Universitat de Lleida

Discrete and Continuous Symmetries in Planar Vector Fields

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Memòria presentada per aspirar al grau de doctora.

Departament de Matemàtica
Universitat de Lleida
Lleida, Juliol de 2008.

El Dr. Jaume Giné Mesa CERTIFICA que aquesta memòria ha estat realitzada per Susanna Maza Sabido sota la meva direcció.

Lleida, Juliol de 2008.

A la meva mare, Fátima.

Preface

Dynamical systems is the branch of mathematics describing processes in motion. Typically, a dynamical system is given as the flow of a vector field, or the solution of a differential or a difference equation. Once the system of differential equations can be solved, given an initial state, it is possible to determine all the future states of the phenomena. Thus, once the differential equations modelling certain observed phenomena are formulated, the next problem is to solve these equations, that is, to find an explicit expression for which the states of the dependent variables maybe predicted by the independent variables.

To find a closed form solution of a differential equation is a difficult task. Moreover, most nonlinear differential equation cannot be solved. It is for that reason that H. Poincaré initiates the study of the qualitative properties of a differential equation in *Sur les courbes definiés par les équations différentielles* (1880). The *qualitative theory of differential equations* gives information about the behavior of the solutions of a differential equation without the explicit knowledge of them.

Normal form theory is an old subject in the qualitative study of differential equations, having been created by H. Poincaré in his Thesis (1890), but is still one of the most useful tools in the study of dynamical systems. Among the numerous analytical and numerical techniques available in the field of dynamical systems, a classical point of view advocates an initial normal forms analysis. Under transformation, a mathematician seeks to reduce a nonlinear dynamical system to its simplest form, eliminating inessential nonlinearities or degrees of freedom.

The so called *linearization* problem of a vector field consists in determining the necessary and sufficient conditions for the existence of a coordinate change near a singularity bringing the vector field into its linear part. Other case of normal form occurs when the vector field is orbitally equivalent to its linear part after a change of coordinates. To determine the necessary and sufficient conditions for the existence of such a change of coordinates is known as the *Orbital linearization* problem. By definition, two vector fields are orbitally equivalent if one of them can be brought to the other by a local diffeomorphism and multiplication by a non-vanishing function.

Symmetry methods for differential equations have become increasingly popular in recent years and have been applied to a variety of problems. Roughly speaking, a symmetry of a geometrical object is a transformation whose action leaves the object apparently unchanged. It is useful in order to understand the concept of symmetry to consider symmetries of simple objects. For instance, consider an equilateral triangle and a circle rotating both about their centers. After a rotation of $2\pi/3$ the triangle looks the same as it did before the rotation, so the rotation is a symmetry. However not all the rotations are symmetries: a rigid triangle has a finite number of symmetries. On the contrary, the circle has an infinite set of symmetries. The infinite set of symmetries of the circle is a one–parametric continuous group of transformations. This kind of symmetries is very useful in order to solve many differential equations. More than a century ago S. Lie understood differential equations as a geometric object and put forward many of the fundamental ideas behind symmetry methods. He focused his attention in symmetries belonging to one–parametric continuous groups of transformations. We refer to such symmetries as *Lie symmetries*.

However, this approach was neglected over several decades because the complexity of the computations needed to determine the complete list of symmetries of a given differential equation. Thanks to the applications in fluid dynamics of Birkhoff the interest for the symmetry grew again and nowadays symmetry theory plays a special role, in that it gives general properties of the system.

A special kind of discrete symmetry that some dynamical systems can exhibit is the so called *reversibility*. Briefly, a dynamical system is called reversible if it is invariant under the action of an involutive spatial diffeomorphism and a reversion in time's arrow. From a dynamical point of view, one of the main consequences is that the involution maps trajectories to other trajectories but the timedirection of the two trajectories is reversed. For instance, consider the motion of an ideal pendulum and film its evolution with a projector, we could not distinguish from the images if we are playing the film forwards or backwards. This is because the motion on the reverse film also corresponds to a possible motion of the same pendulum. Namely, the reverse motion satisfies the same laws of motion as the forward motion. If for a motion picture of a mechanical system one cannot decide whether it is shown in the forward or reverse direction, the system is said to have a *reversal symmetry.* The reversal symmetry described in the former example arises very frequently in classical mechanics as well as in quantum mechanics.

The present work deals with several topics within symmetry methods and qualitative theory of planar vector fields. The first one is about the role of Lie symmetries in finding the change of coordinates that transforms a given vector field into its normal form. More precisely, in Chapters 2,3, and 4 we focus our attention in finding the change of variables that linearizes (resp. orbitally linearizes) a given planar vector field from the knowledge of a Lie symmetry of the vector field. The obtained results will be applied in Chapter 5 where we study a one–parameter family of quadratic systems belonging to the so called *Lotka–volterra* systems. Chapter 6 is devoted to the study of some aspects of the dynamics in the phase plane of smooth Newtonian systems given by a scalar autonomous second order differential equations possessing a *r–dimensional Lie point symmetry algebra* $\mathcal{L}_r$ with $r \geq 2$. Here, we focus on the existence, nonexistence and localization of periodic orbits. Finally, we emphasize on the particular case of *polynomial Liénard* systems possessing a $\mathcal{L}_r$ with $r \geq 2$. For such polynomial Liénard systems, we prove the nonexistence of limit cycles. The last part of this report is about the topic of reversibility. In Chapter 7 we consider analytic systems having a degenerate center at the origin. From the knowledge of a local smooth first integral we prove the existence of a smooth map that transforms the system into a reversible linear system (after rescaling the time). Moreover, by using such a smooth map, we give a procedure to detect, in some cases, reversing symmetries in degenerate centers.

Agraïments

Primer de tot voldria agraïr al meu director de tesi, el professor J. Giné, l'oportunitat que em va donar d'ésser la seva alumna, sense la qual aquesta memòria no hauria pogut fer-se realitat. Tampoc oblido el temps que m'ha dedicat, el recolçament que m'ha donat, així com el fet que hagi compartit amb mi els seus coneixements.

D'altra banda, aquesta memòria no pot començar sense destacar el suport incondicional del meu marit, el professor I.A. García. Vull agrair-li i, alhora, disculpar-me per les incontables converses (a la una de la nit, a les vuit del matí, en diumenge, en vacances..) sobre equacions diferencials amb les que l'he assetjat i el temps que l'hi he robat.

Agraïr a la professora M. Grau la seva companyonia, la seva amistat i els coneixements que m'ha ofert en cada uns dels seminaris impartits en la Universitat de Lleida.

Per finalitzar, voldria agraïr a la meva mare i als meus sogres tot el temps en qué han tingut cura de la meva filla mentre jo escrivia aquest treball.

Contents

Chapter 1

Introduction

1.1 An overview of differential equations

Throughout this dissertation we will consider first order autonomous systems in the plane

$$\dot{x} = \frac{\mathrm{d}x}{\mathrm{d}t} = P(x, y) \ , \quad \dot{y} = \frac{\mathrm{d}y}{\mathrm{d}t} = Q(x, y) \ , \tag{1.1}$$

where P and Q are smooth C^∞ functions defined in an open set $U \subseteq \mathbb{F}^2$ called *phase space*. In what follows, we denote by $\mathbb{F}$ the fields either $\mathbb{R}$ or $\mathbb{C}$ in according with the system (1.1) be real or complex, respectively. The independent "time" variable t will be always real along this work. In some chapters of this work we will take P and Q to be analytic C^ω in U or even coprime polynomials with coefficients in $\mathbb{F}$. In this last case, we say that $m = \max\{\deg P, \deg Q\}$ is the *degree* of polynomial system (1.1).

We associate to system (1.1) the vector field

$$\mathcal{X} = P(x, y)\ \partial_x + Q(x, y)\ \partial_y. \tag{1.2}$$

The existence of solutions for system (1.1) is given by the existence and uniqueness theorem of solutions for an ordinary differential system. A *solution* of (1.1) is a smooth function $\varphi : I \to \mathbb{F}^2$ where $I \subseteq \mathbb{R}$ is the maximal time open interval in which the solution is defined such that $d\varphi(t)/dt = \mathcal{X}(\varphi(t))$, for every $t \in I$. Let φ be a solution of system (1.1). Then, the set of points $\Gamma = \{\varphi(t) \in \mathbb{F}^2 \ : \ t \in I\}$ is called an *orbit* or *trajectory* or *integral curve* of system (1.1) or of the vector field $\mathcal{X}$.

A *phase portrait* of a real system (1.1) defined in $U \subseteq \mathbb{R}^2$ is a plot of several special orbits corresponding to different initial conditions in the phase

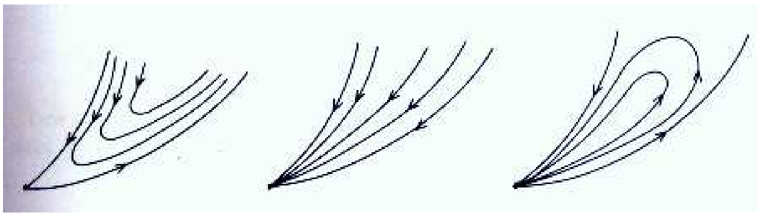


Figure 1.1: Hyperbolic, parabolic and elliptic sectors.

plane. Phase portrait is an invaluable tool in the study of the real dynamical systems. It consists of a plot of key orbits and reveals topological information about system (1.1). In general, a vector field admits a simple classification of its topological type. We will say that two vector fields $\mathcal{X}$ and $\mathcal{Y}$, defined on open subsets U and W of $\mathbb{F}^2$ respectively, are *topologically equivalent* when there exists a homeomorphism $\phi : U \to W$ which sends orbits of $\mathcal{X}$ to orbits of $\mathcal{Y}$ preserving the orientation. If $\mathcal{X}$ and $\mathcal{Y}$ are topologically equivalent, one says that they have the same phase portrait. In fact one can identify the phase portrait of a vector field $\mathcal{X}$ with its equivalence class. Now, we define the orbits (if they exist) that perform the skeleton of a phase portrait.

- A *singular point* of system (1.1) is a point $(x_0, y_0) \in U$ such that $P(x_0, y_0) = Q(x_0, y_0) = 0$. Of course, singular points are special solutions of system (1.1) called stationary solutions because $\varphi(t) = (x_0, y_0)$ for all t. Singular point are defined in the same way when $U \subseteq \mathbb{C}^2$. We say that a singular point is *isolated* if there is no other singular point in a punctured neighborhood of it.

- An isolated singular point (x_0, y_0) of an analytic system (1.1) possesses a neighborhood that is divided (by orbits of (1.1) that approach (x_0, y_0) along defined tangents) into a finite number of open regions called sectors. Canonical sectors of elliptic, hyperbolic, or parabolic type are shown in Figure 1.1. The orbits of (1.1) which lie on the boundary of a hyperbolic sector are called *separatrices.*

- A *periodic orbit* is an orbit $\gamma = \{\varphi(t) \in \mathbb{R}^2 \ : \ t \in \mathbb{R}\}$ different from a singular point such that $\varphi(t)$ is a periodic solution of system (1.1), that is, there exists a positive time $T \in \mathbb{R}^+$ called the period and satisfying $\varphi(t + T) = \varphi(t)$ for all t. Hence, a periodic orbit Γ is, in particular, a closed invariant curve without singular points of system (1.1). An isolated periodic orbit Γ is called a *limit cycle*. Therefore, there is always a neighborhood of a limit cycle free of other periodic orbits. Otherwise, non–isolated periodic orbits belong to a *period annulus*.

- A regular solution $\varphi(t)$ of (1.1) defines a *homoclinic orbit* if $\varphi(t) \to p_0$ as $t \to \pm\infty$ for some singular point p_0 and defines a *heteroclinic orbit* connecting two singular points p_1 and p_2 if $\varphi(t) \to p_1$ as $t \to \infty$ and $\varphi(t) \to p_2$ as $t \to -\infty$. A *graphic* $\Gamma = \cup_{i=1}^{k} \Gamma_i \cup \{p_1, \ldots, p_k\}$ is formed by k singular points $p_1, \ldots, p_k$, $p_{k+1} = p_1$ and k oriented regular orbits $\Gamma_1, \ldots, \Gamma_k$, connecting them such that Γ_i is an unstable characteristic orbit of p_i and a stable characteristic orbit of p_{i+1}.

A point $p \in U$ is an ω–limit point of a solution $\varphi(t)$ of (1.1) if there is time–sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to \infty$ as $n \to \infty$ such that $\varphi(t_n) \to p$ as $n \to \infty$. Similarly, if $t_n \to -\infty$ as $n \to \infty$ and $\varphi(t_n) \to q \in U$ as $n \to \infty$, then the point q is termed α–limit point of the solution $\varphi(t)$. The set of all ω–limit points (resp. α–limit points) of an orbit Γ is called the ω–limit set (resp. α–limit set) of Γ and denoted by $\omega(\Gamma)$ (resp. $\alpha(\Gamma)$). It is easy to see that $\omega(\Gamma)$ and $\alpha(\Gamma)$ are closed sets and invariant by the flow of (1.1). For real systems (1.1), the α or ω–limit sets of any orbit are relatively simple: singular points, limit cycles or graphics. This is a consequence of the celebrated Poincaré–Bendixson Theorem for real planar dynamical systems.

1.2 Invariant curves and inverse integrating factors

Let $\Gamma_i := \{\varphi_i(t) \in \mathbb{F}^2 \ : \ t \in I_i\}$ for $i = 1, 2, \ldots$, be a collection of orbits of system (1.1). The set $\bigcup_i \Gamma_i$ is an invariant curve of system (1.1) because it is invariant by the flow of (1.1). Sometimes, this invariant curve can be implicitly described by the equation $f(x, y) = 0$, for certain function $f : U \to \mathbb{F}$. When $f \in \mathbb{F}[x, y]$ is a polynomial with coefficients in the field $\mathbb{F}$, then $f = 0$ is called an *algebraic invariant curve* of system (1.1). Assume now that $f \in C^1(U)$. From the invariant condition, the vector field $\mathcal{X}$ is tangent to the invariant curve $f = 0$ except at the possible singular points of (1.1) on $f = 0$. Therefore, on the (nonsingular) points of the invariant curve $f = 0$, the gradient vector $\nabla f = (\partial f/\partial x, \partial f/\partial y)$ of the curve is orthogonal to the vector field $\mathcal{X}$. Therefore,

$$\mathcal{X} f|_{f=0} := P\frac{\partial f}{\partial x} + Q\frac{\partial f}{\partial y}\bigg|_{f=0} = 0 \ .$$

We are leaded directly to the following definition.

Definition 1.1 *A* C^1 invariant curve *of system* (1.1) *is a set of points in* $U \subseteq \mathbb{F}^2$ *satisfying an equation* $f(x,y) = 0$ *with* $f \in C^1(U)$ *and such that*

$$\mathcal{X} f = Kf \ , \tag{1.3}$$

for some continuous function $K(x,y)$*, called* cofactor*.*

We say that a C^1 non–constant single–valued function $H : U \subset \mathbb{F}^2 \to \mathbb{F}$ is a *first integral* of system (1.1) if H is constant on each solution of this system, i.e., $H(\varphi(t))$ is constant for all values of $t \in I$ for which the solution $\varphi(t) \in U$ is defined. In other words, a first integral of system (1.1) in U will be a $C^1(U)$ solution of the homogeneous linear partial differential equation $\mathcal{X} H \equiv 0$. Notice that, for systems defined on $U \subseteq \mathbb{F}^2$, the existence of a first integral completely determines its phase portrait because the level sets $\{H(x,y) = h\}$ define a decomposition of U as union of orbits of (1.1).

We define the divergence of the vector field $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ as $\mathrm{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y$.

Definition 1.2 *An* inverse integrating factor *for system* (1.1) *in* U *is a non–locally null* $\mathcal{C}^1$ *solution* $V : U \subset \mathbb{R}^2 \to \mathbb{R}$ *of the linear partial differential equation*

$$\mathcal{X} V \ = \ V \,\mathrm{div}\mathcal{X} \ . \tag{1.4}$$

We observe that, if V is an inverse integrating factor of system (1.1) then the zero–set of V, $V^{-1}(0) := \{(x,y) \in U \mid V(x,y) = 0\}$, is composed of orbits of (1.1). In particular, the cofactor of the invariant curve $V(x,y) = 0$ is just $\mathrm{div}\mathcal{X}$. Notice that, in $U\backslash\{V^{-1}(0)\}$, the vector field $\mathcal{X}/V$ is divergence free, i.e., $\mathrm{div}(\mathcal{X}/V) \equiv 0$. Therefore, in any simply connected subset of $U\backslash\{V^{-1}(0)\}$, the vector field $\mathcal{X}/V$ is hamiltonian. More precisely, in $U\backslash\{V^{-1}(0)\}$, we have $\dot{x} = P(x,y)/V(x,y) = \partial H/\partial y$, $\dot{y} = Q(x,y)/V(x,y) = -\partial H/\partial x$, where the first integral H associated to the inverse integrating factor V can be computed from the well defined line integral

$$H(x,y) = \int \frac{Q(x,y)dx - P(x,y)dy}{V(x,y)} \ . \tag{1.5}$$

The zero–set of $V(x, y)$ contains most of the relevant orbits of the phase portrait of $\mathcal{X}$ in U such as singular points, separatrices, limit cycles and graphics provided they are α or ω–limit sets.

It is known that the existence of invariant algebraic curves can be used to prove topological properties of polynomial systems (1.1). In short, invariant algebraic curves and integrability have a narrow relationship for planar polynomial systems like it is clearly shown in the Darboux theory. Darboux showed in "*Mémoire sur les équations différentielles algébriques du premier ordre et du premier degré*" (1878) how first integrals of polynomial systems possessing sufficient invariant algebraic curves are constructed. In particular, he proved that if a polynomial system of degree m has at least $[m(m+1)/2]+1$ invariant algebraic curves, then it has a first integral. Darboux's first idea consists in searching a first integral for the polynomial system (1.1) as a function of the form

$$\prod_{i=1}^{q} f_i^{\lambda_i}(x, y) \ , \tag{1.6}$$

for suitable $\lambda_i \in \mathbb{C}$ not all zero and $f_i(x, y) = 0$ being invariant algebraic curves of system (1.1). The first integral (1.6) is called *Darboux first integral.* More precisely, denoting by K_i de cofactor of f_i, Darboux's theory says:

- If $q \geq m(m + 1)/2 + 1$ then (1.6) is a first integral of (1.1).
- If $q = m(m+1)/2$ then (1.6) is a first integral or an inverse integrating factor of (1.1).
- If $q < m(m + 1)/2$ and either $\sum_{i=1}^{q} \lambda_i K_i = 0$ or $\sum_{i=1}^{q} \lambda_i K_i = \mathrm{div}\mathcal{X}$ then (1.6) is a first integral or an inverse integrating factor of (1.1) respectively.

For instance, a proof of the first item is the following. Since $\mathcal{X} f_i = K_i f_i$ for $i = 1, \ldots, q$ and $\mathcal{X}$ is a polynomial vector field of degree m, it follows that the cofactors K_i are polynomials of $\deg K_i \leq m - 1$. So K_i has at most $m(m + 1)/2$ coefficients. But, since $q > m(m + 1)/2$, we have that the set $\{K_1, \ldots, K_q\}$ is linearly dependent and therefore there are complex numbers λ_i not all zero for $i = 1, \ldots, q$ such that $\sum_{i=1}^{q} \lambda_i K_i \equiv 0$. Finally, taking the function $H(x, y)$ defined like (1.6), one has

$$\mathcal{X} H = H \sum_{i=1}^{q} \lambda_i \frac{\mathcal{X} f_i}{f_i} = H \sum_{i=1}^{q} \lambda_i K_i \equiv 0 \ .$$

We see that this proof also works if the invariant curves $f_i = 0$ are not algebraic but their associated cofactors K_i are polynomials of $\deg K_i \leq m-1$.

1.3 Normal forms theory

First of all, we recall the notion of equivalence between two vector fields which will allow us to compare their phase portraits. Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector fields defined on open subsets U and W of $\mathbb{F}^2$, respectively. We say that $\mathcal{X}$ is *topologically equivalent* (resp. *C^r–equivalent*) to $\mathcal{Y}$ when there exists a homeomorphism (resp. a diffeomorphism of class C^r) $\phi : U \to W$ which sends orbits of $\mathcal{X}$ to orbits of $\mathcal{Y}$ preserving the orientation. Such a ϕ is called a *topological equivalence* (resp. *C^r–equivalence*) between $\mathcal{X}$ and $\mathcal{Y}$.

Let $\Phi_{\mathcal{X}}(t;(x,y))$ and $\Phi_{\mathcal{Y}}(t;(x,y))$ be the flows generated by $\mathcal{X}$ and $\mathcal{Y}$ in U and W, respectively. We say that $\mathcal{X}$ is *topologically conjugate* (resp. *C^r–conjugate*) to $\mathcal{Y}$ when there exists a homeomorphism (resp. a diffeomorphism of class C^r) $\phi : U \to W$ such that $\phi(\Phi_{\mathcal{X}}(t;(x,y))) = \Phi_{\mathcal{Y}}(t;\phi(x,y))$ for every $(x,y) \in U$ and all t where this make sense. Such a ϕ is called a *topological conjugacy* (resp. *C^r–conjugacy*) between $\mathcal{X}$ and $\mathcal{Y}$. Notice that, any conjugacy is clearly also an equivalence. Moreover, a topological equivalence maps singular points into singular points and periodic orbits into periodic orbits. Conjugacy, in addition, preserves the period of the periodic orbits. In practice, a C^1–conjugacy is characterized by $D\phi(p)\mathcal{X}(p) = \mathcal{Y}(\phi(p))$ for all $p \in U$, where $D\phi$ denotes the differential of ϕ. Notice that a C^r–conjugacy with $r \geq 1$ is just a *change of coordinates* which operates on a vector field $\mathcal{X}$ to change it into $\mathcal{Y}$ of the form

$$\mathcal{Y}(p) = D\phi(\phi^{-1}(p))\mathcal{X}(\phi^{-1}(p)) \ .$$

We will simply write the above expression as $\mathcal{Y} = \phi_* \mathcal{X}$ or $\mathcal{X} = \phi^* \mathcal{Y}$ where ϕ_* and ϕ^* are the *push–forward* and *pull–back* defined by the diffeomorphism ϕ. Another form to express how a change of variables $\phi(x,y) = (u(x,y), v(x,y))$ acts on a vector field $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ is given by $\phi_* \mathcal{X} = (\mathcal{X}u)\partial_u + (\mathcal{X}v)\partial_v$.

Let $\mathcal{X}$ be a $C^\infty(U)$ planar vector field and $p_0 \in U$ a regular point of $\mathcal{X}$, that is, $\mathcal{X}(p_0) \neq 0$. From the Flow Box Theorem, we know that there exists a neighborhood $\mathcal{V} \subset U$ of p and a $C^\infty(\mathcal{V})$ diffeomorphism ϕ such that it is a C^∞–conjugacy in $\mathcal{V}$ between $\mathcal{X}$ and the constant vector field $\mathcal{Y} = \partial_x$. Therefore, after a suitable change of coordinates ϕ, in a neighborhood of a regular point, the flow of $\mathcal{X}$ is rectified.

The celebrated Hartman–Grobman Theorem shows us that, in a neighborhood of a *hyperbolic* singular point p_0 of $\mathcal{X}$ (that is, when the real part of the eigenvalues of the Jacobian matrix $D\mathcal{X}(p_0)$ are different from zero), the qualitative behavior of the orbits of system (1.1) is determined by its linear part $D\mathcal{X}(p_0)$. More precisely, a vector field $\mathcal{X}$ in a neighborhood of a hyperbolic singularity is C^0–conjugated to its linear part.

From classical theory of differential equations, we know that, after a linear change of coordinates, the linear part of system (1.1) can be put into Jordan canonical form. Therefore, without lost of generality, we can assume that system (1.1) has the form $\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{F}(\mathbf{x})$ with $\mathbf{x} = (x, y)^T \in U \subseteq \mathbb{F}^2$, and where the function $\mathbf{F} : U \to U$ represents the nonlinear terms of the system. Here, $J \in \mathcal{M}_2(\mathbb{F})$ is a 2×2 matrix with entries in $\mathbb{F}$. The aim of the formal normal form theory is to determine, given the eigenvalues of J, the class of nonlinearities $\mathcal{N}$ as small as possible such that the system (1.1) can be brought to the form $\dot{\mathbf{y}} = J\mathbf{y} + \tilde{\mathbf{F}}(\mathbf{y})$ with $\tilde{\mathbf{F}}(\mathbf{y}) \in \mathcal{N}$ by means of a near–identity formal coordinate change of the form $\mathbf{x} = \mathbf{y} + \mathbf{f}(\mathbf{y})$ where $\mathbf{f}(\mathbf{y}) = \mathcal{O}(\|\mathbf{y}\|^2)$.

The method of reducing system (1.1) to its normal form dates back to the Ph.D thesis of Poincaré (1890). The coordinate transformation is usually obtained by means of iterative techniques so that, in general, it is actually a formal power series transformation. The nonlinear terms of the differential system which cannot be eliminated are called *resonant terms*. In short, it can be proved that the characterizing property of the Poincaré normal form can be stated as follows: If v_1 and v_2 is an eigenbasis of J with associated eigenvalues λ_1 and λ_2, then $\tilde{\mathbf{F}}(\mathbf{y})$ where $\mathbf{y} = (w, z) \in \mathbb{F}^2$ is a formal series in those monomials $w^{m_1} z^{m_2} v_j$ satisfying

$$m_1\lambda_1 + m_2\lambda_2 = \lambda_j \ . \tag{1.7}$$

We say that the eigenvalues are in *resonance* if they satisfy the relation (1.7) with nonnegative integers m_i such that $m_1 + m_2 \geq 2$. We recall here that, in general, neither the normal form nor the normalizing transformation are unique. Anyway, it is well known that normal forms have been proven to be among the most useful tools in the qualitative and quantitative local analysis of ordinary differential equations.

In general, the normalizing transformation is just a formal power series and only special conditions can ensure its local convergence. Therefore, the normal form of an analytic system (1.1) is analytic only under some conditions. The most simple case that gives a sufficient condition which ensures the convergence of the normal form is given by the following criterium of Poincaré.

Theorem 1.3 *If the eigenvalues λ_1 and λ_2 of the matrix J of the linear part of an analytic system (1.1) near the origin belong to the Poincaré domain (that is, the convex hull of the points $\{\lambda_1, \lambda_2\}$ in the complex plane $\mathbb{C}$ does not contain zero), then there is an analytic normalizing transformation and so an analytic normal form.*

A particular case of a normal form transformation is the so called *linearization*. System (1.1) given in the form $\dot{\mathbf{x}} = J\mathbf{x} + \mathbf{F}(\mathbf{x})$ is said to be smoothly *linearizable* if there exists a near–identity smooth coordinates change around the origin of the form $\mathbf{x} = \mathbf{y} + \mathbf{f}(\mathbf{y})$, where $\mathbf{f}(\mathbf{y}) = \mathcal{O}(\|\mathbf{y}\|^2)$, bringing system (1.1) into $\dot{\mathbf{y}} = J\mathbf{y}$. On the other hand, when system (1.1) can be brought under the former trnasformation to the form $\dot{\mathbf{y}} = J\mathbf{y}h(\mathbf{y})$, with h a scalar function such that $h(\mathbf{0}) = 1$, system (1.1) is said to be smoothly *orbitally linearizable.*

The next theorem contains a significant part of the local theory of planar analytic systems. Its proof or references to them can be found in [2].

Theorem 1.4 (Normal Forms) *Let $\dot{x} = \lambda_1 x + \cdots$, $\dot{y} = \lambda_2 x + \cdots$, be an analytic system near the isolated elementary ($\lambda_2 \neq 0$) singular point at the origin (here the dots denote higher order terms). There exists a smooth near–identity change of coordinates $(X, Y) = \phi(x, y) = (x + \cdots, y + \cdots)$ such that:*

(a) *If $\lambda_1/\lambda_2 \notin \mathbb{Q}$ (non–resonant point), then exactly two analytic invariant curves (separatrices) pass through the origin. Moreover, the system is smoothly equivalent to its linear part $\dot{X} = \lambda_1 X$, $\dot{Y} = \lambda_2 Y$.*

(b) *If $\lambda_1/\lambda_2 = n \in \mathbb{Z}^+$ positive integer ($n : 1$ resonant node) then $\phi(x, y)$ is analytic and the system is analytically equivalent to $\dot{X} = nX + \delta Y^n$, $\dot{Y} = Y$ with δ a constant.*

(c) *If* $\lambda_1/\lambda_2 = p/q \in \mathbb{Q}^+\backslash\mathbb{Z}^+$ *with* $p/q > 1$ *and* p *and* q *relatively primes positive integers (*$p : q$ *resonant node) then* $\phi(x, y)$ *is analytic and the system is analytically equivalent to its linear part* $\dot{X} = pX$, $\dot{Y} = qY$.

(d) *If* $\lambda_1/\lambda_2 = -q/p \in \mathbb{Q}^-$ *with* p *and* q *relatively primes (*$p : q$ *resonant saddle) then the system has two analytic invariant curves passing through the origin. Moreover, it is smoothly orbitally equivalent to*

$$\dot{X} = qX\left[1 + \delta(U^k + aU^{2k})\right] \ , \ \dot{Y} = -pY \ ,$$

with $U = X^pY^q$, $a \in \mathbb{R}$, $k \in \mathbb{Z}^+$ *and* $\delta \in \{0, \pm 1\}$.

(e) *If* $\lambda_1 = 0 \neq \lambda_2$ *(semi–hyperbolic point) then there are two distinguished trajectories: an analytic separatrix corresponding to the eigenvalue* λ_2 *and a center manifold which may be not analytic and not unique. If the center manifold is analytic then it is unique (others are not analytic). Moreover, the system is smoothly orbitally equivalent to* $\dot{X} = X^{\ell+1}/\ell$, $\dot{Y} = Y(1 + bX^\ell)$ *with* b *a constant and* ℓ *a positive integer.*

1.4 Lie groups and differential equations

Roughly speaking, a symmetry for a given differential equation is a special transformation that leaves the differential equation invariant. Most well-known techniques for solving differential equations are based in symmetry methods. The Norwegian mathematician Sophus Lie was the founder of the theory of continuous groups and their applications to the theory of differential equations. His investigations led to one of the major branches of 20th-century mathematics, the theory of Lie groups and Lie algebras. Continuous groups, now called after him Lie groups, have had a profound impact on many areas of mathematics as well as physics.

The application of Lie's continuous symmetry groups include such diverse fields as differential geometry, invariant theory, bifurcation theory, classical and quantum mechanics and so on.

When dealing with differential equations $dy/dx = w(x, y)$, one very often tries to obtain an appropriate change of variables $x^* = x^*(x, y)$, $y^* = y^*(x, y)$, in such a way that the differential equation remains invariant. This means that $dy^*/dx^* = w(x^*, y^*)$. In this case, the diffeomorphism $x^* = x^*(x, y)$,

$y^* = y^*(x, y)$ is called a symmetry point transformation for the differential equation. The symmetry maps points (x, y) into points (x^*, y^*).

In the context of Lie symmetries, we have to consider point transformations that depend on (at least) one arbitrary real parameter ϵ, that is, $x^* = x^*(x, y; \epsilon)$, $y^* = y^*(x, y; \epsilon)$. Moreover, we will need to introduce the group structure in those 1–parameter point transformations. In what follows we give a proper definition of a continuous group of transformations.

Let $\mathbf{x} = (x, y) \in D \subset \mathbb{R}^2$. The set of transformations $\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon)$ defined for each $\mathbf{x}$ in D, depending on a parameter $\epsilon \in I \subset \mathbb{R}$ with a law of composition $\phi : I \times I \to I$, forms a *continuous group of transformations* on D if

(i) For each parameter $\epsilon \in I$ the transformations are one-to-one onto D. In particular $\mathbf{x}^* \in D$.

(ii) The interval I with the law of composition ϕ forms a group.

(iii) $\mathbf{x}^* = \mathbf{x}$ when ϵ is the identity element of the group (I, ϕ).

(iv) If $\mathbf{x}^* = \mathbf{X}(\mathbf{x}; \epsilon)$ and $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}^*; \delta)$, then $\mathbf{x}^{**} = \mathbf{X}(\mathbf{x}; \phi(\epsilon, \delta))$.

A continuous group of transformations defines a *one-parameter Lie group of transformations* if in addition to the above axioms **(i)-(iv)** it satisfies

(v) $\mathbf{X} \in C^\infty(D)$ with respect to $\mathbf{x}$ and moreover is analytic on I with respect to ϵ.

(vi) $\phi(\epsilon, \delta)$ is an analytic function of ϵ and δ.

The symmetry is regarded as a mapping of the (x, y) plane to itself, called the action of the symmetry on the (x, y) plane. The action of a one–parameter Lie group can be seen as motion in the (x, y) plane. The orbit of the Lie group through (x_0, y_0) is the set of points to which (x_0, y_0) can be mapped by a suitable choice of ϵ. Figure 1.2 represents the action of a one–parameter Lie group of transformations. The picture shows how the point (x_0, y_0) is transformed along some planar curve when the parameter ϵ varies. Repeating this with different initial points we can visualize the orbits of the one–parameter Lie group of transformations. Hence, an one-parameter Lie group of transformations in effect defines a stationary flow and the orbits of

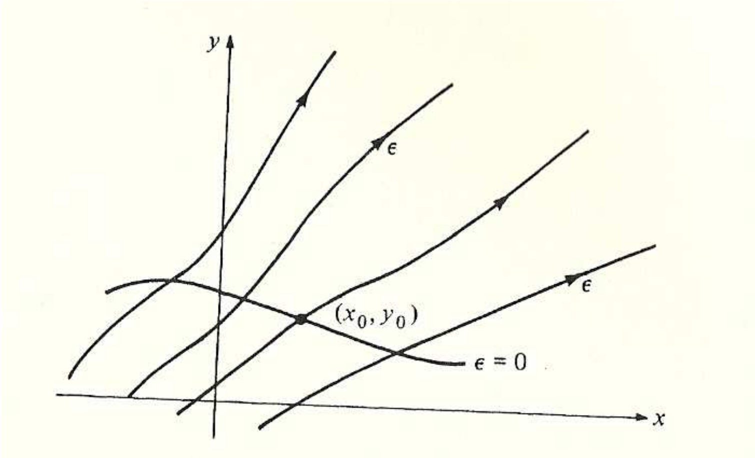


Figure 1.2: Action of a 1–parameter group of transformations.

the one–parameter Lie group can be characterized by the field of its tangent vectors.

More precisely, the first fundamental theorem of Lie is the following one.

Theorem 1.5 *There exists a parametrization $\tau(\epsilon)$ such that the Lie group of transformations $\mathbf{x}^* = \mathbf{X}(\mathbf{x};\epsilon)$ is equivalent to the solution of the initial value problem*

$$\frac{d\mathbf{x}^*}{d\tau} = \mathcal{Y}(\mathbf{x}^*)\ , \quad \mathbf{x}^*(0) = \mathbf{x}\ .$$

In view of this theorem, from now on, without loss of generality, we assume $\phi(\epsilon,\delta) = \epsilon + \delta$. The vector field $\mathcal{Y}(\mathbf{x}) = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ is called *infinitesimal generator* of the one-parameter Lie group. The main advantage in using the generators instead of the finite group transformations is that the generators are linear operators.

For the differential equation $dy/dx = w(x,y) = Q(x,y)/P(x,y)$ with P and Q of class $C^1(U)$, we can always introduce a new time independent variable t and rewrite the differential equation as the C^1-differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$. We call $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ the vector field associate to the former differential system. In this situation, it is well known that the Lie group of point transformations $x^* = x^*(x,y;\epsilon)$, $y^* = y^*(x,y;\epsilon)$ with infinitesimal generator $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ is a Lie point symmetry of the differential system $\dot{x} = P(x,y)$, $\dot{y} = Q(x,y)$ if and only if the commutation relation $[\mathcal{X}, \mathcal{Y}] = \nu(x,y)\mathcal{X}$ holds in U for some scalar function ν. Here $[\mathcal{X}, \mathcal{Y}] = \mathcal{X}\mathcal{Y} - \mathcal{Y}\mathcal{X}$ denotes the *Lie bracket* of two C^1–vector fields. Taking coordinates, the definition of Lie bracket is the following.

$$[\mathcal{X}, \mathcal{Y}] = \left(P\frac{\partial \xi}{\partial x} - \xi\frac{\partial P}{\partial x} + Q\frac{\partial \xi}{\partial y} - \eta\frac{\partial P}{\partial y} \right) \partial_x + \left(P\frac{\partial \eta}{\partial x} - \xi\frac{\partial Q}{\partial x} + Q\frac{\partial \eta}{\partial y} - \eta\frac{\partial Q}{\partial y} \right) \partial_y.$$

From a classical result, see the book of Olver [51], in the particular case that $[\mathcal{X}, \mathcal{Y}] \equiv 0$ in U, the local flows generated by the vector fields $\mathcal{X}$, $\mathcal{Y}$ *commute*, that is, one has $\phi(t, \psi(\tau, x_0, y_0)) = \psi(\tau, \phi(t, x_0, y_0))$ in U for any couple of real numbers t and τ whenever both $\phi(t, \psi(\tau, x_0, y_0))$ and $\psi(\tau, \phi(t, x_0, y_0))$ exist, where $\phi(t, x_0, y_0)$ and $\psi(\tau, x_0, y_0)$ are the flows generated by the vector fields $\mathcal{X}$, $\mathcal{Y}$.

It is well known, see for instance [51], that if $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ is an infinitesimal generator of a Lie symmetry of $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$, then the wedge product $\mathcal{X} \wedge \mathcal{Y} := P\eta - Q\xi$ is an inverse integrating factor V of $\mathcal{X}$ defined in U provided $V(x,y) \not\equiv 0$. So, in the particular case of commutation ($\nu \equiv 0$), we get that $\mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor of both $\mathcal{X}$ and $\mathcal{Y}$. Conversely, given an inverse integrating factor V of $\mathcal{X}$, we can get an infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of $\mathcal{X}$ as

$$\mathcal{Y} = \frac{1}{\mathrm{div}\mathcal{X}} \left(-\frac{\partial V}{\partial y}\partial_x + \frac{\partial V}{\partial x}\partial_y\right), \tag{1.8}$$

which is defined in $U \backslash \{(x,y) \in U : \mathrm{div}\mathcal{X} = 0\}$, see [63].

1.5 Symmetries in the center problem

Consider the analytic system (1.1) defined in an open set $U \subset \mathbb{R}^2$ of an isolated singular point that we can choose, without loss of generality, to be at the origin. Assume that the singular point is *nondegenerate*, that is, the eigenvalues of the Jacobian matrix $D\mathcal{X}(0,0)$ are different from zero.

A singular point of system (1.1) is called *monodromic* if there are no orbits tending to or leaving it forming a certain angle. For analytic systems, a monodromic singular point is always a center or a focus. This claim has been proved at the beginning of the nineties, see [43]. We recall that the origin is a *focus* if there exists a neighborhood of it such that each trajectory spirals around it and we say that the origin is a *center* if there exists a neighborhood of it such that each trajectory is a periodic orbit surrounding it.

It is well-known, see for instance [52], that if both eigenvalues of $D\mathcal{X}(0,0)$ are non zero complex conjugates $\alpha \pm i\beta$ then, the singular point is a center or a focus. More specifically, if $\alpha \neq 0$ the origin is a *strong focus*. Otherwise, if $\alpha = 0$, the origin is a center or a *weak focus*.

When the origin is a nondegenerate singular point and the eigenvalues of $D\mathcal{X}(0,0)$ are not pure imaginary, the Hartman –Grobman Theorem shows us that, in a sufficiently small neighborhood of it, the system is topologically equivalent to its linear part. That is, when the origin is a weak focus or a center, the Hartman –Grobman Theorem does not work.

The *center problem* asks for the conditions under which system (1.1) having eigenvalues of $D\mathcal{X}(0,0)$ pure imaginary at the origin, really is a center. There exists a second dynamical problem which consists in discerning when the periodic orbits surrounding the center have the same period. In that case the center is said to be *isochronous*.

If the origin of system (1.1) is monodromic and nondegenerate, we can always make a linear transformation bringing system (1.1) to the form,

$$\dot{x} = -y + \lambda x + p(x,y), \ \dot{y} = x + \lambda y + q(x,y), \tag{1.9}$$

where $p(x,y)$ and $q(x,y)$ are analytic in a neighborhood $U \subset \mathbb{R}^2$ of the origin and starting at order equal or greater than two. Let $\mathcal{X} = (-y + \lambda x + p(x,y))\partial_x + (x + \lambda y + q(x,y))\partial_y$ be its associated vector field. The case $\lambda = 0$ correspond to a weak focus or a center.

The study of the center problem has been the focus of attention of many researchers along decades. Some important methods have been developed to attack the problem in order to find necessary conditions to have a center, that is, computing the *Lyapunov constants*, see [41] and [53]. The main purpose of this section is to summarize and to show the role that have been played the Lie symmetries in distinguishing between a center or a focus in system (1.9).

The following result, proved in [4], is due to Algaba, Freire and Gamero and characterizes centers in terms of Lie brackets.

Theorem 1.6 *System* (1.9) *with* $\lambda = 0$ *has a center at the origin if, and only if, there exists an analytic vector field in a neighborhood* $U \subset \mathbb{R}^2$ *of the* *origin of the form* $\mathcal{Y} = (x + A(x,y))\partial_x + (y + B(x,y))\partial_y$ *with* $A(x,y)$ *and* $B(x,y)$ *of order equal or greater than two such that* $[\mathcal{X}, \mathcal{Y}] = \nu\mathcal{X}$ *with* $\nu(x,y)$ *an analytic function in* U *such that* $\nu(0,0) = 0$.

Villarini in [61] showed that if the analytic functions $P(x,y)$ and $Q(x,y)$ of system (1.1) are conjugated harmonic functions, i.e., satisfying the Cauchy-Riemann equations $P_x = Q_y$ and $P_y = -Q_x$, then the local flows of the vector field $\mathcal{X} = P\partial_x + Q\partial_y$ and its orthogonal vector field $\mathcal{X}^\perp = Q\partial_x - P\partial_y$ commute, that is $[\mathcal{X}, \mathcal{X}^\perp] \equiv 0$. After that, Sabatini improved in [54] that result replacing the orthogonal condition by the transversal one, i.e, $\mathcal{X} \wedge \mathcal{Y} \neq 0$ in $U \backslash \{(0,0)\}$. He gives the following version of Theorem 1.6 for isochronous centers showing the equivalence between commutation and isochronicity.

Theorem 1.7 *System* (1.9) *with* $\lambda = 0$*, has an isochronous center at the origin if, and only if, there exists an analytic vector field* $\mathcal{Y}$ *in* U *transversal to* $\mathcal{X}$ *in* U *such that* $[\mathcal{X}, \mathcal{Y}] \equiv 0$.

The result of Sabatini, that establishes the equivalence between isochronicity and the existence of transversal commutators, was extended recently for Freire, Gasull and Guillamon in [28]. They show that not only commutators are connected to isochronicity, but also a subset of Lie symmetries of $\mathcal{X}$ characterizes the isochronicity of a center. Given an infinitesimal generator $\mathcal{Y}$ of a Lie Symmetry of $\mathcal{X}$, i.e, $[\mathcal{X}, Y] = \nu\mathcal{X}$, they provided the following necessary and sufficient isochronicity condition based on ν.

Theorem 1.8 *Assume that the vector field* $\mathcal{X}$ *associated to system* (1.9) *with* $\lambda = 0$ *has a center at the origin with period annulus* $\mathcal{P} \subset \mathbb{R}^2$*. Let* $\mathcal{Y}$ *be a* C^1 *vector field transversal to* $\mathcal{X}$ *in* $\mathcal{P} \backslash \{(0,0)\}$ *such that* $[\mathcal{X}, \mathcal{Y}] = \nu\mathcal{X}$ *with* ν *some* C^1 *scalar function. Let* $\gamma(t) \in \mathcal{P}$ *be any periodic orbit of* $\mathcal{X}$ *and denote by* T_γ *its period. Hence, the center is isochronous if and only if there is a neighborhood of the origin such for any* γ *contained in it,* $\int_0^{T_\gamma} \nu(\gamma(t))dt = 0$.

A survey about isochronicity is [19]. The following theorem goes back to Poincaré and Liapunov.

Theorem 1.9 *System* (1.9) *with* $\lambda = 0$ *has a center at the origin if and only if there exists a near-identity analytic change of coordinates* $(u,v) = (x + o(|(x,y)|), y + o(|(x,y)|))$ *transforming system* (1.9) *into the normal form* $\dot{u} = -v[1 + \Psi(u^2 + v^2)]$, $\dot{v} = u[1 + \Psi(u^2 + v^2)]$*, with* Ψ *an analytic function near the origin such that* $\Psi(0) = 0$.

In the particular case that $\Psi(u^2+v^2) \equiv 0$, the transformed system is $\dot{u} = -v$, $\dot{v} = u$. Then, the origin of system (1.9) is an isochronous center because all the orbits in the period annulus have the same period coinciding with the period 2π of the harmonic oscillator. For a proof of this fact see also [47].

As a consequence of the former theorems one has the two following corollaries.

Corollary 1.10 *System* (1.9) *with* $\lambda = 0$ *is analytically orbitally linearizable around the origin if there exists an analytic vector field near the origin of the form* $\mathcal{Y} = (x + A(x,y))\partial_x + (y + B(x,y))\partial_y$, *with* $A(x,y)$ *and* $B(x,y)$ *of order equal or greater than two, and an analytic function* $\nu(x,y)$ *such that* $[\mathcal{X}, \mathcal{Y}] = \nu\mathcal{X}$ *with* $\nu(0,0) = 0$. *Moreover, the origin becomes a center.*

Corollary 1.11 *A center of system* (1.9) *is analytically linearizable if there exists an analytic vector field near the origin of the form* $\mathcal{Y} = (x+A(x,y))\partial_x + (y + B(x,y))\partial_y$, *with* $A(x,y)$ *and* $B(x,y)$ *of order equal or greater than two such that* $[\mathcal{X}, \mathcal{Y}] = 0$. *Moreover, the center is isochronous.*

The above corollary shows the equivalence between analytic linearization for centers and commutation, but to obtain an explicit linearizing change of coordinates is not an easy task. In Chapter 2 we study this problem and present a procedure to obtain the linearization of analytic isochronous centers from a given commutator. Moreover we show the linearization of some examples extracted from the existent literature.

1.6 The role of symmetries in normal forms

In this section we will assume always that system (1.1) is written in the form $\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{F}(\mathbf{x})$ with $\mathbf{x} = (x,y)^T \in U \subseteq \mathbb{R}^2$, where function $\mathbf{F} : U \to U$ is analytic near the origin and represents the nonlinear terms of the system. Here, $A \in \mathcal{M}_2(\mathbb{R})$ is a 2×2 semisimple non–zero matrix with entries in $\mathbb{R}$. Moreover, we write a normal form of system (1.1) as $\dot{\mathbf{x}} = A\mathbf{x} + \tilde{\mathbf{F}}(\mathbf{x})$ with $\tilde{\mathbf{F}}(\mathbf{x})$ having only resonant terms.

We shall use moreover the following notation. $\mathcal{X}_A$ will be the linear vector field with associated matrix $A = (a_{ij}) \in \mathcal{M}_2(\mathbb{R})$, that is, $\mathcal{X}_A = (a_{11}x + a_{12}y)\partial_x + (a_{21}x + a_{22}y)\partial_y$. Therefore, the analytic vector field $\mathcal{X}$ associated to system (1.1) is expressed as $\mathcal{X} = \mathcal{X}_A + \mathcal{X}_{\mathbf{F}}$ where $\mathcal{X}_{\mathbf{F}}$ is the vector field containing only the nonlinear terms of (1.1). Analogously, $\tilde{\mathcal{X}}$ will denote the formal vector field associated to the formal normal form of system (1.1), that is, $\tilde{\mathcal{X}} = \mathcal{X}_A + \mathcal{X}_{\tilde{\mathbf{F}}}$ where $\mathcal{X}_{\tilde{\mathbf{F}}}$ is the formal vector field containing only the nonlinear resonant terms.

There is another characterization of the resonant terms of a normal form in terms of Lie symmetries, see [24].

Proposition 1.12 *The vector field $\tilde{\mathcal{X}} = \mathcal{X}_A + \mathcal{X}_{\tilde{\mathbf{F}}}$ is in normal form if and only if $[\tilde{\mathcal{X}}, \mathcal{X}_A] = 0$.*

A particular consequence of Proposition 1.12 is that the vector fields $\tilde{\mathcal{X}} = \mathcal{X}_A + \mathcal{X}_{\tilde{\mathbf{F}}}$ with $\mathcal{X}_{\tilde{\mathbf{F}}} \not\equiv 0$ which are in normal form always admit the nontrivial linear commuting vector field $\mathcal{X}_A$. Otherwise, when $\tilde{\mathcal{X}} = \mathcal{X}_A$ the same property holds because $[\mathcal{X}_A, \mathcal{X}_I] = 0$ with I the identity matrix 2×2. In fact, we can say more since $[\mathcal{X}_A, \mathcal{X}_{A^k}] = 0$ for any integer $k \geq 0$.

The following is a fundamental property of normal forms. Walcher stated and proved it in [62].

Proposition 1.13 *Let $\tilde{\mathcal{X}} = \mathcal{X}_A + \mathcal{X}_{\tilde{\mathbf{F}}}$ be a vector field in normal such that $[\tilde{\mathcal{X}}, \mathcal{Y}] = 0$, where $\mathcal{Y} = \mathcal{X}_B + \mathcal{X}_{\mathbf{G}}$. Then $[\mathcal{X}_A, \mathcal{Y}] = 0$. Moreover, every first integral of $\tilde{\mathcal{X}}$ is also a first integral of $\mathcal{X}_A$.*

The following result is an illustration of how normal forms may be influenced by Lie symmetries, see [5].

Proposition 1.14 *The analytic vector field $\mathcal{X} = \mathcal{X}_A + \mathcal{X}_{\mathbf{F}}$ can be formally linearized if and only if there is a formal vector field $\mathcal{Y} = \mathcal{X}_I + \mathcal{X}_{\mathbf{G}}$ such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$. If in addition $\mathcal{Y}$ is analytic, then $\mathcal{X}$ is analytically linearizable.*

In the two–dimensional setting the next theorem is a central result about convergence of normal forms, see [24].

Theorem 1.15 *Let $\mathcal{X} = \mathcal{X}_A + \mathcal{X}_{\mathbf{F}}$ be a planar analytic vector field having an elementary singularity at the origin, that is, the matrix A has, at least, an eigenvalue different from zero. If there exists a nontrivial analytic commuting vector field $\mathcal{Y} = \mathcal{X}_B + \mathcal{X}_{\mathbf{G}}$ with $[\mathcal{X}, \mathcal{Y}] \equiv 0$, then $\mathcal{X}$ possesses a convergent normalizing transformation.*

Finally, in the context of the orbital normal forms for planar analytic vector fields, one has the next result.

Proposition 1.16 *Let $\mathcal{X} = \mathcal{X}_A + \mathcal{X}_{\mathbf{F}}$ be a planar analytic vector field where the eigenvalues λ_1 and λ_2 of A satisfy either $\lambda_2/\lambda_1 = -q/p$ with p and q positive and relatively prime integers or $\lambda_1 \neq 0$ and $\lambda_2 = 0$. Then, there is an analytic scalar function h with $h(0,0) \neq 0$ such that $h\mathcal{X}$ admits a convergent transformation to normal form, if and only if, $\mathcal{X}$ admits an analytic inverse integrating factor near the origin.*

Recently, Giné and Grau have generalized the criteria to detect centers stated in Corollary 1.10 in order to detect smooth (analytic) orbital linearization for other types of singular points in smooth or analytic systems. The result, stated and proved in [35], gives the equivalence between the orbital linearizability of a smooth (analytic) vector field and the fact of having a smooth (analytic) infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of the form $\mathcal{Y} = \mathcal{X}_I + \mathcal{X}_{\mathbf{G}}$.

Theorem 1.17 *Consider the smooth (analytic) vector field $\mathcal{X} = \mathcal{X}_A + \mathcal{X}_{\mathbf{F}}$ on $\mathbb{C}^2$ defined in a neighborhood of an isolated nondegenerate singular point. Then, $\mathcal{X}$ is smoothly (analytically) orbitally linearizable if, and only if, there exists a smooth (analytic) vector field near the singular point of the form $\mathcal{Y} = \mathcal{X}_I + \mathcal{X}_{\mathbf{G}}$ such that $[\mathcal{X}, \mathcal{Y}] \equiv \nu\mathcal{X}$ with ν a smooth (analytic) scalar function such that $\nu(0,0) = 0$.*

In the same work, the authors generalize Propositon 1.14 for smooth (analytic) systems. The proofs given in [35] of these facts are not constructive, that is, the authors do not find a constructive way to build this change of coordinates. In Chapter 3, we give a constructive procedure of the smooth linearizing change of coordinates around an isolated nondegenerate singular point of a smooth system from symmetries. Thus, in Chapter 3 we extend the procedure obtained in Chapter 2 for isochronous centers to other types of singular points. Chapter 4 is devoted to the smooth orbital linearization problem; a method to get the normal form transformation is given.

1.7 The algebra of Lie point symmetries

Let us consider a second order differential equations

$$\ddot{x} = w(t, x, \dot{x}) \ , \tag{1.10}$$

with w a smooth function in all of its arguments and $(t, x) \in \mathbb{R} \times U$ with $U \subset \mathbb{R}$ an open interval. The ideas explained here can be extended to differential equations of higher order, but for simplicity we only state it in the second order setting.

A symmetry of (1.10) is a diffeomorphism $\Phi : (t, x) \mapsto (\bar{t}, \bar{x})$ that maps the set of solutions of (1.10) into itself. Therefore, the symmetry condition of (1.10) reads for

$$\bar{x}'' = w(\bar{t}, \bar{x}, \bar{x}') \tag{1.11}$$

where the prime denotes the derivative $' = d/d\bar{t}$. In general, this symmetry condition is too hard to be solved for Φ.

When the symmetry is a 1–parameter Lie group of point transformations Φ_ϵ, then

$$\bar{t} = t + \epsilon\xi(t, x) + O(\epsilon^2) \ , \ \bar{x} = x + \epsilon\eta(t, x) + O(\epsilon^2) \ , \tag{1.12}$$

for ϵ close to zero, and the vector field $\mathcal{Y} = \xi(t, x)\partial_t + \eta(t, x)\partial_x$ is called the *infinitesimal generator* of the 1–parameter Lie group of point transformations Φ_ϵ. In this case, the Lie symmetries (and this is the main difference with respect to discrete symmetries) are obtained by linearizing (1.11) near $\epsilon = 0$ taking into account (1.10). To do this, we need the first terms in the expressions of $\bar{x}'$ and $\bar{x}''$ as Taylor power series of ϵ. In short, we have

$$\begin{aligned}
\bar{x}' &= \frac{d\bar{x}}{d\bar{t}} = \frac{dx + \epsilon\left(\frac{\partial\eta}{\partial t}dt + \frac{\partial\eta}{\partial x}dx\right) + O(\epsilon^2)}{dt + \epsilon\left(\frac{\partial\xi}{\partial t}dt + \frac{\partial\xi}{\partial x}dx\right) + O(\epsilon^2)} = \frac{\dot{x} + \epsilon\left(\frac{\partial\eta}{\partial t} + \frac{\partial\eta}{\partial x}\dot{x}\right) + O(\epsilon^2)}{1 + \epsilon\left(\frac{\partial\xi}{\partial t}1 + \frac{\partial\xi}{\partial x}\dot{x}\right) + O(\epsilon^2)} \\
&= \dot{x} + \eta^{[1]}(t, x, \dot{x})\epsilon + O(\epsilon^2) \ , \\
\bar{x}'' &= \frac{d\bar{x}'}{d\bar{t}} = \frac{d\dot{x} + \epsilon\left(\frac{\partial\eta^{[1]}}{\partial t}dt + \frac{\partial\eta^{[1]}}{\partial x}dx + \frac{\partial\eta^{[1]}}{\partial\dot{x}}d\dot{x}\right) + O(\epsilon^2)}{dt + \epsilon\left(\frac{\partial\xi}{\partial t}dt + \frac{\partial\xi}{\partial x}dx\right) + O(\epsilon^2)} \\
&= \frac{\ddot{x} + \epsilon\left(\frac{\partial\eta^{[1]}}{\partial t} + \frac{\partial\eta^{[1]}}{\partial x}\dot{x} + \frac{\partial\eta^{[1]}}{\partial\dot{x}}\ddot{x}\right) + O(\epsilon^2)}{1 + \epsilon\left(\frac{\partial\xi}{\partial t}1 + \frac{\partial\xi}{\partial x}\dot{x}\right) + O(\epsilon^2)} \\
&= \ddot{x} + \eta^{[2]}(t, x, \dot{x}, \ddot{x})\epsilon + O(\epsilon^2) \ ,
\end{aligned}$$

where we have defined the functions

$$\begin{aligned}
\eta^{[1]}(t, x, \dot{x}) &= \frac{\partial\eta}{\partial t} + \left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial t}\right)\dot{x} - \frac{\partial\xi}{\partial x}\dot{x}^2 \ , \\
\eta^{[2]}(t, x, \dot{x}, \ddot{x}) &= \frac{\partial^2\eta}{\partial t^2} + \left(2\frac{\partial^2\eta}{\partial t\partial x} - \frac{\partial^2\xi}{\partial t^2}\right)\dot{x} + \left(\frac{\partial^2\eta}{\partial x^2} - 2\frac{\partial^2\xi}{\partial t\partial x}\right)\dot{x}^2 - \frac{\partial^2\xi}{\partial x^2}\dot{x}^3
\end{aligned}$$

$$+\left(\frac{\partial\eta}{\partial x}-2\frac{\partial\xi}{\partial t}-3\frac{\partial\xi}{\partial x}\dot{x}\right)\ddot{x}\ .$$

Using the former expressions we obtain the next Taylor expansion near $\epsilon = 0$.

$$\begin{aligned} w(\bar{t},\bar{x},\bar{x}') &= w(t+\epsilon\xi(t,x)+O(\epsilon^2), x+\epsilon\eta(t,x)+O(\epsilon^2), \dot{x}+\eta^{[1]}(t,x,\dot{x})\epsilon+O(\epsilon^2)) \\ &= w(t,x,\dot{x})+\epsilon\left(\frac{\partial w}{\partial t}\xi+\frac{\partial w}{\partial x}\eta+\frac{\partial w}{\partial \dot{x}}\eta^{[1]}\right)+O(\epsilon^2)\ . \end{aligned}$$

Finally, substituting these formulas into (1.11) and recalling (1.10), we obtain equating the coefficients of ϵ that $\eta^{[2]} = \frac{\partial w}{\partial t}\xi+\frac{\partial w}{\partial x}\eta+\frac{\partial w}{\partial \dot{x}}\eta^{[1]}$. This condition is called the determining equations for Lie point symmetries of (1.10) and can be rewritten (see for instance the books [9, 38, 39, 51, 57]) in compact form as

$$\mathcal{Y}^{[2]}(\ddot{x}-w(t,x,\dot{x})) = 0 \text{ when } \ddot{x} = w(t,x,\dot{x})\ , \tag{1.13}$$

where $\mathcal{Y}^{[2]} = \mathcal{Y}+\eta^{[1]}(t,x,\dot{x})\partial_{\dot{x}}+\eta^{[2]}(t,x,\dot{x},\ddot{x})\partial_{\ddot{x}}$ is the so–called *second prolongation* of the infinitesimal generator $\mathcal{Y}$ and $\eta^{[1]}(t,x,\dot{x}) = D_t\eta-\dot{x}D_t\xi$, $\eta^{[2]}(t,x,\dot{x},\ddot{x}) = D_t\eta^{[1]}-\ddot{x}D_t\xi$ where $D_t = \partial_t+\dot{x}\partial_x+\ddot{x}\partial_{\dot{x}}$ is the operator total derivative with respect to t.

On the contrary, the expanded expression of the *determining equations* is given by

$$\begin{aligned} &\frac{\partial^2\eta}{\partial t^2}+\left(2\frac{\partial^2\eta}{\partial t\partial x}-\frac{\partial^2\xi}{\partial t^2}\right)\dot{x}+\left(\frac{\partial^2\eta}{\partial x^2}-2\frac{\partial^2\xi}{\partial t\partial x}\right)\dot{x}^2-\frac{\partial^2\xi}{\partial x^2}\dot{x}^3 = \\ &-\left(\frac{\partial\eta}{\partial x}-2\frac{\partial\xi}{\partial t}-3\frac{\partial\xi}{\partial x}\dot{x}\right)w+\xi\frac{\partial w}{\partial t}+\eta\frac{\partial w}{\partial x}+\left[\frac{\partial\eta}{\partial t}+\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial t}\right)\dot{x}-\frac{\partial\xi}{\partial x}\dot{x}^2\right]\frac{\partial w}{\partial\dot{x}}. \end{aligned} \tag{1.14}$$

Although (1.14) looks complicated, in most cases it is easily solved because both ξ and η are independent of $\dot{x}$. For instance, when $w(t,x,\dot{x})$ is a polynomial in $\dot{x}$, the determining equation is usually solved splitting into a system of partial differential equations by equating powers of $\dot{x}$.

In the particular case that (1.10) is autonomous, that is, $w(x,\dot{x})$ does not depend explicitly on the time t, it always admits the generator $\mathcal{Y} = \partial_t$ of a Lie point symmetry.

Let $\mathcal{L}_r$ be the set of all the infinitesimal generators of 1–parameter Lie group of point symmetries of a given differential equation (1.10). Then, $\mathcal{Y} = \xi(t,x)\partial_t+\eta(t,x)\partial_x \in \mathcal{L}_r$ if and only if $(\xi(t,x),\eta(t,x))$ is a pair of functions satisfying the determining equations (1.14). Since the determining equations (1.14) are linear with respect to ξ and η, it follows that if $\mathcal{Y}_i \in \mathcal{L}_r$ for $i = 1, 2$, then the linear combination $\alpha_1\mathcal{Y}_1 + \alpha_2\mathcal{Y}_2 \in \mathcal{L}_r$ for all $\alpha_i \in \mathbb{R}$. Therefore, $\mathcal{L}_r$ is a real vector space. In the following we denote the dimension of $\mathcal{L}_r$ by $r = \dim \mathcal{L}_r$. We observe that $\mathcal{L}_r$ is finite dimensional because the dimension r is just the number of arbitrary constants that appear in the general solution of the determining equations (1.14). The order of a differential equation places restrictions on the possible values of r. For example (see [38, 57]), for second order differential equation like (1.10), one has $r \in \{0, 1, 2, 3, 8\}$. Additionally, $r = 8$ if and only if (1.10) is linear or linearizable by a point transformation $(t, x) \mapsto (\bar{t}(t, x), \bar{x}(t, x))$ defined in certain domain.

Given $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{L}_r$, we define its *Lie bracket* or *commutator* in the usual way $[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Y}_1\mathcal{Y}_2 - \mathcal{Y}_2\mathcal{Y}_1$. It is easy to see that, the Lie bracket is independent of the coordinate system. Moreover, one can easily prove that if $[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Z}$, then $[\mathcal{Y}_1^{[2]}, \mathcal{Y}_2^{[2]}] = \mathcal{Z}^{[2]}$.

There is a new characterization of the symmetry condition (1.14) that leads to the following important result.

Theorem 1.18 *Given $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{L}_r$, then $[\mathcal{Y}_1, \mathcal{Y}_2] \in \mathcal{L}_r$.*

The proof of Theorem 1.18 is as follows. Given $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{L}_r$, equation (1.14) must be satisfied, that is, $\mathcal{Y}_i^{[2]}(\ddot{x} - w(t, x, \dot{x})) = 0$ when $\ddot{x} = w(t, x, \dot{x})$. It is easy to see that $\mathcal{Y}_i^{[2]}(\ddot{x} - w(t, x, \dot{x})) = \eta_i^{[2]}(t, x, \dot{x}, \ddot{x}) - \mathcal{Y}_i^{[2]}(w(t, x, \dot{x}))$. In addition, $\eta_i^{[2]}$ is linear in $\ddot{x}$, whereas w, and thus, $\mathcal{Y}_i^{[2]}(w)$, is independent of $\ddot{x}$. This means that (1.14) is satisfied if and only if $\mathcal{Y}_i^{[2]}(\ddot{x} - w(t, x, \dot{x})) = \lambda_i(\ddot{x} - w(t, x, \dot{x}))$, with $\lambda_i = \lambda_i(t, x, \dot{x}) = \partial\eta_i^{[2]}/\partial\ddot{x}$. Therefore if we define $\mathcal{Z}$ as the Lie bracket $[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Z}$, we get

$$\begin{aligned} \mathcal{Z}^{[2]}(\ddot{x} - w) &= [\mathcal{Y}_1^{[2]}, \mathcal{Y}_2^{[2]}](\ddot{x} - w) = \mathcal{Y}_1^{[2]}(\lambda_2(\ddot{x} - w)) - \mathcal{Y}_2^{[2]}(\lambda_1(\ddot{x} - w)) \\ &= (\mathcal{Y}_1^{[2]}\lambda_2 - \mathcal{Y}_2^{[2]}\lambda_1)(\ddot{x} - w) \ . \end{aligned}$$

Hence, $\mathcal{Z}^{[2]}(\ddot{x} - w) = 0$ when $\ddot{x} - w = 0$, and therefore $\mathcal{Z} \in \mathcal{L}_r$ proving Theorem 1.18.

Since Theorem 1.18 means that $\mathcal{L}_r$ is closed under the commutator operation, it follows that $\mathcal{L}_r$ is not only a vector space; it is a real *Lie algebra.*

In short, denoting by $\{\mathcal{Y}_1, \ldots, \mathcal{Y}_r\}$ a basis of $\mathcal{L}_r$, we will have $[\mathcal{Y}_i, \mathcal{Y}_j] = \sum_{k=1}^{r} c_{ij}^k \mathcal{Y}_k$ where the constants $c_{ij}^k \in \mathbb{R}$ are called the *structure constants* of the Lie algebra $\mathcal{L}_r$. In particular, if all the structure constants are zero then $\mathcal{L}_r$ is called Abelian.

Using Lie brackets we can write a different characterization of the symmetry condition (1.14) as follows. First, we associate to equation (1.10) the vector field $\mathcal{X} = \partial_t + \dot{x}\partial_x + w(t, x, \dot{x})\partial_{\dot{x}}$. Then, for any $\mathcal{Y}_i = \xi_i(t, x)\partial_t + \eta_i(t, x)\,\partial_x \in \mathcal{L}_r$, easily one can check that $[\mathcal{X}, \mathcal{Y}_i^{[1]}] = \mu_i(t, x, \dot{x})\mathcal{X}$ where $\mu_i(t, x, \dot{x}) = \mathcal{X}\xi_i$ and $\mathcal{Y}_i^{[1]} = \mathcal{Y}_i + \eta_i^{[1]}(t, x, \dot{x})\partial_{\dot{x}}$ is the first prolongation of $\mathcal{Y}$.

Sometimes it is difficult to know the maximal Lie point symmetry algebra $\mathcal{L}_r$ with maximal r for a given differential equation (1.10). Anyway, perhaps we can find a special solution $\mathcal{Y} = \xi(t, x)\partial_t + \eta(t, x)\,\partial_x$ of the determining equations (1.14). Even in this case, one can perform different approaches to simplify or integrate (1.10). For example one can make a reduction of the order of (1.10) or even to do a stepwise integration of (1.10). This systematic method is explained in all the text books on Lie symmetries.

1.8 Reversibility in dynamical systems

Reversibility is one of the fundamental symmetries discussed in natural sciences. Consequently, it arises in many physically motivated dynamical systems. A very good survey on this subject is [44].

The conventional notion of reversibility relates to observations of physical phenomena. A simple example is given by the classical ideal pendulum without friction. In the hamiltonian formulation of classical mechanics, the system is described with variables $(q, p) \in \mathbb{R}^{2n}$, position and momentum, whose evolution satisfies the hamiltonian equations of the motion $dq/dt = \partial H/\partial p$, $dp/dt = -\partial H/\partial q$. Then, the reversibility of the system is directly related to the fact that $H(q, p) = H(q, -p)$. The diffeomorphism $R(q, p) = (q, -p)$ is called a reversing symmetry for the pendulum. In other words, the hamiltonian equations of the motion are invariant under the transformation $(q, p, t) \mapsto (q, -p, -t)$. This implies that if $(q(t), p(t))$ is a trajectory in the phase space with initial condition $(q(0), p(0)) = (q_0, p_0)$ then so is $(q(-t), -p(-t))$ with initial condition $(q_0, -p_0)$. This last solution is just what we see when we play the above film on the pendulum in reverse time.

We recall that a pendulum of length ℓ and mass m in a gravitational constant field g possesses the hamiltonian $H(\theta, \dot{\theta}) = 1/2m\ell^2\dot{\theta}^2 + mg\ell(1-\cos\theta)$ and therefore $H(\theta, \dot{\theta}) = H(\theta, -\dot{\theta})$.

Devaney in [26] generalizes the concept of reversing symmetry allowing that any involution can be a reversing transformation. A precise definition of a planar reversible dynamical system is the following. We say that the smooth differential system (1.1) defined in an open set $U \subset \mathbb{R}^2$ is C^k*–reversible* with $k \in \{1, 2, \ldots, \infty, w\}$ if there is a C^k diffeomorphism $R : U \to U$ which is an involution, that is $R \circ R = Id$, and $R_* \mathcal{X} = -\mathcal{X} \circ R$, where $\mathcal{X}$ is the vector field associated to system (1.1). Notice that, if we denote by $\Phi_t : U \to U$ the associated flow of the differential system, then it follows $R \circ \Phi_t = \Phi_{-t} \circ R$ for all t.

Along this dissertation, when system (1.1) be reversible by means of an involution R, we will say that R is a reversing symmetry for system (1.1). If $\mathcal{X}$ is C^w–reversible, we will also say that $\mathcal{X}$ is analytically reversible.

After Devaney's result, Sevryuk remarked that reversible systems need not to have an involutory reversing symmetry. If the reversing symmetry is not required to be an involution, the dynamical system is called *weakly* reversible. Sevryuk and Arnold found that many results for reversible systems also hold for weakly reversible systems, see [56] and [1]. In this report we only deal with systems (1.1) having a reversing involutory symmetry whose action is such that the fixed points manifold $\mathrm{Fix}(R) = \{\, (x, y) \in \mathbb{R}^2 \mid R(x, y) = (x, y)\}$ is assumed to be of dimension 1.

The first examples of reversible systems were given by Poincaré, see [53]. Systems which are analytically reversible for the involution $R_0(x, y) = (-x, y)$ are called *time–reversible* systems. Notice that system (1.1) is time–reversible if and only if P is even and Q odd in x. Not every involution has such a simple form as in the example above. However, it is well known by Montgomery–Bochner Theorem in [49] that any smooth involution R is C^∞–conjugated to $R_0 = (-x, y)$.

One of the most important topics when one deals with reversible systems is looking for periodic orbits. A set which is invariant under the action of an involution R is called *symmetric* with respect to R. Thus, we say that a periodic orbit γ of a dynamical system is a symmetric periodic orbit with respect to R when the orbit is setwise invariant under R. If $\mathcal{X}$ is a reversible vector field and $u(t)$ is a solution of $\mathcal{X}$, then $R(u(-t))$ is also a solution of the system. This fact implies that if an orbit of a reversible system meets $\mathrm{Fix}(R)$ a two distinct points then it is a periodic orbit and symmetric respect to R.

Birkhoff in 1915 described the use of reversibility to find periodic orbits in his study of the restricted three–body problem. In 1958 Devogelaere described a method for searching symmetric periodic orbits of reversible systems by computer, see [25].

A particular property of reversible systems, is that their symmetric periodic orbits are typically not isolated. Periodic orbits in reversible systems generically arise in continuous families. Taking this into account and the fact that the reversibility seems to imply certain geometric symmetries, several authors have tried to establish the relation between reversibility and the center problem.

The following condition due to Poincaré assures us that the origin of system (1.9) is a center.

Theorem 1.19 *If the vector field $\mathcal{X}$ associated to system (1.9) is symmetric respect to the x–axis, then, the origin is a center.*

The above condition means that system (1.9) is invariant respect to the change $(x, y, t) \mapsto (x, -y, -t)$. In other words, if the vector field $\mathcal{X}$ of system (1.9) with $\lambda = 0$ is an analytically reversible vector field satisfying $p(x, y) = -p(x, -y)$ and $q(x, y) = q(x, -y)$, it has a center at the origin.

Recalling the local normal form of system (1.9) having a center at the origin given in Theorem 1.9, that is $\dot{u} = -v[1 + \Psi(u^2 + v^2)]$, $\dot{v} = u[1 + \Psi(u^2 + v^2)]$, with Ψ an analytic function near the origin and $\Psi(0) = 0$, it is clear that any analytic system having a nondegenerate center at the origin is time–reversible in suitable coordinates. Then, the original system (1.9) is reversible with respect to the involution $R = \phi^{-1} \circ R_0 \circ \phi$, where $\phi(x, y)$ is the near–identity analytic change of variables transforming system (1.9) into the normal form and $R_0(u, v) = (-u, v)$. This assertion is also stated by Teixeira and Yang in [60]. Thus, we have the following result.

Theorem 1.20 *System (1.9) has a center at the origin if and only if it is analytically reversible.*

Let us consider again the normal form of system (1.9) having a center at the origin stated in Theorem 1.9. Performing the change $z = u^2$, the normal form becomes, after removing a common factor, $\dot{z} = -2v$, $\dot{v} = 1$, which is a non-singular system at the origin. The curve of non-invertibility of the map $\Psi_0(u, v) = (z, v)$ is the v–axis, which is the set $\mathrm{Fix}(R_0)$ of fixed points of R_0. Notice that $\Psi_0 \circ R_0 = \Psi_0$. This means that the pull–back induced by Ψ_0 creates a symmetric system with respect to the v–axis. More precisely, the pull–back of Ψ_0 discards the left side of the plane (z, v) and creates a symmetric system by folding the right side into the left side of its phase portrait. In short, by pulling back a non-singular differential equation by means of the map Ψ_0 we obtain a symmetric differential system which has a center at the origin. We call Ψ_0 the fold mapping realizing the reversibility in the center.

Once settled the strong relationship between reversibility and the fact of having a nondegenerate center, the natural problem that arises is to determine the relation between reversibility and the fact of having a nilpotent center. We recall that a singular point is called nilpotent when zero is a double eigenvalue of the Jacobian matrix $D\mathcal{X}(0, 0)$ but the system has non vanishing linear part.

From the duality between vector fields and differential 1–forms, Moussu proved in [42] the existence of a C^∞ normal form possessing the reversing symmetry $R_0 = (-x, y)$ that can be achieved for any analytic nilpotent center after a rescaling of time. Generalizing that result, Berthier and Moussu show in [8] that there exists an analytic change of variables such that system (1.1) having a nilpotent center at the origin is transformed, after a reparametrization of the time, into a system invariant by the change of variables $(x, y, t) \mapsto (-x, y, -t)$.

It is worth to point out that the definitions of reversibility stated in the works of Moussu and Berthier imply that the foliations defined by the orbits of $\mathcal{X}$ are symmetric respect to the symmetry axis defined by $\mathrm{Fix}(R_0)$. In other words, they show that system (1.1) having a nilpotent center at the origin is orbitally equivalent to a time–reversible system, but observe that orbital reversibility does not implies reversibility.

Other authors have been proved analogous results recently. Stróżyna and Żołądek have proved in [58] that there exists an analytic change of coordinates near the origin transforming system (1.1) having a nilpotent singular point at the origin into a generalized Liénard system $\dot{x} = y$, $\dot{y} = a(x) + y\bar{b}(x)$ with $a(x) = a_s x^s + \cdots$, $s \geq 2$, and $\bar{b}(0) = 0$. In fact, if the nilpotent singularity is monodromic, then $s = 2n - 1$ with $n \geq 2$ and after the change $x \mapsto u$ with $u(x) = (2n \int_0^x a(z)dz)^{1/(2n)} = x(a_{2n-1} + \mathcal{O}(x))^{1/(2n)}$ and the reparametrization of the time $t \mapsto \tau$ with $dt/d\tau = u^{2n-1}/a(x) = a_{2n-1}^{-1/(2n)} + \mathcal{O}(x)$ the above normal form leads to $\dot{x} = -y$, $\dot{y} = x^{2n-1} + yb(x)$, where $b(x) = \sum_{j\geq\beta} b_j x^j$. Moreover, if we take into account the center conditions for the Liénard systems, the center problem for nilpotent singularities reduce to the study of the parity of the function $b(x)$. The following theorem can be established.

Theorem 1.21 *Suppose that the analytic system* (1.1) *has a nilpotent center at the origin. Then, there exists an analytic change of variables and an unity time rescaling such that it can be written as*

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1} + yb(x) \ , \tag{1.15}$$

with $n \geq 2$ an integer and $b(x)$ an analytic odd function.

It is straightforward to check that system (1.15) is reversible by means of the involution $R_0(x, y) = (-x, y)$.

To conclude the discussion about the relation between reversibility and analytic systems having center type singularities, the case that remains to study is the case of a system having a degenerate center at the origin. The study of this case is far from to be complete. The main reason is the lack of a normal form theory developed for degenerate singularities. Nevertheless, Cerveau, Berthier and Lins Neto proved in [7] the existence of analytic systems with a degenerate center without nontrivial reversal symmetry, i.e., without an associated local analytic involution.

Chapter 7 is devoted to the topic of the reversal symmetries in dynamical systems. First, we consider analytic systems having degenerated centers at the origin and show the existence of a smooth map around the center bringing the system into a linear reversible system (after rescaling the time). From the knowledge of the reversing symmetries of the reversible linear system we obtain reversing symmetries in the degenenerate center. We also deals with the integrability problem around singularities for some reversible systems.

1.9 Publications

The following is a list of the papers already published in journals with impact factor. All of them are part of this memory. More precisely, the articles published correspond to the works developed in Chapters 2, 3 and 4 of this dissertation.

1. I.A. García & S. Maza, *Linearization of analytic isochronous centers from a given commutator*, J. Math. Anal. Appl. **339** (2008), no. 1, 740–745.

2. I.A. García, J. Giné & S. Maza, *Linearization of smooth planar vector fields around singular points via commuting flows*, Commun. Pure Appl. Math. **7** (2008), no. 6, 1415 – 1428.

3. J. Giné & S. Maza, *Lie symmetries for the orbital linearization of smooth planar vector fields around singular points*, J. Math. Anal. Appl. **345** (2008), no. 1, 63–69.

   The results of the rest of the chapters have been submitted and they actually are under referee. In summary, we have submitted the following papers.

4. J. Giné & S. Maza, *Integrability and Linearizability in the quadratic Lotka-Volterra systems around singular points via Lie symmetries* .

5. I.A. García, J. Giné & S. Maza, *Dynamics of newtonian systems with 2–dimensional Lie point symmetry algebra: emphasis on polynomial Liénard systems*

6. J. Giné & S. Maza, *The reversibility and the center problem* .

Chapter 2

Linearization of Isochronous Centers From Symmetries

Summary. In this chapter we propose a procedure to get the change of variables that linearizes an analytic isochronous center from a given commutator. Moreover, we use this method in order to obtain the linearization of some isochronous centers of the existent literature.

2.1 Introduction

We consider two-dimensional analytic differential systems defined in a neighborhood $U \subset \mathbb{R}^2$ of an isolated singular point of non–degenerate *center* type, i.e. an isolated singular point with a punctured neighborhood filled of periodic orbits and with associated eigenvalues different from zero. We can do a translation of coordinates such that the critical point is located at the origin. Finally, making a linear change of coordinates (and a time rescaling if necessary), it is well known that the system can be written in the form

$$\dot{x} = -y + f(x,y) \ , \ \dot{y} = x + g(x,y) \ , \tag{2.1}$$

with f and g analytic functions in U starting in at least second order terms, i.e. such that $f(0,0) = g(0,0) = 0$ and $\partial_x f(0,0) = \partial_y f(0,0) = \partial_x g(0,0) = \partial_y g(0,0) = 0$. The vector field associated to system (2.1) will be denoted by $\mathcal{X} = (-y + f(x,y))\partial_x + (x + g(x,y))\partial_y$.

We are mainly concerned about the *isochronicity problem*, i.e., to determine whether the periodic orbits around the center have the same period (in this case the center is called *isochronous center*) or not. The main methods used in order to study isochronous center are classified in two categories. The first one, due to Sabatini and Villarini in [54] and [61] respectively, says that a center of an analytic system is isochronous if and only if there exists a commuting analytic vector field of the form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$. Here commuting means $[\mathcal{X}, \mathcal{Y}] \equiv 0$ where the bracket used is the Lie bracket. The second one says that a center of an analytic system is isochronous if and only if there exists an analytic near–identity change of variables $(u,v) = \phi(x,y) = (x + o(x,y)), y + o(x,y))$ that linearizes $\mathcal{X}$, i.e. such that $\phi_* \mathcal{X} = -v\partial_u + u\partial v$ where ϕ_* and ϕ^* is the push–forward and pull–back defined by the analytic diffeomorphism ϕ. This last approach has been mainly used in [46] and [47]. We emphasize that, even for concrete isochronous systems, it is not always an easy task to obtain an explicit commutator or an explicit linearization change. Although, looking for commutators, a exception is given in [13] for f and g polynomials and system (2.1) having either a polynomial or a rational first integral. See [35] for the equivalence between linearizability and commutation in the more general setting of smooth or analytic vector fields in a neighborhood of non–degenerate singular points in $\mathbb{C}^n$.

Before stating the main result of this chapter we remark that, from a linearizing change of coordinates ϕ, it is easy to get a commutator $\mathcal{Y}$ of $\mathcal{X}$. This commutator $\mathcal{Y}$ is just the vector field obtained by applying the inverse change of variables ϕ^{-1} to the radial field $\bar{\mathcal{Y}} = u\partial_u + v\partial_v$, that is,

$$\mathcal{Y} = \phi^* \bar{\mathcal{Y}} = \phi^*(u\partial_u + v\partial_v) \ .$$

This is because the Lie bracket is coordinates free, i.e., $\phi_*[\mathcal{X}, \mathcal{Y}] = [\phi_* \mathcal{X}, \phi_* \mathcal{Y}]$. As far as we know and as the authors in [27] comment, the inverse process (that is, to obtain the linearizing change of coordinates ϕ from a given commutator $\mathcal{Y}$) is an open problem. In this work we study this inverse process and we present a procedure to obtain the linearization of analytic isochronous centers from a given commutator, see Theorem 2.3. We conclude the work showing for the first time the linearization of some isochronous systems studied in the literature by other authors.

2.2 Lie symmetries for non–degenerate analytic centers

The following result, proved in [4] using the machinery of truncated normal forms, characterizes the centers of (2.1) in terms of Lie symmetries.

Theorem 2.1 *System (2.1) has a center at the origin if and only if there exists an infinitesimal generator of a Lie symmetry* $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ *analytic in a neighborhood* U *of the origin such that* $[\mathcal{X}, \mathcal{Y}] = \mu(x, y)\mathcal{X}$*, where* $\mathcal{X}$ *is the associated vector field to system (2.1) and* μ *is an analytic scalar function in* U *such that* $\mu(0, 0) = 0$*.*

The set $N(\mathcal{X})$ of *normalizers* of $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ is defined as the set of all $\mathbb{C}^1(U)$ infinitesimal generators $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ of the Lie group of symmetries of $\mathcal{X}$. In other words, $N(\mathcal{X}) = \{\mathcal{Y} : [\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}\}$ for some scalar function $\mu(x, y)$. The structure of $N(\mathcal{X})$ is well known (see for instance [29]): if $\bar{\mathcal{Y}} \in N(\mathcal{X})$, it can be written as $\bar{\mathcal{Y}} = H\mathcal{Y} + g\mathcal{X}$ where H is a first integral of $\mathcal{X}$ or a non–zero constant and g is any C^1 function. Moreover $[\mathcal{X}, \bar{\mathcal{Y}}] = \bar{\mu}\mathcal{X}$, with $\bar{\mu} = H\mu + \mathcal{X}g$.

For the isochronous case, there is a restricted version of Theorem 2.1 proved for the first time in [54].

Theorem 2.2 *System (2.1) has an isochronous center at the origin if and only if there exists a vector field* $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ *analytic in a neighborhood* U *of the origin such that* $[\mathcal{X}, \mathcal{Y}] = 0$*, where* $\mathcal{X}$ *is the associated vector field to system (2.1).*

2.3 The main theorem

In this section we give the main result of this chapter.

Theorem 2.3 *Let* $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y = (-y + \cdots)\partial_x + (x + \cdots)\partial_y$ *and* $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ *be two analytic vector fields in a neighborhood* U *of the origin such that* $[\mathcal{X}, \mathcal{Y}] = 0$*. Then, a near–identity change of variables* $u = x + \cdots$*,* $v = y + \cdots$*, analytic in* U *that linearizes* $\mathcal{X}$ *is obtained as follows:*

$$u = \frac{\sqrt{f(H)}g(I)}{1 + g^2(I)} \ , \ v = \frac{\sqrt{f(H)}}{1 + g^2(I)} \ , \tag{2.2}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, associated with the inverse integrating factor $\mathcal{X} \wedge \mathcal{Y}$ and f and g are two functions such that $f(H(x,y)) = x^2 + y^2 + \cdots$ and $g(I(x,y)) = (y + \cdots)/(x + \cdots)$.

Proof. The existence of the linearizing change of variables $(x,y) \to (u,v)$ is known. The new result is how to obtain the linearizing change of variables from the knowledge of the commutator $\mathcal{Y}$.

From (2.2) we have that $f(H) = u^2+v^2$ and $g(I) = v/u$ are first integrals of $\mathcal{X}$ and $\mathcal{Y}$ respectively, i.e. $\mathcal{X}f(H) = \mathcal{Y}g(I) \equiv 0$. It follows

$$u\mathcal{X}(u) + v\mathcal{X}(v) \equiv 0 \ , \ u\mathcal{Y}(v) - v\mathcal{Y}(u) \equiv 0 \ .$$

Therefore we have

$$\mathcal{X}(u) = -v\Lambda \ , \ \mathcal{X}(v) = u\Lambda \ , \mathcal{Y}(u) = u\Omega \ , \ \mathcal{Y}(v) = v\Omega \ , \tag{2.3}$$

with $\Lambda(x,y)$ and $\Omega(x,y)$ analytic functions in a neighborhood of the origin. From the above equations we observe that, if we prove that Λ is a non–vanishing constant then the change $(x,y) = (u(x,y), v(x,y))$ linearizes the vector field $\mathcal{X}$.

On the other hand, since $[\mathcal{X}, \mathcal{Y}] = 0$, in particular we have

$$\mathcal{X}\mathcal{Y}(u) - \mathcal{Y}\mathcal{X}(u) \equiv 0 \ , \ \mathcal{X}\mathcal{Y}(v) - \mathcal{Y}\mathcal{X}(v) \equiv 0 \ .$$

Introducing (2.3) in the former relations we get

$$u\mathcal{X}(\Omega) + v\mathcal{Y}(\Lambda) = 0 \ , \ v\mathcal{X}(\Omega) - u\mathcal{Y}(\Lambda) = 0 \ .$$

This is a linear homogeneous algebraic system for the unknowns $\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Lambda)$ with associated determinant $-(u^2+v^2)$ which is different from zero out of the origin. So the unique solution is the trivial one $\mathcal{X}(\Omega) = \mathcal{Y}(\Lambda) \equiv 0$. From this last equality, Λ is either a constant or a first integral of $\mathcal{Y}$ in a neighborhood of the origin. But, this point is a singular point of $\mathcal{Y}$ of type node and so there are no continuous first integral of $\mathcal{Y}$ in that neighborhood in contradiction with the fact that Λ is analytic. Hence the only possibility is Λ equal to a constant different from zero because otherwise, from (2.3), u and v would be analytic first integrals of $\mathcal{X}$ which is impossible. ∎

2.4 Examples

Example 1. Quadratic isochronous centers were classified by Loud [45]. Hence, it is known that the quadratic system

$$\dot{x} = -y - \frac{4}{3}x^2 \ , \ \dot{y} = x\left(1 - \frac{16}{3}y\right) \ , \tag{2.4}$$

has an isochronous center at the origin. Moreover, it is shown (see page 34 of [19]) that the associated vector field $\mathcal{X}$ to system (2.4) commutes with $\mathcal{Y}_1 = 3x(9-24y+32x^2)\partial_x + (3y+4x^2)(9-24y+32x^2)\partial_y$. Taking $\mathcal{Y} = \mathcal{Y}_1/27$ we have $\mathcal{Y} = (x+\cdots)\partial_x + (y+\cdots)\partial_y$ and $[\mathcal{X}, \mathcal{Y}] = 0$. Then $V = \mathcal{X} \wedge \mathcal{Y} = (9+32x^2-24y)(9x^2+16x^4-24x^2y+9y^2)$ up to multiplicative constants and $H(x,y)$ and $I(x,y)$ can be computed (and simplified after deleting some arctan and log functions) to

$$H(x,y) = \frac{9x^2+16x^4-24x^2y+9y^2}{(9+32x^2-24y)^2} \ , \ I(x,y) = \frac{-3y+4x^2}{3x} \ .$$

Since $H(x,y) = (x^2+y^2)/9 + \cdots$ we take $f(H) = 9H$. Moreover $g(I) = -I$ so that

$$g(I(x,y)) = \frac{y - \frac{4}{3}x^2}{x} \ .$$

Finally, from (2.2), we get the linearizing change of coordinates

$$u = \frac{9x}{9+32x^2-24y} \ , \ v = \frac{3(3y-4x^2)}{9+32x^2-24y} \ ,$$

according to [19]. We note that this change of variables linearizes both vector fields $\mathcal{X}$ and $\mathcal{Y}$. ∎

The authors of the survey paper [19] have carried out a quite exhaustive classification of several families of isochronous systems. Also, they give the linearizing changes and commutators in many cases. In any way, some incomplete examples exist in their tables. In the following two examples we obtain the linearizing change that lacks in some of these systems by using our Theorem 2.3.

Example 2. In Table 10 of [19] the authors present the cubic reversible system

$$\dot{x} = -y(1-x)(1-2x) \ , \ \dot{y} = x - 2x^2 + y^2 + 2x^3 \ , \tag{2.5}$$

having an isochronous center at the origin. Moreover, they prove that the associated vector field $\mathcal{X}$ to system (2.5) commutes with $\mathcal{Y} = (1-x)(x - 2x^2 + 2y^2 + 2x^3 - 2xy^2)/(1-2x)\partial_x + y(1-x)(1-4x+6x^2+2y^2)/(1-2x)\partial_y = (x+\cdots)\partial_x + (y+\cdots)\partial_y$. Then $V = \mathcal{X} \wedge \mathcal{Y} = (1-x)(x^2+y^2)(1-4x+8x^2 - 8x^3 + 4x^4 + 4y^2 - 8xy^2 + 4x^2y^2)/(-1+2x)$ is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$. The first integrals $H(x,y)$ and $I(x,y)$ associated with V of $\mathcal{X}$ and $\mathcal{Y}$ respectively are then calculated (and simplified after deleting arctan and arctanh functions). We get

$$\begin{aligned} H(x,y) &= \frac{1-4x+12x^2-16x^3+8x^4+8(-1+x)^2y^2}{(1-2x)^2} , \\ I(x,y) &= \frac{y}{x-2x^2+2x^3-2y^2+2xy^2} . \end{aligned}$$

Since $H(x,y) = 1 + 8(x^2+y^2) + \cdots$ we take $f(H) = (H-1)/8 = x^2+y^2+\cdots$. Moreover $g(I) = I$ so that, taking into account (2.2), we get the linearizing change of coordinates

$$u = (x + 2(-x^2 + x^3 - y^2 + xy^2))\Delta(x,y) \ , \ v = y\Delta(x,y) \ ,$$

where $\Delta(x,y) := (x-1)/[(1-2x)\sqrt{(1+2x(x-1))^2 + 4(x-1)^2y^2}]$. We note that this change of variables only linearizes $\mathcal{X}$. ∎

Example 3. The system labeled as $H4_4$ in Table 13 of [19] is the following linear center perturbed by a homogeneous polynomial of fourth degree

$$\dot{x} = -y - \frac{4}{9}x^4 - \frac{20}{9}x^2y^2 \ , \ \dot{y} = x + \frac{40}{9}x^3y + \frac{16}{9}xy^3 \ . \tag{2.6}$$

The origin is an isochronous center and in [19] is proved that its associated vector field $\mathcal{X}$ commutes with $\mathcal{Y}_1 = f_1(x,y)[x(3+8x^2y)\partial_x + (3y - 12x^4 - 4x^2y^2)\partial_y]$ where $f_1(x,y) = 9 + 24y(x^2+y^2) + 32x^2(x^2+y^2)^2$. We define $\mathcal{Y} = \mathcal{Y}_1/27$ so that $\mathcal{Y} = (x+\cdots)\partial_x + (y+\cdots)\partial_y$ and $[\mathcal{X},\mathcal{Y}] = 0$. Then $V = \mathcal{X} \wedge \mathcal{Y} = (x^2+y^2)f_1(x,y)f_2(x,y)$ up to multiplicative constants is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$. Here $f_2(x,y) = 9 + 16x^6 + 24x^2y + 16x^4y^2$. The first integral of $\mathcal{X}$ associated with V is $H(x,y) = (x^2+y^2)^3 f_1^{-2} f_2$. Hence a first integral good for our purpose is just $f(H) = 3^{2/3}H^{1/3} = x^2 + y^2 + \cdots$. In addition, $\mathcal{Y}$ possesses, associated to V, the first integral $I(x,y) = (3y + 4x^4 + 4x^2y^2)/(3x) = (y+\cdots)/(x+\cdots)$

(after simplifying an arctan function). In short, $g(I) = I$, and due to (2.2), the change of coordinates

$$u = 3^{4/3}x(f_1 f_2)^{-1/3} \ , \ v = 3^{1/3}[3y + 4x^2(x^2 + y^2)](f_1 f_2)^{-1/3} \ ,$$

linearizes both $\mathcal{X}$ and $\mathcal{Y}$. ∎

The last example comes form the work [28]. In that paper it is proved that the potential vector field $\mathcal{X} = -y\partial_x + V'(x)\partial_y$ has an isochronous center at the origin for the rational potential function

$$V(x) = \frac{x^2(x-2)^2}{(x-1)^2} \ , \tag{2.7}$$

and they comment that it is difficult to obtain an explicit commutator and a linearizing change of coordinates. We will obtain in the next example both characterizations of isochronicity for such system.

Example 4. Let us consider the potential systems

$$\mathcal{X} = -y\partial_x + V'(x)\partial_y \ ,$$

with hamiltonian function of type kinetic plus potential $H = y^2/2 + V(x)$ and $V(x) = x^2/2 + o(x^2)$ analytic function having a minimum at the origin. Hence, the origin is a center of $\mathcal{X}$. One can check by straightforward calculations (this formula appears in [28]) that $[\mathcal{X}, \mathcal{Y}] = \mu\mathcal{X}$, where

$$\mathcal{Y} = \frac{V(x)}{V'(x)}\partial_x + \frac{y}{2}\partial_y \ , \ \mu(x) = \frac{(V'(x))^2 - 2V(x)V''(x)}{2(V'(x))^2} \ .$$

First of all we note that if $g(x, y)$ is a C^1 solution of the partial differential equation $\mathcal{X}g = -\mu H$, then $\mathcal{Y}^* = H\mathcal{Y} + g\mathcal{X}$ satisfies $[\mathcal{X}, \mathcal{Y}^*] = 0$. Assuming that $V(x)$ is given by (2.7) one obtains

$$g(x, y) = \frac{(1-x)yA(x, y)}{2(2 - 2x + x^2)B(x, y)} \ ,$$

where $A(x, y) = 8x^2 + y^2 - 8x^3 + 2x^4 - 2xy^2 + x^2y^2$ and $B(x, y) = 8 - 16x + 16x^2 + y^2 - 8x^3 + 2x^4 - 2xy^2 + x^2y^2$. Therefore

$$\mathcal{Y}^* = \frac{A(x, y)C(x, y)}{4(x-1)B(x, y)}\partial_x + \frac{yA(x, y)(8 + C(x, y))}{4(x-1)^2B(x, y)}\partial_y$$

where $C(x,y) = -8x + 12x^2 + y^2 - 8x^3 + 2x^4 - 2xy^2 + x^2y^2$. Since $\mathcal{Y}^* = (xy^2/4 + 2x^3 + \cdots)\partial_x + (y^3/4 + 2x^2y + \cdots)\partial_y$ we take

$$\bar{\mathcal{Y}} = \frac{\mathcal{Y}^*}{H} = \frac{(x-1)C(x,y)}{2B(x,y)}\partial_x + \frac{y(8+C(x,y))}{2B(x,y)}\partial_y = \left(\frac{x}{2} + \cdots\right)\partial_x + \left(\frac{y}{2} + \cdots\right)\partial_y\ ,$$

which also satisfies $[\mathcal{X}, \bar{\mathcal{Y}}] = 0$ and has a star node at the origin.

Now taking into account that $\mathcal{X} \wedge \bar{\mathcal{Y}}$ is an inverse integrating factor of $\bar{\mathcal{Y}}$ it is easy to obtain for $\bar{\mathcal{Y}}$ the next first integral

$$I(x,y) = \frac{y(1-x)^3}{D(x,y)} = \frac{y + \cdots}{8x + \cdots}\ ,$$

where $D(x,y) = 8x - 12x^2 + y^2 + 8x^3 - 2x^4 - 2xy^2 + x^2y^2$. Finally, taking $f(H) = 2H = 8x^2 + y^2 + \cdots$, and solving the system $8u^2 + v^2 = f(H)$, $v/(8u) = I$ we obtain that the change of coordinates

$$u(x,y) = \frac{-D(x,y)}{2\sqrt{2}(x-1)\sqrt{B(x,y)}} = x + \cdots\ ,\ v(x,y) = \frac{2\sqrt{2}y(x-1)^2}{\sqrt{B(x,y)}} = y + \cdots$$

brings $\mathcal{X}$ to the linear vector field $-v\partial_u + 8u\partial_v$. ∎

Chapter 3

Linearization of Planar Vector Fields From Symmetries

Summary. This chapter represents a continuation of the work begun in chapter 1. We propose a constructive procedure to get the change of variables that linearizes a smooth planar vector field on $\mathbb{C}^2$ around an elementary singular point (i.e., a singular point with associated eigenvalues $\lambda, \mu \in \mathbb{C}$ satisfying $\mu \neq 0$) or a nilpotent singular point from a given commutator. Moreover, it is proved that the near–identity change of variables that linearizes the vector field $\mathcal{X} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ is unique and linearizes simultaneously all the centralizers of $\mathcal{X}$.

3.1 Introduction

We study the classical problem of linearizing a nonlinear smooth planar vector field $\mathcal{X}$ around a singular point by means of a local C^∞ transformation. This problem goes back to Poincaré in the more general context of Poincaré–Dulac normal forms. Anyway, if the linearizing transformation is not polynomial, then Poincaré procedure (a step by step construction) is not useful to get the change of variables because involves infinite many steps. Moreover, the presence of resonances in the linear part of the vector field is another problem in the algorithm. In any case, this normal form procedure requires extensive computations usually. Thus we will look for a different approach based on Lie symmetries of $\mathcal{X}$.

We consider two-dimensional autonomous smooth differential systems defined in an open set $U \subset \mathbb{C}^2$ having either an elementary singular point (i.e., a singular point with associated eigenvalues $\lambda, \mu \in \mathbb{C}$ satisfying $\mu \neq 0$) or a nilpotent singular point (that is, a singular point with both eigenvalues zero but with non–vanishing associated linear part). We can do a translation of coordinates such that the singular point is located at the origin and the system has the form

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix} , \tag{3.1}$$

with f and g analytic functions in U starting in at least second order terms, i.e. such that $f(0,0) = g(0,0) = 0$ and $\partial_x f(0,0) = \partial_y f(0,0) = \partial_x g(0,0) = \partial_y g(0,0) = 0$. Finally, making a linear change of coordinates, the system can be written with its linear part in Jordan form, that is, we can suppose that A is of the form:

$$(\mathbf{i})\ A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix} , \ (\mathbf{ii})\ A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} . \tag{3.2}$$

Thus, A is called *semisimple* if it adopts the above form (i) and is the sum of a semisimple and a nilpotent matrix in case (ii).

We shall use the following notation in this work. $\mathcal{X}_A$ will be the linear vector field with associated matrix $A = (a_{ij}) \in \mathcal{M}_2(\mathbb{C})$, that is,

$$\mathcal{X}_A = (a_{11}x + a_{12}y)\partial_x + (a_{21}x + a_{22}y)\partial_y .$$

Therefore, a smooth vector field $\mathcal{X}$ in $\mathbb{C}^2$ with linear part $\mathcal{X}_A$ is expressed as $\mathcal{X} = \mathcal{X}_A + \cdots$, where the dots denote a smooth vector field without linear terms.

We are mainly concerned about the local *linearization problem*, i.e., to determine whether there exists a neighborhood $\mathcal{U} \subset U$ of the origin and a smooth near–identity change of variables $\phi : \mathcal{U} \to \mathbb{C}^2$ of the form $\phi(x,y) = (u(x,y), v(x,y)) = (x + o(x,y)), y + o(x,y))$ that linearizes the associated vector field $\mathcal{X} = \mathcal{X}_A + \cdots$ of system (3.1) in $\mathcal{U}$. This means that $\phi_*\mathcal{X} = \mathcal{X}_A$ where ϕ_* and ϕ^* is the push–forward and pull–back defined by the smooth diffeomorphism ϕ. In the *orbital linearization* case one has $\phi_*\mathcal{X} = h\mathcal{X}_A$, where $h(x,y) = 1 + \cdots$ is a smooth scalar function defined on $\mathcal{U}$. In [20], sufficient conditions to construct linearizing changes of variables for polynomial vector fields based on generalizations of the standard Darboux integrability theory are given.

It is known (see [35]) that, the existence of such a ϕ is equivalent to the existence of a smooth commuting vector field of the form $\mathcal{Y} = (x+o(x,y))\partial_x + (y+o(x,y))\partial_y$. Here commuting means $[\mathcal{X}, \mathcal{Y}] \equiv 0$ where the bracket used is the Lie bracket.

Before stating the main results of this paper we want to comment that, from a linearizing change of coordinates ϕ, it is easy to get a commutator $\mathcal{Y}$ of $\mathcal{X}$. In fact, it is just the vector field obtained by applying the inverse change of variables ϕ^{-1} to the radial field $\bar{\mathcal{Y}} = u\partial_u + v\partial_v$, that is, $\mathcal{Y} = \phi^* \bar{\mathcal{Y}}$. This is because the Lie bracket is a coordinates free geometrical object, i.e., $\phi_*[\mathcal{X}, \mathcal{Y}] = [\phi_* \mathcal{X}, \phi_* \mathcal{Y}]$. As far as we know, the inverse process (that is, to obtain the linearizing change of coordinates ϕ from a given commutator $\mathcal{Y}$) is an open problem, see [27]. In this work we study this inverse process and we present a method to obtain the linearization change from a given commutator, see Theorem 3.6, 3.7, 3.11. Therefore we extend to any linearizable singular point of smooth vector fields in $\mathbb{C}^2$ the ideas introduced in Chapter 2 for the isochronous center case of analytic vector fields in $\mathbb{R}^2$. We conclude the work showing the linearization of some extracted examples of the existent literature.

3.2 Structure of centralizers in planar vector fields

Let $U \subset \mathbb{C}^2$ be an open set. The set $C(\mathcal{X})$ of (nontrivial) *centralizers* of $C^1(U)$ planar vector fields $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ is defined as the set of all $C^1(U)$ infinitesimal generators $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ of commuting (nontrivial) Lie symmetries of $\mathcal{X}$. Here nontrivial means that $\mathcal{X}$ and $\mathcal{Y}$ are transversal vector fields, i.e., $\mathcal{X} \wedge \mathcal{Y} := P\eta - Q\xi \not\equiv 0$. In short, $C(\mathcal{X}) = \{\mathcal{Y} \in C^1(U) : [\mathcal{X}, \mathcal{Y}] = 0,\ \mathcal{X} \wedge \mathcal{Y} \not\equiv 0\}$. In [13] it is described an algorithmic method to get an infinitesimal generator $\mathcal{Y}$ provided that $\mathcal{X}$ has a rational first integral or integrating factor. The set $C(\mathcal{X})$ has the structure of a module over the ring of first integrals of $\mathcal{X}$, see [23, 62]. We prove this fact again for sake of completeness.

Proposition 3.1 *$\mathcal{Y}, \bar{\mathcal{Y}} \in C(\mathcal{X})$ if and only if there are two C^1 scalar functions $f \not\equiv 0$ and g such that $\bar{\mathcal{Y}} = f(H)\mathcal{Y} + g(H)\mathcal{X}$ where H is a first integral of $\mathcal{X}$.*

Proof. Let us see that $[\mathcal{X}, \bar{\mathcal{Y}}] = 0$. In short,

$$[\mathcal{X}, f(H)\mathcal{Y} + g(H)\mathcal{X}] = \mathcal{X}(f(H))\mathcal{Y} + f(H)[\mathcal{X}, \mathcal{Y}] + \mathcal{X}(g(H))\mathcal{X} \ . \tag{3.3}$$

Using the fact that $f(H)$ and $g(H)$ are first integrals of $\mathcal{X}$ and $[\mathcal{X}, \mathcal{Y}] = 0$, we have that (3.3) vanishes, hence, $\bar{\mathcal{Y}} \in C(\mathcal{X})$. To prove the converse, we note that since $\mathcal{X}$ and $\mathcal{Y}$ are transversal, then there are two C^1 functions α and β such that $\bar{\mathcal{Y}} = \alpha\mathcal{Y} + \beta\mathcal{X}$. Due to the fact that we want $[\mathcal{X}, \bar{\mathcal{Y}}] = 0$, we see that α must be either a first integral of $\mathcal{X}$ or a non-zero constant (to preserve transversality) and β must be a first integral or a constant. ∎

3.3 Linearizing changes of coordinates from commutators

The following theorem is proved in [35] and it gives the equivalence between the linearizability of a smooth vector field and the fact of having a smooth commuting vector field of the form $\mathcal{Y} = \mathcal{Y}_I + \cdots$ where $I \in \mathcal{M}_2(\mathbb{C})$ is the identity matrix. The version presented here is a generalization for any type of singular point, which has, in fact, the same proof of the theorem proved in [35].

Theorem 3.2 *Consider the smooth (resp. analytic) vector field $\mathcal{X} = \mathcal{X}_A + \cdots$ on $\mathbb{C}^2$ defined in a neighborhood of the origin with $A \neq 0$. Then, $\mathcal{X}$ is linearizable if, and only if, there exists a smooth (resp. analytic) vector field of the form $\mathcal{Y} = \mathcal{Y}_I + \cdots$, such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$.*

Remark 3.3 It is well known that, see for instance [51, 62], if $\mathcal{Y} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ is a normalizer of $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$, i.e., $[\mathcal{X}, \mathcal{Y}] = \nu\mathcal{X}$, then the wedge product $\mathcal{X} \wedge \mathcal{Y} := P\eta - Q\xi$ is an inverse integrating factor of $\mathcal{X}$. So, in the particular case of commutation ($\nu \equiv 0$), we get that $\mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor of both $\mathcal{X}$ and $\mathcal{Y}$. This will be a key point in the proof of our main results.

The next two propositions are preliminary results that we will need in order to prove the main results of this paper.

Proposition 3.4 *Let $\mathcal{X} = \mathcal{X}_A + \cdots$ be a smooth vector field in $\mathbb{C}^2$ with non–vanishing linear part $\mathcal{X}_A$. Then, $\mathcal{X}$ cannot have a smooth first integral H in a neighborhood of the origin with non–vanishing linear part unless A is either semisimple with eigenvalues $\lambda \neq \mu = 0$ or nilpotent, in which cases, $H(x,y) = y + \cdots$ up to multiplicative constants.*

Proof. Let $\mathcal{X} = \mathcal{X}_A + \cdots$ be a smooth planar vector field with linear part $\mathcal{X}_A$ associated to the non–vanishing matrix $A \in \mathcal{M}_2(\mathbb{C})$. Let $H(x,y) = \sum_{i\geq 1} H_i(x,y)$, with H_i homogeneous polynomial of degree i, be a smooth first integral of $\mathcal{X}$ satisfying $H_1 \not\equiv 0$.

First, assume that A is semisimple and then has been transformed to the case (i) of equation (3.2), that is, $\mathcal{X} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$ with λ and μ not both zero. Imposing $\mathcal{X}H \equiv 0$ we have

$$\lambda x \frac{\partial H_1}{\partial x} + \mu y \frac{\partial H_1}{\partial y} = 0 \ . \tag{3.4}$$

Applying Euler Theorem to the homogeneous polynomial $H_1(x,y)$ we have

$$x \frac{\partial H_1}{\partial x} + y \frac{\partial H_1}{\partial y} = H_1 \ . \tag{3.5}$$

If $\lambda \neq \mu$, from the above two relations we get

$$\frac{\partial H_1}{\partial x} = \frac{\mu H_1}{(\mu - \lambda)x} \ , \ \frac{\partial H_1}{\partial y} = \frac{-\lambda H_1}{(\mu - \lambda)y} \ ,$$

which implies, if $\lambda\mu \neq 0$, that both x and y must divide H_1 in contradiction with the fact that H_1 is a non–vanishing homogeneous polynomial of degree 1. Otherwise, when $\lambda \neq \mu = 0$ we obtain $H_1(x,y) = cy$ with a constant $c \in \mathbb{C}$. Of course, if $\lambda = \mu \neq 0$, then, from (3.4) and (3.5) we get $H_1 \equiv 0$.

Finally, assume that A is not semisimple. Thus, in certain linear coordinates, A has the form (ii) of equation (3.2), that is, $\mathcal{X} = (\lambda x + y + \cdots)\partial_x + (\lambda y + \cdots)\partial_y$ with $\lambda \in \mathbb{C}$. Imposing again $\mathcal{X}H \equiv 0$, the first terms must satisfy

$$(\lambda x + y)\frac{\partial H_1}{\partial x} + \lambda y \frac{\partial H_1}{\partial y} = 0 \ . \tag{3.6}$$

Solving (3.5) and (3.6) leads to

$$\frac{\partial H_1}{\partial x} = \frac{-\lambda H_1}{y} \ , \ \frac{\partial H_1}{\partial y} = \frac{(\lambda x + y) H_1}{y^2} \ .$$

Hence, these restrictions cannot be satisfied by the homogeneous linear polynomial H_1 except when $\lambda = 0$ in which case $H_1(x, y) = cy$ with a constant $c \in \mathbb{C}$ and A is nilpotent proving the proposition. ∎

Proposition 3.5 *Let $\mathcal{X} = \mathcal{X}_A + \cdots$ be a smooth vector field in $\mathbb{C}^2$ with A semisimple having eigenvalues satisfying either $\lambda = \mu \neq 0$ or $0 \neq \lambda \neq \mu \neq 0$ with $\lambda/\mu \notin \mathbb{Q}^-$. Then, $\mathcal{X}$ cannot have a smooth first integral in a neighborhood of the origin.*

Proof. Without loss of generality we have $\mathcal{X} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$. Let $H(x, y) = \sum_{i\geq 1} H_i(x, y)$, with H_i homogeneous polynomial of degree i, be a smooth first integral of $\mathcal{X}$. We shall prove the proposition showing that $H_i \equiv 0$ for all $i \geq 1$ by induction over i.

Under the assumptions over the eigenvalues λ and μ, we can repeat verbatim the beginning of the proof of Proposition 3.4, that is, imposing $\mathcal{X}H \equiv 0$ and using Euler Theorem for H_1, to get $H_1 \equiv 0$. Now, assume that $H_i \equiv 0$ for $i = 1, \ldots, m-1$. We are going to show that $H_m \equiv 0$ finishing the proof. Since now $H(x, y) = \sum_{i\geq m} H_i(x, y)$, imposing $\mathcal{X}H \equiv 0$ and using Euler Theorem for H_m we obtain

$$\lambda x \frac{\partial H_m}{\partial x} + \mu y \frac{\partial H_m}{\partial y} = 0 \ , \tag{3.7}$$

and

$$x \frac{\partial H_m}{\partial x} + y \frac{\partial H_m}{\partial y} = m H_m \ . \tag{3.8}$$

If $\lambda = \mu \neq 0$, these equations are compatible if and only if $H_m \equiv 0$. So we continue solving them in the case $0 \neq \lambda \neq \mu \neq 0$ with $\lambda/\mu \notin \mathbb{Q}^-$ getting

$$\frac{\partial H_m}{\partial x} = \frac{a H_m}{x} \ , \ \frac{\partial H_m}{\partial y} = \frac{b H_m}{y} \ ,$$

where

$$a = \frac{-m\mu}{\lambda - \mu} \ , \ b = \frac{m\lambda}{\lambda - \mu} \ .$$

These partial differential equations have as general solutions

$$H_m(x, y) = x^a f(y) \ , \ H_m(x, y) = y^b g(x) \ ,$$

respectively, with f and g arbitrary functions. Of course, since H_m must be an homogeneous polynomial of degree m, the only possibility is either $f = g \equiv 0$ and so $H_m \equiv 0$ or

$$H_m(x, y) = cx^a y^b \ , \ \text{with } c \in \mathbb{R} \ , \ a, b \in \mathbb{Z}^+ \text{ and } a + b = m \ .$$

The conditions $a, b \in \mathbb{Z}^+$ give the contradiction

$$\frac{\lambda}{\mu} = \frac{a - m}{a} = \frac{b}{b - m} \in \mathbb{Q}^- \ .$$

Therefore, the only possibility is the first one $H_m \equiv 0$ and, by induction, $H \equiv 0$ finishing the proof. ∎

The next two theorems are part of our main results. They give a method to get linearizing changes of variables when the linear part of the vector field is either semisimple with different associated eigenvalues or is the sum of a semisimple and a nilpotent part.

Theorem 3.6 *Let $\mathcal{X} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$ with $\lambda \neq \mu$ and $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ be two smooth vector fields in a neighborhood $U \subset \mathbb{C}^2$ of the origin such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$. Then, a smooth near–identity change of variables $u = x + \cdots$, $v = y + \cdots$, that linearizes $\mathcal{X}$ and orbitally linearizes the vector field $\mathcal{Y}$ is obtained as follows:*

$$u = g(I) \left(\frac{f(H)}{g^\mu(I)} \right)^{\frac{1}{\mu - \lambda}} \ , \ v = \left(\frac{f(H)}{g^\mu(I)} \right)^{\frac{1}{\mu - \lambda}} \ , \tag{3.9}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, associated with the inverse integrating factor $\mathcal{X} \wedge \mathcal{Y}$ and f and g are two functions such that $f(H(x, y)) = (x + \cdots)^\mu / (y + \cdots)^\lambda$ and $g(I(x, y)) = (x + \cdots)/(y + \cdots)$.

Proof. The existence of the linearizing change of variables $(x, y) \mapsto (u, v)$ is known from Theorem 3.2. The new result is how to obtain the linearizing change of variables from the knowledge of the commutator $\mathcal{Y}$. From (3.9) we have that $f(H) = u^\mu / v^\lambda$ and $g(I) = u/v$ are first integrals of $\mathcal{X}$ and $\mathcal{Y}$ respectively, i.e. $\mathcal{X} f(H) = \mathcal{Y} g(I) \equiv 0$. It follows

$$\mu v \mathcal{X}(u) - \lambda u \mathcal{X}(v) \equiv 0 \ , \ v \mathcal{Y}(u) - u \mathcal{Y}(v) \equiv 0 \ .$$

Therefore we have

$$\mathcal{X}(u) = \lambda u \Lambda \ , \ \mathcal{X}(v) = \mu v \Lambda \ , \mathcal{Y}(u) = u\Omega \ , \ \mathcal{Y}(v) = v\Omega \ , \tag{3.10}$$

with $\Lambda(x, y)$ and $\Omega(x, y)$ smooth functions in a neighborhood of the origin. From the above equations we observe that, if we prove that Λ is a non–vanishing constant then the change $(x, y) \mapsto (u(x, y), v(x, y))$ linearizes the vector field $\mathcal{X}$.

On the other hand, since $[\mathcal{X}, \mathcal{Y}] \equiv 0$, in particular we have

$$\mathcal{X}\mathcal{Y}(u) - \mathcal{Y}\mathcal{X}(u) \equiv 0 \ , \ \mathcal{X}\mathcal{Y}(v) - \mathcal{Y}\mathcal{X}(v) \equiv 0 \ .$$

Introducing (3.10) in the former relations we get

$$u\mathcal{X}(\Omega) - \lambda u \mathcal{Y}(\Lambda) = 0 \ , \ v\mathcal{X}(\Omega) - \mu v \mathcal{Y}(\Lambda) = 0 \ .$$

This is a linear homogeneous algebraic system for the unknowns $\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Lambda)$ with associated determinant $(\lambda - \mu)uv$ which, for $\lambda \neq \mu$, is different from zero out of the set $\Sigma = \{(u, v) \in \mathbb{C}^2 \ : \ uv = 0\}$. So the unique solution in $\mathbb{C}^2 \backslash \Sigma$ is the trivial one $\mathcal{X}(\Omega) = \mathcal{Y}(\Lambda) \equiv 0$. From this last equality, Λ is either a constant or a smooth first integral of $\mathcal{Y}$ in a neighborhood of the origin. But, the second option is not possible due to Proposition 3.5. Hence the only possibility is Λ equal to a constant. Moreover, that constant is different from zero because otherwise, from (3.10), u and v would be smooth first integrals of $\mathcal{X}$. This is impossible from Proposition 3.4 because: (i) if $\lambda\mu \neq 0$ then there is no smooth first integral of $\mathcal{X}$ around the origin; (ii) if $\mu = 0$ then there is a unique (up to constants) smooth first integral of $\mathcal{X}$ around the origin which is just $v(x, y) = y + \cdots$; (iii) if $\lambda = 0$ then this case reduce to the previous one interchanging the role of the variables x and y. Hence, the change $(x, y) \mapsto (u(x, y), v(x, y))$ linearizes the vector field $\mathcal{X}$ and orbitally linearizes the vector field $\mathcal{Y}$ because $\Omega(0, 0) \neq 0$. This is due to the fact that, from $\mathcal{Y}(u) = u\Omega$, that is, $(x+\cdots)\partial_x(x+\cdots)+(y+\cdots)\partial_y(x+\cdots) = (x + \cdots)(\Omega(0, 0) + \cdots)$, at first order we get $\Omega(0, 0) = 1$. ∎

Theorem 3.7 *Let $\lambda \in \mathbb{C}$ and $\mathcal{X} = (\lambda x + y + \cdots)\partial_x + (\lambda y + \cdots)\partial_y$ and $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ be two smooth vector fields in a neighborhood $U \subset \mathbb{C}^2$ of the origin such that $[\mathcal{X}, \mathcal{Y}] \equiv 0$. Then, a smooth near–identity change of variables $u = x + \cdots$, $v = y + \cdots$, that linearizes $\mathcal{X}$ and orbitally linearizes the vector field $\mathcal{Y}$ is obtained as follows:*

$$u = g(I) f(H) \exp[\lambda g(I)] \ , \ v = f(H) \exp[\lambda g(I)] \ , \tag{3.11}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, associated with the inverse integrating factor $\mathcal{X} \wedge \mathcal{Y}$ and f and g are two functions such that $f(H(x,y)) = (y+\cdots)\exp[-\lambda(x+\cdots)/(y+\cdots)]$, $g(I(x,y)) = (x+\cdots)/(y+\cdots)$.

Proof. In order to proof the theorem, we will use similar arguments to the followed in Theorem 3.6. So, the sketch of the proof is as follows. From (3.11) we have that $f(H) = v\exp(-u\lambda/v)$ and $g(I) = u/v$ are first integrals of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Using the fact that $\mathcal{X}f(H) = \mathcal{Y}g(I) \equiv 0$ and $[\mathcal{X},\mathcal{Y}] \equiv 0$ we get

$$\mathcal{X}(u) = \Lambda(v+\lambda u)\ ,\ \mathcal{X}(v) = \lambda v\Lambda\ , \mathcal{Y}(u) = u\Omega\ ,\ \mathcal{Y}(v) = v\Omega\ , \tag{3.12}$$

On the other hand, taking into account $[\mathcal{X},\mathcal{Y}](u) = [\mathcal{X},\mathcal{Y}](v) \equiv 0$ and (3.12) we obtain

$$v\mathcal{X}(\Omega) - \lambda v\mathcal{Y}(\Lambda) = 0\ ,\ u\mathcal{X}(\Omega) - (v+\lambda u)\mathcal{Y}(\Lambda) = 0\ .$$

for some smooth functions $\Omega(x,y)$ and $\Lambda(x,y)$ defined in a neighborhood of the origin. This is a linear homogeneous algebraic system for the unknowns $\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Lambda)$ with associated determinant $-v^2$. So the unique solution in $\mathbb{C}^2\backslash\{(u,v) \in \mathbb{C}^2 : v = 0\}$ is the trivial one $\mathcal{X}(\Omega) = \mathcal{Y}(\Lambda) \equiv 0$. From this last equality, Λ is either a constant or a smooth first integral of $\mathcal{Y}$ in a neighborhood of the origin. The second option cannot occur because Proposition 3.5. Finally, we note that $\Lambda \neq 0$ because, otherwise, u and v would be smooth first integrals of $\mathcal{X}$ which is impossible from Proposition 3.4. Hence, repeating verbatim the last paragraph of the proof of Theorem 3.6 we have that the change $(x,y) \mapsto (u(x,y), v(x,y))$ linearizes the vector field $\mathcal{X}$ and orbitally linearizes the vector field $\mathcal{Y}$. ∎

Remark 3.8 In the nilpotent case ($\lambda = 0$ in Theorem 3.7), it is clear that a necessary condition for linearizability of the origin of $\mathcal{X} = (y+\cdots)\partial_x + (\cdots)\partial_y$ is that the point be a non–isolated singularity of $\mathcal{X}$.

3.4 Some properties of the generator of dilations

Proposition 3.9 *The following statements hold:*

(i) *$[\mathcal{X}_A, \mathcal{X}_B] = 0$ if and only if $[A, B] = AB - BA = 0$ for any pair $A, B \in \mathcal{M}_2(\mathbb{C})$.*

(ii) *The generator of dilations $\mathcal{X}_I = x\partial_x + y\partial y$ satisfies $[\mathcal{X}_I, \mathcal{X}] = 0$ with $\mathcal{X}$ a smooth vector field if and only if $\mathcal{X} = \mathcal{X}_A$ for some $A \in \mathcal{M}_2(\mathbb{C})$.*

(iii) *Let $\mathcal{X} = \mathcal{X}_I + \cdots$ be a smooth vector field where the dots denote smooth vector fields without linear terms. Then, there exists a smooth vector field $\mathcal{Y} = \mathcal{Y}_A + \cdots$ for any $A \in \mathcal{M}_2(\mathbb{C})$ such that $[\mathcal{X}, \mathcal{Y}] = 0$. Moreover, if in addition $A = I$ then $\mathcal{X} = \mathcal{Y}$.*

Proof. Statement (i) is an direct consequence of the Lie bracket property $[\mathcal{X}_A, \mathcal{X}_B] = \mathcal{X}_{[B,A]}$.

As corollary of statement (i) we get $[\mathcal{X}_I, \mathcal{X}_A] = 0$ for any $A \in \mathcal{M}_2(\mathbb{C})$. Hence, in order to prove statement (ii) we only need to show that $[\mathcal{X}_I, \mathcal{X}] \neq 0$ for any nonlinear vector field $\mathcal{X} = \mathcal{X}_A + \cdots$. Let V_i the set of homogeneous vector fields vector fields of degree i. Since $\mathcal{X}$ is smooth, it can be expressed like $\mathcal{X} = \mathcal{X}_A + \sum_{i \geq 2} \mathcal{X}_i$ where $\mathcal{X}_i \in V_i$. Using now the bilinearity of the Lie bracket we have

$$[\mathcal{X}_I, \mathcal{X}] = \sum_{i \geq 2} [\mathcal{X}_I, \mathcal{X}_i] \ ,$$

where $\mathcal{X}_i = P_i(x, y)\partial_x + Q_i(x, y)\partial_y \in V_i$. Notice that $\mathcal{X}_I \in V_1$ and, since $[V_i, V_j] \subset V_{i+j-1}$, it is clear that $[\mathcal{X}_I, \mathcal{X}] = 0$ if and only if $[\mathcal{X}_I, \mathcal{X}_i] = 0$ for all $i \geq 2$. In short, we have

$$[\mathcal{X}_I, \mathcal{X}_i] = \left(x\frac{\partial P_i}{\partial x} + y\frac{\partial P_i}{\partial y} - P_i \right) \partial_x + \left(x\frac{\partial Q_i}{\partial x} + y\frac{\partial Q_i}{\partial y} - Q_i \right) \partial_y$$

Since P_i and Q_i are homogeneous polynomials of degree i, they must satisfy the Euler Theorem for homogeneous functions

$$x\frac{\partial P_i}{\partial x} + y\frac{\partial P_i}{\partial y} = iP_i \ , \ x\frac{\partial Q_i}{\partial x} + y\frac{\partial Q_i}{\partial y} = iQ_i \ .$$

Therefore

$$[\mathcal{X}_I, \mathcal{X}_i] = (i-1)P_i(x, y)\partial_x + (i-1)Q_i(x, y)\partial_y \ .$$

We conclude that $[\mathcal{X}_I, \mathcal{X}_i] = 0$ only for the case $i = 1$ proving thus statement (ii).

The proof of statement (iii) is based on the well–known fact in normal forms theory that smooth vector fields of the form $\mathcal{X} = \mathcal{X}_I + \cdots$ are locally smoothly linearizable. This means that, in a neighborhood of the origin, there exists a smooth near–identity change of variables $\phi(x,y) = (u(x,y), v(x,y)) = (x+o(x,y)), y+o(x,y))$ such that $\phi_* \mathcal{X} = \mathcal{X}_I = u\partial_u + v\partial v$. Therefore, from statement (ii), $[\mathcal{X}_I, \mathcal{Y}_A] = 0$ for any $A \in \mathcal{M}_2(\mathbb{C})$. Since ϕ is a near–identity map, it preserves the linear part of vector fields. Hence, undoing the change of variables we prove the existence of a smooth vector field $\mathcal{Y} = \phi^* \mathcal{Y}_A = \mathcal{Y}_A + \cdots$ such that $[\mathcal{X}, \mathcal{Y}] = 0$.

Of course, when $A = I$ we get $\mathcal{X} = \mathcal{Y}$ and the proposition is proved. ∎

In Theorem 3.6 we have studied the local linearization of a smooth vector field $\mathcal{X} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$ in a neighborhood of the origin with $\lambda \neq \mu$. Let us see what is about the case $\lambda = \mu$. So, we consider now smooth vector fields of the form $\mathcal{X} = (\lambda x + \cdots)\partial_x + (\lambda y + \cdots)\partial_y$ with $\lambda \neq 0$. Without loss of generality we consider the case $\lambda = 1$, that is, $\mathcal{X} = \mathcal{X}_I + \cdots$. Taking statement (iii) of Proposition 3.9 into account, from each matrix A, there is a smooth vector field $\mathcal{Y} = \mathcal{Y}_A + \cdots$ satisfying the commutation $[\mathcal{X}, \mathcal{Y}] = 0$. The following proposition shows how we can always obtain $\mathcal{Y}$ with a concrete associated matrix A to its linear part.

Proposition 3.10 *Let $A = (a_{ij}) \in \mathcal{M}_2(\mathbb{C})$ with $A \neq \beta I$ for any $\beta \in \mathbb{C}$ and $\mathcal{X} = \mathcal{X}_I + \cdots$ and $\mathcal{Y} = \mathcal{Y}_A + \cdots$ be two smooth vector fields satisfying $[\mathcal{X}, \mathcal{Y}] = 0$. Then, we can construct a smooth vector field $\bar{\mathcal{Y}} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$ such that $[\mathcal{X}, \bar{\mathcal{Y}}] = 0$ as follows: $\bar{\mathcal{Y}} = f(H)\mathcal{Y} + g(H)\mathcal{X}$ where $H = (x + \cdots)/(y + \cdots)$ is a first integral of $\mathcal{X}$ and*

$$
\begin{aligned}
f(H) &= \frac{H(\mu - \lambda)}{a_{21}H^2 + (a_{22} - a_{11})H - a_{12}} \,, \\
g(H) &= \frac{a_{21}\lambda H^2 + (a_{22}\lambda - a_{11}\mu)H - a_{12}\mu}{a_{21}H^2 + (a_{22} - a_{11})H - a_{12}} \,.
\end{aligned}
$$

Proof. First of all we observe that, since $A \neq \beta I$ for any $\beta \in \mathbb{C}$, it follows that does not exist a smooth scalar function $\alpha(x,y)$ such that $\mathcal{Y} = \alpha \mathcal{X}$. Hence $\mathcal{X} \wedge \mathcal{Y} \not\equiv 0$ and, since $[\mathcal{X}, \mathcal{Y}] = 0$ by hypothesis, $\mathcal{Y} \in C(\mathcal{X})$. So, from Proposition 3.1, any other $\bar{\mathcal{Y}} \in C(\mathcal{X})$ can be written as

$$
\bar{\mathcal{Y}} = f(H)\mathcal{Y} + g(H)\mathcal{X} \tag{3.13}
$$

where H is a first integral of $\mathcal{X}$. Since $\mathcal{X} = \mathcal{X}_I + \cdots$ is locally linearizable around the origin by means of a change $\phi(x,y) = (u(x,y), v(x,y)) = (x + \cdots, y + \cdots)$ we have $\phi_* \mathcal{X} = \mathcal{X}_I$ such that $[\mathcal{X}_I, \phi_* \mathcal{Y}] = 0$ and $[\mathcal{X}_I, \phi_* \bar{\mathcal{Y}}] = 0$. Hence, from statement (ii) of Proposition 3.9, $\phi_* \mathcal{Y}$ and $\phi_* \bar{\mathcal{Y}}$ are also linear vector fields. In short, $\phi_* \mathcal{Y} = \mathcal{Y}_A$. Moreover, imposing $\bar{\mathcal{Y}} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$, we get $\phi_* \bar{\mathcal{Y}} = \lambda u \partial_u + \mu v \partial_v$. Applying the change of coordinates ϕ in both members of the equality (3.13) leads

$$\lambda u \partial_u + \mu v \partial_v = f(\bar{H})((a_{11}u + a_{12}v)\partial_u + (a_{21}u + a_{22}v)\partial_v) + g(\bar{H})(u\partial_u + v\partial_v)$$

where $\bar{H}(u,v) = H \circ \phi^{-1}$ is a first integral of $u\partial_u + v\partial_v$. From the former equation we have

$$f(\bar{H}) = \frac{uv(\mu - \lambda)}{a_{21}u^2 + (a_{22} - a_{11})uv - a_{12}v^2} \ , \ g(\bar{H}) = \frac{a_{21}\lambda u^2 + (a_{22}\lambda - a_{11}\mu)uv - a_{12}\mu v^2}{a_{21}u^2 + (a_{22} - a_{11})uv - a_{12}v^2} \ .$$

If we take $\bar{H} = u/v$, then

$$f(\bar{H}) = \frac{\bar{H}(\mu - \lambda)}{a_{21}\bar{H}^2 + (a_{22} - a_{11})\bar{H} - a_{12}} \ , \ g(\bar{H}) = \frac{a_{21}\lambda \bar{H}^2 + (a_{22}\lambda - a_{11}\mu)\bar{H} - a_{12}\mu}{a_{21}\bar{H}^2 + (a_{22} - a_{11})\bar{H} - a_{12}} \ .$$

Undoing the change of variables, the proposition is proved. ∎

In the following theorem to be useful to our purposes, the associated matrix A to $\mathcal{Y}$ must be such that $A \neq \beta I$ and $\mathcal{Y}$ cannot possess a smooth first integral in a neighborhood of the origin. According with Proposition 3.5, we will take A semisimple with eigenvalues $0 \neq \lambda \neq \mu \neq 0$ such that $\lambda/\mu \notin \mathbb{Q}^-$.

Theorem 3.11 *Let $\mathcal{X} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ be a smooth vector field in a neighborhood $U \subset \mathbb{C}^2$ of the origin. Then, there exists another smooth vector field in U of the form $\mathcal{Y} = (\lambda x + \cdots)\partial_x + (\mu y + \cdots)\partial_y$ satisfying $0 \neq \lambda \neq \mu \neq 0$ and $\lambda/\mu \notin \mathbb{Q}^-$ and such that $[\mathcal{X}, \mathcal{Y}] = 0$. Moreover, a smooth near–identity change of variables $u = x + \cdots$, $v = y + \cdots$, that linearizes $\mathcal{X}$ is obtained as follows:*

$$u = f(H) \left(\frac{g(I)}{f^{\mu}(H)} \right)^{\frac{1}{\mu - \lambda}} \ , \ v = \left(\frac{g(I)}{f^{\mu}(H)} \right)^{\frac{1}{\mu - \lambda}} \ , \tag{3.14}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, associated with the inverse integrating factor $\mathcal{X} \wedge \mathcal{Y}$ and g and f are two functions such that $g(I(x,y)) = (x + \cdots)^{\mu}/(y + \cdots)^{\lambda}$ and $f(H(x,y)) = (x + \cdots)/(y + \cdots)$.

Proof. The steps followed are analogous to the proof of Theorem 3.6. From (3.14) we have that $g(I) = u^\mu / v^\lambda$ and $f(H) = u/v$ are first integrals of $\mathcal{Y}$ and $\mathcal{X}$ respectively. Using the fact that $\mathcal{X} f(H) = \mathcal{Y} g(I) \equiv 0$, we get

$$\mathcal{Y}(u) = \lambda u \Lambda \ , \ \mathcal{Y}(v) = \mu v \Lambda \ , \mathcal{X}(u) = u \Omega \ , \ \mathcal{X}(v) = v \Omega \ , \tag{3.15}$$

with $\Lambda(x,y)$ and $\Omega(x,y)$ smooth functions in a neighborhood of the origin. We note if we prove that Ω is a non–vanishing constant then the change $(x,y) \mapsto (u(x,y), v(x,y))$ linearizes the vector field $\mathcal{X}$.

On the other hand, taking into account that $[\mathcal{X}, \mathcal{Y}](u) = [\mathcal{X}, \mathcal{Y}](v) \equiv 0$ and (3.15) we get

$$u \mathcal{Y}(\Omega) - \lambda u \mathcal{X}(\Lambda) = 0 \ , \ v \mathcal{Y}(\Omega) - \mu v \mathcal{X}(\Lambda) = 0 \ .$$

This is a linear homogeneous algebraic system for the unknowns $\mathcal{Y}(\Omega)$ and $\mathcal{X}(\Lambda)$ with associated determinant $(\lambda - \mu) uv$ which, for $\lambda \neq \mu$, is different from zero out of the set $\Sigma = \{(u,v) \in \mathbb{C}^2 \ : \ uv = 0\}$. So the unique solution in $\mathbb{C}^2 \backslash \Sigma$ is the trivial one $\mathcal{Y}(\Omega) = \mathcal{X}(\Lambda) \equiv 0$. From this last equality, Ω is either a constant or a smooth first integral of $\mathcal{Y}$ in a neighborhood of the origin. But, the second option is not possible due to Proposition 3.5. Hence the only possibility is Ω equal to a constant. Moreover, that constant is different from zero because otherwise, from (3.15), u and v would be smooth first integrals of $\mathcal{X}$ which is impossible from Proposition 3.4. Moreover, Λ is either a constant or a smooth first integral of $\mathcal{X}$ in a neighborhood of the origin. The second option cannot occur because Proposition 3.5. Finally, we note that $\Lambda \neq 0$ because, otherwise, u and v would be smooth first integrals of $\mathcal{Y}$ which is impossible from Proposition 3.4. Hence, the change $(x,y) \mapsto (u(x,y), v(x,y))$ linearizes both vector fields $\mathcal{X}$ and $\mathcal{Y}$. ∎

Now we shall prove that, in fact, the change of variables obtained in Theorem 3.11 is unique.

Proposition 3.12 *Let $\phi_1 : U \to \mathbb{C}^2$ be a change of variables of the form $\phi_1(x,y) = (u_1(x,y), v_1(x,y))$ that linearizes both $\mathcal{Y} = (x + \cdots)\partial_x + (y + \cdots)\partial_y$ and $\mathcal{X} = (-y + \cdots)\partial_x + (x + \cdots)\partial_y$. Then, ϕ_2 linearizes both, $\mathcal{Y}$ and $\mathcal{X}$ if and only if $\phi_2 = \alpha_1 \phi_1 + \alpha_2 \phi_1^\perp$ being $\alpha_i \in \mathbb{C}$ and $\phi_1^\perp = (v_1, -u_1)$. In particular, there is a unique near-identity change of variables that linearizes simultaneously $\mathcal{Y}$ and $\mathcal{X}$.*

Proof. If there exist $\phi_1(x,y) = (u_1(x,y), v_1(x,y))$ that linearizes both $\mathcal{Y} = (x+\cdots)\partial_x + (y+\cdots)$ and $\mathcal{X} = \mathcal{X}_C + \cdots = (-y+\cdots)\partial_x + (x+\cdots)$, then $\mathcal{X}$ is a centralizer of $\mathcal{Y}$ with linear part of center type. Now, we suppose that there exit two smooth near-identity changes of variables $\phi_i = (u_i(x,y), v_i(x,y))$ $i = 1,2$, that linearizes both, $\mathcal{X}$ and $\mathcal{Y}$. Then

$$\phi_{*i}\mathcal{X} = \mathcal{X}_C, \quad \phi_{*i}\mathcal{Y} = \mathcal{Y}_I \quad i = 1,2,$$

being

$$\mathcal{X}(u_i) = -v_i, \ \mathcal{X}(v_i) = u_i, \ \mathcal{Y}(u_i) = u_i, \ \mathcal{Y}(v_i) = v_i \quad i = 1,2.$$

It follows

$$\mathcal{X}\left(\frac{u_1u_2 + v_1v_2}{u_1v_2 - u_2v_1}\right) = \mathcal{Y}\left(\frac{u_1u_2 + v_1v_2}{u_1v_2 - u_2v_1}\right) \equiv 0,$$

and

$$\mathcal{X}\left(\frac{u_1^2 + v_1^2}{u_2^2 + v_2^2}\right) = \mathcal{Y}\left(\frac{u_1^2 + v_1^2}{u_2^2 + v_2^2}\right) \equiv 0 \ .$$

Then, the only possibility is

$$\left(\frac{u_1u_2 + v_1v_2}{u_1v_2 - u_2v_1}\right) = c, \ \left(\frac{u_1^2 + v_1^2}{u_2^2 + v_2^2}\right) = k, \tag{3.16}$$

where c and k are arbitrary constants. Solving the algebraic system (3.16) respect to u_2 and v_2, we obtain $u_2 = \alpha_1 u_1 + \alpha_2 v_1 \quad v_2 = -\alpha_2 u_1 + \alpha_1 v_1$, where

$$\alpha_1 = c/\sqrt{k(1+c^2)}, \quad \alpha_2 = -1/\sqrt{k(1+c^2)}.$$

In fact,

$$\phi_2 = \alpha_1\phi_1 + \alpha_2\phi_1^{\perp} \tag{3.17}$$

where α_1 and α_2 are arbitrary constants. We know, from Theorem 3.2, that there is a smooth near-identity changes of variables ϕ_1 that linearizes both, $\mathcal{X}$ and $\mathcal{Y}$. Hence, from equation (3.16), it follows that ϕ_1 is unique. ∎

Theorem 3.13 *Let $\mathcal{X} = (x+\cdots)\partial_x + (y+\cdots)\partial_y$ be a smooth vector field in a neighborhood $U \subset \mathbb{C}^2$ of the origin. Then, there is a unique smooth near-identity change of variables ϕ defined in U that linearizes $\mathcal{X}$. In particular, ϕ linearizes simultaneously the set $C(\mathcal{X})$ of all the centralizers of $\mathcal{X}$.*

Proof. Let ϕ_1 and ϕ_2 be two smooth near-identity changes of variables that linearize $\mathcal{X}$. Then $\phi_{*1}\mathcal{X} = \mathcal{X}_I$, $\phi_{*2}\mathcal{X} = \mathcal{X}_I$. Let $\mathcal{Y} = \mathcal{Y}_A + \cdots$ be a centralizer of $\mathcal{X}$, that is, $[\mathcal{X}, \mathcal{Y}] \equiv 0$. It follows $\phi_{*1}\mathcal{Y} = \mathcal{Y}_A$ and $\phi_{*2}\mathcal{Y} = \mathcal{Y}_A$. We know from statement (iii) of Proposition 3.9 that we can always obtain a centralizer of $\mathcal{X}$ with any linear part. Let $\widetilde{\mathcal{Y}} = \mathcal{Y}_C + \cdots = (-y + \cdots)\partial_x + (x + \cdots)\partial_y$ be a centralizer of $\mathcal{X}$ with the linear part of center type. Then $\phi_{*1}\widetilde{\mathcal{Y}} = \mathcal{Y}_C$ and $\phi_{*2}\widetilde{\mathcal{Y}} = \mathcal{Y}_C$. This fact contradicts Proposition 3.12, therefore $\phi_1 = \phi_2 = \phi$ and the uniqueness of ϕ is proved. ∎

Remark 3.14 Let $A = (a_{ij})$ be a 2×2 complex matrix and I the identity matrix. Consider two vector fields $\mathcal{X} = \mathcal{X}_A + \cdots$ and $\mathcal{Y} = \mathcal{Y}_I + \cdots$ smooth in a neighborhood $U \subset \mathbb{C}^2$ of the origin in such a way that $[\mathcal{X}, \mathcal{Y}] \equiv 0$. It is clear, from Theorem 3.2, that there is a smooth near–identity change of variables $(x, y) \mapsto \phi(x, y) = (X(x, y), Y(x, y)) = (x + \cdots, y + \cdots)$ near the origin such that linearizes $\mathcal{X}$, i.e., $\phi_*\mathcal{X} = \mathcal{X}_A$. In order to apply Theorems 3.6, 3.7 and 3.11 to get ϕ, we must first carry out a linear change of coordinates $\phi_1(x, y) = (z(x, y), w(x, y)) \in \mathcal{L}(\mathbb{C}^2)$ bringing the linear part A of $\mathcal{X}$ to Jordan canonical form J as in (3.2). Notice moreover that, since the radial vector field $\mathcal{Y}_I$ is invariant under linear changes of coordinates, and the Lie bracket is coordinates–free, we will have $[\phi_{1*}\mathcal{X}, \phi_{1*}\mathcal{Y}] \equiv 0$, where $\phi_{1*}\mathcal{X} = \mathcal{X}_J + \cdots$, $\phi_{1*}\mathcal{Y} = \mathcal{Y}_I + \cdots$. Now we can construct, using Theorems 3.6, 3.7 or 3.11, a smooth near–identity change of variables $(z, w) \mapsto \phi_2(z, w) = (u(z, w), v(z, w)) = (z + \cdots, w + \cdots)$ near the origin linearizing $\phi_{1*}\mathcal{X}$. Since, in addition, such class of transformations preserves the linear part of vector fields we get $(\phi_{2*} \circ \phi_{1*})\mathcal{X} = \mathcal{X}_J$ and $(\phi_{2*} \circ \phi_{1*})\mathcal{Y} = \mathcal{Y}_I + \cdots$. Finally, we apply the last linear change of coordinates $\phi_1^{-1}(u, v) = (X(u, v), Y(u, v)) \in \mathcal{L}(\mathbb{C}^2)$ in order to obtain $(\phi_{1*}^{-1} \circ \phi_{2*} \circ \phi_{1*})\mathcal{X} = \mathcal{X}_A$. Then clearly, since the composition of push–forward is just the push–forward of the composition, $\phi = \phi_1^{-1} \circ \phi_2 \circ \phi_1$.

3.5 Examples

3.5.1 Quadratic Lotka–Volterra systems

We consider the quadratic Lotka–Volterra systems

$$\dot{x} = x(1 + ax + by) \ , \dot{y} = y(-\nu + cx + dy) \ . \tag{3.18}$$

defined on $\mathbb{C}^2$. Necessary and sufficient conditions for linearizability are already known for the case $\nu \in \mathbb{N}$. More concretely, in [20] it is proved that system (3.18) with $\nu \in \mathbb{N} \setminus \{1\}$ is analitically linearizable if and only if one of the following conditions is satisfied: (i) $ma + c = 0$ with $m = 0, 1, \ldots, \nu - 2$; (ii) $b = d = 0$; (iii) $a - c = b - d = 0$; (iv) $b = (\nu - 1)a + c = 0$. See also [36, 40] for some generalizations of the values of ν.

By using our results we prove the following.

Proposition 3.15 *Let $\mathcal{X}$ be the associated vector field to the quadratic Lotka–Volterra system (3.18) with $\nu \in \mathbb{C}\setminus\{0\}$. Then $\mathcal{X}$ possesses a quadratic polynomial commutator $\mathcal{Y} = \mathcal{Y}_I + \cdots$ if one of the following conditions is satisfied:*

(i) $b = c = 0$. *In this case* $\mathcal{Y} = x(1 + ax)\partial_x + y(1 - d\, y/\nu)\partial_y$.

(ii) $a = c$ *and* $b = d$. *Hence* $\mathcal{Y} = x(1 + cx - d\, y/\nu)\partial_x + y(1 + cx - d\, y/\nu)\partial_y$.

Moreover, the changes of variables $(x, y) \mapsto (u(x, y), v(x, y))$ that linearizes $\mathcal{X}$ with $\nu \in \mathbb{C}\setminus\{0\}$ is

(i) If $b = c = 0$ then

$$u(x, y) = \frac{x}{1 + ax} \;, \;\; v(x, y) = -\frac{\nu y}{dy - \nu} \;. \tag{3.19}$$

(ii) When $a = c$ and $b = d$ then

$$u(x, y) = \frac{x}{1 + cx - dy/\nu} \;, \;\; v(x, y) = \frac{y}{1 + cx - dy/\nu} \;. \tag{3.20}$$

Proof. It is an easy computation to show that cases (i) and (ii) gives the commutators of system (3.18) with $\nu \in \mathbb{C}\setminus\{0\}$.

Now we shall compute, in each case, the linearizing changes of variables $(x, y) \mapsto (u(x, y), v(x, y))$ using our results.

(i) The vector field $\mathcal{X} = x(1 + ax)\partial_x + y(-\nu + dy)\partial_y$ commutes with $\mathcal{Y} = x(1 + ax)\partial_x + y(1 - d\, y/\nu)\partial_y$. Then $V = \mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$. Up to multiplicative constants we get $V(x, y) = xy(1 + ax)(dy - \nu)$. So, we obtain associated to V the following first integrals

$$H(x, y) = y^{-1/\nu}(1 + ax)(dy - \nu)^{1/\nu}x^{-1}, \;\; I(x, y) = xy^{-1}(dy - \nu)(1 + ax)^{-1} \;,$$

of $\mathcal{X}$ and $\mathcal{Y}$, respectively. We take the following new first integrals

$$f(H) = -H^\nu/\nu = \frac{u^{-\nu}(x,y)}{v(x,y)}, \; g(I) = -I/\nu$$

where the change of variables $(x,y) \mapsto (u(x,y), v(x,y))$ that linearizes both $\mathcal{X}$ and $\mathcal{Y}$ are given in (3.19).

(ii) The vector fields $\mathcal{X} = x(1+cx+dy)\partial_x + y(-\nu+cx+dy)\partial_y$ and $\mathcal{Y} = x(1+cx-d\ y/\nu)\partial_x + y(1+cx-d\ y/\nu)\partial_y$ satisfy $[\mathcal{X},\mathcal{Y}] \equiv 0$. Then, from the inverse integrating factor $V = \mathcal{X} \wedge \mathcal{Y}$ of both vector fields $\mathcal{X}$ and $\mathcal{Y}$ we obtain the following first integrals

$$H(x,y) = xy^{\frac{1}{\nu}}[-dy+(1+cx)\nu]^{-\frac{1+\nu}{\nu}}, \; I(x,y) = \frac{x}{y} \; ,$$

of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Notice that $\mathcal{Y}$ is orbitally linearizable (linearizable after time rescaling) and therefore $I(x,y)$ is a trivial first integral. We take the following new first integral

$$f(H) = \nu^{-1-\nu}H^{-\nu} = \frac{u^{-\nu}(x,y)}{v(x,y)}$$

where $u(x,y)$ and $v(x,y)$ are given in (3.20). Of course, u/v is just I. It is straightforward to check that the change of variables $(x,y) \mapsto (u(x,y), v(x,y))$ linearizes both $\mathcal{X}$ and $\mathcal{Y}$. ∎

Remark 3.16 In fact, one can shows that cases (i) and (ii) are the only relationships between the parameters of system (3.18) with

$$\nu \in \mathbb{C}\backslash\{0,1,-2,-3,-1/2,-1/3\}$$

such that there is a polynomial commutator $\mathcal{Y} = \mathcal{Y}_I + \cdots$ of degree at most 3.

3.5.2 Isochronous center

This example comes from the work [12]. In that paper the authors present an exhaustive classification of several families of isochronous centers and, in many cases, their commutators and linearizing changes. In any way, there are some incomplete examples. They present the following cubic reversible system

$$\mathcal{X} = -y(1-x)(1-2x)\partial_x + (x-2x^2+y^2+2x^3)\partial_y,$$

and its commutator $\mathcal{Y}$, where

$$\mathcal{Y}=\frac{(1-x)(x-2x^2+2y^2+2x^3-2xy^2)}{1-2x}\partial_x+\frac{y(1-x)(1-4x+6x^2+2y^2)}{1-2x}\partial_y,$$

but they do not give the linearizing change of variables. Moreover, in Chapter 2 the authors obtain the linearizing change of $\mathcal{X}$ and orbitally linearizes $\mathcal{Y}$. In this example we get the linearizing change of variables that linearizes both, $\mathcal{X}$ and $\mathcal{Y}$, by using ours results.

First of all, we will make a linear change of coordinates

$$\phi_1(x,y)=(z(x,y),w(x,y))$$

to write the linear part of the vector field $\mathcal{X}$ in diagonal form. It is matter of linear algebra to see that the change of coordinates ϕ_1 is the following

$$z(x,y)=\frac{1}{2}(\mathrm{i}\,x+y),\quad w(x,y)=\frac{1}{2}(-\mathrm{i}\,x+y),$$

where $\mathrm{i}^2=-1$. The transformed vectors fields $\widetilde{\mathcal{X}}=\phi_{*1}\mathcal{X}$ and $\widetilde{\mathcal{Y}}=\phi_{*1}\mathcal{Y}$ are

$$\begin{aligned}\widetilde{\mathcal{X}} &= (-\mathrm{i}z+3z^2-zw+2\mathrm{i}z^3-4\mathrm{i}z^2w+2\mathrm{i}zw^2)\,\partial_z\\ &\quad +(\mathrm{i}w+3w^2-zw-2\mathrm{i}w^3+4\mathrm{i}zw^2-2\mathrm{i}z^2w)\,\partial_w\end{aligned}$$

and

$$\widetilde{\mathcal{Y}}=\left(\frac{-\mathrm{i}+z-w}{-\mathrm{i}+2z-2w}\right)(-z((\mathrm{i}-2z)^2-8zw-4w^2)\,\partial_z+w(-(\mathrm{i}+2w)^2+8zw+4z^2)\,\partial_w).$$

We note that, since the radial part of a vector field is preserved under linear change of coordinates, $\widetilde{\mathcal{Y}}$ has radial linear part, too. Taking into account $[\widetilde{\mathcal{X}},\widetilde{\mathcal{Y}}]=0$, we get that $V=\widetilde{\mathcal{X}}\wedge\widetilde{\mathcal{Y}}$ is an inverse integrating factor for both vector fields, $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$. This fact allows us to obtain (after some simplifications) first integrals $H(z,w)$ and $I(z,w)$ of $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$, respectively, as follows

$$H(z,w)=\frac{zw(\mathrm{i}-z+w)^2}{(1+4z(\mathrm{i}-z+w))(1-4w(\mathrm{i}-z+w))}=u(z,w)v(z,w),$$

and

$$I(z,w)=\frac{z(1-4w(\mathrm{i}-z+w))}{w(1+4z(\mathrm{i}-z+w))}=\frac{u(z,w)}{v(z,w)},$$

where

$$u(z,w) = \frac{-z\mathrm{i}(\mathrm{i}-z+w)}{1+4z(\mathrm{i}-z+w)}, \quad v(z,w) = \frac{-w\mathrm{i}(\mathrm{i}-z+w)}{1-4w(\mathrm{i}-z+w)}.$$

Due to (3.14) the change of coordinates $\phi_2(u,v) = (u(z,w), v(z,w))$ linearizes $\widetilde{\mathcal{X}}$ and $\widetilde{\mathcal{Y}}$. Finally, from Remark 3.14 the change of coordinates $\phi(x,y) = (X(x,y), Y(x,y))$ that linearizes $\mathcal{X}$ and $\mathcal{Y}$ is $\phi = {\phi_1}^{-1} \circ \phi_2 \circ \phi_1$, that is,

$$\begin{aligned} X &= \frac{(1-x)((x(1+2(-1+x)x)) + 2(-1+x)y^2)}{(1+2(-1+x)x)^2 + 4(-1+x)^2y^2}, \\ Y &= \frac{y(1-x)}{(1+2(-1+x)x)^2 + 4(-1+x)^2y^2}. \end{aligned}$$

Hence, ϕ brings $\mathcal{X}$ and $\mathcal{Y}$ to the linear vector fields $-Y\partial_X + X\partial_Y$ and $X\partial_X + Y\partial_Y$ respectively.

3.5.3 Linearizable hyperbolic saddle

In [35] the authors present the following 1–parameter family of vector fields $\mathcal{X}_c$ and their commutator $\mathcal{Y}_c$, where

$$\mathcal{X}_c = (2x - 4y^2 + 6cy^4)\,\partial_x - y(1-3cy^2)\,\partial_y, \quad \mathcal{Y}_c = (x + y^2 - 6cy^4)\,\partial_x + y(1-3cy^2)\,\partial_y\,,$$

and $c \in \mathbb{R}$ is an arbitrary parameter. Moreover, they deduce that vector fields $\mathcal{X}_c$ are linearizable in a neighborhood of the origin, which is an hyperbolic saddle. They give the linearizing transformation $(x,y) \mapsto (u,v)$ as follows

$$x = u + \frac{v^2}{1+3cv^2}, \quad y = \frac{v}{\sqrt{1+3cv^2}}. \tag{3.21}$$

We obtain the linearizing change of coordinates by applying our results. Taking into account $[\mathcal{X}_c, \mathcal{Y}_c] = 0$, we get that $V = \mathcal{X}_c \wedge \mathcal{Y}_c = 3y(-x+y^2)(-1+3cy^2)$ is an inverse integrating factor for both vector fields, $\mathcal{X}_c$ and $\mathcal{Y}_c$. The first integrals $H(x,y)$ and $I(x,y)$ associated with V of $\mathcal{X}_c$ and $\mathcal{Y}_c$ respectively (after some simplifications) are given by

$$H(x,y) = \frac{1-3cy^2}{y^2(x-y^2)} = \frac{v^2(x,y)}{u(x,y)}, \quad I(x,y) = \frac{(x-y^2)\sqrt{1-3cy^2}}{y} = \frac{u(x,y)}{v(x,y)},$$

where

$$u = x - y^2, \quad v = \frac{y}{\sqrt{1-3cy^2}}.$$

Taking into account (3.9), the change of coordinates $\phi(u,v) = (u(x,y), v(x,y))$ linearizes $\mathcal{X}_c$. The change of coordinates obtained for us is the inverse change of (3.21). We note that this change of variables also linearizes the commutator $\mathcal{Y}_c$, hence by Theorem 4.5 this is the unique change of variables that linearizes both $\mathcal{X}_c$ and $\mathcal{Y}_c$.

Chapter 4

Orbital Linearization From Symmetries

Summary. This work is a generalization of the method proposed in Chapter 3 of linearization of planar vector fields around singularities. Here we propose a constructive procedure to get the change of variables that orbitally linearizes a smooth planar vector field on $\mathbb{C}^2$ around an elementary singular point (i.e., a singular point with associated eigenvalues $\lambda, \mu \in \mathbb{C}$ satisfying $\lambda \neq 0$) or a nilpotent singular point from a given infinitesimal generator of a Lie symmetry.

4.1 Introduction

In this work we focus our attention on the orbital linearization problem, i.e., to determine the local Poincaré normal form transformation that maps the foliation defined by the solutions of a smooth nonlinear planar system into the foliation of a linear one. We consider two-dimensional smooth differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{4.1}$$

defined in an open set $U \subset \mathbb{C}^2$ containing the origin. Let $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ be its associated vector field. We assume that the origin is either an elementary singular point (i.e., a singular point with associated eigenvalues $\lambda, \mu \in \mathbb{C}$ satisfying $\lambda \neq 0$) or a nilpotent singular point (that is, a singular point with both eigenvalues zero but with non–vanishing associated linear part). Following the notation of Chapter 3, the linear part A of system (4.1) written in Jordan form is of the form

$$\textbf{(i)}\ A = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}\ ,\ \textbf{(ii)}\ A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}\ . \tag{4.2}$$

where $\lambda \neq 0$ for the case (i). $\mathcal{X}_A$ will be the linear vector field with associated matrix $A = (a_{ij}) \in \mathcal{M}_2(\mathbb{C})$ and a smooth vector field $\mathcal{X}$ in $\mathbb{C}^2$ with linear part $\mathcal{X}_A$ is expressed as $\mathcal{X} = \mathcal{X}_A + \cdots$, where the dots denote a smooth vector field without linear terms.

The origin of system (4.1) is said to be *orbitally linearizable* if there exists a smooth near–identity change of coordinates $\phi(x,y) = (u(x,y), v(x,y)) = (x + o(x,y)), y + o(x,y))$ in the neighborhood $U \subset \mathbb{C}^2$ of the origin transforming the system into

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = A \begin{pmatrix} u \\ v \end{pmatrix} h(u,v), \tag{4.3}$$

where $h(u,v)$ is a smooth scalar function defined on U such that $h(0,0) \neq 0$. This means that $\phi_* \mathcal{X} = h\mathcal{X}_A$, where $h(x,y) = 1 + \cdots$ is a smooth scalar function defined on U and ϕ_* is the push–forward defined by the smooth diffeomorphism ϕ. In the *linearization* case one has $\phi_* \mathcal{X} = \mathcal{X}_A$, that is $h(x,y) \equiv 1$. The linearization or orbital linearization problem is treated classically by the normal form theory, see [10, 23]. Given a system (4.1), the necessary and sufficient conditions to become linearizable or orbitally linearizable in U as well as the regularity of the change of variables ϕ are given by normal form theory. Nevertheless, to explicitly determine this change ϕ in closed form is, in general, an open problem. Another approach to the linearization or orbital linearization problem is given in [35], where it is shown that the existence of such a ϕ is equivalent to the existence of a smooth vector field of the form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ such that the Lie bracket $[\mathcal{X}, \mathcal{Y}] = \nu \mathcal{X}$ with ν a smooth scalar function not identically zero in the orbital linearization case and with $\nu \equiv 0$ in the linearization case. The smooth vector field $\mathcal{Y}$ is the infinitesimal generator of a Lie symmetry of system (4.1). The existence of such $\mathcal{Y}$ ensures the existence of the change of coordinates which orbitally linearizes or linearizes the vector field $\mathcal{X}$ in U, but in a similar way that in the normal form theory, the proof given in

[35] of this fact is not constructive, that is, a constructive way to build this change of coordinates is not given. In Chapter 2 the change of variables that linearizes the analytic isochronous centers from a given commutator is obtained. Later, in Chapter 3 we propose a constructive procedure to get the change of variables in closed form that linearizes a smooth planar vector field $\mathcal{X}$ on $\mathbb{C}^2$ around an elementary singular point from a given infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of the smooth vector field $\mathcal{X}$. Thus, the orbital linearization problem is the natural next step to study. In this chapter we present a constructive procedure to obtain the closed form of the orbital linearizing change of coordinates of system (4.1) around the origin from a given Lie symmetry of it. Hence, in this work we assume $\nu \neq 0$, that is, $\mathcal{Y}$ does not commute with $\mathcal{X}$. The problem of how to obtain the orbital linearizing change is reduced to know an infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of system (4.1). The problem of knowing an infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of a given system is discussed in [13] for special integrable cases. Finally, we conclude the work with some instructive examples in which we apply the constructive procedure to obtain the smooth change of variables that orbitally linearizes some smooth systems.

4.2 Some preliminary results

In this section we give some preliminary results needed to establish the main theorems of the chapter. The proof of the following proposition is straightforward from the definition of inverse integrating factor, see [15].

Proposition 4.1 *Consider two vector fields $\mathcal{X}_1$ and $\mathcal{X}_2$ defined in an open subset $U \subset \mathbb{C}^2$, which have the same inverse integrating factor V in U. Then the vector field $\mathcal{X} = \mathcal{X}_1 + \alpha \mathcal{X}_2$ has also the function V as inverse integrating factor for arbitrary values of the parameter $\alpha \in \mathbb{C}$.*

The following propositions are proved in Chapter 3.

Proposition 4.2 *Consider the vector field $\mathcal{X}$ associated to the system (4.1) with A semisimple having eigenvalues λ and μ satisfying either $\lambda = \mu \neq 0$ or $0 \neq \lambda \neq \mu \neq 0$ with $\lambda/\mu \notin \mathbb{Q}^-$. Then, $\mathcal{X}$ cannot have a smooth first integral in a neighborhood of the origin.*

Proposition 4.3 *Let* $\mathcal{Y} = (x+o(x,y))\partial_x + (y+o(x,y))\partial_y$ *be a smooth vector field in a neighborhood* $U \subset \mathbb{C}^2$ *of the origin. Then, there is a unique smooth near-identity change of variables* $\phi : U \to \mathbb{C}^2$ *that linearizes* $\mathcal{Y}$.

Proposition 4.4 *The generator of dilations* $\mathcal{X}_I = x\partial_x + y\partial y$ *satisfies* $[\mathcal{X}_I, \mathcal{X}] = 0$ *with* $\mathcal{X}$ *a smooth vector field if and only if* $\mathcal{X} = \mathcal{X}_A$ *for some* $A \in \mathcal{M}_2(\mathbb{C})$.

4.3 Main results

Given a smooth vector field $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ in U, Proposition 4.3 states that the smooth near–identity change $\phi : U \to \mathbb{C}^2$ that linearizes $\mathcal{Y}$ is unique. The following theorem gives another application of this unique smooth near–identity change ϕ.

Theorem 4.5 *Let* $\mathcal{X}$ *and* $\mathcal{Y} = (x+o(x,y))\partial_x + (y+o(x,y))\partial_y$ *be two smooth vector fields in a neighborhood* $U \subset \mathbb{C}^2$ *of the origin such that* $[\mathcal{X}, \mathcal{Y}] = \nu(x,y)\ \mathcal{X}$ *with* ν *a smooth scalar function satisfying* $\nu(0,0) = 0$. *Then, the unique smooth near-identity change of variables* $\phi : U \to \mathbb{C}^2$ *that linearizes* $\mathcal{Y}$ *also orbitally linearizes* $\mathcal{X}$.

Proof. Using Proposition 4.3, let ϕ be the unique smooth near-identity change of variables that linearizes $\mathcal{Y}$. Then $\phi_*\mathcal{Y} = \mathcal{Y}_I$ where $I \in \mathcal{M}_2(\mathbb{C})$ is the identity matrix. Let $\mathcal{X} = \mathcal{X}_A + \cdots$ be the smooth vector field such that $[\mathcal{X}, \mathcal{Y}] = \nu(x,y)\ \mathcal{X}$ with ν a smooth scalar function satisfying $\nu(0,0) = 0$. Since the Lie bracket is coordinate–free, we will have $[\phi_*\mathcal{X}, \phi_*\mathcal{Y}] = [\phi_*\mathcal{X}, \mathcal{Y}_I] = \bar{\nu}\, \phi_*\mathcal{X}$, where $\bar{\nu}$ is the transformed function of $\nu(x,y)$ by the change of variables ϕ. Moreover, there exists a smooth scalar function $\lambda(x,y)$ with $\lambda(0,0) = 1$ such that $[\lambda(x,y)\phi_*\mathcal{X}, \mathcal{Y}_I] \equiv 0$, see the proof of Theorem 1.17 in [35]. On the other hand, if a smooth vector field $\lambda(x,y)\phi_*\mathcal{X} = \mathcal{X}_A + \cdots$ commutes with the generator of dilations $\mathcal{Y}_I$ this implies that $\lambda(x,y)\phi_*\mathcal{X} = \mathcal{X}_A$ which is the linear part of the vector field, see Proposition 4.4. Therefore, $\phi_*\mathcal{X} = h(x,y)\mathcal{X}_A$, where $h(x,y) = 1/\lambda(x,y)$, which implies that the smooth vector field $\mathcal{X}$ is orbitally linearizable by the change of variables ϕ. ∎

In the following theorems we show a method to obtain the change of coordinates that orbitally linearizes (4.1) when the linear part of the system takes the form (i) of (4.2), that is, when A is semisimple. We consider first the case in which the eigenvalues μ and λ satisfy $\mu \neq \mp\lambda \neq 0$.

Theorem 4.6 *Let $\mathcal{X} = (\lambda x + o(x,y))\partial_x + (\mu y + o(x,y))\partial_y$ with $\mu \neq \mp\lambda \neq 0$ and $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ be two smooth vector fields in a neighborhood $U \subset \mathbb{C}^2$ of the origin such that $[\mathcal{X}, \mathcal{Y}] \equiv \nu\mathcal{X}$ with ν a smooth scalar function satisfying $\nu(0,0) = 0$. Then, a smooth near–identity change of variables $u = x + o(x,y)$, $v = y + o(x,y)$, that orbitally linearizes $\mathcal{X}$ is obtained as follows:*

$$u = g(I)\left(\frac{f(H)}{g^{\mu}(I)}\right)^{\frac{1}{\mu-\lambda}}, \quad v = \left(\frac{f(H)}{g^{\mu}(I)}\right)^{\frac{1}{\mu-\lambda}}, \tag{4.4}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and f and g are two functions such that $f(H(x,y)) = (x + o(x,y))^{\mu}/(y + o(x,y))^{\lambda}$ and $g(I(x,y)) = (x + o(x,y))/(y + o(x,y))$.

Proof. The existence of the orbital linearizing change of variables $(x,y) \mapsto (u,v)$ is known from Theorem 1.17. The new result is how to obtain the orbital linearizing change of variables from the knowledge of an infinitesimal generator $\bar{\mathcal{Y}}$ of a Lie symmetry of $\mathcal{X}$ of the form $\bar{\mathcal{Y}} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$.

Our first purpose is to know the first integrals of the two vector fields $\mathcal{X}$ and $\bar{\mathcal{Y}}$ involved in the symmetry. Notice that $\mathcal{X}$ has the following inverse integrating factor $V(x,y) = \mathcal{X} \wedge \bar{\mathcal{Y}}$ defined in U. Let $H(x,y)$ be the first integral of $\mathcal{X}$ in U associated to V computed via quadrature (1.5). Nevertheless, we are not able to obtain a first integral of $\bar{\mathcal{Y}}$ in U. Therefore, we are looking for another infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of $\mathcal{X}$ of the form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ such that its first integral in U can be obtained. From the knowledge of $V(x,y)$, and applying (1.8) we obtain another infinitesimal generator $\mathcal{Y}_1$ of a Lie symmetry of $\mathcal{X}$, that is,

$$\mathcal{Y}_1 = \frac{1}{(\lambda + \mu) + o(x,y)}((\mu - \lambda)x + o(x,y))\partial_x + ((\lambda - \mu)y + o(x,y))\partial_y).$$

Notice that $V(x,y)/\mathrm{div}\mathcal{X}$ is an inverse integrating factor of $\mathcal{Y}_1$. Since the linear part of $\mathcal{Y}_1$ has not the desired radial form, we get a new infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of $\mathcal{X}$ for the case $\lambda \neq -\mu$ of the form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ as follows, see [29].

$$\mathcal{Y} = \mathcal{Y}_1 + 2\mathcal{X}/\mathrm{div}\mathcal{X}, \tag{4.5}$$

Taking into account Proposition 4.1, it follows that $V(x,y)/\mathrm{div}\mathcal{X}$ is an inverse integrating factor of $\mathcal{Y}$ defined on U. This fact allows us to construct a first integral $I(x,y)$ of $\mathcal{Y}$ in U. Notice, that finally we have obtained an infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of $\mathcal{X}$ with the desired form $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ whose first integral I in U is known.

From (4.4) we have that $f(H) = (x + o(x,y))^\mu/(y + o(x,y))^\lambda = u^\mu/v^\lambda$ and $g(I) = (x + o(x,y))/(y + o(x,y)) = u/v$ are first integrals of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Notice that such first integrals $f(H)$ and $g(I)$ always exist as a consequence of Theorem 4.5. Taking into account $\mathcal{X}f(H) = \mathcal{Y}g(I) \equiv 0$, it follows

$$\mu v\mathcal{X}(u) - \lambda u\mathcal{X}(v) \equiv 0 \ , \ v\mathcal{Y}(u) - u\mathcal{Y}(v) \equiv 0 \ .$$

Therefore we have

$$\mathcal{X}(u) = \lambda u\Lambda \ , \ \mathcal{X}(v) = \mu v\Lambda \ , \ \mathcal{Y}(u) = u\Omega \ , \ \mathcal{Y}(v) = v\Omega \ , \tag{4.6}$$

with $\Lambda(x,y)$ and $\Omega(x,y)$ smooth functions in a neighborhood of the origin.

On the other hand, since $[\mathcal{X}, \mathcal{Y}] = \nu\mathcal{X}$, in particular we have

$$\mathcal{X}\mathcal{Y}(u) - \mathcal{Y}\mathcal{X}(u) = \nu\lambda u\Lambda \ , \ \mathcal{X}\mathcal{Y}(v) - \mathcal{Y}\mathcal{X}(v) = \nu\mu v\Lambda \ .$$

Introducing (4.6) in the former relations we get

$$\mathcal{X}(\Omega) - \lambda\mathcal{Y}(\Lambda) = \nu\lambda\Lambda \ , \ \mathcal{X}(\Omega) - \mu\mathcal{Y}(\Lambda) = \nu\mu\Lambda.$$

This is a linear algebraic system for the unknowns $\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Lambda)$ with associated determinant $\lambda - \mu$ which is different from zero because $\mu \neq \lambda$. So the unique solution is $\mathcal{X}(\Omega) = 0$ and $\mathcal{Y}(\Lambda) = -\nu\Lambda$. From this last equality, Λ is a smooth invariant curve of $\mathcal{Y}$ with cofactor $-\nu$. Hence, the change of variables $(x,y) \to (u,v)$ orbitally linearizes the vector field $\mathcal{X}$ because $\Lambda(0,0) = 1 + \cdots$. This is due to the fact that $\mathcal{X}(u) = \lambda u\Lambda$, that is, $(\lambda x + \cdots)\partial_x(x + \cdots) + (\mu y + \cdots)\partial_y(x + \cdots) = (\lambda x + \cdots)(\Lambda(0,0) + \cdots)$, and at first order we have $\Lambda(0,0) = 1$.

On the other hand, Ω is either a constant or a smooth first integral of $\mathcal{X}$ in a neighborhood of the origin. If Ω is a smooth first integral of $\mathcal{X}$, then, the change $(x,y) \mapsto (u(x,y), v(x,y))$ also orbitally linearizes the vector field $\mathcal{Y}$ because $\Omega(0,0) = 1$. This is due to the fact that, from $\mathcal{Y}(u) = u\Omega$, that is, $(x+o(x,y))\partial_x(x+o(x,y)) + (y+o(x,y))\partial_y(x+o(x,y)) = (x + o(x,y))(\Omega(0,0) + o(x,y))$, at first order we get $\Omega(0,0) = 1$. On the contrary, if Ω is a constant, it is different from zero because otherwise, from (4.6), u and v would be smooth first integrals of $\mathcal{Y}$ and this fact contradicts Proposition 4.2. Therefore, if Ω is a constant the change $(x,y) \mapsto (u(x,y), v(x,y))$

linearizes the vector field $\mathcal{Y}$. ∎

Remark. We recall that given a vector field $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$ which admits an infinitesimal generator $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ of a Lie symmetry of $\mathcal{X}$ in U, we can always construct an inverse integrating factor of $\mathcal{X}$ in U as $V(x,y) := \mathcal{X} \wedge \mathcal{Y} = P\eta - Q\xi$. The converse is not always true. If we know the existence of an inverse integrating factor V of $\mathcal{X}$ in U in some cases we can not construct an infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of $\mathcal{X}$ well defined in U and therefore we can not apply our procedure. We recall that a singular point $p \in U$ of $\mathcal{X}$ is called *weak* if $\mathrm{div}\mathcal{X}(p) = 0$. If there is no weak singularity of $\mathcal{X}$ in U, then we can do at least one of the following constructions:

(i) Prescribe the function $\xi(x,y)$ and solve $\eta(x,y)$ from $V = P\eta - Q\xi$.

(ii) Prescribe the function $\eta(x,y)$ and solve $\xi(x,y)$ from $V = P\eta - Q\xi$.

(iii) Take the rescaled hamiltonian vector field $\mathcal{Y} = \dfrac{1}{\mathrm{div}\mathcal{X}}\,(-\dfrac{\partial V}{\partial y}\partial_x + \dfrac{\partial V}{\partial x}\partial_y)$, defined in $U \backslash \{(x,y) \in U : \mathrm{div}\mathcal{X} = 0\}$.

The equivalence between inverse integrating factors and Lie symmetries for planar vector fields $\mathcal{X}$ is not true, in general, in neighborhoods of weak singular points of $\mathcal{X}$. Of course, in some special weak singular points the equivalence can be done. For instance, in a neighborhood $U \subset \mathbb{R}^2$ of a non-degenerate center (i.e., a type center singular point with eigenvalues different from zero), $\mathcal{X}$ always possesses an analytic first integral, an analytic inverse integrating factor as well as an analytic infinitesimal generator $\mathcal{Y}$ of a Lie symmetry of $\mathcal{X}$ well defined in U, see [34, 64].

In the next theorem we show a method to obtain the change of coordinates that orbitally linearizes (4.1) when A is semisimple and $\mu = -\lambda \neq 0$. Notice that in this case the origin of system (4.1) is a weak singular point. Hence, given an inverse integrating factor of $\mathcal{X}$, we can not obtain applying (1.8) an infinitesimal generator $\mathcal{Y} = \xi(x,y)\partial_x + \eta(x,y)\partial_y$ of a Lie symmetry of $\mathcal{X}$ well defined in U. Therefore, in the following theorem we look for another approach in order to obtain the change of coordinates that orbitally linearizes (4.1) around a weak singular point.

Theorem 4.7 *Let $\mathcal{X} = (\lambda x + o(x,y))\partial_x + (-\lambda y + o(x,y))\partial_y$ with $\lambda \neq 0$ and $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ be two smooth vector fields in a neighborhood $U \subset \mathbb{C}^2$ of the origin such that $[\mathcal{X}, \mathcal{Y}] \equiv \nu \mathcal{X}$ with ν a smooth scalar function satisfying $\nu(0,0) = 0$. Then, a smooth near–identity change of variables $\phi(x,y) = (u,v) = (x+o(x,y), y+o(x,y))$ that orbitally linearizes $\mathcal{X}$ is obtained from $f(H(x,y)) = uv$, where H is a first integral of $\mathcal{X}$.*

Proof. From Theorem 1.17 we know the existence of an orbital linearizing change of variables $(x,y) \mapsto (u,v)$ that brings the vector field $\mathcal{X}$ to $\phi_*\mathcal{X} = \lambda h(u,v)(u\partial_u - v\partial_v)$. Therefore $\tilde{H} = uv$ is a first integral of $\phi_*\mathcal{X}$. Pulling back $\tilde{H}$ to the initial coordinates we obtain a first integral $H(x,y) = (x + o(x,y))(y + o(x,y))$ of $\mathcal{X}$. In the rest of the proof we will see that any factorization of H of the form $H(x,y) = (x + o(x,y))(y + o(x,y))$ gives us a change of variables $u = x+o(x,y)$, $v = y+o(x,y)$ that orbitally linearizes $\mathcal{X}$. Taking into account $\mathcal{X}H \equiv 0$ it follows $v\mathcal{X}(u) + u\mathcal{X}(v) \equiv 0$. Therefore, we have $\mathcal{X}(u) = u\Omega$ and $\mathcal{X}(v) = -v\Omega$ with $\Omega(x,y)$ certain smooth function in a neighborhood of the origin. Hence, the change of variables $(x,y) \mapsto (u,v)$ orbitally linearizes the vector field $\mathcal{X}$ because $\Omega(0,0) = \lambda \neq 0$. This value of Ω at the origin follows by taking the first order term in the expansion of $\mathcal{X}(u) = u\Omega$ given by $(\lambda x + \cdots)\partial_x(x + \cdots) + (-\lambda\ y + \cdots)\partial_y(x + \cdots) = (x + \cdots)(\Omega(0,0) + \cdots)$. ∎

On the other hand, it is known that a resonant hyperbolic saddle (i.e., a singular point which eigenvalues satisfy $\lambda/\mu \in \mathbb{Q}^-$) of an analytic vector field is analytically orbitally linearizable if, and only if, it has an analytic first integral in the neighborhood of it, see [37]. Theorem 4.7 generalizes this result in the case of a weak singularity for smooth vector fields with an infinitesimal generator of a Lie symmetry, giving also the explicit change of variables.

The following proposition clarifies how is the normal form when system (4.1) is analytic and A is semisimple with $\mu = \lambda$.

Proposition 4.8 *Let $\mathcal{X} = (\lambda x + o(x,y))\partial_x + (\lambda y + o(x,y))\partial_y$ be an analytic vector field in a neighborhood $U \subset \mathbb{C}^2$ of the origin. Then, the origin is an analytically linearizable 1:1 resonant node of $\mathcal{X}$.*

The proof of the Proposition 4.8 comes from normal form theory, see [10]. In the light of this result, in this work we do not take into account the case

$\mu = \lambda$ because it is already solved in Chapter 3.

In the next theorem we obtain the method to get the change of coordinates that orbitally linearizes (4.1) when the linear part of the system takes the form (ii) of (4.2).

Theorem 4.9 *Let $\mathcal{X} = (\lambda x + y + o(x,y))\partial_x + (\lambda y + o(x,y))\partial_y$ with $\lambda \neq 0$ and $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ be two smooth vector fields in a neighborhood $U \subset \mathbb{C}^2$ of the origin such that $[\mathcal{X}, \mathcal{Y}] \equiv \nu\mathcal{X}$ with ν a smooth scalar function such that $\nu(0,0) = 0$. Then, a smooth near–identity change of variables $u = x + o(x,y)$, $v = y + o(x,y)$, that orbitally linearizes $\mathcal{X}$ is obtained as follows:*

$$u = g(I) f(H) \exp[\lambda g(I)] \ , \ v = f(H) \exp[\lambda g(I)] \ , \tag{4.7}$$

where H and I are first integrals of $\mathcal{X}$ and $\mathcal{Y}$, respectively, and f and g are two functions such that $f(H(x,y)) = (y + \cdots) \exp[-\lambda(x + \cdots)/(y + \cdots)]$, $g(I(x,y)) = (x + \cdots)/(y + \cdots)$.

Proof. From a given infinitesimal generator $\bar{\mathcal{Y}}$ of a Lie symmetry of $\mathcal{X}$ of the form $\bar{\mathcal{Y}} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ and using the same procedure than in the proof of Theorem 4.6, we get another infinitesimal generator $\mathcal{Y}_1$ of a Lie symmetry of $\mathcal{X}$. Then, taking

$$\mathcal{Y} = \mathcal{Y}_1 + 2\mathcal{X}/\mathrm{div}\mathcal{X},$$

we have an infinitesimal generator $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ of a Lie symmetry of $\mathcal{X}$ with $V(x,y)/\mathrm{div}\mathcal{X}$ as inverse integrating factor of $\mathcal{Y}$. From (4.7) we have that $f(H) = v\exp(-u\lambda/v)$ and $g(I) = u/v$ are first integrals of $\mathcal{X}$ and $\mathcal{Y}$ respectively. Using the fact that $\mathcal{X}f(H) = \mathcal{Y}g(I) \equiv 0$ we get

$$\mathcal{X}(u) = \Lambda(v + \lambda u) \ , \ \mathcal{X}(v) = \lambda v\Lambda \ , \mathcal{Y}(u) = u\Omega \ , \ \mathcal{Y}(v) = v\Omega \ . \tag{4.8}$$

On the other hand, taking into account $[\mathcal{X}, \mathcal{Y}] \equiv \nu\mathcal{X}$ and (4.8) we obtain

$$\mathcal{X}(\Omega) - \lambda\mathcal{Y}(\Lambda) = \nu\lambda\Lambda \ , \ u\mathcal{X}(\Omega) - (v + \lambda u)\mathcal{Y}(\Lambda) = \nu(v + \lambda u)\Lambda$$

for some smooth functions $\Omega(x,y)$ and $\Lambda(x,y)$ defined in a neighborhood of the origin. This is a linear homogeneous algebraic system for the unknowns

$\mathcal{X}(\Omega)$ and $\mathcal{Y}(\Lambda)$ with associated determinant v^2. So the unique solution in $\mathbb{C}^2 \backslash \{(u,v) \in \mathbb{C}^2 : v = 0\}$ is $\mathcal{X}(\Omega) = 0$, $\mathcal{Y}(\Lambda) = -\nu\Lambda$. Therefore Λ is a smooth invariant curve of $\mathcal{Y}$ with cofactor $-\nu$. Hence, repeating verbatim the last paragraph of the proof of Theorem 4.6 we have that the change $(x,y) \mapsto (u(x,y), v(x,y))$ orbitally linearizes the vector field $\mathcal{X}$ and also linearizes or orbitally linearizes the vector field $\mathcal{Y}$ depending if Ω is either a smooth first integral of $\mathcal{X}$ or a constant. ∎

The following proposition shows the local structure of orbitally linearizable nilpotents singular points.

Proposition 4.10 *Consider the smooth vector field $\mathcal{X} = (y + o(x,y))\partial_x + o(x,y)\partial_y$ in a neighborhood $U \subset \mathbb{C}^2$ of the origin, which is a nilpotent singular point. If $\mathcal{X}$ is orbitally linearizable near the origin, then it is a nonisolated singular point of $\mathcal{X}$. In this case, $\mathcal{X}$ must be of the form $\mathcal{X} = g(x,y)(1 + o(x,y)\partial_x + o(x,y)\partial_y)$ with $g(x,y) = y + \cdots$. Moreover, all the infinitesimal generators $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ of a Lie symmetry of $\mathcal{X}$ in U must have the invariant curve $g(x,y) = 0$.*

Proof. The vector field $\mathcal{X} = (y + o(x,y))\partial_x + o(x,y)\partial_y)$ written in the coordinates in which it is orbitally linearized is $\mathcal{X} = v\ h(u,v)\partial_u$. Since $v = 0$ is a line filled of singular points of the vector field $\mathcal{X}$, the singular point of $\mathcal{X}$ in the original coordinates must be a nonisolated singular point. Then, it follows that $\mathcal{X}$ must be of the form $\mathcal{X} = g(x,y)(1 + o(x,y)\partial_x + o(x,y)\partial_y)$ with $g(x,y) = y + \cdots$. On the other hand, since the flow of the vector field $\mathcal{Y}$ maps singular points into singular points of $\mathcal{X}$, all the Lie symmetries $\mathcal{Y} = (x + o(x,y))\partial_x + (y + o(x,y))\partial_y$ of $\mathcal{X}$ in U must have the invariant curve $g(x,y) = 0$. ∎

4.4 Examples

In the following example we obtain the orbital linearizing change of variables $\phi(x,y) = (u,v)$ that orbitally linearizes a vector field $\mathcal{X}$ by using Theorem 4.6.

Example 1. In [35] the authors consider the following vector field

$$\mathcal{X} = (x - 2x^2/q + y^2)\partial_x + (-qy/4 + xy)\partial_y, \tag{4.9}$$

with q a parameter such that $-2 \neq q \neq 0$ which admits an infinitesimal generator of a Lie symmetry $\bar{\mathcal{Y}}$ of $\mathcal{X}$ of the form

$$\bar{\mathcal{Y}} = \left(\frac{x(q-2x)}{q-4x} - \frac{2qy^2}{(2+q)(q-4x)} \right) \partial_x + y\partial_y.$$

From the wedge product of $\mathcal{X}$ and $\bar{\mathcal{Y}}$ we get an inverse integrating factor V of $\mathcal{X}$. Then applying (1.8) we have an infinitesimal generator $\mathcal{Y}_1$ of a Lie symmetry of $\mathcal{X}$ well defined if $q \neq 4$. Using (4.5) we obtain another Lie symmetry of (4.9) generated by

$$\mathcal{Y} = \frac{(2+q)(q-2x)x - 2qy^2}{(q-4x)(2+q)} \partial_x + y\partial_y,$$

which has $V(x,y)/\mathrm{div}\mathcal{X}$ as inverse integrating factor. Integrating both vector fields $\mathcal{X}$ and $\mathcal{Y}$, we take the following first integrals

$$f(H) = \frac{(2q+q^2)^{q/4}}{(q^2x - 4x^2 + 2q(x - x^2 + y^2))^{q/4})y}, \quad g(I) = \frac{(q^2x - 4x^2 + 2q(x - x^2 + y^2)}{(2q+q^2)y}.$$

Since,

$$f(H) = \frac{u^{-q/4}(x,y)}{v(x,y)}, \quad g(I) = \frac{u}{v},$$

from equations (4.4) we obtain the change of variables that orbitally linearizes (4.9) as

$$u(x,y) = \frac{(q^2x - 4x^2 + 2q(x - x^2 + y^2))}{(2q+q^2)}, \quad v(x,y) = y.$$

We notice that the change of variables $(x,y) \mapsto (u(x,y), v(x,y))$ also linearizes both vector fields, $\bar{\mathcal{Y}}$ and $\mathcal{Y}$. Moreover, the change also works for $q = 4$. ∎

In the following example we obtain the orbital linearizing change of variables $\phi(x,y) = (u,v)$ that orbitally linearizes a vector field $\mathcal{X}$ by using Theorem 4.7.

Example 2. In [35] the authors consider the following vector field

$$\mathcal{X} = (x - x^2/2 + y^2)\partial_x + (-y + xy)\partial_y, \tag{4.10}$$

which admits a Lie symmetry $\mathcal{Y}$ given by

$$\mathcal{Y} = \left(\frac{x(4-2x)}{4-4x} - \frac{4y^2}{3(4-4x)} \right) \partial_x + y\partial_y.$$

$H = y(x - x^2/2 - 3x^2)$ is a first integral of $\mathcal{X}$ associated to the inverse integrating factor $V = \mathcal{X} \wedge \mathcal{Y}$. Taking $u = x - x^2/2 - 3x^2$ and $v = y$ we have the change of variables that orbitally linearizes (4.10). ∎

Chapter 5

Orbital Linearization in the Quadratic Lotka-Volterra Systems

Summary. In this chapter we consider the linearizability and orbital linearizability properties of the quadratic Lotka-Volterra system in the neighborhood of a singular point with eigenvalues 1 and $-q$. We will use the procedures explained in Chapters 3 and 4 to get the change of variables that linearizes or orbitally linearizes a smooth planar vector field on $\mathbb{C}^2$ around an elementary singular point from a given Lie symmetry as well as the called Darboux linearization. As a consequence, we generalize from $q \in \mathbb{N}\backslash\{0,1\}$ to $q \in \mathbb{C}\backslash\{0,1\}$ some necessary and sufficient conditions of linearization or orbital linearization for such a family given in previous works.

5.1 Introduction

We consider the quadratic Lotka–Volterra family

$$\dot{x} = x(1 + ax + by), \quad \dot{y} = y(-q + cx + dy) \tag{5.1}$$

defined in $\mathbb{C}^2$ with $q \in \mathbb{C}\backslash\{0,1\}$. We say that $\mathcal{X} = x(1+ax+by)\partial_x + y(-q+cx+dy)\partial_y$ is the vector field associated to the differential system (5.1).

The Lotka–Volterra model appears in a large variety of problems in physics, biology and applied mathematics. Is for that reason that it has been widely studied.

In this work we are mainly concerned about the classical problem of local linearization and orbital linearization of system (5.1) in a neighborhood $U \subset \mathbb{C}^2$ of the origin. More concretely, we give the explicit smooth near–identity change of variables $\phi : U \to \mathbb{C}^2$ of the form $\phi(x,y) = (u(x,y), v(x,y)) = (x + o(x,y)), y + o(x,y))$ that linearizes or orbital linearizes system (5.1). In [20] necessary and sufficient conditions for analytic linearizability and orbital analytic linearizability (5.1) are given for $q \in \mathbb{N}\backslash\{0,1\}$. The case $q \in \{0,1\}$ is studied in [21]. See also [36, 40] for some generalizations of the values of q.

There exists the so called Darboux linearization method for polynomials systems by using invariants curves, see [20]. We want to comment that the innovations presented in this work come from an approach to the linearization (resp. orbital linearization) problem based on Lie symmetries. Thus, we use two different methods. We obtain the linearizing (resp. orbital linearizing) change of coordinates ϕ from a given commutator. In some cases, in order to obtain the linearizing (resp. orbital linearizing) change of coordinates ϕ we use an improved version of the Darboux linearization method.

The next theorem is a straightforward generalization of the version's one given in [20] about Darboux linearization. Before the statement of the theorem, we recall a previous definition.

Definition 5.1 *A smooth function $F(x,y)$ satisfying $\mathcal{X}F = KF$ is called a Darboux factor of $\mathcal{X}$ and the smooth function $K(x,y)$ is called its associated cofactor.*

Theorem 5.2 *System $\dot{x} = \lambda x + o(x,y)$, $\quad \dot{y} = \mu y + o(x,y)$ with λ and μ complex numbers different from zero is orbitally linearizable around the origin if there exist Darboux factors $F_i(x,y)$, for $i = 1,\dots,m$, with F_i defined in a neighborhood U of the origin and numbers $\alpha_j, \beta_j \in \mathbb{C}$, such that $F_1(x,y) = x + o(x,y)$, $F_m(x,y) = y + o(x,y)$, $F_i(0,0) \neq 0$ for $i = 2,\dots,m-1$ in such a way that $\sum_{i=1}^{m-1} \alpha_i K_i(x,y) = \lambda h(x,y)$ and $\sum_{i=2}^{m} \beta_i K_i(x,y) = \mu h(x,y)$ where h is smooth on U and $h(0,0) \neq 0$. Here, K_i is the cofactor of F_i. Under these* *conditions, a change of variables* $(x, y) \mapsto (u, v)$ *that brings the system into its orbitally linearizable normal form is given by* $(u, v) = \left(\prod_{i=1}^{m-1} F_i^{\alpha_i}, \prod_{i=2}^{m} F_i^{\beta_i}\right)$.

Proof. Since $\dot{u} = \mathcal{X}(u) = \sum_{i=1}^{m-1} \alpha_i \mathcal{X}(F_i)u/F_i$, taking into account that $F_i(x, y)$ are Darboux factors, it follows that $\dot{u} = u\sum_{i=1}^{m-1} \alpha_i K_i(x, y)$. Hence, $\dot{u} = \lambda u h(x, y)$. Analogously, $\dot{v} = \mu v h(x, y)$. ∎

5.2 Quadratic Lotka–Volterra family

Theorem 5.3 *For* $q \in \mathbb{C}\backslash\{0, 1\}$ *system* (5.1) *has an analytically linearizable saddle at the origin if one of the conditions listed bellow are satisfied.*

(i) $c = 0$;

(ii) $b = d = 0$;

(iii) $a = c, \quad b = d$;

(iv) $b = (q - 1)\, a + c = 0$.

Moreover, if one of the following conditions are satisfied

(v) $q\, a\, b - (q - 1)\, a\, d - c\, d = 0$;

(vi) $ma + c = 0, \quad m = 0, \ldots, q - 2$;

then, the origin of (5.1) *is analytically orbitally linearizable.*

Proof. We will compute either the analytic linearizing or the orbital analytic linearizing changes of variables $(x, y) \mapsto (u(x, y), v(x, y))$ in each case.

Case (i) $c = 0$. In this case system (5.1) has the following Darboux factors, $F_1(x, y) = x$, $F_2(x, y) = dy - q$, $F_3(x, y) = (1 - dy/q)^{b/d + 1/q} + ax\, {}_2\mathcal{F}_1(-1/q, 1 - b/d - 1/q, 1 - 1/q, dy/q)$ and $F_4(x, y) = y$, with cofactors $K_1 = 1 + ax + by$, $K_2 = dy$, $K_3 = (dy + aqx + byq)/q$ and $K_4 = dy - q$, where ${}_2\mathcal{F}_1\,(a_1, a_2; b; x)$ is the hypergeometric function defined by

$${}_2\mathcal{F}_1\,(a_1, a_2; b; x) = \sum_{k=0}^{\infty} \frac{(a_1)_k (a_2)_k}{(b)_k} \frac{x^k}{k!}.$$

It is straightforward to check the existence of numbers $\alpha_1, \alpha_2, \alpha_3,\ \beta_2, \beta_3, \beta_4$ verifying $\alpha_1 K_1 + \alpha_2 K_2 + \alpha_3 K_3 = 1,\ \beta_2 K_2 + \beta_3 K_3 + \beta_4 K_4 = -q$, being $\alpha_1 = -\alpha_3 = 1,\ \alpha_2 = 1/q,\ \beta_4 = -\beta_2 = 1$, and $\beta_3 = 0$. Hence, applying Theorem 5.2 we obtain the change of variables that, in this case, linearizes (5.1), given by $u(x, y) = x(dy - q)^{1/q}/F_3$, and $v(x, y) = y(dy - q)^{-1}$.

Case (ii) $b = d = 0$. In this case, system (5.1) takes the form

$$\dot{x} = x(1 + ax) \qquad \dot{y} = y(-q + cx). \tag{5.2}$$

The vector field associated to system (5.2) admits a quadratic polynomial commutator $\mathcal{Y}$, given by $\mathcal{Y} = x(1+ax)\partial_x + y(1+cx)\partial_y$. Taking into account that $V = \mathcal{X} \wedge \mathcal{Y}$ is an inverse integrating factor for both vector fields $\mathcal{X}$ and $\mathcal{Y}$, we can integrate them. We take the following first integrals in order to apply Theorem 3.6.

$$f(H) = \frac{(1 + ax)^{q+c/a}\ x^{-q}}{y}, \qquad g(I) = \frac{y\ (1 + ax)^{1-c/a}}{x}.$$

Since $f(H) = u^{-q}/v$ and $g(I) = u/v$, from equation (3.9) we obtain that the linearizing change of variables $u(x, y) = x/(1 + ax)$, $v(x, y) = y/(1 + ax)^{c/a}$. We notice that this change of variables also linearizes $\mathcal{Y}$.

Case (iii) $a = c,\ b = d$. This case is already solved in Chapter 3, but we include it for sake of completeness. In this case, system (5.1) reads for

$$\dot{x} = x(1 + cx + dy) \qquad \dot{y} = y(-q + cx + dy). \tag{5.3}$$

The vector field associated to system (5.3) commute with $\mathcal{Y} = x(1 + cx - d\ y/q)\partial_x + y(1 + cx - d\ y/q)\partial_y$. Then, we obtain the inverse integrating factor $V = \mathcal{X} \wedge \mathcal{Y}$ of both vector fields $\mathcal{X}$ and $\mathcal{Y}$ that allows us to obtain the following first integrals

$$H(x, y) = xy^{\frac{1}{q}}(-dy + (1 + cx)q)^{-\frac{1+q}{q}}, \qquad I(x, y) = \frac{x}{y}\ ,$$

of $\mathcal{X}$ and $\mathcal{Y}$, respectively. Since $\mathcal{Y}$ is orbitally linearizable, then $I(x, y)$ is the trivial first integral u/v. Taking the following new first integral $f(H) = q^{-1-q}H^{-q} = u^{-q}/v$, the change of variables $(x, y) \mapsto (u(x, y), v(x, y))$ where $u(x, y) = x/(1 + cx - dy/q)$, $v(x, y) = y/(1 + cx - dy/q)$, linearizes both

$\mathcal{X}$ and $\mathcal{Y}$. Notice that in the cases (ii) and (iii) for $q = -1$, $\mathcal{X} = \mathcal{Y}$ and consequently $V \equiv 0$. Hence, we can not apply our method. However, as the change of variables in both cases also linearizes $\mathcal{Y}$, the same change linearizes $\mathcal{X}$ when $q = -1$.

Case (iv) $b = (q-1)a + c = 0$. In this case, system (5.1) reads for

$$\dot{x} = x(1+ax) \qquad \dot{y} = y(-q + (1-q)ax + dy). \tag{5.4}$$

The vector field $\mathcal{X}$ associated to system (5.4) has three algebraic invariant curves. This fact allows us to get an inverse integrating factor $V = y^2 x^{q+1}$. Applying (1.8) we obtain from V a vector field, $\bar{\mathcal{Y}}$, such that it is a Lie symmetry of $\mathcal{X}$. Then, using the structure of normalizers of $\mathcal{X}$ we look for a commutator $\mathcal{Y}$ of $\mathcal{X}$ with radial linear part given by $\mathcal{Y} = x(1+cx)\partial_x + y(dy(cxq-1)+(1+cx)q(1+c(x-xq)))/(q+cxq)\partial_y$. The first integrals $H(x,y)$ and $I(x,y)$ associated to V of $\mathcal{X}$ and $\mathcal{Y}$ (after deleting some logarithmic functions) are given by

$$H(x,y) = \frac{-x^{-q}(dy - (1+cx)q)}{qy}, \qquad I(x,y) = \frac{-qy(1+cx)^{1+q}}{x(dy - (1+cx)q)}\ .$$

Hence, it follows $f(H) = H(x,y) = u^{-q}/v$, $g(I) = I(x,y)^{-1} = u/v$ where

$$u(x,y) = \frac{x}{1+cx}, \qquad v(x,y) = \frac{-qy(1+cx)^q}{(dy-(1+cx)q)}.$$

Notice that the change of variables linearizes both, $\mathcal{X}$ and $\mathcal{Y}$.

Case (v) $q\,a\,b - (q-1)\,a\,d - c\,d = 0$. First, we consider the case $d \neq 0$ i.e., we take $c = a(qb - (q-1)d)/d$. In this case, system (5.1) has three Darboux factors, $F_1(x,y) = x$, $F_2(x,y) = -dy + q + aqx$, and $F_3(x,y) = y$, and their respective cofactors are $K_1 = 1 + ax + by$, $K_2 = ax + d$, and $K_3 = -q + cx + dy$. It is straightforward to check the existence of a numbers $\alpha_1, \alpha_2, \beta_2, \beta_3$ verifying $\alpha_1 K_1 + \alpha_2 K_2 = h$, $\beta_2 K_2 + \beta_3 K_3 = -qh$, being $\alpha_1 = \beta_1 = 1$, $\alpha_2 = -1$ $\beta_2 = -(d - dq + bq)/d$ and $h = 1 + by - dy$. Hence, applying Theorem 5.2 we obtain the change of variables that, in this case, orbitally linearizes (5.1), given by

$$u(x,y) = qx(-dy+q+qax)^{-1}, \quad v(x,y) = q^{(d-dq+bq)/d}y(-dy+q+qax))^{-(d-dq+bq)/d}.$$

Let us to compute again a change of variables that orbitally linearizes (5.1) by using Lie symmetries. $V = xy(-dy + q + qax)$ is an inverse integrating factor for system (5.1). Then, proceeding as in the case (iv) we obtain a Lie symmetry $\mathcal{Y}$, with radial linear part $\mathcal{Y} = \mathcal{K}dx(2dy(1+q) + q(1+2by - q + a(1+q)x))\partial_x + \mathcal{K}y(d^2y(q-1) + d(1+4ax-q)q + 2abq^2x)\partial_y$, with $\mathcal{K} = 1/[q(2d^2y + d(1+by-ax(q-3)-q) + abqx)]$. Integrating both vector fields, $\mathcal{X}$ and $\mathcal{Y}$, we get

$$f(H) = \frac{(dy-(1+ax)q)^{(d+bq)/d}}{x^q y}, \quad g(I) = \frac{x(dy-(1+ax)q)^{(-3d-2bq-dq)/(d(q-1))}}{y}.$$

From Theorem 4.6 the change of variables that orbitally linearizes (5.1) and linearizes the Lie symmetry $\mathcal{Y}$, reads for

$$\bar{u}(x,y) = x(dy-(1+ax)q)^{\frac{2d+bq}{d(1-q)}}, \quad \bar{v}(x,y) = y(dy-(1+ax)q)^{\frac{d+(b+1)q}{d(q-1)}}.$$

We now consider the case when $d = 0$. Since the other cases are already solved, if $d = 0$ it follows $a = 0$. We shall use the same procedure than in the case $d \neq 0$. In this case system (5.1) has three Darboux factors, $F_1(x,y) = x$, $F_2(x,y) = \exp(-cx+by)$, and $F_3(x,y) = y$. Their respective cofactors are $K_1 = 1+by$, $K_2 = -(cx+byq)$, and $K_3 = -q+cx$. There exits numbers $\alpha_1, \alpha_2, \beta_2, \beta_3$ verifying $\alpha_1 K_1 + \alpha_2 K_2 = h$, $\beta_2 K_2 + \beta_3 K_3 = -qh$. It is easy to see that $\alpha_1 = \alpha_2 = \beta_1 = 1$, $\beta_2 = 1-\lambda$ and $h = 1 - cx + by - bqy$. Hence, applying Theorem 5.2 we obtain the change of variables that orbitally linearizes system (5.1)

$$u(x,y) = x\exp(-cx+by), \quad v(x,y) = y\exp((1-q)(-cx+by)).$$

In the following computations we obtain the change from a given symmetry $\mathcal{Y}$. $V = xy$ is an inverse integrating factor for system (5.1). Moreover, as in the former cases, we obtain a Lie symmetry $\mathcal{Y}$, with radial linear part

$$\mathcal{Y} = \frac{x(1+2cy-q)}{1+cx+by-q}\partial_x + \frac{y(1+2bx-q)}{1+cx+dy-q}\partial_y.$$

Integrating $\mathcal{X}$ and $\mathcal{Y}$ we obtain

$$f(H) = \frac{\exp(cx-by)}{x^q y}, \quad g(I) = \frac{x\exp\left(\frac{2(cx-by)}{1-q}\right)}{y}.$$

From Theorem 4.6 we obtain the change of variables that orbitally linearizes (5.1) and also linearizes the Lie symmetry $\mathcal{Y}$, that is

$$u(x,y) = x\exp\left(\frac{by-cx}{q-1}\right), \quad v(x,y) = y\exp\left(\frac{by-cx}{1-q}\right).$$

Case (vi) $ma + c = 0$ with $m = 0, \dots, q-2$. In this case system (5.1) has the following Darboux factors, $F_1(x,y) = x$, and $F_3(x,y) = y$, with cofactors $K_1 = (m - cx + bmy)/m$, and $K_3 = cx + dy - q$. There exits another Darboux factor $F_2(x,y)$ satisfying $qK_1 + K_2 + K_3 = 0$. This fact allows us to compute the cofactor of $F_2(x,y)$ without the explicit knowledge of it. Thus, solving the former expression for K_2 we obtain $K_2 = (cx(q-m) - ym(d+bq))/m$. Applying Theorem 5.2 we get the numbers $\alpha_1 = 1, \quad \alpha_2 = 1/(q-m), \quad \beta_2 = m/(m-q), \quad \beta_3 = 1$. Hence, the change of variables that orbitally linearizes system (5.1) is $u(x,y) = xF_2(x,y)^{1/(q-m)}$, $v(x,y) = yF_2(x,y)^{m/(m-q)}$, where $F_2(0,0) \neq 0$, see [30]. ∎

In fact, the necessary and sufficient conditions (cases (v) and (vi)) for analytic integrability, i.e., systems with an analytic first integral defined in a neighborhood of the origin, were given in [30]. The case (vi) is also linearizable for $q \in \mathbb{N}\backslash\{1\}$, see [20].

Chapter 6

Newtonian Systems with 2-Dimensional Lie Symmetry Algebra

Summary. In this chapter we study some aspects of the dynamics in the phase plane of smooth second order differential equations $\ddot{x} = w(x, \dot{x})$ possessing an r–dimensional Lie point symmetry algebra $\mathcal{L}_r$ with $r \geq 2$, focusing on the existence, nonexistence and localization periodic orbits. Finally, it is proved that the polynomial Liénard systems $\ddot{x} = f(x)\dot{x} + g(x)$ with $f, g \in \mathbb{R}[x]$ having a $\mathcal{L}_r$ with $r \geq 2$ do not have limit cycles. As far as we know, this is the first work that relates Lie point symmetries and periodic orbits.

6.1 Introduction

We consider Newtonian systems, that is, autonomous second order differential equations

$$\ddot{x} = w(x, \dot{x}) \ , \tag{6.1}$$

with $w \in C^\infty(U)$ and $U \subset \mathbb{R}^2$ an open set. We shall write (6.1) as a first order planar system defined on U in the usual way

$$\dot{x} = y \ , \ \dot{y} = w(x, y) \ . \tag{6.2}$$

We associate to equations (6.1) and (6.2) the vector fields $\mathcal{X} = \partial_t + \dot{x}\partial_x + w(x,\dot{x})\partial_{\dot{x}}$ and $\bar{\mathcal{X}} = y\partial_x + w(x,y)\partial_y$, respectively. In the following, $\mathrm{div}\mathcal{X}$ stands for the divergence of the vector field $\mathcal{X}$. A *singular point* of $\bar{\mathcal{X}}$ is a point $(x_0, 0) \in U$ such that $w(x_0, 0) = 0$.

All the general concepts and the specific ones about Lie theory introduced in this section can be generalized to the nonautonomous case $\ddot{x} = w(t,x,\dot{x})$, but we omit them. A C^1 nonconstant function $I(t,x,y)$ is called an *invariant* (or non–autonomous first integral) of system (6.2) in U if it is constant along the solutions of (6.2). In other words, $\mathcal{X}I \equiv 0$ must be satisfied in U. Of course, we can find at most two functionally independent invariants of (6.2).

A symmetry of (6.1) is a diffeomorphism $\Phi : (t,x) \mapsto (\bar{t},\bar{x})$ that maps the set of solutions of (6.1) into itself. Therefore, the symmetry condition for (6.1) is just $\bar{x}'' = w(\bar{t},\bar{x},\bar{x}')$, where the prime denotes the derivative $' = d/d\bar{t}$. When the symmetry is a 1–parameter Lie group of point transformations Φ_ϵ, then

$$\bar{t} = t + \epsilon\xi(t,x) + O(\epsilon^2)\ ,\ \ \bar{x} = x + \epsilon\eta(t,x) + O(\epsilon^2)\ ,$$

for ϵ close to zero, and the vector field $\mathcal{Y} = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ is called the *infinitesimal generator* of the 1–parameter Lie group of point transformations Φ_ϵ. It is well known (see for instance the books [9, 38, 39, 51, 57]) that, the determining equations for Lie point symmetries can be obtained from the linearized condition

$$\mathcal{Y}^{[2]}(\ddot{x} - w(x,\dot{x})) = 0 \text{ when } \ddot{x} = w(x,\dot{x})\ , \tag{6.3}$$

where $\mathcal{Y}^{[2]} = \mathcal{Y} + \eta^{[1]}(t,x,\dot{x})\partial_{\dot{x}} + +\eta^{[2]}(t,x,\dot{x},\ddot{x})\partial_{\ddot{x}}$ is the so–called *second prolongation* of the infinitesimal generator $\mathcal{Y}$ and $\eta^{[1]}(t,x,\dot{x}) = D_t\eta - \dot{x}D_t\xi$, $\eta^{[2]}(t,x,\dot{x},\ddot{x}) = D_t\eta^{[1]} - \ddot{x}D_t\xi$ where $D_t = \partial_t + \dot{x}\partial_x + \ddot{x}\partial_{\dot{x}}$ is the operator total derivative with respect to t. More precisely,

$$\begin{aligned}
\eta^{[1]}(t,x,\dot{x}) &= \frac{\partial\eta}{\partial t} + \left(\frac{\partial\eta}{\partial x} - \frac{\partial\xi}{\partial t}\right)\dot{x} - \frac{\partial\xi}{\partial x}\dot{x}^2\ ,\\
\eta^{[2]}(t,x,\dot{x},\ddot{x}) &= \frac{\partial^2\eta}{\partial t^2} + \left(2\frac{\partial^2\eta}{\partial t\partial x} - \frac{\partial^2\xi}{\partial t^2}\right)\dot{x} + \left(\frac{\partial^2\eta}{\partial x^2} - 2\frac{\partial^2\xi}{\partial t\partial x}\right)\dot{x}^2 - \frac{\partial^2\xi}{\partial x^2}\dot{x}^3\\
&\quad + \left(\frac{\partial\eta}{\partial x} - 2\frac{\partial\xi}{\partial t} - 3\frac{\partial\xi}{\partial x}\dot{x}\right)\ddot{x}\ .
\end{aligned}$$

Introducing these expressions of $\eta^{[1]}(t,x,\dot{x})$ and $\eta^{[2]}(t,x,\dot{x},\ddot{x})$ into (6.3) we get the *determining equations*

$$\frac{\partial^2\eta}{\partial t^2}+\left(2\frac{\partial^2\eta}{\partial t\partial x}-\frac{\partial^2\xi}{\partial t^2}\right)\dot{x}+\left(\frac{\partial^2\eta}{\partial x^2}-2\frac{\partial^2\xi}{\partial t\partial x}\right)\dot{x}^2-\frac{\partial^2\xi}{\partial x^2}\dot{x}^3 \tag{6.4}$$
$$=-\left(\frac{\partial\eta}{\partial x}-2\frac{\partial\xi}{\partial t}-3\frac{\partial\xi}{\partial x}\dot{x}\right)w+\eta\frac{\partial w}{\partial x}+\left[\frac{\partial\eta}{\partial t}+\left(\frac{\partial\eta}{\partial x}-\frac{\partial\xi}{\partial t}\right)\dot{x}-\frac{\partial\xi}{\partial x}\dot{x}^2\right]\frac{\partial w}{\partial\dot{x}}\,.$$

Several equations (6.1) that arise from applications have the property that $w(x,\dot{x})$ is a polynomial in $\dot{x}$. So, since both ξ and η are independent of $\dot{x}$, the determining equation are solved usually splitting into a system of partial differential equations by equating powers of $\dot{x}$. Of course, since (6.1) is autonomous, it always admits the generator $\mathcal{Y}=\partial_t$ of a Lie point symmetry.

Let $\mathcal{L}_r$ denote the set of all infinitesimal generators of 1–parameter Lie groups of point symmetries of the differential equation (6.1). It is known that $\mathcal{L}_r$ is a finite dimensional real vector space. In the following we denote $r=\dim\mathcal{L}_r$ and $\{\mathcal{Y}_1,\ldots,\mathcal{Y}_r\}$ a basis of $\mathcal{L}_r$ with $\mathcal{Y}_i=\xi_i(t,x)\partial_t+\eta_i(t,x)\partial_x$ for $i=1,\ldots,r$. Moreover, see for instance [38, 57], for autonomous second order differential equation like (6.1), $r\in\{1,2,3,8\}$. Additionally, $r=8$ if and only if (6.1) is linear or linearizable by a point transformation $(t,x)\mapsto(\bar{t}(t,x),\bar{x}(t,x))$ defined in certain domain.

Given $\mathcal{Y}_1,\mathcal{Y}_2\in\mathcal{L}_r$, we define its *Lie bracket* or *commutator* in the usual way $[\mathcal{Y}_1,\mathcal{Y}_2]=\mathcal{Y}_1\mathcal{Y}_2-\mathcal{Y}_2\mathcal{Y}_1$. It is easy to see that, the Lie bracket is independent of the coordinate system. Additionally, if $\mathcal{Y}_1,\mathcal{Y}_2\in\mathcal{L}_r$, then $[\mathcal{Y}_1,\mathcal{Y}_2]\in\mathcal{L}_r$. This means that $\mathcal{L}_r$ is closed under the commutator operation and so $\mathcal{L}_r$ is not only a vector space; it is a real *Lie algebra*. In short, $[\mathcal{Y}_i,\mathcal{Y}_j]=\sum_{k=1}^r c_{ij}^k\mathcal{Y}_k$ where the constants $c_{ij}^k\in\mathbb{R}$ are called the *structure constants* of the Lie algebra $\mathcal{L}_r$. In particular, if all the structure constants are zero then $\mathcal{L}_r$ is called Abelian.

For any $\mathcal{Y}_i=\xi_i(t,x)\partial_t+\eta_i(t,x)\ \partial_x\in\mathcal{L}_r$, easily one can check that $[\mathcal{X},\mathcal{Y}_i^{[1]}]=\mu_i(t,x,\dot{x})\mathcal{X}$ where $\mu_i(t,x,\dot{x})=\mathcal{X}\xi_i$ and $\mathcal{Y}_i^{[1]}=\mathcal{Y}_i+\eta_i^{[1]}(t,x,\dot{x})\partial_{\dot{x}}$ is the first prolongation of $\mathcal{Y}$. If $r\geq 2$, we define the functions

$$V_{ij}(t,x,\dot{x})=\det\{\mathcal{X},\mathcal{Y}_i^{[1]},\mathcal{Y}_j^{[1]}\}=\begin{vmatrix}1 & \dot{x} & w(x,\dot{x})\\ \xi_i(t,x) & \eta_i(t,x) & \eta_i^{[1]}(t,x,\dot{x})\\ \xi_j(t,x) & \eta_j(t,x) & \eta_j^{[1]}(t,x,\dot{x})\end{vmatrix} \tag{6.5}$$

for $i,j\in\{1,\ldots,r\}$ with $1\leq i<j\leq r$. In this work, we will show that, the zero–set of these functions V_{ij} will play a fundamental role in the dynamics

of system (6.2).

When a Lie symmetry acts in the phase plane (x, y) instead of the plane (t, x) it is called an *orbital Lie symmetry*. In this different context, the infinitesimal generator $\bar{\mathcal{Y}} = \xi(x, y)\partial_x + \eta(x, y)\partial_y$ of the orbital Lie symmetry of a vector field $\bar{\mathcal{X}} = P(x, y)\partial_x + Q(x, y)\partial_y$ has the property that the local flow of $\bar{\mathcal{Y}}$ sends orbits of $\bar{\mathcal{X}}$ into orbits of $\bar{\mathcal{X}}$. This assertion for $C^1(\Omega)$ transversal vector fields $\bar{\mathcal{X}}$ and $\bar{\mathcal{Y}}$ in $\Omega \subseteq \mathbb{R}^2$ is equivalent to the condition $[\bar{\mathcal{X}}, \bar{\mathcal{Y}}] = \mu(x, y)\bar{\mathcal{X}}$ for certain scalar function μ.

The $C^1(\Omega)$ function $V(x, y) = \det\{\bar{\mathcal{X}}, \bar{\mathcal{Y}}\} = P\eta - Q\xi$, which satisfies $\bar{\mathcal{X}}V = V\mathrm{div}\bar{\mathcal{X}}$ is called *inverse integrating factor* of $\bar{\mathcal{X}}$ in Ω. It is clear that the zero–set of $V(x, y)$ given by $V^{-1}(0) := \{(x, y) \in \Omega : V(x, y) = 0\}$ is an invariant curve of $\bar{\mathcal{X}}$. Moreover, see [6, 31, 33] for more details, the set $V^{-1}(0)$ contains most of the relevant orbits of the phase portrait of $\bar{\mathcal{X}}$ in Ω such as singular points, separatrices, limit cycles and graphics provided they are α or ω–limit sets of $\bar{\mathcal{X}}$.

The aim of this work is to generalize the concept of inverse integrating factor $V(x, y)$ of system (6.2) via the functions $V_{ij}(t, x, y)$ defined in (6.5). In fact, in the autonomous particular case $\partial V_{ij}/\partial t \equiv 0$, we get that V_{ij} is just an inverse integrating factor of (6.2). On the contrary, when $\partial V_{ij}/\partial t \not\equiv 0$, we will show that the zero–sets $V^{-1}(0)$ and $V_{ij}^{-1}(0)$ have similar properties.

We want to remark here that while the search of an orbital symmetry $\bar{\mathcal{Y}}$ or inverse integrating $V(x, y)$ factor in closed form for system (6.2) is neither algorithmic nor easy in general, to obtain the generators $\mathcal{Y}_i$ for the eventual Lie point symmetry algebra $\mathcal{L}_r$ of (6.2) is often a systematic work.

As far as we know, this is the first time that from the knowledge of a Lie algebra $\mathcal{L}_r$ with $r \geq 2$ of point symmetries of the differential equation (6.1) some dynamical consequences (such as existence or nonexistence and location of periodic orbits) are deduced.

Let $\mathcal{L}_2$ be a Lie algebra with generators $\mathcal{Y}_1, \mathcal{Y}_2 \in \mathcal{L}_2$. Of course, the basis of $\mathcal{L}_2$ can be changed to new generators $\widehat{\mathcal{Y}}_1, \widehat{\mathcal{Y}}_2$ as follows $\widehat{\mathcal{Y}}_i = \sum_{j=1}^{2} a_{ij}\mathcal{Y}_j$ for $i = 1, 2$, such that $\det A \neq 0$ with $A = (a_{ij}) \in \mathcal{M}_2(\mathbb{R})$. Two Lie algebras of equal dimension $\mathcal{L}_r$ and $\bar{\mathcal{L}}_r$ are *isomorphic* if there is a linear one-to-one map $f : \mathcal{L}_r \to \bar{\mathcal{L}}_r$ such that $f([\mathcal{Y}_1, \mathcal{Y}_2]) = [f(\mathcal{Y}_1), f(\mathcal{Y}_2)]$. Two finite dimensional Lie algebras are isomorphic if and only if they have the same structure constants for certain basis. Let $\mathcal{L}_2$ be a 2–dimensional Lie algebra generated by $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Then it is well known that $\mathcal{L}_2$ is isomorphic to either an Abelian Lie algebra or to a Lie algebra satisfying the commutation $[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Y}_1$. On the other hand, two Lie algebras $\mathcal{L}_r$ and $\bar{\mathcal{L}}_r$ of vector fields on $\mathbb{R}^n$ are *similar* if one is obtained from the other by a change of variables. If two r–dimensional Lie algebras of vector fields on $\mathbb{R}^n$ are similar, then they are isomorphic. The converse is not true. In summary, we have the following well–known classification of 2–dimensional non–similar Lie algebras, see for instance [57].

Type	Structure	Canonical Form
I	$[\mathcal{Y}_1, \mathcal{Y}_2] = 0$, $\mathcal{Y}_1 \wedge \mathcal{Y}_2 \neq 0$	$\mathcal{Y}_1 = \frac{\partial}{\partial s}$, $\mathcal{Y}_2 = \frac{\partial}{\partial r}$
II	$[\mathcal{Y}_1, \mathcal{Y}_2] = 0$, $\mathcal{Y}_1 \wedge \mathcal{Y}_2 = 0$	$\mathcal{Y}_1 = \frac{\partial}{\partial s}$, $\mathcal{Y}_2 = r\frac{\partial}{\partial s}$
III	$[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Y}_1$, $\mathcal{Y}_1 \wedge \mathcal{Y}_2 \neq 0$	$\mathcal{Y}_1 = \frac{\partial}{\partial s}$, $\mathcal{Y}_2 = r\frac{\partial}{\partial r} + s\frac{\partial}{\partial s}$
IV	$[\mathcal{Y}_1, \mathcal{Y}_2] = \mathcal{Y}_1$, $\mathcal{Y}_1 \wedge \mathcal{Y}_2 = 0$	$\mathcal{Y}_1 = \frac{\partial}{\partial s}$, $\mathcal{Y}_2 = s\frac{\partial}{\partial s}$

Figure 6.1: Non–similar 2–dimensional Lie algebras $\mathcal{L}_2$ and canonical forms of generators.

Here, the wedge product is defined as $\mathcal{Y}_1 \wedge \mathcal{Y}_2 := \det\{\mathcal{Y}_1, \mathcal{Y}_2\}$. We emphasize that, in the classification of canonical forms given in this table we have used the so–called *canonical coordinates* (r, s). Let $\mathcal{Y}_1 = \xi_1(t,x)\partial_t + \eta_1(t,x)\partial_x$. The functions $r(t,x)$, $s(t,x)$, are defined as solutions of the linear partial differential system $\mathcal{Y}_1(r) = 0$ and $\mathcal{Y}_1(s) = 1$ such that the jacobian of $(r(t,x), s(t,x))$ is different from zero in order to be invertible the change of coordinates. It is clear that, in these canonical coordinates we have $\mathcal{Y}_1 = \partial_s$. Of course, canonical coordinates cannot be defined in some neighborhood of a singular point $(t_0, x_0) \in \mathbb{R}^2$ of $\mathcal{Y}_1$, i.e., satisfying $\xi_1(t_0,x_0) = \eta_1(t_0,x_0) = 0$. Moreover, canonical coordinates are not uniquely defined because, if (r, s) are canonical coordinates then $(\bar{r}, \bar{s}) = (f(r), s + g(r))$ are canonical coordinates for arbitrary smooth functions f and g such that $f'(r) \neq 0$. In this work we do not use canonical coordinates because, in general, they only have a local definition domain and we want to study global dynamical properties.

6.2 The zero–set of V_{ij}

Although the next result is not new, we provide a proof for sake of completeness.

Proposition 6.1 *Assume that system (6.2) possesses an r–dimensional Lie point symmetry algebra with $r \geq 2$ and define the functions $V_{ij}(t,x,\dot{x})$ as in (6.5).*

(i) *V_{ij} satisfies the linear partial differential equation $\mathcal{X}V_{ij} = V_{ij}\,\mathrm{div}\mathcal{X}$, where $\mathcal{X} = \partial_t + \dot{x}\partial_x + w(x,\dot{x})\partial_{\dot{x}}$.*

(ii) *If $r \geq 3$ then, the ratio of any two nonzero V_{ij} is either a constant or an invariant of (6.2).*

(iii) *If $V_{ij} \equiv 0$, then $(\eta_i - y\xi_i)/(\eta_j - y\xi_j)$ is an invariant of system (6.2).*

Proof. We associate to each infinitesimal generator $\mathcal{Y}_i = \xi_i(t,x)\partial_t + \eta_i(t,x)\partial_x$ its *characteristic* $Q_i(t,x,\dot{x}) = \eta_i - \dot{x}\xi_i$. Then, straightforward calculations shows that the functions V_{ij} can be written as

$$V_{ij} = Q_i\mathcal{X}Q_j - Q_j\mathcal{X}Q_i \ . \tag{6.6}$$

Again, it is tedious but straightforward to check that, the determining equations (6.4) for the infinitesimal generator $\mathcal{Y}_i$ expressed via the characteristic is just

$$\mathcal{X}^2Q_i - \frac{\partial w}{\partial \dot{x}}\mathcal{X}Q_i - \frac{\partial w}{\partial x}Q_i = 0 \ . \tag{6.7}$$

Hence, using (6.6) and (6.7), we obtain

$$\mathcal{X}V_{ij} = Q_i\mathcal{X}^2Q_j - Q_j\mathcal{X}^2Q_i = V_{ij}\frac{\partial w}{\partial \dot{x}} = V_{ij}\ \mathrm{div}\mathcal{X} \ ,$$

proving thus statement (i). Statement (ii) is proved by computing

$$\mathcal{X}\left(\frac{V_{ij}}{V_{k\ell}}\right) = \frac{V_{k\ell}\mathcal{X}V_{ij} - V_{ij}\mathcal{X}V_{k\ell}}{V_{k\ell}^2} \equiv 0 \ ,$$

where, in the last step, we have used (i).

Finally, when $V_{ij} \equiv 0$, then from (6.6) we have

$$\mathcal{X}\left(\frac{Q_i}{Q_j}\right) \equiv 0 \ ,$$

where Q_i/Q_j is not a constant because $\mathcal{Y}_i$ and $\mathcal{Y}_j$ are linearly independent. Hence, statement (iii) holds. ∎

Remark 6.2 Consider the functions $V_{ij}(t,x,\dot{x}) = \det\{\mathcal{X}, \mathcal{Y}_i^{[1]}, \mathcal{Y}_j^{[1]}\}$ for all $i,j \in \{1,\ldots,r\}$ with $1 \le i < j \le r$. Then, any linear combination

$$\widehat{V}_{12}(t,x,\dot{x}) = \sum_{i,j} c_{ij} V_{ij}$$

with $c_{ij} \in \mathbb{R}$, satisfies $\mathcal{X}\widehat{V}_{12} = \widehat{V}_{12}\mathrm{div}\mathcal{X}$. In fact, the underlying idea consists on to perform a change of basis in $\mathcal{L}_r$. For instance, take any linearly independent couple $\widehat{\mathcal{Y}}_1, \widehat{\mathcal{Y}}_2 \in \mathcal{L}_r$, next construct $\widehat{V}_{12}(t,x,\dot{x}) = \det\{\mathcal{X}, \widehat{\mathcal{Y}}_1^{[1]}, \widehat{\mathcal{Y}}_2^{[1]}\}$ and take into account that $\widehat{\mathcal{Y}}_i = \sum_{j=1}^r a_{ij}\mathcal{Y}_j$ with $a_{ij} \in \mathbb{R}$ for $i = 1,2$. Recall that the set of C^1 solutions of the partial differential equation $\mathcal{X}V = V\mathrm{div}\mathcal{X}$ is a linear subspace of the set of C^1 functions from $\mathbb{R}^3$ to $\mathbb{R}$.

Every curve $\mathcal{C}$ given in parametric form by the graph $(t, x(t))$ in the (t,x)–plane that is invariant under the Lie group generated by $\mathcal{Y}_i$ must satisfies $Q_i(t,x,\dot{x})|_{\mathcal{C}} \equiv 0$ with Q_i the characteristic of $\mathcal{Y}_i$. Hence, in order to derive solutions of (6.1) that are invariant under the 1–parameter Lie group generated by $\mathcal{Y}_i$, first we have to solve the first order differential equation $\eta_i - \dot{x}\xi_i = 0$ and then, check which (if any) of these solutions satisfy (6.1). Notice that, curves $\mathcal{C}$ such that $\det\{\mathcal{Y}_1, \mathcal{Y}_2\}|_{\mathcal{C}} \equiv 0$ are candidates to invariant solutions of both generators $\mathcal{Y}_1$ and $\mathcal{Y}_2$. From (6.6) we have the next four sufficient conditions in order to obtain a function $x(t)$ such that $V_{12}(t, x(t), \dot{x}(t)) \equiv 0$ for all t.

1. $Q_1(t,x(t),\dot{x}(t)) = Q_2(t,x(t),\dot{x}(t)) \equiv 0$. This means that all the common solutions $x(t)$ of the first order differential equations $\eta_i - \dot{x}\xi_i = 0$ with $i = 1,2$, are such that $V_{12}(t,x(t),\dot{x}(t) \equiv 0$. Of course, if $\mathcal{Y}_1 = \partial_t$, then only constant functions $x(t) = x_0$ are allowed.

2. $Q_1(t,x(t),\dot{x}(t)) = \mathcal{X}Q_1(t,x(t),\dot{x}(t)) \equiv 0$. If $\mathcal{Y}_1 = \partial_t$, then $Q_1(t,x,\dot{x}) = -\dot{x}$ and $\mathcal{X}Q_1 = -w(x,\dot{x})$. Therefore, $V_{12}(t,x,y)$ vanishes in all the singular points of $\bar{\mathcal{X}} = y\partial_x + w(x,y)\partial_y$.

3. $\mathcal{X}Q_1(t,x(t),\dot{x}(t)) = \mathcal{X}Q_2(t,x(t),\dot{x}(t)) \equiv 0$. If $\mathcal{Y}_1 = \partial_t$, in order to get all the functions $x(t)$ belonging to this case, first we solve the differential equation $\mathcal{X}Q_1 = w(x,\dot{x}) = 0$. Let $\phi_t(x_0)$ be the solution such that $\phi_0(x_0) = x_0$. Finally, if there is a particular solution $\phi_t(\bar{x}_0)$ for some $\bar{x}_0 \in \mathbb{R}$ such that $\mathcal{X}Q_2|_{\phi_t(\bar{x}_0)} \equiv 0$, then the function $x(t) = \phi_t(\bar{x}_0)$ vanishes V_{12}. But, the former function $x(t)$ will be a solution of $\ddot{x} =$

$w(x, \dot{x})$ if and only if $\ddot{x}(t) = 0$ and therefore $x(t)$ is not a periodic solution.

4. $Q_2(t, x(t), \dot{x}(t)) = \mathcal{X}Q_2(t, x(t), \dot{x}(t)) \equiv 0$. In this work we will see that, if $\mathcal{Y}_1 = \partial_t$, then this case does not contain periodic solutions $x(t)$ of (6.1) that are limit cycles (isolated periodic solutions) of (6.2).

The next theorem is a general result about the invariant curves of $\bar{\mathcal{X}}$ contained in the zero–set of the functions V_{ij}. We put special emphasis on periodic orbits of (6.2) of any kind (isolated and, therefore, limit cycles or nonisolated and so belonging to a period annulus). Recall here that a limit cycle γ parameterized by $\{(x(t), y(t)) \in U : 0 \leq t < T\}$ is called *hyperbolic* if $\oint_\gamma \mathrm{div}\bar{\mathcal{X}}(x(t), y(t))dt \neq 0$. On the other hand, a C^1 curve $f(x, y) = 0$ defined on U is an invariant curve of $\bar{\mathcal{X}}$ if $\bar{\mathcal{X}}f = Kf$ for some function $K(x, y)$ called the associated *cofactor* of $f = 0$.

Theorem 6.3 *Let $U \subset \mathbb{R}^2$ be an open set and assume that $\ddot{x} = w(x, \dot{x})$ with w smooth in U admits an r–dimensional Lie point symmetry algebra $\mathcal{L}_r$ with $r \geq 2$. Consider the functions $V_{ij}(t, x, \dot{x})$ defined in (6.5) for $i, j \in \{1, \ldots, r\}$ with $1 \leq i < j \leq r$. Suppose that $\gamma = (x(t), y(t)) \subset U$ is a T–periodic orbit of (6.2). Then the next statements hold:*

(i) *If $V_{ij}(t, x, \dot{x}) = V(x, \dot{x}) \not\equiv 0$, with $V \in C^1(U)$, then $V(x, y)$ is an inverse integrating factor of system (6.2) in U. In particular, if γ is a limit cycle, then $\gamma \subset \{V(x, y) = 0\}$.*

(ii) *If $V_{ij}(t, x, \dot{x}) = F(t)G(x, \dot{x}) \not\equiv 0$ with non–constants F and $G \in C^1(U)$, then $\dot{F} = \alpha F$ with $\alpha \in \mathbb{R}\backslash\{0\}$ and $G(x, y) = 0$ is an invariant curve of system (6.2). Moreover, we have:*

- **(ii.1)** *If $\gamma \subset \{G = 0\}$ and G is analytic on U, then G is not square–free, i.e., $G(x, y) = g^n(x, y)u(x, y)$ with a positive integer $n > 1$ and g and u are analytic functions on U satisfying $\gamma \subset \{g = 0\}$ and $\gamma \not\subset \{u = 0\}$.*
- **(ii.2)** *If $\gamma \not\subset \{G = 0\}$ then γ is hyperbolic and $\alpha T = \oint_\gamma \mathrm{div}\bar{\mathcal{X}}(x(t), y(t))dt$.*

Proof. From statement (i) of Proposition 6.1, the function V_{ij} satisfies the linear partial differential equation $\mathcal{X}V_{ij} = V_{ij}\mathrm{div}\mathcal{X}$. Moreover, $\mathcal{X} = \partial_t + \bar{\mathcal{X}}$ so that $\mathrm{div}\mathcal{X} = \mathrm{div}\bar{\mathcal{X}}$. Therefore, if V_{ij} does not depend on t, then it verifies

$\bar{\mathcal{X}}V_{ij} = V_{ij}\mathrm{div}\bar{\mathcal{X}}$ and therefore V_{ij} is an inverse integrating factor of system (6.2) in U proving the statement (i).

To prove statement (ii), assume first that $V_{ij}(t,x,y) = F(t)G(x,y)$. Then, from $\mathcal{X}V_{ij} = V_{ij}\mathrm{div}\mathcal{X}$, we get $\dot{F}G + F\bar{\mathcal{X}}G = FG\mathrm{div}\bar{\mathcal{X}}$. Hence we deduce that $\dot{F} = \alpha F$ with $\alpha \in \mathbb{R}\backslash\{0\}$ and

$$\bar{\mathcal{X}}G = (-\alpha + \mathrm{div}\bar{\mathcal{X}})G \ . \tag{6.8}$$

This last equations means that $G = 0$ is an invariant curve of system (6.2) with associated cofactor $K_G = -\alpha + \mathrm{div}\bar{\mathcal{X}}$.

From now, we denote by $\gamma = (x(t), y(t)) \subset U$, one T-periodic orbit of (6.2) of minimal period $T > 0$.

Let us assume that G is square–free inside the class of analytic functions on U. Let $p \in \gamma$ a point such that $G(p) = \nabla G(p) = 0$, that is, p is a critical point of the invariant curve $G = 0$. Then, p is also a singular point of system (6.2) which is impossible since γ is periodic orbit of (6.2). Since $G = 0$ is an invariant curve of (6.2), if $G(p) = 0$ then $G|_\gamma = 0$. Therefore, the condition $\nabla G|_\gamma = 0$ is equivalent to the fact that G is not square–free, that is, $G(x,y) = g^n(x,y)u(x,y)$ with an integer $n > 1$ and analytic functions g and u on U.

In order to prove statement (ii.1), assume by contradiction that $\gamma \subset \{G = 0\}$ and $\nabla G|_\gamma \neq 0$. Then, using the main result of [32] we have

$$\oint_\gamma K_G(x(t), y(t)) \ dt = \oint_\gamma \mathrm{div}\bar{\mathcal{X}}(x(t), y(t)) \ dt \ ,$$

from which we deduce the contradiction $\alpha T = 0$, proving thus the point (ii.1).

Finally, assume $\gamma \not\subset \{G = 0\}$. Then, dividing both members of (6.8) by G and integrating over γ one have

$$0 = \oint_\gamma \frac{\bar{\mathcal{X}}G}{G}(x(t), y(t))dt = -\alpha T + \oint_\gamma \mathrm{div}\bar{\mathcal{X}}(x(t), y(t))dt \ ,$$

from which we obtain $\oint_\gamma \mathrm{div}\bar{\mathcal{X}}(x(t), y(t))dt \neq 0$ and therefore γ must be a hyperbolic periodic orbit of (6.2). ∎

Corollary 6.4 *Assume that $\ddot{x} = w(x, \dot{x})$, with w smooth in the open set $U \subset \mathbb{R}^2$, admits an r–dimensional Lie point symmetry algebra $\mathcal{L}_r$ with $r \geq 2$.*

Consider the functions $V_{ij}(t,x,\dot{x})$ for $i,j \in \{1,\dots,r\}$ with $1 \le i < j \le r$. If there is one $V_{ij}(t,x,y) = F(t)G(x,y) \not\equiv 0$ with non–constants F and $G \in C^1(U)$, then system (6.2) does not have period annulus in U.

Proof. Assume the existence of a period annulus $\mathcal{A} \subset U$ for (6.2). It is clear that $\mathcal{A} \not\subset \{G=0\}$, so there is a periodic orbit $\gamma \subset \mathcal{A}$ such that $\gamma \not\subset \{G=0\}$. Then, from statement (ii.2) of Theorem 6.3, γ must be hyperbolic, in contradiction with the fact that $\gamma \subset \mathcal{A}$. ∎

In the sequel, we concentrate our attention in the 2–dimensional case $\mathcal{L}_2$. In this case, the following classification shows that, if $\partial_t \in \mathcal{L}_2$ then, the autonomous or separate time–variable forms of $V_{ij}(t,x,\dot{x})$ given in Theorem 6.3 are the only possibilities.

Proposition 6.5 *Assume that $\ddot{x} = w(x,\dot{x})$ with w smooth in the open set $U \subset \mathbb{R}^2$ admits a 2–dimensional Lie point symmetry algebra $\mathcal{L}_2$ spanned by C^1 vector fields $\mathcal{Y}_1$ and $\mathcal{Y}_2$ in its domain of definition. Then, for $\mathcal{Y}_1 = \partial_t$, we have*

$$\mathcal{Y}_2 = \begin{cases} (c_1 t + \alpha(x))\partial_t + \beta(x)\partial_x & \text{if } \ c_2 = 0 \ , \\ (\alpha(x)\exp(c_2 t) - c_1/c_2)\partial_t + \beta(x)\exp(c_2 t)\partial_x & \text{if } \ c_2 \neq 0 \ , \end{cases}$$

for some C^1–functions α and β and where $[\mathcal{Y}_1,\mathcal{Y}_2] = c_1\mathcal{Y}_1 + c_2\mathcal{Y}_2$ with structure constants c_i. In addition, $V_{12}(t,x,\dot{x}) = \dot{x}^2[c_1 + \dot{x}\alpha'(x) - \beta'(x)] + \beta(x)w(x,\dot{x})$ if $c_2 = 0$ and $V_{12}(t,x,\dot{x}) = \exp(c_2 t)[\dot{x}(-c_2\beta(x) + c_2\dot{x}\alpha(x) + \dot{x}^2\alpha'(x) - \dot{x}\beta'(x)) + \beta(x)w(x,\dot{x})]$ if $c_2 \neq 0$.

Proof. Since $\ddot{x} = w(x,\dot{x})$ is autonomous, it always possesses the infinitesimal generator $\mathcal{Y}_1 = \partial_t$ of a Lie point symmetry. Assuming $\mathcal{Y}_1 \in \mathcal{L}_2$ and denoting $\mathcal{Y}_2 = \xi_2(t,x)\partial_t + \eta_2(t,x)\partial_x$, a straightforward calculation shows that

$$[\mathcal{Y}_1,\mathcal{Y}_2] = \frac{\partial \xi_2}{\partial t}\partial_t + \frac{\partial \eta_2}{\partial t}\partial_x \ .$$

It must be satisfied $[\mathcal{Y}_1,\mathcal{Y}_2] = c_1\mathcal{Y}_1 + c_2\mathcal{Y}_2$ for certain structure constants $c_i \in \mathbb{R}$. Hence we get

$$c_1 = -\frac{\xi_2}{\eta_2}\frac{\partial \eta_2}{\partial t} + \frac{\partial \xi_2}{\partial t} \ , \ c_2 = \frac{1}{\eta_2}\frac{\partial \eta_2}{\partial t} \ . \tag{6.9}$$

From the second above condition we have

$$\eta_2(t,x) = \beta(x)\exp(c_2 t) \ ,$$

for certain function β. Now, the first condition in (6.9) is just

$$c_1 = -c_2\xi_2 + \frac{\partial \xi_2}{\partial t} \ ,$$

from which,

$$\xi_2(t,x) = \begin{cases} c_1 t + \alpha(x) & \text{if} \quad c_2 = 0 \ , \\ \alpha(x)\exp(c_2 t) - c_1/c_2 & \text{if} \quad c_2 \neq 0 \ , \end{cases}$$

for some function α. The expressions of V_{12} are straightforward from its definition. Hence, the proposition is proved. ∎

Notice that, from Proposition 6.5, if $\mathcal{Y}_1 = \partial_t$ and $\mathcal{Y}_2$ generate a 2–dimensional Lie point symmetry algebra $\mathcal{L}_2$ of differential equation (6.1), then the generators are analytic for all $t \in \mathbb{R}$. Therefore we define the domain of definition of the generators as the unbounded open strip $\Xi = \{(t,x) \in \mathbb{R} \times \mathbb{X}\} \subset \mathbb{R}^2$.

Theorem 6.6 *Assume that $\ddot{x} = w(x,\dot{x})$ with w smooth in $U \subset \mathbb{R}^2$ admits a 2–dimensional Lie point symmetry algebra $\mathcal{L}_2$ spanned by the $C^1(\Xi)$ vector fields $\mathcal{Y}_1 = \partial_t$ and $\mathcal{Y}_2$ such that $[\mathcal{Y}_1, \mathcal{Y}_2] = c_1\mathcal{Y}_1 + c_2\mathcal{Y}_2$.*

(i) *If $c_2 = 0$ and $\mathcal{Y}_2 \in C^2(\Xi)$, then $V_{12}(t,x,\dot{x}) = G(x,\dot{x})$ with $G(x,y) = y^2[c_1 + y\alpha'(x) - \beta'(x)] + \beta(x)w(x,y)$ an inverse integrating factor of $\mathcal{X}$ in $W = U \cap \{\mathbb{X} \times \mathbb{R}\}$ provided that $G \not\equiv 0$. Moreover, for analytic vector fields $\mathcal{Y}_2$ in Ξ, $\mathcal{X}$ has no limit cycles in W.*

(ii) *If $c_2 \neq 0$ then, changing the basis of $\mathcal{L}_2$ given in Proposition 6.5 in the form $\bar{\mathcal{Y}}_1 = c_1/c_2\mathcal{Y}_1 + \mathcal{Y}_2$, $\bar{\mathcal{Y}}_2 = -\mathcal{Y}_1/c_2$, one has $[\bar{\mathcal{Y}}_1, \bar{\mathcal{Y}}_2] = \bar{\mathcal{Y}}_1$ and $\bar{V}_{12}(t,x,\dot{x}) = \exp(c_2 t)\bar{G}(x,\dot{x})$ with $\bar{G}(x,\dot{x}) = \dot{x}[c_2\dot{x}\alpha(x) - c_2\beta(x) + \dot{x}^2\alpha'(x) - \dot{x}\beta'(x)] + \beta(x)w(x,\dot{x})$. In addition, $\partial w/\partial x \equiv 0$ or $\beta(x) \equiv 0$. If $\bar{G} \not\equiv 0$ and U is a simply connected domain, then $\bar{\mathcal{X}}$ has no periodic orbits in U and all the α or ω–limit sets of $\bar{\mathcal{X}}$ are contained in the invariant curve $\bar{G}(x,y) = 0$ of $\bar{\mathcal{X}}$.*

Proof. The first part of statement (i) is a corollary of Proposition 6.5 and statement (i) of Theorem 6.3. Now we will show the nonexistence of limit cycles in W for $\bar{\mathcal{X}}$. We proceed, by contradiction, assuming the existence of a limit cycle $\gamma = (x(t), y(t)) \subset W$ of $\bar{\mathcal{X}} = y\partial_x + w(x,y)\partial_y$. Since $\gamma \subset \{G(x,y) = 0\}$, first we get $\beta(x) \not\equiv 0$ (otherwise the curve $G = 0$ cannot have ovals) and second one has $G(x(t), y(t)) = 0$ for all $t \in [0, T)$ with $T > 0$ the minimum period of γ. Hence, the T–periodic functions $(x(t), y(t))$ satisfy

$$\dot{x}(t) = y(t) \ , \ \dot{y}(t) = w(x(t), y(t)) = \frac{-y^2(t)[c_1 + y(t)\alpha'(x(t)) - \beta'(x(t))]}{\beta(x(t))} \ ,$$

where $\beta(x(t)) \not\equiv 0$ due to the analiticity of β. But, this takes to contradiction because the differential system

$$\dot{x} = y \ , \ \dot{y} = \frac{-y^2[c_1 + y\alpha'(x) - \beta'(x)]}{\beta(x)} \ ,$$

does not possesses periodic orbits since it has the first integral $H(x,y) = A(x) + B(x)/y$ with

$$B(x) = \beta(x) \exp\left(-c_1 \int 1/\beta(x) \ dx\right) \ , \ A(x) = -\int B(x)\alpha'(x)/\beta(x) \ dx \ .$$

Hence, the level sets of $H(x,y)$ has not ovals.

Now we shall prove statement (ii). After the change of basis of $\mathcal{L}_2$ we have

$$\bar{\mathcal{Y}}_1 = \bar{\xi}_1(t,x)\partial_t + \bar{\eta}_1(t,x)\partial_x = \exp(c_2 t)(\alpha(x)\partial_t + \beta(x)\partial_x) \ , \ \bar{\mathcal{Y}}_2 = -1/c_2\partial_t \ ,$$

from where, the expression of $\bar{V}_{12}(t,x,\dot{x}) = \det\{\mathcal{X}, \bar{\mathcal{Y}}_1^{[1]}, \bar{\mathcal{Y}}_2^{[1]}\} = \exp(c_2 t)\bar{G}(x,\dot{x})$ given in the theorem is obtained.

On the other hand, it is easy to check that $\bar{\mathcal{Y}}_1$ satisfies the determining equations if and only if $\beta(x) \equiv 0$ or $\partial w/\partial x \equiv 0$. In either case, taking into account the determining equations that $\bar{\mathcal{Y}}_2$ satisfies, it follows that, if $\bar{G}(x,\dot{x}) \not\equiv 0$, then $\Omega_1/\bar{V}_{12}$ is a closed 1–form, where

$$\Omega_1 := \begin{vmatrix} dt & dx & d\dot{x} \\ 1 & \dot{x} & w(x,\dot{x}) \\ \bar{\xi}_1(t,x) & \bar{\eta}_1(t,x) & \bar{\eta}_1^{[1]}(t,x,\dot{x}) \end{vmatrix} = \exp(c_2 t)\Omega(t,x,\dot{x}) \ ,$$

with $\Omega(t,x,\dot{x}) = -c_2\bar{G}(x,\dot{x})\ dt + \bar{H}(x,\dot{x})\ dx + \bar{L}(x,\dot{x})\ d\dot{x}$ and

$$\begin{aligned} \bar{H}(x,\dot{x}) &= c_2\dot{x}\alpha(x) - c_2\beta(x) + \alpha(x)w(x,\dot{x}) + \dot{x}^2\alpha'(x) - \dot{x}\beta'(x) \ , \\ \bar{L}(x,\dot{x}) &= \beta(x) - \dot{x}\alpha(x) \ . \end{aligned}$$

In short, in simply connected domains $\mathbb{R}\times U$, the 1–form $\Omega_1/\bar{V}_{12}$ is exact and then there is a function $I(t,x,\dot{x})$ such that $dI = \Omega_1/\bar{V}_{12} = \Omega/\bar{G}$.

We claim that $I(t,x,\dot{x})$ is an invariant of $\ddot{x} = w(x,\dot{x})$. This property is true because, if $\bar{V}_{12}(t,x,\dot{x}) \not\equiv 0$ then, the system of partial differential equations

$$\mathcal{X}(I) = 0 \ , \ \bar{\mathcal{Y}}_1^{[1]}(I) = 0 \ , \ \bar{\mathcal{Y}}_2^{[1]}(I) = 1 \ , \tag{6.10}$$

always has a unique solution $I(t,x,\dot{x})$ up to an additive constant. To see that, develop (6.10) as the linear algebraic system

$$I_t + \dot{x}I_x + wI_{\dot{x}} = 0 \ , \ \bar{\xi}_1 I_t + \bar{\eta}_1 I_x + \bar{\eta}_1^{[1]} I_{\dot{x}} = 0 \ , \ \bar{\xi}_2 I_t + \bar{\eta}_2 I_x + \bar{\eta}_2^{[1]} I_{\dot{x}} = 1 \ ,$$

for the unknowns I_t, I_x and $I_{\dot{x}}$, where the subindex means partial derivative. Since the associated determinant is just $\bar{V}_{12} \not\equiv 0$, we solve the linear algebraic system and obtain, using that $\Omega_1/\bar{V}_{12}$ is exact, that I is computed as the line integral

$$I(t,x,\dot{x}) = \int \frac{\Omega_1}{\bar{V}_{12}} \ .$$

Of course, $I(t,x,\dot{x})$ is an invariant of $\ddot{x} = w(x,\dot{x})$ due to the first equation in (6.10).

In our particular case, I must be of the form $I(t,x,\dot{x}) = -c_2 t + J(x,\dot{x})$ where $J(x,\dot{x})$ is well–defined except on the set $\{\bar{G}(x,\dot{x}) = 0\}$.

Let $\Gamma \subset U$ be the α or ω–limit set of some orbit $(x(t),y(t))$ of $\bar{\mathcal{X}}$. Then $I(t,x(t),y(t))$ is a constant for all t. Since $(x(t),y(t)) \to \Gamma$ as $t\to\infty$ or $t\to-\infty$, we have that $J(x(t),y(t))$ is not defined as $t\to\infty$ or $t\to-\infty$. Therefore, the only possibility is that $\Gamma \subset \{\bar{G}(x,y) = 0\}$.

The fact that no periodic orbits exists for $\bar{\mathcal{X}}$ is trivial if $\partial w/\partial x \equiv 0$. For the other case, namely $\beta(x) \equiv 0$, no limit cycles can exists because of the expression of $\bar{G}(x,y)$ and there are no period annulus from Corollary 6.4. ∎

Theorem 6.7 *Assume that $\ddot{x} = w(x,\dot{x})$ with w smooth in $U \subseteq \mathbb{R}^2$ admits a 2–dimensional Lie point symmetry algebra $\mathcal{L}_2$ spanned by the $C^1(\Xi)$ vector fields $\mathcal{Y}_1 = \partial_t$ and $\mathcal{Y}_2$ such that $V_{12} \equiv 0$. Then $\bar{\mathcal{X}}$ has not periodic solutions in U and its α or ω–limit sets are contained into $\{y = 0\}$ which is fulfilled of singular points of $\bar{\mathcal{X}}$.*

Proof. From Proposition 6.5 we know the expression of $\mathcal{Y}_2$ and V_{12}. First of all, we note that the condition $\beta(x) \equiv 0$ is incompatible with the hypothesis $V_{12} \equiv 0$ due to the linear independence of $\mathcal{Y}_1$ and $\mathcal{Y}_2$. Hence, $V_{12} \equiv 0$ implies that

$$\ddot{x} = w(x, \dot{x}) = \begin{cases} -\frac{\dot{x}^2[c_1+\dot{x}\alpha'(x)-\beta'(x)]}{\beta(x)} & \text{if} \quad c_2 = 0 \ , \\ -\frac{\dot{x}[c_2\dot{x}\alpha(x)-c_2\beta(x)+\dot{x}^2\alpha'(x)-\dot{x}\beta'(x)]}{\beta(x)} & \text{if} \quad c_2 \neq 0 \ . \end{cases}$$

Therefore, the associated vector field $\bar{\mathcal{X}} = y\partial_x - y\Psi(x,y)\partial_y$ has the line $y = 0$ fulfilled of singular points. Out of this straight line, the phase portrait of $\bar{\mathcal{X}}$ in U is the same as the phase portrait of $\partial_x - \Psi(x,y)\partial_y$ which obviously has not periodic solutions. In addition, since $V_{12} \equiv 0$, from statement (iii) of Proposition 6.1, $(\eta_2 - y\xi_2)/(\eta_1 - y\xi_1)$ is an invariant of $\bar{\mathcal{X}}$. Hence, we get the invariants

$$I(t,x,y) = \begin{cases} -c_1 t + (y\alpha(x) - \beta(x))/y & \text{if} \quad c_2 = 0 \ , \\ \exp(c_2 t)(y\alpha(x) - \beta(x))/y & \text{if} \quad c_2 \neq 0 \ . \end{cases}$$

Therefore, the α or ω–limit sets of $\bar{\mathcal{X}}$ in U are contained in the continuum of singular points on $y = 0$. ∎

6.3 Lie point symmetries for polynomial Liénard systems

The subject of this section is the polynomial Liénard family

$$\ddot{x} = -f(x)\dot{x} - g(x) \ , \tag{6.11}$$

with real polynomials $f(x)$ and $g(x)$ of degrees n and m respectively. Regarding the existence of invariant algebraic curves of (6.11), Żołądek in [68] has shown that for $m > n$ there exist systems (6.11) with an invariant algebraic curve. But generic Liénard systems do not have such curves. Moreover, for $2 < n+1 < m$ but $(n,m) \neq (2,4)$ there exist systems with an algebraic limit cycle. In the cases $(n,m) = (0,m)$, $(1,m)$ for $m \neq 3$, $(4,2)$ and $(n+1,n)$ there cannot exist algebraic limit cycles. Moreover, for the remaining case $(n,m) = (1,3)$, in [14] the authors show that (6.11) cannot have simultaneously an algebraic invariant curve and a limit cycle.

Now, we present a characterization of the maximal dimension r of the Lie point symmetry algebra $\mathcal{L}_r$ of (6.11) in terms of the couple (n,m).

Theorem 6.8 *Let $\mathcal{L}_r$ be the maximal r–dimensional Lie point symmetry algebra associated to the polynomial Liénard equation (6.11). Then, for* $\deg f = n$ *and* $\deg g = m$*, the next holds:*

(i) *If $(n,m) \in \{(0,0),(0,1)\}$ then $r = 8$. In addition, there are families with $(n,m) = (1,3)$ and $r = 8$.*

(ii) *If either $m = n+1$ or $m = 2n+1$ or $n = 0$ and $m \geq 2$, then $r = 2$ can exists.*

(iii) *$r = 1$ otherwise.*

The proof of Theorem 6.8 is given in Appendix 6.4. There you can see the precise conditions (algebraic relations between the coefficients of $f(x)$ and $g(x)$) to get $\mathcal{L}_r$ and its infinitesimal generators.

Corollary 6.9 *Assume the polynomial Liénard equation (6.11) possesses r–dimensional Lie point symmetry algebra $\mathcal{L}_r$ with $r \geq 2$. Then, there is a 2–dimensional subalgebra $\mathcal{L}_2 \subseteq \mathcal{L}_r$ such that $\partial_t \in \mathcal{L}_2$.*

Proof. If $r \geq 2$, from the proof of Theorem 6.8, we know that $r \in \{2,8\}$ and the corollary follows when $r = 2$.

When $r = 8$ it is well known, see for instance [39], the existence of a change of coordinates $(t,x) \mapsto (r(t,x), s(t,x))$ defined in some region of the (t,x)–plane such that the Liénard equation (6.11) becomes $d^2s/dr^2 = 0$. Moreover, this linear equation admits a 8–dimensional Lie point symmetry algebra $\mathcal{L}_8$ with generators $\mathcal{Y}_1 = \partial_r$, $\mathcal{Y}_2 = r\partial_r$, $\mathcal{Y}_3 = s\partial_r$, $\mathcal{Y}_4 = sr\partial_r + s^2\partial_s$, $\mathcal{Y}_5 = r^2\partial_r + rs\partial_s$, $\mathcal{Y}_6 = \partial_s$, $\mathcal{Y}_7 = r\partial_s$ and $\mathcal{Y}_8 = s\partial_s$. The commutators table of this $\mathcal{L}_8$ is the following one.

	$\mathcal{Y}_1$	$\mathcal{Y}_2$	$\mathcal{Y}_3$	$\mathcal{Y}_4$	$\mathcal{Y}_5$	$\mathcal{Y}_6$	$\mathcal{Y}_7$	$\mathcal{Y}_8$
$\mathcal{Y}_1$	0	$\mathcal{Y}_1$	0	$\mathcal{Y}_3$	$2\mathcal{Y}_2 + \mathcal{Y}_8$	0	$\mathcal{Y}_6$	0
$\mathcal{Y}_2$		0	$-\mathcal{Y}_3$	0	$\mathcal{Y}_5$	0	$\mathcal{Y}_7$	0
$\mathcal{Y}_3$			0	0	$\mathcal{Y}_4$	$-\mathcal{Y}_1$	$\mathcal{Y}_8 - \mathcal{Y}_2$	$-\mathcal{Y}_3$
$\mathcal{Y}_4$				0	0	$-\mathcal{Y}_2 - 2\mathcal{Y}_8$	$-\mathcal{Y}_5$	$-\mathcal{Y}_4$
$\mathcal{Y}_5$					0	$-\mathcal{Y}_7$	0	0
$\mathcal{Y}_6$						0	0	$\mathcal{Y}_6$
$\mathcal{Y}_7$							0	$\mathcal{Y}_7$
$\mathcal{Y}_8$								0

We observe that, for any generator $\mathcal{Y}_j$ of $\mathcal{L}_8$ there is a 2–dimensional subalgebra $\mathcal{L}_2 \subset \mathcal{L}_8$ such that $\mathcal{Y}_j \in \mathcal{L}_2$. Since the structure constants of a Lie algebra are preserved under changes of variables and $\partial_t \in \mathcal{L}_8$, we conclude that there exists $\mathcal{L}_2 \subseteq \mathcal{L}_8$ with $\partial_t \in \mathcal{L}_2$. ∎

Theorem 6.10 *The polynomial Liénard differential equations (6.11) having a r–dimensional Lie point symmetry algebra $\mathcal{L}_r$ with $r \geq 2$ has no limit cycles in $\mathbb{R}^2$.*

Proof. As a consequence of Corollary 6.9, there is a 2–dimensional subalgebra $\mathcal{L}_2 \subseteq \mathcal{L}_r$ generated by $\mathcal{Y}_1 = \partial_t$ and $\mathcal{Y}_2$. Moreover, $\mathcal{Y}_2$ is an analytic vector field in $\mathbb{R}^2$. The proof of this fact is given by comparing the expression of $\mathcal{Y}_2 = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ in Proposition 6.5 and the general expression (6.16) of the components of an infinitesimal generator of the polynomial Liénard differential equations (6.11). Therefore, applying Theorems 6.6 and 6.7 the result follows. ∎

Remark 6.11 Another proof of Theorem 6.10 not based on the analyticity of the generator $\mathcal{Y}_2$ of $\mathcal{L}_2$ is given by the following reasoning. Applying Theorems 6.6 and 6.7, it follows that necessary conditions to have limit cycles are $[\mathcal{Y}_1, \mathcal{Y}_2] = c_1\mathcal{Y}_1$ and $V_{12}(t,x,\dot{x}) = \det\{\mathcal{X}, \mathcal{Y}_1^{[1]}, \mathcal{Y}_2^{[1]}\} = G(x,\dot{x}) \not\equiv 0$ with $G(x,y)$ an inverse integrating factor of the polynomial Liénard vector field $\bar{\mathcal{X}} = y\partial_x - [f(x)y + g(x)]\partial_y$. Since $\mathcal{Y}_2 = (c_1 t + \alpha(x))\partial_t + \beta(x)\partial_x$, one has

$$G(x,y) = \alpha'(x)y^3 + [c_1 - \beta'(x)]y^2 - f(x)\beta(x)y - g(x)\beta(x) \ . \tag{6.12}$$

Imposing G to be an inverse integrating factor of $\bar{\mathcal{X}}$ we obtain $\sum_{i=0}^{3} A_i(x)y^i \equiv 0$, that is, $A_i(x) \equiv 0$ for $i = 0, \ldots, 3$, where

$$\begin{aligned}
A_3(x) &= -\alpha''(x) \ , \\
A_2(x) &= 2f(x)\alpha'(x) + \beta''(x) \ , \\
A_1(x) &= c_1 f(x) + 3g(x)\alpha'(x) + \beta(x)f'(x) \ , \\
A_0(x) &= 2c_1 g(x) - g(x)\beta'(x) + \beta(x)g'(x) \ .
\end{aligned}$$

Therefore, from $A_3(x) \equiv 0$ we have $\alpha(x) = a_0 + a_1 x$ with arbitrary constants a_i. We will split the proof in two cases according to the vanishing or not of a_1:

i) We assume $a_1 \neq 0$. From $A_2(x) = A_1(x) \equiv 0$, we can express $f(x) = -\beta''(x)/(2a_1)$ and $g(x) = [c_1\beta''(x) + \beta(x)\beta'''(x)]/(6a_1^2)$. In particular, these relations means that $\beta(x)$ is a polynomial. Hence, we write

$$\beta(x) = \sum_{i=0}^{d} b_i x^i \ , \tag{6.13}$$

with $b_d \neq 0$. The last equation to solve is

$$A_0(x) = 2c_1^2\beta''(x) - c_1\beta'(x)\beta''(x) + 3c_1\beta(x)\beta'''(x) + \beta^2(x)\beta^{(4)}(x) \equiv 0 \ . \tag{6.14}$$

We must investigate whether this fourth order nonlinear differential equations admits polynomial solutions. Introducing (6.13) into (6.14) and assuming $d \geq 4$ we have that $A_0(x)$ has degree $3d-4$ with leading coefficient $d(d-1)(d-2)(d-3)b_d^3$ and we get a contradiction. Therefore, the upper bound $d \leq 3$ is satisfied. In fact, if $d < 3$ then the Liénard vector field $\bar{\mathcal{X}}$ becomes linear and no limit cycles can exist. So $d = 3$ and $\bar{\mathcal{X}}$ is a cubic Liénard vector field with linear damping, i.e., f and g have degrees 1 and 3 respectively. But, this family cannot have simultaneously an algebraic invariant curve (in our case $G(x,y) = 0$) and a limit cycle, see [14].

ii) We suppose $a_1 = 0$. Hence $A_2(x) = \beta''(x)$ and $A_2(x) \equiv 0$ implies $\beta(x) = b_0 + b_1 x$. We note here that $\beta(x) \not\equiv 0$ because, otherwise, from the expression of G in (6.12), the invariant curve $G(x,y) = 0$ has not ovals in contradiction with the existence of any limit cycle of $\bar{\mathcal{X}}$. Now, $f(x)$ and $g(x)$ satisfy the differential equations $A_1(x) = c_1 f(x) + (b_0 + b_1 x)f'(x) = 0$ and $A_0(x) = (2c_1 - b_1)g(x) + (b_0 + b_1 x)g'(x) = 0$ respectively. Non–trivial polynomial solutions f and g are only possible for $b_1 \neq 0$. More concretely, if n is the degree of f, then

$$f(x) = K_0(b_0 + b_1 x)^{-c_1/b_1} \ , \ g(x) = K_1(b_0 + b_1 x)^{1-2c_1/b_1} \ ,$$

with K_i arbitrary constants and $c_1 = -nb_1$. But then

$$G(x,y) = -b_1(n+1)y^2 - K_0(b_0 + b_1 x)^{n+1}y - K_1(b_0 + b_1 x)^{2(n+1)} \ ,$$

and the invariant algebraic curve $G(x,y) = 0$ has not ovals proving Theorem 6.10.

6.3.1 The Wilson system

One interesting example inside the polynomial Liénard families is the Wilson system, see [65]. The degrees of f and g are $(n,m)=(2,5)$ and the system is

$$\dot{x}=y, \quad \dot{y}=-\mu(x^2-1)y-x-\frac{\mu^2x^3}{16}(x^2-4), \tag{6.15}$$

with $\mu\neq 0$. System (6.15) has the invariant algebraic curve

$$\phi:=[y+(\mu/4)x(x^2-4)]^2+x^2-4=0,$$

as a limit cycle when $0<|\mu|<2$. For $|\mu|\geq 2$ the invariant algebraic curve ϕ turns out to contain a singular point, and so it cannot be a limit cycle. Moreover, system (6.15) has two additional invariant algebraic curves

$$\phi_\pm:=y+\frac{\mu}{4}x(x^2-2)\pm\frac{\sqrt{\mu^2-4}}{2}x=0\ ,$$

and the Darboux first integral $H(x,y)=\frac{\phi}{\phi_+\phi_-}\left(\frac{\phi_+}{\phi_-}\right)^{\frac{\mu}{\lambda}}$. It is straightforward to see, according with statement (ii) of Theorem 6.8, that Wilson system possesses a 2–dimensional Lie point symmetry algebra $\mathcal{L}_2$ if and only if $\mu=\pm 4/\sqrt{3}$. Of course, no limit cycle appears for these values of the parameter μ according with Theorem 6.10. In short, the generators of $\mathcal{L}_2$ are $\mathcal{Y}_1=\partial_t$ and $\mathcal{Y}_2=1/2\exp(\mp 2t/\sqrt{3})[\mp\sqrt{3}\partial_t-x\partial_x]$. Taking the new base $\bar{\mathcal{Y}}_1=\mathcal{Y}_1/k$ and $\bar{\mathcal{Y}}_2=k\mathcal{Y}_2$ with $k=\pm 2/\sqrt{3}$ we get $[\bar{\mathcal{Y}}_1,\bar{\mathcal{Y}}_2]=\bar{\mathcal{Y}}_2$ and its associated $\bar{V}_{12}(t,x,y)=\exp(-kt)\bar{G}(x,y)$ with $\bar{G}(x,y)=3x^2-4x^4+x^6\mp 6\sqrt{3}xy\pm 4\sqrt{3}x^3y+9y^2$ according to statement (ii) of Theorem 6.6. In summary, although system (6.15) is integrable for all values of μ, the Lie point symmetry method does not reflect this integrability.

6.4 Appendix

Proof of Theorem 6.8. We shall use the notation $f(x)=\sum_{i=0}^n f_ix^i$ and $g(x)=\sum_{i=0}^m g_ix^i$ with $f_i,g_i\in\mathbb{R}$ such that $f_n\neq 0$ and $g_m\neq 0$. Since (6.11) is autonomous, it is obvious that always admits the Lie point symmetry $\mathcal{Y}=\partial_t$. Hence, if we denote by $\mathcal{L}_r$ the r–dimensional Lie point symmetry algebra associated to (6.11), we have $r\geq 1$.

Let $\mathcal{Y} = \xi(t,x)\partial_t + \eta(t,x)\partial_x \in \mathcal{L}_r$. Then, the functions $\xi(t,x)$ and $\eta(t,x)$ must satisfy the determining equations (6.4). Such equation is a polynomial of degree 3 in the variable $\dot{x}$, i.e., $\sum_{i=0}^{3} \Lambda_i(t,x)\dot{x}^i \equiv 0$ where

$$\begin{aligned}
\Lambda_3(t,x) &= -\frac{\partial^2 \xi}{\partial x^2} \, , \\
\Lambda_2(t,x) &= 2f(x)\frac{\partial \xi}{\partial x} + \frac{\partial^2 \eta}{\partial x^2} - 2\frac{\partial^2 \xi}{\partial t \partial x} \, , \\
\Lambda_1(t,x) &= \eta f'(x) + 3g(x)\frac{\partial \xi}{\partial x} + f(x)\frac{\partial \xi}{\partial t} + 2\frac{\partial^2 \eta}{\partial t \partial x} - \frac{\partial^2 \xi}{\partial t^2} \, , \\
\Lambda_0(t,x) &= \eta g'(x) + g(x)\left(2\frac{\partial \xi}{\partial t} - \frac{\partial \eta}{\partial x}\right) + f(x)\frac{\partial \eta}{\partial t} + \frac{\partial^2 \eta}{\partial t^2} \, .
\end{aligned}$$

Hence, from the vanishing of the cubic and quadratic coefficients $\Lambda_3(t,x) = \Lambda_2(t,x) \equiv 0$, the following expressions follows

$$\xi(t,x) = A(t)x + B(t) \, , \; \eta(t,x) = \dot{A}(t)x^2 + C(t)x + D(t) - 2A(t)\int F(x)\,dx \, , \tag{6.16}$$

for some functions $A(t)$, $B(t)$, $C(t)$ and $D(t)$. Here $F(x)$ is a primitive of $f(x)$, that is, $F'(x) = f(x)$. Hence, in order to solve the determining equations, the only that remains to do is to impose $\Lambda_1(t,x) = \Lambda_0(t,x) \equiv 0$. Such a coefficients Λ_i are polynomials in the variable x. So, we have to annul all its coefficients.

(i) Assume $m > 2n+1$ (in particular $m > 1$) and $n > 1$. Then, $\deg_x \Lambda_1(t,x) = m$ with leading coefficient $3g_m A(t)$. So we have $A(t) \equiv 0$. Now, $\deg_x \Lambda_1(t,x) = n$ with leading coefficient $f_n(nC(t) + \dot{B}(t))$. Hence we put $C(t) = -\dot{B}(t)/n$. Therefore, $\deg_x \Lambda_1(t,x) = n-1$ with leading coefficient $[n^2 f_n D(t) + f_{n-1}\dot{B}(t)]/n$, from which we have $D(t) = -f_{n-1}\dot{B}(t)/[n^2 f_n]$.

Hence, $\deg_x \Lambda_0(t,x) = m$ with leading coefficient $(2n+1-m)g_m\dot{B}(t)/n$ and so we get $\dot{B}(t) = 0$, that is, the constant $B(t) = b_0 \in \mathbb{R}$. Finally, we have $\xi(t,x) = b_0$ and $\eta(t,x) = 0$ or equivalently $\dim \mathcal{L}_r = 1$.

(ii) Assume $m < n+1$ and $n > 0$. Then, $\deg_x \Lambda_1(t,x) = 2n+1$ with leading coefficient $-2nf_n^2 A(t)/[(n+1)(n+2)]$. So we have $A(t) \equiv 0$. From now, we may simply repeat verbatim the first paragraph of part (i) obtaining thus $C(t) = -\dot{B}(t)/n$ and $D(t) = -f_{n-1}\dot{B}(t)/[n^2 f_n]$.

Moreover, $\deg_x \Lambda_0(t,x) = n+1$ with leading coefficient $-f_n\ddot{B}(t)/n$ and so we get $\ddot{B}=0$, that is, $B(t) = b_0 + b_1 t$ with $b_i \in \mathbb{R}$. Under these assumptions, $\deg_x \Lambda_0(t,x) = m$ with leading coefficient $b_1 g_m(2n+1-m)/n$. Then, the only option is $b_1 = 0$ and therefore $\xi(t,x) = b_0$ and $\eta(t,x) = 0$, that is, $\dim \mathcal{L}_r = 1$.

(iii) Assume $n+1 \le m < 2n+1$. The unbounded set $(n,m) \in \mathbb{N}^2$ in the degrees space with these restriction may be studied by using the straight lines parametrization $m = n+k$ with $n \ge k$ and $k \in \mathbb{N}\backslash\{0\}$.

We put $m = n+k$. Since $n \ge k$, we have $\deg_x \Lambda_1(t,x) = 2n+1$ with leading coefficient $-2nf_n^2 A(t)/[(n+1)(n+2)]$. Therefore we get $A(t) \equiv 0$. With exact arguments, we repeat verbatim the first paragraph of part (i) obtaining thus $C(t) = -\dot{B}(t)/n$ and $D(t) = -f_{n-1}\dot{B}(t)/[n^2 f_n]$.

From now we have two possibilities, namely, either $k=1$ or $k>1$.

(a) If $k > 1$ then $\deg_x \Lambda_2(t,x) = n+k$ with leading coefficient $(n+1-k)g_{n+k}\dot{B}(t)/n$. This implies $\dot{B}(t) \equiv 0$ because $g_{n+k} \neq 0$ by hypothesis and $n+1-k > 0$ due to the condition $n \ge k$. Hence, we take the constant $B(t) = b_0 \in \mathbb{R}$. But, in this case we get $\mathcal{Y} = \xi(t,x)\partial_t + \eta(t,x)\partial_x = b_0\partial_t$ so that $\dim \mathcal{L}_r = 1$.

(b) If $k=1$ then $\deg_x \Lambda_2(t,x) = n+1$ with leading coefficient $[ng_{n+1}\dot{B}(t) - f_n\ddot{B}(t)]/n$. The vanishing of this coefficient gives a linear ordinary differential equation for $B(t)$ which general solution is

$$B(t) = b_1 + b_0 \frac{f_n}{ng_{n+1}} \exp\left(\frac{ng_{n+1}}{f_n} t\right) ,$$

with b_0 and b_1 arbitrary real constants. Moreover, one must impose $b_0 \neq 0$ in order to have $\dim \mathcal{L}_r > 1$. In short, we have that $\Lambda_i(t,x) \equiv 0$ if and only if the polynomials $P_i(x) \equiv 0$ for $i = 1,2$ where

$$\begin{aligned} P_1(x) &= -[f_1 f_{n-1} + n^2(n+2)g_{n+1}] + \sum_{i=0}^{n-1}[n(n-i)f_n f_i - (i+1)f_{n-1}f_{i+1}]x^i , \\ P_2(x) &= [nf_n^3 g_0 + 2n^2 f_n^3 g_0 - f_{n-1}f_n^2 g_1 - nf_0 f_{n-1} f_n g_{n+1} - n^2 f_{n-1} g_{n+1}^2] \\ &\quad - f_n[-2n^2 f_n^2 g_1 + 2f_{n-1}f_n g_2 + nf_{n-1}f_1 g_{n+1} + n^2 f_n f_0 g_{n+1} + n^3 g_{n+1}^2]x + \\ &\quad \sum_{i=2}^{n} f_n[nf_n^2 g_i(1-i+2n) - f_n f_{n-1} g_{i+1}(1+i) - ng_{n+1}(f_i f_{n-1} + nf_{i-1}f_n)]x^i, \end{aligned}$$

Moreover, the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t}\ ,\ \mathcal{Y}_2 = \exp\left(\frac{n g_{n+1}}{f_n} t\right)\left(\frac{f_n}{n g_{n+1}}\frac{\partial}{\partial t} - \frac{f_{n-1} + n f_n\, x}{n^2 f_n}\frac{\partial}{\partial x}\right)\ .$$

(iv) We suppose $m = 2n+1$ and $n > 0$ (in particular $m > 2$). Then, $\deg_x \Lambda_1(t,x) = 2n+1$ with leading coefficient $A(t)\Psi_1$, where $\Psi_1 := 3g_{2n+1} - 2nf_n^2/(2+3n+n^2)$. So we have two options, either $A(t) \equiv 0$ or $\Psi_1 = 0$.

(a) Taking $A(t) \equiv 0$ we get, repeating verbatim the first paragraph of part (i), $C(t) = -\dot{B}(t)/n$ and $D(t) = -f_{n-1}\dot{B}(t)/[n^2 f_n]$. Under these conditions we have

$$\mathcal{Y} = B(t)\frac{\partial}{\partial t} - \frac{f_{n-1} + n f_n x}{n^2 f_n}\dot{B}(t)\frac{\partial}{\partial x}\ ,$$

so that, a necessary condition to have $\dim \mathcal{L}_r > 1$ is $\dot{B}(t) \not\equiv 0$. Then, we have

$$\Lambda_1(t,x) = -\frac{(f_{n-1} + n f_n x)\dot{B}(t)}{n^2 f_n} f'(x) + \dot{B}(t) f(x) - \frac{2+n}{n}\ddot{B}(t)\ .$$

Solving the first order differential equation $\Lambda_1(t,x) = 0$ with respect to $f(x)$ we obtain

$$f(x) = k_0(f_{n-1} + n f_n x)^n + \frac{n+2}{n}\frac{\ddot{B}(t)}{\dot{B}(t)}\ ,$$

with k_0 nonvanishing real constant. Since $f(x)$ does not depend on t, it follows $\ddot{B}(t)/\dot{B}(t) = k$ with $k \in \mathbb{R}$. Hence, the general solution for $B(t)$ is

$$B(t) = \begin{cases} b_0 + \frac{b_1 \exp(kt)}{k} & \text{if} \quad k \neq 0\ , \\ b_0 + b_1 t & \text{if} \quad k = 0\ , \end{cases}$$

with arbitrary constants b_i. So we have two cases:

(a.1) When $k \neq 0$, we have

$$\Lambda_0(t,x) = \frac{-b_1 \exp(kt)}{n^3 f_n}\bar{\Lambda}_0(x)\ ,$$

where

$$\begin{aligned}\bar{\Lambda}_0(x) &= n(f_{n-1}+nf_nx)g'(x) - n^2(1+2n)f_ng(x)\\ &\quad +k(f_{n-1}+nf_nx)(2k(n+1)+k_0n(f_{n-1}+nf_nx)^n)\end{aligned}$$

Solving the first order differential equation $\bar{\Lambda}_0(x) = 0$ with respect to $g(x)$ we obtain

$$g(x) = (f_{n-1}+nf_nx)\left[k_1(f_{n-1}+nf_nx)^{2n} + \frac{k^2(1+n)+kk_0n(f_{n-1}+nf_nx)^n}{n^3f_n}\right]$$

with constant $k_1 \in \mathbb{R}\backslash\{0\}$. We conclude that the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t}\ ,\ \mathcal{Y}_2 = \exp(kt)\left(\frac{1}{k}\frac{\partial}{\partial t} - \frac{f_{n-1}+nf_n\ x}{n^2f_n}\frac{\partial}{\partial x}\right)\ .$$

(a.2) If $k = 0$, then

$$\Lambda_0(t,x) = \frac{b_1}{n^2f_n}\bar{\Lambda}_0(x)\ ,$$

where $\bar{\Lambda}_0(x) = -(f_{n-1}+nf_nx)g'(x) + n(2n+1)f_ng(x)$. Since $\dot{B}(t) = b_1 \neq 0$ in order to have $\dim \mathcal{L}_r > 1$, $\Lambda_0(t,x) = 0$ if and only if $\bar{\Lambda}_0(x) = 0$ from which we get

$$g(x) = k_1(f_{n-1}+nf_nx)^{2n+1}\ ,$$

with k_1 a real constant different from zero. Finally we obtain that the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t}\ ,\ \mathcal{Y}_2 = t\frac{\partial}{\partial t} - \frac{f_{n-1}+nf_n\ x}{n^2f_n}\frac{\partial}{\partial x}\ .$$

(b) If $A(t) \not\equiv 0$, then we can solve $g(x)$ from the equation $\Lambda_1(t,x) = 0$ and we obtain

$$\begin{aligned}g(x) &= [4F(x)\dot{A}(t) - f(x)(x\dot{A}(t)+\dot{B}(t)) - 2\dot{C}(t) - \{D(t) - \\ &\quad 2A(t)\int F(x)dx + x(C(t)+x\dot{A}(t))\}f'(x) - 3x\ddot{A}(t) + \ddot{B}(t)]/[3A(t)],\end{aligned} \tag{6.17}$$

where $F'(x) = f(x)$. From this expression, we deduce for $n > 0$ that $\deg_x g(x) = n^2+n+1$ with nonvanishing leading coefficient $[3nf_n^2]/[3(n+1)(n+2)] \neq 0$. But, our hypothesis is $\deg_x g(x) = m = 2n+1$. This is only possible if $n = 1$ and this case is further analyzed in case (viii). The remaining case, namely $n = 0$ and $m = 1$, is trivial because the Liénard equation is linear and so $\dim \mathcal{L}_r = 8$.

After the analysis performed in (i)–(iv), only remains open the cases $n = 0$ and $n = 1$ with $m > 3$. Of course, when $(n, m) \in \{(0, 0), (0, 1)\}$ then the Liénard equation is linear and it has a 8–dimensional Lie point symmetry algebra. Let us study the remaining cases.

(v) Suppose $n = 0$ and $m > 2$. Then, $\deg_x \Lambda_1(t, x) = m$ with leading coefficient $3g_m A(t)$. So we have $A(t) \equiv 0$. Therefore $\Lambda_1(t, x) = \bar{\Lambda}_1(t) = f_0\dot{B}(t) - \ddot{B}(t) + 2\dot{C}(t)$. On the other hand, $\deg_x \Lambda_0(t, x) = m$ with leading coefficient $g_m[2\dot{B}(t) + (m-1)C(t)]$ so that $\dot{B}(t) = (1-m)C(t)/2$ and, solving the differential equation $\bar{\Lambda}_1(t) = 0$, we get $C(t) = c_0 \exp[(m-1)f_0 t/(m+3)]$ with arbitrary constant c_0. Now, $\deg_x \Lambda_0(t, x) = m-1$ with leading coefficient $m g_m D(t) - g_{m-1}C(t)$. Hence we put $D(t) = g_{m-1}C(t)/(m g_m)$. We note here that, a necessary condition to have $\dim \mathcal{L}_r > 1$ is $c_0 \neq 0$. Under these conditions,

$$\Lambda_0(t, x) = \frac{c_0 \exp[(m-1)f_0 t/(m+3)]}{m(m+3)^2 g_m} \bar{\Lambda}_0(x) \ ,$$

where $\bar{\Lambda}_0(x)$ is the next polynomial

$$\begin{aligned}\bar{\Lambda}_0(x) &= m^2(3+m)^2 g_0 g_m - g_{m-1}[2(m^2-1)f_0^2 + (3+m)^2 g_1] \\ &\quad + \left\{m(m-1)g_m[(3+m)^2 g_1 - 2(m+1)f_0^2] - 2(3+m)^2 g_2 g_{m-1}\right\} x \\ &\quad + \sum_{i=2}^{m-1} (3+m)^2[m(m-i)g_m g_i - (i+1)g_{m-1}g_{i+1}]x^i \ .\end{aligned}$$

Finally, if $\bar{\Lambda}_0(x) \equiv 0$, then the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t} \ , \ \mathcal{Y}_2 = -\exp\left(\frac{(m-1)f_0 t}{m+3}\right)\left(\frac{m+3}{2f_0}\frac{\partial}{\partial t} - \frac{g_{m-1} + m g_m x}{m g_m}\frac{\partial}{\partial x}\right) \ .$$

(vi) Suppose $n = 0$ and $m = 2$. Then, $\deg_x \Lambda_1(t, x) = 2$ with leading coefficient $3g_2 A(t)$. So we have $A(t) \equiv 0$. Now, $\deg_x \Lambda_0(t, x) = 2$ with leading coefficient $g_2[2\dot{B}(t) + C(t)]$ so that $C(t) = -2\dot{B}(t)$. Therefore $\deg_x \Lambda_0(t, x) = 1$ with leading coefficient $2[D(t)g_2 + g_1\dot{B}(t) - f_0\ddot{B}(t) - B^{(3)}(t)]$, so that $D(t) = [-g_1\dot{B}(t) + f_0\ddot{B}(t) + B^{(3)}(t)]/g_2$. Now, since $\Lambda_1(t, x) = f_0\dot{B}(t) - 5\ddot{B}(t)$, we vanish $\Lambda_1(t, x)$ obtaining $B(t) = b_0 + 5b_1 \exp[t f_0/5]/f_0$ with arbitrary constants b_i. Hence

$$\Lambda_0(t, x) = \frac{b_1 \exp[t f_0/5]}{625 g_2}(36 f_0^4 - 625 g_1^2 + 2500 g_0 g_2) \ .$$

We emphasize here that, a necessary condition to have $\dim \mathcal{L}_r > 1$ is $b_1 \neq 0$. Finally, if the parameter relation $36f_0^4 - 625g_1^2 + 2500g_0g_2 = 0$ is satisfied, then the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t}\ , \ \mathcal{Y}_2 = \exp\left(\frac{f_0 t}{5}\right)\left(\frac{5}{f_0}\frac{\partial}{\partial t} - \frac{25g_1 - 6f_0^2 + 50g_2x}{25g_2}\frac{\partial}{\partial x}\right)\ .$$

(vii) Suppose $n = 1$ and $m > 3$. Then, $\deg_x \Lambda_1(t,x) = m$ with leading coefficient $9g_mA(t)$. So we have $A(t) \equiv 0$. Now, $\deg_x \Lambda_1(t,x) = 1$ with leading coefficient $f_1[\dot{B}(t) + C(t)]$ so that $C(t) = -\dot{B}(t)$. Therefore $\Lambda_1(t,x) = f_1D(t) + f_0\dot{B}(t) - 3\ddot{B}(t)$ so that $D(t) = [3\ddot{B}(t) - f_0\dot{B}(t)]/f_1$. Now, $\deg_x \Lambda_0(t,x) = m$ with leading coefficient $(m-3)f_1g_m\dot{B}(t)$. Hence, $B(t) = b_0 \in \mathbb{R}$ and we get a 1–dimensional Lie point symmetry algebra.

(viii) Assume $n = 1$ and $m = 3$. Then, $\deg_x \Lambda_1(t,x) = 3$ with leading coefficient $A(t)(9g_3 - f_1^2)/3$. So we have two possibilities:

(a) Let $A(t) \equiv 0$ and $9g_3 - f_1^2 \neq 0$. Hence, $\deg_x \Lambda_1(t,x) = 1$ with leading coefficient $f_1[\dot{B}(t) + C(t)]$ so that $C(t) = -\dot{B}(t)$. Therefore $\Lambda_1(t,x) = f_1D(t) + f_0\dot{B}(t) - 3\ddot{B}(t)$ so that $D(t) = [3\ddot{B}(t) - f_0\dot{B}(t)]/f_1$. Now, $\deg_x \Lambda_0(t,x) = 2$ with leading coefficient $[(f_1g_2 - 3f_0g_3)\dot{B}(t) + (9g_3 - f_1^2)\ddot{B}(t)]/f_1$. From the vanishing of this coefficient, two cases arise:

(a.1) If $f_1g_2 - 3f_0g_3 \neq 0$ then

$$B(t) = b_0 + \frac{b_1(f_1^2 - 9g_3)}{f_1g_2 - 3f_0g_3}\exp\left(\frac{(f_1g_2 - 3f_0g_3)t}{f_1^2 - 9g_3}\right)$$

with arbitrary constants b_i. We remark here that, a necessary condition to have $\dim \mathcal{L}_r > 1$ is $b_1 \neq 0$. Then, $\Lambda_0(t,x) \equiv 0$ if and only if the parameter relations

$$\begin{aligned} 0 &= f_1^2(f_1^2g_1 - 2f_0f_1g_2 + 4g_2^2) + 3[f_1^2(f_0^2 - 6g_1) + 4f_0f_1g_2 - 9g_2^2]g_3 \\ &\quad +9(9g_1 - 2f_0^2)g_3^2\ , \\ 0 &= 3f_1^6g_0 - f_0f_1^5g_1 + 27g_3^2(4f_0^2g_2 + 9g_1g_2 - 81g_0g_3) - f_1^4[g_2(f_0^2 - 3g_1) \\ &\quad +81g_0g_3] + f_0f_1^3[2g_2^2 + 3g_3(f_0^2 + 6g_1)] - 9f_0f_1g_3[5g_2^2 + g_3(4f_0^2 + 9g_1)] \\ &\quad +3f_1^2[g_2^3 + 2g_2g_3(f_0^2 - 9g_1) + 243g_0g_3^2]\ , \end{aligned}$$

are satisfied. In short, the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t}\ ,$$

$$\mathcal{Y}_2 \;=\; \exp\left(\frac{(f_1g_2-3f_0g_3)t}{f_1^2-9g_3}\right)\left(\frac{f_1^2-9g_3}{f_1g_2-3f_0g_3}\frac{\partial}{\partial t}+\frac{3g_2-f_0f_1+(9g_3-f_1^3)x}{f_1^2-9g_3}\frac{\partial}{\partial x}\right) .$$

(a.2) If $f_1g_2 - 3f_0g_3 = 0$ then, taking $g_2 = 3f_0g_3/f_1$, $\deg_x \Lambda_0(t,x) = 2$ with leading coefficient $(9g_3 - f_1^2)\ddot{B}(t)/f_1$. So $\ddot{B}(t) \equiv 0$, that is, $B(t) = b_0 + b_1 t$. In order to have $\dim \mathcal{L}_r > 1$ we need $b_1 \neq 0$. Finally, $\Lambda_0(t,x) \equiv 0$ if and only if

$$0 = f_1^2 g_1 - 3f_0^2 g_3 \ , \ 0 = 3f_1g_0 - f_0g_1 \ .$$

In summary, the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\mathcal{Y}_1 = \frac{\partial}{\partial t} \ , \ \mathcal{Y}_2 = t\frac{\partial}{\partial t} - \frac{f_0 + f_1 x}{f_1}\frac{\partial}{\partial x} \ .$$

(b) Let $A(t) \not\equiv 0$. Then we can solve $g(x)$ from the equation $\Lambda_1(t,x) = 0$ and we obtain (6.17). More precisely, $g(x) = f_1^2x^3/9 + f_0f_1x^2/3 + \bar{g}_1(t)x + \bar{g}_0(t)$ with $\bar{g}_1(t) = -[f_1C(t) - 3f_0\dot{A}(t) + f_1\dot{B}(t) + 3\ddot{A}(t)]/[3A(t)]$ and $\bar{g}_0(t) = -[f_1D(t) + f_0\dot{B}(t) + 2\dot{C}(t) - \ddot{B}(t)]/[3A(t)]$. This means

$$g_3 = \frac{f_1^2}{9} \ , \ g_2 = \frac{1}{3}f_0f_1 \ ,$$

and, from $\bar{g}_i(t) = g_i$ with $i = 0, 1$, we solve $C(t)$ and $D(t)$ as a function of $A(t)$, $B(t)$ and some of their derivatives. Finally, $\deg_x \Lambda_1(t,x) = 1$ and $\Lambda_1(t,x) \equiv 0$ if and only if $A(t)$ and $B(t)$ satisfy the following two coupled linear differential equations of order 4 with constant coefficients

$$\begin{aligned} -9A^{(iv)} + 6f_0A^{(iii)} + 3(f_0^2 - 3g_1)\ddot{A} + 3(3f_1g_0 - f_0g_1)\dot{A} &= \Omega_1(B) \ , \\ 3f_1B^{(iv)} - 6f_1B^{(iii)} + f_1(3f_1g_0 - f_0g_1)\dot{B} &= \Omega_2(A) \ , \end{aligned} \tag{6.18}$$

with $\Omega_1 = 2f_1(3g_1 - f_0^2)\dot{B} + 6f_1B^{(iii)}$ and $\Omega_2 = 6(f_0f_1g_0 - g_1^2)\dot{A} + 6(f_0^2 - 2g_1)A^{(iii)} - 6A^{(v)}$. Hence, the general solution $A(t)$ depends linearly on 4 arbitrary constants (and similarly for $B(t)$). In short, the Lie point symmetry algebra is 8–dimensional. Here, we do not obtain the closed form of the generators $\mathcal{Y}_i$, but their analyticity in all $\mathbb{R}^2$ is guaranteed because the functions $A(t)$ and $B(t)$ are analytic in $\mathbb{R}$ since they are solutions of the linear system (6.18).

(ix) We suppose $n = 1$ and $m = 2$. Then, $\deg_x \Lambda_1(t,x) = 3$ with leading coefficient $-f_1^2 A(t)/3$. So $A(t) \equiv 0$. Now, following verbatim the first paragraph of part (viii)-(a) we get $C(t) = -\dot{B}(t)$ and $D(t) = [3\ddot{B}(t) - f_0\dot{B}(t)]/f_1$. Now, $\deg_x \Lambda_0(t,x) = 2$ with leading coefficient $g_2\dot{B}(t) - f_1\ddot{B}(t)$. Therefore,

$$B(t) = b_0 + \frac{b_1 f_1}{g_2} \exp\left(\frac{g_2 t}{f_1}\right) ,$$

with arbitrary constants b_i. Moreover, $b_1 \neq 0$ if we want $\dim \mathcal{L}_r > 1$. Finally, $\Lambda_0(t,x) \equiv 0$ if and only if the parameter relations

$$\begin{aligned} 0 &= f_1^2 g_1 + 2g_2(2g_2 - f_0 f_1) , \\ 0 &= 3f_1^4 g_0 - f_0 f_1^3 g_1 - f_1^2 g_2(f_0^2 - 3g_1) + g_2^2(2f_0 f_1 + 3g_2) , \end{aligned}$$

are satisfied. Hence, under these conditions, the Lie point symmetry algebra has $\dim \mathcal{L}_r = 2$ and it is spanned by

$$\begin{aligned} \mathcal{Y}_1 &= \frac{\partial}{\partial t} , \\ \mathcal{Y}_2 &= \exp\left(\frac{g_2 t}{f_1}\right)\left(\frac{f_1}{g_2}\frac{\partial}{\partial t} + \frac{f_0 f_1 - 3g_2 + f_1^2 x}{f_1^2}\frac{\partial}{\partial x}\right) . \end{aligned}$$

(x) We suppose $n = 1$ and $m = 1$. The analysis is totally analogous to the case (ix), obtaining $A(t) \equiv 0$, $C(t) = -\dot{B}(t)$ and $D(t) = [3\ddot{B}(t) - f_0\dot{B}(t)]/f_1$. Now, $\deg_x \Lambda_0(t,x) = 2$ with leading coefficient $-f_1\ddot{B}(t)$. Therefore, $B(t) = b_0 + b_1 t$ with arbitrary constants b_i. Moreover, we need $b_1 \neq 0$ for $\dim \mathcal{L}_r > 1$. But, $\deg_x \Lambda_0(t,x) = 1$ with leading coefficient $2b_1 g_1$ which never vanishes. Hence $\dim \mathcal{L}_r = 1$.

(xi) We take $n = 1$ and $m = 0$. The analysis is the same as case (x) and we get $A(t) \equiv 0$, $C(t) = -\dot{B}(t)$, $D(t) = [3\ddot{B}(t) - f_0\dot{B}(t)]/f_1$, $B(t) = b_0 + b_1 t$ with arbitrary constants b_i and $b_1 \neq 0$ if we have $\dim \mathcal{L}_r > 1$. Finally, $\Lambda_0(t,x) = 3b_1 g_0 \neq 0$ and therefore $\dim \mathcal{L}_r = 1$. ∎

Chapter 7

Reversibility and the Center Problem

Summary. In this work we study the narrow relation between reversibility and the center problem and also between reversibility and the integrability problem. It is well known that an analytic system having either a non-degenerate or nilpotent center at the origin is analytically reversible or orbitally analytically reversible, respectively. Here we prove the existence of a smooth map that transforms an analytic system having a degenerate center at the origin into a reversible linear system (after rescaling the time). Moreover, if the degenerate center has an analytic or a $\mathcal{C}^\infty$ reversing symmetry, the transformed system by the map has also a reversing symmetry. From the knowledge of a first integral we give a procedure to detect reversing symmetries.

7.1 Introduction

This chapter is focused in the planar differential systems with a reversing symmetry. A reversing symmetry is one of the fundamental symmetries in natural science and it arises in many branches in physics, see for instance [44, 59] and references therein. More specifically, this work is devoted to the study of two problems arising in the theory of analytic reversible systems in the plane. The first one is the study of the narrow relation between reversibility and the center problem, see [50, 60]. It is well–known that an analytic system having a non-degenerate (resp. nilpotent) center at the origin is analytically reversible (resp. analytically orbitally reversible), see [8, 53, 58]. Nevertheless the relation between reversibility and the center problem for degenerate singular points is not established. Here we prove the existence of a smooth map that transforms an analytic system having a degenerate center at the origin into a reversible linear system (after rescaling the time). Moreover, if the degenerate center has an analytic or a $\mathcal{C}^\infty$ reversing symmetry, the transformed system by the map has also a reversing symmetry. From the knowledge of a local smooth first integral near the center we give a procedure to detect reversing symmetries in some cases. The second problem in which we focus our attention is the study of the existence of a local analytic first integral in a neighborhood of a non-degenerated singular point for a reversible system and also for certain degenerate singular points.

The work is organized as follows: In Section 7.2 we summarize some general definitions and results about reversible systems. In Section 7.3 we study the relation between reversibility and the center problem presenting our results for degenerate singular points. In Section 7.4 we show that an orbitally analytically reversible system has always a local analytic first integral in a neighborhood of a non-degenerate singular point. Moreover, we prove the existence of a local analytic first integral in a neighborhood of a degenerate singular point for some reversible systems.

7.2 Background

Consider two-dimensional analytic differential systems

$$\dot{x} = P(x, y), \quad \dot{y} = Q(x, y), \tag{7.1}$$

defined in a neighborhood $U \subset \mathbb{R}^2$ of the origin such that $P(0, 0) = Q(0, 0) = 0$. We denote by $\mathcal{X} = P(x, y)\partial_x + Q(x, y)\partial_y$ the associated vector field to system (7.1) and $\mathrm{div}\mathcal{X} = \partial P/\partial x + \partial Q/\partial y$ its divergence.

Recall that an analytic *involution* $R : U \to \mathbb{R}^2$ is an analytic diffeomorphism different from the identity such that $R \circ R = Id$, where Id is the identity map. The following proposition can be deduced from the works developed by Montgomery and Zippin. In [49] the proposition is given for $\mathcal{C}^\infty$–involutions. For sake of completeness we give its proof.

Proposition 7.1 *After a linear change of coordinates, any analytic involution different from the identity takes the form* $R(x,y) = (R_1, R_2) = (x + r_1(x,y), -y + r_2(x,y))$ *with analytic functions* r_i *without constant nor linear terms. Moreover, the analytic near–identity change of coordinates* $\phi(x,y) = (u,v) = (x + \cdots, y + \cdots)$ *with*

$$u = \frac{x + R_1}{2}, \;\; v = \frac{y - R_2}{2}$$

linearizes the involution, that is, $R_0(u,v) = \phi \circ R \circ \phi^{-1}(u,v) = (u, -v)$.

Proof. Since R is a diffeomorphism near the origin, it is clear the linear part of R is not null. Moreover, if we denote by $B \in \mathcal{M}_2(\mathbb{R})$ the non vanishing matrix associated to the linear part of R, it is easy to check that $R \circ R = Id$, implies $B^2 = I_2$, the identity matrix. Hence, B is of the form

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

with $a^2 + bc = cb + d^2 = 1$ and $b(a+d) = c(a+d) = 0$. Hence, we have two cases: (i) $a = -d$ and $d^2 + bc = 1$; (ii) $b = c = 0$ and $a^2 = d^2 = 1$. Notice that, when a and d have different sign, then case (ii) is included in case (i). It is easy to see that no nonlinear involution R can exists in case (ii) with equal sign of a and d. More precisely, in case $b = c = 0$ and $a = d = \pm 1$ we have $R(x,y) = (\pm x + f(x,y), \pm y + g(x,y))$ with f and g functions without linear nor constant terms. Then, the condition $R(\pm x + f(x,y), \pm y + g(x,y)) = Id$ does not have solution for any f and g.

From now we only study case (i) which is equivalent to the conditions $\mathrm{Tr} B = 0$ and $\det B = -1$. Therefore, doing a linear change of coordinates, we can assume

$$B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

without loss of generality. This implies that $R(x,y) = (x + r_1(x,y), -y + r_2(x,y))$.

Finally, taking into account $R \circ R = Id$ and the explicit expression of ϕ, it is easy to see that

$$\begin{aligned} R_0(u,v) &= \phi \circ R \circ \phi^{-1}(u,v) = \phi(R(x,y)) = \phi(R_1, R_2) \\ &= (u(R_1,R_2), v(R_1,R_2)) = \left(\frac{R_1 + R_1 \circ R}{2}, \frac{R_2 - R_2 \circ R}{2} \right) \\ &= \left(\frac{R_1 + x}{2}, \frac{R_2 - y}{2} \right) = (u, -v) \ , \end{aligned}$$

finishing the proof. ∎

Definition 7.2 *System* (7.1) *is* C^k*–reversible* with $k \in \{1, 2, \ldots, \infty, w\}$ if there is a C^k diffeomorphism $R : U \to U$ which is an involution such that $R_* \mathcal{X} = -\mathcal{X} \circ R$, where $\mathcal{X}$ is the vector field associated to system (7.1).

When system (7.1) is reversible by means of an involution R, we will say that R is a reversing symmetry for system (7.1). The manifold Fix(R) of fixed points of R is defined as $\mathrm{Fix}(R) = \{(x,y) \in \mathbb{R}^2 | R(x,y) = (x,y)\}$ and it is assumed to be of dimension 1. If $\mathcal{X}$ is C^w–reversible, we also say that $\mathcal{X}$ is analytically reversible. We will call orbitally analytically reversible to a system (7.1) when in the set of all systems which are analytically orbitally equivalent to it there is one which is analytically reversible.

The first examples of reversible systems were given by Poincaré, see [53]. Systems which are reversible by means of the involution $R_0(x,y) = (x, -y)$ are called *time–reversible* systems. Notice that system (7.1) is time–reversible if and only if $P(x,-y) = -P(x,y)$ and $Q(x,-y) = Q(x,y)$.

Remark. We note that the analytic near–identity change of coordinates $\phi(x,y) = (u,v)$ that linearizes the involution $R(x,y)$ transforms the reversible system (7.1) into a time–reversible system

$$\dot{u} = v\tilde{P}(u, v^2), \quad \dot{v} = \tilde{Q}(u, v^2). \tag{7.2}$$

Proposition 7.3 *The critical point at the origin of a reversible system* (7.1) *is always weak, that is,* $\mathrm{div}\mathcal{X}(0,0) = 0$. *In particular, if the origin is a nondegenerated singular point, then it is either a center or a weak hyperbolic saddle.*

Proof. We shall prove that, if A is the matrix associated to the linear part of a reversible system (7.1), then $\mathrm{Tr}A = 0$. If $A = 0$ it is obvious. So, we assume $A \neq 0$. We take

$$A = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix},$$

and the involution $R(x, y) = (ax + by + \cdots, cx - ay + \cdots)$ with $a^2 + bc = 1$. Imposing the reversibility condition $R_*\mathcal{X} = -\mathcal{X} \circ R$, we obtain at first order the condition $\alpha + \delta = 0$. Hence $\mathrm{div}\mathcal{X}(0, 0) = 0$. ∎

7.2.1 The non-invertible map Ψ

Let Ψ be a non-invertible map such that $\Psi \circ R = \Psi$ where R is an involution. We have the following definition depending on the functional class of Ψ.

Definition 7.4 *A reversible system (7.1) by means of the involution $R(x, y)$ defined in U is said to be algebraically reversible if $\Psi = (f(x, y), g(x, y))$ being f and g algebraic over $\mathbb{C}(x, y)$. Additionally, if f and g are rational over $\mathbb{C}(x, y)$. Then, the system is said to be rationally reversible.*

Definition 7.5 *A reversible system (7.1) by means of the involution $R(x, y)$ defined in U is reducible at the origin via the map $\Psi = (f(x, y), g(x, y))$, with f and g analytic functions (real or complex) around of it, if the differential equation associated to the system is the pull-back associated to Ψ of a differential equation without singularities on $\Psi(U)$.*

We want to remark that in [22] and [66] the authors define an algebraically (resp. rational) reversible system as a reversible reducible system via the algebraic (resp. rational) map Ψ. The standard example of rationally reversible system is the time–reversible system $\dot{x} = -y + P(x^2, y)$, $\dot{y} = x(1 + Q(x^2, y))$ which has the reversing symmetry $R_0 = (-x, y)$. In this case, $\Psi_0 = (x^2, y)$ is the map realizing the reversibility. Notice that $\Psi_0 \circ R_0 = \Psi_0$. The curve of non-invertibility of Ψ is the so-called *fold curve*. The pull–back induced by Ψ creates a symmetric system respect to the fold curve, that is, of the y axis. The next proposition shows the behavior of the the map Ψ under a change of coordinates. Let system (7.1) be reversible by means of an involution $R(x, y)$. After apply the change of variables ϕ that linearizes the involution R, system (7.1) is transformed into system (7.2). Then, the following result holds.

Proposition 7.6 *If system* (7.2) *is reducible at the origin via the map* Ψ_0*, then system (7.1) is reducible at the origin via the map* $\Psi = \phi^{-1} \circ \Psi_0 \circ \phi$ *such that* ϕ *is the change of variables that linearizes the involution* R *of system* (7.1).

Proof. Let $\mathcal{X}$ and $\bar{\mathcal{X}}$ be the vector fields associated to system (7.1) and system (7.2), respectively. In other words, $\phi_* \mathcal{X} = \bar{\mathcal{X}}$.

We will show that if the map Ψ_0 reduces the vector field $\bar{\mathcal{X}}$ to a non–singular vector field $\bar{\mathcal{Y}}$ at the origin (that is, $\bar{\mathcal{Y}}$ is a rescaling of $\Psi_{0*}\bar{\mathcal{X}}$) then the map $\phi^{-1} \circ \Psi_0 \circ \phi$ reduces the vector field $\mathcal{X}$ to the vector field $\mathcal{Y}$, which is non–singular at the origin after a rescaling. Let $\Psi = \phi^{-1} \circ \Psi_0 \circ \phi$. Then, it follows

$$\mathcal{Y} = \Psi_* \mathcal{X} = \Psi_*(\phi^* \bar{\mathcal{X}}) = (\Psi \circ \phi^{-1})_* \bar{\mathcal{X}} = (\phi^{-1} \circ \Psi_0)_* \bar{\mathcal{X}} = \phi^*(\Psi_{0*} \bar{\mathcal{X}}) \ .$$

Finally, since $\Psi_{0*}\bar{\mathcal{X}}$ is orbitally equivalent to a non–singular vector field at the origin and ϕ is a diffeomorphism, we get that $\mathcal{Y}$ is orbitally equivalent to a non–singular vector field at the origin. ∎

Hence, given an analytic involution R and using the change of variables that linearizes this involution we always can find the non-invertible map Ψ, see Lemma 3.6 in [59]. The reciprocal problem, that is to find R from the knowledge of Ψ, is a strong problem based in the relation $\Psi \circ R = \Psi$.

7.3 Reversibility and the center problem

The origin of system (7.1) is called *monodromic* if there are no orbits tending to or leaving the origin with a certain angle. For analytic systems, a monodromic singular point is always a center or a focus, see [43]. Poincaré shown that if the origin of system (7.1) is monodromic and satisfying $R_{0*}\mathcal{X} = -\mathcal{X} \circ R_0$, with R_0 the involution $R_0 = (-x, y)$, then the system has a center at the origin (the symmetry principle), see [50]. Time–reversibility implies some geometric properties, y–axis is a line of symmetry for the orbits of the system, hence, no trajectory around the origin can be a spiral.

In this section we discuss the relation between reversibility and the systems having a center type singularity at the origin. Let us consider the system

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = A_i \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f(x,y) \\ g(x,y) \end{pmatrix}, \tag{7.3}$$

with A_i a real 2×2 matrix. The system is defined in an open set $U \subset \mathbb{R}^2$ having a center at the origin, where f and g are analytic functions in U starting in at least second order terms. Doing a linear change of coordinates and a rescaling of time (if necessary), the system can be written with its linear part into the Jordan form, that is, A_i must be of the form:

$$\textbf{(i)}\ A_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},\ \textbf{(ii)}\ A_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ \textbf{(iii)}\ A_3 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \qquad (7.4)$$

When the origin of (7.3) is a center, the class (i) is called non-degenerated center, the class (ii) nilpotent center and (iii) degenerated center. According to Poincaré, system (7.3) with linear part A_1 has a center at the origin if, and only if, there exists a near–identity analytic change of coordinates

$$(u, v) = \phi(x, y) = (x + o(|(x, y)|), y + o(|(x, y)|)) \ ,$$

transforming system (7.3) with linear part A_1 into the normal form

$$\dot{u} = -v[1 + \psi(u^2 + v^2)] \ , \ \dot{v} = u[1 + \psi(u^2 + v^2)] \ , \qquad (7.5)$$

with ψ an analytic function near the origin such that $\psi(0) = 0$. It is clear that the transformed system (7.5) is time–reversible. Then, the original system (7.3) with linear part A_1 is reversible by means of the involution $R = \phi^{-1} \circ R_0 \circ \phi$. It follows that all system (7.3) with linear part A_1 having a center is reversible. This assertion is also stated in [60]. Thus, we have the following result.

Theorem 7.7 *System* (7.3) *with linear part A_1 is a center if, and only if, it is analytically reversible.*

Strózyna and Żołądek have proved in [58] that there exits an analytic change of coordinates near the origin transforming the nilpotent system (7.3) with linear part A_2 into a generalized Liénard system $\dot{x} = y$, $\dot{y} = a(x) + y\bar{b}(x)$ with $a(x) = a_s x^s + \cdots$, $s \geq 2$, and $\bar{b}(0) = 0$. In fact, following [58], if the nilpotent singularity is monodromic then there is a change of variables and a time rescaling leading to $\dot{x} = y$, $\dot{y} = -x^{2n-1} + yb(x)$ with $n \geq 2$. Hence, the center problem for nilpotent singularities reduce to the study of the parity of the function $b(x)$ according with the center conditions for the Liénard systems, see also [8]. Thus the following theorem can be established.

Theorem 7.8 *Suppose that the analytic system* (7.3) *with linear part* A_2 *has a center at the origin. Then, there exists an analytic change of variables and a unity time rescaling such that it can be written as*

$$\dot{x} = y, \quad \dot{y} = -x^{2n-1} + yb(x) \ , \tag{7.6}$$

with $n \geq 2$ *an integer and* $b(x)$ *an analytic odd function.*

It is straightforward to check that system (7.6) is analytically reversible by means of the involution $R(x, y) = (-x, y)$. Hence, we can establish the following result.

Theorem 7.9 *System* (7.3) *with linear part* A_2 *satisfying the monodromy conditions is a center if, and only if, is orbitally analytically reversible.*

The problem that remains open is to characterize when an analytic system having a degenerate center is C^k–reversible. A contribution towards the solution of this problem is the objective of the following section.

7.3.1 Reversibility for degenerated centers

In the previous subsection, we have seen that all non-degenerated and nilpotent analytic centers are analytically reversible or orbitally analytically reversible, respectively. However, in [7] is proved the existence of analytic systems with a degenerated center that admits neither a Liouville first integral nor a nontrivial reversal symmetry. A natural open question is if any degenerated center admits a $\mathcal{C}^k(U)$ involution where $U \subset \mathbb{R}^2$ containing the center with $k \in \mathbb{N} \cup \{\infty\}$ or perhaps a non-continuous involution R, see [59]. In the following we investigate the existence of a $\mathcal{C}^\infty$ map Φ bringing system (7.1) having a degenerated center at the origin into a reversible linear system. Moreover, we study the existence of an analytic or a $\mathcal{C}^\infty$ involution R and the relation between them.

Assume that system (7.1) has a degenerated center at the origin. In [48] the authors prove the following results.

Theorem 7.10 *System* (7.1) *has a center at the origin if and only if there exits a first integral of class* $\mathcal{C}^\infty$ *with an isolated minimum at the origin in a neighborhood of it.*

Proposition 7.11 *System* (7.1) *has a center at the origin if and only if there exits an invariant Lebesgue measure with density* μ *of class* $\mathcal{C}^\infty$ *defined in a neighborhood of it.*

From a classical Liouville result, it is known that a measure with density μ is invariant for the flow of $\mathcal{X}$ if and only if $\mathcal{X}(\mu) + \mu \operatorname{div}\mathcal{X} = 0$. Hence, Proposition 7.11 proves the existence of an integrating factor μ of class $\mathcal{C}^\infty$ near the center at the origin of system (7.1). Moreover, we can establish the following result.

Proposition 7.12 *System* (7.1) *with a* $\mathcal{C}^\infty$ *first integral* H *defined in a neighborhood of the origin has always an inverse integrating factor* V *of class* $\mathcal{C}^\infty$ *in a neighborhood of the origin.*

Proof. Taking $V = PH/(\partial H/\partial y) = -QH/(\partial H/\partial x)$, it is straightforward to see, by construction, that this V is of class $\mathcal{C}^\infty$. ∎

Remark. Assume that system (7.1) has a center at the origin. We claim that there always exists a $\mathcal{C}^\infty$ integrating factor μ of Proposition 7.11 different from $1/V$, being V the $\mathcal{C}^\infty$ inverse integrating factor of Proposition 7.12. This fact follows because, if we assume $\mu = 1/V$, then we can always construct another $\mathcal{C}^\infty$ integrating factor $\bar{\mu}$ of the form $\bar{\mu} = \mu H$ with H a $\mathcal{C}^\infty$ first integral of Theorem 7.10. Hence, $\bar{\mu} \neq 1/V$. This point will be useful in the proof of the next theorem.

In the next theorem we present the main result of this section.

Theorem 7.13 *Assume that the analytic system* (7.1) *has a degenerated center at the origin. Then, there exits a map* $\Phi(x, y)$ *of class* $\mathcal{C}^\infty$ *in a neighborhood* U *of the origin such that transforms (after rescaling the time) system* (7.1) *into a linear time reversible system.*

Proof. We will split the proof in two parts according to wether system (7.1) is or not hamiltonian.

We consider first the case in which system (7.1) is not hamiltonian. From Theorem 7.10 we know the existence of a smooth first integral of system (7.1) in a neighborhood of the origin. Applying Proposition 7.12 it follows that there exits an inverse integrating factor V of class $\mathcal{C}^\infty$ in a neighborhood of the origin. Moreover, from Proposition 7.11 we also know the existence of an integrating factor μ of class $\mathcal{C}^\infty$ in a neighborhood of the origin. Then, it is well known that we can construct another first integral H of class $\mathcal{C}^\infty$ in a neighborhood of the origin of system (7.1) of the form $H = \mu V$. Since $\mathcal{X}V = (\mathrm{div}\mathcal{X})V$ and $\mathcal{X}\mu = -(\mathrm{div}\mathcal{X})\mu$, we can define a map of class $\mathcal{C}^\infty$ in a neighborhood of the origin as $\Phi_1 = (V, \mu)$ such that system (7.1) in the new coordinates (V, μ) takes the form $\dot{V} = (\mathrm{div}\mathcal{X})V$ and $\dot{\mu} = -(\mathrm{div}\mathcal{X})\mu$. Performing the linear change of variables $\Phi_2 = (r, s) = (V - \mu, V + \mu)$ the system becomes, after rescaling the time, the linear system $r' = s$ and $s' = r$ which is a reversible system by means of the involution $R(s, r) = (-s, r)$. Therefore, the map $\Phi = \Phi_2 \circ \Phi_1$ of class $\mathcal{C}^\infty$ in a neighborhood of the origin transforms (after rescaling the time) system (7.1) into a time–reversible system.

Assume now that the analytic system (7.1) is hamiltonian. In this case we can not follow the same reasonings of the previous case because any integrating factor or inverse integrating factor is of the form $f(H)$, where H is a first integral of system (7.1). System (7.1) has an analytic first integral $H(x, y)$ around the origin such that $H(0, 0) = 0$. Since the origin is a center for system (7.1), it has no invariant straight line through it. It follows that neither x nor y can divide $H(x, y)$. Hence, there is a positive integer m such that $\partial^k H(0, 0)/\partial y^k = 0$ for $k = 1, \ldots, m-1$ and $\partial^m H(0, 0)/\partial y^m \neq 0$. From Weierstrass Preparation Theorem, see [11], it follows that in a neighborhood of the origin, the function $H(x, y)$ can be represented as $H(x, y) = w(x, y)u(x, y)$ where $w(x, y) = \sum_{i=0}^m \varphi_i(x) y^i$ with $\varphi_m \equiv 1$ and $\varphi_i(x)$ analytic functions near the origin for $0 \leq i \leq m - 1$ and u is a unity, that is, $u(x, y)$ is an analytic function in a neighborhood of the origin satisfying $u(0, 0) \neq 0$.

We can always factorize $w(x, y) = \prod_{i=1}^k (y - g_i(x))^{m_i}$ with $\sum_{i=1}^k m_i = m$ where $g_i(x)$ are the analytic branches of $w(x, y)$ near the origin and the positive integers m_i their multiplicities, see [11]. Notice that, since $w(x, y)$ has an isolated zero at $(0, 0)$, it follows that no branches $g_i(x)$ can be real–valued. In fact, branches appear in complex conjugates couples. Thus, we can write $w(x, y) = v(x, y)\bar{v}(x, y)$ where we define $v(x, y) = \prod_{i=1}^{k/2} (y - g_i(x))^{m_i}$ and the overbar denotes complex conjugated operation, that is, $\bar{v}(x, y) = \prod_{i=k/2+1}^{k} (y - g_i(x))^{m_i}$. Of course, this means that $H(x, y) = (f^2(x, y) + g^2(x, y))u(x, y)$ where f and g are real valued functions satisfying $v = f + ig$ with $i^2 = -1$. Hence, one can take a C^∞ map in a neighborhood of the origin as $\Phi = (r, s) = (f\sqrt{u}, g\sqrt{u})$ that transforms (after rescaling the time) system (7.1) into the time–reversible system $r' = -s$ and $s' = r$. In order to characterize the time–rescaling realized above, we must to compute either $\mathcal{X}(f\sqrt{u})$ or $\mathcal{X}(g\sqrt{u})$ with $\mathcal{X} = P(x,y)\partial_x + Q(x,y)\partial_y$. Taking into account that that $\partial H/\partial x = Q/V$ and $\partial H/\partial y = -P/V$ where $V(x,y)$ is the inverse integrating factor of system (7.1) with associated first integral $H(x,y) = (f^2(x,y) + g^2(x,y))u(x,y)$, it is straightforward to see that $\mathcal{X}(f\sqrt{u}) = -2VJ_\Phi g\sqrt{u}$ where $J_\Phi(x,y)$ is the jacobian determinant of the map Φ. Then, the time–rescaling realized is $t \mapsto \tau$ with $d\tau = 2VJ_\Phi\, dt$. ∎

In fact, the former result shows how system (7.1) having a degenerated center at the origin is transformed, after rescaling the time, into a time-reversible system having either a linear saddle or the linear center at the origin (see the proof of Theorem 7.13). Let $\tilde{\mathcal{R}}(f,g)$ be any analytic involution near the origin. Let $\bar{\mathcal{X}}$ be either the linear center $-g\partial_f + f\partial_g$ or the linear saddle $g\partial_f + f\partial_g$. Assume that $\bar{\mathcal{X}}$ is reversible under $\tilde{\mathcal{R}}$. Then, $\tilde{\mathcal{R}}(f,g) = (af + cg + \cdots, cf - ag + \cdots)$ with $a^2 + c^2 = 1$ in the center case and $\tilde{\mathcal{R}}(f,g) = (af - cg + \cdots, cf - ag + \cdots)$ with $a^2 - c^2 = 1$ for the saddle. The proof of this fact is just to take a formal power series around the origin for $\tilde{\mathcal{R}}$ whose linear part be a linear involution and to obtain its coefficients by imposing the reversibility condition $\tilde{\mathcal{R}}_* \bar{\mathcal{X}} = -\bar{\mathcal{X}} \circ \tilde{\mathcal{R}}$.

The next proposition states that the condition of being reversible $\mathcal{X}$ is invariant under the map Φ of Theorem 7.13. Moreover, Proposition 7.15 shows that the the condition of being reversible $\Phi_*\mathcal{X}$ defined on $\Phi(U)$ is invariant after rescaling the time.

Proposition 7.14 *Assume that the analytic system* (7.1) *has a degenerated center at the origin and is $\mathcal{C}^\infty$–reversible by means of the involution R. Let Φ be a smooth map defined on U such that the curve of noninvertibility* $\det(d\Phi) = 0$ *divide U in open sets U_i. Then, the vector field $\Phi_*\mathcal{X}$ defined on $\Phi(U)$ is reversible by means of the involution $\mathcal{R}_a = \Phi \circ R \circ \Phi_i^{-1}$ where Φ_i^{-1} is the inverse of the diffeomorphism $\Phi|_{U_i}$ for some i.*

Proof. Using the notation of the proposition, the open neighborhood U of the origin is given by the union

$$U = \cup_i U_i \cup \{(x,y) \in U : \det(d\Phi(x,y)) = 0\} .$$

Then, from the inverse function theorem, we have that the restriction of the smooth map $\Phi : U \to \Phi(U)$ to the open subset U_i denoted by $\Phi|_{U_i}$ is a smooth diffeomorphism for all i having as inverse $\Phi_i^{-1} : \Phi(U) \to U_i$.

Let $\mathcal{X}$ be an analytic vector field defined on U which is reversible by means of the involution R. We define the map $\mathcal{R}_a : \Phi(U) \to \Phi(U)$ as $\mathcal{R}_a = \Phi \circ R \circ \Phi_i^{-1}$. We note that, in fact, $\mathcal{R}_a$ is an involution. Finally, if we define $\mathcal{Y}$ the vector field on $\Phi(U)$ given by $\mathcal{Y} = \Phi_* \mathcal{X}$, we claim that $\mathcal{R}_{a*}\mathcal{Y} = -\mathcal{Y} \circ \mathcal{R}_a$ proving thus the proposition. We have

$$\begin{aligned} \mathcal{R}_{a*}\mathcal{Y} &= (\mathcal{R}_a \circ \Phi)_* \mathcal{X} = (\Phi \circ R \circ \Phi_i^{-1} \circ \Phi)_* \mathcal{X} = (\Phi \circ R)_* \mathcal{X} = \Phi_*(R_* \mathcal{X}) \\ &= -\Phi_*(\mathcal{X} \circ R) = -\Phi_* \mathcal{X} \circ \Phi_* R = -\mathcal{Y} \circ \mathcal{R}_a \ , \end{aligned}$$

and the claim is proved. ∎

Proposition 7.15 *Assume that a vector field $h\mathcal{X}$ is reversible under an involution R where h is a scalar function. Hence, the reescaled vector field $\mathcal{X}$ is reversible under R.*

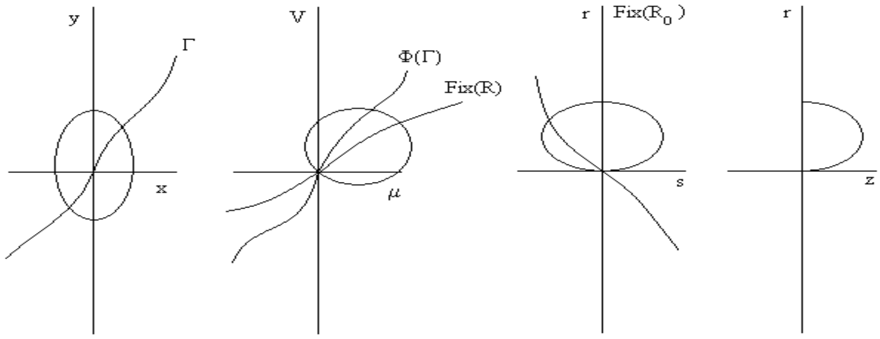
Proof. Since $h\mathcal{X}$ is a reversible vector field under R, we have $R_*(h\mathcal{X}) = -(h\mathcal{X}) \circ R$. Therefore $(h \circ R^{-1}) R_* \mathcal{X} = -(h \circ R)(\mathcal{X} \circ R)$. Taking into account $h \circ R = h \circ R^{-1} = h$, the proposition is proved. ∎

The following result shows that given an involution R on U there exists a fold mapping Ψ, that is, satisfying $\Psi \circ R = \Psi$ and it shows that from the knowledge of Φ we can look for an analytic or a $\mathcal{C}^\infty$ involution R.

Proposition 7.16 *Assume that the analytic system* (7.1) *has a degenerated center at the origin. Assume also that there exists an involution R of class $\mathcal{C}^\infty$ such that $\Phi \circ R = \mathcal{R}_a \circ \Phi$ and $R_* \mathcal{X} = -\mathcal{X} \circ R$, where $\mathcal{R}_a \in \tilde{\mathcal{R}}$. Then, system* (7.1) *is a $\mathcal{C}^\infty$–reversible center in a neighborhood of the origin by means of the map $\Psi = \Psi_0 \circ \varphi \circ \Phi$ where φ is the change of coordinates that brings the involution $\mathcal{R}_a$ to $R_0 = (-x, y)$, that is, $\mathcal{R}_a = \varphi^{-1} \circ R_0 \circ \varphi$.*

Proof. Since $\Psi_0 \circ R_0 = \Psi_0$, it follows $\Psi_0 \circ \varphi \circ \mathcal{R}_a = \Psi_0 \circ \varphi$. Taking into account $\Phi \circ R = \mathcal{R}_a \circ \Phi$, we get $\Psi_0 \circ \varphi \circ \Phi \circ R = \Psi_0 \circ \varphi \circ \Phi$ which means $\Psi \circ R = \Psi$. ∎

The following figure clarifies how a neighborhood U of the origin containing part of the curve of non-invertibility $\Gamma = \det(d\Phi) = 0$ and the curve of fixes points of R i.e., the fold curve $\mathrm{Fix}(R)$, is transformed by the map $\Psi = \Psi_0 \circ \varphi \circ \Phi$ realizing the reversibility.



7.3.2 The functional class of the map Ψ realizing the reversibility

Notice that, given a reversible analytic system with a degenerated center at the origin, the functional class of the map Ψ of Proposition 7.16 realizing the reversibility on U is deeply related with the functional class of the first integral H defined around of the center. In the light of the result of Theorem 7.13 the following result holds.

Proposition 7.17 *Assume that the analytic system* (7.1) *be a reversible hamiltonian system with a degenerated center at the origin. Then, the map Ψ of Proposition 7.16 realizing the reversibility on U is analytic.*

Proof. From Theorem 7.13 it follows that the two factors of the first integral of (7.1) around the origin are analytic which implies that Φ is analytic. Hence, since the map $\Psi = \Psi_0 \circ \varphi \circ \Phi$ is a composition of analytic maps, Ψ is analytic. ∎

In the next proposition we show the relationship between the functional class of the map Ψ realizing the reversibility for system (7.1) having a degenerated center at the origin and the fact that system (7.1) possess a Darboux first integral defined in a neighborhood of the degenerated center, see [16, 17].

Proposition 7.18 *Assume that the analytic system* (7.1) *be a reversible system with a degenerated center at the origin. Moreover, assume also that system* (7.1) *possesses a Darboux first integral of the form $H = \prod_{i=1}^{k} f_i^{\alpha_i}$ where $f_i = 0$ are irreducible invariant algebraic curves and $\alpha_i \in \mathbb{Q}$. Then, the map Ψ of Proposition 7.16 realizing the reversibility is rational.*

Proof. It is straightforward to see that a system which admits a Darboux first integral around the origin with $\alpha_i \in \mathbb{Q}$ admits a Darboux first integral around the origin of the form $H = \prod_{i=1}^{k} f_i^{\beta_i}$ with $\beta_i \in \mathbb{Z}$. Moreover, always exists a factorization of H such that the factors are rational. Hence, the map Φ and the map Ψ realizing the reversibility are rational functions. ∎

In the next section we apply the results obtained in Theorem 7.13 and Proposition 7.16 in order to determine the reversibility condition for systems having a degenerated center at the origin.

7.3.3 Examples

The method works as follows. First, we factorize the $\mathcal{C}^\infty$ first integral around the center of system (7.1) in order to obtain the map Φ which brings the system into a time–reversible system (after rescaling the time). Second, we solve the equation $\Phi \circ R = \tilde{\mathcal{R}} \circ \Phi$ to find an involution R such that $R_*\mathcal{X} = -\mathcal{X} \circ R$, where $\mathcal{X}$ is the vector field associated to system (7.1). Here $\tilde{\mathcal{R}}$ is a family of reversing symmetries of the linear center. According with Proposition 7.15 any involution $\mathcal{R}_a$ admitted by $\Phi_*\mathcal{X}$ is an involution of the linear center, i.e., $\mathcal{R}_a \in \tilde{\mathcal{R}}$. Finally, applying Proposition 7.1 we get the analytic change of variables that transform system (7.1) into a time–reversible system.

Example 1. Consider the hamiltonian system

$$\dot{x} = -6y^2(x^3 + 2y^3), \quad \dot{y} = 6x^2(x^3 + y^3). \tag{7.7}$$

which has the analytic first integral $H = 2y^6 + x^6 + 2x^3y^3$. The origin of system (7.7) is a center because is a monodromic singular point with an analytic first integral around of it. We can factorize $H = (x^3 + (1+i)y^3)(x^3 + (1-i)y^3)$. Taking $\Phi = (f, g) = (x^3 + y^3, y^3)$, H becomes $H = f^2 + g^2$ and system (7.7) is transformed into

$$\dot{f} = -g\,k(f, g), \quad \dot{g} = f\,k(f, g)\ , \tag{7.8}$$

where $k = 9(f-g)^{2/3}g^{2/3}$. Notice that rescaling the time we obtain the linear center which admits the family of linear involutions given by $\tilde{\mathcal{R}}(f, g) = (af + cg, cf - ag)$ with $c^2 = 1 - a^2$. We are looking (if it exists) for a real value $\tilde{a}$ such that $k \circ \mathcal{R}_{\tilde{a}} = k$. Recall that, if there exists such $\tilde{a}$, then system (7.8) is reversible by means of the involution $\mathcal{R}_{\tilde{a}}$. Finally, we will search for an involution R of class $\mathcal{C}^\infty$ such that $\Phi \circ R = \mathcal{R}_{\tilde{a}} \circ \Phi$ and $R_* \mathcal{X} = -\mathcal{X} \circ R$ where $\mathcal{X}$ is the vector field associated to system (7.7).

It is easy to see that $k(f,g) = k(af + cg, cf - ag)$ when $a = c = \sqrt{2}/2$. Therefore system (7.8) is a reversible system by means of the involution $\mathcal{R}_{\sqrt{2}/2}(f,g) = \sqrt{2}/2(f+g, f-g)$. Moreover, the involution $R = (2^{1/6}y, 2^{-1/6}x)$ satisfies $\Phi \circ R = \mathcal{R}_{\sqrt{2}/2}\Phi$ and $R_*\mathcal{X} = -\mathcal{X} \circ R$. Hence, we have showed that system (7.7) is reversible by means of R.

Since the analytic change of variables $\phi = (z, w) = 1/2(-2^{-1/6}x + y, 2^{-1/6}x + y)$ brings the involution R to the canonical Jordan form $R_0 = (-z, w)$, it follows that ϕ transforms system (7.7) into the time reversible system

$$\begin{aligned} \dot{z} &= w((1+\sqrt{2})w^4 + 2(-1+5\sqrt{2})w^2z^2 + (1+5\sqrt{2})z^4), \\ \dot{w} &= -z((-1+5\sqrt{2})w^4 + 2(1+5\sqrt{2})w^2z^2 + (-1+\sqrt{2})z^4). \end{aligned}$$

Moreover, applying Proposition 7.16 we obtain the map realizing the reversibility as $\Psi = \Psi_0 \circ \varphi \circ \Phi = ((2-\sqrt{2})(x^3 - \sqrt{2}y^3)^2, (\sqrt{2+\sqrt{2}})(x^3 + \sqrt{2}y^3))$with $\varphi = (((\sqrt{2-\sqrt{2}})/2)f - ((\sqrt{2+\sqrt{2}})/2)g, \ \ ((\sqrt{2+\sqrt{2}})/2)f + ((\sqrt{2-\sqrt{2}})/2)g)$ because $\Psi \circ R = \Psi$. Notice that system (7.7) is reducible at the origin via the map Ψ. ∎

Example 2. Consider the hamiltonian system

$$\dot{x} = -y^2(3x^2 + y + 9x^2y^3 - 9xy^6 + 3y^9), \quad \dot{y} = (x+y^3)^3 \ , \tag{7.9}$$

which has the polynomial first integral $H = y^4 + (x+y^3)^4$ around the origin. The origin of system (7.9) is a center because it is a monodromic singular point with an analytic first integral around of it. We can factorize $H = ((x+y^3)^2 - iy^2)((x+y^3)^2 + iy^2)$. Taking $\Phi = (f,g) = ((x+y^3)^2, y^2)$, system (7.9) is transformed into $\dot{f} = 2\sqrt{fg}\ g$ and $\dot{g} = -\sqrt{fg}\ f$ which is a reversible system by means of the involution $\mathcal{R}_0(f,g) = (g,f)$, i.e., $\mathcal{R}_a(f,g)$ with $a = 0$ and consequently $c = 1$. Moreover, the involution $R = (y - (x+y^3)^3, x+y^3)$ satisfies $\Phi \circ R = \mathcal{R}_0\Phi$ and $R_*\mathcal{X} = -\mathcal{X} \circ R$, where $\mathcal{X}$ is the vector field associated to system (7.9). Hence, we have showed that system (7.9) is reversible by means of R. ∎

In the next example we consider an analytic system with a degenerated center at the origin which has a C^∞ first integral around of it.

Example 3. Consider the system

$$\begin{aligned}\dot{x} &= (x^5-y)(2x^2+x^4+x^{10}+2x^{12}+x^{20}-2x^5y-4x^7y-4x^{15}y+y^2 \\ &\quad +2x^2y^2+6x^{10}y^2-4x^5y^3+y^4), \qquad (7.10)\\ \dot{y} &= x(2x^2+2x^4+11x^{10}+9x^{12}+5x^{18}+12x^{20}+5x^{28}-12x^5y-13x^7y \\ &\quad -15x^{13}y-38x^{15}y-25x^{23}y+y^2+4x^2y^2+15x^8y^2+42x^{10}y^2 \\ &\quad +50x^{18}y^2-5x^3y^3-18x^5y^3-50x^13y^3+2y^4+25x^8y^4-5x^3y^5),\end{aligned}$$

which has a C^∞ first integral $H=\exp[-1/(x^2+(x^5-y)^2)](2x^2+(x^5-y)^2)$ around the origin. The origin of system (7.10) is a center because is a monodromic singular point with a first integral of class C^∞ around of it. Factorizing H, we obtain the map $\Phi=(f,g)=\exp[-1/\ (2(x^2+(x^5-y)^2)]\ (\sqrt{2}\,x,\ x^5-y)$ of class C^∞ around the origin that transforms system (7.10) into the system $\dot{f}=\sqrt{2}\ (1+W(s))/W(s)^2\ g$ and $\dot{g}=-\sqrt{2}\ (1+W(s))/W(s)^2\ f$, where $W(s)$ is the Lambert function and $s=2/(f^2+2g^2)$. The Lambert W function, is the inverse function of $f(w)=we^w$ where e^w is the natural exponential function and w is any complex number. Notice that the transformed system is a time–reversible system by means of the involution $\mathcal{R}_1(f,g)=(f,-g)$. Moreover,

$$\begin{aligned}R &= \Big(x((x^2+(x^5-y)^2)\ W(r)\ )^{-1/2},\ \ \exp[1/2((-1/(x^2+(x^5-y)^2)+W(r)\ )] \\ &\quad (x^5-y+x^5\ ((x^2+(x^5-y)^2)\ W(r)\ )^{-2})\Big),\end{aligned}$$

where $r=\exp[1/(x^2+(x^5-y)^2)]\ (x^2+(x^5-y)^2)$, satisfies the equation $\Phi\circ R=\mathcal{R}_1\circ\Phi$. In fact, using the property $W(\exp(\alpha)\ \alpha)=\alpha$, we can simplify the expression of R as $R=(x,-y+2x^5)$, which is an involution satisfying $R_*\mathcal{X}=-\mathcal{X}\circ R$, where $\mathcal{X}$ is the vector field associated to system (7.10). From Proposition 7.1, we get the analytic change of variables $\phi=(z,w)=(x,-x^5+y)$ bringing system (7.10) into the time–reversible system $\dot{z}=-w(2z^2+w^2+(z^2+w^2)^2)$, $\dot{w}=z(2z^2+w^2+2(z^2+w^2)^2)$. ∎

In [3] the authors present the following polynomial system having a non analytically reversible degenerated center at the origin, i.e., the system does not admits as reversing symmetry a local analytic involution near the center. We shall obtain the map Φ which transforms such system (after rescaling the time) into a time–reversible system and we will use it to confirm the aforementioned property.

Example 4. Consider the following hamiltonian function $H = (1+b^2)x^{10} - 2bx^5y^3 + y^6$ with $b > 0$. Associated to H, one has the following hamiltonian system

$$\dot{x} = 6y^2(bx^5 - y^3), \quad \dot{y} = 10x^9 + 10b^2x^9 - 10ax^4y^3, \tag{7.11}$$

having a degenerated center at the origin. Since, we can write $H = (y^3 - bx^5)^2 + y^6$, we take $\Phi = (f, g) = (y^3 - bx^5, y^3)$. Performing the map Φ we obtain (after rescaling the time) the linear center $f' = -g$, $g' = f$. The time–rescaling realized is given by $d\tau = (15/2^{11/15}\, g^{2/5}(f+bg)^{4/3})dt$. On the other hand, the set of analytic reversing symmetries admitted for the linear center is given by the following 2–parameter family $\tilde{\mathcal{R}}(f, g) = (af + cg + K(x, y)), cf - ag + M(x, y))$, with $a^2 + c^2 = 1$ and $M(x, y)$ and $K(x, y)$ starting at order greater or equal than two. Therefore, the family of maps $\mathcal{R}(x, y) = (2^{-1/3}((\,(-2ba \pm \sqrt{1-a^2}\,(1-b^2)\,)x^5 + (a \pm \sqrt{1-a^2}\,b\,)y^3 + K(-bx^5 + y^3, x^5) + bM(-bx^5 + y^3, x^5)))^{1/3}, (2^{-1/5}(\pm\sqrt{1-a^2}\,y^3 - (a \pm b\,\sqrt{1-a^2})\,x^5\,M(-bx^5 + y^3, x^5)))^{1/5})$ satisfies the equation $\Phi \circ \mathcal{R} = \tilde{\mathcal{R}} \circ \Phi$. It is clear that there is no involutions inside the family $\mathcal{R}$. ∎

7.4 Analytic integrability via reversibility

Consider the systems which are time–reversible with $R_0 = (x, -y)$, that is

$$\dot{x} = yP(x, y^2), \quad \dot{y} = Q(x, y^2). \tag{7.12}$$

Taking $z = y^2$, we obtain (after ignoring a common factor) the reduced system

$$\dot{x} = P(x, z), \quad \dot{z} = 2Q(x, z). \tag{7.13}$$

When the linear part of system (7.12) takes the form A_1, then the reduced system (7.13) is a non-singular system at the origin because $P(0,0) \neq 0$. Hence (7.13) has a local analytic first integral near the origin. Pulling back this first integral we obtain an analytic first integral around the origin of system (7.12). In this example, the map $\Psi_0 = (x, y^2)$ provides a tool for detecting an analytic first integral of system (7.12). In the following results we prove the existence of an analytic first integral for some reversible systems by using this method. Let us consider first system (7.1) with a non-degenerated singular point.

Proposition 7.19 *If system* (7.1) *is an analytically reversible differential system with a hyperbolic weak saddle (i.e., with real eigenvalues* λ_1 *and* λ_2*, satisfying* $\lambda_1/\lambda_2 = -1$*) at the origin, then it has a local analytic first integral at the origin.*

Proof. If system (7.1) is an analytically reversible system by means of an involution $R(x, y)$, then applying Proposition 7.1 there exists an analytic change of variables $(x, y) \mapsto (u, v)$ that transforms system (7.1) into system (7.2). More precisely, since the origin is a hyperbolic weak saddle, we have $\dot{u} = (av + v\tilde{P}(u, v^2))h(u, v)$, $\dot{v} = (bu + \tilde{Q}(u, v^2))h(u, v)$ with $ab > 0$ where $\tilde{P}(0, 0) = 0$ and $\tilde{Q}(0, 0) = \partial\tilde{Q}/\partial u(0, 0) = 0$. After ignoring the common factor $h(u, v)$, we take $z = v^2$ and (deleting a common factor again) we obtain the reduced system $\dot{u} = (a + \tilde{P}(u, z))$, $\dot{z} = 2(bu + \tilde{Q}(u, z))$. Since $\tilde{P}(0, 0) = 0$, the reduced system is non–singular at the origin. Then, we can get a local analytic first integral $H(u, z)$ around the origin. Pulling back to the initial coordinates, we obtain a local analytic first integral $H(x, y)$ around the origin of system (7.1). ∎

Proposition 7.20 *Assume that the analytic system* (7.1) *has an analytically integrable weak saddle at the origin, then it is an orbitally analytically reversible system.*

Proof. It is well known that a weak saddle of an analytic system has a local analytic first integral around it if and only if all the saddle quantities vanish. In particular, the saddle must be analytically orbitally linearizable, see [37]. Hence, there exists an analytic change of variables $\phi(x, y) = (u, v) = (x + \cdots, y + \cdots)$ that transforms system (7.1) into $\dot{u} = \lambda u h(u, v)$, $\dot{v} = -\lambda v h(u, v)$ with $\lambda \in \mathbb{R}\backslash\{0\}$ and $h(0, 0) \neq 0$, which is orbitally equivalent to an analytically reversible system with analytic reversing symmetry $\tilde{R}(u, v) = (v, u)$. ∎

Theorem 7.21 *Assume that the analytic system* (7.1) *has a non–degenerated weak singular point at the origin. Then, it has a local analytic first integral around the origin if and only if it is orbitally analytically reversible.*

Proof. We recall that, a topological obstruction for the existence of an analytic first integral in a neighborhood of a singular point is the existence of at least one parabolic or elliptic sector, see [52]. Hence, taking into account the kinds of elementary singular points, we conclude that the origin of system (7.1) will be analytically integrable only in the cases of centers and saddles. Centers are analytically reversibles and analytically integrable from Poincaré's result. On the other hand, taking into account Propositions 7.19 and 7.20, we prove the result. ∎

In the next result we present an improvement of a theorem stated for nilpotent centers in [18]. We want to emphasize that our approach gives a short proof using elementary methods. Moreover, our theorem is a generalization that can be applied to other types of nilpotent singular points.

Theorem 7.22 *Consider the time–reversible analytic differential system with a nilpotent singularity at the origin of the form*

$$\dot{x} = y(1 + P(x, y^2)), \quad \dot{y} = Q(x, y^2), \tag{7.14}$$

with P and Q analytic functions in a neighborhood of the origin such that $P(0,0) = 0$, $Q(0,0) = \partial Q/\partial x(0,0) = 0$. Then, system (7.14) *has a local analytic first integral at the origin.*

Proof. Taking $z = y^2$, we obtain (after ignoring a common factor) the reduced system

$$\dot{x} = 1 + P(x, z), \quad \dot{z} = 2Q(x, z)$$

which is a non–singular system at the origin. Hence, it has a local analytic first integral $H(x, z)$. Doing the pull–back we obtain a local analytic first integral $H(x, y^2)$ of system (7.14). ∎

Notice that Theorem 7.22 also works with other types of nilpotent singularities like for instance the family $\dot{x} = y$, $\dot{y} = ax^{2m}(1 + f(x)) + y^2 g(x, y^2)$, where $a \in \mathbb{R}$, $m \in \mathbb{N}$ and with f and g analytic functions around the origin such that $f(0) = 0$. This family has a cusp at the origin. In the following theorem we show a generalization of Theorem 7.22 for degenerated singularities, i.e., singularities with null eigenvalues.

Proposition 7.23 *Consider the analytic time–reversible system with a degenerated singular point at the origin of the form*

$$\dot{x} = y^{2n-1} P(x, y^{2n}), \quad \dot{y} = Q(x, y^{2n}), \tag{7.15}$$

such that $P(0,0) \neq 0$, $Q(0,0) = \partial Q/\partial x(0,0) = 0$ and $n \geq 2$ an integer number. Then, system (7.15) *has a local analytic first integral at the origin.*

Proof. Taking $z = y^{2n}$, we obtain (after deleting a common factor) the reduced system

$$\dot{x} = P(x,z), \quad \dot{z} = 2n\, Q(x,z)$$

which is a non-singular system at the origin and therefore it has a local analytic first integral $H(x,z)$. Doing the pull–back, we obtain a local analytic first integral $H(x, y^{2n})$ of system (7.15). ∎

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