Universitat Ramon Llull

DOCTORAL THESIS

Title	**A CONTRIBUTION TO MULTIVARIATE VOLATILITY MODELING WITH HIGH FREQUENCY DATA**
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To my mother, Maria

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Abbreviations

AIG	American International Group
AXP	American Express
AR	Autoregressive
ARCH	Autoregressive Conditional Heteroskedastic
ARFIMA	Autoregressive Fractionally Integrated Moving Average
ARMA	Autoregressive Moving Average
BAC	Bank of America
BEKK	Baba, Engle, Kraft, and Kroner
Bivariate EGARCH	Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic
Bivariate EGARCH-X	Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure
Bivariate Realized EGARCH	Bivariate Realized Exponential Generalized Autoregressive Conditional Heteroskedastic
Bivariate Realized GARCH	Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic
Bivariate Realized GARCH(2,2)	Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic (2,2)
CHARMA	Conditional Heteroskedastic Autoregressive Moving Average

DJIA	Dow Jones Industrial Average
EGARCH	Exponential Generalized Autoregressive Conditional Heteroskedastic
EGARCH-X	Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure
FCGARCH	Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic
GARCH	Generalized Autoregressive Conditional Heteroskedastic
GARCH-X	Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure
HAR	Heterogeneous Autoregressive
HARCH	Heterogeneous ARCH
HAR-RV	Heterogeneous AutoRegressive Realized Volatility
HARST	Multiple Regime Smooth Transition Heterogeneous Autoregressive
HEAVY	High Frequency Based Volatility
H-L	High-Low
JPM	J.P. Morgan
LMSV	Long-Memory Stochastic Volatility
MA	Moving Average
MAE	Mean Absolute Error
MAPE	Mean Absolute Percentage Error
MEM	Multiplicative Error Model

M-GARCH	Multiplicative Generalized Autoregressive Conditional Heteroskedastic
MIDAS	Mixed Data Sampling
MLE	Maximum Log-Likelihood
OGARCH	Orthogonal Generalized Autoregressive Conditional Heteroskedastic
PC	Principal Component
PC Bivariate EGARCH	Principal Component Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic
PC Bivariate EGARCH-X	Principal Component Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure
PC Bivariate Realized EGARCH	Principal Component Bivariate Realized Exponential Generalized Autoregressive Conditional Heteroskedastic
PC Bivariate Realized GARCH	Principal Component Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic
PC Bivariate Realized GARCH(2,2)	Principal Component Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic (2,2)
PC-EGARCH	Principal Component Exponential Generalized Autoregressive Conditional Heteroskedastic
PC-EGARCHX	Principal Component Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure

PC-GARCH	Principal Component Generalized Autoregressive Conditional Heteroskedastic
PC-Realized EGARCH	Principal Component Realized Exponential Generalized Autoregressive Conditional Heteroskedastic
PC-Realized GARCH	Principal Component Realized Generalized Autoregressive Conditional Heteroskedastic
PC-Realized GARCH(2,2)	Principal Component Realized Generalized Autoregressive Conditional Heteroskedastic (2,2)
RCA	Random Coefficient Autoregressive
Realized EGARCH	Realized Exponential Generalized Autoregressive Conditional Heteroskedastic
Realized GARCH	Realized Generalized Autoregressive Conditional Heteroskedastic
Realized GARCH(2,2)	Realized Generalized Autoregressive Conditional Heteroskedastic (2,2)
RK	Realized Kernel
RMSE	Root Mean Squared Error
RV	Realized Variance
S&P	Standard & Poor’s
SV	Stochastic Volatility
TARCH	Threshold Autoregressive Conditional Heteroskedastic

Chapter one: Introduction

The literature written on the topic of volatility forecasting has been lately enriched by the access to high frequency data, documented that it would enhance the modeling problem in a number of ways. As such, high frequency data allows a better understanding of dynamic properties of highly persistent volatility, it allows dissemination of announcements in markets that trigger shocks in volatility, it provides more accurate forecasts, and the realized measures computed from high frequency data were found as good estimators of conditional future volatility, and may boost the complex volatility models by reducing the parameter uncertainty. Due to the large contribution to volatility field of high frequency data topic, this work concentrates on further disseminating on the existing high frequency volatility models and the realized measures used in their compilation, as well as on proposing new opportunities of research by employing alternative models that would allow improved volatility estimation in univariate and multivariate assets. As such, it aligns to the two major lines of research in the literature, that of ranking models and proposing alternative methods to model conditional volatility.

The current thesis focuses on the topic of volatility forecasting of financial (stock) time series, and searches to reach three objectives. The first scope constitutes of the proposal of a new method of volatility forecasting that follows a recently developed research line that pointed to using measures of intraday volatility and also of measures of night volatility. It starts from an idea developed by Hansen, Huang and Shek (2010b), who proposed a partial form of a Bivariate Realized GARCH (Realized Generalized Autoregressive Conditional Heteroskedastic) model, and applies it to other realized GARCH-type models to obtain new bivariate versions of them. The need for new models is given by the question whether adding measures of night volatility[1] improves day volatility estimations. New models will be proposed: Bivariate EGARCH (Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic), Bivariate EGARCH-X (Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure), Bivariate Realized EGARCH (Bivariate Realized Exponential Generalized Autoregressive Conditional Heteroskedastic), Bivariate GARCH (complete form) (Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic)

[1] Defined as incremental increase or decrease of the price at the market opening as compared to the price at the market closing the previous trading day

and Bivariate Realized GARCH(2,2) (Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic (2,2)).

The second scope is to propose a methodology to forecast multivariate day volatility with autoregressive models that use day and night volatility estimates, as well as high frequency information when that is available. For this, the Principal Component (PC) algorithm will be applied to the univariate realized GARCH-type of models discussed in chapter three (in first instance), and to the bivariate models proposed in chapter five (in second instance). This method was inspired from Burns' (2005) PC GARCH (Principal Component Generalized Autoregressive Conditional Heteroskedastic), that estimated a multivariate simple GARCH model by estimating univariate GARCH models of the principal components of the initial variables. In first instance, following the methodology of Burns, the Principal Component algorithm will be applied to a multivariate EGARCH, EGARCH-X, Realized GARCH, Realized EGARCH and Realized GARCH(2,2) model, as such there would result new versions of the PC-GARCH type of models that would accommodate not only to day, but also to intraday volatility data. New models will emerge: PC EGARCH (Principal Component Exponential Generalized Autoregressive Conditional Heteroskedastic), PC EGARCHX (Principal Component Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure), PC Realized GARCH (Principal Component Realized Generalized Autoregressive Conditional Heteroskedastic), PC Realized EGARCH (Principal Component Realized Exponential Generalized Autoregressive Conditional Heteroskedastic), and PC Realized GARCH(2,2) (Principal Component Realized Exponential Generalized Autoregressive Conditional Heteroskedastic (2,2)), that will be used to forecast multivariate volatility by using high frequency information obtained from intraday stock returns.

In second instance, the contribution will be the application of the PC algorithm to multivariate realized-GARCH models so that it would allow the multivariate asset volatility forecasting with bivariate autoregressive models proposed in chapter five, that allow using day, intraday and night volatility data. New models will result: PC Bivariate EGARCH (Principal Component Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic), PC Bivariate EGARCHX (Principal Component Bivariate Exponential Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure), PC Bivariate Realized GARCH (Principal

Component Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic), PC Bivariate Realized EGARCH (Principal Component Bivariate Realized Exponential Generalized Autoregressive Conditional Heteroskedastic), PC Bivariate Realized GARCH(2,2) (Principal Component Bivariate Realized Generalized Autoregressive Conditional Heteroskedastic (2,2)).

The third goal of the thesis is to test the gain or loss in performance of existing or newly proposed volatility forecasting models, as well the accuracy of the intraday measures used in the estimations of the realized models. For this purpose, based on the estimations, there will be undertaken rankings of the models, also of the intraday measures used. With regard to the models, there will be ranked the realized models already proposed (together with the simple EGARCH model), following different criteria and methodologies. The scope is to see if some models persistently rank better or worse, across the various types of ranks. As well, there will be ranked the bivariate realized models, and also the bivariate versions will be compared to the univariate ones in order to investigate whether using night volatility measurements in the models' equations improves volatility estimation or not. Finally, the PC realized models and PC bivariate realized models will be estimated and their performances will be ranked; improvements the PC methodology brings in high frequency multivariate modeling of stock returns will also be discussed.

Specifically, the scopes of the work may be disseminated from the resumed description of the chapters that follows.

In chapter two it will be undertaken a literature review of the topic, starting from the papers that proposed the basic models to the more recent papers that include the foundation of the current work. The description explains the development of the field towards integrating high frequency data in the estimation exercise, the need for accessing measures of volatility that would describe the latent phenomenon in shorter than a day periods, offering a conceptual framework of the attempt to propose new models for estimating and forecasting volatility. The ranking addresses to the new models' power assessment, attempting to offer a contribution to the existing debate on the identification of the prevalent models.

The third chapter considers two types of models: two simple models that do not use measures of intraday volatility (GARCH(1,1) (proposed by Bollerslev (1986) and EGARCH(1,1)) (proposed by Nelson (1991)), and four models that incorporate measures of intraday volatility (EGARCH-X(1,1) (a version of the GARCH-X model proposed by Engle (2002)), Realized GARCH(1,1), Realized EGARCH(1,1) and Realized GARCH(2,2) of Hansen, Huang and Shek (2010a). These models were already proposed, the scope of this chapter being to estimate and rank them for the four stocks (AIG - American International Group, AXP - American Express, BAC - Bank of America and JPM – J. P. Morgan) considered, as well as to rank the six measures of intraday volatility used (realized kernels, high-low, and realized variance sampled at 15 seconds, 5 minutes, 15 minutes, and 20 minutes). The models will be estimated in sample and out of sample, for each of the realized measures mentioned above and the performance will be measured by the size of their maximized loglikelihood functions and also by the values of three loss functions (RMSE – Root Mean Squared Error, MAE – Mean Absolute Error and MAPE – Mean Absolute Percentage Error). In order to make the rankings comparable across criteria, the size of the loglikelihood and loss functions will be normalized. The rankings will be combined in order to build up a general ranking for all the estimations made.

The fourth chapter, in its first instance, will propose new methods that attach an algorithm (Principal Component Analysis) to a class of multivariate realized and non-realized models, offering an improved alternative to the multivariate volatility estimation of stocks. The method takes advantage of a methodology proposed by Alexander (2000) for Orthogonal GARCH, also of a methodology proposed by Burns (2005) that indicated a method to estimate a multivariate GARCH model by estimating univariate GARCH models of the principal components. The method has been called PC GARCH. This chapter will adapt this methodology to the class of models that uses measures of intraday volatility, also to a simple model (EGARCH), the result being a new PC-class of models: PC EGARCH(1,1), PC EGARCHX(1,1), PC Realized GARCH(1,1), PC Realized EGARCH(1,1) and PC Realized GARCH(2,2).

In its second stance, the fourth chapter will comprise a performance assessment of the new PC models. As such, the models will be estimated in sample and out of sample, for each of the intraday volatility measures considered (realized kernels, high-low, realized variance sampled at 15 seconds, 5 minutes, 15 minutes, and 20 minutes), and their accuracy will be assessed by the size of the three loss functions considered (RMSE, MAE and MAPE). Models will be compared to their peers (the other PC models) and will be ranked. As well, the six measures of intraday volatility will be assessed with regards to the size of the loss functions of the PC-models they will be used for. In order to combine the rankings across the criteria, the loss and loglikelihood functions will be normalized for making them comparable.

The fifth chapter will have, as well, a dual scope. The first one is to propose new bivariate models that use not just measures of day volatility, but also measures of night volatility. The idea sparked from the observation that the price at the stock market closing of one share differs from the price of the same share at the opening of the stock market the next day. This suggests that during the night there also exists latent volatility, although no trades take place when the market is closed. As such, a bivariate model that would encumber measures of day and intraday volatility and also of night volatility, could be formulated and estimated, as it could improve the estimation of day volatility. The first such model was proposed by Hansen, Huang and Shek (2010b), being a partial version of a Bivariate Realized GARCH model, with an exogenous realized measure. This chapter continues the idea of Hansen, Huang and Shek (2010b) and proposes the formulation of the bivariate version of one simple EGARCH model, as well bivariate versions of the realized models discussed in chapter three. As such, new models will emerge: a Bivariate EGARCH, a Bivariate EGARCH-X, a Bivariate Realized EGARCH, a Bivariate GARCH model (in its complete form – with an endogenous realized measure) and a Bivariate Realized GARCH(2,2) model. These models' estimation will differ from that of the models in chapter three by the fact that it will not be estimated the volatility of a univariate vector, but that of a bivariate vector formed from two univariate vectors correlated with a different from zero correlation factor. The newly proposed models will be estimated in sample and out of sample, using one measure of intraday volatility, the realized kernels. In chapter five there will be also undertaken (the second scope of the chapter) a ranking of the new bivariate models and it will be checked the gain or loss in accuracy when compared to the models' univariate formulations. The forecasting performance will be expressed with regards to the size of the maximized loglikelihood functions, and also with regards to the three loss functions considered (RMSE, MAE and MAPE); a general ranking will be obtained through normalization of the functions' values.

Finally, chapter six will attach the Principal Component Algorithm to the bivariate realized models from chapter five, in order to solve the problem of multivariate estimation of stock volatility, by offering an alternative to multivariate GARCH estimations through univariate models. The chapter takes as well advantage of the methodology offered by Alexander (2000) for Orthogonal GARCH and of the one of Burns (2005) for PC GARCH. New models will be obtained: a PC Bivariate EGARCH(1,1) model, a PC Bivariate EGARCH-X(1,1) model, a PC Bivariate Realized EGARCH(1,1) model, a PC Bivariate Realized GARCH(1,1) (partial) and a PC Bivariate Realized GARCH(1,1) (complete) model, as well as a PC Realized GARCH(2,2) model. The new models will be estimated in sample and out of sample, having the realized kernels as measures of intraday volatility. The models will be ranked after the size of their loss functions.

Chapter two: Literature review

Volatility represents an intensely researched topic in Time Series Econometrics for several decades. The interest came following the numerous applications in real life, as volatility played a central role in a broad range of activities, from portfolio allocation to density forecasting in risk management. The number and size of various crashes and crises forced that the attention of practitioners, researchers and regulators move from traditional research in Financial Economics which pointed to assessing the mean in stock market returns, and to the level and stationarity of volatility of stock prices, towards developing econometric tools that better model price volatility. With time, it became evident that returns at high frequencies were difficult to accurately predict and that returns' volatility was easier to model. This explains why Financial Econometrics dedicated that much attention to modeling financial volatility, which gained an essential role in modern pricing and risk management theory. In Financial Economics it was found that the distributional pattern of returns was essential in describing the fluctuation of any financial or economic time series. Conclusions on conditional distributions may say a lot on how to price a specific instrument, how to allocate funds according to a specific portfolio, how to measure risk and performance and how to undertake the management decision process. The distributional pattern is highly connected to other features of a portfolio, like conditional return fractiles that determined the probability that extreme jumps occur in portfolio value.

Correct econometric modeling became essential to a large panel of activities like risk management, investments, security valuation, asset pricing, and monetary policy. Poon and Granger (2003) found that volatility forecasting became a central tool in option forecasting due to the constantly increasing use of derivative securities trading, in financial risk management due to banking cross-boundary globalization, in market timing decisions, portfolio management and in variance modeling of asset prices. Andersen, Bollerslev, Christoffersen and Diebold (2005) resumed the applications of volatility forecasting in three types of activities: 1) generic forecasting applications that included point forecasting, interval forecasting, probability forecasting including sign forecasting and density forecasting; 2) financial applications like risk management ones (specifically value-at-risk and expected shortfall, covariance risk assessment: time-varying betas and conditional Sharpe ratios, asset allocations with time-varying covariances, option valuation with dynamic volatility); and 3) applications residing outside the financial field such as medicine, weather forecasting, and agriculture.

Given the various contexts in which volatility played a role, the concept received various definitions. A less rigorous definition of volatility is that of a series of fluctuations that describes a phenomenon over a specific period of time. Economics describes volatility in a more formal way, as the variation of a random component of a time series, without necessarily specifying a certain implied metric. Further narrowing the concept of volatility to confine it to Financial Economics, Andersen et al. (2005) described volatility as an instantaneous standard deviation of a random Wiener-driven component in a continuous-time diffusion model. Campbell, Lo and MacKinlay (1997) observed that one pattern that mainly distinguished Financial Economics from microeconomics was the role the uncertainty played in both theory and empirics. In the absence of uncertainty, the problems in Financial Economics reduce to simple microeconomic exercises. Andersen, Bollerslev, Christoffersen, and Diebold (2005) observed that one main characteristic of financial data, as compared to microeconomic data, was the latent, or inherently unobserved character of volatility that evolved stochastically along the time. Since volatility could not be directly measured, but rather estimated, an intensive research was done towards more accurate modeling of this process. The high degree of uncertainty and volatility's hidden (latent) character in financial data, transformed the return variance estimation and forecasting problem into a filtering problem in which the "true" volatility cannot be determined exactly, but extracted with some degree of error.

Volatility may be described in both discrete and continuous time patterns, contingent to data availability or model use. Security prices follow a continuous rather than a discrete pattern, given the high liquid markets in which transactions are made at every second. For this, it is natural to consider the stock market time series as arising through discrete observations of an underlying continuous time process. However, the models that describe continuous data may be formulated in discrete time as this would allow, for example, deducing distributional implications of a time series that evolves under a continuous time pattern. Discrete and continuous formulations do not contradict each other and both pose considerable econometric challenges.

Given the topic's high relevance in Finance, the literature has developed in two directions: compounding new models that would result in higher estimation and forecasting performances and, due to the numerous formulations of volatility models, better ranking of the existing models, mainly in terms of predictions' accuracy. With respect to the modeling development process, models range from basic versions of historical volatility models (random walk, historical averages of squared returns or of absolute returns, moving averages, exponential weights, autoregressive or fractionally integrated autoregressive absolute return models) to the more complex GARCH models and their extensions (Exponential GARCH (EGARCH) model of Nelson (1991), Nonlinear GARCH (NGARCH) model of Higgins and Bera (1992) and Engle and Bollerslev (1986), Threshold GARCH (TGARCH) model of Zakoian (1994), Glosten-Jagannathan-Runkle GARCH (GJRGARCH) model of Glosten, Jaganathan and Runkle (1993), Power ARCH (PARCH) model of Taylor (1986) and Schwert (1989), Augmented GARCH (AUGGARCH) model of Duan (1997). Other important volatility forecasting models that found useful in both applied and theoretical contexts were the random coefficient autoregressive (RCA) model proposed by Nicholls and Quinn (1982), the conditional heteroskedastic autoregressive moving average (CHARMA) model proposed by Tsay (1987), the stochastic volatility (SV) models (based on Black-Scholes model and different generalizations of it) compiled by Melino and Turnbull (1990), Taylor (1994), Harvey, Ruiz and Shephard (1994), and Jacquier, Polson and Rossi (1994), or the implied standard deviation models. Each model has proved its own strengths and weaknesses when tested with various types of data, and having at hand such a large number of models, all designed to serve to the same scope, it became highly relevant to correctly distinguish between various models in order to find the appropriate ones that would provide the highest accuracy in the forecasting exercise.

Comprehensive reviews of the literature on volatility modeling are the works written by Bollerslev, Chou and Kroner (1992), Bera and Higgins (1993) and Bollerslev, Engle and Nelson (1994) and more recently Andersen, Bollerslev, Christoffersen and Diebold (2005). Poon and Granger (2003) investigated 93 published and working papers written on the topic of volatility forecasting (out of which only 66 were considered relevant) searching for a comprehensive insight on how the literature ranked the volatility models, also revealing an apparent lack of consensus in the literature in identifying the most accurate model in predicting the latent volatility.

With respect to models' ranking, the obvious conclusion is that the literature has not arrived yet to a consensus as regards which models performed better. This results from the observation that the literature contains contradictory evidence as regards the performance of volatility forecasting models. The subjectivism arises from various sources, starting from the fact that conditional evidence is unobserved and there is no natural and intuitive way to model the conditional heteroskedasticity, reason for which each model will capture features that its author thought to be important and, ultimately, from the fact that models with poor forecasting capacities in all empirical tests have not been identified yet. As such, ranking volatility forecasting models may vary, conditional to factors related to the models themselves, or to the methodology used (in-sample or out-of-sample methods), to the type of asset whose underlying volatility is estimated (volatility of the exchange rates, commodities or volatility of the stock returns, etc.), to the forecasting horizon or to the error statistic choice. As an example, Brailsford and Faff (1996) concluded that models' performance ranking was sensitive to the choice of the error statistic, for each such statistic being identified different rankings structures.

Literature on this subject may be characterized, as such, as a framework of a mixed set of findings, a conclusion that may be grasped from this lack of consensus being that volatility forecasting is a notoriously complicated undertaking. There is evidence that underlines the superiority of more complex models such as GARCH models, while equally weighted evidence points to the superiority of simpler alternatives. This is seen as an extremely problematic fact due to the difficulty that this contradiction rises in choosing the appropriate model in volatility forecasting in decision-making and analysis activities. However, despite literature obvious complexity and lack of homogeneity, Matei (2009) found that, in the pool of all volatility forecasting models, GARCH model was an appropriate model to use when one has to evaluate the volatility of the returns of stock groups with large amounts (thousands) of observations.

2.1 Literature on modeling realized volatility

The serial correlation in the volatility of financial asset returns paved the way to an extremely rich literature written on the topic of volatility modeling and forecasting. Such volatility is typically modeled in empirical contexts with daily data starting from GARCH-type of models or stochastic volatility processes that consider volatility as a latent, unobserved variable. Although returns may be measured with minimal measurement error and analyzed with ordinary time series techniques, being constructed from face prices of assets, volatility needs more careful and complex computational modeling due to the latency property. A common approach to deal with the fundamental latency problem of return variance is to conduct inferences on volatility through strong parametric assumptions. Another option is to employ models designed for option pricing in order to transform back derivatives' prices into forecasts of implied latent volatility over a specific horizon. One important drawback of such methods is that they rely heavily on the type of models chosen - forecasts may vary significantly according to the choice of the model. Another drawback is that they include in the estimated measure a volatility risk premium that fluctuates with time, the effect being that their forecasts on the underlying asset volatility are often biased. Another flaw results from the backward looking methodology employed. The current and future volatilities are estimated starting from the return standard derivation of backward looking rolling samples, returns which most often are calculated from past daily observations. Because of this, models are less prepared to represent volatility shocks that currently happen and even less to anticipate them. However, backward looking models are not meaningless, as volatility is persistent and thus offers some useful information on ongoing patterns, but volatility also has a mean reverting character due to which the unit root type forecasts of it are not optimal, since conditionally biased given the history of the past returns.

Despite the large variety of the models which seek solutions to relatively similar questions, most of the models designed to estimate and forecast latent volatility fail to describe adequately significant issues as regards the fluctuation of financial returns (Bollerslev (1987), Carnero, Peña and Ruiz (2004) and Malmsten and Teräsvirta (2004)). One such important feature of latent volatility which is not satisfactorily encapsulated in the models is the low, though diminishing autocorrelation in the squared returns related to the high excess kurtosis of returns. Adequate modeling of return dynamics is required as accurate forecasting is essential in risk management or decision taking processes. As such, the assumption of the existence of Gaussian standardized returns has been contested in many studies, being replaced by heavy-tailed distributions. By investigating the literature written on the topic of volatility forecasting, it can be observed that its largest part has concentrated on obtaining and using highly restrictive and complex parametric versions of GARCH models or of stochastic volatility models, which soon have found their limitation in terms of predictability, especially at higher frequency distributions of returns.

The increasing opportunity to get access to high frequency data allowed researchers to experiment more straightforward methods to model volatility by constructing daily time series out of intraday data. This step allows treating "volatility" as observed rather than latent, to which standard time-series techniques may be applied. Since the addition of intraday sampled squared returns provides a consistent estimator of the actual daily volatility, the forecasting performance of the estimated models could be evaluated more accurately than in the case of methodologies that employ squared daily returns as volatility measures. Merton (1980) observed that the conditional variance over a fixed period can be expressed, arbitrarily but still satisfactorily accurate, under a sum of squared realizations, when data is available at high sampling frequency. Andersen and Bollerslev (1998) also provided an argument for the same conclusion, saying that ex post daily foreign exchange volatility can be optimally estimated by aggregating 288 squared five-minute returns. Among all existing models, the Autoregressive Fractionally Integrated Moving Average (ARFIMA) and the Heterogeneous Autoregressive (HAR) models emerged as the most popular models in literature, capable of capturing the observed long-memory pattern of volatility and empirically outperforming more traditional counterparts such as GARCH and stochastic volatility models.

Realized volatility notion solves many drawbacks of the traditional methods that use squared returns. In the presence of no transaction costs, with continuously observed prices, the realized return variance may be modeled with no error by using realized returns. When we control for the measurement error, the ex post volatility eventually becomes from latent, observable, which allows it to be modeled directly rather than being estimated from a latent process. Moreover, the realized variance is correlated with the concept of cumulative expected variation of the returns over a specific horizon for a large set of underlying no- arbitrage diffusive data generating processes. On contrary, over the short term it is not possible to link the actual realized returns to the expected returns if not making supplementary assumptions.

Andersen and Bollerslev (1998), Patton (2005), and Hansen and Lunde (2005) used realized volatility measures in order to estimate the out-of-sample forecasting performance of GARCH models. Starting from pioneering studies led by Barndorff-Nielsen and Shephard (2002), Meddahi (2002) and Andersen, Bollerslev, Diebold and Labys (2003), more recent papers proposed methodologies to isolate information obtained from realized volatility measures constructed from high frequency data, and integrate it in daily return modeling. Aït-Sahalia, Mykland and Zhang (2005), Zhang, Mykland and Aït-Sahalia (2005), Bandi and Russell (2005, 2006), and Hansen and Lunde (2006) also provided solutions to the inconsistency problem.

Works written by Anderson, Bollerslev, Diebold and Labys (2001) and Barndorff-Nielsen and Shephard (2001) argue that the realized volatility measures are not just unbiased ex-post estimators of daily volatility, but also asymptotically free of any measurement error. Barndorff-Nielsen and Shephard (2004, 2006) and Jiang and Oomen (2008) proposed nonparametric methods to detect the existence of jumps in intraday financial time series. Jiang et al. (2010) used the bipower variation developed by Barndorff-Nielsen and Shephard (2004, 2006) and the variance swap approach of Jiang and Oomen (2008) to identify the price jumps in the 2-, 3-, 5-, 10-year notes and 30-year bonds, finding that macroeconomic news were often precipitated by market volatility increase and by liquidity withdrawal, while liquidity shocks played an important role for price jumps in US Treasury market. This work created an opportunity in modeling the volatility of energy prices as a common feature of energy futures prices is that they are very volatile and often exhibit jumps during announcement periods. Bjursell et al. (2009) were the firsts to apply a nonparametric method based on realized and bipower variations calculated from intraday data in order to identify jumps in daily futures prices of crude oil, heating oil and natural gas contracts traded at the New York Mercantile Exchange, finding that in terms of realized volatility the natural gas was the most volatile and that the large volatility days may be often associated with large jump components.

Parametric approaches were extensively used in order to model and identify jumps in financial stock data. Chan and Maheu (2002) proposed an autoregressive conditional jump intensity anchored in a common GARCH setting in order to detect jumps in a seventy-two year long time series, finding significant time variation in the conditional jump intensity and evidence of time variation in the jump size distribution. Maheu and McCurdy (2004) modeled conditional variance of returns by using jumps coupled with smoothly changing components.

Nonparametric approaches have been employed by Huang and Tauchen (2005) who used the Monte Carlo analysis on various jump test statistics developed by Barndorff-Nielsen and

Shephard (2004, 2006) and by Andersen, Bollerslev and Diebold (2004). They concluded that newer developed z-tests performed very well, with an appropriate size and power properties, accurate in identifying the days on which jumps occurred. As well, the theoretical and Monte Carlo analysis indicated that microstructure noise biased the tests against detecting jumps, and that a simple lagging strategy corrected the bias. Using a similar nonparametric approach, Andersen et al. (2007) provided evidence that the volatility jump component was both highly significant and less persistent than the continuous sample path component.

Sampling at higher frequency has some disadvantages also. It has been proved that employing intraday volatility estimates is a trade-off between higher accuracy in latent volatility description, theoretically optimized when the frequency sampling is the highest as possible, and the microstructure noise that may arise through bid-ask bounce, asynchronous trading, price discreteness, and infrequent trading. Some references discussing this issue are Madhavan (2000), Biais, Glosten and Spatt (2005), and Hansen and Lunde (2005).

However, in the last decade, the advancement in volatility modeling has stalled in some aspects. The larger access to high-frequency data moved away the attention from further modeling of volatility with daily data, and thus the effect of better data inputs was negligible in what concerned improving model designs. It has been empirically demonstrated that the standard models designed to provide estimates by using daily observations were improper in functioning with intraday values. As well, new models specified for the intraday data failed in capturing the information of the interdaily movements as well, reason for which their daily forecasts were not as precise as expected. In the context of not having an empirically superior proved alternative as concerns modeling day or intraday volatility by using high-frequency data, the standard practice continued to use the traditional modeling tools in order to obtain relatively good estimates of daily values, although intraday data was available. The emphasis has continuingly been placed on low-dimensional volatility modeling, and mainly univariate. Although multivariate variants of the existing ARCH and GARCH models were already proposed by Bollerslev, Engle and Nelson (1994), Ghysels, Harvey and Renault (1996), and Kroner and Ng (1998), the multitude of constraints and computational problems they raised made them computationally difficult to be applied in empirical contexts. Therefore, few applications dealt with more assets in the same time. As a consequence, practitioners avoided to search for solutions to highly practical relevant multidimensional problems, and continued to rely on simple exponential smoothing methods combined with the assumption of conditional normally distributed returns. An example of models that found large applicability in business contexts is the RiskMetrics proposed by J.P. Morgan; although it employs counterfactual assumptions and proves to be most of the times suboptimal, it functions sufficiently well, its main quality being the model's feasibility, simplicity, and short implementation time characteristic to high-dimensional contexts.

In this context, the realized volatility methodology improves modeling process in two regards: 1. it proposes a rigorous methodology that fully exploits the information contained in the high-frequency data and which proves efficient in forecasting daily return values; and 2. the models distinguish through simplicity and facile implementation in high-dimensional environments.

High frequency models proved that incorporating any measure of intraday volatility significantly improves the modeling of jumps in time series data. GARCH-X model (with an exogenous realized measure) with realized variance (Engle, 2002) or realized variance and bipower variation (Bandorff-Nielsen and Shephard, 2007) as realized measures empirically proved a significant gain in fit when jumps occurred. Engle and Gallo (2006) proposed the Multiplicative Error Model (MEM) which was first to contain a separate equation for the realized measure. A similar complete model nested in an MEM setting was the High Frequency Based Volatility (HEAVY) model proposed by Shephard and Sheppard (2010). Both MEM and HEAVY models are difficult to use, as they work with multiple latent processes.

A different completed high frequency version of the GARCH model was the Realized GARCH model proposed by Hansen, Huang and Shek (2010a), model which combined a GARCH structure for returns with a model that used realized measures of volatility. As compared to MEM and HEAVY models, the Realized GARCH model takes advantage of the natural relationship between the realized measure and the conditional variance and proposes, instead of introducing additional latent factors, a single measurement equation in which the realized measure is a consistent estimator of the integrated variance. Besides its elegant mathematical structure, the Realized GARCH model is easy to estimate, captures the return-volatility dependence (leverage effect) and was empirically proven that it outperformed the conventional GARCH.

Papers written on the topic of realized volatility are heterogeneous in their scope and indexing methodologies. Some papers comprise extensive reviews of the written literature, rather limiting to a general discussion of volatility, like Poon and Granger (2003) and Andersen, Bollerslev, Christoffersen and Diebold (2006). A commonality in the general reviews is not discussing the microstructure noise problem. Instead, this problem has been discussed in Bandi and Russell (2006), paper which places an emphasis on the economic determinant of the noise component. McAleer and Medeiros (2008) addressed to the problem of measurement error. Barndorff-Nielsen and Shephard (2007) reviewed the papers by putting a stronger emphasis on nonparametric estimation of volatility and on the frictionless case with/without jump effects.

By reviewing the volatility literature, it can be observed that there have been extensive studies on the expected return volatility but little in what concerns the expected mean return from high frequency asset prices. This perspective produced a significant effort from the researchers as regards obtaining and empirically using realized volatility estimates disseminated from high frequency data. As such, in the today markets, the realized volatility field became a well-established practice to use intraday returns to build up ex-post volatility measures. Due to larger access to high-quality transaction data over a well-diversified panel of financial assets, it's unavoidable that this topic be further investigated and more tested in wider empirical contexts over the future.

Modeling with various sampling schemes

The realized volatility approximates the quadratic variation pretty well as the sampling frequency increases. Nevertheless, this simple statement complicates further the problem according to the following two stances. The first one is that even for the most liquid assets a continuous price is not available. This constraint leads to an unavoidable discretization error in the estimates of the realized volatility which determines us to recognize the existence of a measurement error. Although by subsequent reiteration we may estimate the magnitude of such errors, according to the continuous asymptotic theory, this inference is always subject to sampling distortions and is totally true only when price jumps are disregarded. The second issue refers to the large panel of microstructure effects which induces spurious autocorrelations in the high frequency sampled return series. In this category there are included the effects of rounding, price discreteness, bid-ask bounces, the trades which occur on various markets, the steady (gradual) response of process to a block trade, asymmetric information contained in order of different size, spreads positioning according to the dealer inventory control, strategic order flows and data recording flaws. The spurious autocorrelations emerging from these sources may increase the estimates of the realized variance and thus generate a traditional type of bias-variance trade off. Although the general recommendation is to use the highest sampling frequency as its optimal for efficiency captured signal, this also tends to bias the estimate of the realized volatility.

The above described tradeoff is often plotted through a volatility signature diagram which illustrates the sample mean of the realized volatility estimator over a long time period as a function of the sampling frequency. As such, the long time duration diminishes the impact of sampling variability and therefore, when the microstructure noise is not considered, the plot should appear as an approximately horizontal line. Nevertheless, it is observed in empirical applications that in plots with transaction data sampled from highly liquid stocks we will find spikes at high sampling frequencies and more moderate reductions in order to stabilize at frequencies at 5-40 minute range. On contrary, the reversal occurs for returns built up from bid-ask quote midpoints as asymmetric adjustments of the spread determines positive serial correlation and biases the signature diagram downward at the highest sampling frequency. As such, for the case of the illiquid stocks, the inactive trading produces positive return serial autocorrelation, which induces the signature diagram increase at lower sampling frequencies. Aït-Sahalia, Mykland and Zhang (2006), Bandi and Russell (2007) and Andersen, Bollerslev, Diebold and Labys (2003) have further developed this topic by trading off efficient sampling with bias-inducing noise in order that optimal sampling schemes be obtained.

Another solution proposed in order to deal with the tradeoff described above is to use alternative quadratic variance estimators that would be more efficient and less sensitive to the microstructure noise. Huang and Tauchen (2005) and Andersen, Bollerslev and Diebold (2007) are among them, suggesting that staggered returns and realized bipower variation (the latter for non-parametrical measurement of the jump component in asset return volatility) be used, effective in noise reduction, while Andersen, Bollerslev, Frederiksen and Nielsen (2006) extended the signature diagrams in order to count also for power and h-skip bipower variation.

An alternative realized variance like high-low measure has been used by Brandt and Jones (2006), Alizadeh, Brandt and Diebold (2002), Brandt and Diebold (2006), Gallant, Hsu and Tauchen (1999), Yang and Zhang (2000), Schwert (1990), Parkinson (1980), and Garman and Klass (1980). Moreover, Christensen and Podolskij (2006) and Dobrev (2007) generalized the high-frequency data estimator in various ways, and discussed its link to the realized variance topic. Zhou (1996) sought a method to correct the bias of the realized variance estimators by explicitly accounting for the covariance in the lagged squared return observations. Hansen and Lunde (2006) extended the work began by Zhou for the case of non-independent and identically-distributed noise. Aït-Sahalia, Mykland and Zhang (2006) examined the necessary correction when the noise was independent and identically normally distributed, while Zhang, Mykland and Aït-Sahalia (2005) came with a consistent volatility estimator which considered all the data available, averaging realized variances through forming different sub-samples and correcting for the remaining bias. Aït-Sahalia, Mykland and Zhang (2005) extended further this work and proposed a method to account for some serial correlated errors. Barndorff-Nielsen, Hansen, Lunde and Shephard (2006) proposed kernel estimators as realized measures.

In a traditional setting, prices are observed at discrete and unevenly spaced intervals, reason which determines one to look for different sampling schemes. An interval [0,1] is subdivided in n_t sub-periods and the observation times are defined under the form of a set $\Upsilon_{t=}\{\varphi_0, \dots, \varphi_{n_t}\}$ with $0 < \varphi_0 < \varphi_2 < \ \dots < \varphi_{n_t} = 1$ where $\rho_{1,n_t} = \varphi_i - \varphi_{i-1}$ is the length of each subinterval. Naturally, such length should decrease while the number of observations in a day increases. Then the intraday variance over each subperiod may be defined as

$$IV_{i,t} = \int_{\varphi_{i-1}}^{\varphi_i} \delta^2(t + \varphi - 1) d\varphi$$

McAleer and Medeiros (2008) distinguish four sampling schemes, as it follows:

1) *The calendar time sampling* in which the intervals have equal length in calendar time, meaning that $\rho_{1,n_t} = \frac{1}{n_t}$, $\forall\ i$. One example is sampling prices at each 5, 10 or 15 minutes. A methodology for this type of sampling has been offered by Wasserfallen and Zimmermann (1985), Andersen and Bollerslev (1997), and Dacorogna, Gencay, Muller and Pictet (2001), motivated by the fact that intraday data is irregularly spaced, with no

fixed period spacing, so that for most of the data sampled observations must be built upon artificially. Hansen and Lunde (2006) found that the *previous tick* method (method which adds values of the last observations in the missing gaps) is a straightforward and competitive method to sample prices according to calendar time. More exactly, this method samples only the first observation of a five-minute interval.

2) Another sampling method is the *transaction time sampling* in which prices are sampled with every transaction made.
3) A third alternative is the *business time sampling* in which sampling times are selected in such a way that $IV_{i,t} = \frac{IV_t}{n_t}$.
4) Finally, there is the *tick time sampling* in which prices are recorded at each change.

To be mentioned that in the first sampling choice the observations are latent, while in the last three ones the sampled data is observed, each sampling choice producing effects in the estimated integration variance.

The conditional return variation and the concept of realized volatility

The following section is dedicated to the natural liaison existing between quadratic variation and the integrated variance, in order to cover some practical aspects as regards the estimations of the realized volatility and of the variance of conditional return. Assuming an invariable drift and volatility coefficients, both conditional and unconditional variance in returns will equal the quadratic variation of the log price. On contrary, if assuming volatility as a stochastic process, then precise distinction between conditional variance (which stands for the expected size of the innovations of the squared returns over a specific interval) and the quadratic variation over a specific time horizon, is needed. Therefore, the difference may be expressed as an expectation against future realizations of the volatility of stock returns. Theoretically, the realized volatility would express only the actual realizations, and not their previous expectations. However, the realized volatility estimates are efficient in capturing the conditional return variation as one may build up accurate forecasts/conditional expectations of return volatility out of a financial or economic time series formed from past realized volatility.

The above assertions may be even strengthened under a simplified setting. If the instant return is a continuous-time process and the return, average and volatility series are low or not correlated processes, then the conditional expectation of the return should be normally distributed, conditional to the cumulative drift and on the quadratic variation. Therefore, the distribution of the return series is mixed Gaussian with the mixture ruled out by the integrated variance realizations, along with their integrated mean. Realization jumps from the integrated variation process make the outliers of the returns become probable while the persistence in the integrated variance process may determine volatility clustering. Furthermore, over short horizons, when the conditional mean is very low as compared to the cumulative absolute return innovations, the integrated variance process may be intrinsically linked to the conditional variance.

Because the realized variance is roughly unbiased for the related unobserved quadratic variation, the realized volatility estimate comes as the natural point of reference against which to estimate the volatility forecasts accuracy. There may be also undertaken tests of goodness-of-fit on the residuals resulted from subtracting the forecast from the realized volatility measures.

The realized volatility topic is also related to the return variation estimated over a discrete time period rather than with the spot (instant) volatility. The distinction appears due to the differentiation between realized volatility concept and a whole range of literature written in the search of spot volatility estimation from discrete observations, mainly in a setting with a constant diffusion coefficient. Although theoretically the measurement of realized volatility can be adapted easily to spot volatility estimation, in practice this is not feasible as frequent sampling over very small intervals may amplify the effects of microstructure noise.

Modeling and forecasting realized volatility

A well-known fact in the literature is that when GARCH and SV models are employed, the standardized returns do not exhibit a Gaussian distribution. Instead, the standardized returns present an excess kurtosis, thing that justifies the employment of heavy-tailed distributions. Andersen, Bollerslev, Diebold and Labys (2000, 2001, 2003) proved that when modeling has been made with the employment of realized variance measures, the distribution of standardized exchange rates is approaching the properties of a typical Gaussian. A similar application with stock returns run by Andersen, Bollerslev, Diebold and Labys (2001) arrived to similar conclusions.

The log-realized variance is significantly persistent, but stationary, with long memory properties, traditionally expressed like an ARFIMA(p,d,q) process. Various models have been proposed to catch the properties of such time series. One of them is the Multiplicative Error Model (MEM) proposed by Engle and Gallo (2006) that is consistent and asymptotically normal under a wide range of specifications for the error density function. The MEM model is best suited to model the conditional behavior of positively valued variables choosing a convenient GARCH-type structure when modeling variance and persistence. Another model is the HEAVY model (Shephard and Sheppard (2010)), a high frequency based volatility model of daily asset return volatility based on measures constructed from high frequency data. The authors proved that such models perform more robust to level breaks in the volatility than conventional GARCH models, adjusting to the new level much faster. Supplementary, although such model shows mean reversion, it exhibits as well momentum, a feature that misses from classical models.

Another model which uses realized measures is the Heterogeneous AutoRegressive Realized Volatility (HAR-RV) model proposed by Corsi, Zumbach, Muller and Dacorogna (2001) and Corsi (2003), model that has at its fundament the Heterogeneous ARCH (HARCH) model proposed by Müller, Dacorogna, Davé, Olsen, Puctet and von Weizsäcker (1997). The HAR-RV model represents an additive cascade of different volatility components produced by the actions of the participants in the market that produces remarkably good out-of-sample forecasting performance. The HAR-RV model is in such a way built up that the additive volatility cascade leads to an AR (autoregressive) - type model in the realized volatility, considering volatilities realized over different sampling sizes.

Subsequently, McAleer and Medeiros (2006) offered a multiple regime smooth transition generalization of the HAR-RV model (called Multiple Regime Smooth Transition Heterogeneous Autoregressive HARST), by coming with a flexible model able to capture the non-linearities and long-range dependence in time series dynamics. The model has been designed to describe concurrently long memory and size and sign asymmetries.

In volatility forecasting topic, sources of long memory have been intensively searched, because shorter memory of a model, better forecasting performances may be produced. For example, Hyung, Poon and Granger (2005) evidentiated that numerous nonlinear short memory models, especially those which present infrequent breaks, may generate long memory patterns. Some of these models are the regime switching model of Hamilton and Susmel (1994), the volatility component model of Engle and Lee (1999), the model proposed by Diebold and Inoue (2001), the break model developed by Granger and Hyung (2004), and the multiple regime-switching model of Medeiros and Veiga (2004). The latter one is developed to aim describing size and sign asymmetries in financial volatility as well as intermittent dynamics and excess kurtosis. Hilledebrand (2005) and Hilledebrand and Medeiros (2006) evidenced the statistical consequences of neglecting structural breaks and regime switches in autoregressive and GARCH models, proposing two solutions to remedy the problem: the identification of those regimes with constant unconditional volatility that use a change point detector and then estimate a separate GARCH model on each of the separate resulting segments, and the estimation of a multiple-regime GARCH model, like that of the type of a FCGARCH (Flexible Coefficient Generalized Autoregressive Conditional Heteroskedastic).

Scharth and Medeiros (2006) came with a new model built up on regression trees that described the realized volatility dynamics for some DJIA (Dow Jones Industrial Average) stocks. They presented empirical evidence that additive price changes convey meaningful information as regards multiple regimes in the realized volatility of stocks, whereas large rises (falls) occurred in prices are highly dependent on persistent regimes of low (high) variance in stocks. Therefore, past cumulated daily returns incorporated as source of regimes' switches accounts for high empirical values of long memory parameter estimates. The nonlinear model has been found to be superior to the other long memory models, ARFIMA and HAR-RV.

In all previously mentioned references, volatility has been assumed to refer only to short memory between breaks on each component of volatility and within each regime. A significant improvement of this approach came from Martens, van Dijk and Pooter (2004) who considered a model that combined the long memory properties with nonlinearity, particularly relevant in modeling asymmetries and leverage effects. The model they proposed is a nonlinear model for realized volatility which accommodated level shifts, day-of-the-week effects, leverage effects and volatility level effects. Deo, Hurvich and Lu (2006) proposed a long-memory stochastic volatility model (LMSV) which is found as a very good competitor to the method which predicted realized variance by using a long memory stochastic volatility model applied to high frequency return data while accounting for significant gradually varying intra-day seasonality in volatility. Koopman, Jungbacker and Hol (2005) established a model which joined unobserved elements and long-memory, while Hillebrand and Medeiros (2008) documented a model that joined long memory with different features of nonlinearity.

Despite such a rich literature written on the emerging field of realized volatility topic, open questions reside as regards the sources of long memory characteristic to the realized volatility and as regards the extension of benefits in terms of volatility predictability from combining long-memory with nonlinear models (Ohanissian, Russell and Tsay (2004)).

Treating the same topic of long-memory, Lieberman and Philips (2008) offered some analytical explanations on the reasoning according to which realized volatility series typically display long range dependence with a memory parameter (d) of around 0.4. They found that long-memory properties were an effect of the accumulation of realized variance and offered some solutions to refine the statistical inference as regards the parameter d in ARFIMA(p,d,q) models.

Aït-Sahalia and Mancini (2006) compared the out-of-sample relative capacity of forecasting of realized variance in different contexts. Ghysels and Sinko (2006) assessed the extent to which the correction for microstructure noise improved forecasting future volatility using Mixed Data Sampling (MIDAS) and found that the conditional optimal sampling works reasonably well in practice. As well, they found that within the class of quadratic variation measures, the subsampling and averaging approach (Zhang, Mykland and Aït-Sahalia (2005)) represents the class of estimators that best predicts volatility at five minute sampling schemes. Furthermore, Corradi, Distaso and Swanson (2006) estimated and forecasted conditional predictive density and confidence intervals for integrated volatility by newly proposed nonparametric kernel estimators, built upon various realized volatility measures constructed using *ex post* variation of asset prices. Corsi, Kretschmer, Mittnik and Pigorsch (2008) showed that the residuals of the commonly used time-series models for realized volatility exhibited non-Gaussianity and volatility clustering, proposing extensions to explicitly account for these properties and assess their relevance when modeling realized volatility. Moreover, they demonstrated that allowing for time-varying volatility of realized volatility leads to significant improvement of model fit and of the predictive performance as well, while the distributional assumption for residuals proved to be crucial in density forecasting.

Another important topic in the context of realized volatility is that, regardless the microstructure noise presence, the realized volatility is an estimated quantity rather than a true, daily, value of volatility or of the integrated variance, while integrated quarticity may be replaced by realized quarticity. This fact opens the perspective of employing generated regressors and generated variables in forecasting exercises, associated with critical questions on the efficient estimation and invalid inferences that may occur when biased (asymptotic) standard errors are used (Pagan (1984, 1986), McKenzie and McAleer (1997)).

Andersen, Bollerslev and Meddahi (2004, 2005) built up a general model-free adjustment method aimed at estimating the unbiased volatility loss functions starting from practically feasible realized volatility benchmarks. According to them, an efficient measurement error accounting in the evaluations of volatility forecasts may lead to markedly higher estimates for the true degree of return-volatility predictability. Corradi and Distaso (2006) proposed a procedure to test for the correct specification of the functional form of the volatility process within the class of eigenfunction stochastic volatility models. The procedure starts from the comparison of the moments of realized volatility measures with the corresponding ones of integrated volatility implied by the model under the null hypothesis. They first provided primitive conditions as regards the measurement error associated to the realized measure, which would allow to construct asymptotically valid specification tests. Then, they established those regularity conditions under which the realized measures (realized volatility, bipower variation, and modified subsampled realized volatility) satisfy the given primitive assumptions.

Multivariate empirical studies

One of the most cited papers which discussed the topic of realized variance in applications with multivariate models is de Pooter, Martens and van Dijk (2008). In this paper it is investigated the merits of high-frequency intraday data when forming mean-variance efficient stock portfolios with daily rebalancing from the individual stock components of the S&P100 index. They focused on the problem of establishing the optimal sampling frequency as revealed by the performance of these portfolios. Surprisingly, the authors found that the optimal frequency is not the highest frequency one, but it ranges between 30 and 65 minutes, significantly lower than the popular five-minute one, which typically is motivated by the aim of maintaining a balance between the variance and bias in covariance matrix estimates due to market microstructure effects like non-synchronous trading and bid-ask bounce. Another important finding is that bias-correction procedures, based on combining covariance matrix estimates with low-frequency and high-frequency, and on the summing of leads and lags, do not significantly influence the optimal sampling frequency or the portfolio performance. This is also robust to presence of transaction costs and to the portfolio rebalancing frequency.

Another paper that discusses in multivariate context the functioning of realized variance modeling is Bauer and Vorkink (2006) which proposes a new matrix logarithm model of the realized covariance of stock returns, by employing latent factors as functions of both lagged volatility and returns. The model proves advantageous as it is parsimonious, does not require imposing parametric restrictions, and yields a positive definite covariance matrix. The model is empirically tested with a covariance matrix of size sorted stock returns and two factors are isolated as satisfactory to capture most of the dynamics. A new method to track down an index using the model of the realized volatility covariance matrix proposed was also introduced.

2.2 The autoregressive models

In what follows, we will make a description of the basic models that this work addresses to, which are the main autoregressive models. An autoregressive model, in its simplest form, is a model in which one uses the statistical properties of the past behavior of a variable y_t to predict its behavior in the future. In other words, we can predict the value of the variable y_{t+1} by just taking into account the sum of the weighted values that y_t took in the previous period plus the error term ε_t.

The basic autoregressive models were the ARMA models. They were built in order to shape up the basic particularities of volatility. With respect to financial time series, one main such characteristic is that volatility is not uniformly dispersed in time, but higher or lower in some periods as compared to other periods, phenomenon known as the clustering effect. Then, as previously mentioned, another characteristic is the continuous evolvement in time as jumps are rarely seen. The third characteristic is that volatility does not diverge to infinity, but varies with some fixed range, describing its stationarity. Finally, it is the leverage effect that describes a different reaction when a price largely increases or decreases.

Updates of the volatility forecasting models try to encompass such characteristics as the earlier models have failed to capture such features. One example would be the EGARCH model which was developed in order to capture the asymmetry in volatility induced by large "positive" and "negative" asset returns. Some financial time series might be serially uncorrelated, but dependent. Volatility models try to reveal such dependence in the return series of financial data.
The ARMA(p,q) model may be defined as it follows

$$r_t = \mu_t + a_t$$

$$\mu_t = \varphi_0 + \sum_{i=1}^{p} \varphi_i r_{t-i} - \sum_{i=1}^{q} \theta_i \varepsilon_{t-i}$$

where p and q are non-negative integers. ε_t is called shock or innovation of an asset r_t while μ_t stands for the mean equation for r_t .

It results that $\sigma_t = Var(r_t|F_{t-1}) = Var(\varepsilon_t|F_{t-1})$

The autoregressive moving-average (ARMA) models join the concepts of AR and MA models with having as the main scope keeping the number of parameters small. Their importance in finance is given mainly for their use in explaining ARCH and GARCH models, the generalized autoregressive conditional heteroskedastic model being seen as a non-standard ARMA model for an ε_t^2 series. The ARMA model has been firstly proposed by Box, Jenkins and Reinsel (1994).

Modeling conditional heteroskedasticity is equivalent to creating a dynamic equation which reproduces the evolution in time of the conditional variance of the asset return. Conditional heteroskedastic models may be grouped in two categories: one is the one in which h_t is modeled by an exact function, while the other is comprised by models that use a stochastic equation to describe h_t. Examples would be GARCH model, for the first category, and stochastic volatility models for the second one.

The ARCH(m) model was proposed by Engle (1982) and has the following form

$$\mu_t = E_{t-1}(r_t) = E_{t-1}(\sqrt{\sigma_t}\psi_t), \sigma_t = var(r_t) = \omega + \alpha_1 r_{t-1}^2 + \cdots + \alpha_m r_{t-m}^2 \text{ or}$$

$$\sigma_t = var(r_t) = \omega + \sum_{i=1}^{m} \alpha_i r_{t-i}^2$$

where ψ_t is a sequence of independent and identically distributed random variables with mean zero and variance 1, $\omega > 0$, and $\alpha_i \geq 0$ for $i \geq 1$. As such, ARCH model expresses the variance in returns σ_t as a function of the past squared returns. In order to ensure finite unconditional variance of r_t, α_i's must satisfy some regularity conditions so that the unconditional variance r_t be finite. In practice, ψ_t is frequently assumed to follow the standard normal or a standardized Student-t distribution or a generalized error distribution.

From the model's structure, we can see that large squared shocks in the past $\{\varepsilon_{t-1}^2\}_{i=1}^{m}$ result into a large conditional variance σ_t for the return r_t. Therefore, r_t tends to assume large values (in modulus). Consequently, in ARCH terms, a large shock tends to be followed by another large shock. This is similar to clusters observed in asset returns.

ARCH models are simple and easy to handle, and take care of clustered errors, as well as of nonlinearities. One characteristic of ARCH models is the "random coefficients problem": the power of forecast changes from one period to another.

Among the weaknesses of the ARCH model, could be mentioned the following:

1. The model assumes the fact that both positive and negative shocks produce similar effects on volatility as it depends on the square of the previous shocks, while in the real world the price of a financial asset shows different (most often opposite) effects when affected by negative and positive shocks.
2. The ARCH model is rather restrictive. This is due to the fact that α_1^2 must find in different restricted intervals, depending of the series' moment. Thus, in an ARCH(1)

model, α_1^2 must be in the $\left[0,\frac{1}{3}\right]$ interval if the series has a finite fourth moment. The constraint becomes more difficult to establish for higher order ARCH models. In the real world, such characteristic limits the ability of ARCH models with Gaussian innovations to capture excess kurtosis.

3. Another weakness of the model is that it doesn't help in understanding the source of variations of a financial time series. However, the only contribution is that it provides a mechanical method of linking the past variations to the present ones, thus depicting the time-varying conditional variance. But the causes of such behavior are not better illustrated.
4. Finally, ARCH models, in most of the instances, overpredict volatility because they respond slowly to large isolated shocks to the return series.

Although the ARCH model has a basic form, one of its characteristics is that it requires many parameters to describe appropriately the volatility process of an asset return. Thus, alternative models must be further searched, one of them being the one developed by Bollerslev (1986) who proposed a useful extension known as the Generalized ARCH.

As compared to the ARCH model, the Generalized Autoregressive Centralized Heteroskedastic (GARCH) model has only three parameters that allow for an infinite number of squared roots to influence the current conditional variance. This feature allows GARCH be more parsimonious than the ARCH model, feature that explains the wide preference for use in practice, as against ARCH.

While ARCH incorporates the feature of autocorrelation observed in return volatility of most financial assets, GARCH improves ARCH by adding a more general feature of conditional heteroskedasticity. Simple models - low values of parameters p and q in GARCH(p,q) - are frequently used for modeling the volatility of financial returns; these models generate good estimates with few parameters. Like everything else, however, GARCH is not a "perfect model", and thus could be improved - these improvements are observed in the form of the alphabet soup that uses GARCH as its prime ingredient: TARCH (Threshold Autoregressive Conditional Heteroskedastic), OGARCH (Orthogonal Generalized Autoregressive Conditional

Heteroskedastic), M-GARCH (Multiplicative Generalized Autoregressive Conditional Heteroskedastic), PC-GARCH etc.

Similar to the ARCH model, the conditional variance determined through GARCH is a weighted average of past residuals. The weights decline but never reach zero. Essential to GARCH is the fact that it permits the conditional variance to be dependent upon previous own lags.

The GARCH model proposed by Bollerslev (1986) expressed the same variance σ_t as a function of past squared returns and past variance. As such, Bollerslev added to ARCH equation for variance, the past variance of the same time series:

$$\sigma_t = var(r_t) = \omega + \sum_{i=1}^{m} \alpha_i r_{t-i}^2 + \sum_{j=1}^{s} \alpha_j \sigma_{t-j}$$

The GARCH(1,1) version of the model is

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$\sigma_t = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}$$

One of the shortcomings of GARCH is that this model takes into account only the size of the movement of the returns (magnitude), not the direction as well. Investors behave and plan their actions differently depending on whether a share moves up or down which explains why the volatility is not symmetric in the stance of the directional movements. Market declines forecast higher volatility than comparable market increases. This represents the leverage effect described by Gourieroux and Jasiak (2002). Both GARCH and ARCH have this limitation that impedes them from very accurate forecasts.

All GARCH models necessitate lots of data. Simulations (both univariate and multivariate) proved that 1000 observations is a small sample, and fewer than this does not provide any picked up signal. 5000 observations is not as well a very large sample in terms of accuracy with which parameters are estimated. GARCH models require several years of daily data in order to be trustworthy.

As observed by Andersen and Bollerslev (1998) and Andersen et al. (1999), a source of failure of the autoregressive conditional heteroskedastic models in modeling volatility could be the latent (hidden) character of volatility, stochastically evolving through time. Due to the fact that models, until then, were compounded to describe volatility of time series formed from daily returns (usually the returns were formed from prices registered at the last transaction of the trading day), having the purpose of a better approximation of the underlying (true) volatility, maybe the failure of the GARCH-class of models to provide good forecasts was not a failure of the models themselves, but rather a failure to specify correctly the true volatility measures against which the forecasting performance was measured. According to Andersen and Bollerslev, the standard way of using ex post daily squared returns as the measure of "true" volatility for daily forecasts was flawed as such measure comprised a large and noisy independent zero mean constant variance error term which was unrelated to the actual volatility. As such, Andersen and Bollerslev suggested that cumulative squared-returns from intra-day data be used as an alternative way to express such "true" volatility. Such measure, called "integrated volatility" offered the opportunity of a more meaningful and accurate volatility forecast evaluation. This represented a step forward in forecasting problem as it indicates the necessity of using high frequency data in empirical estimations.

This observation opened the perspective towards modeling in autoregressive conditional heteroskedastic frameworks by also using measures of intraday data. It was Engle (2002) who proposed first a new version of GARCH models to include measures of intraday volatility. The GARCH-X model of Engle (2002) was a standard GARCH model to which it was added an exogenous component x_{t-l} that described the intraday volatility of each trading day. As such, the GARCH-X model took the form

$$\sigma_t = var(r_t) = \omega + \sum_{i=1}^{m} \alpha_i r_{t-1}^2 + \sum_{j=1}^{m} \alpha_j \sigma_{t-j} + \sum_{l=1}^{m} \gamma_j x_{t-l}$$

Introducing measures of intraday volatility in order to better describe the latent volatility, proved a better empirical fit of a GARCH-type model, highlighted by the fact that when jumps in (true) volatility occurred, the GARCH-X model updated its estimated volatility to the new level much faster than a standard GARCH model.

Among the weaknesses of this model, the most important one was the fact that in a GARCH-X model, the intraday measure of volatility x_t was treated as an exogenous variable, posting no dependence x_t to σ_t, when actually that existed. As for the measures of intraday volatility used, Engle (2002) employed the realized variance, while Barndorff-Nielsen and Shephard (2007) used the realized variance and the bipower variation.

In the empirical exercise to follow we will use the univariate version of an EGARCH-X model in its logarithmic form:

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$log\sigma_t = \omega + \beta log\sigma_{t-1} + \gamma log x_{t-1} + \tau_1 z_{t-1} + \tau_2 (z_{t-1}^2 - 1)$$

in which x_{t-1} is treated as exogenous since we do not link it to any other variable.

Some models endogenized x_t trying to explain the measure of intraday volatility by creating an additional latent volatility process for each intraday measure included in the model. The univariate Multiplicative Error Model (MEM) of Engle and Gallo (2006) proposed a separate volatility process for returns and realized measures, but employed two realized measures besides squared returns (intraday range and realized variance). As such, MEM formulated a separate GARCH equation for each of the realized measures, introducing a latent volatility process for each of them.

$$z_t = \frac{r_t - \mu_r}{\sqrt{\sigma_t}}$$

$$\sigma_t = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1} + \delta r_{t-1} + \varphi R_{t-1}^2$$

$$z_{R,t} = \frac{R_t - \mu_R}{\sqrt{\sigma_{R,t}}}$$

$$\sigma_{R,t} = \omega_R + \alpha_R R_{t-1}^2 + \beta_R \sigma_{R,t-1} + \delta_R r_{t-1}$$

$$z_{RV,t} = \frac{RV_t - \mu_{RV}}{\sqrt{\sigma_{RV,t}}}$$

$$\sigma_{RV,t} = \omega_{RV} + \alpha_{RV} RV_{t-1}^2 + \beta_{RV}\sigma_{RV,t-1} + \delta_{RV} r_{t-1} + \vartheta_{RV} RV_{t-1} 1_{(r_{t-1<0})} + \varphi_{RV} r_{t-1}^2$$

where σ_t describes volatility of the daily returns, while $\sigma_{R,t}$ and $\sigma_{RV,t}$ describe the volatility processes of the intraday range and of the realized variance.

The univariate High Frequency Based Volatility (HEAVY) model proposed by Shephard and Sheppard (2010) was nested in the MEM framework, but unlike MEM model, it contains a separate conditional volatility process for only one intraday volatility measure used (that is realized kernels) (two latent volatility processes as against MEM that had three such processes):

$$z_t = \frac{r_t - \mu_r}{\sqrt{\sigma_t}}$$

$$\sigma_t = \omega + \alpha r_{t-1}^2 + \beta\sigma_{t-1} + \gamma x_{t-1}$$

$$z_t = \frac{r_t - \mu_r}{\sqrt{\sigma_t}}$$

$$\nu_t = \omega_R + \alpha_R x_{t-1} + \beta_R \mu_{t-1}$$

where σ_t represents the volatility process of the asset returns and ν_t represents the volatility process of the realized measures used.

The fact that such models work with more latent volatility processes by employing a parallel GARCH structure infers that a high number of coefficients need to be estimated (four for each of the two or three processes), which makes the estimating problem rather complicated.

An important step in the field of volatility forecasting by using intraday measures has been made by Hansen, Huang and Shek (2010a) who proposed a new GARCH-type model with an endogenous realized measure of intraday volatility linked to the return variance by a measurement equation. The proposed Realized GARCH model maintains a single volatility-factor structure of the traditional GARCH model and, as compared to MEM and HEAVY models which introduced separate volatility equations for each of the realized measure, proposes an additional equation which models the natural relationship between the realized measure and the conditional return variance produced by the model, avoiding as such to introduce additional latent volatility factors. In its linear formulation, the Realized GARCH(1,1) model takes the form

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$\sigma_t = \omega + \beta\sigma_{t-1} + \gamma x_{t-1}$$

$$x_t = \xi + \varphi\sigma_t + \tau(z_t) + u_t$$

where $\sigma_t = var(r_t|F_{t-1})$ represents the conditional variance, x_t the realized measure as a consistent estimator of the integrated variance, $z_t \sim iid(0, \sigma_u^2)$ the studentized returns, and u_t are the random innovations. The first equation is called the return equation and the second one is called the GARCH equation. As it can be easily observed, the first two equations form a typical GARCH-X model, if x_t is taken as exogenous. The novelty of the Realized GARCH model introduced by Hansen, Huang and Shek (2010a) is the third equation (that is called the measurement equation) that links x_t to σ_t. The advantages of including into a GARCH structure of a measurement equation that defines the realized measure through a linear relationship with the conditional variance, instead of regressing it against its past lagged values, nests the model in a simple, tractable GARCH structure and offers an elegant formulation of the dependence between shocks to returns and shocks to volatility, known as the leverage effect. A simple but effective formulation of the leverage function $\tau(z)$ is

$$\tau(z) = \tau_1 z + \tau_2(z^2 - 1)$$

that allows for an asymmetric response in volatility to return shocks. A convenient version to work with of the Realized GARCH(1,1) model is the log-linear specification

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$log\sigma_t = \omega + \beta log\sigma_{t-1} + \gamma log x_{t-1}$$

$$log x_t = \xi + \varphi log\sigma_t + \delta z_t + u_t$$

and we shall confine to this formulation in the empirical exercise to follow. To this model it can be written the volatility shock formula

$$\gamma\{\delta z_t + u_t\}$$

with the coefficients defined as in the previous model.

The Realized EGARCH model proposed by Hansen, Huang and Shek (2010a) slightly extends the Realized GARCH model. The difference constitutes in the way the leverage function in the measurement equation is defined, as it contains two parts instead of one. The log-linear form of the Realized EGARCH(1,1) model that we will use is defined as follows

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$log\sigma_t = \omega + \beta log\sigma_{t-1} + \left(\tau_1 z_{t-1} + \tau_2 (z_{t-1}^2 - 1)\right) + \gamma log x_{t-1}$$

$$log x_t = \xi + \varphi log\sigma_t + \delta z_t + u_t$$

The leverage function sums the leverage effects (shocks which can be explained by returns) $\tau_1 z_{t-1}$ and those shocks that are uncorrelated to the leverage effects $\tau_2(z_{t-1}^2 - 1)$.

The volatility shock may be calculated as it follows

$$\tau z_t + \gamma\{\delta z_t + u_t\} = \tau_1 z_t + \tau_1(z_t^2 - 1) + \gamma\{\delta_1 z_t + \delta_2(z_t^2 - 1) + u_t\}$$

The Realized EGARCH(1,1) model represents an improved version of the EGARCH(1,1) model (Nelson (1991)), built upon estimations of day and intraday volatility:

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$\sigma_t = \omega + \sum_{k=1}^{p} \beta_k \sigma_{t-k} + \sum_{k=1}^{p} (\tau_1 z_{t-k} + \tau_2(|z_{t-k}| - E(|z_{t-k}|))$$

We will use the EGARCH(1,1) model in its logarithmic form

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$log\sigma_t = \omega + \beta log\sigma_{t-1} + \tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)$$

Another model we will use in the empirical exercise is the Realized GARCH(2,2) model, which is built on a 2-lag Realized GARCH structure.

$$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$$

$$log\sigma_t = \omega + \alpha log(max(r_{t-1}^2, 10^{-20})) + \beta_1 log\sigma_{t-1} + \beta_2 log\sigma_{t-2} + \gamma_1 logx_{t-1} + \gamma_2 logx_{t-2}$$

$$logx_t = \xi + \varphi log\sigma_t + (\tau_1 z_t + \tau_2(z_t^2 - 1)) + u_t$$

The volatility shock formula for the Realized GARCH(2,2) model is

$$\gamma_1 * \{\tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1) + u_{t-1}\} + \gamma_2 * \{\tau_1 z_{t-2} + \tau_2(z_{t-2}^2 - 1) + u_{t-2}\}.$$

Chapter three: Estimating a class of realized volatility models

3.1 Introduction

Chapter three proposes to undertake an extensive benchmarking that will seek to compare the performances of some of the recently proposed realized models, as well the gain in accuracy obtained when using one type of intraday realized measures against another type. In chapter's first stance, the models will be estimated across different criteria. There will be considered two simple models, with no measurement of intraday volatility - GARCH(1,1)) (proposed by Bollerslev (1986)) and EGARCH(1,1) (proposed by Nelson (1991)), one realized model with an exogenous measurement of intraday volatility – EGARCH-X (a version of the GARCH-X model proposed by Engle (2002)), and three realized models with endogenous measurements of intraday volatility – Realized EGARCH(1,1), Realized GARCH(1,1), Realized GARCH(2,2) (Hansen, Huang and Shek (2010a)). These models will be estimated in sample and out of sample, the realized ones using six measures of intraday volatility: high-low, realized kernels and realized variances sampled at 15 seconds, 5 minutes, 15 minutes, and 20 minutes. The models will be estimated separately for each of the four stocks considered: AIG, AXP, BAC and JPM, and will be measured, with each estimation, the maximized loglikelihood function and three loss functions (RMSE, MAE, and MAPE).

In second instance, there will be undertaken rankings of the models for each measure of intraday volatility used, and according to the maximized loglikelihood function and the three loss functions calculated. The functions' values will be normalized, as described in the methodological section of the chapter, and will be combined in order to obtain general rankings of the models. As well, there will be obtained rankings of the realized measures for each estimated model, having as measure of fit the maximized loglikelihood functions and the three loss functions mentioned above. The functions' values will be also normalized in order to combine the rankings. A general ranking of the realized measures used will be obtained.

3.2 The models

The table that follows resumes the list of the models that will be used in the first empirical exercise. GARCH(1,1) and EGARCH(1,1) models are simple models with no intraday measures, the others will comprise various types of measures of intraday volatility.

Model*	**Return Equation***	**GARCH Equation***	**Measurement equation***
GARCH(1,1)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$\sigma_t = \omega + \alpha r_{t-1}^2 + \beta \sigma_{t-1}$	-
EGARCH(1,1) (loglikelihood form)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)$	-
EGARCH-X(1,1) (loglikelihood form)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \gamma log x_{t-1} + \tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)$	-
Realized GARCH(1,1) (loglikelihood form)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \gamma log x_{t-1}$	$log x_t = \xi + \varphi log\sigma_t + \delta z_t + u_t$
Realized EGARCH(1,1) (loglikelihood form)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \left(\tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)\right) + \gamma log x_{t-1}$	$log x_t = \xi + \varphi log\sigma_t + \delta z_t + u_t$
Realized GARCH(2,2) (loglikelihood form)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \alpha log(max(r_{t-1}^2, 10^{-20})) + \beta_1 log\sigma_{t-1} + \beta_2 log\sigma_{t-2} + \gamma_1 log x_{t-1} + \gamma_2 log x_{t-2}$	$log x_t = \xi + \varphi log\sigma_t + (\tau_1 z_t + \tau_2(z_t^2 - 1)) + u_t$

*Notations were maintained as they appear in Hansen, Huang and Shek (2010a)

3.3 Data and Methodology

The models will be estimated in sample and out of sample, for each estimation being considered six measures of intraday volatility – high-low, realized kernel, realized variance sampled at 15 seconds, 5 minutes, 15 minutes and 20 minutes. The models will be estimated by maximizing the loglikelihood function, and will be evaluated according to four criteria: the value of the maximum loglikelihood function, and the values of three loss functions: the root mean squared error (RMSE), the mean absolute error (MAE) and the mean absolute percentage error (MAPE).

Each of the twelve tests will be assessed according to each of the four above mentioned criteria. There will be made classifications that will highlight: how models rank against each other for each measure of intraday volatility considered and how the measures of intraday volatility rank for each estimated model.

The source of the data used is the Department of Economics of the Stanford University, through the kind help of Professor Peter Reinhard Hansen and of Professor Zhuo (Albert) Huang, and has been made available to me during the research stage I spent at Stanford University during April-June 2010 period. The price quotations, including the measures of intraday data (realized kernels, realized variance sampled at 15 seconds, 5, 15 and 20 minutes) were provided through the kind help of them, thing for which I feel very indebted. The High-Low data was calculated as a simple difference between the daily highest and lowest prices for each of the stocks considered.

The data represents 3436 observations long daily time series comprising price information of four stocks (AIG - American International Group, AXP - American Express, BAC - Bank of America and JPM – J.P. Morgan) over January 4, 1995 – September 30, 2008 period. The daily price data used had multiple expressions: open prices, close prices, opening and closing times, number of trades taking place each day, various measures of intraday volatility, like realized kernels, highest and lowest prices of each trading day, and the realized variance sampled at 15 seconds, 5 minute, 15 minutes and 20 minutes.

In order to avoid the outliers that would result from 'quiet' days, the data was cleaned by removing the half trading days around the Christmas and the Thanksgiving Days, the length of the trading day being assessed after the number of trades taken place that day with each stock considered. All time series were equally adjusted so that they have equal length and correspondence of the prices for each day. This means that if one day was excluded for a stock, the same day was excluded for the other stocks as well.

The daily returns were calculated in logarithmic form, as follows:

$$r_t = 100 * (\log(closing\ price_t) - \log(opening\ price_t))$$

for each trading day t, $1 \leq t \leq 3436$.

The option towards the calculation of the daily returns as price variation from the opening to the closing of each trading day, instead as price variation from the closing of one trading day to the closing of the next trading day, may be justified as follows: in some days, varying from one company to another, the stocks were split in a number of stocks with smaller values. Since the variation in price from the last trading of one day to the first trading of the next day was not due to the underlying latent volatility of that asset but due to the administrative decisions related to the number of shares each company held, such price variations should have not been modeled by the models. As such, returns were calculated by measuring the variation of price from the beginning of each trading day to the price corresponding to the closing of the same trading day, as the trading day volatility was considered only.

However, in chapter five, when there will be proposed bivariate models, such restriction does not necessarily need to hold anymore (although we will keep it for the bivariate estimations) since those models will be specifically designed to measure both the day and night volatility. To avoid days in which price variations from one day to another existed due to stock splits, the time series were shortened as such during the sampled period no such price drops would have taken place (only for bivariate estimations).

As mentioned, one GARCH(1,1) model, one EGARCH(1,1) model, one EGARCH-X(1,1) model, one Realized GARCH(1,1) model, one Realized EGARCH(1,1) model, and one Realized GARCH(2,2)) will be estimated in sample and out of sample for each of the four stocks, and for each measure of intraday volatility (realized kernels, high-low, and realized variance sampled at 15 seconds, 5 minutes, 15 minutes and 20 minutes) . The in sample estimation was made by maximizing the loglikelihood function over the whole sample (considering all 3436 observations), subject to some constraints, and then by measuring the size of the maximized loglikelihood function over the same 3436-long sample. As such, there will be estimated coefficients that allow the best fit of the model over the whole sample.

The out of sample estimation was made by maximizing the loglikelihood function over shorter time series of sampled data (the first 2000 observations/days). This is considered as a better predictive test as it actually finds the estimators for a best fit over an interval, but then actually measures the forecasting accuracy of the model for the next period, not included in the model's estimation.

In both cases, the model's predictive capacity was measured by the value of the maximum log-likelihood function (corresponding to the returns), and by the size of three loss functions: root mean squared error (RMSE), mean absolute error (MAE), and mean absolute percentage error (MAPE), calculated as follows:

$$RMSE = \sqrt{\frac{\sum_{t=1}^{n}(\sigma_t - r_t^2)^2}{n}}$$

$$MAE = \frac{\sum_{t=1}^{n}|\sigma_t - r_t^2|}{n}$$

$$MAPE = \frac{\sum_{t=1}^{n}\left|\frac{\sigma_t - r_t^2}{r_t^2}\right|}{n}.$$

Higher the maximum loglikelihood function and lower the loss functions (over the whole period for in sample estimations and over the last 1436 days for out of sample estimations), better fit of the model over the population the sampled data belongs to.

The log-likelihood function is a joint likelihood function formed from two partial functions. The first partial log-likelihood function (L_1) represents the model fit to the return values, while the second partial function (L_2) measures the fit to the realized measures, x_t. With respect to the models that do not comprise measures of intraday data (like GARCH and EGARCH), the total loglikelihood function is L_1. With respect to all models, the loglikelihood function to be maximized will be $L = L_1 + L_2$, but what will be actually measured and accounted for ranking purposes, will be L_1, since what is compared is the model's capacity in describing and forecasting volatility of returns. As such, in each estimation it will be maximized L, but what we account for in order to measure the fitness of the model will be L_1 only.

The loglikelihood functions corresponding to the above models are $l(r,x) = -\frac{1}{2}\sum_{t=1}^{n}\left\{log(2\pi) + log(\sigma_t) + \frac{(r_t-\mu)^2}{\sigma_t}\right\}$ (EGARCH and EGARCH-X) and $l(r,x) =$

$-\frac{1}{2}\sum_{t=1}^{n}\left\{2\,log(2\pi)+log(\sigma_t)+\frac{(r_t-\mu)^2}{\sigma_t}+log(\sigma_u^2)+\frac{u_t^2}{\sigma_u^2}\right\}$ (Realized EGARCH, Realized GARCH and Realized GARCH(2,2)).

The rankings to be obtained will rank the models (in first instance) and then the realized measures (in second instance), according to the corresponding maximized loglikelihood and to each of the three loss functions. In order to combine the rankings, the values of the functions will be normalized according to the formula

$$x_{normalized,t}=\frac{x_t-a_t}{b_t-a_t}$$

where x_t represents the value of the corresponding calculated function, a_t the function value of the highest ranked model or realized measure, and b_t the value of the lowest ranked model or realized value. By normalizing the loglikelihood function and the loss function values, in each ranking, we will be able afterwards to add them across the rankings for the criteria considered. After normalization, the values of each model and realized measure will be summed across the rankings, in order to obtain the general rankings. The sum of the normalized values of each model and of each realized measure will give the position of that model and measure in the final rankings.

3.4 Results

The maximized loglikelihood functions, and the loss functions calculated with each estimation, are summarized in the following tables:

Model	AIG											
	Maximum log-likelihood function											
	In sample						Out of sample					
	H-L	RK	15’’	5’	15’	20’	H-L	RK	15’’	5’	15’	20’
EGARCH(1,1)	-6.258,9	-6.258,9	-6.258,9	-6.258,9	-6.258,9	-6.258,9	-949,9	-949,9	-949,9	-949,9	-949,9	-949,9
EGARCH-X	-6.239,4	-6.153,4	-6.207,6	-6.164,5	-6.138,1	-6.149,7	-977,9	-900,6	-918,3	-901,7	-897,2	-900,4
GARCH(1,1)	-6.215,1	-6.215,1	-6.215,1	-6.215,1	-6.215,1	-6.215,1	-921,7	-921,7	-921,7	-921,7	-921,7	-921,7
Realized EGARCH(1,1)	-6.429,3	-6.173,3	-6.244,6	-6.188,0	-6.151,3	-6.168,5	-2.623,0	-926,1	-920,5	-928,3	-910,6	-917,5

Realized GARCH(2,2)	-6.848,2	-6.145,1	-6.219,6	-6.164,4	-6.129,7	-6.147,9	-3.469,9	-933,6	-920,0	-940,4	-923,2	-932,5
Realized GARCH(1,1)	-7.562,8	-6.161,7	-6.233,4	-6.177,9	-6.146,0	-6.162,8	-3.303,3	-900,5	-899,5	-902,2	-901,8	-904,1

Model	**AXP**											
	Maximum log-likelihood function											
	In sample						**Out of sample**					
	H-L	**RK**	**15’’**	**5’**	**15’**	**20’**	**H-L**	**RK**	**15’’**	**5’**	**15’**	**20’**
EGARCH(1,1)	-6.472,2	-6.472,2	-6.472,2	-6.472,2	-6.472,2	-6.472,3	-915,9	-962,2	-915,9	-915,9	-915,9	-915,9
EGARCH-X	-6.466,4	-6.397,1	-6.461,9	-6.430,6	-6.408,6	-6.408,3	-916,3	-899,3	-913,0	-903,1	-899,3	-902,1
GARCH(1,1)	-6.488,9	-6.488,9	-6.488,9	-6.488,9	-6.488,9	-6.488,9	-919,0	-919,0	-919,0	-919,0	-919,0	-919,0
Realized EGARCH(1,1)	-6.640,3	-6.404,4	-6.546,3	-6.438,0	-6.415,5	-6.415,4	-945,1	-898,6	-900,5	-898,3	-897,9	-900,5
Realized GARCH(2,2)	-6.620,0	-6.369,1	-6.523,3	-6.410,1	-6.389,4	-6.392,3	-944,8	-898,8	-901,2	-900,1	-897,8	-901,3
Realized GARCH(1,1)	-6.643,6	-6.408,9	-6.558,5	-6.447,9	-6.428,6	-6.430,1	-951,2	-898,6	-900,7	-898,2	-897,2	-900,6

Model	**BAC**											
	Maximum log-likelihood function											
	In sample						**Out of sample**					
	H-L	**RK**	**15’’**	**5’**	**15’**	**20’**	**H-L**	**RK**	**15’’**	**5’**	**15’**	**20’**
EGARCH(1,1)	-6.131,3	-6.131,3	-6.131,3	-6.131,3	-6.131,3	-6.131,3	-884,0	-884,0	-884,0	-884,0	-884,0	-884,0
EGARCH-X	-6.122,3	-6.086,4	-6.095,5	-6.083,3	-6.087,8	-6.090,3	-889,9	-858,6	-868,8	-857,1	-861,0	-863,4
GARCH(1,1)	-6.147,3	-6.147,3	-6.147,3	-6.147,3	-6.147,3	-6.147,3	-890,5	-890,5	-890,5	-890,5	-890,5	-890,5
Realized EGARCH(1,1)	-6.222,7	-6.088,5	-6.111,4	-6.088,0	-6.094,1	-6.100,7	-1.072,6	-854,4	-859,7	-855,6	-856,9	-859,2
Realized GARCH(2,2)	-6.297,6	-6.052,8	-6.094,5	-6.058,4	-6.056,5	-6.069,1	-1131,9	-850,8	-858,6	-853,8	-854,6	-856,0
Realized GARCH(1,1)	-6.351,3	-6.100,0	-6.128,7	-6.100,3	-6.105,6	-6.116,7	-1.110,5	-853,4	-856,5	-853,6	-854,4	-855,8

Model	**JPM**											
	Maximum log-likelihood function											
	In sample						**Out of sample**					
	H-L	**RK**	**15’’**	**5’**	**15’**	**20’**	**H-L**	**RK**	**15’’**	**5’**	**15’**	**20’**
EGARCH(1,1)	-6.452,1	-6.452,1	-6.452,1	-6.452,1	-6.452,1	-6.452,1	-921,7	-921,7	-921,7	-921,7	-921,7	-921,7
EGARCH-X	-6.439,1	-6.396,0	-6.412,0	-6.387,4	-6.388,7	-6.395,2	-922,0	-898,1	-901,4	-896,5	-897,2	-899,4

GARCH(1,1)	-6.471,7	-6.471,7	-6.471,7	-6.471,7	-6.471,7	-6.471,7	-922,9	-922,9	-922,9	-922,9	-922,9	-922,9
Realized EGARCH(1,1)	-6.716,9	-6.404,1	-6.432,5	-6.392,2	-6.390,5	-6.397,4	-958,5	-895,0	-891,9	-894,3	-895,7	-897,4
Realized GARCH(2,2)	-6.951,0	-6.376,3	-6.411,9	-6.363,3	-6.364,0	-6.375,1	-974,2	-894,6	-891,4	-893,1	-893,0	-894,4
Realized GARCH(1,1)	-6.825,8	-6.421,9	-6.451,8	-6.409,5	-6.409,1	-6.419,3	-989,6	-896,7	-893,3	-896,3	-896,7	-899,0

Model	**AIG**																	
	Errors																	
	In sample																	
	H-L			**RK**			**15''**			**5'**			**15'**			**20'**		
	RMSE	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**
EGARCH(1,1)	109,978	7,200	91,668	109,978	7,200	91,669	109,978	7,200	91,668	109,978	7,200	91,668	109,978	-1,465	7,200	91,668	109,978	7,200
EGARCH-X	110,955	7,436	82,658	107,371	7,033	79,635	119,619	8,537	80,875	107,322	6,969	76,896	108,297	-0,936	7,022	75,570	107,664	6,862
GARCH(1,1)	109,866	6,953	81,602	109,866	6,953	81,602	109,866	6,953	81,602	109,866	6,953	81,602	109,866	-1,141	6,953	81,602	109,866	6,953
Realized EGARCH(1,1)	111,256	7,112	82,232	107,455	6,382	79,366	106,923	6,436	82,177	107,718	6,365	77,414	107,657	-2,277	6,487	75,069	107,723	6,436
Realized GARCH(2,2)	113,429	9,293	130,225	107,776	6,326	78,228	107,130	6,388	82,333	108,045	6,351	76,356	108,249	-2,391	6,468	74,264	108,150	6,392
Realized GARCH(1,1)	113,881	9,970	167,169	108,764	7,321	80,492	112,315	7,758	82,570	107,752	7,194	77,919	109,776	-0,800	7,247	76,364	108,641	7,141

Model	**AIG**																	
	Errors																	
	Out of sample																	
	H-L			**RK**			**15''**			**5'**			**15'**			**20'**		
	RMSE	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**
EGARCH(1,1)	308,680	33,457	162,269	308,680	33,457	162,269	308,680	33,457	162,269	308,680	33,457	162,269	308,680	33,457	162,269	308,680	33,457	162,269
EGARCH-X	309,942	32,098	120,727	303,272	33,463	134,337	306,515	33,760	156,403	303,894	33,423	129,100	303,730	33,866	125,610	304,268	33,495	127,927
GARCH(1,1)	306,730	33,830	159,941	306,730	33,830	159,941	306,730	33,830	159,941	306,730	33,830	159,941	306,730	33,830	159,941	306,730	33,830	159,941
Realized EGARCH(1,1)	318,500	33,650	137,646	306,100	32,082	137,238	303,748	32,281	155,271	306,557	32,080	133,778	304,771	32,693	127,307	305,670	32,414	134,399
Realized GARCH(2,2)	318,290	33,627	138,881	308,553	31,715	132,371	304,869	31,777	155,405	309,263	31,754	128,675	308,812	32,115	121,721	309,147	31,810	125,542
Realized GARCH(1,1)	318,619	33,638	109,373	303,002	33,393	136,283	300,538	35,315	159,358	303,442	33,153	131,694	305,234	33,916	125,537	305,267	33,523	130,586

Model	AXP																	
	Errors																	
	In sample																	
	H-L			RK			15’’			5’			15’			20’		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE
EGARCH(1,1)	7,470	3,604	81,854	7,470	3,603	81,839	7,470	3,604	81,854	7,470	3,604	81,877	7,470	3,604	81,854	7,471	3,599	81,429
EGARCH-X	7,466	3,599	81,544	7,290	3,529	80,941	7,439	3,583	78,833	7,314	3,531	74,028	7,323	3,528	76,818	7,332	3,530	76,533
GARCH(1,1)	7,524	3,656	80,685	7,524	3,656	80,685	7,524	3,656	80,685	7,524	3,656	80,685	7,524	3,656	80,685	7,524	3,656	80,685
Realized EGARCH(1,1)	8,359	4,262	96,256	7,305	3,516	79,334	7,472	3,693	73,975	7,318	3,519	72,602	7,337	3,515	75,780	7,342	3,530	75,588
Realized GARCH(2,2)	8,437	4,311	94,548	7,317	3,506	76,007	7,521	3,722	72,185	7,335	3,541	72,493	7,372	3,517	72,478	7,377	3,531	72,485
Realized GARCH(1,1)	8,004	3,988	94,177	7,335	3,555	79,334	7,501	3,722	73,812	7,344	3,558	72,063	7,376	3,559	75,297	7,382	3,559	74,901

Model	AXP																	
	Errors																	
	Out of sample																	
	H-L			RK			15’’			5’			15’			20’		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE
EGARCH(1,1)	9,959	5,128	167,936	12,498	5,683	154,289	9,959	5,128	167,937	9,959	5,128	167,937	9,959	5,128	167,937	9,959	5,128	167,937
EGARCH-X	9,980	5,077	168,064	9,444	5,132	192,768	9,891	5,123	168,698	9,672	5,003	166,571	9,591	5,042	179,150	9,698	5,075	179,160
GARCH(1,1)	9,946	5,208	164,936	9,946	5,208	164,936	9,946	5,208	164,936	9,946	5,208	164,936	9,946	5,208	164,936	9,946	5,208	164,936
Realized EGARCH(1,1)	10,429	4,977	179,342	9,445	5,095	184,196	9,492	4,941	170,022	9,544	4,919	164,278	9,567	5,016	173,782	9,648	5,080	175,361
Realized GARCH(2,2)	10,376	5,006	164,756	9,430	5,015	166,912	9,548	4,926	160,594	9,558	4,844	147,590	9,552	4,914	154,535	9,624	4,978	155,181
Realized GARCH(1,1)	10,480	4,957	172,299	9,385	5,157	188,268	9,462	4,946	168,918	9,480	4,941	164,513	9,495	5,062	175,605	9,615	5,096	174,553

Model	BAC																	
	Errors																	
	In sample																	
	H-L			RK			15’’			5’			15’			20’		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE
EGARCH(1,1)	10,743	3,605	83,150	10,743	3,605	83,150	10,743	3,605	83,150	10,743	3,605	83,150	10,743	3,605	83,150	10,743	3,605	83,151
EGARCH-X	10,765	3,574	81,843	10,305	3,546	87,703	10,346	3,532	73,989	10,269	3,513	83,668	10,356	3,540	85,291	10,398	3,544	84,893
GARCH(1,1)	10,788	3,717	87,312	10,788	3,717	87,312	10,788	3,717	87,312	10,788	3,717	87,312	10,788	3,717	87,312	10,788	3,717	87,312
Realized EGARCH(1,1)	11,166	3,839	91,930	10,319	3,497	88,839	10,477	3,486	76,042	10,348	3,457	84,272	10,385	3,477	86,601	10,411	3,470	87,081
Realized GARCH(2,2)	11,544	4,188	107,268	10,320	3,499	91,893	10,509	3,511	77,713	10,381	3,461	86,074	10,493	3,481	91,496	10,487	3,489	92,819
Realized GARCH(1,1)	11,535	4,009	104,661	10,397	3,573	92,926	10,463	3,552	77,711	10,357	3,539	88,292	10,450	3,579	92,925	10,411	3,579	95,018

Model	BAC																	
	Errors																	
	Out of sample																	
	H-L			RK			15’’			5’			15’			20’		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE
EGARCH(1,1)	25,565	8,172	93,437	25,565	8,172	93,437	25,565	8,172	93,437	25,565	8,172	93,437	25,565	8,172	93,437	25,565	8,172	93,437
EGARCH-X	25,931	7,723	80,977	24,379	8,003	111,922	25,110	7,870	96,174	24,481	7,775	101,233	24,678	7,967	103,307	24,810	7,997	101,778
GARCH(1,1)	25,551	8,504	95,643	25,551	8,504	95,643	25,551	8,504	95,643	25,551	8,504	95,643	25,551	8,504	95,643	25,551	8,504	95,643
Realized EGARCH(1,1)	27,837	7,732	53,212	24,166	7,778	111,482	25,023	7,416	88,625	24,432	7,525	96,929	24,428	7,707	104,877	24,578	7,756	103,663
Realized GARCH(2,2)	27,958	7,727	47,461	24,202	7,656	107,684	25,133	7,363	85,278	24,590	7,432	92,874	24,769	7,553	98,816	24,867	7,652	98,169
Realized GARCH(1,1)	27,905	7,744	51,084	24,218	7,973	117,059	24,844	7,443	90,560	24,263	7,701	101,494	24,380	7,920	112,971	24,409	7,946	111,773

Model	JPM																	
	Errors																	
	In sample																	
	H-L			RK			15’’			5’			15’			20’		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE
EGARCH(1,1)	14,876	4,232	91,832	14,876	4,232	91,832	14,876	4,232	91,832	14,876	4,232	91,832	14,876	4,232	91,832	14,876	4,232	91,832
EGARCH-X	14,747	4,166	88,796	14,550	4,109	99,946	14,475	4,161	94,243	14,517	4,098	95,287	14,648	4,106	95,249	14,616	4,127	94,452
GARCH(1,1)	15,190	4,337	94,224	15,190	4,337	94,224	15,190	4,337	94,224	15,190	4,337	94,224	15,190	4,337	94,224	15,190	4,337	94,224
Realized EGARCH(1,1)	15,384	4,756	105,189	14,486	4,075	101,907	14,511	4,051	94,010	14,476	4,059	95,075	14,633	4,103	95,598	14,588	4,127	95,222
Realized GARCH(2,2)	15,335	4,987	150,324	14,641	4,099	101,593	14,642	4,097	94,318	14,639	4,079	94,511	14,791	4,124	95,093	14,743	4,140	94,835
Realized GARCH(1,1)	15,476	4,622	107,646	14,738	4,102	103,434	14,690	4,178	96,372	14,736	4,126	97,031	14,845	4,146	96,856	14,822	4,159	96,410

Model	JPM																	
	Errors																	
	Out of sample																	
	H-L			RK			15’’			5’			15’			20’		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE	RMSE	MAE	MAPE
EGARCH(1,1)	17,076	6,291	209,547	17,076	6,291	209,547	17,076	6,291	209,547	17,076	6,291	209,547	17,076	6,291	209,547	17,076	6,291	209,547
EGARCH-X	17,138	6,136	197,810	16,491	6,623	272,792	16,632	6,430	231,639	16,456	6,574	261,150	16,558	6,588	253,632	16,583	6,564	242,926
GARCH(1,1)	16,962	6,502	207,430	16,962	6,502	207,430	16,962	6,502	207,430	16,962	6,502	207,430	16,962	6,502	207,430	16,962	6,502	207,430

Realized EGARCH(1,1)	17,876	5,849	193,311	16,227	6,661	305,740	16,217	6,287	250,710	16,302	6,536	275,149	16,482	6,646	262,457	16,476	6,661	252,356
Realized GARCH(2,2)	18,019	5,868	195,067	16,070	6,731	288,947	16,148	6,284	236,838	16,177	6,581	255,354	16,458	6,747	247,157	16,287	6,731	237,353
Realized GARCH(1,1)	18,117	5,913	206,839	16,293	6,821	314,819	16,241	6,544	259,438	16,520	6,821	288,650	16,576	6,866	268,751	16,504	6,853	255,878

The models were ranked according to two goals: to rank the realized models for each intraday volatility measure used (plus GARCH and EGARCH models which don't have a measure of intraday volatility), and to rank the measures of intraday volatility for each type of realized model. The models and the realized volatility measures were ranked with respect to the size of the maximized loglikelihood functions (the higher the better) and with respect to the size of the three loss functions considered (the lower the better).

After normalization, the loglikelihood and loss functions will be:

	AIG		**AXP**	
	in sample	**out of sample**	**in sample**	**out of sample**
EGARCH	5,03	5,01	3,45	4,38
EGARCHX	0,17	0,40	0,85	1,12
GARCH	2,57	2,18	4,41	4,41
Realized EGARCH	1,73	2,75	3,00	0,87
Realized GARCH(2,2)	0,70	4,02	1,50	1,02
Realized GARCH(1,1)	2,05	1,11	3,60	1,01

	BAC		**JPM**	
	in sample	**out of sample**	**in sample**	**out of sample**
EGARCH	4,00	4,10	3,92	4,80
EGARCHX	1,27	1,08	0,87	0,88
GARCH	5,11	5,03	5,06	5,02
Realized EGARCH	2,29	1,17	1,92	0,81
Realized GARCH(2,2)	0,77	1,08	1,00	0,77
Realized GARCH(1,1)	3,77	0,98	3,20	1,53

	AIG		**AXP**	
	in sample	**out of sample**	**in sample**	**out of sample**
RK	0,262	0,056	0,000	0,062
H-L	4,000	4,000	4,000	4,000
RV5m	0,463	0,075	0,955	0,297

RV15m	0,000	0,002	0,378	0,005
RV15s	1,208	0,267	2,789	0,996
RV20m	0,214	0,050	0,391	0,358

	BAC		JPM	
	in sample	**out of sample**	**in sample**	**out of sample**
RK	0,083	0,046	0,262	0,182
H-L	4,000	4,000	4,000	4,000
RV5m	0,024	0,017	0,006	0,088
RV15m	0,199	0,146	0,026	0,137
RV15s	0,770	0,420	0,789	0,192
RV20m	0,407	0,241	0,217	0,289

	AIG						**AXP**					
	In sample			**Out of sample**			**Out of sample**			**Out of sample**		
	RMSE	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**
EGARCH	4,27	4,28	5,12	4,94	4,46	6,00	3,33	2,61	5,08	5,02	4,56	3,07
EGARCHX	1,55	3,89	0,23	1,13	4,00	0,62	0,00	0,33	2,70	1,80	2,65	5,02
GARCH	4,05	2,92	1,43	3,08	5,53	5,36	5,06	4,60	4,35	4,07	5,29	2,43
Realized EGARCH	0,55	0,23	0,36	2,96	1,99	1,23	1,52	1,79	2,51	1,37	1,25	5,22
Realized GARCH(2,2)	1,80	0,78	0,72	5,48	0,88	0,58	2,66	2,18	0,93	1,31	0,20	0,33
Realized GARCH(1,1)	3,46	5,56	1,68	1,50	5,20	1,04	2,14	2,70	2,18	1,00	1,57	4,78

	BAC						**JPM**					
	In sample			**Out of sample**			**Out of sample**			**Out of sample**		
	RMSE	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**
EGARCH	4,50	2,71	0,74	5,01	3,85	1,77	2,76	2,98	0,05	5,10	0,70	1,16
EGARCHX	0,03	1,20	0,93	1,44	2,01	4,41	0,22	0,67	3,15	1,94	3,16	3,50
GARCH	5,06	5,23	3,23	4,94	6,00	2,60	5,61	5,21	2,28	4,31	3,38	0,87
Realized EGARCH	1,11	0,43	2,04	1,52	0,57	2,80	0,92	0,72	3,73	1,44	2,45	4,41
Realized GARCH(2,2)	2,16	1,23	4,41	2,40	0,00	1,14	2,03	1,48	4,21	0,91	2,98	3,29
Realized GARCH(1,1)	1,87	2,52	5,18	1,02	1,45	4,56	2,79	1,69	5,31	2,17	5,10	5,83

	AIG						AXP					
	In sample			**Out of sample**			**Out of sample**			**Out of sample**		
	RMSE	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**
RK	0,39	0,19	1,29	0,57	0,88	1,59	0,00	0,01	1,70	0,00	4,00	4,00
H-L	3,30	3,34	4,00	4,00	2,22	0,88	4,00	4,00	4,00	4,00	1,92	2,03
RV5m	0,33	0,09	0,57	0,77	0,77	1,12	0,18	0,09	0,01	0,75	0,00	0,00
RV15m	0,76	0,34	0,00	0,69	1,95	0,46	0,33	0,02	0,66	0,63	1,82	1,78
RV15s	1,74	1,33	1,95	0,49	2,10	4,00	1,44	1,66	0,78	1,08	1,56	1,23
RV20m	0,52	0,12	0,42	0,86	1,22	0,99	0,40	0,08	0,60	1,10	2,97	1,85

	BAC						JPM					
	In sample			**Out of sample**			**Out of sample**			**Out of sample**		
	RMSE	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**	**RMSE**	**MAE**	**MAPE**
RK	0,11	0,77	2,85	0,00	3,80	4,00	0,35	0,23	2,46	0,09	3,93	4,00
H-L	4,00	4,00	3,57	4,00	2,44	0,00	4,00	4,00	3,00	4,00	0,00	0,00
RV5m	0,08	0,00	1,90	0,25	1,16	2,92	0,21	0,06	0,74	0,25	3,51	2,97
RV15m	0,47	0,61	2,52	0,46	3,10	3,40	1,22	0,33	0,78	0,69	3,91	2,49
RV15s	0,59	0,49	0,00	1,12	0,53	2,33	0,04	1,08	0,49	0,30	2,28	1,89
RV20m	0,55	0,68	2,64	0,62	3,66	3,30	0,96	0,71	0,63	0,59	3,85	2,03

These enable us to obtain the general rankings across the models and realized measures used:

	AIG				**AXP**			
	in sample		**out of sample**		**in sample**		**out of sample**	
Highest ranked	EGARCHX	0,17	EGARCHX	0,40	EGARCHX	0,85	Realized EGARCH	0,87
↑	Realized GARCH(2,2)	0,70	Realized GARCH	1,11	Realized GARCH(2,2)	1,50	Realized GARCH	1,01
	Realized EGARCH	1,73	GARCH	2,18	Realized EGARCH	3,00	Realized GARCH(2,2)	1,02
	Realized GARCH	2,05	Realized EGARCH	2,75	EGARCH	3,45	EGARCHX	1,12
↓	GARCH	2,57	Realized GARCH(2,2)	4,02	Realized GARCH	3,60	EGARCH	4,38
Lowest ranked	EGARCH	5,03	EGARCH	5,01	GARCH	4,41	GARCH	4,41

	BAC				JPM			
	in sample		**out of sample**		**in sample**		**out of sample**	
Highest ranked	Realized GARCH(2,2)	0,77	Realized GARCH	0,98	EGARCHX	0,87	Realized GARCH(2,2)	0,77
	EGARCHX	1,27	EGARCHX	1,08	Realized GARCH(2,2)	1,00	Realized EGARCH	0,81
	Realized EGARCH	2,29	Realized GARCH(2,2)	1,08	Realized EGARCH	1,92	EGARCHX	0,88
	Realized GARCH	3,77	Realized EGARCH	1,17	Realized GARCH	3,20	Realized GARCH	1,53
Lowest ranked	EGARCH	4,00	EGARCH	4,10	EGARCH	3,92	EGARCH	4,80
	GARCH	5,11	GARCH	5,03	GARCH	5,06	GARCH	5,02

	AIG				AXP			
	in sample		**out of sample**		**in sample**		**out of sample**	
Highest ranked	rv15m	0,000	rv15m	0,002	rk	0,000	rv15m	0,005
	rv20m	0,214	rv20m	0,050	rv15m	0,378	rk	0,062
	Rk	0,262	rk	0,056	rv20m	0,391	rv5m	0,297
	rv5m	0,463	rv5m	0,075	rv5m	0,955	rv20m	0,358
Lowest ranked	rv15s	1,208	rv15s	0,267	rv15s	2,789	rv15s	0,996
	h-l	4,000	h-l	4,000	h-l	4,000	h-l	4,000

	BAC				JPM			
	in sample		**out of sample**		**in sample**		**out of sample**	
Highest ranked	rv5m	0,024	rv5m	0,017	rv5m	0,006	rv5m	0,088
	rk	0,083	rk	0,046	rv15m	0,026	rv15m	0,137
	rv15m	0,199	rv15m	0,146	rv20m	0,217	rk	0,182
	rv20m	0,407	rv20m	0,241	rk	0,262	rv15s	0,192
Lowest ranked	rv15s	0,770	rv15s	0,420	rv15s	0,789	rv20m	0,289
	h-l	4,000	h-l	4,000	h-l	4,000	h-l	4,000

	AIG											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	Realized EGARCH	0,55	Realized EGARCH	0,23	EGARCHX	0,23	EGARCHX	1,13	Realized GARCH(2,2)	0,88	Realized GARCH(2,2)	0,58
	EGARCHX	1,55	Realized GARCH(2,2)	0,78	Realized EGARCH	0,36	Realized GARCH(1,1)	1,50	Realized EGARCH	1,99	EGARCHX	0,62
	Realized GARCH(2,2)	1,80	GARCH	2,92	Realized GARCH(2,2)	0,72	Realized EGARCH	2,96	EGARCHX	4,00	Realized GARCH(1,1)	1,04

	Realized GARCH(1,1)	3,46	EGARCHX	3,89	GARCH	1,43	GARCH	3,08	EGARCH	4,46	Realized EGARCH	1,23
	GARCH	4,05	EGARCH	4,28	Realized GARCH(1,1)	1,68	EGARCH	4,94	Realized GARCH(1,1)	5,20	GARCH	5,36
Lowest ranked	EGARCH	4,27	Realized GARCH(1,1)	5,56	EGARCH	5,12	Realized GARCH(2,2)	5,48	GARCH	5,53	EGARCH	6,00

	AXP											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	EGARCHX	0,00	EGARCHX	0,33	Realized GARCH(2,2)	0,93	Realized GARCH(1,1)	1,00	Realized GARCH(2,2)	0,20	Realized GARCH(2,2)	0,33
	Realized EGARCH	1,52	Realized EGARCH	1,79	Realized GARCH(1,1)	2,18	Realized GARCH(2,2)	1,31	Realized EGARCH	1,25	GARCH	2,43
	Realized GARCH(1,1)	2,14	Realized GARCH(2,2)	2,18	Realized EGARCH	2,51	Realized EGARCH	1,37	Realized GARCH(1,1)	1,57	EGARCH	3,07
	Realized GARCH(2,2)	2,66	EGARCH	2,61	EGARCHX	2,70	EGARCHX	1,80	EGARCHX	2,65	Realized GARCH(1,1)	4,78
	EGARCH	3,33	Realized GARCH(1,1)	2,70	GARCH	4,35	GARCH	4,07	EGARCH	4,56	EGARCHX	5,02
Lowest ranked	GARCH	5,06	GARCH	4,60	EGARCH	5,08	EGARCH	5,02	GARCH	5,29	Realized EGARCH	5,22

	BAC											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	EGARCHX	0,03	Realized EGARCH	0,43	EGARCH	0,74	Realized GARCH(1,1)	1,02	Realized GARCH(2,2)	0,00	Realized GARCH(2,2)	1,14
	Realized EGARCH	1,11	EGARCHX	1,20	EGARCHX	0,93	EGARCHX	1,44	Realized EGARCH	0,57	EGARCH	1,77
	Realized GARCH(1,1)	1,87	Realized GARCH(2,2)	1,23	Realized EGARCH	2,04	Realized EGARCH	1,52	Realized GARCH(1,1)	1,45	GARCH	2,60
	Realized GARCH(2,2)	2,16	Realized GARCH(1,1)	2,52	GARCH	3,23	Realized GARCH(2,2)	2,40	EGARCHX	2,01	Realized EGARCH	2,80
	EGARCH	4,50	EGARCH	2,71	Realized GARCH(2,2)	4,41	GARCH	4,94	EGARCH	3,85	EGARCHX	4,41
Lowest ranked	GARCH	5,06	GARCH	5,23	Realized GARCH(1,1)	5,18	EGARCH	5,01	GARCH	6,00	Realized GARCH(1,1)	4,56

	JPM											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	EGARCHX	0,22	EGARCHX	0,67	EGARCH	0,05	Realized GARCH(2,2)	0,91	EGARCH	0,70	GARCH	0,87
	Realized EGARCH	0,92	Realized EGARCH	0,72	GARCH	2,28	Realized EGARCH	1,44	Realized EGARCH	2,45	EGARCH	1,16

	Realized GARCH(2,2)	2,03	Realized GARCH(2,2)	1,48	EGARCHX	3,15	EGARCHX	1,94	Realized GARCH(2,2)	2,98	Realized GARCH(2,2)	3,29
	EGARCH	2,76	Realized GARCH(1,1)	1,69	Realized EGARCH	3,73	Realized GARCH(1,1)	2,17	EGARCHX	3,16	EGARCHX	3,50
	Realized GARCH(1,1)	2,79	EGARCH	2,98	Realized GARCH(2,2)	4,21	GARCH	4,31	GARCH	3,38	Realized EGARCH	4,41
Lowest ranked	GARCH	5,61	GARCH	5,21	Realized GARCH(1,1)	5,31	EGARCH	5,10	Realized GARCH(1,1)	5,10	Realized GARCH(1,1)	5,83

	AIG											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	RV5m	0,33	RV5m	0,09	RV15m	0,00	RV15s	0,49	RV5m	0,77	RV15m	0,46
	Rk	0,39	RV20m	0,12	RV20m	0,42	Rk	0,57	Rk	0,88	h-l	0,88
	RV20m	0,52	Rk	0,19	RV5m	0,57	RV15m	0,69	RV20m	1,22	RV20m	0,99
	RV15m	0,76	RV15m	0,34	Rk	1,29	RV5m	0,77	RV15m	1,95	RV5m	1,12
	RV15s	1,74	RV15s	1,33	RV15s	1,95	RV20m	0,86	RV15s	2,10	Rk	1,59
Lowest ranked	h-l	3,30	h-l	3,34	h-l	4,00	h-l	4,00	h-l	2,22	RV15s	4,00

	AXP											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	Rk	0,00	Rk	0,01	RV5m	0,01	Rk	0,00	RV5m	0,00	RV5m	0,00
	RV5m	0,18	RV15m	0,02	RV20m	0,60	RV15m	0,63	RV15s	1,56	RV15s	1,23
	RV15m	0,33	RV20m	0,08	RV15m	0,66	RV5m	0,75	RV15m	1,82	RV15m	1,78
	RV20m	0,40	RV5m	0,09	RV15s	0,78	RV15s	1,08	h-l	1,92	RV20m	1,85
	RV15s	1,44	RV15s	1,66	Rk	1,70	RV20m	1,10	RV20m	2,97	h-l	2,03
Lowest ranked	h-l	4,00	h-l	4,00	h-l	4,00	h-l	4,00	Rk	4,00	Rk	4,00

	BAC											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	RV5m	0,08	RV5m	0,00	RV15s	0,00	Rk	0,00	RV15s	0,53	h-l	0,00
	Rk	0,11	RV15s	0,49	RV5m	1,90	RV5m	0,25	RV5m	1,16	RV15s	2,33
	RV15m	0,47	RV15m	0,61	RV15m	2,52	RV15m	0,46	h-l	2,44	RV5m	2,92
	RV20m	0,55	RV20m	0,68	RV20m	2,64	RV20m	0,62	RV15m	3,10	RV20m	3,30
	RV15s	0,59	Rk	0,77	Rk	2,85	RV15s	1,12	RV20m	3,66	RV15m	3,40
Lowest ranked	h-l	4,00	h-l	4,00	h-l	3,57	h-l	4,00	Rk	3,80	Rk	4,00

	JPM											
	In sample						**Out of sample**					
	RMSE		**MAE**		**MAPE**		**RMSE**		**MAE**		**MAPE**	
Highest ranked	RV15s	0,04	RV5m	0,06	RV15s	0,49	Rk	0,09	h-l	0,00	h-l	0,00
	RV5m	0,21	Rk	0,23	RV20m	0,63	RV5m	0,25	RV15s	2,28	RV15s	1,89
↕	Rk	0,35	RV15m	0,33	RV5m	0,74	RV15s	0,30	RV5m	3,51	RV20m	2,03
	RV20m	0,96	RV20m	0,71	RV15m	0,78	RV20m	0,59	RV20m	3,85	RV15m	2,49
Lowest	RV15m	1,22	RV15s	1,08	Rk	2,46	RV15m	0,69	RV15m	3,91	RV5m	2,97
ranked	h-l	4,00	h-l	4,00	h-l	3,00	h-l	4,00	Rk	3,93	Rk	4,00

3.5 Conclusions

The first conclusion that may be grasped by looking to the results is that models' ranking is sensitive to the type of estimation employed (in sample or out of sample), to the stock choice and to the type of criterion chosen for comparing the models' performance. However, common patterns may be found and general conclusions may be grasped.

With regards to the AIG stock, when models were ranked after the size of the maximized loglikelihood, the EGARCHX model ranked the best, while the EGARCH ranked worst, for both in sample and out of sample estimations. On a second place, Realized GARCH model ranked fairly well (better in the out of sample estimations). If looking to the size of the loss functions, we see that EGARCHX model ranked very well again, while EGARCH and GARCH models ranked consistently low at almost each criterion. Realized EGARCH ranks well as well across all criteria, as well Realized GARCH(2,2) model for out of sample estimations (excepting for out of sample, RMSE criterion). On the fourth place, comes the Realized GARCH model.

With regards to the AXP stock, using MLE as ranking criterion, EGARCHX performs well (the best for in sample estimations), remarking also the Realized EGARCH which comes into a third position for in sample ranking and the first as out of sample ranking. Realized GARCH(2,2) performs as well under both types of estimations. When the loss function sizes are considered, EGARCHX comes the best under RMSE and MAE criteria (for in sample estimations), while Realized GARCH(2,2) ranks the best for three other criteria. Realized EGARCH and Realized

GARCH as well rank among the best as well. The GARCH and EGARCH models were the worst ranked, for five out of six criteria.

With respect to the BAC stock, EGARCHX ranks consistently on the second position (for MLE criterion), while Realized GARCH(2,2) ranks the first and the third alternatively, indicating also good forecasting performance. Realized GARCH also ranks well, especially when out of sample estimation is considered. EGARCH and GARCH models rank the worst under both types of estimations. When loss functions are used as comparison criteria, EGARCHX ranks very well again (between the best ranked for four out of six criteria), Realized EGARCH and Realized GARCH(2,2) rank among the best two for three and, respectively, two, criteria. GARCH and EGARCH rank the worst under four (out of six criteria), and Realized GARCH(1,1) ranks the worst under two criteria.

With respect to the JPM stock, EGARCHX ranks the best when in sample estimation is considered, with Realized GARCH(2,2) on the second spot, while EGARCH and GARCH models ranked the worst. For out of sample estimations, the Realized GARCH(2,2) and EGARCH models ranked the best, and EGARCH and GARCH models ranked on the last two positions. For loss function criteria, in sample estimations, EGARCHX, Realized GARCH(2,2) and Realized EGARCH ranked the best, while GARCH and EGARCH models ranked the worst. For out of sample estimations, Realized GARCH(1,1) and Realized EGARCH ranked well, while GARCH and EGARCH ranked poor.

One general conclusion on the models ranking is that EGARCHX model ranks the best under most of the criteria, followed closely by the Realized GARCH(2,2) and Realized EGARCH models. However, this conclusion contradicts the previous belief that by adding a measurement equation in the realized models that links the realized measures to the conditional volatility, the estimation performances would improve. EGARCHX model doesn’t formulate such a link, leaving the realized measures exogenous, and this actually enhances its forecasting performance.

A second general conclusion is that, indeed, incorporating the measures of intraday volatility in the GARCH equations enhances the modeling problem. This is evidenced by the fact that under most of the estimations, EGARCHX, Realized EGARCH and Realized GARCH(2,2) ranked the best, while GARCH and EGARCH models ranked the worst, with only few exceptions.

When the realized measures' ranking is considered, a first observation may be made when looking to the worst ranked, across all stocks, in sample and out of sample, for MLE criterion. That conclusion is that H-L ranks the worst, with no exception, under each criterion. Then, a second observation is that realized variance sampled at 5 minutes and 15 minutes ranks the highest, with RK coming on a third position. The high ranking of RV5m and RV15m, against a higher ranking of RV15s, may be explained due to the existence of a high microstructure noise at very high frequencies. This confirms the earlier theories according to which more frequent sampling the better, but too frequent sampling decreases the data accuracy due to higher microstructure noise.

When loss functions are used as ranking criterion, we may see again that H-L measures ranks the worst under almost all criteria. The best ranked are RV5m, RV15m and RV5s, while RK ranks better and worse alternatively, being very sensitive to the loss function, estimation method and stock choices.

The general conclusion is that H-L is not sufficiently accurate to measure the intraday volatility, while the realized variance, sampled at not too high or too low frequencies, improves the best the forecasting abilities.

Chapter four: Principal Component Analysis with high frequency volatility models

4.1 Introduction

In applied contexts, a frequent problem is not just estimating the univariate volatility of individual stocks, but also the multivariate volatility modeling of a multiple stock asset. This poses some difficulties as compared to the one stock case, because the modeling has to consider not just the conditional variance of each stock in the asset, but also the correlations between the individual stocks' variances. Volatility modeling of multiple stock assets requires a multivariate approach to be considered. The leading multivariate volatility models are PC GARCH (Principal Component GARCH) (Burns, 2005), BEKK (Baba, Engle, Kraft, and Kroner) (Engle and Kroner, 1995), (Engle and Mezrich, 1996), DCC (Dynamic Conditional Correlations) (Engle, 2002), (Engle and Sheppard, 2001), Orthogonal GARCH (Alexander, 2000), and GO GARCH (Generalized Orthogonal GARCH) (van der Weide, 2002).

The PC-GARCH model has kept attention because it offers a straightforward method of modeling a multivariate problem through the estimation of a series of univariate models applied to principal components. It has been empirically proven that attaching a Principal Component algorithm to a GARCH(1,1) model brought significant benefits in both performance and costs involved in forecasting problems of highly correlated stock assets. The PC-GARCH model not just minimizes computational efforts by reducing significantly the computational time and by getting rid of any problem that may arise from complex data manipulations, but also ensures a better fit of the model to data as it ensures a tight control of the amount of "noise" and thus results in more stable correlation estimates.

In what follows, the modeling problem of a multivariate asset will be adapted to the methodology of Burns (2005), accommodating to a high frequency context. This method (called PC GARCH) allowed to solve the modeling problem of multiple stock assets (a multivariate GARCH problem) through univariate GARCH estimations of the principal components. Starting from Burns' (2005) PC-GARCH model, we propose an updated method to solve multivariate volatility forecasting problems of multiple stock assets by employing high-frequency data models. New models will be proposed: PC EGARCH(1,1), PC EGARCHX(1,1), PC Realized GARCH(1,1), PC Realized EGARCH(1,1), and PC Realized GARCH(2,2). These models follow to be estimated for a four-stock asset, and each model's forecasting accuracy will be evaluated. As well, it will be documented the gain in asset volatility forecasting performance as compared to aggregated forecasts made on individual stocks. Methods on how the investors would take advantage of the newly proposed models to better portfolio management, will also be discussed.

Besides Burns' (2005) paper, this chapter takes also advantage of a similar methodology proposed by Alexander (2000) for Orthogonal GARCH (OGARCH), of the work of Hansen, Huang and Shek (2010a) who proposed the Realized GARCH and Realized EGARCH (Realized Exponential Generalized Autoregressive Conditional Heteroskedastic) models, and also of the work of Nelson (1991) and Engle (2002) who proposed EGARCH and GARCH-X (Generalized Autoregressive Conditional Heteroskedastic with an exogenous realized measure) models, respectively.

Principal Component Analysis (PCA) – algorithm description

Stating the problem in general terms first, let's consider $x_1, x_2, \dots, x_k$ variables that represent the innovations of k stocks obtained from employing a high frequency GARCH model. These vectors form a $T \times N$ matrix $\boldsymbol{R}$, such that $\boldsymbol{R}'\boldsymbol{R}$ be a symmetric matrix, with one on the diagonal. We assume that each column in the stationary matrix has mean zero and variance one, after previously subtracted the sample mean and divided by the sample standard deviation. Let's consider as well a matrix $\boldsymbol{\Omega} = \boldsymbol{R}'\boldsymbol{R}$ the variance-covariance matrix of $\boldsymbol{R}$, therefore positive definite. Matrix $\boldsymbol{R}$ will have on each column the time series formed by the N innovations obtained above.

The goal is to obtain linear combinations of the k innovations such that these combinations be orthogonal to each other: $p_m = l_{1,m}x_1 + l_{2,m}x_2 + \cdots + l_{k,m}x_k$, $1 \le m \le k$.

The above equation may be written in matrix form: $\boldsymbol{P}_{\boldsymbol{T}\times\boldsymbol{N}} = \boldsymbol{R}_{\boldsymbol{T}\times\boldsymbol{N}}\boldsymbol{L}_{\boldsymbol{N}\times\boldsymbol{N}}$, with each column of $\boldsymbol{P}_{\boldsymbol{T}\times\boldsymbol{N}}$ orthogonal to any other columns of $\boldsymbol{P}_{\boldsymbol{T}\times\boldsymbol{N}}$. In other words, what are sought are those factor loadings l's that multiplied by the corresponding x's give p's that are orthogonal. Restating further, what will be obtained will be that matrix $\boldsymbol{L_{N \times N}} = \begin{pmatrix} l_{11} & \dots & l_{1N} \\ \dots & \dots & \dots \\ l_{N1} & \dots & l_{NN} \end{pmatrix}$ that, multiplied with the matrix $\boldsymbol{R_{T \times N}}$ will give the matrix $\boldsymbol{P_{T \times N}}$ with columns orthogonal to each other.

$\boldsymbol{L_{N \times N}}$ will be called the matrix of the eigenvectors of **Ω.** The weights l_{ij} of each vector x_j will be chosen from a set of eigenvectors of the correlation matrix **Ω** such that:

a) The principal components be orthogonal, obtained by multiplying the matrix $\boldsymbol{L_{N \times N}}$ to the matrix $\boldsymbol{R_{T \times N}}$.
b) The first principal component to explain the maximum amount of total variation in **R,** the second component to explain the maximum remaining variation, etc.

As known from matrix algebra, if a matrix $\boldsymbol{L}$ is set to be composed of orthogonal unit eigenvectors of $\mathbf{R'R}$**,** then the resulting principal components will be orthogonal. It results that the necessary condition for $\boldsymbol{P}$ be orthogonal is that columns of $\boldsymbol{L}$ be orthogonal. So, what is searched is an orthogonal matrix $\boldsymbol{L}$ that multiplied by $\boldsymbol{R}$ gives an orthogonal matrix.

The **P** matrix can be written as it follows: $= \begin{pmatrix} p_{11} & p_{12} & \dots & p_{1N} \\ p_{21} & p_{22} & \dots & p_{2N} \\ \dots & \dots & \dots & \dots \\ p_{T1} & p_{T2} & \dots & p_{TN} \end{pmatrix}$, with $P_1, \dots, P_N$ defined as the orthogonal and unit-length eigenvectors of each principal component of the matrix $\boldsymbol{P}$: $\boldsymbol{P} = \left(\overset{\boldsymbol{P_1}}{\begin{bmatrix} p_{11} \\ \dots \\ p_{T1} \end{bmatrix}} \quad \overset{\boldsymbol{P_2}}{\begin{bmatrix} p_{12} \\ \dots \\ p_{T2} \end{bmatrix}} \quad \overset{\dots}{\begin{bmatrix} \dots \\ \dots \\ \dots \end{bmatrix}} \quad \overset{\boldsymbol{P_N}}{\begin{bmatrix} p_{1N} \\ \dots \\ p_{TN} \end{bmatrix}} \right)$.

As well, a matrix $\boldsymbol{\Gamma}_{N \times N} = \begin{pmatrix} var(P_1, P_1) & var(P_1, P_2) & \dots & var(P_1, P_N) \\ var(P_2, P_1) & var(P_2, P_2) & \dots & var(P_2, P_N) \\ \dots & \dots & \dots & \dots \\ var(P_N, P_1) & var(P_N, P_2) & \dots & var(P_N, P_N) \end{pmatrix}$ is defined as the variance-covariance matrix of **P,** with $var(P_i, P_i)$ the variance-covariance between P_i with itself and $var(P_i, P_j)$ the variance-covariance between P_i with P_j.

Since the principal components need to be orthogonal, then $var(P_i, P_j)$ with $i \neq j$ should equal zero, while $var(P_i, P_i)$ should equal the σ 's of P's. It means that $\boldsymbol{\Gamma}_{N \times N}$ is a diagonal matrix with the variance of P's on the diagonal, and all other values null: $\boldsymbol{\Gamma}_{N\times N} = \begin{pmatrix} \sigma^{(P_1)} & 0 & \dots & 0 \\ 0 & \sigma^{(P_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma^{(P_N)} \end{pmatrix}$.

As such, in order to obtain a matrix $\boldsymbol{L}$ that multiplied by **R** gives an orthogonal matrix $\boldsymbol{P}$**,** we need to find out the factor loadings l's that form a matrix $\boldsymbol{L}$ that**,** when multiplied by **R,** gives a matrix $\boldsymbol{P}$ whose variance-covariance matrix ($\boldsymbol{\Gamma}$) is diagonal (equivalent to orthogonal $\boldsymbol{P}$ eigenvectors).

Since $\boldsymbol{\Gamma}$ is the variance-covariance matrix of $\boldsymbol{P}$**,** then $\boldsymbol{\Gamma} = \boldsymbol{P}'\boldsymbol{P} = (\boldsymbol{RL})'\boldsymbol{RL} = \boldsymbol{L}'\boldsymbol{R}'\boldsymbol{RL} = \boldsymbol{L}'\boldsymbol{\Omega}\boldsymbol{L}.$

Because $\boldsymbol{L}$ is orthogonal, $\boldsymbol{L}' = \boldsymbol{L}^{-1}$ and $\boldsymbol{P} = \boldsymbol{RL}$ becomes $\boldsymbol{R} = \boldsymbol{PL}^{-1} = \boldsymbol{PL}'.$ It means that when it will be found the matrix $\boldsymbol{L}$ that by multiplication with $\boldsymbol{R}$ gives an orthogonal matrix $\boldsymbol{P}$**,** then it will be possible to obtain the matrix $\boldsymbol{R}$ whose each column X_i will be a linear combination of P's and other factor loadings w's: $X_i = w_{i1}P_1 + w_{i2}P_2 + \cdots + w_{ik}P_k$, where X_i and P_i denote the columns of $\boldsymbol{R}$ and $\boldsymbol{P}$ respectively.

Thus each data vector will be a linear combination of the principal components. The proportion of the total variation in $\boldsymbol{R}$ that is explained by the m^{th} principal component is λ_m/(sum of the eigenvalues). Thus, the operation of scaling the original variables with the matrix of orthogonal unit eigenvectors $\boldsymbol{L}$ gives us uncorrelated components (PCs) that we could use to reduce the earlier multivariate GARCH problem to a set of univariate GARCH problems.

4.2 Data

The same data as in the previous chapter will be used, spanning over the same period: four stocks (AIG, AXP, BAC and JPM) with daily frequency data (open prices, close prices, highest and lowest prices of each trading day, the opening and the closing time information and various measures of intraday volatility, like realized kernels and realized variance sampled at 15 seconds, 5 minute, 15 minutes and 20 minutes). The sample is 3436 observations long (between January 4, 1995 – September 30, 2008). The data has been cleaned by excluding the half-trading days, and the returns in their logarithmic forms were calculated as follows:

$$r_t = 100 * (\log(closing\ price_t) - \log(opening\ price_t))$$

for each trading day t, $1 \leq t \leq 3436$.

4.3 Methodology

The methodology that follows is a restatement of the methodology given by Burns (2005) for PC GARCH, but adapted to new formulations of models that include measures of intraday volatility. Seven steps need to be made in order to define the PC EGARCH, PC Realized GARCH, PC Realized EGARCH, PC EGARCH-X and PC Realized GARCH(2,2) models.

There are given N stocks with T price observations. Time series of their returns will be formed according to the formula

$$r_t = 100 * (\log(closing\ price_t) - \log(opening\ price_t))$$

for each trading day t, $1 \leq t \leq T$.

The daily stock returns were plot in a matrix $\boldsymbol{X}_{T\times N} = \begin{pmatrix} x_1^{(1)} & x_1^{(2)} & \dots & x_1^{(N)} \\ x_2^{(1)} & x_2^{(2)} & \dots & x_2^{(N)} \\ \dots & \dots & \dots & \dots \\ x_T^{(1)} & x_T^{(2)} & \dots & x_T^{(N)} \end{pmatrix}$. In return matrix $\boldsymbol{X} = (x_t^i)_{1\leq t\leq T; 1\leq i\leq N}$, each variable is observed in time. The purpose is to estimate the volatility of the N-stock asset $\boldsymbol{X}$**,** that is equivalent to finding the variance-covariance matrix of $\boldsymbol{X}$**.** We call this variance-covariance matrix $\boldsymbol{\Omega}_{N\times N}$. In a multivariate problem we look for the matrix $\boldsymbol{\Omega}_{N\times N}$ that is equivalent to σ in a univariate problem.

$$\boldsymbol{\Omega}_{N\times N} = \begin{pmatrix} \sigma^{(1)} & cov(x^{(1)}, x^{(2)}) & \dots & cov(x^{(1)}, x^{(N)}) \\ cov(x^{(2)}, x^{(1)}) & \sigma^{(2)} & \dots & cov(x^{(2)}, x^{(N)}) \\ \dots & \dots & \dots & \dots \\ cov(x^{(N)}, x^{(1)}) & cov(x^{(N)}, x^{(2)}) & \dots & \sigma^{(N)} \end{pmatrix}$$

in which $\sigma^{(1)}, \dots, \sigma^{(N)}$ are the univariate variance of each stock $x^{(i)}$.

The seven step procedure to find $\boldsymbol{\Omega}_{N\times N}$ follows.

Step 1

A univariate GARCH-type of model (one of the realized models we considered) is employed for each stock returns $x^{(i)}$. This will provide us with the σ's that are on the diagonal of the variance-covariance matrix $\boldsymbol{\Omega}_{N\times N}$. ***Step 1*** provides us the diagonal of the matrix $\boldsymbol{\Omega}_{N\times N}$, what follows (**Steps 2** to **7**) will be undertaken in order to find the covariances between the stocks.

Step 2

Step 1 gives us the day and night standardized returns z_t. We center and reduce the variables by constructing a matrix $\boldsymbol{R}_{\boldsymbol{pop},T\times N}$ having as elements $\frac{(x_t^{(i)}-\mu^{(i)})}{\sqrt{\sigma^{(i)}}}$. These elements are called standardized innovations and will be noted as $z_t^{(i)}$, being calculated as standardized returns with the estimated variance and mean process for each of the stocks. These standardized residuals (returns) are observed in time at each t for each variable *i*. Because $\mu^{(i)}$ and $\sigma^{(i)}$ are not observable, they need to be estimated. Their estimation is undertaken by employing the considered realized GARCH-type of models, and their estimations will be noted as $m^{(i)}$ and ${s^{(i)}}^2$. The matrix $\boldsymbol{R}$ having as elements $\frac{(x_t^{(i)}-m^{(i)})}{s^{(i)}}$ will be calculated. Because elements of $\boldsymbol{R_{pop}}$ are unknown, as they contain $\mu^{(i)}$ and $\sigma^{(i)}$ (the average and variance of population), $\boldsymbol{R}$ will be an estimator of $\boldsymbol{R_{pop}}$.

The variance-covariance matrix of the columns of $\boldsymbol{X}_{T\times N}$ is $\boldsymbol{\Omega}_{N\times N}$. The variance-covariance matrix of the columns of $\boldsymbol{R}_{\boldsymbol{pop},T\times N}$ will be the $\boldsymbol{V}_{N\times N}$, which has $\frac{(x_t^{(i)}-\mu^{(i)})}{\sqrt{\sigma^{(i)}}}$ on its diagonal and $cov(\frac{x_i-\mu_i}{\sqrt{\sigma_i}},\frac{x_j-\mu_j}{\sqrt{\sigma_j}})$, $i\neq j$, in the other cells of the matrix. But $var\left(\frac{x_i-\mu_i}{\sqrt{\sigma_i}}\right)=1$ and $cov\left(\frac{x_i-\mu_i}{\sqrt{\sigma_i}},\frac{x_j-\mu_j}{\sqrt{\sigma_j}}\right)=\frac{1}{\sqrt{\sigma_i}\sqrt{\sigma_j}}cov\left(x_i-\mu_i,x_j-\mu_j\right)=\frac{1}{\sqrt{\sigma_i}\sqrt{\sigma_j}}cov\left(x_i,x_j\right)=corr\left(x_i,x_j\right)$.

In other words, $var\left(\boldsymbol{R_{pop}}\right)=corr(\boldsymbol{X})$, that is the variance-covariance matrix of $\boldsymbol{R}_{\boldsymbol{pop},T\times N}$ is equal to the correlation matrix of $\boldsymbol{X}_{T\times N}$. As such, $\boldsymbol{V}_{N\times N}=Corr(\boldsymbol{X}_{T\times N})$.

It results that an estimator of the correlation matrix of $\boldsymbol{X}$ may be obtained by finding an estimator for $\boldsymbol{V}_{N\times N} = var(\boldsymbol{R}_{\boldsymbol{pop}})$. Since an estimator for $\boldsymbol{R}_{\boldsymbol{pop}}$ is $\boldsymbol{R}$, it results that an estimator for $\boldsymbol{V}_{N\times N} = var(\boldsymbol{R}_{\boldsymbol{pop}})$ is $var(\boldsymbol{R})$. It results that an estimator for $corr(\boldsymbol{X})$ is $var(\boldsymbol{R})$.

Step 3

We apply Principal Component algorithm on the day standardized returns (columns of matrix $\boldsymbol{R}_{T\times N}$). PCA creates new variables called principal components. The Matlab gives us the weights (l's) that, by multiplying to each of the standardized returns, form the principal components. In matrix form, the matrix of weights ($\boldsymbol{L}_{N\times N}$) that multiplied by the matrix $\boldsymbol{R}_{T\times N}$ of the standardized returns, gives us the matrix of the principal components. The result will be another matrix of the principal components, $\boldsymbol{P}_{T\times N}$, in which the columns of $\boldsymbol{P}$ (the first characteristic) will be linear combinations of the columns of $\boldsymbol{R}$. As such, we may write

$$\boldsymbol{P}_{T\times N} = \boldsymbol{R}_{T\times N} \times \boldsymbol{L}_{N\times N}$$

in which $\boldsymbol{L}$ is the principal component loadings' (or weights') matrix ($\boldsymbol{L}$ is a matrix formed from eigenvectors of the variance-covariance matrix of the variables on which PC is applied; as such, $\boldsymbol{L}$ will be a matrix having as columns eigenvectors of the matrix $var(\boldsymbol{R})$). Because $var(\boldsymbol{R})$ is symmetric, its eigenvectors will be orthogonal two by two, with module 1. It results that matrix $\boldsymbol{L}$ verifies the following relationship:

$$\boldsymbol{L}'\boldsymbol{L} = \boldsymbol{L}\boldsymbol{L}' = \boldsymbol{I}$$

By multiplying the $\boldsymbol{P} = \boldsymbol{R}\boldsymbol{L}$ relationship with $\boldsymbol{L}'$, we will obtain $\boldsymbol{P}\boldsymbol{L}' = \boldsymbol{R} \Rightarrow var(\boldsymbol{R}) = var(\boldsymbol{P}\boldsymbol{L}') = \boldsymbol{L}var(\boldsymbol{P})\boldsymbol{L}'$. As such we obtained $var(\boldsymbol{R}) = \boldsymbol{L}var(\boldsymbol{P})\boldsymbol{L}'$.

The second characteristic of the principal components is that they are orthogonal two by two, that is equivalent to that the covariance of any two columns of $\boldsymbol{P}_{T\times N}$ is 0. That means that the variance-covariance matrix of $\boldsymbol{P}_{T\times N}$ will be

$$\boldsymbol{\Gamma}_{N\times N} = \begin{pmatrix} \sigma^{(P_1)} & 0 & \dots & 0 \\ 0 & \sigma^{(P_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma^{(P_N)} \end{pmatrix}$$

The above relationship becomes $var(\boldsymbol{R}) = \boldsymbol{L\Gamma L'}$**.** Since the relationship is valid for any *t* moment, we can write $var(\boldsymbol{R})_{\boldsymbol{t}} = \boldsymbol{L\Gamma_t L'}$**.**

As such, $var(\boldsymbol{R})_{\boldsymbol{t}} = \begin{pmatrix} l_{11} & l_{12} & \dots & l_{1N} \\ l_{21} & l_{22} & \dots & l_{2N} \\ \dots & \dots & \dots & \dots \\ l_{N1} & l_{N2} & \dots & l_{NN} \end{pmatrix} \begin{pmatrix} \sigma_t^{(P_1)} & 0 & \dots & 0 \\ 0 & \sigma_t^{(P_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma_t^{(P_N)} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \dots & l_{N1} \\ l_{12} & l_{22} & \dots & l_{N2} \\ \dots & \dots & \dots & \dots \\ l_{1N} & l_{2N} & \dots & l_{NN} \end{pmatrix} =$

$$\begin{pmatrix} l_{11}\sigma_t^{(P_1)} & l_{12}\sigma_t^{(P_2)} & \dots & l_{1N}\sigma_t^{(P_N)} \\ l_{21}\sigma_t^{(P_1)} & l_{22}\sigma_t^{(P_2)} & \dots & l_{2N}\sigma_t^{(P_N)} \\ \dots & \dots & \dots & \dots \\ l_{N1}\sigma_t^{(P_1)} & l_{N2}\sigma_t^{(P_2)} & \dots & l_{NN}\sigma_t^{(P_N)} \end{pmatrix} \begin{pmatrix} l_{11} & l_{21} & \dots & l_{N1} \\ l_{12} & l_{22} & \dots & l_{N2} \\ \dots & \dots & \dots & \dots \\ l_{1N} & l_{2N} & \dots & l_{NN} \end{pmatrix} =$$

$$\begin{pmatrix} \sum_{k=1}^N l_{1k}^2 \sigma_t^{(P_k)} & \sum_{k=1}^N l_{1k} l_{2k} \sigma_t^{(P_k)} & \dots & \sum_{k=1}^N l_{1k} l_{Nk} \sigma_t^{(P_k)} \\ \sum_{k=1}^N l_{2k} l_{1k} \sigma_t^{(P_k)} & \sum_{k=1}^N l_{2k}^2 \sigma_t^{(P_k)} & \dots & \sum_{k=1}^N l_{2k} l_{Nk} \sigma_t^{(P_k)} \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^N l_{Nk} l_{1k} \sigma_t^{(P_k)} & \sum_{k=1}^N l_{Nk} l_{2k} \sigma_t^{(P_k)} & \dots & \sum_{k=1}^N l_{Nk}^2 \sigma_t^{(P_k)} \end{pmatrix}$$

The main difference that Burns' methodology takes advantage of (for GARCH models, but the advantage remains even for the case of realized GARCH models), is that $\boldsymbol{\Gamma}_{N\times N}$ is a diagonal matrix (because the principal components are orthogonal two by two), while $\boldsymbol{\Omega}_{N\times N}$ is not. On its diagonal, $\boldsymbol{\Gamma}$ will have the variances of the principal components.

At ***Step 4***, it will be obtained an estimator of $\boldsymbol{\Gamma}_{N\times N}$, by using univariate realized GARCH-type of models.

Step 4

We employ a univariate realized GARCH model on each column of $\boldsymbol{P}_{T\times N}$, step which provides us with the $\sigma^{(P_i)}$'s on the diagonal of $\boldsymbol{\Gamma}_{N\times N}$. By finding an estimator of $\boldsymbol{\Gamma}_{N\times N}$ through univariate realized GARCH-type estimations, due to the fact that $var(\boldsymbol{R}) = \boldsymbol{L\Gamma L'}$**,** it will be obtained an estimator for $\boldsymbol{var}(\boldsymbol{R})$**.** And, as said before, an estimator of $var(\boldsymbol{R})$ is an estimator of $\boldsymbol{V}_{N\times N}$, and thus we obtain an estimate of the correlation matrix $\boldsymbol{X}$ (as $\boldsymbol{V}_{N\times N} = corr(\boldsymbol{X}_{T\times N})$).

Step 5

We return to the initial space $\boldsymbol{V}_{N\times N}$ (variance-covariance matrix of $\boldsymbol{R}_{T\times N}$) through a relationship specific to the principal component theory (and which has been obtained in the previous section):

$$\boldsymbol{V}_{N\times N} = \boldsymbol{L}_{N\times N} \times \boldsymbol{\Gamma}_{N\times N} \times \boldsymbol{L}_{N\times N}^{\boldsymbol{T}}$$

where $\boldsymbol{L}_{N\times N}^{\boldsymbol{T}}$ is the transposed matrix of $\boldsymbol{L}_{N\times N}$.

But, $\widehat{\boldsymbol{V}}_{N\times N} = \widehat{\boldsymbol{Corr}}(\boldsymbol{X}_{T\times N}) = \widehat{\boldsymbol{C}}_t$, where $\widehat{\boldsymbol{C}}_t$ is an estimator of the correlation matrix of $\boldsymbol{X}$. However, $\widehat{\boldsymbol{V}}_{N\times N}$, the estimator of $\boldsymbol{V}_{N\times N}$, is known now from ***Step 4***. So, we know now $\widehat{\boldsymbol{C}}_t$.

$$\widehat{\boldsymbol{V}}_{N\times N} = \widehat{\boldsymbol{Corr}}(\boldsymbol{X}_{T\times N}) = \begin{pmatrix} \hat{\boldsymbol{c}}_1 & \hat{\boldsymbol{c}}_{12} & \dots & \hat{\boldsymbol{c}}_{1N} \\ \hat{\boldsymbol{c}}_{21} & \hat{\boldsymbol{c}}_2 & \dots & \hat{\boldsymbol{c}}_{2N} \\ \dots & \dots & \dots & \dots \\ \hat{\boldsymbol{c}}_{N1} & \hat{\boldsymbol{c}}_{N2} & \dots & \hat{\boldsymbol{c}}_N \end{pmatrix}, \quad \text{with} \quad \hat{\boldsymbol{c}}_{ij} = \sum_{k=1}^{N} l_{ik} l_{jk} \sigma^{(P_k)}, i, j = 1, \dots, N, \hat{\boldsymbol{c}}_i = \hat{\boldsymbol{c}}_{ii}, i = 1, \dots, N, \hat{\boldsymbol{c}}_j = \hat{\boldsymbol{c}}_{jj}, j = 1, \dots, N.$$

The correlation matrix of $\boldsymbol{X}$ needs to have 1 on its diagonal. However, there is no guarantee that the elements on the diagonal of $\widehat{\boldsymbol{C}}_t$ will be equal to 1. That is why $\widehat{\boldsymbol{C}}_t$ will be transformed as such to look like a correlation matrix ($\widetilde{\boldsymbol{C}}_t$). Between $\widehat{\boldsymbol{C}}_t$ and $\widetilde{\boldsymbol{C}}_t$ there will be only very minor differences. $\widetilde{\boldsymbol{C}}_t$ will be an estimator of $corr(\boldsymbol{X})_t$ and will be obtained at the next ***Step***.

Step 6

At ***Step 6***, $\widetilde{\boldsymbol{C}}_t$ will be obtained from $\widehat{\boldsymbol{C}}_t$. This is the same operation through which a variance-covariance matrix is transformed into a correlation matrix.

$\widehat{\boldsymbol{C}}_t$ is an estimator of the correlation matrix of $\boldsymbol{X}_{T\times N}$, and an estimator of the variance-covariance matrix of $\boldsymbol{R}_{T\times N}$:

$$\widehat{\boldsymbol{C}}_t = \begin{pmatrix} \hat{\boldsymbol{c}}_1 & \hat{\boldsymbol{c}}_{12} & \dots & \hat{\boldsymbol{c}}_{1N} \\ \hat{\boldsymbol{c}}_{21} & \hat{\boldsymbol{c}}_2 & \dots & \hat{\boldsymbol{c}}_{2N} \\ \dots & \dots & \dots & \dots \\ \hat{\boldsymbol{c}}_{N1} & \hat{\boldsymbol{c}}_{N2} & \dots & \hat{\boldsymbol{c}}_N \end{pmatrix}$$

$\widetilde{\boldsymbol{C}}_t$ is the correlation matrix of $\boldsymbol{X}_{T\times N}$.

$$\widetilde{\boldsymbol{C}}_t = \begin{pmatrix} \frac{1}{\sqrt{\hat{\boldsymbol{c}}_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_N}} \end{pmatrix} \widehat{\boldsymbol{C}}_t \begin{pmatrix} \frac{1}{\sqrt{\hat{\boldsymbol{c}}_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_N}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 & \frac{\hat{\boldsymbol{c}}_{12}}{\sqrt{\hat{\boldsymbol{c}}_1}\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & \frac{\hat{\boldsymbol{c}}_{1N}}{\sqrt{\hat{\boldsymbol{c}}_1}\sqrt{\hat{\boldsymbol{c}}_N}} \\ \frac{\hat{\boldsymbol{c}}_{21}}{\sqrt{\hat{\boldsymbol{c}}_2}\sqrt{\hat{\boldsymbol{c}}_1}} & 1 & \dots & \frac{\hat{\boldsymbol{c}}_{2N}}{\sqrt{\hat{\boldsymbol{c}}_2}\sqrt{\hat{\boldsymbol{c}}_N}} \\ \dots & \dots & \dots & \dots \\ \frac{\hat{\boldsymbol{c}}_{N1}}{\sqrt{\hat{\boldsymbol{c}}_N}\sqrt{\hat{\boldsymbol{c}}_1}} & \frac{\hat{\boldsymbol{c}}_{N2}}{\sqrt{\hat{\boldsymbol{c}}_N}\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & 1 \end{pmatrix}$$

$$\widetilde{\boldsymbol{C}}_t = \begin{pmatrix} 1 & \tilde{\boldsymbol{c}}_{12} & \dots & \tilde{\boldsymbol{c}}_{1N} \\ \tilde{\boldsymbol{c}}_{21} & 1 & \dots & \tilde{\boldsymbol{c}}_{2N} \\ \dots & \dots & \dots & \dots \\ \tilde{\boldsymbol{c}}_{N1} & \tilde{\boldsymbol{c}}_{N2} & \dots & 1 \end{pmatrix}, \text{ with } \tilde{\boldsymbol{c}}_{ij} = \frac{\sum_{k=1}^{N} l_{ik} l_{jk} \sigma^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{ik}^2 \sigma^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{jk}^2 \sigma^{(P_k)}}}$$

Step 7

We need an estimator of $\boldsymbol{\Omega}_{N\times N}$, while $\boldsymbol{\Omega}_{N\times N}$ is an estimator of the variance-covariance matrix of $\boldsymbol{X}_{T\times N}$. We already have the diagonal of $\boldsymbol{\Omega}_{N\times N}$, that is the $\sigma^{(i)}$'s obtained at ***Step 1.*** Now we move in the opposite direction than we did in ***Step 6***, that is we move from $\widetilde{\boldsymbol{C}}_t$ to $\boldsymbol{\Omega}_{N\times N}$. This is done through the following iteration

$$\widehat{\boldsymbol{\Omega}}_{N\times N} = \begin{pmatrix} \sigma^{(1)} & cov(x^{(1)}, x^{(2)}) & \dots & cov(x^{(1)}, x^{(N)}) \\ cov(x^{(2)}, x^{(1)}) & \sigma^{(2)} & \dots & cov(x^{(2)}, x^{(N)}) \\ \dots & \dots & \dots & \dots \\ cov(x^{(N)}, x^{(1)}) & cov(x^{(N)}, x^{(2)}) & \dots & \sigma^{(N)} \end{pmatrix} =$$

$$\begin{pmatrix} \sqrt{\sigma^{(1)}} & 0 & \dots & 0 \\ 0 & \sqrt{\sigma^{(2)}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\sigma^{(N)}} \end{pmatrix} \widetilde{\boldsymbol{C}}_t \begin{pmatrix} \sqrt{\sigma^{(1)}} & 0 & \dots & 0 \\ 0 & \sqrt{\sigma^{(2)}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\sigma^{(N)}} \end{pmatrix} =$$

$$\begin{pmatrix} \sigma^{(1)} & \sqrt{\sigma^{(1)}}\sqrt{\sigma^{(2)}}\tilde{\boldsymbol{c}}_{12} & \dots & \sqrt{\sigma^{(1)}}\sqrt{\sigma^{(N)}}\tilde{\boldsymbol{c}}_{1N} \\ \sqrt{\sigma^{(2)}}\sqrt{\sigma^{(1)}}\tilde{\boldsymbol{c}}_{21} & \sigma^{(2)} & \dots & \sqrt{\sigma^{(2)}}\sqrt{\sigma^{(N)}}\tilde{\boldsymbol{c}}_{2N} \\ \dots & \dots & \dots & \dots \\ \sqrt{\sigma^{(N)}}\sqrt{\sigma^{(1)}}\tilde{\boldsymbol{c}}_{N1} & \sqrt{\sigma^{(N)}}\sqrt{\sigma^{(2)}}\tilde{\boldsymbol{c}}_{N2} & \dots & \sigma^{(N)} \end{pmatrix}$$

$\widetilde{\boldsymbol{C}}_t$ is obtained at ***Step 6***, while $\sigma^{(1)}$'s are obtained at ***Step 1***. Then,

$$\widehat{\boldsymbol{\Omega}}_{N\times N}$$

$$= \begin{pmatrix} \sigma_t^{(1)} & \sqrt{\sigma_t^{(1)}}\sqrt{\sigma_t^{(2)}}\frac{\sum_{k=1}^{N} l_{1k}l_{2k}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}} & \dots & \sqrt{\sigma_t^{(1)}}\sqrt{\sigma_t^{(N)}}\frac{\sum_{k=1}^{N} l_{1k}l_{Nk}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}} \\ \sqrt{\sigma_t^{(2)}}\sqrt{\sigma_t^{(1)}}\frac{\sum_{k=1}^{N} l_{2k}l_{1k}\sigma^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}} & \sigma_t^{(2)} & \dots & \sqrt{\sigma_t^{(2)}}\sqrt{\sigma_t^{(N)}}\frac{\sum_{k=1}^{N} l_{2k}l_{Nk}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}} \\ \dots & \dots & \dots & \dots \\ \sqrt{\sigma_t^{(N)}}\sqrt{\sigma_t^{(1)}}\frac{\sum_{k=1}^{N} l_{Nk}l_{1k}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}} & \sqrt{\sigma_t^{(N)}}\sqrt{\sigma_t^{(2)}}\frac{\sum_{k=1}^{N} l_{Nk}l_{2k}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}} & \dots & \sigma_t^{(N)} \end{pmatrix}$$

GARCH equations of standardized returns

The findings at ***Step 3*** may be otherwise exploited. As such, the relationship $\boldsymbol{L\Gamma L'} = \boldsymbol{\Omega}$ is equivalent to

$$\begin{pmatrix} l_{11} & l_{12} & \dots & l_{1N} \\ l_{21} & l_{22} & \dots & l_{2N} \\ \dots & \dots & \dots & \dots \\ l_{N1} & l_{N2} & \dots & l_{NN} \end{pmatrix} * \begin{pmatrix} \sigma^{(P_1)} & 0 & \dots & 0 \\ 0 & \sigma^{(P_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sigma^{(P_N)} \end{pmatrix}$$

$$* \begin{pmatrix} l_{11} & l_{21} & \dots & l_{N1} \\ l_{12} & l_{22} & \dots & l_{N2} \\ \dots & \dots & \dots & \dots \\ l_{1N} & l_{2N} & \dots & l_{NN} \end{pmatrix} = \begin{pmatrix} \sigma^{(1)} & cov(x^{(1)}, x^{(2)}) & \dots & cov(x^{(1)}, x^{(N)}) \\ cov(x^{(2)}, x^{(1)}) & \sigma^{(2)} & \dots & cov(x^{(2)}, x^{(N)}) \\ \dots & \dots & \dots & \dots \\ cov(x^{(N)}, x^{(1)}) & cov(x^{(N)}, x^{(2)}) & \dots & \sigma^{(N)} \end{pmatrix}$$

This gives us representations of the conditional volatility of the standardized returns as a function of past volatility of the principal components. As such, we have a representation of the volatility of the returns of one stock not just as a function of its past volatility, but also as a function of the past volatility of other stocks (here other stocks' volatilities being expressed as the variance of principal components).

As such, for each of the models considered, we may write the conditional volatility equations of these standardized returns, as follows:

GARCH(1,1) model:

$$\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 (\omega_{P_j} + \alpha_{P_j} r_{P_{j,t-1}}^2 + \beta_{P_j} \sigma_{P_{j,t-1}}) \right], i = 1, \dots, k$$

that is equivalent to

$$\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j} + l_{ij}^2 \alpha_{P_j} r_{P_{j,t-1}}^2 + l_{ij}^2 \beta_{P_j} \sigma_{P_{j,t-1}}) \right), i = 1, \dots, k$$

EGARCH(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 (\omega_{P_j} + \beta_{P_j} log\sigma_{P_{j,t-1}} + \tau_1^{P_j} z_{P_{j,t}} + \tau_2^{P_j} \left(z_{P_{j,t}}^2 - 1 \right)) \right], i = 1, \dots, k$$

that is equivalent to

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 \omega_{P_j} + l_{ij}^2 \beta_{P_j} log\sigma_{P_{j,t-1}} + l_{ij}^2 \tau_1^{P_j} z_{P_{j,t}} + l_{ij}^2 \tau_2^{P_j} \left(z_{P_{j,t}}^2 - 1 \right) \right], i = 1, \dots, k$$

EGARCH-X(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 (\omega_{P_j} + \beta_{P_j} log\sigma_{P_{j,t-1}} + \gamma_{P_j} logx_{P_{j,t-1}} + \tau_1^{P_j} z_{P_{j,t}} + \tau_2^{P_j} \left(z_{P_{j,t}}^2 - 1 \right)) \right], i = 1, \dots, k \text{ and exogenous } x.$$

that is equivalent to

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 \omega_{P_j} + l_{ij}^2 \beta_{P_j} log\sigma_{P_{j,t-1}} + l_{ij}^2 \gamma_{P_j} logx_{P_{j,t-1}} + l_{ij}^2 \tau_1^{P_j} z_{P_{j,t}} + l_{ij}^2 \tau_2^{P_j} \left(z_{P_{j,t}}^2 - 1 \right) \right], i = 1, \dots, k \text{ and exogenous } x.$$

Realized GARCH(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 (\omega_{P_j} + \beta_{P_j} log\sigma_{P_{j,t-1}} + \gamma_{P_j} logx_{P_{j,t-1}} \right], i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 \omega_{P_j} + l_{ij}^2 \beta_{P_j} log\sigma_{P_{j,t-1}} + l_{ij}^2 \gamma_{P_j} log x_{P_{j,t-1}} \right], i = 1, \dots, k \text{ and endogenous } x.$$

Realized EGARCH(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 (\omega_{P_j} + \beta_{P_j} log\sigma_{P_{j,t-1}} + \gamma_{P_j} log x_{P_{j,t-1}} + \tau_1^{P_j} z_{P_{j,t}} + \tau_2^{P_j} \left(z_{P_{j,t}}^2 - 1 \right)) \right], i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left[l_{ij}^2 \omega_{P_j} + l_{ij}^2 \beta_{P_j} log\sigma_{P_{j,t-1}} + l_{ij}^2 \gamma_{P_j} log x_{P_{j,t-1}} + l_{ij}^2 \tau_1^{P_j} z_{P_{j,t}} + l_{ij}^2 \tau_2^{P_j} \left(z_{P_{j,t}}^2 - 1 \right) \right], i = 1, \dots, k \text{ and endogenous } x.$$

Realized GARCH(2,2) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \Big[l_{ij}^2 (\omega_{P_j} + \alpha_{P_j} log(max\left(r_{P_{j,t-1}}^2, 10^{-20} \right)) + \beta_1^{P_j} \, log\sigma_{P_{j,t-1}} + \beta_2^{P_j} \, log\sigma_{P_{j,t-2}} + \gamma_1^{P_j} log x_{P_{j,t-1}} + \gamma_2^{P_j} log x_{P_{j,t-2}}) \Big], i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$log\sigma_{x_i} = \sum_{j=1}^{k} \Big[l_{ij}^2 \omega_{P_j} + l_{ij}^2 \alpha_{P_j} lo\,g \left(max \left(r_{P_{j,t-1}}^2, 10^{-20} \right) \right) + l_{ij}^2 \beta_1^{P_j} \, log\sigma_{P_{j,t-1}} + l_{ij}^2 \beta_2^{P_j} \, log\sigma_{P_{j,t-2}} + l_{ij}^2 \gamma_1^{P_j} log x_{P_{j,t-1}} + l_{ij}^2 \gamma_2^{P_j} log x_{P_{j,t-2}} \Big], i = 1, \dots, k \text{ and endogenous } x.$$

In matrix form, the general models will be:

GARCH(1,1)

$$\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j}\right) + \begin{pmatrix} l_{i1}^2 \alpha_{P_1} \\ l_{i2}^2 \alpha_{P_2} \\ \dots \\ l_{ik}^2 \alpha_{P_k} \end{pmatrix}' \begin{pmatrix} r_{P_1,t-1}^2 \\ r_{P_2,t-1}^2 \\ \dots \\ r_{P_k,t-1}^2 \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \beta_{P_1} \\ l_{i2}^2 \beta_{P_2} \\ \dots \\ l_{ik}^2 \beta_{P_k} \end{pmatrix}' \begin{pmatrix} \sigma_{P_1,t-1} \\ \sigma_{P_2,t-1} \\ \dots \\ \sigma_{P_k,t-1} \end{pmatrix}, 1 \leq i \leq k$$

EGARCH(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j}\right) + \begin{pmatrix} l_{i1}^2 \beta_{P_1} \\ l_{i2}^2 \beta_{P_2} \\ \dots \\ l_{ik}^2 \beta_{P_k} \end{pmatrix}' \begin{pmatrix} log\sigma_{P_1,t-1} \\ log\sigma_{P_2,t-1} \\ \dots \\ log\sigma_{P_k,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \tau_1^{P_1} \\ l_{i2}^2 \tau_1^{P_2} \\ \dots \\ l_{ik}^2 \tau_1^{P_k} \end{pmatrix}' \begin{pmatrix} z_{P_1,t} \\ z_{P_2,t} \\ \dots \\ z_{P_k,t} \end{pmatrix}$$
$$+ \begin{pmatrix} l_{i1}^2 \tau_2^{P_1} \\ l_{i2}^2 \tau_2^{P_2} \\ \dots \\ l_{ik}^2 \tau_2^{P_k} \end{pmatrix}' \begin{pmatrix} z_{P_1,t}^2 - 1 \\ z_{P_2,t}^2 - 1 \\ \dots \\ z_{P_k,t}^2 - 1 \end{pmatrix}, 1 \leq i \leq k$$

EGARCH-X(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j}\right) + \begin{pmatrix} l_{i1}^2 \beta_{P_1} \\ l_{i2}^2 \beta_{P_2} \\ \dots \\ l_{ik}^2 \beta_{P_k} \end{pmatrix}' \begin{pmatrix} log\sigma_{P_1,t-1} \\ log\sigma_{P_2,t-1} \\ \dots \\ log\sigma_{P_k,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \gamma_{P_1} \\ l_{i2}^2 \gamma_{P_2} \\ \dots \\ l_{ik}^2 \gamma_{P_k} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_k,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} l_{i1}^2 \tau_1^{P_1} \\ l_{i2}^2 \tau_1^{P_2} \\ \dots \\ l_{ik}^2 \tau_1^{P_k} \end{pmatrix}' \begin{pmatrix} z_{P_1,t} \\ z_{P_2,t} \\ \dots \\ z_{P_k,t} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \tau_2^{P_1} \\ l_{i2}^2 \tau_2^{P_2} \\ \dots \\ l_{ik}^2 \tau_2^{P_k} \end{pmatrix}' \begin{pmatrix} z_{P_1,t}^2 - 1 \\ z_{P_2,t}^2 - 1 \\ \dots \\ z_{P_k,t}^2 - 1 \end{pmatrix}, 1 \leq i \leq k \text{ and exogenous } x.$$

Realized GARCH(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j}\right) + \begin{pmatrix} l_{i1}^2 \beta_{P_1} \\ l_{i2}^2 \beta_{P_2} \\ \dots \\ l_{ik}^2 \beta_{P_k} \end{pmatrix}' \begin{pmatrix} log\sigma_{P_1,t-1} \\ log\sigma_{P_2,t-1} \\ \dots \\ log\sigma_{P_k,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \gamma_{P_1} \\ l_{i2}^2 \gamma_{P_2} \\ \dots \\ l_{ik}^2 \gamma_{P_k} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_k,t-1} \end{pmatrix}, 1 \leq i \leq k \text{ and endogenous } x.$$

Realized EGARCH(1,1) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j}\right) + \begin{pmatrix} l_{i1}^2 \beta_{P_1} \\ l_{i2}^2 \beta_{P_2} \\ \dots \\ l_{ik}^2 \beta_{P_k} \end{pmatrix}' \begin{pmatrix} log\sigma_{P_1,t-1} \\ log\sigma_{P_2,t-1} \\ \dots \\ log\sigma_{P_k,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \gamma_{P_1} \\ l_{i2}^2 \gamma_{P_2} \\ \dots \\ l_{ik}^2 \gamma_{P_k} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_k,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l_{i1}^2 \tau_1^{P_1} \\ l_{i2}^2 \tau_1^{P_2} \\ \dots \\ l_{ik}^2 \tau_1^{P_k} \end{pmatrix}' \begin{pmatrix} z_{P_1,t} \\ z_{P_2,t} \\ \dots \\ z_{P_k,t} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \tau_2^{P_1} \\ l_{i2}^2 \tau_2^{P_2} \\ \dots \\ l_{ik}^2 \tau_2^{P_k} \end{pmatrix}' \begin{pmatrix} z_{P_1,t}^2 - 1 \\ z_{P_2,t}^2 - 1 \\ \dots \\ z_{P_k,t}^2 - 1 \end{pmatrix}, 1 \le i \le k \text{ and endogenous } x.$$

Realized GARCH(2,2) model (loglikelihood form):

$$log\sigma_{x_i} = \sum_{j=1}^{k} \left(l_{ij}^2 \omega_{P_j}\right) + \begin{pmatrix} l_{i1}^2 \alpha_{P_1} \\ l_{i2}^2 \alpha_{P_2} \\ \dots \\ l_{ik}^2 \alpha_{P_k} \end{pmatrix}' \begin{pmatrix} log(\left(max\left(r_{P_1,t-1}^2, 10^{-20}\right)\right) \\ log(\left(max\left(r_{P_2,t-1}^2, 10^{-20}\right)\right) \\ \dots \\ log(\left(max\left(r_{P_k,t-1}^2, 10^{-20}\right)\right) \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \beta_1^{P_1} \\ l_{i2}^2 \beta_1^{P_2} \\ \dots \\ l_{ik}^2 \beta_1^{P_k} \end{pmatrix}' \begin{pmatrix} log\sigma_{P_1,t-1} \\ log\sigma_{P_2,t-1} \\ \dots \\ log\sigma_{P_k,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l_{i1}^2 \beta_2^{P_1} \\ l_{i2}^2 \beta_2^{P_2} \\ \dots \\ l_{ik}^2 \beta_2^{P_k} \end{pmatrix}' \begin{pmatrix} log\sigma_{P_1,t-2} \\ log\sigma_{P_2,t-2} \\ \dots \\ log\sigma_{P_k,t-2} \end{pmatrix} + \begin{pmatrix} l_{i1}^2 \gamma_1^{P_1} \\ l_{i2}^2 \gamma_1^{P_2} \\ \dots \\ l_{ik}^2 \gamma_1^{P_k} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_k,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l_{i1}^2 \gamma_2^{P_1} \\ l_{i2}^2 \gamma_2^{P_2} \\ \dots \\ l_{ik}^2 \gamma_2^{P_k} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-2} \\ logx_{P_2,t-2} \\ \dots \\ logx_{P_k,t-2} \end{pmatrix}, 1 \le i \le k \text{ and endogenous } x.$$

In the specific setting of our problem with the four stocks (AIG, AXP, BAC and JPM), $N = 4$ and

$$x_1 = AIG, x_2 = AXP, x_3 = BAC, x_4 = JPM, P_1 = P_{AIG}, P_2 = P_{AXP}, P_3 = P_{BAC}, P_4 = P_{JPM}$$

the above equations of conditional volatility may be re-written by replacing the corresponding indices. We can thus express conditional volatility of each stock (e.g. AIG) as a function of returns, volatilities and intraday volatility measures of the principal components of all stocks (AIG, AXP, BAC and JPM).

4.4 Results

The final forms of the PC EGARCH, PC EGARCHX, PC Realized EGARCH, PC Realized GARCH and PC Realized GARCH(2,2) models will be the $\widehat{\boldsymbol{\Omega}}_{N\times N}$ matrix below, having as $\sigma_t^{(k)}$ and $\sigma_t^{(P_k)}$ formulas the GARCH equations of EGARCH, EGARCHX, Realized EGARCH, Realized GARCH and Realized GARCH(2,2) models estimated for the stock returns and principal components of the stock standardized returns.

$$\widehat{\boldsymbol{\Omega}}_{N\times N} = \begin{pmatrix} \sigma_t^{(1)} & \sqrt{\sigma_t^{(1)}}\sqrt{\sigma_t^{(2)}}\frac{\sum_{k=1}^{N} l_{1k} l_{2k}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}} & \cdots & \sqrt{\sigma_t^{(1)}}\sqrt{\sigma_t^{(N)}}\frac{\sum_{k=1}^{N} l_{1k} l_{Nk}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}} \\ \sqrt{\sigma_t^{(2)}}\sqrt{\sigma_t^{(1)}}\frac{\sum_{k=1}^{N} l_{2k} l_{1k}\sigma^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}} & \sigma_t^{(2)} & \cdots & \sqrt{\sigma_t^{(2)}}\sqrt{\sigma_t^{(N)}}\frac{\sum_{k=1}^{N} l_{2k} l_{Nk}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}} \\ \cdots & \cdots & \cdots & \cdots \\ \sqrt{\sigma_t^{(N)}}\sqrt{\sigma_t^{(1)}}\frac{\sum_{k=1}^{N} l_{Nk} l_{1k}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{1k}^2\sigma_t^{(P_k)}}} & \sqrt{\sigma_t^{(N)}}\sqrt{\sigma_t^{(2)}}\frac{\sum_{k=1}^{N} l_{Nk} l_{2k}\sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{Nk}^2\sigma_t^{(P_k)}}\sqrt{\sum_{k=1}^{N} l_{2k}^2\sigma_t^{(P_k)}}} & \cdots & \sigma_t^{(N)} \end{pmatrix}$$

In the forecasting activity, they may be used not just to forecast the 1-day ahead volatility of an *n*-stock asset, but also to better weight one portfolio components in order to minimize the portfolio variation. As such, when a risk averse investor will know what the variance of stocks and the covariance between stocks will be for the next period, he/she will choose to include in the portfolio the stocks with the lowest variation and the lowest covariation, as they will determine lower values of next day variance-covariance matrix $\widehat{\boldsymbol{\Omega}}_{N\times N}$. This comes in support of the idea that the portfolio risk (here represented by volatility) may be reduced by comprising in the portfolio, assets that are opposite in their fluctuations (as this translates in negative covariations and, as such, in lower portfolio omega).

The results obtained above may be even further exploited. According to the variance-covariance matrix formula obtained at the Methodological section, there have been calculated the 4×4 variance-covariance matrices for each day of the sample, according to each model used. According to each day variance-covariance matrix, it has been calculated the variance of the portfolio, considering an equal share of each stock in the portfolio composition. As such, there can be calculated the daily estimated portfolio volatilities, as well the real portfolio volatilities, according to the following formulas:

$$\sigma_t^{(N\ asset\ portfolio,\ \ estimated)} = \sum_{i=1}^{N} a_{i,t}^2 \sigma_{it} + 2 * \sum_{i,j=1}^{N} a_i a_j \left(\sqrt{\sigma_t^{(i)}} \sqrt{\sigma_t^{(j)}} \frac{\sum_{k=1}^{N} l_{ik} l_{jk} \sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{ik}^2 \sigma_t^{(P_k)}} \sqrt{\sum_{k=1}^{N} l_{jk}^2 \sigma_t^{(P_k)}}}\right)$$

$$\sigma_t^{(N\ asset\ portfolio,\ \ real)} = \sum_{i=1}^{N} a_{i,t}^2 r_{it}^2 + 2 * \sum_{i,j=1}^{N} a_i a_j (r_{it} r_{jt})$$

Then, for each day, there will be calculated the squared errors, and for the whole sample the RMSE. The results will be:

	RMSE		RMSE
PC EGARCH	**15,2650**	EGARCH	**111,7485**
PC EGARCHX	**14,8765**	EGARCHX	**109,0850**
PC GARCH	**15,2555**	GARCH	**111,6882**
PC Realized EGARCH	**14,9362**	Realized EGARCH	**39,8860**
PC Realized GARCH(2,2)	**15,1692**	Realized GARCH(2,2)	**109,4996**
PC Realized GARCH	**14,9731**	Realized GARCH	**110,4932**

By comparing these portfolio errors with those of the individual stocks, it can be observed that by making portfolios of stocks it is much easier to predict variations (risk) than predicting individually the stocks. This allows us conclude that it is more effective from forecasting point of view to make portfolios of stocks and predict their return volatility than predicting individually the stock returns.

4.5 Conclusions

In the chapter we offered an adaptation of one existing method (PCA applied to a GARCH model) to a class of conditional autoregressive models that use high frequency data (realized GARCH models). The result was new PC models that comprise measures of intraday volatility: PC Realized GARCH, PC Realized EGARCH, PC Realized GARCH(2,2), PC EGARCHX. As well, a standard PC EGARCH model that uses only daily data was formulated. These models were used in order to assess the volatility of an *n*-stock asset, problem that must take into consideration not only individual volatilities but also correlations between stock returns. By reducing the problem to the case of a four-stock asset, the newly composed models were estimated and their forecasting accuracy was assessed. It was found that the new PC high frequency models estimate and forecast very well the volatility of multiple stock assets, and that by forming portfolios with those assets a risk averse investor could weigh the stocks into the portfolio according to the forecasts, such that the portfolio overall volatility be reduced (by allowing higher weights to those stocks estimated to vary the least on a 1-day ahead horizon coupled simultaneously with negative covariances between stocks). This confirms the Finance theory according to which diversification with low volatility stocks that commove in opposed directions (correlations close to unit) reduces portfolio overall variance.

Furthermore, it was found that by putting stocks into a portfolio in order to forecast the portfolio variance instead of forecasting individual stocks' variance, it delivers significantly lower summed errors (for the first case against the second). This indicated that the best tool for an investor in order to reduce the risk is to choose low volatility stocks, with negative and close to unit correlation, to put them in portfolios and forecast portfolio volatility by using one of the PC-Realized GARCH type of models that assesses multivariate variance by using high frequency data.

Chapter five: The Bivariate Realized models

5.1 Introduction

This chapter aims to develop a class of realized models fitted to a larger context that allows using information on night volatility. In the third chapter of the thesis, besides the simple non-realized models (GARCH(1,1) and EGARCH(1,1)), it was also considered a set of realized models that used daily measures of intraday volatility in the formulation of daily volatility equations (besides daily squared returns and daily variance). As such, there were considered for testing the following models: EGARCH-X(1,1), Realized EGARCH(1,1), Realized GARCH(1,1) and Realized GARCH(2,2). These models estimate the daily volatility by considering return time series formed of either close-to-close or of open-to-close return time series. They were estimated and their accuracy in forecasting was assessed by using a double methodology (in sample and out of sample) and various measures of intraday volatility. Their accuracy was assessed by comparing the size of the maximum loglikelihood function and the size of three error measurements (MAE, RMSE and MAPE).

In chapter four it was proposed a class of PC-GARCH models that allowed forecasting volatility of multivariate assets formed of highly correlated stocks. By applying Principal Component algorithm to the recently proposed realized GARCH models, there were proposed new composite models (PC EGARCH(1,1), PC EGARCHX(1,1), PC Realized GARCH(1,1), PC Realized EGARCH(1,1) and PC Realized GARCH(2,2)). General models were proposed, as well as four-dimensional models were specifically defined to accommodate to the empirical exercise with the four stocks (AIG, AXP, BAC, and JPM). Following a similar methodology as for the non-PC models, the PC-variants were estimated in sample and out of sample, for six types of realized measures of intraday variance (RK, RV sampled at 15 seconds, 5 minutes, 15 minutes and 20 minutes, H-L) and their accuracy capacity was assessed according to the size of the maximum loglikelihood functions, and the size of RMSE, MAE and MAPE loss functions. As well, it was investigated the gain (or loss) in accuracy obtained by employing a PCA procedure in realized volatility modeling.

In the chapter to follow, the general scope is to further develop the models that use measures of intraday volatility (the realized models) by reformulating the previous univariate models (which are models that compound daily volatility from one vector information that has relevance to daily volatility – returns, variance, realized measures) to a double component formula that would also consider information on night volatility. This idea sparked from an observation on financial stock time series: not only the prices at the opening of the trading market differ from those at the market closing the same day (day volatility), but also prices at the market closing differ from those at the market opening the following trading day. This is even further surprising as during the night the market is closed and thus no transactions with stocks take place. As such, although no trades occur, some hidden volatility still exists, thing which indicates that volatility modeling could be further developed by extending the existing day models to formulations that would also consider measurements of night volatility.

The current work starts from an idea belonging to Hansen, Huang and Shek (2010b) who proposed a partial form (with exogenous realized measures) of a Bivariate Realized GARCH model. We use this idea in order to further propose bivariate versions of other univariate realized models. This defines the first goal of the chapter that reformulates a class of realized GARCH-type of models so that they also encumber measures of night volatility. As such, starting from an existing model (Bivariate GARCH partial model), we propose new realized models and one non-realized model that also enclose measures of night volatility: Bivariate Realized GARCH(1,1) (with an endogenous component of realized measure and therefore with a separate measurement equation, that we will call a full version model), Bivariate EGARCH-X, Bivariate Realized EGARCH (1,1), Bivariate Realized GARCH(2,2) and Bivariate EGARCH(1,1).

The second goal of the chapter is to estimate the new models (and also the already proposed Bivariate Realized GARCH partial model) for each of the four stocks (AIG, AXP, BAC and JPM), estimation that will be undertaken in sample and out of sample, by using the realized kernels as measures of the intraday volatility (for the realized models).

The third goal of the chapter is to undertake a performance assessment of the new models (and of the Bivariate Realized GARCH partial model), as it will be assessed the accuracy gain (or loss) in forecasting of the bivariate models when compared to the univariate versions; the performance will be evaluated by comparing the maximum loglikelihood functions and the loss functions

(RMSE, MAE and MAPE). The bivariate models will also be ranked, in order to determine the better performers among their peers.

The following models will be considered for bivariate transformation:

Model*	**Return Equation***	**GARCH Equation***	**Measurement equation***
EGARCH(1,1) (loglikelihood form) (Nelson, 1991)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \left(\tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)\right)$	-
EGARCH-X(1,1) (loglikelihood form) (Engle, 2002)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \gamma log x_{t-1} + \tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)$	-
Realized GARCH(1,1) (loglikelihood form) (Hansen et al., 2010)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \gamma log x_{t-1}$	$log x_t = \xi + \varphi log\sigma_t + \delta z_t + u_t$
Realized EGARCH(1,1) (loglikelihood form) (Hansen et al., 2010)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \beta log\sigma_{t-1} + \left(\tau_1 z_{t-1} + \tau_2(z_{t-1}^2 - 1)\right) + \gamma log x_{t-1}$	$log x_t = \xi + \varphi log\sigma_t + \delta z_t + u_t$
Realized GARCH(2,2) (loglikelihood form) (Hansen et al., 2010)	$z_t = \frac{r_t - \mu}{\sqrt{\sigma_t}}$	$log\sigma_t = \omega + \alpha log(max(r_{t-1}^2, 10^{-20})) + \beta_1 log\sigma_{t-1} + \beta_2 log\sigma_{t-2} + \gamma_1 log x_{t-1} + \gamma_2 log x_{t-2}$	$log x_t = \xi + \varphi log\sigma_t + (\tau_1 z_t + \tau_2(z_t^2 - 1)) + u_t$

where z_t are studentized returns, r_t are daily returns, x_t are the intraday volatility measure and u_t are the errors.

The loglikelihood functions corresponding to the above models are $l(r,x) = -\frac{1}{2}\sum_{t=1}^{n}\left\{log(2\pi) + log(\sigma_t) + \frac{(r_t-\mu)^2}{\sigma_t}\right\}$ (EGARCH and EGARCH-X) and $l(r,x) = -\frac{1}{2}\sum_{t=1}^{n}\left\{2\,log(2\pi) + log(\sigma_t) + \frac{(r_t-\mu)^2}{\sigma_t} + log(\sigma_u^2) + \frac{u_t^2}{\sigma_u^2}\right\}$ (Realized EGARCH, Realized GARCH and Realized GARCH(2,2)).

5.2 The models

As mentioned in the introductory part, we use the idea of Hansen, Huang and Shek (2010b) who formulated a partial version of the Bivariate Realized GARCH model in order to obtain bivariate versions of the models mentioned in the above table. However, we will also enclose this model in our estimation part and also in the performance assessment exercise.

Hansen, Huang and Shek's (2010b) partial Bivariate Realized GARCH model is:

$$r_t = r_t^{\bullet} + r_t^{\circ}, z_t^{\circ} = \frac{r_t^{\circ} - \mu}{\sqrt{\sigma_t^{\circ}}}, z_t^{\bullet} = \frac{r_t^{\bullet} - \mu}{\sqrt{\sigma_t^{\bullet}}}$$

$$log\sigma_t^{\circ} = \omega_{\circ} + \tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ}) + \beta_{\circ} logh_{t-1}^{\circ} + \gamma_{\circ} logx_{t-1}$$

$$log\sigma_t^{\bullet} = \omega_{\bullet} + \tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ}) + \beta_{\bullet} logh_{t-1}^{\bullet} + \gamma_{\bullet} logx_{t-1}$$

where $\bullet$ denotes the night information, and ° denotes the day information of the vector.

The loglikelihood function

The data is a bivariate vector compounded of two univariate vectors that refer to uncorrelated sets of information (we consider first that night volatility is uncorrelated to day one):

$$\begin{pmatrix} r_t^{\bullet} \\ r_t^{\circ} \end{pmatrix} |F_{t-1} \sim N(\mathbf{0}, \begin{pmatrix} \sigma_t^{\bullet} & 0 \\ 0 & \sigma_t^{\circ} \end{pmatrix})$$

Accordingly, the random vector $\begin{pmatrix} r_t^{\bullet} \\ r_t^{\circ} \end{pmatrix}$ depends solely on the information set available at time *t-1*, and has a normal distribution with $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ mean (equivalent to the fact that r_t° and $r_t^{\bullet}$ have both zero mean), and a variance equal to the variance-covariance matrix $\begin{pmatrix} \sigma_t^{\bullet} & 0 \\ 0 & \sigma_t^{\circ} \end{pmatrix}$. The latter one is equivalent to $var(r_t^{\circ}) = \sigma_t^{\circ}$, $var(r_t^{\bullet}) = \sigma_t^{\bullet}$ and $cov(r_t^{\circ}, r_t^{\bullet}) = 0$.

The total volatility is $r_t = r_t^{\circ} + r_t^{\bullet}$. A theoretical result says that when a random vector (as $\begin{pmatrix} r_t^{\bullet} \\ r_t^{\circ} \end{pmatrix}$) is normally distributed, then its components will be normal as well (this time unidimensional normal since each component is normal). From $\begin{pmatrix} r_t^{\bullet} \\ r_t^{\circ} \end{pmatrix} |F_{t-1} \sim N(\mathbf{0}, \begin{pmatrix} \sigma_t^{\bullet} & 0 \\ 0 & \sigma_t^{\circ} \end{pmatrix})$ it results that $r_t^{\bullet} | F_{t-1}, \sim N(0, \sigma_t^{\bullet})$ and $r_t^{\circ} | F_{t-1} \sim N(0, \sigma_t^{\circ})$. Since a sum of two normal variables is a normal variable with the average equal to the arithmetical sum of the two component averages,

$E(r_t|F_{t-1}) = E(r_t^\bullet|F_{t-1}) + E\left(r_t^\circ|F_{t-1}\right) \sim 0 + 0 = 0,$ and $var(r_t|F_{t-1}) = var(r_t^\bullet|F_{t-1}) + var\left(r_t^\circ|F_{t-1}\right) + 2cov(r_t^\bullet|F_{t-1}, r_t^\circ|F_{t-1})$. This is why we will have[2] $r_t|F_{t-1} \sim N\left(0, \sigma_t^\bullet + \sigma_t^\circ\right)$.

If $r_t|F_{t-1} \sim N(0, \sigma_t^\bullet + \sigma_t^\circ)$, then the density function of $r_t|F_{t-1}$ will have the form of a normal variable, that is:

$$f(r_t) = \frac{1}{\sqrt{\sigma_t^*}\sqrt{2\pi}} e^{\frac{r_t^2}{2\sigma_t^*}}$$

where σ_t^* is the variance of r_t, and, as mentioned before, $\sigma_t^* = \sigma_t^\bullet + \sigma_t^\circ$. Since *n* observations of *t=1, … ,n* are made, the likelihood function is the $\begin{pmatrix} r_1 \\ \cdots \\ r_n \end{pmatrix}$ vector's density, and $r_1, \ldots, r_n$ are independent to each other, the likelihood function will be

$$l(r_t) = \prod_{t=1}^{n} f(r_t) = \prod_{t=1}^{n} \frac{1}{\sqrt{\sigma_t^*}\sqrt{2\pi}} e^{\frac{r_t^2}{2\sigma_t^*}} = \left(\frac{1}{\sqrt{2\pi}}\right)^n \left(\prod_{t=1}^{n} \frac{1}{\sqrt{\sigma_t^*}}\right)^{-\frac{1}{2}\sum_{t=1}^{n}\frac{r_t^2}{\sigma_t^*}}$$

Taking the log of this expression and using the logarithm properties, the loglikelihood function of the total returns r_t will become

$$logl(r_t) = log\left(\prod_{t=1}^{n}\left(\left(\frac{1}{\sqrt{2\pi}}\right)^n\right) + log\left(\left(\prod_{t=1}^{n}\frac{1}{\sqrt{\sigma_t^*}}\right)\right) - \frac{1}{2}\sum_{t=1}^{n}\frac{r_t^2}{\sigma_t^*}\right.$$

$$= -\frac{n}{2}log(2\pi) - \frac{1}{2}\sum_{t=1}^{n} log(\sigma_t^*) - \frac{1}{2}\sum_{t=1}^{n}\frac{r_t^2}{\sigma_t^*} = \frac{1}{2}\left[\sum_{t=1}^{n} log(2\pi) + log(\sigma_t^*) + \frac{r_t^2}{\sigma_t^*}\right]$$

If we consider a more complete model with a non-null correlation between r_t° and $r_t^\bullet$ (meaning that the night volatility influences the day one), that is $corr\left(r_t^\circ, r_t^\bullet\right) = \rho \neq 0$, the formulation of the loglikelihood function slightly changes. Let's observe first that ρ does not depend on *t*, that is the correlation is not time dependant. Then, the covariance will be

$$cov\left(r_t^\circ, r_t^\bullet\right) = corr\left(r_t^\circ, r_t^\bullet\right)\sqrt{var(r_t^\bullet)var\left(r_t^\circ\right)} = \rho\sqrt{\sigma_t^\bullet \sigma_t^\circ}$$

[2] Note: In all the above notations we kept the conditionality $|F_{t-1}$ because the variables come from a bidimensional variable that is also $|F_{t-1}$ conditioned.

That means that in the new model (with a non-null correlation), the variance-covariance matrix will take the form

$$\begin{pmatrix} \sigma_t^{\bullet} & \rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} \\ \rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} & \sigma_t^{\circ} \end{pmatrix}$$

having on the first diagonal the variances of r_t° and $r_t^{\bullet}$, and on the second diagonal the covariance between r_t° and $r_t^{\bullet}$, that is $cov(r_t^{\circ}, r_t^{\bullet})$ (since $cov(r_t^{\circ}, r_t^{\bullet}) = cov(r_t^{\bullet}, r_t^{\circ})$). As such, the $|F_{t-1}$ conditioned distribution of the $\binom{r_t^{\bullet}}{r_t^{\circ}}$ vector will be:

$$\binom{r_t^{\bullet}}{r_t^{\circ}} |F_{t-1} \sim N(\mathbf{0}, \begin{pmatrix} \sigma_t^{\bullet} & \rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} \\ \rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} & \sigma_t^{\circ} \end{pmatrix})$$

The conditional variance of r_t, $var(r_t|F_{t-1})$, will be $var(r_t|F_{t-1}) = var(r_t^{\bullet}|F_{t-1}) + var(r_t^{\circ}|F_{t-1}) + 2cov(r_t^{\bullet}|F_{t-1}, r_t^{\circ}|F_{t-1}) = \sigma_t^{\bullet} + \sigma_t^{\circ} + 2\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}$, that is $\sigma_t^{*} = \sigma_t^{\bullet} + \sigma_t^{\circ} + 2\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}$.

The loglikelihood function of $r_t = r_t^{\bullet} + r_t^{\circ}$ will be the same as the one iterated for the null correlation case, with the only difference that the variance will enclose the correlation term:

$$\sigma_t^{*} = \sigma_t^{\bullet} + \sigma_t^{\circ} + 2\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}$$

However, we want to consider the loglikelihood function of the bivariate vector $\binom{r_t^{\bullet}}{r_t^{\circ}}$, and not the one of the univariate one $r_t = r_t^{\bullet} + r_t^{\circ}$. As such, in order to define the new loglikelihood function, we will consider the density function of the bidimensional normal $\binom{r_t^{\bullet}}{r_t^{\circ}}$.

The general form of a p dimensional normal vector $N_p(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ (a matrix with $\boldsymbol{\mu}$ vector average and $\boldsymbol{\Sigma}$ variance-covariance matrix) takes the form:

$$f(x) = \frac{1}{(\sqrt{2\pi})^p} \frac{1}{\sqrt{det(\Sigma)}} e^{-\frac{1}{2}(x-\mu)'\Sigma^{-1}(x-\mu)}$$

in which **x** is any vector to which it has been calculated the density function with p arguments, $det(\Sigma)$ is the determinant of the variance-covariance matrix Σ and $(x-\mu)'\Sigma^{-1}(x-\mu)$ is the matrix product between the transpose of the $(x-\mu)$ vector, the inverse of matrix Σ and the $(x-\mu)$ vector . As such, with p=2 for the particular case of a bidimensional vector $\binom{r_t^{\bullet}}{r_t^{\circ}}$, the density function will be

$$f\left(r_t^{\bullet}, r_t^{\circ}\right) = \frac{1}{(\sqrt{2\pi})^2} \frac{1}{\sqrt{det(\Sigma)}} e^{-\frac{1}{2}(r_t^{\bullet}, r_t^{\circ})\Sigma^{-1}\binom{r_t^{\bullet}}{r_t^{\circ}}}$$

in which $\mu = 0$ and $\Sigma = \begin{pmatrix} \sigma_t^{\bullet} & \rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} \\ \rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} & \sigma_t^{\circ} \end{pmatrix}$

Since $det(\Sigma) = \sigma_t^{\bullet}\sigma_t^{\circ} - \rho^2\sigma_t^{\bullet}\sigma_t^{\circ} = \sigma_t^{\bullet}\sigma_t^{\circ}(1-\rho^2)$, then the log form of it will be $log(det(\Sigma)) = log(\sigma_t^{\bullet}) + log(\sigma_t^{\circ}) + log(1-\rho^2)$.

The inverse matrix of the variance-covariance matrix will be

$$\Sigma^{-1} = \frac{1}{\sigma_t^{\bullet}\sigma_t^{\circ}(1-\rho^2)} \begin{pmatrix} \sigma_t^{\bullet} & -\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} \\ -\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}} & \sigma_t^{\circ} \end{pmatrix}$$

As such, the product $-\frac{1}{2}\left(r_t^{\bullet}, r_t^{\circ}\right)\Sigma^{-1}\binom{r_t^{\bullet}}{r_t^{\circ}}$ will become

$$-\frac{1}{2}\left(r_t^{\bullet}, r_t^{\circ}\right)\Sigma^{-1}\binom{r_t^{\bullet}}{r_t^{\circ}} = -\frac{1}{2}\frac{{r_t^{\bullet}}^2\sigma_t^{\circ} + {r_t^{\circ}}^2\sigma_t^{\bullet} - 2r_t^{\bullet}r_t^{\circ}\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}}{\sigma_t^{\bullet}\sigma_t^{\circ}(1-\rho^2)}$$

Thus, the loglikelihood function $logl(r_t^{\bullet}, r_t^{\circ})$ will be obtained through multiplying the functions $f\left(r_t^{\bullet}, r_t^{\circ}\right)$ for the *t=1,…,n*, and taking the log of the resulting product:

$$logl(r_t^{\bullet}, r_t^{\circ}) = -\frac{1}{2}\sum_{t=1}^{n}\left\{2log(2\pi) + lo\,g(1-\rho^2) + log(\,\sigma_t^{\bullet}) + log(\,\sigma_t^{\circ}) + \frac{r_t^{\bullet 2}\sigma_t^{\circ} + r_t^{\circ 2}\sigma_t^{\bullet} - 2r_t^{\bullet}r_t^{\circ}\rho\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}}{\sigma_t^{\bullet}\sigma_t^{\circ}(1-\rho^2)}\right\}$$

By doing some simple iterations in the expression above, we obtain the final form of the bivariate loglikelihood function as

$$logl(r_t^{\bullet}, r_t^{\circ}) = -\frac{1}{2}\sum_{t=1}^{n}\left\{2log(2\pi) + log(1-\rho^2) + log(\,\sigma_t^{\bullet}) + log(\,\sigma_t^{\circ}) + \frac{r_t^{\bullet 2}/\sigma_t^{\bullet} + r_t^{\circ 2}/\sigma_t^{\circ}}{(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)}\frac{r_t^{\bullet}r_t^{\circ}}{\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}}\right\}$$

New bivariate models

In the section to follow we will formulate the new bivariate models. There will be proposed a Bivariate EGARCH(1,1), a Bivariate Realized GARCH(1,1) model in its complete form (containing also a measurement equation of the realized measure), a Bivariate Realized EGARCH(1,1) model, a Bivariate EGARCH-X(1,1) model and a Realized GARCH(2,2) model. These models will contain night and day volatility information and (excepting the first one) a measurement of intraday volatility.

We propose the following bivariate models:

Model	**Return Equations**	**GARCH Equations**	**Measurement equation**
Bivariate Realized GARCH(1,1), full form	$r_t = r_t^{\bullet} + r_t^{\circ}$ $z_t^{\circ} = \frac{r_t^{\circ} - \mu^{\circ}}{\sqrt{\sigma_t^{\circ}}}$ $z_t^{\bullet} = \frac{r_t^{\bullet} - \mu^{\bullet}}{\sqrt{\sigma_t^{\bullet}}}$	$log\sigma_t^{\circ} = \omega_{\circ} + \tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ}) + \beta_{\circ} log\sigma_{t-1}^{\circ} + \gamma_{\circ} logx_{t-1}$ $log\sigma_t^{\bullet} = \omega_{\bullet} + \tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ}) + \beta_{\bullet} log\sigma_{t-1}^{\bullet} + \gamma_{\bullet} logx_{t-1}$	$logx_t = \xi + \varphi log\sigma_t^{\circ} + \vartheta log\sigma_t^{\bullet} + \delta^{(\circ 1)} z_t^{\bullet} + \delta^{(\circ 2)} z_t^{\circ} + u_t$
Bivariate Realized EGARCH(1,1)	$r_t = r_t^{\bullet} + r_t^{\circ}$ $z_t^{\circ} = \frac{r_t^{\circ} - \mu^{\circ}}{\sqrt{\sigma_t^{\circ}}}$ $z_t^{\bullet} = \frac{r_t^{\bullet} - \mu^{\bullet}}{\sqrt{\sigma_t^{\bullet}}}$	$log\sigma_t^{\circ} = \omega_{\circ} + \varepsilon^{(\circ 1)}[\tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ})] + \varepsilon^{(\circ 2)}\{[\tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ})]^2 - 1\} + \beta_{\circ} log\sigma_{t-1}^{\circ} + \gamma_{\circ} logx_{t-1}$ $log\sigma_t^{\bullet} = \omega_{\bullet} + \varepsilon^{(\bullet 1)}[\tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ})] + \varepsilon^{(\bullet 2)}\{[\tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ})]^2 - 1\} + \beta_{\bullet} log\sigma_{t-1}^{\bullet} + \gamma_{\bullet} logx_{t-1}$	$logx_t = \xi + \varphi log\sigma_t^{\circ} + \vartheta log\sigma_t^{\bullet} + \delta^{(\circ 1)} z_t^{\bullet} + \delta^{(\circ 2)} z_t^{\circ} + u_t$
Bivariate EGARCH(1,1)	$r_t = r_t^{\bullet} + r_t^{\circ}$ $z_t^{\circ} = \frac{r_t^{\circ} - \mu^{\circ}}{\sqrt{\sigma_t^{\circ}}}$ $z_t^{\bullet} = \frac{r_t^{\bullet} - \mu^{\bullet}}{\sqrt{\sigma_t^{\bullet}}}$	$log\sigma_t^{\circ} = \omega_{\circ} + \varepsilon^{(\circ 1)}[\tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ})] + \varepsilon^{(\circ 2)}\{[\tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ})]^2 - 1\} + \beta_{\circ} log\sigma_{t-1}^{\circ}$ $log\sigma_t^{\bullet} = \omega_{\bullet} + \varepsilon^{(\bullet 1)}[\tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ})] + \varepsilon^{(\bullet 2)}\{[\tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ})]^2 - 1\} + \beta_{\bullet} log\sigma_{t-1}^{\bullet}$	
Bivariate EGARCH-X	$r_t = r_t^{\bullet} + r_t^{\circ}$ $z_t^{\circ} = \frac{r_t^{\circ} - \mu^{\circ}}{\sqrt{\sigma_t^{\circ}}}$ $z_t^{\bullet} = \frac{r_t^{\bullet} - \mu^{\bullet}}{\sqrt{\sigma_t^{\bullet}}}$	$log\sigma_t^{\circ} = \omega_{\circ} + \varepsilon^{(\circ 1)}[\tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ})] + \varepsilon^{(\circ 2)}\{[\tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ})]^2 - 1\} + \beta_{\circ} log\sigma_{t-1}^{\circ} + \gamma_{\circ} logx_{t-1}$ $log\sigma_t^{\bullet} = \omega_{\bullet} + \varepsilon^{(\bullet 1)}[\tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ})] + \varepsilon^{(\bullet 2)}\{[\tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ})]^2 - 1\} + \beta_{\bullet} log\sigma_{t-1}^{\bullet} + \gamma_{\bullet} logx_{t-1}$	
Bivariate Realized GARCH(2,2)	$r_t = r_t^{\bullet} + r_t^{\circ}$ $z_t^{\circ} = \frac{r_t^{\circ} - \mu^{\circ}}{\sqrt{\sigma_t^{\circ}}}$ $z_t^{\bullet} = \frac{r_t^{\bullet} - \mu^{\bullet}}{\sqrt{\sigma_t^{\bullet}}}$	$log\sigma_t^{\circ} = \omega_{\circ} + \tau^{(\circ 1)}(z_{t-1}^{\bullet}) + \tau^{(\circ 2)}(z_{t-1}^{\circ}) + \alpha^{\circ} log(max({r_{t-1}^{\circ}}^2, 10^{-20})) + \beta^{(\circ 1)} log\sigma_{t-1}^{\circ} + \beta^{(\circ 2)} log\sigma_{t-2}^{\circ} + \gamma^{(\circ 1)} logx_{t-1} + \gamma^{(\circ 2)} logx_{t-2}$ $log\sigma_t^{\bullet} = \omega_{\bullet} + \tau^{(\bullet 1)}(z_{t-1}^{\bullet}) + \tau^{(\bullet 2)}(z_{t-1}^{\circ}) + \alpha^{\bullet} log(max({r_{t-1}^{\bullet}}^2, 10^{-20})) + \beta^{(\bullet 1)} log\sigma_{t-1}^{\bullet} + \beta^{(\bullet 2)} log\sigma_{t-2}^{\bullet} + \gamma^{(\bullet 1)} logx_{t-1} + \gamma^{(\bullet 2)} logx_{t-2}$	$logx_t = \xi + \varphi log\sigma_t^{\circ} + \vartheta log\sigma_t^{\bullet} + \varepsilon_1[\delta^{(\circ 1)} z_t^{\bullet} + \delta^{(\circ 2)} z_t^{\circ}] + \varepsilon_2\{[\delta^{(\circ 1)} z_t^{\bullet} + \delta^{(\circ 2)} z_t^{\circ}]^2 - 1\} + u_t$

According to the loglikelihood function iterated previously, the loglikelihood functions of the bivariate models will be:

Model	Loglikelihood functions
• Bivariate Realized GARCH(1,1), full form • Bivariate Realized EGARCH(1,1) • Bivariate Realized GARCH(2,2)	$l(r_t^{\bullet}, r_t^{\circ}, x_t) = -\frac{1}{2}\sum_{t=1}^{n}\Big\{2\log(2\pi) + log(1-\rho^2) + log\sigma_t^{\bullet} + log\sigma_t^{\circ} + \frac{(r_t^{\bullet}-\mu^{\bullet})^2/\sigma_t^{\bullet} + (r_t^{\circ}-\mu^{\circ})^2/\sigma_t^{\circ}}{(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)}\frac{(r_t^{\bullet}-\mu^{\bullet})(r_t^{\circ}-\mu^{\circ})}{\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}}\Big\} + -\frac{1}{2}\sum_{t=1}^{n}\{log(2\pi) + lo\,g(\sigma_u^2) + u_t^2/\sigma_u^2\}$
• Bivariate EGARCH(1,1) • Bivariate EGARCH-X	$l(r_t^{\bullet}, r_t^{\circ}) = -\frac{1}{2}\sum_{t=1}^{n}\Big\{2\log(2\pi) + log(1-\rho^2) + log\sigma_t^{\bullet} + log\sigma_t^{\circ} + \frac{(r_t^{\bullet}-\mu^{\bullet})^2/\sigma_t^{\bullet} + (r_t^{\circ}-\mu^{\circ})^2/\sigma_t^{\circ}}{(1-\rho^2)} - \frac{2\rho}{(1-\rho^2)}\frac{(r_t^{\bullet}-\mu^{\bullet})(r_t^{\circ}-\mu^{\circ})}{\sqrt{\sigma_t^{\bullet}\sigma_t^{\circ}}}\Big\}$

5.3 Methodology

To estimate the models, we have considered shorter time series as compared to those used in chapters three and four, due to the following reason: as explained before, along the time series over January 4, 1995 – September 30, 2008 period, some stocks, at various points in time, were split in at least two parts: as such each stock became two or more stocks, and, as such, each new stock had half (or less) of the price of the old stock. For example, on June 18, 1997, the BAC stock had at the closing of the trading day the \$134,75 price. At the opening of the next trading day, the stock had \$67.5 price. Due to the large drop, we presume that such a split decision was put in place, and the variation in price was not the effect of the stock conditional volatility. In the four time series we use, we can observe drops in prices by different large amounts (30%, 40% or 50%) from one day to another, and we assume they are due to these administrative decisions. As we do not want to include this variance (effect of administrative decisions) in the volatility modeling since it’s not part of latent volatility process of the stock, but rather a decision of the boards towards stocks’ splitting, we decided to restrict the time series only to a common period in which no such events occurred. In chapters three and four we could consider the whole time series as we could avoid this type of variation by considering only daily (open-to-close) returns. However, in this exercise we cannot avoid anymore this problem, since we have to include also night (close-to-open) volatility. As such, we need to consider only that period that has no such stock-splits along. By investigating the data available, it came natural to consider that August 30, 2004 – September 30, 2008 (1023 daily observations) would be the appropriate period for the models’ estimation. Half trading days around the Christmas and Thanksgiving days were excluded. The daily returns were calculated in logarithmic form, as follows:

$$r_t^{\circ} = 100 * (\log(closing\ price_t) - \log(opening\ price_t))$$

$$r_t^{\bullet} = 100 * (\log(opening\ price_t) - \log(closing\ price_t))$$

for each trading day t, $1 \leq t \leq 1023$.

We have estimated the proposed bivariate models by using one type of intraday measure only – realized kernels, and the estimation has been done in sample, as well out of sample. The cut point for the out of sample estimations was chosen such that we kept the proportionality in chapters three and four. In these two chapters, for the out of sample estimations, we maximized the loglikelihood function for the 1^{st} (January 4, 1995) to the 2999^{th} (December 28, 2006) observation period, while the loglikelihood and loss functions were measured for the rest of the sample (from the 3000^{th} observation (January 3, 2007) to the 3436^{th} one (September 30^{th}, 2008)). This means that the cut-point was at the 87% of the observations. Taking the same proportionality between the number of observations on which we maximize the loglikelihood functions and the sample we use to assess the models’ forecasting accuracy, we chose that for the estimations in chapters five and six, the cut point be at the 894^{th} observation. As such, we maximized the loglikelihood functions for August 30, 2004 to March 26, 2008 period, and we measured the loglikelihood and loss functions for the March 27, 2008 – September 30, 2008 period. For making possible the performance comparison between the bivariate models with the univariate ones, we have estimated again the univariate models, in sample and out of sample, with realized kernels as measures of intraday variance, this time over the shorter time period (1023 days long). The same as in chapter three, although the loglikelihood functions composed of both loglikelihood functions of the returns and of the intraday measures were maximized, when we compared the models having as criteria the size of the loglikelihood function, we considered only the corresponding loglikelihood function for returns (although we maximize the loglikelihood function composed by the partial loglikelihood functions for returns and for the realized measures). As such, the estimation has been done by maximizing the total loglikelihood functions (MLE) (sum of partial loglikelihood functions for the returns and for the intraday measures), and the ranking criterion with respect to the MLE was the partial loglikelihood function for returns solely.

The bivariate models will be ranked after the size of the partial MLE for returns and after the three loss functions ($RMSE = \sqrt{\frac{\sum_{t=1}^{n}(\sigma_t - r_t^2)^2}{n}}$, $MAE = \frac{\sum_{t=1}^{n}|\sigma_t - r_t^2|}{n}$, $MAPE = \frac{\sum_{t=1}^{n}\left|\frac{\sigma_t - r_t^2}{r_t^2}\right|}{n}$). In each ranking, each function that served as criterion for ranking (MLE, RMSE, MAE, or MAPE) was normalized according to the formula

$$x_{normalized,t} = \frac{x_t - a_t}{b_t - a_t}$$

where a_t and b_t represent the function values of the highest/lowest ranked model. Normalizing the MLE or the loss function values will enable us to make the functions comparable across the rankings, independent of the criteria considered, and their sum will allow us to obtain the general rankings.

5.4 Results

The results were as follows:

Model		Maximum loglikelihood (in sample)			
		AIG	AXP	BAC	JPM
EGARCH	Univariate	-1730,2073	-1676,6592	-1511,6476	-1660,7353
	Bivariate	-2895,1909	-2749,5632	-2503,8515	-2755,4350
EGARCH-X	Univariate	-1713,2371	-1646,1439	-1478,3047	-1625,4193
	Bivariate	-2828,6248	-2794,2029	-2440,3491	-2701,7384
Realized EGARCH	Univariate	-1717,6155	-1649,0006	-1481,3114	-1628,1639
	Bivariate	-2847,5983	-2846,4539	-2445,9272	-2704,8329

Realized GARCH	Univariate	-1718,2407	-1649,1724	-1482,3424	-1629,4545
	Bivariate (complete)	-2882,7995	-2857,1460	-2449,4971	-2717,8488
	Bivariate (partial)	-2880,0910	-2861,1102	-2446,5110	-2705,1120
Realized GARCH(2,2)	Univariate	-1707,7811	-1646,5174	-1473,6082	-1619,1382
	Bivariate	-2854,2452	-2906,9021	-2443,4573	-2689,5946

Model		**Maximum loglikelihood (out of sample)**			
		AIG	**AXP**	**BAC**	**JPM**
EGARCH	Univariate	-415,6748	-326,1583	-358,6989	-339,1702
	Bivariate	-1038,6760	-615,0292	-665,3093	-618,1814
EGARCH-X	Univariate	-397,4277	-317,2314	-352,4266	-332,9467
	Bivariate	-801,0685	-577,7278	-696,0684	-600,3923
Realized EGARCH	Univariate	-399,1439	-316,4594	-360,6550	-331,3792
	Bivariate	-762,7673	-579,1729	-684,3183	-591,1353
Realized GARCH	Univariate	-387,7238	-315,2809	-352,3031	-331,3937
	Bivariate (complete)	-783,2343	-579,2929	-684,5237	-590,2800
	Bivariate (partial)	-791,8152	-577,1349	-681,4939	-588,6861
Realized GARCH(2,2)	Univariate	-390,2788	-313,1411	-349,9115	-328,9576
	Bivariate	-741,6102	-581,3847	-681,7825	-591,8240

However, in the above tables, the bivariate models have differently composed log-likelihood functions as compared to the univariate versions: they maximized a bidimensional vector $\begin{pmatrix} r_t^{\bullet} \\ r_t^{\circ} \end{pmatrix}$ formed of two subvectors $(r_t^{\bullet}, r_t^{\circ})$, and a non-null correlation factor was considered, that is ρ. Due to this, the loglikelihood functions are not fully comparable to those of the univariate models (in which we maximized loglikelihood functions of univariate models, with no correlation factor). A conclusion on the superiority of one of the two models' categories would not be consistent. For this scope we use the following results.

The forecasting accuracy was compared by calculating the three loss functions, and the results were:

In sample		**EGARCH**		**EGARCH-X**		**Realized EGARCH**		**Realized GARCH**				**Realized GARCH(2,2)**	
		Univ	**Biv**	Univ	**Biv**	Univ	**Biv**	Univ	**Biv (com)**	Univ	**Biv (par)**	Univ	**Biv**
AIG	RMSE	202,98	**189,01**	203,12	**195,97**	189,71	**196,51**	253,59	**218,68**	253,59	**220,38**	189,92	**250,63**
	MAE	17,51	**15,27**	19,69	**17,08**	16,77	**15,83**	22,72	**21,72**	22,72	**21,70**	16,97	**22,10**
	MAPE	122,86	**131,77**	115,55	**114,68**	108,43	**118,15**	114,09	**123,17**	114,09	**122,39**	105,57	**125,39**
AXP	RMSE	6,57	**6,54**	6,26	**6,70**	6,27	**6,18**	6,24	**6,17**	6,24	**6,19**	6,22	**7,42**

	MAE	2,77	**2,72**	2,67	**2,94**	2,66	**2,70**	2,67	**2,68**	2,67	**2,69**	2,68	**3,09**
	MAPE	132,88	**119,14**	127,41	**137,54**	124,83	**124,45**	124,55	**123,79**	124,55	**124,78**	122,76	**131,22**
BAC	RMSE	16,25	**16,32**	15,76	**15,80**	15,84	**15,89**	15,95	**15,90**	15,95	**15,87**	15,72	**15,42**
	MAE	3,86	**4,00**	3,73	**3,72**	3,55	**3,65**	3,77	**3,78**	3,77	**3,74**	3,60	**3,63**
	MAPE	84,01	**89,53**	83,84	**84,17**	78,78	**82,15**	82,87	**85,58**	82,87	**82,49**	81,08	**82,48**
JPM	RMSE	11,10	**11,17**	10,66	**10,66**	10,58	**10,74**	10,60	**10,50**	10,60	**10,66**	10,55	**10,52**
	MAE	3,26	**3,21**	3,15	**3,13**	3,06	**3,15**	3,12	**3,11**	3,12	**3,15**	3,13	**3,11**
	MAPE	146,36	**171,58**	158,83	**158,80**	155,04	**163,82**	156,35	**161,78**	156,35	**156,67**	153,29	**160,88**

Out of sample		**EGARCH**		**EGARCH-X**		**Realized EGARCH**		**Realized GARCH**				**Realized GARCH(2,2)**	
		Univ	**Biv**	Univ	**Biv**	Univ	**Biv**	Univ	**Biv (com)**	Univ	**Biv (par)**	Univ	**Biv**
AIG	RMSE	565,12	**574,73**	551,61	**533,64**	543,67	**567,07**	552,58	**572,62**	552,58	**573,91**	538,26	**585,47**
	MAE	108,62	**100,96**	106,57	**104,57**	103,33	**103,03**	121,50	**103,61**	121,50	**103,24**	104,25	**121,94**
	MAPE	168,40	**46,63**	146,78	**97,37**	113,39	**99,87**	158,06	**105,25**	158,06	**99,76**	109,64	**221,38**
AXP	RMSE	14,19	**14,70**	13,61	**13,76**	13,61	**13,69**	13,51	**13,23**	13,51	**13,26**	13,39	**13,38**
	MAE	8,69	**8,44**	8,49	**9,31**	8,47	**8,38**	8,39	**8,45**	8,39	**8,55**	8,85	**8,46**
	MAPE	131,67	**94,74**	118,28	**136,50**	120,22	**114,13**	116,36	**117,35**	116,36	**118,39**	149,39	**121,45**
BAC	RMSE	43,50	**43,55**	42,71	**43,04**	43,64	**42,96**	42,89	**42,70**	42,89	**42,84**	42,88	**43,69**
	MAE	19,06	**18,69**	17,97	**18,08**	17,69	**17,53**	17,81	**18,40**	17,81	**17,65**	17,73	**17,73**
	MAPE	112,38	**106,59**	117,38	**117,72**	90,42	**98,10**	111,86	**140,66**	111,86	**97,83**	110,99	**90,79**
JPM	RMSE	26,08	**26,51**	25,48	**25,62**	25,04	**24,62**	24,97	**25,46**	24,97	**25,29**	24,88	**25,48**
	MAE	12,08	**12,08**	12,34	**12,35**	11,47	**11,35**	11,56	**12,45**	11,56	**12,07**	11,85	**12,02**
	MAPE	99,45	**107,58**	86,01	**92,93**	63,46	**57,36**	58,74	**76,08**	58,74	**65,48**	66,79	**74,73**

To assess the accuracy of the models, we compare the three loss functions corresponding to the bivariate models with the loss functions with respect to the univariate models: a lower loss function, the better the model is. The results were normalized according to the methodology described in chapter three, having obtained the following general rankings.

	AIG, Bivariate models					
	in sample			out of sample		
Highest ranked	EGARCH-X	-2.828,62	**0,00**	Realized GARCH(2,2)	-741,61	**0,00**
	Realized EGARCH	-2.847,60	**0,29**	Realized EGARCH	-762,77	**0,07**
	Realized GARCH(2,2	-2.854,25	**0,38**	Realized GARCH – complete	-783,23	**0,14**
	Realized GARCH - partial	-2.880,09	**0,77**	Realized GARCH - partial	-791,82	**0,17**
	Realized GARCH - complete	-2.882,80	**0,81**	EGARCH-X	-801,07	**0,20**
Lowest ranked	EGARCH	-2.895,19	**1,00**	EGARCH	-1.038,68	**1,00**

	AXP, Bivariate models					
	in sample			out of sample		
Highest ranked	EGARCH	-2.749,56	**0,00**	Realized GARCH - partial	-577,13	**0,00**
↕	EGARCH-X	-2.794,20	**0,28**	EGARCH-X	-577,73	**0,02**
	Realized EGARCH	-2.846,45	**0,62**	Realized EGARCH	-579,17	**0,05**
	Realized GARCH - complete	-2.857,15	**0,68**	Realized GARCH - complete	-579,29	**0,06**
	Realized GARCH - partial	-2.861,11	**0,71**	Realized GARCH(2,2)	-581,38	**0,11**
Lowest ranked	Realized GARCH(2,2)	-2.906,90	**1,00**	EGARCH	-615,03	**1,00**

	BAC, Bivariate models					
	in sample			out of sample		
Highest ranked	EGARCH-X	-2.440,35	**0,00**	EGARCH	-665,31	**0,00**
↕	Realized GARCH(2,2)	-2.443,46	**0,05**	Realized GARCH - partial	-681,49	**0,53**
	Realized EGARCH	-2.445,93	**0,09**	Realized GARCH(2,2)	-681,78	**0,54**
	Realized GARCH - partial	-2.446,51	**0,10**	Realized EGARCH	-684,32	**0,62**
	Realized GARCH - complete	-2.449,50	**0,14**	Realized GARCH - complete	-684,52	**0,62**
Lowest ranked	EGARCH	-2.503,85	**1,00**	EGARCH-X	-696,07	**1,00**

	JPM, Bivariate models					
	in sample			out of sample		
Highest ranked	Realized GARCH(2,2)	-2.689,59	**0,00**	Realized GARCH - partial	-588,69	**0,00**
↕	EGARCH-X	-2.701,74	**0,18**	Realized GARCH - complete	-590,28	**0,05**
	Realized EGARCH	-2.704,83	**0,23**	Realized EGARCH	-591,14	**0,08**
	Realized GARCH – partial	-2.705,11	**0,24**	Realized GARCH(2,2)	-591,82	**0,11**
	Realized GARCH - complete	-2.717,85	**0,43**	EGARCH-X	-600,39	**0,40**
Lowest ranked	EGARCH	-2.755,44	**1,00**	EGARCH	-618,18	**1,00**

	AIG, Bivariate models					
	in sample			out of sample		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE
Highest ranked	EGARCH	EGARCH	EGARCH-X	EGARCH-X	EGARCH	EGARCH
	EGARCH-X	Realized EGARCH	Realized EGARCH	Realized EGARCH	Realized EGARCH	EGARCH-X
	Realized EGARCH	EGARCH-X	Realized GARCH – partial	Realized GARCH - complete	Realized GARCH - partial	Realized GARCH - partial
	Realized GARCH – complete	Realized GARCH – partial	Realized GARCH - complete	Realized GARCH - partial	Realized GARCH - complete	Realized EGARCH
	Realized GARCH - partial	Realized GARCH - complete	Realized GARCH(2,2)	EGARCH	EGARCH-X	Realized GARCH - complete
Lowest ranked	Realized GARCH(2,2)	Realized GARCH(2,2)	EGARCH	Realized GARCH(2,2)	Realized GARCH(2,2)	Realized GARCH(2,2)

	AXP, Bivariate models					
	in sample			out of sample		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE
Highest ranked	Realized GARCH – complete	Realized GARCH – complete	EGARCH	Realized GARCH - complete	Realized EGARCH	EGARCH
	Realized EGARCH	Realized GARCH – partial	Realized GARCH – complete	Realized GARCH - partial	EGARCH	Realized EGARCH
	Realized GARCH – partial	Realized EGARCH	Realized EGARCH	Realized GARCH(2,2)	Realized GARCH - complete	Realized GARCH - complete
	EGARCH	EGARCH	Realized GARCH – partial	Realized EGARCH	Realized GARCH(2,2)	Realized GARCH - partial
	EGARCH-X	EGARCH-X	Realized GARCH(2,2)	EGARCH-X	Realized GARCH - partial	Realized GARCH(2,2)
Lowest ranked	Realized GARCH(2,2)	Realized GARCH(2,2)	EGARCH-X	EGARCH	EGARCH-X	EGARCH-X

	BAC, Bivariate models					
	in sample			out of sample		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE
Highest ranked	Realized GARCH(2,2)	Realized GARCH(2,2)	Realized EGARCH	Realized GARCH - complete	Realized EGARCH	Realized GARCH(2,2)
	EGARCH-X	Realized EGARCH	Realized GARCH(2,2)	Realized GARCH - partial	Realized GARCH - partial	Realized GARCH – partial
	Realized GARCH – partial	EGARCH-X	Realized GARCH – partial	Realized EGARCH	Realized GARCH(2,2)	Realized EGARCH
	Realized EGARCH	Realized GARCH – partial	EGARCH-X	EGARCH-X	EGARCH-X	EGARCH
	Realized GARCH - complete	Realized GARCH - complete	Realized GARCH – complete	EGARCH	Realized GARCH - complete	EGARCH-X
Lowest ranked	EGARCH	EGARCH	EGARCH	Realized GARCH(2,2)	EGARCH	Realized GARCH - complete

	JPM, Bivariate models					
	in sample			out of sample		
	RMSE	MAE	MAPE	RMSE	MAE	MAPE
Highest ranked	Realized GARCH – complete	Realized GARCH(2,2)	Realized GARCH – partial	Realized EGARCH	Realized EGARCH	Realized EGARCH
	Realized GARCH(2,2)	Realized GARCH – complete	EGARCH-X	Realized GARCH - partial	Realized GARCH(2,2)	Realized GARCH - partial
	Realized GARCH – partial	EGARCH-X	Realized GARCH(2,2)	Realized GARCH - complete	Realized GARCH - partial	Realized GARCH(2,2)
	EGARCH-X	Realized GARCH – partial	Realized GARCH – complete	Realized GARCH(2,2)	EGARCH	Realized GARCH - complete
	Realized EGARCH	Realized EGARCH	Realized EGARCH	EGARCH-X	EGARCH-X	EGARCH-X
Lowest ranked	EGARCH	EGARCH	EGARCH	EGARCH	Realized GARCH - complete	EGARCH

5.5 Conclusions

The first outcome of the chapter was the proposal of a methodology that led to formulating four bivariate realized GARCH models (Bivariate EGARCHX, Bivariate Realized GARCH (complete form), Bivariate Realized GARCH(2,2) and Bivariate Realized EGARCH) and one bivariate non-realized model (Bivariate EGARCH). The novelty of this method is the incorporation in the models of a night measure of volatility, computed from price changes between the closing and opening of the trading market.

The first observation that may be grasped by looking to the data is that rankings are sensitive to the stock choice, ranking criterion and estimation methodology. However, although results are diverse, some conclusions still may be grasped with regards to improvements in forecasting capability of the bivariate models.

For the in sample modeling, when RMSE is taken as the ranking criterion, we may observe that the Bivariate Realized GARCH model, in both partial and complete formulations, is a better forecaster than comparing to its univariate version, while Realized EGARCH clearly surpasses the bivariate version. If MAE is taken as criterion, Bivariate EGARCH is found as a better forecaster than simple EGARCH, as well as Bivariate EGARCH-X as against EGARCH-X. Bivariate Realized EGARCH performs poorer than Realized EGARCH, the same for Bivariate Realized GARCH(2,2). Finally, when MAPE is the criterion, EGARCH, Realized EGARCH, Realized GARCH (partial and complete forms) and Realized GARCH(2,2) are better than the bivariate versions.

For the in sample estimations, when RMSE is taken as the ranking criterion, EGARCH, EGARCH-X and Realized GARCH(2,2) are better than their bivariate versions, while Realized EGARCH and Realized GARCH (partial and complete) post similar cumulative errors. When MAE is the criterion, EGARCH and Realized EGARCH perform better when defined as bivariate models, while EGARCH-X and Realized GARCH (complete version) seems to work better without night volatility measures. Finally, when MAPE is the ranking criterion, Bivariate EGARCH and Realized EGARCH work better than EGARCH and Realized EGARCH, respectively, while EGARCH-X seems to surpass clearly the bivariate version.

For both in sample and out of sample estimations it may be concluded that the bivariate versions may improve the forecasting capacity of the simple and realized GARCH models, dependent on the methodology choices and on the error measurement choices. However, the bivariate models do not prove totally inferior to their univariate counterparts, they exhibit in numerous instances superior performances as compared to the univariate ones and as such they may be used in the forecasting exercise together with the univariate models for more reliable and precise estimates.

When ranking the bivariate models, it may be also observed that such rankings are sensitive to methodology, ranking criterion and stock choice. As such, for AIG, Bivariate EGARCHX and Bivariate EGARCH models rank among the first, while Bivariate Realized GARCH(2,2) consistently ranks the worst. According to the AXP stock, Bivariate Realized GARCH(2,2) model ranks low, as well Bivariate EGARCHX model. Bivariate Realized GARCH (complete), Bivariate Realized EGARCH and Bivariate EGARCH models rank among the best ones. For the BAC stock, Bivariate EGARCH model ranks almost consistently the lowest, while Bivariate Realized GARCH(2,2), Bivariate Realized and Bivariate GARCH (partial and complete) rank among the best ones. For JPM stock, Bivariate EGARCH and Bivariate EGARCHX rank the worst, while Bivariate Realized GARCH (partial and complete), Bivariate Realized EGARCH and Bivariate Realized EGARCH(2,2) rank the best.

A general conclusion with respect to the ranking part is that Bivariate Realized GARCH (partial and complete) and Bivariate Realized EGARCH models are good forecasters in any of the four stock choices, while Bivariate Realized GARCH(2,2), Bivariate EGARCH and Bivariate EGARCHX models may prove as well good modeling choices, dependent to the stock choice made.

Chapter six: The PC Bivariate Realized models

6.1 Introduction

Chapter six proposes, as general objective, to solve the volatility forecasting problem of multivariate stock assets, when bivariate GARCH modeling and high frequency data are considered. This involves multivariate modeling, but as explained in the fourth chapter, this poses high computational difficulties. One solution proposed to solve similar problems was the one belonging to Burns (2005) who suggested a method (PC-GARCH) that solved multivariate GARCH problems by univariate GARCH estimation of the principal components. Starting from this idea, in the chapter that follows it will be proposed an adaptation of this method to bivariate realized GARCH models that use high frequency data. The three specific objectives of the chapter are:

a. To propose a solution to solving the volatility forecasting problem of a multivariate asset by employing a Principal Component procedure to a class of bivariate realized (and one non-realized) models discussed in chapter five. New models would emerge: a PC Bivariate EGARCH(1,1) model, a PC Bivariate EGARCH-X(1,1) model, a PC Bivariate Realized EGARCH(1,1) model, a PC Bivariate Realized GARCH(1,1) (partial) and a PC Bivariate Realized GARCH(1,1) (complete) model, as well a PC Bivariate Realized GARCH(2,2) model.
b. To estimate in sample and out of sample the new PC bivariate models with data from four stocks.
c. To assess the forecasting capacity of the new PC bivariate models, and investigate their use in portfolio asset allocation.

6.2 Methodology

We follow a similar methodology as in chapter four. The PC algorithm has been described there and may be used for reference as regards the computations that follow to be used in the current chapter.

The seven steps mentioned in chapter four need to be reiterated here. Since we want to perform the PC algorithm on the bivariate models, we will first run the bivariate models in chapter five, in sample and out of sample, for each of the four stocks. As such, we perform a bivariate GARCH model (each of the previous chapter's models) on each of the four stocks considered. This will give the standardized returns z_t for day and night, that will be stuck into two matrices $\boldsymbol{R}^{\circ}_{T\times N} = \frac{(x_t^{\circ(i)} - m^{\circ(i)})}{s^{\circ(i)}}$ and $\boldsymbol{R}^{\bullet}_{T\times N} = \frac{(x_t^{\bullet(i)} - m^{\bullet(i)})}{s^{\bullet(i)}}$. $\boldsymbol{R}^{\circ}_{T\times N}$ will be an estimator of $\boldsymbol{R}^{\circ}_{pop,T\times N}$, and $\boldsymbol{R}^{\bullet}_{T\times N}$ will be an estimator of $\boldsymbol{R}^{\bullet}_{pop,T\times N}$. We call $\boldsymbol{V}^{\circ}_{N\times N} = var\left(\boldsymbol{R}^{\circ}_{T\times N}\right) = var\left(\boldsymbol{R}^{\circ}_{pop,T\times N}\right)$ and it can be proved that $\boldsymbol{V}^{\circ}_{N\times N} = Corr(\boldsymbol{X}^{\circ}_{T\times N})$ (*Step 2*).

We perform the PCA algorithm (*Step 3*) on the night and day standardized returns (columns of $\boldsymbol{R}^{\circ}_{T\times N}$ and of $\boldsymbol{R}^{\bullet}_{T\times N}$). The Matlab will deliver us the weights (*l*'s) that, multiplied to each of the standardized returns, will form the principal components. In other words, Matlab gives us that matrix of weights ($\boldsymbol{L}_{N\times N}$) that multiplied by the matrix $\boldsymbol{R}_{T\times N}$ of the standardized returns, gives us the matrix of the principal components. As such, by employing PCA, we will obtain the principal components that will be linear functions of the four standardized return series. This will be done for the day standardized returns and also for the night standardized returns.

$$\boldsymbol{P}^{\circ}_{T\times N} = \boldsymbol{R}^{\circ}_{T\times N} \times \boldsymbol{L}^{\circ}_{N\times N}$$

$$\boldsymbol{P}^{\bullet}_{T\times N} = \boldsymbol{R}^{\bullet}_{T\times N} \times \boldsymbol{L}^{\bullet}_{N\times N}$$

As well, the following relationship will hold

$$var(\boldsymbol{R}^{\circ}) = \boldsymbol{L}^{\circ} var(\boldsymbol{P}^{\circ}) \boldsymbol{L}^{\circ\prime}$$

We perform then the bivariate GARCH models on the principal components obtained above, in order to obtain the GARCH equations of the four principal components (*Step 4*). The columns of $\boldsymbol{P}^{\circ}_{T\times N}$ and $\boldsymbol{P}^{\bullet}_{T\times N}$ will be the day and night return series on which the bivariate models will be run.

Step 4 will deliver us $\boldsymbol{\Gamma}^{\circ} = var(\boldsymbol{P}^{\circ})$, which means, according to the above relationship, that we will have an estimate of $var(\boldsymbol{R}^{\circ})$. Since $\boldsymbol{V}^{\circ}_{N\times N} = var\left(\boldsymbol{R}^{\circ}_{T\times N}\right) = corr(\boldsymbol{X}^{\circ}_{T\times N})$, we will have then an estimate of the correlation matrix $\boldsymbol{X}^{\circ}$.

As such, as *Step 5*, we will obtain

$$\widehat{\boldsymbol{V}^{\circ}_{N\times N}} = \widehat{var(\boldsymbol{R}^{\circ}_{T\times N})} = \widehat{corr(\boldsymbol{X}^{\circ}_{T\times N})} = \widehat{\boldsymbol{C}}_t$$

$$= \begin{pmatrix} \sum_{k=1}^{N} l^{\circ 2}_{1k}\sigma_t^{(P_k)} & \sum_{k=1}^{N} l^{\circ}_{1k}l^{\circ}_{2k}\sigma_t^{(P_k)} & \dots & \sum_{k=1}^{N} l^{\circ}_{1k}l^{\circ}_{Nk}\sigma_t^{(P_k)} \\ \sum_{k=1}^{N} l^{\circ}_{2k}l^{\circ}_{1k}\sigma_t^{(P_k)} & \sum_{k=1}^{N} l^{\circ 2}_{2k}\sigma_t^{(P_k)} & \dots & \sum_{k=1}^{N} l^{\circ}_{2k}l^{\circ}_{Nk}\sigma_t^{(P_k)} \\ \dots & \dots & \dots & \dots \\ \sum_{k=1}^{N} l^{\circ}_{Nk}l^{\circ}_{1k}\sigma_t^{(P_k)} & \sum_{k=1}^{N} l^{\circ}_{Nk}l^{\circ}_{2k}\sigma_t^{(P_k)} & \dots & \sum_{k=1}^{N} l^{\circ 2}_{Nk}\sigma_t^{(P_k)} \end{pmatrix}$$

However, the correlation matrix of $\boldsymbol{X}^{\circ}_{T\times N}$ needs to have 1 on its diagonal. However, there is no guarantee that the elements on the diagonal of $\widehat{\boldsymbol{C}}_t$ will equal 1. That is why $\widehat{\boldsymbol{C}}_t$ will be transformed into a correlation matrix ($\widetilde{\boldsymbol{C}}_t$), between $\widehat{\boldsymbol{C}}_t$ and $\widetilde{\boldsymbol{C}}_t$ being only very minor differences. $\widetilde{\boldsymbol{C}}_t$ will be an estimator of $corr(\boldsymbol{X})_t$ and will be obtained at the next *Step*.

Step 6 will obtain an estimator of the correlation matrix, $\widetilde{\boldsymbol{C}}_t$:

$$\widetilde{\boldsymbol{C}}_t = \begin{pmatrix} \frac{1}{\sqrt{\hat{\boldsymbol{c}}_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_N}} \end{pmatrix} \widehat{\boldsymbol{C}}_t \begin{pmatrix} \frac{1}{\sqrt{\hat{\boldsymbol{c}}_1}} & 0 & \dots & 0 \\ 0 & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \frac{1}{\sqrt{\hat{\boldsymbol{c}}_N}} \end{pmatrix} = \begin{pmatrix} 1 & \frac{\hat{\boldsymbol{c}}_{12}}{\sqrt{\hat{\boldsymbol{c}}_1}\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & \frac{\hat{\boldsymbol{c}}_{1N}}{\sqrt{\hat{\boldsymbol{c}}_1}\sqrt{\hat{\boldsymbol{c}}_N}} \\ \frac{\hat{\boldsymbol{c}}_{21}}{\sqrt{\hat{\boldsymbol{c}}_2}\sqrt{\hat{\boldsymbol{c}}_1}} & 1 & \dots & \frac{\hat{\boldsymbol{c}}_{2N}}{\sqrt{\hat{\boldsymbol{c}}_2}\sqrt{\hat{\boldsymbol{c}}_N}} \\ \dots & \dots & \dots & \dots \\ \frac{\hat{\boldsymbol{c}}_{N1}}{\sqrt{\hat{\boldsymbol{c}}_N}\sqrt{\hat{\boldsymbol{c}}_1}} & \frac{\hat{\boldsymbol{c}}_{N2}}{\sqrt{\hat{\boldsymbol{c}}_N}\sqrt{\hat{\boldsymbol{c}}_2}} & \dots & 1 \end{pmatrix} ==$$

$$\begin{pmatrix} 1 & \tilde{\boldsymbol{c}}_{12} & \dots & \tilde{\boldsymbol{c}}_{1N} \\ \tilde{\boldsymbol{c}}_{21} & 1 & \dots & \tilde{\boldsymbol{c}}_{2N} \\ \dots & \dots & \dots & \dots \\ \tilde{\boldsymbol{c}}_{N1} & \tilde{\boldsymbol{c}}_{N2} & \dots & 1 \end{pmatrix} \text{with } \tilde{\boldsymbol{c}}_{ij} = \frac{\sum_{k=1}^{N} l^{\circ}_{ik}l^{\circ}_{jk}\sigma^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l^{\circ 2}_{ik}\sigma^{(P_k)}}\sqrt{\sum_{k=1}^{N} l^{\circ 2}_{jk}\sigma^{(P_k)}}}$$

At *Step 7*, we will obtain the variance-covariance matrix of $\boldsymbol{X}$, from the correlation matrix:

$$
\widehat{\boldsymbol{\Omega}}_{N\times N} = \begin{pmatrix} \sigma^{(1)} & cov\left(x^{(1)}, x^{(2)}\right) & \dots & cov\left(x^{(1)}, x^{(N)}\right) \\ cov\left(x^{(2)}, x^{(1)}\right) & \sigma^{(2)} & \dots & cov\left(x^{(2)}, x^{(N)}\right) \\ \dots & \dots & \dots & \dots \\ cov\left(x^{(N)}, x^{(1)}\right) & cov\left(x^{(N)}, x^{(2)}\right) & \dots & \sigma^{(N)} \end{pmatrix}
$$

$$
= \begin{pmatrix} \sqrt{\sigma^{(1)}} & 0 & \dots & 0 \\ 0 & \sqrt{\sigma^{(2)}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\sigma^{(N)}} \end{pmatrix} \widetilde{\boldsymbol{C}}_t \begin{pmatrix} \sqrt{\sigma^{(1)}} & 0 & \dots & 0 \\ 0 & \sqrt{\sigma^{(2)}} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \sqrt{\sigma^{(N)}} \end{pmatrix}
$$

$$
= \begin{pmatrix} \sigma^{(1)} & \sqrt{\sigma^{(1)}}\sqrt{\sigma^{(2)}}\tilde{\boldsymbol{c}}_{12} & \dots & \sqrt{\sigma^{(1)}}\sqrt{\sigma^{(N)}}\tilde{\boldsymbol{c}}_{1N} \\ \sqrt{\sigma^{(2)}}\sqrt{\sigma^{(1)}}\tilde{\boldsymbol{c}}_{21} & \sigma^{(2)} & \dots & \sqrt{\sigma^{(2)}}\sqrt{\sigma^{(N)}}\tilde{\boldsymbol{c}}_{2N} \\ \dots & \dots & \dots & \dots \\ \sqrt{\sigma^{(N)}}\sqrt{\sigma^{(1)}}\tilde{\boldsymbol{c}}_{N1} & \sqrt{\sigma^{(N)}}\sqrt{\sigma^{(2)}}\tilde{\boldsymbol{c}}_{N2} & \dots & \sigma^{(N)} \end{pmatrix}
$$

$$
= \begin{pmatrix} \sigma_t^{\circ(1)} & \sqrt{\sigma_t^{\circ(1)}}\sqrt{\sigma_t^{\circ(2)}}\frac{\sum_{k=1}^{N} l_{1k}^{\circ} l_{2k}^{\circ} \sigma_t^{(\mathring{P}_k)}}{\sqrt{\sum_{k=1}^{N} l_{1k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}\sqrt{\sum_{k=1}^{N} l_{2k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}} & \dots & \sqrt{\sigma_t^{\circ(1)}}\sqrt{\sigma_t^{\circ(N)}}\frac{\sum_{k=1}^{N} l_{1k}^{\circ} l_{Nk}^{\circ} \sigma_t^{(\mathring{P}_k)}}{\sqrt{\sum_{k=1}^{N} l_{1k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}\sqrt{\sum_{k=1}^{N} l_{Nk}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}} \\ \sqrt{\sigma_t^{\circ(2)}}\sqrt{\sigma_t^{\circ(1)}}\frac{\sum_{k=1}^{N} l_{2k}^{\circ} l_{1k}^{\circ} \sigma_t^{(\mathring{P}_k)}}{\sqrt{\sum_{k=1}^{N} l_{2k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}\sqrt{\sum_{k=1}^{N} l_{1k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}} & \sigma_t^{\circ(2)} & \dots & \sqrt{\sigma_t^{\circ(2)}}\sqrt{\sigma_t^{\circ(N)}}\frac{\sum_{k=1}^{N} l_{2k}^{\circ} l_{Nk}^{\circ} \sigma_t^{(\mathring{P}_k)}}{\sqrt{\sum_{k=1}^{N} l_{2k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}\sqrt{\sum_{k=1}^{N} l_{Nk}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}} \\ \dots & \dots & \dots & \dots \\ \sqrt{\sigma_t^{\circ(N)}}\sqrt{\sigma_t^{\circ(1)}}\frac{\sum_{k=1}^{N} l_{Nk}^{\circ} l_{1k}^{\circ} \sigma_t^{(\mathring{P}_k)}}{\sqrt{\sum_{k=1}^{N} l_{Nk}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}\sqrt{\sum_{k=1}^{N} l_{1k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}} & \sqrt{\sigma_t^{\circ(N)}}\sqrt{\sigma_t^{\circ(2)}}\frac{\sum_{k=1}^{N} l_{Nk}^{\circ} l_{2k}^{\circ} \sigma_t^{(\mathring{P}_k)}}{\sqrt{\sum_{k=1}^{N} l_{Nk}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}\sqrt{\sum_{k=1}^{N} l_{2k}^{\circ 2}\sigma_t^{(\mathring{P}_k)}}} & \dots & \sigma_t^{\circ(N)} \end{pmatrix}
$$

GARCH equations of standardized returns

The findings at *Step 3* may be otherwise exploited. We further reiterate the expression obtained in chapter four, that is $\boldsymbol{L\Gamma L'} = \boldsymbol{\Omega}$, in which $\boldsymbol{L}$ is the matrix of the weights, $\boldsymbol{\Gamma}$ is the variance-covariance matrix of the principal components and $\boldsymbol{\Omega}$ is the variance-covariance matrix of the standardized returns. This relationship gives the variance-covariance matrix of the standardized returns as an expression in terms of the weight matrix and the variance-covariance matrix of the principal components.

Since both $\boldsymbol{\Omega}$ and $\boldsymbol{\Gamma}$ are diagonal matrices, with variances on the diagonal, it means that the above relationship expresses the volatility of the standardized returns in terms of principal components and of the weights *l*’s.

As such, the sigma's of the x's (of the standardized returns), may be expressed as follows (in general terms):

$$\sigma_{x_i} = \sum_{j=1}^{k} l_{ij}^{\circ\,2} \sigma_{P_j}, \qquad i = 1, \dots, k$$

that describes the volatility of the standardized innovations as a function of the volatility of the principal components.

For our particular case, the above formulated volatility equations with regard to the four stocks may be rewritten by replacing the corresponding indices with $k = 4$

and

$$x_1 = AIG, x_2 = AXP, x_3 = BAC, x_4 = JPM, P_1 = P_{AIG}, P_2 = P_{AXP}, P_3 = P_{BAC}, P_4 = P_{JPM}$$

By replacing the sigma's of the principal components with their equivalents from the GARCH equations of each bivariate model, we will obtain the following GARCH equations of standardized returns:

PC Bivariate Realized GARCH(1,1)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} \left\{ l_{ij}^{\circ 2} \left[\omega^{\circ}_{P_j} + \tau_{P_j}^{(\circ 1)} \left(z^{\bullet}_{P_j,t-1} \right) + \tau_{P_j}^{(\circ 2)} \left(z^{\circ}_{P_j,t-1} \right) + \beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \gamma^{\circ}_{P_j} logx_{P_j,t-1} \right] \right\}, i = 1, \dots, k \text{ and endogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} \left\{ l_{ij}^{\bullet 2} \left[\omega^{\bullet}_{P_j} + \tau_{P_j}^{(\bullet 1)} \left(z^{\bullet}_{P_j,t-1} \right) + \tau_{P_j}^{(\bullet 2)} \left(z^{\circ}_{P_j,t-1} \right) + \beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1} + \gamma^{\bullet}_{P_j} logx_{P_j,t-1} \right] \right\}, i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\circ 2} \omega^{\circ}_{P_j} + \sum_{j=1}^{k} l_{ij}^{\circ 2} \tau_{P_j}^{(\circ 1)} \left(z^{\bullet}_{P_j,t-1} \right) + \sum_{j=1}^{k} l_{ij}^{\circ 2} \tau_{P_j}^{(\circ 2)} \left(z^{\circ}_{P_j,t-1} \right) + \sum_{j=1}^{k} l_{ij}^{\circ 2} \beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \sum_{j=1}^{k} l_{ij}^{\circ 2} \gamma^{\circ}_{P_j} logx_{P_j,t-1}, i = 1, \dots, k \text{ and endogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{j,t-1}\right) + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right) + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1} + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\gamma^{\bullet}_{P_j} logx_{P_j,t-1}, i = 1, \dots, k, \text{and endogenous } x.$$

PC Bivariate Realized EGARCH(1,1)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\left\{\omega^{\circ}_{P_j} + \varepsilon^{(\circ 1)}_{P_j}\left[\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right] + \varepsilon^{(\circ 2)}_{P_j}\left[\left(\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right] + \beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \gamma^{\circ}_{P_j} logx_{P_j,t-1}\right\}\right\}, i = 1, \dots, k \text{ and endogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k}\left\{l^{\bullet 2}_{ij}\left\{\omega^{\bullet}_{P_j} + \varepsilon^{(\bullet 1)}_{P_j}\left[\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right] + \varepsilon^{(\bullet 2)}_{P_j}\left[\left(\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right] + \beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1} + \gamma^{\bullet}_{P_j} logx_{P_j,t-1}\right\}\right\} \; i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{p_j} + \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\varepsilon^{(\circ 1)}_{P_j}\left[\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right\} + \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\varepsilon^{(\circ 2)}_{P_j}\left[\left(\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right]\right\} + \sum_{j=1}^{k} l^{\circ 2}_{ij}\beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \sum_{j=1}^{k} l^{\circ 2}_{ij}\gamma^{\circ}_{P_j} logx_{P_j,t-1}, i = 1, \dots, k \text{ and endogenous } x.$$

$$
\begin{aligned}
log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} \; &+ \sum_{j=1}^{k}\left\{l^{\bullet 2}_{ij}\varepsilon^{(\bullet 1)}_{P_j}\left[\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right\} \\
&+ \sum_{j=1}^{k}\left\{l^{\bullet 2}_{ij}\varepsilon^{(\bullet 2)}_{P_j}\left[\left(\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2-1\right]\right\} + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1} \\
&+ \sum_{j=1}^{k} l^{\bullet 2}_{ij}\gamma^{\bullet}_{P_j} logx_{P_j,t-1}, i=1,\dots,k \text{ and endogenous } x.
\end{aligned}
$$

PC Bivariate EGARCH(1,1)

$$
\begin{aligned}
log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k}\Bigg\{l^{\circ 2}_{ij}\Bigg\{&\omega^{\circ}_{P_j} + \varepsilon^{(\circ 1)}_{P_j}\left[\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\circ 2)}_{P_1}\left(z^{\circ}_{P_j,t-1}\right)\right] \\
&+ \varepsilon^{(\circ 2)}_{P_j}\left[\left(\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2-1\right] + \beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1}\Bigg\}\Bigg\}, i=1,\dots,k
\end{aligned}
$$

$$
\begin{aligned}
log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k}\Bigg\{l^{\bullet 2}_{ij}\Bigg\{&\omega^{\bullet}_{P_j} + \varepsilon^{(\bullet 1)}_{P_j}\left[\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\bullet 2)}_{j}\left(z^{\circ}_{P_j,t-1}\right)\right] \\
&+ \varepsilon^{(\bullet 2)}_{P_j}\left[\left(\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2-1\right] + \beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1}\Bigg\}\Bigg\}, i=1,\dots,k
\end{aligned}
$$

that is equivalent to

$$
\begin{aligned}
log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{P_j} &+ \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\varepsilon^{(\circ 1)}_{P_j}\left[\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right\} \\
&+ \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\varepsilon^{(\circ 2)}_{P_j}\left[\left(\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2-1\right]\right\} + \sum_{j=1}^{k} l^{\circ 2}_{ij}\beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1}, i \\
&= 1,\dots,k
\end{aligned}
$$

$$
\begin{aligned}
log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} &+ \sum_{j=1}^{k}\left\{l^{\bullet 2}_{ij}\varepsilon^{(\bullet 1)}_{P_j}\left[\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right\} \\
&+ \sum_{j=1}^{k}\left\{l^{\bullet 2}_{ij}\varepsilon^{(\bullet 2)}_{P_j}\left[\left(\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right)+\tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2-1\right]\right\} + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1}, i \\
&= 1,\dots,k
\end{aligned}
$$

PC Bivariate EGARCH-X(1,1)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\left\{\omega^{\circ}_{P_j} + \varepsilon^{(\circ 1)}_{P_j}\left[\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right.\right.$$
$$\left.\left. + \varepsilon^{(\circ 2)}_{P_j}\left[\left(\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right] + \beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \gamma^{\circ}_{P_j} logx_{P_j,t-1}\right\}\right\}, i = 1, \ldots, k \text{ and exogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k}\left\{l^{\bullet 2}_{ij}\left\{\omega^{\bullet}_{P_j} + \varepsilon^{(\bullet 1)}_{P_j}\left[\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right.\right.$$
$$\left.\left. + \varepsilon^{(\bullet 2)}_{P_j}\left[\left(\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right] + \beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1} + \gamma^{\bullet}_{P_j} logx_{P_j,t-1}\right\}\right\}, i = 1, \ldots, k \text{ and exogenous } x.$$

that is equivalent to

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{P_j} + \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\varepsilon^{(\circ 1)}_{P_j}\left[\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]\right\}$$
$$+ \sum_{j=1}^{k}\left\{l^{\circ 2}_{ij}\varepsilon^{(\circ 2)}_{P_j}\left[\left(\tau^{(\circ 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\circ 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right]\right\} + \sum_{j=1}^{k} l^{\circ 2}_{ij}\beta^{\circ}_{P_j} log\sigma^{\circ}_{P_j,t-1}$$
$$+ \sum_{j=1}^{k} l^{\circ 2}_{ij}\gamma^{\circ}_{P_j} logx_{P_j,t-1}, i = 1, \ldots, k \text{ and exogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\varepsilon^{(\bullet 1)}_{P_j}\left[\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right]$$
$$+ \sum_{j=1}^{k} l^{\bullet 2}_{ij}\varepsilon^{(\bullet 2)}_{P_j}\left[\left(\tau^{(\bullet 1)}_{P_j}\left(z^{\bullet}_{P_j,t-1}\right) + \tau^{(\bullet 2)}_{P_j}\left(z^{\circ}_{P_j,t-1}\right)\right)^2 - 1\right] + \sum_{j=1}^{k} l^{\bullet 2}_{ij}\beta^{\bullet}_{P_j} log\sigma^{\bullet}_{P_j,t-1}$$
$$+ \sum_{j=1}^{k} l^{\bullet 2}_{ij}\gamma^{\bullet}_{P_j} logx_{P_j,t-1}, i = 1, \ldots, k \text{ and exogenous } x.$$

PC Bivariate Realized GARCH(2,2)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} \left\{ l^{\circ 2}_{ij} \left[\omega^{\circ}_{P_j} + \tau^{(\circ 1)}_{P_j} \left(z^{\bullet}_{P_j,t-1} \right) + \tau^{(\circ 2)}_{P_j} \left(z^{\circ}_{P_j,t-1} \right) + \alpha^{\circ}_{P_j} log(max\left(r^{\circ 2}_{P_j,t-1}, 10^{-20} \right)) + \beta^{(\circ 1)}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \beta^{(\circ 2)}_{P_j} log\sigma^{\circ}_{P_j,t-2} + \gamma^{(\circ 1)}_{P_j} logx_{P_j,t-1} + \gamma^{(\circ 2)}_{P_j} logx_{P_j,t-2} \right] \right\}, i = 1, \dots, k \text{ and endogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} \left\{ l^{\bullet 2}_{ij} \left[\omega^{\bullet}_{P_j} + \tau^{(\bullet 1)}_{P_j} \left(z^{\bullet}_{j,t-1} \right) + \tau^{(\bullet 2)}_{P_j} \left(z^{\circ}_{P_j,t-1} \right) + \alpha^{\bullet}_{P_j} log(max\left(r^{\bullet 2}_{P_j,t-1}, 10^{-20} \right)) + \beta^{(\bullet 1)}_{P_j} log\sigma^{\bullet}_{P_j,t-1} + \beta^{(\bullet 2)}_{P_j} log\sigma^{\bullet}_{P_j,t-2} + \gamma^{(\bullet 1)}_{P_j} logx_{P_j,t-1} + \gamma^{(\bullet 2)}_{P_j} logx_{P_j,t-2} \right] \right\}, i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij} \omega^{\circ}_{P_j} + \sum_{j=1}^{k} l^{\circ 2}_{ij} \left[\tau^{(\circ 1)}_{P_j} \left(z^{\bullet}_{P_j,t-1} \right) + \tau^{(\circ 2)}_{P_j} \left(z^{\circ}_{P_j,t-1} \right) \right] + \sum_{j=1}^{k} l^{\circ 2}_{ij} \alpha^{\circ}_{P_j} log(max\left(r^{\circ 2}_{P_j,t-1}, 10^{-20} \right)) + \sum_{j=1}^{k} l^{\circ 2}_{ij} \beta^{(\circ 1)}_{P_j} log\sigma^{\circ}_{P_j,t-1} + \sum_{j=1}^{k} l^{\circ 2}_{ij} \beta^{(\circ 2)}_{P_j} log\sigma^{\circ}_{P_j,t-2} + \sum_{j=1}^{k} l^{\circ 2}_{ij} \gamma^{(\circ 1)}_{P_j} logx_{P_j,t-1} + \sum_{j=1}^{k} l^{\circ 2}_{ij} \gamma^{(\circ 2)}_{P_j} logx_{P_j,t-2}, i = 1, \dots, k \text{ and endogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij} \omega^{\bullet}_{P_j} + \sum_{j=1}^{k} l^{\bullet 2}_{ij} \left[\tau^{(\bullet 1)}_{P_j} \left(z^{\bullet}_{P_j,t-1} \right) + \tau^{(\bullet 2)}_{P_j} \left(z^{\circ}_{P_j,t-1} \right) \right] + \sum_{j=1}^{k} l^{\bullet 2}_{ij} \alpha^{\bullet}_{P_j} log(max\left(r^{\bullet 2}_{P_j,t-1}, 10^{-20} \right)) + \sum_{j=1}^{k} l^{\bullet 2}_{ij} \beta^{(\bullet 1)}_{P_j} log\sigma^{\bullet}_{P_j,t-1} + \sum_{j=1}^{k} l^{\bullet 2}_{ij} \beta^{(\bullet 2)}_{P_j} log\sigma^{\bullet}_{P_j,t-2} + \sum_{j=1}^{k} l^{\bullet 2}_{ij} \gamma^{(\bullet 1)}_{P_j} logx_{P_j,t-1} + \sum_{j=1}^{k} l^{\bullet 2}_{ij} \gamma^{(\bullet 2)}_{P_j} logx_{P_j,t-2}, i = 1, \dots, k \text{ and endogenous } x.$$

We may rewrite the above GARCH equations of the above PC Bivariate models, in matrix form, as follows:

PC Bivariate Realized GARCH(1,1)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{P_j} + \begin{pmatrix} l^{\circ 2}_{i1}\tau^{(\circ 1)}_{P_1} \\ l^{\circ 2}_{i2}\tau^{(\circ 1)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\tau^{(\circ 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} \overset{\bullet}{z}_{P_1,t-1} \\ \overset{\bullet}{z}_{P_2,t-1} \\ \dots \\ \overset{\bullet}{z}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\tau^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\tau^{(\circ 2)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\tau^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \overset{\circ}{z}_{P_1,t-1} \\ \overset{\circ}{z}_{P_2,t-1} \\ \dots \\ \overset{\circ}{z}_{P_i,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} l^{\circ 2}_{i1}\beta^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\beta^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\beta^{\circ}_{P_k} \end{pmatrix}' \begin{pmatrix} log\overset{\circ}{\sigma}_{P_1,t-1} \\ log\overset{\circ}{\sigma}_{P_2,t-1} \\ \dots \\ log\overset{\circ}{\sigma}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\gamma^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\gamma^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\gamma^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} + \begin{pmatrix} l^{\bullet 2}_{i1}\tau^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\tau^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\tau^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} \overset{\bullet}{z}_{P_1,t-1} \\ \overset{\bullet}{z}_{P_2,t-1} \\ \dots \\ \overset{\bullet}{z}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\tau^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\tau^{(\bullet 2)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\tau^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \overset{\circ}{z}_{P_1,t-1} \\ \overset{\circ}{z}_{P_2,t-1} \\ \dots \\ \overset{\circ}{z}_{P_i,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} l^{\bullet 2}_{i1}\beta^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\beta^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log\overset{\bullet}{\sigma}_{P_1,t-1} \\ log\overset{\bullet}{\sigma}_{P_2,t-1} \\ \dots \\ log\overset{\bullet}{\sigma}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\gamma^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\gamma^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\gamma^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.$$

PC Bivariate Realized EGARCH(1,1)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{p_j} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 1)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 1)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} \tau^{(\circ 1)}_{P_1}\left(\overset{\bullet}{z}_{P_1,t-1}\right) + \tau^{(\circ 2)}_{P_1}\left(\overset{\circ}{z}_{P_1,t-1}\right) \\ \tau^{(\circ 1)}_{P_2}\left(\overset{\bullet}{z}_{P_2,t-1}\right) + \tau^{(\circ 2)}_{P_2}\left(\overset{\circ}{z}_{P_2,t-1}\right) \\ \dots \\ \tau^{(\circ 1)}_{P_i}\left(\overset{\bullet}{z}_{P_i,t-1}\right) + \tau^{(\circ 2)}_{P_i}\left(\overset{\circ}{z}_{P_i,t-1}\right) \end{pmatrix}$$
$$+ \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \left(\tau^{(\circ 1)}_{P_1}\left(\overset{\bullet}{z}_{P_1,t-1}\right) + \tau^{(\circ 2)}_{P_1}\left(\overset{\circ}{z}_{P_1,t-1}\right)\right)^2 - 1 \\ \left(\tau^{(\circ 1)}_{P_2}\left(\overset{\bullet}{z}_{P_2,t-1}\right) + \tau^{(\circ 2)}_{P_2}\left(\overset{\circ}{z}_{P_2,t-1}\right)\right)^2 - 1 \\ \dots \\ \left(\tau^{(\circ 1)}_{P_i}\left(\overset{\bullet}{z}_{P_i,t-1}\right) + \tau^{(\circ 2)}_{P_i}\left(\overset{\circ}{z}_{P_i,t-1}\right)\right)^2 - 1 \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\beta^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\beta^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\beta^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} log\overset{\circ}{\sigma}_{P_1,t-1} \\ log\overset{\circ}{\sigma}_{P_2,t-1} \\ \dots \\ log\overset{\circ}{\sigma}_{P_i,t-1} \end{pmatrix}$$
$$+ \begin{pmatrix} l^{\circ 2}_{i1}\gamma^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\gamma^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\gamma^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.$$

$$\log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} + \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} \tau^{(\bullet 1)}_{P_1}\left(z^{\bullet}_{P_1,t-1}\right) + \tau^{(\bullet 2)}_{P_1}\left(z^{\circ}_{P_1,t-1}\right) \\ \tau^{(\bullet 1)}_{P_2}\left(z^{\bullet}_{P_2,t-1}\right) + \tau^{(\bullet 2)}_{P_2}\left(z^{\circ}_{P_2,t-1}\right) \\ \dots \\ \tau^{(\bullet 1)}_{P_i}\left(z^{\bullet}_{P_i,t-1}\right) + \tau^{(\bullet 2)}_{P_i}\left(z^{\circ}_{P_i,t-1}\right) \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 2)}_{P_1} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \left(\tau^{(\bullet 1)}_{P_1}\left(z^{\bullet}_{P_1,t-1}\right) + \tau^{(\bullet 2)}_{P_1}\left(z^{\circ}_{P_1,t-1}\right)\right)^2 - 1 \\ \left(\tau^{(\bullet 1)}_{P_2}\left(z^{\bullet}_{P_2,t-1}\right) + \tau^{(\bullet 2)}_{P_2}\left(z^{\circ}_{P_2,t-1}\right)\right)^2 - 1 \\ \dots \\ \left(\tau^{(\bullet 1)}_{P_i}\left(z^{\bullet}_{P_i,t-1}\right) + \tau^{(\bullet 2)}_{P_i}\left(z^{\circ}_{P_i,t-1}\right)\right)^2 - 1 \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\beta^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\beta^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} \log\sigma^{\bullet}_{P_1,t-1} \\ \log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ \log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\bullet 2}_{i1}\gamma^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\gamma^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\gamma^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} \log x_{P_1,t-1} \\ \log x_{P_2,t-1} \\ \dots \\ \log x_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.$$

that is equivalent to

$$\log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{p_j} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 1)}_{P_1}\tau^{(\circ 1)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 1)}_{P_2}\tau^{(\circ 1)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 1)}_{P_i}\tau^{(\circ 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 1)}_{P_1}\tau^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 1)}_{P_2}\tau^{(\circ 2)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 1)}_{P_i}\tau^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 1)2}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 1)2}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i}\tau^{(\circ 1)2}_{P_i} \end{pmatrix}' \begin{pmatrix} {z^{\bullet}_{P_1,t-1}}^2 \\ {z^{\bullet}_{P_2,t-1}}^2 \\ \dots \\ {z^{\bullet}_{P_i,t-1}}^2 \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 2)2}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 2)2}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i}\tau^{(\circ 2)2}_{P_i} \end{pmatrix}' \begin{pmatrix} {z^{\circ}_{P_1,t-1}}^2 \\ {z^{\circ}_{P_2,t-1}}^2 \\ \dots \\ {z^{\circ}_{P_i,t-1}}^2 \end{pmatrix}$$

$$+ 2\begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 1)}_{P_1}\tau^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_2}\tau^{(\circ 1)}_{P_2}\tau^{(\circ 2)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i}\tau^{(\circ 1)}_{P_i}\tau^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}z^{\circ}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1}z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}z^{\circ}_{P_i,t-1} \end{pmatrix} - \sum_{j=1}^{k} l^{\circ 2}_{ij}\varepsilon^{(\circ 2)}_{P_j} + \begin{pmatrix} l^{\circ 2}_{i1}\beta^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\beta^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\beta^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} \log\sigma^{\circ}_{P_1,t-1} \\ \log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ \log\sigma^{\circ}_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\circ 2}_{i1}\gamma^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\gamma^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\gamma^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} \log x_{P_1,t-1} \\ \log x_{P_2,t-1} \\ \dots \\ \log x_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.$$

$$
\begin{aligned}
log\sigma^{\bullet}_{x_i} = {} & \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{p_j} + \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 1)}_{P_1}\tau^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 1)}_{P_2}\tau^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 1)}_{P_i}\tau^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 1)}_{P_1}\tau^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 1)}_{P_2}\tau^{(\bullet 2)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 1)}_{P_i}\tau^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1} \end{pmatrix} \\
& + \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 2)}_{P_1}\tau^{(\bullet 1)2}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 2)}_{P_1}\tau^{(\bullet 1)2}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 2)}_{P_i}\tau^{(\bullet 1)2}_{P_i} \end{pmatrix}' \begin{pmatrix} {z^{\bullet}_{P_1,t-1}}^2 \\ {z^{\bullet}_{P_2,t-1}}^2 \\ \dots \\ {z^{\bullet}_{P_i,t-1}}^2 \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 2)}_{P_1}\tau^{(\bullet 2)2}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 2)}_{P_1}\tau^{(\bullet 2)2}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 2)}_{P_i}\tau^{(\bullet 2)2}_{P_i} \end{pmatrix}' \begin{pmatrix} {z^{\circ}_{P_1,t-1}}^2 \\ {z^{\circ}_{P_2,t-1}}^2 \\ \dots \\ {z^{\circ}_{P_i,t-1}}^2 \end{pmatrix} \\
& + 2\begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 2)}_{P_1}\tau^{(\bullet 1)}_{P_1}\tau^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 2)}_{P_1}\tau^{(\bullet 1)}_{P_2}\tau^{(\bullet 2)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 2)}_{P_i}\tau^{(\bullet 1)}_{P_i}\tau^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}z^{\circ}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1}z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}z^{\circ}_{P_i,t-1} \end{pmatrix} - \sum_{j=1}^{k} l^{\bullet 2}_{ij}\varepsilon^{(\bullet 2)}_{P_j} \\
& + \begin{pmatrix} l^{\bullet 2}_{i1}\beta^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\beta^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\gamma^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\gamma^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\gamma^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i \\
& = 1, \dots, k \text{ and endogenous } x.
\end{aligned}
$$

PC Bivariate EGARCH(1,1)

$$
\begin{aligned}
log\sigma^{\circ}_{x_i} = {} & \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{P_j} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 1)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 1)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} \tau^{(\circ 1)}_{P_1}\left(z^{\bullet}_{P_1,t-1}\right) + \tau^{(\circ 2)}_{P_1}\left(z^{\circ}_{P_1,t-1}\right) \\ \tau^{(\circ 1)}_{P_2}\left(z^{\bullet}_{P_2,t-1}\right) + \tau^{(\circ 2)}_{P_2}\left(z^{\circ}_{P_2,t-1}\right) \\ \dots \\ \tau^{(\circ 1)}_{P_i}\left(z^{\bullet}_{P_i,t-1}\right) + \tau^{(\circ 2)}_{P_i}\left(z^{\circ}_{P_i,t-1}\right) \end{pmatrix} \\
& + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_1} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \left(\tau^{(\circ 1)}_{P_1}\left(z^{\bullet}_{P_1,t-1}\right) + \tau^{(\circ 2)}_{P_1}\left(z^{\circ}_{P_1,t-1}\right)\right)^2 - 1 \\ \left(\tau^{(\circ 1)}_{P_2}\left(z^{\bullet}_{P_2,t-1}\right) + \tau^{(\circ 2)}_{P_2}\left(z^{\circ}_{P_2,t-1}\right)\right)^2 - 1 \\ \dots \\ \left(\tau^{(\circ 1)}_{P_i}\left(z^{\bullet}_{P_i,t-1}\right) + \tau^{(\circ 2)}_{P_i}\left(z^{\circ}_{P_i,t-1}\right)\right)^2 - 1 \end{pmatrix} \\
& + \begin{pmatrix} l^{\circ 2}_{i1}\beta^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\beta^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\beta^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\circ}_{P_1,t-1} \\ log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ log\sigma^{\circ}_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k
\end{aligned}
$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} + \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} \tau^{(\bullet 1)}_{P_1}\left(z^{\bullet}_{P_1,t-1}\right) + \tau^{(\bullet 2)}_{P_1}\left(z^{\circ}_{P_1,t-1}\right) \\ \tau^{(\bullet 1)}_{P_2}\left(z^{\bullet}_{P_2,t-1}\right) + \tau^{(\bullet 2)}_{P_2}\left(z^{\circ}_{P_2,t-1}\right) \\ \dots \\ \tau^{(\bullet 1)}_{P_i}\left(z^{\bullet}_{P_i,t-1}\right) + \tau^{(\bullet 2)}_{P_i}\left(z^{\circ}_{P_i,t-1}\right) \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\bullet 2}_{i1}\varepsilon^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\varepsilon^{(\bullet 2)}_{P_1} \\ \dots \\ l^{\bullet 2}_{ii}\varepsilon^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \left(\tau^{(\bullet 1)}_{P_1}\left(z^{\bullet}_{P_1,t-1}\right) + \tau^{(\bullet 2)}_{P_1}\left(z^{\circ}_{P_1,t-1}\right)\right)^2 - 1 \\ \left(\tau^{(\bullet 1)}_{P_2}\left(z^{\bullet}_{P_2,t-1}\right) + \tau^{(\bullet 2)}_{P_2}\left(z^{\circ}_{P_2,t-1}\right)\right)^2 - 1 \\ \dots \\ \left(\tau^{(\bullet 1)}_{P_i}\left(z^{\bullet}_{P_i,t-1}\right) + \tau^{(\bullet 2)}_{P_i}\left(z^{\circ}_{P_i,t-1}\right)\right)^2 - 1 \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\bullet 2}_{i1}\beta^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\beta^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k$$

that is equivalent to

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l^{\circ 2}_{ij}\omega^{\circ}_{p_j} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 1)}_{P_1}\tau^{(\circ 1)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 1)}_{P_2}\tau^{(\circ 1)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 1)}_{P_i}\tau^{(\circ 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 1)}_{P_1}\tau^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 1)}_{P_2}\tau^{(\circ 2)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 1)}_{P_i}\tau^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 1)2}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 1)2}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i}\tau^{(\circ 1)2}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}{}^2 \\ z^{\bullet}_{P_2,t-1}{}^2 \\ \dots \\ z^{\bullet}_{P_i,t-1}{}^2 \end{pmatrix} + \begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 2)2}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 2)2}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i}\tau^{(\circ 2)2}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1}{}^2 \\ z^{\circ}_{P_2,t-1}{}^2 \\ \dots \\ z^{\circ}_{P_i,t-1}{}^2 \end{pmatrix}$$

$$+ 2\begin{pmatrix} l^{\circ 2}_{i1}\varepsilon^{(\circ 2)}_{P_1}\tau^{(\circ 1)}_{P_1}\tau^{(\circ 2)}_{P_1} \\ l^{\circ 2}_{i2}\varepsilon^{(\circ 2)}_{P_2}\tau^{(\circ 1)}_{P_2}\tau^{(\circ 2)}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\varepsilon^{(\circ 2)}_{P_i}\tau^{(\circ 1)}_{P_i}\tau^{(\circ 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}z^{\circ}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1}z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}z^{\circ}_{P_i,t-1} \end{pmatrix} - \sum_{j=1}^{k} l^{\circ 2}_{ij}\varepsilon^{(\circ 2)}_{P_j}$$

$$+ \begin{pmatrix} l^{\circ 2}_{i1}\beta^{\circ}_{P_1} \\ l^{\circ 2}_{i2}\beta^{\circ}_{P_2} \\ \dots \\ l^{\circ 2}_{ii}\beta^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\circ}_{P_1,t-1} \\ log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ log\sigma^{\circ}_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k$$

$$
\begin{aligned}
log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\bullet 2}\omega^{\bullet}_{p_j} &+ \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 1)}\tau_{P_1}^{(\bullet 1)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_2}^{(\bullet 1)}\tau_{P_2}^{(\bullet 1)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 1)}\tau_{P_i}^{(\bullet 1)} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 1)}\tau_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_2}^{(\bullet 1)}\tau_{P_2}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 1)}\tau_{P_i}^{(\bullet 2)} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1} \end{pmatrix} \\
&+ \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_1}^{(\bullet 1)2} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_2}^{(\bullet 1)2} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)}\tau_{P_i}^{(\bullet 1)2} \end{pmatrix}' \begin{pmatrix} {z^{\bullet}_{P_1,t-1}}^2 \\ {z^{\bullet}_{P_2,t-1}}^2 \\ \dots \\ {z^{\bullet}_{P_i,t-1}}^2 \end{pmatrix} + \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_1}^{(\bullet 2)2} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_2}^{(\bullet 2)2} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)}\tau_{P_i}^{(\bullet 2)2} \end{pmatrix}' \begin{pmatrix} {z^{\circ}_{P_1,t-1}}^2 \\ {z^{\circ}_{P_2,t-1}}^2 \\ \dots \\ {z^{\circ}_{P_i,t-1}}^2 \end{pmatrix} \\
&+ 2\begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_1}^{(\bullet 1)}\tau_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_2}^{(\bullet 1)}\tau_{P_2}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)}\tau_{P_i}^{(\bullet 1)}\tau_{P_i}^{(\bullet 2)} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}z^{\circ}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1}z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}z^{\circ}_{P_i,t-1} \end{pmatrix} - \sum_{j=1}^{k} l_{ij}^{\bullet 2}\varepsilon_{P_j}^{(\bullet 2)} \\
&+ \begin{pmatrix} l_{i1}^{\bullet 2}\beta^{\bullet}_{P_1} \\ l_{i2}^{\bullet 2}\beta^{\bullet}_{P_2} \\ \dots \\ l_{ii}^{\bullet 2}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k
\end{aligned}
$$

PC Bivariate EGARCH-X(1,1)

$$
\begin{aligned}
log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\circ 2}\omega^{\circ}_{P_j} &+ \begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\varepsilon_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 1)} \end{pmatrix}' \begin{pmatrix} \tau_{P_1}^{(\circ 1)}\left(z^{\bullet}_{P_1,t-1}\right) + \tau_{P_1}^{(\circ 2)}\left(z^{\circ}_{P_1,t-1}\right) \\ \tau_{P_2}^{(\circ 1)}\left(z^{\bullet}_{P_2,t-1}\right) + \tau_{P_2}^{(\circ 2)}\left(z^{\circ}_{P_2,t-1}\right) \\ \dots \\ \tau_{P_i}^{(\circ 1)}\left(z^{\bullet}_{P_k,t-1}\right) + \tau_{P_i}^{(\circ 2)}\left(z^{\circ}_{P_i,t-1}\right) \end{pmatrix} \\
&+ \begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 2)} \end{pmatrix}' \begin{pmatrix} \left(\tau_{P_1}^{(\circ 1)}\left(z^{\bullet}_{P_1,t-1}\right) + \tau_{P_1}^{(\circ 2)}\left(z^{\circ}_{P_1,t-1}\right)\right)^2 - 1 \\ \left(\tau_{P_2}^{(\circ 1)}\left(z^{\bullet}_{P_2,t-1}\right) + \tau_{P_2}^{(\circ 2)}\left(z^{\circ}_{P_2,t-1}\right)\right)^2 - 1 \\ \dots \\ \left(\tau_{P_i}^{(\circ 1)}\left(z^{\bullet}_{P_i,t-1}\right) + \tau_{P_i}^{(\circ 2)}\left(z^{\circ}_{P_i,t-1}\right)\right)^2 - 1 \end{pmatrix} + \begin{pmatrix} l_{i1}^{\circ 2}\beta^{\circ}_{P_1} \\ l_{i2}^{\circ 2}\beta^{\circ}_{P_2} \\ \dots \\ a_{ii}^{\circ 2}\beta^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\circ}_{P_1,t-1} \\ log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ log\sigma^{\circ}_{P_i,t-1} \end{pmatrix} \\
&+ \begin{pmatrix} a_{i1}^{\circ 2}\gamma^{\circ}_{P_1} \\ a_{i2}^{\circ 2}\gamma^{\circ}_{P_2} \\ \dots \\ l_{ii}^{\circ 2}\gamma^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and exogenous } x.
\end{aligned}
$$

$$
log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\bullet 2}\omega^{\bullet}_{P_j} + \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 1)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_2}^{(\bullet 1)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 1)} \end{pmatrix}' \begin{pmatrix} \tau_{P_1}^{(\bullet 1)}\left(z^{\bullet}_{P_1,t-1}\right) + \tau_{P_1}^{(\bullet 2)}\left(z^{\circ}_{P_1,t-1}\right) \\ \tau_{P_2}^{(\bullet 1)}\left(z^{\bullet}_{P_2,t-1}\right) + \tau_{P_2}^{(\bullet 2)}\left(z^{\circ}_{P_2,t-1}\right) \\ \dots \\ \tau_{P_i}^{(\bullet 1)}\left(z^{\bullet}_{P_i,t-1}\right) + \tau_{P_i}^{(\bullet 2)}\left(z^{\circ}_{P_i,t-1}\right) \end{pmatrix}
$$

$$
+ \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)} \end{pmatrix}' \begin{pmatrix} \left(\tau_{P_1}^{(\bullet 1)}\left(z^{\bullet}_{P_1,t-1}\right) + \tau_{P_1}^{(\bullet 2)}\left(z^{\circ}_{P_1,t-1}\right)\right)^2 - 1 \\ \left(\tau_{P_2}^{(\bullet 1)}\left(z^{\bullet}_{P_2,t-1}\right) + \tau_{P_2}^{(\bullet 2)}\left(z^{\circ}_{P_2,t-1}\right)\right)^2 - 1 \\ \dots \\ \left(\tau_{P_i}^{(\bullet 1)}\left(z^{\bullet}_{P_i,t-1}\right) + \tau_{P_i}^{(\bullet 2)}\left(z^{\circ}_{P_i,t-1}\right)\right)^2 - 1 \end{pmatrix} + \begin{pmatrix} l_{i1}^{\bullet 2}\beta^{\bullet}_{P_1} \\ l_{i2}^{\bullet 2}\beta^{\bullet}_{P_2} \\ \dots \\ l_{ii}^{\bullet 2}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} l_{i1}^{\bullet 2}\gamma^{\bullet}_{P_1} \\ l_{i2}^{\bullet 2}\gamma^{\bullet}_{P_2} \\ \dots \\ l_{ii}^{\bullet 2}\gamma^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and exogenous } x.
$$

that is equivalent to

$$
log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\circ 2}\omega^{\circ}_{p_j} + \begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 1)}\tau_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\varepsilon_{P_2}^{(\circ 1)}\tau_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 1)}\tau_{P_i}^{(\circ 1)} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 1)}\tau_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\varepsilon_{P_2}^{(\circ 1)}\tau_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 1)}\tau_{P_i}^{(\circ 2)} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)}\tau_{P_1}^{(\circ 1)2} \\ l_{i2}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)}\tau_{P_2}^{(\circ 1)2} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 2)}\tau_{P_i}^{(\circ 1)2} \end{pmatrix}' \begin{pmatrix} {z^{\bullet}_{P_1,t-1}}^2 \\ {z^{\bullet}_{P_2,t-1}}^2 \\ \dots \\ {z^{\bullet}_{P_i,t-1}}^2 \end{pmatrix} + \begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)}\tau_{P_1}^{(\circ 2)2} \\ l_{i2}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)}\tau_{P_2}^{(\circ 2)2} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 2)}\tau_{P_i}^{(\circ 2)2} \end{pmatrix}' \begin{pmatrix} {z^{\circ}_{P_1,t-1}}^2 \\ {z^{\circ}_{P_2,t-1}}^2 \\ \dots \\ {z^{\circ}_{P_i,t-1}}^2 \end{pmatrix}
$$

$$
+ 2\begin{pmatrix} l_{i1}^{\circ 2}\varepsilon_{P_1}^{(\circ 2)}\tau_{P_1}^{(\circ 1)}\tau_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\varepsilon_{P_2}^{(\circ 2)}\tau_{P_2}^{(\circ 1)}\tau_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\varepsilon_{P_i}^{(\circ 2)}\tau_{P_i}^{(\circ 1)}\tau_{P_i}^{(\circ 2)} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}z^{\circ}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1}z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}z^{\circ}_{P_i,t-1} \end{pmatrix} - \sum_{j=1}^{k} l_{ij}^{\circ 2}\varepsilon_{P_j}^{(\circ 2)} + \begin{pmatrix} l_{i1}^{\circ 2}\beta^{\circ}_{P_1} \\ l_{i2}^{\circ 2}\beta^{\circ}_{P_2} \\ \dots \\ l_{ii}^{\circ 2}\beta^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\circ}_{P_1,t-1} \\ log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ log\sigma^{\circ}_{P_i,t-1} \end{pmatrix}
$$

$$
+ \begin{pmatrix} l_{i1}^{\circ 2}\gamma^{\circ}_{P_1} \\ l_{i2}^{\circ 2}\gamma^{\circ}_{P_2} \\ \dots \\ l_{ii}^{\circ 2}\gamma^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and exogenous } x.
$$

$$log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\bullet 2}\omega^{\bullet}_{p_j} + \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 1)}\tau_{P_1}^{(\bullet 1)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_2}^{(\bullet 1)}\tau_{P_2}^{(\bullet 1)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 1)}\tau_{P_i}^{(\bullet 1)} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 1)}\tau_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_2}^{(\bullet 1)}\tau_{P_2}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 1)}\tau_{P_i}^{(\bullet 2)} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_1}^{(\bullet 1)2} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_2}^{(\bullet 1)2} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)}\tau_{P_i}^{(\bullet 1)2} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}{}^2 \\ z^{\bullet}_{P_2,t-1}{}^2 \\ \dots \\ z^{\bullet}_{P_i,t-1}{}^2 \end{pmatrix} + \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_1}^{(\bullet 2)2} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_2}^{(\bullet 2)2} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)}\tau_{P_i}^{(\bullet 2)2} \end{pmatrix}' \begin{pmatrix} z^{\circ}_{P_1,t-1}{}^2 \\ z^{\circ}_{P_2,t-1}{}^2 \\ \dots \\ z^{\circ}_{P_i,t-1}{}^2 \end{pmatrix}$$

$$+ 2 \begin{pmatrix} l_{i1}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_1}^{(\bullet 1)}\tau_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\varepsilon_{P_1}^{(\bullet 2)}\tau_{P_2}^{(\bullet 1)}\tau_{P_2}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\varepsilon_{P_i}^{(\bullet 2)}\tau_{P_i}^{(\bullet 1)}\tau_{P_i}^{(\bullet 2)} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1}z^{\circ}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1}z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}z^{\circ}_{P_i,t-1} \end{pmatrix} - \sum_{j=1}^{k} a_{ij}^{\bullet 2}\varepsilon_{P_j}^{(\bullet 2)}$$

$$+ \begin{pmatrix} l_{i1}^{\bullet 2}\beta^{\bullet}_{P_1} \\ l_{i2}^{\bullet 2}\beta^{\bullet}_{P_2} \\ \dots \\ l_{ii}^{\bullet 2}\beta^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l_{i1}^{\bullet 2}\gamma^{\bullet}_{P_1} \\ l_{i2}^{\bullet 2}\gamma^{\bullet}_{P_2} \\ \dots \\ l_{ii}^{\bullet 2}\gamma^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}, i = 1, \dots, k \text{ and exogenous } x.$$

PC Bivariate Realized GARCH(2,2)

$$log\sigma^{\circ}_{x_i} = \sum_{j=1}^{k} l_{ij}^{\circ 2}\omega^{\circ}_{P_j} + \begin{pmatrix} l_{i1}^{\circ 2} \\ l_{i2}^{\circ 2} \\ \dots \\ l_{ii}^{\circ 2} \end{pmatrix}' \begin{pmatrix} \tau_{P_1}^{(\circ 1)}(z^{\bullet}_{P_1,t-1}) + \tau_{P_1}^{(\circ 2)}(z^{\circ}_{P_1,t-1}) \\ \tau_{P_2}^{(\circ 1)}(z^{\bullet}_{P_2,t-1}) + \tau_{P_2}^{(\circ 2)}(z^{\circ}_{P_2,t-1}) \\ \dots \\ \tau_{P_i}^{(\circ 1)}(z^{\bullet}_{P_i,t-1}) + \tau_{P_i}^{(\circ 2)}(z^{\circ}_{P_i,t-1}) \end{pmatrix}$$

$$+ \begin{pmatrix} l_{k1}^{\circ 2}\alpha^{\circ}_{P_1} \\ l_{k2}^{\circ 2}\alpha^{\circ}_{P_2} \\ \dots \\ l_{ii}^{\circ 2}\alpha^{\circ}_{P_i} \end{pmatrix}' \begin{pmatrix} log(max(r^{\circ 2}_{P_1,t-1}, 10^{-20})) \\ log(max(r^{\circ 2}_{P_2,t-1}, 10^{-20})) \\ \dots \\ log(max(r^{\circ 2}_{P_i,t-1}, 10^{-20})) \end{pmatrix} + \begin{pmatrix} l_{i1}^{\circ 2}\beta_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\beta_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\beta_{P_i}^{(\circ 1)} \end{pmatrix}' \begin{pmatrix} log\sigma^{\circ}_{P_1,t-1} \\ log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ log\sigma^{\circ}_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l_{i1}^{\circ 2}\beta_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\beta_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\beta_{P_i}^{(\circ 2)} \end{pmatrix}' \begin{pmatrix} log\sigma^{\circ}_{P_1,t-2} \\ log\sigma^{\circ}_{P_2,t-2} \\ \dots \\ log\sigma^{\circ}_{P_i,t-2} \end{pmatrix} + \begin{pmatrix} l_{i1}^{\circ 2}\gamma_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\gamma_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\gamma_{P_i}^{(\circ 1)} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix}$$

$$+ \begin{pmatrix} l_{i1}^{\circ 2}\gamma_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\gamma_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\gamma_{P_i}^{(\circ 2)} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-2} \\ logx_{P_2,t-2} \\ \dots \\ logx_{P_i,t-2} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.$$

$$
log\sigma^{\bullet}_{x_i}=\sum_{j=1}^{k} l_{ij}^{\bullet 2}\omega^{\bullet}_{P_j}+\begin{pmatrix} l_{i1}^{\bullet 2} \\ l_{i2}^{\bullet 2} \\ \dots \\ l_{ii}^{\bullet 2}\end{pmatrix}'\begin{pmatrix}\tau_{P_1}^{(\bullet 1)}\left(z^{\bullet}_{P_1,t-1}\right)+\tau_{P_1}^{(\bullet 2)}\left(z^{\circ}_{P_1,t-1}\right) \\ \tau_{P_2}^{(\bullet 1)}\left(z^{\bullet}_{P_2,t-1}\right)+\tau_{P_2}^{(\bullet 2)}\left(z^{\circ}_{P_2,t-1}\right) \\ \dots \\ \tau_{P_i}^{(\bullet 1)}\left(z^{\bullet}_{P_i,t-1}\right)+\tau_{P_i}^{(\bullet 2)}\left(z^{\circ}_{P_i,t-1}\right)\end{pmatrix}
$$

$$
+\begin{pmatrix} l_{i1}^{\bullet 2}\alpha^{\bullet}_{P_1} \\ l_{i2}^{\bullet 2}\alpha^{\bullet}_{P_2} \\ \dots \\ l_{ii}^{\bullet 2}\alpha^{\bullet}_{P_i}\end{pmatrix}'\begin{pmatrix} log(max\left(r^{\bullet 2}_{P_1,t-1},10^{-20}\right)) \\ log(max\left(r^{\bullet 2}_{P_2,t-1},10^{-20}\right)) \\ \dots \\ log(max\left(r^{\bullet 2}_{P_i,t-1},10^{-20}\right))\end{pmatrix}+\begin{pmatrix} l_{i1}^{\bullet 2}\beta_{P_1}^{(\bullet 1)} \\ l_{i2}^{\bullet 2}\beta_{P_2}^{(\bullet 1)} \\ \dots \\ l_{ii}^{\bullet 2}\beta_{P_i}^{(\bullet 1)}\end{pmatrix}'\begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1}\end{pmatrix}
$$

$$
+\begin{pmatrix} l_{i1}^{\bullet 2}\beta_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\beta_{P_2}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\beta_{P_i}^{(\bullet 2)}\end{pmatrix}'\begin{pmatrix} log\sigma^{\bullet}_{P_1,t-2} \\ log\sigma^{\bullet}_{P_2,t-2} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-2}\end{pmatrix}+\begin{pmatrix} l_{i1}^{\bullet 2}\gamma_{P_1}^{(\bullet 1)} \\ l_{i2}^{\bullet 2}\gamma_{P_2}^{(\bullet 1)} \\ \dots \\ l_{ii}^{\bullet 2}\gamma_{P_i}^{(\bullet 1)}\end{pmatrix}'\begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1}\end{pmatrix}
$$

$$
+\begin{pmatrix} l_{i1}^{\bullet 2}\gamma_{P_1}^{(\bullet 2)} \\ l_{i2}^{\bullet 2}\gamma_{P_2}^{(\bullet 2)} \\ \dots \\ l_{ii}^{\bullet 2}\gamma_{P_i}^{(\bullet 2)}\end{pmatrix}'\begin{pmatrix} logx_{P_1,t-2} \\ logx_{P_2,t-2} \\ \dots \\ logx_{P_i,t-2}\end{pmatrix}, i=1,\dots,k \text{ and endogenous } x.
$$

that is equivalent to

$$
log\sigma^{\circ}_{x_i}=\sum_{j=1}^{k} l_{ij}^{\circ 2}\omega^{\circ}_{P_j}+\begin{pmatrix} l_{i1}^{\circ 2}\tau_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\tau_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\tau_{P_i}^{(\circ 1)}\end{pmatrix}'\begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1}\end{pmatrix}+\begin{pmatrix} l_{i1}^{\circ 2}\tau_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\tau_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\tau_{P_i}^{(\circ 2)}\end{pmatrix}'\begin{pmatrix} z^{\circ}_{P_1,t-1} \\ z^{\circ}_{P_2,t-1} \\ \dots \\ z^{\circ}_{P_i,t-1}\end{pmatrix}
$$

$$
+\begin{pmatrix} l_{k1}^{\circ 2}\alpha^{\circ}_{P_1} \\ l_{k2}^{\circ 2}\alpha^{\circ}_{P_2} \\ \dots \\ l_{ii}^{\circ 2}\alpha^{\circ}_{P_i}\end{pmatrix}'\begin{pmatrix} log(max\left(r^{\circ 2}_{P_1,t-1},10^{-20}\right)) \\ log(max\left(r^{\circ 2}_{P_2,t-1},10^{-20}\right)) \\ \dots \\ log(max\left(r^{\circ 2}_{P_i,t-1},10^{-20}\right))\end{pmatrix}+\begin{pmatrix} l_{i1}^{\circ 2}\beta_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\beta_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\beta_{P_i}^{(\circ 1)}\end{pmatrix}'\begin{pmatrix} log\sigma^{\circ}_{P_1,t-1} \\ log\sigma^{\circ}_{P_2,t-1} \\ \dots \\ log\sigma^{\circ}_{P_i,t-1}\end{pmatrix}
$$

$$
+\begin{pmatrix} l_{i1}^{\circ 2}\beta_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\beta_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\beta_{P_i}^{(\circ 2)}\end{pmatrix}'\begin{pmatrix} log\sigma^{\circ}_{P_1,t-2} \\ log\sigma^{\circ}_{P_2,t-2} \\ \dots \\ log\sigma^{\circ}_{P_i,t-2}\end{pmatrix}+\begin{pmatrix} l_{i1}^{\circ 2}\gamma_{P_1}^{(\circ 1)} \\ l_{i2}^{\circ 2}\gamma_{P_2}^{(\circ 1)} \\ \dots \\ l_{ii}^{\circ 2}\gamma_{P_i}^{(\circ 1)}\end{pmatrix}'\begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1}\end{pmatrix}
$$

$$
+\begin{pmatrix} l_{i1}^{\circ 2}\gamma_{P_1}^{(\circ 2)} \\ l_{i2}^{\circ 2}\gamma_{P_2}^{(\circ 2)} \\ \dots \\ l_{ii}^{\circ 2}\gamma_{P_i}^{(\circ 2)}\end{pmatrix}'\begin{pmatrix} logx_{P_1,t-2} \\ logx_{P_2,t-2} \\ \dots \\ logx_{P_i,t-2}\end{pmatrix}, i=1,\dots,k \text{ and endogenous } x.
$$

$$
\begin{aligned}
log\sigma^{\bullet}_{x_i} = \sum_{j=1}^{k} l^{\bullet 2}_{ij}\omega^{\bullet}_{P_j} &+ \begin{pmatrix} l^{\bullet 2}_{i1}\tau^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\tau^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\tau^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} z^{\bullet}_{P_1,t-1} \\ z^{\bullet}_{P_2,t-1} \\ \dots \\ z^{\bullet}_{P_i,t-1} \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\tau^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\tau^{(\bullet 2)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\tau^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} \overset{\circ}{z}_{P_1,t-1} \\ \overset{\circ}{z}_{P_2,t-1} \\ \dots \\ \overset{\circ}{z}_{P_i,t-1} \end{pmatrix} \\
&+ \begin{pmatrix} l^{\bullet 2}_{i1}\alpha^{\bullet}_{P_1} \\ l^{\bullet 2}_{i2}\alpha^{\bullet}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\alpha^{\bullet}_{P_i} \end{pmatrix}' \begin{pmatrix} log(max(r^{\bullet 2}_{P_1,t-1}, 10^{-20})) \\ log(max(r^{\bullet 2}_{P_2,t-1}, 10^{-20})) \\ \dots \\ log(max(r^{\bullet 2}_{P_i,t-1}, 10^{-20})) \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\beta^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\beta^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\beta^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-1} \\ log\sigma^{\bullet}_{P_2,t-1} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-1} \end{pmatrix} \\
&+ \begin{pmatrix} l^{\bullet 2}_{i1}\beta^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\beta^{(\bullet 2)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\beta^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} log\sigma^{\bullet}_{P_1,t-2} \\ log\sigma^{\bullet}_{P_2,t-2} \\ \dots \\ log\sigma^{\bullet}_{P_i,t-2} \end{pmatrix} + \begin{pmatrix} l^{\bullet 2}_{i1}\gamma^{(\bullet 1)}_{P_1} \\ l^{\bullet 2}_{i2}\gamma^{(\bullet 1)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\gamma^{(\bullet 1)}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-1} \\ logx_{P_2,t-1} \\ \dots \\ logx_{P_i,t-1} \end{pmatrix} \\
&+ \begin{pmatrix} l^{\bullet 2}_{i1}\gamma^{(\bullet 2)}_{P_1} \\ l^{\bullet 2}_{i2}\gamma^{(\bullet 2)}_{P_2} \\ \dots \\ l^{\bullet 2}_{ii}\gamma^{(\bullet 2)}_{P_i} \end{pmatrix}' \begin{pmatrix} logx_{P_1,t-2} \\ logx_{P_2,t-2} \\ \dots \\ logx_{P_i,t-2} \end{pmatrix}, i = 1, \dots, k \text{ and endogenous } x.
\end{aligned}
$$

6.3 Results

The first main result is the variance covariance matrix that gives the volatility of the multivariate asset:

$$
\widehat{\boldsymbol{\Omega}}_{N\times N} = \begin{pmatrix}
\sigma_t^{\circ(1)} & \sqrt{\sigma_t^{\circ(1)}}\sqrt{\sigma_t^{\circ(2)}}\frac{\sum_{k=1}^{N} \overset{\circ}{l}_{1k}\overset{\circ}{l}_{2k}\sigma_t^{(\overset{\circ}{P}_k)}}{\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{1k}\sigma_t^{(\overset{\circ}{P}_k)}}\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{2k}\sigma_t^{(\overset{\circ}{P}_k)}}} & \dots & \sqrt{\sigma_t^{\circ(1)}}\sqrt{\sigma_t^{\circ(N)}}\frac{\sum_{k=1}^{N} \overset{\circ}{l}_{1k}\overset{\circ}{l}_{Nk}\sigma_t^{(\overset{\circ}{P}_k)}}{\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{1k}\sigma_t^{(\overset{\circ}{P}_k)}}\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{Nk}\sigma_t^{(\overset{\circ}{P}_k)}}} \\
\sqrt{\sigma_t^{\circ(2)}}\sqrt{\sigma_t^{\circ(1)}}\frac{\sum_{k=1}^{N} \overset{\circ}{l}_{2k}\overset{\circ}{l}_{1k}\sigma_t^{(\overset{\circ}{P}_k)}}{\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{2k}\sigma_t^{(\overset{\circ}{P}_k)}}\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{1k}\sigma_t^{(\overset{\circ}{P}_k)}}} & \sigma_t^{\circ(2)} & \dots & \sqrt{\sigma_t^{\circ(2)}}\sqrt{\sigma_t^{\circ(N)}}\frac{\sum_{k=1}^{N} \overset{\circ}{l}_{2k}\overset{\circ}{l}_{Nk}\sigma_t^{(\overset{\circ}{P}_k)}}{\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{2k}\sigma_t^{(\overset{\circ}{P}_k)}}\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{Nk}\sigma_t^{(\overset{\circ}{P}_k)}}} \\
\dots & \dots & \dots & \dots \\
\sqrt{\sigma_t^{\circ(N)}}\sqrt{\sigma_t^{\circ(1)}}\frac{\sum_{k=1}^{N} \overset{\circ}{l}_{Nk}\overset{\circ}{l}_{1k}\sigma_t^{(\overset{\circ}{P}_k)}}{\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{Nk}\sigma_t^{(\overset{\circ}{P}_k)}}\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{1k}\sigma_t^{(\overset{\circ}{P}_k)}}} & \sqrt{\sigma_t^{\circ(N)}}\sqrt{\sigma_t^{\circ(2)}}\frac{\sum_{k=1}^{N} \overset{\circ}{l}_{Nk}\overset{\circ}{l}_{2k}\sigma_t^{(\overset{\circ}{P}_k)}}{\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{Nk}\sigma_t^{(\overset{\circ}{P}_k)}}\sqrt{\sum_{k=1}^{N} \overset{\circ}{l}{}^{2}_{2k}\sigma_t^{(\overset{\circ}{P}_k)}}} & \dots & \sigma_t^{\circ(N)}
\end{pmatrix}
$$

where $\sigma_t^{\circ(k)}$ and $\sigma_t^{(\overset{\circ}{P}_k)}$, *k=1,…,N*, are the conditional volatilities estimated by one of the bivariate realized models proposed in chapter five. In order to calculate the volatility of a portfolio formed of the stocks considered for multivariate volatility modeling, there will be calculated the real and conditional $\sigma's$ as follows, considering equal weights of stocks (a's):

$$\sigma_t^{(N\ asset\ portfolio,\ \ estimated)} = \sum_{i=1}^{N} a_{i,t}^2 \sigma_{it} + 2 * \sum_{i,j=1}^{N} a_i a_j \left(\sqrt{\sigma_t^{(i)}} \sqrt{\sigma_t^{(j)}} \frac{\sum_{k=1}^{N} l_{ik} l_{jk} \sigma_t^{(P_k)}}{\sqrt{\sum_{k=1}^{N} l_{ik}^2 \sigma_t^{(P_k)}} \sqrt{\sum_{k=1}^{N} l_{jk}^2 \sigma_t^{(P_k)}}} \right)$$

$$\sigma_t^{(N\ asset\ portfolio,\ \ real)} = \sum_{i=1}^{N} a_{i,t}^2 r_{it}^2 + 2 * \sum_{i,j=1}^{N} a_i a_j (r_{it} r_{jt})$$

The RMSEs calculated over the whole sample for each of the models considered, are as follows. As well there were calculated the RMSEs of the individual stocks whose volatility was estimated by using correspondent bivariate models.

	RMSE		RMSE
PC Bivariate EGARCH	**25,7556**	Bivariate EGARCH	**190,1578**
PC Bivariate EGARCHX	**26,5245**	Bivariate EGARCHX	**197,0068**
PC Bivariate Realized EGARCH	**25,8015**	Bivariate Realized EGARCH	**197,5427**
PC Bivariate Realized GARCH (complete)	**26,3228**	Bivariate Realized GARCH (complete)	**219,5970**
PC Bivariate Realized GARCH (partial)	**27,9438**	Bivariate Realized GARCH (partial)	**221,2924**
PC Bivariate Realized GARCH(2,2)	**28,6432**	Bivariate Realized GARCH(2,2)	**251,4380**

It may be seen that, by including the stocks into a portfolio, the volatility exercise becomes more precise, as compared to modeling individual stocks' volatility and summing their errors over the whole sample.

6.4 Conclusions

In this chapter we offered a method to forecast multivariate volatility of multiple stock assets by using a method (Principal Component Algorithm) adapted to autoregressive conditional heteroskedastic models that use measures of night and day volatility, but also measures of intraday volatility (bivariate realized GARCH type of models). We offered bivariate formulations of the new models for an *n*-stock asset (PC Bivariate EGARCH, PC Bivariate EGARCHX, PC Bivariate Realized EGARCH, PC Bivariate Realized GARCH (complete and partial forms) and PC Bivariate Realized GARCH(2,2)). By reducing the *n*-multivariate to a 4-stock dimension, we estimated the new models, and assessed their 1-day ahead forecasting performance. We found the models to be very effective, the portfolio forecasting error over the whole sample being significantly lower than the summed errors of the bivariate models applied to the individual stocks composing the portfolio. According to the methodology presented in the chapter, a risk-averse investor could best use the new models by taking a number of stocks, putting them into a portfolio, forecasting the variance of the individual stocks and the correlations between them according to the methodology described in the chapter; then, according to the forecasted volatility and correlations, the investor would increase in the portfolio the weights of the stocks with the lowest variance and that commove opposingly. Thus, the resulted variance of the portfolio will be significantly lower, equivalent to a reduced risk. The new portfolio variance could be then assessed by using one of the newly proposed PC Bivariate Realized GARCH models.

Chapter seven: Conclusions and further research

Conclusions

The current work proposed to offer some insights on the volatility forecasting topic, by addressing to two types of objectives. The first type was to enhance the volatility modeling problem by proposing alternative (bivariate and multivariate) models to forecast volatility of individual stocks or of multiple stock assets. As such it extended a method proposed by Hansen, Huang and Shek (2010b) for bivariate modeling of the Realized GARCH model (that we called the partial model), method that allowed the inclusion in the GARCH equations of night volatility measurements, to a class of realized GARCH models. New models emerged: a Bivariate Realized EGARCHX, a complete form of the Bivariate Realized GARCH model, a Bivariate Realized EGARCH model, and a Bivariate GARCH(2,2) model. A non-realized Bivariate EGARCH model was proposed.

To the same objective it addressed the proposal of a method that targeted volatility modeling of multiple stock assets, taking advantage of the method suggested by Burns (2005) for the PC GARCH model. As such, in the current work it was proposed an adaptation of Burns' methodology, applied to realized GARCH models. The goal was to forecast volatility of multivariate assets by a method that attached a Principal Component Algorithm to the realized volatility models, taking advantage of the availability of high frequency data. New models emerged: PC EGARCHX, PC Realized GARCH, PC Realized EGARCH, and PC Realized GARCH(2,2). As well, a non-realized model PC-EGARCH was proposed. They were estimated according to various methodologies and their forecasting accuracy was assessed, across various criteria.

To the completion of the first objective, a method that attached the Principal Component algorithm to the bivariate realized models was proposed. This method aims to take advantage of the realized volatility modeling with high frequency data and with night volatility data, as well as to the Principal Component Analysis, in order to model and forecast volatility of multivariate stock assets. New models were proposed, namely: PC Bivariate EGARCHX, PC Bivariate Realized EGARCH, PC Bivariate Realized GARCH (complete and partial forms) and PC Bivariate Realized GARCH(2,2), as well as a non-realized model, PC Bivariate EGARCH.

These models were estimated and their forecasting performance was evaluated, across four criteria.

The second type of objectives addressed to an existing problem in the literature and in various applied contexts. That results from the coexistence of a high number of volatility models and from a lack of consensus in literature on which models perform better. As such, finding a better model or group of models emerged as a central research topic. The second objective was to rank some of the recently proposed realized GARCH models, and also to rank the models proposed in the current research. A general conclusion was that ranking proved to be sensitive to the methodology employed, realized volatility measures used, stock choice, and criterion used for performance ranking. However, consistent conclusions emerged. As such, it was found that EGARCHX model, Realized EGARCH and Realized GARCH(2,2) persistently ranked better, while the non-realized models GARCH and EGARCH performed poor in each stance almost. This allowed us to conclude that incorporating measures of intraday volatility enhances the modeling problem.

As regards the ranking of the bivariate models, it was found that the Bivariate Realized GARCH (partial and complete) and Bivariate Realized EGARCH models performed well in any of the four stock choices, followed by the Bivariate Realized GARCH(2,2), Bivariate EGARCH and Bivariate EGARCHX models.

With respect to the second objective, it was also assessed the gain or loss obtained when night volatility measures were employed. As such, it has been found that the bivariate models surpass the univariate ones when specific methodology, ranking criteria and stocks are used. The results are mixed, allowing us to conclude that the bivariate models did not prove totally inferior to their univariate counterparts, being a good alternative to be used in the forecasting exercise together with the univariate models for more reliable and precise estimates.

Rankings have been made as well with regards to the measures of intraday volatility. H-L was found to perform poor in most estimations, while the realized variance sampled at 5 minutes and 15 minutes ranked the best. This confirms the fact that higher the frequency, better the sampling, with the amendment that at too frequent sampling the microstructure noise may be too high for a precise intraday estimate. Realized kernels ranked as well good.

With respect to the principal component models it was found that they were a good alternative in estimating the volatility of highly correlated stock assets, and that putting stocks in one portfolio allowed for a more accurate volatility estimator than modeling volatility as separate univariate processes. As well, multivariate modeling with univariate estimations of the principal components allowed risk-averse investors to form low-risk portfolios by selecting stocks with negative covariances and smaller one-day ahead volatility forecasts.

Further research

Further research could be extended to the usage of high frequency data to other classes of models and to other measures of intraday volatility (like bipower variation, quadratic variance with Markov chains etc.). Measurements of intra-night volatility could be simulated by using Monte Carlo simulation methods, and incorporated in new realized volatility models. The studies could be extended to other types of data, like commodity data (oil, gold, etc.), or to the exchange rates, while multi-period horizon forecasts could be considered as well.

Principal component extensions may be considered to larger classes of stocks, that may be restrained to a reduced number of variables according to the highest relevant principal components. As such, a possible application would be to consider the 500 stocks compounding the S&P index, to apply the PC algorithm to all 500 variables, finding the 500 orthogonal principal components and then selecting only those with the highest impact on the initial variables. As such, it may obtained a new index, with much fewer variables than S&P500, but close in terms of relevance, with which it may be further worked. It may be formed a new portfolio of stocks with the new variables to which the new PC-models may be applied in order to find their variance-covariance matrix, and consequently to find their volatility.

Bivariate modeling can be enhanced by proposing a method to generate intranight measures of volatility. New models may be proposed, like models that would include night and day, as well as intranight and intraday volatility estimates.

Finally, the current work could be extended by estimating different univariate and multivariate GARCH models with Bayesian statistics techniques. Further research could also include extensions of using high frequency information, or of night volatility information to other multivariate GARCH models, like VEC, BEKK, CCC, TVC and DCC models, and the employment of new developments like semi-parametric estimation, more flexible DCC and factor models, finite mixtures of GARCH models.

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