suppose that there exists a subgraph G_i^S which is not of the type described above. Call S_i^S and the set of sellers and buyers, respectively, in G_i^S. Call $n_i^s = \left|S_i^S\right|$ and $m_i^s = \left|B_i^S\right|$.

Case a.1) $n_i^s > m_i^s + 1$.

Consider now one seller in G_i^S, call it s_1. Then, consider all buyers linked to s_1, the set $N_{G_i^S}(s_1)$. Consider now one buyer, call it b_1, with $b_1 \in N_{G_i^S}(s_1)$. Consider now the subgraphs that we get if we remove the connection $s_1 : b_1$ from G_1^S.

If $\left|N_{G_i^S}(s_1)\right| = 1$, then when removing the connection $s_1 : b_1$ from G_1^S we get, on the one hand, a subgraph which consists of the isolated seller s_1, call it G_1, and on the other hand the subgraph induced by the set of nodes $S_i^S \setminus s_1 \cup B_i^S$ in G_i^S, call it G_2. Clearly, G_1 is of type G^S. Now, recall that since G_i^S is of type G^S, any subset of sellers is almost non-deficient. In particular, the $n_i^S - 1$ sellers in $S_i^S \setminus s_1$ are collectively linked to the m_i^S buyers, and any subset of sellers in $S_i^S \setminus s_1$ is collectively linked to a set of buyers of at least size $\min\left\{n_i^S - 1, m_i^S\right\}$. Therefore, subgraph G_2 is also of type G^S since it consists of $n_i^S - 1$ sellers and m_i^S buyers, with the set of $n_i^S - 1$ sellers being almost non deficient..

Otherwise, if $\left|N_{G_i^S}(s_1)\right| > 1$, when removing the connection $s_1 : b_1$ from G_1^S we get a connected subgraph, call it G_1. Now, consider the subgraph induced by the set of nodes $S_i^S \setminus s_1 \cup B^S$ in G_1, call it G_{11}. Call G_{12} the subgraph formed by the isolated node s_1. Again, we can show that G_1 decomposes as G_{11} and G_{12}, with both G_1 and G_2 being of type G^S.

Therefore, we have shown that we can remove one connection inside G_i^S, and still all members of G_i^S are of type G^S. That is, g is not minimal.

Case a.2) $n_i^s = m_i^s + 1$. Suppose now that we have a graph G_i^S which is connected, of type G^S and has $n_i^s = m_i^s + 1$.

We want to show that G_i^S is minimal $\Leftrightarrow$ each buyer $b_j \in G_i^S$ is such that $\left|N_{G_i^S}(b_j)\right| = 2$.

$\Rightarrow$) Since in a G^S any set of buyers is strictly non-deficient, we know that for any buyer b_j belonging to G_i^S it will be the case that $\left|N_{G_i^S}(b_j)\right| \geq 2$.

Step 0: Suppose that we have a connected graph $g = \langle S \cup B, L \rangle$, with $n+1$ sellers and n buyers and such that all buyers b in B fulfill $|N_g(b)| = 2$. Then, all nodes and connections in g will be present in the union of another graph g', which is connected, has n sellers and $n-1$ buyers and is such that buyers b in g fulfill $|N_g(b)| = 2$, plus a linked pair of a seller $\overline{s}$ and buyer $\overline{b}$, plus a link joining $\overline{b}$ to a seller in g'.

Proof of step 0: Note that in g all buyers have two links coming out of them. This means there are $2n$ links in g. Moreover, there are $n+1$ sellers in g. This implies that there is at least one seller, we will call him $\overline{s}$, who is linked to only one buyer in g, let us call him $\overline{b}$. Therefore, we can split nodes and connections in g into two groups: on the one hand, the pair $\overline{s} : \overline{b}$, on the other hand the subgraph induced by the set of nodes $S \cup B \setminus \{\overline{s}, \overline{b}\}$ in g, (let us call it g'), and then a connection from buyer $\overline{b}$ to one seller in g' (since buyer $\overline{b}$ must have exactly two connections).

It remains to show that g' fulfills the conditions. Indeed, there are n sellers and $n-1$ buyers in g'. It is also true that g' is connected. To see why, recall that g will be connected if there exists a path in g that links any two nodes in g (see section 2.2). We know that g is connected, and this implies in particular that we can find a path in g connecting any node in g' with a node in g'. If this path uses nodes $\overline{s}$ and $\overline{b}$, then that would mean that the path uses a seller in g', call him s', then goes to $\overline{b}$, then goes to $\overline{s}$, comes back to $\overline{b}$, and then necessarily goes to g' again by linking $\overline{b}$ and s'. This implies that we could rule out this loop and still get a path connecting any two nodes in g' and using only nodes in g'. That is, g' is a connected graph.

We can also check that g' is such that all its buyers b in g' fulfill $|N_{g'}(b)| = 2$. Clearly that was the case in g, and in g' we have all the buyer which were in g excepting $\overline{b}$, and we have not ruled out any of the connections of buyer in g'.

Step 1: What we will show now is that any connected graph g with $n+1$ sellers and n buyers and such that all buyers b in g are such that $|N_g(b)| = 2$ is always of type G^S.

Proof of step 1: To show step 1 we will use induction.

($n = 1$) We can check that for $n = 1$ the condition is fulfilled, since the only graph with 2 sellers, 1 buyer and with the buyer having two connections is the one in which the two sellers are connected to the buyer, and it is of type G^S.

($n = t + 1$) Let us suppose that the result is true for $n = t$. We will show that the result holds for $n = t + 1$.

Take a graph $g = \langle S \cup B, L \rangle$ with $t + 2$ sellers, $t + 1$ buyers, and such that every buyer in g is linked to exactly 2 sellers. By using step 0 we know that all nodes and connections in g are present in the union of another graph g', which is connected, has $t + 1$ sellers and t buyers and is such that buyers b in g fulfill $|N_g(b)| = 2$, plus a linked pair of a seller $\overline{s}$ and buyer $\overline{b}$, plus a link joining $\overline{b}$ to a seller in g'. Call the set of sellers and the set of buyers in g' respectively as $S' = S \setminus \overline{s}$ and $B' = B \setminus \overline{b}$. Then, by the induction step we know that g' is of type G^S. This implies that any subset of k sellers in S', let us call it S'_k, is such that $|N_{g'}(S'_k)| \geq k$ for $k \leq t$. To check that g is also of type G^S we must show that any subset of k sellers in S, let us call it S_k, is such that $|N_g(S_k)| \geq k$ for $k \leq t + 1$. Take such a subset S_k.

case i) If $|S_k| \leq t$. Then either $S_k \subseteq S'$ or $S_k \subseteq S' \cup \overline{s} = S$. In the first case it is immediate that S_k is non deficient in g. Otherwise, if $S_k \subseteq S$, then it must be the case that $S_k = S''_k \cup \overline{s}$, with $S''_k \subseteq S'$. Then, we know that $N_g(S_k) = N_g(S''_k) \cup N_g(\overline{s})$, with $N_g(S''_k)$ being completely in B', and of size $|N_g(S''_k)| \geq |S''_k|$ and with $N_g(\overline{s}) = \left\{\overline{b}, b'\right\}$ with $b' \in B'$. Then we will have that $|N_g(S_k)| \geq |S''_k| + 1 = |S_k|$. That is, we have shown that S_k is non deficient in g.

case ii) If $|S_k| = t + 1$ then either $S_k = S'$ or $S_k \subseteq S' \cup \overline{s} = S$. In the first case, we know that $|N_g(S_k)| = |N_g(S')| = \left|B' \cup \overline{b}\right| = t + 1$. The second case can be solved as case i).

Step 2: Suppose that we have a connected graph $g = \langle S \cup B, L \rangle$ of type G^S, with $|S| = |B| + 1$, such that there exists at least one buyer, let us call it $b \in B$, such that $|N_g(b)| > 2$. We will now show that we can rule out one of the links which come out of b, and we still get a graph g_1 involving $S \cup B$ with $|S| = |B| + 1$ which is connected.

Proof of step 2: Select two sellers out of the set $N_g(b)$ and call them s_1 and s_2. Now, define the graph $g' = \langle S \cup B, L \setminus \{s_1 : b, s_2 : b\}\rangle$, that is, the graph that results when we take out two connections in g. This graph g' may be connected, may be the union of two or three connected subgraphs. We analyze each case separately.

case i) If g' is connected, then clearly $g'' = g' + \{s_2 : b\} = g - \{s_1 : b\}$ will also be connected. Then we have shown that we can rule out connection $\{s_1 : b\}$ from g and still get a connected graph.

case ii) Now suppose that g' is composed of two connected subgraphs, call them g'_1 and g'_2. Then, either s_1 and s_2 belong to two different subgraphs or they both belong to the same one. In the first case, suppose w.l.o.g that b and s_1 belong to g'_1 and that s_2 belongs to g'_2. Then, consider the graph $g'' = g' + \{s_2 : b\} = g - \{s_1 : b\}$. Given that g'_1 is connected, there exists a path linking any two nodes in g'_1, without using the connection $\{s_1 : b\}$ (which is not present in g'_1). Then, in particular, any node in g'_1 can be linked to b' without using the connection $\{s_1 : b\}$ through a path in g'_1. Given that g'' contains the connection $\{s_2 : b\}$, we will therefore be able to construct a path linking any two nodes in g''. Therefore g'' is connected, so we have been able to rule out a connection from g and still get a connected graph.

case iii) Suppose that g' is composed of three connected subgraphs, call them g'_1, g'_2 and g'_3, such that s_1 belongs to g'_1, b belongs to g'_3 and s_2 belongs to g'_2. Given that g is connected but g' is not, it is the case that $s_1 : b$ is the only link joining g'_1 to g'_3 in g, that $s_2 : b$ is the only link joining g'_2 to g'_3 in g, and that g'_1 and g'_2 are not linked to each other in g. See Figure 9 below.

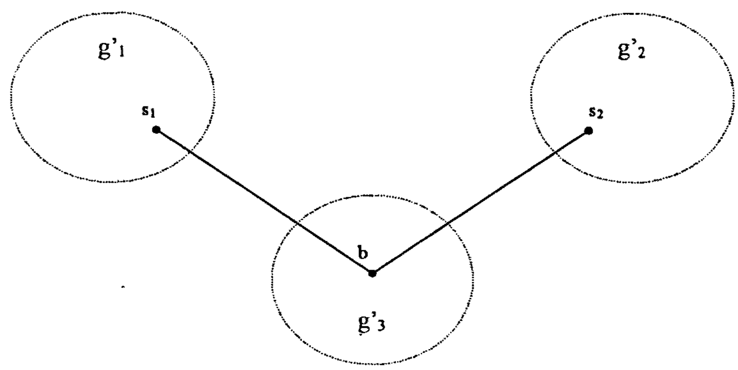


Figure 9

Now, call n'_1 and m'_1 the number of buyers and sellers and buyers in g'_1 and similarly n'_2 and m'_2 and n'_3 and m'_3 for g'_2 and g'_3. Given that g is a subgraph of type G^S, the set composed by the $n'_1+n'_3$ sellers in g'_1 and g'_3, the set composed by the $n'_1+n'_2$ the sellers in g'_1 and g'_2 and the $n'_2+n'_3$ sellers in g'_2 and g'_3 have to be non-deficient in g (collectively linked to at least the same number of buyers). Since the $n'_1+n'_3$ sellers in g'_1 and g'_3 are only linked to the buyers in g'_1 and g'_3, we have that $m'_1+m'_3 \geq n'_1+n'_3$. In a similar way we can deduce that $m'_2+m'_3 \geq n'_2+n'_3$. Finally, note that the $n'_1+n'_2$ are linked to the $m'_1+m'_2$ buyers in graphs g'_1 and g'_2 but also to buyer b. That is,

$$\begin{aligned} m'_1+m'_3 &\geq n'_1+n'_3 \\ m'_1+m'_2+1 &\geq n'_1+n'_2 \\ m'_2+m'_3 &\geq n'_2+n'_3 \end{aligned}$$

Now, adding up all equations:

$$2(m'_1+m'_2+m'_3)+1 \geq 2(n'_1+n'_2+n'_3) \tag{5.1}$$

Recall that g has $|S|$ sellers and $|B|$ buyers with $|S| = |B|+1$. Then, $m'_1+m'_2+m'_3 = |B|$ and $n'_1+n'_2+n'_3 = |S|$, so that the inequality in 5.1 is $2|B|+1 \geq 2|S| = 2|B|+2$, which is a contradiction.

Step 3: Aiming for a contradiction, suppose that we have a connected graph $g = \langle S \cup B, L\rangle$ of type G^S, with $|S| = |B|+1$, such that there exists at least one buyer, let us call it $b \in B$, such that $|N_g(b)| > 2$ and such that g is minimal. Then, by step 2 we know that there exists another graph g_1 involving the same nodes but such that b has one connection less. We can repeat the procedure in step 2 with all buyers that have strictly more than 2 connections, until we reach a graph g_t which involves the same nodes ($S \cup B$ with $|S| = |B|+1$), is connected, and such that each buyer in B is connected in g_t to exactly 2 sellers. Applying step 1 we know that g_t is of type G^S. Therefore, we will have shown that starting from g we can rule out several connections and reach a graph g_t with the same nodes and still G^S. Think of the same procedure on the opposite direction, starting from g_t. All we do is adding more and more links to g_t. But, given proposition 16, the type of the graph will not change, so in any case we will always have a subgraph of type G^S. Therefore, graph g_1 is also of type G^S, and we have shown that g can not be minimal.

b) Suppose that there exists a subgraph G_i^B which is not of the type described above. The proof is symmetric to a).

c) Suppose that there exists a subgraph G_i^E which is not of the type described above. Call S_i^E and B_i^E the set of sellers and buyers, respectively, in G_i^E, and call L_i^E the set of links in G_i^E. Call $n_i^E = \left|S_i^E\right| = \left|B_i^E\right|$.

By the definition of a subgraph of type G^E, there exists a matching involving all players in G_i^E. Choose one of the possible matchings involving all players and relabel them so that the matching is $M = \left\{s_1 : b_1, s_2 : b_2, ..., s_{n_i^E} : b_{n_i^E}\right\}$. Now, given that G_i^E is not of the type described above, it must be the case that there exists at least one connection linking a $s_i \in \left\{s_1, s_2, ..., s_{n_i^E}\right\}$ with a $b_j \in \left\{b_1, b_2, ..., b_{n_i^E}\right\}$, with $i \neq j$. Now, consider the subgraph that results when we remove the connection $s_i : b_j$ from subgraph G_i^E, call it $G_1 = \left\langle S_i^E \cup B_i^E, L_i^E \setminus s_i : b_j\right\rangle$. Either G_1 is a disconnected subgraph or it is not. If it is a connected subgraph, then it is clear of the type G^E, since a maximum matching exists. If it is not a connected subgraph, then G_1 is the union of two connected subgraphs, both of type G^E.

We have ruled out a connection from G_i^E but all its agents keep being of type G^E. Therefore g is not minimal. ■

Note finally that constructing minimal subgraphs is very easy. Indeed, if they are of type G^E they are simply linked pairs. Otherwise, they are isolated subgraphs or graphs of type G^S with one more sellers than buyer. (or symmetric for G^B). To construct such a subgraph, take as an starting point the simple graph in which two sellers are linked to a buyer (call it g). To construct a G^S with 3 sellers and 2 buyers we simply have to add to graph g a new buyer, call it b', with two links, one link joining b' with a seller in g and one link joining b' with a new seller s'. We would now call the resulting graph g, and would repeat the procedure. All minimal subgraphs of type G^S can be constructed in this way.

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