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SPATIAL HETEROGENEITY AND EQUILIBRIUM

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ABSTRACT

This thesis consists of five chapters, based on four different articles. All of them are devoted to different aspects of spatial heterogeneity and its impact on economic equilibrium in space. The concept of heterogeneous continuous space is discussed in the introductory chapter.

The first model "Equilibrium in Continuous Space under Decentralized Production" addresses the issue of the impact of differences across locations in exogeneous productivity on the structure of equilibrium prices, production and trade. The goal is to describe the general equilibrium in a spatially decentralized economy, when production, consumption and markets are distributed in continuous space and transportation costs are essentially linear. It is shown that an autarky equilibrium can exist only if transport costs are high enough. In the general case, the general equilibrium in this model includes some endogeneously determined trade areas, with flows of goods across space, and autarky areas where production and consumption activities take place only at the same point. An analytical solution in explicit functions is obtained; it contains equilibrium prices, labor supply and flows of goods as functions of the spatial variable. The model can be applied to a set of practical questions in regional economics. In particular, it is able to describe persistent price differentials across regions and non-local consequences of road construction and transportation cost shocks for the economy. The differences across locations in population density may have either historical or economic reasons.

The second model "Hotelling's Revival" extends a well-known research of H.Hotelling (1929) to the two-dimensional case with spatially heterogeneous demand density, preserving the rest of his classical assumptions. It is shown that the problem of demand discontinuity in the one-dimensional model, which was discovered by d'Aspremont, Gabszewich and Thisse (1979), disappears in this case. This also holds for any bounded distribution of consumers on any compact set on a plane, which can describe real geographical situations. Demand continuity still holds for any transport costs, strictly increasing in distance and not necessarily linear. Although this is sufficient for the existence of Nash equilibrium in mixed strategies, in pure strategies it exists only for some subset of cases. Examples of both existence and non-existence are constructed, and for some family of densities the separation

point between the two cases is found.

The third model addresses locational choice of heterogeneous consumers, when land is also heterogeneous in quality. It is based on two articles. The first, "Dacha Pricing", is presented in chapter 4 and studies the problem of locational rent in a city-neighbourhood when utility includes both the impact of transport costs and time for transportation. For the case of identical agents the problem is solved explicitly and comparative statics with respect to exogeneous changes in transport cost and speed is studied. For the case of agents who are heterogeneous with respect to their income, a solution is also obtained. The model explains some evidence about dacha pricing in Russia and its dynamics during the transition period. The second article related to this model is "Location and Land Size Choice by Heterogeneous Agents". It generalizes the first one and form a separate chapter 5. A new approach about the general equilibrium allocation of heterogeneous divisible good (like land) among a continuum of heterogeneous consumers is proposed. The model is based on continuity of primitives which allow not only to find a general equilibrium solution in a class of continuous functions, but also to treat the solution to a continuous problem as the limit of the corresponding sequence of discrete problems. This solves one of Berliant's paradoxes, related to spatial economics. The multiplicity of equilibria is shown to take place.

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Contents

Chapter 1

Introduction

1.1 The Link between Chapters

1. The real economic world contains space as a structural element. The first economic paper related to spatial aspect of economy appeared almost two centuries ago (Von Thünen's "An Isolated State" (1828)), and many authors have made significant contributions to this topic since that time. Although modern microeconomic theory neglects the spatial aspects to a large extent, in the real economy they play quite a significant role. This role emerges from the existence of transport costs, which make different locations asymmetric with respect to each other. This is the first source of spatial heterogeneity. The second source is exogeneous heterogeneity, determined outside the economy: geographical and historical factors.

2. This spatial heterogeneity includes at least the following three aspects, which will be adressed in this research.

a) Differences across locations in exogeneous productivity factors for the same economic inputs (like different agricultural productivity of land in different locations, for given labor and capital, optimally invested in these locations).

b) Differences across locations in population density. It is defined either by historical factors (for example, nobody would reoptimize the location of Paris, for example, even if everybody would agree to move it, let say 200 km to the south - too high fixed costs) or by endogeneous agglomeration (something like Silicon Valley phenomenon).

c) Differences in land quality used for residential purposes. There are at least two factors here: different land quality per se, and difference with respect to location to some center (Central Business District model).

3. Three chapters of my thesis are addressing these three questions. While each model has its own economic interest and circle of covered questions (as well as its place inside economic literature), both in pure and applied theory, they are linked together through this framework of spatial heterogeneity.

4. The approach used everywhere is either a general equilibrium framework for a particular choice of primitives (the 2-nd and the 4-th chapters) or locational games of the Hotelling type (chapter 3). The choice of primitives involves different levels of deepness of reasoning, but is always based on some reasonable approximations to the real economic world.

While in migration models, wage differentials are often exogeneously given, but mobility is free, in the model of chapter 2 mobility of consumers is prohibited. Instead incomes are endogeneously determined by producers' choice, who have not only the freedom to choose the labor supply, but also to deliver goods to the markets of their neighbours.

The second model deals with two-dimensional Hotelling competition. Here agents are also immobile and exogeneously distributed with some density in a two-dimensional space with Euclidean metric and some transport costs, strictly increasing in distance.

In the third model agents can choose locations, either taking into account some already existing center (a version of CBD model, with application to dacha pricing in Russia), or basing it on a given distribution of land quality in space. This model deals also with heterogeneous agents with respect to their income, and the properties of the general equilibrium mapping from the space of types into the space of locations with endogeneous choice of land size makes it an interesting exercise of general equilibrium modelling.

5. All densities in my models have some properties of continuity and differentiability. This enables to construct an exact solution by solving an array (infinite in chapters 2 and 4, and finite in chapter 3) of the first order conditions for the optimization problems, using differentiability and the method of differential equations. The models with a continuum of agents are viewed as the limits of their discrete approximations, which are used for intuition about interaction.

6. All the models are computable, i.e. the solution is constructed in the form of compact formulae which can be calibrated on real data and used for the solution of real economic problems, which may include such issues as:
a) the impact of integration on regional employment and welfare;
b) two-dimensional geographical competition between real firms taking into account real spatial demand density;
c) the impact of technological advancements and oil price shocks on locational rental prices of land.

1.2 The Philosophy of Continuous Space

Why spatial heterogeneity is important? First, it exists in the real world. Second, it is a potential source of disequilibrium. In physics, for example, spatial gradients becomes driving forces for wave motion. In geography, relief of mountains is relatively stable, but for sand piles it can be unstable. Thus, the existence of spatial equilibrium is not a trivial question in natural sciences. Why it should be trivial in economics? There is an intuition from natural sciences, that equilibrium may be consistent with spatial heterogeneity when either different spatial parts are not interacting, or when there exists some sort of friction, neutralizing the forces of interaction. In economics, this concept becomes wider, since different bundles can bring the same utility. This concept will be exploited in chapter 4 (Dacha Pricing). No interaction becomes a basic characteristic feature when autarky economies are studied (see the beginning of chapter 2, about equilibrium in continuous space when trade is forbidden). But in most cases some sort of friction makes a spatially heterogeneous system stable. In economics the role of friction is played by transport costs, which are viewed here as a sort of tax paid to nature. Due to friction existing in physics there is no way to run any transportation at zero cost. This minimal cost emerges from technology, and thus the question of competition between transportation firms is neglected. It does not matter whether the transport sector is operated by the central planner or is perfectly competitive - in both cases at zero profit level it exhibits non-zero costs. And these costs result in a special topology of interaction between different agents through their location. This will be the main idea of modelling spatial economic systems in this thesis.

While the idea of space is the essential part of all physical models, in the majority of economic papers it is either taken in a very simple way, like few isolated points, or not considered at all. Thus something should be said in favour of my selection of continuous space to be a structural element of all models considered here. Continuous space should be viewed here as geometric, or geographical space. It has a natural order of points (elements) and the idea of a neighbourhood is already inside the concept. Transportation costs, which are proportional to distance in almost all of the models here, bring additional economic properties to this topological concept of neighbouhood [1]. Economic agents in different locations become asymmetric to each other, since they have to pay different transport cost. These transport costs are not derived from any optimal economic activity, they should be taken as given. Although economists think that economic activity is completely self-dependent, it is not true, since economic structures can exist only on the basis of some physical and biological structures, which are more simple forms of organization of *materia*. For the current economic decisions the existing geographical structure should be taken as given: economy is still too weak to change it significantly. And this structure has these properties, discussed above, which should become an essential part of economic models.

Now, why the space should be treated continuously, and not discretely. There is some sense in discrete approximation. If you look at any map, it is easy to discover that the majority of population is living in cities, where most of economic activity is taking place now, and city size is usually much smaller than the distance between cities. So, why not take cities as discrete points? Here the issue of complexity arises. Yes, we can consider formally a discrete model with thousands of cities. But such models can be solved only with superpowerfull computers, if they can be solved at all. On the other hand, when the number of elements goes to infinity, many of such discrete models have their continuous analogies, which are the limits of discrete approximations. In this limit, the solution is decribed by well behaved an-

[1] Transportation costs cannot be eliminated by the competitive structure of the economy, since they capture the cost structure of transport firms. Under any competition policy, transport costs for economic agents would at least include costs of transport firms, like costs of petrol and competitive wage of drivers, which should be proportional to distance under optimal exploitation of transport vehicles. Clearly, transport costs might also include non-linear terms and fixed cost. But what is important is that linear terms should also be there, and often their role is dominant

alytical functions, which allow for further analysis, like comparative statics, etc. When one deals with thousands of numbers, which appear in any computer solution, it is very difficult to see the economic properties of it and and economic intuition behind. That is why my choice was an analytical study of economies in continuous space.

1.3 The Goals

The objectives of research are different for different chapters, and they are discussed separately there. Because the literature related to each chapter is quite different, its surveys can be found inside introductions to chapters. Here I would like to formulate some general goals, which are aimed by this dissertation as a whole.

First of all, the models considered here bring some new contributions to economic theory. They try to capture the effects emerging in spatially decentralized economy, when spatial topology is taken into account. A significant amount of economic effects are discovered; they are discussed in the conclusive part of each chapter. Here I will stress only some important technical points.

One of them is the concept of competition in a continuous space, which becomes significantly different from a discrete one. While in the discrete case we can have "atomic markets", with an infinity of buyers and sellers in a point, leading to the case of perfect competition, in the continuous case the situation becomes different. Let us zoom the space with an imaginary microscope. There always exist so small pieces of space, that no economic activity can be discovered there. Changing the scale a bit, it is possible to cover space with a set of neighbourhoods that each would contain zero or one agent (which can be a producer or a consumer). In this space each producer can be surrounded by two closest consumers, and should compete for them. How to escape a trap of emerging monopoly or oligopoly? It is possible to introduce a market which is the neighbourhood of a point. In real life there are usually more consumers then producers. What is formally done in chapter 2, is an aggregation of all consumers from the neighbourhood of a producer in their location in a demand, emerging at this location. Then a flow of goods may occur via the border between neighbourhoods as a result of the optimizing activity of different producers. Note that continuity is an important property to be able to construct such a model.

A similar concept of local market aggregation via neighbourhood is used in chapter 5 (Location Choice). But there also the issue of mobility is added.

Secondly, all models can be applied to the solution of real economic problems. The reason why the thesis is completely theoretical and does not include econometric calculations is not because this is pure theory, but mainly because its requires very tiny data for its verification, which either are not available or not made public because of their strategic importance for firms. For example, the calibration of "Equilibrium in continuous space" on some agricultural example would require the maps with land fertility for a particular good, or at least database with hundreds or thousands of values related to different geographical points. The data on trade flows are also usually aggregated, at least on regional level; here the flow between villages is of importance. As for prices, I have discovered prices for a particular fruit in a particular day in 5 spanish cities, but this is too small a database to talk about any reasonable econometric work. Moreover, if somebody would do such work in future, its volume might be easily comparable with the volume of this dissertation. For the model "Dacha pricing" I did some calibration on the aggregate database (in form of maps) about land price in different locations.

The third goal is philosophical. Like any science, economics in its childhood used mainly words to describe its ideas. Later, with maturity more words were replaced by formulae and abstract mathematical concepts. However, probably for historical reasons, mathematicians had higher influence on economic theory than physicists. Thus, economics still has less ideas and concepts from natural sciences. If it aims to describe the real world, it should make realistic assumptions about its physical properties or at least argue that their importance is negligible. Under the philosophy of physicists, models should be built asymptotically, so that the exactness of solutions and exactness of assumptions should be of the same order of magnitude. In these models I argue that neglection of the spatial structure can bring higher errors to economic models than it is usually believed. Also, these models may serve as a tool to find the range of parameters (like transport costs, the level of heterogeneity, etc) where we can safely operate with traditional, spaceless, economic models. All the models developed here are essentially quantative. This means that if we have good data, we can test their precision. It is not only the sign of derivative which is important, but also the relative value of different derivatives in comparing opposite effects. The trouble of economics is that utility is not observable and that the main data sets have very high variance. This decreases the possibility to test those tiny quantative effects, which can be predicted by these models. As for qualitative effects, they are formulated and discussed in chapters.

1.4 Further Directions of Research

The concept of equilibrium in heterogeneous space, which was elaborated in this dissertation, opens several directions of further research. In fact, each of the chapters can be considered not as an ended story, but just as a window to a particular direction of research.

In “Equilibrium in continuous space” a one-dimensional model is solved. Its generalization to two-dimensions is demanding. Another generalization of this model can be an attempt to see how the trade areas were growing along with technological development. Comparative statics with respect to the value of transport cost in this case is like a film: the whole spatial pattern reacts non-linearly to its change, and the film is quite different for different initial conditions.

The chapter on “Hotelling’s revival” contains practically all potential generalizations in a two-dimensional euclidean space. However, more dimensions can be considered. But the addition of non-geographical dimensions would require the introduction of a new topology; the euclidean structure might be not very realistic in this case, and some hybrid version of it might be better.

“Dacha pricing” has already an extension in the chapter on “Location choice”. But both of these chapters can be extended further. In “Dacha” paper the equilibrium is unique, and this allows for computation, calibration and practical applications of the model. There are too many restrictive assumptions, and the extension may go in the direction to eliminate them.

For example, allowing the frequency of travelling to dacha is an interesting extension. It is only partially considered in this chapter.

On contrast, in "Location" paper, an additional assumption about land size choice gives rise to multiplicity of equilibria. It is not clear, whether this result holds for all utility functions, or for some utilities the solution may be unique. The Pareto optimality of these equilibria was not investigated. An interesting link may come from the study of the relationship between these equilibria and those obtained by auction mechanism.

Chapter 2

Equilibrium in Continuous Space

The goal of this paper is to describe the general equilibrium of a spatially decentralized economy, when production, consumption and markets are distributed in a continuous space and transportation costs are essentially linear. It is shown that an autarky equilibrium can exist only if transportation costs are high enough. If transportation is cheap, it forces local interaction and destroys local monopoly power. In the general case, the general equilibrium includes some endogeneously determined trade areas, with flows of goods across space, and autarky areas where production and consumption activities take place at the same point. I obtain an analytical solution in explicit functions, which contains equilibrium prices, labor supply and flows of goods in a continuous one-dimensional space. The model can be applied to a set of practical questions in regional economics. In particular, it is able to describe persistent price differentials across regions and non-local consequences of road construction and transportation cost shocks for the economy.

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2.1 Introduction

2.1.1 Evidence and Applications

The prices for physically identical goods may vary across regions. For example, Fig.2.1, taken from [18,20], shows the isoprice lines for potatoes in the USA. The range of spatial price variation is comparable with their average value. It is also easy to see that prices were higher in the southern regions of the USA. Can this be explained in a general equilibrium framework? Before going into modelling, it is useful to consider the possible reasons for this effect. The USA is a country with well-developed market mechanisms. It is unlikely that tastes can differ so much across regions. Neither is it purely an income effect, which is likely to be smaller and to have the reverse direction [1]. That is why a potential explanation should include regional differences in geographic conditions (land productivity) and also the possibility to transport physically identical goods across regions for resale.

Another example comes from Russia which is on its way to transition to a market economy. Table 1 shows the price variation across regions for several basic consumption goods (data are taken from [19]). Before transition, in the 1980-ies, all these prices were almost identical across regions, because they were fixed by the state. It is surprising that after price liberalization they did not move all together to new equilibrium, but instead have moved to a spatially heterogeneous pattern. The shortages have disappeared, which suggests that prices in each region stayed at least in autarky equilibrium. But, according to classical microeconomics, if there are arbitrage opportunities, then all prices should move to the same value which corresponds to an equilibrium. These arbitrage opportunities could be blocked by either transportation costs or tariffs. Both play similar roles, and hence only transportation costs will be considered further on. Therefore an interesting question arises: when can autarky equilibrium exist so that no trade between regions occurs, and when does the equilibrium imply trade flows?

The existing literature on spatial economics has provided insights only on

[1]For many years the South was poorer than the North of the USA. As the demand for a normal good (here potatoes are likely to be a normal good) rises with income, for a fixed supply the prices in high-income areas should be higher. However, the picture shows higher prices for potatoes in southern regions, where income is lower.

Table 2.1: Price Dispersion over Regions in Russia in 1993. Source: [19]

Good	Average	Min	Max	Ratio Max/min
Beef	597.85	143.03	1026.00	7.17
Boiled sausage	829.84	350.00	1700.00	4.86
Butter	1221.59	98.91	2580.00	26.08
Cheese (brine)	1279.25	350.00	3000.00	8.57
Eggs (per 10)	211.53	59.24	515.00	8.69
Bread black (loaf)	34.06	4.70	93.00	19.79
Bread white (loaf)	43.02	1.18	113.78	96.42

very partial aspects of this question, but it has approached it from different sides. For the sake of mathematical simplicity, an autarky model will be presented first and a model with trade afterwards.

The model presented here addresses the question of price variation in space, and thus it has applications for regional economics and interregional trade. But it also presents an interesting example of how local interaction can establish competitive markets in an environment with a continuous heterogeneous space populated by a continuum of immobile agents. The general equilibrium solution has an interesting mathematical structure which enriches the results obtained before.

The model has several practical applications. Despite its one-dimensionality, it may describe real-world price patterns quite well. The first application is the spatial price microstructure for agricultural goods. Land productivity for a given good depends on geographic conditions, mainly on latitude. For equal capital and labor investment, banana crops will be much lower in Canada than in Equador. The case of potato prices in the USA is a more realistic example, because potato growing is worthwhile everywhere in the USA, while nobody would grow bananas in Canada: the crop will be so low that it is much cheaper to import them from Equador. The model can be also applied to the use of natural resources - such as mining, forestry, hunting. In some sense, this example is similar to the previous one, because the difference in regional land productivity is even more pronounced in this case. A third application is the possibility of forecasting the economic consequences of road construction, or the provision of other infrastructure, affecting transportation costs. For example, bananas in remote areas of Central Africa or Amazonia could be quite cheap, when these areas stay in natural autarky as a consequence of the technical impossibility to transport them to other regions. Road construction can influence banana prices, trade flows and employment in banana industry in these regions. The fourth application is the historical path of trade and autarky areas in the world, as the consequence of a gradual decrease in transport costs due to technological progress. The model shows that the patterns of production and trade are highly sensitive to changes in productivity parameters and transportation costs. This might be one of the possible explanations of high instability of rural regional economies which is an effect, mentioned in [8]. There are also several potential policy applications for the impacts of regional taxation policy and price regulation on the equilibrium in spatial markets.

The model also has a theoretical impact. While there were models with a continuum of agents in a continuous space, they did not address the question of exogeneous spatial heterogeneity. Also, the mechanism of local interaction was not applied for this type of models.

Finally, a few words about the structure of this article. A brief literature survey is provided in the next subsection. Then, the formal assumptions and definitions are introduced. Further on, the model is derived. Some examples, providing additional intuition and interesting results, which cannot be seen from the solution directly, are considered. Finally, conclusions and applications are discussed. Some proofs are transferred to the appendix.

2.1.2 Survey of the Literature

A literature survey is not an easy task since all existing models have some significant differences from it. The literature relating to this problem can be divided into three groups: the classical papers on regional economics, recent surveys about current problems in this area and general equilibrium models with a spatial consideration. I would like to make a particular accent on the literature related to the basic particular features of the model: price variation, continuous space with a continuum of agents and transport costs, and local interaction.

The classical literature starts from the pioneering work of Von Thünen [7], which has a very significant impact upon modern research in this area (see [1], for example). Von Thünen describes a single town, surrounded by a continuous agricultural plane. Because of transportation costs, agricultural activity takes place not further away than a certain distance from the city, and the equilibrium wage also depends on the distance. Thus, von Thünen shows the impact of spatial heterogeneity on price variation. Though there is no given center in my model, its philosophy is not very far from Von Thünen.

Hotelling [3] describes an oligopoly between two producers, who have different spatial locations and compete for a continuum of consumers. Actually, this model has had a number of applications in the theory of industrial organization and political science. His model has been generalized by many authors (see [9], for example). Lösch [21] extended Hotelling's model to a two-dimensional space, and found a hexagon structure there being optimal. However, he did not show the emergence of this structure from a decentralized process (Krugman [23]). Alonso [15] mentions that the difficulties encounted in trying to solve the problem of interaction of several producers might have deep mathematical grounding. In this paper I show that with a continuum of producers having a "chain interaction", this problem can be solved analytically.

Kantorovich [5] and Beckmann [6] considered the problem of optimal transportation of masses in a continuous space. However, they have treated it basically as a central planner's problem. Samuelson's model [4] is essentially discrete. He puts buyers and sellers of a good at nodes of a transport network, assigns demand and supply function to each node and defines the transportation costs between each pair of nodes. Then he finds equilibrium prices and flows of goods. The model to be presented has two basic differences with [4]: first, it is generalized to a continuous space; second, it deals only with one good and a numeraire. This enables studying spatial patterns theoretically.

The second strand of literature consists of recent surveys and papers in regional economics [2]. The incorporation of space in a general equilibrium

[2]The state of the art in this area is presented in "Handbook of Regional and Urban

model is still absent, as has been noted recently (Nijkamp, [13]). Though the present model is far from being a full answer to this question, it intends to provide a step in this direction.

Susan Scotchmer and Jacques Thisse [1] provide a survey of the current problems in the economics of space. I will touch upon only one of the issues. The idea of Arrow and Debreu [25] about commodities may explain the persistent price difference for physically identical goods by naming them different commodities, with the only difference in location. The concept of convexity in preferences is crucial for the construction of general equilibrium. But if we use it for these commodities, a rational consumer should buy a bundle of small quantities of good in all places instead of buying one unit of that good in one place, which normally does not happen [1].

Paul Krugman and Anthony Venables [22] "are not aware of any formal models of world trade" where regions are not discrete points and believe that such continuous models "will be entirely untractable". Their own solution, however, is based on "extremely unrealistic assumptions about both natural geography and the motives for trade" [22]. Only in his last book [23] Krugman has started to treat space continuously, before it consisted of two points. He also used Dixit-Stiglitz preferences which do not allow for the same variety to be produced in two different points. But there are many examples (especially raw materials) in which consumers cannot distinguish between the place of origin of a particular good.

The third group is related to the general equilibrium in space. William Alonso [17] derives the demand for housing, manufacture and agriculture, and then finds the equilibrium rental price. In his model the rental price depends on location, and particularly on the distance to a city center. The issue is that land is almost the only good with a fixed supply which is also immobile, and hence his model cannot be used for the case considered here, when supply depends on producers' decisions and the output can be transported in space.

Walter Isard in a series of of influential articles [14] considers such phenomena as agglomeration of industrial location, models of transition pro-

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Economics"[2]

cesses in space, interregional trade theory and general equilibrium in multiregional setting. In his general equilibrium approach, he introduces several regions with production functions, input-output specification and transportation costs between each pair of regions. However, he reaches such a high level of complexity that only some abstract equilibrium equations can be formulated. As is mentioned by Andreu Mas-Colell and William Zame in [11], the cost of this abstract approach is that "much interesting economics lies in the details of particular models". All results depend on the specification of the model, and quite little intuition about its behaviour can thus be obtained. This seems to be a particular feature of discrete models, which normally require computer use and some specific input data in order to gain any further intuition about the results.

Andreu Mas-Colell [12] considers an equilibrium model with differentiated commodities. Hugo Sonnenschein [10] has developed the general equilibrium model in an abstract space of product differentiation. Like William Alonso, his treatment is also continuous, and all the continuum of agents, who have fixed locations over a circle, have free choice to produce and to buy what they want. Still they have one dimension of freedom: to buy one unit of good of the variety that maximizes their utility. In the model presented here, the good is physically identical, but production and consumption takes place in a continuum of locations. But this is not the only and the main difference. In order to achieve price equalization, Sonnenschein relies strongly on transportation costs that are quadratic in distance.

This structure of transportation costs is a rather important issue and deserves a separate discussion. Harold Hotelling [3] considers oligopolistic competition in space with a continuum of consumers and linear transportation costs, which is a realistic assumption. Later on, his followers used his model for the space of product differentiation and introduced quadratic costs to avoid the demand discontinuity [9], which led in some cases to equilibrium non-existence. However, this cost structure does not seem to be in line with reality in spatial models. In this paper, the demand discontinuity problem does not appear, because the production is also distributed continuously. Also, it is shown that a combination of a linear and quadratic term (where quadratic can be infinitely small - see appendix) makes it possible to have both convexity and realism.

Beckmann and Puu [18] have created the theory of densities and flows in spatial economics - the tools that are used throughout this model as well. But they consider an optimal transportation flow for the whole economy, and this approach is not necessarily consistent with the decentralized optimizing behaviour of producers.

Despite this quite heterogenous description of the existing relevant literature (which nevertheless seems to be necessary), it is possible to establish a methodological dimension to find the relative position of this study. There are two basic models in continuous space which are close to this one. The model of Alonso [17] is related to land rent and is based on the idea of a central business district (CBD). In his model, commuting (transportation) costs are linear, in this model they are asymptotically linear for small distances. In Alonso's CBD model, each of the continuum of agents commutes with one CBD; thus, there is a continuum of possible communications. The model of Sonnenschein [10] has quadratic transportation costs, and hence agents are induced to interact locally. Because there is not a single center, but a continuum of centers (all points are equal), this reduces the number of possible interactions to continuum [3]. Therefore, the levels of complexity of Alonso's and Sonnenshein's models are similar. To the author's knowledge, nobody tried to consider a continuum of interactions for the continuum of points in spatial economics yet. The basic difference of this model from Sonnenschein, as was already mentioned, is that transportation costs have not only a quadratic, but also a linear term. This enables having both local interaction (as optimal choice) and different prices in different locations (like in Alonso's model).

2.2 The Structure of the Model

The traditional general equilibrium approach considers the market as a central clearing house, with costless access of all producers and consumers to it. The historical grounding for this concept might be a central market square in a medieval town. At that time the world was composed of many local autarky economies. The great geographical discoveries opened America, India and China to Europe, and all the advantages of international trade were exploited. This was the basis for international trade theory. In fact, initially

[3] from the squared continuum

trade only took place with goods not producible in the domestic economy (tea or coffee in Europe, for example). Only since the middle of the XIX-th century, after globalization of trade, did competition among physically identical goods became a reality, but taxes, imposed by the government, could easily protect domestic producers. Free trade agreements, economic integration processes and technological achievements in transportation technologies work in the direction of decreasing trade costs - and hence erode autarky economies. Tariff elimination brings up the concept of a continuous space. This idea is not new, and to some extent was developed starting from Von Thünen [7] and Hotelling [3]. Both have shown that the price which consumers in different points are willing to pay for a good which is physically the same might be different. However, general equilibrium theorists (see [10], for example) beleived that in the long run prices would finally be equalized. However, empirical evidence shows persistent price differentials across regions, which do not converge to zero.

The idea of Arrow and Debreu about commodities may explain this difference by introducing different commodities, whose only difference is location. However, this contradicts with the concept of preference convexity as no consumer would optimally prefer to give up one loaf of bread from the nearest bakery for a bundle (of the same weight) of bread from all bakeries in the world! The existence of simple transportation technology makes it possible for both producers and consumers to convert different commodities into the same just by paying some transportation fee. The introduction of a transportation technology is a formal difference between autarky and non-autarky economies. But this technology would not always be optimally utilized. The idea of this chapter is to describe one particular example of a continuum of spatially separated autarkies. In the next chapter it will be shown, under what conditions the equilibrium in an integrated economy will coincide with autarky equilibrium. In other words, when firms will optimally choose no-action in transportation activity. The next chapters will deal with the description of other possible equilibria in an integrated economy. The particular preferences and technologies (though of rather general and typical macroeconomic form) are chosen in order to obtain an explicit solution, to study comparative statics between an autarky and an integrated economy and to obtain important economic conclusions.

The model has some elements which are not very common in the economic literature. That is why it is useful to introduce several definitions in order to clarify some further ideas and conclusions.

2.2.1 Some Useful Definitions

The first three definitions give names to some subsets of the whole economy. They will play an important role for the mathematical structure of the solution.

Definition 2.1 *A* supereconomy *is the whole object in the considered model. Mathematically, it is represented by an interval* $[0, 1]$ *(or another one-dimensional topological object in continuous space like a ray or a circle)).*

Definition 2.2 *A* subeconomy *is any subset of the supereconomy, defined by some common properties of the points inside it. In most cases it is represented by a subinterval* $[a, b] \subset [0, 1]$*.*

Definition 2.3 *A* microeconomy *is any point inside a supereconomy,* $x \in [0, 1]$*. We assume that each microeconomy has one producer with a measure zero output capacity. Intuitively, it may be thought of as a village with some production and consumption inside it, but with an infinitely small impact on the whole supereconomy, and even on any subeconomy.*

Autarky-1 and autarky-2 are two new concepts. Index 1 is related to the assumption, while index 2 to the result. Thus, autarky-1 is exogeneously postulated, while autarky-2 emerges endogeneously. The same word "autarky" should not be misleading; it is kept to stress the similarity of the mathematical form of the solution. Empirically, these two autarkies might be undistinguishable. One should see the reason why trade is not going on.

Definition 2.4 *The supereconomy is said to stay in* autarky-1*, if all of its microeconomies have no technical possibility to access the market of their neighbours. In autarky-1 a microproducer enjoys monopolistic power over the local market.*

Definition 2.5 *A subeconomy (supereconomy) is said to stay in* autarky-2*, if there exists a technical possibility to access other spatial markets, but any producer rationally chooses to sell all the output locally, because transportation costs are too high.*

Note that the local monopoly power emerging in autarky-2 is not a usual one, since all producers are facing competition. The chance to behave like a local monopolist in a small neighbourhood of a general equilibrium solution is imposed by the structure of transport costs and price-taking behaviour of all participants of the economy.

These definitions are mathematical and some of the economic intuition behind them is as follows. An interval is a good approximation for an economy, which either has a geographical structure, close to one-dimensional (Chile or the US western coast) or when there is an obvious symmetry with respect to the second coordinate (for example, climatic conditions, which are essential for agricultural crops, usually depend on latitude, but in some cases their dependence on longitute can be neglected). A subeconomy can be thought of as some part of the whole economy with the spatial borders determined endogeneously.

A microeconomy can be thought of as a very small location. Assuming that there is essentially one producer in each microeconomy is a matter of convenience, for it allows us to abstract from competition within each location, which would only complicate the analysis without providing deeper results. Another reason for doing this is related to the link between the continuous and the discrete versions of the model which will be discussed subsequently.

2.2.2 Basic Assumptions

In this model there is a continuum of producers and continuum of consumers. All are immobile and distributed in space with some density. There are two goods: a composite good without transportation costs, and a commodity which can be moved from one place to another with some transportation costs. Producers have an access to transport the goods. All consumers have identical preferences, but may be facing different prices. Producers are different from consumers. They use their own labor and location-specific capital to produce a specific good. Consumers' utility depends on this specific and composite good. Producers consume only this composite good (remember the joke: a shoemaker without shoes), but leisure also enters their utility. Formally, the assumptions of this model are as follows.

Assumption 1. There is a continuum of microeconomies on a line with a Cobb-Douglas function in production: $Y(x) = A(x)K(x)^{1-\alpha}L(x)^{\alpha}$, where $x \in [0,1]$ is a point, $A(x)$ - technological productivity in this point, $K(x)$ - capital endowment, $L(x)$ - labor[4]. The supereconomy $[0,1]$ is a composition of these microeconomies, together with the topological links between them. Further, it will be assumed that the capital is fixed at every point (land endowment, for example) and the decision about labor is taken by producer, who is considered to be self-employed (farmer). Hence, $Y(x) = \gamma(x)L(x)^{\alpha}$.

Assumption 2. There is a continuum of consumers (different from producers) with the demand density $D(x) = \delta(x)p(x)^{-\epsilon}$, where $p(x)$ is the local price [5].

Assumption 3. All producers have Cobb-Douglas preferences about their income C, which is the price multiplied by output, and leisure: $U = C^{\beta}(1-L)^{1-\beta}$.

Assumption 4. Producers choose the labor supply so as to maximize their utility.

For the purpose of the next section we will need:

Assumption 5. There is no possibility of transportation, i.e. goods can be sold only on the local market.

In the model with trade this assumption will be relaxed.

In order to bring about more intuition about these assumptions, several pictures are introduced. Fig. 2.2 shows the output curves as the function of labor input for two different microeconomies $x_{1,2}$, which are some arbitrary points of the productivity parameter as a function of the spatial coordinate x (Fig. 2.3). The demand density is also an arbitrary continuous function of x (Fig. 2.5), and it generates the family of demand curves (Fig. 2.4).

[4]As a point has measure zero, productivity, capital and labor should be thought of as densities.

[5]This demand can be derived from the consumer's utility, which is additive and linear in the numeraire.

The continuous structure of this economy is essential for getting mathematical results. It is a good approximation of a real economy when the number of agents is very large, and when population density and productivity factors change slowly between locations. Producers and consumers are actually different agents here. The whole economy, which produces thousands of goods, can be decomposed into the economy producing one particular good (which is studied here) and the rest of economy, so that the impact of this one-good economy on the rest is negligible. Consumers in this model might be the producers in the rest of the sectors, but they are represented here just by their demand.

2.2.3 A Discrete Analog of the Model

Before going further, it is useful to introduce the discrete analog of this model. The discussion in [24] about the problems arising from the difference in the structure of set of equilibria in a continuous model and its discrete approximation in models with location and a transportation technology, is not relevant here for the following reasons. First, in the examples of Berliant and ten Raa [24] agents themselves are spaceless, but demand some land, while here the locations are fixed and land endowments are equal. Second, in different metric spaces the transition between the sequence of discrete models and their continuous limits may occur differently. In this paper the limits for the densities are considered in the class of continuous functions of space, while in [24] the wider space L_1 (of functions for which Lebesgue integrals exist) is considered.

Imagine that the number of firms is finite, and they are located equidistantly on an interval. If the differences in productivity and demand coefficients between any two neighbours are much smaller than the values of these coefficients, then the differences can be replaced with differentials, and the continuous model is a natural limit of such a sequence of discrete models. The structure of the interaction has to be kept when a continuous model is considered. Here, like in all the models of classical physics, first the structure is considered for a discrete case, and then all discrete-valued functions are replaced by their continuous limits. For all the models it is crucial to assume that the spatial structure is locally homogeneous, so that if we look at it with a microscope, we discover that all functions are locally constant. Thus, the transition to formally continuous models occurs via a sequence of grids (chains), the difference between the solutions for these chains is small, when the distance between elements vanishes to zero, and the continuous model is claimed to be the limit of the Cauchy sequence of these discrete models [6].

While speaking about continuous models, one has to bear in mind their discrete analogs. This is useful in order to understand why the autarky producer in a continuous economy should be a local monopolist, and why the spatial competition turns out to be oligopolistic. Another reason for referring to discrete analog is the practical use of the model. In reality only discrete producers exist, but studying discrete models will end up with quite big matrices of data, where one can hardly catch some regularity. Going from discrete to continuous data can be done via interpolation (it is always possible to construct polynomial functions, for example, which have any set of values in any finite set of fixed points).

2.3 Autarky-1

Now we consider the solution for an autarky-1 economy. Each producer is a monopolist here, because in the discrete analog he is a monopolist [7]. That is why he takes into account the impact of the quantity sold on price. As he is a producer at the same time, he plans his optimal labor supply to reach the highest utility, taking into account its impact both on output and price. In autarky-1, the supply of this particular good in any microeconomy should be equal to demand. The market clearing condition for this model is given by:

$$\gamma(x)L(x)^{\alpha} = \delta(x)p(x)^{-\epsilon}. \tag{2.1}$$

Proposition 2.1 *In the autarky-1 case producers behave as local monopolists, i.e. they choose the optimal output, which is the function of their labor supply, taking into account that prices depend on the quantity that they produce.*

---

[6] In the space where each discrete solution corresponds to some continuous function, which can be uniquely associated to it, according to some interpolation algorythm

[7] That is why he is a monopolist in his point. He also avoids spatial competition from neighbours because of Assumption 5.

PROOF:
Producers are monopolists because there is only one producer in each microeconomy (see Definition 2.3) and there is no access of other producers to this market (Definition 2.4, Assumption 5). Utility is a function of $C(p(x), \delta(x), \gamma(x))$ *and* L*. The parameters* $\delta(x), \gamma(x)$ *are exogeneous for the producer, but he can decide optimally on prices and labor supply.*

The price can be expressed as a function of labor and densities:

$$p(x) = (\frac{\delta(x)}{\gamma(x)})^{\frac{1}{\epsilon}} L(x)^{-\frac{\alpha}{\epsilon}}. \tag{2.2}$$

The utility of a producer, after the substitution of the market clearing condition, can be expressed in the terms of his labor only:

$$U = hL^{\mu}(1 - L)^{\nu}, \tag{2.3}$$

$$h \equiv \delta^{\frac{\beta}{\epsilon}} \gamma^{\beta - \frac{\beta}{\epsilon}}, \mu \equiv \alpha\beta(1 - 1/\epsilon), \nu \equiv 1 - \beta. \tag{2.4}$$

The solution to his maximization problem is given by the formula:

$$L(x) = L^* = \frac{\alpha\beta(1 - 1/\epsilon)}{1 - \beta + \alpha\beta(1 - 1/\epsilon)}. \tag{2.5}$$

Note that the labor supply appears to be dependent only on the parameters of Cobb-Douglas functions and the elasticity of demand, but not on the density of producers and consumers. Thus, the production level is proportional to the productivity parameter γ (which is a function of spatial point x).

However, the price will also depend on these densities:

$$p_0(x) = (\frac{\delta(x)}{\gamma(x)})^{\frac{1}{\epsilon}} (L^*)^{-\frac{\alpha}{\epsilon}}. \tag{2.6}$$

The results obtained for the autarky-1 economy are summarized in the following Proposition.

Proposition 2.2 *In the autarky-1 model, under given assumptions 1-5, the optimal labor supply is the same in all microeconomies. It does not depend on production and consumption densities and depends only on Cobb-Douglas and elasticity parameters* α, β, ϵ*. The equilibrium price depends on the spatial point and is given by the formula (2.6).*

Lemma 2.1 *If the production and consumption densities are continuous and differentiable functions and if the production density is positive in all of the points, then the equilibrium price is also a continuous and differentiable function of a spatial point x.*

PROOF:
The price is given by an algebraic function of two differentiable functions. Thus, its derivative can be calculated explicitely. It exists in all points, except for those where $\gamma(x) = 0$.

Thus, for any continuous functions $\gamma(x), \delta(x)$ the autarky-1 equilibrium price function $p_0(x)$ and labor supply can be constructed. The solution is given by the price function, continuously depending on the spatial coordinate, and the constant labor supply over space (Fig. 2.6).

2.4 Autarky-2

The assumption of no trade due to no transportation possibilities seems to be unrealistic and plays mainly the role of analytical device [8]. Starting from this section, assumption 5 is replaced by assumption 5a:

Assumption 5a. There exists a possibility to transport the good to spatially different markets. The marginal transportation cost is equal to t [9].

Therefore, transportation now is possible and the cost of transportation of one unit of good per one unit of distance costs t monetary units.

If the transportation cost is higher than the price differential, nobody will find it optimal to use transportation, and the system will stay in autarky, despite the possibility of transportation. Thus, the global condition for staying in autarky is, using (2.6):

[8] Although, in some particular cases (trade legally prohibited, producer is surrounded by mountains and has no access to air transport, etc) it becomes realistic, they do not play an important role in modern trade.

[9] This means that the transportation cost of one unit of good between two locations with infinitely small distance dx is equal to tdx. The more detailed structure of transportation costs will be specified subsequently.

$$\max_x |\frac{dp(x)}{dx}| = \max_x |\frac{L^*}{\epsilon}(\frac{\delta(x)}{\gamma(x)})^{\frac{1}{\epsilon}-1}\frac{\delta'(x)\gamma(x) - \delta(x)\gamma'(x)}{\gamma(x)^2}| < t. \tag{2.7}$$

Proposition 2.3 *If the assumptions 1-4 and 5a hold, the functions $\delta(x), \gamma(x)$ are such that for the whole supereconomy the inequality (2.7) holds, then this supereconomy will stay in autarky-2. All producers are still local monopolists.*

PROOF:
The inequality (2.7) guarantees the absence of any two points with a price differential less than transportation costs. Any transportation procedure can only decrease this price differential. Hence, nobody finds it optimal to utilize this transportation possibility, and all the supereconomy stays in autarky-2. Every producer knows that it happens in equilibrium, and thus he can behave as a monopolist on his local market.

It is very important to discuss the autarky concept of this model and to make the distinction from similar concepts, studied in literature. The analogy of autarky-1, discussed above, would be the short-run equilibrium allocation of producers and consumers in the paper of Sonnenschein [10], which results in some equilibrium price as the function of the point in a commodity space. There, in the short run, every market is cleared only locally, without taking into account the possibility to adjust production. In order to proceed with this analogy, we should think about this model in a dynamic way. Initially, nobody realizes that transportation is possible: we have the autarky-1 price function as a result. Then producers start utilizing arbitrage opportunities, if they exist, by transporting their goods to the neighbouring markets. Condition (2.7) shows under what conditions these arbitrage opportunities will never arise. The general equilibrium with trade, which will be discussed in the next section, is an analogy of the long-run equilibrium in [10]. Actually, in [10] this short-run price function can be any continuous function (which can be uniquely derived from the initial density of firms). In this model the autarky price function may also be an arbitrary function (which is derived from the densities of demand coefficients and productivity coefficients). Though consumers in this model cannot choose the location of consumption (as they do in [10]), instead they can choose its level, and producers can do it also (even in the short run), for each point in space.

The described differences are not conceptually crucial in the sense that they lead to a similar equilibrium structure. The basic distinction between these models, which brings new conceptual ideas, is the following. When Sonnenschein [10] speaks about price dynamics, he means the specification of some adjustment rule on the way of transition from the short-run to long-run equilibrium. In his framework, the only feasible long-run equilibrium is full price equalization. It is useful to determine which of his assumptions leads to this outcome. The first is topological - he considers only a circle. It is possible to show that for an interval the result may be different (it depends on how the boundary condition is chosen). What is more important, and this will be shown later, is the local price of transportation between points. In his model it is just quadratic (and thus vanishes for an infinitesimal move). The analogy in a spatial context is how the unit transportation price depends on distance. Is it realistic to assume it quadratic? Yes, but not just quadratic; it is necessary to add a linear term. Then for infinitesimally small move the price of transportation will be infinitesimally small of the same order. The quadratic term eliminates the benefits from long-distance trade, and forces the producer to sell only on the neighboring market.

The main consequence from the assumption about linear transport cost is the possibility to stay in autarky in the long run also, given that the short-run equilibrium price gradient is not sufficiently high (see formula (2.7)). The structure of all possible long-run equilibria in this model is essentially richer and will be described in the next section.

2.5 Local Trade and Endogeneous Integration

In this section the possibility of market integration will be introduced via the access of all producers to all markets. The basic difference from the autarky-2 case is that now the autarky-1 price function is assumed to violate the restriction (2.7), at least at some points. Hence, after the "opening of autarky-1" the arbitrage possibilities will emerge, and the whole supereconomy will move to a new equilibrium. Since the structure of equilibrium will become more complicated, it is useful to introduce new definitions.

Definition 2.6 *An* integrated economy-1 *is the supereconomy with the po-*

tential access of all producers to all markets.

This definition is related to new formulation of the model in its extended version (roads are constructed). In order to distinguish the type of solution where trade is possible but not chosen from the case when it is chosen, it is useful to introduce the concept of integrated economy-2.

Definition 2.7 *An* integrated economy-2 *is a general equilibrium solution for an integrated economy-1, in which trade actually occurs across locations. This name will be used for both the description of the solution for subeconomies, but if no subeconomy has autarky-2 solution, then it can be used also for the solution of supereconomy.*

It is important to see the link between two types of autarkies and two types of integrated economy. There are two possible points of view on the question. Mathematically, autarky-1 and autarky-2 are analogies, because mathematically they are described by the same formulae, but they are the solutions to different economic problems. Economically, an integrated economy-1 and integrated economy-2 are analogies, because they are the solutions to the same economic problem. However, the solution of integrated economy-2 type is conceptually simpler, and thus it can be used as a brick for the description of solutions to integrated economy-1 model, along with autarky-2, which might be a solution for a complementary subeconomy.

Formally, in order to go to an integrated economy-1 model, we have Assumption 5a instead of 5, and restriction (2.7) is no longer valid. Though consumers remain immobile in this model, this is not a crucial assumption. It is easy to show that their mobility will not alter this model in any way under the assumption that their transportation costs are higher than those of producers for all distances.

Assume that local trade is profitable, i.e. there exists at least one point with a price gradient above transportation costs. In this case, the producer at this point (who is assumed to start from autarky-1 prices) might be better off by selling some of the output on the neighboring market. It is clear that this opportunity will immediately blow up the autarky equilibrium in all the economy (or at least, in some parts of it). The reason is that as trade will change the price in both the original and the neighbouring market, there might be an incentive for the other neighbour to ship the good to the market of the trade initiator, and so on. If we allow for the possibility to trade in higher distances, the complexity of such a system will increase dramatically [10]. However, there exists a simple and rather realistic way to eliminate this possibility. If we assume that marginal transportation costs are increasing functions of distance, then for a given price gradient the local trade in the infinitesimal distance will be the cheapest option. Formally, the following transportation unit cost can be considered: $T(x, y) = t|x - y| + R(x - y)^2$, where M can be a small positive constant; $R = F(\delta(x), \gamma(x))$ (see the Appendix for details). This is not an unrealistic assumption, if not only the ticket price (which is likely to be about linear in distance), but also the opportunity cost of forgiven transportation time is taken into account. From now on, we shall make we shall make this assumption to ensure the optimality of local trade.[11]

This assumption seems to be consistent with stylized facts. Some evidence on the volume of transportation as a function of distance is given in [16]. According to [16], railroad shipment volume drops about 4 times as the distance doubles. This means that most of the transportation is short-distant. It seems that the statistical data for transportation of similar goods might be even more impressive, as some long-distance transportation occurs for the goods that are not produced in the region of destination.

If all initial densities are continuous and differentiable functions, the autarky (short-run) solution is described also by a continuous and differentiable function. Without loss of generality, the interval where this function is defined, might be divided into a finite number of subintervals, where the sign of a derivative is constant. The boundaries between these intervals will have zero derivative. Some of them are price maxima. But there the producers would be unable to extract any profit from trade at any distance, because of the specification of the transportation cost structure. So, a producer having any arbitrage opportunity would be able to exploit it at a marginally small distance and only in one direction.

[10] This has been discussed already in the literature survey

[11] Note that another possibility to support the obtained general equilibrium might be the assumption of local interaction: all producers know only the information about equilibrium price derivative at their point, and can sell their goods only in the neighbouring markets.

Inside each of these subintervals there might be some other subintervals, where the absolute value of the price derivative is above t. When the possibility of transportation is introduced, only these intervals are a source of non-stability. If they do not exist, the system will stay in autarky.

Further on, the equations will be derived for this representative subinterval, which is characterized by a constant sign (take it as positive) of the price spatial derivative in autarky. All functions in this model are assumed to be continuous and differentiable. Hence, it is possible to consider a small part of the economy, centered at a point x, with total length dx: $[x-dx/2, x+dx/2]$. As dx can be chosen as arbitrarily small, all continuous functions inside this economy can be approximated by constants (with any precision). The production of this part is $Y(x)dx$, demand $D(x)dx$, where Y and D are given by the same formulae as in the autarky model. Let $\Phi(x)$ be the flow of good at point x. The idea of flows is taken originally from physics; it was developped for economics by Beckmann and Puu [18]. The intuition behind the flow is as follows. Suppose that $[x - dx/2, x + dx/2]$ is a small economy. The flow of good $\Phi(x - dx/2)$ is the quantity of good which arrives to this economy by crossing its left border, and $\Phi(x + dx/2)$ is the flow, which this economy "exports" through its right border $x + dx/2$. When the good moves in the other direction, the flow is assumed to be negative. The feasibility constraint *(which becomes a sort of law of account)* implies that the net outflow should be equal to the difference between production and consumption of this good in the economy:

$$Y(x)dx - D(x)dx = \Phi(x + dx/2) - \Phi(x - dx/2) = \Phi'(x)dx(1 + o(dx)) \quad (2.8)$$

($o(dx)$ is an infinitely small number of the order dx). Hence, in the limit,

$$Y(x) - D(x) = \Phi'(x). \quad (2.9)$$

Now the objective function of each producer is to maximize his utility with respect to the optimal labor and optimal trade flow. His consumption now is given by the revenue at the local market, revenue at the neighboring market (with different price) minus transportation costs:

$$C(x)dx = p(x)[Y(x)dx - \Phi(x)] + p(x + dx)\Phi(x) - \Phi(x)tdx. \quad (2.10)$$

Because dx is infinitely small, the linear part of the Taylor expansion can be used, giving:

$$C(x) = p(x)Y(x) + (p'(x) - t)\Phi(x). \quad (2.11)$$

The infinitely small terms are neglected in this expression. So, in the limit $dx \to 0$ this formula becomes exact.

Substituting $C(x)$ into utility function, we get:

$$U = (p(x)Y(x) + (p'(x) - t)\Phi(x))^{\beta}(1 - L(x))^{1-\beta}. \tag{2.12}$$

Now consider the case, when $p'(x) < 0$. In this case the producer will choose the same volume of trade flow, but it will go in the opposite direction. Summarizing this, we find that almost each producer [12] maximizes the value of his utility with respect to the labor supply and the absolute value of trade flow $|\Phi(x)|$, taking the price function as given. For the output there is a direct equation $Y(x) = \gamma(x)L(x)^{\alpha}$. Substituting this expression into the utility function leads to the following maximization problem:

$$\max M(x) \equiv \max(p(x)\gamma(x)L(x)^{\alpha} + (|p'(x)| - t)|\Phi(x)|)^{\beta}(1 - L(x))^{1-\beta} \tag{2.13}$$

A rational agent, indexed by x, maximizes the value of M with respect to his control variables $L(x), |\Phi(x)|$, taking all other parameters, including price $p(x)$, as given [13]. The price is implicitly determined by the market clearing condition which is obtained by substituting assumptions 1 and 2 into the equation (2.9):

$$\delta(x)p(x)^{-\epsilon} = \gamma(x)L^{\alpha}(x) - \Phi'(x). \tag{2.14}$$

The derivative $\Phi'(x)$ is determined by the collective action of the producer and his neighbours. For each x, the first order conditions have the following form:

$$\begin{aligned}\frac{\partial M}{\partial L} &= \beta[p\gamma L^{\alpha} + (|p'| - t)|\Phi|]^{\beta-1}p\gamma\alpha L^{\alpha-1}(1 - L)^{1-\beta} \\ &\quad -[p\gamma L^{\alpha} + (|p'| - t)|\Phi|]^{\beta}(1 - \beta)(1 - L)^{-\beta} = 0, &(2.15)\\ \frac{\partial M}{\partial |\Phi|} &= \beta[p\gamma L^{\alpha} + (|p'| - t)|\Phi|]^{\beta-1}(|p'| - t) = 0. &(2.16)\end{aligned}$$

[12] There may exist a finite number of points (total measure zero), where the derivative of the price function does not exist. Particularly, the right- and left-side derivatives may exist and be equal to $+t$ or $-t$, in equilibrium.

[13] The producer is a price-taker here. He can not decide about the price himself, but has to take into account the behaviour of his neighbours. While in a discrete analog of the model there may be is a chain of oligopolists, in a continuous model they become price-takers. The structure of transport costs implies that producers trade only locally, but the prices and flows propagate over space through this mechanism of local interaction.

Because the expression in square brackets has the meaning of producer's consumption, it can not be equal to zero at the optimum. Formally, it can be shown that the substitution of $C = 0$ into the first F.O.C. gives either $L = 0$, or $L = 1$ (both are minima). Hence, the second Kuhn-Tucker condition reduces to the form:

$$(|p'(x)| - t)|\Phi(x)| = 0. \tag{2.17}$$

The meaning is the following: For all points where $|p'(x)| < t$, the flow is zero: $\Phi(x) = 0$. Non-zero flows can be consistent only with $p'(x) = t$.

Lemma 2.2 *The case $|p'(x)| > t$ is not consistent with equilibrium.*

PROOF:
Let $\exists y : p'(y) > t$. Let $p'(y) = t + \sigma$. Then the producer in y can make additional profit by transporting and selling a very small amount of output additionally to the neighbouring market. This additional flow can be made so small, that the price derivative will change by less than σ. By this activity this producer will be better off, which is not consistent with equilibrium.

If $p' = \pm t$, then L can be determined, giving:

$$L(x) = L^{**} = \frac{\alpha\beta}{1 - \beta + \alpha\beta} \tag{2.18}$$

The comparison of this result with formula (2.5) shows that:
1) in the local trade model the labor supply is also constant (this is an interesting result; later (see subsection 2.7.3) it will be exploited as an indicator of different degrees of competitiveness);
2) it is higher in comparison with the autarky model (the difference depends on the demand elasticity, as well as on the parameters of the Cobb-Douglas functions).

Now the price is determined up to a constant, and the market clearing condition, which gives the expression for $\Phi'(x)$, enables us to reconstruct the function $\Phi(x)$ by integration:

$$p(x) = p(0) + tx, \tag{2.19}$$

$$\Phi(x) = \Phi(0) + \int_0^x dx[\gamma(x)(L^{**})^\alpha - \delta(x)(p(0) + tx)^{-\epsilon}]. \tag{2.20}$$

Proposition 2.4 *The whole supereconomy is endogeneously decomposed into two subeconomies: "autarky-2" and "trade". Formulae (2.18)-(2.20) describe the structure of solutions inside the trade part of the supereconomy.*

PROOF:
From the equation (2.17) it follows, that either $p'(x) = \pm t$, *or* $\Phi(x) = 0$. *The first case gives the subset of points, where the flow is non-zero, i.e. trade occurs. The second case gives the subset of points with zero flow, which corresponds to autarky-2.*

Lemma 2.3 *The trade flow is a continuous function on the whole supereconomy. At the boundary* s *of the trade subeconomy the following condition should hold:* $\Phi(s) = 0$.

PROOF:
1) $\Phi(x)$ *is a continuous function, because it is given by the integral (2.20) of a function with a limited variation (here the possibility of discontinuous changes in labor supply is also taken into account, and the property of the Stieltjes-Riemann integral is used). Hence, the flow is continuous inside the trade subeconomy. In the autarky subeconomy it is continuous, because it is zero. Now it is enough to prove the continuity on the border of two subeconomies.*
2) From Gauss theorem, the variation of the flow at a point is equal to excess demand at this point. In the case of continuous densities this excess demand is infinitely small. Hence, the flow is continuous in all "non-atomic" points. [14] *This is the reason why there is a zero border condition:* $\Phi(s) = 0$.

Considering the whole supereconomy, it is also necessary to add the outside border conditions, in a and b:

$$\Phi(a) = 0, \Phi(b) = 0, \Phi(s) = 0. \tag{2.21}$$

Here a is the left border of the economy, b is its right border, and s is a border between an "autarky" part of the economy (with optimally zero transportation) and a "non-autarky" part. The guess for the sign can be inferred from the sign of the spatial derivative of $p_0(x)$ - the autarky model solution. To find the borders between different parts might be a more difficult problem. Some examples will be considered later.

[14] Atomic point is a point with positive measure.

Consider the example of economy [0,1] which, without the possibility of transportation, has $p'_0 > t$ at all points. Such an economy will not have any parts which will stay in autarky after the opening of trade. Hence, the prices will be given by:

$$p(x) = p(0) + tx. \tag{2.22}$$

For the flows the following is valid:

$$\Phi'(x) = \gamma L^{\alpha} - \delta p^{-\epsilon}, \tag{2.23}$$

$$\Phi(0) = 0, \Phi(1) = 0. \tag{2.24}$$

Integration gives the following result:

$$\Phi(x) = \int_0^x dx[\gamma(x)(L^{**})^{\alpha} - \delta(x)(p(0) + tx)^{-\epsilon}]. \tag{2.25}$$

This expression has only one free parameter $p(0)$, which has to be determined from the second border condition $\Phi(1) = 0$. The obtained result may be formulated as a Proposition:

Proposition 2.5 *When the equilibrium prices in the corresponding autarky-1 economies are such that the price spatial derivatives are above the unit transportation costs at all points, then the opening of a possibility to trade will bring this system to a new equilibrium, which is characterized by the formulae (2.18), (2.22), (2.25) and has the following properties:*
i) the equilibrium labor supply is different from the autarky case (it is constant and higher for $\epsilon > 1$);
ii) the price spatial derivative decreases in absolute value to the value of transportation costs;
iii) the trade flows are represented by a function, depending on a spatial point, which is different from zero, in the general case.
The general equilibrium for the economy with trade can be uniquely constructed, and algebraic formulae can be written for any initial densities of production and consumption factors.

Note that the description of the space topology was essential for obtaining a solution. Two boundary conditions were used to define a unique solution for an interval. The case of a circle will be considered later. Other possible one-dimensional objects are: a) a line and b) a ray. A line has no ends, but it is of no economic interest. A ray may be a model "city-countryside", in the style of Von Thünen [7]. The introduction of an atomic point (city) makes is possible to imitate the supply of the city with agricultural goods. Let the city be located at $x = 0$ and has a price $p(0) = a$. The city may have a positive inflow of goods. The spatial price derivative will be equal to $-t$ in some neighbourhood. But as it cannot be negative, at some point $x = b, b \in (0, 1)$ there should be a border point with an autarky area. This is exactly in line with Von Thünen's arguments. But this is a more general conclusion, because it allows the existence of local markets at all points (villages).

2.6 Simple Examples

This section will present a simple model which shows the discontinuous behavior of an economy as a response to a marginal change in transportation costs. This example illustrates the possibility of high output volatility in rural areas as a response to small changes in economic variables.
Consider the case of an interval [0,1] with a constant productivity factor ($\gamma = \gamma_0 = const$) and a growing demand density:

$$\delta(x) = \gamma_0 t^{\epsilon} (L^*)^{\alpha} (x - x_0)^{\epsilon} \tag{2.26}$$

As was shown in the sections about autarky-1 and autarky-2, when the spatial price gradient is below or equal to t, the labor supply is fixed at the level L^*. It is easy to see that for this case the price gradient is exactly equal to t at each point, in autarky equilibrium. Let $x_0 = 0$, for simplicity. Now assume that the transportation costs have decreased marginally, thus allowing for an opening of the possibility to exploit spatial price arbitrage. All producers will immediately start selling part of their output to a neighbour with a higher price. Finally, a new equilibrium, described in the section about local trade, will occur. It may correspond to the same spatial form of price function, but the trade flows will be non-zero and the labor supply, L^{**}, will be higher than before. There is a possibility for many equilibria to occur. The point $x = 1$ represents the accumulation point of trade flows. One solution arises with the same prices. In this case the trade flow function is given by the formula

$$\Phi(x) = \int_0^x dx[\gamma_0 (L^{**})^{\alpha} - \delta(x)(p(0) + tx)^{-\epsilon}], \tag{2.27}$$

and $\Phi(1) > 0$. Hence, if prices stay the same, there should be additional consumption of the good at the point $x = 1$ at the same price. Imagine that there is a city with price control. Before it imported this good, but now the flow from the rural neighbourhood will substitute this import.

Another case assumes no additional consumption. Therefore, by definition, $\Phi(1) = 0$. As production is higher now, this can be sustainable only with a general price drop. Hence, a marginal shift in transportation costs might give rise to new equilibria with very significant differences from the initial autarky, corresponding to slighly higher t.

The question of Pareto optimality might also be studied. In general, deviating from autarky does not necessarily improve the welfare of all agents. For example, in the second case producers will work more and sell their output at a lower price. Whether their utility will go up or down depends on parameters α, β, ϵ. It may also go up for some group of agents and down for other groups. For example, producers in the area may win and in others - loose from the opening of trade possibility. Note that in this model the possibility of trade is endogeneous and depends not only on transportation costs, but also on the spatial distribution of productivity and demand factors.

Another example also considers an interval [0,1] with constant productivity, equal to 1, and the demand density:

$$\delta(x) = (L^*)^\alpha (1 + x^2)^\epsilon. \tag{2.28}$$

This demand density is also growing non-linearly. But in this case it leads to a non-constant price derivative in autarky-1 equilibrium. It is easy to show that the autarky-1 equilibrium price will be $p_0(x) = 1 + x^2$, and $p_0'(x) = 2x$. In this case for any $t \in [0, 2]$ there exists a point $x_0 \in [0, 1]$, such that $p_0'(x_0) = t$ (it is $x_0 = t/2$). If trade is allowed in such an economy, the interval $[x_0, 1]$ would definitely have some trade flows and new prices. But the rest of the economy would not necessarily stay in autarky. In the general case, only the part $[0, y]$, where $y \in [0, x_0]$, will stay without trade flows. The equilibrium prices and flows will be:

$$p(x) = 1 + x^2, 0 \leq x \leq y; \tag{2.29}$$

$$p(x) = 1 + y^2 + t(x - y), y \leq x \leq 1; \tag{2.30}$$

$$\Phi(x) = 0, 0 < x < y; \tag{2.31}$$

$$\Phi(x) = \int_y^x dx[L^{**\alpha} - \delta(x)(1 + y^2 + t(x - y))^{-\epsilon}], y < x < 1. \tag{2.32}$$

Note that these equations define two areas in the economy. The interval $[0, y)$ is characterized by autarky-2 (trade flows are zero), while the zone $(y, 1]$ is characterized by local trade everywhere. The point y itself has a mixed property: it is influenced by trade flows only from one side. The producer there faces one-side competition, and thus cannot enjoy local monopoly power.

The point y is determined by the equation $\Phi(1) = 0$. For example, in the case $\epsilon = 2, \alpha = \beta = 1/2$ the point y is determined by the equation

$$\int_y^1 dx[\frac{1}{\sqrt{3}} - \frac{1}{\sqrt{5}} \frac{(1 + x^2)^2}{[1 + y^2 + t(x - y)]^2}] = 0. \tag{2.33}$$

This equation also defines y as the function of parameter t. So, a one- parametric family of general equilibria is constructed. With the change of t the following occurs:
a) the point y, which separates the "autarky-2" and the "trade" parts of economy, also moves;
b) the labor supply inside the region, belonging to different states (autarky-2, non-autarky) for these two values of t has a shock (discontinuous change);
c) prices in the "trade" part of the economy move, but in the autarky-2 part stay the same.

These results can be combined in the following proposition.

Proposition 2.6 *In the general case, the supereconomy, staying in autarky-1, after the introduction of transportation technology, breaks into two subeconomies: the autarky-2 subeconomy and the trade subeconomy. In principle, there may be no spatial links between some parts of these subeconomies, but autarky-2 always has a border with the trade subeconomy.*
The border (it may consist of several disconnected points) between these two subeconomies is characterized by a spatial discontinuity in labor supply, but prices remain continuous.
When the price of transportation moves, the border moves, and prices inside the trade subeconomy move also. But some microeconomies near this border

would have a discontinuous shock in labor supply as the result from the shift from spatial monopoly to spatial oligopoly, and visa versa.

Other examples, with spatially disconnected trade zones, can also be easily constructed. The general behaviour of the solution (equilibrium price, labor supply and trade flow as the functions of spatial coordinate) is shown in Fig. 2.7. The price and trade flow are always continuous functions of x, while the equilibrium labor supply takes only two values, and the shift occurs discontinuously. The spatial derivative of the equilibrium price function is bounded above and below, and the trade flow is different from zero only where $p'(x) = \pm t$.

2.7 Extensions

The model described above can be extended in different directions. Replacing utility or production functions by different families of functions is not likely to change crucially the results if concavity properties are preserved. Replacing the transport cost function by a more general form is more interesting, and this will be discussed subsequently. Another extention is related to the possibility of two-dimensional generalizations. The third subsection considers the extension to perfectly competitive markets in every microeconomy. It discusses the erosion of local monopoly power through local interaction, the limit when transport costs are zero and some particular features of transportation technology.

2.7.1 More General Form of Transport Cost Function

Suppose that the transportation cost function is essentially linear for small distances (this is always valid for differentiable functions with a strictly positive derivative, because of Taylor's formula), but with marginal transport costs different in different locations: $t = t(x)$. This problem can also be solved with the method discussed above. Consider the transformation $y(x) = \int_0^x t(x)dx$, where y defines a new coordinate. It is strictly monotonous in x and thus is a one-to-one map from x into y. This mapping preserves all topological links between neighbourhoods and thus the initial model can be expressed in this new variable y. Note that $dy(x)/dx = t(x)$. It is easy to check that the new densities

$$\bar{\delta}(y) = \frac{\delta(x(y))}{t(x(y))}, \qquad \bar{\gamma}(y) = \frac{\gamma(x(y))}{t(x(y))} \tag{2.34}$$

correspond to the same problem, but in the new coordinate y, where the transport cost per distance dy is always dy, and thus $\bar{t}(y) \equiv 1$. Since all functions are differentiable, the new densities would be also differentiable, and the method used before can also be applied. Thus, replacing constant transport cost t by a function of location does not change any qualitative results of the model.

2.7.2 Problems Related to Two-Dimensional Generalizations

Although there exists a well-developed theory of flows in a two-dimensional space [18], it cannot be easily applied to this model. Briefly speaking, Beckmann and Puu [18] have written the equations for flows in a two-dimensional space which can solve the problem of finding flows which minimize the overall flow (and thus, total transport cost) of the supereconomy for any given excess demand as a function of a spatial point. However, this excess demand cannot be easily calculated when both production in a point and transportation decisions are taken by producers at each point independently. The borders between trade and autarky areas have to be endogeneously determined. In a general two-dimensional case this seems to become a rather sophisticated mathematical problem. However, some symmetric two-dimensional problems can be solved.

The first trivial generalization is adding the second coordinate y, while keeping γ and δ as functions of only one coordinate x. Then the whole solution will also depend only on x, all transportation flows would go along x and all borders between autarky and trade zones would be intervals, corresponding to some level of x (the same as in one-dimensional model!), covering all values of y.

Another solvable generalization is a radially symmetric model (in Von Thünen' style). Here production and consumption densities would become the functions of the radial variable r only, keeping constant values with the change of the angular variable ϕ. If they are initially expressed as functions of Cartesian coordinates, the transformation of variables should be provided which will make them some different functions. However, they may become singular in a point $r = 0$, and thus either the problem should be studied on a disc with a hole (excluding the neighbourhood of $r = 0$) or some special boundary conditions should be formulated in $r = 0$. The intuition obtained from the solution of the one-dimensional problem, suggests that autarky zones would emerge as rings $a < r < b$, where a, b are constants, and trade flows would have a radial direction either towards the center or away from it (possible both ways, but in different trade rings). The price gradient would also have a radial direction, limited by t in absolute value.

2.7.3 Local Monopoly Power and Competition through Local Interaction

This model is an example of local interaction. In autarky-1 we have a continuum of microeconomies. The unique producer in each microeconomy enjoys a local monopoly power. He chooses lower level of labor supply and produces lower output than he would do in a competitive microeconomy. Road construction allows for transportation technology and links these microeconomies into a chain. Then each producer is facing a potential competition from two of his neighbours on both sides. While prices for the physically identical good are different in each location, transportation cost limits the competition. For high transportation costs nothing happens, competitors rationally choose not to enter to neighbouring markets, and autarky-1 becomes autarky-2. The case of low transportation costs is of higher interest, since it creates the possibility of a competitive behaviour. The chain of local monopolies does not behave as an oligopoly, it behaves competitively. Each producer takes prices as given. Given by the aggregate behaviour of a chain, where each producer and each consumer are maximizing their utilities.

In order to see to what extent competitive behaviour can be ensured by local interaction, it is useful to compare this model and one with perfect competition. There are two possible benchmark models. The spatial structure of the economy creates several differences with a spaceless economy: a) differences in productivity and demand parameters; b) existence of a chain of neighbours; c) necessity to pay transport cost.

Consider the first benchmark model, which difference is in the existence of competitive behaviour in microeconomy. Suppose that we have a continuum of producers and consumers in each point. They are replacing a subeconomy in such a way, that all consumers and producers are having now some average characteristics $\gamma(x), \delta(x)$ and no longer pay transport cost. Hence, we abstract from their heterogeneity (which was one of the conditions for trade to exist) and concentrate on elimination of chain and the necessity to pay transport cost. Each of them would choose labor supply taking the price as given. The producer optimization problem in autarky-1 with perfect competition, its solution L_∞ and the equilibrium price p_∞ are given by the following formulae:

$$\max_L p^\beta \gamma^\beta L^{\alpha\beta}(1-L)^{1-\beta}; \tag{2.35}$$

$$L_\infty(x) = \frac{\alpha\beta}{1-\beta+\alpha\beta}; \tag{2.36}$$

$$p_\infty = (\frac{\delta(x)}{\gamma(x)})^{1/\epsilon} L_\infty^{-\alpha/\epsilon}. \tag{2.37}$$

Note that for such a model labor supply increases and the price becomes lower than in the microeconomy with the same parameters. The main observation is that the labor supply coincides with the one given by the formula (2.18): $L_\infty = L^{**}$. It means that the equilibrium labor supply (hence, the output of each producer) is exactly the same as in integrated economy-2. Prices and consumption are different in these models, because paying transport costs perturbs optimal choices.

Proposition 2.7 *Under considered assumptions about preferences, technologies and sufficiently low transport costs the local interaction of a chain of microeconomies creates the same production decisions as in a benchmark model, where the same producers and consumers are put together in a point to form a perfectly competitive microeconomy.*

It is not important that agents in the first benchmark model became homogeneous: we can always consider a very small subeconomy in a neighbourhood of some point, so that the preferences and productivity parameters are asymptotically the same (because of continuity). The first benchmark model concentrates on local interaction and abstracts from global interaction. It was shown that local interaction in a chain can ensure competition.

Consider another benchmark model: with zero transport costs and heterogeneity. Formally, we put $t = 0$ in the original model, keeping its spatial structure. It is the same as to have a spaceless economy with heterogeneous preferences and technologies. Producers will take their decisions competitively, but now they take the price determined by the aggregate demand and supply as given. As before, the labor supply would be L_∞, but the price will take into account global interaction:

$$p_0 = (\frac{\int_0^1 \gamma(x)dx L_\infty^\alpha}{\int_0^1 \delta(x)dx})^{-1/\epsilon}. \quad (2.38)$$

When $t \to 0$, then spatial supereconomy converges to a usual perfectly competitive economy, where all producers and consumers are in the same point. This limit eliminates the effect of transport cost, but brings global interaction. As we have seen, there is no difference in the choice of labor supply: in all three cases (integrated economy-2 and two benchmarks) producers behave competitively, choosing high level of labor supply and output. It is important to note that this competitive behaviour occurs not only in the limit $t \to 0$, but also in the whole range of transport costs, below some threshold point.

It is possible to mention the analogy between transport costs and transaction cost. Integrated economy can be viewed as an example of a market, where heterogeneous producers and consumers can enter the common market after paying transaction cost (which depends on their relative location). The results of this model suggest that there might be conditions when sufficiently small transaction costs do not perturb the competitiveness of the market.

2.8 Summary

Existing general equilibrium theory clearly has some uncompleteness, which limits its application to the spatial analysis. That is why a new theory should be developed. The demand for such a theory clearly arises from the integration processes which occur to greater extent in the world. Not only globalization takes place, regions also loose economic boundaries.

The method of spatial general equilibrium opens the possibility to study such effects as:

1) the impact of a change in transportation costs for the equilibrium structure of production, regional differences in prices and employment;
2) the comparison of regional welfare analysis for different transportation networks and different transportation costs;
3) the comparison of the optimal solution for the Central Planner problem with the market solution,
4) the change in locational structure of production and transportation flows under the transition from a planned to a market economy.
The local transportation model developed here shows also the technical difficulties which may arise in the process of solution. Here the discrete approach which creates enormous complexity is given up in favor of a continuous model, which might have explicit analytical solutions. The advantage of having an analytical solution even for some special cases is the possibility to study comparative statics, which is quite difficult to do by computer, especially when the number of parameters is not small.

The main results of this paper are the following:

a) The introduction of transportation technologies, converting spatially different commodities into each other, is the main tool for constructing the general equilibrium in space.
b) The construction of autarky equilibria in space shows that identical physical goods differentiated only by location might have different prices. This is consistent with the theory of commodities.
c) Transportation technologies link all autarkies into an integrated economy, formally by the possibility to convert some commodities into each other. The optimization choice of a producer thus becomes more dimensional, as he takes into account the possibility of an access to spatially different markets. The general equilibrium of an integrated economy includes all local productions, all local consumptions, all local trade flows and all local prices. In autarky-1 general equilibrium all trade flows are zero, by definition. In an integrated economy they might be or might not be zero.
d) If the transportation cost function has a strictly positive derivative at the point zero, the general equilibrium of an integrated economy, in the general case, will not imply price equalization across space.

e) Despite the price difference across locations in equilibrium, this model can be viewed as an example of a competitive equilibrium. Following the traditions of Arrow and Debreu, it is possible to think of a good in different locations as being a different commodity, and of transporting as a transformation function, converting one commodity into another. Then producers have access to certain technologies and take prices as given. Here the price vector is replaced by a function $p(x)$, which is an infinite-dimensional vector.
f) Another way to think about this model is to consider it as a chain oligopoly, where each producer faces the competition with his neighbours, but he is infinitely small and cannot do anything better than to accept the price in his location as given.
g) In each point there is only one producer, but not one seller. The number of sellers is determined endogeneously, and in equilibrium there might be 1, 2 or 3 sellers. If there is only one seller, he is a local monopolist, if two or three - there is an oligopoly. Nobody has the power to decide about being a local monopolist, unless this decision is suggested by the whole outcome of this network interaction through space.

2.9 Appendix

The following Lemma formally proves that there exists constant R in the formula for the transportation costs as the function of distance (see section 4) makes only local trade (in the small neighbourhood of a point) an optimal decision for any producer. It is useful to note that for negative R the trade with distant neighbours becomes optimal. For $R = 0$ the following situation may occur. If $p_0'(x) = t$, for some subinterval, any producer becomes indifferent at what distance to trade. Formally, a game between a continuum of producers will take place, with coordination of actions being the main problem. It may have an infinite number of solutions, with rather complicated structure. In order to escape this possibility, the assumption of positive R is used in this model.

Lemma 2.4 *Suppose that $\delta(x), \gamma(x)$ are continuous and differentiable functions. Then $\forall \delta(x), \forall \gamma(x) \exists R^*$, such that $\forall R, R > R^*$, the transportation costs $T(x, y) = t|x - y| + R(x - y)^2$ would make it unprofitable for any producer to trade at any positive distance, except for infinitely small.*

PROOF.
1) Note first, that any $|dp/dx| > t$ *is not consistent with general equilibrium, as it enables for the producer at* x *to resell more output and to gain more profits. Hence, it always should be the case:* $|dp/dx| \leq t$.
2) For any function $p(x), x \in [a, b]$, *such that* $|dp/dx| \leq t$, *and* $\forall x, y \in [a, b]$,

$$|p(x) - p(y)| \leq t|x - y| < T(x, y). \tag{2.39}$$

The equality holds only in the limit $y \to x$. *Hence, the trade may occur in equilibrium only for infinitely small distances. (In reality, they would be the distances between neighbouring producers).*

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Chapter 3

Hotelling's Revival

The problem of demand discontinuity in the original Hotelling (1929) model with linear transport costs and a uniform distribution of consumers on an interval can be solved by either introducing quadratic transport costs (d'Aspremont et al, 1979) or by going to a two-dimensional space (Economides, 1986). In the present paper the two-dimensional results are generalized for any bounded distribution of consumers on any compact set on a plane, which can describe real geographical situations. These results still hold for any transport costs strictly increasing in distance. However, continuity does not guarantee the existence of a Nash equilibrium in pure strategies for all cases. Examples of both existence and non-existence are constructed, and for some family of densities the separation point between the two cases is found.

JEL Classification: L13, R10, R30.
KEYWORDS: Two-dimensional Hotelling model, demand continuity, existence of Nash equilibrium.

3.1 Introduction

Harold Hotelling's paper [1] had a big impact on different branches of economic theory, specially on spatial economics and industrial organization. In fact, he accomplished several things. First, he introduced linear transportation costs along with spatially dispersed consumers. Second, he considered oligopolistic competition in space. And last, he mentioned the possibility of using his model not only for real space, but also for other abstract spaces, like in the product differentation literature.

The first idea was developed later by spatial economists, while the last two had an impact on the theory of industrial organization. Starting from the late 70's, critisism of Hotelling's approach from the position of game theory has been raised [2,3,4] based on the non-existence of Nash equilibria in pure strategies. One of them was about the assumption of a linear transport cost function. D'Aspremont et al [3] proposed to replace the linear function by a quadratic one. Under these assumption a Nash equilibrium was shown to exist. There was a further development of this idea [5] with the result that under quadratic transport costs (or, equivalently, quadratic consumers' preferences for product variety) a Nash equilibrium exists in a two-stage game, where first firms choose locations, and later prices.

This approach reoriented the development of economic research and models with linear transportation costs were basically neglected. Most of the literature from the early 80's up to the present is based on quadratic transport costs [1]. Since the assumption of quadratic costs is counter-intuitive in the real space, research focused mainly on the space of product differentiation, where distances are not measurable, and everybody can be equally happy with any structure of transport cost.

Veendorp and Majeed [17] studied a two-dimensional generalization of Hotelling's model with quadratic transport costs. They considered a two-stage locational game and found that maximal differentiation in one dimension along with minimal differentiation in another is an optimal solution. The result was obtained by computation. They explain it as a tendency to

[1] Although, one should mention [4], where a mixture of linear and quadratic transport costs was introduced. A paper of d'Aspremont and Motta [19] is another example of current research related to linear transport costs.

minimize the length of a curve with indifferent consumers, in order to reduce the severeness of competition.

With quadratic costs the indifferent consumers are always located in a straight line. With linear costs they are located in hyperbolas. This makes all computations more complicated. In this paper we show that, with linear costs, in any Nash equilibrium we will get maximal differentiation in one dimension. But in this case the existence of a Nash equilibrum becomes a non-trivial question. It will be also shown, that the decrease in the measure of indifferent consumers tends to lead to a failure in the existence of a Nash equilibrium. Thus the intuition obtained from models with quadratic costs cannot be applied to models with linear costs. In a similar vein, Irmen and Thisse [16] proved equilibria to display maximal differentiation in one dimension only for a multidimensional model with quadratic cost. However, they assumed different weights for distances in different dimensions.

Beath and Katsoulacos [2, p.24] mention that the difficulties with Hotelling's model arise from his "special assumptions of infinitely inelastic demands and constant marginal transport costs". As we know, to assume quadratic transport costs is a way to avoid these difficulties. Even though the literature on industrial organization was very concerned about this demand discontinuity, the paper of Economides [8] received little attention. He proves that in a two-dimensional space, with linear transport costs and different metric spaces (including the Euclidean space), demand is continuous and a Nash equilibrium always exists. He proves it[2] for a uniform distribution of consumers over a disc in R^2.

In this paper I try to obtain more general results. The disc can be replaced first by a square, which is more attractive for the problem of product differentiation. This will be done in Lemma 3.2. I later extend this result to any compact set in a two-dimensional space, with any bounded density of consumers, not necessarily uniform. This is much more realistic for spatial competition with real geography. The existence of an equilibrium in this case is also proved.

[2]Klein [10] goes in other direction and studies the existence of a symmetric equilibrium on a disc for elastic demand from each consumer

It is interesting to mention that this agenda was already set by Hotelling himself, who wrote that "instead of a uniform distribution of consumers along a line we might have assumed varying density... instead of linear market we might suppose the buyers spread out in a plane... if transportation is in straight lines at a cost proportional to the distance, the boundary will be a hyperbola... if cost is given by such a complicated function as a railroad freight schedule, the boundaries will be of another kind..." [1]. This paper is called "Hotelling's Revival", because it revives the initial direction of research proposed by Hotelling. The main goal is to see to what extent we can get rid of the problems that have been discovered fifty years after the publication of the original paper.

Several examples of the non-existence of a Nash equilibrium in pure strategies in Hotelling-type models have been constructed in [20]. At the same time, sufficient conditions for its existence have also been obtained [18]. Hence, in some cases a Nash equilibrium in pure strategies will exist, while in other cases it will not. This study shows that both things might take place in two-dimemsional Hotelling model with the original assumptions [3]. The main tool for proving Nash equilibrium existence is the theorem of Maskin and Dasgupta [12]. It requires the upper semicontinuity and quasiconcavity of profit functions. It will be shown later that continuity can be obtained for many two-dimensional generalizations of Hotelling's model. This gives Nash existence in mixed strategies. However, as Economides notes, quasiconcavity may fail even for a disc area with uniformly distributed consumers. He manages to prove Nash existence in pure strategies, but the standard methods are no longer useful here. Hence, less ambitious methods based on particular functional forms of the profit function should be applied.

This paper has several goals. First, it aims to explore the potential of the original Hotelling model by extending it to a two-dimensional space. This is the reason why I keep the assumption of linear transport costs in the beginning. Second, it studies the impact of the structure of transport cost on the continuity of the demand function in the two-dimensional case with a euclidean metric. The main result of this paper is to show the continuity of demand in a two-dimensional Hotelling model with Euclidean metric, contin-

[3]Many authors have realtered these assumptions, keeping Hotelling's name. This may be a reason for some confusions

uous density of consumers and monotonously increasing transport cost. This guarantees the existence of a Nash equilibrium at least in mixed strategies. But in pure strategies it fails to exist in many cases. The construction of examples of both Nash existence and non-existence and the determination of sufficient conditions for its existence can be considered as the third goal of this research.

The paper is organized as follows. Section 2 is about demand continuity in the two-dimensional case. It starts from basic assumptions of the model, and then continuity is proved for a sequence of problems with increasing generality. In section 3 the question of existence of a Nash equilibrium is studied. The problem turns out to be non-trivial, since continuity of demand does not guarantee its existence. For some particular class of functions the threshold point, separating the two possible cases (existence and non-existence) is found explicitely. It is also possible to establish some sufficient conditions for the existence of a Nash equilibrium. Later, in section 4, the cases of quadratic and more general transport costs are also considered in R^2. It is shown that continuity of demand can be proved for a quite general structure of transport costs. Conclusions are drawn in section 5. Most of the proofs are done in the appendix, which consists of three subsections. The first part of appendix contains the proofs related to the continuity of demand. The second part of the appendix proves that maximal differentiation in one dimension is the solution for a model with a two-dimensional square uniformly populated with consumers, linear transport costs and a Euclidean metric. The third part of the appendix formulates the problem in elliptic coordinates, relates the two-dimensional density to a one-dimensional density and formulates three lemmas about sufficient conditions for the existence of a Nash equilibrium in pure strategies.

3.2 Demand Continuity in a Two-Dimensional Model

In a model of Cournot competition among two firms on an interval, Nash equilibrium may fail to exist when demand is not continuous[6, p.393]. In this section it will be proved that in a two-dimensional extension of Hotelling's model of Cournot competition among two firms with fixed locations, the total demand of a firm is a continuous function of its pricing strategy. I keep the same assumptions as Hotelling [1], except for space dimensionality. I will start with a set of restrictive assumptions on the structure of the spatial primitives, which will be relaxed later (see Theorem 3.1 below).

ASSUMPTIONS.

1. Consumers are located with uniform density 1 over the finite square $S = \{(x, y) \in R^2 | -L < x < L \text{ and } -L < y < L\}$. (This assumption will be relaxed later).

2. There are two firms, which are symmetrically located at points $(-c, 0)$ and $(c, 0)$ of the plane (x, y) with $c < L$, so that firms are located in the interior of the square S.

3. Each consumer inelastically demands one unit of the good from the closest firm.

4. A consumer has to pay the price p_i and the transport cost, which is linear in distance and equal to tr, where t is the given unit cost and $r = \sqrt{(x - x_f)^2 + (y - y_f)^2}$ is the distance (in Euclidean metric) to the firm located at (x_f, y_f).

5. Firms have zero production costs. Their locations are fixed. They set their prices, p_i, trying to maximize their profits.

For any pair of pricing policies, (p_1, p_2), there exists a separating border between the demand areas of the firms. By assumption 2, the euclidean distance between the consumer at (x, y) and the firms is given by $r_i = \sqrt{y^2 + (x \pm c)^2}$. Then the line of "border consumers", who are indifferent between the two firms is given by the equation [4]

$$p_1 + tr_1 = p_2 + tr_2. \tag{3.1}$$

Let $a = \frac{|p_1 - p_2|}{2t}$. Then we get the hyperbola equation for this border: $r_2 - r_1 = 2a$. The border is represented by the right branch of a hyperbola, if firm 1 charges lower price, and vice versa.

[4] Strictly speaking, by the intersection of the set of solutions with the square where consumers are located

Lemma 3.1 *The border between consumers buying from different firms, is described by the hyperbola equation*

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \tag{3.2}$$

where the coefficients $a = \frac{|p_2 - p_1|}{2t}$ *and* $b = \sqrt{c^2 - a^2}$ *depend parametrically on prices.*

The equation of a hyperbola has a standard mathematical form and can be found in any elementary geometry textbook. The points $(c, 0), (-c, 0)$ are focuses, and a is the point where the hyperbola intersects the horizontal axis. Obviously, a is a linear decreasing function of p_1, when p_2 is fixed. Thus, this intersection moves continuously to the right when the price of the left firm, p_1, is decreasing. The upper branch of the hyperbola is given by the equation $y(x) = \sqrt{c^2 - a^2}\sqrt{x^2/a^2 - 1}$, which shows that for any $a' > a$ for all sets of feasible x, $|y(x, a')| < |y(x, a)|$. This means that the hyperbolas are "inside one another" and never intersect.

Lemma 3.2 *The total demand from one firm in a square-Hotelling model is a continuous function of its own and its rival's prices.*
PROOF: See Appendix 1.

Note that, in contrast to what happened in the one-dimensional model, demand is also continuous at point $p_1 - p_2 = 2tc$. The intuition is that, in a one-dimensional framework, when the consumers located behind a firm are captured by the rival firm, they add up to a positive measure. But in a two-dimensional case they have a zero measure.

Finally, let us consider the general case of any compact subset in a two-dimensional space.

Theorem 3.1 *With a uniform density of consumers on any compact subset of a two-dimensional space, the aggregate demand from one firm is still a continuous function of its and its rival prices.*
PROOF: See Appendix 1.

Now we can consider a further generalization of Hotelling's model, namely, the case of a non uniform density of consumers. The conclusion of the previous theorem will still hold, unless this density has no atomic points, i.e. as long as the integral of the density over any zero-measure subset is zero.

Theorem 3.2 *Aggregate demand is a continuous function, when two firms are located in any two separated points, even when the consumers are distributed with any continuous density over any compact set on a plane.*
PROOF: See Appendix.

In fact, the continuity of demand density is not a necessary condition for the continuity of the total demand and can be replaced by boundedness.

Theorem 3.3 *Total demand is still a continuous function for any upper-bounded demand density function defined on any compact set on a plane.*
PROOF: See Appendix.

The last theorem makes it possible to consider problems with real geographical data. Indeed, usually maps do not contain continuous densities, but their approximations in step functions.

3.3 Nash Equilibrium

In Hotelling-type games the existence of pure strategy Nash equilibria is not a trivial question. The sufficient conditions formulated by Dasgupta and Maskin [12] very often do not hold. Roberts and Sonnenschein [14] have constructed examples of Nash existence failure due to the possibility of non-concave profit functions generated by standard economic assumptions. H.Dierker and Grodal [20] have given other examples of non-existence: it arises from the multiplicity of market clearing prices which leads to discontinuous price selections. E.Dierker [21] stressed the essential role of the quasiconcavity of profit functions for the existence of a Nash equilibrium in pure strategies and proved some results related to log-concavity of market shares.

This section is organized as follows. First the case of uniform square is studied: a symmetric Nash equilibrium is explicitly constructed and maximal differentiation in one dimension is obtained. Then we discuss cases in which cornering the market is an inferior strategy and how the nonexistence of Nash equilibria may emerge even when profit is a continous function of both prices. Later the modified Hotelling model with a non-uniform density of consumers is studied. In the beginning a particular example of Nash failure is constructed for a non-uniform density of consumers. Then for a uniparametric family of densities the threshold point between Nash existence and non-existence is discovered. It is important to note that in some cases a Nash equilibrium exists despite the non-concavity of profit functions. Finally, the comparison of one-dimensional and two-dimensional Hotelling models is provided. A sufficient condition for Nash existence in the case of logarithmic one-dimensional densities, generated by aggregation of consumers in elliptic coordinates, is formulated and proved in Appendix 3.

3.3.1 The Uniform Square Case

For simplicity, in this subsection we consider the case of a uniform distribution of consumers over a square. The profits of the firms are given by the functions:

$$\pi_1 = p_1 S_1, \qquad \pi_2 = p_2 S_2, \tag{3.3}$$

where S_i denotes the area served by firm i. Let us take p_2 as a parameter and express S_1 as a function of both prices for all possible cases. Let $g_i(p_2)$ be a function such that when p_1 is equal to it, the separating hyperbola passes through the corners of the square (left for g_1 and right for g_2). Then

$$\begin{aligned}
S_1 &= 0 && \text{if} \quad p_1 > p_2 + 2ct \\
S_1 &= S_b && \text{if} \quad g_1(p_2) < p_1 < p_2 + 2ct \\
S_1 &= 2L^2 - S_a && \text{if} \quad p_2 < p_1 < g_1(p_2) \\
S_1 &= 2L^2 + S_a && \text{if} \quad g_2(p_2) < p_1 < p_2 \\
S_1 &= 4L^2 - S_b && \text{if} \quad p_2 - 2ct < p_1 < g_2(p_2) \\
S_1 &= 4L^2 && \text{if} \quad 0 < p_1 < p_2 - 2ct
\end{aligned}$$

Here S_i for $i \in \{a, b\}$ is given by the algebraic expressions obtained in the proof of lemma 2 in Appendix 1. Note that $S_2 = 4L^2 - S_1$.

A Nash equilibrium satisfies the following system of equations:

$$\begin{aligned} \frac{\partial \pi_1}{\partial p_1} &= 0, \\ \frac{\partial \pi_2}{\partial p_2} &= 0 \end{aligned} \tag{3.4}$$

Profits are continuous functions, differentiable almost everywhere (except for the points linking different formulas for S_1). For the range of possible prices, both profits start from 0 and end in 0, , thus having at least a maximum in between. Consider the family of profit curves for firm 1 as a function of its own price, depending parametricaly on the rival's price (which picks an element from this family). If all the elements of the family have a maximum, then the function corresponding to the optimal rival's policy also has it. We take this curve and find its maximum. This pair of prices will correspond to a Nash equilibrium.

Usually the simpler case of a symmetric location is considered. Then it is possible to find the solution explicitly. Economides [8] does it for a disc area. We extend his results for other areas. It is important to stress here that it is also possible to consider problems with an asymmetric geometry of consumers (the theorems can be used), for which only asymmetric location makes sense. In this case, a Nash equilibrium is also asymmetric. The system of nonlinear algebraic equations can be solved only numerically, but these algorithms may solve applied problems in the real geographic space.

I also want to stress the difference between the initial Hotelling model, formulated for the real space, and a mathematically equivalent formulation in a product space, where the shape of corresponding transport costs can be derived from the shape of the utility function. First of all, it is possible to modify consumer's utility in the real space model, and then we deal with two separate effects - the shape of individual demand and the shape of transport cost, which makes the problem in real space mathematically richer. Second, disjoint areas with a non-uniform consumer density, which are typical in the real space, might be rather counterintuitive in the product space.

Economides [8] has proved the existence of a symmetric Nash equilibrium on a disc uniformly populated by consumers, by finding an explicit solution. In the case of a square it is possible to show (see Appendix 2 for details) that the Nash equilibrium price for a square with side $2L$ and locations at a distance c from the center is given by the following formula:

$$P^*(c) = \frac{2L^2}{\frac{L\sqrt{L^2+c^2}}{2ct} + \frac{c}{2t}\ln\frac{L+\sqrt{L^2+c^2}}{c}}. \quad (3.5)$$

In the general case, symmetric reply is not necessarily optimal [5], and it is necessary to calculate other local maxima and to compare payoffs there with the payoff in the symmetric Nash equilibrium. In the case of a square, the payoff for cornering market is equal to $\hat{\pi} = 4L^2(p_2 - 2ct)$, while the symmetric Nash payoff is equal to $\pi = 2L^2 p_2$. It is easy to see that cornering is inferior, if $p_2 < 4ct$. In symmetric Nash $p_2 = P^*(c) < 2ct$, and thus a symmetric Nash strategy dominates cornering for all c.

This result is very much similar to those of [8] and suggests that not very much depends on the shape of the market, at least in the symmetric case. As $c \to 0$, the prices also approach zero in a linear way, and for the infinite increase in c they converge to a finite limit. It is possible to show that this optimal price strictly increases with the increase of parameter c. Thus, the maximum is reached for the maximal possible value (location at the edge of the square). This suggests that the maximal differentiation principle is valid in 2-dimensional Euclidean space.

Proposition 3.1 *In a two-dimensional two-stage locational game on a uniform square the principle of maximal differentiation is valid, i.e. firms prefer to locate first at the highest possible distance, in order to be able to set highest prices in the symmetrical Nash equilibrium.*
PROOF: See Appendix 2.

3.3.2 The General Case

The question of equilibrium existence can be approached by different mathematical methods: direct maximization, the proof by continuity along with convexity, and algebraic topology. The first method is useful only when all functions are specified, and can be applied in some particular cases (e.g. a square). The existing theorems for general functional forms [12] require

[5] Although there is a tradition in the industrial organization literature to consider symmetric Nash for symmetric problems

too many conditions (continuity, convexity, etc), which very often are not satisfied even for quite regular geometrical shapes. In the case of the original Hotelling model with linear transport cost and a uniform distribution of consumers, both continuity and convexity fail and this leads to Nash non-existence. In the two-dimensional case continuity can be easily recovered, as has been shown above. But non-convexity of payoffs can exist even for very regular geometrical shapes (even for a disc [8]). That is why we have existence theorems for mixed strategy Nash equilibria, while the proof of existence in pure strategies remains quite a painstaking task even for particular geometric forms.

It is interesting to consider an asymmetric case, which has received little attention in the literature (except for [13]). For the case of an arbitrary compact set with any density of consumers there is no reason to expect the existence of a symmetric equilibrium. In the case of a disc, Economides [8] shows that the aggregate demand can be represented by a continuous function with two local maxima, and not differentiable at some points. This suggests that in a more general case of asymmetric equilibria, which is studied here, it is natural to expect several local maxima. What is the reason for this? Although the aggregate demand from one firm is a continuous non-increasing function of its own price, the product of demand and price which represents profits may have several maxima. Everything depends on the relative location of the curves $D(p)/D(p_0)$, where p_0 is "cornering price" (the price at which one firm wins all consumers), and the curve p_0/p (here the rival's price is taken as a parameter, fixed for each curve).

Proposition 3.2 *1. If $d(pD(p))/dp > 0$ at $p = p_0$, then cornering the market is never an optimal strategy.*
2. If in addition $D(p)$ is concave for $p > p_0$, then the profit function is also concave and has a unique maximum.
3. If the problem is symmetric, then there exists symmetric Nash equilibrium. If not, a Nash equilibrium may be asymmetric in prices.

PROOF:
1) Since the right derivative of profit is positive at p_0, the cornering price is not profit-maximizing.
2) Recall that $D'(p) \leq 0$ always in Hotelling-type competition, since any price increase may only lead to shift of some consumers to the other firm.

Since concavity implies $D''(p) < 0$, the second derivative of profits $\pi''(p) = pD'' + 2D' < 0$. Thus the profit function is concave. Because profits are continuous, their own price derivative at p_0 is positive, and since they vanish to zero at $p_0 + 4ct$ (rival's cornering), their exists a unique maximum.
3) As this happens for all rival's prices, for some price, which is optimal for rival, it will also occur. In the symmetric case this will give a unique symmetric Nash equilibrium, while in asymmetric case this equilibrium may be asymmetric.

Note that this proposition gives only some very specific sufficient conditions for Nash equilibrium existence. As it was shown in [8], Nash equilibrium may exist despite the presence of several local maxima. The phenomenon of several local maxima (at least two) in aggregation problems was mentioned also in [15].

Before proceeding with a theorem, let us consider a simple example, which suggests what we should expect in the general case.

Consider a function with two local maxima. Let each function be a member of a family of functions, so that the dependence on the parameter, characterizing the element of the family, is continuous. Then for almost all the values of this parameter this function has unique global maximum. However,as the parameter changes the global maximizer shifts continuosly except at a point where the two local maximum values are equal: there the global maximizer can jump to a strategy which is not necessarily close as a consequence of an arbitrarily small change in parameter. If we allow for mixed strategies only at the point where this jump occurs, then there exists a continuous path between all optimal replies to the rival's strategy. This path consists of some curves $x = F(y)$, which are linked through the space of mixed strategies between each other. Note that the payoff remains a continuous function of the rival's strategy (the firm is indifferent to choose any probability to play a strategy mix at this point). Similarly, an optimal rival's reply $y = G(x)$ can be constructed. Suppose for a moment that the maximum is a continuous function of pure strategies for both firms, and mixed are not necessary. Then we can substitute one equation into another, and get the fixed point problem: $x = F(G(x))$. The composition of F and G is a continuous function, the set of strategies [0,1] is compact, and by Brouwer's theorem, there exists a fixed point $x^* \in [0, 1]$. Then $(x^*, G(x^*))$ is a Nash equilibrium in pure strategies.

Note that the continuous dependence of maximum on the rival's strategy is indeed a condition for the existence of Nash equilibrium in pure strategies that cannot be dispensed with. There exist several theorems about Nash existence in the case of a continuum of strategies [11,12]. One of them claims that although continuity may be replaced by upper semicontinuity, continuity of the maximum and quasiconcavity of payoff with respect to strategy is required for the existence of Nash in pure strategies. The theorems provide some sufficient conditions, and the examples with adisc [8] and a square show that the existence of several maxima, which violates the quasiconcavity of payoffs, may still be consistent with the the existence of a Nash equilibrium in pure strategies. In the case of a continuous demand the quasiconcavity of payoffs is not necessary, if the maximum depends continuously on both strategies.

It is possible to show that in the generic case the shift between several maxima may occur discontinuously in rival's strategy papameter, and thus Nash equilibrium in pure strategies might fail to exist. There is another theorem about the existence of a Nash equilibrium in mixed strategies. Its conditions might be satisfied in a more general case. But what is mixed equilibrium with a continuum of strategies from a practical point of view? Observing time series, we may observe complicated sequences of pricing policies, which is not distinguishable from chaos. Calculating statistics, one may find distributions and relate them to probabilities to play particular strategies in a Nash equilibrium in mixed strategies. Thus, depending on the configuration, only in some cases with an asymmetric consumer location we can have a Nash equilibrium in pure strategies. Often it exists in mixed strategies.

Lemma 3.3 *In a 2-dimensional generalization of Hotelling's model with a nonuniform distribution of consumers over a compact set in space and linear transport costs, a Nash equilibrium does not necessarily exist in pure strategies, despite the continuity of profits in both prices.*

PROOF:
Consider the family of payoffs for the first firm, $f(x, y)$*,as a function of its own strategy* $x \in [0, 1]$*, parametrically dependent on the strategy of its rival* $y \in [0, 1]$*. Let* $f(x, y)$ *be a continuous function of both variables. Assume also,*

that $f(0,y) = f(1,y) = 0, \forall y \in (0,1)$ *and* $f(x,y) > 0, \forall x \in (0,1), \forall y \in [0,1]$. *Assume similar conditions for the payoff of the second firm* $g(x,y)$*, which might have a completely different shape. As the range of price variation can always be normalized to 1, with zero payoffs for both firms on both ends (either due to zero pricing or due to the complete capture of the market by the rival firm), this insures the existence of internal maximum, because of continuity. But in a quite general case each of these functions has a finite number of local maxima (in the firm's own strategy), which in the generic case have the identical values only for a set of parameters of measure zero. This gives rise to the potential possibility for the discontinuity of the global maximum with respect to rival's price. Although this does not necessarily imply the absence of Nash equilibrium in pure strategies, in some particular cases this may occur.*

3.3.3 A particular Example

Here an example of a function with two maxima will be provided. Consider a one-dimensional modified Hotelling model in which two firms are located at the points $x = -1$ and $x = 1$, while consumers are located with the positive density $1/2a$ only inside the following intervals: $[-2a, -a]$ and $[a, 2a]$, where a is an arbitrary parameter. Note that it is possible to arrange infinitely many specifications of the two-dimensional Hotelling model, which are mathematically described by the same demand functions that will arise in this one-dimensional case.

Theorem 3.4 *Every one-dimensional Hotelling model with a bounded demand density defined on the interval connecting the fixed locations of thetwo firms, can be represented as a mathematically equivalent two-dimensional model with some bounded demand density over some compact set. This can be done in infinitely many ways. If in the one-dimensional model some consumers are located behind the firms then there is no mathematically equivalent two-dimensional representation via bounded demand density functions.*

PROOF:
In the two-dimensional case, any price shift dp will generate the shift of indifferent consumers between two corresponding hyperbolas. Their location depends only on the price differential. If we set the density function along each hyperbola in such a way that the integral of this density along the intersection

with the bounded set of consumers is always equal to the one-dimensional demand density at a point, then the arising demand functions in the two cases will be the same. This can be done in infinitely many ways, as we have only one constraint via the value of each integral in the space of functions. But the image of the interval behind the firm is also the interval behind the firm, and thus no equivalent correspondence in the class of bounded measures is possible.

The most natural way to calculate the corresponding one-dimensional demand density is to use elliptic coordinates (see Appendix 3). Due to the above theorem, this one-dimensional example is mathematically equivalent to the corresponding two-dimensional examples. Direct calculations show that the profits of the first firm are given by the following function of its own price p_1 and the price of its rival p_2:

$$\begin{aligned}
\pi = p_1 &\quad \text{for} \quad 0 < p_1 < p_2 - 4a; \\
\pi = p_1 \frac{p_2 - p_1}{4a} &\quad \text{for} \quad p_2 - 4a < p_1 < p_2 - 2a; \\
\pi = 0.5 p_1 &\quad \text{for} \quad p_2 - 2a < p_1 < p_2 + 2a; \\
\pi = p_1 (1 - \frac{p_1 - p_2}{4a}) &\quad \text{for} \quad p_2 + 2a < p_1 < p_2 + 4a; \\
\pi = 0 &\quad \text{for} \quad p_1 > p_2 + 4a.
\end{aligned}$$

Consider the maxima of the function $\pi(p_1)$, taking the dependence on p_2 and a as parametrical. It is easy to show that this function always has a maximum in the point $p_1 = p_2 + 2a$, equal to $Max2 = 0.5p_2 + a$. Another maximum may exist in a point $p_1 = p_2 - 4a$, if $p_2 > 8a$, and is equal to $Max1 = p_2 - 4a$. If $p_2 < 8a$, this maximum shifts to a point $p_1 = 0.5p_2$ and is equal to $Max1 = p_2^2/16a$. Direct calculations show that $Max1 > Max2$, for $p_2 > 10a$, and $Max1 < Max2$ for $p_2 < 10a$. When $Max1$ is not located at the point where the first firm corners the market (cornering occurs for $p_1 = p_2 - 4a; p_2 > 8a$), we always have $Max2 > Max1$. This result implies the following optimal replies $pi[pj]$:

$$p_1[p_2] = p_2 - 4a \quad \text{for} \quad p_2 > 5; \tag{3.6}$$

$$\begin{aligned}
p_1[p_2] &= p_2 + 2a \quad \text{for} \quad p_2 < 5; \\
p_2[p_1] &= p_1 - 4a \quad \text{for} \quad p_1 > 5;
\end{aligned}$$

$$p_2[p_1] = p_1 + 2a \quad \text{for} \quad p_1 < 5. \tag{3.7}$$

The minimal pricing policy is zero, and though the maximal is theoretically not limited, nothing interesting or relevant to equilibrium occurs at very high prices. Fig. 3.11 shows the graphs of optimal replies for $a = 1/2$ (in this case firms are located just behind two disjoint sets of consumers). A Nash equilibrium in pure strategies fails to exist in this case (and also in the corresponding family of two-dimensional problems). This shows that demand continuity is still not sufficient to guarantee the existence of a Nash equilibrium in Hotelling-type games.

3.3.4 An example with a Non-Uniform Density of Consumers

The example that we discuss here may be of some help in understanding why a Nash equilibrium in pure strategies may fail to exist and when one can be sure about its existence. It turns out that not much depends on the particular shape of the two-dimensional area, in the sense that regular shapes do not guarantee the existence of a Nash equilibrium in pure strategies, while it may nevertheless exist for less regular shapes. The issue is that only the behaviour of the aggregate demand from a firm matters [6], and this demand depends on the behaviour of an integral of the demand density over some area on a plane. Thus, as it was shown in the previous subsection, any two-dimensional problem is mathematically equivalent to a corresponding one-dimensional problem with firms at the ends of an interval and a non-uniform density of consumers inside. Thus, the study of such problems with different densities gives the key to the solution of the two-dimensional problems.

It is well known that a Nash equilibrium exists for a uniform density, when firms are located at the end of an interval and transportation costs are linear in distance. In the previous example it was shown that a Nash equilibrium fails to exist when consumers are concentrated in two disjoint clusters. An interesting question arises: What is crucial for the existence of a Nash equilibrium? Can it fail to exist when the demand density is continuous? The example described below shows that the answer to this question

[6] Not in the sense of continuity, which was already proved, but in the sense of belonging to some class of smooth functions, with derivatives between some bounds, which guarantee the continuous behaviour of the global maximum of the profit function.

is positive.

Consider an interval $[-1, 1]$ with the demand density (Fig. 3.10a,b)

$$\rho(x) = a + b|x|, x \in [-1, 1]. \tag{3.8}$$

It is easy to show that in this case the profit function of the first firm (located at $x = -1$) is given by the formula:

$$\begin{aligned} \Pi = p_1(2a + b) \quad &\text{for} \quad 0 < p_1 < p_2 - 2; \\ \Pi = p_1[a + \frac{b}{2} + ay + \frac{b}{2}y^2] \quad &\text{for} \quad y \equiv \frac{p_1 - p_2}{2}; p_2 - 2 < p_1 < p_2; \\ \Pi = p_1[a + \frac{b}{2} + ay - \frac{b}{2}y^2] \quad &\text{for} \quad p_2 < p_1 < p_2 + 2; \\ \Pi = 0; p_1 > p_2 + 2. & \end{aligned} \tag{3.9}$$

Consider the case $p_1 \in [p_2 - 2, p_2]$, which corresponds to the location of the indifferent consumer in the right half of the interval ($y \in [0, 1]$). Then the critical points of the profit function can be found from $d\Pi/dp_1 = 0$, which leads to the equation:

$$a + \frac{b}{2} + \frac{a}{2}(p_2 - p_1) + \frac{b}{8}(p_2 - p_1)^2 - p_1[\frac{a}{2} - \frac{b}{4}(p_2 - p_1)] = 0. \tag{3.10}$$

In this case the second firm's profit is $\Pi_2 = p_2(2a + b - D_1)$, where D_1 is the demand from the first firm. Thus, its profit maximization condition is given by

$$a + \frac{b}{2} - \frac{a}{2}(p_2 - p_1) - \frac{b}{8}(p_2 - p_1)^2 - p_2[\frac{a}{2} + \frac{b}{4}(p_2 - p_1)] = 0. \tag{3.11}$$

The system of the last two equations determines all possible Nash equilibria and deserves further study. The easiest way to proceed with it is to transform this system, first adding and then substracting these equations from each other. After some calculations it can be reduced to the system:

$$(p_2 - p_1)[3a + b(p_2 - p_1)] = 0; \tag{3.12}$$

$$2a + b = (p_2 + p_1)[\frac{a}{2} + \frac{b}{4}(p_2 - p_1)]. \tag{3.13}$$

From the first equation it follows that either (a) $p_1 = p_2$, or (b) $p_1 - p_2 = 3a/b$ should hold.

The case (a) represents symmetric Nash equilibrium, while the case (b) the asymmetric one. First consider the symmetric case. Then,

$$p_1 = p_2 = 2 + \frac{b}{a}. \tag{3.14}$$

Since the profit function is not differentiable at the corner point, which occurs for firm 1 at price $p^c = b/a$, it is necessary to compare the candidate solution for the symmetric Nash with the profits when the market is cornered by the first firm, to be sure that Nash dominates. Otherwise, a pure strategy Nash equilibrium may not exist [7]. In this case the profit of the first firm at the corner Π^c and its profit at the symmetric candidate-for-Nash price Π^{SN} are given by the formulas:

$$\Pi^c = \frac{b}{a}(2a + b); \Pi^{SN} = \frac{a}{2}(2 + \frac{b}{a})^2. \tag{3.15}$$

Thus, symmetric Nash dominates cornering when $b/a \in [-2; 2]$. But the problem considered makes no economic sense for $b/a < -1$, since $\rho(x)$ becomes negative at some points. Hence, the following lemma can be formulated.

Lemma 3.4 *If $a \geq 0; b/a \in [-1; 2]$, then the problem with the density $\rho(x) = a + b|x|; x \in [-1, 1]$ has an economic meaning, and a symmetric Nash equilibrium exists. It is given by the formula $p_1 = p_2 = 2 + b/a$. If $b/a > 2$, a symmetric Nash equilibrium fails to exist since cornering the market dominates it.*

Consider now the asymmetric Nash equilibrium. Since $p_1 = p_2 + 3a/b$, and for $y > 0$ we need $p_1 < p_2$, the ratio a/b can take only negative values. As it was shown before, the problem in this case has an economic sense only if $b \in [-a; 0]$. The solution for the prices is given by the formulas

$$p_1 = \frac{3a}{2b} - 4 - \frac{2b}{a}; p_2 = p_1 - \frac{3a}{b}. \tag{3.16}$$

It is easy to check that, in the given interval of parameter b, prices are always positive.

Lemma 3.5 *When consumers are more concentrated to the center of the market, i.e. $\rho(x) = a + b|x|; a > 0, b \in [-1; 0]$, an asymmetric Nash equilibrium also exists.*

[7] One should also check whether an asymmetric Nash can exist before coming to a final conclusion about Nash existence in pure strategies

3.3.5 Comparison of One- and Two-Dimensional Hotelling Models

The one-dimensional Hotelling model with linear transport cost has several shortcomings. The first is that in the real world geography is two-dimensional with a heterogeneous density of consumers across space and Euclidean distance. The second is more technical: the aggregate demand from on firm is discontinuous in price, which leads to the non-existence of a Nash equilibrium in many cases.

We have shown that the two-dimensional generalization of Hotelling's model not only makes it more realistic, and thus the results can be applied in economic geography, but also eliminates the main technical problem of demand discontinuity. However, it still cannot guarantee the existence of Nash equilibrium in pure strategies for all cases. The most that can be done is to provide sufficient conditions. However, numerical calculations may prove Nash existence in very tiny cases. The natural question arises: Why it is so? Generically, the demand function can be any continuous function with a non-positive derivative. It is possible to generate any downward sloping continuous function by an appropriate choice of densities and areas, and for any function there exists an infinite quantity of possibilies to do it. But this demand, being multiplied by price, can have any number of maxima, and the problem of Nash non-existence is due to the possibility of shifts across these local maxima.

Nevertheless, some gains from going to two-dimensional models can be obtained even in the question of Nash equilibrium existence in pure strategies. First of all, it is useful to note that all two-dimensional problems can be represented as some equivalent one dimensional problem with the location of firms at the ends of the interval. It is well known that there is no discontinuity in this case, since price undercutting cannot gain a positive measure of the market. However, a Nash equilibrium in pure strategies can fail to exist even in this one-dimensional problem for an appropriate choice of densities. An example is given above. Note that density discontinuity is not the reason of failure: a similar example can be provided with a continuous density. The remarkable thing is that for any location of two firms in two-dimensional space the problem is mathematically equivalent to a one-dimensional problem with a non-uniform density, but with firms always located at the endpoints of the interval! This is very interesting, since in many cases such one-dimensional problem has a Nash equilibrium, and each corresponding density generates an infinite amount of corresponding two-dimensional problems, which also have Nash existence!

There exists also another property of the two-dimensional model. If the two-dimensional demand density is given by a continuous function, then the corresponding one-dimensional density would be also a continuous function. And if it is given by a bounded function, in a one-dimensional model the density will be bounded in all interior points of the interval $(-1, 1)$. This issue is discussed with more detail in Appendix 3.

3.4 Other Transport Cost Structures

3.4.1 Quadratic Transport Cost

Consider first a quadratic transport cost function $T(r) = tr^2$. The border between consumers buying from different firms will now be described by the following equation: $p_2 - p_1 = t[(x+c)^2 + y^2 - (x-c)^2 - y^2]$, or, $p_2 - p_1 = 4ctx$. Here, isolines are vertical. Demand also shifts continuously with any price change, and thus Lemma 3.2 and Theorems 3.1, 3.2, 3.3 hold as well. In fact, the generalization of Hotelling's model with quadratic transport cost, introduced in [3], from one to two dimensions is a trivial one, as it does not bring any new geometrical structure.

3.4.2 The General Case

Let us consider the role of different transport cost structures for demand continuity. In the one-dimensional Hotelling model, a linear transport cost leads to a demand discontinuity, while for quadratic cost the problem disappears [3]. Is linearity pathological in some sense, or does a quadratic structure have some specially good topological properties? Consider the general case of any strictly increasing cost $T(x)$ in distance: $T'(x) > 0; T(-x) = T(x)$. Here x has the meaning of a distance in any metric space.

Consider the one-dimensional model first. Let two firms be located at the points 0 and y. The equation which describes the set of indifferent consumers at given prices is:

$$\begin{aligned}\Delta T(x;y) &= p_2 - p_1; \\ \Delta T(x;y) &\equiv T(|x|) - T(|x-y|). \end{aligned} \quad (3.17)$$

A simple example (Fig. 3.3) shows, that this equation might have several solutions, and thus the demand area from one firm might be a set of disjoint intervals. This happens, for example, if $T(x)$ has a concave-convex form. Intuitively, if y is located in the area of concavity, $\Delta T(x;y)$ is decreasing for $y < x < y + a, a > 0$, and then starts to increase, when x enters the convex part of the graph $T(x)$.

Lemma 3.6 *If the transport cost function is twice differentiable, and if its graph as a function of distance is strictly convex, i.e.* $T'(x) > 0, T''(x) > 0$, *then the set of indifferent consumers in not wider than a point for any pricing policies and any location* $y, y \neq 0$. *When one of the prices moves in a direction, the separating border between consumers also moves in a direction.*

PROOF:
Consider the function $\Delta T(x)$ *(Fig. 3.4). For* $x \in [0, y]$ *it is always increasing, because both terms have positive derivative. For* $x > y$, $T'(x) > T'(x-y)$ *and* $T(x) > T(x-y)$. *Hence,* $\Delta T'(x) > 0$. *For* $x < 0$, *both arguments are negative,* $T(|x|) > T(|x-y|)$, *and* $T'(x) > T'(x-y)$, *being negative. Then, again,* $\Delta T'(x) > 0$. *So, for the whole set of* y *and* x, $\Delta T(x)$ *is an increasing function of its argument, and equation (3.17) has a unique solution for any set of prices on the real line. This solution depends monotonously on each price, and thus the separating border moves in one direction.*

The following theorem is a simple corollary of this lemma.

Theorem 3.5 *Total demand of a firm is a continuous function of its price for any different locations* ($y \neq 0$), *when transport cost is strictly convex in distance.*
In particular, demand is continuous for the original Hotelling model if we add any positive quadratic term to the linear transport cost (even very small). However, a pure strategy Nash equilibrium does not always exist.

PROOF:
First of all, a linear-quadratic transport cost guarantees demand continuity

in a one-dimensional Hotelling's model. This guarantees the existence of a Nash equilibrium at least in mixed strategies. To insure its existence in pure strategies it is necessary to check whether the maximum depends continuously on both strategies. As it was shown in [4], for small distances between firms, a Nash equilibrium in pure strategies may not exist.

It is easy to see that if we require concavity (instead of convexity), the solution of equation (3.17) is no longer unique. Depending on the price differential, we may have 2, 1 or 0 solutions (Fig. 3.5). Thus, when the price changes, these borders may move in different directions and collapse at some point. Nevertheless, demand can be shown to be continuous even in this case. What really matters is the number of solutions to the equation $\Delta T(x) = 0$. If it is finite for every x, demand is still continuous. It looses continuity (in the one-dimensional case) if this set of solutions has a positive measure, at least for some x. It is easy to see, for example, that if transport cost behaves linearly at least at some subinterval, this case may occur.

What happens in the two-dimensional case? Can total demand be discontinuous for some transport cost which is monotonously increasing? [8] To study this question, we should first investigate whether the set of solutions to the equation $T(r) - T(r, q) = a$ (here r, q are elements of a metric space, and a is a scalar parameter denoting a price differential) has the local structure of a one-dimensional manifold.

Theorem 3.6 *If the transport cost function is differentiable and strictly increasing in distance, ($T'(r) > 0$), where r is the Euclidean distance in the two-dimensional space, then for any point away from the line connecting the locations of two firms, there is only one curve, $\Delta T(r; q) = a$, passing through any solution of this equation. Thus, the set of solutions to this equation on any compact subset of the two-dimensional plane has measure zero.*
PROOF: See Appendix 1.

Consider now a square with a uniform distribution of consumers in a space with the Euclidean metric and any transport cost function, monotonously increasing in distance.

[8] If not, for sure this may happen. Consider, for example, a transport cost which is the same for some interval of distances. Then on a plane the solution to (3.17) is the intersection of two rings, which may have positive measure.

Theorem 3.7 *Given any strictly increasing transport cost function, and for all firms, located at different points, the total demand from consumers, uniformly distributed on a square, is a continuous function of the price of any firm.*
PROOF: See Appendix 1.

It is also easy to show that replacing the square by any compact set on a plane and the demand density by any bounded function, cannot lead to the demand discontinuity. Thus,

Theorem 3.8 *The total demand of any firm in a two-dimensional Hotelling's model with any bounded density of consumers on any compact set and any strictly increasing transport cost is still a continuous function of prices, if firms are located at any two different points.*
PROOF: See Appendix 1.

3.5 Conclusions

The main results of this paper can be summarized as follows.

1. The original Hotelling model is extended to any compact area in two-dimensional space. Demand continuity is proved for any bounded density of consumers.

2. The general case of transport cost functions strictly increasing in distance is considered. First, it is shown that in the strictly convex case the separating border between consumers has a unique point even on the line which connects firms. This reveals the reason for discontinuity in the original Hotelling's model and the ways to recover it. Secondly, it is shown that in a very general case the set of indifferent consumers has two-dimensional measure zero, and this insures the continuity of aggregate demand.

3. Finally, total demand from any firm is a continuous function of both prices in the whole range of their potential variation for a quite general setting: any bounded consumer density on any compact set on a plane, in the environment of a quite general class of strictly monotonous differentiable transport cost functions.

4. The Nash equilibrium, which can be also asymmetric, always exists at least in mixed strategies. Its existence in pure strategies depends crucially on the fact whether the maximum of the profit function of one firm with respect to its own strategy is a continuous function of its rival's strategy. In some particular cases there may be several local maxima, and the transition between them may occur in a discontinuous way. This also suggests that the observed time sequences of prices by rival firms in real space may be of two types:
a) fixed on equal or different level (the case of symmetric or asymmetric Nash),
b) appearing as a random sequence, close to chaotic, which nevertheless may represent Nash equilibrium in mixed strategies.

5. All the results can also be translated into the language of product differentiation models, where the role of transport cost is played by the shape of individual utility functions, defined on the continuous space of varieties. For example, the case of strictly increasing transport cost corresponds to the case of utility which strictly decreases with the distance from the optimal point in variety space (but neither concave or convex). The "islands" and nonuniform densities of consumers are much less intuitive in product differentiation spaces, but they also can be introduced, and all the results of this paper are valid for those cases.

3.6 Appendix 1

PROOF OF LEMMA 3.2.
For an infinitely small price differential the separating hyperbola is close to the vertical axis, and thus intersects the upper and lower borders of the square $y = \pm L$. Let us consider separately two cases: a) when the hyperbola intersects $y = \pm L$, and b) when it intersects $x = L$ (see Fig. 3.1). For equal prices the separating hyperbola transforms into a vertical line, and for a small price differential it is useful to express the equation of the hyperbola as $x(y) = \pm a\sqrt{1 + y^2/b^2}$. If the price of the left firm is lower, the hyperbola lies inside the right hemi-square. Let S_a denote the area between the vertical line $x = 0$ and this hyperbola, which represents the additional demand attracted

by the left firm by pricing below equal prices. We have:

$$S_a = 2\int_0^L x(y)dy = \frac{2a}{b}\int_0^L \sqrt{y^2+b^2}dy = \frac{a}{b}(L\sqrt{L^2+b^2} + b^2\ln(\frac{L+\sqrt{L^2+b^2}}{b})) \tag{3.18}$$

$$b = \sqrt{c^2 - \frac{(p_1-p_2)^2}{4t^2}} \tag{3.19}$$

Note that b is a continuous function of both prices on the whole area where it is defined. S_a is a continuous function of b, except for $b = 0$. The case of b close to zero will be considered separately because it corresponds to the case where the hyperbola is almost horizontal and in this case it cannot intersect $y = \pm L$. Thus, we have proved that in the case a) total demand is a continuous function of both prices.

Consider now the case b) in which this hyperbola intersects $x = L$. Total demand of the right firm in this case is given by the area inside the hyperbola:

$$S_b = \frac{2b}{a}\int_a^L \sqrt{x^2-a^2}dx = \frac{b}{a}(L\sqrt{L^2-a^2} - a^2\ln(\frac{L+\sqrt{L^2-a^2}}{a})), \tag{3.20}$$

where $a = \frac{|p_1-p_2|}{2t}$. Here it is important to check continuity for $a \to c$. In this case, transport cost between 2 firms is exactly equal to the price difference. This was precisely the reason of total demand discontinuity in the one-dimensional Hotelling model. In a two-dimensional space, on contrary, there is no discontinuity at this point, as can be easily shown. If the square covers firms' locations, i.e. $L > c$, then also $L > a$, and the square root is defined. When $|p_1 - p_2| \to 2ct$, prices cannot be close to equal, and thus $a \neq 0$. This eliminates the potential discontinuity due to the presence of a zero in denominator of the expression S_b. Finally, the whole expression is proportional to b, which vanishes to zero in this case, and thus the demand from the right firm goes to zero continuously. This eliminates the price undercutting effect when one firm captures all the customers behind the second firm just by a marginal decrease in price.

PROOF OF THEOREM 3.1.
Consider any compact area D in R^2 with a homogeneous demand distribution

(See Fig. 3.2). In a metric space, a compact set is always bounded. Thus there exists an L high enough to cover this compact set by a square. The border between the consumers buying from the left or right firm will lie on the intersection of the hyperbola (see the previous lemmas) with this compact set. This hyperbola moves continuously with changes in any price. The additional demand due to a change in prices is given by the measure of the intersection of the area between the two hyperbolas corresponding to the two pricesand the compact set. additional demand is the . Denote by $1_D(x, y)$ the support of the set D, that is,the function which takes the value 1 on the set, and zero otherwise. When the price moves from p to p', the demand increase is given by the formula:

$$S(p') - S(p) = 2\int_0^L \int_{x(y)}^{x'(y)} 1_D(x, y)dxdy < 2\int_0^L (x'(y) - x(y))dy < \epsilon, \quad (3.21)$$

where ϵ can be any small number (the same, which is used in Lemma 2, when $|p - p'| < \delta$). The first inequality follows from the fact that the support of D is smaller than the support function of the square. The second is the continuity of total demand from one firm, which was proved in Lemma 2. Hence,in a two-dimensional Hotelling competition between firms at fixed locations, total demand of a firm is a continuous function of both prices.

PROOF OF THEOREM 3.2.
A continuous function on a compact set takes its maximal value at some point. Thus, there exists some maximal density of consumers (atomic points with positive mass in a point are thus excluded). Consider a corresponding problem, where the density function is replaced by its maximal value on its support. By Theorem 3.1, the resulting demand for such a problem is a continuous function. Note now that the integrals which enter the expression for additional demand, arising from marginal shift in prices for the original density of consumers, can be majorated with the integrals with this uniform distribution, and continuity can be proved in a way, similar to Theorem 3.1.

PROOF OF THEOREM 3.3.
In the proof of Theorem 3.2, continuity was used only to prove the existence of a maximum of this density function on a compact set. When continuity is replaced by boundedness, the corresponding integrals can be majorated in a similar way.

PROOF OF THEOREM 3.6.
Consider the neighbourhood of any solution to the equation (3.17) outside the line, which connects two firms (Fig. 3.6). Let $T(r) = c_1$, and $T(r, q) = c_2$ in this point. Each of these equations defines some circle with the center in the location of the corresponding firm. If transport costs are strictly increasing in distance, then any circle with greater radius will correspond to a higher cost of transportation. Let us increase both parameters $c_{1,2}$ by a small $\delta > 0$. This will not change equation (3.17). If circles have different tangency lines in their point of intersection (and this always happens, if we stay away from the straight line, connecting the locations of firms), then for each parameter δ there exists a unique solution to the equation (3.17) in the neighbourhood of the initial point. It depends continuously on δ (Fig. 3.7). Thus, we have a unique curve, passing through the point r and describing the solutions to (3.17) in the neighbourhood of r (Fig. 3.9). Except for the very pathological case, when the set of these solutions is not measurable (in Lebesgue sense), its two-dimensional measure is zero, if we take their intersetion with any compact area in R^2. As the measure of an interval is zero, the local behaviour on a straight line which passes through firms' locations does not matter.

PROOF OF THEOREM 3.7.
Let us change one of the prices marginally. Consider again any point r, as in the previous theorem. We can easily find two solutions of the new equation in the neighbourhood of r (one on the first circle, when we move the second, and one on the second). Both points move continuously with price change. As transport costs are differentiable, for very small shift, the solutions would lie close to the line, connecting these points. Thus, the curve of solutions shifts continuously in the neighbourhood of each solution, away from the line described above. Except for very pathological cases, the total length of this curve inside any compact set on R^2 is finite. Hence, the total additional demand for one firm arises from infinitely small area, if the price shift is infinitely small. Thus, the total demand from any firm is a continuous function of its price.

PROOF OF THEOREM 3.8.
When density function is bounded, we can substitute this upper bound M instead of density function and find a new upper bound for a demand shift, which can be done infinitely small for infinitely small price shift. With the

demand from a compact set we can also do the procedure, similar to the proof of theorem 3.1.

3.7 Appendix 2

SYMMETRIC NASH EQUILIBRIUM FOR THE UNIFORM SQUARE.
Let $p \equiv p_1 - p_2$, and $T(p)$ be the demand from the first firm, calculated in the neighbourhood of the symmetric equilibrium, i.e. near $p = 0$. Proceeding similarly to [8] (his notations are also kept), it is possible to show that in this case $p^*(c) = -T(0)/T'(0)$. Direct calculations show the validity of the formula in the main text. Further on, it is useful to consider normalizations: $L = 1; p = 1$. Then

$$p^*(c) = \frac{2}{\frac{\sqrt{1+c^2}}{2c} + \frac{c}{2}\ln(\frac{1+\sqrt{1+c^2}}{c})}. \tag{3.22}$$

Consider the function $q(c) \equiv 4/p^*(c)$. It is easy to show that $q'(c) = ln(\frac{1+\sqrt{1+c^2}}{c}) - \frac{\sqrt{1+c^2}}{c}$. For all $c \in [0, \infty]$, it is possible to show that $q'(c) < 0$. This statement is not so obvious, and needs several estimations (upper inequalities). First of all, for $c > 1$,

$$q'(c) = \ln(1 + \frac{\sqrt{1+c^2}}{c} - \frac{\sqrt{1+c^2}}{c}) < \frac{1-c}{c} < 0. \tag{3.23}$$

For $c \in [0, 1]$ consider $f(c) \equiv \sqrt{1+c^2}\frac{c-1}{c} - \ln c$. Note that

$$q'(c) = f(c) + \ln(1 + \sqrt{1+c^2}) - \sqrt{1+c^2} < f(c) \tag{3.24}$$

for $c \in [0, 1]$ (inequality $\ln(1 + x) < x, \forall x > -1$ was used). It is enough to show that $f(c) < 0, c \in (0, 1)$. Note that $f(1) = 0$. If $f'(c) > 0, \forall c \in (0, 1)$, then $f(c) < 0$ inside this interval. Direct calculations show that

$$f'(c) = \frac{1 + c^3 - c\sqrt{1+c^2}}{c^2 - \sqrt{1+c^2}}. \tag{3.25}$$

Note that $1 + c^3 > c^{3/2}, c \neq 1$, and $c\sqrt{1+c^2} < 2c^{3/2}$ for $2 - \sqrt{3} < c < 2 + \sqrt{3}$ (the latter can be shown by the solution of the corresponding quadratic inequality $c^2 - 4c + 1 < 0$). For $0 < c < 2 - \sqrt{3}$ it is possible to show directly that $c\sqrt{1+c^2} < 1 < 1 + c^3$. Thus, $f'(c) > 0, \forall c \in [0,1]$, hence $f(c) < 0$ and $q'(c) < 0$ for $0 < c < 1$. Then $q(c)$ reaches its minimum at the out-border of possible locations, and the equilibrium Nash price reaches its maximum there.

3.8 Appendix 3

3.8.1 Elliptic Coordinates

The most natural way to approach the issue of competition of two firms with linear transport cost for consumers, distributed over some area in two-dimensional space, is to introduce elliptic coordinates (Fig. 3.12). These coordinates are locally orthogonal and they are used for the solutions of some problems in mathematical physics, when the geometry includes ellipses or hyperbolas. In this problem, the hyperbola is the separating curve between two sets of consumers, buying from this or that firm. This one-parametic family of hyperbolas moves as the price moves. If the set of consumers is an ellipse, then all integrals can be easily calculated, but the solution can be written at least in the form of integral for any density of consumers over any two-dimensional set.

Consider the family of hyperbolas and ellipses, given by the following equations (here $\tau \in [-1, 1]$ parametrizes the hyperbola, while $\sigma \in [1, \infty]$ is a parameter for the family of ellipses):

$$\frac{x^2}{\tau^2} - \frac{y^2}{1-\tau^2} = 1, \tag{3.26}$$

$$\frac{x^2}{\sigma^2} + \frac{y^2}{\sigma^2 - 1} = 1. \tag{3.27}$$

It is possible to express the Cartesian coordinates x, y via these elliptic τ, σ: $x = \pm\sigma\tau; y = \pm\sqrt{(1-\tau 2)(\sigma^2 - 1)}$. The Jacobian of this transformation is $J = (\sigma^2 - \tau^2)/\sqrt{(\sigma^2-1)(1-\tau^2)}$. If the firms are located in the focal points ($x = \pm 1, y = 0$) (which correspond to ($\tau = \pm 1, \sigma = 0$) in elliptic coordinates), then the set of indifferent consumers is located at the hyperbola $\tau = p_2 - p_1$ (here transport cost is normalized to 1/2, and the first firm is located to the left of the origin). If $P \equiv p_2 - p_1$, $\rho(\tau, \sigma)$ is the density of consumers in new coordinates and $\sigma = \sigma_\pm(\tau)$ are the equations of the upper and lower border of consumers (Fig. 3.13), then the total demand from the first firm is given by the formula:

$$D_1(P) = \int_{-1}^{P} d\tau \int_{\sigma_-(\tau)}^{\sigma_+(\tau)} d\sigma\rho(\tau,\sigma)|J(\tau,\sigma)|. \qquad (3.28)$$

Although this integral can be calculated analytically for many cases, the simplest case of uniform density $\rho = 1$ and elliptic borders $1 < \sigma < R$ will be considered here (Fig. 3.13). Then the profit of a firm is given by the following function:

$$\begin{aligned} \Pi(p_1,p_2) = \pi p_1 R\sqrt{R^2-1}; 0 < p_1 < p_2 - 1; \\ \Pi = R\sqrt{R^2-1}(p_1 Arcsin(p_2-p_1) + 0.5\pi p_1) + \\ p_1(p_2-p_1)\sqrt{1-(p_2-p_1)^2}\ln(R+\sqrt{R^2-1}); |p_1 - p_2| < 1; \\ \Pi = 0; p_1 > p_2 + 1. \end{aligned} \qquad (3.29)$$

The behaviour of this function is not quite obvious, and computations are necessary for some conclusions about its shape. Some results are discussed in the next subsection. But before it is useful to discuss some analytical properties of the demand densities.

Lemma 3.7 *If the two-dimensional density $\rho(x,y)$ is a continuous function of both variables, then the corresponding one-dimensional density is a continuous function of its variable.*

PROOF:
Since $\rho(\tau) = \int d\sigma\rho(x(\sigma,\tau), y(\sigma,\tau))J(\sigma,\tau)$, the only problem which can emerge is the singularity of Jacobian in the point $\sigma = 1$. Note that this point corresponds to the degenerated ellipse $(-1 < x < 1; y = 0)$. But it is easy to show that this singularity is integrable, i.e. the integral over a small neighbourhood of $\sigma = 1$ is infinitely small for any bounded two-dimensional density. (Note that continuity on a compact set insures boundedness.)

Formally, the following chain of inequalities proves the continuity:

$$|\rho_1(\tau+\Delta) - \rho_1(\tau)| = |\int_1^{\sigma_0} \frac{\rho(\tau,\sigma) - \rho(\tau)}{\sqrt{\sigma^2-1}} d\sigma| \leq$$

$$\max|\rho(\tau+\Delta,\sigma) - \rho(\tau,\sigma)|| \int_1^{\sigma_0} \frac{d\sigma}{\sqrt{\sigma^2-1}}| = C_\Delta I. \qquad (3.30)$$

(3.31)

The constant $C_\Delta \to 0$ for $\Delta \to 0$ because of continuity of the two-dimensional density, while the integral I is bounded, because in the neighbourhood of $\sigma = 1$ it can be majorated by $\sqrt{2(\sigma - 1)}$, which goes to zero as $\sigma \to 1$.

3.8.2 The Results of Computations

This subsection provides some results of numerical computations of profit functions as the functions of both prices for different elliptic areas (see the previous subsection of the Appendix). Some values of R were fixed, and the profit function was calculated on a grid of prices with the step 0.1. For any given p_2 it is possible to draw a graph of profits as the function of own pricing policy. For small R, equal to 1.5, the maximum M was unique, at least for $p_2 < 1.5$, and the symmetric Nash equilibrium was unique: it occurs at the prices $p_1 = p_2 = 1.0$ (approximately). However, for ($p_2 = 2.5$ there were two maxima (one corresponding to cornering market). With the further rival price increase, only cornering the market was a local (and global) maximum for the first firm (it should just set the price below the rival's).
For high values of R it was possible to observe the dynamics of these maxima in more details. For $R = 15$ and $p_2 = 2.0$ there were two maxima: $max1 = 705$ (corner at $p_1 = 1$) and $max2 = 717$ (at $p_2 = 1.7$). For $p_2 = 2.2$ the situation has reversed: the corner maximum ($max1 = 846$, obtained for $p_2 = 1.1$) became higher than the non-corner maximum ($max2 = 803$, for $p_2 = 1.7$) (Fig. 3.12). As the profit function is continuous for both prices, it is possible to infer that for some intermediate price of the rival two maxima are equal, and the shift between them occurs discontinuously.

3.8.3 Sufficient Conditions for Nash Existence

This subsection contains several lemmas about sufficient conditions for the existence of Nash equilibrium in pure strategies for one-dimensional density with firms located at the ends of an interval $[-1, 1]$. Since every two-dimensional model is equivalent to this one-dimensional specification, the results can be applied for that model also. These sufficient conditions exploit the property of profit concavity and thus describes not all the cases of existence of Nash equilibrium.

Lemma 3.8 *Let the first firm be located in the point $x = -1$ and the second* *in $x = 1$. Let transport costs be t and let the consumers be distributed with a continuous density $\rho(x)$, satisfying the property of symmetry $\rho(x) = \rho(-x)$ with the total mass of consumers normalized to one. Then if a symmetric Nash equilibrium exists, it is given by the formula $p_1^* = p_2^* = t/\rho(0)$.*

PROOF:
The profits of firms are given by the formulae

$$\Pi_{1,2} = p_{1,2}(\frac{1}{2} \pm \int_0^{x_0} \rho(x)dx), x_0 \equiv \frac{p_2 - p_1}{2t}, \tag{3.32}$$

and thus the first-order conditions for Nash equilibrium are:

$$\begin{aligned} \int_{-1}^{x_0} \rho(x)dx - \frac{p_1}{2t}\rho(x_0) = 0, \\ \int_{x_0}^{1} \rho(x)dx - \frac{p_2}{2t}\rho(x_0) = 0. \end{aligned} \tag{3.33}$$

Since density is symmetric, the candidate for a Nash equlibrium should correspond to $x_0 = 0$, with equal prices. Substitution $p_1 = p_2$ gives the value of the price $p^* = t/\rho(0)$.

Lemma 3.9 *Symmetric Nash equilibrium exists, if $\rho'(0) < 4\rho^2(0)$.*

PROOF:
The second derivative of the first firm's profit function with respect to its own price in the point $p_1 = p_2 = p^*$ is given by the expression

$$\Pi'' = \frac{\rho'(0)}{4t\rho(0)} - \frac{\rho(0)}{t}, \tag{3.34}$$

which is negative if and only if $\rho'(0) < 4\rho^2(0)$.

However, this condition guarantees only the existence of a local maximum in the point of a symmetric Nash equilibrium. The condition of concavity of the profit function is sufficient to guarantee its globality. The following lemma states this condition.

Lemma 3.10 *If $\rho(x)$ is such that*

$$|\frac{d}{dx}(\ln \rho(x))| < \frac{2}{1+\frac{1}{2\rho(0)}}, \qquad \forall x \in (-1,1), \tag{3.35}$$

then the profit function is concave, and symmetric Nash equilibrium represents a global maximium.

PROOF:
Since the second derivative of the first firm's profit function with respect to its price is given by the formula

$$\Pi'' = \frac{p_1}{4t^2}\rho'(x_0) - \frac{\rho(x_0)}{t},$$

it takes negative values iff $p_1\rho'(x_0) < 4t\rho(x_0)$. Thus, the condition

$$\frac{d}{dx_0}(\ln \rho(x_0)) \leq \frac{4t}{p_1}$$

would guarantee concavity of the profit function. If the rival firm sets its price at p_2, the set of possible reactions to it is $p_1 \in (0, p_2 + 2t)$ (it is because for other prices profits cannot be positive). Since p_1 is in the denominator of the upper bound for the logarithmic derivative of the density, the most restrictive case occurs when this price takes its maximal value. If the second firm plays the symmetric Nash strategy $p^* = t/\rho(0)$, then $1/p_1 \geq 1/(p^* + 2t)$ for all possible p_1, and thus $2/(1+1/2\rho(0))$ is an upper bound for possible variation of the logarithmic derivative of the density function over the whole interval which still guarantees the concavity of the profit function. The absolute value should be taken, because a similar procedure for the second firm gives the negative lower bound for the derivative of density.

This lemma gives the following intuition about the concavity of a profit function. High values of the derivative of the density function may destroy the concavity of the firms profit functions. And the same effect may arise from very low densities in the middle: when $\rho(0)$ is small, the range of acceptable variations of density derivatives also shrinks.

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Chapter 4

Dacha Pricing

The problem of location rent in a city-neighbourhood is considered on the basis of the utility surplus due to transportation price and transportation time simultaneously. For the case of identical agents the problem is solved explicitly and comparative statics is studied. For the case of heterogeneous agents with respect to their income an implicit analytical solution is obtained. The model explains the evidence about dacha pricing in Russia and its dynamics during the transition period.

JEL classification: D58; R31.

KEYWORDS: location, rent, general equilibrium, dacha.

4.1 Introduction

People living in big cities often demand a small country-side house with a piece of land ("dacha"), where they can enjoy nature. Dachas are used in Russia both for family recreation and small agricultural activity. In the summer period some family members have to commute to the city where they normally have a job. They spend both time and part of their income for this commuting. The relative importance of these factors for utility loss in Russia was quite different in pre-transition and post-transition period. To capture this effect, a special form of utility function will be introduced in this model. In the case of big Russian centers, like Moscow and St.Petersburg, the price of land in the neighbourhood was relatively low (because of low agricultural productivity and state control). Hence, a lot of people could get these pieces of land and construct a house there. It is necessary to mention several features of contemporary Russian economy, which make this model especially useful to explain the available evidence. First, a quick transition from the planned to a market economy exposed some planned structure of dacha locations to a persistent shock in transport costs, caused by the significant increase in fuel prices. Comparative statics of the general equilibrium model provided below, gives a lot of insights about this issue. Second, the relative spatial homogeneity of locations around big northern cities in Russia (Moscow and St.Petersburg) allows to neglect location-specific utility in the basic model.

The main objective of this article is to derive a highly stylized model which can explain real effects. The general equilibrium for this model will be derived: prices of dachas will become a function of the distance to the city and also will reflect equilibrium between demand and supply, given an exogeneous wage distribution in the city and some density of land spots for potential dachas.

The question of residential rent as a function of space is of high practical importance and thus was studied by many authors (see, for example, [1-8]). Alonso [6] considers the influence of distance on leisure, commuting time and commuting costs. He assumes that the utility of households depends on the distance to the center, the amount of land and on other consumption goods (numeraire), and solves the utility maximization problem. This model usually forms the basis of other urban theory models. For example, Beckmann

[9] and Henderson [10] have reconsidered the distance in terms of leisure loss, but have omitted monetary costs of transportation. Further attempts to derive the utility function and to apply it for obtaining rent curves can be found in [7]. Beckmann [8] studies the role of residential density on urban rent. In his other paper [17] he introduces the idea of mapping of agents from income to location intervals and finds an analytical solution to the problem of endogeneous city size, assuming that richer agents would locate further. Fujita [14] constructs significant extensions of CBD theory, including comparative statics with the change in transport costs and the equilibrium location of different types of household, but does not consider the cumulative effect of transportation on both disponsible income and leisure losses. This paper attempts to obtain a closed analytical-form general-equilibrium solution for a particular utility function which takes into account two basic properties - loss in leisure and increase in transport expenditures, imposed by the chosen location.

It is necessary to mention the paper of Brown [15], devoted to the comparative statics of Alonso's model. She considers a utility function, which includes not only composite good and time, but also housing space, and derives some analytical results for the demand change emerging from the change in income, transportation cost and transportation speed. Since the assumptions of this paper are very similar to [15], it is necessary to stress the differences. The present paper considers not only demand change, but also the general equilibrium effect, emerging from the simultaneous behaviour of all heterogeneous agents. General equilibrium effects include also the change in prices as the result of the change of the total extent of the market when transport speed or price varies. The solution of the present model also allows to see the effect of a change in wealth distribution on equilibrium prices. Although the effect of variation in housing space is neglected in the present model, due to the complexity of the analysis in a general equilibrium framework, it can be justified by the considered application for the pricing of dachas in Russia (where there is little variation in size of the major part of private land). Also, the analysis in [15] relies strongly on the assumption that all cross-derivatives of the utility function are equal to zero, which is not assumed in the present model.

As it is shown by Berliant and ten Raa [13], there are theoretical problems with the correspondence between continuous models of locational choice with its discrete analogs. In the model with a continuum of heterogeneous agents, choosing across heterogeneous locations, every market is formally a point, having measure zero. Thus, even when a solution is given by a continuous mapping from the space of agents' heterogeneity to the space of locations, the measure of agents in a small neighbourhood of a point may not be preserved under the mapping. On the other hand, in discrete models it is preserved, because each agent has a non-zero measure of land. The present model solves this paradox by requiring a mass-preserving mapping as a part of the general equilibrium solution concept.

Besides its purely theoretical interest, the model can explain interesting phenomena about Russian transition, when the relative price of transportation increased and wealth distribution became much more non-equal (the inequality coefficient, measured as a ratio of the incomes of top 10th to bottom 10th percentile, has increased from a typical European level of 3-4 in 80s up to well-above-American level of 10-15 in 90s [16]). This paper addresses the impact of these two effects and thus neglects some other geographical effects. I abstract from such effects like pollution and access to nature; all points out of the city are assumed to bring the same recreative utility. It also does not take into account the topology of road infrastructure [1]. The main goal of the model is to see how the relative rental structure reacts to changes in transportation time, transportation costs and income distribution.

The change in transportation costs had a negative impact for dachas: the total extent of the market[2] collapsed to less distant areas. To isolate this effect, a simple model with one type of agents is discussed in the section 2. The main observation is that while before transition dachas were available for practically all consumers, after the number of dachas feasible for consumers (in mathematical sense it can be treated as supply) has endogeneously declined, and the market had to regulate that shortage. The

[1]The model can be also applied to a complicated road topology; distance from the city can be calculated along road, with further integration inside equidistant isolines (Fig. 4.1)

[2]A new concept of the *extent of the market* is used in this paper to compare different number of heterogeneous commodities which are owned in the market equilibrium. The term *market size* is different, it usually applies to the number of consumers at the market for a homogeneous good. Here there is an analogy between these two concepts due to the fact that shrinking of the *extent of the market* goes together with the decline of the number of consumers who participate in this market

increase of a relative richness of the rich in Russia has created high demand for the best locations, but also has cut the total demand (as more people are poor now). To study the effect of an impact of income distribution change a general equilibrium model is developed in section 3, and some examples are studied in section 4. The analytical solution can be obtained for any income distribution. The goal of section 5 is to introduce some empirical evidence, which allows to calibrate the model (section 6). Finally, the conclusions are formulated in section 7.

4.2 A Simple Model: Identical Agents

The goal of this section is to explain on a simple model the effect of endogeneous shrinking of the market for dachas as the consequence of increase of transport costs in Russia during the transition. The utility includes both consumption and leisure, since in different periods different effects were determining the market size for dachas in Russia. The model considered here is similar to that considered in Wheaton [11], in the sense that all points in space give the same utility to all agents. Later, with consideration of heterogeneous agents, this will no longer be valid. The structure of the simple and general model differs only in the assumptions about wealth distribution.

4.2.1 Assumptions about Preferences. Equilibrium Rental Price

Consider the following model. There is a two-dimensional radially-symmetric land. The city occupies the territory between 0 and R_0. Assume that all agents are identical: they have the same income in the city, w, and their preferences are the same. The preferences are Cobb-Douglas in consumption and leisure with equal role of both factors: $\alpha = 1/2$. The exogeneous income w can be spent on a composite good C, a rental price of a dacha $P(R)$ and a transportation cost to get to dacha:

$$U = \sqrt{Cl}; \qquad C \equiv w - P - 2bR; \qquad l \equiv l_0 - \frac{2R}{V}. \tag{4.1}$$

Here R is the distance to the city, V is the speed of transportation (in Km/h), b is the price of 1 km of transportation and l_0 is the basic endowment of daily leisure. By assumption, all agents can enjoy their leisure only when it takes place on a dacha (like when the head of a household has to work in the city in the summer period and has to commute to a dacha where his family spends vacations). All dachas have equal size which is fixed exogeneously. This was a realistic assumption for Russia, the source of data for model calibration [3]. Including the possibility to choose dacha size leads to a multiplicity of equilibria (see, for example, Beckmann [17]), while adds little to the economics of this model: richer people in Russia always choose dachas closer to the city (in contrast to a typical pattern of location in western metropolitan areas).

The equilibrium renting price $P(R)$ is determined in such a way that agents are indifferent (for given w, b, V, l_0) across locations. It can be obtained from the indifference curve in the distance-rent space $U_0^2 = C(R)l(R)$:

$$P(R) = w - 2bR - \frac{U_0^2}{l_0 - \frac{2R}{V}}. \tag{4.2}$$

The obtained indirect utility U_0 may be zero or positive in equilibrium. The rental price can take the values only above some positive constant A, which corresponds to the daily rent covering just construction cost.

4.2.2 Geometry of the Model. Different Types of Equilibria

Assume now that the total number of citizens is N, and the number of feasible sites for dachas in the interval of distances $[R, R + dR]$ is given by a function $\nu(R)dR$. It can be obtained from real geographical data (or map) in a following way. If there are several roads, going out from a city to countryside, one may divide them into small intervals (let say, 1 km long areas) and then count all lots which are actually used (or can be used) as dachas (country-houses). The distribution of quantities in each 1-km interval as the function of distance in the limit when intervals are going to zero determines the function $\nu(R)$ (Fig. 4.1). In a model with radially symmetric homogeneous plane with a dense radial network of roads the density of lots is constant in two-dimensional space, but is proportional to the distance: $\nu(R) = \nu_0 R$ (Fig. 4.2).

[3] Similar assumption can also be justified for Germany, with the flower gardens around small houses

The total demand for dachas is given by the quantity of agents N, while the total supply N_s is

$$N_s = \int_{R_0}^{R^*} \nu(R)dR, \tag{4.3}$$

where $R^* = \min\{\frac{w-A}{2b}; l_0V/2\}$. This endogeneous boundary R^* deserves a special consideration. The budget constraint determines endogeneously the "maximal budget distance" $R_w \equiv (w-A)/2b$, and the leisure constraint defines the "maximal leisure distance" $R_l \equiv l_0V/2$ (Fig. 4.3). The budget constraint becomes binding at the distance R_w, where it is just sufficient to cover house construction and transport cost. The time constraint becomes binding when the distance to dacha is so high (or transport speed is so low) that all leisure time is spent on transportation. In both cases an agent has zero utility, which is the minimum in this model. Land is a free good at distances higher than R^*, since nobody can afford the project of its use as a dacha.

If the supply N_s is higher than demand N, all agents get positive utility in equilibrium. If it is less, rationing takes place, and this is only consistent with $U_0 = 0$. If $R_w < R_l$, then

$$P(R) = w - 2bR; \qquad R \le R_w = (w-A)/2b. \tag{4.4}$$

If $R_l < R_w$, the "leisure constraint" is binding, and all agents again get zero utility in equilibrium. For low transportation costs and high speeds it is possible to show that all agents can get dachas and thus a positive utility.

Consider the case when all of agents can have dachas at distances lower than critical: $R_{max} \equiv \bar{R} < R^*$. Then $N = 0.5\nu_0(\bar{R}^2 - R_0^2)$. The fact that land is a free good at higher distances suggests that at the border only minimal construction costs A are covered: $P(\bar{R}) = A$. Then, equalizing the utilities in each of locations gives the following equation: $U_0^2 = (w-A-2b\bar{R})(l_0-2\bar{R}/V)$. For all $R, R_0 < R < \bar{R} < R^*$, the daily rent price for dacha is given by the formula:

$$P(R) = w - 2bR - (w - A - 2b\bar{R})\frac{l_0V - 2\bar{R}}{l_0V - 2R}. \tag{4.5}$$

Consider the comparative statics of formula (4.5):

$$\frac{\partial P(R)}{\partial w} = 1 - \frac{l_0V - 2\bar{R}}{l_0V - 2R} > 0, \tag{4.6}$$

$$\frac{\partial P(R)}{\partial V} = -\frac{2l_0(\bar{R}-R)}{(l_0V-2R)^2}(w-A-2b\bar{R}) < 0, \tag{4.7}$$

$$\frac{\partial P(R)}{\partial b} = -2R + 2\bar{R}\frac{l_0V-2\bar{R}}{l_0V-2R}, \tag{4.8}$$

$$\frac{dP(R)}{dR} = -2b - \frac{2(w-A-2b\bar{R})(l_0V-2\bar{R})}{(l_0V-2R)^2}. \tag{4.9}$$

Wealth increase will increase dacha prices at each location. The increase of transportation speed will decrease the prices of dachas at all locations. The third of partial derivatives may have different signs. For $R < R_2 \equiv l_0V/2-\bar{R}$, $dP/db > 0$, and for $R_2 < R < \bar{R}$, the derivative is negative. It means that the increase of transportation costs will increase the prices for dachas in the neighbourhood of the city, but decrease them in more distant areas. The increase of transportation costs also decreases R^*, and thus, after some point, the total number of affordable dacha sites might become less than N. The further increase of b will decrease prices, because for zero utility from (4.4) P depends negatively on b.

Proposition 4.1 *The equilibrium number of dachas increases with the increase of wealth and speed of transportation, and this increases public welfare, giving everybody higher utility. The increase of transport prices decreases the equilibrium number of dachas, and thus the public welfare.*

4.2.3 The Effect of Variable Frequency of Travelling

An interesting and reasonable extension of the simple model can be to include the possibility of choosing the frequency of travelling to dacha. Having in mind, that agents (heads of household) have to commute to the dacha (where their families live in the summer period) not more often than daily and assuming that they get zero utility when they are unable to do it, consider the following utility function [4]:

$$U_f = \Omega(w - P(R) - 2bR\Omega)^\alpha \left(l_0 - \frac{2R}{V}\right)^{1-\alpha}; 0 < \alpha < 1; 0 < \Omega < 1. \tag{4.10}$$

In a general equilibrium framework, agents are taking prices $P(R)$ as given, and choose an optimal Ω. The first order condition gives the following ex-

[4] For $\Omega = 1$ and $\alpha = 1/2$ it coincides with one considered above

pression for it:

$$\Omega^* = \frac{w - P(R)}{\alpha + 2bR}, \tag{4.11}$$

when $w - P(R) - 2bR < \alpha$. Since $\Omega \leq 1$, we can have a corner solution $\Omega^* = 1$, if $w - P(R) - 2bR > \alpha$. Because $P(R) \geq A$ (A - construction cost of dacha), there always exists so high R_1, that for $R > R_1$, the solution for Ω will be interior. The interior circle around the city will correspond to the corner solution $\Omega^* = 1$, surrounded by the zone with a variable frequency of communicating with dacha. The iso-utility curve can also be constructed. Thus, we have:

Proposition 4.2 *If the frequency of travelling to dacha is considered as a choice variable, the equilibrium includes two zones. In the closest zone agents travel to dacha with maximal possible frequency (every day), while in the outer zone they travel less often. The border of the second zone can be determined, either by total demand (then all agents get positive utility), or by the income or leisure borders.*

4.3 General Equilibrium with Heterogeneous Agents

Now let agents be heterogeneous with respect to their income. The income distribution is characterized by a density $f(w)$, so that

$$\int_0^\infty f(w)dw = N. \tag{4.12}$$

For any finite number of agents N the set of wealths $w_i; i = 1, 2, ...N$ is also finite. It is possible to construct a hystogramm and then to approximate it by a continuous density function $f(w)$ by a spline method. The continuity of the density is required in order to work with differential equations, which will be derived in the process of solving the problem. The wealth distribution function $f(w)$ is used in this model as a primitive. Let us define what is the mathematical concept of an equilibrium.

Definition 4.1 *A general equilibrium is a mapping of agents of type s with different endowments $w(s)$ and preferences $U(s)$ into the locations $R(s)$ and*

a set of prices $P(R)$, such that the following requirements are satisfied:
1) each agent of type s maximizes his/her utility taking the price function $P(R)$ as given;
2) the supply is equal to demand in any small neighbourhood of each type (this means that the measure of any small neighbourhood of each agent is preserved under this transformation).

Note that the equality between supply and demand is important not only on aggregate and not only in each point, but also in each neighbourhood. The intuition is explained in Fig. 4.4. Because each type has measure zero, a one-to-one mapping (here from the income distribution into spatial distribution) does not necessarily preserve measure. It is well-known result in mathematics that one-to-one continuous correspondence is possible across the sets of different measure. When the functions in each point are equalized, this does not imply equalization of integrals. It may happen that the problem described by Berliant and ten Raa [13] emerges for this reason. That is why we require that in equilibrium the distribution densities are such that the total mass of agents in any neighbourhood of a particular type is equal to the mass of agents in the image of this neighbourhood.

Since here preferences do not depend explicitly on location, but enter through transportation losses only, for equal prices in space greater distance implies less utility. Agents with less wealth are willing to give up more leisure for an additional unit of consumption, and thus any given price schedule $P(R), P'(R) < 0$, which will make some agent with intermediate wealth indifferent across locations will imply their choice at higher distance. Note that $P'(R) > 0$ is impossible since everybody would choose the closest location. It is possible to construct the price schedule $P_s(R; s)$, which makes each agent of type s indifferent across locations. His choice in space will be unique if the equilibrium price schedule lies above his indifference schedule, and is tangent to it only at one point (Fig. 4.5).

Since in this model it is possible to find a one-to-one correspondence between income $w(s)$ and location $R(s)$ for each type s, it is possible to exclude s from consideration, and study the mappings $w \to R$. Hence, equilibrium is given by a pair of functions $(P(R), w(R))$, such that every agent maximizes his utility with respect to the choice of dacha locations at given prices and the demand and supply clears the market for each location. The wage is mapped into R one-to-one, because for higher wages time becomes more valuable, and agents prefer a closer location even at higher price[5]. Thus $w(R)$ is a monotonically decreasing function. We assume that it is differentiable (it can be proved for any function $f(w)$, smooth enough).

The general equilibrium formulation of the problem includes N individual optimizations:

$$\max_R [(w_i - P(R) - 2bR)(l_0 - \frac{2R}{V})], \tag{4.13}$$

where $P(R)$ is the equilibrium price function which is taken by agents as given. While for finite N we have also N locations for dachas and a finite set of prices, it is possible to consider the limit as $N \to \infty$, and to define $P(R)$ as a continuous function.

The market clearing condition for this problem can be written in a differential form as:

$$f(w)dw = \pm\nu(R)dR. \tag{4.14}$$

As a similar approach was used by Beckmann [17] but is rarely used by economists. This equation says that the number of agents inside income bracket $[w, w + dw]$ should be the same as their number in the "location bracket" $[R, R + dR]$. The density $\nu(R)$ comes from the supply condition. The differential equation $dw/dR = \nu(R)/f(w)$ locally defines the mapping $F : W \to R$ from the space of wealths into the space of locations. The economic meaning is that every location for a dacha can be chosen by only one agent. In this model it is clear that richer agents would prefer closer locations, since the "price" of leisure for them is higher.

Definition 4.2 *An individual price schedule $P_i(R)$ is a set of prices for different locations, which would make an agent i indifferent across them.*

[5]Suppose that we have 2 agents of different wealth at a point for any given set of prices, and for the poorer this is an optimal point. It means that the price gradient at this point equalizes marginal gains in utility from higher leisure and lower consumption, and vice versa. But at the same point this is no longer true for the richer agent, and he would be better off by moving towards the city, having a relatively higher valuation of leisure.

In other words, $P_i(R)$ defines an individual indifference curve, corresponding to some fixed level of utility, in the space (R, P). At any point of this curve, the gradient of utility is tangent to it. It is always oriented towards the origin $(0, 0)$ (for $P'(R) < 0$), since $\forall R$, $\frac{dU}{dP} < 0$ and $\frac{dU}{dR} < 0$ (Fig. 4.5). Thus, an individual would choose such R_i, where his individual price schedule has a tangency point with the equilibrium price function:

$$\frac{dP_i(R_i)}{dR} = \frac{dP(R_i)}{dR}; \qquad \forall i. \tag{4.15}$$

It means that the equilibrium price function can be constructed as an envelope of individual price schedules. From differential geometry we know that if a family of curves has a form $g(x, y; C) = 0$, where C is a parameter of each curve, then the envelope curve can be found by excluding C from the system:

$$g(x, y, C) = 0; \qquad \frac{dg(x, y, C)}{dC} = 0. \tag{4.16}$$

Here the role of g is played by utility, x - by R, y - by P and the parameter C is individual wealth w_i. The individual price schedule enters the iso-utility curve:

$$(w_i - 2bR - P_i(R))(l_0 - \frac{2R}{V}) - U_i^2 = 0. \tag{4.17}$$

Formally we should differentiate it with respect to the parameter i. Note that the general equilibrium solution would include also the equilibrium mapping: $i \to w_i \to R_i$. Thus, it does not matter, whether we differentiate w.r.t. i or R_i (the tangency point of the individual curve). Thus, we get:

$$\left[-2b - \frac{dP_i}{dR_i}\right]\left[l_0 - \frac{2R_i}{V}\right] = \frac{2}{V}\left[w_i - 2bR_i - P_i(R_i)\right]. \tag{4.18}$$

Now we can eliminate the index i, since all points of the curve $P(R)$ are the points $P_i(R_i)$ of corresponding individual curves (they are the tangency points). Finally we get the following system of first-order differential equations, which defines the general equilibrium:

$$(P'(R) + 2b)(l_0 - \frac{2R}{V}) = \frac{2}{V}(P(R) + 2bR - w(R)), \tag{4.19}$$

$$w'(R) = -\frac{\nu(R)}{f(w)}. \tag{4.20}$$

The second equation of this system is the law of mass preserving. It has a negative sign, because richer agents will always locate closer to the city. The first equation comes from an infinite sequence of individual optimizations. It defines implicitly a mapping $R \to (w, P)$, which satisfies the envelope condition (tangency between $P_w(R)$ and $P(R)$). This system of equations together with the border conditions

$$P(R_{max}) = A, \qquad w(R_0) = w_{max}. \tag{4.21}$$

determines the general equilibrium.

The solution not only exists, but can also be constructed analytically for a big class of differentiable distributions. For any given $f(w)$ and $\nu(R)$ the condition of putting the wealthiest agent at R_0 determines $w(R)$ uniquely. The market size is determined by the outer border R_{max}. This border is the solution of the equation $w(R) = A + 2bR$. Some conclusions can be drawn already for this general case. First, the increase in transport prices always makes the market for dachas smaller. But the change in wealth distribution has an ambigous impact: all depends on how the perturbation of income distribution affects the income of the "border" consumer. If dachas are available for a small fraction of consumers, then the rise in wealth inequality make them affordable for a higher fraction of consumers. Contrary, if dachas are available for almost all consumers (which was the case of pre-transition Russia), then the rise of wealth inequality tends to shrink the number of dachas, as they become too expensive for a higher fraction of consumers.

4.4 Particular Cases

While the system of differential equations, derived in the previous section, may have an analytical solution in a general case, it is difficult to identify some particular effects unless the income distribution is specified. In the case of Russia the income distribution during transition became much more uneven (see details in the section devoted to model calibration). Its upper tail which is most important for this paper was not estimated at all [16]. That is why a case with two types of agents, significantly different in their wealth, may be of interest for this model. Another interesting example is the power distribution $f(w) = w^{-a}$, which was studied by Beckmann [17] in a model without leisure consideration. Usually wealth distribution has a peak, but for contemporary dacha market in Russia only the upper tail is important (people with wealth below some threshold stay away from the dacha market now). There are some estimations that for Russia this tail has power distribution, with a close to one.

4.4.1 An Example with Two Types of Agents

This section presents an example with two types of agents, which differ only in their incomes (rich and poor). Let t be the share of rich in the society. Consider a radially symmetrical model with the total number of agents N. Rich agents will be located at $R_0 < R < R_1$, where $R_1 = \sqrt{\frac{2tN}{\nu} + R_0^2}$, and have utility U_r, which will be defined on the rent continuity basis. For the poor, three situations considered in section 2 are possible: a) all can find a suitable site for dacha, b) the longest distance will be determined by the leisure constraint; c) the longest distance will be determined by the budget constraint of the poor. Consider the case c) as the most realistic for post-transition Russia. The most distant location of poor agents is: $R_2 = \frac{w_1 - A}{2b}$. In the interval $[R_1, R_2]$, $P(R) = w_1 - 2bR$. Then, $U_r^2 = (l_0 - 2R_1/V)(w_2 - w_1)$, from the price continuity condition at R_1. Finally, the rental price is given by the formula:

$$P(R) = w_2 - 2bR - (w_2 - w_1)\frac{l_0 V - 2R_1}{l_0 V - 2R}; R_0 < R < R_1; \quad (4.22)$$

$$P(R) = w_1 - 2bR; R_1 < R < R_2. \quad (4.23)$$

The price gradient in the poor region is equal $-2b$ everywhere. In the rich region it is higher in absolute value, having an additional term, proportional to the income differential and depending on the distance:

$$\frac{dP(R)}{dR} = -2b - 2(w_2 - w_1)\frac{l_0 V - 2R_1}{(l_0 V - 2R)^2}. \quad (4.24)$$

Proposition 4.3 *The price gradient is nonlinear and is higher in absolute value near the city, where richer agents would locate their dachas.*

4.4.2 The Case of Power Distribution

Let $\nu(R) = R$ (radial symmetry) and $f(w) = w^{-a}; w > w_0; a > 1$ (w_0 is the lowest income, so that $\int f(w)dw = 1$). Then, from the mass preserving condition $w'(R) = -Rw^a$ and the boundary condition $w(R_0) = \infty$ it is possible to reconstruct the mapping:

$$w(R) = [\frac{a-1}{2}(R^2 - R_0^2)]^{\frac{1}{1-a}}. \tag{4.25}$$

For the price function there is a differential equation

$$P' + h(R)P = g(R); \qquad h(R) \equiv \frac{-2}{l_0V - 2R};$$
$$g(R) \equiv -2b + \frac{4bR - 2w(R)}{l_0V - 2R}. \tag{4.26}$$

It has a one-dimensional class (with parameter D) on analytical solutions:

$$P(R) = e^{-\int h(R)dR}[D + \int g(R)e^{\int h(R)dR}dR]. \tag{4.27}$$

The border condition $P(R_{max}) = A$, where R_{max} is determined as the solution to the equation $w(R_{max}) = A + 2b_{max}$, allows to determine constant D and hence to find the unique solution.

Consider a particular case $a = 2; R_0 = 0$. Then $w(R) = 2/R^2$, and the solution to the problem has a form:

$$P(R) = -2bR + \frac{2(DR+2)}{(l_0V - 2R)R}; \tag{4.28}$$
$$D \equiv (A + 2br)(l_0V - 2r)/2 - 2/r,$$

where r is the endogenous border of the market. It is easy to show that $P(R) \to +\infty$ and $P'(R) \to -\infty$, when $R \to 0$. This explains how the high price gradient near the city emerges from the competition between the rich.

4.5 Some Empirical Evidence

There are two goals of this section. The first is to give some facts about the role of dachas in Russia and the history of the dacha market structure, starting from the pre-transition period. The second is to capture some stylized facts about dacha prices nowadays.

There were several historical reasons why many Russian citizens had dachas. The first one is the very low opportunity cost of land in northern regions of Russia near big cities (Moscow and St.Petersburg), which has allowed the central authorities to autorize the construction of dachas near the city for practically all interested persons. The failure to solve agricultural problems was another reason to encourage citizens to have small gardens. Historically these land lots were of fixed size (normally 0.06 or 0.12 hectars), which is another factor which simplifies the estimation of land rent in the dacha price. Though the legal market for selling land did not exist in Russia before the transition, the black market almost immediately incorporated these land rents. The advertisements about dacha sales which were published before the transition, according to the author's personal observations did not incorporate the enormously high price differences related to the distances which are observed now. During the last few years this land rent differentiation across regions became so obvious that even maps with price isolines appeared in journals. One of them was taken by the author from [3] (Fig. 4.6 (Map 1)). This map contains several points in the Moscow region with prices for 0.01 hectars of land. Also this map contains the areas which have exhibited price changes during one year. There was no impact on prices in very distant areas, the prices in the circle closest to Moscow have increased (in USD). The transportation price was quite low (even in comparison to wages) ten years ago in Russia, and increased significantly during the transition both in absolute dollar (to eliminate the inflation effect) terms, and as a percentage of the average wage [6]. It means that according to this model, the demand for the closest locations pushed the rents higher, while the more distant areas (about 100 km from the city) became much less valuable, because it means too high a loss of both time and money to go there very often. The nonlinearity of rent gradients can be clearly seen from this map.

Another data set was available for the neighbourhood of St.Petersburg [12] (see Fig. 4.7 (Map 2)). Table 1 gives the land prices as the function of distance to the city center.

[6] After price liberalization in 1992 the wages first dropped in dollar terms, but then recovered to about 100 dollars per month in one year, and stays at this level for the last two years. Meanwhile, the petrol prices were increasing from just few cents for liter of petrol to about quarter of a dollar, which is close to the prices in the USA

Table 4.1: Land Prices for Dachas near St.Petersburg, in USD per 100 sq.m, 1996

Distance ,R, km	30	50	80	110	140
Direction 1 (Vs)	800	400	100	n/a	n/a
Direction 2 (Pr)	300	280	120	80	70
Direction 3 (Vy)	400	300	200	100	80
Direction 4 (Ga)	400	280	100	50	30
Direction 5 (Lo)	350	200	80	30	n/a

The first observation is that different directions differ: this can be explained in aggregate direction externalities, like pollution, climate, access to forests and lakes, which are neglected in the model. The second observation relates to how prices depend on distance in certain directions. It is immediately obvious that for all directions the price gradient near the city is essentially non-linear (much higher), which suggests that low valuation of distant areas occurs not only because of direct transportation costs. The sharp price gradient near the city cannot be explained in the framework of Alonso's model, but it can be explained by the general equilibrium effect of income differential across agents having different opportunity cost of leisure. Richer people, who are likely to settle closer to the city (and this is not only a theoretical prediction, but also a stylized fact for Russian dachas, in contract to some western evidence) have a higher opportunity cost for leisure. The high rental gradient nonlinearity suggests that income distribution in "new" Russia became significantly unequal. The third observation is that at the distance around 100 km the price gradient becomes very small. It may mean that the value of this land is no longer determined by the concept of dacha, with its almost-every-day commuting to the big city. Alternatively, people may decide to exploit distant dachas in a regime with a low frequency of communication with the city.

4.6 Calibration of the Model

In order to apply model to Russian data, taking into account radical economic changes over last years, it is useful to make some preliminary estimations. In the 1980-s the average wage in Russia was a bit more than 200 roubles per month, and in 1993-95 it was about 100 dollars [7]. In the 80-ies a bus trip for 200 km costed about 4 roubles, and 1 liter of petrol - 0.40 rbl. Taking into account that most of russian cars consumed 10 liters of petrol for 100 km, the fuel cost of travelling of 2 persons in a car was equivalent to their travelling in bus. It means that a person who has a dacha 50 km away from the city, spent about 2 rbl. for round-trip per day. The typical price for construction of dacha that time was about 10 000 rbl. (small summer house). If we convert this into daily rent, assuming that discount and depreciation factors are such that daily rent is 0.0001 of total cost [8], then daily rent would be about 1 rbl. Taking daily wage as 7 rbl., according to this model, it is possible to have locational daily rent between 4 rbl. and zero. Travelling costs were not binding, travelling time was binding before transition. That is why many people got dachas at distances 100-150 km from St.Petersburg.

Now consider the current situation. In the beginning of 1996 the transportation costs near St.Petersburg were the following [9]: petrol price 1600-1900 rbl/liter, bus - about 2000 rbl. per 10 km, electric train - about 400 rbl. per 1 zone (7-10 km). The exchange rate from May 1995 till May 1996 was fairly stable at the level 4500-5000 rbl./dollar. The average dollar wage in 1994-95, according to international statistics, in Russia was about 100 dollars/month. On the other hand, income distribution was very unequal, so that many people had incomes below 50 dollars, and a relatively small group had about 500-1000 dollars, and higher. This situation was relatively stable during 1995-98, in the sense that neither income distribution nor transport prices were changing significantly. Assume that the construction price is 10,000 dollars, and the daily rent is 1 dollar (the same discount rate as before). Take the transportation costs as 0.04 dollar/km [10]. Then, a 100 km-trip (which is round-trip for dacha, located 50 km away) would cost 4 dollars. Taking into account that the average daily wage is only 3.3 dollars, we can easily conclude that even these small distances became too expensive for a

[7] Calculation in dollars is more convinient because of inflation, even despite the fact that the purchasing power of dollar for domestic goods was declining

[8] We can always do that, playing with discount rate. In Russia future is very uncertain. Nevertheless, the ratio of rental and buying prices is such, that renting cost for 20-30 years might be equal to buying costs. The similar ratio holds for many countries.

[9] These data were obtained from the citizens directly

[10] This is good estimation for buses, for electric trains it is cheaper, for cars more expensive, as costs should include car depreciation as well

representative citizen. That is why the dachas for every-day trip are used now only by a small share of population; particularly, the high-income group. Those from the low-income group, who already have them at moderate distances, might have a difficulty to sell them, because the high-income group will demand closer locations. That is why it is likely to expect a change of patterns of uses of dacha: the low income group with dachas at long distances will use them for fewer trips, either for vacations or for occasional visits, thus minimizing transport costs. Taking this into account, it is easy to understand why the price gradient is so low at distances above 80 km (the same effect is observed for the Moscow region): transportation is taking place less often. Thus, we come to the following conclusion: in the 1990-ies the radius of using dachas for almost-everyday transportation decreased and is determined not by the leisure loss (as in the 1980-ies), but by transportation costs.

Why is the gradient so high in the closest neighbourhood to the city? It cannot be explained by additional transportation costs, but by income distribution. The recent data about income distribution in Russia [16] show that still 40.9 % of the population has the monthly income below 600 rbl. (of 1998, when 1 USD = 6.1 rbl), and only 4.5% has it above 2000 rbl (330 USD). There are no official statistical data about the upper tail of the income distribution (which would be quite interesting for this paper), but the same article mentions an opinion that 1.5% of the richest russians own 65% of the total wealth and have a monthly income above 5000 USD. These people certainly have a very high opportunity cost of leisure. According to the results of the present article they are likely to select locations close to the city. Then the sharp price gradient near the city can be easily explained in the framework of the model with two types of agents considered above.

4.7 Conclusions

1. A sequence of general equilibrium model of renting prices for dachas has been developed. The simplest model takes into account such factors as both income and leisure loss during transportation. In the case of Russia, the leisure factor was more important before transition, while income factor becomes more important now, when relative prices for transportation have significantly increased. The effect of the shrinking market for dachas is discussed in this framework.

2. The variable frequency of commuting to the dacha is also considered. In the closer circle to the city it takes its highest value (every day communication), but in the outer circle it becomes variable. This is consistent with the observed decline in the price gradient at distances around 100 km from the city, where commuting to dachas can occur less frequently.

3. The more general model allows to take into account the effect of wealth distribution. It allows to obtain an analytical expression for renting prices for any income distribution. The model predicts higher absolute value of price gradient near the city, because of a higher opportunity cost of leisure for richer agents. This is consistent with the evidence of dacha prices near Moscow and St.Petersburg (Russia) in the post-transition period.

4. The model also describes the negative impact of the relative increase of transport costs for public welfare. In particular, the total number of dachas, which is exogeneous in this model, has declined, because the previous owners could not afford to travel to the remote locations often enough to use them as dachas efficiently. Also, the richer group of people would not buy these dachas because they prefer and can afford to buy them in closer locations.

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Chapter 5

Location and Land Size Choice

A new approach for the general equilibrium allocation of a heterogeneous divisible good (like land) among an infinity of heterogeneous consumers is proposed. Consumers have different initial wealth and choose both quality and quantity of land; finally they have land in a form of intervals with endogeneously defined borders. A solution for this problem exists and can be explicitely constructed, but it involves indeterminacy related to order in space.

JEL Classification: C68, D58, R33.
KEYWORDS: heterogeneous divisible good, general equilibrium.

5.1 Introduction

This paper addresses the question of general equilibrium allocation of heterogeneous land across heterogeneous agents, when it is possible to choose both land quality and quantity. This formulation of the problem is a conceptual generalization of an equilibrium with differentiated commodities, proposed by Mas-Colell in 1975, where only the possibility to consume integer quantities was considered. It is also different from the approach developed later by Mas-Colell and Zame (1991), where different problems with infinite-dimensional models were discussed. Here the functional classes are restricted from all measurable functions to continuous functions, in order to eliminate mathematical problems and still keep economic intuition. The basic difference between this paper and the product differentiation literature is in the possibility of having an independent choice of both qualities and quantities of these qualities, varying continuously. On the other hand, this paper can also be seen as an extension of the models of Alonso and Fujita from a partial to a general equilibrium framework.

The assumptions of the model were made so as to solve two puzzles (paradoxes) in the location literature. The first paradox, studied by Berliant (1985), is that in a model with a continuum of agents and a continuum of land, it is impossible to allocate land in such a way that everyone holds land parcels with positive measure. If land holdings have zero measure, the problem of quantity selection cannot be studied properly.

Berliant's own proposed strategy is to concentrate on models where both agents and land are discrete. Discrete models have been studied in a general equilibrium framework, e.g. by Schweizer *et al* (1976). There land is modeled as a finite number of discrete commodities, and the Arrow-Debreu paradigm applies. However, in discrete models of land choice the size of land parcels is restricted exogeneously, while in reality it may be chosen continuously and therefore emerges endogeneously.

As Berliant and ten Raa (1991) showed, a second paradox is that the results from discrete modelling do not automatically carry over to continuous models. Considering the traditional approach in general equilibrium literature, they constructed an example of non-convergence of a sequence of discrete models to a continuous one. To overcome these difficulties, I adopt a slightly different strategy which has not been followed in the literature so far. I assume that the set of agents is countable but possibly infinite, while I retain the intuitively appealing assumption about land continuity[1].

To ensure the analytical tractability of the problem, I will employ techniques that were developed in urban economics. Alonso (1964) introduced the concept of a rental bid curve, which can be derived for a particular type of agent. Originally it was used in central business district theory for deriving a price function, which makes an agent indifferent across a continuum of locations. With several types of agents one needs to study the upper envelope of these curves in order to find an equilibrium price. Fujita (1989) constructed an equilibrium solution with a finite number of types. However, these models are more of a partial equilibrium nature. Beckmann (1969) used a technique which will be called "mass preserving condition" and will play an important role in this paper.

Heterogeneous land will be treated as a continuum of qualities, with a continuous dependence on a spatial coordinate. Agents are assumed to have identical preferences, and to differ only in initial wealth. The utility function I consider is Cobb-Douglas in preferences for a composite good and land size, and is a linear functional on the space of qualities. This allows for the possibility of substitution between different quantities of land of different quality, decreasing marginal utility of land, and also makes an agent indifferent between having a bundle of commodities (interval of land) of similar quality and a particular size of land of average quality.

The general equilibrium problem can be formulated in the following way: there is a big landowner (King) who values only the composite good (gold), and heterogeneous agents with some initial wealth distribution coming from endowments in gold with preferences for both gold and land. The land market is assumed to have a natural restriction: land can be consumed only in the form of intervals (connected sets).

The definition of a general equilibrium with an infinite number of agents

[1]The idea of land as a continuum in the sense of set power is not important here. In this paper "land continuity" means two properties: 1) land is an interval; b) the property (quality) of land depends continuously on the spatial coordinate, which parametrizes this interval

is not an easy question. The description of an allocation involves a mapping of agents from the set of wealths to the space of locations. In a discrete model, the feasibility condition can be naturally formalized by requiring this mapping to be a bijection. Excess demand leads necessarily to a congestion of agents: several agents have to be mapped into the same piece of land. But in a model with infinitely many agents this is not necessarily the case because there exist many ways of defining a one-to-one map from the set of wealths onto the land space. In particular, not every one-to-one correspondence between two intervals can be generated as the limit of a sequence of one-to-one correspondences between two discrete sets which approximate it in a grid of points.

In order to design a continuous model that can be represented as the limit of a discrete model without congestion, we therefore need to restrict the densities of the continuous model in a suitable way. To do this, the usual market clearing condition is reformulated here. We require from our continuous model that the measure of agents in any given interval of the wealth distribution is identical to the measure of agents in the image of this interval in the land space. This "mass-preserving" condition was used by Beckmann (1969) in a similar context, and it ensures that the continuous model can emerge as the limit of a sequence of discrete models without congestion, such that Berliant's second paradox cannot arise.

Existence of equilibrium is not a problem in this model, the problem is uniqueness. Sometimes there are too many solutions, resulting from two sources of indeterminacy - indeterminacy in the value of a particular land lot and indeterminacy in the order of agents in the land space. In the case of a finite number of agents both indeterminacies work, so that neither price functions nor borders can be uniquely determined. In fact, there exists a continuum of solutions. However, when the number of agents goes to infinity, the range of variation of individual wealth, resulting from indeterminacy of prices at which he can possess a particular land lot, goes to zero. In the limit, there is only one indeterminacy - the order of agents in space. For any given order in space, the equilibrium price function, the size of chosen land lots and the quantity of the composite good can be uniquely determined. However, the general market mechanism studied above does not guarantee a unique order in space if we start from any particular wealth distribution of agents. The multiplicity of orders in space, which can take place even for the same price set, suggests that the market mechanism adopted here may not be sufficient for the segregation of agents in space according to their wealth.

Equilibrium existence is shown by computing a solution for a particular class of functions. For some particular orders in space, the analytical formulae describing the mapping from the space of wealth to location space, which is consistent with an individual land quality and land size choice, are obtained.

The article is organized as follows. Section 2 contains the formal assumptions of the model. Section 3 elaborates the idea of a general equilibrium for this model. The case with a finite number of agents (both homogeneous and heterogeneous case) is considered in Section 4. Section 5 is the central part of this paper: it is devoted to equilibrium allocations with an infinite number of agents.

5.2 The Model

5.2.1 Set of Agents and Initial Endowments

There are two fundamentally different types of agents: one King (denoted as K)[2] and the integer set I of agents, $i \in I$. I is assumed to be countable, i.e. either finite or infinite. To each agent i we assign an initial wealth endowment w_i in such a way that $w_i \geq w_j, \forall i > j$, i.e. we order agents by wealth.

In an environment of differentiated commodities the assumption that the set of agents is countable and land is a continuum guarantees that we always have more commodities than agents. Thus, agents can choose more than one point in a commodity space, and this allows to introduce the choice of quantity [3]. This assumption is not usual in general equilibrium and product

[2]The necessity to introduce a special type of agents, who hold land, but do not value it, emerges from the impossibility to assign uniquely wealth from land, owned by a finite number of "normal" agents, who value both land and composite good. Although one King will introduce a possibility of monopolistic behaviour, it will not take place here, because of special preferences of King. It is also possible to replace King by an infinite number of equal landowners, to ensure competitive bahaviour. However, in the case of finite number of normal agents this looks a bit asymmetric

[3]Under a special assumption about the utility function, which is discussed later

differentiation literature, where the total number of agents is usually higher or equal to the number of varieties.

The model has very different properties depending on whether this countable number of agents is finite or infinite. Both cases will be considered separately. A special agent (King, landowner) is also introduced to close the model: he differs in preferences and initial allocation from the rest of the agents. The King initially holds all the land, but has no positive endowment of composite good. Agents have heterogeneous initial holdings w_i of composite good, which gives rise to the endowment distribution. Generically, all agents have different endowments; hence, they can be ordered with respect to their wealth, w_i. In the case of an infinite number of agents, for mathematical reasons it will also be necessary to represent them with a density function $f(w)$.

5.2.2 Commodities

There are two commodities: a composite good C and a differentiated commodity L (land). The composite good is assumed to be a numeraire, with price normalized to one. The supply C_0 of the numeraire is fixed by the initial endowments: $C_0 = \sum w_i$. Land L is the continuous set of locations x inside the unit interval on the real line, $L = \{x : x \in [0, 1]\}$. Locations may also differ by their quality $v(x)$, which is assumed to be a strictly positive differentiable function on the unit interval. If $v(x) = const$, land is said to be homogeneous in quality. Alternatively, land is a continuum of differentiated commodities.

5.2.3 Land Quality and Preferences

With respect to preferences, we have two types of agents. The King values only the composite good and derives no positive utility from land (thus $U_K = C$)[4], while all the other agents value both the consumption good and land, and do not differ among each other in their preferences.

[4] We can even impose a small disutility from holding land for him, in order to eliminate the strategies of not selling all land, which may emerge from his behaviour as a monopolist. In the case of many kings, behaving competitively, this is not necessary

An allocation may be described by the quantity of the composite good, C, and the land interval, $[a, b]$, owned by the agent. The utility of this allocation is given by

$$U = C|b-a|^{\alpha-1}\int_a^b v(x)dx, \tag{5.1}$$

where $v(x)$ represents the land quality preferences and α is a parameter, $\alpha \in (0, 1)$. This utility is linear in qualities (thus proportional to the average quality), but has the property of a decreasing marginal utility to the volume of a bundle. If ψ represents land size, $\psi = b - a$, the utility can be written as

$$U = C\psi^{\alpha-1}\int_a^b v(x)dx. \tag{5.2}$$

Denoting the average land quality by v^*, the utility is $U = C\psi^\alpha v^*$. For $v = const$ it reduces to the usual Cobb-Douglas utility for C and land size ψ. It has some analogy with the Dixit-Stiglitz preferences which include the L_p-space norm of variety bundle $||v||_p = [\int v^p(x)dx]^{1/p}$. Here $p = 1$, and this guarantees linearity. However, because we want a decreasing marginal utility to the volume of the bundle of closely substitutable commodities, we depart from considering utility as a linear functional on the whole bundle (see the examples by Mas-Colell and Zame (1991)). For small land heterogeneity, $||v(x) - v_0||_1 \to 0$, and this allows for a continuous transition from the problem with homogeneous land to the problem with heterogeneous land. This functional form is very useful for the problem of simultaneous selection of qualities and quantities (the relevant lemma will be proved later).

The problem allows for a simple generalization. Imagine that land is two-dimensional, heterogeneous along one axis x, but homogeneous along the second dimension y. Then we can only consider heterogeneous locations, and pool together all land of the same quality. This gives rise to land density $L(x)$, which is assumed to be a continuous function. Then the land quantity inside the interval $[a, b]$ of heterogeneous commodities is equal to $\int_a^b L(x)dx$. If $L(x) = 1$, we return to the initial formulation of the model, without different land densities. Later we will restrict the consumption of land only to the form of intervals of differentiated commodities: consumption of two varieties would automatically require the consumption of all intermediate varieties. On such bundles the utility can be represented by the Stilties integral

$$U = C\psi^{\alpha-1} \int_a^b v(x)dl(x), \tag{5.3}$$

where $dl(x) = L(x)dx$ and $\psi \equiv \int_a^b L(x)dx$.

The utility was chosen of multiplicative (Cobb-Douglas) type in land and composite good in order to avoid problems with boundary solutions. They emerge, for example, for the utility additive in land and numeraire. Different classes of utility might give rise to different properties of the solution.

5.3 The General Equilibrium

The general equilibrium of this model is a competitive equilibrium, i.e. all agents are taking prices as given. All of them are maximizing their utility subject to a budget constraint. However, there are several differences with the general equilibrium problems usually considered in the literature. First, the market clearing condition takes a special form here, which we will call “mass preserving condition”. It is necessary to ensure the convergence of a sequence of models with finite number of agents to the limit model with an infinity of agents. The advantage of this approach is also shown in one example in Appendix 2. Second, because of lack of ex post competition at any market a King is introduced, who initially owns all land but does not value it. Economically, he is necessary to ensure an objective valuation of bundles of land with different quality. (The assumption of price-taking behaviour in this model is quite natural, because there is no room for ex post bargaining on land between neighbours after the equilibrium allocations are obtained). While the King may potentially create monopolistic behaviour, under certain conditions (discussed later) his utility maximization will coincide with an optimal behaviour as a market regulator. There is no difference between the models with one King and infinitely many Kings; in this model monopolistic behaviour cannot perturb a perfectly competitive outcome.

The economic concepts and mathematical tools used for the description of a general equilibrium allocation are quite different for the case of finite and infinite number of agents. While in the finite case land ownership is represented by intervals, agents can be represented by an ordered sequence in wealth space and markets can be cleared at every point, the limit transition to the infinite number of agents would require the introduction of densities, mappings and a new form of market clearing condition. An appropriate choice of the limit economy would allow us to escape the second paradox of Berliant and to have a continuous transition between finite and infinite models.

5.3.1 Allocations

All the definitions will be given initially for a finite economy. Later the discussion on how to use them for an economy with an infinite number of agents will be provided. For the sake of simplicity, we consider $L(x) = 1$. It is easy to generalize these definitions for any $L(x)$, just by using $\psi_i = \int_{a_i}^{bi} L(x)dx$.

An allocation of agent i includes a quantity of composite good C_i and an interval of land $[a_i, b_i[$. Intersection of land intervals belonging to different agents is not allowed. Thus, any equilibrium allocation is a partition of the total land interval $X \equiv [0, 1]$ into $\cup_i [a_i, b_i[$, where $b_i = a_{i+1}$. The alternative representation of this interval includes its center $x_i \equiv (b_i + a_i)/2$ and its length $\psi_i \equiv b_i - a_i$. This can be summarized in:

Definition 5.1 *An allocation for agent i is the set $A_i \equiv \{x_i, \psi_i, C_i\}$, where $x_i \in X, \psi_i \in R_+, C_i \in R_+$.*

Later it will be shown that an agent is indifferent across a bundle of similar land qualities and some quantity of the average quality. This allows for double representation of allocation: by the pair of numbers x_i, ψ_i, or the pair of borders $[a_i, b_i]$. The restriction to have land only in the form of intervals is an important assumption of this paper. It implies a very special market structure. The notions of individual feasibility and social feasibity are introduced below.

Definition 5.2 *The allocation is called* individually feasible *if it contains a connected set (interval of land $[a, b] \in X$) and some quantity of composite good, and the budget constraint is satisfied.*

Usually the set of allocations is considered in an economy with the number of dimensions I times higher than the dimensionality of an individual allocations space.

Definition 5.3 *A set of allocations defines a vector mapping between the set of agents I and the allocation space $(X \times R \times R)^I$: $\{x_i = F(w_i), \psi_i = \Phi(w_i), C_i = G(w_i); i \in I\}$.*

Now it is useful to reduce the dimensionality of the allocation space, keeping the only "copy" of the location space X. The land quantity is an interval, symmetrically located around the central point x_i: $[F(w_i) - \Phi(w_i)/2, F(w_i) - \Phi(w_i)/2]$ (the possibility of such a treatment will be discussed later).

Definition 5.4 *An allocation $\mathcal{A} = \cup A_i$ is called* socially feasible*, if it contains the union of disjoint intervals $[a_i, b_i[$, which correspond to individual allocations and cover the whole space: $\cup[a_i, b_i[= X$ (Fig. 5.1).*

Thus, any socially feasible allocation contains a partition of the land interval into the union of subintervals.

When we consider the limit economy, we usually consider the limit properties of very large economies. Any economy with any finite number of agents above some fixed N_0 has allocations in the form of intervals, so all definitions are still valid. But the limit economy itself is better described by densities. There are two densities to be introduced in this model: the density of consumers in wealth space $f(w)$ and the slot density $S(x)$ (which will be defined later). It is also possible to introduce these densities for finite N: first we can construct a hystogramm by dividing the interval into about $\sqrt{N}$ bands and calculating the number of consumers inside. Later a spline method can be used to reconstruct a continuous density on the basis of a hystogramm. Later the mapping between these two continuous densities can be introduced.

In a model with a finite number of agents the wealth is assigned only to a finite subset of the wealth interval. Since at equilibrium every point in land space X is assigned to some agent, we have a correspondence between wealth space W and location space X, with some points $w \in W$ having many images and the rest having zero. We are interested in the possibility of constructing a continuous mapping F between the wealth and location spaces and to assign to it some reasonable meaning for an infinite number of agents. We will study the properties of the mapping $F : W \to X$ (Fig. 5.2). Up to now, the wealth was defined only on a subset of wealth space, containing a countable number of points w_i. In Appendix 3 it will be shown that the wealth distribution allows for double mathematical representation: as an infinite set w_i and as a wealth distribution $f(w)$. We need this double representation, in order to have a bijection from the wealth space to the set of chosen locations and to the set of land allocations.

Generally speaking, this mapping may not be a bijection and not preserve measure of agents. Later it will be shown that these properties are required by a mass preserving condition. When N is finite, the usual market clearing condition works well. But for the infinite N (and for big N asymptotically) is should be replaced by a stronger condition of mass preserving mapping from the set of agents $i \in I$ (or from the wealth space) into the set of locations $x_i \in X$. The mass preservation would require the consistency of location choice F with the land size choice, given by the mapping Φ. The set of locations is chosen by agents and it is a countable subset of the unit interval. It defines a set of land parcels ψ_i in a unique way. In contrast to the set of locations, the union of the elements of the set of land parcels coincides with the unit interval X.

5.3.2 Market Clearing and the Mass Preserving Condition

The usual market clearing condition, which requires markets to clear for each commodity, is not sufficient for the model with an infinite number of agents. This is due to the congestion problem mentioned in the introduction: what can be represented in a model with continuous land cannot always be translated back into a discrete model. While in a continuous model agents of different types could choose very close points, in its discrete approximation they may end up with the same slot. This gives rise to the congestion problem underlying Berliant's paradox. An example in appendix 2 reveals the economic intuition about the congestion problem. Its mathematical origin is in the fact that a point has a measure zero. The market clearing condition for any particular commodity is an equation in a point. Aggregation of these conditions can be done in many different ways, and the correct way is one which is based on the mass preservation of agents under mapping. The continuity of this mapping (almost everywhere) is an important requirement.

The mass preserving condition can be written as $f(w)dw = dxL(x)/S(x)$.

Here $L(x)$ is the density function, describing the measure of land in location x, and $S(x)$ is the land size in x, chosen by agent of type w, where w is the inverse image of x. Geometrically, it can be thought as a rectangular $[x - S/(2L(x)), x + S/(2L(x))] \times [0, L(x)]$. Thus, the measure preserving condition indicates that the measure of agents in the interval dw of the wealth distribution should be equal to the measure of the image of this distribution in the land space. The differential form of market clearing condition, was already used by Beckmann (1969). It is actually stronger than the traditional condition of market clearing in each point (which would be of measure zero here) as it requires market clearing in any neighbourhood.

The mass-preserving condition is a continuous form of a "correct" market clearing condition, when the number of agents goes to infinity. In the following lemmas it will be shown, that it is not only a correct form of market clearing for an infinite number of agents, but also is an asymptotic approximation to a discrete, usual form of market clearing condition. For a finite number of agents the clearing of each market at location x means that land in this location in equilibrium can belong to only one owner. A solution to such a problem would include a price function and allocations. Allocations can be described as a correspondence between the set of agents and intervals where they locate. When the number of agents goes to infinity, the set of points which they occupy in wealth space can be approximated by a continuous wealth distribution function $f(x)$. What we need to prove is that small perturbations of the wealth distribution (including replacing a discrete distribution by a continuous one for a finite but large N) have small impact on the solution as a whole. As it will be shown, there exists indeterminacy related to order in space. Thus, closeness of solutions would mean a small difference between 2 solutions of different problems (continuous and discrete), but generated by the same order in space.

Lemma 5.1 *The equilibrium price depends only on average wealth of all agents and not on its distribution.*

PROOF: See Appendix 1.

Definition 5.5 *Let $\{w_i, i = 1, ...N\}$ be the set of initial wealths (later: discrete wealth distribution) for a problem with finite number of agents N. We can construct a hystogramm, dividing the wealth interval into $\sqrt{N}$ subintervals, and then smooth this distribution using spline method. As a result we* *get a continuous distribution $f_N(w)$, which corresponds to a discrete problem with N agents.*

Definition 5.6 *Consider a sequence of discrete models with discrete distributions of wealth and increasing N in such a way that the total wealth is preserved: $W = \sum_{i=1}^{N} w_i = const, \forall N$. This sequence will be denoted $\{M_{d,N}\}$, and any of its elements as $M_{d,N}$.*

Definition 5.7 *Consider a sequence of discrete models with continuous wealth distribution $f_N(w)$, increasing N and preserved wealth: $W = \int_0^\infty f_N(w)dw = const, \forall N$. This sequence will be denoted as $\{M_{c,N}\}$, and any of its elements as $M_{c,N}$.*

This $f_N(w)$ can be treated as probability distribution of wealth, so that the agent's location in wealth space is smoothed between its central position w_i and the centers of intervals connecting him with his left and right neighbours in wealth space (see Appendix 3 for more details). We want to establish asymptotical equivalence of descriptions by discrete and continuous wealth distributions for big N. For this, we need to show that the difference between the solutions for any realization $\{w_i(\xi), i = 1, ...N\}$ (ξ - random parameter, fixing realization) is small in an appropriate normed functional space (the details will be provided later). This allows us to make no distinction between discrete and continuous distributions already for big finite N. Then both sequences $M_{d,N}$ and $M_{c,N}$ would generate Cauchy sequences in the space of norms of the solutions. As it is known from functional analysis, there exists a limit for such sequences. We will assign the topological properties of discrete wealth model (allocations as intervals, etc) to this limit, but we will also use functional properties of the continuous functions. In such a manner, the mass preserving condition would become a description of market clearing condition when $N \to \infty$.

Lemma 5.2 *When $N \to \infty$, the difference between the problem with discrete wealth distribution $\{w_i\}$ and its continuous analog $f_N(w)$ becomes infinitely small. Thus, a model with continuous wealth distribution is an asymptotical approximation of the model with discrete wealth distribution.*

PROOF: See Appendix 1.

Lemma 5.3 *Consider all realizations $\{w_i(\xi)\}$ of a distribution $f_N(w)$. For big N the solutions for all these realizations are close to each other.*

PROOF: See Appendix 1.

Proposition 5.1 *All the solutions of any particular realization of the wealth distribution with some particular order in space are in the ϵ-ball of the solution to the continuous problem, obtained on the basis of the mass-preserving condition.*

PROOF:
As it was shown in Lemmas 5.2 and 5.3, we can choose N_0 such that $\forall N, N > N_0$: $|M_{d,N} - M_{c,N}| < \epsilon/3$, $M_{c,N} - M_{c,\infty}| < \epsilon/3$ and $|M_{d,N,\xi} - M_{d,N}| < \epsilon/3$. Using the triangular theorem, we can estimate the difference between solutions for all realizations of the random wealth distribution: $|M_{d,N,\xi} - M_{c,\infty}| < \epsilon$. For the problem $M_{c,\infty}$ we use mass-preserving condition.

Proposition 5.2 *1. For a finite number of agents, when we restrict our attention to the class of solutions, preserving order in space, the market clearing condition can be written also in the form $f(w)\Delta w = (L(x)/S(x))\Delta x$.*
2. The mass preserving condition $f(w)dw = (L(x)/S(x))dx$ is equivalent to social feasibility for a problem with infinite number of agents, and thus represents a correct form of market clearing condition.

PROOF:
1. For a finite N we choose Δw and Δx in such a way, that all agents from the wealth bracket Δw choose to occupy the land interval Δx. Since $f(w)\Delta w$ by construction is the number of agents in this wealth bracket, the same number should be preserved in the land interval Δx. The market clearing condition requires to the allocation of only one agent to each land commodity, and it holds here.

2. Now we will proceed with a continuous model, but we will also talk about asymptotics for high N. Let $f(w)$ and $g(x)$ be continuous functions describing densities of probability distributions, and let $x = x(w)$ be a continuous mapping between the sets $\{X : x \in X\}$ and $\{W : w \in W\}$. The condition $f(w) = g(x(w))dx(w)/dw$ holds for any differentiable monotonous function of a random variable (see Korn [20], for example). Consider a discrete model with N sufficiently large (Fig. 5.2). Let w_i be a random variable *with the support* $[(w_{i-1}+w_i)/2;(w_i+w_{i+1})/2]$. *Then* $\forall\epsilon > 0$ *there exists so high* $N(\epsilon)$, *that any realization of this wealth over the support would be a small perturbation of the initial problem. Hence, the difference between the solution of the problem where* $f(w)$ *has a continuous distribution or a particular discrete realization of it, will be infinitely small (See Prop.1).*

3. Suppose that we have individual and social feasibility, i.e. all land is the union of intervals, corresponding to indivial land allocations. Then for any socially feasible allocation we can construct a mapping $x = F(w)$, *first on a discrete set of points, and then by associating a continuous wealth distribution* $f(w)$ *to any discrete wealth distribution* w_i *(see Appendix 3). The mass preserving condition should hold as the mathematical theorem about transformation of densities under continuous monotonous mappings.*

4. The mass preserving condition states that the measure of agents in any neighbourhood of any point in the wealth space and in its image in the space of locations should coincide. Suppose that a neighbourhood dw *exists, such that there are more agents in its image* dx, *than in itself. Then there will be at least some points occupied by more than one agent, and this contradicts feasibility. Suppose that there exists a neighbourhood* dw, *such that there are more agents in it, than in its image* dx. *Then some agents from this neighbourhood will choose to have no land, which is never possible for the considered utility function (for muliplicative utility we always have in equilibrium an interior allocation for positive wealth and positive price for any land with strictly positive quality).*

The mass preserving condition is an important instrument to solve the second paradox of Berliant. As it is shown in the appendix, in a model with a continuum of agents there may be many solutions in a form of a one-to-one correspondence, which cannot be generated as the limits of some sequence of discrete problems with such a property. Although we have only an infinity of agents here (not a continuum), this infinity can be represented by a continuous wealth distribution function (see Appendix 3); and this is already a continuum.

5.3.3 Price System P(x)

Since we have a continuum of commodities, we need a continuum of prices. For any commodity the price is a positive number: $\forall x \in X : P(x) \in R_{++}$. Since we assumed $v(x)$ to be continuous, we will require $P(x)$ to be a continuous function of x[5]. Any allocation can be supported by some selection of prices $P(x)$. The price of a land parcel $[a, b]$ is defined as $\bar{P}([a, b]) = \int_a^b P(x)dx$. By continuity, for small parcels we have the following fomula:

$$\bar{P}([x, x + dx]) = P(x)dx = P(x)\psi. \tag{5.4}$$

5.3.4 Definition of Equilibrium

In the case of a finite number of agents the general equilibrium does not differ from the one considered in the literature [21]. It includes utility maximization for every agent subject to the budget constraint and market clearing condition. For the considered utility function, expenditures on a composite good would always form a fixed fraction of the wealth of each agent, hence of the total wealth W. Thus, it is possible to set the price of the composite good to one (it will be a numeraire), when the total amount of this good at the market represents this particular fraction of the total wealth. Market clearing for land should be provided in each location, i.e. each location x can be assigned to one and only one owner.

Before the formal mathematical definition of the general equilibrium for the model with an infinite number of agents it is necessary to clarify the equality $S(x_i) = \psi(w_i)$ used there. While ψ is the land size chosen by an agent with wealth w_i, in an equilibrium it will lead to occupation of the land interval of size S: $[x_i - S/2, x_i + S/2]$ by this agent. The sizes of S and ψ are equal, and therefore this equality is true. However, these functions have different arguments. For $N = \infty$ it is possible to introduce slot density in a point x. While each agent would formally occupy zero land, it is possible to keep track of the relative density of agents in land space (after they have made their choices) $\rho(x)$. The number of agents in the interval dx is equal to $\rho(x)dx$. It is also equal to all land inside this interval, $L(x)dx$, divided by the slot density $S(x)$. Hence, we have: $\rho(x) = L(x)/S(x)$. Now we can define the general equilibrium for the model with an infinite number of agents.

[5] The continuity of the price system was also required by Mas-Colell and Zame (1991)

Definition 5.8 *The general equilibrium for the model with an infinite number of heterogeneous agents is a distribution of land slots $S(x)$, a mapping $F(w)$ of agents with different endowments w into slots $S(x)$ in chosen locations $x(w)$, and a set of prices $P(x)$ per unit of land in location x, such that the following requirements are satisfied:*
1) each agent of type w maximizes his utility subject to his budget constraint, taking price schedule $P(x)$ as given;
2) the simultaneous choice of $x(w)$ and $\psi(w)$ generates a mapping $x = F(w)$ and the partition of the interval $[0,1]$ into slot density $S(x)$, where $S(x) = \psi(w)$;
3) the supply is equal to demand in any small neighbourhood of any type: $f(w)dw = dxL(x)/S(x)$ (this means that the measure of any small neighbourhood of each agent is preserved under this transformation);
4) the market for the composite good C clears.

There are two interesting and significantly different cases: with finite and infinite number of agents. The case with a finite number of agents is interesting for bringing the intuition for two different indeterminacies into this model. One of them can be eliminated when we move from the finite to an infinite case. The next section contains a sequence of small models related to the finite number of agents. The infinite case will be considered later.

5.4 Finite Number of Agents

5.4.1 Homogeneous Land

We consider first the problem with homogeneous land ($v(x) = 1$) and a finite number of consumers. We can assume without loss of generality that $L(x) = 1$, since only the total amount of land (and not its geographical map) matter here. Suppose that we have N agents with different wealths $w_i, i = 1, 2, ...N$ with preferences $U = C\psi^{\alpha}$, and a King with preferences $U = C$, who initially owns the whole land interval $[0,1]$. Since locations do not enter preferences, agents are indifferent across them; they choose only land volume. King who is willing to get composite good (which is numeraire with price 1) sets such a price P for land (which does not depend on location), so that all land can be sold[6]. It is easy to show that under these preferences

[6] It can be easily shown that he cannot get more revenue by monopolisitc behaviour, because of unit elasticity of demand for land

the demand of agents is given by the following formulae:

$$C_i = \frac{w_i}{1+\alpha}; \psi_i = \frac{\alpha w_i}{(1+\alpha)P}. \tag{5.5}$$

The price can be determined from the market clearing condition: $\sum_i \psi_i = 1$. It is always a constant fraction of the total wealth of agents $W \equiv \sum_i w_i$: $P = \frac{\alpha}{1+\alpha} W$.

We consider land here as one commodity, disregarding location, and thus equilibrium is unique. If we allow land ownership only in the form of intervals, this equilibrium gives rise to as many different allocations in space as different orders in space (it is $N!$, where N is the number of agents). This will be one of the reasons for indeterminacy.

5.4.2 Heterogeneous Land. Basic Indeterminacy.

Let $L(x) = 1$. While introducing $L(x)$ makes it possible to consider two-dimensional geographical maps for land, for any $v(x), v'(x) > 0$ there always exist a possibility of a nonlinear transformation of coordinate x into a new coordinate $z = z(x)$, so that land quality becomes another function $\hat{v}(z)$ with similar properties and $\hat{L}(z) = 1$.

The utility function allows for a possibility of a continuous transition between homogeneous and heterogeneous cases. Although we formally have a continuum of commodities instead of one, for a small heterogeneity (as a small norm of differences between homogeneous and heterogeneous quality in a functional space) the difference in utility is also small. However, heterogeneous land brings another sort of indeterminacy into the model. It will be shown that due to the lack of competition for "inside land" the price of it cannot be uniquely defined. Hence, the wealth from its possession is also undetermined, and this leads to a difficulty in defining borders, even if some order in space is chosen. But with the increase of number of agents the relative weight (in the sense of functional norm) of this indeterminacy declines, and in the limit there is no wealth indeterminacy. We can also talk about the convergence of a sequence of discrete models to one with an infinity of agents. Although an indeterminacy exists for any finite number of agents and disappears in the limit, the set of possible solutions for any discrete model can be put inside a compact ball, and the sequence of these balls converges to a solution of the problem with an infinity of agents. There are still many equilibria (it will be shown in the next section that they can be classified by the order of agents in space), but the same indeterminacy exists for discrete models as well. If we separate the balls, which correspond to each order (taking it for example as wealth preserving), then we can talk about this convergence.

Let us consider formally a model with two agents and heterogeneous land. Let land quality $v(x)$ be a positive monotonously increasing function. Assume that the first agent is located to the left, between $x \in [0, z]$, and agent 2 - between $x \in [z, 1]$, where the border z has to be determined endogeneously. If $v(x) = x$, the problem is even simpler, because for a linear quality function the utility from the bundle of lands of different qualities is exactly equal to the utility in a central point, multiplied by the power α of land size. Then the Lagrangians are:

$$\mathcal{L}_1 = C_1 v(z/2) z^\alpha + \lambda_1 [w_1 - C_1 - \int_0^z P(x)dx]; \tag{5.6}$$

$$\mathcal{L}_2 = C_2 v((1-z)/2)(1-z)^\alpha + \lambda_2 [w_2 - C_2 - \int_z^1 P(x)dx]. \tag{5.7}$$

Consider an example with $v(x) = x$. The first order conditions lead to the following equations:

$$\lambda_1 = \frac{z^{\alpha+1}}{2}; \quad P(z)z = C_1(\alpha + 1); \tag{5.8}$$

$$C_1 = w_1 - \int_0^z P(x)dx; \tag{5.9}$$

$$\lambda_2 = \frac{1+z}{2}(1-z)^\alpha; \quad P(z)(1-z^2) = C_2[(1+\alpha)z - 1 + \alpha]; \tag{5.10}$$

$$C_2 = w_2 - \int_z^1 P(x)dx. \tag{5.11}$$

While we need to define the whole function $P(z)$, we only have a few algebraic equations, which allow us to find the border price $P(z)$ if we know $P(x)$ in the rest of the points. Exclusion of $P(z)$ gives an algebraic equation for z

$$\frac{\alpha+1}{z}[w_1 - \int_0^z P(x)dx] = \frac{(1+\alpha)z - 1 + \alpha}{1-z^2}[w_2 - \int_z^1 P(x)dx]. \tag{5.12}$$

which has as many solutions, as many possibilities we have to choose the function $P(x)$. Adding a market clearing condition does not help much: still the class of possible price fields includes a continuum of elements. Thus, we have:

Proposition 5.3 *The considered problem has a continuum of solutions, and the shape of equilibrium price function $P(x)$ cannot be determined uniquely from the optimization problem.*

Note that we have not yet considered the indeterminacy related to order in space. This issue will be considered in details in the problem with an infinite number of agents.

5.4.3 The Degree of Indeterminacy. Convergence

The indeterminacy of the problem with a finite number of agents is so high, that not even individual wealth can be uniquely determined. Is there some chance to get rid of it when the number of agents grows unboundedly? What is the reason for this indeterminacy?

Suppose that we have a King who initially owns all land, with a monotonously increasing and continuous $v(x)$. He can set prices in such a way that the buyers would be indifferent across locations. With the preferences considered here he just needs to set $P(x) = Av^{1/\alpha}(x)$.

Will the final allocation be stable after purchases at these prices have taken place? Not necessarily. Note that we are facing a game theoretical situation, since the King no longer can be a market maker, having neither land nor preferences for it. Each agent i would have some bundle of land with minimal quality v_1 and maximal quality v_2. The prices can be determined only at the borders at such a level that no agent would be willing to resell small pieces of land to their neighbours. All prices inside should clearly be a monotonous function of quality, but there is no market mechanism which would prevent agents from overvaluation of his internal land while bargaining with neighbours (Fig. 5.3). Another problem is that prices at which an agent can buy or sell land may be different, because of hedonic valuation: the less land of the same quality an agent has, the higher his marginal valuation will be. All this gives rise to a multiplicity of equilibria, related to the indeterminacy of the price function. But some estimates can be made. The prices of the worst land (left border a_i) and the best land (right border b_i) are determined by the market. Thus, the upper bound of the wealth of agent i in land is $P(b_i)(b_i - a_i)$, while the lower bound is $P(a_i)(b_i - a_i)$. When N is increasing, the ratio of the land quality variation to an average quality is vanishing to zero for all agents. Thus, the ratio of price indeterminacy to the price at each point is also vanishing to zero. The following lemma summarizes these results.

Lemma 5.4 *Let $P_\infty(x)$ be the equilibrium land price for an infinite number of agents, and let $P_{N,k}(x)$ be one of the price functions at which an equilibrium can be supported for N agents (there are many possibilities, indexed by k). Let ϵ_N be the degree of indeterminacy of the land price function, formally defined as*

$$\epsilon_N = \frac{\max_k ||P_{N,k}(x) - P_\infty(x)||}{||P_\infty(x)||}, \tag{5.13}$$

where $||f||$ is the L_p norm of f. Then $\epsilon_N \to 0$, when $N \to \infty$, and the price indeterminacy asymptotically disappears (Fig. 5.4).
PROOF: See Appendix 1.

Note that P_∞ will coincide with the King's valuation of land, which would make him indifferent across locations, if he would like to possess land. The absence of an interest in land makes him a non-strategic player, and he is actually a market maker, while he wants to get maximal revenue from land sales. Because of the last lemma, it makes sense to study the problem with an infinite number of agents.

5.5 Equilibrium Allocations with an Infinite Number of Agents

5.5.1 Problem Formulation and the Equivalence Theorem

Before solving the problem of land choice formally, it is useful to employ the equivalence theorem, which allows to reduce the dimensionality of the problem (from continuum to infinity) and to have a utility function which has both quality and quantity in its arguments.

Lemma 5.5 *For $N \to \infty$, when an agent changes his choice marginally moving one of the borders, the impact of land volume change is unboundedly larger then the impact of quality change.*
PROOF: See Appendix 1.

Theorem 5.1 *For $N \to \infty$, the problem where each individual chooses a bundle represented by a land interval $[a, b]$ (and some quantity of a composite good) for a utility $U = C\psi^{\alpha} \int_a^b v(x)dx$ with differentiable $v(x)$ is asymptotically equivalent to a problem with the utility function $\bar{U} = C\psi^{\alpha-1}v(x)$, where each agent has to choose only x and ψ. This allows to reduce the dimensionality of varieties from a continuum to infinity, and gives the freedom to specify this infinite (basic) sets of varieties in many possible ways. For the last problem, agents have to choose locations in space and the quantity of land of the quality corresponding to these locations.*
PROOF:
By Lemma 1, $U = \bar{U}(1 + O(1/N))$, and thus both utilities asymptotically coincide.

For a sequence of agents ordered by wealth, consider the following optimization problem:

$$\max_{C_i, x_i, \psi_i} U_i, \qquad U_i = C_i v(x_i)\psi_i^{\alpha}; \tag{5.14}$$

$$s.t. \qquad C_i \geq 0; \psi_i \geq 0;$$

$$C_i + P(x_i)\psi_i = w_i. \tag{5.15}$$

All agents $i \in I$ are maximizing their utility taking prices as given. As was mentioned before, we can introduce an infinite number of Kings, each initially owning the same land endowments [7]. Then a King cannot play strategically with prices as a monopolist. But even one King, under our assumption about his utility, cannot do it. If he would deviate from the price $P(x) = Bv^{1/\alpha}(x)$ (it will be obtained further as an equilibrium price, under which there would be no secondary trade among agents) by setting lower B, he will get less revenues. But setting B higher will not allow him to sell all land because of the unit elastic demand for it. Adding a small disutility from unsold land (maintainence cost) would make the King choose optimally that particular

[7] Although this is a standard assumption of perfectly competitive markets, it sounds not very realistic. However, historical stories with crusaders of conquistadors selling land to buy weapon might serve as a reasonable background

B, which would make this equilibrium stable and optimal (there would exist no Pareto improving trade across agents). Formally, we can forget that the King is a utility maximizer and think of him as a market maker. We also add the market clearing condition in its special form of “mass preserving condition” for the reasons discussed above.

5.5.2 The Solution Method

Consider the array of Lagrangians, formally associated with agents:

$$L_i = C_i v(x_i)\psi_i^{\alpha} + \lambda_i[w_i - C_i - P(x_i)\psi_i], i = 1, 2, ... \tag{5.16}$$

The First order conditions give 4 infinite sequences of equations:

$$\frac{\partial L_i}{\partial C_i} = v(x_i)\psi_i^{\alpha} - \lambda_i = 0; \tag{5.17}$$

$$\frac{\partial L_i}{\partial \psi_i} = C_i v(x_i)\alpha\psi_i^{\alpha-1} - \lambda_i P(x_i) = 0; \tag{5.18}$$

$$\frac{\partial L_i}{\partial x_i} = C_i \psi_i^{\alpha} v'(x_i) - \lambda_i P'(x_i)\psi_i = 0; \tag{5.19}$$

$$\frac{\partial L_i}{\partial \lambda i} = w_i - C_i - P(x_i)\psi_i = 0. \tag{5.20}$$

It is easy to show that

$$C_i = \frac{w_i}{1+\alpha}; \psi_i = \frac{\alpha w_i}{(1+\alpha)P(x_i)}. \tag{5.21}$$

Land expenditure is a constant fraction of wealth for each agent i.

Definition 5.9 *The isoutility curve in space $(x, \psi(x))$ for any given price function $P(x)$ (under which an agent would choose the land size $\psi(x)$ in a location x) is a relation between x and $\psi(x)$, such that utility from land is constant: $v(x)\psi^{\alpha}(x) = const$ (Fig. 5.5).*

The isoutility curve is a compact representation of an indifference curve, which is a continuum of equalities across bundles $[a(y), b(y)]; y \in X$.

Definition 5.10 *The isoexpenditure curve $P(x)\psi(x) = const$ space is a compact form of a budget line, which includes only constant land expenditures on different bundles of land (Fig. 5.6).*

The new terms are used here because the budget constraint usually includes all expenditures. Fortunately, for this utility the fraction of expenditures on land does not depend on the bundle, and it allows us to consider the land subbudget, which gives rise to the isoexpenditure curve.

Lemma 5.6 *For the problem considered, the isoutility curves coincide with the isoexpenditure curves $P(x)\psi(x) = const$, for each individual independently of wealth.*

PROOF:
For Cobb-Douglas preferences in quality and quantity $V = v(x)\psi^{\alpha}$, the expenditures are fixed fraction of an individual wealth. Thus, $\psi(x) = \gamma w_i/P(x); \gamma = \alpha/(1+\alpha)$. For each individual i, we can find a $P_i(x)$, such that makes his utility constant at every location, given that he spends a fixed fraction of his wealth.

The sequence of F.O.C. arising from location choice formally gives the differential equation, which can be reconstructed from the equalities in an infinite set of points:

$$\frac{v'(x)}{v(x)} = \alpha \frac{P'(x)}{P(x)}. \tag{5.22}$$

Its integral is equal to $P(x) = Bv^{1/\alpha}(x)$. The parameter B should be a constant; otherwise it cannot be an equilibrium, by an arbitrage argument. Note, that since isoutility curves have the same functional form, this price field would generate the same utility from a small, arbitrary fixed land size ψ_0 in all locations for all agents. It can be used as a candidate for an equilibrium price function. The constant B can be selected on the basis of the mass preserving condition. If the same B would be obtained for different orders in space, then we have a muliplicity of locational equilibria, associated with this particular price function. This will later be shown to be the case for our problem, but for different utility functions it would not necessarily be the case.

The problem of simultaneous choice of land quality and quantity by an infinite number of agents can be solved in the following way:
a) isoutility curves (analogies of rental bid curves) are constructed for each agent i; they define individual prices at which agent i is indifferent across

bundles in different locations;
b) for any given order of agents in space the mass-preserving condition is used to determine corresponding x_i and ψ_i; this will uniquely determine endogeneous borders a_i, b_i;
c) chosen locations x_i will determine a particular realization of an infinite-dimensional space of varieties, and then Theorem 1 can be used.

Now consider the sequence of market clearing conditions. For each small neighbourhood of agent i in the wealth and location space the mass preserving equality should be satisfied. It can be transformed using the already obtained expressions:

$$wf(w)dw = B\frac{1+\alpha}{\alpha}L(x)v^{1/\alpha}(x)dx. \tag{5.23}$$

The integration of this differential equation will give a two-parametric family of functions $G(x, w; A, B) = 0$, which are candidates for the continuous mapping. Now there are two more conditions, related to the correspondence between the ending points of the intervals in the space of wealths and the space of locations. There are many possibilities to construct a solution: the simplest are the order preserving and the order reversing mappings. Each of them determines uniquely A and B, and thus the mapping. Note that the condition linking the integral of wealth and the integral of prices will appear automatically, on the basis of individual choice and the mass preserving condition. Both parts of the equation (20) can be integrated for some ordered correspondence across wealths and chosen locations. Thus, the mapping becomes monotonous and defined for all elements of the set. This generates the equilibrium price function and the size of chosen land for every agent in a unique way. Thus, we have:

Theorem 5.2 *For any differentiable monotonous function $v(x)$ and any integrable functions $f(w)$ and $L(x)$ the equilibium exists. Generically, it is not unique, since for different classes of orders in space the problem may have a separate solution.*

5.5.3 Multiplicity of Equilibria

In this model we can obtain a family of price functions of a particular shape related to function $v(x)$: $P(x) = Bv^{1/\alpha}(x)$. Since all agents optimally spend a fixed fraction of their income on land, the integral of the price function over space should always be equal to a constant fraction of the total wealth: $\int_0^1 P(x)dx = \gamma \int f(w)dw$. This uniquely determines the constant B, which is consistent with equilibrium. However, at this price we have as many equilibria as orders in space among agents: $N!$, where $N \to \infty$.

5.5.4 An Example of a Particular Solution

Consider a uniform wealth distribution: $f(w) \equiv 1$, $w \in [0, 1]$, and quality linearly increasing in location: $v(x) = x$. Let $L(x) \equiv 1$, since as was mentioned earlier, the choice of a non-uniform $L(x)$ is equivalent to some monotonous perturbation of this land quality function.

First, we will determine the mapping. Suppose that it is order preserving, i.e. $w = 0$ maps into $x = 0$, and $w = 1$ maps into $x = 1$. Then

$$x = F(w) = w^{\frac{2\alpha}{\alpha+1}}. \tag{5.24}$$

The set of prices and sizes of chosen land, corresponding to this mapping are:

$$P(x) = \frac{1}{2}x^{1/\alpha}; \tag{5.25}$$

$$S(x) = \frac{2\alpha}{1+\alpha}x^{\frac{\alpha-1}{2\alpha}}. \tag{5.26}$$

We can also construct a solution with the reversed order. It corresponds to the same price function, but the mapping will be different:
$x = F(w) = (1 - w^2)^{\alpha/(\alpha+1)}$.

5.6 Summary

This paper extends the locational results of Alonso [5] and Fujita [6] to the case of an infinity of types of agents, when the difference is in wealth. It also extends the theory of differentiated commodities for the case when the consumption of different quantities of different qualities may take place. Here the problem with a countable number of agents heterogeneous in their wealth is studied. When agents have both freedom to choose location and land size, the general equilibrium solution involves a mapping from the space of types into locations and slots of land, supported by a set of prices in such a way, that each agent maximizes his utility and the supply is equal to demand in any small neighbourhood of each type.

In the most general formulation of the utility function the problem can not be solved explicitely, and it is not clear whether is has a solution. That is why the preferences were narrowed to a class of utilities multiplicative in land quality and land size. The existence of a solution was proven by construction. However, this problem has a multiplicity of solutions, different in the order of agents in space.

The first cause of indeterminacy is the impossibility to price land uniquely when the number of agents is finite. This indeterminacy is shown to disappear asymptotically when the number of agents becomes infinite. The second indeterminacy is more fundamental, since it persists with an infinite number of agents. It is related to the coincidence of isoutility and isoexpenditure curves, thus allowing for different equilibrium orders of agents in space.

This model explains why in a general equilibrium poor and rich agents may be mixed in space, although the price function reflecting the land quality would make them to choose different sizes of land in neighbouring locations. Some interesting related questions (Pareto optimality of equilibrium allocations, alternative mechanisms in land markets) deserve further research.

5.7 Appendix

5.7.1 Proofs of Lemmas

PROOF OF LEMMA 5.1.
Consider equation (5.23), obtained as a particular form of the mass-preserving condition for the model with an infinite number of agents. If we integrate it over the whole interval for the order-preserving mapping, we get an expression for B:

$$B = \frac{\alpha}{1+\alpha} \frac{\int_{w_{min}}^{w_{max}} w f(w) dw}{\int_a^b L(x) v^{1/\alpha}(x) dx}. \tag{5.27}$$

Given all primitives for land $(L(x), v(x), a, b)$, the equilibrium price $P(x)$ in (22) is the same for all problems with $\int w f(w) dw = const$. Thus, for any fixed number of agents N, the problems with different wealth distribution but the same average wealth give the same equilibrium price function.

Note that the average wealth should not be preserved for a sequence of models with variable N, but should change inversely proportionally to N, to keep the total wealth constant for a sequence of models. On the other hand, the preservation of average wealth would imply that small perturbations of continuous wealth distribution have no impact on equilibrium price.

PROOF OF LEMMA 5.2.

1. Let us introduce the Banach space H with elements $h = (P(x), x \in [a,b]; a_i, i = 1,...N)$ and a norm $||h|| = |\int_a^b P(x)dx| + (1/N)[|\sum_{i=1}^N a_i| + \sum_{i=1}^N C_i]$.
2. A solution to the general equilibrium problem contains consumptions of the composite good C_i, a price function $P(x)$ and the borders a_i between land slots. The solutions for two different problems can be compared by calculating the norm of their difference. If for any fixed N the average difference between prices across space, between borders and consumptions of the composite good across agents is small, then two solutions are close, and the norm of their difference is small. If N is variable, it is difficult to compare the borders and consumptions sets (since the sets of agents are different), but it is possible to compare prices. Then only the first term of the norm should be kept.
3. Consider the discrete and continuous wealth distributions $f_N(w)$ for sufficiently large N. Although there is price multiplicity, according to Lemma 4, the difference between equilibrium price functions vanishes to zero. Therefore, it does not matter what particular price function is chosen for a model with a finite number of agents. On the other hand, a continuous function $f_N(w)$ gives rise to a unique price function $P_N(x)$, as if the number of agents is infinite. Two price functions are close to each other (see the proof in Lemma 4). We agreed that it is sufficient to compare only prices to prove the closeness of solutions for this case. It is obvious that $||M_{d,N} - M_{c,N}|| \to 0$ as $N \to \infty$.

PROOF OF LEMMA 5.3.

1. Consider the order preserving solution (mapping $x = F(w)$ is a strictly increasing function) for any realization $\{w_i(\xi), i = 1,...N\}$ of the distribution $f_N(w)$. By construction, any individual wealth may fluctuate in a small

neighbourhood of w_i, corresponding to the initial discrete problem. If the range of all wealth variation for the whole set of agents is normalized to one, then these fluctuations will be of order $1/N$. Since the fluctuations of each individual are independent, for a large number of agents the fluctuations of the total wealth would be of order $1/\sqrt{N}$. Since for a continuous model (we can always think that a continuous function $f_N(w)$ corresponds to an artificial problem with an infinite number of agents) the equilibrium price $P_\infty(x)$ depends continuously on total wealth W, the impact of total wealth fluctuations would have a negligible impact on equilibrium prices for $N \to \infty$.
2. Consider now the mapping $x = F_\xi(w)$, which is defined for any particular realization ξ on a discrete set of points $w_i(\xi)$ into the set of disjoint intervals in the space of locations. Agent n will have $n-1$ previous neighbours in the wealth space, and they will also be his previous neighbours in the space of locations. As we know, the individual wealth fluctuations would have an impact of order $1/N$ on wealth, the indeterminacy related to the price function $P_N(x)$ is also of order $1/N$ in every point (see Lemma 5.4). Consider now the impact of these fluctuations on the set of borders a_i. Note that for our utility function agents always spend a fixed part of their wealth on land. Since the quality of land varies continuously in space, the ratio of price differences to prices at each land slot and across neighbours would be small (see Lemma 4). The relative shift of the border $a_2(\xi)$ for any problem with $\{w_i(\xi)\}$ with respect to the problem with $\{w_i\}$ would be of order: $a_2(\xi) - a_2 = a_2(1 + O(1/N))$. If we consider the border a_i, its shift would not accumulate less than proportionally to N (the Lemma in Appendix 3 proves the order of this estimate to be $N^{3/4}$ (interplay of independency of fluctuations with "slow" variation in space due to continuity). Hence, $a_i(\xi) - a_i = a_i(1 + O(N^{-1/4})$. This will allow us to estimate the norm of the difference across solutions for different ξ with any small ϵ when N is high enough. The part of the norm associated with the average difference of the borders for a_i for different realizations will vanish to zero as $N^{-1/4}$, when $N \to \infty$. This gives an estimation of the speed of the convergence to zero and also proves convergence.

PROOF OF LEMMA 5.4.
Let $W \equiv \sum w_i$. When $N \to \infty$, for each i: $w_i/W \to \infty$. If the land quality is bounded by positive constants both from below and from above, everybody can buy only an infinitely small interval $|b_i - a_i| \to 0$. Because $v(x)$ is a continuous function, $\forall i : |v(a_i) - v(b_i)|/v(a_i) \to 0$. Since $P_\infty(x) = Av^{1/\alpha}(x)$

and since $P_{N,k}(a_i) = P_\infty(a_i)$, we can also conclude that $\forall k, \forall x \in [a_i, b_i], \forall i$: $|P_{N,k}(x) - P_\infty(x)|/P_\infty(x) \to 0$. We can integrate this limit over x, getting:

$$\frac{\int_0^1 |P_{N,k}(x) - P_\infty|dx}{\int_0^1 P_\infty dx} \to 0. \tag{5.28}$$

Taking the maximium over k, we get finally: $\epsilon_N \to 0$.

PROOF OF LEMMA 5.5.
Consider a finite but large number of agents. The utility of an agent owning an interval $[y, z]$ is $U = CV$, where $V \equiv \int_y^z v(x)dx$ - the utility factor derived from land. Consider the change in V when the border z moves marginally. Let $z - y = \Delta x$. We have:

$$\frac{dV}{dz} = v(z)(\Delta x)^\alpha + \alpha(\Delta x)^{\alpha-1} \int_y^z v(x)dx. \tag{5.29}$$

Now we can use a Taylor expansion for v: $v(x) = v(y) + v'(y)(x-y) + O((x-y)^2)$. Substitution gives an asymptotical expression for the derivative:

$$\frac{dV}{dz} = [v(z) + \alpha v(y)](\Delta x)^\alpha + \frac{\alpha v'(y)}{2}(\Delta x)^{\alpha+1} + O((\Delta x)^{\alpha+2}). \tag{5.30}$$

When the number of agents goes to infinity, the last formula represents an asymptotical series of dV/dz in the powers of Δx. The first term describes the quantity effect, since quality enters it only in border points. It becomes infinitely times more important, and the quality effect can be asymptotically neglected in comparison with the quantity effect.

5.7.2 An Example Revealing the Impact of Congestion

The following example aims to reveal the difference between the market clearing condition and the mass preserving condition, and its role for escaping Berliant's paradox about the non-convergence of a sequence of discrete models to a continuous one.

Consider a continuum of agents, indexed by $i \in [0, 1]$. Let the set of differentiated commodities be represented by the same interval and indexed by j. Let preferences of agents be given by the expression:

$$U_i(j) = 1 - (j - i^2)^2. \tag{5.31}$$

Then every agent would maximize his utility at a point $j = i^2$. The market clearing condition at every point tells that every variety is chosen by only one agent. The equilibrium allocation will be represented by a map: $i \to i^2$. This allocation may be supported by a zero price, since there is no competition in any of the markets.

Consider now a sequence of discrete problems, with agents in a finite number N of points $x_i = i/N$, and varieties $y_j = j/N$. It is easy to see that the continuous solution is no longer sustainable, since in the limit half of agents $x_i \in [0; 1/2]$ would map into a quarter of the total number of varieties $y_j \in [0; 1/4]$. We have a congestion property of this mapping. In the discrete case competition in a congested area can be regulated by higher prices, and thus agents will not be able to realize their first best choices.

A proper way to deal with such a model will be to introduce a numeraire C, give agents initial allocations of it, and all land to one big owner, who does not value it. Then utilities will be $U_i(j) = C_i - (j - i^2)^2$ for "small" agents, and $U = C$ for the big owner. The usual market clearing condition fails to capture congestion (still every agent is choosing the different point). Thus it has to be replaced by a mass preserving condition, which is one of the important parts of the general equilibrium concept throughout this paper.

5.7.3 Two Equivalent Mathematical Representations of Wealth Distribution

A solution to the considered problem includes the mapping from the space of individual wealths to locations. This mapping assigns to each individual with w_i some location x_i in space. Since the solution is constructed by using methods of differential calculus, it is necessary to define the mapping in all intermediate points, i.e. to go from a discrete to a continuous mapping. But in the beginning we had an infinity of agents, with a discrete wealth distribution. It is possible to formally associate a smooth function with small support around the central point of wealth, and then add these functions to get a continuous wealth distribution. The possibility to obtain this equivalence lies in the fact that the space of all continuous functions is only infinite-dimensional (on contrast to the space of all measurable functions); this can be easily shown by Fourier transformation.

A formal way to proceed from discrete to continuous wealth distributions is the following. Assume that agents do not have a certain wealth, but a wealth distributed with probability density $f(w)$ in the interval $(w_{i-1} + w_i)/2, (w_i + w_{i+1})/2$. We can start from a uniform $f(w)$ in each of these intervals, but then use the spline method to create a differentiable $f(w)$. It is important that the difference between all solutions, corresponding to different $f(w)$ reconstructed from discrete w_i, will be negligibly small as $N \to \infty$.

Suppose that we have a set of discrete models with an increasing number N of agents. Let the set of individual wealths be obtained as a realization of the theoretical wealth distribution with smooth properties. When statisticians work with hystogramms, it is optimal to define the number of wealth brackets as $\sqrt{N}$, where N is the number of observations. Then the law of big numbers would push to zero the relative fluctuations inside each wealth bracket, and the wealth density would also change continuously between brackets.

The following Lemma proves that the fluctuations of borders between agents are small for all realizations of discrete wealth distribution, given by a continuous density function. This is a formal justification of the asymptotical validity of double representation of wealth distribution (by sequence and continuous function) for big N.

Lemma 5.7 *Let $\{\psi_i(\xi), i = 1, ...N\}$ be a set of interval lengths chosen by agents in any particular order preserving solution for the problem with a random realization of wealths $\{w_i(\xi), i = 1, ...N\}$, and let $\{\psi_{i,0}\}$ be a "central" solution, associated with an initial discrete wealth distribution. Then for all the borders between agents' land allocations in equilibrium we get an estimate: $a_{i,\xi} = a_i(1 + O(N^{-1/4}))$.*

PROOF: 1. We can represent $\psi_i(\xi) = \psi_i + u_i(\xi)$, where u_i is a random variable and ξ fixes its realization. Since land size in each location depends continuously on wealth and since land quality also depends continuously on location leading to price continuity, u_i are independently distributed. Without large error, we can assume that they are normally distributed with zero average and standard deviation σ_i proportional to c_i/N. This is because the average interval size is $1/N$, and c_i is some constant of order 1 for fixed i

and a slowly varying function of i.
2. Since $a_{i+1,\xi} = \sum_{k=1}^{i} \psi_k(\xi) = \sum \psi_{k,0} + \sum u_k(\xi) = a_{i+1,0} + \sum_{k=1}^{i} u_k(\xi)$, *the difference between borders is small if the sum of* u_k *is small. Let us divide all sum into several sums of the length* $\sqrt{N}$. *For large* N, *due to continuity, the variances of each* u_k *are almost the same, and we can use it to apply the theorem of the sum of independently distributed variables: the variance of* $\sum_{k=1}^{l} u_k$ *will be proportional to the length of the sequence* l. *Hence, the standard deviation of this sum will be of order* $O(N^{-3/4})$ *(each standard deviation is already proportional to* $1/N$*; the term associated with the small differences in variances of the elements of this sample is of order* $O(N^{-3/2})$ *and thus can be neglected). Since we have not more than* $\sqrt{N}$ *sequences (each sequence had less than* N *elements originally), we finally get an estimate:* $a_{i,\xi} - a_i = a_i O(N^{-1/4})$.

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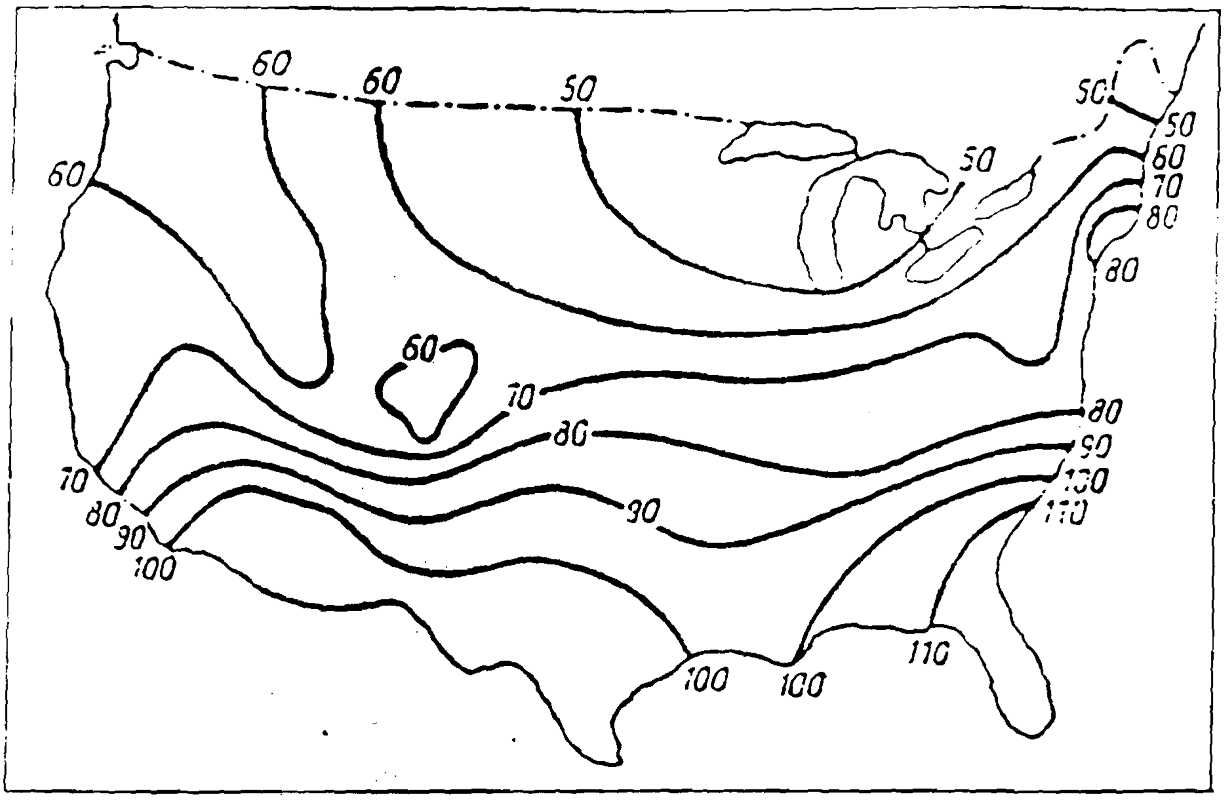


Fig. 2.1. **Producer prices for potatoes** in the United states, in cents per bushel on December 1, **for 1906-1915.** (From H. Working "Factors Determining the Price of Potatoes in St.Paul **and Minneapolis**, 1922.)

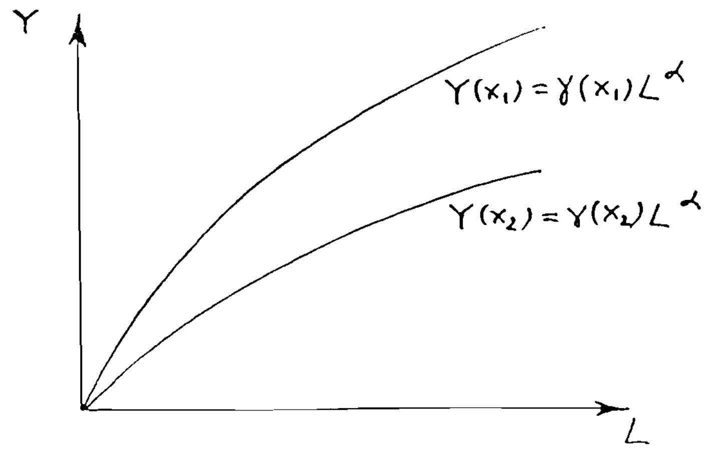


Fig. 2.2. The output curves.

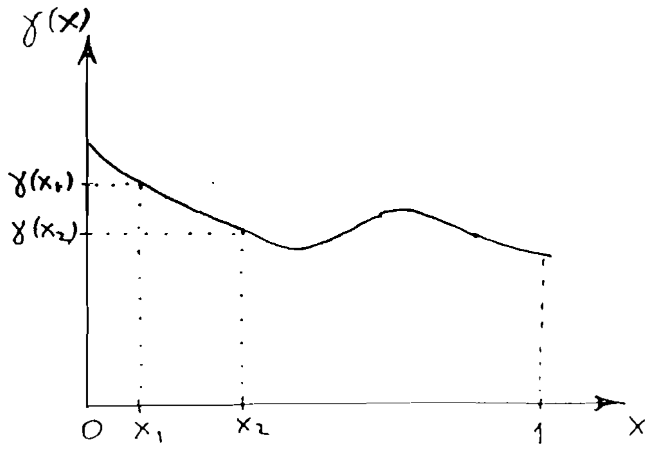


Fig. 2.3. Productivity parameter $\gamma(x)$.

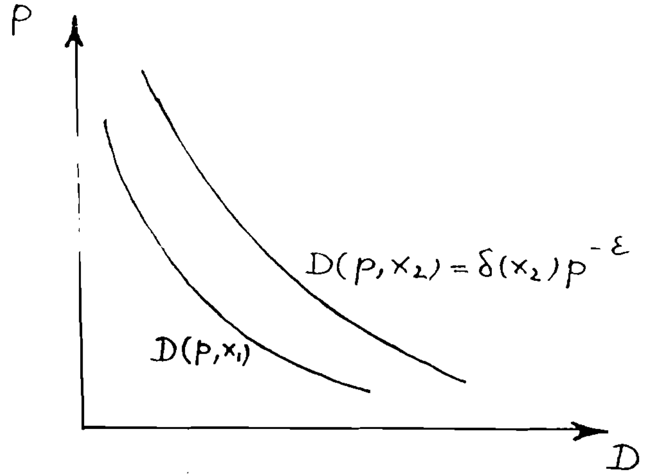


Fig. 2.4. The demand curves.

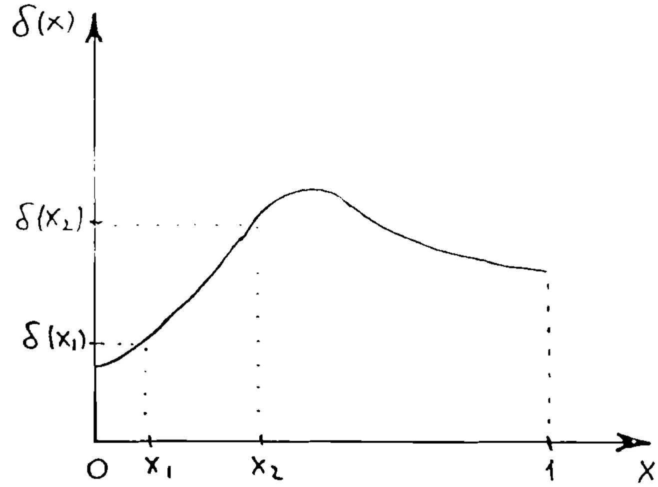


Fig. 2.5. The demand density $\delta(x)$.

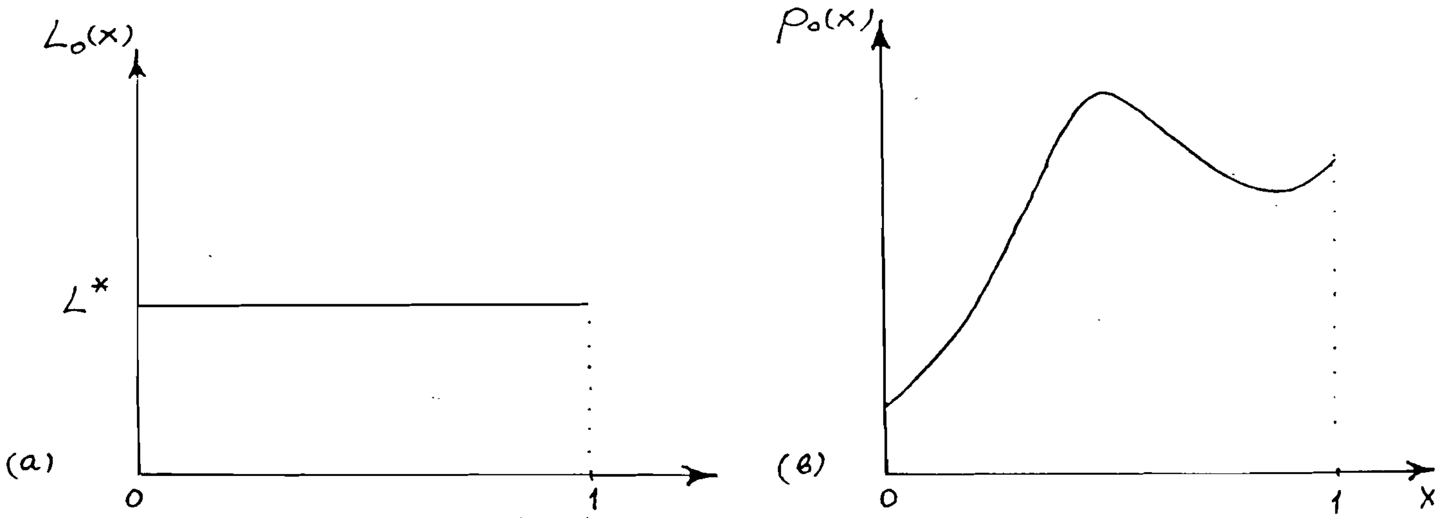


Fig. 2.6. The autarky solution.
(a) Equilibrium labor supply.
(b) Equilibrium price as the function of location x.

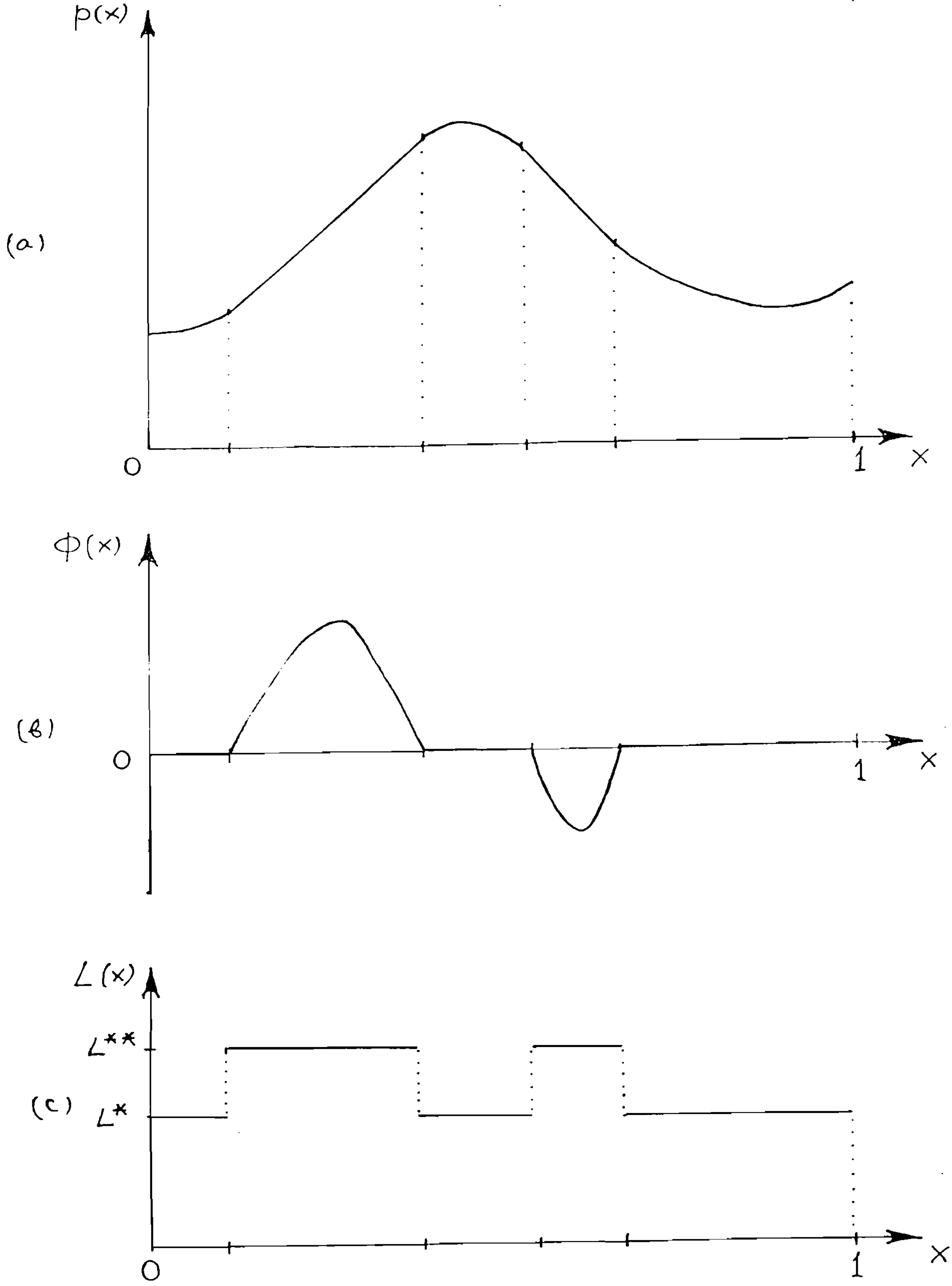


Fig. 2.7. The general behaviour of the solution for the integrated economy.
(a) Equilibrium pricing.
(b) Equilibrium trade flow
(c) equilibrium labor supply

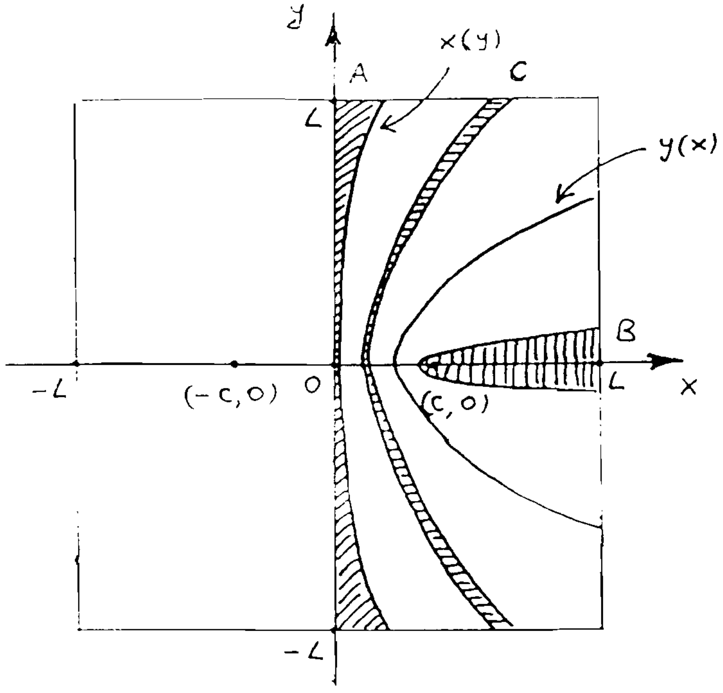


Fig. 3.1. Illustration for the proof of Lemma 2. The shaded areas show consumers, shifting between firms for small price increase/decrease, when:

A - prices are almost equal
B - price difference is near critical
C - an intermediate case

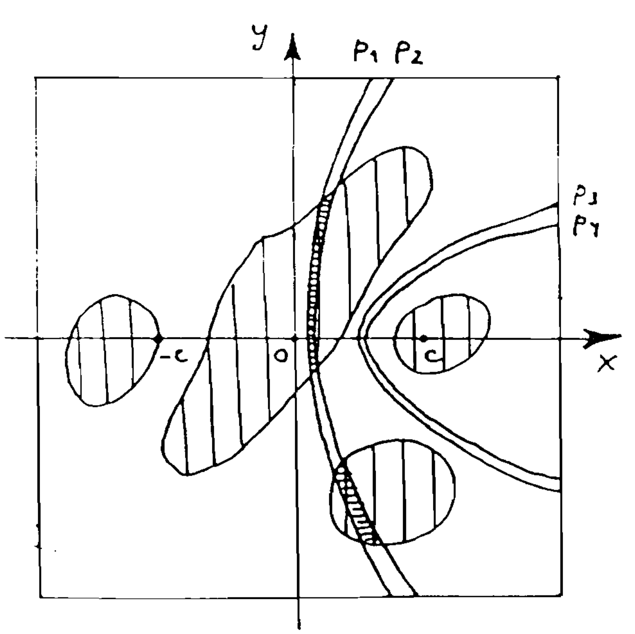
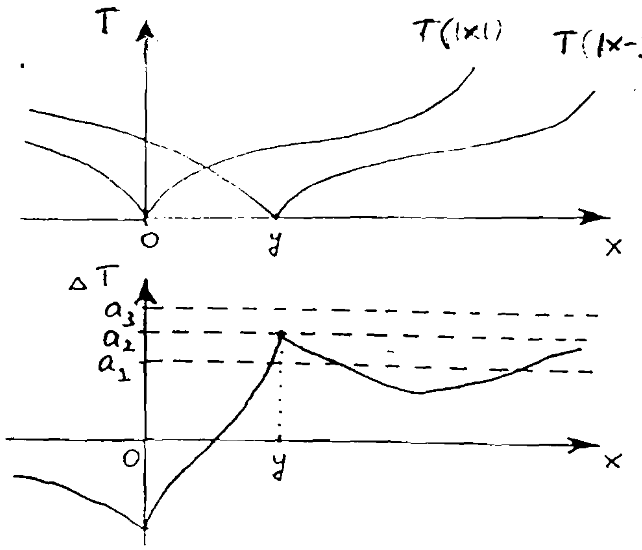


Fig. 3.2. Illustration for the proof of Theorem 1. Demand is continuous when consumers are inside any compact set of **R2**. Price shift may affect the aggregate demand from one firm (p1 → p2) or may not (p3 → p4).



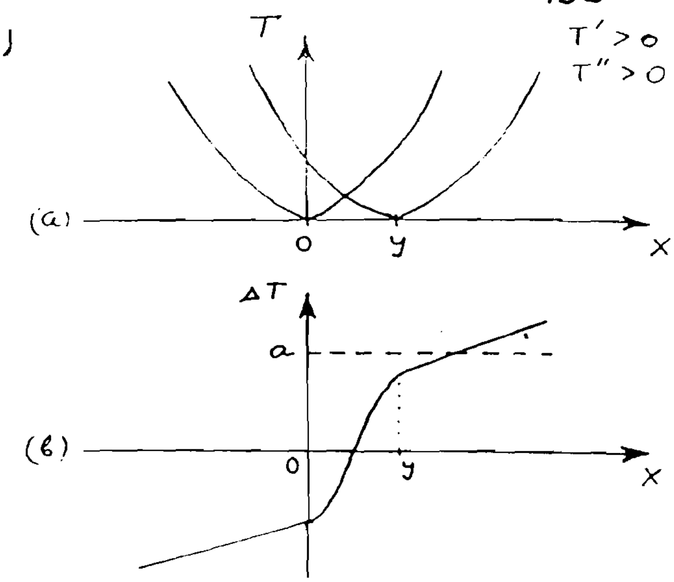
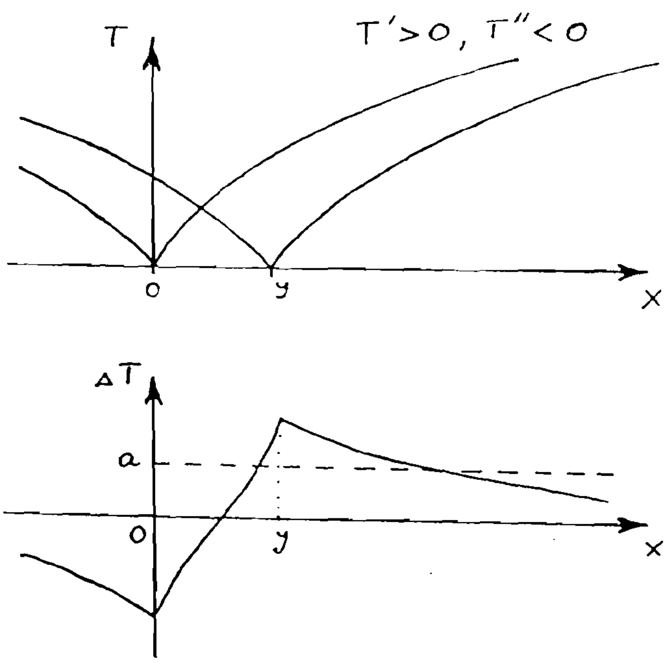


Fig. 3.3. For different values of parameter **a** there may be 0, 1 or more solutions.

Fig. 3.4. Convex transport cost. The solution to the equation **ΔT = a** is unique.



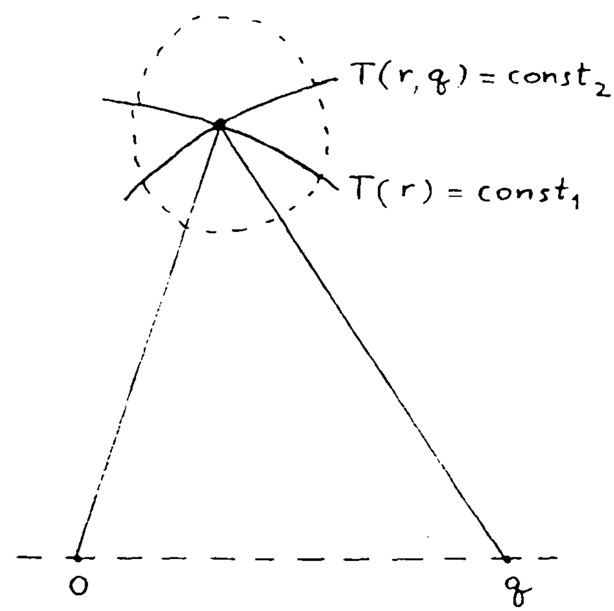


Fig. 3.5. Concave transport cost. The solution to the equation **ΔT = a** may be not unique.

Fig. 3.6. Iso-transport cost curves: $\mathbf{T(r) = c_1}$; $\mathbf{T(r,q) = c_2}$. Firms are located in the points **0** and **q**.

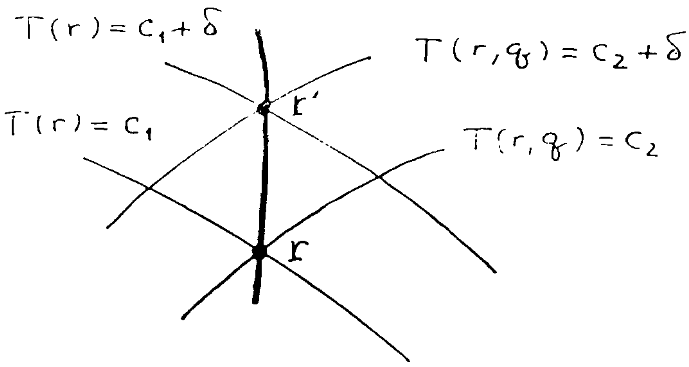


Fig. 3.7. The uniqueness of the solution curve,

passing through any solution.

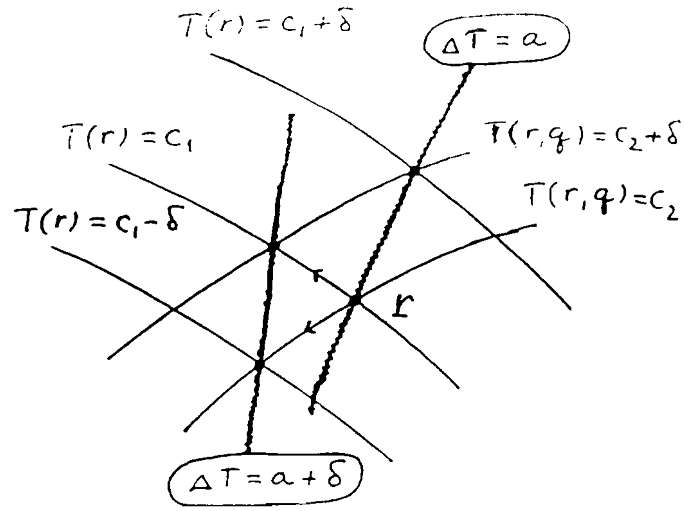


Fig. 3.8. The curve of indifferent consumers

depends continuously on price change.

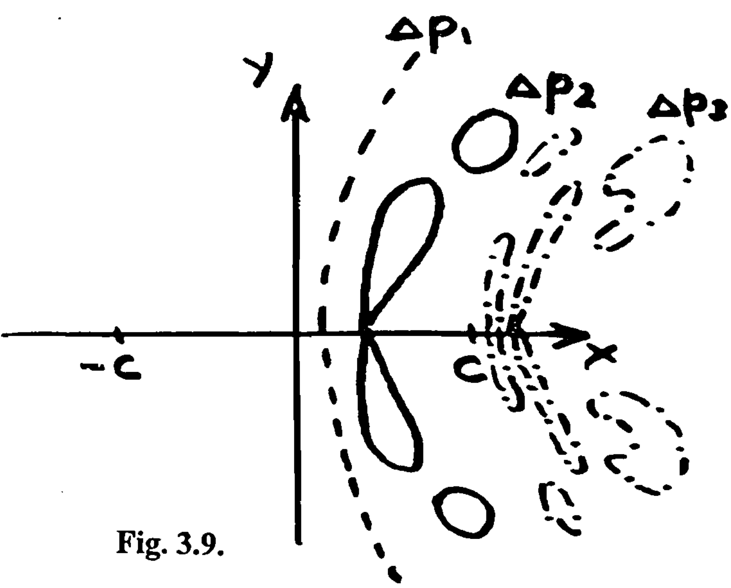


Fig. 3.9.

A set of indifferent consumers. ΔT =Δp .

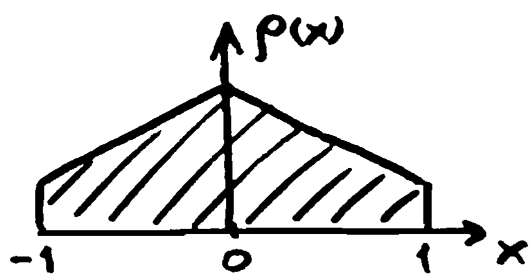
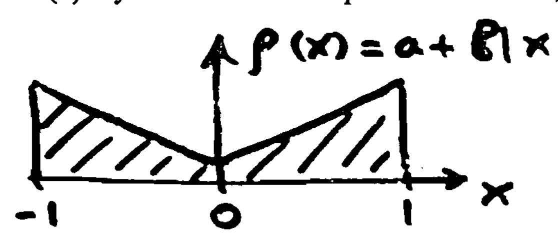


Fig. 3.10. The density functions, for which:
(a) Symmetric Nash equilibrium exists,



(b) Symmetric Nash equilibrium does not exist

(cornering dominates, if **b/a>2**).

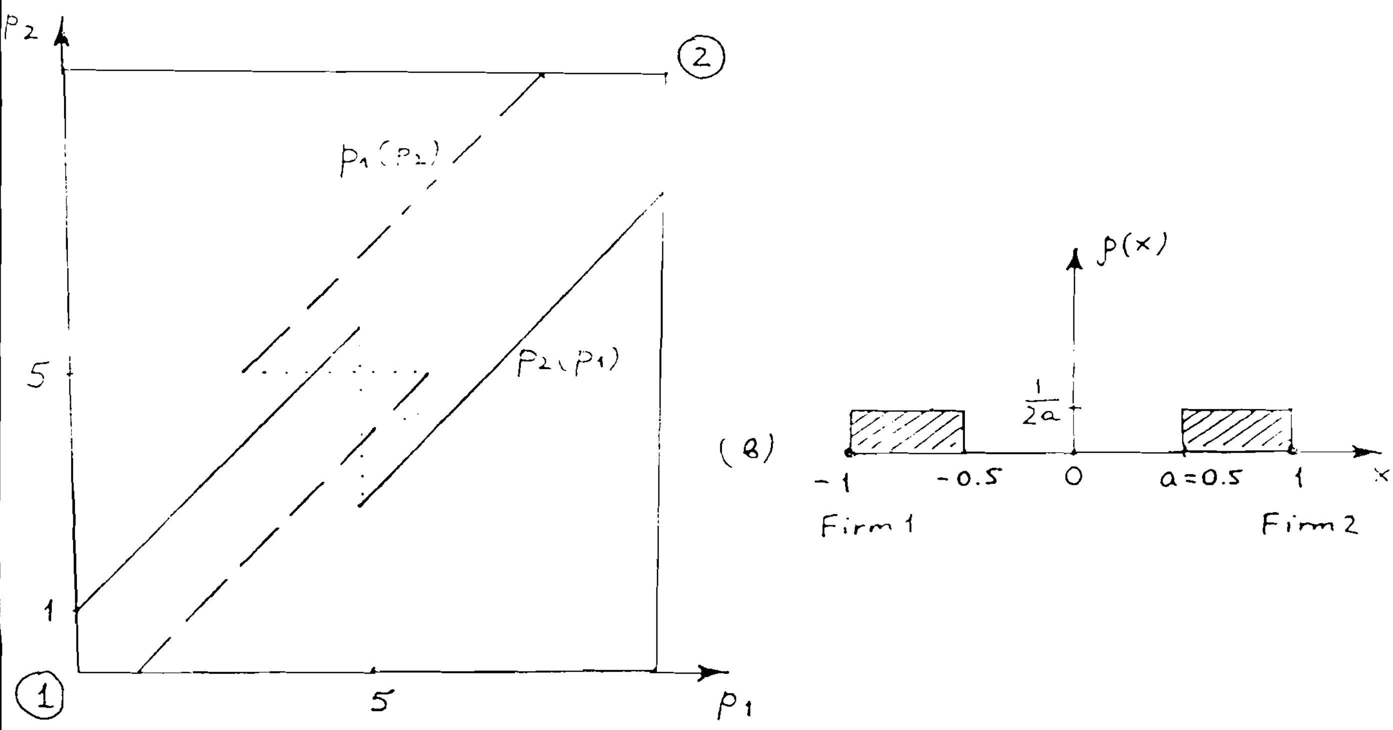


Fig. 3.11. Nash equilibrium in pure strategies does not exist (a), when the distribution of consumers ρ has a form (b).

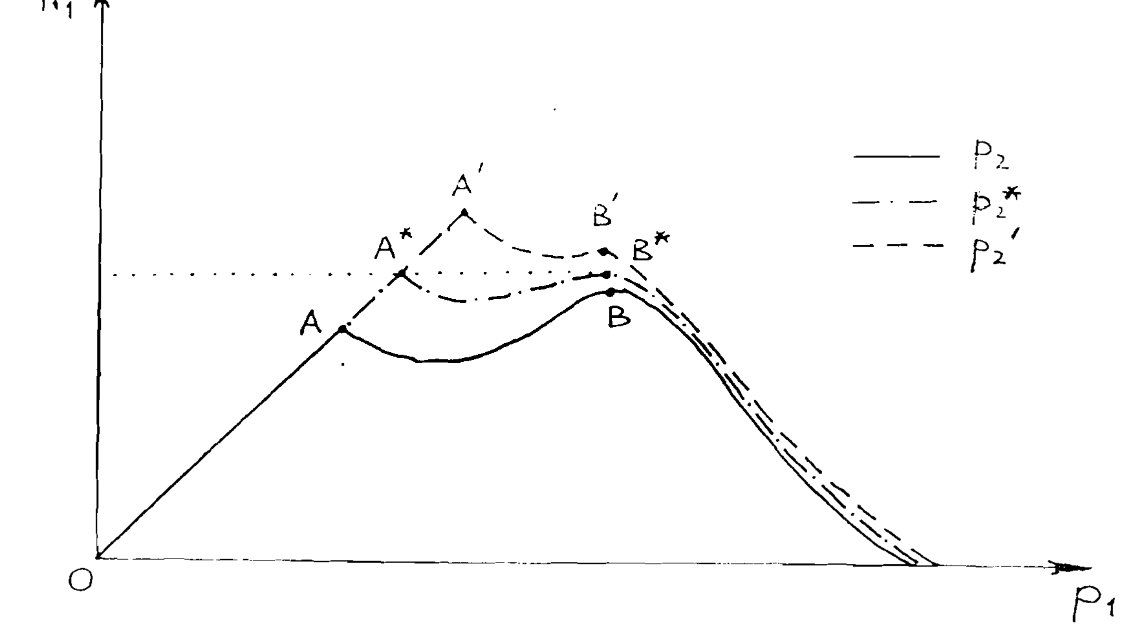


Fig. 3.12. Numerical calculations show the possibility of a discontinuous shift across the two local maxima **A***, **B***, when the rival's price takes the critical value $\mathbf{p_2^*}$.

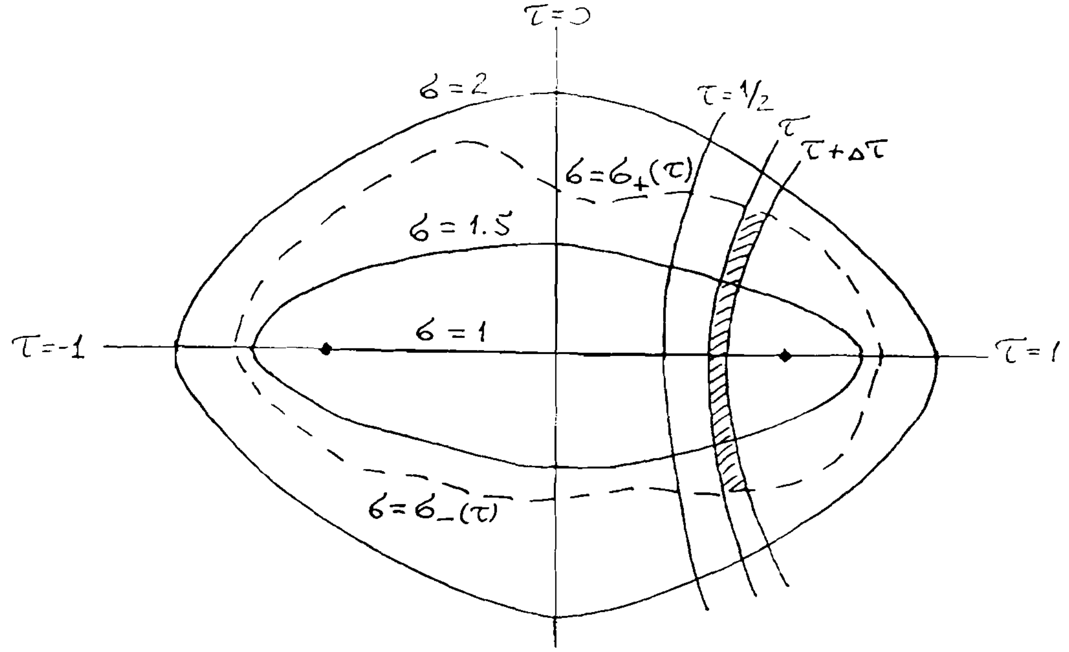


Fig. 3.13. Elliptic coordinates: σ, τ . Integration of $\rho_2(\tau, \sigma)$ along τ = **const** gives one-dimensional density $\rho_1(\tau)$.

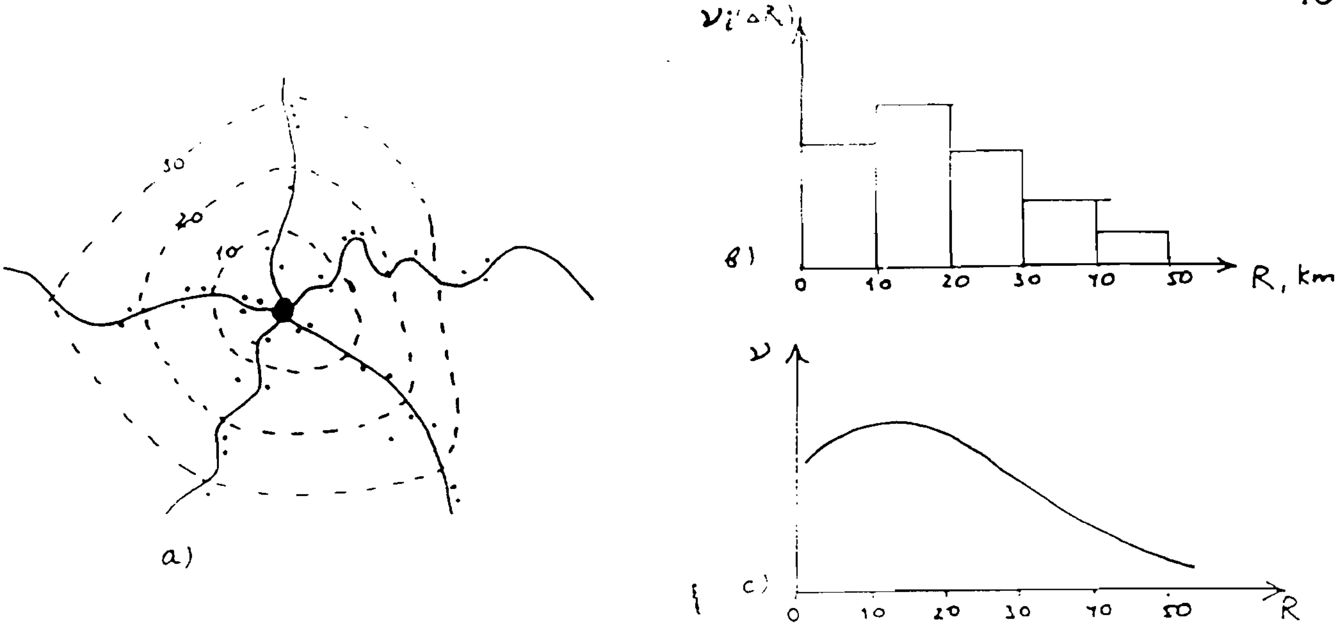


Fig. 4.1. Reconstructing density $\nu(R)$ from real geographical maps.
(a) Map with a city, roads, dacha lots and equilibrium isolines;
(b) Discrete density: quantity of lots inside distance ranges;
(c) Continuous density, which is the limit of discrete density.

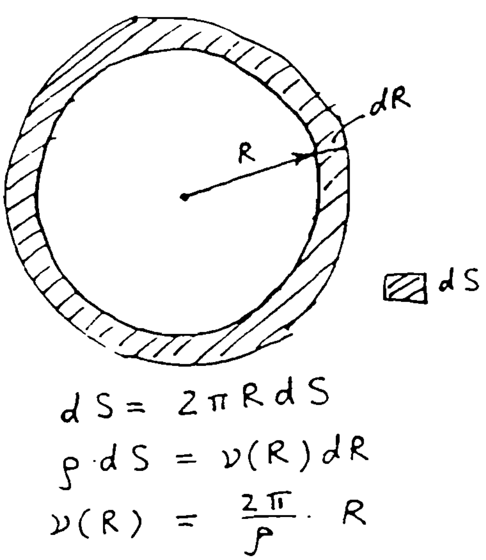


Fig. 4.2. Radially symmetric model. Derivation of radial density.

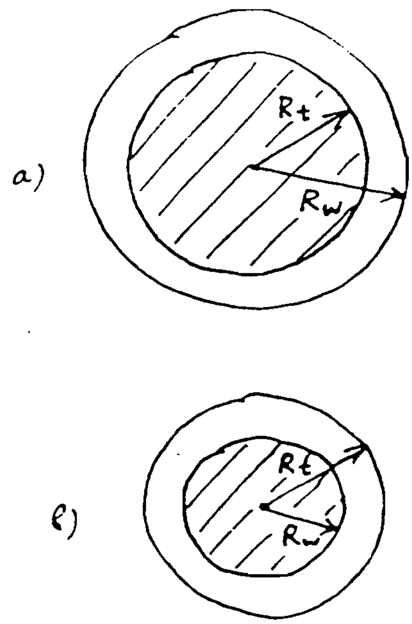


Fig. 4.3. Endogeneous dacha area, when:
(a) time is binding ($R_t < R_w$);
(b) income is binding ($R_w < R_t$).

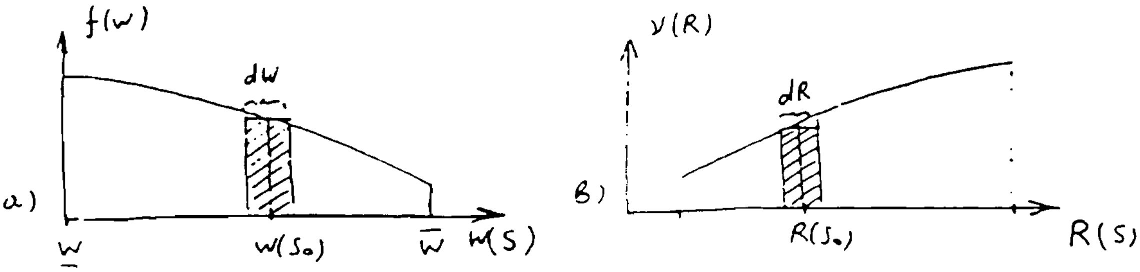


Fig. 4.4. Mapping from the space of types s via income distribution **w(s)** into geographical location **R(s)**. The law of mass preserving implies: **f(w)dw = ν(R)dR**.

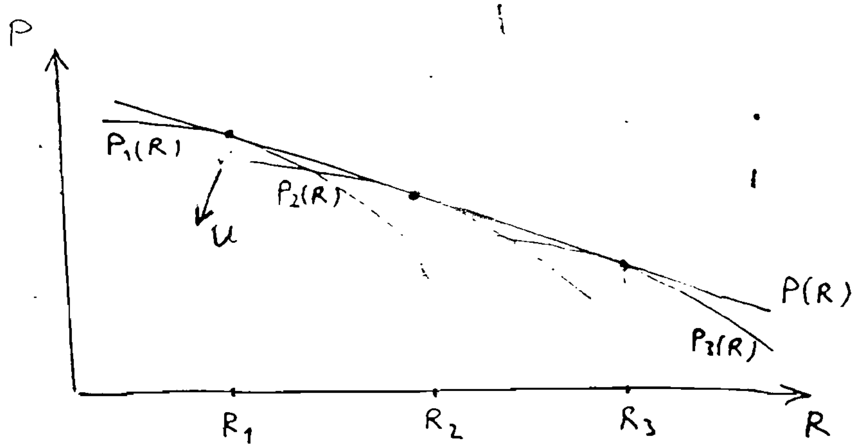


Fig. 4.5. Individual price schedules $\mathbf{P_i(R)}$, = 1, 2, 3, making agents indifferent across locations, and equilibrium price schedule **P(R)**.

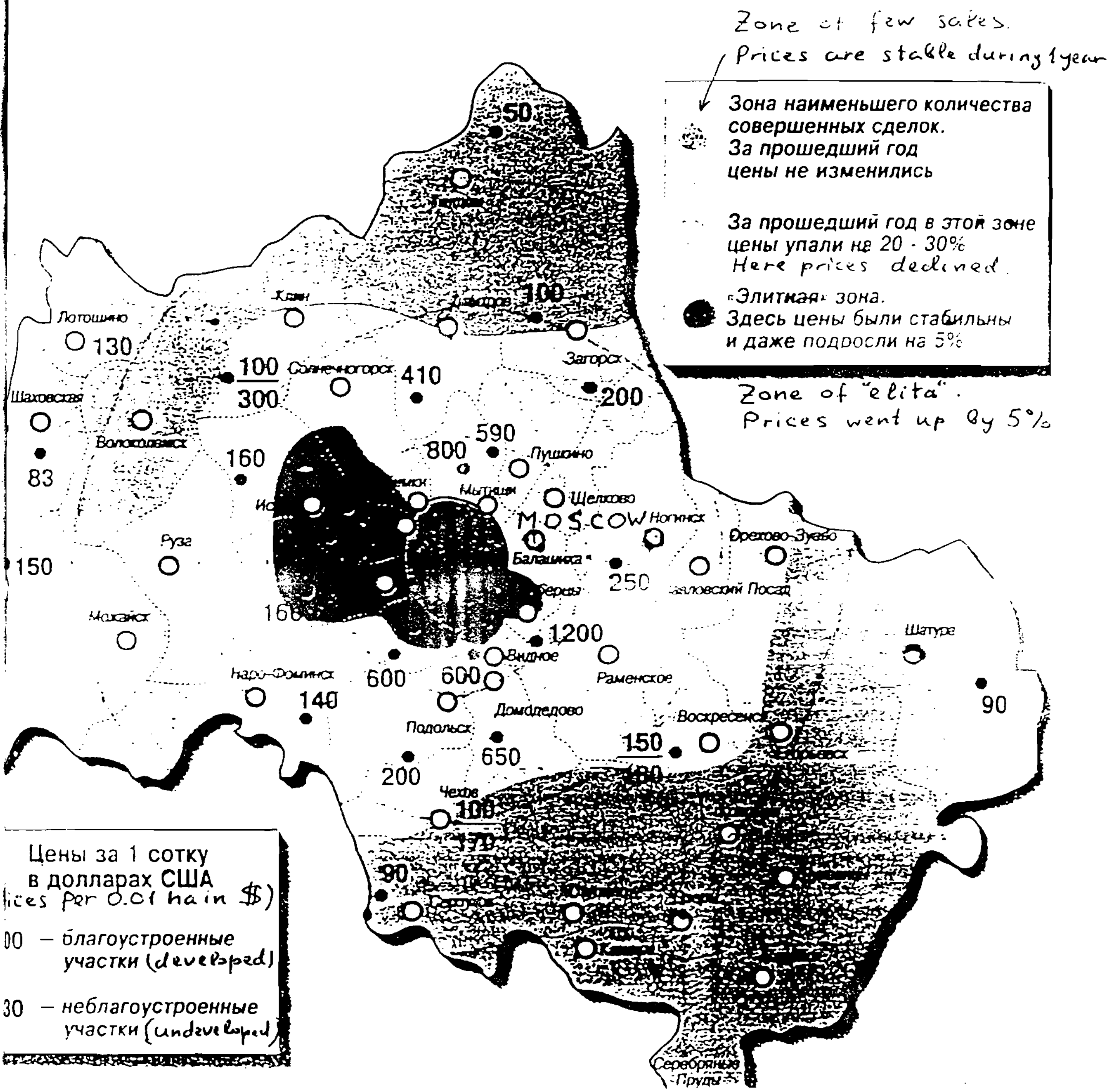


Fig. 4.6. Map 1. Moscow district. Prices per 100 sq.m inUSD. Dark color shows the zone of "elita", where the prices were growing from 1994 to 1995. Gray color shows the zone of stable prices. The prices in white color zone have dropped by 20-30% during one year.

Fig. 4.7. Map 2. The region of St.Petersburg. White color shows the area of high demand (concentrated along roads). The graph shows the price per 100 sq.m in USD in 1996 along different directions (shown by color) at different distances from the city center (in km).

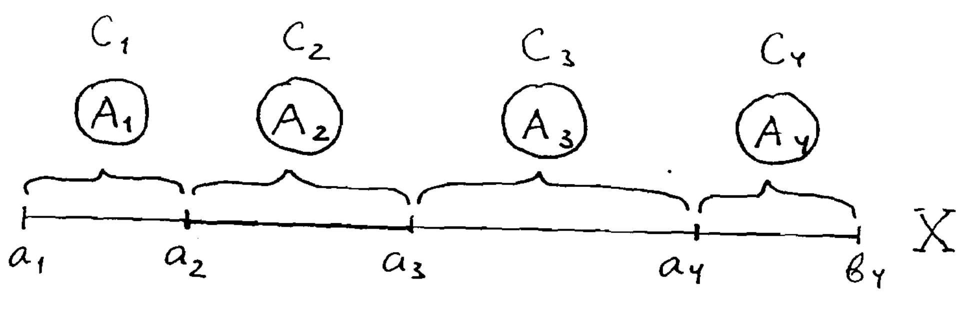


Fig. 5.1. Socially feasible allocation.

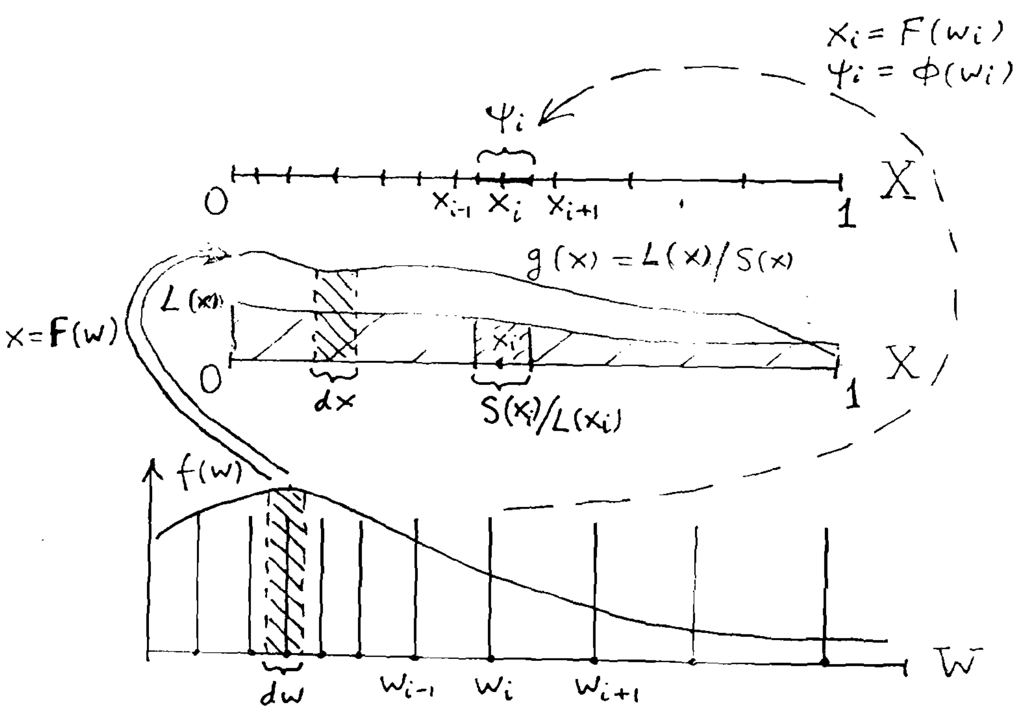


Fig. 5.2. Mappings $x_i = F(w_i)$, $\psi_i = \Phi(w_i)$, $x = F(w)$.
Mass preserving condition: $f(w)dw = g(x)dx$.

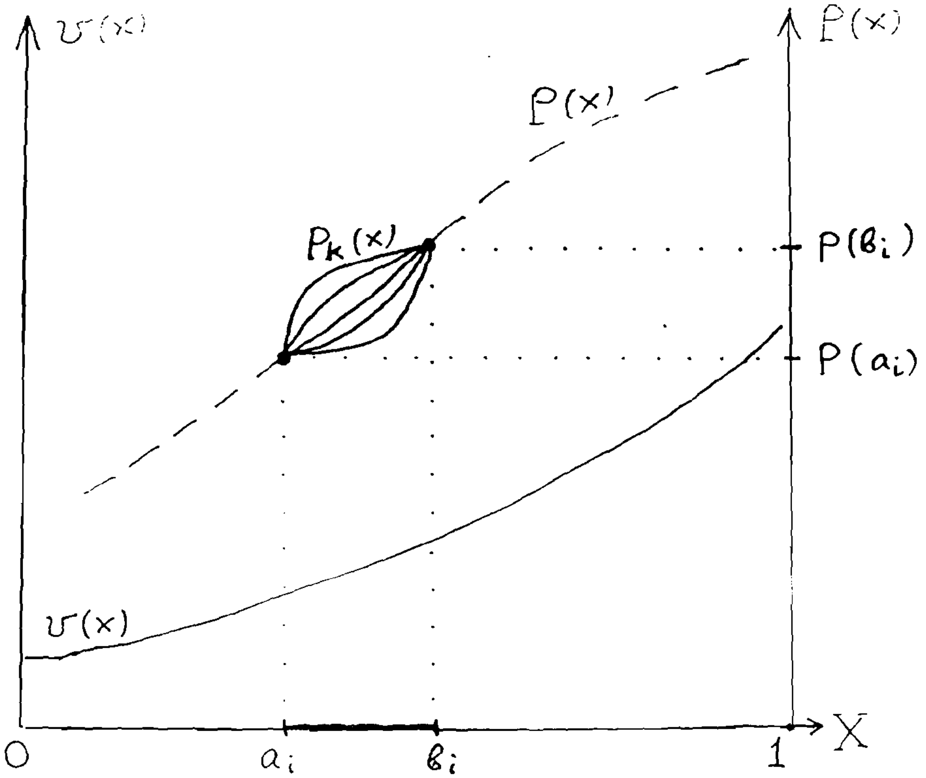


Fig. 5.3. An illustration why pricing of land **$P_k(x)$** has indeterminacy for finite N.

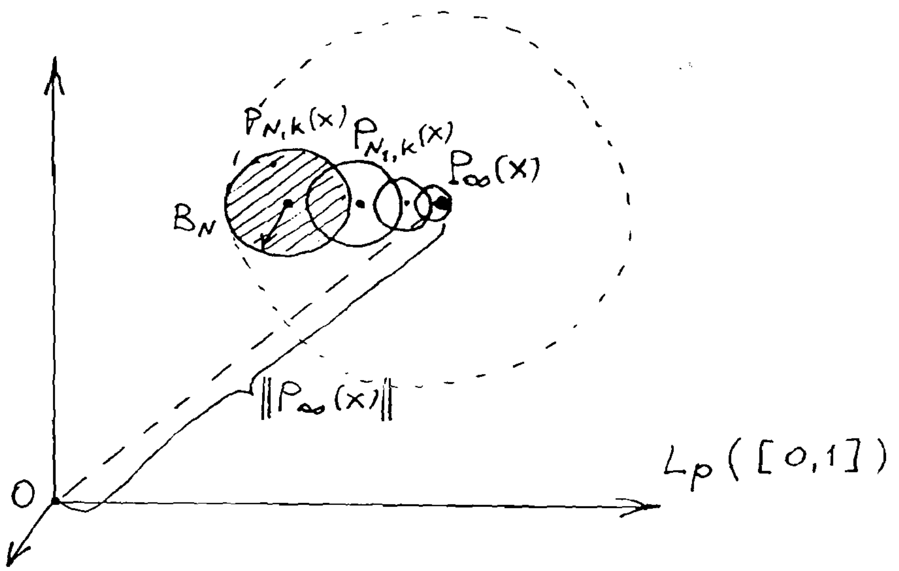


Fig. 5.4. The indeterminacy of price function **$P_N(x)$**, supporting a problem with finite N, can be represented as a "ball" **B_N** in functional space. When N goes to infinity , the ball's size goes to zero, and all its points stay close to **$P_\infty(x)$**. Indeterminacy asymptotically disappears.

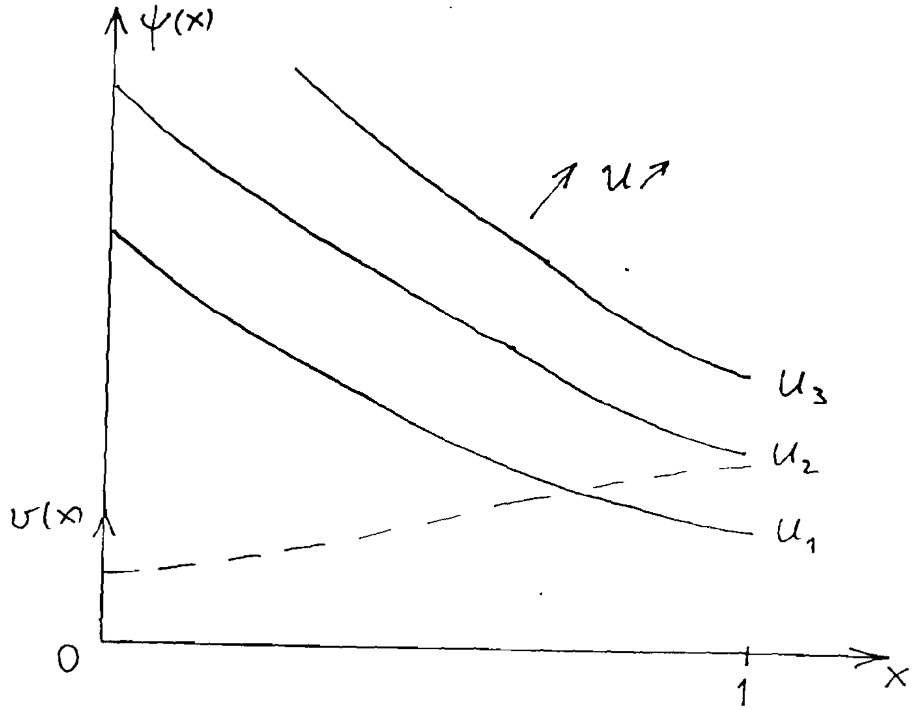


Fig. 5.5. Isoutility curves: $v(x)\psi^{\alpha}(x) = const.$

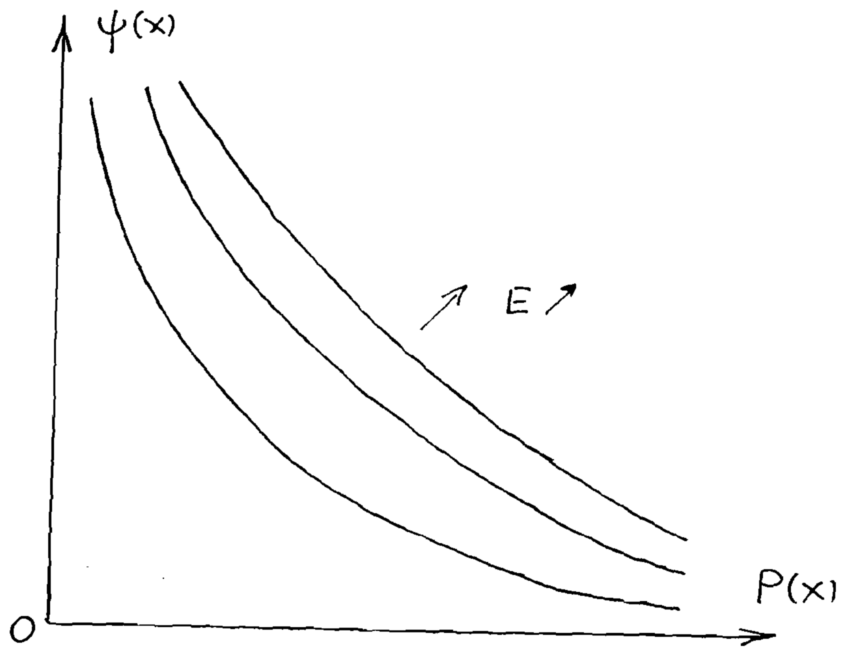


Fig. 5.6. Isoexpenditure curves: $P(x)\psi(x) = const.$