# UNIVERSITAT DE LLEIDA 

Departament de Matemàtica

## Ph.D. THESIS

in the Applied Mathematics Area:

The Jacobi identities for finite-dimensional Poisson structures: a P.D.E. based analysis of some new constructive results and solution families

Author: Dr. Benito Hernández Bermejo<br>Licenciado y Doctor en Ciencias Físicas<br>Advisor: Dr. Isaac Antonio García Rodríguez<br>Profesor Titular de Universidad, Área de Matemática Aplicada (UdL)

El abajo firmante, Dr. Isaac Antonio García Rodríguez, Profesor Titular de Universidad en el Área de Matemática Aplicada del Departamento de Matemática de la Universidad de Lleida

## CERTIFICO

que la presente Memoria de Tesis Doctoral ha sido realizada por el doctorando D. Benito Hernández Bermejo bajo mi dirección.

Firmado:

Lleida, Enero de 2008.

A mis padres,
Juan y Francisca.
"Todo lo que somos es fruto de lo que pensamos."
SIDDHARTA GAUTAMA, BUDA (S. V-IV a. de C.)
"Creo que nuestro futuro depende enteramente de nuestro conocimiento de este Cosmos en el que flotamos como una mota de polvo en el cielo de la mañana."

CARL SAGAN (Cosmos, 1980)
"La física es matemática no debido a que sepamos tanto sobre el mundo físico, sino debido a que sabemos tan poco; sólo podemos descubrir sus propiedades matemáticas."

BERTRAND RUSSELL (An Outline of Philosophy, 1927)
"En la medida en que se refieren a la realidad, las proposiciones de la matemática no son seguras $y$, viceversa, en la medida en que son seguras, no se refieren a la realidad."

ALBERT EINSTEIN (Conferencia pronunciada ante la Academia Prusiana de Ciencias el 27 de Enero de 1921)

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## INDEX

RESUMEN (ABSTRACT IN SPANISH) ..... 1
RESUM (ABSTRACT IN CATALAN) ..... 3
ABSTRACT ..... 5
ESTRUCTURA Y RESULTADOS DE LA TESIS ..... 7

1. INTRODUCTION ..... 29
1.1. Classical Hamiltonian systems ..... 29
1.1.1. Introductory aspects of classical Hamiltonian systems ..... 30
1.1.2. The canonical Poisson bracket ..... 31
1.1.3. Canonical transformations in the symplectic formalism ..... 33
1.1.4. Liouville's theorem ..... 35
1.2. Poisson structures and Poisson systems ..... 37
1.2.1. Generalized Liouville's theorem and conservative systems ..... 37
1.2.2. Poisson structures and finite-dimensional Poisson systems ..... 38
1.2.3. Phase space structure: symplectic foliation and Darboux' theorem ..... 42
1.2.4. Changes of coordinates and time reparametrizations ..... 45
1.2.5. Introductory examples of Poisson structures and systems ..... 47
1.2.6. Interest and applications of Poisson structures and systems ..... 50
1.3. The Jacobi partial differential equations ..... 52
1.3.1. General properties ..... 52
1.3.2. A perspective on some important families of solutions ..... 54
1.3.3. The subject of this work: motivation, methods, aims and applications ..... 57
1.4. Miscellaneous background and terminology ..... 61
1.4.1. Some results on matrix algebra ..... 61
1.4.2. Functional independence and related topics ..... 62
1.4.3. Equivalence between dynamical systems of the ODE type ..... 65
1.4.4. Integrability of dynamical systems of the ODE type ..... 67
2. THREE-DIMENSIONAL SOLUTIONS ..... 71
2.1. Introduction ..... 71
2.2. A threefold three-dimensional family ..... 73
2.2.1. Introduction ..... 73
2.2.2. First case: characterization and global Darboux analysis ..... 74
2.2.3. Second case: characterization and global Darboux analysis ..... 80
2.2.4. Third case: characterization and global Darboux analysis ..... 84
2.2.5. Case-by-case examples ..... 85
2.3. Cyclic solutions of type I ..... 88
2.3.1. Characterization of the family ..... 88
2.3.2. Casimir invariants and global Darboux analysis ..... 90
2.3.3. Examples ..... 94
2.4. Cyclic solutions of type II ..... 97
2.4.1. Characterization of the family ..... 97
2.4.2. Casimir invariants and global Darboux analysis ..... 99
2.4.3. Examples ..... 103
2.5. Use of an ansatz for the simplification of the Jacobi equations ..... 108
2.5.1. Description of the method and case classification ..... 108
2.5.2. Case-by-case examples ..... 112
2.5.3. Some remarks ..... 116
2.6. A perspective of the three-dimensional solutions considered ..... 118
3. FOUR-DIMENSIONAL SOLUTIONS ..... 119
3.1. Introduction ..... 119
3.2. A four-dimensional family ..... 120
3.2.1. Characterization of the family ..... 120
3.2.2. Casimir invariants and global Darboux analysis ..... 124
3.2.3. Examples and relationship with other Poisson structures ..... 133
3.2.4. Some remarks about the case classification ..... 135
3.3. Insights provided by the four-dimensional family analyzed ..... 137
4. SOLUTIONS OF ARBITRARY DIMENSION ..... 139
4.1. Introduction ..... 139
4.2. Separable solutions ..... 142
4.2.1. Characterization of the family ..... 142
4.2.2. Casimir invariants and global Darboux analysis ..... 143
4.2.3. Examples ..... 147
4.3. Multiseparable solutions ..... 152
4.3.1. Characterization of the family ..... 152
4.3.2. Casimir invariants and global Darboux analysis ..... 155
4.3.3. Examples ..... 159
4.3.4. Comparison with the separable family ..... 165
4.4. Distinguished solutions (D-solutions) ..... 171
4.4.1. Distinguished Jacobi equations and distinguished Poisson structures ..... 171
4.4.2. The family of $\mathrm{D}_{\psi}$-solutions ..... 174
4.4.3. Discussion and examples ..... 178
4.4.4. Further remarks about D-solutions ..... 183
4.4.5. Comparison with the multiseparable family ..... 184
4.4.6. Comparison with the separable family ..... 185
4.5. Hemiseparable solutions of type I ..... 187
4.5.1. Characterization of the family ..... 187
4.5.2. Casimir invariants and global Darboux analysis ..... 189
4.5.3. Examples ..... 194
4.6. Hemiseparable solutions of type II ..... 199
4.6.1. Characterization of the family ..... 199
4.6.2. Casimir invariants and global Darboux analysis ..... 201
4.6.3. Examples ..... 205
4.7. The role of time reparametrizations ..... 209
4.7.1. Introduction and formulation of the problem ..... 209
4.7.2. First family of reparametrization factor solutions ..... 213
4.7.3. Second family of reparametrization factor solutions ..... 214
4.7.4. Miscellaneous examples ..... 221
4.7.5. Application to the generalization of separable solutions ..... 222
4.7.6. Application to the generalization of multiseparable solutions ..... 224
4.7.7. Application to the generalization of distinguished solutions ..... 226
4.7.8. Global Darboux reduction for Poisson structures of rank two ..... 227
4.8. A perspective of the results for arbitrary dimension ..... 231
5. CONCLUSIONS ..... 233
APPENDIX 1. THE PFAFFIAN METHOD ..... 237
A1.1. Description of the method ..... 237
A1.2. Examples ..... 240
A1.3. Concluding remarks ..... 246
APPENDIX 2. MATHEMATICAL NOTATION ..... 249
APPENDIX 3. ABBREVIATIONS ..... 253
APPENDIX 4. PUBLICATIONS WITH IMPACT FACTOR ..... 255
BIBLIOGRAPHY ..... 257

## RESUMEN (ABSTRACT IN SPANISH)

Las ecuaciones de Jacobi constituyen un conjunto de ecuaciones diferenciales parciales no lineales que surgen al implementar, en un sistema de coordenadas arbitrario, una estructura de Poisson definida en una variedad finito-dimensional suave. En esta disertación se investigan ciertas soluciones antisimétricas de estas ecuaciones. Esto se hace desde una doble perspectiva que incluye tanto la determinación de nuevas familias de soluciones, como la construcción de nuevos análisis globales de Darboux para estructuras de Poisson. Los resultados más generales investigados se refieren al caso de soluciones de dimensión arbitraria. Esta perspectiva tiene interés en vista del número, relativamente modesto, de familias de soluciones de este tipo tratadas en la literatura. Adicionalmente, el análisis global de Darboux de matrices de estructura conlleva, en primera instancia, la determinación global de conjuntos completos de invariantes distinguidos funcionalmente independientes, la cual proporciona una descripción global de la estructura simpléctica del espacio de fases de cualquier sistema de Poisson asociado; y en segundo término, requiere la determinación global y constructiva de la forma canónica de Darboux. Este tipo de análisis es de interés ya que la construcción de las coordenadas de Darboux es conocida sólamente para una muestra limitada de estructuras de Poisson y, además, el hecho de llevar a cabo globalmente dicha reducción mejora el alcance del teorema de Darboux, que sólo garantiza en principio la existencia local de las coordenadas de Darboux. En este trabajo, tales reducciones hacen uso en ocasiones de reparametrizaciones temporales, por tanto en consonancia con las definiciones usuales de equivalencia entre sistemas. De hecho, las reparametrizaciones temporales desempeñan un papel destacado en la comprensión de las condiciones bajo las cuales la forma canónica de Darboux se puede construir globalmente, cuestión esta que también se investiga en detalle en esta disertación. Las implicaciones de tales resultados en relación a propiedades de integrabilidad se consideran asimismo en este contexto.

La disertación se estructura como sigue. El Capítulo 1 se dedica a la revisión de diversos resultados clásicos que describen el marco básico de la investigación. Las contribuciones originales de la tesis se incluyen en los Capítulos 2 a 4. Finalmente, el trabajo concluye en el Capítulo 5 con la presentación de conclusiones.

## RESUM (ABSTRACT IN CATALAN)

Les equacions de Jacobi constitueixen un conjunt d'equacions diferencials parcials no lineals que sorgeix de l'aplicació en un sistema arbitrari de coordenades d'una estructura de Poisson definida en una varietat llisa de dimensió finita. Certes solucions antisimètriques d'aquestes equacions s'investiguen en aquesta dissertació. Això es fa des d'una perspectiva doble incloenthi tant la determinació de famílies de solucions noves com la construcció global d'anàlisis de Darboux de les estructures de Poisson. La majoria dels resultats generals investigats es refereixen al cas de solucions de dimensió arbitraria. La perspectiva així obtinguda és d'interès en vista del relativament modest nombre de famílies de solucións d'aquesta classe comunicades en la literatura. Així mateix, l'anàlisi global de Darboux de les matrius d'estructura dona, en primer lloc, la determinació global de conjunts complets d'invariants distingits funcionalment independents, proporcionant així una descripció global de l'estructura simplèctica de l'espai de fases de qualsevol sistema de Poisson associat; i en segon lloc, la determinació constructiva i global de la forma canònica de Darboux. Aquest tipus d'anàlisi és d'interès perquè la construcció de les coordenades de Darboux és només coneguda per una mostra limitada d'estructures de Poisson i, a més a més, el fet de globalment realitzar tal reducció millora l'abast del teorema de Darboux, les úniques garanties del qual en principi son l'existència local de les coordenades de Darboux. En aquest treball, aquestes reduccions a vegades fan ús de reparametritzacions del temps, així en acord amb les definicions usuals d'equivalència de sistemes. De fet, les reparametritzacions del temps juguen un paper significatiu en la comprensió de les condicions sota les quals la forma canònica de Darboux es pot implementar globalment, una qüestió també investigada en detall en aquesta tesi. Les implicacions d'aquests resultats dins la connexió amb la integrabilitat són també considerades en aquest context.

La tesi s'estructura de la manera següent. El Capítol 1 és una revisió de diversos resultats clàssics i coneguts que descriuen el marc bàsic de la investigació. Les contribucions originals de la tesi s'inclouen en els Capítols 2, 3 i 4. Finalment, el treball acaba en el Capítol 5 amb la presentació d'algunes conclusions.

## ABSTRACT

Jacobi equations constitute a set of nonlinear partial differential equations which arise from the implementation in an arbitrary system of coordinates of a Poisson structure defined on a finite-dimensional smooth manifold. Certain skew-symmetric solutions of such equations are investigated in this dissertation. This is done from a twofold perspective including both the determination of new solution families as well as the construction of new global Darboux analyses of Poisson structures. The most general results investigated refer to the case of solutions of arbitrary dimension. The perspective thus obtained is of interest in view of the relatively modest number of solution families of this kind reported in the literature. In addition, the global Darboux analysis of structure matrices deals, in first place, with the global determination of complete sets of functionally independent distinguished invariants, thus providing a global description of the symplectic structure of phase space of any associated Poisson system; and secondly, with the constructive and global determination of the Darboux canonical form. Such kind of analysis is of interest because the construction of the Darboux coordinates is a task only known for a limited sample of Poisson structures and, in addition, the fact of globally performing such reduction improves the scope of Darboux' theorem, which only guarantees in principle the local existence of the Darboux coordinates. In this work, such reductions sometimes make use of time reparametrizations, thus in agreement with the usual definitions of system equivalence. In fact, time reparametrizations play a significant role in the understanding of the conditions under which the Darboux canonical form can be globally implemented, a question also investigated in detail in this dissertation. The implications of such results in connection with integrability issues are also considered in this context.

The dissertation is structured as follows. Chapter 1 is devoted to the revision of diverse classical and well-known results that describe the basic framework of the investigation. The original contributions of the thesis are included in Chapters 2 to 4. Finally, the work ends in Chapter 5 with the presentation of some conclusions.

## ESTRUCTURA Y RESULTADOS DE LA TESIS

Esta disertación compendia una investigación sobre ciertas soluciones antisimétricas de las ecuaciones en derivadas parciales de Jacobi para estructuras de Poisson finito-dimensionales. Tales ecuaciones diferenciales parciales, junto a las condiciones de antisimetría que las acompañan, surgen de la implementación de una estructura de Poisson definida en una variedad finito-dimensional suave, en un sistema de coordenadas arbitrario. Por tanto, en esta tesis se asume la formulación del problema en términos de ecuaciones en derivadas parciales, definiéndose así como una investigación de las propiedades de las soluciones escritas en forma de matrices de estructura.

La caracterización y el análisis de nuevas familias de soluciones de estructuras de Poisson están bien justificados desde varios puntos de vista. Desde una visión puramente matemática, este es un ejemplo no trivial de sistema de ecuaciones diferenciales parciales no lineales y acopladas, por lo cual tiene un interés intrínseco (que incluye tanto aspectos analíticos como de clasificación de soluciones) que ha merecido atención en la literatura durante varias décadas. Adicionalmente, la determinación de una matriz de estructura adecuada es un paso insoslayable en el problema de reexpresar (si ello es posible) un sistema dinámico dado (más exactamente, un sistema de ecuaciones diferenciales ordinarias de primer orden descrito en términos de un campo vectorial) como un sistema de Poisson finito-dimensional. En tal sentido, llevar a cabo dicha reescritura abre la posibilidad de emplear gran variedad de métodos analíticos y numéricos específicamente desarrollados para sistemas de Poisson. Más aun, el teorema de Darboux asegura la equivalencia dinámica (al menos local) entre los sistemas de Poisson y los sistemas Hamiltonianos clásicos. En consecuencia, si la transformación de Darboux existe y puede construirse (y especialmente si esto puede hacerse globalmente) es también posible la transferencia de información entre las ecuaciones de movimiento de Poisson y de Hamilton, en la cual todas las herramientas y resultados bien conocidos existentes en la teoría Hamiltoniana pueden aplicarse al análisis del sistema de Poisson en consideración. Por último, el interés de las estructuras y sistemas de Poisson es comprensible en vista de su frecuente presencia en multitud de dominios de la matemática aplicada, la física y la ingeniería.

Los resultados desarrollados en esta tesis consideran tanto la determinación de nuevas
familias de soluciones antisimétricas de las ecuaciones de Jacobi, como la construcción de nuevos análisis globales de estructuras de Poisson. Los resultados más generales investigados se refieren al caso de soluciones de dimensión arbitraria. Esta perspectiva tiene interés en vista del número, relativamente modesto, de familias de soluciones de dimensión arbitraria tratadas en la literatura. Adicionalmente, el análisis global de matrices de estructura conlleva, en primera instancia, la determinación global de conjuntos completos de invariantes distinguidos funcionalmente independientes, la cual proporciona una descripción global (que además es independiente del Hamiltoniano) de la estructura simpléctica del espacio de fases de cualquier sistema de Poisson asociado; y en segundo término, requiere la determinación global y constructiva de la forma canónica de Darboux. Este tipo de análisis es de interés ya que la construcción de las coordenadas de Darboux es conocida sólamente para una muestra limitada de estructuras de Poisson y, además, el hecho de llevar a cabo globalmente dicha reducción mejora el alcance del teorema de Darboux, que sólo garantiza en principio la existencia local de las coordenadas de Darboux. En este trabajo, tales reducciones hacen uso en ocasiones de reparametrizaciones temporales, por tanto en consonancia con las definiciones usuales de equivalencia entre sistemas. De hecho, las reparametrizaciones temporales desempeñan un papel destacado en la comprensión de las condiciones bajo las cuales la forma canónica de Darboux se puede construir globalmente, cuestión esta que también se investiga en detalle en esta disertación. Las implicaciones de tales resultados en relación a propiedades de integrabilidad se consideran asimismo en este contexto. Para concluir, merece la pena indicar que a lo largo del trabajo se plantea y analiza un número significativo de ejemplos de interés aplicado.

Tras esta visión de conjunto, en lo que sigue vamos a desarrollar el resumen de los resultados que se exponen a lo largo de esta disertación. Dicho resumen va a seguir el mismo esquema organizativo que la tesis en su conjunto, en cuanto a la ordenación por capítulos y secciones. Asimismo conviene señalar que la exposición que sigue va a ser fundamentalmente de tipo descriptivo, es decir se van a obviar buena parte de los aspectos y condiciones más técnicos para centrarse en una presentación en la que primen la visión conceptual y la exposición concisa y generalista de las aportaciones incluidas en este trabajo. Como es lógico, para un tratamiento formal con la totalidad de los detalles (tales como hipótesis, suposiciones y requisitos técnicos para la validez de un resultado, etc.) se dirige al lector a la presentación completa tal y como se desarrolla en los capítulos posteriores. En lo que sigue tampoco se hará referencia a los numerosos ejemplos aplicados que se detallan a lo largo del trabajo, remitiendo
al lector a los mismos a fin de complementar en tal sentido la sinopsis que aquí se expone.

En el Capítulo 1 se presenta una introducción cuya finalidad es hacer que la obra sea, en la medida de lo posible, autocontenida. Para ello se exponen los prerrequisitos necesarios junto con algunos resultados clásicos que serán de utilidad posteriormente.

Así, en la Sección 1.1 se hace una revisión somera de los aspectos más básicos relativos a los sistemas Hamiltonianos, con especial énfasis en el corchete canónico de Poisson, en las transformaciones canónicas y en el teorema de Liouville. Estos tres son los aspectos de mayor importancia para lo que ha de venir más tarde: en el caso del corchete canónico de Poisson, porque el concepto más amplio de estructura de Poisson se basa en la generalización del corchete canónico; en cuanto a las transformaciones canónicas, conviene notar que las limitaciones asociadas a las mismas se van a superar en buena medida en el contexto más amplio de los sistemas de Poisson, lo cual se verá en el marco del formalismo simpléctico o matricial para dichas transformaciones; por último, el teorema de Liouville es la base para una generalización posterior importante como es la idea de sistema conservativo, que es una de las motivaciones conceptuales más importantes para la introducción de los sistemas de Poisson.

La Sección 1.2 se ocupa de hacer una presentación autocontenida de los sistemas y estructuras de Poisson, incluyendo todos los elementos necesarios para el tratamiento posterior. Se comienza motivando la necesidad de generalizar los sistemas Hamiltonianos clásicos a la luz del teorema generalizado de Liouville y la idea de sistema conservativo. Seguidamente se introducen los conceptos más importantes de la teoría Poissoniana, comenzando con las definiciones básicas de corchete de Poisson general y sus propiedades, así como las de estructura y sistema de Poisson finito-dimensionales, junto con otras definiciones y resultados asociados como los de la matriz de estructura, las identidades (ecuaciones en derivadas parciales) de Jacobi, y los invariantes distinguidos (o de Casimir) y las relaciones (ecuaciones diferenciales parciales) que los caracterizan. En todo lo anterior, el concepto de sistema de Poisson ocupa una posición central. Dichos sistemas tienen la forma

$$
\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)
$$

o equivalentemente, por componentes

$$
\frac{\mathrm{d} x_{i}}{\mathrm{~d} t}=\sum_{j=1}^{n} J_{i j}(x) \partial_{j} H(x)=\left\{x_{i}, H\right\}, \quad i=1, \ldots, n
$$

donde $\mathcal{J}=\left(J_{i j}\right)$ constituye la matriz de estructura, y $H$ es el Hamiltoniano (que es por construcción una integral primera). Con posterioridad se analiza la estructura del espacio
de fases (la foliación simpléctica) para los sistemas de Poisson, descripción esta que culmina con la exposición del teorema de Darboux, verdadera base de la equivalencia dinámica entre los sistemas de Poisson y los sistemas Hamiltonianos. Por la utilidad y ubicuidad de las transformaciones en este trabajo, se presentan también los resultados prácticos fundamentales relativos a cómo afectan las transformaciones de coordenadas y las reparametrizaciones temporales (o NTTs, por "new-time transformations") a los sistemas de Poisson. Esta es una vertiente en la que los sistemas de Poisson muestran su amplitud como generalización, ya que en un sentido muy literal, en la dinámica de Poisson todas las transformaciones pueden considerarse canónicas (preservan el formato) lo cual supone un gran avance operativo con respecto al caso Hamiltoniano. La sección concluye con la presentación de algunos ejemplos clásicos de sistemas de Poisson, que permiten ilustrar las definiciones y conceptos anteriores, junto con algunas consideraciones en torno al interés fundamental y aplicado de los sistemas de Poisson.

En la Sección 1.3 se presenta un enfoque específico sobre las ecuaciones diferenciales parciales de Jacobi para las matrices de estructura, que constituyen el núcleo conceptual y operacional del estudio desarrollado en esta disertación,

$$
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{j l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i j}\right)=0, \quad i, j, k=1, \ldots, n
$$

junto a las condiciones de antisimetría que las acompañan:

$$
J_{i j}=-J_{j i}, \quad i, j=1, \ldots, n
$$

Como es sabido, estos dos conjuntos de ecuaciones constituyen las condiciones necesarias y suficientes para que una matriz sea matriz de estructura. En esta sección se consideran en primer lugar las propiedades más básicas de estas ecuaciones, como cuestiones de notación y terminología, un análisis del número de ecuaciones e incógnitas independientes, y el falso problema de la sobredeterminación. Seguidamente se desarrolla una perspectiva en torno al estado de la cuestión relativo a algunas de las principales soluciones ya conocidas, además de la evolución histórica del problema. Todo esto proporciona la base para delimitar con mayor claridad el propósito de este trabajo, que de forma breve puede resumirse como la búsqueda y el análisis global de soluciones antisimétricas de las ecuaciones de Jacobi, entendiendo análisis global en el doble sentido de (a) la determinación global de los invariantes de Casimir; y (b) la construcción, también global, de la forma canónica de Darboux. Las motivaciones para ello son claras: el interés intrínseco de las ecuaciones de Jacobi como ejemplo de sistema de ecuaciones diferenciales parciales no lineales acopladas; con propósitos de clasificación de soluciones y
de identificación de nuevas estructuras de Poisson; la unificación conceptual y operativa de distintos sistemas de Poisson que pueden verse desde una perspectiva común al encontrar una nueva familia de matrices de estructura que abarque a las anteriores; la utilidad aplicada para el problema de reescribir un campo vectorial dado como sistema de Poisson (si es posible) con el número de herramientas de análisis que ello aporta; la transferencia de información y técnicas de análisis que se pueden emplear si se sabe construir globalmente la forma canónica de Darboux, y por tanto la formulación Hamiltoniana clásica a la que se reduce el sistema de Poisson inicial. Esta sección se concluye con algunas consideraciones metodológicas.

La Sección 1.4 se dedica a una revisión de resultados matemáticos misceláneos que son de uso frecuente a lo largo de la disertación. En primer lugar, y dada la importancia de las matrices de estructura y sus propiedades algebraicas, se revisan algunas propiedades del álgebra matricial, con especial énfasis en congruencia de matrices antisimétricas. Seguidamente se pasa a propiedades funcionales, entre las que destacan resultados clásicos sobre dependencia e independencia funcionales, así como el teorema de la función inversa para funciones de varias variables. Seguidamente se presentan algunas consideraciones sobre un concepto de interés como es la equivalencia entre sistemas dinámicos dados por ecuaciones diferenciales ordinarias (esto es campos vectoriales). En este contexto se presentan las nociones de equivalencia orbital topológica (TOE) y equivalencia orbital suave (SOE), haciéndose un hincapié especial en el papel del tiempo y sus reparametrizaciones (NTTs) en estas definiciones. La presente sección finaliza con un apartado dedicado a algunas definiciones básicas sobre integrabilidad, principalmente los conceptos de constante del movimiento, así como de integrabilidad en los sentidos algebraico y de Liouville, todos los cuales también serán de aplicación posterior.

Una vez presentados en el Capítulo 1 diversos resultados clásicos bien conocidos, en los Capítulos 2 a 4 se exponen los resultados originales de la tesis. Estos siguen un orden de complejidad creciente, de forma que en los Capítulos 2,3 y 4 se describen los resultados para dimensiones 3,4 y arbitraria $n$, respectivamente.

En el Capítulo 2 se consideran distintas contribuciones que tienen como denominador común el trabajo sobre estructuras de Poisson tridimensionales. En este contexto, los diversos análisis globales que se van a llevar a cabo, junto a la investigación de cómo el uso de un determinado ansatz puede simplificar notablemente las ecuaciones de Jacobi, comenzarán a sentar las bases para los desarrollos de capítulos posteriores.

En la Sección 2.1 se hace una breve introducción al capítulo y se describe la estructura general del mismo.

Las primeras contribuciones del capítulo se presentan en la Sección 2.2. En la misma se hace una investigación sistemática de ciertas soluciones tridimensionales globalmente analizables. El resultado es una familia que se divide en tres subfamilias, de ahí el nombre de triple familia ("threefold family") dado a la misma.

La primera subfamilia de la Sección 2.2 viene dada por las expresiones:

$$
\left\{\begin{array}{l}
u(x)=\eta(x) \psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right) \phi_{3}\left(x_{3}\right) \\
v(x)=\eta(x) \psi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) \psi_{3}\left(x_{3}\right) \\
w(x)=\eta(x) \phi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right) \psi_{3}\left(x_{3}\right)
\end{array}\right.
$$

O bien, de forma más breve:

$$
J_{i j}(x)=\eta(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \sum_{k=1}^{3} \epsilon_{i j k} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3
$$

Para este tipo de soluciones que, bajo las hipótesis establecidas, tienen rango 2 en todo el dominio, se puede construir de manera global el único invariante de Casimir independiente, que puede expresarse de la forma siguiente:

$$
D(x)=\sum_{i=1}^{3} \int \frac{\phi_{i}\left(x_{i}\right)}{\psi_{i}\left(x_{i}\right)} \mathrm{d} x_{i}
$$

Esto permite la reducción global a la forma canónica de Darboux por medio de un algoritmo en dos etapas sucesivas, la primera de las cuales es un cambio de coordenadas y la segunda una reparametrización temporal. Todo lo anterior permite demostrar la integrablidad algebraica y de Liouville (esta última para el sistema Hamiltoniano resultante tras la reducción) para esta primera subfamilia.

La segunda subfamilia de la Sección 2.2 aparece a su vez estructurada en tres subcasos diferentes, que brevemente pueden expresarse como sigue:

$$
\begin{aligned}
& \text { Subcaso } 1 \Rightarrow\left(u=0, v=\eta(x), w=\eta(x) \xi\left(x_{1}, x_{2}\right)\right) \\
& \text { Subcaso } 2 \Rightarrow\left(v=0, w=\eta(x), u=\eta(x) \zeta\left(x_{1}, x_{3}\right)\right) \\
& \text { Subcaso } 3 \Rightarrow\left(w=0, u=\eta(x), v=\eta(x) \chi\left(x_{2}, x_{3}\right)\right)
\end{aligned}
$$

Estos tres subcasos de la segunda subfamilia se pueden analizar globalmente en la situación denominada separable, que por ejemplo en el Subcaso 1 anterior consiste en que la función $\xi\left(x_{1}, x_{2}\right)$ tenga la forma $\xi\left(x_{1}, x_{2}\right)=\xi_{1}\left(x_{1}\right) / \xi_{2}\left(x_{2}\right)$, y análogamente en los Subcasos 2 y 3 ,
mutatis mutandis. Estando satisfecha la hipótesis de separabilidad, y siguiendo con el ejemplo del Subcaso 1, un invariante de Casimir global vendría dado por:

$$
D\left(x_{1}, x_{2}\right)=\int \xi_{1}\left(x_{1}\right) \mathrm{d} x_{1}+\int \xi_{2}\left(x_{2}\right) \mathrm{d} x_{2}
$$

La determinación de este invariante permite nuevamente la reducción global y constructiva del sistema a la forma canónica de Darboux por medio de un cambio de coordenadas seguido de una reparametrización temporal. Lo anterior conduce también a establecer la doble integrabilidad (algebraica, y de Liouville para el campo vectorial Hamiltoniano resultante) para tales sistemas de Poisson.

La tercera subfamilia de la Sección 2.2 es comparativamente simple, siendo idénticamente nulas dos de sus tres componentes independientes, lo cual hace que la determinación global del invariante distinguido, la reducción global a la forma canónica de Darboux y el establecimiento de la integrabilidad, sean todos ellos muy sencillos de llevar a cabo.

En la Sección 2.3 se estudia una familia tridimensional diferente, denominada cíclica de tipo I. Esta familia tiene una propiedad de interés, a saber que su análisis global requiere de una clasificación de casos, empleándose en cada uno de tales casos un invariante de Casimir diferente. Esto es indicativo de que la mera caracterización de un conjunto completo de invariantes de Casimir puede no ser suficiente para construir la forma canónica de Darboux con toda generalidad para una familia dada. Las soluciones cíclicas de tipo I tienen la forma siguiente

$$
J_{i j}(x)=\eta(x)\left(\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}\right) \sum_{k=1}^{3}\left(\epsilon_{i j k}\right)^{2} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3
$$

donde para cada $i=1,2,3$, la función $\psi_{i}\left(x_{i}\right)$ denota una de las primitivas de $\phi_{i}\left(x_{i}\right)$, y las constantes $\kappa_{i j}$ verifican las propiedades

$$
\begin{gathered}
\kappa_{i j}+\kappa_{j i}=0, \text { para todo } i, j \\
\kappa_{12}+\kappa_{23}+\kappa_{31}=0
\end{gathered}
$$

Equivalentemente podemos escribir:

$$
\left\{\begin{array}{l}
J_{12}(x)=\eta(x)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}\right) \phi_{3}\left(x_{3}\right) \\
J_{23}(x)=\eta(x)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)+\kappa_{23}\right) \phi_{1}\left(x_{1}\right) \\
J_{31}(x)=\eta(x)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)+\kappa_{31}\right) \phi_{2}\left(x_{2}\right)
\end{array}\right.
$$

Los tres invariantes de Casimir complementarios que se necesitan para estudiar la reducción a la forma canónica de Darboux son:

$$
D_{1}(x)=\frac{\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)+\kappa_{31}}{\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)+\kappa_{23}}=\frac{\chi_{31}\left(x_{3}, x_{1}\right)}{\chi_{23}\left(x_{2}, x_{3}\right)} \quad \text { si } \quad \chi_{23}\left(x_{2}, x_{3}\right) \neq 0
$$

$$
\begin{aligned}
& D_{2}(x)=\frac{\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}}{\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)+\kappa_{31}}=\frac{\chi_{12}\left(x_{1}, x_{2}\right)}{\chi_{31}\left(x_{3}, x_{1}\right)} \quad \text { si } \quad \chi_{31}\left(x_{3}, x_{1}\right) \neq 0 \\
& D_{3}(x)=\frac{\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)+\kappa_{23}}{\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}}=\frac{\chi_{23}\left(x_{2}, x_{3}\right)}{\chi_{12}\left(x_{1}, x_{2}\right)} \quad \text { si } \quad \chi_{12}\left(x_{1}, x_{2}\right) \neq 0
\end{aligned}
$$

Gracias a los mismos, es posible hacer una clasificación de reducciones globales a la forma canónica de Darboux, todas las cuales hacen uso sucesivo de una transformación de coordenadas y de una reparametrización temporal. Con ello se demuestra también la integrabilidad, tanto algebraica como en el sentido de Liouville (esta para el sistema Hamiltoniano obtenido tras efectuar la reducción) de los sistemas de Poisson basados en este tipo de matrices de estructura.

Lo sucedido con las familias cíclicas de tipo I no es excepcional. Ello se demuestra en la Sección 2.4 en la que se estudian las familias cíclicas tipo II. Estas son en gran medida análogas (pero diferentes como soluciones) a las de tipo I. Por ello, las observaciones hechas para estas en párrafos anteriores van a permanecer válidas ahora. Las matrices de estructura cíclicas de tipo II tienen la forma siguiente

$$
J_{i j}(x)=\eta(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)\left(\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right) \sum_{k=1}^{3}\left(\epsilon_{i j k}\right)^{2} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3
$$

donde para cada $i=1,2,3$, la función $\psi_{i}\left(x_{i}\right)$ denota una de las primitivas de $\phi_{i}\left(x_{i}\right)$. O, de manera equivalente,

$$
\left\{\begin{array}{l}
J_{12}(x)=\eta(x) \psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right) \phi_{3}\left(x_{3}\right) \\
J_{23}(x)=\eta(x) \psi_{2}\left(x_{2}\right) \psi_{3}\left(x_{3}\right)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)\right) \phi_{1}\left(x_{1}\right) \\
J_{31}(x)=\eta(x) \psi_{3}\left(x_{3}\right) \psi_{1}\left(x_{1}\right)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)\right) \phi_{2}\left(x_{2}\right)
\end{array}\right.
$$

En este caso, nuevamente aparecen tres invariantes distinguidos complementarios que debemos tener en cuenta, y que vienen dados por las expresiones siguientes:

$$
\begin{aligned}
& D_{1}(x)=\frac{\psi_{2}\left(x_{2}\right)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)\right)}{\psi_{1}\left(x_{1}\right)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)\right)}=\frac{\psi_{2}\left(x_{2}\right) \omega_{31}\left(x_{3}, x_{1}\right)}{\psi_{1}\left(x_{1}\right) \omega_{23}\left(x_{2}, x_{3}\right)} \quad \text { si } \quad \omega_{23}\left(x_{2}, x_{3}\right) \neq 0 \\
& D_{2}(x)=\frac{\psi_{3}\left(x_{3}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right)}{\psi_{2}\left(x_{2}\right)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)\right)}=\frac{\psi_{3}\left(x_{3}\right) \omega_{12}\left(x_{1}, x_{2}\right)}{\psi_{2}\left(x_{2}\right) \omega_{31}\left(x_{3}, x_{1}\right)} \quad \text { si } \quad \omega_{31}\left(x_{3}, x_{1}\right) \neq 0 \\
& D_{3}(x)=\frac{\psi_{1}\left(x_{1}\right)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)\right)}{\psi_{3}\left(x_{3}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right)}=\frac{\psi_{1}\left(x_{1}\right) \omega_{23}\left(x_{2}, x_{3}\right)}{\psi_{3}\left(x_{3}\right) \omega_{12}\left(x_{1}, x_{2}\right)} \quad \text { si } \quad \omega_{12}\left(x_{1}, x_{2}\right) \neq 0
\end{aligned}
$$

Estos tres invariantes permiten construir sendas reducciones globales a la forma canónica de Darboux para cada caso, todas ellas establecidas a partir de un cambio de coordenadas seguido de una reparametrización temporal. A su vez, esta reducción global a la forma Hamiltoniana clásica implica la demostración de la doble integrabilidad, de manera análoga a la ya vista en otras familias de soluciones, de los sistemas de Poisson establecidos sobre la base de estas
matrices de estructura. Estamos por tanto ante una familia paralela a la cíclica de tipo I, que lleva a conclusiones similares, lo cual permite completar una perspectiva de los condicionantes, aun en dimensión 3, que se dan a la hora de plantear el análisis global de una familia de estructuras de Poisson.

Para concluir el capítulo, en la Sección $\mathbf{2 . 5}$ se plantea una idea que muestra cómo mediante un enfoque relativamente directo es posible simplificar notablemente la complejidad del problema consistente en construir soluciones antisimétricas de las ecuaciones de Jacobi. El mayor interés del enfoque desarrollado en esta sección es de tipo conceptual, ya que el plantear esta clase de procedimiento sienta las bases para una de las secciones de mayor importancia de esta disertación, como es la que aparece en el Capítulo 4 dedicada a las reparametrizaciones temporales, las cuales tienen en el plano formal una gran relación con el planteamiento objeto de la Sección 2.5. Por tanto esta sección sirve para establecer e ilustrar de forma sencilla las bases de una filosofía de trabajo cuya adaptación posterior bajo la forma de reparametrizaciones temporales será en gran medida la culminación de buena parte de los resultados desarrollados a lo largo de la presente disertación. La idea básica de la Sección 2.5 es, por consiguiente, simplificar el problema de la determinación de matrices de estructura por medio de un ansatz de tipo aditivo. Es decir, partiendo de una solución ya conocida $\left(u_{0}(x), v_{0}(x)\right.$, $\left.w_{0}(x)\right)$, se plantea una generalización de la forma

$$
\left(u_{0}(x), v_{0}(x), w_{0}(x)\right) \longrightarrow\left(u_{0}(x)+\xi(x), v_{0}(x)+\xi(x), w_{0}(x)+\xi(x)\right)
$$

donde $\xi(x)$ es una función arbitraria. Este planteamiento lleva inmediatamente a que la (única) ecuación de Jacobi independiente para el caso tridimensional

$$
u \partial_{1} v-v \partial_{1} u+w \partial_{2} u-u \partial_{2} w+v \partial_{3} w-w \partial_{3} v=0
$$

se simplifica notablemente, quedando reducida a

$$
\left(u_{0}-v_{0}\right) \partial_{1} \xi+\left(w_{0}-u_{0}\right) \partial_{2} \xi+\left(v_{0}-w_{0}\right) \partial_{3} \xi=\lambda(x) \xi
$$

donde

$$
\lambda(x)=\partial_{1}\left(u_{0}-v_{0}\right)+\partial_{2}\left(w_{0}-u_{0}\right)+\partial_{3}\left(v_{0}-w_{0}\right)
$$

Vemos así que se obtiene una triple simplificación del problema, a saber:
(a) Se transforma un problema no lineal en un problema lineal.
(b) La ecuación en derivadas parciales de Jacobi se reduce a un problema en ecuaciones diferenciales ordinarias, esto es el definido por las ecuaciones características para la ecuación lineal que debe satisfacer $\xi(x)$.
(c) Reducimos el número de incógnitas de tres a una.

El problema de determinar $\xi(x)$ se puede resolver en general para todos los casos posibles, que resultan ser tres. La resolución detallada de cada uno de ellos se presenta en la Sección 2.5, y dado que su descripción es relativamente técnica se omite aquí por brevedad. En este punto queda completada la presentación de resultados del Capítulo 2 que, como ya se ha venido viendo, tiene por objeto el caso de dimensión $n=3$.

Por último, el capítulo se cierra con la Sección 2.6 que recapitula brevemente los resultados obtenidos y enumera algunas de las cuestiones que quedan planteadas para su examen en etapas posteriores del trabajo.

En el Capítulo 3 se pasa a considerar el caso inmediatamente superior desde el punto de vista dimensional, esto es $n=4$. Si bien las estructuras de Poisson han sido estudiadas y desarrolladas en la literatura con una preferencia especial por el escenario tridimensional (el más sencillo que no es trivial) también es cierto que los casos de dimensiones más altas como $n=4 \mathrm{y} n=5$ son bastante frecuentes (aunque menos) en muy diversas aplicaciones. Además, en dimensión cuatro hay una mayor riqueza dinámica y de posibles valores no triviales del rango de la matriz de estructura, todo lo cual parece indicar que detenerse en un problema de dimensión cuatro no sólo está justificado, sino que puede proporcionarnos algunas claves de interés en la transición entre la situación $n=3$, más sencilla, y el caso en que $n$ es completamente arbitraria, en que la complejidad y generalidad del problema son las mayores posibles.

Sobre esta base, la Sección 3.1 presenta una breve introducción al capítulo, planteando algunas de las ideas anteriormente expuestas que justifican la necesidad de una etapa de transición entre la situación tridimensional y la puramente $n$-dimensional. Como se verá a la luz de los resultados obtenidos, esta estrategia estará bien justificada. Según se indica en la Sección 3.1, todo el Capítulo 3 está dedicado a la caracterización y análisis de una familia concreta de soluciones antisimétricas de las ecuaciones de Jacobi.

El grueso de las contribuciones del Capítulo 3 se desarrolla en la Sección 3.2. La familia de matrices de estructura considerada tiene la forma

$$
J_{i j}(x)=\sigma_{i j} \eta(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \sum_{k, l=1}^{4} \epsilon_{i j k l} \phi_{l}\left(x_{l}\right), \quad i, j=1, \ldots, 4
$$

donde, entre otras propiedades, cabe destacar que las constantes $\sigma_{i j}$ son simétricas

$$
\sigma_{i j}=\sigma_{j i} \quad \text { para cada par }(i, j), \quad i \neq j
$$

y cumplen la relación

$$
\sigma_{12} \sigma_{34}=\sigma_{13} \sigma_{24}=\sigma_{14} \sigma_{23}
$$

Por tanto, se trata de analizar las matrices de estructura de la forma:

$$
\mathcal{J}=\eta\left(\begin{array}{cccc}
0 & \sigma_{12} \psi_{1} \psi_{2}\left(\phi_{4}-\phi_{3}\right) & \sigma_{13} \psi_{1} \psi_{3}\left(\phi_{2}-\phi_{4}\right) & \sigma_{14} \psi_{1} \psi_{4}\left(\phi_{3}-\phi_{2}\right) \\
\sigma_{12} \psi_{1} \psi_{2}\left(\phi_{3}-\phi_{4}\right) & 0 & \sigma_{23} \psi_{2} \psi_{3}\left(\phi_{4}-\phi_{1}\right) & \sigma_{24} \psi_{2} \psi_{4}\left(\phi_{1}-\phi_{3}\right) \\
\sigma_{13} \psi_{1} \psi_{3}\left(\phi_{4}-\phi_{2}\right) & \sigma_{23} \psi_{2} \psi_{3}\left(\phi_{1}-\phi_{4}\right) & 0 & \sigma_{34} \psi_{3} \psi_{4}\left(\phi_{2}-\phi_{1}\right) \\
\sigma_{14} \psi_{1} \psi_{4}\left(\phi_{2}-\phi_{3}\right) & \sigma_{24} \psi_{2} \psi_{4}\left(\phi_{3}-\phi_{1}\right) & \sigma_{34} \psi_{3} \psi_{4}\left(\phi_{1}-\phi_{2}\right) & 0
\end{array}\right)
$$

Bajo las condiciones que se asumen para estas matrices de estructura, puede demostrarse que las mismas tienen rango 2 globalmente en el dominio de definición. Esta constancia del rango proporciona la base para considerar la caracterización global de los invariantes distinguidos (de los que evidentemente habrá dos, funcionalmente independientes) así como la reducción global y constructiva a la forma canónica de Darboux. Todo ello se considera en detalle en la Sección 3.2 cuya descripción ahora nos ocupa. Sucede que resolver este problema en toda su extensión lleva a un análisis bastante prolijo en casos y subcasos, tanto para los invariantes distinguidos a emplear como para las posteriores reducciones, por lo cual no parece oportuna la reproducción de todo ello en esta sinopsis. Baste decir a este respecto, que nuevamente ocurre en todos los casos que la reducción a la forma canónica de Darboux se produce siempre en las dos etapas ya citadas, siendo la primera un cambio de coordenadas, en tanto que la segunda es una reparametrización temporal. Conceptualmente, y expresado en términos muy simples, la situación que encontramos en este contexto es similar a la que se daba con las familias cíclicas (de ambos tipos, I y II). Esto es: el mero conocimiento de un conjunto completo de invariantes de Casimir independientes no garantiza la construcción de una vez por todas de la forma canónica de Darboux, aun en el caso de rango constante, debido a que la complejidad de la familia de soluciones puede hacer que sean necesarias distintas descomposiciones en las que se empleen diversos invariantes, requiriéndose así un análisis sistemático y por casos en orden a demostrar que la familia realmente puede reducirse de manera global a la forma canónica de Darboux. Como resultado de esta clasificación, se demuestra también en todos los casos que los sistemas de Poisson construidos sobre la base de estas matrices de estructura son integrables algebraicamente y en el sentido de Liouville, en este último caso referido al sistema Hamiltoniano resultante tras la reducción, como es lógico.

El Capítulo 3 termina en la Sección 3.3, en la cual se recapitula acerca de las distintas reducciones construidas hasta ese punto de la disertación, destacando la pauta común que puede apreciarse, según la cual es la combinación de una transformación de coordenadas y
una reparametrización temporal la que permite la construcción global de la forma canónica de Darboux. Esto tiene relación con el hecho de que las reparametrizaciones temporales preservan la estructura de Poisson en todos los casos considerados. Es conocida la propiedad de que para $n \leq 3$ las NTTs preservan la estructura de Poisson, lo cual no sucede necesariamente para $n \geq 4$, pese a que sí sea el caso en la familia 4 -d considerada en este capítulo. El hecho de que las reparametrizaciones temporales no preserven siempre la estructura de Poisson en dimensión arbitraria ( $n \geq 4$ ) suscita asimismo el interrogante acerca de la posible construcción global de la forma canónica de Darboux para familias de dimensión $n$ general con la ayuda de este tipo de transformaciones. Queda así planteado este estado de cosas, que se clarificará en gran medida en el capítulo siguiente. Se intuye, ya en este estadio, que el papel de las reparametrizaciones temporales será destacado en el contexto de la construcción global de la forma canónica de Darboux, al menos en ciertos casos, intuición esta que se confirmará posteriormente.

En el Capítulo 4 se pasa ya al problema de investigar las soluciones antisimétricas de las ecuaciones de Jacobi en su forma más general correspondiente a dimensión arbitraria n. Es, por tanto, el capítulo más largo de la disertación, y sus resultados son también los de mayor amplitud e interés, por varias razones, entre las que cabe citar que es en este caso en el que las soluciones de las ecuaciones de Jacobi son peor conocidas y menos tratadas en la literatura, además de que la situación de dimensión arbitraria incluye como casos particulares a los considerados en los Capítulos 2 y 3 . De hecho, se verá que bastantes resultados considerados en los capítulos anteriores se van a ver generalizados en el actual, que ahora se inicia.

La exposición comienza en la Sección 4.1, en la cual se presenta una introducción al capítulo. En la misma se enfatiza el interés del caso de dimensión arbitraria, al que ya nos hemos referido en el párrafo anterior, además de anticipar en líneas generales los contenidos por venir, que fundamentalmente consisten: (a) en la caracterización de nuevas soluciones antisimétricas de las ecuaciones de Jacobi, que serán de un interés especial si, además de tener dimensión arbitraria, permiten también valores cualesquiera del rango y están definidas en términos de funciones de no linealidad arbitraria; (b) en el análisis global y constructivo de las correspondientes familias de soluciones, esto es familias de matrices de estructura; y (c) en un estudio específico de las reparametrizaciones temporales, orientado a su empleo para los problemas (a) y (b) recién enumerados. En relación a todo lo anterior cabe mencionar que en los resultados expuestos en el Capítulo 4 los ítems (a) y (b) no van necesariamente juntos,
dándose casos en que esto es así, pero también reportándose resultados en los que (a) no va acompañado de (b), y viceversa, según se irá viendo.

La Sección 4.2 se dedica a la caracterización y análisis global de una nueva familia de estructuras de Poisson denominadas separables. Las mismas tienen matrices de estructura de la forma siguiente

$$
J_{i j}(x)=s_{i j} \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right), \quad i, j=1, \ldots, n
$$

siendo $S \equiv\left(s_{i j}\right)$ una matriz real, antisimétrica y constante. A pesar de su simplicidad formal, las matrices de estructura separables son muy generales, en el sentido de aparecer con gran frecuencia en aplicaciones de muy diversos campos, como se constata por medio de ejemplos. Además, cumplen los requisitos anteriormente expuestos de tener dimensión arbitraria, admitir valores cualesquiera del rango, y estar definidas en términos de funciones de no linealidad arbitraria. Bajo las hipótesis consideradas para esta familia, puede verse que el rango de estas matrices es igual al rango de $S$. Esto tiene la consecuencia de asociar de manera natural el problema de determinar un conjunto completo de invariantes de Casimir globales e independientes con la caracterización de una base del núcleo de la matriz constante $S$, obteniéndose así que dicho conjunto completo puede expresarse en la forma

$$
D_{i}(x)=\sum_{j=1}^{n} k_{j}^{[i]} \int \frac{\mathrm{d} x_{j}}{\psi_{j}\left(x_{j}\right)}, \quad i=r+1, \ldots, n
$$

siendo $r$ el rango de $S$, y $\left(k^{[r+1]}, \ldots, k^{[n]}\right)$ una base de $\operatorname{Ker}(S)$, donde $k^{[i]}=\left(k_{1}^{[i]}, \ldots, k_{n}^{[i]}\right)^{T}$ para $i=r+1, \ldots, n$. Es interesante comprobar que estos resultados permiten la determinación global y constructiva de la forma canónica de Darboux para la familia separable, y además esto es así exclusivamente mediante transformaciones de coordenadas, es decir sin recurrir al uso de reparametrizaciones temporales. En el caso de rango 2, lo anterior permite asimismo demostrar la doble integrabilidad, algebraica y de Liouville (esta última en el sentido ya habitual para el sistema Hamiltoniano relacionado) asociada naturalmente a este tipo de estructuras de Poisson.

Otro nuevo tipo de matrices de estructura con propiedades cualitativamente bastante similares es el considerado en la Sección 4.3. Dichas soluciones se han denominado multiseparables. Su definición es, sin embargo, algo más elaborada. Para la misma se introducen dos matrices $n \times n$ reales y regulares $(n \geq 2) A=\left(a_{i j}\right)$ y $B=\left(b_{i j}\right)$, con $A=B^{-1}$. Llamaremos además $B_{i} \equiv\left(b_{i 1}, \ldots, b_{i n}\right)$ a la $i$-ésima fila de $B$, para $i=1, \ldots, n$. Adicionalmente, si $2 \leq r \leq n$ es un entero par, y si denotamos por $\psi_{i}(x)$, con $i=1, \ldots, r$, a $r$ funciones de la forma $\psi_{i}(x)=\varphi_{i}\left(B_{i} \cdot x\right)$, entonces las matrices de estructura multiseparables van a estar
definidas según la expresión

$$
J_{i j}(x)=\sum_{k=1}^{r / 2} \Lambda_{i j}^{2 k-1,2 k} \psi_{2 k-1}(x) \psi_{2 k}(x), \quad i, j=1, \ldots, n
$$

donde por definición

$$
\Lambda_{i j}^{k l} \equiv\left|\begin{array}{cc}
a_{i k} & a_{i l} \\
a_{j k} & a_{j l}
\end{array}\right|=a_{i k} a_{j l}-a_{i l} a_{j k}, \quad i, j, k, l=1, \ldots, n
$$

Para las matrices de estructura multiseparables puede demostrarse que su rango es igual al número $r$ globalmente en todo el dominio, y además un conjunto completo de invariantes distinguidos funcionalmente independientes viene dado por:

$$
D_{i}(x)=\sum_{j=1}^{n} b_{i j} x_{j}, \quad i=r+1, \ldots, n
$$

Por tanto estamos de nuevo ante matrices de estructura de dimensión y rango arbitrarios, definidas en términos de funciones no lineales generales. Una propiedad interesante en este sentido es que toda matriz de estructura multiseparable tiene un conjunto completo de invariantes de Casimir lineales, según se ve, pero el recíproco no es cierto. Además, gracias a estos invariantes es ahora posible construir globalmente en todo el dominio la reducción a la forma canónica de Darboux, lo cual se lleva a cabo nuevamente sólo por medio de transformaciones de coordenadas, por tanto sin recurrir a reparametrizaciones temporales. Al igual que para las matrices separables, la doble integrabilidad (en el sentido ya habitual del término) se demuestra para el caso $r=2$. En esta sección se incluye también una comparativa entre las familias separable y multiseparable, a fin de clarificar su relación mutua, llegándose a la conclusión de que ambas familias son diferentes, es decir ninguna está contenida en la otra, si bien tienen intersección no nula.

La Sección 4.4 se ocupa de una nueva familia de soluciones antisimétricas de las ecuaciones de Jacobi, adicional a las anteriores, y que se ha denominado familia de matrices de estructura distinguidas, o bien D-soluciones. Estas vienen dadas como las soluciones de las ecuaciones de Jacobi distinguidas, que son las siguientes:

$$
\left\{\begin{array}{l}
\mathcal{J}^{T}=-\mathcal{J} \\
\mathcal{J} \cdot \nabla J_{i j}=0 \quad \text { para todo } \quad i, j=1, \ldots, n
\end{array}\right.
$$

O equivalentemente:

$$
\left\{\begin{array}{l}
J_{i j}=-J_{j i} \\
\sum_{l=1}^{n} J_{k l} \partial_{l} J_{i j}=0
\end{array} \quad, \quad \text { para todo } i, j, k=1, \ldots, n\right.
$$

No es complicado demostrar que toda solución de las ecuaciones de Jacobi distinguidas es solución también de las ecuaciones de Jacobi, y por tanto matriz de estructura. Más aun, dada una solución distinguida $\mathcal{J}(x)$, todas sus componentes $J_{i j}(x)$ son invariantes distinguidos de $\mathcal{J}(x)$, de ahí el nombre que recibe esta familia. En otras palabras, y expresándolo de forma muy sucinta, una matriz de estructura distinguida es la que está formada en su totalidad por funciones de estructura que, al mismo tiempo, son invariantes de Casimir de la propia matriz. Debido a este carácter especial, es posible demostrar que las D-soluciones tienen una serie de propiedades que permiten con gran facilidad obtener una infinidad de nuevas soluciones distinguidas a partir de una dada.

El carácter tan peculiar de las D-soluciones plantea la cuestión de su misma existencia. Esta se responde afirmativamente dentro de la propia Sección 4.4 mediante la construcción explícita de las denominadas $\mathrm{D}_{\psi}$-soluciones. Dados dos enteros positivos $n$ y $\rho \leq n$, se consideran las $(n-\rho)$ funciones

$$
D_{l}(x)=x_{l}-\sum_{k=1}^{\rho} a_{l k} x_{k}, \quad l=\rho+1, \ldots, n
$$

donde $a_{l k}$ son constantes reales para todo $l, k$. Adicionalmente, para $i, j=1, \ldots, \rho$, sean $\psi_{i j}\left(y_{1}, \ldots, y_{n-\rho}\right)$ funciones antisimétricas en sus subíndices, esto es

$$
\psi_{i j}\left(y_{1}, \ldots, y_{n-\rho}\right)=-\psi_{j i}\left(y_{1}, \ldots, y_{n-\rho}\right)
$$

para todo $i, j$. Finalmente, sea $\mathcal{J} \equiv\left(J_{i j}\right)$ la matriz $n \times n$ definida como:

$$
J_{i j}(x)= \begin{cases}\psi_{i j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , i, j=1, \ldots, \rho \\ \sum_{k=1}^{\rho} a_{j k} \psi_{i k}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , i=1, \ldots, \rho ; j=\rho+1, \ldots, n \\ \sum_{k=1}^{\rho} a_{i k} \psi_{k j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , i=\rho+1, \ldots, n ; j=1, \ldots, \rho \\ \sum_{k, l=1}^{\rho} a_{i k} a_{j l} \psi_{k l}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , i, j=\rho+1, \ldots, n\end{cases}
$$

Entonces, $\mathcal{J}$ es una D-solución globalmente definida en $\mathbb{R}^{n}$ y cuyo rango es menor o igual que $\rho-\rho \bmod 2$ para todo $x \in \mathbb{R}^{n}$. Además, las $(n-\rho)$ funciones $D_{l}(x)$ constituyen en todo $\mathbb{R}^{n}$ un conjunto de invariantes de Casimir de $\mathcal{J}$ funcionalmente independientes.

En la sección se demuestran también otras propiedades, por ejemplo que toda matriz antisimétrica constante es una D -solución (y una $\mathrm{D}_{\psi}$-solución también). La peculiaridad de las D-soluciones, tanto por su forma funcional como por el hecho de no garantizarse la
constancia del rango, hace que en esta familia no resulte clara la posibilidad de construir globalmente la forma canónica de Darboux. Asimismo, en la Sección 4.4 se compara la familia distinguida con las familias separable y multiseparable. En ambos casos se demuestra que la familia distinguida es diferente a las otras dos (no está contenida en ninguna de ellas, ni las contiene) pero, también en ambos casos, las D-soluciones tienen una intersección no nula con ambas familias.

La disertación prosigue en la Sección 4.5 con el tratamiento de una familia adicional, denominada hemiseparable de tipo I. Como sucedió en el caso de dimensión tres con las familias cíclicas, en este contexto vamos a tener que recurrir a una clasificación de invariantes de Casimir para poder resolver en todos los casos posibles el análisis global de estas estructuras de Poisson. La familia hemiseparable de tipo I viene dada por:

$$
J_{i j}(x)=\eta(x) \varphi_{i}\left(x_{i}\right) \varphi_{j}\left(x_{j}\right) \chi_{i j}\left(x_{i}, x_{j}\right), \quad i, j=1, \ldots, n
$$

Adicionalmente, la caracterización de la familia hemiseparable de tipo I se basa en la definición de las funciones

$$
\psi_{i}\left(x_{i}\right)=\int \frac{\mathrm{d} x_{i}}{\varphi_{i}\left(x_{i}\right)}, \quad i=1, \ldots, n
$$

cada una de las cuales denota una primitiva de $1 / \varphi_{i}\left(x_{i}\right)$, así como en la definición de las funciones $\chi_{i j}\left(x_{i}, x_{j}\right)$ conforme a

$$
\chi_{i j}\left(x_{i}, x_{j}\right)=\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}, \quad i, j=1, \ldots, n
$$

siendo $\kappa_{i j}, i, j=1, \ldots, n$, constantes reales arbitrarias que son antisimétricas

$$
\kappa_{i j}+\kappa_{j i}=0, \quad i, j=1, \ldots, n
$$

y satisfacen las condiciones de suma cero siguientes:

$$
\kappa_{i j}+\kappa_{j k}+\kappa_{k i}=0, \quad i, j, k=1, \ldots, n
$$

En función de las demás propiedades con las cuales se definen estas matrices de estructura, se puede probar que las mismas tienen rango 2 globalmente. Esto permite identificar por casos un conjunto completo de invariantes distinguidos globalmente definidos, como sigue: sea $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ para el par $(i, j)$ en el dominio de interés, entonces dicho conjunto completo viene dado por:

$$
D_{k}(x)=\frac{\psi_{j}\left(x_{j}\right)-\psi_{k}\left(x_{k}\right)+\kappa_{j k}}{\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}}=\frac{\chi_{j k}\left(x_{j}, x_{k}\right)}{\chi_{i j}\left(x_{i}, x_{j}\right)}, \quad k=1, \ldots, n ; \quad k \neq i, j
$$

La existencia de estos invariantes demuestra al mismo tiempo la integrabilidad algebraica para sistemas de Poisson con matrices de estructura de este tipo. Por tanto, en función de qué conjunto de invariantes sea el que se utiliza, puede demostrarse la posibilidad de construir globalmente la forma canónica de Darboux, esta vez por medio de la combinación de un cambio de coordenadas y una reparametrización temporal. A su vez, este resultado implica la integrabilidad en el sentido de Liouville (para el sistema Hamiltoniano que se obtiene tras llevar a cabo la reducción) asociada a este tipo de estructuras de Poisson. Un resultado adicional de interés que se demuestra en la Sección 4.5, en forma de ejemplo, es que la familia hemiseparable de tipo I contiene como caso especial a la familia cíclica de tipo I ya tratada en el capítulo sobre resultados en dimensión tres.

Con una estructura muy similar a la recién vista, en la Sección 4.6 se estudia una familia de soluciones que es formalmente paralela a la anterior, razón por la cual se la denominará familia de matrices de estructura hemiseparables de tipo II. Nuevamente, va a suceder que la construcción de la forma canónica de Darboux requiere un tratamiento por casos, con lo que se muestra que este comportamiento no debe considerarse como algo excepcional. Por supuesto, a pesar de las similitudes aludidas entre ambas clases, las matrices de estructura hemiseparables de los tipos I y II constituyen familias diferentes. Las matrices de estructura hemiseparables de tipo II tienen la forma

$$
J_{i j}(x)=\eta(x) \varphi_{i}\left(x_{i}\right) \varphi_{j}\left(x_{j}\right) \omega_{i j}\left(x_{i}, x_{j}\right), \quad i, j=1, \ldots, n
$$

donde se define que

$$
\psi_{i}\left(x_{i}\right)=a_{i} \exp \left(\int \frac{\mathrm{~d} x_{i}}{\varphi_{i}\left(x_{i}\right)}\right), \quad i=1, \ldots, n
$$

siendo las $a_{i} \neq 0, i=1, \ldots, n$, constantes reales arbitrarias (no nulas) y cada exponente

$$
\int \frac{\mathrm{d} x_{i}}{\varphi_{i}\left(x_{i}\right)}, \quad i=1, \ldots, n
$$

denota una primitiva de $1 / \varphi_{i}\left(x_{i}\right)$. Finalmente, las funciones $\omega_{i j}\left(x_{i}, x_{j}\right)$ se definen como

$$
\omega_{i j}\left(x_{i}, x_{j}\right)=\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right), \quad i, j=1, \ldots, n
$$

En esta ocasión, suponiendo que $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ para el par $(i, j)$ en el dominio en cuestión, se tiene que el rango de estas matrices de estructura es 2 con carácter global, y además un conjunto completo de invariantes de Casimir globales para las mismas viene dado por:

$$
D_{k}(x)=\frac{\psi_{i}\left(x_{i}\right)\left[\psi_{j}\left(x_{j}\right)-\psi_{k}\left(x_{k}\right)\right]}{\psi_{k}\left(x_{k}\right)\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right]}=\frac{\psi_{i}\left(x_{i}\right) \omega_{j k}\left(x_{j}, x_{k}\right)}{\psi_{k}\left(x_{k}\right) \omega_{i j}\left(x_{i}, x_{j}\right)}, \quad k=1, \ldots, n ; \quad k \neq i, j
$$

Claramente, este resultado implica la integrabilidad algebraica para los sistemas de Poisson basados en estas estructuras de Poisson. Adicionalmente, la caracterización por casos que se ha obtenido para los conjuntos completos de invariantes de Casimir anteriores permite construir sendas reducciones globales a la forma canónica de Darboux, las cuales proceden nuevamente en dos etapas, que son un cambio de coordenadas y una reparametrización temporal que le sigue. Con ello se demuestra en todos los casos que los sistemas de Poisson basados en matrices de estructura hemiseparables de tipo II llevan también a sistemas Hamiltonianos integrables en el sentido de Liouville. Dentro de los ejemplos aplicados que se presentan en esta sección cabe destacar la demostración de que las matrices de estructura cíclicas de tipo II, vistas en el Capítulo 2, son un caso especial de la familia hemiseparable de tipo II. Con ello resulta que las cuatro familias (cíclicas y hemiseparables, de los tipos I y II) mantienen una relación conceptual especial dentro de esta disertación, pues aportan por duplicado un tipo de casuística de especial interés, como es la constatación de que el mero conocimiento de un conjunto completo de invariantes distinguidos independientes puede no bastar para construir en todos los casos posibles la reducción global a la forma canónica de Darboux para una familia dada de estructuras de Poisson.

Como ya se ha anticipado, el papel de las reparametrizaciones temporales (o NTTs) es importante tanto en la obtención de nuevas estructuras de Poisson a partir de una dada, como en la construcción de la forma canónica de Darboux, al menos en muchos de los casos tratados a lo largo del trabajo. Por este motivo el objetivo de la Sección 4.7 es clarificar esta cuestión, por tanto desarrollando una investigación específica sobre las reparametrizaciones temporales la cual, como veremos, será de utilidad para comprender aspectos relevantes de cuanto se ha ido desarrollando a lo largo de la disertación. Para ello se introduce el concepto de factor de reparametrización para una matriz de estructura $\mathcal{J}(x)$ de rango constante, que es una función $\eta(x)$ suave y que no se anula tal que $\eta(x) \mathcal{J}(x)$ sigue siendo matriz de estructura. Es interesante notar que al multiplicar por un factor de reparametrización una matriz de estructura, los invariantes distinguidos no cambian, y la reducción global a la forma canónica de Darboux de $\eta(x) \mathcal{J}(x)$ es inmediata si previamente suponemos conocido el mismo tipo de reducción para $\mathcal{J}(x)$. Como sucedió al estudiar el uso de un ansatz aditivo en el caso tridimensional, podemos ahora considerar la investigación de los factores de reparametrización (o lo que es equivalente, de reparametrizaciones temporales) como un ansatz multiplicativo que se aplica sobre una solución (matriz de estructura) previamente conocida. Al igual que sucedió en el caso aditivo, se comprueba ahora que el problema general de investigar las ecuaciones de

Jacobi $n$-dimensionales se simplifica notablemente, llevando al siguiente sistema de ecuaciones diferenciales parciales para $\eta(x)$ :

$$
\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=0, \quad i, j, k=1, \ldots, n
$$

O lo que es equivalente, según puede comprobarse:

$$
\sum_{\substack{l=1 \\
l \neq i, j, k}}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=0, \quad\left\{\begin{array}{l}
i, j, k=1, \ldots, n \\
i \neq j ; i \neq k ; j \neq k
\end{array}\right.
$$

En última instancia, el núcleo de la investigación de los factores de reparametrización va a consistir en buscar familias de soluciones (en dimensión arbitraria) del sistema de ecuaciones en derivadas parciales para $\eta(x)$.

En la Sección 4.7 se reportan dos familias de factores de reparametrización. La primera de tales familias está relacionada con los invariantes distinguidos, ya que los mismos (si no se anulan en el dominio de interés) son siempre factores de reparametrización válidos en cualquier dimensión $n$ y para cualquier valor $r$ del rango de la matriz de estructura. Por este motivo los invariantes de Casimir que no se anulan se denominan en este ámbito "primera familia de factores de reparametrización". En el mismo contexto, el segundo resultado de interés que se desarrolla en la Sección 4.7 muestra que si el rango de la matriz de estructura (que se está suponiendo constante) es igual a 2 entonces, independientemente de la dimensión, cualquier factor de reparametrización es admisible, esto es preserva el carácter de sistema de Poisson. Y de manera complementaria, si la matriz de estructura es simpléctica con $n \geq 4$, entonces los únicos factores de reparametrización que admite son las constantes. En este contexto, cuando se habla de "segunda familia de factores de reparametrización" nos referimos a todas las funciones suaves y que no se anulan en el dominio de interés, que son factores de reparametrización válidos cuando el rango es 2 , para dimensión arbitraria $n$.

Los resultados sobre factores de reparametrización tienen dos aplicaciones importantes que se resumen a continuación.

La primera de ellas es la generalización de las matrices de estructura separables, multiseparables y distinguidas, ya que la multiplicación de las mismas por factores de reparametrización adecuados hace que el resultado ya no sea, respectivamente, una matriz de estructura separable, multiseparable o de tipo D-solución. Dado que en esta etapa se dispone de factores de reparametrización explícitos, únicamente dependientes del rango, ello hace que este tipo de generalización sea en gran medida directa. Por tanto esta primera aplicación se refiere a la
caracterización de nuevas matrices de estructura $n$-dimensionales, tomando como base las ya conocidas, conforme a la filosofía anteriormente expuesta.

La segunda aplicación que surge como consecuencia del estudio de factores de reparametrización es la determinación global y constructiva, bajo condiciones muy generales, de la forma canónica de Darboux para sistemas de Poisson de dimensión arbitraria con matrices de estructura de rango constante e igual a 2. Esto es posible en base a un algoritmo de dos etapas, la primera de las cuales es un cambio de coordenadas, mientras que la segunda es una reparametrización temporal. Concretamente, sea un sistema de Poisson $n$-dimensional,

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x), \quad x \in \mathbb{R}^{n}
$$

con matriz de estructura $n \times n$, que denotaremos $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$, y tal que el rango de $\mathcal{J}$ es 2 en todas partes. Sea $\left(D_{3}(x), \ldots, D_{n}(x)\right)$ un conjunto completo de invariantes de Casimir de $\mathcal{J}(x)$, independientes en el dominio de interés. Adicionalmente, se consideran dos funciones suaves y arbitrarias $\left(d_{1}(x), d_{2}(x)\right)$ tales que la transformación

$$
\begin{cases}y_{i}=d_{i}(x) \quad, \quad & i=1,2 \\ y_{j}=D_{j}(x), & j=3, \ldots, n\end{cases}
$$

sea uno a uno globalmente, y su matriz Jacobiana $M$ verifique:

$$
|M|=\left|\frac{\partial\left(d_{1}(x), d_{2}(x), D_{3}(x), \ldots, D_{n}(x)\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}\right| \neq 0, \quad \text { para todo } x
$$

Entonces el sistema de Poisson se reduce a la forma canónica de Darboux en las coordenadas $\left(y_{1}, \ldots, y_{n}\right)$ anteriores y en el nuevo tiempo $\tau$, dado por la reparametrización temporal:

$$
\mathrm{d} \tau=\left.\left(\left\{d_{1}(x), d_{2}(x)\right\}_{\mathcal{J}}\right)\right|_{x(y)} \mathrm{d} t=\left.\left[\left(\nabla_{x} d_{1}(x)\right)^{T} \cdot \mathcal{J}(x) \cdot\left(\nabla_{x} d_{2}(x)\right)\right]\right|_{x(y)} \mathrm{d} t \equiv \eta(y) \mathrm{d} t
$$

En la reducción anterior, el hecho de que la matriz de estructura tenga rango 2, y por tanto cualquier función suave y que no se anule sea admisible como factor de reparametrización, hace que podamos aplicar a conveniencia una reparametrización temporal arbitraria, lo cual permite completar satisfactoriamente la construcción de la forma canónica de Darboux. Como corolario, esto nos muestra que los sistemas de Poisson cuya matriz de estructura tiene rango 2 son, bajo las hipótesis establecidas al efecto, algebraicamente integrables y reducibles a un sistema Hamiltoniano integrable en el sentido de Liouville. Es evidente entonces en este punto que muchos de los ejemplos considerados con anterioridad son de hecho casos particulares de esta reducción, lo cual es indicativo también de la generalidad y aplicabilidad práctica del presente resultado.

En este momento es ya posible tener una perspectiva clara del papel relativo desempeñado por varios de los resultados desarrollados a lo largo de la disertación. Este es el propósito de la Sección 4.8. En la misma se pone de manifiesto cómo los distintos tipos de reducciones considerados, aparentemente diversos, en realidad se ciñen a lo permitido por los resultados sobre factores de reparametrización identificados en la sección precedente. Así, en los capítulos 2 y 3 (dimensiones 3 y 4, respectivamente) siempre era posible proceder combinando cambios de coordenadas con reparametrizaciones temporales, ya que las estructuras de Poisson consideradas eran siempre de rango 2. Por el contrario, en el caso de dimensión general la situación es más rica: cuando se trabaja con matrices de estructura de rango arbitrario (separables y multiseparables) entonces las reducciones a la forma canónica de Darboux, si son completamente generales para la familia en cuestión, necesariamente han de emplear únicamente transformaciones de coordenadas, ya que las reparametrizaciones temporales arbitrarias no van a ser admisibles para rango $r \geq 4, \mathrm{y}$ de hecho van a estar limitadas a la mínima expresión en el caso simpléctico (existente tanto para las estructuras de Poisson separables como para las multiseparables). En cambio, todavía en el contexto $n$-dimensional, la combinación de transformaciones de coordenadas y reparametrizaciones temporales está vigente en casos como los de las familias hemiseparables de ambos tipos (I y II) dado que en situaciones como esas tenemos matrices de estructura de rango 2. De esta forma el estudio de las NTTs ha clarificado en un grado muy amplio el sentido de las reducciones globales a la forma canónica de Darboux en situaciones distintas y en buena medida complementarias. En la Sección 4.8 este argumento se emplea también como hilo conductor para mostrar los aspectos comparativos más generales de las distintas familias de estructuras de Poisson investigadas a lo largo del capítulo, que concluye de esta manera.

Una vez finalizada la presentación de resultados, se procede en el Capítulo 5 a sistematizar brevemente las principales conclusiones que pueden obtenerse de todo lo anterior. En la disertación se ha procedido a desarrollar una doble línea de investigación, que combina tanto (a) la búsqueda de nuevas familias de estructuras de Poisson; como (b) la construcción de nuevos análisis globales (esto es, determinación global de los invariantes distinguidos y reducción global a la forma canónica de Darboux) para matrices de estructura. Ambos enfoques, que son evidentemente complementarios, han estado presentes de manera simultánea en algunos casos (como las matrices separables y multiseparables) mientras que en otros es sólo uno de los dos el que se aporta, bien sea en un sentido (caso de las D-soluciones) o bien en el
contrario (como en las familias hemiseparables de ambos tipos). De cualquier modo, los resultados obtenidos tienden a combinar el interés matemático fundamental con una proyección aplicada, lo que parece claro a la luz de las implicaciones en integrabilidad y cálculo de invariantes, en la presentación de resultados que mejoran el alcance del teorema de Darboux, en la caracterización y clasificación de nuevas soluciones antisimétricas de las ecuaciones de Jacobi, en la comprensión del papel del parámetro temporal en la existencia de la estructura de Poisson como tal, en relación a distintos problemas aplicados ya mencionados, así como por la diversidad e interés de los muchos ejemplos tratados a lo largo del trabajo. Otras consideraciones en torno a implicaciones adicionales de los resultados examinados, así como a posibles líneas futuras de investigación en este campo, son también objeto de comentario en el Capítulo 5. Con ello finaliza la presente disertación como tal.

Por cuestiones técnicas de mero interés operacional se ha incluido el Apéndice 1. En efecto, las relaciones que caracterizan a los invariantes de Casimir constituyen un sistema de ecuaciones diferenciales parciales acopladas. Dado que la determinación de tales invariantes es una necesidad frecuente en este trabajo, se ha incluido en el citado Apéndice 1 una descripción del método Pfaffiano para el cálculo de dichas constantes del movimiento. El método Pfaffiano proporciona una alternativa demostrablemente más eficaz para la determinación de los invariantes distinguidos, preferible al procedimiento habitual (antes aludido) consistente en resolver el sistema de ecuaciones diferenciales parciales que caracteriza a estos invariantes. Como es la norma a lo largo de toda la disertación, la descripción del método Pfaffiano en este apéndice se acompaña de ejemplos detallados que lo ilustran.

Finalmente, los Apéndices 2 y 3, por su parte, se limitan a cuestiones de notación y terminología matemática empleadas a lo largo de la presente tesis, en el caso del primero, y a la enumeración de las abreviaturas de uso común en este trabajo, en el segundo. En cuanto al Apéndice 4, en el mismo se incluyen a modo de indicios de calidad las publicaciones en revistas con índice de impacto derivadas (hasta el momento) de la investigación descrita en esta tesis.

## CHAPTER 1.

## INTRODUCTION

### 1.1. CLASSICAL HAMILTONIAN SYSTEMS

It is difficult to overemphasize the scope and relevance of Hamiltonian dynamics. Hamiltonian theory is a very rich subject with deep mathematical implications in diverse domains, specially differential equations, geometry and integrability theory. The very specific form of Hamilton's equations has allowed also the development of a plethora of specialized methods of great applied projection, the canonical perturbation theory or the specifically adapted numerical methods (symplectic integrators) being some of the most celebrated. Hamilton's equations are also the basis for the understanding of very different applied systems, from classical to quantum mechanics, from celestial mechanics to electrodynamics, from optics to plasma physics or fluid dynamics. The diversity of dynamical behaviors embraced by Hamiltonian dynamics include not only regular motions, but also chaotic dynamics (the celebrated KAM theorem being one of the paramount results in this field). Even the mere mention, if exhaustive, of all of the previously cited aspects of the Hamiltonian theory would be (by far) out of the scope of this section. Our purpose here is just to provide a very concise introduction of some of the most significant results regarding Hamilton's formalism, in order to make this dissertation self-contained. For such purpose, in this section we shall be mainly inspired by the classical treatments of Goldstein [57] and Landau and Lifshitz [108]. For their central interest for the generalization from Hamilton to Poisson systems, special emphasis will be put on the significance of Poisson brackets and in the symplectic (or matrix) formulation of Hamilton's equations. The reader interested in more detailed presentations of Hamiltonian theory is referred to the classic literature in the field $[1,9,21,57,108,110,159,161,164,167,173]$ and references therein.

### 1.1.1. Introductory aspects of classical Hamiltonian systems

We shall not enter here in the subject of how the canonical equations can be derived from Lagrange's formulation by means of Legendre transformations. This issue is extensively considered in the literature and is not essential for this introduction. For practical purposes, our starting point here will be to consider Hamilton's canonical equations of motion, which are given by the following set of ODEs (see Appendix 3 for the different abbreviations used in this work):

$$
\begin{equation*}
\dot{q}_{i}=\frac{\partial H(q, p, t)}{\partial p_{i}}, \quad \dot{p}_{i}=-\frac{\partial H(q, p, t)}{\partial q_{i}}, \quad i=1, \ldots, s \tag{1}
\end{equation*}
$$

(the notation $\dot{x}$ will denote the time derivative of a function $x(t)$, namely $\dot{x}=\mathrm{d} x / \mathrm{d} t$; the reader is referred to Appendix 2 for mathematical notation). In system (1) the variables $q_{i}$ are generalized coordinates, while the $p_{i}$ are termed generalized momenta. Together, variables $\left(q_{i}, p_{i}\right)$ are known as canonical variables, and they constitute a coordinate system of a phase space of dimension $n=2 s$. Function $H(q, p, t)$ often corresponds to the energy of the system, and is known as a Hamiltonian (or Hamilton's) function. Thanks to the structure of equations (1) it is easy to verify that:

$$
\frac{\mathrm{d} H}{\mathrm{~d} t}=\frac{\partial H}{\partial t}+\sum_{i=1}^{s} \frac{\partial H}{\partial q_{i}} \dot{q}_{i}+\sum_{j=1}^{s} \frac{\partial H}{\partial p_{j}} \dot{p}_{j}=\frac{\partial H}{\partial t}
$$

Therefore if the Hamiltonian is time-independent, energy is conserved for the system. In what follows, the name classical Hamiltonian system or simply Hamiltonian system will always be referred to a set of canonical equations of the form (1). In addition, a Hamiltonian system of the form (1) is frequently termed in the literature a Hamiltonian system of $s$ degrees of freedom.

A form of reexpressing the canonical equations that will be of great importance for the rest of the work is known as the symplectic or matrix notation. Consider that we denote the canonical coordinates by a single column vector $x \equiv\left(q_{1}, \ldots, q_{s}, p_{1}, \ldots, p_{s}\right)^{T}$. We then define the symplectic matrix $\mathbb{S}_{n}$ as:

$$
\mathbb{S}_{n}=\left(\begin{array}{c|c}
\mathbb{O}_{s} & \mathbb{I}_{s}  \tag{2}\\
\hline-\mathbb{I}_{s} & \mathbb{O}_{s}
\end{array}\right)
$$

Then it is clear that system (1) can be equivalently written in matrix form as:

$$
\begin{equation*}
\dot{x}=\mathbb{S}_{n} \cdot \nabla_{x} H \tag{3}
\end{equation*}
$$

Expression (3) for Hamilton's equations is known as their symplectic or matrix form. The most relevant properties of $\mathbb{S}_{n}$ at this stage are that it is skew-symmetric and invertible for any $n$, namely $\operatorname{Rank}\left(\mathbb{S}_{n}\right)=n$.

### 1.1.2. The canonical Poisson bracket

Let $f(q, p, t)$ be a given smooth function of the canonical variables and time. The total time derivative of $f(q, p, t)$ is, according to its standard definition [57,108]:

$$
\frac{\mathrm{d} f}{\mathrm{~d} t}=\frac{\partial f}{\partial t}+\sum_{k=1}^{s}\left(\frac{\partial f}{\partial q_{k}} \dot{q}_{k}+\frac{\partial f}{\partial p_{k}} \dot{p}_{k}\right)
$$

Making use of the canonical equations (1) this is equivalent to:

$$
\frac{\mathrm{d} f}{\mathrm{~d} t}=\frac{\partial f}{\partial t}+\{f, H\}_{c}
$$

where $\{f, H\}_{c}$ denotes the canonical Poisson bracket of $f$ and $H$, defined as [57,108]:

$$
\begin{equation*}
\{f, H\}_{c}=\sum_{k=1}^{s}\left(\frac{\partial f}{\partial q_{k}} \frac{\partial H}{\partial p_{k}}-\frac{\partial f}{\partial p_{k}} \frac{\partial H}{\partial q_{k}}\right) \tag{4}
\end{equation*}
$$

If we consider (see Subsection 1.4.4 for additional details) that a first integral is a function $I(x, t)$ such that its total time derivative is zero

$$
\frac{\mathrm{d} I}{\mathrm{~d} t}=0
$$

we find that such condition can equivalently be written as:

$$
\frac{\mathrm{d} I}{\mathrm{~d} t}=\frac{\partial I}{\partial t}+\{I, H\}_{c}=0
$$

In particular, if we are dealing with a time-independent first integral, then the condition is simply $\{I, H\}_{c}=0$. For two arbitrary smooth functions $f(q, p, t)$ and $g(q, p, t)$ the definition is the same:

$$
\begin{equation*}
\{f, g\}_{c}=\sum_{k=1}^{s}\left(\frac{\partial f}{\partial q_{k}} \frac{\partial g}{\partial p_{k}}-\frac{\partial f}{\partial p_{k}} \frac{\partial g}{\partial q_{k}}\right) \tag{5}
\end{equation*}
$$

The canonical Poisson bracket (5) has the following fundamental properties (in a sense to be specified later) that can be deduced from its definition:

- Skew-symmetry:

$$
\{f, g\}_{c}=-\{g, f\}_{c}
$$

- Bilinearity:

$$
\left\{k_{1} f_{1}+k_{2} f_{2}, g\right\}_{c}=k_{1}\left\{f_{1}, g\right\}_{c}+k_{2}\left\{f_{2}, g\right\}_{c}, \quad\left\{f, k_{1} g_{1}+k_{2} g_{2}\right\}_{c}=k_{1}\left\{f, g_{1}\right\}_{c}+k_{2}\left\{f, g_{2}\right\}_{c}
$$

for all constants $k_{1}, k_{2} \in \mathbb{R}$.

- Leibnitz' rule:

$$
\left\{f, g_{1} g_{2}\right\}_{c}=g_{2}\left\{f, g_{1}\right\}_{c}+g_{1}\left\{f, g_{2}\right\}_{c}
$$

- Jacobi identity:

$$
\left\{\{f, g\}_{c}, h\right\}_{c}+\left\{\{g, h\}_{c}, f\right\}_{c}+\left\{\{h, f\}_{c}, g\right\}_{c}=0
$$

Actually, if we consider that the canonical Poisson bracket of two functions $f$ and $g$ defines a product between those functions, then it can be seen that the previous properties imply that the canonical Poisson bracket induces a structure of a Lie algebra [57]. We shall return to this issue later.

Notice in particular that for the canonical coordinates we have:

$$
\begin{equation*}
\left\{q_{i}, q_{j}\right\}_{c}=\left\{p_{i}, p_{j}\right\}_{c}=0, \quad\left\{q_{i}, p_{j}\right\}_{c}=\delta_{i j}, \quad i, j=1, \ldots, s \tag{6}
\end{equation*}
$$

where $\delta_{i k}$ accounts for Kronecker's delta. Regarding the symplectic (or matrix) formulation (3) it is interesting to note that the canonical Poisson bracket of two functions can be equivalently expressed in terms of column gradient vectors as:

$$
\begin{equation*}
\{f, g\}_{c}=(\nabla f)^{T} \cdot \mathbb{S}_{n} \cdot \nabla g \tag{7}
\end{equation*}
$$

Following the same column vector terminology $x=\left(x_{1}, \ldots, x_{n}\right)^{T} \equiv\left(q_{1}, \ldots, q_{s}, p_{1}, \ldots, p_{s}\right)^{T}$ introduced before for the symplectic form of the equations, note in particular that according to (6) and (7) we have:

$$
\begin{equation*}
\left\{x_{i}, x_{j}\right\}=\left(\mathbb{S}_{n}\right)_{i j}, \quad i, j=1, \ldots, n \tag{8}
\end{equation*}
$$

An important property of the Poisson bracket (known as Poisson's theorem) is that if $I_{1}(q, p, t)$ and $I_{2}(q, p, t)$ are first integrals of the system, then their Poisson bracket is also a first integral:

$$
\left\{I_{1}, I_{2}\right\}_{c}=\text { constant }
$$

Of course, by a successive application of Poisson's theorem there is no guarantee that new nontrivial first integrals are to be determined. The reason is clear: the number of functionally independent global first integrals is limited to $(2 s-1)$, where $s$ is the number of degrees of freedom, as indicated. In some cases the outcome is a trivial first integral, namely an actual constant. In other cases, the first integral obtained is simply a function of the initial first integrals $I_{1}$ and $I_{2}$. If none of these two possibilities takes place, then the Poisson bracket of $I_{1}$ and $I_{2}$ is an additional (independent of $I_{1}$ and $I_{2}$ ) first integral of the Hamiltonian system.

### 1.1.3. Canonical transformations in the symplectic formalism

One of the advantages of Hamilton's equations is that the choice of the generalized coordinates is not limited, as they can be any set of variables determining the state of the system. However the canonical equations (1) are not formally invariant when a general smooth change of coordinates

$$
\begin{equation*}
q_{i}^{*} \equiv q_{i}^{*}(q, p, t), \quad p_{i}^{*} \equiv p_{i}^{*}(q, p, t), \quad i=1, \ldots, s \tag{9}
\end{equation*}
$$

is performed. This explains why in the canonical formalism, a key issue is the determination of the conditions under which the form of Hamilton's equations remains unaltered after such a transformation is applied. In other words, the goal is to determine the conditions by which after the change of variables (9) applied to (1) the new system remains formally

$$
\dot{q}_{i}^{*}=\frac{\partial H^{*}\left(q^{*}, p^{*}, t\right)}{\partial p_{i}^{*}}, \quad \dot{p}_{i}^{*}=-\frac{\partial H^{*}\left(q^{*}, p^{*}, t\right)}{\partial q_{i}^{*}}, \quad i=1, \ldots, s
$$

with a new Hamiltonian $H^{*}\left(q^{*}, p^{*}, t\right)$. When this is actually the case, the transformation is termed canonical. In practice (and also for our purposes in this work) the canonical transformations that do not explicitly depend on time have a special interest, namely those of the kind:

$$
\begin{equation*}
q_{i}^{*} \equiv q_{i}^{*}(q, p), \quad p_{i}^{*} \equiv p_{i}^{*}(q, p), \quad i=1, \ldots, s \tag{10}
\end{equation*}
$$

Canonical transformations of the form (10) are known as restricted canonical transformations.
There are two equivalent points of view [57,108,110,161,167,173] for the study of canonical transformations. One of them is based on the use of generating functions. Such method is not natural for our purposes in this work, and consequently it will not be described here. The second possibility works directly on the symplectic form of the canonical equations. This point of view is the one to be adopted in what follows. For this, consider the canonical equations written in the symplectic (or matrix) form (3), namely $\dot{x}=\mathbb{S}_{n} \cdot \nabla_{x} H$. Similarly to what was done for the definition of vector $x$, let $y \equiv y(x)$ be a new system of canonical coordinates defined in terms of $x$ by means of a smooth transformation. Let

$$
\begin{equation*}
M=\frac{\partial\left(y_{1}, \ldots, y_{n}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)} \tag{11}
\end{equation*}
$$

be the Jacobian matrix of such transformation. Then it is not difficult to prove that after the change of coordinates, the transformed system has the form:

$$
\dot{y}=M \cdot \mathbb{S}_{n} \cdot M^{T} \cdot \nabla_{y} H^{*}(y)
$$

where $H^{*}(y)=H(x(y))$. It is known that, in the case of restricted canonical transformations, the Hamiltonian function of the transformed system is just the one of the original system reexpressed in terms of the new variables, i.e.

$$
\dot{y}=\mathbb{S}_{n} \cdot \nabla_{y} H^{*}(y)
$$

again with $H^{*}(y)=H(x(y))$. Accordingly, the condition for the transformation to be canonical is simply:

$$
\begin{equation*}
\mathbb{S}_{n}=M \cdot \mathbb{S}_{n} \cdot M^{T}, \quad \text { or equivalently } \quad \mathbb{S}_{n}=M^{T} \cdot \mathbb{S}_{n} \cdot M \tag{12}
\end{equation*}
$$

Actually, it can be shown [57] that (12) is a necessary and sufficient condition for the change of coordinates $y=y(x)$ to be a restricted canonical transformation. The corresponding calculations are more complicated in the case of general (or time-dependent) canonical transformations. However, it is also possible to prove [57] that condition (12) is also necessary and sufficient for the transformation to be canonical even in the case (9) in which the time appears explicitly

These results have some implications in connection with Poisson brackets. Let us first recall the identities (6-8). As before, consider now a smooth change of variables $y \equiv y(x)$ of Jacobian matrix (11). Then, substitution in equation (7) allows computing the canonical Poisson brackets of the new variables $y$ in terms of the initial variables $x$. The outcome can be shown to be:

$$
\left\{y_{i}, y_{j}\right\}_{c}=\left(\nabla_{x} y_{i}\right)^{T} \cdot \mathbb{S}_{n} \cdot \nabla_{x} y_{j}=\left(M \cdot \mathbb{S}_{n} \cdot M^{T}\right)_{i j}, \quad i, j=1, \ldots, n
$$

Of course, if the transformation is canonical, then equation (12) implies that

$$
\begin{equation*}
\left\{y_{i}, y_{j}\right\}_{c}=\left(\mathbb{S}_{n}\right)_{i j}, \quad i, j=1, \ldots, n \tag{13}
\end{equation*}
$$

which is actually analogous to (8) for the new variables. And reciprocally, if (13) holds, then the transformation is canonical. The Poisson brackets among the canonical variables, such as (8) or (13), are known as fundamental Poisson brackets.

Therefore these results imply that the fundamental Poisson brackets always have the same value in any system of canonical variables, namely they are invariant under restricted canonical transformations (actually they are invariant under general canonical transformations, but we shall not be concerned with this issue in the present work, in which we shall only deal with time-independent transformations of coordinates). Thus, the invariance of the fundamental Poisson brackets is equivalent to the canonical nature of the transformation. In fact, it is possible to prove that the canonical Poisson bracket between any two functions is also invariant
under canonical transformations (either restricted or not). This is why all canonical Poisson brackets (for any two functions) are canonical invariants, namely they always take the same value, independently of the system of canonical variables in which the bracket is computed. This reason explains why the canonical Poisson brackets are denoted by the symbol $\{,\}_{c}$, without reference to the set of canonical variables being employed.

Thus, we see that Hamilton's equations are invariant under canonical transformations. In an analogous fashion, canonical invariance of Poisson brackets implies that the equations expressed in terms of Poisson brackets are also form-invariant under canonical transformations. As it can be seen [57], it is possible to develop an entire coordinate-free formulation of classical mechanics, parallel to Hamilton's formalism, that is expressed only by means of Poisson brackets. Such formulation has the same form in all the systems of canonical coordinates, and is specially useful for the transition from classical to quantum mechanics by means of the correspondence principle.

### 1.1.4. Liouville's theorem

We of course do not intend here to provide a review of classical Hamiltonian systems, but only an outline of some properties relevant to what is to follow. Many important aspects of the theory are thus being disregarded. For its conceptual interest for future developments, one of the issues that is worth recalling, even briefly, is Liouville's theorem. For this, note that every point of the $2 s$-dimensional phase space corresponds to a definite state of the system. When the system evolves, such a point moves in phase space describing a phase trajectory. The differential $\mathrm{d} \nu$ given by

$$
\mathrm{d} \nu=\mathrm{d} q_{1} \ldots \mathrm{~d} q_{s} \mathrm{~d} p_{1} \ldots \mathrm{~d} p_{s}
$$

can be physically interpreted as a volume element in phase space $\mathbb{R}^{2 s}$. Consider then the Riemann integral $\int \mathrm{d} \nu$ extended to a given region $\Omega$ of phase space, and thus representing the volume of that region. It is possible to show [108] that such integral has the property of being invariant with respect to canonical transformations $q^{*} \equiv q^{*}(q, p, t)$ and $p^{*} \equiv p^{*}(q, p, t)$, namely the value of the integral is independent of the system of canonical variables used,

$$
\int \ldots \int_{\Omega} \mathrm{d} q_{1} \ldots \mathrm{~d} q_{s} \mathrm{~d} p_{1} \ldots \mathrm{~d} p_{s}=\int \ldots \int_{\Omega^{*}} \mathrm{~d} q_{1}^{*} \ldots \mathrm{~d} q_{s}^{*} \mathrm{~d} p_{1}^{*} \ldots \mathrm{~d} p_{s}^{*}
$$

with $\Omega^{*}$ denoting the region $\Omega$ expressed in terms of the new canonical variables $\left(q^{*}, p^{*}\right)$. Actually, this is a consequence of the fact that the Jacobian determinant of a canonical transformation is always equal to one, or $|M|=1$ in the previous notation. Moreover, it is possible
to prove that the time evolution of a Hamiltonian system, namely the evolution in time of the canonical variables $(q, p)$, can be considered as a canonical transformation, as we shall see now. To be precise, this means the following: let $q(t)$ and $p(t)$ be the values of the canonical variables at time $t$, and let $q(t+\varrho)$ and $p(t+\varrho)$ be the values at time $t+\varrho$, with $\varrho$ a parameter corresponding to the time interval elapsed. Then the latter are functions of the former (and of the time interval $\varrho$ as a parameter):

$$
\begin{equation*}
q(t+\varrho)=q(q(t), p(t), \varrho), \quad p(t+\varrho)=p(q(t), p(t), \varrho) \tag{14}
\end{equation*}
$$

Thus the transformation (14) of the canonical variables associated with the time evolution of the system is in fact a canonical transformation, as it can be established [57,108]. This implies that a region $\Omega$ of phase space that evolves according to the time evolution of the system can of course get deformed but always maintains a constant volume:

$$
V(t)=\int_{\Omega} \mathrm{d} \nu=\text { constant }
$$

This result is known as Liouville's theorem, and is one of the most distinctive features of Hamiltonian dynamics. In the framework of this thesis, it is conceptually relevant because its generalization will lead to the important concept of conservative system (Subsection 1.2.1).

As indicated, it is unavoidable to skip here many significant aspects of the Hamiltonian theory. Some classical topics at this level are the generating function formalism for canonical transformations, the Hamilton-Jacobi equation, the principle of least action, the canonical perturbation theory, etc. The interested reader is referred to the classical literature on the subject.

### 1.2. POISSON STRUCTURES AND POISSON SYSTEMS

We shall now focus on the kind of dynamical systems providing the framework of this thesis, which are finite-dimensional Poisson systems [21,110,139,159,163,167]. This type of systems of ODEs arise from the general concept of Poisson structure, which is essentially a smooth manifold endowed with an abstract Poisson bracket operation (to be defined in what follows). Here we shall only provide a brief introduction to Poisson structures, which are analyzed in detail in different classic references $[1,21,109,139,159,167,171]$. As we shall see also in the following introduction, Poisson systems constitute for several reasons a natural and wide generalization (in a sense to be specified) of Hamiltonian systems, while retaining the essential aspects of the latter after the generalization process. The reader interested in further basic aspects of the theory of finite-dimensional Poisson systems is referred to the classical works just mentioned, which have been the basis of the present introductory review.

### 1.2.1. Generalized Liouville's theorem and conservative systems

A relevant result to be discussed at this stage is the generalized Liouville's theorem [96]. Consider a general $n$-dimensional nonautonomous system of ODEs:

$$
\begin{equation*}
\dot{x}=f(x, t), \quad x \in \mathbb{R}^{n} \tag{15}
\end{equation*}
$$

Let $\Omega(0) \subset \mathbb{R}^{n}$ be a domain in phase space at time $t=0$, and let $\Omega(t)$ represent this domain at time $t$ as transformed by the time evolution of the system. In other words, $\Omega(t)$ is the set of all points $x(t)$ such that $x(0) \in \Omega(0)$, according to the time evolution defined by equations (15). We then consider the volume integral (in the Riemann sense) of some smooth function $m(x, t): \mathbb{R}^{n} \times \mathbb{R} \rightarrow \mathbb{R}$ over this moving domain $\Omega(t):$

$$
\begin{equation*}
I(t)=\int \ldots \int_{\Omega(t)} m(x, t) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{n} \tag{16}
\end{equation*}
$$

The integral $I(t)$ is called an integral invariant of system (15) if its value is constant in time, namely if it is a constant of motion:

$$
\frac{\mathrm{d} I}{\mathrm{~d} t}=0
$$

It is then possible to establish the generalized Liouville's theorem, which states that the integral $I(t)$ in (16) is an integral invariant of system (15) if and only if:

$$
\frac{\partial m(x, t)}{\partial t}+\nabla \cdot[m(x, t) f(x, t)]=0
$$

This is the general form of the so-called Liouville equation. This greatly generalizes the result seen for classical Hamiltonian systems, for which it is simple to check that it is $m(x, t)=1$. In such case, the generalized Liouville theorem reduces to $\nabla_{p, q} \cdot f=0$, a result often mentioned in the framework of classical Hamiltonian systems. More generally, in the case in which $m(x, t)$ is time-independent and nonnegative, such function can be seen as a weight associated with every point of phase space in a measure $\mu$ such that $\mathrm{d} \mu=m(x) \mathrm{d} x_{1} \ldots \mathrm{~d} x_{n}$. Then the generalized Liouville's theorem states that the total measure $\mu(\Omega(t))$ of the set $\Omega(t)$ is constant in time if and only if $\nabla \cdot[m(x) f(x, t)]=0$. Such measure-preserving (also known as conservative) flows are of interest because they share a basic but fundamental constraint on the trajectories in phase space. It is then clear that conservative systems provide a generalization of classical Hamiltonian systems: notice, for instance, that odd-dimensional flows are not excluded in the definition of conservative or measure-preserving systems. Moreover, also in the even-dimensional case conservative systems (non-negative $m(x)$ ) are significantly more general than Hamiltonian systems $(m(x)=1)$. Actually, the appearance in the applications of non-Hamiltonian systems of ODEs which however display measure-preserving properties that strongly suggest a parallelism with Hamiltonian systems, has motivated the search of generalizations of the latter able to account for odd-dimensional flows and being more general than Hamiltonian systems also in the even-dimensional case. One of the paramount extensions of Hamilton's equations is provided by Poisson systems, which are the subject of this work. They are described in the following subsections.

### 1.2.2. Poisson structures and finite-dimensional Poisson systems

As it was mentioned at the end of Subsection 1.1.3, it is possible to develop coordinatefree formulations of classical Hamiltonian systems, only based on the use of canonical Poisson brackets. Precisely, the use of such kind of approach allows naturally the definition of Poisson structures and systems [139]. Due to this reason, and following such reference, this is the point of view that will be adopted in this subsection.

Definition 1.2.2.1. (Poisson bracket). A Poisson bracket on a smooth manifold $\mathbb{M}$ is an operation that assigns a smooth real-valued function $\{f, g\}$ on $\mathbb{M}$ to each pair of smooth and real-valued functions $f$ and $g$ defined on $\mathbb{M}$, according to the properties:
(a) Skew-symmetry:

$$
\{f, g\}=-\{g, f\}
$$

(b) Bilinearity:

$$
\left\{k_{1} f_{1}+k_{2} f_{2}, g\right\}=k_{1}\left\{f_{1}, g\right\}+k_{2}\left\{f_{2}, g\right\}, \quad\left\{f, k_{1} g_{1}+k_{2} g_{2}\right\}=k_{1}\left\{f, g_{1}\right\}+k_{2}\left\{f, g_{2}\right\}
$$

$$
\text { for all constants } k_{1}, k_{2} \in \mathbb{R}
$$

(c) Leibnitz' rule:

$$
\left\{f, g_{1} g_{2}\right\}=g_{2}\left\{f, g_{1}\right\}+g_{1}\left\{f, g_{2}\right\}
$$

(d) Jacobi identity:

$$
\{\{f, g\}, h\}+\{\{g, h\}, f\}+\{\{h, f\}, g\}=0
$$

In all these equations, $f, g$ and $h$ denote arbitrary smooth functions on $\mathbb{M}$.

Now some remarks are in order. In first place, we note that for the purposes of this work the manifolds of interest are $\mathbb{R}^{n}$ or some domain $\Omega \subset \mathbb{R}^{n}$, all of which are smooth manifolds. Secondly, it is interesting to recall that properties (a) to (d) in Definition 1.2.2.1 do coincide with those already seen in Subsection 1.1.2 for the canonical Poisson bracket.

A manifold $\mathbb{M}$ endowed with a Poisson bracket is called a Poisson manifold, $\mathbb{M}_{\mathbb{P}}$, the bracket defining a Poisson structure on such manifold. The notion of a Poisson manifold is more general than that of a manifold with a classical (canonical) Poisson bracket, as it will be proved in brief. In particular, a Poisson manifold needs not be even-dimensional, as we shall see now in the following:

## Example.

Consider the canonical Poisson bracket (4) defined in an even-dimensional real space $\mathbb{M}_{\mathbb{P}}=$ $\mathbb{R}^{n}, n=2 s$, with canonical variables $\left(q_{1}, \ldots, q_{s}, p_{1}, \ldots, p_{s}\right)$ :

$$
\begin{equation*}
\{f, g\}=\sum_{k=1}^{s}\left(\frac{\partial f}{\partial q_{k}} \frac{\partial g}{\partial p_{k}}-\frac{\partial f}{\partial p_{k}} \frac{\partial g}{\partial q_{k}}\right) \tag{17}
\end{equation*}
$$

As we know, all the properties (a)-(d) indicated in Definition 1.2.2.1 are verified by (17). More generally, a Poisson bracket can also be defined on $\mathbb{R}^{n}$ for every value of $n$. Let $(q, p, z)=$ $\left(q_{1}, \ldots, q_{s}, p_{1}, \ldots, p_{s}, z_{1}, \ldots, z_{l}\right)$ with $2 s+l=n$, and define the Poisson bracket between two functions $f(q, p, z)$ and $g(q, p, z)$ by the same expression (17). It can be seen that the same properties (a)-(d) in Definition 1.2 .2 .1 are still satisfied. Actually this example plays an important role in what is to follow: Darboux' theorem will show that locally (except at
singular points, to be defined) every Poisson bracket can be reduced to this one. Note also that every function depending only on the $z$ variables commutes (in the sense of the bracket) with any other function. In other words, if $f \equiv f(z)$, then $\{f, g\}=0$ for any function $g$, in the sense of the Poisson bracket (17). Such functions $f(z)$, and in particular the variables $z_{1}, \ldots, z_{l}$, are known as distinguished functions or Casimir functions, and are characterized by the property that their Poisson bracket with any other function is always zero. Notice that now the basic relationships (6) are supplemented by the additional ones:

$$
\left\{q_{i}, z_{k}\right\}=\left\{p_{i}, z_{k}\right\}=\left\{z_{j}, z_{k}\right\}=0, \quad i=1, \ldots, s, \quad j, k=1, \ldots, l
$$

The previous example motivates thus the following definition:

Definition 1.2.2.2. (Casimir or distinguished function). Let $\mathbb{M}_{\mathbb{P}}$ be a Poisson manifold. A smooth, real-valued function $D: \mathbb{M}_{\mathbb{P}} \rightarrow \mathbb{R}$ is called a distinguished (or Casimir) function if the Poisson bracket of $D$ with any other real-valued function vanishes identically, namely $\{D, f\}=0$ for all $f: \mathbb{M}_{\mathbb{P}} \rightarrow \mathbb{R}$.

In the case of the canonical Poisson bracket (on $\mathbb{M}_{\mathbb{P}}=\mathbb{R}^{2 s}$ ) the only distinguished functions are the constants, which always verify the requirements of the definition. On the other extreme, if the Poisson bracket is completely trivial (namely $\{f, g\}=0$ for every $f$ and $g$ ) then every function is distinguished.

As it was anticipated in Subsection 1.1.2, Poisson brackets have the structure of a Lie algebra. Actually, it can be proved that there exists an important connection between the Poisson bracket of two functions and the Lie bracket of the vector fields induced by such functions [139]. This issue will not be considered here for the sake of conciseness. However, this relationship allows the establishment of the analysis of Poisson manifolds in terms of coordinate systems. Let $x=\left(x_{1}, \ldots, x_{n}\right)$ be local coordinates on $\mathbb{M}_{\mathbb{P}}$, then it can be proved [139] that the Poisson bracket of two functions $f(x)$ and $g(x)$ takes the form:

$$
\begin{equation*}
\{f, g\}=\sum_{i, j=1}^{n}\left\{x_{i}, x_{j}\right\} \frac{\partial f}{\partial x_{i}} \frac{\partial g}{\partial x_{j}} \tag{18}
\end{equation*}
$$

Then, in order to compute the Poisson bracket of any pair of functions in a system of local coordinates, is suffices to know the Poisson brackets between all pairs of coordinate functions $x_{i}$. Such basic brackets will be termed structure functions:

$$
J_{i j}(x)=\left\{x_{i}, x_{j}\right\}, \quad i, j=1, \ldots, n
$$

Thus, the structure functions of the Poisson manifold $\mathbb{M}_{\mathbb{P}}$ relative to the coordinates $x_{i}$ uniquely determine the Poisson structure itself. For convenience, the structure functions are grouped into a skew-symmetric $n \times n$ matrix $\mathcal{J}$, called structure matrix. Therefore, in the usual column gradient vector notation formula (18) can be expressed as:

$$
\begin{equation*}
\{f, g\}=(\nabla f)^{T} \cdot \mathcal{J} \cdot \nabla g \tag{19}
\end{equation*}
$$

From a constructive point of view, it is also possible to show from (19) that the distinguished functions are the solution set of the following system of PDEs:

$$
\begin{equation*}
\mathcal{J}(x) \cdot \nabla D(x)=0 \tag{20}
\end{equation*}
$$

It can be seen [139] that, in the system of coordinates $x$, the vector field associated with a smooth function $H(x)$ for a Poisson bracket of structure matrix $\mathcal{J}(x)$ takes the form (in evident notation):

$$
\begin{equation*}
\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)=\{x, H\} \tag{21}
\end{equation*}
$$

Equation (21) gives the general form of a Poisson system. In terms of the individual coordinates:

$$
\begin{equation*}
\frac{\mathrm{d} x_{i}}{\mathrm{~d} t}=\sum_{j=1}^{n} J_{i j}(x) \partial_{j} H(x)=\left\{x_{i}, H\right\}, \quad i=1, \ldots, n \tag{22}
\end{equation*}
$$

We see then that we obtain a generalization of classical Hamiltonian systems, in which odd dimensions are allowed. As before, $H(x)$ plays the role of Hamiltonian, and $\mathcal{J}(x)$ is a generalization of the symplectic matrix. For instance, in the case of bracket (17) defined on $\mathbb{R}^{n}$ for an arbitrary value of $n$ and coordinates $(q, p, z)$ with $2 s+l=n$, the structure matrix is:

$$
\begin{equation*}
\mathcal{J}(x)=\mathbb{S}_{2 s} \oplus \mathbb{O}_{l} \tag{23}
\end{equation*}
$$

We then obtain the canonical Poisson bracket for classical Hamiltonian systems in the particular (even-dimensional) case $l=0$. Of course, it is necessary to know which skew-symmetric matrices $\mathcal{J}(x)$ are structure matrices for Poisson brackets. In other words, we need to characterize those matrices such that, according to (18), verify properties (a)-(d) of Definition 1.2.2.1. The answer is the following one [139]:

Theorem 1.2.2.3. (Structure matrices). Let $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ be an $n \times n$ matrix of functions $J_{i j}(x): \Omega \rightarrow \mathbb{R}$ defined over an open subset $\Omega \subset \mathbb{R}^{n}$. Then $\mathcal{J}(x)$ is the structure matrix for a Poisson bracket $\{f, g\}=(\nabla f)^{T} \cdot \mathcal{J} \cdot \nabla g$ over $\Omega$ if and only if it has the following properties:
(a) Skew-symmetry:

$$
\begin{equation*}
J_{i j}=-J_{j i}, \quad i, j=1, \ldots, n \tag{24}
\end{equation*}
$$

(b) Jacobi identities (Jacobi partial differential equations):

$$
\begin{equation*}
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{j l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i j}\right)=0, \quad i, j, k=1, \ldots, n \tag{25}
\end{equation*}
$$

for all $x \in \Omega$.

Notice that equations (25) form a set of nonlinear coupled PDEs which must be verified by the structure functions. In particular, any constant skew-symmetric matrix trivially satisfies the system (24-25) and thus determines a Poisson bracket. This remark, in spite of its simplicity, already constitutes a wide generalization of the classical symplectic matrices (2) and also of their noncanonical generalization (23) which thus become particular cases of the family of constant structure matrices.

### 1.2.3. Phase space structure: symplectic foliation and Darboux' theorem

In order to understand the phase space geometry induced by a general Poisson structure on a manifold, a closer look at the structure matrix $\mathcal{J}(x)$ which determines the local coordinate form of the Poisson bracket is unavoidable. As it will be explained, the rank is the most important invariant of a structure matrix. We are going to see that in the case of maximal rank we are in the situation of a symplectic Poisson structure, well-known in classical Hamiltonian theory. In the more general case of nonmaximal rank, the Poisson manifold $\mathbb{M}_{\mathbb{P}}$ is going to be foliated into symplectic submanifolds in such a way that any Poisson system on $\mathbb{M}_{\mathbb{P}}$ naturally restricts to any one of the symplectic submanifolds and hence, by restriction, returns us to the canonical case, the classical scenario of Hamiltonian dynamics. However, for many problems it may be more natural to remain in the larger Poisson manifold itself, specially when the noncanonical variables are more natural for that specific problem (e.g. for physical reasons) or when one is interested in the collective behavior of systems depending on parameters, with the underlying symplectic structure varying with the parameters themselves.

Given a Poisson manifold $\mathbb{M}_{\mathbb{P}}$ in which a set of local coordinates $x$ is defined, it is necessary to introduce the following important definition:

Definition 1.2.3.1. (Rank of a Poisson manifold). The rank of a Poisson manifold at a point $x$ is equal to the rank of the structure matrix $\mathcal{J}$ at that point.

Actually, it can be shown that the rank of a Poisson manifold is independent of the choice of the coordinates. According to Theorem 1.4.1.1, we know immediately that the rank of a Poisson manifold at any point is always even. In this sense, notice (according to equation (20) for the Casimir invariants) that such invariants are directly related to the kernel of the structure matrix, and do not exist (apart from the trivial solutions, namely the real constants) if $\operatorname{Rank}(\mathcal{J})=n$. This case is usually denoted by means of a specific term:

Definition 1.2.3.2. (Symplectic manifold). A Poisson manifold of dimension $n$ is said to be symplectic if its Poisson structure has maximal (and thus even) rank $n$ everywhere.

Symplectic manifolds will be denoted by $\mathbb{M}_{\mathbb{S}}$ and they lie at the basis, for instance, of classical Hamiltonian systems (notice that the symplectic matrix $\mathbb{S}_{n}$ is of maximal rank $n$ everywhere). A structure matrix is said to be trivial if it has zero rank everywhere in the Poisson manifold.

When $\mathcal{J}$ has constant rank $r$, it can be seen [139] that each Poisson manifold naturally splits into a collection of even-dimensional symplectic submanifolds, what is known as the symplectic foliation of a Poisson manifold. Therefore, phase space is split in a family of evendimensional symplectic leaves, the dimension of each leaf being equal to $r=\operatorname{Rank}(\mathcal{J})$. In fact the symplectic leaves can be seen to be obtained as the slices, or level sets, of a set of $(n-r)$ functionally independent Casimir invariants, namely by the equations $\left\{D_{i}(x)=a_{i} \mid a_{i} \in\right.$ $\mathbb{R}, i=1, \ldots, n-r\}$. Since $(n-r)$ is the maximal number of functionally independent Casimir invariants, such a set is termed a complete set of independent Casimir invariants. If the rank of the Poisson manifold is not constant, then the symplectic leaves will have varying dimension. In this work, we shall be concerned only with constant-rank structure matrices, for which Darboux' theorem (to be introduced shortly) is applicable and the symplectic foliation does not have variable dimension. Thus, in the constant-rank case, the dynamics of every Poisson system can be naturally restricted to any symplectic leaf, thus becoming a symplectic system. The precise meaning of this assertion will be clear shortly with Darboux' theorem. Actually, in spite that Poisson systems are clearly a formal generalization of classical Hamiltonian systems, it is not evident at this stage that this generalization preserves the Hamiltonian (or
conservative, in general terms) nature of the dynamics. In other words, we need a result accounting for the dynamical equivalence between classical Hamiltonian and Poisson systems in order to state that the latter are a rightful generalization of the former. The central result that justifies such gap is Darboux' theorem [109,139,171]:

Theorem 1.2.3.3. (Darboux' theorem). Let $\mathbb{M}_{\mathbb{P}}$ be an $n$-dimensional Poisson manifold of constant rank $r=2 s$ everywhere. Then, in a neighborhood of each $x_{0} \in \mathbb{M}_{\mathbb{P}}$ there exist local variables $(q, p, z) \equiv\left(q_{1}, \ldots, q_{s}, p_{1}, \ldots, p_{s}, z_{1}, \ldots, z_{l}\right)$, with $n=r+l$, in terms of which the Poisson bracket takes the form:

$$
\begin{equation*}
\{f, g\}=\sum_{k=1}^{s}\left(\frac{\partial f}{\partial q_{k}} \frac{\partial g}{\partial p_{k}}-\frac{\partial f}{\partial p_{k}} \frac{\partial g}{\partial q_{k}}\right) \tag{26}
\end{equation*}
$$

The leaves of the symplectic foliation intersect the coordinate chart in the slices

$$
z_{1}=k_{1}, \ldots, z_{l}=k_{l}
$$

determined by the distinguished coordinates $z$.

In other words, every Poisson system for which the rank of the Poisson structure is constant can be (locally at least) reduced in the neighborhood of each point to a Poisson system of the form:

$$
\left\{\begin{array}{l}
\dot{q}_{i}=\frac{\partial H}{\partial p_{i}}, \quad \dot{p}_{i}=-\frac{\partial H}{\partial q_{i}}, \quad i=1, \ldots, s  \tag{27}\\
\dot{z}_{j}=0, \quad j=1, \ldots, l
\end{array}\right.
$$

We thus see that system (27) is essentially an $s$-degree of freedom classical Hamiltonian system plus $l=(n-r)$ decoupled (and constant) Casimir invariants given by $z_{1}, \ldots, z_{l}$. Equations (27) are known as the Darboux canonical form of the Poisson system. Therefore, the Darboux canonical form corresponds to the structure matrix $\mathcal{J}=\mathbb{S}_{2 s} \oplus \mathbb{O}_{l}$ regarded in (23). Darboux' theorem provides a (local) link between Poisson systems and classical Hamiltonian systems accounting for their dynamical equivalence (apart from the embedding corresponding to the symplectic foliation, which is decoupled in the Darboux canonical form). The explicit construction of the Darboux coordinates may be a complicated task in general, specially in the case of their global determination, in which the transfer of results between the Poisson and the classical Hamiltonian formats is optimal for the applications. The global construction of the Darboux canonical form for different families of Poisson structures will be one of the central issues of this thesis. Accordingly, it is worth recalling that, unless otherwise specified, we shall systematically work with structure matrices of constant rank in the region of interest.

For the sake of clarity, in most of what is to follow, the Darboux canonical form will be written in a way slightly different from that corresponding to (23), which is obtained simply by reordering the canonical variables $(q, p)$ which are to be grouped in canonical pairs, while the decoupled Casimir invariants remain at the same places. Thus the new ordering is $\left(q_{1}, p_{1}, \ldots, q_{s}, p_{s}, z_{1}, \ldots, z_{l}\right)$. The fact of grouping the variables in canonical pairs can be sometimes advantageous from the point of view of clarity, while of course the Darboux canonical form is exactly the same: the new equations for the Darboux canonical form are still those in (27), but the new order for the variables has the consequence of introducing a permutation of rows and columns in the structure matrix, which changes from (23) to:

$$
\mathcal{J}_{D[n, r]} \equiv\left(\begin{array}{cc}
0 & 1  \tag{28}\\
-1 & 0
\end{array}\right) \overbrace{\oplus \ldots \oplus}^{s}\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \oplus \mathbb{O}_{1} \overbrace{\oplus \ldots \oplus \mathbb{O}_{1}}^{(n-r)}
$$

The notation $\mathcal{J}_{D[n, r]}$ given by (28) will be maintained throughout the work. Notice that in (28) the expression $A \overbrace{\oplus \ldots \oplus}^{k} A$, for any matrix $A$, denotes the direct sum of $k$ matrices $A$.

### 1.2.4. Changes of coordinates and time reparametrizations

In practice, if we wish to investigate issues such as the global determination of the Darboux canonical form for Poisson systems, it is necessary to make use of certain types of transformations that ensure system equivalence (see Subsection 1.4.3 for a discussion of this concept). In this paragraph, two important types of transformations and their effect on Poisson systems are considered.

The first kind of transformation of interest in our context is provided by smooth coordinate transformations. Consider a general Poisson system (22) defined in a given domain $\Omega \subset \mathbb{R}^{n}$ in terms of a set of coordinates $x$. Let $y \equiv y(x)$ be a general smooth coordinate transformation, and let $M \equiv\left(M_{i j}\right)$ be the associated Jacobian matrix, namely:

$$
M=\frac{\partial\left(y_{1}, \ldots, y_{n}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}
$$

Then it can be proved that, after such a change of coordinates, the transformed ODEs obtained from (22) still constitute a Poisson system in terms of the $y$ coordinates in $\Omega^{*}=y(\Omega)$ :

$$
\begin{equation*}
\dot{y}=\mathcal{J}^{*}(y) \cdot \nabla H^{*}(y) \tag{29}
\end{equation*}
$$

Now in (29) the new structure matrix $\mathcal{J}^{*}(y)$ is given by:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{30}
\end{equation*}
$$

or equivalently,

$$
\mathcal{J}^{*}=M \cdot \mathcal{J} \cdot M^{T}
$$

And the new Hamiltonian is simply $H^{*}(y)=H(x(y))$. This is a significant feature of Poisson systems: in a sense, all smooth transformations can be regarded as "canonical" now, as far as they preserve the key property of having the form of a Poisson system for the transformed set of ODEs. When compared to the restrictions imposed by canonical transformations on classical Hamiltonian systems (see Subsection 1.1.3) we obtain an indication about the generalization achieved by means of Poisson systems.

The second kind of transformations to be considered in this paragraph corresponds to time reparametrizations (also known in the literature as new-time transformations, or NTTs) for Poisson systems [21], which are transformations of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{31}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a smooth function in $\Omega$ which does not vanish in $\Omega$. Thus, again if (22) is an arbitrary Poisson system defined in $\Omega$, then every time reparametrization (31) leads from (22) to the system:

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{32}
\end{equation*}
$$

Equations (32) are not necessarily of Poisson type because $\eta \mathcal{J}$ may lose the property of being a structure matrix, in spite that $\mathcal{J}$ is. Of course, time reparametrizations do not alter the topology of trajectories in phase space, their only dynamical role amounts to modify the "speed" at which every point moves on the system trajectories. A well-known result [70] is that the system (32) obtained from (22) after a general time reparametrization is always a Poisson system in the cases of dimensions $n=2$ and $n=3$. On the contrary, this is not necessarily the case for dimensions $n \geq 4$. Time reparametrizations will play an important role both for the determination of the Darboux canonical form and for the understanding of some detailed conditions under which such a construction can be carried out. In a sense, the results to be developed in connection with such kind of transformations will provide a unifying perspective of the results given in this thesis. This reason explains that precisely the end of this work is devoted to time reparametrizations (Section 4.7). Additional aspects regarding the connection between time reparametrizations and system equivalence are considered in Subsection 1.4.3.

### 1.2.5. Introductory examples of Poisson structures and systems

In order to clarify and illustrate the previous theory as well as an advance of some typical features to be found throughout the work, two simple and well-known instances of Poisson structures of applied interest are briefly presented in what follows. The first one corresponds to a mechanical system (Euler top) and is clearly close to the physical roots of classical Hamiltonian systems, in the sense that the Hamiltonian corresponds to the system energy and the constancy of the Casimir invariants also reflect the conservation of angular momentum. The second example, on the contrary, has a biomathematical inspiration and describes a population dynamics system (Lotka-Volterra equations) for which neither the Hamiltonian nor the distinguished functions have a clear physical interpretation, thus suggesting conceptually new domains of applicability of Poisson systems. Both examples chosen are three-dimensional, as far as the visualization of the phase space structure (namely, of a nontrivial symplectic foliation) should be simpler in such case. The reduction to the Darboux canonical form can be globally carried out for both systems. However, it will not be provided here because both reductions are particular cases of more general results to be presented in Chapter 2. Consequently, for the moment we limit ourselves here to an illustrative description of the systems in order to appreciate the naturalness of the Poisson description. Of course, many other examples are going to appear along this work.

## Example 1. Euler equations for the rigid body (Euler top)

We shall consider the following system of ODEs known as Euler equations [139], which describe the rotation of a rigid body, or Euler top:

$$
\left\{\begin{array}{l}
\dot{x}_{1}=\frac{\mu_{2}-\mu_{3}}{\mu_{2} \mu_{3}} x_{2} x_{3} \\
\dot{x}_{2}=\frac{\mu_{3}-\mu_{1}}{\mu_{3} \mu_{1}} x_{3} x_{1} \\
\dot{x}_{3}=\frac{\mu_{1}-\mu_{2}}{\mu_{1} \mu_{2}} x_{1} x_{2}
\end{array}\right.
$$

Here $x_{i}$ denotes the $i$-th component of angular momentum, and constants $\mu_{i}$ are the moments of inertia about the coordinate axes, both for $i=1,2,3$. Energy is conserved for this system, but of course the flow is odd-dimensional and a classical Hamiltonian formulation is excluded.

However this system is of Poisson kind in terms of the following structure matrix:

$$
\mathcal{J}(x)=\left(\begin{array}{ccc}
0 & -x_{3} & x_{2} \\
x_{3} & 0 & -x_{1} \\
-x_{2} & x_{1} & 0
\end{array}\right)
$$

Notice that the rank of the structure matrix is 2 everywhere in $\mathbb{R}^{3}$ except at the origin, in which the rank vanishes. The Hamiltonian is the total energy (kinetic energy, in this case):

$$
H(x)=\frac{1}{2}\left(\frac{x_{1}^{2}}{\mu_{1}}+\frac{x_{2}^{2}}{\mu_{2}}+\frac{x_{3}^{2}}{\mu_{3}}\right)
$$

Since the rank is 2 (excluding the origin from the analysis) there must be $(n-r)=1$ independent Casimir invariants. Recall that these invariants are the solutions of the system of PDEs (20), or:

$$
\mathcal{J}(x) \cdot \nabla D(x)=0
$$

In our case, the independent distinguished function can be chosen to be:

$$
\begin{equation*}
D(x)=\left(x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)^{1 / 2}=\|x\| \tag{33}
\end{equation*}
$$

Namely, the distinguished invariant (33) is the Euclidean norm of the angular momentum, which is a conserved quantity during the system rotation. Of course, any other smooth function of $D(x)$ in (33) will be also a Casimir invariant. Therefore, the symplectic foliation is given by:

$$
\begin{equation*}
x_{1}^{2}+x_{2}^{2}+x_{3}^{2}=\text { constant } \tag{34}
\end{equation*}
$$

which are concentric spheres in $\mathbb{R}^{3}$. We thus see that the symplectic leaves (34) are evendimensional (two-dimensional, in this case) and therefore Darboux' theorem ensures that on the symplectic leaves the dynamics is Hamiltonian in the classical sense, at least locally (in the neighborhood of each point). The actual trajectories of the system in phase space are obtained by the intersection of the symplectic leaves with the energy level sets

$$
H(x)=\frac{1}{2}\left(\frac{x_{1}^{2}}{\mu_{1}}+\frac{x_{2}^{2}}{\mu_{2}}+\frac{x_{3}^{2}}{\mu_{3}}\right)=\text { constant }
$$

which in geometric terms are ellipsoids in phase space.

## Example 2. 3-d Lotka-Volterra equations

Let us now consider the system of ODEs [63,136]:

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{1}\left(\lambda_{1}+c x_{2}+x_{3}\right)  \tag{35}\\
\dot{x}_{2}=x_{2}\left(\lambda_{2}+x_{1}+a x_{3}\right) \\
\dot{x}_{3}=x_{3}\left(\lambda_{3}+b x_{1}+x_{2}\right)
\end{array}\right.
$$

This is a $3-\mathrm{d}$ instance of Lotka-Volterra equations, which are models of common use in mathematical biology for the description of population interactions. The number of individuals of every species corresponds to the system variables $x_{i}>0$ for all $i=1,2,3$. Constants $\lambda_{1}, \lambda_{2}$ and $\lambda_{3}$ account for the specific growth rates, and constants $a, b$ and $c$ account for interspecific interactions. System (35) is Poisson for certain values of the parameters $\left(a, b, c, \lambda_{1}, \lambda_{2}, \lambda_{3}\right)$. Here we shall consider one of such cases [136] in which the system parameters verify:

$$
a b c=-1, \quad \lambda_{3}=\lambda_{2} b-\lambda_{1} a b
$$

When these conditions are satisfied, one possibility for the structure matrix is:

$$
\mathcal{J}(x)=\left(\begin{array}{ccc}
0 & c x_{1} x_{2} & b c x_{1} x_{3} \\
-c x_{1} x_{2} & 0 & -x_{2} x_{3} \\
-b c x_{1} x_{3} & x_{2} x_{3} & 0
\end{array}\right)
$$

Again, the rank of $\mathcal{J}$ is 2 . In this case, the Hamiltonian function is:

$$
\begin{equation*}
H(x)=a b x_{1}+x_{2}-a x_{3}+\lambda_{3} \ln x_{2}-\lambda_{2} \ln x_{3} \tag{36}
\end{equation*}
$$

Contrarily to the previous example, now the Hamiltonian (36) does not have an evident interpretation, let it be physical or biological. Since $\operatorname{Rank}(\mathcal{J})=2$, there must be one independent Casimir invariant, and the symplectic leaves are two-dimensional. In this case, one possible choice is given by the following quasimonomial function:

$$
D(x)=\left(x_{1}\right)^{1 / c}\left(x_{2}\right)^{b}\left(x_{3}\right)^{-1}
$$

As it was here the case with $H(x)$, this distinguished invariant does not seem to have a direct interpretation in terms of the model itself. Again, the system trajectories in phase space will be given by the intersection of the level surfaces of functions $H(x)$ and $D(x)$.

### 1.2.6. Interest and applications of Poisson structures and systems

Today, finite-dimensional Poisson systems (e.g. see [126,139] and references therein for an overview and a historical discussion) are ubiquitous in the literature in most fields of applied mathematics, physics, engineering and mathematical biology, such as in mechanics $[1,4,5,36,37,38,39,40,41,65,91,106,107,119,122,128,141,151,152,153,155]$, dynamical systems theory $[25,29,33,63,83,86,117,143]$, fluid mechanics [134], electromagnetism [31,42,94,112,113], plasma physics [142], population dynamics $[53,56,83,88,101,102,103,135,136,143,144,145,146]$, optics $[6,42,43,93,94]$, quantization [150], control theory [19,148], delay equations [118], network analysis $[129,154]$, field theories [52], etc. Of course, the investigation of such applied systems has provided the basis for a very diverse mathematical research, both fundamental and applied, that has been active for decades. Such activity has led to the development of very varied methods and results: describing a given system in terms of a Poisson structure allows the obtainment of a wide range of information which may be in the form of perturbative solutions [31] and use of generating functions [45], invariants [84,175], bifurcation properties and characterization of chaotic behavior [12,35,43,142], efficient numerical integration $[97,99,130,131,149]$, use of variational principles [32,34,115], integrability results [7,20,21,44,58,100,118,120,138], reductions [6,42,43,70,76,77,78,79,83,85], as well as stability analysis in terms of either the energy-Casimir algorithm $[18,86,92,93,107,140,147,172]$ or the energy-momentum method [155], to cite a sample.

There are several reasons justifying the importance and flexibility of Poisson systems. The first one, already mentioned, is that it provides a wide generalization of classical Hamiltonian systems, allowing not only for odd-dimensional vector fields, but also because (even in the symplectic case) a structure matrix verifying (24-25) admits a great diversity of forms apart from the classical constant symplectic matrix [57]. The determination of Casimir invariants and their use in order to carry out a reduction (local, in principle) is the cornerstone of the (at least local) equivalence between Poisson systems and classical Hamiltonian systems, as stated by Darboux' theorem [139,171]. This justifies that Poisson systems can be regarded, to a large extent, as a rightful generalization of classical Hamiltonian systems. This connection is an additional and important advantage of Poisson systems, as far as it accounts for the potential transfer of results and techniques from classical Hamiltonian theory once a given system has been recognized as a Poisson one and the Darboux canonical form has been constructed, specially if this can be achieved globally.

On the other hand, it was already indicated that sometimes the Poisson representation
is preferable to a Darboux reduction to classical Hamiltonian form. There are several different reasons that can account for such preference. One is that in many applications it is found that the noncanonical variables are desirable, for instance because they are more natural on a physical or modelling basis. A simple instance is provided by the LotkaVolterra model of the previous subsection as well as diverse models in population dynamics [63,83,101,102,103,135,136,143,144,145,146]. Other instances well-known in the literature are, for example, $[112,113]$ in electromagnetism, the Euler top in mechanics [139] just seen in Subsection 1.2 .5 , etc. Moreover, many methods of analysis have been specifically developed for the Poisson representation and their use may be advantageous in terms of the original Poisson system format. Some instances of this are given by noncanonical perturbation theory [31], by the energy-Casimir [18,86,92,93,107,140,147,172] and energy-momentum [155] methods, a variety of numerical methods $[97,99,130,131,149]$, etc. The energy-Casimir method is specially illustrative in this sense, because its application requires the use of the Casimir invariants of the Poisson system, thus being excluded in practice in the case of the reduced Hamiltonian system. To conclude, it was also mentioned the case of systems depending on parameters, with the underlying symplectic structure varying with the parameters themselves, where the use of the Poisson formulation allows a joint analysis of the different symplectic leaves [81].

What has been said in this subsection up to now concerns Poisson systems of the ODE type, namely finite-dimensional Poisson systems. Of course, the reason is that this is the subject of the present work. However, it is interesting to stress that the concept of Poisson structure does not incorporate any dimensional limit. Well the opposite, it can be said that the domain in which Poisson systems theory has achieved the greatest successes is the one of evolution equations (partial differential equations) for nondissipative systems, for which the underlying Poisson manifold has an infinite dimension. In such case, the concept of Poisson bracket admits a direct generalization that can be carried out with the help of some mathematical modifications. Despite this field is not the subject of this thesis, and therefore will be mentioned only briefly here, it is worth recalling the plethora of application domains and results [139,163] obtained in the last years in fluid dynamics [11,27,49,92,121,132,137,160], plasma physics $[11,68,133]$, continuum mechanics [48,160], field theories [54,125], etc. This enumeration is of course not exhaustive (as it is not the list of references given, which is merely illustrative). The interested reader is referred to these works and references therein for further details.

### 1.3. THE JACOBI PARTIAL DIFFERENTIAL EQUATIONS

We have seen that the Jacobi PDEs (25) together with the skew-symmetry conditions (24) are one essential aspect of the formulation of a dynamical system as a finite-dimensional Poisson system. Actually, such equations constitute the basis for the determination of the structure matrix, and consequently of the Poisson structure itself when expressed in an arbitrary system of coordinates. The investigation of the Jacobi equations, including the determination of solutions and the analysis of such solutions, is the central subject of the present thesis. It is thus natural that after a general treatment of finite-dimensional Poisson systems we now focus our description on the more specific subject of the Jacobi PDEs. In what follows, terms such as "Jacobi equations" or "skew-symmetric Jacobi equations" will systematically denote the joint system composed by (24-25) unless otherwise specified. In addition, expressions such as " $n$-dimensional Poisson structure" or " $n$-dimensional solution of the Jacobi equations" shall refer to Poisson structures for which $n$ is the dimension of the associated Poisson manifold (and of the corresponding vector field given by the respective Poisson system); in terms of coordinates, such expressions allude to $n \times n$ structure matrices.

### 1.3.1. General properties

Let us then focus on the main subject of this thesis, which is primarily given by the Jacobi partial differential equations. We thus look at the system:

$$
\begin{equation*}
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{j l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i j}\right)=0, \quad i, j, k=1, \ldots, n \tag{37}
\end{equation*}
$$

As we know, this is a set of coupled nonlinear partial differential equations, in which the solutions form the entries of a skew-symmetric $n \times n$ matrix. This implies a degree of redundancy related to the fact that $J_{i j}=-J_{j i}$ for all $i \neq j$. Taking this into account, the number of nonredundant unknowns in (37) is:

$$
N_{\text {unknowns }}=\frac{n(n-1)}{2}=1,3,6,10,15, \ldots \quad(\text { for } n \geq 2)
$$

Consequently, for large $n$ the number of unknowns increases quadratically. Let us look, on the other hand, at the number of equations in (37). Simply speaking, the number of equations is of course $n^{3}$. However, this is not the actual number of independent equations. The reason is clear if we examine the situation when two or more indexes coincide. For instance, let us look at system (37) if $i=j$ (for any value of $k$ ). Then the equation $(i, i, k)$ vanishes identically:

$$
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{i k}+J_{i l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i i}\right)=\sum_{l=1}^{n}\left(J_{i l} \partial_{l}\left[J_{i k}+J_{k i}\right]\right)=0, \quad i, k=1, \ldots, n
$$

Clearly, the cyclic symmetry of $(i, j, k)$ in equations (37) implies that this is the situation whenever at least any two indexes $(i, j, k)$ coincide. Therefore, it is necessary to determine the number of independent equations, and such number corresponds to those for which the three indexes $(i, j, k)$ are all different. In addition, equations (37) are invariant under permutations of the three indexes $(i, j, k)$. Taking this into account, the number of independent equations in (37) is thus [26]:

$$
N_{\text {equations }}=\binom{n}{3}=\frac{n!}{3!(n-3)!}=\frac{n(n-1)(n-2)}{6}=0,1,4,10,20, \ldots \quad(\text { for } n \geq 2)
$$

Therefore, for $n=2$ the Jacobi PDEs are identically satisfied, while for large $n$ the number of independent equations grows as the cube of $n$. We shall return to this issue later in this subsection and also in the next one. But before, for illustrative purposes it is interesting to explicitly write the (independent) Jacobi PDEs arising in the simplest nontrivial cases $n=3$ and $n=4$. To begin with, consider the case $n=3$, in which a single independent equation is present, as we know in advance. If we choose the nonredundant unknown functions as $J_{12}$, $J_{23}$ and $J_{31}$, it is not difficult to verify that system (37) amounts to one equation that can be written as:

$$
J_{12} \partial_{1} J_{31}-J_{31} \partial_{1} J_{12}+J_{23} \partial_{2} J_{12}-J_{12} \partial_{2} J_{23}+J_{31} \partial_{3} J_{23}-J_{23} \partial_{3} J_{31}=0
$$

Some aspects of the three-dimensional case will be analyzed in Chapter 2 of this work. Similarly, let us look at the case $n=4$, for which we know in advance that there are 4 independent equations (with 6 nonredundant unknown functions). With a choice of the unknowns consisting in using those $J_{i j}$ for which $i<j$, the system can be written as:

$$
\begin{gathered}
J_{12} \partial_{1} J_{13}-J_{13} \partial_{1} J_{12}+J_{12} \partial_{2} J_{23}-J_{23} \partial_{2} J_{12}+J_{13} \partial_{3} J_{23}-J_{23} \partial_{3} J_{13}+ \\
J_{14} \partial_{4} J_{23}+J_{34} \partial_{4} J_{12}-J_{24} \partial_{4} J_{13}=0 \\
J_{12} \partial_{1} J_{14}-J_{14} \partial_{1} J_{12}+J_{12} \partial_{2} J_{24}-J_{24} \partial_{2} J_{12}+J_{14} \partial_{4} J_{24}-J_{24} \partial_{4} J_{14}+ \\
J_{13} \partial_{3} J_{24}-J_{23} \partial_{3} J_{14}-J_{34} \partial_{3} J_{12}=0 \\
J_{13} \partial_{1} J_{14}-J_{14} \partial_{1} J_{13}+J_{13} \partial_{3} J_{34}-J_{34} \partial_{3} J_{13}+J_{14} \partial_{4} J_{34}-J_{34} \partial_{4} J_{14}+ \\
J_{12} \partial_{2} J_{34}+J_{23} \partial_{2} J_{14}-J_{24} \partial_{2} J_{13}=0 \\
J_{23} \partial_{2} J_{24}-J_{24} \partial_{2} J_{23}+J_{23} \partial_{3} J_{34}-J_{34} \partial_{3} J_{23}+J_{24} \partial_{4} J_{34}-J_{34} \partial_{4} J_{24}- \\
J_{12} \partial_{1} J_{34}+J_{13} \partial_{1} J_{24}-J_{14} \partial_{1} J_{23}=0
\end{gathered}
$$

Some considerations and results on the case $n=4$ will be the goal of Chapter 3. We see then that the Jacobi equations seem to become progressively complex as dimensionality increases.

Actually, the brief analysis just carried out already shows two interesting facts, worth being stressed, about the Jacobi equations:
(a) The Jacobi equations are identically satisfied in dimension $n=2$. This is a well-known result, as it will be mentioned in the next subsection.
(b) The number of independent equations increases (for growing $n$ ) faster than the number of unknowns, in the sense that:

$$
\frac{N_{\text {unknowns }}}{N_{\text {equations }}}=\frac{3}{n-2} \rightarrow 0, \quad \text { when } \quad n \rightarrow \infty
$$

This is the reason that explains the belief, even in relatively recent literature [66], that the Jacobi equations could be overdetermined for sufficiently large $n$, apart from some simple known instances valid for arbitrary $n$. This issue will be also discussed in the next subsection.

Some of the questions posed after the analysis presented in this subsection suggest that it would be convenient to review aspects such as what are the most important known solutions of the Jacobi equations, the existence of solutions of arbitrary dimension and rank, and the progression in the generality of the solution families reported in the literature. In addition, this will provide a framework for the contributions to be presented in this thesis. The next paragraph is devoted to such issues.

### 1.3.2. A perspective on some important families of solutions

We shall now briefly review some well-known families of structure matrices together with some related issues. As mentioned before, the comparatively faster increase in the number of equations with respect to the number of unknowns has motivated that even in relatively recent times [66] some authors have considered that in general the Jacobi equations form an overdetermined system for large $n$. On the other hand, some specific families defined for arbitrary dimension have been known for long, as we shall recall in this subsection.

Generally speaking, given that equations (25) constitute a set of coupled nonlinear partial differential equations, the characterization of solutions of $(24-25)$ has proceeded by means of either suitable ansatzs [29,63,71] or through a diversity of other (sometimes ad hoc) approaches $[28,36,42,45,53,55,63,66,70,76,77,78,79,81,85,89,90,98,119,135,136,141,143]$ which have led to the determination of the general solution for certain particular cases $[13,62,64,111,127]$ that
shall be discussed in this subsection. Roughly speaking, these efforts have led to the progressive determination of certain families of solutions of increasing complexity in two senses: in first place, a dimensional increase, and in second term, an increase in the degree of nonlinearity. Regarding the dimensional increase, the growing complexity of the Jacobi equations (25) as the dimension $n$ increases has determined that the analysis is very often focused on three-dimensional solutions $[13,42,53,55,62,63,64,66,70,71,75,76,81]$, while the characterization of solutions of dimensions four $[58,65,77,151,156]$, five $[94,122,151]$, etc. is significantly less frequent. In addition, some wide families of $n$-dimensional solutions have also been analyzed in the literature $[16,17,36,78,79,85,111,143]$. In parallel, the advances in the investigation of the Jacobi PDEs have led to known solutions of increasing nonlinearity. Starting with the most basic, which are the constant structure matrices (of which the symplectic matrices [57] are just a particular case), we can also speak of linear (or Lie-Poisson) [111,139], affine-linear [16], quadratic $[17,104,116,143]$, and cubic [63] structures, together with some solution families which comprise functions of arbitrary nonlinearity $[13,55,62,64,66,70,71,76,77,78,79,85,127]$. On the other hand, when we focus on the important issue of structure matrices for which the Darboux canonical form has been constructed globally, the number of instances falls significantly $[6,42,70,73,76,77,78,79,83,85,102,103,114,142,153,171]$. Of course, the previous enumerations are not intended to be exhaustive, since this would greatly exceed the scope of this work. More modestly, they just illustrate from a very generic perspective some of the main trends in the constructive investigation of the Jacobi PDEs.

As anticipated, some solution families deserve an explicit mention in this introduction, both because of their special relevance and generality, as well as due to their comparative interest for this work. In some sense, they can be regarded as the most general solution families known (but of course, this does not prevent that other ad hoc instances of structure matrices not comprised in such categories might have been determined in different contexts). Such families of interest are now enumerated and briefly discussed:
(a) Constant skew-symmetric matrices. It is evident in the form of the Jacobi equations that every constant skew-symmetric real matrix is a structure matrix. In spite of being apparent, such solutions are interesting as far as they generalize the symplectic matrices $\mathbb{S}_{n}$ and the structure matrices $\mathcal{J}_{D[n, r]}$ associated with the Darboux canonical form. In addition, constant skew-symmetric matrices have relevant applications in different domains, see for instance $[57,101,102,103,142]$.
(b) Lie-Poisson structure matrices. A logical step after the constant structure matrices is
the search of linear ones. A nice result in this context is that every linear skew-symmetric matrix of entries

$$
\begin{equation*}
J_{i j}(x)=\sum_{k=1}^{n} c_{i j}^{k} x_{k}, \quad i, j=1, \ldots, n \tag{38}
\end{equation*}
$$

is a structure matrix if and only if the $c_{i j}^{k}$ are the structure constants of a Lie algebra [62,109,111,139,171]. This reason, as well as the fact that the structure matrices of the form (38) are due to Lie himself [111], account for the name of Lie-Poisson given to such Poisson structures.
(c) Structure matrices of arbitrary dimension and rank 2. An interesting result is the general form $[13,62,64]$ of structure matrices of rank 2 (at most) and dimension $n \geq 3$ : given a set of $(n-1)$ smooth functions $\mathbb{R}^{n} \rightarrow \mathbb{R}$ denoted by $\left(\eta(x), D_{1}(x), \ldots, D_{n-2}(x)\right)$ then

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \sum_{k_{1}, \ldots, k_{n-2}=1}^{n} \epsilon_{i, j, k_{1}, \ldots, k_{n-2}} \prod_{l=1}^{n-2} \partial_{k_{l}} D_{l}(x), \quad i, j=1, \ldots, n \tag{39}
\end{equation*}
$$

defines a structure matrix $\mathcal{J}(x)$ such that $\operatorname{Rank}(\mathcal{J}(x)) \leq 2$ at every point $x$, where $\epsilon_{i, j, k_{1}, \ldots, k_{n-2}}$ denotes the Levi-Civita symbol. In addition, functions $\left(D_{1}(x), \ldots, D_{n-2}(x)\right)$ are Casimir invariants of $\mathcal{J}$ in (39). Conversely, for every structure matrix of rank at most 2, there exist smooth functions $\left(\eta(x), D_{1}(x), \ldots, D_{n-2}(x)\right)$ such that (39) holds. From the point of view of this thesis, in which the global construction of the Darboux canonical form is of central interest, this form of the solution is to some extent inconvenient because the constancy of the rank is not guaranteed, and therefore the applicability of Darboux' theorem is not ensured. In spite that one of the results of this work is the global Darboux reduction for $n$-dimensional Poisson structures of rank 2 (see Chapter 4) the form (39) is not employed for the establishment of the result due to such reason.
(d) Symplectic structure matrices. It can be seen that a matrix $\mathcal{J}(x)$ determines a symplectic Poisson structure in a manifold $\mathbb{M} \subset \mathbb{R}^{n}$ if and only if its inverse $\mathcal{K}(x)=[\mathcal{J}(x)]^{-1}$ is skew-symmetric and verifies the following system of linear PDEs:

$$
\begin{equation*}
\partial_{k} K_{i j}+\partial_{j} K_{k i}+\partial_{i} K_{j k}=0, \quad i, j, k=1, \ldots, n \tag{40}
\end{equation*}
$$

Precisely, equations (40) are the Jacobi equations for the inverse matrix $\mathcal{K}(x) \equiv\left(K_{i j}(x)\right)$. The general solution of these linear equations is known to be [127] of the form of a "generalized curl"

$$
\begin{equation*}
K_{i j}(x)=\partial_{i} f_{j}(x)-\partial_{j} f_{i}(x), \quad i, j=1, \ldots, n \tag{41}
\end{equation*}
$$

for any given set of smooth functions $\left(f_{1}(x), \ldots, f_{n}(x)\right)$.

The previous considerations may be illustrative in order to provide some hints about the present-day state of the art in what regards to the determination of Poisson structures and their global Darboux analysis. Such issues are, generally speaking, the purpose of this work. It is worth, however, giving more detailed explanations about the subject and aims of this dissertation. This is precisely the goal of the next paragraph.

### 1.3.3. The subject of this work: motivation, methods, aims and applications

As indicated, the study of solution families of the Jacobi equations (24-25) and the construction of global Darboux reductions for them is the generic framework of this thesis. Before giving a more detailed description of such matters, it is interesting to add some background on the motivation underlying this kind of analysis.

In first place, it is clear that the issue of describing a given vector field $\dot{x}=f(x)$ not yet explicitly written in the form (21) in terms of a Poisson structure is a fundamental question in this context. This can be regarded as a sort of "inverse problem" which still remains as an open issue (for instance, see $[22,29,30,42,63,66,81,83,89,90,100,105,119,127,135,136,141,142,143,153]$ and references therein). Expressing a vector function defining a vector field, $f(x): \mathbb{R}^{n} \rightarrow$ $\mathbb{R}^{n}$, in the form $f(x)=\mathcal{J}(x) \cdot \nabla H(x)$, with $\mathcal{J}(x)$ being a structure matrix, is a nontrivial decomposition to which important efforts have been devoted in past years in a variety of approaches. The source of the difficulty is obviously twofold: first, a known constant of motion $H(x)$ of the system able to play the role of the Hamiltonian is required. The problem of constructing first integrals has been extensively investigated in the literature (for instance, see [60] and references therein for a survey on the subject). Apart from the first integral $H(x)$, it is clearly necessary to find a suitable structure matrix for the vector field. Consequently, finding a solution of the identities (25) complying also with the skew-symmetry conditions (24) is unavoidable. Compared to the determination of first integrals (which is a consolidated branch of research) this is a subject still in progress. Therefore, the need for a structure matrix in the decomposition $f(x)=\mathcal{J}(x) \cdot \nabla H(x)$ just mentioned provides a first explanation about the attention deserved in the literature by the obtainment and classification of skewsymmetric solutions of the Jacobi equations (in this sense, see the discussion in Subsection 1.3.2 and references therein). Accordingly, the determination of new solutions of arbitrary dimension will be an important part of this thesis.

In addition, in this context it will be of special interest the characterization of $n$-dimensional solution families (with $n$ arbitrary) leading to Poisson systems for which the Darboux con-
struction can be globally determined. The reason is twofold: in one hand, we have that while three-dimensional Poisson structures have been widely investigated, the number of known families of arbitrary dimension is quite limited (in this sense, recall again the discussion in Subsection 1.3.2). Additionally, we have already seen that (depending on the kind of application) the mere knowledge of a given family of structure matrices may be not sufficient for applied purposes if the Darboux canonical form cannot be globally constructed, thus providing a full connection between the corresponding Poisson system and a classical Hamiltonian formulation. Detailed reasons for this have been exposed in Subsection 1.2.6. In this sense, it is worth emphasizing that the number of $n$-dimensional families for which an explicit and global construction of the Darboux canonical form has been determined is quite limited, as it was detailed in the previous subsection. Consequently, the construction of global Darboux reductions for different Poisson structures plays an important role in what follows, as we shall see. Actually, the strategy of finding and analyzing solution families has the additional benefit of providing a conceptual and operational unification of (seemingly) very different Poisson structures and systems that, when identified as particular cases of such a family, can be regarded from a more elegant and economic standpoint. In fact, the general procedures developed for the entire family now become directly applicable to every member of it, thus avoiding the need of a case-by-case analysis. Various examples of this conceptual and operational unification shall be seen in this work.

Together, the combined problem of determining new solutions of the Jacobi equations (2425) and the global Darboux analysis of those solutions (which includes the determination of its Casimir invariants and thus the construction of the symplectic foliation for the system) constitute a kind of nonlinear analysis to be applied over a set of coupled nonlinear PDEs. Of course, such analysis entails an intrinsic mathematical interest which seems worth being explored.

Therefore, in the next chapters attention will be paid to the combined problem of the determination and the Darboux analysis of different Poisson structures. In this sense, several comments are to be done. In Chapters 2 and 3 the most important conceptual contribution lies in the Darboux analyses because the general solution of the Jacobi equations is already known, as indicated in equations (39) and (41) of Subsection 1.3.2. However, this statement is to some extent misleading because the knowledge of (39) and (41) is not of real usefulness for the purposes of this work, and specifically for the construction of the Darboux canonical form, due to the facts that (i) the constancy of the rank of solutions (39) is not ensured; and
(ii) solution (41) is defined for the matrix inverse of $\mathcal{J}$. This implies a specific interest in the determination of suitable solution families of the Jacobi equations amenable to analysis. Chapter 2 also includes, in Section 2.5, an illustration of a technique allowing a very significant simplification of the Jacobi equations. The presentation of this methodology is justified as far as it will provide the conceptual basis for the analysis of time reparametrizations in Chapter 4. In fact, the different Darboux reductions considered in Chapters 2 and 3 constitute also the guide for the more general developments of Chapter 4 . In this one, it is possible to combine the determination of new solutions with their global analysis. This is done for different families that are considered in detail. D-solutions are an exception to this trend since (due to their remarkable mathematical character) their global Darboux analysis seems to be a very specific issue not possible in general. Nevertheless, D-solutions display very general properties regarding the generation of new Poisson structures from a given one, a subject considered in detail. Chapter 4 also makes use of the philosophy anticipated in Section 2.5 in order to provide additional results, based on time reparametrizations, which allow the generalization of solution families previously determined in the chapter, as well as a better understanding of when and how the Darboux analysis is actually possible. Specifically, some of the most interesting results allowed by such methodology are the general global Darboux reduction for $n$-dimensional Poisson structures of rank 2 (again, the use of (39) is to be excluded for this purpose) and the generalization of the separable, multiseparable and D-solutions. In fact, the perspectives developed from the concept of time reparametrization greatly unify most lines of research considered along the work, which naturally converge on it. Accordingly, the contributions based on the use of NTTs constitute the natural ending of this thesis.

From a purely methodological perspective, some significant issues deserve at least a short explanation, which is briefly presented here. From the point of view of the general structure of the dissertation, it can be emphasized that it basically consists in first place of an Introduction (Chapter 1) devoted to recalling classical and well-known results necessary in order to make this thesis self-contained. This is followed by Chapters 2,3 and 4 , which comprise the original contributions of the work in an order of increasing complexity (and increasing dimension). Finally, Chapter 5 contains the conclusions. It is worth indicating that the different sections of Chapters 2 to 4 of this thesis have been written, as much as possible, in such a way that they are also self-contained and readable independently, once a common terminological and operational basis has been established in the present chapter.

It is necessary to specify that the tensor notation sometimes used in the investigation
of the Jacobi equations (specially in those analyses relying on geometric techniques) will be completely avoided here for the sake of simplicity, given that it is not necessary for the PDE perspective adopted in this work, in which a standard subindex notation is simpler and equally effective. Of course, it is well-known that Poisson structures have been widely investigated on the basis of geometric techniques in which, for instance, the structure matrix is described as a differential form rather than as a skew-symmetric matrix. As indicated, the point of view of the present dissertation is the one of DEs. However, for the interested reader it is worth mentioning some classical and modern references that develop in detail the geometric setting of Poisson structures and systems $[1,2,11,47,126,165,166]$.

In addition, still in a methodological context, it is precise to note that the majority of the calculations performed in this dissertation can be carried out by hand. In this sense, useful references frequently employed are $[3,8,26,158]$. However, many of such calculations have been done (or verified) by means of the symbolic program Mathematica@ (for instance, see [ 15,174$]$ for a reference regarding this working environment).

### 1.4. MISCELLANEOUS BACKGROUND AND TERMINOLOGY

The aim of this section is to provide a brief outline of some useful concepts that will be useful throughout the forthcoming chapters. The motivation for doing this is twofold: in first place, this is convenient in order to achieve a self-contained presentation; and in second term, it also helps in fixing the notation and the terminology to be employed.

The structure of the section is the following. In Subsection 1.4.1 the domain of matrix algebra is considered. Classical results on functional independence are regarded in Subsection 1.4.2. The important concept of equivalence between dynamical systems is the subject of Subsection 1.4.3. To conclude, some definitions of integrability and related topics are presented in Subsection 1.4.4.

### 1.4.1. Some results on matrix algebra

In this work real matrices will be used frequently. Accordingly, we review here some linearalgebraic elementary results regarding such matrices [14,69,123,158,168]. Without exception, all matrices considered in this work are real, and all definitions and properties also refer to real matrices.

A square matrix will be termed regular or invertible if it has maximal rank. Otherwise it will be named singular or degenerate matrix. If a matrix has $n$ rows and $m$ columns it will be termed an $n \times m$ matrix. If a matrix is square with $n$ rows and columns, we shall equivalently say that such matrix is $n \times n$ or that it has order $n$.

Specifically, skew-symmetric matrices play an important role in what is to follow. A square matrix $S$ is termed skew-symmetric if $S^{T}=-S$, with $S^{T}$ denoting the transpose of $S$. The next one is an important property of these matrices:

Theorem 1.4.1.1. (Rank of a skew-symmetric matrix). If $S$ is a real skew-symmetric matrix, then $\operatorname{Rank}(S)$ is an even number.

In spite of its simplicity (the proof is straightforward) this property will play a very significant role in the next chapters, both from a mathematical and from a dynamical perspective.

We now define the direct sum of square matrices. Let $A_{1}, A_{2}, \ldots, A_{k}$ be square matrices of arbitrary orders $n_{1}, n_{2}, \ldots, n_{k}$, respectively. The direct sum of such matrices will be denoted either by $A_{1} \oplus A_{2} \oplus \ldots \oplus A_{k}$ or also by $\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{k}\right)$, and is given by the block-diagonal
matrix:

$$
A_{1} \oplus A_{2} \oplus \ldots \oplus A_{k}=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{k}\right)=\left(\begin{array}{cccc}
A_{1} & \mathbb{O}_{n_{1} \times n_{2}} & \cdots & \mathbb{O}_{n_{1} \times n_{k}} \\
\mathbb{O}_{n_{2} \times n_{1}} & A_{2} & \cdots & \mathbb{O}_{n_{2} \times n_{k}} \\
\vdots & \vdots & \ddots & \vdots \\
\mathbb{O}_{n_{k} \times n_{1}} & \mathbb{O}_{n_{k} \times n_{2}} & \cdots & A_{k}
\end{array}\right)
$$

Sometimes it is convenient to use the expression $A_{1} \overbrace{\oplus \ldots \oplus}^{k} A_{k}$ instead of $A_{1} \oplus A_{2} \oplus \ldots \oplus A_{k}$. A useful property of the direct sum of matrices is that if $A=\operatorname{diag}\left(A_{1}, A_{2}, \ldots, A_{k}\right)$ and $B=$ $\operatorname{diag}\left(B_{1}, B_{2}, \ldots, B_{k}\right)$, where $A_{i}$ and $B_{i}$ have the same orders for all $i=1, \ldots, k$, then for the product we have that $A \cdot B=\operatorname{diag}\left(A_{1} \cdot B_{1}, A_{2} \cdot B_{2}, \ldots, A_{k} \cdot B_{k}\right)$.

Two real square matrices $A$ and $B$ of the same order $n$ are said to be congruent on $\mathbb{R}$ if there exists a regular real matrix $L$ such that $B=L \cdot A \cdot L^{T}$. It can be proved that two real matrices congruent on $\mathbb{R}$ always have the same rank. In addition we have the following result, specific for skew-symmetric matrices [14]:

Theorem 1.4.1.2. (Congruence of skew-symmetric matrices). Let $S$ be a real skewsymmetric matrix of order $n$ and rank $r=2 s$. Then:
(a) Every real matrix congruent on $\mathbb{R}$ with $S$ is also skew-symmetric.
(b) Matrix $S$ is congruent on $\mathbb{R}$ with a canonical matrix $\mathcal{J}_{D[n, r]}$ of the form:

$$
\mathcal{J}_{D[n, r]}=\left(\begin{array}{cc}
0 & 1  \tag{42}\\
-1 & 0
\end{array}\right) \overbrace{\oplus \ldots \oplus}^{s}\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \oplus \mathbb{O}_{1} \overbrace{\oplus \ldots \oplus \mathbb{O}_{1}}^{(n-r)}
$$

(c) Two real skew-symmetric matrices of the same order are congruent on $\mathbb{R}$ if and only if they have the same rank.

The use of the notation $\mathcal{J}_{D[n, r]}$ for the canonical form (42) should be clear after the results discussed in Section 1.2. In particular, subindex $D$ refers to the fact that $\mathcal{J}_{D[n, r]}$ is the Darboux canonical form structure matrix.

### 1.4.2. Functional independence and related topics

It is worth recalling also some classical concepts and results that will be of use in the work, mainly regarding the inverse function theorem and the concept of functional independence
[50,95,124,139]. As usual, we shall say that a real function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ is of class $C^{p}$ if all the partial derivatives up to order $p$ exist and are continuous. The same function is termed smooth or of class $C^{\infty}$ if it is of class $C^{p}$ for all the positive integers $p$. The functions considered throughout the dissertation will be usually defined in open subsets, or in domains (namely open and connected subsets) of $\mathbb{R}^{n}$.

Let us begin by the inverse function theorem. In general, if we have $n$ functions of the form

$$
\left\{\begin{array}{cl}
y_{1}= & f_{1}\left(x_{1}, \ldots, x_{n}\right) \\
\vdots & \vdots \\
y_{n}= & f_{n}\left(x_{1}, \ldots, x_{n}\right)
\end{array}\right.
$$

then the problem consists of studying whether or not to every $\left(y_{1}, \ldots, y_{n}\right)$ there is associated a unique $\left(x_{1}, \ldots, x_{n}\right)=\left(g_{1}\left(y_{1}, \ldots, y_{n}\right), \ldots, g_{n}\left(y_{1}, \ldots, y_{n}\right)\right)$, in such a way that

$$
\left\{\begin{array}{cl}
x_{1}= & g_{1}\left(f_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, f_{n}\left(x_{1}, \ldots, x_{n}\right)\right) \\
\vdots & \vdots \\
x_{n}= & g_{n}\left(f_{1}\left(x_{1}, \ldots, x_{n}\right), \ldots, f_{n}\left(x_{1}, \ldots, x_{n}\right)\right)
\end{array}\right.
$$

and also

$$
\left\{\begin{array}{cl}
y_{1}= & f_{1}\left(g_{1}\left(y_{1}, \ldots, y_{n}\right), \ldots, g_{n}\left(y_{1}, \ldots, y_{n}\right)\right) \\
\vdots & \vdots \\
y_{n}= & f_{n}\left(g_{1}\left(y_{1}, \ldots, y_{n}\right), \ldots, g_{n}\left(y_{1}, \ldots, y_{n}\right)\right)
\end{array}\right.
$$

The classical result in this sense is provided by the inverse function theorem [50,124]:

Theorem 1.4.2.1. (Inverse function theorem). Let $f(x) \equiv\left(f_{1}(x), \ldots, f_{n}(x)\right): \Omega \rightarrow \mathbb{R}^{n}$, be a function of type $C^{1}(\Omega)$, where $\Omega$ is an open subset of $\mathbb{R}^{n}$, and let $x_{0} \in \Omega$ be a point such that the determinant of the associated Jacobian matrix is different from zero at $x_{0}$ :

$$
\left|\frac{\partial\left(f_{1}, \ldots, f_{n}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}\left(x_{0}\right)\right| \neq 0
$$

Then there exists a neighborhood $\Upsilon_{x_{0}}$ of $x_{0}$ in $\Omega$ and an open neighborhood $\Upsilon_{y_{0}}$ of $y_{0} \equiv f\left(x_{0}\right)$, such that $f\left(\Upsilon_{x_{0}}\right)=\Upsilon_{y_{0}}$ and the restriction of $f$ to $\Upsilon_{x_{0}}$ has a $C^{1}$ inverse $f^{-1}(y) \equiv g(y)=$ $\left(g_{1}(y), \ldots, g_{n}(y)\right): \Upsilon_{y_{0}} \rightarrow \Upsilon_{x_{0}}$. In addition, for $y \in \Upsilon_{y_{0}}$ and $x=f^{-1}(y)=g(y)$ we have:

$$
\frac{\partial\left(g_{1}, \ldots, g_{n}\right)}{\partial\left(y_{1}, \ldots, y_{n}\right)}=\left(\frac{\partial\left(f_{1}, \ldots, f_{n}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}\right)^{-1}
$$

Moreover, if $f$ is of class $C^{p}, p \geq 1$, so is $g=f^{-1}$.

It is thus clear that the inverse function theorem is an existence theorem: it guarantees the existence of the inverse function, but it does not provide an explicit expression for it. Actually, such an expression may not exist in terms of elementary functions even in simple cases. In addition it is important to note that, under the hypotheses of the theorem, the transformation leading from $x$ to $f(x)$ is locally invertible, in a neighborhood of the point $x_{0}$ in which the determinant of the Jacobian is not zero. But, in general, such transformation will not be globally invertible, namely on the entire domain of definition of the function $f$. For instance, consider the function:

$$
f\left(x_{1}, x_{2}\right)=\binom{f_{1}\left(x_{1}, x_{2}\right)}{f_{2}\left(x_{1}, x_{2}\right)}=\binom{e^{x_{1}} \cos x_{2}}{e^{x_{1}} \sin x_{2}}
$$

The determinant of the Jacobian is now:

$$
\left|\frac{\partial\left(f_{1}, f_{2}\right)}{\partial\left(x_{1}, x_{2}\right)}\right|=\left|\begin{array}{cc}
e^{x_{1}} \cos x_{2} & -e^{x_{1}} \sin x_{2} \\
e^{x_{1}} \sin x_{2} & e^{x_{1}} \cos x_{2}
\end{array}\right|=e^{x_{1}} \neq 0
$$

for all $\left(x_{1}, x_{2}\right) \in \mathbb{R}^{2}$. Then, according to the inverse function theorem such function is injective in a neighborhood of every point of $\mathbb{R}^{2}$. However, the function is not injective on the entire plane $\mathbb{R}^{2}$ since obviously $f\left(x_{1}, x_{2}\right)=f\left(x_{1}, x_{2}+2 \pi\right)$.

Let us now turn to the notions of functional dependence and independence. These concepts correspond in an intuitive way to the idea of a given function being, at the same time, expressible as a function of one or more other functions, an issue that often appears in practice in a great variety of situations. To be precise, let $f_{1}(x), \ldots, f_{m}(x)$ (with $x=\left(x_{1}, \ldots, x_{n}\right)$ ) be a set of $m$ functions of class $C^{1}$ defined in an open set $\Omega \subset \mathbb{R}^{n}$. It is said that a function $f_{k}$ depends functionally in $\Omega$ on the functions $f_{i}(x)$, for $i=1, \ldots, m, i \neq k$, if there exists a class $C^{1}$ function $\psi$ such that $f_{k}(x)=\psi\left(f_{1}(x), \ldots, f_{k-1}(x), f_{k+1}(x), \ldots, f_{m}(x)\right)$ for every $x \in \Omega$. It is also said that the set of functions $f_{1}(x), \ldots, f_{m}(x)$ are functionally dependent in $\Omega$ if at least one of them depends functionally on the rest. If this is not the case, the functions $f_{1}(x), \ldots, f_{m}(x)$ are termed functionally independent. The following result provides the classical necessary condition for functional dependence [50]:

Theorem 1.4.2.2. (Functional dependence). Let $f_{1}(x), \ldots, f_{m}(x)$ be $m$ functions of class $C^{1}$ in an open set $\Omega \subset \mathbb{R}^{n}$. If $f_{1}(x), \ldots, f_{m}(x)$ are functionally dependent in $\Omega$, then the rank of the Jacobian matrix

$$
\frac{\partial\left(f_{1}, \ldots, f_{m}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}
$$

is strictly less than $m$ at every point $x \in \Omega$.

Notice in particular that in the case $m=n$, Theorem 1.4.2.2 states that in order that $n$ functions $f_{1}(x), \ldots, f_{n}(x)$ of $n$ variables $x=\left(x_{1}, \ldots, x_{n}\right)$ are functionally dependent in $\Omega$ it is necessary that the Jacobian determinant vanishes at every point $x \in \Omega$. From Theorem 1.4.2.2 we also obtain easily the following corollary [50]:

Corollary 1.4.2.3. (Functional independence). Let $f_{1}(x), \ldots, f_{m}(x)$ be $m$ functions of class $C^{1}$ in an open set $\Omega \subset \mathbb{R}^{n}$. If the rank of the Jacobian matrix

$$
\frac{\partial\left(f_{1}, \ldots, f_{m}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}
$$

is equal to $m$ at some point $x_{0} \in \Omega$, then the functions $f_{1}(x), \ldots, f_{m}(x)$ are functionally independent in $\Omega$. In particular, if the determinant of the Jacobian matrix of $n$ functions $f_{1}(x), \ldots, f_{n}(x)$ with respect to $x=\left(x_{1}, \ldots, x_{n}\right)$ is different from zero at some point $x_{0} \in \Omega$, then the functions $f_{1}(x), \ldots, f_{n}(x)$ are functionally independent in $\Omega$.

Additional necessary and sufficient conditions can be found in the literature. However, the previous statements are the most used in practice, and they suffice in order to make this work self-contained.

### 1.4.3. Equivalence between dynamical systems of the ODE type

One issue of great importance in this thesis regards the concept of system equivalence, when applied to the comparison of smooth dynamical systems of the ODE type (as it is the situation in the case of Poisson systems). There is not a unique, systematically applied, answer to this question [21]. However, one of the most commonly accepted is the concept of topological orbital equivalence (TOE) [10,21,96]. The basic notion underlying TOE is that two systems are regarded as equivalent if the phase portrait of one of them can be continuously deformed into the other, possibly retaining the sense of motion (namely the orientation) in the phase space. TOE can be formally expressed in the following way: recall that a homeomorphism is an association of the points $x \in \mathbb{R}^{n}$ and $y \in \mathbb{R}^{n}$ which is an one-to-one, continuous map with a continuous inverse. Namely, we have $f: X \rightarrow Y$ or $y=f(x)$, where $f(x)$ is single-valued and continuous. Moreover, the inverse $x=f^{-1}(y)$ is likewise single-valued and continuous. A set in $X$ is said to be topologically equivalent to a set in $Y$ if the two sets can be mapped into each other by a homeomorphism. This equivalence is too general for dynamic purposes,
and in particular for TOE. A homeomorphism is called a diffeomorphism if both $f$ and $f^{-1}$ are differentiable at all points. In addition, a homeomorphism is orientation-preserving if a right-handed coordinate system in $X$ is mapped into a right-handed system in $Y$. (Recall [46] that a right-handed coordinate system in $\mathbb{R}^{n}$ of basis $\left\{B_{i}=\left(b_{i 1}, \ldots, b_{i n}\right) \mid i=1, \ldots, n\right\}$ expressed in terms of the canonical basis of $\mathbb{R}^{n}$, is the one for which $|B|>0$, where matrix $B$ is defined as $B \equiv\left(b_{i j}\right)$ for $\left.i, j=1, \ldots, n\right)$. If the map is a diffeomorphism, then it is orientation-preserving if the determinant of the Jacobian matrix is everywhere positive:

$$
\left|\frac{\partial\left(f_{1}(x), \ldots, f_{n}(x)\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}\right|>0, \quad \text { everywhere }
$$

If phase portraits (i.e. all orbits) of two given smooth ODE systems $\dot{x}=f(x)$, with $x \in \mathbb{R}^{n}$, and $\dot{y}=g(y)$, also with $y \in \mathbb{R}^{n}$, can be related by an orientation-preserving homeomorphism, then the two systems are TOE. In addition, if the transformation is also a diffeomorphism this kind of equivalence is known as smooth orbital equivalence or SOE [21].

An interesting aspect of system equivalence is the fact that the preservation of the parameter on the trajectories (the time variable) is not required. In other words, for the purpose of equivalence every trajectory is considered as a curve without parametrization but with the orientation induced by the flow. Let us comment further on these issues. Consider two smooth systems of ODEs:

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=f(x), \quad \frac{\mathrm{d} y}{\mathrm{~d} \tau}=g(y)
$$

with both $x \in \mathbb{R}^{n}$ and $y \in \mathbb{R}^{n}$. Then the concept of equivalence means that both systems are TOE (respectively, SOE) if there exists an orientation-preserving homeomorphism (respectively, diffeomorphism) $y=y(x)$ together with a time reparametrization $\mathrm{d} \tau=\eta(x) \mathrm{d} t$, with $\eta(x)>0$, which allow transforming each system into the other one. (See Subsection 1.2.4 for the definition of time reparametrizations, also known as new-time transformations, or NTTs for short).

For the conceptual framework of this thesis, SOE is the definition of system equivalence adopted. In practice, however, the requirement to the diffeomorphism of being orientationpreserving will be often disregarded in what is to follow. The reason is that the properties to be analyzed here in connection with the concept of system equivalence (typically, the determination of Casimir invariants of the global reduction to the Darboux canonical form) are, without exception, not affected by possible changes in the orientation of the flow. This explains that for practical purposes the equivalence between two systems will be often relaxed in this work to the existence of a diffeomorphism and a time reparametrization connecting those systems, with the understanding that inversions of the orientation may be present in
such a case, but they do not alter the scope of the results considered. This explains also that in Subsection 1.2.4 the function $\eta(x)$ defining a time reparametrization was defined to be $C^{\infty}$ and nonvanishing (instead of being strictly positive). In what follows, every time that two systems of ODEs are related, the exact form of the relationship between them will be specified.

### 1.4.4. Integrability of dynamical systems of the ODE type

Integrability is in its own right an important branch of nonlinear science in general, and of Hamiltonian (and Poisson) systems in particular [9,21,60,110,159,161,164,167]. Needless to say, the purpose of this subsection is not to account for the state of the art (such a thing would be impossible here) but merely to fix the terminology as well as to provide a concise review of the main results required for this work.

As in the case of system equivalence, there is not a unique or precise meaning for the term "integrable system". However some definitions and criteria are of very common use, and these will be the ones adopted here.

Given an ODE-type dynamical system $\dot{x}=f(x)$, with $x \in \mathbb{R}^{n}$, a first integral (also called a constant of motion) defined on an open subset $U \subset \mathbb{R}^{n}$ and interval $T \subset \mathbb{R}$ is a $C^{1}$ real-valued function $I(x, t): U \times T \rightarrow \mathbb{R}$ such that its time derivative is zero for all $(x, t) \in U \times T$ :

$$
\frac{\mathrm{d} I(x, t)}{\mathrm{d} t}=\frac{\partial I(x, t)}{\partial t}+\sum_{i=1}^{n} f_{i}(x) \frac{\partial I(x, t)}{\partial x_{i}}=0
$$

Often the functions of interest are time-independent first integrals, namely those of the form $I(x)$. This case is of special dynamical interest because, when present, the system solutions lie on the level sets of the time-independent constants of motion. A first integral is called trivial if $I(x, t)$ is actually a real number, namely it is independent of $x$ and $t$. Clearly, if $I$ is a first integral so is $g(I)$, where $g$ is any $C^{1}$ function. Obviously, $I$ and $g(I)$ are not functionally independent first integrals. In practice, a set of first integrals is useful for the dynamical investigation of the system as far as they are functionally independent.

Commonly, integrability is understood as the existence of sufficiently many first integrals to render the global integration of the differential equations possible. In this context, the term "global" is crucial: the question of local integrability is somehow trivial in the sense that locally a sufficient number of independent first integrals always exists, as the following result shows [60]:

Theorem 1.4.4.1. Let $f$ be $C^{0}$ on an open subset $V \subset(U \times \mathbb{R})$. If the initial value problem $\dot{x}=f(x)$ with $x\left(t_{0}\right)=x_{0}$ has a unique $C^{1}$ solution, then the vector field has $n$ independent first integrals of class $C^{1}$ in a neighborhood of a point $\left(x_{0}, t_{0}\right) \in V$.

Therefore, the problem of finding local first integrals is trivial in the sense that the initial value problem always provides local first integrals that can be built from the unique local solution. Thus, the problem of integrability is generally understood as the problem of finding globally defined first integrals. In what follows, this global character is assumed in connection with integrability, and actually the global investigation of some integrability properties of Poisson systems will be an important subject of this work. As said, different notions of integrability and integrable systems can be found in the literature. We shall review here two of special interest for what is to follow. The first one is valid for general dynamical systems (namely systems that are not necessarily Hamiltonian) [60]:

Definition 1.4.4.2. (Algebraic integrability). A vector field $\dot{x}=f(x)$, with $x \in \mathbb{R}^{n}$, is algebraically integrable if there exist $(n-1)$ independent first integrals $I_{i}$, with $i=1, \ldots,(n-1)$, which are algebraic over $\mathbb{R}$.
(Recall that a real-valued function $I(x)$, with $x \in \mathbb{R}^{n}$, is said to be algebraic over $\mathbb{R}$ if there exist polynomials in $x$ with real coefficients, $v_{0}(x), v_{1}(x), \ldots, v_{k}(x)$, with $k>0$, such that: $\left.v_{0}+v_{1} I+\ldots+v_{k} I^{k}=0\right)$.

A second standard definition, this time valid for classical Hamiltonian systems, is the following one [60]:

Definition 1.4.4.3. (Liouville integrability). A Hamiltonian system of $s$ degrees of freedom $(s=n / 2)$ and of Hamiltonian $H(p, q)$ is Liouville integrable if there exist $s$ independent analytic first integrals $I_{1}=H(p, q), I_{2}, \ldots, I_{s}$, in involution (namely, $\left\{I_{i}, I_{j}\right\}_{c}=0$ for all $i, j=1, \ldots, s)$.

Moreover, if the manifolds defined by the intersection of their level sets

$$
\bigcap_{i=1}^{s}\left\{I_{i}=a_{i} \mid(p, q) \in \mathbb{R}^{n}\right\}
$$

are compact and connected, then a theorem due to Arnol'd [9] states that they are topologically real tori, namely they are homeomorphic to real tori. (In the context of Definition 1.4.4.3,
recall that a real-valued function $I(x)$, with $x \in \mathbb{R}^{n}$, is termed analytic in an open set $\Omega \subset \mathbb{R}^{n}$ if it is smooth in $\Omega$ and its Taylor series expansion at any point $x_{0} \in \Omega$ is convergent to the value of $I(x)$ everywhere in a neighborhood of $x_{0}$ ).

Compared to algebraic integrability, Liouville integrability is dimensionally advantageous. In the case of algebraic integrability, it is intuitively clear that one first integral can be used to reduce the dynamics of an $n$-dimensional system to a system evolving on a manifold of dimension ( $n-1$ ). Nicely, the structure of Hamiltonian systems is such that a first integral in an $n$-dimensional phase space can be used to reduce the dynamics to an $(n-2)$-dimensional phase space. If more first integrals in involution are known, the Liouville reduction can be repeated to lower the dimension of the system. If the number of first integrals in involution is $(s-1)$, then the system can be reduced to an one degree of freedom (hence, integrable) Hamiltonian system. The construction of action-angle variables $[57,110]$ is a well-known reduction of this kind. The condition that the first integrals must be in involution is important for the existence of the canonical coordinates that allow the Liouville integration of the Hamiltonian system. It is interesting to note also that such involution condition implies that the first integrals employed are isolating (the definition of isolating first integral is somehow technical and will not be precise in this work; the interested reader is referred to [110]). For instance, in the generalized case of Poisson systems (to which the concept of Liouville integrability can be extended) the Casimir invariants are not isolating first integrals. Therefore the dimensional reduction obtained from the use of Casimir functions must be based on the idea of algebraic integrability, rather than on the one of Liouville integrability.

Remark 1.4.4.4. In the rest of this work it will be assumed that, if required, the smooth first integrals (Casimir invariants and Hamiltonian functions) considered in connection with the different problems examined, satisfy the functional properties that might be necessary for the application of Definitions 1.4.4.2 and 1.4.4.3. To be precise, it is assumed that:
(a) When the property of algebraic integrability (Definition 1.4.4.2) is discussed for a Poisson system, all the members of the complete set of Casimir invariants under consideration as well as the Hamiltonian, are algebraic functions everywhere in the domain of interest.
(b) When the property of Liouville integrability (Definition 1.4.4.3) is discussed for a Hamiltonian system, its Hamiltonian is an analytic function everywhere in the domain of interest.

## CHAPTER 2.

## THREE-DIMENSIONAL SOLUTIONS

### 2.1. INTRODUCTION

This chapter is devoted to the global analysis of three-dimensional solutions of the Jacobi equations. As we shall see along the work, the justification for this study is mainly twofold. In one hand, this investigation will be the source of new results, mainly the explicit and global characterization of the symplectic structure of the Poisson structures under consideration, which leads to new global and constructive determinations of the Darboux canonical form, a procedure only known for a very limited sample of Poisson structures. Another line of investigation that is regarded in this chapter is the generation of new solutions starting from a known one, a strategy that seems to be very fruitful as far as it leads to a significant simplification of the Jacobi equations. These contributions provide interesting results and methods that will lead to important clues for the investigation of the Jacobi equations in the more general $n$-d framework in Chapter 4. Thus, a second fundamental justification for the investigation of the 3-d scenario (regarding both the global determination of Casimir invariants and of Darboux reductions, as well as the generation of solution families from a known one) is the potential usefulness of these ideas, necessary for the development of generalized methods and results in the $n$-d case.

The structure of the present chapter is the following. In Section 2.2 an investigation of the 3-d Jacobi equations is performed. This leads to the determination of a threefold family of solutions for which the global analysis is possible. Such analysis, including the global characterization of the distinguished invariants and the global construction of the Darboux canonical form, is presented on a case-by-case basis. In Sections 2.3 and 2.4 two additional and formally very similar one each other (but different, namely not equivalent) families of 3-d Poisson structures are also characterized and globally analyzed in detail. To conclude, in Section 2.5 it is proved how the knowledge of a given 3-d solution of the Jacobi equations can
be used as a basis for the determination of an infinite family of additional solutions, in such a way that the problem consisting in solving the Jacobi equations is significantly simplified: this kind of philosophy will be very useful in the $n$-d context, as we shall see. Throughout the different subsections, very diverse applied examples are presented in detail. The chapter ends in Section 2.6 with some conclusions and final comments.

### 2.2. A THREEFOLD THREE-DIMENSIONAL FAMILY

In this section, a systematic investigation of the skew-symmetric solutions of the threedimensional Jacobi equations is presented. As a result, three disjoint and complementary subfamilies of solutions are characterized and globally analyzed [70].

The structure of the section is as follows. Subsection 2.2 .1 provides an introduction in which some notation is fixed. In Subsections 2.2 .2 to 2.2.4, respectively, three different, disjoint and complementary subfamilies of solutions are investigated including their characterization, symplectic properties and global reduction to the Darboux canonical form. To conclude, Subsection 2.2.5 contains some case-by-case instances reported in the literature.

### 2.2.1. Introduction

As indicated, the development of a systematic investigation of the three-dimensional Jacobi equations (24-25) is the purpose of this section. As we shall see, three disjoint categories of solutions of the problem appear naturally. For each of them, a subfamily of solutions is found. Such subfamilies are remarkably general. This explains that many well-known threedimensional Poisson structures and dynamical systems now happen to appear embraced as particular cases of a wider family, as we shall show in detail. Therefore, a first outcome is that of the unification of many different Poisson structures seemingly unrelated. Moreover, this unification is not only conceptual. In fact, the new families are amenable to explicit and detailed analysis, in spite of their generality. In particular, it is possible to develop algorithms for the determination of important properties such as the symplectic structure and the Darboux canonical form. The advantage of these common strategies is that they are simultaneously valid for all the particular cases which can now be analyzed in a unified and very economic way, instead of using a case-by-case approach. In addition, the methods developed are valid globally in phase space, thus improving the usual scope of Darboux' theorem which does only guarantee, in principle, a local reduction. The possibility of constructing the Darboux canonical form is also remarkable in view that the practical determination of the Darboux coordinates is a complicated task in general, which has been carried out only for a very limited sample of systems, as discussed in the previous chapter.

For the sake of conciseness, in what follows we shall use the following notation for the entries of three-dimensional structure matrices:

$$
\begin{equation*}
u(x) \equiv J_{12}(x), \quad v(x) \equiv J_{31}(x), \quad w(x) \equiv J_{23}(x) \tag{43}
\end{equation*}
$$

Now, if in the case $n=3$ we simplify the Jacobi identities (25) with the help of (24) and substitute also definition (43), then the joint system (24-25) takes the form:

$$
\begin{equation*}
u \partial_{1} v-v \partial_{1} u+w \partial_{2} u-u \partial_{2} w+v \partial_{3} w-w \partial_{3} v=0 \tag{44}
\end{equation*}
$$

The three-dimensional version of system (24-25) shall be written in the compact form (44) in the rest of this section.

### 2.2.2. First case: characterization and global Darboux analysis

For the characterization of the first subfamily of solutions, it is convenient to begin with the establishment of an important general property [63,70] of equation (44):

Proposition 2.2.2.1. Let $(u(x), v(x), w(x))$ be a set of $C^{\infty}(\Omega)$ functions solution of equation (44) in a domain $\Omega \subset \mathbb{R}^{3}$, and let $\eta(x): \Omega \rightarrow \mathbb{R}$ be an arbitrary $C^{\infty}(\Omega)$ function. Then $\left(u^{*}(x), v^{*}(x), w^{*}(x)\right)=(\eta(x) u(x), \eta(x) v(x), \eta(x) w(x))$ is also a solution of equation (44).

Proof. After substitution of $\left(u^{*}(x), v^{*}(x), w^{*}(x)\right)=(\eta(x) u(x), \eta(x) v(x), \eta(x) w(x))$ into equation (44) we arrive at:

$$
\begin{gathered}
\eta\left[u \partial_{1}(\eta v)-v \partial_{1}(\eta u)+w \partial_{2}(\eta u)-u \partial_{2}(\eta w)+v \partial_{3}(\eta w)-w \partial_{3}(\eta v)\right]= \\
\eta\left[\eta u \partial_{1} v+u v \partial_{1} \eta-\eta v \partial_{1} u-u v \partial_{1} \eta+\eta w \partial_{2} u+u w \partial_{2} \eta-\right. \\
\left.\eta u \partial_{2} w-u w \partial_{2} \eta+\eta v \partial_{3} w+v w \partial_{3} \eta-\eta w \partial_{3} v-v w \partial_{3} \eta\right]=0
\end{gathered}
$$

This completes the proof.
Q.E.D.

It is important to stress that this proposition is not valid in general in dimensions higher than three, as it can be easily verified [70]. In order to physically interpret the result contained in Proposition 2.2.2.1 it is necessary to first recall the concept of time reparametrization (see Subsection 1.2.4). For the moment, here we provide the following specific definition, valid for the rest of the chapter:

Definition 2.2.2.2. Let $\Omega \subset \mathbb{R}^{3}$ be a domain. A reparametrization of time is defined as a transformation of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{45}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$.

In addition, let

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{46}
\end{equation*}
$$

be an arbitrary three-dimensional Poisson system defined in a domain $\Omega \subset \mathbb{R}^{3}$. Then, every reparametrization of time of the form (45) leads from (46) to the differential system:

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{47}
\end{equation*}
$$

Consequently, in the three-dimensional case reparametrizations (45) preserve the existence of a Poisson structure in the system, this time with structure matrix $\eta \mathcal{J}$ in (47). On the contrary, such transformations in general destroy the Poisson structure in higher dimensions because for a given $\mathcal{J}$ which is a structure matrix, $\eta \mathcal{J}$ is not necessarily a solution of (24-25).

We proceed now to characterize a first subfamily of solutions of equation (44). For this, we shall assume that none of the solution functions $(u(x), v(x), w(x))$ is identically zero (the relaxation of this condition will lead to the other two subfamilies of solutions, as we shall see in Subsections 2.2.3 and 2.2.4).

Definition 2.2.2.3. For every domain $\Omega \subset \mathbb{R}^{3}$, we shall denote by $\Gamma_{[u, v, w]}(\Omega)$ the set of solutions of equation (44) defined in $\Omega$ which are of the form $(u(x), v(x), w(x))$, with $u(x)$, $v(x)$ and $w(x)$ nonvanishing in $\Omega$ and $C^{\infty}(\Omega)$.

We now have the following result:

Theorem 2.2.2.4. Consider the family of functions of the form

$$
\left\{\begin{array}{l}
u(x)=\eta(x) \psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right) \phi_{3}\left(x_{3}\right)  \tag{48}\\
v(x)=\eta(x) \psi_{1}\left(x_{1}\right) \phi_{2}\left(x_{2}\right) \psi_{3}\left(x_{3}\right) \\
w(x)=\eta(x) \phi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right) \psi_{3}\left(x_{3}\right)
\end{array}\right.
$$

defined in a domain $\Omega \subset \mathbb{R}^{3}$, where $\left(\eta, \psi_{i}, \phi_{i}\right), i=1,2,3$, are arbitrary $C^{\infty}(\Omega)$ functions of their respective arguments which do not vanish in $\Omega$. Then the family of functions (48) belongs to $\Gamma_{[u, v, w]}(\Omega)$.

Proof. For solutions belonging to $\Gamma_{[u, v, w]}(\Omega)$, we can equivalently write (44) as:

$$
\begin{equation*}
u^{2} \partial_{1}\left(\frac{v}{u}\right)+w^{2} \partial_{2}\left(\frac{u}{w}\right)+v^{2} \partial_{3}\left(\frac{w}{v}\right)=0 \tag{49}
\end{equation*}
$$

From (49) it is clear that $(u(x), v(x), w(x))$ are solutions if we take:

$$
\begin{align*}
& \frac{v}{u}=\alpha\left(x_{2}, x_{3}\right) \Longrightarrow \begin{cases}u & =u_{1}\left(x_{2}, x_{3}\right) \psi_{1}(x) \\
v & =v_{1}\left(x_{2}, x_{3}\right) \psi_{1}(x)\end{cases}  \tag{50}\\
& \frac{u}{w}=\beta\left(x_{1}, x_{3}\right) \Longrightarrow\left\{\begin{array}{lll}
u & =u_{2}\left(x_{1}, x_{3}\right) \psi_{2}(x) \\
w & = & w_{2}\left(x_{1}, x_{3}\right) \psi_{2}(x)
\end{array}\right.  \tag{51}\\
& \frac{w}{v}=\gamma\left(x_{1}, x_{2}\right) \Longrightarrow\left\{\begin{array}{lll}
v & = & v_{3}\left(x_{1}, x_{2}\right) \psi_{3}(x) \\
w & = & w_{3}\left(x_{1}, x_{2}\right) \psi_{3}(x)
\end{array}\right. \tag{52}
\end{align*}
$$

In (50-52) the functions $\left(\alpha, \beta, \gamma, u_{i}, v_{i}, w_{i}, \psi_{j}\right)$, with $i \in\{1,2,3\}$ and $j=1,2,3$, are $C^{\infty}(\Omega)$ and nonvanishing arbitrary functions of their respective arguments. A family of solutions of equations (50-52) is found if we assume that $\psi_{j}(x) \equiv \psi_{j}\left(x_{j}\right)$ for all $j=1,2,3$. Then, taking also into account Proposition 2.2.2.1 and Definition 2.2.2.3 we arrive at result (48). Q.E.D.

Therefore, we can equivalently state:

Corollary 2.2.2.5. For every domain $\Omega \subset \mathbb{R}^{3}$, solution (48) can be written as:

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \sum_{k=1}^{3} \epsilon_{i j k} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3 \tag{53}
\end{equation*}
$$

where $\left(\eta, \psi_{i}, \phi_{i}\right)$ are arbitrary $C^{\infty}(\Omega)$ functions of their respective arguments which do not vanish in $\Omega$ and $\epsilon$ is the Levi-Civita symbol.

Now it is convenient to introduce the following:

Definition 2.2.2.6. For every domain $\Omega \subset \mathbb{R}^{3}$, the subset of $\Gamma_{[u, v, w]}(\Omega)$ composed of those solutions of equation (44) given in Theorem 2.2.2.4 will be denoted $\Delta(\Omega)$.

As anticipated in Subsection 2.2.1, the generality of solutions (53) is not an obstacle in what regards the characterization of their main properties. We begin by the symplectic structure and the Casimir invariant.

Proposition 2.2.2.7. For every domain $\Omega \subset \mathbb{R}^{3}$, the rank of the structure matrices belonging to $\Delta(\Omega)$ is constant in $\Omega$ and equal to 2, and a Casimir function of the family of solutions (53) forming $\Delta(\Omega)$ is

$$
\begin{equation*}
D(x)=\sum_{i=1}^{3} \int \frac{\phi_{i}\left(x_{i}\right)}{\psi_{i}\left(x_{i}\right)} \mathrm{d} x_{i} \tag{54}
\end{equation*}
$$

Moreover, the Casimir invariant (54) is globally defined in $\Omega$.

Proof. The rank is constant in $\Omega$ and has value 2 as a consequence of the nonvanishing properties of functions $\left(\eta, \psi_{i}, \phi_{i}\right)$. In addition, according to the Pfaffian method for the determination of Casimir invariants (see Appendix 1 and references therein), which is the simplest in this case, the Casimir function is found to be the solution of the system

$$
\sum_{i=1}^{3} \frac{\phi_{i}\left(x_{i}\right)}{\psi_{i}\left(x_{i}\right)} \mathrm{d} x_{i}=0
$$

The integration is immediate and leads to (54). The remaining properties of the Casimir invariant also arise from those of functions $\phi_{i}$ and $\psi_{i}$.

It is interesting to note that $\eta(x)$ does not affect neither the symplectic structure nor the form of the Casimir invariant. This is to be expected from the fact that it is a common factor of the structure functions. An additional consequence of Proposition 2.2.2.7 is:

Corollary 2.2.2.8. Consider that the assumptions of Remark 1.4.4.4 hold. Every threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ and such that the structure matrix $\mathcal{J}(x)$ belongs to $\Delta(\Omega)$, is an algebraically integrable system in $\Omega$.

We proceed now to construct globally the Darboux canonical form.

Theorem 2.2.2.9. For every three-dimensional Poisson system

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x)
$$

defined in a domain $\Omega \subset \mathbb{R}^{3}$ and such that $\mathcal{J} \in \Delta(\Omega)$, the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\left(z_{1}, z_{2}, z_{3}\right)$ and the new time $\tau$, where
$\left(z_{1}, z_{2}, z_{3}\right)$ is related to $\left(x_{1}, x_{2}, x_{3}\right)$ by the diffeomorphism globally defined in $\Omega$

$$
\left\{\begin{aligned}
z_{1}\left(x_{1}\right) & =\int \frac{\phi_{1}\left(x_{1}\right)}{\psi_{1}\left(x_{1}\right)} \mathrm{d} x_{1} \\
z_{2}\left(x_{2}\right) & =\int \frac{\phi_{2}\left(x_{2}\right)}{\psi_{2}\left(x_{2}\right)} \mathrm{d} x_{2} \\
z_{3}(x) & =\sum_{i=1}^{3} \int \frac{\phi_{i}\left(x_{i}\right)}{\psi_{i}\left(x_{i}\right)} \mathrm{d} x_{i}
\end{aligned}\right.
$$

and the new time $\tau$ is given by a time reparametrization of the form:

$$
\mathrm{d} \tau=\eta(x(z)) \phi_{1}\left(x_{1}(z)\right) \phi_{2}\left(x_{2}(z)\right) \phi_{3}\left(x_{3}(z)\right) \mathrm{d} t
$$

Proof. We begin by noticing that Darboux' theorem (Theorem 1.2.3.3) is applicable to family (53) because its members have constant rank 2 everywhere in $\Omega$, as seen in Proposition 2.2.2.7. This is a key necessary condition which is verified in the case of $\Delta(\Omega)$. Recall also that, after a general smooth change of coordinates $y=y(x)$, a given structure matrix $\mathcal{J}(x)$ is transformed into another one $\mathcal{J}^{*}(y)$ according to the rule (30):

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{55}
\end{equation*}
$$

The reduction can be carried out in three steps:

Step 1. We perform a first change of variables, which is globally diffeomorphic in $\Omega$ :

$$
\begin{equation*}
y_{i}\left(x_{i}\right)=\int \frac{\phi_{i}\left(x_{i}\right)}{\psi_{i}\left(x_{i}\right)} \mathrm{d} x_{i}, \quad i=1,2,3 \tag{56}
\end{equation*}
$$

The diffeomorphic character of $(56)$ is a direct consequence of the fact that both $y_{i}\left(x_{i}\right)$ and its inverse $x_{i}\left(y_{i}\right)$ are differentiable for every $i=1,2,3$, and in addition we have $y_{i}^{\prime}\left(x_{i}\right) \neq 0$ and $x_{i}^{\prime}\left(y_{i}\right) \neq 0$ everywhere. According to (55) we arrive at:

$$
\mathcal{J}^{*}(y)=\tilde{\eta}(y)\left(\begin{array}{ccc}
0 & 1 & -1  \tag{57}\\
-1 & 0 & 1 \\
1 & -1 & 0
\end{array}\right)
$$

where $\tilde{\eta}(y)=\eta(x(y)) \phi_{1}\left(x_{1}\left(y_{1}\right)\right) \phi_{2}\left(x_{2}\left(y_{2}\right)\right) \phi_{3}\left(x_{3}\left(y_{3}\right)\right)$.

Step 2. We can make use of the Casimir $D(y)=y_{1}+y_{2}+y_{3}$ of $\mathcal{J}^{*}(y)$ in (57) and perform a second change of variables globally diffeomorphic in $\Omega^{*}=y(\Omega) \subset \mathbb{R}^{3}$ :

$$
\left\{\begin{align*}
z_{1} & =y_{1}  \tag{58}\\
z_{2} & =y_{2} \\
z_{3} & =y_{1}+y_{2}+y_{3}
\end{align*}\right.
$$

In the case of (58) the diffeomorphic nature of the transformation in $\mathbb{R}^{3}$ is clear since it is a linear and invertible transformation. The new structure matrix can be found again by means of (55):

$$
\mathcal{J}^{* *}(z)=\hat{\eta}(z)\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

where $\hat{\eta}(z)=\tilde{\eta}(y(z))$.
Step 3. Finally, we can carry out a reparametrization of time of the form (45), namely d $\tau=$ $\hat{\eta}(z) \mathrm{d} t$, where $\tau$ is the new time and $\hat{\eta}(z)$ is easily seen to be nonvanishing in $\Omega^{* *}=$ $z(y(\Omega))$ and $C^{\infty}\left(\Omega^{* *}\right)$. The result is, according to Proposition 2.2.2.1 and (46-47), a new Poisson system with structure matrix

$$
\mathcal{J}_{D[3,2]}(z)=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{59}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

and time $\tau$. Consequently, the structure matrix $\mathcal{J}_{D[3,2]}$ in (59) is already the one corresponding to the Darboux canonical form.

The reduction is thus globally completed.
Q.E.D.

According to the remarks made in Section 1.2 in connection with Darboux' theorem, it is worth noting an interesting corollary of Theorem 2.2.2.4, namely that all the diverse instances shown in Subsection 2.2 .5 can actually be seen as the global representation of the same basic Poisson structure (namely the Darboux one) in different systems of coordinates. This is obviously a consequence of the transformation rule (55). However, in the case of Theorem 2.2.2.9 this equivalence is proved globally in $\Omega$, thus exceeding the usual scope of Darboux' theorem. Notice also how this is founded on the fact that the rank of the structure matrix remains constant in $\Omega$, which is ensured by the nonvanishing conditions verified by the structure functions. Consideration of a possible variability in the value of the rank would
lead to additional geometric issues $[109,114,171]$ not regarded in this work for the sake of conciseness. In addition, the global Darboux reduction implies the following:

Corollary 2.2.2.10. Consider that the assumptions of Remark 1.4.4.4 hold. Every threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ and such that the structure matrix $\mathcal{J}(x)$ belongs to $\Delta(\Omega)$, can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

The description of the first subfamily of solutions is thus completed. We now proceed to examine the second possibility.

### 2.2.3. Second case: characterization and global Darboux analysis

The second subfamily of solutions arises when we consider the case in which one of the structure functions $(u, v, w)$ is identically zero, while the remaining two are not.

Definition 2.2.3.1. For every domain $\Omega \subset \mathbb{R}^{3}$, we shall denote by $\Gamma_{[v, w]}(\Omega), \Gamma_{[u, w]}(\Omega)$ and $\Gamma_{[u, v]}(\Omega)$ the sets of solutions $(u, v, w)$ of equation (44) defined in $\Omega$ which are of the forms $(0, v(x), w(x)),(u(x), 0, w(x))$ and $(u(x), v(x), 0)$, respectively, where $u(x), v(x)$ and $w(x)$ are, when present, $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$.

Then, for the present context we have the following result:

Theorem 2.2.3.2. For every domain $\Omega \subset \mathbb{R}^{3}$, the general solutions of equation (44) corresponding to $\Gamma_{[v, w]}(\Omega), \Gamma_{[u, w]}(\Omega)$ and $\Gamma_{[u, v]}(\Omega)$ are, respectively,

$$
\begin{align*}
\Gamma_{[v, w]}(\Omega) & \Rightarrow \quad\left(u=0, v=\eta(x), w=\eta(x) \xi\left(x_{1}, x_{2}\right)\right)  \tag{60}\\
\Gamma_{[u, w]}(\Omega) & \Rightarrow \quad\left(v=0, w=\eta(x), u=\eta(x) \zeta\left(x_{1}, x_{3}\right)\right)  \tag{61}\\
\Gamma_{[u, v]}(\Omega) & \Rightarrow \quad\left(w=0, u=\eta(x), v=\eta(x) \chi\left(x_{2}, x_{3}\right)\right) \tag{62}
\end{align*}
$$

where functions $(\eta, \xi, \zeta, \chi)$ appearing in (60-62) are arbitrary, $C^{\infty}(\Omega)$ with regard to their respective arguments and nonvanishing in $\Omega$.

Proof. It is immediate after substitution in equation (44).
Q.E.D.

Accordingly, for example in the case $u=0$ we have found structure matrices of the form

$$
\mathcal{J}=\eta(x)\left(\begin{array}{ccc}
0 & 0 & -1  \tag{63}\\
0 & 0 & \xi\left(x_{1}, x_{2}\right) \\
1 & -\xi\left(x_{1}, x_{2}\right) & 0
\end{array}\right)
$$

where $\eta(x)$ and $\xi\left(x_{1}, x_{2}\right)$ are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. As it can be seen, the overall factor considered in Proposition 2.2.2.1 already appears explicitly in (63) and needs not be added a posteriori.

Following the same scheme than in the previous subsection, we now proceed to develop the main properties of the solutions just found. For the sake of conciseness this shall be done only for the case $\Gamma_{[v, w]}(\Omega)$, given that all the corresponding algorithms and results are entirely analogous for $\Gamma_{[u, w]}(\Omega)$ and $\Gamma_{[u, v]}(\Omega)$.

We shall begin with the symplectic structure and Casimir invariants. Again, the Pfaffian method (see Appendix 1) seems to be the simplest one in order to characterize these properties. From (63) the Pfaffian system to be solved is easily seen to be $\xi\left(x_{1}, x_{2}\right) \mathrm{d} x_{1}+\mathrm{d} x_{2}=0$. Clearly, this equation cannot be solved without some additional information because it is very generic. In order to restrict the problem, it is worth introducing an additional condition:

Definition 2.2.3.3. Let $\Omega \subset \mathbb{R}^{2}$ be a domain and let $\xi: \Omega \rightarrow \mathbb{R}$ be a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$. We shall say that $\xi\left(x_{1}, x_{2}\right)$ is separable in $\Omega$ if it can be written in the form

$$
\begin{equation*}
\xi\left(x_{1}, x_{2}\right)=\frac{\xi_{1}\left(x_{1}\right)}{\xi_{2}\left(x_{2}\right)} \tag{64}
\end{equation*}
$$

for all $\left(x_{1}, x_{2}\right) \in \Omega$, where $\xi_{1}\left(x_{1}\right)$ and $\xi_{2}\left(x_{2}\right)$ are $C^{\infty}(\Omega)$ and do not vanish in $\Omega$.

Now note that all specific systems found in practice (see Subsection 2.2 .5 ) verify the property that $\xi\left(x_{1}, x_{2}\right)$ is separable (notice that the only exception is the case of 3 -d systems with a known first integral [81], but this is not a specific system but a generic situation which does not correspond to any particular vector field, and therefore it does not affect the generality of (64)). An analogous property is verified for all examples of functions $\zeta\left(x_{1}, x_{3}\right)$ and $\chi\left(x_{2}, x_{3}\right)$. Consequently, it seems well justified to conclude that, typically, $\xi, \zeta$ and $\chi$ will be separable in the form indicated in Definition 2.2.3.3.

Proposition 2.2.3.4. For every domain $\Omega \subset \mathbb{R}^{3}$, if a solution of the form (60) belonging to $\Gamma_{[v, w]}(\Omega)$ has a function $\xi\left(x_{1}, x_{2}\right)$ which is separable in $\Omega$ according to (64), then the rank of such structure matrix is constant in $\Omega$ and has value 2, and a Casimir function of the Poisson structure is

$$
\begin{equation*}
D\left(x_{1}, x_{2}\right)=\int \xi_{1}\left(x_{1}\right) \mathrm{d} x_{1}+\int \xi_{2}\left(x_{2}\right) \mathrm{d} x_{2} \tag{65}
\end{equation*}
$$

In addition, the Casimir invariant (65) is globally defined in $\Omega$.

Proof. The rank is constant and of value 2 in $\Omega$ due to the nonvanishing properties of $\eta, \xi_{1}$ and $\xi_{2}$. Additionally, taking (64) into account the Pfaffian system to be solved (see Appendix 1) becomes $\xi_{1}\left(x_{1}\right) \mathrm{d} x_{1}+\xi_{2}\left(x_{2}\right) \mathrm{d} x_{2}=0$. This leads to the Casimir function immediately. The remaining properties of the Casimir invariant are a direct consequence of those of $\xi_{1}$ and $\xi_{2}$. Q.E.D.

Thus, only the knowledge of the Casimir invariant (65) leads to:

Corollary 2.2.3.5. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider a threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (60) belonging to $\Gamma_{[v, w]}(\Omega)$ and has a function $\xi\left(x_{1}, x_{2}\right)$ which is separable in $\Omega$ according to (64). Then such Poisson system is algebraically integrable in $\Omega$.

In addition, the Darboux canonical form can also be computed under similar assumptions:

Theorem 2.2.3.6. For every three-dimensional Poisson system

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x)
$$

defined in a domain $\Omega \subset \mathbb{R}^{3}$ and such that $\mathcal{J} \in \Gamma_{[v, w]}(\Omega)$ is given by (60) and $\xi\left(x_{1}, x_{2}\right)$ in (60) is separable in $\Omega$ according to (64), the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\left(y_{1}, y_{2}, y_{3}\right)$ and the new time $\tau$, where $\left(y_{1}, y_{2}, y_{3}\right)$ is related to $\left(x_{1}, x_{2}, x_{3}\right)$ by the diffeomorphism globally defined in $\Omega$

$$
\left\{\begin{array}{l}
y_{1}=\int \xi_{1}\left(x_{1}\right) \mathrm{d} x_{1}+\int \xi_{2}\left(x_{2}\right) \mathrm{d} x_{2}  \tag{66}\\
y_{2}=x_{2} \\
y_{3}=x_{3}
\end{array}\right.
$$

and the new time $\tau$ is given by a time reparametrization of the form:

$$
\mathrm{d} \tau=\eta(x(y)) \frac{\xi_{1}\left(x_{1}(y)\right)}{\xi_{2}\left(y_{2}\right)} \mathrm{d} t
$$

Proof. Notice first that Darboux' theorem is applicable in this case because solutions of $\Gamma_{[v, w]}(\Omega)$ of the form (63-64) have constant rank 2 everywhere in $\Omega$, as anticipated in Proposition 2.2.3.4. The reduction can be carried out in two steps:

Step 1. The change of variables (66), which is globally diffeomorphic in $\Omega$, is introduced. To see that it is actually a global diffeomorphism, notice that it is globally one-to-one as a consequence that $\xi_{1}\left(x_{1}\right) \neq 0$ everywhere, and in addition the transformation is globally onto (since $\Omega^{*}=y(\Omega)$ by definition). Thus transformation (66) is a global bijection: therefore its global inverse exists and is unique, and moreover both are differentiable everywhere because transformation (66) is smooth and its Jacobian is regular everywhere in $\Omega$. Notice that (66) is not the only possibility but it would be similar, for instance, to choose $\left(y_{1}=x_{1}, y_{2}=D\left(x_{1}, x_{2}\right), y_{3}=x_{3}\right)$. From (55), (63), (64) and (66) we are led to:

$$
\mathcal{J}^{*}(y)=\tilde{\eta}(y)\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{array}\right)
$$

where $\tilde{\eta}(y)=\eta(x(y)) \xi(x(y))=\eta(x(y)) \xi_{1}\left(x_{1}(y)\right) / \xi_{2}\left(y_{2}\right)$.

Step 2. A reparametrization of time of the kind (45), i.e. $\mathrm{d} \tau=\tilde{\eta}(y) \mathrm{d} t$, where $\tau$ is the new time and $\tilde{\eta}(y)$ is clearly nonvanishing in $\Omega^{*}=y(\Omega)$ and $C^{\infty}\left(\Omega^{*}\right)$. The resulting structure matrix is

$$
\tilde{\mathcal{J}}_{D[3,2]}(y)=\left(\begin{array}{ccc}
0 & 0 & 0  \tag{67}\\
0 & 0 & 1 \\
0 & -1 & 0
\end{array}\right)
$$

Since $\tilde{\mathcal{J}}_{D[3,2]}$ in (67) corresponds to the Darboux canonical form (given that it coincides with $\mathcal{J}_{D[3,2]}$ apart from a simple reordering of the variables) we have that the reduction has been accomplished globally.
Q.E.D.

This reduction implies also:

Corollary 2.2.3.7. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider a threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (60) belonging to $\Gamma_{[v, w]}(\Omega)$ and has a function $\xi\left(x_{1}, x_{2}\right)$ which is separable in $\Omega$ according to (64). Then such Poisson system can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

The present analysis of the solutions of (44) can now be completed without difficulty. This is the aim of the next subsection.

### 2.2.4. Third case: characterization and global Darboux analysis

Following the previous considerations, the last possibility is to look for solutions of (44) such that two of the three functions $(u, v, w)$ are identically zero, while the remaining one is not.

Definition 2.2.4.1. The sets of solutions $(u, v, w)$ of equation (44) defined in a domain $\Omega \subset \mathbb{R}^{3}$ which are of the forms $(u(x), 0,0),(0, v(x), 0)$ and $(0,0, w(x))$, where $u(x), v(x)$ and $w(x)$ are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$, will be denoted $\Gamma_{[u]}(\Omega), \Gamma_{[v]}(\Omega)$ and $\Gamma_{[w]}(\Omega)$, respectively.

Since all the results which are going to be examined are completely analogous for $\Gamma_{[u]}(\Omega)$, $\Gamma_{[v]}(\Omega)$ and $\Gamma_{[w]}(\Omega)$, we shall concentrate without lack of generality on the analysis of $\Gamma_{[w]}(\Omega)$.

Theorem 2.2.4.2. For every domain $\Omega \subset \mathbb{R}^{3}$, the general solution of equation (44) corresponding to $\Gamma_{[w]}(\Omega)$ consists of the sets of functions of the form $(u=0, v=0, w(x))$, where $w(x)$ is an arbitrary function of class $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. Analogous results hold for $\Gamma_{[u]}(\Omega)$ and $\Gamma_{[v]}(\Omega)$.

Proof. It is immediate from equation (44).
Q.E.D.

Accordingly, for example in the case of $\Gamma_{[w]}(\Omega)$ we have arrived to solutions of the form

$$
\mathcal{J}(x)=\eta(x)\left(\begin{array}{ccc}
0 & 0 & 0  \tag{68}\\
0 & 0 & 1 \\
0 & -1 & 0
\end{array}\right)
$$

with $\eta(x)$ a function $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. Notice that the multiplication by a global factor considered in Proposition 2.2.2.1 needs not be taken into account here, since it is already explicit in (68). Note also that solutions described by Theorem 2.2.4.2 correspond to structure matrices which are just time reparametrizations of the Darboux canonical form (in particular, the Casimir invariant $D(x)=x_{1}$ appears as already decoupled, and therefore it is explicit). Consequently, this kind of solutions is quite simple and is only considered here for the sake of completeness: the analysis of features such as the Casimir invariants (which are decoupled) or the construction of the Darboux canonical form (which amounts to performing a time reparametrization) becomes a straightforward version of those considered in Subsections 2.2.2 and 2.2.3, and can therefore be omitted. To conclude, we also have the following statement about integrability issues:

Corollary 2.2.4.3. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider a threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of one of the three kinds $\Gamma_{[u]}(\Omega), \Gamma_{[v]}(\Omega)$ or $\Gamma_{[w]}(\Omega)$ characterized in Theorem 2.2.4.2. Then such Poisson system is algebraically integrable in $\Omega$, and it can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

We conclude this section by regarding to some instances of Poisson structures of the three kinds just analyzed.

### 2.2.5. Case-by-case examples

The number of Poisson structures embraced by the previous three subfamilies is very significant. For this reason, we shall not attempt to provide instances of how the constructive procedures developed work (namely the determination of Casimir invariants or the reduction to the Darboux canonical form) since in this case it seems more illustrative of the generality of the results achieved to provide a brief enumeration of the very diverse Poisson structures and systems that hereafter appear to be unified in the common methodological and conceptual framework established after the solution subfamilies investigated.

In first place, it can be said that the family of solutions $\Delta(\Omega)$ is very general, therefore containing numerous previously known structure matrices of very diverse three-dimensional systems as particular cases, as we shall see in detail now. Of special relevance are the Lie-Poisson structure matrix associated with the Lie algebra so(3) (for which $\psi_{i}\left(x_{i}\right)=1, \phi_{i}\left(x_{i}\right)=x_{i}$ and
$\eta=1$, such as in the Euler top [139]) as well as the separable matrices [85] $\left(\phi_{i}\left(x_{i}\right)=\right.$ constant, $\eta=1$ ). Among other instances of well-known Poisson structures described by the solution subfamily (48) just analyzed, we can mention the Kermack-McKendrick model for epidemics [63,135], several integrable cases of the Lorenz system [55], the Lotka-Volterra equations from population dynamics [63,136,143], their QP (or Quasi-Polynomial) Poisson generalization [83], the Maxwell-Bloch equations [42], different formulations of the Rabinovich system [55], the RTW interaction equations [55], or the spin system [114]. A detailed list analyzing these examples can be found in [70] (see Table I of such reference). It is worth recalling that the time dependence of some of the structure matrices enumerated in the previous list of examples is immaterial in this context, since the Jacobi equations are time-independent and therefore time plays the only role of a parameter in the solutions.

Analogously, numerous well-known systems from diverse fields have structures matrices which are particular cases of $(60),(61)$ or $(62)$. Let us cite a sample of each kind.

As particular cases of $\Gamma_{[v, w]}(\Omega)$, we can mention the Poisson structure for the system of circle maps [63], the May-Leonard equations [63], several formulations of the Rabinovich system [55], or the Poisson structure for 3-d systems with a known first integral [81].

Regarding $\Gamma_{[u, w]}(\Omega)$, some instances are found in connection with the Kermack-McKendrick model for epidemics $[63,135]$, some integrable cases of the Lorenz system $[55,63]$ and of the Maxwell-Bloch equations [42,63], and the two energy level system equations [63].

And in connection with $\Gamma_{[u, v]}(\Omega)$, we have instances in the context of the system of circle maps [63], the Lorenz system [63], the Maxwell-Bloch equations [42], the May-Leonard system in population dynamics [63] or the Rabinovich equations [55].

Detailed tables analyzing these three families of examples can be found in [70], specifically Table II for $\Gamma_{[v, w]}(\Omega)$, Table III for $\Gamma_{[u, w]}(\Omega)$ and Table IV for $\Gamma_{[u, v]}(\Omega)$. The reader is referred to such tables for further details regarding the functional structure of the previous examples.

To conclude, it can be said that in spite of their simplicity, examples of Poisson structures corresponding to $\Gamma_{[u]}(\Omega), \Gamma_{[v]}(\Omega)$ or $\Gamma_{[w]}(\Omega)$ are not uncommon in the literature [42,70]. In addition, it is worth mentioning that there is an important category of particular cases of (68) which are present in diverse applications, namely the Lie-Poisson structure matrices associated with the Lie algebra so(3) when expressed in certain systems of noncartesian coordinates
$\left(y_{1}, y_{2}, y_{3}\right)$. The simplest possibility is perhaps that of spherical coordinates [139]:

$$
\mathcal{J}_{s o(3)}\left(y_{1}, y_{2}, y_{3}\right)=-\frac{1}{y_{1} \sin y_{3}}\left(\begin{array}{ccc}
0 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{array}\right)
$$

Additional instances of (68) arising from the Lie algebra so(3) for other choices of the coordinate system are also of customary use [114].

### 2.3. CYCLIC SOLUTIONS OF TYPE I

In this section, an additional three-dimensional family of solutions of the Jacobi equations is characterized and globally analyzed [76]. In particular, it is worth noting that in spite of its general form (defined in terms of functions of an arbitrary nonlinearity) it is possible the explicit and global determination of its main features, such as the case-classification of the Casimir invariants and the global construction of the Darboux canonical form. As we shall see, the analysis of this family has a special interest arising from the fact that a complete understanding of the global analysis of these solutions implies a case classification such that a different distinguished invariant is to be used in each case. In other words, the mere knowledge of one nontrivial distinguished invariant now does not guarantee the reduction to the Darboux canonical form, in spite that we are dealing with a structure matrix of constant rank 2. Accordingly, the symplectic analysis of this kind of Poisson structures requires the use of a case-dependent global Casimir invariant and subsequent global Darboux reduction. Such classification is carried out in the present section.

The structure of the section is the following. In Subsection 2.3.1 the solution family is characterized. The classification of the Casimir invariants and the associated Darboux reductions are developed in Subsection 2.3.2. The analysis concludes in Subsection 2.3.3 with some examples.

### 2.3.1. Characterization of the family

The first result to be presented is the following one:

Theorem 2.3.1.1. Let $\left(\eta(x), \phi_{1}\left(x_{1}\right), \phi_{2}\left(x_{2}\right), \phi_{3}\left(x_{3}\right)\right)$ be a set of functions defined in a domain $\Omega \subset \mathbb{R}^{3}$, all of which are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. In addition, let $\kappa_{i j}, i, j=1,2,3$, be arbitrary real constants that are skew-symmetric

$$
\begin{equation*}
\kappa_{i j}+\kappa_{j i}=0, \text { for all } i, j \tag{69}
\end{equation*}
$$

and satisfy the zero-sum condition

$$
\begin{equation*}
\kappa_{12}+\kappa_{23}+\kappa_{31}=0 \tag{70}
\end{equation*}
$$

Then $\mathcal{J}=\left(J_{i j}\right)$ is a family of 3-d structure matrices which are globally defined in $\Omega$, with

$$
\begin{equation*}
J_{i j}(x)=\eta(x)\left(\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}\right) \sum_{k=1}^{3}\left(\epsilon_{i j k}\right)^{2} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3 \tag{71}
\end{equation*}
$$

where $\epsilon_{i j k}$ is the Levi-Civita symbol, and for every $i=1,2,3$, function $\psi_{i}\left(x_{i}\right)$ denotes one of the primitive functions of $\phi_{i}\left(x_{i}\right)$.

Proof. Skew-symmetry is already explicit in expression (71). Recall that for $n=3$, system (24-25) actually consists of the following independent nonlinear equation:

$$
\begin{equation*}
J_{12} \partial_{1} J_{31}-J_{31} \partial_{1} J_{12}+J_{23} \partial_{2} J_{12}-J_{12} \partial_{2} J_{23}+J_{31} \partial_{3} J_{23}-J_{23} \partial_{3} J_{31}=0 \tag{72}
\end{equation*}
$$

Consider first family (71) in the particular case $\eta(x)=1$. For this, let $\hat{J}_{i j}(x) \equiv J_{i j}(x) / \eta(x)$ in (71). Then, substitution in (72) produces after some algebra:

$$
\begin{gathered}
\hat{J}_{12} \partial_{1} \hat{J}_{31}-\hat{J}_{31} \partial_{1} \hat{J}_{12}+\hat{J}_{23} \partial_{2} \hat{J}_{12}-\hat{J}_{12} \partial_{2} \hat{J}_{23}+\hat{J}_{31} \partial_{3} \hat{J}_{23}-\hat{J}_{23} \partial_{3} \hat{J}_{31}= \\
\left(\psi_{1}-\psi_{2}+\kappa_{12}\right) \phi_{3}\left(-\phi_{1}\right) \phi_{2}-\left(\psi_{3}-\psi_{1}+\kappa_{31}\right) \phi_{2} \phi_{1} \phi_{3}+\left(\psi_{2}-\psi_{3}+\kappa_{23}\right) \phi_{1}\left(-\phi_{2}\right) \phi_{3}- \\
\left(\psi_{1}-\psi_{2}+\kappa_{12}\right) \phi_{3} \phi_{2} \phi_{1}+\left(\psi_{3}-\psi_{1}+\kappa_{31}\right) \phi_{2}\left(-\phi_{3}\right) \phi_{1}-\left(\psi_{2}-\psi_{3}+\kappa_{23}\right) \phi_{1} \phi_{3} \phi_{2}= \\
-2 \phi_{1} \phi_{2} \phi_{3}\left(\kappa_{12}+\kappa_{23}+\kappa_{31}\right)=0
\end{gathered}
$$

This proves the result for the case $\eta=1$. For general $\eta$ it suffices to recall Proposition 2.2.2.1 in which it is shown that in the $3-\mathrm{d}$ case $\eta \mathcal{J}$ is a structure matrix for every arbitrary nonvanishing $C^{\infty}$ function $\eta(x)$ and for every structure matrix $\mathcal{J}$. The proof is thus complete. Q.E.D.

Now some remarks are in order. In first place, it is useful for what is to follow to give the explicit form of the components of $\mathcal{J}$ for family (71), which are:

$$
\left\{\begin{array}{l}
J_{12}(x)=\eta(x)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}\right) \phi_{3}\left(x_{3}\right) \\
J_{23}(x)=\eta(x)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)+\kappa_{23}\right) \phi_{1}\left(x_{1}\right) \\
J_{31}(x)=\eta(x)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)+\kappa_{31}\right) \phi_{2}\left(x_{2}\right)
\end{array}\right.
$$

As indicated in Theorem 2.3.1.1, for every $i$ the primitive $\psi_{i}\left(x_{i}\right)$ of $\phi_{i}\left(x_{i}\right)$ must be chosen to be one and the same for all the entries of $\mathcal{J}$. However, the specific choice is actually arbitrary. To see this it suffices to notice that if a different integration constant is selected, for instance after replacing $\psi_{i}\left(x_{i}\right)$ by $\psi_{i}\left(x_{i}\right)+k_{i}$ for every $i$, then the outcome is also a member of the solution family, this time with constants $\tilde{\kappa}_{i j}=\kappa_{i j}+k_{i}-k_{j}$, which also verify (69-70). Thus conditions (69-70) express in a generalized form this degree of freedom associated with the choice of the primitives of functions $\phi_{i}\left(x_{i}\right)$.

Secondly, notice that the form of the Poisson structures we are dealing with is such that only two possibilities exist regarding the vanishing of the independent entries $\left(J_{12}, J_{23}, J_{31}\right)$ at
a given point, namely: (i) either none or one of them vanishes (case of rank two), or (ii) all of them vanish (case of zero rank). To see this, it is convenient to define the functions:

$$
\chi_{i j}\left(x_{i}, x_{j}\right) \equiv \psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}, \quad i, j=1,2,3
$$

Thus it is clearly not possible that only two of such entries $\left(J_{12}, J_{23}, J_{31}\right)$ vanish at the same point, as a consequence of the zero-sum relation $\chi_{12}\left(x_{1}, x_{2}\right)+\chi_{23}\left(x_{2}, x_{3}\right)+\chi_{31}\left(x_{3}, x_{1}\right)=$ 0 . Accordingly, when dealing with the nontrivial case in which the rank of these structure matrices is equal to 2 , we know in advance that for every point $x \in \Omega$ there are at least two pairs $(i, j) \in\{(1,2),(2,3),(3,1)\}$ for which function $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$. These features will be useful in the next subsection.

To conclude, it is interesting for what is to come to recall the physical interpretation of the degree of freedom corresponding to the factor $\eta(x)$, namely the fact that in the 3-d case $\eta \mathcal{J}$ is a structure matrix if and only if $\mathcal{J}$ is [63,70]. Such result is not generally valid for dimension $n \geq 4$, as already mentioned. The interpretation of such three-dimensional feature is naturally associated with time reparametrizations [70], which are transformations of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{73}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$. Thus, if

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{74}
\end{equation*}
$$

is an arbitrary three-dimensional Poisson system defined in $\Omega$, then every time reparametrization (73) leads from (74) to the system:

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{75}
\end{equation*}
$$

Therefore, in the 3 -d case time reparametrizations (73) preserve the Poisson structure, this time with structure matrix $\eta \mathcal{J}$ in (75). This is not the case in general for $n \geq 4$, as indicated.

### 2.3.2. Casimir invariants and global Darboux analysis

We can now characterize some of the properties of the family identified in Theorem 2.3.1.1. In first place, we focus on the following:

Theorem 2.3.2.1. Let $\mathcal{J}=\left(J_{i j}\right)$ be a structure matrix of the form (71) characterized in Theorem 2.3.1.1, which is defined in a domain $\Omega \subset \mathbb{R}^{3}$ and such that for a given pair $(i, j)$ it is $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$. Then $\operatorname{Rank}(\mathcal{J})=2$ in $\Omega$ and a Casimir invariant for $\mathcal{J}$ is

$$
\begin{equation*}
D_{k}(x)=\frac{\psi_{j}\left(x_{j}\right)-\psi_{k}\left(x_{k}\right)+\kappa_{j k}}{\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}}=\frac{\chi_{j k}\left(x_{j}, x_{k}\right)}{\chi_{i j}\left(x_{i}, x_{j}\right)} \tag{76}
\end{equation*}
$$

where $(i, j, k)$ is a cyclic permutation of (1,2,3). Moreover, every Casimir invariant (76) is globally defined in $\Omega$.

Proof. After some algebra it is not difficult to show that

$$
\partial_{i} D_{a}(x)=-\frac{\phi_{i}\left(x_{i}\right) \chi_{j k}\left(x_{j}, x_{k}\right)}{\chi_{b c}^{2}\left(x_{b}, x_{c}\right)}=-\frac{J_{j k}(x)}{\eta(x) \chi_{b c}^{2}\left(x_{b}, x_{c}\right)}
$$

where both $(a, b, c)$ and $(i, j, k)$ are cyclic permutations of $(1,2,3)$. With the help of this property the result can be directly shown through the verification of the fact that $\mathcal{J} \cdot \nabla D_{k}=0$ for each of the three cases $k=1,2,3$ indicated. The statement is completed taking into account the $C^{\infty}(\Omega)$ property of the $\phi_{i}\left(x_{i}\right)$.
Q.E.D.

Therefore it is possible to give the explicit list of Casimir invariants corresponding to the three complementary cases just analyzed:

$$
\begin{aligned}
& D_{1}(x)=\frac{\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)+\kappa_{31}}{\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)+\kappa_{23}}=\frac{\chi_{31}\left(x_{3}, x_{1}\right)}{\chi_{23}\left(x_{2}, x_{3}\right)} \quad \text { if } \quad \chi_{23}\left(x_{2}, x_{3}\right) \neq 0 \quad \text { in } \Omega \\
& D_{2}(x)=\frac{\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}}{\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)+\kappa_{31}}=\frac{\chi_{12}\left(x_{1}, x_{2}\right)}{\chi_{31}\left(x_{3}, x_{1}\right)} \quad \text { if } \quad \chi_{31}\left(x_{3}, x_{1}\right) \neq 0 \quad \text { in } \Omega \\
& D_{3}(x)=\frac{\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)+\kappa_{23}}{\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}}=\frac{\chi_{23}\left(x_{2}, x_{3}\right)}{\chi_{12}\left(x_{1}, x_{2}\right)} \quad \text { if } \quad \chi_{12}\left(x_{1}, x_{2}\right) \neq 0 \quad \text { in } \Omega
\end{aligned}
$$

Notice the symmetry of such a choice, since $D_{1} D_{2} D_{3}=1$ when all of them are defined in $\Omega$. We can also state:

Corollary 2.3.2.2. Consider that the assumptions of Remark 1.4.4.4 hold. Every threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (71) characterized in Theorem 2.3.1.1 and such that for a given pair $(i, j)$ it is $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$, is an algebraically integrable system in $\Omega$.

The previous results allow the constructive and global determination of the Darboux canonical form for this kind of Poisson structures:

Theorem 2.3.2.3. Let $\Omega \subset \mathbb{R}^{3}$ be a domain where a Poisson system (22) with $n=3$ is defined everywhere, for which $\mathcal{J}=\left(J_{i j}\right)$ is a structure matrix of the form (71) characterized in Theorem 2.3.1.1, and such that for a given pair $(i, j)$ it is $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$. Then such Poisson system can be globally reduced in $\Omega$ to an one degree of freedom Hamiltonian system and the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\left(y_{1}, y_{2}, y_{3}\right)$ and the new time $\tau$, where $\left(y_{1}, y_{2}, y_{3}\right)$ are given by the diffeomorphism globally defined in $\Omega$

$$
\left\{\begin{array}{l}
y_{i}(x)=x_{i}  \tag{77}\\
y_{j}(x)=x_{j} \\
y_{k}(x)=-D_{k}(x)
\end{array}\right.
$$

in which $(i, j, k)$ is a cyclic permutation of $(1,2,3)$ and $D_{k}(x)$ is the Casimir invariant (76); while the new time $\tau$ is defined by a time reparametrization of the form:

$$
\begin{equation*}
\mathrm{d} \tau=J_{i j}(x(y)) \mathrm{d} t \tag{78}
\end{equation*}
$$

Proof. Only the situation $\chi_{12}\left(x_{1}, x_{2}\right) \neq 0$ will be considered here, since the analysis of the other two cases is analogous. Note that, according to Theorem 2.3.2.1, Darboux' theorem is applicable because $\mathcal{J}$ has constant rank 2 everywhere in $\Omega$. Recall also that, after a general smooth coordinate change $y=y(x)$, an arbitrary structure matrix $\mathcal{J}(x)$ is transformed into another one $\mathcal{J}^{*}(y)$ as:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{79}
\end{equation*}
$$

The reduction can be carried out in two steps. We first perform the change of variables (77), which in this case is

$$
\left\{\begin{array}{l}
y_{1}=x_{1}  \tag{80}\\
y_{2}=x_{2} \\
y_{3}=-D_{3}(x)
\end{array}\right.
$$

where $D_{3}(x)$ is given by (76). For what is to come it is necessary to explicitly write the transformation inverse of (80) which is:

$$
\left\{\begin{array}{l}
x_{1}=y_{1}  \tag{81}\\
x_{2}=y_{2} \\
x_{3}=\zeta_{3}\left[\psi_{2}\left(y_{2}\right)+\kappa_{23}+\left(\psi_{1}\left(y_{1}\right)-\psi_{2}\left(y_{2}\right)+\kappa_{12}\right) y_{3}\right]
\end{array}\right.
$$

where function $\zeta_{3}$ is the inverse function of $\psi_{3}\left(x_{3}\right)$. Note that $\zeta_{3}$ exists and is differentiable in $\tilde{\Omega}=\psi_{3}(\Omega)$. In fact, since $\psi_{i}^{\prime}=\phi_{i}$ is smooth and nonvanishing for every $i$, we have that $\zeta_{3}^{\prime}=1 / \psi_{3}^{\prime}=1 / \phi_{3}$ is also smooth and nonvanishing everywhere in $\tilde{\Omega}$, and therefore $\zeta_{3}$ is everywhere a smooth and strictly monotonous function. Then the examination of (80-81) easily shows that the variable transformation (80) to be performed exists and is a diffeomorphism everywhere in $\Omega$ as a consequence that by hypothesis we have $\chi_{12}\left(x_{1}, x_{2}\right) \neq 0$ and $\phi_{3}\left(x_{3}\right) \neq 0$ in $\Omega$. Then, according to (76) and (80), and taking (79) into account, after some algebra we are led to

$$
\mathcal{J}^{*}(y)=J_{12}(x(y))\left(\begin{array}{ccc}
0 & 1 & 0  \tag{82}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

where from equations (71) and (81) we have

$$
\begin{equation*}
J_{12}(x(y))=\eta\left(y_{1}, y_{2}, x_{3}(y)\right)\left(\psi_{1}\left(y_{1}\right)-\psi_{2}\left(y_{2}\right)+\kappa_{12}\right) \phi_{3}\left(x_{3}(y)\right) \tag{83}
\end{equation*}
$$

The explicit dependence of $x_{3}(y)$ is obviously the one given in (81) and was not displayed in (83) for the sake of clarity. Note that $J_{12}(x(y))$ is nonvanishing in $\Omega^{*}=y(\Omega)$ and $C^{\infty}\left(\Omega^{*}\right)$. These properties allow the accomplishment of the second step of the reduction which is a reparametrization of time. Thus, making use of (83) in equation (78), the transformation $\mathrm{d} \tau=J_{12}(x(y)) \mathrm{d} t$ is performed. According to (73-75) this leads from the structure matrix (82) to the Darboux one:

$$
\mathcal{J}_{D[3,2]}(y)=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{84}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

The reduction is thus globally completed.
Q.E.D.

After the previous global reduction, we can consequently establish:

Corollary 2.3.2.4. Consider that the assumptions of Remark 1.4.4.4 hold. Every threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (71) characterized in Theorem 2.3.1.1 and such that for a given pair $(i, j)$ it is $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$, can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

The previous results can be now illustrated by means of some instances. This is the aim of the next subsection.

### 2.3.3. Examples

We shall consider three examples. The first one involves a Poisson structure reported for the Halphen equations. Example 2 regards a very similar structure matrix, this time relevant to the study of the system of circle maps. To conclude, the third instance refers to a classical mechanical system, the Euler top.

## Example 1. Poisson structure for the Halphen equations

Let us first consider the following structure matrix which has deserved some attention regarding the analysis of the Halphen system [63]:

$$
\mathcal{J}(x)=\eta(x)\left(\begin{array}{ccc}
0 & x_{1}-x_{2} & x_{1}-x_{3}  \tag{85}\\
x_{2}-x_{1} & 0 & x_{2}-x_{3} \\
x_{3}-x_{1} & x_{3}-x_{2} & 0
\end{array}\right)
$$

with

$$
\begin{equation*}
\eta(x)=\left[2\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)\right]^{-1} \tag{86}
\end{equation*}
$$

It can be seen that the structure matrix (85-86) belongs to the family (71) with $\psi_{i}\left(x_{i}\right)=x_{i}$ and $\kappa_{i j}=0$ for all $i, j=1,2,3$, provided $x_{i} \neq x_{j}$ in $\Omega$ for every pair $i \neq j$. If this is the case, function $\eta(x)$ is $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. Note that this condition also implies $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ (and therefore $J_{i j}(x) \neq 0$ ) in $\Omega$ for every pair $i \neq j$. In order to perform the Darboux reduction it should be noted that every Casimir invariant (76) is now defined in $\Omega$ and can thus be employed. For instance, we can focus on $D_{3}(x)$ :

$$
\begin{equation*}
D_{3}(x)=\frac{x_{2}-x_{3}}{x_{1}-x_{2}} \tag{87}
\end{equation*}
$$

Therefore the reduction to Darboux form now makes use of the following diffeomorphism

$$
\left\{\begin{array}{l}
y_{1}=x_{1} \\
y_{2}=x_{2} \\
y_{3}=-D_{3}(x)
\end{array}\right.
$$

with $D_{3}(x)$ given by (87). The inverse of this transformation is then:

$$
\left\{\begin{array}{l}
x_{1}=y_{1} \\
x_{2}=y_{2} \\
x_{3}=y_{2}+\left(y_{1}-y_{2}\right) y_{3}
\end{array}\right.
$$

After applying (79) the outcome is that $\mathcal{J}$ in (85-86) is transformed into:

$$
\mathcal{J}^{*}(y)=\left(y_{1}-y_{2}\right) \eta\left(y_{1}, y_{2}, y_{2}+y_{3}\left(y_{1}-y_{2}\right)\right)\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \equiv \tilde{J}_{12}(y)\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

with $\tilde{J}_{12}(y)=\left[2\left(y_{1}-y_{2}\right)^{2} y_{3}\left(1-y_{3}\right)\right]^{-1}$. The reduction is then completed by means of the time reparametrization $\mathrm{d} \tau=\tilde{J}_{12}(y) \mathrm{d} t$, which finally leads to the Darboux canonical form (84) with $y_{3}$ acting as the decoupled Casimir function and $\left(y_{1}, y_{2}\right)$ as classical Hamiltonian canonical variables.

## Example 2. Poisson structure for the system of circle maps

In addition, it is worth mentioning in this context the Poisson structure appearing in the study of the system of circle maps [63]. The structure matrix is of the form (85), but this time with

$$
\eta(x)=-\left[\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)\right]^{-1}
$$

Thus the conditions for the regularity of the functions are exactly the same, the functions $\phi_{i}\left(x_{i}\right)$ and $\psi_{i}\left(x_{i}\right)$ retain their definitions, and the constants $\kappa_{i j}$ have the same zero values, than in the case of the Poisson structure for the Halphen system. The difference existing in $\eta(x)$ does not induce variations in the form of the Casimir invariants, in the diffeomorphic changes of variables leading to the Darboux reduction, or in the conditions indicating when all of them are properly defined. Consequently these results also remain valid in the context of the Poisson structure for the system of circle maps.

## Example 3. Euler top

As a third example, the following cubic and homogeneous structure matrix appearing [63] in the analysis of the Euler equations for a triaxial top will be considered:

$$
\left\{\begin{array}{l}
J_{12}(x)=\left(\alpha_{2} x_{1}^{2}-\alpha_{1} x_{2}^{2}\right) x_{3}  \tag{88}\\
J_{23}(x)=\left(\alpha_{3} x_{2}^{2}-\alpha_{2} x_{3}^{2}\right) x_{1} \\
J_{31}(x)=\left(\alpha_{1} x_{3}^{2}-\alpha_{3} x_{1}^{2}\right) x_{2}
\end{array}\right.
$$

where the $\alpha_{i}$ are real constants related to the principal moments of inertia $\mu_{i}$ of the top according to the expressions:

$$
\alpha_{1}=\frac{\mu_{2}-\mu_{3}}{\mu_{2} \mu_{3}}, \quad \alpha_{2}=\frac{\mu_{3}-\mu_{1}}{\mu_{1} \mu_{3}}, \quad \alpha_{3}=\frac{\mu_{1}-\mu_{2}}{\mu_{1} \mu_{2}}
$$

Assuming that $\alpha_{1} \alpha_{2} \alpha_{3} \neq 0$, equations (88) can be equivalently written as:

$$
\left\{\begin{array}{l}
J_{12}(x)=\frac{1}{2 \alpha_{1} \alpha_{2} \alpha_{3}}\left(\alpha_{2} \alpha_{3} x_{1}^{2}-\alpha_{1} \alpha_{3} x_{2}^{2}\right)\left(2 \alpha_{1} \alpha_{2} x_{3}\right)  \tag{89}\\
J_{23}(x)=\frac{1}{2 \alpha_{1} \alpha_{2} \alpha_{3}}\left(\alpha_{1} \alpha_{3} x_{2}^{2}-\alpha_{1} \alpha_{2} x_{3}^{2}\right)\left(2 \alpha_{2} \alpha_{3} x_{1}\right) \\
J_{31}(x)=\frac{1}{2 \alpha_{1} \alpha_{2} \alpha_{3}}\left(\alpha_{1} \alpha_{2} x_{3}^{2}-\alpha_{2} \alpha_{3} x_{1}^{2}\right)\left(2 \alpha_{1} \alpha_{3} x_{2}\right)
\end{array}\right.
$$

Expressed in this way, the structure matrix (89) can be recognized as a member of family (71) with

$$
\eta=\left(2 \alpha_{1} \alpha_{2} \alpha_{3}\right)^{-1}, \quad \psi_{1}\left(x_{1}\right)=\alpha_{2} \alpha_{3} x_{1}^{2}, \quad \psi_{2}\left(x_{2}\right)=\alpha_{1} \alpha_{3} x_{2}^{2}, \quad \psi_{3}\left(x_{3}\right)=\alpha_{1} \alpha_{2} x_{3}^{2}
$$

and $\kappa_{i j}=0$ for all $i, j=1,2,3$. Since functions $\phi_{i}\left(x_{i}\right)$ must be nonvanishing in $\Omega$, this implies that in what follows the structure matrix (89) is to be analyzed in a domain of the set $\left\{\left(x_{1}, x_{2}, x_{3}\right) \in \mathbb{R}^{3} \mid x_{1} x_{2} x_{3} \neq 0\right\}$. In addition, according to (76) we can employ different forms for the Casimir invariant. For instance, if $\chi_{12}\left(x_{1}, x_{2}\right)=\alpha_{2} \alpha_{3} x_{1}^{2}-\alpha_{1} \alpha_{3} x_{2}^{2} \neq 0$ in $\Omega$, we have:

$$
\begin{equation*}
D_{3}(x)=\frac{\alpha_{1} \alpha_{3} x_{2}^{2}-\alpha_{1} \alpha_{2} x_{3}^{2}}{\alpha_{2} \alpha_{3} x_{1}^{2}-\alpha_{1} \alpha_{3} x_{2}^{2}} \tag{90}
\end{equation*}
$$

Then, in this case a transformation leading to the Darboux canonical form is defined by (80) and (90), and its inverse is a diffeomorphism in $y(\Omega)$ given by:

$$
\left\{\begin{aligned}
x_{1} & =y_{1} \\
x_{2} & =y_{2} \\
x_{3} & =\varsigma_{3}\left[\frac{\alpha_{3}}{\alpha_{2}} y_{2}^{2}+\left(\frac{\alpha_{3}}{\alpha_{1}} y_{1}^{2}-\frac{\alpha_{3}}{\alpha_{2}} y_{2}^{2}\right) y_{3}\right]^{1 / 2}
\end{aligned}\right.
$$

where $\varsigma_{3} \equiv \operatorname{sign}\left(x_{3}\right)$ denotes the usual sign function, namely $\varsigma_{3}$ is a constant of value $(+1)$ if $x_{3}>0$ and $(-1)$ if $x_{3}<0$ (recall that $x_{3} \neq 0$ in $\Omega$ ). The rest of the Darboux reduction does not present special features apart from the ones indicated in the proof of Theorem 2.3.2.3, and therefore is omitted for the sake of conciseness.

### 2.4. CYCLIC SOLUTIONS OF TYPE II

The present section is to a great extent parallel to the previous one. As it was the case there, another three-dimensional family of solutions of the Jacobi equations is characterized and globally analyzed. This solution family (denoted as cyclic of type II) is formally very similar to the cyclic of type I, a parallelism accounting for such terminology. Apart from this, it is worth noting that both families are of course different, as a simple examination shows. On the other hand, the cyclic solutions of type II also have a very general form (defined in terms of functions of arbitrary nonlinearity) in spite of which it is possible the explicit and global determination of their main features, such as the case-classification of the Casimir invariants and the global construction of the Darboux canonical form. Again, the analysis of this family displays a remarkable feature, namely that a complete understanding of its global analysis implies a case classification such that a different distinguished invariant is to be used in each case. Therefore, we again find a situation in which the knowledge of one nontrivial distinguished invariant now does not guarantee the reduction to the Darboux canonical form, in spite of working with a structure matrix of constant rank 2 . Thus, as it was the case in the previous section, now the symplectic analysis of this family of Poisson structures requires the use of both case-dependent global Casimir invariants and global Darboux reductions. Such classification is done in what follows.

The structure of the section is now anticipated. In Subsection 2.4.1 the solution family is characterized. In Subsection 2.4.2, the classification of the Casimir invariants and the associated Darboux reductions are constructed. To conclude, some examples are provided in Subsection 2.4.3.

### 2.4.1. Characterization of the family

In first place, the following result is presented:

Theorem 2.4.1.1. Let $\left(\eta(x), \phi_{1}\left(x_{1}\right), \phi_{2}\left(x_{2}\right), \phi_{3}\left(x_{3}\right)\right)$ be a set of functions defined in a domain $\Omega \subset \mathbb{R}^{3}$, all of which are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. In addition, let $\psi_{i}\left(x_{i}\right)$ denote for every $i=1,2,3$, one primitive function of $\phi_{i}\left(x_{i}\right)$ defined in such a way that $\psi_{i}\left(x_{i}\right)$ is nonvanishing in $\Omega$. Then $\mathcal{J}=\left(J_{i j}\right)$ is a family of 3-d structure matrices which are globally defined in $\Omega$, with

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)\left(\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right) \sum_{k=1}^{3}\left(\epsilon_{i j k}\right)^{2} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3 \tag{91}
\end{equation*}
$$

where $\epsilon_{i j k}$ denotes the Levi-Civita symbol.

Proof. Skew-symmetry can be directly verified in (91). Regarding the Jacobi equations, recall that for $n=3$, system (24-25) can be reduced to the following independent nonlinear equation:

$$
\begin{equation*}
J_{12} \partial_{1} J_{31}-J_{31} \partial_{1} J_{12}+J_{23} \partial_{2} J_{12}-J_{12} \partial_{2} J_{23}+J_{31} \partial_{3} J_{23}-J_{23} \partial_{3} J_{31}=0 \tag{92}
\end{equation*}
$$

Let us first consider family (91) in the particular case $\eta(x)=1$. For this, let $\hat{J}_{i j}(x) \equiv$ $J_{i j}(x) / \eta(x)$ in (91). Then, substitution in (92) produces after some algebra:

$$
\begin{gathered}
\hat{J}_{12} \partial_{1} \hat{J}_{31}-\hat{J}_{31} \partial_{1} \hat{J}_{12}+\hat{J}_{23} \partial_{2} \hat{J}_{12}-\hat{J}_{12} \partial_{2} \hat{J}_{23}+\hat{J}_{31} \partial_{3} \hat{J}_{23}-\hat{J}_{23} \partial_{3} \hat{J}_{31}= \\
\phi_{1} \phi_{2} \phi_{3} \psi_{1} \psi_{2} \psi_{3}\left[\left(\psi_{1}-\psi_{2}\right)\left(\psi_{3}-\psi_{1}\right)-\psi_{1}\left(\psi_{1}-\psi_{2}\right)-\left(\psi_{1}-\psi_{2}\right)\left(\psi_{3}-\psi_{1}\right)-\right. \\
\psi_{1}\left(\psi_{3}-\psi_{1}\right)+\left(\psi_{2}-\psi_{3}\right)\left(\psi_{1}-\psi_{2}\right)-\psi_{2}\left(\psi_{2}-\psi_{3}\right)-\left(\psi_{1}-\psi_{2}\right)\left(\psi_{2}-\psi_{3}\right)- \\
\left.\psi_{2}\left(\psi_{1}-\psi_{2}\right)+\left(\psi_{3}-\psi_{1}\right)\left(\psi_{2}-\psi_{3}\right)-\psi_{3}\left(\psi_{3}-\psi_{1}\right)-\left(\psi_{2}-\psi_{3}\right)\left(\psi_{3}-\psi_{1}\right)-\psi_{3}\left(\psi_{2}-\psi_{3}\right)\right]=0
\end{gathered}
$$

The result is thus proved for the case $\eta=1$. For general $\eta$ it is sufficient to make use of Proposition 2.2.2.1, according to which in the $3-\mathrm{d}$ case $\eta \mathcal{J}$ is a structure matrix for an arbitrary nonvanishing $C^{\infty}$ function $\eta(x)$ and for every structure matrix $\mathcal{J}$. This completes the proof.
Q.E.D.

It is now convenient to provide some comments. In first place, we can display the explicit form of the components of $\mathcal{J}$ for family (91), which are:

$$
\left\{\begin{array}{l}
J_{12}(x)=\eta(x) \psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right) \phi_{3}\left(x_{3}\right) \\
J_{23}(x)=\eta(x) \psi_{2}\left(x_{2}\right) \psi_{3}\left(x_{3}\right)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)\right) \phi_{1}\left(x_{1}\right) \\
J_{31}(x)=\eta(x) \psi_{3}\left(x_{3}\right) \psi_{1}\left(x_{1}\right)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)\right) \phi_{2}\left(x_{2}\right)
\end{array}\right.
$$

As mentioned in Theorem 2.4.1.1, for every $i$ the primitive $\psi_{i}\left(x_{i}\right)$ of $\phi_{i}\left(x_{i}\right)$ must be chosen to be one and the same for all the entries of $\mathcal{J}$. However, the specific choice is actually arbitrary as far as the nonvanishing character of such function is respected.

In second term, it is worth noting that the form of the structure matrices (91) is such that there are only two possibilities regarding the vanishing of the independent entries $\left(J_{12}, J_{23}, J_{31}\right)$ at a given point, namely: (i) either none or one of them vanishes (case of rank two), or (ii) all of them vanish (case of zero rank). To see this as well as for future use, it is convenient to define the functions:

$$
\omega_{i j}\left(x_{i}, x_{j}\right) \equiv \psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right), \quad i, j=1,2,3
$$

Then, clearly it is not possible that just two of such entries $\left(J_{12}, J_{23}, J_{31}\right)$ vanish at the same point, as a consequence of the zero-sum relation $\omega_{12}\left(x_{1}, x_{2}\right)+\omega_{23}\left(x_{2}, x_{3}\right)+\omega_{31}\left(x_{3}, x_{1}\right)=0$. As a consequence, in the analysis of the nontrivial case in which the rank of the structure matrices is equal to 2 , we can make use of the feature that for every point $x \in \Omega$ there are at least two pairs $(i, j) \in\{(1,2),(2,3),(3,1)\}$ for which function $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$. This kind of property will be employed in the next subsection.

To finish, in order to make the section self-contained we shall recall again the physical interpretation of the degree of freedom corresponding to the factor $\eta(x)$, namely the fact that in the 3 -d case $\eta \mathcal{J}$ is a structure matrix if and only if $\mathcal{J}$ is [63,70]. Such result is not generally valid for dimension $n \geq 4$, as we already mentioned. As we know, the interpretation of such three-dimensional feature is naturally associated with time reparametrizations [70], which are transformations of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{93}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$. Accordingly, if

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{94}
\end{equation*}
$$

is an arbitrary three-dimensional Poisson system defined in $\Omega$, then every time reparametrization (93) leads from (94) to the system:

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{95}
\end{equation*}
$$

Then, we see that in the 3 -d case time reparametrizations (93) preserve the Poisson structure, this time with structure matrix $\eta \mathcal{J}$ in (95).

### 2.4.2. Casimir invariants and global Darboux analysis

It is now possible to determine the main properties of the family identified in Theorem 2.4.1.1:

Theorem 2.4.2.1. Let $\mathcal{J}=\left(J_{i j}\right)$ define a structure matrix of the form (91) characterized in Theorem 2.4.1.1, which is defined in a domain $\Omega \subset \mathbb{R}^{3}$ and such that for a given pair $(i, j)$ it is $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$. Then $\operatorname{Rank}(\mathcal{J})=2$ in $\Omega$ and a Casimir invariant for $\mathcal{J}$ is

$$
\begin{equation*}
D_{k}(x)=\frac{\psi_{i}\left(x_{i}\right)\left(\psi_{j}\left(x_{j}\right)-\psi_{k}\left(x_{k}\right)\right)}{\psi_{k}\left(x_{k}\right)\left(\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right)}=\frac{\psi_{i}\left(x_{i}\right) \omega_{j k}\left(x_{j}, x_{k}\right)}{\psi_{k}\left(x_{k}\right) \omega_{i j}\left(x_{i}, x_{j}\right)} \tag{96}
\end{equation*}
$$

where $(i, j, k)$ is a cyclic permutation of ( $1,2,3$ ). Moreover, every Casimir invariant (96) is globally defined in $\Omega$.

Proof. It is not difficult to show after some calculations that

$$
\partial_{i} D_{a}(x)=-\frac{\phi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \psi_{k}\left(x_{k}\right) \omega_{j k}\left(x_{j}, x_{k}\right)}{\left(\psi_{a}\left(x_{a}\right) \omega_{b c}\left(x_{b}, x_{c}\right)\right)^{2}}=-\frac{J_{j k}(x)}{\eta(x)\left(\psi_{a}\left(x_{a}\right) \omega_{b c}\left(x_{b}, x_{c}\right)\right)^{2}}
$$

where both $(a, b, c)$ and $(i, j, k)$ are arbitrary cyclic permutations of $(1,2,3)$. Making use of these equalities, the simplest procedure to prove the result is through the direct verification of the fact that $\mathcal{J} \cdot \nabla D_{k}=0$ for each of the three cases $k=1,2,3$ indicated. The proof is completed by taking into account the $C^{\infty}(\Omega)$ property of the $\phi_{i}\left(x_{i}\right)$.
Q.E.D.

Accordingly, we can now give the explicit list of Casimir invariants corresponding to the three complementary cases just analyzed:

$$
\begin{aligned}
& D_{1}(x)=\frac{\psi_{2}\left(x_{2}\right)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)\right)}{\psi_{1}\left(x_{1}\right)\left(\psi_{2}\left(x_{2}\right)-\psi_{3}\left(x_{3}\right)\right)}=\frac{\psi_{2}\left(x_{2}\right) \omega_{31}\left(x_{3}, x_{1}\right)}{\psi_{1}\left(x_{1}\right) \omega_{23}\left(x_{2}, x_{3}\right)} \quad \text { if } \quad \omega_{23}\left(x_{2}, x_{3}\right) \neq 0 \quad \text { in } \Omega \\
& D_{2}(x)=\frac{\psi_{3}\left(x_{3}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right)}{\psi_{2}\left(x_{2}\right)\left(\psi_{3}\left(x_{3}\right)-\psi_{1}\left(x_{1}\right)\right)}=\frac{\psi_{3}\left(x_{3}\right) \omega_{12}\left(x_{1}, x_{2}\right)}{\psi_{2}\left(x_{2}\right) \omega_{31}\left(x_{3}, x_{1}\right)} \quad \text { if } \omega_{31}\left(x_{3}, x_{1}\right) \neq 0 \quad \text { in } \Omega \\
& D_{3}(x)=\frac{\psi_{1}\left(x_{1}\right)\left(\psi_{2}\left(x_{2}-\psi_{3}\left(x_{3}\right)\right)\right.}{\psi_{3}\left(x_{3}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right)}=\frac{\psi_{1}\left(x_{1}\right) \omega_{23}\left(x_{2}, x_{3}\right)}{\psi_{3}\left(x_{3}\right) \omega_{12}\left(x_{1}, x_{2}\right)} \quad \text { if } \quad \omega_{12}\left(x_{1}, x_{2}\right) \neq 0 \quad \text { in } \Omega
\end{aligned}
$$

It is worth noting the symmetry of the choice made for the complementary forms of the Casimir invariant, since $D_{1} D_{2} D_{3}=1$ when all of them are simultaneously defined in $\Omega$. The case characterization of the independent Casimir invariant leads to:

Corollary 2.4.2.2. Consider that the assumptions of Remark 1.4.4.4 hold. Every threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (91) characterized in Theorem 2.4.1.1 and such that for a given pair $(i, j)$ it is $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$, is an algebraically integrable system in $\Omega$.

The results just shown make possible the constructive and global determination of the Darboux canonical form for this family of Poisson structures:

Theorem 2.4.2.3. Let $\Omega \subset \mathbb{R}^{3}$ be a domain where a Poisson system (22) with $n=3$ is defined everywhere, for which $\mathcal{J}=\left(J_{i j}\right)$ is a structure matrix of the form (91) characterized
in Theorem 2.4.1.1, and such that for a given pair $(i, j)$ it is $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$. Then such Poisson system can be globally reduced in $\Omega$ to an one degree of freedom Hamiltonian system and the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\left(y_{1}, y_{2}, y_{3}\right)$ and the new time $\tau$, where $\left(y_{1}, y_{2}, y_{3}\right)$ are given by the diffeomorphism globally defined in $\Omega$

$$
\left\{\begin{array}{l}
y_{i}(x)=x_{i}  \tag{97}\\
y_{j}(x)=x_{j} \\
y_{k}(x)=-D_{k}(x)
\end{array}\right.
$$

in which $(i, j, k)$ is a cyclic permutation of $(1,2,3)$ and $D_{k}(x)$ is the Casimir invariant (96); while the new time $\tau$ is defined by a time reparametrization of the form:

$$
\begin{equation*}
\mathrm{d} \tau=J_{i j}(x(y)) \mathrm{d} t \tag{98}
\end{equation*}
$$

Proof. Here we can consider only the case $\omega_{12}\left(x_{1}, x_{2}\right) \neq 0$, since the analysis of the other two possibilities is analogous. Notice that, according to Theorem 2.4.2.1, Darboux' theorem is applicable because $\mathcal{J}$ has constant rank 2 everywhere in $\Omega$. Recall also that, after a general smooth coordinate transformation $y=y(x)$, an arbitrary structure matrix $\mathcal{J}(x)$ is converted into another one $\mathcal{J}^{*}(y)$ as:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{99}
\end{equation*}
$$

It is possible to carry out the reduction in two steps. We first perform the change of variables (97), which in this case is

$$
\left\{\begin{array}{l}
y_{1}=x_{1}  \tag{100}\\
y_{2}=x_{2} \\
y_{3}=-D_{3}(x)
\end{array}\right.
$$

where $D_{3}(x)$ is given by (96). For what is to come it is necessary to explicitly write the transformation inverse of (100) which is:

$$
\left\{\begin{array}{l}
x_{1}=y_{1}  \tag{101}\\
x_{2}=y_{2} \\
x_{3}=\zeta_{3}\left(\frac{\psi_{1}\left(y_{1}\right) \psi_{2}\left(y_{2}\right)}{\psi_{1}\left(y_{1}\right)-\omega_{12}\left(y_{1}, y_{2}\right) y_{3}}\right)
\end{array}\right.
$$

where function $\zeta_{3}$ is the inverse function of $\psi_{3}\left(x_{3}\right)$. We see that $\zeta_{3}$ exists and is differentiable in $\tilde{\Omega}=\psi_{3}(\Omega)$. Actually, since $\psi_{i}^{\prime}=\phi_{i}$ is smooth and nonvanishing for every $i$, we have
that $\zeta_{3}^{\prime}=1 / \psi_{3}^{\prime}=1 / \phi_{3}$ is also smooth and nonvanishing everywhere in $\tilde{\Omega}$, and therefore $\zeta_{3}$ is everywhere a smooth and strictly monotonous function. Then the examination of (100101) easily shows that the variable transformation (100) to be performed exists and is a diffeomorphism everywhere in $\Omega$ as a consequence that by hypothesis we have $\omega_{12}\left(x_{1}, x_{2}\right) \neq 0$ and $\phi_{3}\left(x_{3}\right) \neq 0$ in $\Omega$. In particular, notice that the denominator of the argument inside $\zeta_{3}$ in equation (101) does not vanish, since:

$$
\psi_{1}\left(y_{1}\right)-\omega_{12}\left(y_{1}, y_{2}\right) y_{3}=\frac{\psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right)}{\psi_{3}\left(x_{3}\right)}
$$

Thus, according to (96) and (100), and taking (99) into account, after some algebra we are led to

$$
\mathcal{J}^{*}(y)=J_{12}(x(y))\left(\begin{array}{ccc}
0 & 1 & 0  \tag{102}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

where from equations (91) and (101) we have

$$
\begin{equation*}
J_{12}(x(y))=\eta\left(y_{1}, y_{2}, x_{3}(y)\right) \psi_{1}\left(y_{1}\right) \psi_{2}\left(y_{2}\right) \omega_{12}\left(y_{1}, y_{2}\right) \phi_{3}\left(x_{3}(y)\right) \tag{103}
\end{equation*}
$$

The explicit dependence of $x_{3}(y)$ is obviously the one given in (101) and was not displayed in (103) for the sake of clarity. Notice that $J_{12}(x(y))$ is nonvanishing in $\Omega^{*}=y(\Omega)$ and $C^{\infty}\left(\Omega^{*}\right)$. These properties allow the accomplishment of the second step of the reduction which is a reparametrization of time. Thus, making use of (103) in equation (98), the transformation $\mathrm{d} \tau=J_{12}(x(y)) \mathrm{d} t$ is to be performed. According to (93-95) this leads from the structure matrix (102) to the Darboux one:

$$
\mathcal{J}_{D[3,2]}(y)=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{104}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Therefore the reduction is globally completed.
Q.E.D.

We then arrive at the following corollary:

Corollary 2.4.2.4. Consider that the assumptions of Remark 1.4.4.4 hold. Every threedimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{3}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (91) characterized in Theorem 2.4.1.1 and such that for a given pair $(i, j)$ it is $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$, can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

The aim of the next subsection is to provide some examples of application of the previous results.

### 2.4.3. Examples

In what follows some instances of the solution family just analyzed are considered. The first one deals with the Poisson formulation for Lotka-Volterra systems, which is later generalized to the framework of the Quasi-Polynomial equations in the second example. Finally, the third instance considers a Poisson structure related to the system of circle maps.

## Example 1. Lotka-Volterra system

In first place we shall consider a Poisson structure employed in the analysis of the 3-d Lotka-Volterra (LV) equations. The following LV system has received some attention in the literature $[63,136]$,

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{1}\left(\lambda_{1}+a_{2} x_{2}+x_{3}\right)  \tag{105}\\
\dot{x}_{2}=x_{2}\left(\lambda_{2}+x_{1}+a_{3} x_{3}\right) \\
\dot{x}_{3}=x_{3}\left(\lambda_{3}+a_{1} x_{1}+x_{2}\right)
\end{array}\right.
$$

in which $x_{i}>0$ for all $i$. Notice that system (105) is the same considered in Example 2 of Subsection 1.2.5. However, for convenience (related to the generalization to be performed in the next example) we now give different names to the system constants. Among several classical integrable cases of interest, the following one is to be considered [63]:

$$
\begin{equation*}
a_{i}=1, \quad \lambda_{i}=0, \quad i=1,2,3 \tag{106}
\end{equation*}
$$

System (105-106) is Poisson, in terms of the structure matrix:

$$
\begin{equation*}
J_{i j}(x)=x_{i} x_{j}\left(x_{i}-x_{j}\right), \quad i, j=1,2,3 \tag{107}
\end{equation*}
$$

And the following first integral plays the role of Hamiltonian:

$$
\begin{equation*}
H(x)=\ln \left[\left(\frac{x_{3}}{x_{1} x_{2}}\left(x_{1}-x_{2}\right)^{2}\right)^{-k}\left(\frac{x_{1}}{x_{2} x_{3}}\left(x_{2}-x_{3}\right)^{2}\right)^{k-1}\right] \tag{108}
\end{equation*}
$$

for arbitrary $k \in \mathbb{R}$. It can be seen that the structure matrix (107) belongs to the family (91) with $\eta(x)=1, \phi_{i}\left(x_{i}\right)=1$ and $\psi_{i}\left(x_{i}\right)=x_{i}$ for $i=1,2,3$. Since $x_{i}>0$ for all $i$, such
structure matrix is defined in every domain $\Omega \subset \mathbb{R}_{+}^{3}$. In addition, now $\omega_{i j}\left(x_{i}, x_{j}\right)=\left(x_{i}-x_{j}\right)$ for every pair $(i, j)$. Thus if $x_{i} \neq x_{j}$ in $\Omega$ for a pair $i \neq j$, then $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ (and therefore $\left.J_{i j}(x) \neq 0\right)$ in $\Omega$. Depending on $i$ and $j$, we have to employ according to (96) different forms for the Casimir invariant. For instance, if $\omega_{12}\left(x_{1}, x_{2}\right) \neq 0$ in $\Omega$, we have:

$$
\begin{equation*}
D_{3}(x)=\frac{x_{1}\left(x_{2}-x_{3}\right)}{x_{3}\left(x_{1}-x_{2}\right)} \tag{109}
\end{equation*}
$$

Thus the reduction to Darboux form begins with the following diffeomorphism

$$
\left\{\begin{array}{l}
y_{1}=x_{1}  \tag{110}\\
y_{2}=x_{2} \\
y_{3}=-D_{3}(x)
\end{array}\right.
$$

with $D_{3}(x)$ given by (109). The inverse of this transformation is then:

$$
\left\{\begin{array}{l}
x_{1}=y_{1}  \tag{111}\\
x_{2}=y_{2} \\
x_{3}=\frac{y_{1} y_{2}}{y_{1}-\left(y_{1}-y_{2}\right) y_{3}}
\end{array}\right.
$$

Notice that $y_{1}-\left(y_{1}-y_{2}\right) y_{3}=x_{1} x_{2} / x_{3}$ and consequently does not vanish, as expected. Thus, after the application of (99) the result is that $\mathcal{J}(x)$ in (107) is transformed into:

$$
\mathcal{J}^{*}(y)=y_{1} y_{2}\left(y_{1}-y_{2}\right)\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right) \equiv \tilde{J}_{12}(y)\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Finally, the reduction is completed by means of the time reparametrization $\mathrm{d} \tau=\tilde{J}_{12}(y) \mathrm{d} t$, which produces the Darboux canonical form (104) with $\left(y_{1}, y_{2}\right)$ in the role of canonical Hamiltonian variables and $y_{3}$ as a decoupled Casimir invariant.

## Example 2. A nonstandard Quasi-Polynomial generalization of the

## Lotka-Volterra system

In this second example the previous LV system is generalized as a Quasi-Polynomial (QP) flow in such a way that its associated Poisson structure is also generalized, while remaining in the framework of the family characterized in Theorem 2.4.1.1. The reader is referred to $[23,24,51,60,61,72,73,80,82,83,86,87]$ and references therein for an approach to QP systems and their related formalism.

Let us thus consider system (105-106) and perform the quasimonomial transformation

$$
\begin{equation*}
x_{i}=y_{i}^{c_{i}}, \quad i=1,2,3 ; \quad c_{1} c_{2} c_{3} \neq 0 \tag{112}
\end{equation*}
$$

followed by the time reparametrization

$$
\begin{equation*}
\mathrm{d} \tau=\left(\prod_{i=1}^{3} c_{i} y_{i}^{c_{i}-1}\right)^{-1} \mathrm{~d} t \tag{113}
\end{equation*}
$$

The outcome is the following QP generalization of the LV flow, this time defined in terms of variables $y_{i}$ and time $\tau$ :

$$
\left\{\begin{array}{l}
\dot{y}_{1}=c_{2} c_{3} y_{1}^{c_{1}} y_{2}^{c_{2}-1} y_{3}^{c_{3}-1}\left(y_{2}^{c_{2}}+y_{3}^{c_{3}}\right)  \tag{114}\\
\dot{y}_{2}=c_{1} c_{3} y_{1}^{c_{1}-1} y_{2}^{c_{2}} y_{3}^{c_{3}-1}\left(y_{1}^{c_{1}}+y_{3}^{c_{3}}\right) \\
\dot{y_{3}}=c_{1} c_{2} y_{1}^{c_{1}-1} y_{2}^{c_{2}-1} y_{3}^{c_{3}}\left(y_{1}^{c_{1}}+y_{2}^{c_{2}}\right)
\end{array}\right.
$$

Both transformations (112) and (113) become identical in the case $c_{1}=c_{2}=c_{3}=1$, and system (114) is thus reduced to (105-106) in such situation. On the other hand, equations (114) still conform to a Poisson system. Actually, the Hamiltonian (108) is directly generalized as:

$$
H^{*}(y)=\ln \left[\left(\frac{y_{3}^{c_{3}}}{y_{1}^{c_{1}} y_{2}^{c_{2}}}\left(y_{1}^{c_{1}}-y_{2}^{c_{2}}\right)^{2}\right)^{-k}\left(\frac{y_{1}^{c_{1}}}{y_{2}^{c_{2}} y_{3}^{c_{3}}}\left(y_{2}^{c_{2}}-y_{3}^{c_{3}}\right)^{2}\right)^{k-1}\right]
$$

for arbitrary $k \in \mathbb{R}$. Finally, both the quasimonomial transformation and the time reparametrization transform the structure matrix (107) leading to the more general form:

$$
\begin{equation*}
J_{i j}^{*}(y)=y_{i}^{c_{i}} y_{j}^{c_{j}}\left(y_{i}^{c_{i}}-y_{j}^{c_{j}}\right) \sum_{k=1}^{3}\left(\epsilon_{i j k}\right)^{2} c_{k} y_{k}^{c_{k}-1}, \quad i, j=1,2,3 \tag{115}
\end{equation*}
$$

Structure matrix (115) belongs to family (91) with $\eta(y)=1$ and $\psi_{i}\left(y_{i}\right)=y_{i}^{c_{i}}$. Since we have $y_{i}>0$ for all $i,(115)$ is correctly defined in $\mathbb{R}_{+}^{3}$ without further assumptions. Now the reduction to the Darboux canonical form can also be performed globally and it is a generalization of the one for the LV case. For this, note first that if $(i, j, k)$ is a cyclic permutation of $(1,2,3)$ and $\omega_{i j}=\left(y_{i}^{c_{i}}-y_{j}^{c_{j}}\right) \neq 0$ in $\Omega$, then the Casimir invariant $D_{k}(y)$ is:

$$
D_{k}(y)=\frac{y_{i}^{c_{i}}\left(y_{j}^{c_{j}}-y_{k}^{c_{k}}\right)}{y_{k}^{c_{k}}\left(y_{i}^{c_{i}}-y_{j}^{c_{j}}\right)}
$$

Then, use of the corresponding transformation can be made in order to carry out the reduction to Darboux form. For instance, if $\omega_{12} \neq 0$ :

$$
\left\{\begin{array}{l}
z_{1}=y_{1} \\
z_{2}=y_{2} \\
z_{3}=\frac{y_{1}^{c_{1}}\left(y_{3}^{c_{3}}-y_{2}^{c_{2}}\right)}{y_{3}^{c_{3}}\left(y_{1}^{c_{1}}-y_{2}^{c_{2}}\right)}
\end{array}\right.
$$

And the inverse of this transformation is also a generalization of (111):

$$
\left\{\begin{array}{l}
y_{1}=z_{1} \\
y_{2}=z_{2} \\
y_{3}=\left(\frac{z_{1}^{c_{1}} z_{2}^{c_{2}}}{z_{1}^{c_{1}}-\left(z_{1}^{c_{1}}-z_{2}^{c_{2}}\right) z_{3}}\right)^{1 / c_{3}}
\end{array}\right.
$$

The remaining details of the Darboux reduction are essentially similar to those of the LV case and are not given. To conclude, it is worth noticing that extensive families of QP Poisson systems have been analyzed in detail in the literature [73,83,86], but always in terms of a different (in fact separable [85] and quadratic) kind of Poisson brackets. In such context the family (115) is not only new, but in addition it provides an instance of non-standard family of Poisson structures for QP systems.

## Example 3. Poisson structure for the system of circle maps

As a last example the following structure matrix, which is of interest for the analysis of the system of circle maps [63], will be considered:

$$
\begin{equation*}
J_{i j}(x)=\eta(x) x_{i} x_{j}\left(x_{i}-x_{j}\right), \quad i, j=1,2,3 \tag{116}
\end{equation*}
$$

where

$$
\begin{equation*}
\eta(x)=-\left[\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)\right]^{-1} \tag{117}
\end{equation*}
$$

This structure matrix is to a great extent similar to the one in Example 1, apart from the factor $\eta(x)$ which nevertheless introduces some differences. As before, we have $\psi_{i}\left(x_{i}\right)=x_{i}$ and $\phi_{i}\left(x_{i}\right)=1$ for every $i=1,2,3$. But according to Theorem 2.4.1.1, now the structure is defined provided that in $\Omega$ we have $x_{i} \neq 0$ for every $i$, and $x_{i}-x_{j} \neq 0$ for every pair $i \neq j$. If this is the case, function $\eta(x)$ is $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. Note that the same conditions also imply $\omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ (and $J_{i j}(x) \neq 0$ ) in $\Omega$ for every pair $i \neq j$. Consequently, Theorem 2.4.2.1 implies that now every alternative form (96) of the Casimir invariant is simultaneously defined in $\Omega$, namely:

$$
D_{1}(x)=\frac{x_{2}\left(x_{3}-x_{1}\right)}{x_{1}\left(x_{2}-x_{3}\right)}, \quad D_{2}(x)=\frac{x_{3}\left(x_{1}-x_{2}\right)}{x_{2}\left(x_{3}-x_{1}\right)}, \quad D_{3}(x)=\frac{x_{1}\left(x_{2}-x_{3}\right)}{x_{3}\left(x_{1}-x_{2}\right)}
$$

Therefore, in order to perform the Darboux reduction of (116-117) either expression can be employed. For instance, if we focus again on $D_{3}(x)$, which coincides with (109), then transformation (110) is also the same. The rest of the reduction is thus analogous to the one
in the Lotka-Volterra case, just with minor differences due to the presence of $\eta(x)$ as given by (117). Since such reduction does not present any feature not mentioned in the proof of Theorem 2.4.2.3, the rest will be omitted for the sake of conciseness.

### 2.5. USE OF AN ANSATZ FOR THE SIMPLIFICATION OF THE JACOBI EQUATIONS

In this section we provide a relatively simple method for the determination of additional 3-d skew-symmetric solutions of the Jacobi equations from a known one. More precisely, it is proved that the knowledge of a single and arbitrary solution of the three-dimensional Jacobi equations allows determining an infinity of new solutions, which can be generally and explicitly constructed [71]. The main motivation underlying this kind of methodology is that it is very illustrative of how the previous knowledge of a single solution can help much in the study of the Jacobi equations, leading to a remarkable simplification of the problem. In particular, this philosophy will be very useful in Chapter 4 , in which it will be the basis for the study of the new-time transformations (NTTs). In fact, the use of NTTs for the generation of new solutions from a single one will help much in the obtainment of a deeper understanding of important cases in which the Darboux canonical form can be globally obtained, as well as in the generalization of known solution families, thus leading to aspects of central interest for this work.

The structure of the section proceeds as follows. In Subsection 2.5.1 the method is presented and developed, analyzing and solving all the different possibilities existing. Case-bycase examples are presented in Subsection 2.5.2. We finish in Subsection 2.5.3 with some conclusions and final remarks.

### 2.5.1. Description of the method and case classification

For the sake of conciseness, the following simplified notation for the entries of 3-d structure matrices will be used again:

$$
\begin{equation*}
u(x) \equiv J_{12}(x), \quad v(x) \equiv J_{31}(x), \quad w(x) \equiv J_{23}(x) \tag{118}
\end{equation*}
$$

In the case $n=3$, system (24-25) actually consists of a single independent equation. If we make use of the definition (118) we know that such equation can be written in the form:

$$
\begin{equation*}
u \partial_{1} v-v \partial_{1} u+w \partial_{2} u-u \partial_{2} w+v \partial_{3} w-w \partial_{3} v=0 \tag{119}
\end{equation*}
$$

The equation corresponding to the 3 -d version of system (24-25) will be written in the form (119) in the rest of the section.

Now let $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$ be a known solution of (119). In what follows it will be assumed that it is a regular and nontrivial solution, i.e. that the rank of the structure matrix
represented by $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$ is constant and equal to 2 everywhere in the domain of interest. Then we can look for new solutions according to the ansatz $\left(u_{0}(x)+\xi(x), v_{0}(x)+\right.$ $\xi(x), w_{0}(x)+\xi(x)$, where $\xi(x)$ is an arbitrary smooth function to be determined. If we substitute the ansatz in (119) we see after some algebra that all the nonlinear terms are of the form $\xi \partial_{i} \xi$, for $i=1,2,3$, and in fact all such terms do cancel out. Consequently, we arrive at the following linear PDE for $\xi$ :

$$
\begin{equation*}
\left(u_{0}-v_{0}\right) \partial_{1} \xi+\left(w_{0}-u_{0}\right) \partial_{2} \xi+\left(v_{0}-w_{0}\right) \partial_{3} \xi=\lambda(x) \xi \tag{120}
\end{equation*}
$$

where

$$
\begin{equation*}
\lambda(x)=\partial_{1}\left(u_{0}-v_{0}\right)+\partial_{2}\left(w_{0}-u_{0}\right)+\partial_{3}\left(v_{0}-w_{0}\right) \tag{121}
\end{equation*}
$$

Therefore, the mere knowledge of one solution of (119) allows a threefold simplification of the problem of finding new solutions:
(a) We can transform a nonlinear problem into a linear problem.
(b) We can transform the PDE problem (119) into an ODE problem, namely the one given by the characteristic equations of (120-121).
(c) We can reduce the number of unknowns from three to one.

We can now proceed to analyze equation (120-121). Three cases must be distinguished:
I. Case $\lambda(x)=0$

It is relatively frequent, as we shall see in the examples section, that $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$ are such that $\lambda(x)$ in (120-121) vanishes at every point of the domain of interest. In such case the characteristic equations of (120) are:

$$
\begin{equation*}
\frac{\mathrm{d} x_{1}}{u_{0}-v_{0}}=\frac{\mathrm{d} x_{2}}{w_{0}-u_{0}}=\frac{\mathrm{d} x_{3}}{v_{0}-w_{0}}, \quad \mathrm{~d} \xi=0 \tag{122}
\end{equation*}
$$

Obviously, we need two constants of motion of (122) in order to have the general solution of (120-121). It is not difficult to verify that two such constants can be chosen as $I_{1}(x)=$ $x_{1}+x_{2}+x_{3}$ and $I_{2}(x)=D(x)$, where $D(x)$ is a Casimir invariant of the known solution $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$. Accordingly, the general solution of (120-121) now is:

$$
\begin{equation*}
\xi(x)=\Psi\left(x_{1}+x_{2}+x_{3}, D(x)\right) \tag{123}
\end{equation*}
$$

where $\Psi$ is an arbitrary smooth function of its two real arguments. Therefore we have arrived to the following new family of solutions of (119):

$$
\begin{gathered}
(u(x), v(x), w(x))= \\
\left(u_{0}(x)+\Psi\left(I_{1}(x), I_{2}(x)\right), v_{0}(x)+\Psi\left(I_{1}(x), I_{2}(x)\right), w_{0}(x)+\Psi\left(I_{1}(x), I_{2}(x)\right)\right)
\end{gathered}
$$

We can now consider the second possibility.

## II. Case $\lambda(x) \neq 0$, with invertible $I_{1}(x)$ and $I_{2}(x)$

This time the characteristic equations are:

$$
\begin{equation*}
\frac{\mathrm{d} x_{1}}{u_{0}-v_{0}}=\frac{\mathrm{d} x_{2}}{w_{0}-u_{0}}=\frac{\mathrm{d} x_{3}}{v_{0}-w_{0}}=\frac{\mathrm{d} \xi}{\lambda \xi} \tag{124}
\end{equation*}
$$

We need three constants of motion of (124) in order to find the general integral of (120-121). However, the two constants known from Case I, i.e. $I_{1}(x)=x_{1}+x_{2}+x_{3}$ and $I_{2}(x)=D(x)$ (where $D(x)$ is a Casimir invariant of the known solution $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$ being used) are also first integrals of (124). The third constant of motion required is then evident and takes the form of a quadrature: Assuming that the standard invertibility conditions (see Theorem 1.4.2.1) arise for $I_{1}(x)$ and $I_{2}(x)$, it will be possible to make use of both invariants and express two independent variables in terms of the remaining one, $I_{1}$ and $I_{2}$. For instance:

$$
\begin{equation*}
x_{2}=\alpha\left(x_{1}, I_{1}, I_{2}\right) \quad, \quad x_{3}=\beta\left(x_{1}, I_{1}, I_{2}\right) \tag{125}
\end{equation*}
$$

Then it is immediate to write:

$$
\frac{\mathrm{d} \xi}{\xi}=\frac{\lambda \mathrm{d} x_{1}}{u_{0}-v_{0}} \equiv \nu\left(x_{1}, I_{1}, I_{2}\right) \mathrm{d} x_{1}
$$

After integration we finally obtain the third constant of motion:

$$
I_{3}(x)=\frac{\xi}{h\left(x_{1}, I_{1}(x), I_{2}(x)\right)}
$$

where

$$
\begin{equation*}
h\left(x_{1}, x_{2}, x_{3}\right)=\exp \left(\int \nu\left(x_{1}, x_{2}, x_{3}\right) \mathrm{d} x_{1}\right) \tag{126}
\end{equation*}
$$

Consequently, from (126) we arrive at the following general integral for $\xi$ in the case $\lambda \neq 0$ :

$$
\Phi\left(I_{1}(x), I_{2}(x), \frac{\xi}{h\left(x_{1}, I_{1}(x), I_{2}(x)\right)}\right)=0
$$

where $\Phi$ is an arbitrary smooth function of its three real arguments.
We finally have:
III. Case $\lambda(x) \neq 0$, with non-invertible $I_{1}(x)$ and $I_{2}(x)$

It may still happen that the usual invertibility conditions are not satisfied for $I_{1}(x)$ and $I_{2}(x)$, i.e. it is not possible to determine equations of the form (125). In this case it is still feasible to easily construct infinite families of solutions from a given one. As usual, $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$ denote the known solution, corresponding to a structure matrix $\mathcal{J}(x)$. It is well-known that after a smooth change of variables $y=y(x)$ a structure matrix $\mathcal{J}(x)$ is transformed into another structure matrix $\mathcal{J}^{*}(y)$ according to the rule (30):

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{3} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1,2,3 \tag{127}
\end{equation*}
$$

In principle, the change of variables (127) needs not be globally defined on the domain of interest for what is to follow. However, for the sake of simplicity the global character of the transformation shall be assumed. In the case of a change of variables restricted to a subdomain the procedure described below is not affected, the only difference being that we would arrive at new families of solutions of the Jacobi equations defined on that subdomain of the initial domain of definition.

Therefore, a new system of coordinates in which (120-121) can be solved for $\mathcal{J}^{*}(y)$ is to be introduced. This is very simple to do, but obviously there is not a unique choice. For instance, a straightforward possibility is the Darboux canonical form of matrix $\mathcal{J}(x)$, i.e.

$$
\mathcal{J}^{*}(y)=\mathcal{J}_{D[3,2]}=\left(\begin{array}{ccc}
0 & 1 & 0  \tag{128}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

According to (118) and (127), in the case of the choice (128) we are mapping the functions $\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)$ into

$$
\left(u_{0}^{*}(y), v_{0}^{*}(y), w_{0}^{*}(y)\right)=\left(J_{12}^{*}(y), J_{31}^{*}(y), J_{23}^{*}(y)\right)=(1,0,0)
$$

Notice that equations (120-121) become trivial for (128) because in the Darboux form we are in Case I, actually. Then, the general solution of (120-121) for matrix (128) is $\xi(y)=$ $\Psi\left(y_{1}+y_{2}+y_{3}, y_{3}\right)$. In this way we have arrived to the family of solutions:

$$
\begin{gather*}
\left(u^{*}(y), v^{*}(y), w^{*}(y)\right)=\left(u_{0}^{*}(y), v_{0}^{*}(y), w_{0}^{*}(y)\right)+\Psi\left(y_{1}+y_{2}+y_{3}, y_{3}\right)(1,1,1)= \\
(1,0,0)+\Psi\left(y_{1}+y_{2}+y_{3}, y_{3}\right)(1,1,1) \tag{129}
\end{gather*}
$$

in evident notation. Then we can make use of (127) and transform back the solution family $\left(u^{*}(y), v^{*}(y), w^{*}(y)\right)$ in (129) into the original coordinates $x$. We thus arrive at:

$$
\begin{gather*}
(u(x), v(x), w(x))=\left(u_{0}(x), v_{0}(x), w_{0}(x)\right)+ \\
\Psi\left(y_{1}(x)+y_{2}(x)+y_{3}(x), y_{3}(x)\right)\left(Q_{12}(x), Q_{31}(x), Q_{23}(x)\right) \tag{130}
\end{gather*}
$$

where

$$
Q_{i j}(x)=\sum_{k, l=1}^{3} \frac{\partial x_{i}}{\partial y_{k}} a_{k l} \frac{\partial x_{j}}{\partial y_{l}} \quad, \quad i, j=1,2,3
$$

with the $a_{k l}$ being the entries of

$$
A=\left(\begin{array}{ccc}
0 & 1 & -1 \\
-1 & 0 & 1 \\
1 & -1 & 0
\end{array}\right)
$$

This completes the procedure of Case III. Of course, the whole method remains entirely identical in the case of choices different from (128).

The result just described in Case III is interesting for several reasons:
(a) The first one is that we are producing solutions such that the terms added to $u_{0}, v_{0}$ and $w_{0}$ in (130) now are not one and the same due to the presence of the functions $Q_{i j}$. Clearly this is due to the fact that we are making a previous transformation of variables. Therefore the procedure allows determining solutions which are not only those of the form $\left(u_{0}+\xi, v_{0}+\xi, w_{0}+\xi\right)$, actually. In other words, we see that the method is in fact more general that it seemed in principle, leading to more general families of solutions than those originally expected.
(b) The second one is that the procedure described in Case III is, in fact, also applicable to Cases I and II, because the verification of condition (125) is not essential for the introduction of a new coordinate system. Therefore the generality of the method, as considered in the previous item, applies to the three Cases I to III and is an intrinsic feature of this approach to the construction of solutions.

### 2.5.2. Case-by-case examples

We can now proceed to see some examples illustrating each of the three possibilities previously analyzed.

## Example 1. Constant structure matrices

Constant structure matrices are, in spite of their simplicity, ubiquitous in very diverse problems [101,102,103,142], one important example being the Darboux representation of 3d Poisson structures. Obviously, every constant 3-d skew-symmetric matrix is a structure matrix. Therefore, let $\left(u_{0}, v_{0}, w_{0}\right)$ be constants, not all equal to zero. We then have $\lambda=0$ in (120-121). Two cases must be distinguished:
(i) $u_{0}=v_{0}=w_{0} \neq 0$. In this situation equation (120-121) becomes trivial and every smooth $\xi(x)$ is a solution.
(ii) $u_{0}, v_{0}$ and $w_{0}$ are not equal. This is the generic case. According to (123) we only need to find a Casimir invariant of $\left(u_{0}, v_{0}, w_{0}\right)$. It is straightforward to check that one choice is $D(x)=w_{0} x_{1}+v_{0} x_{2}+u_{0} x_{3}$. Consequently we arrive at the family of solutions:

$$
(u, v, w)=\left(u_{0}, v_{0}, w_{0}\right)+\Psi\left(x_{1}+x_{2}+x_{3}, w_{0} x_{1}+v_{0} x_{2}+u_{0} x_{3}\right)(1,1,1)
$$

in evident notation, with $\Psi$ arbitrary and smooth. Notice that $I_{1}(x)=x_{1}+x_{2}+x_{3}$ and $I_{2}(x)=w_{0} x_{1}+v_{0} x_{2}+u_{0} x_{3}$ are independent when $u_{0}, v_{0}$ and $w_{0}$ are not equal, as we are now assuming by hypothesis.

Thus starting from a simple, constant solution we have arrived to a nonconstant family of solutions just by finding one Casimir invariant.

## Example 2. Lie-Poisson so(3) and Hamiltonian ray optics Poisson structures

Another important Poisson structure is given by the Lie-Poisson bracket associated with the Lie algebra so(3), namely $\left(u_{0}, v_{0}, w_{0}\right)=\left(x_{3}, x_{2}, x_{1}\right)$. For instance, see [114,139,162]. In this case we again have $\lambda=0$, and the resulting PDE (120-121) is:

$$
\left(x_{3}-x_{2}\right) \partial_{1} \xi+\left(x_{1}-x_{3}\right) \partial_{2} \xi+\left(x_{2}-x_{1}\right) \partial_{3} \xi=0
$$

It is well-known that a Casimir invariant of this Poisson structure is $D(x)=x_{1}^{2}+x_{2}^{2}+x_{3}^{2}$. The general solution is then $\xi=\Psi\left(x_{1}+x_{2}+x_{3}, x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)$. Therefore we have arrived to the family:

$$
(u, v, w)=\left(x_{3}, x_{2}, x_{1}\right)+\Psi\left(x_{1}+x_{2}+x_{3}, x_{1}^{2}+x_{2}^{2}+x_{3}^{2}\right)(1,1,1)
$$

Consequently, the Lie-Poisson so(3) structure can now be seen as a particular case of a wider set.

Many other 3-d Poisson structures of the Lie-Poisson kind can also be generalized in a very similar way. For instance, we can consider the Hamiltonian ray optics structure [93] given by $\left(u_{0}, v_{0}, w_{0}\right)=\left(4 x_{3},-2 x_{1},-2 x_{2}\right)$. In this case we again have $\lambda=0$. The Casimir invariant is $D(x)=x_{1} x_{2}-x_{3}^{2}$ and consequently we arrive at the family of solutions:

$$
(u, v, w)=\left(4 x_{3},-2 x_{1},-2 x_{2}\right)+\Psi\left(x_{1}+x_{2}+x_{3}, x_{1} x_{2}-x_{3}^{2}\right)(1,1,1)
$$

Therefore the treatment is completely similar to that of so(3), as anticipated.

## Example 3. Kermack-McKendrick model Poisson structure

We now take as our starting point the Kermack-McKendrick structure matrix [63,135] given by $\left(u_{0}, v_{0}, w_{0}\right)=\left(-b x_{1} x_{2}, 0,-a x_{2}\right)$, with $a$ and $b$ real constants. In this case $\lambda$ is not identically zero, and system (120-121) becomes:

$$
\begin{equation*}
-b x_{1} x_{2} \partial_{1} \xi+\left(b x_{1} x_{2}-a x_{2}\right) \partial_{2} \xi+a x_{2} \partial_{3} \xi=\left[b\left(x_{1}-x_{2}\right)-a\right] \xi \tag{131}
\end{equation*}
$$

Of course we know a first invariant $I_{1}(x)=x_{1}+x_{2}+x_{3}$, and a second one which is given by a Casimir function of $\left(u_{0}, v_{0}, w_{0}\right)$. This is easily found to be $I_{2}(x)=D(x)=x_{3}+(a / b) \ln x_{1}$. Therefore we only need the third invariant $I_{3}$ in order to have the general integral of (131). We can first make use of $I_{1}$ and $I_{2}$ to find relationships of the type (125). In our case, after some algebra we obtain:

$$
\begin{gather*}
x_{1}=\exp \left(\frac{b}{a}\left(I_{2}-x_{3}\right)\right)  \tag{132}\\
x_{2}=I_{1}-x_{3}-\exp \left(\frac{b}{a}\left(I_{2}-x_{3}\right)\right) \tag{133}
\end{gather*}
$$

Substituting (132-133) in the characteristic equations we arrive at:

$$
\begin{equation*}
\frac{\mathrm{d} \xi}{\xi}=\frac{b\left(x_{1}-x_{2}\right)-a}{a x_{2}} \mathrm{~d} x_{3}=\frac{b x_{3}-b I_{1}-a+2 b \exp \left(\frac{b}{a}\left(I_{2}-x_{3}\right)\right)}{a I_{1}-a x_{3}-a \exp \left(\frac{b}{a}\left(I_{2}-x_{3}\right)\right)} \mathrm{d} x_{3} \tag{134}
\end{equation*}
$$

Integrating (134) and simplifying we can set:

$$
I_{3}(x)=\frac{\xi}{x_{2}} \exp \left(\frac{b x_{3}}{a}\right)
$$

Therefore the general integral of equation (131) is

$$
\begin{equation*}
\Phi\left[x_{1}+x_{2}+x_{3}, x_{3}+\left(\frac{a}{b}\right) \ln x_{1}, \frac{\xi}{x_{2}} \exp \left(\frac{b x_{3}}{a}\right)\right]=0 \tag{135}
\end{equation*}
$$

with $\Phi$ an arbitrary smooth function of three variables. In spite of its seeming complexity, equation (135) contains families of very simple solutions. As an example of this assertion, it is straightforward to verify that, for instance, the solution $\xi(x)=k x_{1} x_{2}, k \in \mathbb{R}$, belongs to the general integral (135).

## Example 4. Lotka-Volterra and Quasi-Polynomial Poisson structures

As a final example we consider the LV $[63,136,142,143,144,145,146]$ and $\mathrm{QP}[83,86]$ structure matrices of the form

$$
\begin{equation*}
\left(u_{0}, v_{0}, w_{0}\right)=\left(a_{12} x_{1} x_{2}, a_{31} x_{1} x_{3}, a_{23} x_{2} x_{3}\right) \tag{136}
\end{equation*}
$$

where the $a_{i j}$ are real constants for all $i, j$, and $x_{i}>0$ for all $i$ (i.e. the domain of definition of these structure matrices is the interior of the positive orthant, namely $\mathbb{R}_{+}^{3}$ ). For the sake of conciseness, we shall consider here the generic case in which none of the $a_{i j}$ is zero. It is then easy to check that

$$
\begin{equation*}
\lambda=\left(a_{31}-a_{12}\right) x_{1}+\left(a_{12}-a_{23}\right) x_{2}+\left(a_{23}-a_{31}\right) x_{3} \tag{137}
\end{equation*}
$$

does not vanish in general. Therefore we have to make use of the two invariants, namely $I_{1}=x_{1}+x_{2}+x_{3}$ and the Casimir $I_{2}=x_{1}^{a_{23}} x_{2}^{a_{31}} x_{3}^{a_{12}}$ and find two relationships of the form (125). Clearly this is not possible in this case, as anticipated in Subsection 2.5.1. Consequently we have to apply the procedure of Case III. For this we shall perform a suitable change of variables which is diffeomorphic and globally defined in the interior of the positive orthant of $\mathbb{R}^{3}$ : we define $\left(y_{1}, y_{2}, y_{3}\right)=\left(x_{1}^{\alpha}, x_{2}^{\beta}, x_{3}^{\gamma}\right)$. According to equation (127) it is not difficult to show that always there exist suitable values of $\alpha, \beta$ and $\gamma$ such that $\left(u_{0}^{*}(y), v_{0}^{*}(y), w_{0}^{*}(y)\right)$ become either $\left(y_{1} y_{2}, y_{1} y_{3}, y_{2} y_{3}\right)$ or $\left(-y_{1} y_{2},-y_{1} y_{3},-y_{2} y_{3}\right)$. In both cases, equation (137) is still applicable in the new variables and now we do have $\lambda(y)=0$. Therefore in the variables $\left(y_{1}, y_{2}, y_{3}\right)$ we are in Case I of Subsection 2.5.1 and we are thus led to the general solution $\xi(y)=\Psi\left(y_{1}+y_{2}+y_{3}, y_{1} y_{2} y_{3}\right)$, with arbitrary smooth $\Psi$. If we now transform back these results into the original variables $\left(x_{1}, x_{2}, x_{3}\right)$ we arrive at the family of solutions given by:

$$
\begin{gather*}
(u, v, w)=\left(a_{12} x_{1} x_{2}, a_{31} x_{1} x_{3}, a_{23} x_{2} x_{3}\right)+ \\
\Psi\left(x_{1}^{\alpha}+x_{2}^{\beta}+x_{3}^{\gamma}, x_{1}^{\alpha} x_{2}^{\beta} x_{3}^{\gamma}\right)\left(a_{12} x_{1}^{1-\alpha} x_{2}^{1-\beta}, a_{31} x_{1}^{1-\alpha} x_{3}^{1-\gamma}, a_{23} x_{2}^{1-\beta} x_{3}^{1-\gamma}\right) \tag{138}
\end{gather*}
$$

As suggested in Subsection 2.5.1 (Case III) the derivation of solutions taking (136) as starting point and making use of the Darboux canonical form (128) is another possible line
of action. Here it has been preferred the use of a different choice in order to emphasize the multiplicity of suitable possibilities for the determination of new solutions. However, the use of the Darboux canonical form is an equally good and simple alternative in this case. It will be omitted here for the sake of brevity, although it is straightforward in the present example (see [85] for the algorithm of global reduction of the structure matrices (136) to the Darboux form). It is worth recalling that the solutions found in the case of the Darboux reduction are, of course, different from (138) since the multiplicity of choices obviously reflects the multiplicity of solutions that can be determined in this way.

### 2.5.3. Some remarks

We have described a method according to which the knowledge of one solution of the Jacobi equations greatly simplifies the procedure of determining additional solutions in the 3 -d case. In particular, this is possible due to two main reasons:
(i) Knowledge of a given solution allows reformulating the problem into a linear one (as it can be verified without difficulty, this property only holds in dimension 3). This is not the first time that such kind of simplification is presented in the literature regarding 3-d Poisson systems [55,66,81]. However, relevant advantages of the present method when compared to $[55,66,81]$ are that:
(a) Now we operate directly on the Poisson structure independently of the form of the Hamiltonian, while in $[55,66,81]$ an specific Hamiltonian is to be assumed.
(b) The present method produces in a straightforward way a large number of new solution families. In this sense, it seems to be more effective and simpler to apply than $[55,66,81]$.
(ii) The second is that the use of a known solution as starting point allows reducing the number of unknowns from three to one. Again, this type of reduction in the number of unknowns is not new in the field, a good example being the so called conformal invariance of the solutions of the 3 -d Jacobi equations [63], which is a property equivalent to the preservation of the Poisson structure after an arbitrary NTT. However, the quantity and richness of solutions produced by the method described in the previous subsections is remarkable when compared to the single, multiplicative family of solutions that the conformal invariance generates.

Therefore the present approach can be regarded quite naturally in the framework of the analysis of the Jacobi identities. However, it is the simultaneous combination of the two previous properties in the present method what actually makes it fruitful and simple to apply for the determination of solutions.

There are some additional comments of interest in connection with the procedure developed in this section, and more precisely with its possible generalization. An evident question in this sense is the possible extension of additive ansatzs to dimensions higher than 3 . This issue is not considered here for the sake of conciseness, but an examination of the problem suggests that the powerful advantages found in the 3 -d case are not present in higher dimensions, in which the associated equations are significantly more complicated. Nevertheless, there is another possible and natural line of research (just mentioned) that provides a logical extension of the present one, which is the use of a multiplicative (instead of an additive) ansatz. In such case we find an evident connection with the operation of time reparametrization discussed in Subsections 1.2.4 and 1.4.3, and systematically applied throughout the present chapter. In this sense, the interest is twofold: not only as an alternative procedure for the idea of constructing an infinity of solutions from a known one, but also as an operation of great interest for the global determination of the Darboux canonical form. The advance in this direction will be the purpose of Section 4.7, already in the general $n$-d framework.

### 2.6. A PERSPECTIVE OF THE THREE-DIMENSIONAL SOLUTIONS CONSIDERED

We can now present some unifying perspectives regarding the different 3-d results obtained.
Among the many features that might be emphasized, for what is to come in future chapters it is important to stress now an issue common to all the Darboux reductions presented, namely the fact that a concluding NTT is always applicable. Of course, at this stage it is not clear why this should be the case. In fact, in Chapter 4 we shall analyze some $n$-d families in which a mere diffeomorphism suffices for a global reduction to the Darboux canonical form. On the other hand, along the present chapter a combination of diffeomorphism and time reparametrization was always possible. In fact, it is evident that in the 2-d case only an NTT (without diffeomorphism) suffices in order to achieve the global Darboux reduction. Needless to say, all the previous possibilities are in complete agreement with the criteria for system equivalence presented in Subsection 1.4.3. These issues will be progressively investigated in the following chapters.

In addition, in Section 2.5 the method based on the use of an additive ansatz has been presented. It might seem that this line of analysis has no relationship with the construction of global Darboux reductions. As indicated in the previous subsection, this is not the case, at least on a conceptual level. In fact, this kind of strategy will be of fundamental importance for such reductions, since the idea of constructing an infinite family of solutions from a known one can (and will) be directly applied in the framework of NTTs, and consequently in the investigation of an essential aspect of Darboux reductions, in the general $n$-d case. This point of view, already examined in the discussion provided in Subsection 2.5.3, will be developed and should become apparent in Chapter 4.

Before proceeding to the study of the general $n$-d problem, in Chapter 3 we shall still focus on dimensionally specific problems, this time with $n=4$. The investigation of the $n$-d situation will be the goal of Chapter 4.

## CHAPTER 3.

## FOUR-DIMENSIONAL SOLUTIONS

### 3.1. INTRODUCTION

The 3 -d results investigated in Chapter 2 are to be complemented in the 4 -d context in the present chapter. There are several reasons accounting for the convenience of extending the 3 -d analysis of Chapter 2 with a detailed study of a 4 -d solution family.

The first of such reasons is that in the 4-d case there is a wider dynamical freedom from the point of view of the symplectic structure of the phase space. This will confirm some interesting aspects (already reported in the framework of the cyclic families of both types I and II) related to the need of a case classification for the global Darboux reductions. This issue has some relevant implications, already mentioned in the 3 -d context, and it will be reexamined in the 4-d framework in Subsection 3.2.4.

In second term, we already know that the applied use of solutions of the Jacobi equations has focused mostly in the 3 -d case, also with a significant (but numerically less important) sample of applied Poisson structures in the 4-d domain, while the instances in dimensions 5 and higher become progressively less frequent. Therefore, a detailed consideration of a 4 -d solution family can provide significant complementary elements allowing a better understanding of the features reported in dimension 3. As it will be clear in Chapter 4, the analysis of solutions in dimensions 3 and 4 provides the background required for the general investigation in arbitrary dimension $n$, which is the purpose of such chapter.

The structure of the present chapter includes two additional sections, namely Section 3.2 in which the 4-d family is characterized and analyzed in full detail [77], followed by Section 3.3 which is oriented towards the discussion and the conceptual analysis of the results obtained.

### 3.2. A FOUR-DIMENSIONAL FAMILY

The structure of this section is the following. In Subsection 3.2.1 the main theorem giving the characterization of the 4 - d solution family is provided. This is the basis for the analysis presented in Subsection 3.2.2, in which the different possible complete sets of independent Casimir invariants are classified, together with the subsequent global reductions to the Darboux canonical form. Some instances are the subject of Subsection 3.2.3. The section is concluded with some final remarks in Subsection 3.2.4.

### 3.2.1. Characterization of the family

We begin this paragraph with one of the main results of the chapter:

Theorem 3.2.1.1. Consider the family of functions of the form

$$
\begin{equation*}
J_{i j}(x)=\sigma_{i j} \eta(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \sum_{k, l=1}^{4} \epsilon_{i j k l} \phi_{l}\left(x_{l}\right), \quad i, j=1, \ldots, 4 \tag{139}
\end{equation*}
$$

defined in a domain $\Omega \subset \mathbb{R}^{4}$, where $\epsilon_{i j k l}$ denotes the Levi-Civita symbol and such that:
(a) Constants $\sigma_{i j} \in \mathbb{R}$ are defined for every pair $(i, j), i \neq j$.
(b) $\sigma_{i j}=\sigma_{j i}$ for every pair $(i, j), i \neq j$.
(c) $\sigma_{i j} \neq 0$ for at least one pair $(i, j), i \neq j$.
(d) $\eta(x), \psi_{i}\left(x_{i}\right)$ and $\phi_{i}\left(x_{i}\right)$ are $C^{\infty}(\Omega)$ functions of their respective arguments for every $i$.
(e) $\eta(x)$ and $\psi_{i}\left(x_{i}\right)$ are nonvanishing in $\Omega$ for every $i$.
(f) The differences $\left(\phi_{i}\left(x_{i}\right)-\phi_{j}\left(x_{j}\right)\right)$ are nonvanishing in $\Omega$ for every pair $(i, j), i \neq j$.

Then the set of functions $J_{i j}(x)$ defined in (139) constitutes in $\Omega$ a skew-symmetric solution of the four-dimensional Jacobi identities

$$
\begin{equation*}
\sum_{l=1}^{4}\left(J_{i l} \partial_{l} J_{j k}+J_{k l} \partial_{l} J_{i j}+J_{j l} \partial_{l} J_{k i}\right)=0, \quad i, j, k=1, \ldots, 4 \tag{140}
\end{equation*}
$$

and therefore $\mathcal{J}=\left(J_{i j}\right)$ is a four-dimensional structure matrix in $\Omega$, if and only if:

$$
\begin{equation*}
\sigma_{12} \sigma_{34}=\sigma_{13} \sigma_{24}=\sigma_{14} \sigma_{23} \tag{141}
\end{equation*}
$$

Proof. Consider first functions (139) in the case $\eta=1$. Substitution of (139) in equation (140) of indexes $(i, j, k)$ leads after some algebra to:

$$
\begin{gather*}
\sum_{l=1}^{4}\left(J_{i l} \partial_{l} J_{j k}+J_{k l} \partial_{l} J_{i j}+J_{j l} \partial_{l} J_{k i}\right)= \\
\psi_{i} \psi_{j} \psi_{k} \sum_{r_{1}, r_{2}, s_{1}, s_{2}=1}^{4}\left\{\left(\sigma_{i j} \sigma_{j k} \epsilon_{i j r_{1} r_{2}} \epsilon_{j k s_{1} s_{2}}+\sigma_{k j} \sigma_{i j} \epsilon_{k j r_{1} r_{2}} \epsilon_{i j s_{1} s_{2}}\right)\left(\partial_{j} \psi_{j}\right) \phi_{r_{2}} \phi_{s_{2}}+\right. \\
\left(\sigma_{k i} \sigma_{i j} \epsilon_{k i r_{1} r_{2}} \epsilon_{i j s_{1} s_{2}}+\sigma_{j i} \sigma_{k i} \epsilon_{j i r_{1} r_{2}} \epsilon_{k i s_{1} s_{2}}\right)\left(\partial_{i} \psi_{i}\right) \phi_{r_{2}} \phi_{s_{2}}+ \\
\left(\sigma_{i k} \sigma_{j k} \epsilon_{i k r_{1} r_{2}} \epsilon_{j k s_{1} s_{2}}+\sigma_{j k} \sigma_{k i} \epsilon_{j k r_{1} r_{2}} \epsilon_{k i s_{1} s_{2}}\right)\left(\partial_{k} \psi_{k}\right) \phi_{r_{2}} \phi_{s_{2}}+ \\
\left.\left(\sigma_{i s_{2}} \sigma_{j k} \epsilon_{i s_{2} r_{1} r_{2}} \epsilon_{j k s_{1} s_{2}}+\sigma_{k s_{2}} \sigma_{i j} \epsilon_{k s_{2} r_{1} r_{2}} \epsilon_{i j s_{1} s_{2}}+\sigma_{j s_{2}} \sigma_{k i} \epsilon_{j s_{2} r_{1} r_{2}} \epsilon_{k i s_{1} s_{2}}\right) \psi_{s_{2}} \phi_{r_{2}}\left(\partial_{s_{2}} \phi_{s_{2}}\right)\right\}= \\
\psi_{i} \psi_{j} \psi_{k} \sum_{r_{1}, r_{2}, s_{1}, s_{2}=1}^{4}\left\{\left(\sigma_{i s_{2}} \sigma_{j k} \delta_{j k s_{1} s_{2}}^{i s_{2} r_{1} r_{2}}+\sigma_{k s_{2}} \sigma_{i j} \delta_{i j s_{1} s_{2}}^{k r_{1} r_{2}}+\sigma_{j s_{2}} \sigma_{k i} \delta_{k i s_{1} s_{2}}^{j s_{2} r_{1} r_{2}}\right) \psi_{s_{2}} \phi_{r_{2}}\left(\partial_{s_{2}} \phi_{s_{2}}\right)\right\} \tag{142}
\end{gather*}
$$

where the $\delta$ symbol denotes the generalized Kronecker delta according to its standard definition [ $8,59,123,157]$ namely: given $q$ superindexes $\left(i_{1}, \ldots, i_{q}\right)$ and $q$ subindexes $\left(j_{1}, \ldots, j_{q}\right)$ all of them taking values in the range $(1, \ldots, n)$, then $\delta_{j_{1} \ldots j_{q}}^{i_{1} \ldots i_{q}}$ is defined by the properties: (a) it is totally antisymmetric in the superindexes; (b) it is totally antisymmetric in the subindexes; (c) if the superindexes are all different (this is, $i_{a_{1}} \neq i_{a_{2}}$ if $a_{1} \neq a_{2}$ ) and the subindexes are a permutation of the superindexes, then $\delta_{j_{1} \ldots i_{q}}^{i_{1} \ldots i_{q}}$ takes the value +1 (respectively, -1 ) if $\left(i_{1}, \ldots, i_{q}\right)$ and $\left(j_{1}, \ldots, j_{q}\right)$ are permutations of the same (of different) sign; (d) the value of $\delta_{j_{1} \ldots j_{q}}^{i_{1} \ldots i_{q}}$ is zero otherwise. Consequently, it can be verified that the expression in (142) vanishes if two of the three indexes $(i, j, k)$ are equal. Consider then the case in which $i, j$ and $k$ are different. If $m$ is the integer, $1 \leq m \leq 4$, such that $(i, j, k, m)$ is a permutation of $(1,2,3,4)$, we arrive at:

$$
\begin{gather*}
\sum_{l=1}^{4}\left(J_{i l} \partial_{l} J_{j k}+J_{k l} \partial_{l} J_{i j}+J_{j l} \partial_{l} J_{k i}\right)= \\
\psi_{i} \psi_{j} \psi_{k} \psi_{m}\left(\partial_{m} \phi_{m}\right)\left\{\sigma_{i m} \sigma_{j k}\left(\phi_{k}-\phi_{j}\right)+\sigma_{k m} \sigma_{i j}\left(\phi_{j}-\phi_{i}\right)+\sigma_{j m} \sigma_{k i}\left(\phi_{i}-\phi_{k}\right)\right\}= \\
\psi_{i} \psi_{j} \psi_{k} \psi_{m}\left(\partial_{m} \phi_{m}\right)\left\{\left(\sigma_{j m} \sigma_{k i}-\sigma_{k m} \sigma_{i j}\right) \phi_{i}+\left(\sigma_{k m} \sigma_{i j}-\sigma_{i m} \sigma_{j k}\right) \phi_{j}+\left(\sigma_{i m} \sigma_{j k}-\sigma_{j m} \sigma_{k i}\right) \phi_{k}\right\} \tag{143}
\end{gather*}
$$

Now let $p$, where $0 \leq p \leq 4$, be the number of functions $\phi_{i}$ which have constant value everywhere in $\Omega$. Taking into account hypothesis (f) of the theorem, there are five different possibilities to be examined for equation (143):
$p=0$ : in this case it is straightforward that (143) vanishes if and only if (141) holds.
$p=1$ : the analysis and the result are similar to those of the case $p=0$.
$p=2$ : assume without loss of generality that $\phi_{k}$ and $\phi_{m}$ are constant in $\Omega$ while $\phi_{i}$ and $\phi_{j}$ are not. Then expression (143) vanishes if and only if:

$$
\sigma_{i m} \sigma_{j k}-\sigma_{j m} \sigma_{i k}=\left(\sigma_{i m} \sigma_{j k}-\sigma_{i j} \sigma_{k m}\right) \phi_{k}+\left(\sigma_{i j} \sigma_{k m}-\sigma_{i m} \sigma_{j k}\right) \phi_{m}=0
$$

Given that $\phi_{k} \neq \phi_{m}$, these equations are equivalent to (141).
$p=3$ : suppose without loss of generality that $\phi_{i}, \phi_{j}$ and $\phi_{k}$ are constant in $\Omega$, while $\phi_{m}$ is not. Then expression (143) is equal to zero if and only if:

$$
\left\{\left(\sigma_{j m} \sigma_{k i}-\sigma_{k m} \sigma_{i j}\right) \phi_{i}+\left(\sigma_{k m} \sigma_{i j}-\sigma_{i m} \sigma_{j k}\right) \phi_{j}+\left(\sigma_{i m} \sigma_{j k}-\sigma_{j m} \sigma_{k i}\right) \phi_{k}\right\} \partial_{m} \phi_{m}=0
$$

Taking into account that $\partial_{m} \phi_{m}$ does not vanish everywhere in $\Omega$, and that $\phi_{i}, \phi_{j}$ and $\phi_{k}$ are arbitrary (as far as hypothesis ( f ) of the theorem is respected) the outcome is again that (141) is necessary and sufficient for the vanishing of (143).
$p=4$ : equations (143) vanish because $\partial_{m} \phi_{m}=0$ for all possible values of $m$. This is to be expected because in this case we are dealing with a separable Poisson structure [85].

Then conditions (141) are necessary and sufficient for the vanishing of (143) when $0 \leq p \leq 3$. For $p=4$ expression (143) is always zero. This concludes the analysis of the case $\eta=1$.

Let us now turn to the general form (139) of the solution, namely to general $\eta$. To analyze this case, consider an arbitrary four-dimensional skew-symmetric solution $J_{i j}(x)$ of the Jacobi equations. If such solution is multiplied by a $C^{\infty}(\Omega)$ function $\eta(x)$ the resulting set of functions $\hat{J}_{i j}(x)=\eta(x) J_{i j}(x)$ will be a skew-symmetric solution of (140) if and only if $\eta$ verifies:

$$
\begin{equation*}
\left(J_{i m} J_{j k}+J_{k m} J_{i j}+J_{j m} J_{k i}\right) \partial_{m} \eta=0 \tag{144}
\end{equation*}
$$

where again $(i, j, k, m)$ denotes every permutation of $(1,2,3,4)$. We now apply condition (144) to the functions $J_{i j}$ in (139) for which $\eta=1$, just considered in the first part of this proof. It can thus be seen that:

$$
\begin{gather*}
J_{i m} J_{j k}+J_{k m} J_{i j}+J_{j m} J_{k i}=  \tag{145}\\
\psi_{i} \psi_{j} \psi_{k} \psi_{m} \sum_{p, q, r, s=1}^{4} \phi_{q} \phi_{s}\left\{\sigma_{i m} \sigma_{j k} \delta_{i m p q}^{j k r s}+\sigma_{i j} \sigma_{k m} \delta_{k m p q}^{i j r s}+\sigma_{j m} \sigma_{k i} \delta_{j m p q}^{k i r s}\right\}
\end{gather*}
$$

To evaluate this expression, consider first the cases $0 \leq p \leq 3$, which are verified if and only if (141) is valid. In such situations equation (145) becomes

$$
\begin{gathered}
J_{i m} J_{j k}+J_{k m} J_{i j}+J_{j m} J_{k i}= \\
\psi_{i} \psi_{j} \psi_{k} \psi_{m} \sigma_{i m} \sigma_{j k} \sum_{p, q, r, s=1}^{4} \phi_{q} \phi_{s}\left\{\delta_{i m p q}^{j k r s}+\delta_{k m p q}^{i j r s}+\delta_{j m p q}^{k i r s}\right\}=0
\end{gathered}
$$

and the result is proved. For the remaining case $p=4$ it can be seen after some algebra that (145) amounts to:

$$
\begin{gather*}
J_{i m} J_{j k}+J_{k m} J_{i j}+J_{j m} J_{k i}=\psi_{i} \psi_{j} \psi_{k} \psi_{m}\left\{\left(\sigma_{i m} \sigma_{j k}-\sigma_{j m} \sigma_{k i}\right)\left(\phi_{i} \phi_{j}+\phi_{k} \phi_{m}\right)+\right.  \tag{146}\\
\left.\left(\sigma_{i j} \sigma_{k m}-\sigma_{i m} \sigma_{j k}\right)\left(\phi_{i} \phi_{k}+\phi_{j} \phi_{m}\right)+\left(\sigma_{j m} \sigma_{k i}-\sigma_{i j} \sigma_{k m}\right)\left(\phi_{i} \phi_{m}+\phi_{j} \phi_{k}\right)\right\}
\end{gather*}
$$

This expression must vanish everywhere in $\Omega$ if (139) is to be a solution for arbitrary $\eta$ in this case. Since $p=4$ (namely all $\phi_{i}$ are constant in $\Omega$ ) then hypothesis (f) implies that there are two possibilities: either $\phi_{i} \neq 0$ for every $i=1, \ldots, 4$; or $\phi_{i}=0$ for just one value of $i$. It can be shown in both situations that (146) vanishes if and only if (141) is verified. Consequently, the inclusion of function $\eta$ implies that conditions (141) are also necessary and sufficient in the case $p=4$. This completes the proof of Theorem 3.2.1.1.
Q.E.D.

Therefore the family of Poisson structures just characterized has the matrix form

$$
\mathcal{J}=\eta\left(\begin{array}{cccc}
0 & \sigma_{12} \psi_{1} \psi_{2}\left(\phi_{4}-\phi_{3}\right) & \sigma_{13} \psi_{1} \psi_{3}\left(\phi_{2}-\phi_{4}\right) & \sigma_{14} \psi_{1} \psi_{4}\left(\phi_{3}-\phi_{2}\right)  \tag{147}\\
\sigma_{12} \psi_{1} \psi_{2}\left(\phi_{3}-\phi_{4}\right) & 0 & \sigma_{23} \psi_{2} \psi_{3}\left(\phi_{4}-\phi_{1}\right) & \sigma_{24} \psi_{2} \psi_{4}\left(\phi_{1}-\phi_{3}\right) \\
\sigma_{13} \psi_{1} \psi_{3}\left(\phi_{4}-\phi_{2}\right) & \sigma_{23} \psi_{2} \psi_{3}\left(\phi_{1}-\phi_{4}\right) & 0 & \sigma_{34} \psi_{3} \psi_{4}\left(\phi_{2}-\phi_{1}\right) \\
\sigma_{14} \psi_{1} \psi_{4}\left(\phi_{2}-\phi_{3}\right) & \sigma_{24} \psi_{2} \psi_{4}\left(\phi_{3}-\phi_{1}\right) & \sigma_{34} \psi_{3} \psi_{4}\left(\phi_{1}-\phi_{2}\right) & 0
\end{array}\right)
$$

where additionally $\sigma_{12} \sigma_{34}=\sigma_{13} \sigma_{24}=\sigma_{14} \sigma_{23}$. For what is to follow, the next definition will be necessary:

Definition 3.2.1.2. For every domain $\Omega \subset \mathbb{R}^{4}$, the set of structure matrices defined in $\Omega$ and of the kind (139) characterized in Theorem 3.2.1.1 will be denoted $\Theta(\Omega)$.

To provide the basis for the analysis of the symplectic structure and Darboux reduction in Subsection 3.2.2, and also in order to complete the description of these Poisson structures, the following result is important:

Proposition 3.2.1.3. Let $\Omega \subset \mathbb{R}^{4}$ be a domain, then every structure matrix $\mathcal{J} \in \Theta(\Omega)$ has constant rank of value 2 everywhere in $\Omega$.

Proof. The determinant of $\mathcal{J}$ in (147) is:

$$
\begin{gathered}
|\mathcal{J}|=\eta\left(\psi_{1} \psi_{2} \psi_{3} \psi_{4}\right)^{2}\left[\left(\sigma_{14} \sigma_{23}-\sigma_{13} \sigma_{24}\right)\left(\phi_{1} \phi_{2}+\phi_{3} \phi_{4}\right)+\right. \\
\left.\left(\sigma_{12} \sigma_{34}-\sigma_{14} \sigma_{23}\right)\left(\phi_{1} \phi_{3}+\phi_{2} \phi_{4}\right)+\left(\sigma_{13} \sigma_{24}-\sigma_{12} \sigma_{34}\right)\left(\phi_{1} \phi_{4}+\phi_{2} \phi_{3}\right)\right]^{2}
\end{gathered}
$$

Due to identities (141) we obtain that $|\mathcal{J}|=0$. Therefore the rank cannot be 4 , but only 2 or 0 . The fact that the rank is 2 everywhere in $\Omega$ is implied by conditions (c), (e) and (f) of Theorem 3.2.1.1.
Q.E.D.

Proposition 3.2.1.3 provides the basis for the explicit and global determination of the symplectic structure and Darboux reduction of these Poisson structures. This is the purpose of the next subsection.

### 3.2.2. Casimir invariants and global Darboux analysis

Before developing the main issues of this subsection it is convenient to recall a known definition (see Subsection 1.2.4) that will be needed for their establishment. For this reason, as we did in Chapter 2, now a definition specifically adapted for the present chapter is provided:

Definition 3.2.2.1. Let $\Omega \subset \mathbb{R}^{4}$ be a domain. A reparametrization of time is defined as a transformation of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{148}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$.

Recall that the sense of this definition is the following: let

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{149}
\end{equation*}
$$

be an arbitrary four-dimensional Poisson system defined in a domain $\Omega \subset \mathbb{R}^{4}$. Then, every reparametrization of time of the form (148) leads from (149) to the differential system:

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{150}
\end{equation*}
$$

Note however that such transformation often destroys the Poisson structure for systems of dimension higher than three $[63,70]$, because for a given $\mathcal{J}$ which is a structure matrix, $\eta \mathcal{J}$ is not necessarily a solution of (24-25) as it has been discussed in the proof of Theorem 3.2.1.1 in connection with the four-dimensional case.

The main purpose of this subsection is the investigation of the symplectic structure of family $\Theta(\Omega)$. The central result in this sense corresponds to the next theorem, for which the proof is constructive and completely classifies the different cases arising in the explicit determination of the Casimir invariants and the global reduction to the Darboux canonical form for the members of $\Theta(\Omega)$ :

Theorem 3.2.2.2. For every four-dimensional Poisson system

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x)
$$

defined in a domain $\Omega \subset \mathbb{R}^{4}$ and such that $\mathcal{J} \in \Theta(\Omega)$, both a complete set of independent Casimir invariants as well as the reduction to the Darboux canonical form, can be globally constructed in $\Omega$.

Proof. The proof begins with an auxiliary result:

Lemma 3.2.2.3. Let $\Omega \subset \mathbb{R}^{4}$ be a domain, then every $\mathcal{J} \in \Theta(\Omega)$ can be transformed into a structure matrix $\mathcal{J}^{*}$ defined in a domain $\Omega^{*}$, of rank constant and equal to 2 in $\Omega^{*}$ and components of the form

$$
\begin{equation*}
J_{i j}^{*}(y)=\sigma_{i j} \eta^{*}(y) \sum_{k, l=1}^{4} \epsilon_{i j k l} \phi_{l}^{*}\left(y_{l}\right), \quad i, j=1, \ldots, 4 \tag{151}
\end{equation*}
$$

Moreover, $\mathcal{J}^{*}$ is obtained through the change of variables globally diffeomorphic in $\Omega$

$$
\begin{equation*}
y_{i}\left(x_{i}\right)=\int \frac{\mathrm{d} x_{i}}{\psi_{i}\left(x_{i}\right)}, \quad i=1, \ldots, 4 \tag{152}
\end{equation*}
$$

and $\Omega^{*}=y(\Omega)$ is the diffeomorphic image of $\Omega$ through transformation (152).

Proof of Lemma 3.2.2.3. Recall that after a general smooth coordinate change $y \equiv y(x)$, a given structure matrix $\mathcal{J}(x)$ is transformed into another one $\mathcal{J}^{*}(y)$ according to the rule (30). The use of (30) with transformation (152) on $\mathcal{J}$ leads to (151) with $\eta^{*}(y)=\eta(x(y))$ and $\phi_{i}^{*}(y)=\phi_{i}(x(y))$ for $i=1, \ldots, 4$. The fact that the rank of (151) is constant and of value 2 everywhere in $\Omega^{*}$ is a direct consequence of Proposition 3.2.1.3 and identity (30). Transformation (152) is a global diffeomorphism as a consequence that both $y_{i}\left(x_{i}\right)$ and its inverse $x_{i}\left(y_{i}\right)$ are differentiable for every $i=1, \ldots, 4$, and in addition $y_{i}^{\prime}\left(x_{i}\right) \neq 0$ and $x_{i}^{\prime}\left(y_{i}\right) \neq 0$ everywhere.
Q.E.D.

The structure matrix (151) will be the starting point for the rest of the proof. Now two complementary cases are to be distinguished:

CASE I: $\sigma_{i j} \neq 0$ for all pairs $(i, j), i \neq j$. The analysis of this case must begin with a definition and some preliminary results:

Definition 3.2.2.4. Given a domain $\Omega \subset \mathbb{R}^{4}$, a structure matrix belonging to $\Theta(\Omega)$ is said to be $\sigma$-positive if all its constants $\sigma_{i j}$ can be chosen to be positive, where $i, j=$ $1, \ldots, 4$ and $i \neq j$.

The previous definition leads to:

Lemma 3.2.2.5. Let $\Omega \subset \mathbb{R}^{4}$ be a domain, and let $\mathcal{J} \in \Theta(\Omega)$ be a structure matrix for which $\sigma_{i j} \neq 0$ for every pair $i \neq j$, where $i, j=1, \ldots, 4$. Then $\mathcal{J}$ is $\sigma$-positive and can be expressed in terms of the set of constants $\tilde{\sigma}_{i j}=\left\|\sigma_{i j}\right\|$ for $i, j=1, \ldots, 4$, with $i \neq j$.

Proof of Lemma 3.2.2.5. We define $\sigma \equiv \sigma_{12} \sigma_{34}=\sigma_{13} \sigma_{24}=\sigma_{14} \sigma_{23}$ from now on in this section (recall equation (141)). Four main cases can be distinguished:

Case 1: $\sigma_{i j}>0$ for all $i \neq j$. The matrix is already in $\sigma$-positive form.
Case 2: $\sigma_{i j}<0$ for all $i \neq j$. This is reduced to Case 1 by redefining $\phi_{i}\left(x_{i}\right)$ as $\tilde{\phi}_{i}\left(x_{i}\right)=-\phi_{i}\left(x_{i}\right)$ for every $i$.

Case 3: $\sigma>0$ with constants $\sigma_{i j}$ both positive and negative. There are two subcases:
Case 3.1: There are two negative and four positive constants $\sigma_{i j}$ with $i<j$.
Case 3.1.1: $\sigma_{12}<0$ and $\sigma_{34}<0$.
Case 3.1.2: $\sigma_{13}<0$ and $\sigma_{24}<0$.
Case 3.1.3: $\sigma_{14}<0$ and $\sigma_{23}<0$.
The three subcases 3.1.x are reduced in two steps:
Step 1: redefine $\phi_{i}\left(x_{i}\right)$ as $\tilde{\phi}_{i}\left(x_{i}\right)=-\phi_{i}\left(x_{i}\right)$ for every $i$.
Step 2: redefine $\psi_{i}\left(x_{i}\right)$ as $\tilde{\psi}_{i}\left(x_{i}\right)=-\psi_{i}\left(x_{i}\right)$ for $i=3,4$ in subcase 3.1.1, for $i=1,3$ in subcase 3.1.2 and for $i=1,4$ in subcase 3.1.3.

Case 3.2: There are two positive and four negative constants $\sigma_{i j}$ with $i<j$. These are three possible cases that coincide with the ones appearing after Step 1 of items 3.1.1, 3.1.2 and 3.1.3 and therefore their reduction corresponds to the transformations indicated in Step 2 of those three subcases.

Case 4: $\sigma<0$. Clearly it can be assumed without loss of generality that $\sigma_{12}<0$. Then there are four possibilities:

Case 4.1: $\sigma_{13}<0$ and $\sigma_{14}<0$. Redefining $\tilde{\psi}_{1}\left(x_{1}\right)=-\psi_{1}\left(x_{1}\right)$ it is reduced to Case 1.

Case 4.2: $\sigma_{13}>0$ and $\sigma_{14}>0$. Redefining $\tilde{\psi}_{2}\left(x_{2}\right)=-\psi_{2}\left(x_{2}\right)$ it is reduced to Case 1.

Case 4.3: $\sigma_{13}>0$ and $\sigma_{14}<0$. Redefining $\tilde{\psi}_{3}\left(x_{3}\right)=-\psi_{3}\left(x_{3}\right)$ it is reduced to Case 2.

Case 4.4: $\sigma_{13}<0$ and $\sigma_{14}>0$. Redefining $\tilde{\psi}_{4}\left(x_{4}\right)=-\psi_{4}\left(x_{4}\right)$ it is reduced to Case 2.

This completes the proof of Lemma 3.2.2.5.
Q.E.D.

A result that complements the last lemma is the next one:

Lemma 3.2.2.6. For every set of positive real constants $\left(\sigma_{12}, \sigma_{13}, \sigma_{14}, \sigma_{23}, \sigma_{24}, \sigma_{34}\right)$ verifying conditions (141) there exists a unique set of positive real constants $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}\right)$ such that the equalities $\sigma_{i j}=\sigma_{i} \sigma_{j}$ are satisfied for every pair $(i, j)$, with $i<j, 1 \leq i \leq 3$, $2 \leq j \leq 4$.

Proof of Lemma 3.2.2.6. The existence of the constants $\sigma_{i}$ can be seen on their explicit expressions

$$
\sigma_{1}=\left(\frac{\sigma_{12} \sigma_{13} \sigma_{14}}{\sigma}\right)^{1 / 2}, \quad \sigma_{2}=\left(\frac{\sigma \sigma_{12}}{\sigma_{13} \sigma_{14}}\right)^{1 / 2}, \quad \sigma_{3}=\left(\frac{\sigma \sigma_{13}}{\sigma_{12} \sigma_{14}}\right)^{1 / 2}, \quad \sigma_{4}=\left(\frac{\sigma \sigma_{14}}{\sigma_{12} \sigma_{13}}\right)^{1 / 2}
$$

where now $\sigma>0$. To prove uniqueness, taking logarithms of equalities $\sigma_{i j}=\sigma_{i} \sigma_{j}$ allows reducing the problem to the investigation of the following linear system:

$$
\left(\begin{array}{llll}
1 & 1 & 0 & 0  \tag{153}\\
1 & 0 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
0 & 0 & 1 & 1
\end{array}\right) \cdot\left(\begin{array}{c}
\ln \sigma_{1} \\
\ln \sigma_{2} \\
\ln \sigma_{3} \\
\ln \sigma_{4}
\end{array}\right)=\left(\begin{array}{c}
\ln \sigma_{12} \\
\ln \sigma_{13} \\
\ln \sigma_{14} \\
\ln \sigma-\ln \sigma_{14} \\
\ln \sigma-\ln \sigma_{13} \\
\ln \sigma-\ln \sigma_{12}
\end{array}\right)
$$

Then the application of the Rouché-Fröbenius theorem [69,168] shows that system (153) has a unique solution for $\left(\sigma_{1}, \sigma_{2}, \sigma_{3}, \sigma_{4}\right)$ and the result is proved.
Q.E.D.

Therefore notice that in Case I, Lemma 3.2.2.5 can be used to assume that all the $\sigma_{i j}>0$. Moreover, Lemma 3.2.2.6 can also be employed to write $\sigma_{i j}=\sigma_{i} \sigma_{j}$ in every case. Then from (151) we have the following type of Poisson matrix:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sigma_{i} \sigma_{j} \eta^{*}(y) \sum_{k, l=1}^{4} \epsilon_{i j k l} \phi_{l}^{*}\left(y_{l}\right), \quad i, j=1, \ldots, 4 \tag{154}
\end{equation*}
$$

with $\sigma_{i}>0$ for $i=1, \ldots, 4$. We can now state:

Lemma 3.2.2.7. For a domain $\Omega \subset \mathbb{R}^{4}$, assume that $\mathcal{J} \in \Theta(\Omega)$ is equivalent after transformation (152) to a structure matrix $\mathcal{J}^{*}$ of the form (154) defined in $y(\Omega)=\Omega^{*} \subset$ $\mathbb{R}^{4}$ and such that $\sigma_{i}>0$ for $i=1, \ldots, 4$. Then a complete set of independent Casimir invariants of such structure matrix $\mathcal{J}^{*}$ which are globally defined in $\Omega^{*}$ is given by:

$$
\begin{gather*}
D_{1}(y)=\sigma_{2} \sigma_{3} \sigma_{4} y_{1}+\sigma_{1} \sigma_{3} \sigma_{4} y_{2}+\sigma_{1} \sigma_{2} \sigma_{4} y_{3}+\sigma_{1} \sigma_{2} \sigma_{3} y_{4}  \tag{155}\\
D_{2}(y)=\sigma_{1} \sigma_{2} \sigma_{3} \sigma_{4} \sum_{i=1}^{4} \int \frac{\phi_{i}\left(y_{i}\right)}{\sigma_{i}} \mathrm{~d} y_{i} \tag{156}
\end{gather*}
$$

Proof of Lemma 3.2.2.7. The result is obtained as an application of the Pfaffian method (see Appendix 1). The simplest proof, however, consists in the direct verification of the identities $\mathcal{J}^{*}(y) \cdot \nabla_{y} D_{i}(y)=0$ for $i=1,2$ in (155-156).
Q.E.D.

We can then proceed to the reduction to the Darboux canonical form in Case I. For this, consider the following change of variables globally defined in $\Omega^{*}$ :

$$
\left\{\begin{align*}
z_{1} & =y_{1}  \tag{157}\\
z_{2} & =y_{2} \\
z_{3} & =D_{1}(y) \\
z_{4} & =D_{2}(y)
\end{align*}\right.
$$

where $D_{1}(y)$ and $D_{2}(y)$ are those in (155) and (156). Transformation (157) is everywhere differentiable and invertible, since $\left(y_{1}, y_{2}, D_{1}(y), D_{2}(y)\right)$ are functionally independent in $\Omega^{*}$ because their Jacobian determinant never vanishes in $\Omega^{*}$. If in addition we assume
that (157) is one-to-one in $\Omega^{*}$, then such change of variables is a global diffeomorphism. When the transformation rule (30) is applied for (157) to matrix (154) the result is:

$$
\mathcal{J}^{* *}(z)=\eta^{* *}(z) \cdot\left(\begin{array}{cccc}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

which is defined in $\Omega^{* *}=z\left(\Omega^{*}\right)$, and where $\eta^{* *}(z)=\sigma_{1} \sigma_{2} \eta^{*}(y(z))\left[\phi_{4}^{*}(y(z))-\phi_{3}^{*}(y(z))\right]$. To conclude, the reduction to the Darboux canonical form is achieved making use of Definition 3.2.2.1 to perform a time reparametrization of the form (148), namely $\mathrm{d} \tau=$ $\eta^{* *}(z) \mathrm{d} t$, where $\tau$ is the new time and $\eta^{* *}(z)$ is clearly nonvanishing in $\Omega^{* *}$ and $C^{\infty}\left(\Omega^{* *}\right)$. According to (149) and (150) the result is a new Poisson system with Darboux-type structure matrix:

$$
\mathcal{J}_{D[4,2]}=\left(\begin{array}{cccc}
0 & 1 & 0 & 0  \tag{158}\\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{array}\right)
$$

The reduction is thus globally completed in Case I.

CASE II: $\sigma_{i j}=0$ for some pair $(i, j), i \neq j$. Again matrix (151) is our starting point. Now notice that $\sigma=0$ and as a consequence of conditions (141) we actually have $\sigma_{i j}=0$ for at least three of the six pairs $(i, j)$, with $i<j, 1 \leq i \leq 3,2 \leq j \leq 4$. This leads to eight possible subcases:

$$
\begin{align*}
& \left\{\left(I I . A .1: \sigma_{14}=\sigma_{24}=\sigma_{34}=0\right), \quad\left(I I . A .2: \sigma_{12}=\sigma_{13}=\sigma_{14}=0\right)\right. \\
& \quad\left(I I . A .3: \sigma_{12}=\sigma_{23}=\sigma_{24}=0\right), \quad\left(I I \cdot A \cdot 4: \sigma_{13}=\sigma_{23}=\sigma_{34}=0\right)  \tag{159}\\
& \quad\left(I I . B .1: \sigma_{13}=\sigma_{14}=\sigma_{34}=0\right), \quad\left(I I \cdot B .2: \sigma_{12}=\sigma_{13}=\sigma_{23}=0\right) \\
& \left.\quad\left(I I . B .3: \sigma_{12}=\sigma_{14}=\sigma_{24}=0\right), \quad\left(\text { II.B.4: } \sigma_{23}=\sigma_{24}=\sigma_{34}=0\right)\right\}
\end{align*}
$$

As it can be seen, these subcases are grouped in two different four-member sets (II.A and II.B). The four members of each set present analogous symplectic structures and similar reduction procedures to Darboux form. Let us start with the II.A possibilities:

Lemma 3.2.2.8. For a domain $\Omega \subset \mathbb{R}^{4}$, assume that $\mathcal{J} \in \Theta(\Omega)$ is equivalent after transformation (152) to a structure matrix $\mathcal{J}^{*}$ of the form (151) defined in $y(\Omega)=\Omega^{*} \subset$ $\mathbb{R}^{4}$ and corresponding to one of the subcases II.A. 1 to II.A. 4 in (159). Then a complete
set of independent Casimir invariants of such structure matrix $\mathcal{J}^{*}$ which are globally defined in $\Omega^{*}$ is, respectively:

$$
\begin{array}{ll}
\text { II.A.1: }: & D_{1}(y)=y_{4} \\
D_{2}(y)= & \sigma_{23} \int \phi_{1}\left(y_{1}\right) \mathrm{d} y_{1}+\sigma_{13} \int \phi_{2}\left(y_{2}\right) \mathrm{d} y_{2}+\sigma_{12} \int \phi_{3}\left(y_{3}\right) \mathrm{d} y_{3}- \\
& \left(\sigma_{23} y_{1}+\sigma_{13} y_{2}+\sigma_{12} y_{3}\right) \phi_{4}\left(y_{4}\right) \\
\text { II.A.2 : } \quad D_{1}(y)= & y_{1} \\
D_{2}(y)= & \sigma_{34} \int \phi_{2}\left(y_{2}\right) \mathrm{d} y_{2}+\sigma_{24} \int \phi_{3}\left(y_{3}\right) \mathrm{d} y_{3}+\sigma_{23} \int \phi_{4}\left(y_{4}\right) \mathrm{d} y_{4}- \\
& \left(\sigma_{34} y_{2}+\sigma_{24} y_{3}+\sigma_{23} y_{4}\right) \phi_{1}\left(y_{1}\right) \\
\text { II.A.3: }: & D_{1}(y)=y_{2} \\
D_{2}(y)=\sigma_{34} \int \phi_{1}\left(y_{1}\right) \mathrm{d} y_{1}+\sigma_{14} \int \phi_{3}\left(y_{3}\right) \mathrm{d} y_{3}+\sigma_{13} \int \phi_{4}\left(y_{4}\right) \mathrm{d} y_{4}- \\
& \left(\sigma_{34} y_{1}+\sigma_{14} y_{3}+\sigma_{13} y_{4}\right) \phi_{2}\left(y_{2}\right) \\
\text { II.A.4: }: & D_{1}(y)=y_{3} \\
D_{2}(y)= & \sigma_{24} \int \phi_{1}\left(y_{1}\right) \mathrm{d} y_{1}+\sigma_{14} \int \phi_{2}\left(y_{2}\right) \mathrm{d} y_{2}+\sigma_{12} \int \phi_{4}\left(y_{4}\right) \mathrm{d} y_{4}- \\
& \left(\sigma_{24} y_{1}+\sigma_{14} y_{2}+\sigma_{12} y_{4}\right) \phi_{3}\left(y_{3}\right)
\end{array}
$$

Proof of Lemma 3.2.2.8. It is similar to the one of Lemma 3.2.2.7.
Q.E.D.

We carry out now the reduction to the Darboux canonical form for subcase II.A. For the sake of conciseness this will be done for the first possibility II.A.1, since the procedure is entirely analogous for the remaining situations II.A. 2 to II.A.4. Thus for II.A. 1 the following change of variables globally defined in $\Omega^{*}$ is defined:

$$
\left\{\begin{array}{l}
v_{1}=y_{1}  \tag{160}\\
v_{2}=y_{2} \\
v_{3}=D_{2}(y) \\
v_{4}=D_{1}(y)
\end{array}\right.
$$

where $D_{1}(y)$ and $D_{2}(y)$ are those in Lemma 3.2.2.8 for subcase II.A. 1 and according to hypothesis (c) of Theorem 3.2.1.1 it is assumed $\sigma_{12} \neq 0$ without loss of generality. Now we see that transformation (160) is everywhere differentiable and invertible, since
$\left(y_{1}, y_{2}, D_{1}(y), D_{2}(y)\right)$ are functionally independent in $\Omega^{*}$ because their Jacobian determinant never vanishes in $\Omega^{*}$. Additionally, if we assume that (160) is one-to-one in $\Omega^{*}$, then the change of variables is a global diffeomorphism. Applying (30) and (160) to such structure matrix it is again obtained a structure matrix of the form $\mathcal{J}^{* *}(v)=\eta^{* *}(v) \mathcal{J}_{D[4,2]}$ defined in $v\left(\Omega^{*}\right)$, where now $\eta^{* *}(v)=\sigma_{12} \eta^{*}(y(v))\left[\phi_{4}^{*}(y(v))-\phi_{3}^{*}(y(v))\right]$ and $\mathcal{J}_{D[4,2]}$ is given in (158). The reduction is concluded by means of a time reparametrization (148) of the form $\mathrm{d} \tau=\eta^{* *}(v) \mathrm{d} t$, where $\eta^{* *}(v)$ is nonvanishing in $v\left(\Omega^{*}\right)$ and $C^{\infty}\left(v\left(\Omega^{*}\right)\right)$. The result is thus a new Poisson system with structure matrix (158) and the reduction is globally completed.

Consider next subcases II.B in (159). For each of them both generic and nongeneric possibilities must be distinguished, according to the following definition:

Definition 3.2.2.9. Given a structure matrix of the kind (151) characterized in Lemma 3.2.2.3 and corresponding to one of the subcases II.B. 1 to II.B. 4 in (159), such structure matrix will be called generic if only three of the six constants $\sigma_{i j}$ vanish, for $i<j, 1 \leq i \leq 3,2 \leq j \leq 4$, while if four or five of such constants are zero the same type of structure matrices will be termed nongeneric.

Obviously the case in which all constants $\sigma_{i j}$ vanish is excluded due to condition (c) of Theorem 3.2.1.1. Now the generic II.B subcases will be treated first. For them we have the following result:

Lemma 3.2.2.10. For a domain $\Omega \subset \mathbb{R}^{4}$, assume that $\mathcal{J} \in \Theta(\Omega)$ is equivalent after transformation (152) to a structure matrix $\mathcal{J}^{*}$ of the form (151) defined in $y(\Omega)=\Omega^{*} \subset$ $\mathbb{R}^{4}$ and corresponding to one of the generic subcases II.B. 1 to II.B. 4 in (159). Then a complete set of independent Casimir invariants of such structure matrix $\mathcal{J}^{*}$ which are globally defined in $\Omega^{*}$ is, respectively:

$$
\begin{array}{ll}
\text { II.B.1: } & D_{1}(y)=\sigma_{23} \sigma_{24} y_{1}+\sigma_{12} \sigma_{24} y_{3}+\sigma_{12} \sigma_{23} y_{4} \\
& D_{2}(y)=\sigma_{23} \sigma_{24} \int \phi_{1}\left(y_{1}\right) \mathrm{d} y_{1}+\sigma_{12} \sigma_{24} \int \phi_{3}\left(y_{3}\right) \mathrm{d} y_{3}+\sigma_{12} \sigma_{23} \int \phi_{4}\left(y_{4}\right) \mathrm{d} y_{4} \\
\text { II.B.2: } & D_{1}(y)=\sigma_{24} \sigma_{34} y_{1}+\sigma_{14} \sigma_{34} y_{2}+\sigma_{14} \sigma_{24} y_{3} \\
& D_{2}(y)=\sigma_{24} \sigma_{34} \int \phi_{1}\left(y_{1}\right) \mathrm{d} y_{1}+\sigma_{14} \sigma_{34} \int \phi_{2}\left(y_{2}\right) \mathrm{d} y_{2}+\sigma_{14} \sigma_{24} \int \phi_{3}\left(y_{3}\right) \mathrm{d} y_{3}
\end{array}
$$

$$
\begin{array}{ll}
\text { II.B.3: } & D_{1}(y)=\sigma_{23} \sigma_{34} y_{1}+\sigma_{13} \sigma_{34} y_{2}+\sigma_{13} \sigma_{23} y_{4} \\
& D_{2}(y)=\sigma_{23} \sigma_{34} \int \phi_{1}\left(y_{1}\right) \mathrm{d} y_{1}+\sigma_{13} \sigma_{34} \int \phi_{2}\left(y_{2}\right) \mathrm{d} y_{2}+\sigma_{13} \sigma_{23} \int \phi_{4}\left(y_{4}\right) \mathrm{d} y_{4} \\
\text { II.B. } 4: & D_{1}(y)=\sigma_{13} \sigma_{14} y_{2}+\sigma_{12} \sigma_{14} y_{3}+\sigma_{12} \sigma_{13} y_{4} \\
& D_{2}(y)=\sigma_{13} \sigma_{14} \int \phi_{2}\left(y_{2}\right) \mathrm{d} y_{2}+\sigma_{12} \sigma_{14} \int \phi_{3}\left(y_{3}\right) \mathrm{d} y_{3}+\sigma_{12} \sigma_{13} \int \phi_{4}\left(y_{4}\right) \mathrm{d} y_{4}
\end{array}
$$

Proof of Lemma 3.2.2.10. It is similar to the one of Lemma 3.2.2.7.
Q.E.D.

Regarding the reduction to the Darboux canonical form for the generic II.B subcases, possibility II.B. 1 will be the only one explicitly considered, since again the procedure is completely analogous for the other cases II.B. 2 to II.B.4. Then for II.B. 1 (generic) the transformation globally defined in $\Omega^{*}$ to be performed is:

$$
\begin{cases}w_{1} & =y_{1}  \tag{161}\\ w_{2} & =y_{2} \\ w_{3} & =D_{1}(y) \\ w_{4} & =D_{2}(y)\end{cases}
$$

where $D_{1}(y)$ and $D_{2}(y)$ are those in Lemma 3.2.2.10 for II.B.1. Once (161) is defined, the rest of aspects and procedures of the reduction for the generic II.B. 1 case are entirely similar to those of subcase II.A.1.

The only remaining situations are the nongeneric II.B subcases. The results to be presented are completely analogous for the four possibilities II.B. 1 to II.B.4, and consequently we shall only deal explicitly with II.B. 1 for the sake of brevity. For this, notice that there are two possible nongeneric situations for II.B.1:
II.B.1.a: One of $\left(\sigma_{12}, \sigma_{23}, \sigma_{24}\right)$ vanishes. These three subcases are retrieved as particular instances of the II.A cases already analyzed, in such a way that the complete set of independent Casimir invariants and the reduction to the Darboux canonical form are also obtained as particular results of the ones given for II.A. Specifically, we may have:

- $\sigma_{12}=0:$ Such matrix is a particular case of II.A. 2 in which $\sigma_{34}=0$.
- $\sigma_{23}=0$ : This is a particular case of II.A. 4 with $\sigma_{14}=0$.
- $\sigma_{24}=0$ : It is a particular case of II.A. 1 with $\sigma_{13}=0$.
II.B.1.b: Two of ( $\sigma_{12}, \sigma_{23}, \sigma_{24}$ ) vanish. Then the Casimir invariants are apparent and only a time reparametrization remains in order to reduce the Poisson system to the Darboux canonical form.

The classification is similar for the nongeneric II.B. 2 to II.B. 4 possibilities. Case II is thus concluded.

The demonstration of Theorem 3.2.2.2 is therefore complete.
Q.E.D.

Thus not only the structure matrices considered but also their possible kinds of Casimir invariants and global reductions to the Darboux canonical form are completely characterized after the previous results. Consequently, we are now able to state:

Corollary 3.2.2.11. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider a four-dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{4}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (139) characterized in Theorem 3.2.1.1. Then such Poisson system is algebraically integrable in $\Omega$, and it can be reduced globally in $\Omega$ to a Liouville integrable Hamiltonian system.

Once the main properties have been considered in detail, it is interesting to put in perspective the family just analyzed, as far as it is closely related to other Poisson structures reported in the literature. This is the aim of the next paragraph.

### 3.2.3. Examples and relationship with other Poisson structures

In this subsection the relationship of the family of solutions investigated with some other well-known Poisson structures is briefly explored. This is useful not only because the family of form (139) characterized in Theorem 3.2.1.1 provides a generalization of other structures or families of structures to be mentioned, but also because pointing up the intersections among different families should be helpful for future investigations regarding the Jacobi equations. Additionally, such illustrations provide interesting examples of the solutions analyzed throughout the section.

Consider first the particular case of members of $\Theta(\Omega)$ for which functions $\eta(x)$ and $\phi_{i}\left(x_{i}\right)$ $(i=1, \ldots, 4)$ have constant values. The result is always a separable structure matrix (see Sec-
tion 4.2 for a detailed description), namely a structure matrix of the form $J_{i j}=s_{i j} \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)$, where the $s_{i j}$ are real constants that constitute the entries of a skew-symmetric matrix $S=\left(s_{i j}\right)$, and the $\psi_{i}\left(x_{i}\right)$ are nonvanishing $C^{\infty}(\Omega)$ functions. Recall that separable matrices are always solutions of the Jacobi equations (24-25) independently of the dimension of the Poisson manifold [85]. There are several interesting kinds of Poisson systems for which separable structures are natural in general dimension $n$, and consequently in the specific case of dimension $n=4$. This is the case of Poisson models arising in the domain of population dynamics (for either Lotka-Volterra systems [63,136,143,144,145,146] or generalizations of the QP type [83,86]), plasma models [142], and systems such as the Toda and relativistic Toda lattices $[4,5,36,37,38,39,40,41]$. The reader is referred to Section 4.2 for further examples and the full details regarding issues such as the determination of the Casimir invariants and the reduction to the Darboux canonical form for separable Poisson structures [85]. Note in addition that according to Proposition 3.2.1.3 the structures belonging to $\Theta(\Omega)$ have constant rank of value 2 everywhere in $\Omega$, while the rank of a separable matrix is the rank of $S$. Then it is interesting to remark that the particular case in which $\eta$ and $\phi_{i}(i=1, \ldots, 4)$ are constant does not comprise all possible four-dimensional separable matrices but only separable structures of rank two, thus illustrating an intersection between two different families of Poisson structures.

As a second example, consider the limit case in which the functions $\psi_{4}\left(x_{4}\right)=\phi_{4}\left(x_{4}\right)=0$ are considered in (147). In the resulting structure matrix, it is clear that $x_{4}$ is a Casimir function. Then if a reduction is carried out to the symplectic leaf $x_{4}=c$, the outcome is the 3-d Poisson structure of matrix:

$$
\mathcal{J}_{3 d}=\tilde{\eta} \cdot\left(\begin{array}{ccc}
0 & \psi_{1} \psi_{2} \tilde{\phi}_{3} & -\psi_{1} \psi_{3} \tilde{\phi}_{2}  \tag{162}\\
-\psi_{1} \psi_{2} \tilde{\phi}_{3} & 0 & \psi_{2} \psi_{3} \tilde{\phi}_{1} \\
\psi_{1} \psi_{3} \tilde{\phi}_{2} & -\psi_{2} \psi_{3} \tilde{\phi}_{1} & 0
\end{array}\right)
$$

where $\tilde{\eta}\left(x_{1}, x_{2}, x_{3}\right)=\eta\left(x_{1}, x_{2}, x_{3}, c\right)$ and $\tilde{\phi}_{i}=\sigma_{j k} \phi_{i}$ for $i=1,2,3$, where $(i, j, k)$ denotes an arbitrary permutation of $(1,2,3)$. Dropping the tildes for the sake of clarity, the resulting structures can also be expressed as:

$$
\begin{equation*}
\left(\mathcal{J}_{3 d}\right)_{i j}\left(x_{1}, x_{2}, x_{3}\right)=\eta\left(x_{1}, x_{2}, x_{3}\right) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \sum_{k=1}^{3} \epsilon_{i j k} \phi_{k}\left(x_{k}\right), \quad i, j=1,2,3 \tag{163}
\end{equation*}
$$

Structure matrices of the form (162-163) have been studied in detail in Chapter 2 (Section 2.2) as well as in the literature [70], and actually they comprise as particular cases very different Poisson matrices employed before in several domains, including the Euler top, the KermackMcKendrick model, certain integrable cases of the Lorenz system, population models such
as those of LV and QP types, the Maxwell-Bloch equations, the Rabinovich system, or the RTW interaction equations. A discussion of these particular instances as well as an analysis of the structure matrices (162-163) including their symplectic structure, Casimir invariants and construction of the Darboux coordinates, are present in the aforementioned reference [70] as well as in Chapter 2. Such family is also interesting from the point of view of the separable Poisson structures considered in the first part of this subsection, since it is evident that three-dimensional separable structure matrices are also particular cases of (163).

It can be thus appreciated how the identification of the solutions characterized in Theorem 3.2.1.1 leads to the establishment of some new links among different families of Poisson structures.

### 3.2.4. Some remarks about the case classification

The case classification arising in the global construction of the Darboux canonical form carried out in Subsection 3.2.2 is, apart from its intrinsic interest, significant in the following sense: the mere knowledge of one possible complete set of independent Casimir invariants globally defined, does not guarantee the construction of such canonical form. Interestingly, even if the rank of the structure matrix remains constant and equal to 2 (as it is always the case in the family reported in the present section) several different decompositions may arise, as we have seen, requiring a systematic analysis in order to prove that the family can actually be reduced globally to the Darboux form in all possible cases. This feature was already present in some of the 3-d families examined (to be precise, in the case of the type I and II cyclic Poisson structures) and now is reported also in dimension 4 (this time with an increasing number of possible subcases, actually).

In a sense, such feature is quite relevant in connection with the $n$-dimensional structure matrices of rank lower or equal to 2 given by equation (39) of Subsection 1.3.2. In spite of being the general solution corresponding to the case of general $n$ and $r \leq 2$, expression (39) is of not practical use for our purposes, as the analysis of the present section shows, for two different reasons: first of all, because the constancy of the rank is not guaranteed in the structure matrices (39), which is a key condition for the applicability of Darboux' theorem. And, in second term, because expression (39) is defined on the basis of a single set of Casimir invariants, on which no further assumptions are imposed. As we have seen, just one complete set of independent Casimir invariants may be not sufficient for practical purposes, including the mere definition of the family according to (39), depending on the actual complexity of the

Poisson structure under consideration. This is the reason why a family such as the one in (139) characterized in Theorem 3.2.1.1 considered in this chapter is not, in practice, described by expression (39). Exactly the same situation was found in Sections 2.3 and 2.4 in the context of both types of 3-d cyclic Poisson structures. We shall see in Chapter 4 (Subsection 4.7.8) the result providing the global Darboux reduction for Poisson structures of arbitrary dimension $n$ and constant rank 2. This explains the inclusion here of the present considerations, and accounts also for the fact that structure matrices expressed in the form (39) are not convenient for the establishment of such result.

### 3.3. INSIGHTS PROVIDED BY THE FOUR-DIMENSIONAL FAMILY ANALYZED

We have seen that features quite similar to those found in the 3-d case now appear in the 4-d situation, but obviously with a greater richness of possibilities associated with the increasing dimension of phase space, while the rank remains constant and equal to 2. In particular, as it was the rule in $3-\mathrm{d}$, now we still have an essentially double-step reduction procedure, consisting in a first change of variables, followed by a final time reparametrization. Now we have the background necessary to investigate in the next chapter the general $n$-dimensional case. There, many of the previous features will be elucidated and put under a common perspective. Specifically, it will be proved that the kind of reductions previously investigated are based on the assumption that all possible NTTs are acceptable in the sense that they preserve the Poisson structure. This is valid in the systems analyzed in Chapters 2 and 3, but not in general. It is known in the literature (and it was mentioned in different points of the exposition in Chapters 2 and 3 ) that all time reparametrizations are acceptable in dimensions 2 and 3, but not in general in dimensions equal or higher than 4 . But, in addition, in the present chapter we have seen that every NTT preserves the Poisson structures associated with matrices (147) in spite of being a 4-d family. This state of affairs will be clarified and notably refined in the next chapter. As a consequence, the conditions for the applicability of this kind of two-step global Darboux reduction procedure will be identified and analyzed in what follows. In other words, the fact that the role of time reparametrizations is important for the perspective adopted in the present work explains that, apart from being used when required, it will be necessary to focus on them as a subject of research on which much of what has been done converges. To this it must be added that in the next chapter we shall see instances of $n$-d solution families for which a global Darboux reduction can be achieved without the need of NTTs. On the other hand, the specific investigation of time reparametrizations will lead to the general and global Darboux reduction for Poisson structures of rank 2. This seeming diversity of situations will be completely understood in relatively simple terms in the next chapter.

## CHAPTER 4.

## SOLUTIONS OF ARBITRARY DIMENSION

### 4.1. INTRODUCTION

We now start the most important chapter of this thesis, in which a study of the general $n$-dimensional situation is carried out. The reasons for this significance are very diverse and of a fundamental nature. Many of them have been already considered in different points of this dissertation, but due to their importance it is worth recalling them, at least briefly, in the present introduction.

First of all, it is evident that the $n$-d case is of course the most important and, on the other hand, it is the most poorly understood. Consequently, progress in this sense is of paramount importance for the understanding of the solutions of the Jacobi equations and their global analysis. In particular, the number of known $n$-d solutions is rather limited, specially if we wish to consider solution families of arbitrary dimension and rank, defined in terms of functional forms not limited to a given degree of nonlinearity. To such requirements we must attach the interest of identifying those kinds of solutions that, in addition, are amenable to a global symplectic analysis as well as to a global and constructive reduction to the Darboux canonical form. The significance of new contributions of this type is increased by taking into account the scarcity of similar results in the literature. In addition, it is worth anticipating that the $n$-d analysis to be presented here will allow a complete understanding of the results presented in Chapters 2 and 3.

Apart from the characterization and global analysis of $n$-d Poisson structures, but in close connection with it, a significant part of the chapter will be devoted to the investigation of NTTs that preserve the Poisson structure, also in the $n$-d framework. For this reason, as well as for the sake of completeness, it is necessary to provide here a general $n$-d definition of time reparametrizations. Such definition will be of great importance in the present chapter:

Definition 4.1.1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain. A reparametrization of time is defined as a transformation of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{164}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$.

Let us recall, in addition, the close relationship of NTTs with the role of time in Poisson systems. Precisely, this is the reason why NTTs are also termed time reparametrizations in this work and in the literature. Let

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{165}
\end{equation*}
$$

be an arbitrary $n$-dimensional Poisson system defined in a domain $\Omega \subset \mathbb{R}^{n}$. Then, every reparametrization of time of the form (164) leads from (165) to the differential system:

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{166}
\end{equation*}
$$

The question regarding whether or not the vector field (166) is a Poisson system will be of interest in this chapter, since it is known that, in general, such transformations destroy the Poisson structure in dimensions $n \geq 4$ because for a given $\mathcal{J}$ which is a structure matrix, $\eta \mathcal{J}$ is not necessarily a solution of the Jacobi equations (24-25).

The structure of the chapter is the following. In Sections 4.2 and 4.3 two $n$-d solution families are investigated (the separable family in the case of 4.2, and the multiseparable family in the case of 4.3). Such families are defined for arbitrary values of $n$ and $r$, and can be globally analyzed (including the reduction to the Darboux canonical form) without resort to the use of NTTs, namely by purely diffeomorphic transformations. A very special kind of Poisson structures, termed distinguished solutions (or D-solutions) will be the subject of Section 4.4. D-solutions are also defined for arbitrary values of $n$ and $r$. In this case, apart from the family characterization, the functional form of D-solutions is mathematically remarkable and permits the investigation of properties associated with them which allow a simple but elegant generalization of any given D-solution. In Sections 4.5 and 4.6 we focus on two formally similar (but different) families of Poisson structures termed hemiseparable solutions (of types I and II, respectively) which are defined for arbitrary $n$ but only admit the value $r=2$, and can also be globally analyzed, including the global reduction to Darboux canonical form. Curiously, the reduction of both hemiseparable families requires the combined use of a diffeomorphism and an ending NTT. After the background acquired in the $n$-dimensional
context, the clarification of this situation will be carried out in Section 4.7, which is devoted to the investigation of families of NTTs valid in arbitrary dimension. The results obtained lead not only to the understanding of the naturalness of the several procedures employed in different situations, but in addition allow the generalization of the separable, multiseparable and D-solution families, as well as a very general result regarding the global construction of the Darboux canonical form for structure matrices of rank 2 and arbitrary dimension. The present chapter is finished in Section 4.8 with some concluding remarks.

### 4.2. SEPARABLE SOLUTIONS

In this section, the characterization of a family [85] of $n$ - $d$ solutions of the Jacobi identities is presented. Such a family is very general, as far as it is defined for arbitrary values of the rank, and it is functionally built on the use of functions of arbitrary nonlinearity. Such solution family can be regarded as the result of applying the classical method of separation of variables $[8,67,170,176]$, a feature which accounts for the denomination given to it. As an outcome of their generality, separable Poisson structures unify in a common framework many different and well-known Poisson systems seemingly unrelated. This unification is not only conceptual, but also allows the development of general and global methods of analysis, including the constructive determination of the Darboux canonical form.

The section begins with the characterization of the family in Subsection 4.2.1. Later, Subsection 4.2 .2 is devoted to the global construction of the Casimir invariants and the Darboux canonical form. The analysis is concluded in Subsection 4.2 .3 with some examples.

### 4.2.1. Characterization of the family

We first provide the characterization of the separable family of structure matrices:

Theorem 4.2.1.1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain, and let $\left(\psi_{1}\left(x_{1}\right), \ldots, \psi_{n}\left(x_{n}\right)\right)$ be a set of functions defined in $\Omega$, which are $C^{\infty}(\Omega)$ and do not vanish in any point of $\Omega$. In addition, let $S \equiv\left(s_{i j}\right)$ be an $n \times n$ constant skew-symmetric real matrix. Then every matrix $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ given by

$$
\begin{equation*}
J_{i j}(x)=s_{i j} \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right), \quad i, j=1, \ldots, n \tag{167}
\end{equation*}
$$

is a structure matrix globally defined in $\Omega$.

Proof. Skew-symmetry is clear in (167) by construction. To complete the proof of the statement, we substitute the entries of matrix $\mathcal{J}$ in (167) into the Jacobi equations (25). We arrive at:

$$
\begin{gather*}
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{j l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i j}\right)= \\
\sum_{l=1}^{n}\left[s_{i l} s_{j k} \psi_{i} \psi_{l}\left(\delta_{j l} \psi_{j}^{\prime} \psi_{k}+\delta_{k l} \psi_{j} \psi_{k}^{\prime}\right)+s_{j l} s_{k i} \psi_{j} \psi_{l}\left(\delta_{k l} \psi_{k}^{\prime} \psi_{i}+\delta_{i l} \psi_{k} \psi_{i}^{\prime}\right)+s_{k l} s_{i j} \psi_{k} \psi_{l}\left(\delta_{i l} \psi_{i}^{\prime} \psi_{j}+\delta_{j l} \psi_{i} \psi_{j}^{\prime}\right)\right] \tag{168}
\end{gather*}
$$

where $\psi_{i}^{\prime}$ means $\mathrm{d} \psi_{i} / \mathrm{d} x_{i}$ for every $i$, and $\delta_{i j}$ is Kronecker's delta. Simplifying the deltas in (168) we arrive at:

$$
\begin{gathered}
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{j l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i j}\right)= \\
\psi_{i} \psi_{j} \psi_{k}\left[\psi_{j}^{\prime}\left(s_{j i} s_{j k}+s_{i j} s_{j k}\right)+\psi_{k}^{\prime}\left(s_{j k} s_{k i}+s_{k j} s_{k i}\right)+\psi_{i}^{\prime}\left(s_{k i} s_{i j}+s_{i k} s_{i j}\right)\right]=0
\end{gathered}
$$

due to the skew-symmetry of $S$. This proves the result.
Q.E.D.

Definition 4.2.1.2. The structure matrices of the form (167) characterized in Theorem 4.2.1.1, as well as the Poisson structures associated with them, will be termed separable.

Clearly, the name separable is related to the fact that we have characterized solutions for which a separation of variables has been carried out, a classical method for the determination of solutions in linear PDEs $[8,67,170,176]$ which also works in the present domain (however, not producing the general solution, as it is often the case in the framework of linear equations). The need for the nonvanishing condition $\psi_{i}\left(x_{i}\right) \neq 0$ in $\Omega$ for all $i=1, \ldots, n$, will become clear in the next subsection. In this sense, the following results are devoted to analyze the main properties of the separable structure matrices, and specifically the symplectic structure and the global construction of the Darboux canonical form.

### 4.2.2. Casimir invariants and global Darboux analysis

We shall start by considering the determination of the Casimir invariants:

Theorem 4.2.2.1. Let $\mathcal{J}$ be a separable structure matrix of the form (167) which is defined in a domain $\Omega \subset \mathbb{R}^{n}$. In addition, let $r=\operatorname{Rank}(S)$, and let $\left(k^{[r+1]}, \ldots, k^{[n]}\right)$ be a basis of $\operatorname{Ker}(S)$, where $k^{[i]}=\left(k_{1}^{[i]}, \ldots, k_{n}^{[i]}\right)^{T}$ for $i=r+1, \ldots, n$. Then Rank $(\mathcal{J})=r$ everywhere in $\Omega$ and the functions

$$
\begin{equation*}
D_{i}(x)=\sum_{j=1}^{n} k_{j}^{[i]} \int \frac{\mathrm{d} x_{j}}{\psi_{j}\left(x_{j}\right)}, \quad i=r+1, \ldots, n \tag{169}
\end{equation*}
$$

form a complete set of independent Casimir invariants of $\mathcal{J}$ which are globally defined in $\Omega$.

Proof. We proceed following three steps, which take the form of auxiliary lemmas.

Lemma 4.2.2.2. Let $\mathcal{J}$ be a separable structure matrix of the form (167) which is defined in a domain $\Omega \subset \mathbb{R}^{n}$. Then, $\operatorname{Rank}(\mathcal{J})=\operatorname{Rank}(S)$ at every point $x \in \Omega$.

Proof of Lemma 4.2.2.2. Note that we can write $\mathcal{J}=Q \cdot S \cdot Q$, where matrix $Q$ is defined by $Q=\operatorname{diag}\left(\psi_{1}\left(x_{1}\right), \ldots, \psi_{n}\left(x_{n}\right)\right)$. As a consequence of the nonvanishing character of functions $\psi_{i}\left(x_{i}\right)$ in $\Omega$ for every $i=1, \ldots, n$, we have that matrix $Q$ is invertible everywhere in $\Omega$. Thus matrices $\mathcal{J}$ and $S$ are congruent on $\mathbb{R}$ at every point of $\Omega$. This completes the proof of Lemma 4.2.2.2.
Q.E.D.

Then, according to Lemma 4.2.2.2, there are $(n-r)$ functionally independent Casimir invariants defined in every point of $\Omega$. In fact they can be globally characterized, which is the aim of the next two lemmas:

Lemma 4.2.2.3. Let $\mathcal{J}$ be a separable structure matrix of the form (167) which is defined in a domain $\Omega \subset \mathbb{R}^{n}$. In addition, let $k=\left(k_{1}, \ldots, k_{n}\right)^{T} \in \mathbb{R}^{n}$ be a vector such that $k \in \operatorname{Ker}(S)$. Then, the function

$$
\begin{equation*}
D(x)=\sum_{j=1}^{n} k_{j} \int \frac{\mathrm{~d} x_{j}}{\psi_{j}\left(x_{j}\right)} \tag{170}
\end{equation*}
$$

is a Casimir invariant of $\mathcal{J}$ globally defined in $\Omega$.

Proof of Lemma 4.2.2.3. Function (170) is obtained by application of the Pfaffian method (see Appendix 1). However, the simplest proof consists in checking that under the hypotheses of the lemma, function $D(x)$ in (170) verifies $\mathcal{J} \cdot \nabla D=0$. The $C^{\infty}(\Omega)$ character of $D(x)$ is a consequence of the fact that functions $\psi_{i}\left(x_{i}\right)$ are $C^{\infty}(\Omega)$ and do not vanish in $\Omega$, for every $i=1, \ldots, n$. This proves Lemma 4.2.2.3.
Q.E.D.

Actually, it can be seen that there exists a natural association between linearly independent vectors in $\operatorname{Ker}(S)$ and functionally independent Casimir invariants of the form (170). The last auxiliary lemma proves this property:

Lemma 4.2.2.4. Let $\mathcal{J}$ be a separable structure matrix of the form (167) which is defined in a domain $\Omega \subset \mathbb{R}^{n}$, and such that $\operatorname{Rank}(S)=r$. Additionally, let $\left(k^{[r+1]}, \ldots, k^{[n]}\right)$ be a basis of $\operatorname{Ker}(S)$, where $k^{[i]}=\left(k_{1}^{[i]}, \ldots, k_{n}^{[i]}\right)^{T}$ for $i=r+1, \ldots, n$. Then the functions (169)

$$
D_{i}(x)=\sum_{j=1}^{n} k_{j}^{[i]} \int \frac{\mathrm{d} x_{j}}{\psi_{j}\left(x_{j}\right)}, \quad i=r+1, \ldots, n
$$

form a complete set of independent Casimir invariants of $\mathcal{J}$ which are globally defined in $\Omega$.

Proof of Lemma 4.2.2.4. From Lemma 4.2.2.3 we know that every function $D_{i}(x)$ in (169) is a Casimir invariant globally defined in $\Omega$. Thus we only need to verify the functional independence of functions (169). This can be seen immediately after evaluating the Jacobian matrix associated with $\left(D_{r+1}(x), \ldots, D_{n}(x)\right)$, which is:

$$
\frac{\partial\left(D_{r+1}(x), \ldots, D_{n}(x)\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}=\left(\begin{array}{ccc}
k_{1}^{[r+1]} / \psi_{1}\left(x_{1}\right) & \ldots & k_{n}^{[r+1]} / \psi_{n}\left(x_{n}\right)  \tag{171}\\
\vdots & & \vdots \\
k_{1}^{[n]} / \psi_{1}\left(x_{1}\right) & \ldots & k_{n}^{[n]} / \psi_{n}\left(x_{n}\right)
\end{array}\right)
$$

Obviously, Jacobian (171) is of maximal rank (equal to $n-r$ ) everywhere in $\Omega$ as a consequence of the linear independence of the vectors $\left(k^{[r+1]}, \ldots, k^{[n]}\right)$, together with the nonvanishing property of the $\psi_{i}\left(x_{i}\right)$ in $\Omega$ for every $i=1, \ldots, n$. Since the number of Casimir functions in the set (169) is equal to $(n-r)$, namely the rank of the Jacobian, it is also proved that $\left(D_{r+1}(x), \ldots, D_{n}(x)\right)$ is a complete set of independent Casimir invariants of $\mathcal{J}$. The proof of Lemma 4.2.2.4 is finished.
Q.E.D.

Together, the three previous lemmas prove Theorem 4.2.2.1.
Q.E.D.

Therefore, the Casimir invariants of the separable structure matrices can be completely determined from the kernel of the constant matrix $S$, which is a significant simplification of the problem with respect to the general case. We shall see several instances of this in the next subsection. We also have the following outcome:

Corollary 4.2.2.5. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider an ndimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is separable and has the form (167) with $\operatorname{Rank}(S)=2$. Then such Poisson system is algebraically integrable in $\Omega$.

We can now examine the global reduction to the Darboux canonical form. Such result is accomplished in the following theorem, of which the proof is constructive:

Theorem 4.2.2.6. Let $\Omega \subset \mathbb{R}^{n}$ be a domain where a Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ is defined everywhere, for which $\mathcal{J}(x)$ is a separable structure matrix. Then, such Poisson system can be globally reduced in $\Omega$ to the Darboux canonical form by means of a coordinate
transformation which is a diffeomorphism in $\Omega$.

Proof. We shall use expression (30) for the transformation of the structure matrix (167) under general smooth coordinate changes, $y \equiv y(x)$ :

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{172}
\end{equation*}
$$

We introduce a first diffeomorphic transformation, which is globally defined in $\Omega$ :

$$
\begin{equation*}
y_{i}=\int \frac{\mathrm{d} x_{i}}{\psi_{i}\left(x_{i}\right)}, \quad i=1, \ldots, n \tag{173}
\end{equation*}
$$

The diffeomorphic character of the coordinate change (173) in $\Omega$ is clear since the transformation is everywhere continuous, one-to-one and differentiable, as well as onto $\Omega^{*}$ by definition (since $\Omega^{*} \equiv y(\Omega)$ ). Accordingly, the inverse exists and is also continuous, one-to-one and differentiable. These properties arise as a consequence of the fact that functions $\psi_{i}\left(x_{i}\right)$ are $C^{\infty}(\Omega)$ and do not vanish in $\Omega$, for every $i=1, \ldots, n$. When (167) and (173) are substituted in (172), we obtain:

$$
\begin{equation*}
J_{i j}^{*}(y)=s_{i j}, \quad i, j=1, \ldots, n \tag{174}
\end{equation*}
$$

In other words, we have transformed the structure matrix in such a way that now $\mathcal{J}^{*}(y)=S$ is a matrix of constant entries. In addition to this, we now apply to the structure matrix (174) a second transformation $z \equiv z(y)$, which is linear and therefore defined globally in $\mathbb{R}^{n}$ :

$$
\begin{equation*}
z_{i}=\sum_{j=1}^{n} l_{i j} y_{j}, \quad i=1, \ldots, n \tag{175}
\end{equation*}
$$

In (175) matrix $L \equiv\left(l_{i j}\right)$ is a constant, $n \times n$ invertible matrix. The diffeomorphic character (everywhere in $\mathbb{R}^{n}$ ) of transformation (175) is evident because it is linear and invertible. According to (172), the structure matrix $\mathcal{J}^{*}(y)$ now is transformed in a new one:

$$
\begin{equation*}
\mathcal{J}^{* *}(z)=L \cdot \mathcal{J}^{*} \cdot L^{T}=L \cdot S \cdot L^{T} \tag{176}
\end{equation*}
$$

It is well-known (recall Theorem 1.4.1.2) that the invertible matrix $L$ in (176) can be chosen in order to have:

$$
\mathcal{J}^{* *}(z)=\mathcal{J}_{D[n, r]}=\left(\begin{array}{cc}
0 & 1  \tag{177}\\
-1 & 0
\end{array}\right) \overbrace{\oplus \ldots \oplus}^{r / 2}\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \oplus \mathbb{O}_{1} \overbrace{\oplus \ldots \oplus \mathbb{O}_{1}}^{(n-r)}
$$

where $r=\operatorname{Rank}(S)$ is an even number because $S$ is skew-symmetric, as we know. With (177) the Poisson system has been reduced globally to the Darboux canonical form, since $\mathcal{J}^{* *}(z)$ is
a direct sum of $r / 2$ symplectic matrices $\mathbb{S}_{2}$ plus $(n-r)$ null $1 \times 1$ matrices $\mathbb{O}_{1}$ associated with the Casimir invariants, which in the Darboux representation are decoupled and correspond to the variables $\left(z_{r+1}, \ldots, z_{n}\right)$. The proof is complete.
Q.E.D.

It is worth emphasizing that the reduction has been completed explicitly and globally in the domain of interest, as anticipated. This is remarkable, since the number of Poisson structures for which this can be done is exceedingly limited. Well on the contrary, in the present case this is possible in a quite natural way. The previous reduction implies also the following result:

Corollary 4.2.2.7. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider an ndimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is separable and has the form (167) with $\operatorname{Rank}(S)=2$. Then such Poisson system can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

In addition to their advantageous manipulation properties, the separable structure matrices embrace and unify many different Poisson structures of very common use in the literature. We shall now see a sample in the following subsection.

### 4.2.3. Examples

The first two instances to be shown deal with separable Poisson structures appearing in very diverse kinds of dynamical systems (Example 1) and in different Toda lattice equations (Example 2). The third instance regards the important family of constant skew-symmetric matrices. The subsection concludes with some varied examples briefly displayed, and grouped under the common denomination of "further examples", which complement the presentation of applied instances of Poisson structures of the separable kind.

Example 1. Lotka-Volterra and Quasi-Polynomial systems

The following kind of separable structure matrices

$$
\begin{equation*}
J_{i j}(x)=s_{i j} x_{i} x_{j}, \quad s_{i j}=-s_{j i} \in \mathbb{R}, \quad i, j=1, \ldots, n \tag{178}
\end{equation*}
$$

were first recognized by Plank [143] in the characterization of Poisson structures of the LotkaVolterra equations, and were important later in the wider case of QP Poisson systems [83,86]. However, particular cases of (178) had been previously found in different contexts, such as plasma physics [142] or population dynamics [29,63,136] (see also the relativistic Toda lattice equations in the next example).

In (178) we have $\psi_{i}\left(x_{i}\right)=x_{i}$, and therefore the Casimir invariants are immediately found to be, according to (170), of the form:

$$
\begin{equation*}
D(x)=\sum_{j=1}^{n} k_{j} \ln \left(x_{j}\right), \quad k=\left(k_{1}, k_{2}, \ldots, k_{n}\right)^{T} \in \operatorname{Ker}(S) \tag{179}
\end{equation*}
$$

In the specific case of Lotka-Volterra equations, the first integrals (179) were already noticed by Volterra himself [169], but they were not generically recognized as Casimir invariants until Plank's work [143]. Being Hamiltonian-independent, they also appear in more general types of models sharing the structure matrix (178), such as those treated in $[83,86]$.

The first transformation (173) necessary to achieve the Darboux canonical form now is:

$$
\begin{equation*}
y_{i}=\int \frac{\mathrm{d} x_{i}}{\psi_{i}\left(x_{i}\right)}=\ln \left(x_{i}\right), \quad i=1, \ldots, n \tag{180}
\end{equation*}
$$

The change of variables (180) is to be followed by the linear transformation (175). This kind of two-step reduction to a classical Hamiltonian formulation has been known for long -outside the framework of Poisson structures - in the particular case of conservative, even-dimensional and symplectic Lotka-Volterra systems [101,102,103]. The realization that such reduction is, in fact, Hamiltonian-independent and inherent to structure matrices of the kind (178) was formalized in [83].

## Example 2. Toda lattice and relativistic Toda lattice

Toda lattice equations when expressed in Flaschka's variables $\left(\alpha_{1}, \ldots, \alpha_{N-1}, \beta_{1}, \ldots, \beta_{N}\right)$ constitute a Poisson system with brackets

$$
\begin{equation*}
\left\{\alpha_{i}, \beta_{i}\right\}=-\alpha_{i}, \quad\left\{\alpha_{i}, \beta_{i+1}\right\}=\alpha_{i}, \quad i=1, \ldots, N-1 \tag{181}
\end{equation*}
$$

while the rest of the elementary brackets vanish $[36,37,38,39,40,41]$. The Poisson bracket (181) corresponds to a separable structure matrix given by the following elements,

$$
\begin{cases}\psi_{i}\left(\alpha_{i}\right)=\alpha_{i}, & i=1, \ldots, N-1  \tag{182}\\ \psi_{j}\left(\beta_{j}\right)=1, \quad j=1, \ldots, N\end{cases}
$$

$$
S=\left(\begin{array}{c|c}
\mathbb{O}_{(N-1)} & R_{(N-1) \times N}  \tag{183}\\
\hline-\left(R_{(N-1) \times N}\right)^{T} & \mathbb{O}_{N}
\end{array}\right)
$$

where the subindexes of the submatrices indicate their sizes, $\mathbb{O}$ denotes as usual the null matrix, and

$$
R_{(N-1) \times N}=\left(\begin{array}{cccccc}
-1 & 1 & 0 & \ldots & 0 & 0 \\
0 & -1 & 1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & 1
\end{array}\right)
$$

It is immediate that the kernel of $S$ is one-dimensional, a basis of which is provided by the vector

$$
k=(\overbrace{0, \ldots, 0}^{N-1}, \overbrace{1, \ldots, 1}^{N})^{T}
$$

Consequently, from (170) there is only one independent Casimir invariant,

$$
D=\sum_{j=1}^{N} \int \mathrm{~d} \beta_{j}=\sum_{j=1}^{N} \beta_{j}
$$

which is the result found in [36]. The reduction to the Darboux form also becomes straightforward, since we have to perform transformation (173)

$$
\begin{cases}\tilde{\alpha}_{i}=\ln \left(\alpha_{i}\right), & \\ i=1, \ldots, N-1 \\ \tilde{\beta}_{j}=\beta_{j}, & \\ j=1, \ldots, N\end{cases}
$$

and then carry out the linear change of variables (175).
Analogously, we consider now the relativistic Toda equations expressed in similar variables [36], namely $\left(\alpha_{1}, \ldots, \alpha_{N-1}, \beta_{1}, \ldots, \beta_{N}\right)$. Again, it is a Poisson system with brackets
while the rest of the elementary brackets vanish. This Poisson bracket corresponds to a separable structure matrix of the form (178) examined in Example 1. Therefore, all the considerations mentioned there hold also in the present instance.

## Example 3. Constant structure matrices

A simple but important example is provided by constant structure matrices, or equivalently by constant $n \times n$ skew-symmetric real matrices of arbitrary rank. Such matrices not only
include the entire classical Hamiltonian theory, but in addition appear frequently in very diverse developments for which the use of noncanonical variables is necessary, such as in the case of population dynamics $[83,101,102,103]$ and plasma physics [142], to mention a sample. Actually, it is immediate that constant structure matrices constitute a particular case of those of the form (167) regarded in the previous subsections. In this situation it is worth noting that $\psi_{i}\left(x_{i}\right)=1$, and thus the Casimir functions (169) are linear. Due to these facts, when globally constructing the Darboux canonical form we find that transformation (173) amounts to the identity, and therefore the reduction to the Darboux form only involves the linear transformation (175-177). The reduction of constant structure matrices to classical Hamiltonian form is a well-known procedure $[56,101,102,103]$ that now is retrieved, however, just as a particular case in the wider framework of separable Poisson structures.

Constant structure matrices will be very useful for the purpose of comparing different families, as we shall see in the next sections. In part this is due to the fact that, in spite of their simplicity, they are defined for all possible values of the dimension $n$ and the rank $r$. Because of this combination of simplicity and generality, they are going to be specially adequate for some of the verifications required in what is to follow.

## Further examples: Kermack-McKendrick model, circle maps, Lotka-Volterra

 equations, $2 \times 2$ gamesWe end the present subsection with a brief enumeration of other examples which have also deserved some interest in the literature. We shall not elaborate on them with the detail of the previous instances, but only outline the most interesting features.

As a first example we touch upon the Kermack-McKendrick model [63,135], which admits a 3-d Poisson structure in terms of matrix:

$$
\mathcal{J}\left(x_{1}, x_{2}, x_{3}\right)=\left(\begin{array}{ccc}
0 & -b x_{1} x_{2} & 0  \tag{184}\\
b x_{1} x_{2} & 0 & -a x_{2} \\
0 & a x_{2} & 0
\end{array}\right)
$$

where the $x_{i}$ denote the system variables and $a, b$ are real constants. We again have a separable matrix with $\left(\psi_{1}\left(x_{1}\right), \psi_{2}\left(x_{2}\right), \psi_{3}\left(x_{3}\right)\right)=\left(x_{1}, x_{2}, 1\right)$. Therefore, this example turns out to be very similar to the nonrelativistic Toda lattice examined before, as it can be seen from (182183). We thus find that seemingly unrelated problems can be analyzed in a general, unifying framework.

Next we shall mention the 3-d Poisson structures appearing in the study of certain circle maps [63], in which we have:

$$
\mathcal{J}\left(x_{1}, x_{2}, x_{3}\right)=\left(\begin{array}{ccc}
0 & 0 & -\left(x_{1}\right)^{2}\left(x_{3}\right)^{2} \\
0 & 0 & -\left(x_{2}\right)^{2}\left(x_{3}\right)^{2} \\
\left(x_{1}\right)^{2}\left(x_{3}\right)^{2} & \left(x_{2}\right)^{2}\left(x_{3}\right)^{2} & 0
\end{array}\right)
$$

We thus find $\psi_{i}\left(x_{i}\right)=\left(x_{i}\right)^{2}$, a more nonlinear kind of function. The evaluation of the Casimir invariants and the Darboux canonical form do not present any special difficulty in this case and are omitted.

The list of structure matrices employed for the study of conservative Lotka-Volterra systems is not limited to those of the form (178) already considered. For instance [143], in the two-dimensional case a symplectic structure matrix for which $J_{12}=x_{1}^{1-l_{1}} x_{2}^{1-l_{2}}$ was employed, with $l_{1}, l_{2} \in \mathbb{R}$. Thus we now have $\psi_{i}\left(x_{i}\right)=x_{i}^{1-l_{i}}$ for $i=1,2$. Of course, no nontrivial Casimir invariants exist now. Note also that $S$ is the $2 \times 2$ symplectic matrix $\mathbb{S}_{2}$, and the linear transformation (175) is not necessary: the reduction to the classical Hamiltonian form only involves transformation (173) or, alternatively, the use of a single and direct NTT. Again, the details of the reduction to the Darboux canonical form are omitted for the sake of conciseness, as far as it does not present particular difficulties for any values of the constants $l_{1}$ and $l_{2}$.

Finally we shall consider a very different kind of 2-d structure matrix found in the context of $2 \times 2$ games [88], in which:

$$
\mathcal{J}\left(x_{1}, x_{2}\right)=\left(\begin{array}{cc}
0 & x_{1}\left(1-x_{1}\right) x_{2}\left(1-x_{2}\right) \\
-x_{1}\left(1-x_{1}\right) x_{2}\left(1-x_{2}\right) & 0
\end{array}\right)
$$

Now we have that $\Omega=\operatorname{Int}\left(\varpi_{1} \times \varpi_{1}\right)$ is the interior of the cartesian product of two probability simplices, and $\psi_{i}\left(x_{i}\right)=x_{i}\left(1-x_{i}\right)$ for $i=1,2$. Obviously there are no nontrivial Casimir functions in this case. Again, $S$ is the symplectic matrix $\mathbb{S}_{2}$, and no linear transformation (175) is required, namely the reduction to the classical Hamiltonian form only makes use of transformation (173):

$$
y_{i}=\int \frac{\mathrm{d} x_{i}}{x_{i}\left(1-x_{i}\right)}=\ln \left(\frac{x_{i}}{1-x_{i}}\right), \quad i=1,2
$$

Notice that all the manipulations are properly defined because functions $\psi_{i}\left(x_{i}\right)$ are smooth and nonvanishing in the domain $\Omega$.

### 4.3. MULTISEPARABLE SOLUTIONS

A family of skew-symmetric solutions of the Jacobi partial differential equations is characterized and analyzed in this section. As it was the case for separable Poisson structures, the new family (termed multiseparable in what follows) has some remarkable properties [78]. In first place, it is defined for arbitrary values of the dimension and the rank. Secondly, it is described in terms of arbitrary smooth functions, namely it is not limited to a given degree of nonlinearity. Additionally, it is possible to determine explicitly the fundamental properties of those solutions, such as their Casimir invariants and the algorithm for the reduction to the Darboux canonical form which, as we know, have been reported only for a very limited sample of finite-dimensional Poisson structures. Moreover, such analysis is carried out globally in phase space, thus improving the usual local scope of Darboux' theorem.

The structure of the section is the following. In Subsection 4.3.1 the multiseparable solution family is characterized. Globally defined Casimir invariants and construction of the Darboux canonical form are provided in Subsection 4.3.2. Some applied examples of multiseparable Poisson structures are later examined in Subsection 4.3.3. To conclude, it is convenient to carry out a comparison with the separable family in order to clarify their respective domains of applicability. This issue is analyzed in Subsection 4.3.4.

### 4.3.1. Characterization of the family

In first place, a preliminary definition is provided:

Definition 4.3.1.1. Let $A=\left(a_{i j}\right)$ and $B=\left(b_{i j}\right)$ be two $n \times n$ real and regular matrices $(n \geq 2)$ such that $A=B^{-1}$. Let also $B_{i} \equiv\left(b_{i 1}, \ldots, b_{\text {in }}\right)$ denote the $i$-th row of $B$, for $i=1, \ldots, n$. In addition, let $\Omega \subset \mathbb{R}^{n}$ be a domain in which a system of local coordinates $x=\left(x_{1}, \ldots, x_{n}\right)$ is defined. If $r$ is an even integer, $2 \leq r \leq n$, we shall denote by $\Omega_{i}^{*} \subset \mathbb{R}$ the subsets $\Omega_{i}^{*} \equiv\left\{B_{i} \cdot x \mid x \in \Omega\right\}$, for $i=1, \ldots, r$. Let also $\psi_{i}(x): \Omega \rightarrow \mathbb{R}$, with $i=1, \ldots, r$, denote $r$ functions which are $C^{\infty}(\Omega)$ and do not vanish at any point of $\Omega$, and such that they can be expressed in the form $\psi_{i}(x)=\varphi_{i}\left(B_{i} \cdot x\right)$, where every function $\varphi_{i}\left(y_{i}\right): \Omega_{i}^{*} \rightarrow \mathbb{R}$ is $C^{\infty}\left(\Omega_{i}^{*}\right)$ and does not vanish in any point of $\Omega_{i}^{*}$. Finally let

$$
\Lambda_{i j}^{k l} \equiv\left|\begin{array}{cc}
a_{i k} & a_{i l}  \tag{185}\\
a_{j k} & a_{j l}
\end{array}\right|=a_{i k} a_{j l}-a_{i l} a_{j k}, \quad i, j, k, l=1, \ldots, n
$$

Then an $n \times n$ matrix $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ defined in $\Omega$ is termed multiseparable if it has the form:

$$
\begin{equation*}
J_{i j}(x)=\sum_{k=1}^{r / 2} \Lambda_{i j}^{2 k-1,2 k} \psi_{2 k-1}(x) \psi_{2 k}(x), \quad i, j=1, \ldots, n \tag{186}
\end{equation*}
$$

Moreover, for every $n \geq 2$, multiseparable matrices will be also defined in $\Omega$ for the additional even value $r=0$ as $J_{i j}(x)=0$ for $i, j=1, \ldots, n$ and for every $x \in \Omega$.

## This definition provides the basis for the following result:

Theorem 4.3.1.2. Let $n \geq 2$ be an integer, and let $\Omega \subset \mathbb{R}^{n}$ be a domain in which a multiseparable matrix $\mathcal{J}$ is defined. Then $\mathcal{J}$ is a structure matrix globally defined in $\Omega$.

Proof. Since the case $r=0$ is clear, we shall focus on the case $r \geq 2$. Skew-symmetry of $\mathcal{J}$ is a consequence of the fact that $\Lambda_{i j}^{2 k-1,2 k}=-\Lambda_{j i}^{2 k-1,2 k}$ for $i, j=1, \ldots, n$ and for $k=1, \ldots, r / 2$ in (185-186). Let us now turn to the Jacobi identities (25). Substitution of (186) into (25) produces after some rearrangements:

$$
\begin{gather*}
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{j l} \partial_{l} J_{k i}+J_{k l} \partial_{l} J_{i j}\right)= \\
\sum_{p, q=1}^{r / 2} \varphi_{2 p-1} \varphi_{2 p}\left[\varphi_{2 q-1}^{\prime} \varphi_{2 q} \sum_{l=1}^{n} b_{2 q-1, l}\left(\Lambda_{i l}^{2 p-1,2 p} \Lambda_{j k}^{2 q-1,2 q}+\Lambda_{j l}^{2 p-1,2 p} \Lambda_{k i}^{2 q-1,2 q}+\Lambda_{k l}^{2 p-1,2 p} \Lambda_{i j}^{2 q-1,2 q}\right)\right. \\
\left.+\varphi_{2 q-1} \varphi_{2 q}^{\prime} \sum_{l=1}^{n} b_{2 q, l}\left(\Lambda_{i l}^{2 p-1,2 p} \Lambda_{j k}^{2 q-1,2 q}+\Lambda_{j l}^{2 p-1,2 p} \Lambda_{k i}^{2 q-1,2 q}+\Lambda_{k l}^{2 p-1,2 p} \Lambda_{i j}^{2 q-1,2 q}\right)\right] \equiv \\
\sum_{p, q=1}^{r / 2} \varphi_{2 p-1} \varphi_{2 p}\left(\varphi_{2 q-1}^{\prime} \varphi_{2 q} T_{1}+\varphi_{2 q-1} \varphi_{2 q}^{\prime} T_{2}\right) \tag{187}
\end{gather*}
$$

where $T_{1}$ and $T_{2}$ are terms to be examined separately. Let us first look at $T_{1}$. Using the definition of the constants $\Lambda_{i j}^{k l}$ given in (185), after some algebra it is found that:

$$
\begin{gathered}
T_{1}=\left|\begin{array}{cc}
a_{i, 2 p-1} & a_{i, 2 p} \\
\delta_{2 q-1,2 p-1} & \delta_{2 q-1,2 p}
\end{array}\right| \cdot\left|\begin{array}{cc}
a_{j, 2 q-1} & a_{j, 2 q} \\
a_{k, 2 q-1} & a_{k, 2 q}
\end{array}\right|+ \\
\left|\begin{array}{cc}
a_{j, 2 p-1} & a_{j, 2 p} \\
\delta_{2 q-1,2 p-1} & \delta_{2 q-1,2 p}
\end{array}\right| \cdot\left|\begin{array}{cc}
a_{k, 2 q-1} & a_{k, 2 q} \\
a_{i, 2 q-1} & a_{i, 2 q}
\end{array}\right|+\left|\begin{array}{cc}
a_{k, 2 p-1} & a_{k, 2 p} \\
\delta_{2 q-1,2 p-1} & \delta_{2 q-1,2 p}
\end{array}\right| \cdot\left|\begin{array}{cc}
a_{i, 2 q-1} & a_{i, 2 q} \\
a_{j, 2 q-1} & a_{j, 2 q}
\end{array}\right|
\end{gathered}
$$

where the symbol $\delta_{i j}$ stands for Kronecker's delta. Notice that in $T_{1}$ it is always $\delta_{2 q-1,2 p}=0$ since $p$ and $q$ are integers. Now consider two complementary cases for $T_{1}$ :

Case 1.1. Assume $p=q$ in $T_{1}$. Then $\delta_{2 q-1,2 p-1}=1$ and $T_{1}$ becomes:

$$
\begin{gathered}
T_{1}=a_{i, 2 p}\left(a_{j, 2 p} a_{k, 2 p-1}-a_{j, 2 p-1} a_{k, 2 p}\right)+a_{j, 2 p}\left(a_{k, 2 p} a_{i, 2 p-1}-\right. \\
\left.a_{i, 2 p} a_{k, 2 p-1}\right)+a_{k, 2 p}\left(a_{i, 2 p} a_{j, 2 p-1}-a_{i, 2 p-1} a_{j, 2 p}\right)=0
\end{gathered}
$$

Case 1.2. Let $p \neq q$ in $T_{1}$. Now $\delta_{2 q-1,2 p-1}=0$ and $T_{1}$ vanishes straightforwardly.

Consequently it is $T_{1}=0$ in all cases. Similarly, let us now examine $T_{2}$. Following an analogous procedure it can be found that:

$$
\begin{gathered}
T_{2}=\left|\begin{array}{cc}
a_{i, 2 p-1} & a_{i, 2 p} \\
\delta_{2 q, 2 p-1} & \delta_{2 q, 2 p}
\end{array}\right| \cdot\left|\begin{array}{cc}
a_{j, 2 q-1} & a_{j, 2 q} \\
a_{k, 2 q-1} & a_{k, 2 q}
\end{array}\right|+ \\
\left|\begin{array}{cc}
a_{j, 2 p-1} & a_{j, 2 p} \\
\delta_{2 q, 2 p-1} & \delta_{2 q, 2 p}
\end{array}\right| \cdot\left|\begin{array}{ll}
a_{k, 2 q-1} & a_{k, 2 q} \\
a_{i, 2 q-1} & a_{i, 2 q}
\end{array}\right|+\left|\begin{array}{ll}
a_{k, 2 p-1} & a_{k, 2 p} \\
\delta_{2 q, 2 p-1} & \delta_{2 q, 2 p}
\end{array}\right| \cdot\left|\begin{array}{ll}
a_{i, 2 q-1} & a_{i, 2 q} \\
a_{j, 2 q-1} & a_{j, 2 q}
\end{array}\right|
\end{gathered}
$$

As before, note that $\delta_{2 q, 2 p-1}=0$ in $T_{2}$ since $p$ and $q$ are integers. Two complementary cases appear now for $T_{2}$ :

Case 2.1. It is $p=q$ in $T_{2}$. Thus $\delta_{2 q, 2 p}=1$ and $T_{2}$ reduces to:

$$
\begin{gathered}
T_{2}=a_{i, 2 p-1}\left(a_{j, 2 p-1} a_{k, 2 p}-a_{j, 2 p} a_{k, 2 p-1}\right)+a_{j, 2 p-1}\left(a_{i, 2 p} a_{k, 2 p-1}-\right. \\
\left.a_{i, 2 p-1} a_{k, 2 p}\right)+a_{k, 2 p-1}\left(a_{i, 2 p-1} a_{j, 2 p}-a_{i, 2 p} a_{j, 2 p-1}\right)=0
\end{gathered}
$$

Case 2.2. Assume $p \neq q$ in $T_{2}$. Then $\delta_{2 q, 2 p}=0$ and it is immediate that $T_{2}$ vanishes.

Therefore we also have $T_{2}=0$ in all cases. Together with the previous result $T_{1}=0$, this implies in (187) that multiseparable matrices verify the Jacobi equations (25) for $r \geq 2$. This completes the proof of Theorem 4.3.1.2.
Q.E.D.

One of the most significant features of the multiseparable family of Poisson structures is that it can be explicitly and globally analyzed both for the determination of its Casimir invariants and for the construction of the Darboux canonical form. The development of such issues is the purpose of the next subsection.

### 4.3.2. Casimir invariants and global Darboux analysis

In what follows, a theorem summarizing the main features of the multiseparable solutions is provided. The proof of such theorem is constructive:

Theorem 4.3.2.1. For every n-dimensional ( $n \geq 2$ ) Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ and such that $\mathcal{J} \equiv\left(J_{i j}\right)$ is a multiseparable structure matrix of the form characterized in Definition 4.3.1.1, we have that:
(a) $\operatorname{Rank}(\mathcal{J})=r$ everywhere in $\Omega$.
(b) The functions

$$
\begin{equation*}
D_{i}(x)=\sum_{j=1}^{n} b_{i j} x_{j}, \quad i=r+1, \ldots, n \tag{188}
\end{equation*}
$$

form globally in $\Omega$ a complete set of functionally independent Casimir invariants of $\mathcal{J}$.
(c) It is possible to perform globally in $\Omega$ the reduction of such Poisson system to the Darboux canonical form by means of a transformation which is a diffeomorphism everywhere in the domain $\Omega$.

Proof. The proof of the theorem begins with an auxiliary result:

Lemma 4.3.2.2. If $\mathcal{J}$ is a multiseparable structure matrix defined in the domain $\Omega \subset \mathbb{R}^{n}$, then functions (188) form a set of functionally independent Casimir invariants of $\mathcal{J}$ in $\Omega$.

Proof of Lemma 4.3.2.2. Functional independence can be seen by direct evaluation of the Jacobian matrix of functions (188):

$$
\frac{\partial\left(D_{r+1}(x), \ldots, D_{n}(x)\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}=\left(\begin{array}{ccc}
b_{r+1,1} & \ldots & b_{r+1, n}  \tag{189}\\
\vdots & & \vdots \\
b_{n, 1} & \ldots & b_{n, n}
\end{array}\right)
$$

Thus the Jacobian (189) has constant rank (equal to $n-r)$ in $\mathbb{R}^{n}$ as a consequence that matrix $B$ is invertible, and accordingly functions (188) are functionally independent in $\Omega$. In addition, let us show that such functions are Casimir invariants. If $r=0$ the result is direct. For $r \geq 2$, we evaluate the $i$-th component of the matrix product $\mathcal{J} \cdot \nabla D_{p}$ for every $p=r+1, \ldots, n$ :

$$
\begin{equation*}
\left(\mathcal{J} \cdot \nabla D_{p}\right)_{i}=\sum_{j=1}^{n} J_{i j} \partial_{j} D_{p}=\sum_{k=1}^{r / 2} \varphi_{2 k-1} \varphi_{2 k} \sum_{j=1}^{n} b_{p j} \Lambda_{i j}^{2 k-1,2 k} \tag{190}
\end{equation*}
$$

After some algebra, (190) amounts to:

$$
\left(\mathcal{J} \cdot \nabla D_{p}\right)_{i}=\sum_{k=1}^{r / 2} \varphi_{2 k-1} \varphi_{2 k}\left|\begin{array}{ll}
a_{i, 2 k-1} & a_{i, 2 k}  \tag{191}\\
\delta_{p, 2 k-1} & \delta_{p, 2 k}
\end{array}\right|
$$

But note that $p=r+1, \ldots, n$, while $1 \leq k \leq(r / 2)$. This implies that in all cases it is $\delta_{p, 2 k-1}=\delta_{p, 2 k}=0$, and the expression in (191) vanishes. Consequently, it is $\mathcal{J} \cdot \nabla D_{p}=0$ for all $p=r+1, \ldots, n$ and the proof of Lemma 4.3.2.2 is complete.
Q.E.D.

A direct outcome of Lemma 4.3.2.2 is that $\operatorname{Rank}(\mathcal{J}) \leq r$ everywhere in $\Omega$. Let us now prove that, in fact, $r$ is the actual value of the rank:

Lemma 4.3.2.3. If $\mathcal{J}$ is a multiseparable structure matrix defined in the domain $\Omega \subset \mathbb{R}^{n}$, then $\operatorname{Rank}(\mathcal{J})=r$ everywhere in $\Omega$.

Proof of Lemma 4.3.2.3. According to Definition 4.3.1.1, the result is verified if $r=0$. For $r \geq 2$, in order to prove this lemma recall first that under a smooth change of variables $y \equiv y(x)$, every structure matrix $\mathcal{J}(x)$ is transformed into a new structure matrix $\mathcal{J}^{*}(y)$ according to the rule:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{192}
\end{equation*}
$$

In our case, we shall perform the following change of variables:

$$
\begin{equation*}
y_{i}=\sum_{j=1}^{n} b_{i j} x_{j}, \quad i=1, \ldots, n \tag{193}
\end{equation*}
$$

In (193) we obviously have $\partial y_{i} / \partial x_{j}=b_{i j}$ for all $i, j=1, \ldots, n$. Taking this into account, substitution of (186) in (192) implies that:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{p=1}^{r / 2} \varphi_{2 p-1}\left(y_{2 p-1}\right) \varphi_{2 p}\left(y_{2 p}\right) \sum_{k, l=1}^{n} b_{i k} b_{j l} \Lambda_{k l}^{2 p-1,2 p} \tag{194}
\end{equation*}
$$

The use of definition (185) in (194) leads after some calculations to:

$$
J_{i j}^{*}(y)=\sum_{p=1}^{r / 2}\left|\begin{array}{cc}
\delta_{i, 2 p-1} & \delta_{i, 2 p}  \tag{195}\\
\delta_{j, 2 p-1} & \delta_{j, 2 p}
\end{array}\right| \varphi_{2 p-1}\left(y_{2 p-1}\right) \varphi_{2 p}\left(y_{2 p}\right)
$$

In (195) three cases can be distinguished:

Case 1. If it is $(i, j)=(2 p-1,2 p)$ we have $J_{i j}^{*}(y)=\varphi_{i}\left(y_{i}\right) \varphi_{j}\left(y_{j}\right)$. This is thus the case for $(i, j)=\{(1,2), \ldots,(r-1, r)\}$.

Case 2. When it is $(i, j)=(2 p, 2 p-1)$ we find $J_{i j}^{*}(y)=-\varphi_{i}\left(y_{i}\right) \varphi_{j}\left(y_{j}\right)$. This happens for $(i, j)=\{(2,1), \ldots,(r, r-1)\}$.

Case 3. In any other situation, it is $J_{i j}^{*}(y)=0$.
Consequently, we have just arrived to the following structure matrix:

$$
\mathcal{J}^{*}(y)=\left(\begin{array}{cc}
0 & \varphi_{1} \varphi_{2}  \tag{196}\\
-\varphi_{1} \varphi_{2} & 0
\end{array}\right) \overbrace{\oplus \cdots \oplus}^{r / 2}\left(\begin{array}{cc}
0 & \varphi_{r-1} \varphi_{r} \\
-\varphi_{r-1} \varphi_{r} & 0
\end{array}\right) \oplus \mathbb{O}_{1} \overbrace{\oplus \ldots \oplus \mathbb{O}_{1}}^{(n-r)}
$$

where $\mathbb{O}_{1}$ denotes the $1 \times 1$ null submatrix. Let us define the set $\Omega^{*} \subset \mathbb{R}^{n}$ according to $\Omega^{*} \equiv$ $\{B \cdot x \mid x \in \Omega\}$. It is clear that $\mathcal{J}^{*}(y)$ in (196) is defined on $\Omega^{*}$. Now let $y^{*}=\left(y_{1}^{*}, \ldots, y_{n}^{*}\right) \in \Omega^{*}$ be a point in which $\mathcal{J}^{*}(y)$ is evaluated. We then have $y^{*}=B \cdot x^{*}$ for some $x^{*} \in \Omega$. But this means that $y_{i}^{*}=B_{i} \cdot x^{*}$ for $i=1, \ldots, r$, which implies that $y_{i}^{*} \in \Omega_{i}^{*}$ for all $i=1, \ldots, r$. On the other hand, it is assumed by Definition 4.3.1.1 that every function $\varphi_{i}\left(y_{i}\right)$ does not vanish in $\Omega_{i}^{*}$ for $i=1, \ldots, r$. We see then that $\operatorname{Rank}\left(\mathcal{J}^{*}\right)=r$ everywhere in $\Omega^{*}$. Since according to transformation (192) matrices $\mathcal{J}(x)$ and $\mathcal{J}^{*}(y)$ are congruent, this implies in particular that $\operatorname{Rank}(\mathcal{J})=r$ at every point of $\Omega$. Lemma 4.3.2.3 is thus proved.
Q.E.D.

As a consequence of Lemmas 4.3.2.2 and 4.3.2.3, we have that the Casimir invariants (188) constitute a complete set. After this remark, the statements (a) and (b) of Theorem 4.3.2.1 are already proved. Let us then regard item (c). The fact that $\operatorname{Rank}(\mathcal{J})=r$ is constant in $\Omega$ implies that Darboux' theorem is applicable. In the case $r=0$ the statement (c) of the theorem is valid since $\mathcal{J}$ does coincide with its Darboux canonical form, the diffeomorphic transformation thus being the identity. Then, in what remains of the proof we shall focus on the case $r \geq 2$. For this, the starting point will be matrix $\mathcal{J}^{*}(y)$ in (196) which was obtained after the diffeomorphic transformation $y=B \cdot x$. Since every function $\varphi_{i}\left(y_{i}\right)$ does not vanish in $\Omega_{i}^{*}$ for $i=1, \ldots, r$, it is possible to perform on $\mathcal{J}^{*}(y)$ an additional transformation of coordinates $z \equiv z(y)$ defined as:

$$
\begin{cases}z_{i}=\int \frac{\mathrm{d} y_{i}}{\varphi_{i}\left(y_{i}\right)} & , \quad i=1, \ldots, r  \tag{197}\\ z_{i}=y_{i} & , \quad i=r+1, \ldots, n\end{cases}
$$

Transformation (197) is globally defined in $\Omega^{*}$, and actually it is not difficult to verify that it is also diffeomorphic: since functions $\varphi_{i}\left(y_{i}\right)$ are $C^{\infty}$ and nonvanishing, both $z_{i}\left(y_{i}\right)$ and its inverse are always differentiable and strictly monotonic for every $i=1, \ldots, n$. The outcome after transformation (197) is a new structure matrix $\mathcal{J}^{* *}(z)$ which is obtained from (192) and
(196) as:

$$
\begin{equation*}
J_{i j}^{* *}(z)=\sum_{k, l=1}^{n} \frac{\partial z_{i}}{\partial y_{k}} J_{k l}^{*}(y) \frac{\partial z_{j}}{\partial y_{l}}=\frac{\mathrm{d} z_{i}}{\mathrm{~d} y_{i}} J_{i j}^{*}(y) \frac{\mathrm{d} z_{j}}{\mathrm{~d} y_{j}}, \quad i, j=1, \ldots, n \tag{198}
\end{equation*}
$$

Now two different cases are to be recognized:

Case 1. If $1 \leq i \leq r$ and $1 \leq j \leq r$, then from (198) we have:

$$
J_{i j}^{* *}(z)=\frac{J_{i j}^{*}(y)}{\varphi_{i}\left(y_{i}\right) \varphi_{j}\left(y_{j}\right)}, \quad i, j=1, \ldots, r
$$

Case 2. In any other case different from the previous one, we obtain $J_{i j}^{* *}(z)=0$ because for all those values of $i$ and $j$ it is $J_{i j}^{*}(y)=0$ in expression (198).

Accordingly, a comparison with (196) shows that:

$$
\mathcal{J}^{* *}(z)=\mathcal{J}_{D[n, r]}=\left(\begin{array}{cc}
0 & 1  \tag{199}\\
-1 & 0
\end{array}\right) \overbrace{\oplus \ldots \oplus}^{r / 2}\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \oplus \mathbb{O}_{1} \overbrace{\oplus \ldots \oplus \mathbb{O}_{1}}^{(n-r)} \overbrace{}
$$

Therefore the Darboux canonical form (199) is globally constructed by means of a diffeomorphism for every $r \geq 2$. The proof of Theorem 4.3.2.1 is complete.
Q.E.D.

Thus the multiseparable Poisson structures considered, as well as their complete families of Casimir invariants and the global reduction to the Darboux canonical form, have been entirely characterized after the previous results, which lead us to the establishment of the following:

Corollary 4.3.2.4. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider an $n$ dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is multiseparable according to (186) and Definition 4.3.1.1, and it has $r=2$. Then such Poisson system is algebraically integrable in $\Omega$, and it can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

At this stage, it is convenient to illustrate by means of some examples the generality of the family just analyzed as well as the different procedures described. This is the purpose of the next subsection.

### 4.3.3. Examples

We shall consider several instances, most of them regarding Poisson structures well-known in the literature, including domains such as population dynamics (Kermack-McKendrick system) or mechanics (Toda lattice). In addition, other examples are useful in order to better explain the nature and scope of the results just developed. This is specially the case in the first two instances analyzed in what follows.

## Example 1. A counterexample on the linearity of Casimir invariants

As we have seen, always there exists a complete set of linear Casimir invariants for multiseparable matrices. The purpose of this instance is to show that such condition is necessary but not sufficient for a structure matrix to be mulstiseparable. As a counterexample of sufficiency, consider the following three-dimensional structure matrix:

$$
\mathcal{J}(x)=\left(\begin{array}{ccc}
0 & \eta(x) & 0  \tag{200}\\
-\eta(x) & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

where $\eta\left(x_{1}, x_{2}, x_{3}\right)$ is a smooth and nonvanishing function. Accordingly, matrix $\mathcal{J}$ in (200) is always a structure matrix, as it can be easily verified, its rank being 2 in all points. In addition, the only independent Casimir invariant can be chosen to be linear, precisely $D(x)=x_{3}$. On the other hand, there are cases in which matrix (200) is not multiseparable. To see this, note that for the multiseparable case with $n=3$ and $r=2$ the general form arising from Definition 4.3.1.1 is just:

$$
\left\{\begin{array}{l}
J_{12}(x)=\Lambda_{12}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)  \tag{201}\\
J_{13}(x)=\Lambda_{13}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right) \\
J_{23}(x)=\Lambda_{23}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)
\end{array}\right.
$$

Consequently, if function $\eta(x)$ is not of the form $\varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)$ then the structure matrix (200) cannot be multiseparable, in spite of having always a complete set of independent linear Casimir invariants.

## Example 2. Constant structure matrices

In spite of their simplicity, constant structure matrices are important both for their practical applications (including the entire classical Hamiltonian theory as well as very diverse
applications based on the use of noncanonical coordinates [101,102,103,142]) and also for purposes relative to the comparison of the separable, multiseparable and distinguished solutions (as we shall see in the next section, constant skew-symmetric matrices have the property of belonging to those three families at the same time). In addition, they are of interest as mere examples due to their generality, since constant structure matrices comprise all possible combinations of dimension and rank.

In order to see that every constant structure matrix is a multiseparable matrix, let us first recall Theorem 1.4.1.2, in which it was shown that every constant $n \times n$ skew-symmetric real matrix $S$ of rank $r$ is congruent in $\mathbb{R}$ with $\mathcal{J}_{D[n, r]}$, namely there exists an $n \times n$ constant invertible real matrix $A$ such that $S=A \cdot \mathcal{J}_{D[n, r]} \cdot A^{T}$. Vice versa, every matrix of the form $A \cdot \mathcal{J}_{D[n, r]} \cdot A^{T}$ is skew-symmetric by construction, as it is easy to check. In other words, the set of constant skew-symmetric real matrices and the set of matrices of the form $A \cdot \mathcal{J}_{D[n, r]} \cdot A^{T}$ coincide. Now if we expand in detail the product $A \cdot \mathcal{J}_{D[n, r]} \cdot A^{T}$ for arbitrary values of $n$ and $r \geq 2$, and for arbitrary $A$, it is immediate to show that the outcome is a matrix $S \equiv\left(s_{i j}\right)$ of the form:

$$
s_{i j}=\sum_{k=1}^{r / 2} \Lambda_{i j}^{2 k-1,2 k}, \quad i, j=1, \ldots, n
$$

And consequently, such matrix $S$ is multiseparable in terms of functions $\psi_{i}(x)=1$ for all $i=1, \ldots, r$. The zero-rank case (namely matrix $\mathbb{O}_{n}$ ) is of course embraced trivially in this argument after Definition 4.3.1.1. It is thus proved that every constant skew-symmetric real matrix is multiseparable.

## Example 3. Kermack-McKendrick system

The following structure matrix is of interest $[63,135]$ for the analysis of the well-known Kermack-McKendrick model for epidemics:

$$
\mathcal{J}(x)=b x_{1} x_{2}\left(\begin{array}{ccc}
0 & 1 & -1  \tag{202}\\
-1 & 0 & 1 \\
1 & -1 & 0
\end{array}\right)
$$

where $b>0$ is a real constant. Since $x_{i}>0$ for all $i=1,2,3$, it is $\operatorname{Rank}(\mathcal{J})=2$, a Casimir invariant being $D(x)=x_{1}+x_{2}+x_{3}$. For what is to follow it is interesting to notice that (202) is not a separable structure matrix. In terms of the elements described in Definition 4.3.1.1,
matrix (202) is multiseparable with:

$$
A=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
-1 & -1 & 1
\end{array}\right) \quad, \quad B=A^{-1}=\left(\begin{array}{ccc}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 1 & 1
\end{array}\right)
$$

and functions $\varphi_{i}\left(y_{i}\right)=\kappa_{i} y_{i}$ for $i=1,2$, where $\kappa_{1}$ and $\kappa_{2}$ are arbitrary real constants verifying the condition $\kappa_{1} \kappa_{2}=b$. We can check how $\mathcal{J}$ in (202) is generated according to Definition 4.3.1.1 and (201):

$$
\begin{gathered}
J_{12}=\Lambda_{12}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)=\left|\begin{array}{cc}
1 & 0 \\
0 & 1
\end{array}\right| \kappa_{1} x_{1} \kappa_{2} x_{2}=b x_{1} x_{2} \\
J_{13}=\Lambda_{13}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)=\left|\begin{array}{cc}
1 & 0 \\
-1 & -1
\end{array}\right| \kappa_{1} x_{1} \kappa_{2} x_{2}=-b x_{1} x_{2} \\
J_{23}=\Lambda_{23}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)=\left|\begin{array}{cc}
0 & 1 \\
-1 & -1
\end{array}\right| \kappa_{1} x_{1} \kappa_{2} x_{2}=b x_{1} x_{2}
\end{gathered}
$$

The calculations for the remaining nonzero entries are entirely similar as far as $\Lambda_{i j}^{k l}=-\Lambda_{j i}^{k l}$ for all $i, j, k, l$. Let us now consider the Darboux canonical form for $\mathcal{J}$. If we apply (192) for the coordinate change (193), namely $y=B \cdot x$, we arrive after some calculations at:

$$
\mathcal{J}^{*}(y)=b y_{1} y_{2}\left(\begin{array}{ccc}
0 & 1 & 0  \tag{203}\\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

To complete the reduction to the Darboux canonical form according to the procedure given in the previous subsection, an additional transformation (197) is to be applied to matrix $\mathcal{J}^{*}(y)$ in (203). Now such transformation amounts to:

$$
\left\{\begin{array}{l}
z_{1}=\int \frac{\mathrm{d} y_{1}}{\kappa_{1} y_{1}}=\frac{1}{\kappa_{1}} \ln y_{1}  \tag{204}\\
z_{2}=\int \frac{\mathrm{d} y_{2}}{\kappa_{2} y_{2}}=\frac{1}{\kappa_{2}} \ln y_{2} \\
z_{3}=y_{3}
\end{array}\right.
$$

Then, the result after the change of coordinates (204) is the Darboux canonical form:

$$
\mathcal{J}^{* *}(z)=\mathcal{J}_{D[3,2]}=\left(\begin{array}{ccc}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & 0
\end{array}\right)
$$

Consequently, the reduction is globally and constructively completed. The diffeomorphic character of all the transformations involved is also evident.

## Example 4. Poisson bracket for the Toda lattice

As a last instance, a Poisson structure which is frequently employed for the study of the Toda system shall be considered $[4,5,36,37,38,39,40,41]$. The Toda lattice ODEs, when expressed in Flaschka's variables $x=\left(x_{1}, \ldots, x_{n}\right)=\left(\alpha_{1}, \ldots, \alpha_{N-1}, \beta_{1}, \ldots, \beta_{N}\right)$ are a Poisson system with noncanonical brackets

$$
\left\{\alpha_{i}, \beta_{i}\right\}=-\alpha_{i}, \quad\left\{\alpha_{i}, \beta_{i+1}\right\}=\alpha_{i}, \quad i=1, \ldots, N-1
$$

while the rest of elementary brackets are zero. Therefore, this is a Poisson structure of dimension $n=(2 N-1)$ and having the following structure matrix
where as usual, $\mathbb{O}$ denotes the square null submatrix of size given by the subindex. This example is conceptually interesting, since matrix (205) is at the same time linear (Lie-Poisson), separable and (as we are going to see) multiseparable. Regarded as a separable matrix, it was already analyzed in this chapter, precisely in Subsection 4.2.3 (Example 2). This double separable and multiseparable nature provides also an additional motivation for a comparison of both families, a task carried out in Subsection 4.3.4. In addition, let us notice again that the rank of $\mathcal{J}$ is $r=n-1=2 N-2$. Consequently, there is only one independent Casimir invariant, which can be chosen to be $D(x)=\sum_{i=1}^{N} \beta_{i}$.

Let us first show that the structure matrix (205) is multiseparable for every $n \geq 3$. In terms of Definition 4.3.1.1, we now have the functions:

$$
\begin{cases}\varphi_{i}\left(y_{i}\right)=-y_{i}, & i=1,3, \ldots, r-1=n-2=2 N-3  \tag{206}\\ \varphi_{i}\left(y_{i}\right)=1, & i=2,4, \ldots, r=n-1=2 N-2\end{cases}
$$

And the matrices $A$ and $B$ are given in what follows. In first place, we have for $A$ :

$$
A=\left(\begin{array}{ccccccccc|c}
-1 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0  \tag{207}\\
0 & 0 & -1 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & -1 & 0 & 0 \\
\hline 0 & 1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 0 & \ldots & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 1 & \ldots & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & -1 & 1
\end{array}\right)
$$

Notice that for the sake of clarity, every row of $A$ is symbolically split in two parts of sizes $2 N-2$ (left) and 1 (right), while vertically every column is also divided schematically in two pieces of sizes $N-1$ (up) and $N$ (down). For $B$ we have:

$$
B=\left(\begin{array}{ccccc|ccccccc}
-1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0  \tag{208}\\
0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & -1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 1 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & -1 & \ldots & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\
0 & 0 & 0 & \ldots & -1 & 0 & 0 & 0 & 0 & \ldots & 0 & 0 \\
0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 1 & 0 \\
\hline 0 & 0 & 0 & \ldots & 0 & 1 & 1 & 1 & 1 & \ldots & 1 & 1
\end{array}\right)
$$

Again, for clarity every row of $B$ has been divided in two parts of sizes $N-1$ (left) and $N$ (right), while vertically every column is also separated in two pieces of sizes $2 N-2$ (up) and 1 (down). It is simple to check that $A$ in (207) and $B$ in (208) are invertible and $A=B^{-1}$. Let us verify that these elements generate the structure matrix (205). According to Definition 4.3.1.1 and equations (206-208) we now have:

$$
\begin{cases}\varphi_{i}\left(B_{i} \cdot x\right)=\alpha_{(i+1) / 2} & , \quad i=1,3, \ldots, r-1=n-2=2 N-3  \tag{209}\\ \varphi_{i}\left(B_{i} \cdot x\right)=1 & , \quad i=2,4, \ldots, r=n-1=2 N-2\end{cases}
$$

Therefore using (186) together with (209) we arrive at:

$$
\begin{equation*}
J_{i j}(x)=\sum_{k=1}^{r / 2} \Lambda_{i j}^{2 k-1,2 k} \alpha_{k}, \quad i, j=1, \ldots, n \tag{210}
\end{equation*}
$$

If we examine matrix $A$ in (207) we see that four cases appear in (210):
Case 1: $1 \leq i \leq(N-1), 1 \leq j \leq(N-1)$. In this case, every determinant $\Lambda_{i j}^{2 k-1,2 k}$ contains at least three zeros, and thus vanishes.

Case 2: $N \leq i \leq(2 N-1), N \leq j \leq(2 N-1)$. Now every determinant $\Lambda_{i j}^{2 k-1,2 k}$ has a null column, and consequently also vanishes.

Case 3: $1 \leq i \leq(N-1), N \leq j \leq(2 N-1)$. Examination of $A$ shows that the coefficient $\Lambda_{i j}^{2 k-1,2 k}$ will be different from zero if and only if for a given $i$ it is $k=i$, and $j$ takes any of the two values $j=(i+N-1)$ or $j=(i+N)$. Then, according to (210) the only entries of $\mathcal{J}$ that do not vanish are the ones associated with those determinants $\Lambda_{i j}^{2 k-1,2 k}$ that are not zero, which are:

$$
\left\{\begin{array}{lll}
\Lambda_{i, i+N-1}^{2 i-1,2 i}=-1 & \Rightarrow & J_{i, i+N-1}=-\alpha_{i} \tag{211}
\end{array}, \quad i=1, \ldots, N-1 . ~(\quad, \quad i=1, \ldots, N-1\right.
$$

Case 4: $N \leq i \leq(2 N-1), 1 \leq j \leq(N-1)$. This case is skew-symmetrical of Case 3 , therefore it is not necessary to repeat the calculations since the argument is entirely similar.

The outcome of the previous classification is precisely matrix $\mathcal{J}$ in (205), as expected.
To conclude the example, let us now turn to the construction of the Darboux canonical form, developed in the last subsection. As we know, the first step is the coordinate transformation (193) of the form $y=B \cdot x$, where $y=\left(y_{1}, \ldots, y_{n}\right)$. From the definition of $B$ in (208) note in particular that we now have:

$$
\begin{equation*}
y_{2 i-1}=-\alpha_{i}, \quad i=1, \ldots, N-1 \tag{212}
\end{equation*}
$$

Making use of (210), (211) and (212), the application to $\mathcal{J}$ in (205) of the transformation rule (192) for the change (193) leads after some algebra to:

$$
\mathcal{J}^{*}(y)=\left(\begin{array}{cc}
0 & -y_{1}  \tag{213}\\
y_{1} & 0
\end{array}\right) \stackrel{\overbrace{}}{\oplus \ldots \oplus}\left(\begin{array}{cc}
0 & -y_{2 N-3} \\
y_{2 N-3} & 0
\end{array}\right) \oplus \mathbb{O}_{1}
$$

We now apply to (213) the second transformation (197) which now becomes:

$$
\begin{cases}z_{i}=-\int \frac{\mathrm{d} y_{i}}{y_{i}}=-\ln y_{i} & , \quad i=1,3, \ldots, 2 N-3  \tag{214}\\ z_{i}=\int \mathrm{d} y_{i}=y_{i} & , \quad i=2,4, \ldots, 2 N-2 \\ z_{i}=y_{i} & , \quad i=2 N-1\end{cases}
$$

Taking (198) into account, the application of transformation (214) to the structure matrix (213) finally leads to the Darboux canonical form:

$$
\mathcal{J}^{* *}(z)=\mathcal{J}_{D[2 N-1,2 N-2]}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \overbrace{\oplus \cdots \oplus}^{(N-1)}\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \oplus \mathbb{O}_{1}
$$

Recall also how the diffeomorphic character of both coordinate transformations (193) and (214) is clear in practice.

### 4.3.4. Comparison with the separable family

As we have seen through the examples, there is to some extent an overlap between the separable and the multiseparable families. Clearly they are very different, but it is interesting anyway to provide some additional details regarding the actual relationship between both kinds of solutions. For reasons that will be clear in the presentation, it is illustrative to follow a progressive treatment in three steps, namely $n=2, n=3$ and finally the general case of practical interest $n \geq 3$.

## Case $n=2$

If $s_{12} \in \mathbb{R}$ is a real constant, in the separable case we have $J_{12}(x)=s_{12} \psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right)$. When $n=2$, it is always possible to suppress formally $s_{12}$ by redefining one function (or both), for example as $\tilde{\psi}_{1}\left(x_{1}\right)=s_{12} \psi_{1}\left(x_{1}\right)$. Thus, we can equivalently say that for the separable case it is $J_{12}(x)=\psi_{1}\left(x_{1}\right) \psi_{2}\left(x_{2}\right)$. On the other hand, for the multiseparable family (with $r=2$ ) now we have:

$$
J_{12}(x)=\Lambda_{12}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)=\left|\begin{array}{cc}
a_{11} & a_{12}  \tag{215}\\
a_{21} & a_{22}
\end{array}\right| \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)
$$

In particular, if we make the choice $A=B=\mathbb{I}_{2}$, then (215) becomes:

$$
J_{12}(x)=\varphi_{1}\left(x_{1}\right) \varphi_{2}\left(x_{2}\right)
$$

Therefore, we obtain that in the case $n=2$, separable solutions are a subset of multiseparable solutions (note the common requirement of smoothness for the structure functions in both families). The converse is not true, because in general (215) is given by:

$$
\begin{equation*}
J_{12}(x)=\Lambda_{12}^{12} \varphi_{1}\left(b_{11} x_{1}+b_{12} x_{2}\right) \varphi_{2}\left(b_{21} x_{1}+b_{22} x_{2}\right) \tag{216}
\end{equation*}
$$

And clearly, the structure function (216) is not separable in general. Consequently, for $n=2$ the separable Poisson structures are properly contained in the multiseparable ones (notice that the case $r=0$, namely matrix $\mathbb{O}_{2}$, is also embraced by the previous statement after Definition 4.3.1.1). Thus, all the two-dimensional separable examples (and the corresponding global analyses carried out for them) considered in Subsection 4.2.3 now become particular instances in the more general framework of multiseparable Poisson structures.

Case $n=3$

We are going to see three simple instances that provide complementary results for this case.

To begin with, an example of a multiseparable structure matrix which is not separable is presented. Such instance was developed previously (matrix (202) of Example 3 from Subsection 4.3.3). The proof of the multiseparable character of (202) is therefore not necessary, since it was already provided at that stage. In addition, it is evident that matrix (202) is not separable, something that can be appreciated by simple inspection (in particular, note that the element $J_{23}=b x_{1} x_{2}$ of (202) does depend on $x_{1}$ ). Consequently, for $n=3$ multiseparable matrices are not a subset of separable matrices.

Let us prove also the converse by means of a second instance. This can be done with the help of Example 3 from Subsection 2.5.2, which is also a structure matrix for the KermackMcKendrick model (a well-known bi-Hamiltonian system [63,135]). Such matrix was also regarded in Subsection 4.2.3 about separable matrices, and in particular it is given by (184). We then see that this matrix is of rank 2. In Subsection 4.2 .3 the separable nature of this structure matrix was already shown. It is interesting to note that a possible choice for the Casimir invariant of (184) is

$$
D\left(x_{1}, x_{2}, x_{3}\right)=x_{3}+\frac{a}{b} \ln x_{1}
$$

which is neither linear nor functionally dependent on a single linear function. This is already a proof in the sense that matrix (184) cannot be multiseparable. Alternatively, this can be
verified directly on the form of the matrix. To see this, notice that in the case $n=3, r=2$, the form of a general multiseparable structure matrix is given by (201). According to such equation, the functional dependence of $J_{12}, J_{23}$ and $J_{31}$ is the same (apart from a multiplicative real constant). Thus, if $J_{12}, J_{23}$ and $J_{31}$ do not vanish then they share the same functional dependence on $x$. On the other hand, in matrix (184) we have $J_{12}=-b x_{1} x_{2}, J_{23}=-a x_{2}$ and $J_{31}=0$, which is not in agreement with a multiseparable functional form. Then, for $n=3$ separable matrices are not a subset of multiseparable matrices.

To conclude the case $n=3$ we go back to Example 4 (the Toda lattice) analyzed in the previous subsection. In such example it was already mentioned that the structure matrix (205) considered is at the same time separable and multiseparable. In particular, matrix (205) is defined for the values $n=3, N=2$ and $r=2$, thus taking the following form:

$$
\mathcal{J}=\left(\begin{array}{c|cc}
0 & -\alpha_{1} & \alpha_{1}  \tag{217}\\
\hline \alpha_{1} & 0 & 0 \\
-\alpha_{1} & 0 & 0
\end{array}\right)
$$

Matrix (217) is defined in terms of variables ( $\alpha_{1}, \beta_{1}, \beta_{2}$ ), and its multiseparable formulation is obtained from matrices

$$
A=\left(\begin{array}{cc|c}
-1 & 0 & 0 \\
\hline 0 & 1 & 0 \\
0 & -1 & 1
\end{array}\right), \quad B=A^{-1}=\left(\begin{array}{c|cc}
-1 & 0 & 0 \\
0 & 1 & 0 \\
\hline 0 & 1 & 1
\end{array}\right)
$$

and functions $\varphi_{1}\left(y_{1}\right)=-y_{1}$ and $\varphi_{2}\left(y_{2}\right)=1$. The separable structure of matrix (217) is, on the other hand, evident.

Then, the three previous examples for the case $n=3$ show that the separable and multiseparable structure matrices have nonempty intersection, but none of the families is a subset of the other one.

In what follows the general case $n \geq 3$ is considered. Of course, this includes the case $n=3$ just seen. However, the separate consideration of the $n=3$ scenario has allowed the presentation of purely three-dimensional examples of interest that otherwise would be excluded from the general $n \geq 3$ analysis that follows. Moreover, considering here the case $n=3$ will allow a more complete presentation of the general situation in which $n \geq 3$, as it will become clear in what follows.

## Case $n \geq 3$

For general $n \geq 3$ we follow a structure similar to the one employed for $n=3$, namely the consideration of three complementary examples. The instances employed are such that they are dimension-independent, namely they are defined for every $n \geq 3$.

In first place, we look at examples of structure matrices that are multiseparable but not separable. Among the infinity of possible choices, we can make use here of the one defined in terms of the following elements:

$$
A=\left(\begin{array}{ccccc}
1 & 0 & 0 & \ldots & 0 \\
-1 & 1 & 0 & \ldots & 0 \\
-1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
-1 & 0 & 0 & \ldots & 1
\end{array}\right), \quad B=A^{-1}=\left(\begin{array}{ccccc}
1 & 0 & 0 & \ldots & 0 \\
1 & 1 & 0 & \ldots & 0 \\
1 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & 0 & 0 & \ldots & 1
\end{array}\right), \quad n \geq 3
$$

In order to comprise at the same time any possible value of $n \geq 3$, we obviously must fix $r=2$. If, to be specific, we choose functions $\varphi_{i}\left(y_{i}\right)=y_{i}^{\gamma_{i}}$ for $i=1,2$, with both $\gamma_{1}, \gamma_{2}>0$, we arrive at structure matrices of the form:

$$
\begin{equation*}
J_{i j}(x)=\Lambda_{i j}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)=\Lambda_{i j}^{12} x_{1}^{\gamma_{1}}\left(x_{1}+x_{2}\right)^{\gamma_{2}}, \quad i, j=1, \ldots, n \tag{218}
\end{equation*}
$$

Evidently, matrix (218) is not separable in general. For instance, the entry $J_{23}$ of the multiseparable matrix (218) is:

$$
J_{23}(x)=\Lambda_{23}^{12} x_{1}^{\gamma_{1}}\left(x_{1}+x_{2}\right)^{\gamma_{2}}=x_{1}^{\gamma_{1}}\left(x_{1}+x_{2}\right)^{\gamma_{2}}
$$

It is clear that a great diversity of analogous examples can be generated in a similar way.
Given that the case $n=3$ was specifically regarded in the previous item, we can also consider now another kind of examples for which $n \geq 4$. This has the advantage of allowing the consideration of the case $r=4$ in the multiseparable framework. Thus if $n \geq 4$ and $r=4$ we have multiseparable structure matrices of the form:

$$
\begin{equation*}
J_{i j}(x)=\Lambda_{i j}^{12} \varphi_{1}\left(B_{1} \cdot x\right) \varphi_{2}\left(B_{2} \cdot x\right)+\Lambda_{i j}^{34} \varphi_{3}\left(B_{3} \cdot x\right) \varphi_{4}\left(B_{4} \cdot x\right), \quad i, j=1, \ldots, n \tag{219}
\end{equation*}
$$

It is also evident now that, in general, structure matrices of the form (219) are not separable, because $J_{i j}$ in (219) consists of a sum of functions, and in addition such entry $J_{i j}$ typically will not depend only on variables $x_{i}$ and $x_{j}$, as it would be the case for a separable structure matrix. We thus conclude that for any $n \geq 3$, the multiseparable structure matrices are not a subset of the separable ones.

Let us now turn to a reciprocal set of examples, namely those of separable structure matrices which are not multiseparable, for any $n \geq 3$. A simple instance is provided by the separable matrices considered in Example 1 of Subsection 4.2.3, and given by expression (178), namely $J_{i j}=s_{i j} x_{i} x_{j}$, with $s_{i j}=-s_{j i}$ for $i, j=1, \ldots, n$. For simplicity it is preferable to restrict ourselves to the case $\operatorname{Rank}(\mathcal{J})=\operatorname{Rank}(S)=(n-1)$. If this is the case, there exists only one independent Casimir invariant, which can be chosen to be:

$$
\begin{equation*}
D(x)=\sum_{i=1}^{n} k_{i} \ln x_{i} \tag{220}
\end{equation*}
$$

where $k=\left(k_{1}, \ldots, k_{n}\right)^{T} \in \operatorname{Ker}(S)$. We see thus that the Casimir invariant (220) is not linear, and in fact it is not linearly dependent on a linear function (apart from particular instances such as those in which $D(x)$ in (220) only depends on a single variable). Since we know that every multiseparable matrix has a complete set of independent linear Casimir invariants, it is then proven that the present kind of separable structure matrices is not in general multiseparable, for any $n \geq 3$.

To conclude, we consider a last family of examples which are, for every $n \geq 3$, separable and multiseparable at the same time. Such family can be generated by means of the multiseparable definition in the following way: according to Definition 4.3.1.1 and expression (186) we introduce the multiseparable family obtained for the choice $A=B=\mathbb{I}_{n}$. The outcome has the form:

$$
\begin{gather*}
J_{i j}(x)=\sum_{k=1}^{r / 2} \Lambda_{i j}^{2 k-1,2 k} \psi_{2 k-1}(x) \psi_{2 k}(x)= \\
\sum_{k=1}^{r / 2}\left|\begin{array}{cc}
\delta_{i, 2 k-1} & \delta_{i, 2 k} \\
\delta_{j, 2 k-1} & \delta_{j, 2 k}
\end{array}\right| \varphi_{2 k-1}\left(x_{2 k-1}\right) \varphi_{2 k}\left(x_{2 k}\right), \quad i, j=1, \ldots, n \tag{221}
\end{gather*}
$$

A multiseparable structure matrix of the form (221) has already been analyzed in the present section, and in particular in the proof of Theorem 4.3.2.1, see the analysis accompanying equations (195-196). According to this, matrix (221) is actually of the form:

$$
\mathcal{J}=\left(\begin{array}{cc}
0 & \varphi_{1} \varphi_{2}  \tag{222}\\
-\varphi_{1} \varphi_{2} & 0
\end{array}\right) \overbrace{\oplus \ldots \oplus}^{r / 2}\left(\begin{array}{cc}
0 & \varphi_{r-1} \varphi_{r} \\
-\varphi_{r-1} \varphi_{r} & 0
\end{array}\right) \oplus \mathbb{O}_{1} \overbrace{\oplus \ldots \oplus \mathbb{O}_{1}}^{(n-r)}
$$

Consequently, the structure matrix (222) is multiseparable by construction, but it is also separable, as it is apparent (note the dependences $\varphi_{i} \equiv \varphi_{i}\left(x_{i}\right)$ for every $\left.i=1, \ldots, r\right)$. And, in addition, structure matrices of the form (222) are defined for every $n \geq 3$, as intended.

Thus for the case $n \geq 3$ we conclude that both families of structure matrices have nonempty intersection, but none of them is contained in the other one. In fact, for $n \geq 3$ there is an infinity of structure matrices verifying either of the following propositions:
(a) They are separable but not multiseparable.
(b) They are multiseparable but not separable.
(c) They are both separable and multiseparable.

The case-by-case comparison is thus complete. We have therefore arrived to some concluding remarks:

## Conclusions of the comparison

Throughout the present subsection the fundamental goal has been to establish that the separable and multiseparable families are different. This was to be expected to some extent, but the purpose of the comparison has allowed an example-based discussion in which use has been made of some instances, useful for a better description of the relationship between both kinds of structure matrices. From the analysis, as well as from the rest of examples in Subsection 4.3.3, it has been learned essentially that:
(a) For every multiseparable structure matrix there is a complete set of independent linear Casimir invariants, but the converse is not true: there exist structure matrices with such kind of complete set that are not multiseparable.
(b) Considered as a whole, both families have a nonempty intersection but none of them is contained in the other one.
(c) However, for $n=2$ the separable family is a subset of the multiseparable family.
(d) In addition, for every $n \geq 3$ both families have a nonempty intersection, but such that none of them is a subset of the other one.

### 4.4. DISTINGUISHED SOLUTIONS (D-SOLUTIONS)

In this section, another new $n$-d family of skew-symmetric solutions of the Jacobi PDEs is investigated. Such family is mathematically remarkable, as far as the functional dependences of the solutions appear to be associated with the Casimir invariants of the solutions themselves. This kind of Poisson structures (termed distinguished solutions or D-solutions due to this reason) are defined for every possible combination of values of the dimension $(n \geq 3)$ and the rank, and are also determined in terms of arbitrary nonlinear smooth functions, properties usually not present at the same time in the already known solution families. In particular, Dsolutions display simple properties allowing the generation of an infinity of D-solutions from a given one, which is a relevant feature, when present, in the framework of the Jacobi equations. Additionally, some families of D-solutions of special significance and complying to the previous requirements are constructively characterized and analyzed. Different properties of interest are discussed with the help of detailed examples.

The section is structured as follows. In Subsection 4.4.1 the family of D-solutions is characterized, and different general properties naturally associated with it are determined. A special (but important) particular subset of D -solutions is given by $\mathrm{D}_{\psi}$-solutions, which are characterized in Subsection 4.4.2. The analysis of D -solutions in general, and $\mathrm{D}_{\psi}$-solutions in particular, is continued by means of a discussion, frequently illustrated with examples, which takes place in Subsections 4.4.3 and 4.4.4. To conclude the section, detailed comparisons with the multiseparable and the separable families are developed in Subsections 4.4.5 and 4.4.6, respectively.

### 4.4.1. Distinguished Jacobi equations and distinguished Poisson structures

We begin with a description of the problem:

Definition 4.4.1.1. Let $\mathcal{J} \equiv\left(J_{i j}\right)$ be an $n \times n$ matrix defined in a domain $\Omega \subset \mathbb{R}^{n}(n \geq 3)$ and composed by $C^{\infty}(\Omega)$ real functions $J_{i j}(x)$. Then $\mathcal{J}$ is said to be a solution of the distinguished Jacobi equations in $\Omega$ if for every $x \in \Omega$ it is skew-symmetric and

$$
\begin{equation*}
\sum_{l=1}^{n} J_{k l} \partial_{l} J_{i j}=0, \quad i, j, k=1, \ldots, n \tag{223}
\end{equation*}
$$

or, equivalently, if $\mathcal{J}^{T}=-\mathcal{J}$ and $\mathcal{J} \cdot \nabla J_{i j}=0$ for all $i, j=1, \ldots, n$, where the superscript $T$ denotes the transpose matrix. Every matrix $\mathcal{J}$ being a solution of the distinguished Jacobi equations (24,223) will be termed a distinguished solution, or a D-solution.

This definition implies two relevant consequences that can be summarized as follows:

Corollary 4.4.1.2. Let $\mathcal{J} \equiv\left(J_{i j}\right)$ be a $D$-solution defined in a domain $\Omega \subset \mathbb{R}^{n}$, then:
(a) $\mathcal{J}$ is a solution of the Jacobi equations (24-25), and therefore it is a structure matrix in $\Omega$. The converse is obviously not true, namely not every structure matrix is a $D$-solution.
(b) Functions $J_{i j}(x)$ are Casimir invariants of $\mathcal{J}$ globally defined in $\Omega$ for all $i, j=1, \ldots, n$.

Therefore, briefly speaking D-solutions can be described as structure matrices defined in terms of their own Casimir invariants. As indicated in Subsection 1.2.2, Casimir invariants are also termed "distinguished functions" in the literature on Poisson systems [139]. This is the reason accounting for the denomination of "distinguished solutions" given here to the present kind of structure matrices entirely composed of Casimir invariants. Later in this section, a wide family of D-solutions will be constructed and characterized in full detail. Before that, it is interesting to further focus on some general properties associated with D-solutions. Such properties are not present in general in Poisson structures, but can be easily determined in D-solutions. In order to see this, a preliminary definition is convenient:

Definition 4.4.1.3. Let $N \equiv\left(N_{i j}(x)\right)$ be an $n \times n$ real matrix defined in a domain $\Omega \subset \mathbb{R}^{n}$ and such that functions $N_{i j}(x)$ are $C^{\infty}(\Omega)$ for all $i, j=1, \ldots, n$. A $C^{\infty}(\Omega)$ function $f(x): \Omega \rightarrow \mathbb{R}$ is said to be kernel-gradient (or $K G$ ) for matrix $N$ if $N \cdot \nabla f=0$ for every $x \in \Omega$.

The previous definition is natural in this context, since a D-solution is just a skewsymmetric matrix for which all the entries belong to the set of its KG functions. This point of view will be useful in brief because a $C^{\infty}$ function $g\left(f_{1}, \ldots, f_{k}\right)$ of one or more KG functions $\left(f_{1}(x), \ldots, f_{k}(x)\right)$ is also a KG function of the same matrix. We can now state a first result:

Theorem 4.4.1.4. Let $\mathcal{J} \equiv\left(J_{i j}\right)$ be an $n \times n$ D-solution defined in a domain $\Omega \subset \mathbb{R}^{n}$ and such that $\operatorname{Rank}(\mathcal{J})=r$ in $\Omega$. Let $\left(D_{r+1}(x), \ldots, D_{n}(x)\right)$ be a complete set of independent Casimir invariants of $\mathcal{J}$ in $\Omega$. In addition, let $A \equiv\left(\alpha_{i j}\right)$ denote an $n \times n$ matrix of entries of the form $\alpha_{i j}\left(D_{r+1}(x), \ldots, D_{n}(x)\right)$ such that the real functions $\alpha_{i j}\left(y_{1}, \ldots, y_{n-r}\right)$ are $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ for all $i, j=1, \ldots, n$. Then:
(a) For every odd real polynomial of $\mathcal{J}$,

$$
\begin{equation*}
P(\mathcal{J})=\sum_{i=0}^{k} a_{2 i+1} \mathcal{J}^{2 i+1}, \quad k \geq 0 \tag{224}
\end{equation*}
$$

and for every skew-symmetric matrix $A$ of the aforementioned kind, the matrix product $(P(\mathcal{J}) \cdot A)^{m} \cdot P(\mathcal{J})$ is a D-solution in $\Omega$ for every integer $m \geq 0$.
(b) For every odd real polynomial $P(\mathcal{J})$ of $\mathcal{J}$ of the form (224) and for every symmetric matrix $A$ of the kind indicated which commutes with $\mathcal{J}$, the product $(A \cdot P(\mathcal{J}))^{m}$ is a $D$-solution in $\Omega$ for every odd integer $m \geq 1$.
(c) For every even real polynomial without constant term of $\mathcal{J}$,

$$
Q(\mathcal{J})=\sum_{i=1}^{k} a_{2 i} \mathcal{J}^{2 i}, \quad k \geq 1
$$

and for every skew-symmetric matrix A of the kind already indicated, the matrix product $(Q(\mathcal{J}) \cdot A)^{m} \cdot Q(\mathcal{J})$ is a $D$-solution in $\Omega$ for every odd integer $m \geq 1$.
(d) If $\eta\left(y_{1}, \ldots, y_{n-r}\right): \mathbb{R}^{n-r} \rightarrow \mathbb{R}$ is an arbitrary $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ real function, then the product $\eta\left(D_{r+1}(x), \ldots, D_{n}(x)\right) \mathcal{J}$ is a $D$-solution in $\Omega$.

Proof. (a) All Casimir invariants of $\mathcal{J}$ are also KG functions for $(P(\mathcal{J}) \cdot A)^{m} \cdot P(\mathcal{J})$. Then, the form of $A$ implies that every entry of $(P(\mathcal{J}) \cdot A)^{m} \cdot P(\mathcal{J})$ is a KG function of the matrix itself. To conclude, notice that $P(\mathcal{J})$ is skew-symmetric:

$$
[P(\mathcal{J})]^{T}=\sum_{i=0}^{k} a_{2 i+1}(-1)^{2 i+1} \mathcal{J}^{2 i+1}=-P(\mathcal{J})
$$

According to this, $(P(\mathcal{J}) \cdot A)^{m} \cdot P(\mathcal{J})$ is also skew-symmetric:

$$
\left[(P(\mathcal{J}) \cdot A)^{m} \cdot P(\mathcal{J})\right]^{T}=-P(\mathcal{J}) \cdot\left[A^{T} \cdot(P(\mathcal{J}))^{T}\right]^{m}=-(P(\mathcal{J}) \cdot A)^{m} \cdot P(\mathcal{J})
$$

(b) The Casimir invariants of $\mathcal{J}$ are KG functions of $(A \cdot P(\mathcal{J}))^{m}$. Due to the functional dependence of $A$, matrix $(A \cdot P(\mathcal{J}))^{m}$ is entirely composed by KG functions of $(A \cdot P(\mathcal{J}))^{m}$. In addition, it can be seen that $(A \cdot P(\mathcal{J}))^{m}$ is skew-symmetric because $P(\mathcal{J})$ is, since:

$$
\left[(A \cdot P(\mathcal{J}))^{m}\right]^{T}=\left([P(\mathcal{J})]^{T} \cdot A^{T}\right)^{m}=(-P(\mathcal{J}) \cdot A)^{m}=-(A \cdot P(\mathcal{J}))^{m}
$$

(c) Every Casimir invariant of $\mathcal{J}$ is a KG function of $(Q(\mathcal{J}) \cdot A)^{m} \cdot Q(\mathcal{J})$, so that taking into account the form of $A$, we have that $(Q(\mathcal{J}) \cdot A)^{m} \cdot Q(\mathcal{J})$ is composed by its own KG functions.

The proof will be complete after verifying the skew-symmetry of $(Q(\mathcal{J}) \cdot A)^{m} \cdot Q(\mathcal{J})$. To see this, note first that $Q(\mathcal{J})$ is symmetric:

$$
[Q(\mathcal{J})]^{T}=\sum_{i=1}^{k} a_{2 i}(-1)^{2 i} \mathcal{J}^{2 i}=Q(\mathcal{J})
$$

And consequently, we have:

$$
\left[(Q(\mathcal{J}) \cdot A)^{m} \cdot Q(\mathcal{J})\right]^{T}=Q(\mathcal{J}) \cdot[-A \cdot Q(\mathcal{J})]^{m}=-(Q(\mathcal{J}) \cdot A)^{m} \cdot Q(\mathcal{J})
$$

(d) Matrix $\eta\left(D_{r+1}(x), \ldots, D_{n}(x)\right) \mathcal{J}$ is obviously skew-symmetric. The Casimir invariants of $\mathcal{J}$ are KG functions of $\eta \mathcal{J}$, and consequently this matrix is fully composed of its own KG functions.

This completes the proof of the theorem.
Q.E.D.

In the previous theorem, note in particular that matrix $A$ can be constant. As indicated, this kind of properties, when present in the context of the Jacobi PDEs, deserve some interest. In particular, Theorem 4.4.1.4 is significant for several reasons. First, because it is not limited from the point of view of the dimension or the rank for which it can be applied, and consequently it is remarkably general. And second, because such kind of results are uncommon in the usual context of finite-dimensional Poisson structures. We see however that they are present in the specific domain of the distinguished problem. Some less general instances of this kind of properties for nondistinguished Poisson structures can be found in [70,71] for certain three-dimensional situations and in [83] for some $n$-dimensional solution families. In this work, this type of results is also found in Sections 2.5 and 4.7.

### 4.4.2. The family of $D_{\psi}$-solutions

The fact that D-solutions are given in terms of their Casimir invariants is mathematically nice but, at the same time, the issue of the practical determination of some representative D-solutions requires further investigation. This is actually possible: as anticipated, after the previous definitions and general properties the aim now is to consider one family of D-solutions which is amenable to constructive characterization. This is the content of the next result:

Theorem 4.4.2.1. Let $n \geq 3$ and $\rho \leq n$ be two positive integers, and consider the ( $n-\rho$ ) functions

$$
\begin{equation*}
D_{l}(x)=x_{l}-\sum_{k=1}^{\rho} a_{l k} x_{k} \quad, \quad l=\rho+1, \ldots, n \tag{225}
\end{equation*}
$$

where $a_{l k}$ are real constants for all $l, k$. In addition, for $i, j=1, \ldots, \rho$, let $\psi_{i j}\left(y_{1}, \ldots, y_{n-\rho}\right)$ be $C^{\infty}\left(\mathbb{R}^{n-\rho}\right)$ functions that are skew-symmetric in their subindexes, namely $\psi_{i j}\left(y_{1}, \ldots, y_{n-\rho}\right)=$ $-\psi_{j i}\left(y_{1}, \ldots, y_{n-\rho}\right)$ for all $i, j$. Finally, let $\mathcal{J} \equiv\left(J_{i j}\right)$ be the $n \times n$ matrix given by:

$$
J_{i j}(x)= \begin{cases}\psi_{i j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , \quad i, j=1, \ldots, \rho  \tag{226}\\ \sum_{k=1}^{\rho} a_{j k} \psi_{i k}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , \quad i=1, \ldots, \rho ; j=\rho+1, \ldots, n \\ \sum_{k=1}^{\rho} a_{i k} \psi_{k j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , i=\rho+1, \ldots, n ; j=1, \ldots, \rho \\ \sum_{k, l=1}^{\rho} a_{i k} a_{j l} \psi_{k l}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right) & , \quad i, j=\rho+1, \ldots, n\end{cases}
$$

Then, $\mathcal{J}$ is a D-solution of the Jacobi equations which is globally defined in $\mathbb{R}^{n}$ and such that $\operatorname{Rank}(\mathcal{J}) \leq \rho-\rho \bmod 2$ for every $x \in \mathbb{R}^{n}$. In addition, the ( $n-\rho$ ) functions $D_{l}(x)$ in (225) constitute everywhere in $\mathbb{R}^{n}$ a set of functionally independent Casimir invariants of $\mathcal{J}$.

Proof. The proof is constructive. For this, let us first consider the submatrix structure of $\mathcal{J}$ as suggested by equation (226), namely:

$$
\mathcal{J} \equiv\left(\begin{array}{c|c}
\mathcal{J}^{[1]} & \mathcal{J}^{[2]}  \tag{227}\\
\hline \mathcal{J}^{[3]} & \mathcal{J}^{[4]}
\end{array}\right)
$$

where $\mathcal{J}^{[1]}, \mathcal{J}^{[2]}, \mathcal{J}^{[3]}$ and $\mathcal{J}^{[4]}$ are submatrices of sizes $\rho \times \rho, \rho \times(n-\rho),(n-\rho) \times \rho$ and $(n-\rho) \times(n-\rho)$, respectively. In the rest of the proof, the entries of $\mathcal{J}^{[k]}$ will be denoted $J_{i j}^{[k]}$ for all $k=1, \ldots, 4$. Therefore, according to (226) and (227) we have $J_{i j}^{[1]}=$ $\psi_{i j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right)$ for all $i, j=1, \ldots, \rho$. Regarding $\mathcal{J}^{[2]}$, note that from (226) we have:

$$
\begin{equation*}
J_{i j}^{[2]}=\sum_{k=1}^{\rho} a_{j k} J_{i k}^{[1]}, \quad i=1, \ldots, \rho, j=\rho+1, \ldots, n \tag{228}
\end{equation*}
$$

Similarly for $\mathcal{J}^{[3]}$, equation (226) implies that:

$$
\begin{equation*}
J_{i j}^{[3]}=\sum_{k=1}^{\rho} a_{i k} J_{k j}^{[1]}, \quad i=\rho+1, \ldots, n, j=1, \ldots, \rho \tag{229}
\end{equation*}
$$

To conclude, notice that from (226) we also obtain for $\mathcal{J}^{[4]}$ :

$$
\begin{equation*}
J_{i j}^{[4]}=\sum_{k, l=1}^{\rho} a_{i k} a_{j l} J_{k l}^{[1]}, \quad i, j=\rho+1, \ldots, n \tag{230}
\end{equation*}
$$

Taking (228) into account, we can express (230) in an alternative form, very similar to that for $\mathcal{J}^{[3]}$ in (229):

$$
\begin{equation*}
J_{i j}^{[4]}=\sum_{k=1}^{\rho} a_{i k} J_{k j}^{[2]}, \quad i, j=\rho+1, \ldots, n \tag{231}
\end{equation*}
$$

For what is to come, it is also necessary to observe that for $\mathcal{J}^{[4]}$ there is another expression analogous to the one displayed in (231). Now such alternative dependence can be obtained from (229) and (230) in terms of $\mathcal{J}^{[3]}$ as:

$$
\begin{equation*}
J_{i j}^{[4]}=\sum_{k=1}^{\rho} a_{j k} J_{i k}^{[3]}, \quad i, j=\rho+1, \ldots, n \tag{232}
\end{equation*}
$$

With the help of expressions (228-232) some auxiliary results can be provided now:

Lemma 4.4.2.2. Matrix $\mathcal{J} \equiv\left(J_{i j}\right)$ in (226) is skew-symmetric.

Proof of Lemma 4.4.2.2. Submatrix $\mathcal{J}^{[1]}$ is skew-symmetric by definition. Let us now prove that submatrices $\mathcal{J}^{[2]}$ and $\mathcal{J}^{[3]}$ also verify skew-symmetry. According to (228) and (229) we have:

$$
J_{i j}^{[2]}+J_{j i}^{[3]}=\sum_{k=1}^{\rho} a_{j k} J_{i k}^{[1]}+\sum_{k=1}^{\rho} a_{j k} J_{k i}^{[1]}=0, \quad i=1, \ldots, \rho, j=\rho+1, \ldots, n
$$

To conclude, making use of (230) for $\mathcal{J}^{[4]}$ we have:

$$
J_{j i}^{[4]}=\sum_{k, l=1}^{\rho} a_{j k} a_{i l} J_{k l}^{[1]}=-\sum_{k^{\prime}, l^{\prime}=1}^{\rho} a_{i k^{\prime}} a_{j l^{\prime}} J_{k^{\prime} l^{\prime}}^{[1]}=-J_{i j}^{[4]}, \quad i, j=\rho+1, \ldots, n
$$

Lemma 4.4.2.2 is thus proved.
Q.E.D.

Lemma 4.4.2.3. Functions $D_{l}$ in (225) are $K G$ for matrix $\mathcal{J}$ in (226) for all $l=\rho+1, \ldots, n$.

Proof of Lemma 4.4.2.3. Consider the first $\rho$ rows of $\mathcal{J}$ (namely those comprising $\mathcal{J}^{[1]}$ and $\left.\mathcal{J}^{[2]}\right)$. Thus for $i=1, \ldots, \rho$ and for $l=\rho+1, \ldots, n$ we have:

$$
\left(\mathcal{J} \cdot \nabla D_{l}\right)_{i}=\sum_{j=1}^{n} J_{i j} \partial_{j} D_{l}=-\sum_{j=1}^{\rho} J_{i j}^{[1]} a_{l j}+\sum_{j=\rho+1}^{n} J_{i j}^{[2]} \delta_{l j}=0
$$

where $\delta_{l j}$ denotes Kronecker's delta and the last equality is a consequence of (228). Analogously, consider now the last $(n-\rho)$ rows of $\mathcal{J}$ (which involve submatrices $\mathcal{J}^{[3]}$ and $\mathcal{J}^{[4]}$ ). Then for $i, l=\rho+1, \ldots, n$ we arrive at:

$$
\left(\mathcal{J} \cdot \nabla D_{l}\right)_{i}=\sum_{j=1}^{n} J_{i j} \partial_{j} D_{l}=-\sum_{j=1}^{\rho} J_{i j}^{[3]} a_{l j}+\sum_{j=\rho+1}^{n} J_{i j}^{[4]} \delta_{l j}=0
$$

where now the last identity arises as a result of (232). The proof of Lemma 4.4.2.3 is therefore complete.
Q.E.D.

Lemma 4.4.2.4. For all $i, j=1, \ldots, n$, functions $J_{i j}(x)$ entering matrix $\mathcal{J}$ in (226) are $K G$ for $\mathcal{J}$.

Proof of Lemma 4.4.2.4. By construction in (226), all entries $J_{i j}$ are either functions $\psi_{i j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right)$, as it is the case of $\mathcal{J}^{[1]}$, or linear combinations with constant coefficients of such functions (as happens for $\mathcal{J}^{[2]}, \mathcal{J}^{[3]}$ and $\mathcal{J}^{[4]}$ ). Therefore, in a compact and unified way it can be said that there exist real constants $b_{i j k l}$ such that:

$$
\begin{equation*}
J_{i j}=\sum_{\substack{k, l=1 \\ l>k}}^{\rho} b_{i j k l} \psi_{k l}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right), \quad i, j=1, \ldots, n \tag{233}
\end{equation*}
$$

From (233) it can be seen that:

$$
\begin{equation*}
\nabla J_{i j}=\sum_{\substack{k, l=1 \\ l>k}}^{\rho} b_{i j k l} \sum_{q=\rho+1}^{n}\left(\frac{\partial \psi_{k l}}{\partial D_{q}}\right) \nabla D_{q}, \quad i, j=1, \ldots, n \tag{234}
\end{equation*}
$$

Finally, as a consequence of (234) and Lemma 4.4.2.3 we obtain $\mathcal{J} \cdot \nabla J_{i j}=0$ for $i, j=1, \ldots, n$. This proves Lemma 4.4.2.4.
Q.E.D.

Therefore it is proved that $\mathcal{J}$ is a D-solution. Then as a consequence of Lemma 4.4.2.3, the $(n-\rho)$ functions $D_{l}$ in (225) are Casimir invariants, for which the functional independence is clear. To complete the proof, note that by construction the rows $\rho+1, \ldots, n$ of $\mathcal{J}$ are linear combinations (with constant coefficients) of the first $\rho$ ones, as implied by (229) and (231). Then it is clear that $\operatorname{Rank}(\mathcal{J}) \leq \rho$. Since the rank of a skew-symmetric matrix is always even, from Lemma 4.4.2.2 this means that necessarily it is $\operatorname{Rank}(\mathcal{J}) \leq \rho-\rho \bmod 2$, which is the bound given. This completes the proof of the theorem.
Q.E.D.

Of course, in the D-solution family described in Theorem 4.4.2.1 the linear dependences among the elements of $\mathcal{J}$ have been chosen in such a way that the $\rho$ first rows and columns (those conforming $\mathcal{J}^{[1]}$ ) span the rest of rows and columns, as it is clear from equations (228-232). Actually, this choice is entirely arbitrary and was used without loss of generality. Analogous families of D-solutions can be generated on the basis of the rest of possible $\rho \times \rho$ submatrices. This fact, the convenience for future use and the notation employed in (226) motivate the following definition:

Definition 4.4.2.5. Every $D$-solution being either of the type constructed in Theorem 4.4.2.1 or a permutation of such construction will be termed a $D_{\psi}$-solution.

After presenting the main definitions and results, the purpose of the next subsection will be to provide some comments and examples aimed at clarifying the content and implications of the previous developments.

### 4.4.3. Discussion and examples

The first basic aspect of D-solutions which is interesting to consider regards the possible rank values of $D$-solutions in general, and of $D_{\psi}$-solutions in particular, for a given value of $n$. The fact that D-solutions are composed by Casimir invariants seems to suggest that such Poisson structures must be degenerate and therefore that the maximal possible rank (namely $n-n \bmod 2)$ should be excluded for D -solutions having an even value of $n$. However such statement is not mathematically correct, as the following example displays.

## Example 1. Constant structure matrices

As mentioned, constant structure matrices play a significant role in the theory of Poisson systems, not only because they comprise as a special case the classical symplectic matrices (and therefore the whole classical Hamiltonian theory) but also because they are the source of very varied noncanonical applications. Constant structure matrices of arbitrary rank are Dsolutions, since they are entirely composed by (trivial, namely constant) Casimir invariants: in fact, constants are Casimir invariants for every structure matrix. This is interesting because constant structure matrices are able to take all possible ranks associated with every given value of $n$. Therefore it is worth emphasizing the next statement:

Corollary 4.4.3.1. For every integer $n \geq 3$ and for every possible nontrivial rank value ( $r$ even, $2 \leq r \leq n-n \bmod 2$ ) there exists an infinity of $n$-dimensional $D$-solutions having constant rank $r$ in $\mathbb{R}^{n}$.

In particular, constant structure matrices also arise from Theorem 4.4.2.1 as $\mathrm{D}_{\psi^{-s o l u t i o n s} \text {. }}$ To see this, it suffices to consider the value $\rho=n$. In such case, there are $(n-\rho)=0$
functions $D_{l}$ of the form (225) and $\mathcal{J}$ in (226) is entirely given by submatrix $\mathcal{J}^{[1]}$, in other words $\mathcal{J}=\mathcal{J}^{[1]}$. Moreover, functions $J_{i j}^{[1]}=\psi_{i j}\left(D_{\rho+1}(x), \ldots, D_{n}(x)\right)$ lose their dependences and become arbitrary constants. Consequently, $\mathcal{J}$ is an arbitrary constant skew-symmetric real matrix, which can have every admissible (even) rank. Thus, in this situation we still have $\mathrm{D}_{\psi}$-solutions, but in the limit case in which no Casimir invariants of the form (225) are implemented.

Let us now present an additional example of $\mathrm{D}_{\psi}$-solution. Such instance has been chosen to illustrate the fact that the upper bound to the rank may not be reached.

## Example 2. A four-dimensional $\boldsymbol{D}_{\psi}$-solution

The following matrix corresponds to the form studied in Theorem 4.4.2.1 with $n=4$, $\rho=2$, and with the two linear Casimir invariants $D_{3}(x)=x_{2}+x_{3}$ and $D_{4}(x)=x_{1}+x_{2}+x_{4}$ of the kind (225) implemented. If $\psi\left(y_{1}, y_{2}\right): \mathbb{R}^{2} \rightarrow \mathbb{R}$ is an arbitrary $C^{\infty}\left(\mathbb{R}^{2}\right)$ function, the matrix is:

$$
\mathcal{J}=\left(\begin{array}{cc|cc}
0 & \psi\left(D_{3}, D_{4}\right) & -\psi\left(D_{3}, D_{4}\right) & -\psi\left(D_{3}, D_{4}\right)  \tag{235}\\
-\psi\left(D_{3}, D_{4}\right) & 0 & 0 & \psi\left(D_{3}, D_{4}\right) \\
\hline \psi\left(D_{3}, D_{4}\right) & 0 & 0 & -\psi\left(D_{3}, D_{4}\right) \\
\psi\left(D_{3}, D_{4}\right) & -\psi\left(D_{3}, D_{4}\right) & \psi\left(D_{3}, D_{4}\right) & 0
\end{array}\right) \equiv\left(\begin{array}{c|c}
\mathcal{J}^{[1]} & \mathcal{J}^{[2]} \\
\hline \mathcal{J}^{[3]} & \mathcal{J}^{[4]}
\end{array}\right)
$$

It is evident that $\mathcal{J}$ in (235) is skew-symmetric, that both $D_{3}(x)$ and $D_{4}(x)$ are KG functions for such matrix, and that all the entries of $\mathcal{J}$ are $C^{\infty}$ functions of $D_{3}(x)$ and $D_{4}(x)$. Moreover, the structure (226) of the matrix (or, more in detail, the identities (228-232) among the four submatrices) can be easily verified. Accordingly, $\mathcal{J}$ in (235) is a $\mathrm{D}_{\psi}$-solution. In addition, for the rank we obtain from Theorem 4.4.2.1 the bound $\operatorname{Rank}(\mathcal{J}) \leq \rho-\rho \bmod 2=2$, which will be in general the case almost everywhere in $\mathbb{R}^{4}$, but not in the hypersurface $\psi\left(D_{3}(x), D_{4}(x)\right)=0$ where $\operatorname{Rank}(\mathcal{J})$ vanishes.

Another aspect of interest of $\mathrm{D}_{\psi}$-solutions is the possible presence of nonlinear Casimir invariants, additional to those implemented in (225). Associated with this issue is again the fact that $\operatorname{Rank}(\mathcal{J})$ can take lower values than the upper bound $\rho-\rho \bmod 2$. These possibilities are illustrated in the next instance.

## Example 3. An additional four-dimensional $D_{\psi}$-solution

The following matrix is a $\mathrm{D}_{\psi}$-solution, this time with $n=4$ and $\rho=3$ :

$$
\mathcal{J}(x)=\left(\begin{array}{ccc|c}
0 & x_{2}+x_{4} & \left(x_{2}+x_{4}\right)^{2} & -x_{2}-x_{4}  \tag{236}\\
-x_{2}-x_{4} & 0 & 0 & 0 \\
-\left(x_{2}+x_{4}\right)^{2} & 0 & 0 & 0 \\
\hline x_{2}+x_{4} & 0 & 0 & 0
\end{array}\right) \equiv\left(\begin{array}{c|c}
\mathcal{J}^{[1]} & \mathcal{J}^{[2]} \\
\mathcal{J}^{[3]} & \mathcal{J}^{[4]}
\end{array}\right)
$$

In this case, the linear function of the form (225) defined is $D_{4}(x)=x_{2}+x_{4}$. Thus $\mathcal{J}$ in (236) is a D -solution because it is skew-symmetric and $D_{4}(x)$ is a KG function for $\mathcal{J}$. Actually, the submatrix structure (227) can be identified without difficulty for all $\mathcal{J}^{[i]}, i=1, \ldots, 4$, according to the constructive steps of the proof of Theorem 4.4.2.1. In addition, the same theorem predicts the bound $\operatorname{Rank}(\mathcal{J}) \leq \rho-\rho \bmod 2=2$. Accordingly, at least one additional functionally independent Casimir invariant must exist. Actually, it can be easily verified that it is given by $D_{3}(x)=x_{3}+x_{2} x_{4}+x_{4}^{2}$, which is obviously nonlinear. In fact, both $D_{3}(x)$ and $D_{4}(x)$ are functionally independent as we can see on their Jacobian:

$$
\frac{\partial\left(D_{3}, D_{4}\right)}{\partial\left(x_{1}, x_{2}, x_{3}, x_{4}\right)}=\left(\begin{array}{cccc}
0 & x_{4} & 1 & x_{2}+2 x_{4}  \tag{237}\\
0 & 1 & 0 & 1
\end{array}\right)
$$

Functional independence is thus proven since Jacobian (237) has rank 2 everywhere in $\mathbb{R}^{4}$. On the other hand, this set of Casimir invariants is not complete when the upper bound $\operatorname{Rank}(\mathcal{J})$ $=2$ is not accomplished. Actually this is possible, since all the entries of $\mathcal{J}$ vanish in the hyperplane $x_{2}+x_{4}=0$. On the contrary, $\operatorname{Rank}(\mathcal{J})=2$ everywhere else in $\mathbb{R}^{4}$, thus complying to the upper bound predicted by Theorem 4.4.2.1. In such case, Casimir invariants $D_{3}(x)$ and $D_{4}(x)$ constitute a complete set.

A relevant question indirectly posed by Theorem 4.4.2.1 regards the construction of Dsolutions for which the implemented Casimir functions are nonlinear. In general, the method should consist of the a priori specification of $(n-\rho)$ future independent Casimir invariants of the form

$$
\begin{equation*}
D_{i}(x)=x_{i}-\mu_{i}\left(x_{1}, \ldots, x_{\rho}\right), \quad i=\rho+1, \ldots, n \tag{238}
\end{equation*}
$$

where the $\mu_{i}\left(x_{1}, \ldots, x_{\rho}\right)$ are smooth functions. Let $R_{i}$ denote the $i$-th row of the matrix. According to the procedure used in Theorem 4.4.2.1 for the case of linear invariants, we have that $R_{i}=\sum_{j=1}^{\rho}\left(\partial_{j} \mu_{i}\right) R_{j}$ for $i=\rho+1, \ldots, n$ in the matrix to be obtained, together with a similar relationship for the columns, so that the $D_{i}(x)$ in (238) are KG functions by
construction. Similarly to what is done in Theorem 4.4.2.1, $\mathcal{J}$ should be splitted in four regions, with $\mathcal{J}^{[1]}$ defined as in (226), so that the entries of $\mathcal{J}^{[1]}$ are KG functions of the resulting matrix. This is exactly the procedure used in Theorem 4.4.2.1, but applied to the nonlinear functions (238). The problem however is that the outcome for $\mathcal{J}^{[2]}, \mathcal{J}^{[3]}$ and $\mathcal{J}^{[4]}$ generally produces functions that are not KG functions of the matrix thus constructed. The reason for this is that now the coefficients of the linear combinations generating the entries of $\mathcal{J}^{[2]}, \mathcal{J}^{[3]}$ and $\mathcal{J}^{[4]}$ are of the form $\partial_{j} \mu_{i}$, namely they are not constant (as in the case of Theorem 4.4.2.1). The outcome is that the entries of $\mathcal{J}^{[2]}, \mathcal{J}^{[3]}$ and $\mathcal{J}^{[4]}$ lose in general their functional dependence with respect to functions (238). On the contrary, such functional dependence is preserved in the case of linear invariants (225) because (in the notation of Theorem 4.4.2.1) in this situation we have that $\partial_{j} \mu_{i}=a_{i j}$ is always a constant. A simple example now illustrates this situation.

## Example 4. A counterexample in dimension 3

Let us consider the case $n=3, \rho=2$ for the implementation of a nonlinear Casimir invariant of the form $D_{3}(x)=x_{3}-\mu\left(x_{1}, x_{2}\right)$. For $D_{3}(x)$ to be by construction a KG function, it has to be $R_{3}=\left(\partial_{1} \mu\right) R_{1}+\left(\partial_{2} \mu\right) R_{2}$, and a similar relationship for the columns. Thus, defining $J_{12}(x)=\psi\left(D_{3}(x)\right)$ we are led to the matrix:

$$
\mathcal{J}(x)=\psi\left(D_{3}(x)\right)\left(\begin{array}{ccc}
0 & 1 & \partial_{2} \mu\left(x_{1}, x_{2}\right)  \tag{239}\\
-1 & 0 & -\partial_{1} \mu\left(x_{1}, x_{2}\right) \\
-\partial_{2} \mu\left(x_{1}, x_{2}\right) & \partial_{1} \mu\left(x_{1}, x_{2}\right) & 0
\end{array}\right)
$$

Now it is evident that $\mathcal{J}$ in (239) is skew-symmetric and that $D_{3}(x), J_{12}(x)$ and $J_{21}(x)$ are KG functions for $\mathcal{J}$, as expected. However, it is straightforward to check that the rest of nondiagonal entries are not in general KG functions of $\mathcal{J}$. Consequently, $\mathcal{J}$ in (239) is generally not a D-solution. Notice that in the particular case (225) of linear Casimir functions the partial derivatives of $\mu\left(x_{1}, x_{2}\right)$ are constant as indicated, and then every entry of $\mathcal{J}$ in (239) is a KG function, thus conforming a $\mathrm{D}_{\psi}$-solution. On the contrary, when functions $\partial_{i} \mu\left(x_{1}, x_{2}\right)$ are not constant, the property of being a D-solution is not necessarily preserved.

Since the previous reasoning is clear, the naturalness of the result contained in Theorem 4.4.2.1 suggests yet another relevant question, namely the possibility that all the solutions of the distinguished Jacobi equations $(24,223)$ are actually $\mathrm{D}_{\psi}$-solutions. This is certainly not the case, as it can be seen in what follows.

## Example 5. A counterexample in arbitrary dimension

In this example D -solutions of arbitrary dimension $n \geq 3$ and not being $\mathrm{D}_{\psi}$-solutions will be considered. In fact, such D-solutions do not have linear Casimir invariants at all, as we shall see in what follows. Consider the following $n \times n$ skew-symmetric matrix:

$$
\mathcal{J}(x)=\left(\begin{array}{c|cccc}
0 & x_{3} / x_{2} & \left(x_{3} / x_{2}\right)^{2} & \ldots & \left(x_{3} / x_{2}\right)^{n-1}  \tag{240}\\
\hline-x_{3} / x_{2} & & & & \\
-\left(x_{3} / x_{2}\right)^{2} & & & & \\
\vdots & & & \mathbb{O}_{(n-1)} & \\
-\left(x_{3} / x_{2}\right)^{n-1} & & & &
\end{array}\right)
$$

In (240) the symbol $\mathbb{O}$ denotes a null submatrix of size indicated by the respective subindex. Matrix (240) will be defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which $x_{2} \neq 0$ and $x_{3} \neq 0$ for every $x \in \Omega$. In such case, we have $\operatorname{Rank}(\mathcal{J})=2$ everywhere in $\Omega$. It is then possible to find $(n-2) \mathrm{KG}$ functions for matrix (240) functionally independent in $\Omega$. These are:

$$
\begin{equation*}
D_{3}(x)=\frac{x_{3}}{x_{2}} ; \quad D_{i}(x)=\frac{x_{3} x_{i-1}}{x_{2}}-x_{i}, \quad i=4, \ldots, n \tag{241}
\end{equation*}
$$

That they are KG functions is simple to verify. Regarding functional independence, notice that we have the Jacobian:

$$
\left(\frac{\partial\left(D_{3}, D_{4}, \ldots, D_{n}\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}\right)^{T}=\left(\begin{array}{cccc}
0 & 0 & \ldots & 0  \tag{242}\\
-x_{3} / x_{2}^{2} & -x_{3}^{2} / x_{2}^{2} & \ldots & -x_{3} x_{n-1} / x_{2}^{2} \\
1 / x_{2} & 2 x_{3} / x_{2} & \ldots & x_{n-1} / x_{2} \\
0 & -1 & \ldots & 0 \\
0 & 0 & \ldots & 0 \\
\vdots & \vdots & & \vdots \\
0 & 0 & \ldots & x_{3} / x_{2} \\
0 & 0 & \ldots & -1
\end{array}\right)
$$

Thus if we choose in $(242)$ the submatrix composed by the last $(n-2)$ rows, we see that it is upper triangular with determinant $\left[(-1)^{n-3} / x_{2}\right] \neq 0$ in $\Omega$. Accordingly, functional independence of $D_{3}(x), \ldots, D_{n}(x)$ holds in $\Omega$ for every $n \geq 3$. Since all the entries of $\mathcal{J}$ in (240) are $C^{\infty}$ functions of $D_{3}(x)$ we have that such matrix is a $D$-solution for every $n \geq 3$, and functions $D_{3}(x), \ldots, D_{n}(x)$ in (241) form a complete set of independent Casimir invariants of
$\mathcal{J}$ in $\Omega$. On the other hand, recall that if a $\mathrm{D}_{\psi}$-solution of constant rank has one or more independent Casimir invariants, then at least one of them can be taken to be linear: in the case $\rho<n$ this is so by construction; and according to Example 1 , in the complementary case $\rho=n$ the matrix is constant, which implies the existence of linear Casimir invariants when such matrix is degenerate. Conversely, a degenerate structure matrix of constant rank without linear Casimir invariants is not a $\mathrm{D}_{\psi}$-solution. This is evidently the case for matrix $\mathcal{J}$ in (240) since the ansatz of a generic linear Casimir invariant $D(x)=\sum_{i=1}^{n} a_{i} x_{i}$ substituted in the identity $\mathcal{J} \cdot \nabla D=0$ immediately leads to $a_{i}=0$ for all $i=1, \ldots, n$. Therefore it is proved that $\mathcal{J}$ in (240) is a D-solution but not a $\mathrm{D}_{\psi}$-solution, for every $n \geq 3$.

Note in addition that the significance of the last example is reinforced in view of the results displayed in Theorem 4.4.1.4. This motivates the following conclusion:

Corollary 4.4.3.2. For every $n \geq 3$, there exists an infinity of $n$-dimensional $D$-solutions that are not $D_{\psi}$-solutions.

In other words, $\mathrm{D}_{\psi}$-solutions do not provide the general solution of the distinguished Jacobi equations $(24,223)$. In the next subsection we conclude the present discussion by briefly regarding some of the previous issues as well as other questions from a more general perspective.

### 4.4.4. Further remarks about D-solutions

As discussed in Chapter 1, skew-symmetric Jacobi equations become increasingly complex as dimension grows. This explains that the characterization of families of arbitrary dimension composed by generic functions (namely not limited to a given degree of nonlinearity) and having arbitrary rank is very uncommon (some instances of the same kind are provided by the separable and multiseparable Poisson structures considered in the two previous sections of this chapter). For this reason, D-solutions may well be regarded as a significant contribution in such sense. When compared to the analyses of the separable and multiseparable families just mentioned, it can be seen that a typical outcome is the global construction of the Darboux canonical form. This is clearly not possible in the case of D-solutions, neither in general nor in the specific case of $\mathrm{D}_{\psi}$-solutions. In both cases, this is mainly due to (a) the generality of the functional form of the entries $J_{i j}(x)$ of such structure matrices; and (b)
the possible lack of constancy of the rank (already illustrated in the examples) which is a necessary condition for the applicability of Darboux' theorem. Together, both reasons seem to exclude a global application of Darboux' theorem in the case of D-solutions, at least in general. To conclude, notice also that in spite of the mathematically nice specialization of the general skew-symmetric Jacobi equations which is provided by the D-solution problem, this does not seem to imply that even a complete identification of D-solutions is at hand, as Corollary 4.4.3.2 points out. Consequently, even in the more specific distinguished version, skew-symmetric Jacobi equations retain their interest as a significant problem to be analyzed.

### 4.4.5. Comparison with the multiseparable family

The D-solutions (specially in the case of $\mathrm{D}_{\psi}$-solutions) and multiseparable structure matrices have in common the important role played by the linear Casimir invariants. This feature suggests and motivates a comparison between both solution families in order to establish their difference as solution sets. As we are going to see now, both families are unlike in spite of such common feature. Let us show it.

In first place, notice that (according to Theorem 4.3.2.1) for multiseparable structure matrices always exists a complete set of independent linear Casimir invariants. On the other hand, for general D-solutions this needs not be the case as it has been seen, for instance, in Example 5. In fact, the same statement remains valid even if we restrict ourselves to $\mathrm{D}_{\psi^{-}}$ solutions, as it can be seen from the previous examples (such as Example 3). According to this, neither D -solutions nor $\mathrm{D}_{\psi}$-solutions are subsets of the set of multiseparable solutions.

Conversely: let us recall that multiseparable solutions are not, in general, structure matrices composed by Casimir invariants since, according to Definition 4.3.1.1 and Theorem 4.3.2.1, these structure matrices depend on $\left(B_{1} \cdot x, \ldots, B_{r} \cdot x\right)$, while their Casimir invariants are $\left(B_{r+1} \cdot x, \ldots, B_{n} \cdot x\right)$. To further clarify this issue, notice in addition that both sets of linear functions $\left(B_{1} \cdot x, \ldots, B_{r} \cdot x\right)$ and $\left(B_{r+1} \cdot x, \ldots, B_{n} \cdot x\right)$ are functionally independent because matrix $B$ is invertible. Therefore, multiseparable solutions are not a subset of D-solutions (and obviously, not a subset of $\mathrm{D}_{\psi}$-solutions, because in such a case they would be also a subset of D-solutions).

Accordingly, the family of multiseparable structure matrices is different from the family of D-solutions, and also different from the family of $\mathrm{D}_{\psi}$-solutions. Nevertheless, the three families have an infinite intersection. For example, we know from previously seen instances (Example 2 in Subsection 4.3.3 and Example 1 in Subsection 4.4.3) that the entire set of
constant structure matrices is contained at the same time in the multiseparable, in the Dsolution and in the $\mathrm{D}_{\psi}$-solution families. This clarifies completely the relationship between the multiseparable and the $\mathrm{D}\left(\right.$ or $\left.\mathrm{D}_{\psi}\right)$ solutions.

### 4.4.6. Comparison with the separable family

For the sake of completeness, a brief comparison between D-solutions and separable solutions is also provided. In this case both families display strong differences that anticipate their radically different nature. Such analysis is interesting, nevertheless, as far as it completes the pairwise comparison among the three $n$-dimensional families already presented in this dissertation. As in the previous subsection, the use of suitable examples will suffice for our purposes.

In one sense, we can make use of matrix (240) from Example 5 in Subsection 4.4.3, which is a D-solution structure matrix defined for arbitrary $n \geq 3$. Moreover, we see that matrix (240) is not separable, as it is evident from simple inspection. We therefore conclude that D-solutions are not a subset of separable solutions.

In the opposite sense, and being also defined for every $n \geq 3$, we can consider the structure matrices provided in Example 1 of Subsection 4.2.3, which are given by the expression (178). The entries of such structure matrices are, in general, clearly functionally independent of the Casimir invariants (179) of such matrices. For instance, let us consider for simplicity the case of rank ( $n-1$ ) in which a single independent Casimir invariant of the form (179) exists. Equivalently we can write the Casimir invariant as

$$
\begin{equation*}
D(x)=\prod_{j=1}^{n} x_{j}^{k_{j}} \tag{243}
\end{equation*}
$$

with $k=\left(k_{1}, \ldots, k_{n}\right)^{T} \in \operatorname{Ker}(S)$. On the other hand, the number of nonredundant entries of the structure matrix is $n(n-1) / 2$, which for instance can be taken to be those over the diagonal of $\mathcal{J}$. It is clear that, in general, at most one of the nonvanishing nonredundant entries of the matrix will be functionally dependent on the Casimir invariant (243), the rest being by construction independent of it, thus showing that the matrix is generically not a D-solution. Therefore, we obtain from this example that separable solutions are not a subset of D-solutions.

To conclude, notice also that both families are not disjoint, since constant structure matrices are both separable (see Example 3 of Subsection 4.2.3) and D-solutions (see Example 1 of

Subsection 4.4.3). Actually, constant skew-symmetric matrices are also $\mathrm{D}_{\psi}$-solutions, as it was shown in the same Example 1 of Subsection 4.4.3. Thus there is an intersection between separable structure matrices and D-solutions, containing an infinity of elements for every $n \geq 3$. The same can be said, of course, between separable structure matrices and $\mathrm{D}_{\psi}$-solutions.

It is worth noting the conceptually relevant role played by constant structure matrices for the purpose of family comparison, because they are at the same time separable structure matrices, multiseparable structure matrices, D-solutions and also $\mathrm{D}_{\psi}$-solutions. Thus, in spite of their simplicity, constant skew-symmetric matrices have been very useful for the pairwise comparisons we were interested in. Such comparisons are now concluded, as far as they will not be required any more in what is to follow for the additional families of Poisson structures to be discussed.

### 4.5. HEMISEPARABLE SOLUTIONS OF TYPE I

In this section, an additional family of $n$-dimensional solutions of the Jacobi equations is characterized and globally analyzed. In particular, it is worth noting that in spite of its general form (defined in terms of functions of an arbitrary nonlinearity) it is possible the explicit and global determination of its main features, such as the case-classification of the Casimir invariants and the global construction of the Darboux canonical form (which is a result known for a limited sample of $n$-d Poisson structures, as we know). This family presents the special feature that a complete understanding of its global analysis implies a case classification such that different distinguished invariants are to be used in each case. Consequently, the mere knowledge of one complete set of distinguished invariants does not guarantee the throughout reduction to the Darboux canonical form, in spite that we are dealing with structure matrices of constant rank. Accordingly, the symplectic analysis of this kind of Poisson structures requires the use of a case-dependent (i) complete set of global Casimir invariants, and (ii) subsequent global Darboux reduction to be constructed. Such classification is carried out in the present section. The family embraces as particular cases different systems of applied interest that are also analyzed as examples.

In Subsection 4.5.1 the analysis begins with the main results regarding the functional characterization of the solution family. Then, in Subsection 4.5.2 the global analysis of the family is provided. The section concludes with some applied examples, which are the subject of Subsection 4.5.3.

### 4.5.1. Characterization of the family

We begin the subsection by providing a first result:

Theorem 4.5.1.1. Let $\eta(x)$ and $\varphi_{i}\left(x_{i}\right)$, for $i=1, \ldots, n$, be functions defined in a domain $\Omega \subset \mathbb{R}^{n}$, all of which are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. Let $\kappa_{i j}, i, j=1, \ldots, n$, be arbitrary real constants that are skew-symmetric

$$
\begin{equation*}
\kappa_{i j}+\kappa_{j i}=0, \quad i, j=1, \ldots, n \tag{244}
\end{equation*}
$$

and satisfy the zero-sum conditions

$$
\begin{equation*}
\kappa_{i j}+\kappa_{j k}+\kappa_{k i}=0, \quad i, j, k=1, \ldots, n \tag{245}
\end{equation*}
$$

In addition let

$$
\begin{equation*}
\psi_{i}\left(x_{i}\right)=\int \frac{\mathrm{d} x_{i}}{\varphi_{i}\left(x_{i}\right)}, \quad i=1, \ldots, n \tag{246}
\end{equation*}
$$

denote one primitive of $1 / \varphi_{i}\left(x_{i}\right)$. Finally, let the functions $\chi_{i j}\left(x_{i}, x_{j}\right)$ be defined by

$$
\chi_{i j}\left(x_{i}, x_{j}\right)=\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}, \quad i, j=1, \ldots, n
$$

and assume that $\chi_{i j}\left(x_{i}, x_{j}\right)$ is nonvanishing in $\Omega$ at least for one pair $(i, j)$. Then $\mathcal{J}=\left(J_{i j}\right)$ is a family of n-dimensional structure matrices globally defined in $\Omega$, where

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \varphi_{i}\left(x_{i}\right) \varphi_{j}\left(x_{j}\right) \chi_{i j}\left(x_{i}, x_{j}\right), \quad i, j=1, \ldots, n \tag{247}
\end{equation*}
$$

Proof. Skew-symmetry is evident in (247). We then substitute $\mathcal{J}$ in (247) into the Jacobi identities (25) and obtain after some algebra:

$$
\sum_{l=1}^{n}\left(J_{l i} \partial_{l} J_{j k}+J_{l j} \partial_{l} J_{k i}+J_{l k} \partial_{l} J_{i j}\right)=\eta T_{1}+\eta^{2} T_{2}
$$

where $T_{1}$ and $T_{2}$ are the following terms, to be examined separately:

$$
\begin{gathered}
T_{1}=\sum_{l=1}^{n} \varphi_{i} \varphi_{j} \varphi_{k} \varphi_{l}\left(\partial_{l} \eta\right)\left(\chi_{i l} \chi_{j k}+\chi_{j l} \chi_{k i}+\chi_{k l} \chi_{i j}\right) \\
T_{2}=\sum_{l=1}^{n}\left\{\varphi_{i} \varphi_{l} \chi_{i l}\left[\delta_{l j} \varphi_{j}^{\prime} \varphi_{k} \chi_{j k}+\delta_{l k} \varphi_{j} \varphi_{k}^{\prime} \chi_{j k}+\varphi_{j} \varphi_{k}\left(\delta_{l j} \frac{1}{\varphi_{j}}-\delta_{l k} \frac{1}{\varphi_{k}}\right)\right]+\right. \\
\varphi_{j} \varphi_{l} \chi_{j l}\left[\delta_{l k} \varphi_{k}^{\prime} \varphi_{i} \chi_{k i}+\delta_{l i} \varphi_{k} \varphi_{i}^{\prime} \chi_{k i}+\varphi_{k} \varphi_{i}\left(\delta_{l k} \frac{1}{\varphi_{k}}-\delta_{l i} \frac{1}{\varphi_{i}}\right)\right]+ \\
\left.\varphi_{k} \varphi_{l} \chi_{k l}\left[\delta_{l i} \varphi_{i}^{\prime} \varphi_{j} \chi_{i j}+\delta_{l j} \varphi_{i} \varphi_{j}^{\prime} \chi_{i j}+\varphi_{i} \varphi_{j}\left(\delta_{l i} \frac{1}{\varphi_{i}}-\delta_{l j} \frac{1}{\varphi_{j}}\right)\right]\right\}
\end{gathered}
$$

Regarding $T_{1}$, if every $\chi_{i j}$ is substituted by its expression $\chi_{i j}=\psi_{i}-\psi_{j}+\kappa_{i j}$ and the result is simplified, it is found that:

$$
\begin{gathered}
\chi_{i l} \chi_{j k}+\chi_{j l} \chi_{k i}+\chi_{k l} \chi_{i j}= \\
\left(\kappa_{j k}+\kappa_{k l}-\kappa_{j l}\right) \psi_{i}+\left(\kappa_{k i}+\kappa_{i l}-\kappa_{k l}\right) \psi_{j}+\left(\kappa_{i j}+\kappa_{j l}-\kappa_{i l}\right) \psi_{k}- \\
\left(\kappa_{j k}+\kappa_{k i}+\kappa_{i j}\right) \psi_{l}+\kappa_{i l} \kappa_{j k}+\kappa_{j l} \kappa_{k i}+\kappa_{k l} \kappa_{i j}
\end{gathered}
$$

In the last identity, the terms multiplied by one of the $\left(\psi_{1}, \ldots, \psi_{n}\right)$ vanish due to the zero-sum relations (245). In addition, using the same equations (245) we have:

$$
\kappa_{i l} \kappa_{j k}+\kappa_{j l} \kappa_{k i}+\kappa_{k l} \kappa_{i j}=
$$

$$
\kappa_{i l}\left(\kappa_{j l}-\kappa_{k l}\right)+\kappa_{j l}\left(\kappa_{k l}-\kappa_{i l}\right)+\kappa_{k l}\left(\kappa_{i l}-\kappa_{j l}\right)=0
$$

It is thus proved that $T_{1}=0$. We proceed now with $T_{2}$ : expanding its expression and cancelling out similar terms, after a suitable rearrangement we arrive at:

$$
T_{2}=2 \varphi_{i} \varphi_{j} \varphi_{k}\left[\chi_{i j}+\chi_{j k}+\chi_{k i}\right]=2 \varphi_{i} \varphi_{j} \varphi_{k}\left[\kappa_{i j}+\kappa_{j k}+\kappa_{k i}\right]=0
$$

Therefore it is also $T_{2}=0$ and the proof is complete.
Q.E.D.

Let us now provide some brief but relevant comments. In first place recall that, as indicated in Theorem 4.5.1.1, for every $i$ the primitive $\psi_{i}\left(x_{i}\right)$ obtained from $\varphi_{i}\left(x_{i}\right)$ in (246) must be chosen to be one and the same for all the entries of $\mathcal{J}$. However, the specific choice is actually arbitrary. In this sense, notice that if a different integration constant is selected, for instance after replacing $\psi_{i}\left(x_{i}\right)$ by $\psi_{i}\left(x_{i}\right)+k_{i}$ for every $i$, then the outcome is also a member of the solution family, this time with constants $\tilde{\kappa}_{i j}=\kappa_{i j}+k_{i}-k_{j}$, which also verify (244-245). Thus conditions (244-245) express in a generalized form this degree of freedom associated with the choice of primitives (246). Secondly, note that by construction the functions $\psi_{i}\left(x_{i}\right)$ and $\chi_{i j}\left(x_{i}, x_{j}\right)$ are $C^{\infty}(\Omega)$. In third place, it is worth observing that the definition (246) allows an alternative expression for the solution family just characterized, namely $\mathcal{J}=\left(J_{i j}\right)$ can also be written as

$$
\begin{equation*}
J_{i j}(x)=\frac{\eta(x)}{\psi_{i}^{\prime}\left(x_{i}\right) \psi_{j}^{\prime}\left(x_{j}\right)} \chi_{i j}\left(x_{i}, x_{j}\right)=\frac{\eta(x)}{\psi_{i}^{\prime}\left(x_{i}\right) \psi_{j}^{\prime}\left(x_{j}\right)}\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}\right], \quad i, j=1, \ldots, n \tag{248}
\end{equation*}
$$

where functions $\psi_{i}^{\prime}\left(x_{i}\right)$ are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$, while the rest of defining properties were already presented in Theorem 4.5.1.1. Under these assumptions, this can be taken as an alternative definition of the solution family of structure matrices. Both ways of expressing such family will be useful for what is to follow.

### 4.5.2. Casimir invariants and global Darboux analysis

We can now characterize some of the properties of the family described in Theorem 4.5.1.1:

Theorem 4.5.2.1. Let $\mathcal{J}$ be a structure matrix of the form (247) characterized in Theorem 4.5.1.1, which is defined in a domain $\Omega \subset \mathbb{R}^{n}$ and such that the pair $(i, j)$ verifies that function $\chi_{i j}\left(x_{i}, x_{j}\right)$ is nonvanishing in $\Omega$. Then $\operatorname{Rank}(\mathcal{J})=2$ everywhere in $\Omega$ and a complete set of independent Casimir invariants for $\mathcal{J}$ is given by:

$$
\begin{equation*}
D_{k}(x)=\frac{\psi_{j}\left(x_{j}\right)-\psi_{k}\left(x_{k}\right)+\kappa_{j k}}{\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}}=\frac{\chi_{j k}\left(x_{j}, x_{k}\right)}{\chi_{i j}\left(x_{i}, x_{j}\right)}, \quad k=1, \ldots, n ; k \neq i, j \tag{249}
\end{equation*}
$$

Moreover, every Casimir invariant in (249) is globally defined in $\Omega$.

Proof. Given that functions $\eta(x)$ and $\varphi_{i}\left(x_{i}\right)$ are nonvanishing in $\Omega$, the use of rank-preserving matrix transformations shows that $\operatorname{Rank}(\mathcal{J})=\operatorname{Rank}(X)$ in $\Omega$, where $X \equiv\left(\chi_{i j}\left(x_{i}, x_{j}\right)\right)$ for every pair $(i, j)$. Since at least one of the entries of $X$ is also nonvanishing in $\Omega$, this implies that $\operatorname{Rank}(\mathcal{J}) \geq 2$ everywhere in $\Omega$. We can now submit matrix $X$ to additional rank-preserving transformations: notice that $\operatorname{Rank}(X)$ is also maintained if we subtract the first row to the rest of rows, and then if on the resulting matrix we subtract the first column to every one of the remaining columns. This leads to a new matrix $X^{*}$ given by:

$$
X^{*}=\left(\begin{array}{cccc}
0 & \chi_{12} & \ldots & \chi_{1 n}  \tag{250}\\
-\chi_{12} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\chi_{1 n} & 0 & \ldots & 0
\end{array}\right)
$$

It is then clear from (250) that $\operatorname{Rank}(\mathcal{J})=\operatorname{Rank}\left(X^{*}\right) \leq 2$ in every point of $\Omega$. Therefore we conclude that $\operatorname{Rank}(\mathcal{J})=2$ in $\Omega$. This proves the first part of the statement. For the second part, notice first that every function $D_{k}(x)$ in (249) always depends on $x_{i}, x_{j}$ and $x_{k}$ (since functions $\psi_{k}\left(x_{k}\right)$ cannot be constant for any $k$, according to the conditions established) and in addition $D_{k}(x)$ does not depend on the rest of variables. This implies immediately the functional independence of the set $\left\{D_{k}(x) \mid k=1, \ldots, n ; k \neq i, j\right\}$. Moreover, since both $\chi_{j k}\left(x_{j}, x_{k}\right)$ and $\chi_{i j}\left(x_{i}, x_{j}\right)$ are $C^{\infty}(\Omega)$ and $\chi_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$, function $D_{k}(x)$ is necessarily $C^{\infty}(\Omega)$. Therefore, to complete the proof it is only required to prove that functions $D_{k}(x)$ are Casimir invariants for every $k$. The simplest procedure to see this is to verify that $\mathcal{J} \cdot \nabla D_{k}=0$ for every $k=1, \ldots, n$, with $k \neq i, j$ (notice that for both values $k=i, j$, function $D_{k}(x)$ is a constant, and then also a Casimir invariant, but trivial). We thus have:

$$
\partial_{i} D_{k}(x)=\frac{\psi_{i}^{\prime} \chi_{k j}}{\left(\chi_{i j}\right)^{2}}, \quad \partial_{j} D_{k}(x)=\frac{\psi_{j}^{\prime} \chi_{i k}}{\left(\chi_{i j}\right)^{2}}, \quad \partial_{k} D_{k}(x)=\frac{\psi_{k}^{\prime} \chi_{j i}}{\left(\chi_{i j}\right)^{2}}, \quad k=1, \ldots n ; k \neq i, j
$$

Then for every $l=1, \ldots, n$ it can be seen that:

$$
\begin{equation*}
\sum_{s=1}^{n} J_{l s} \partial_{s} D_{k}=J_{l i} \partial_{i} D_{k}+J_{l j} \partial_{j} D_{k}+J_{l k} \partial_{k} D_{k}=\frac{\eta \varphi_{l}}{\left(\chi_{i j}\right)^{2}}\left(\chi_{l i} \chi_{k j}+\chi_{l j} \chi_{i k}+\chi_{l k} \chi_{j i}\right) \tag{251}
\end{equation*}
$$

In (251) the last term vanishes for every choice of $i, j, k, l$,

$$
\chi_{l i} \chi_{k j}+\chi_{l j} \chi_{i k}+\chi_{l k} \chi_{j i}=0
$$

as it was already shown in the proof of Theorem 4.5.1.1. Consequently, $\mathcal{J} \cdot \nabla D_{k}=0$ for every $k \neq i, j$. This completes the proof.
Q.E.D.

As a consequence, we have that every Poisson system of this kind has $(n-2)$ independent Casimir invariants, additional to the Hamiltonian. In other words:

Corollary 4.5.2.2. Consider that the assumptions of Remark 1.4.4.4 hold. Every n-dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (247) characterized in Theorem 4.5.1.1, is an algebraically integrable system in $\Omega$.

In order to make the section self-contained, it is necessary to recall the concept of time reparametrization for Poisson systems (see Definition 4.1.1) which are transformations of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{252}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$. Thus, if

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{253}
\end{equation*}
$$

is an arbitrary Poisson system defined in $\Omega$, then every time reparametrization (252) leads from (253) to the system (not necessarily of Poisson type):

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{254}
\end{equation*}
$$

Having this in mind, an additional consequence of the previous results is that they allow the constructive and global determination of the Darboux canonical form for this kind of Poisson systems. This statement is contained in the following:

Theorem 4.5.2.3. Let $\Omega \subset \mathbb{R}^{n}$ be a domain where a Poisson system

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x)
$$

is defined everywhere, for which $\mathcal{J}(x)$ is a structure matrix of the form (247) characterized in Theorem 4.5.1.1, and such that the pair $(i, j)$ verifies that function $\chi_{i j}\left(x_{i}, x_{j}\right)$ is nonvanishing in $\Omega$. Then such Poisson system can be globally reduced in $\Omega$ to an one degree of freedom Hamiltonian system and the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\left(y_{1}, \ldots, y_{n}\right)$ and the new time $\tau$, where $\left(y_{1}, \ldots, y_{n}\right)$ are given by the
diffeomorphism globally defined in $\Omega$

$$
\left\{\begin{align*}
y_{i}(x) & =x_{i}  \tag{255}\\
y_{j}(x) & =x_{j} \\
y_{k}(x) & =D_{k}(x), \quad k=1, \ldots, n ; \quad k \neq i, j
\end{align*}\right.
$$

in which the $D_{k}(x)$ are the Casimir invariants (249); and the new time $\tau$ is defined by the time reparametrization:

$$
\begin{equation*}
\mathrm{d} \tau=J_{i j}(x(y)) \mathrm{d} t \tag{256}
\end{equation*}
$$

Proof. Note that, according to Theorem 4.5.2.1, Darboux' theorem is applicable because $\mathcal{J}$ has constant rank 2 in $\Omega$. For the sake of clarity and without loss of generality, assume that it is $\chi_{12} \neq 0$ everywhere in $\Omega$. Recall also that, after a general smooth change of coordinates $y \equiv y(x)$, an arbitrary structure matrix $\mathcal{J}(x)$ is transformed into another one $\mathcal{J}^{*}(y)$ as:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{257}
\end{equation*}
$$

For the family of interest, the reduction is carried out in two steps. We first perform the change of variables (255), which in this case is

$$
\left\{\begin{array}{l}
y_{1}(x)=x_{1}  \tag{258}\\
y_{2}(x)=x_{2} \\
y_{k}(x)=D_{k}(x), \quad k=3, \ldots, n
\end{array}\right.
$$

where the $D_{k}(x)$ are given by (249), namely:

$$
\begin{equation*}
D_{k}(x)=\frac{\chi_{2 k}\left(x_{2}, x_{k}\right)}{\chi_{12}\left(x_{1}, x_{2}\right)}=\frac{\psi_{2}\left(x_{2}\right)-\psi_{k}\left(x_{k}\right)+\kappa_{2 k}}{\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)+\kappa_{12}}, \quad k=3, \ldots, n \tag{259}
\end{equation*}
$$

Notice that the change of coordinates (258) is invertible everywhere in $\Omega$, its inverse being

$$
\left\{\begin{array}{l}
x_{1}(y)=y_{1}  \tag{260}\\
x_{2}(y)=y_{2} \\
x_{k}(y)=\zeta_{k}\left[\psi_{2}\left(y_{2}\right)+\kappa_{2 k}-y_{k} \chi_{12}\left(y_{1}, y_{2}\right)\right], \quad k=3, \ldots, n
\end{array}\right.
$$

where function $\zeta_{k}$ is the inverse function of $\psi_{k}$ for every $k$. The examination of (258-260) shows that the variable transformation (258) to be performed exists and is a diffeomorphism everywhere in $\Omega$ as a consequence that by hypothesis we have $\chi_{12}\left(x_{1}, x_{2}\right) \neq 0$ and $\psi_{k}^{\prime}\left(x_{k}\right) \neq 0$
in $\Omega$. Then, according to (258) and (259), and taking (257) into account, after some algebra we are led to

$$
\mathcal{J}^{*}(y)=J_{12}(x(y))\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0  \tag{261}\\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right)
$$

where from equations (247) and (260) we have

$$
\begin{equation*}
J_{12}(x(y))=\eta\left(y_{1}, y_{2}, x_{3}(y), \ldots, x_{n}(y)\right) \varphi_{1}\left(y_{1}\right) \varphi_{2}\left(y_{2}\right) \chi_{12}\left(y_{1}, y_{2}\right) \tag{262}
\end{equation*}
$$

The explicit dependences of $\left(x_{3}(y), \ldots, x_{n}(y)\right)$ are obviously the ones given in (260) and were not displayed in (262) for the sake of clarity. Note that $J_{12}(x(y))$ is nonvanishing in $\Omega^{*}=y(\Omega)$ and $C^{\infty}\left(\Omega^{*}\right)$. These properties allow the accomplishment of the second step of the reduction which is a reparametrization of time, which in this case does not suppress the Poisson structure of the vector field. Thus, making use of (262) in equation (256), the transformation $\mathrm{d} \tau=$ $J_{12}(x(y)) \mathrm{d} t$ is performed. According to (252-254) this leads from the structure matrix (261) to the Darboux canonical one:

$$
\mathcal{J}_{D[n, 2]}(y)=\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0  \tag{263}\\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right)=\mathbb{S}_{2} \oplus \mathbb{O}_{(n-2)}
$$

Therefore the reduction is globally completed.
Q.E.D.

The global reduction just achieved also implies the following consequence:

Corollary 4.5.2.4. Consider that the assumptions of Remark 1.4.4.4 hold. Every n-dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (247) characterized in Theorem 4.5.1.1, can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

This concludes the analysis of the family of Poisson structures, since at this stage the reduction directly connects the initial Poisson systems with their classical Hamiltonian formulation.

In what follows, the results just developed are illustrated by means of some applied examples. This provides several instances of physical systems embraced by the previous setting, and also illustrates the procedures introduced. This is the purpose of the next subsection.

### 4.5.3. Examples

Now some illustrations of the solution family (247) are given. The first one deals with certain Poisson structures of interest in connection with the Halphen equations and the system of circle maps. The second one is applied to a Poisson structure arising in the study of the Euler top. Finally, in the last example it is shown that the cyclic Poisson structures of type I analyzed in Section 2.3 are actually a particular 3-d case of the type I hemiseparable solutions considered in the present section.

## Example 1. Generalization of the Poisson structures for the Halphen equations and the system of circle maps

Let us consider a family of three-dimensional structure matrices of the kind:

$$
\begin{equation*}
J_{i j}\left(x_{1}, x_{2}, x_{3}\right)=\eta_{3}\left(x_{1}, x_{2}, x_{3}\right)\left(x_{i}-x_{j}\right), \quad i, j=1,2,3 \tag{264}
\end{equation*}
$$

Poisson structures of this form have deserved some attention both for the study of the Halphen system [63], in which

$$
\eta_{3}\left(x_{1}, x_{2}, x_{3}\right)=\left[2\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)\right]^{-1}
$$

as well as for the Poisson formulation of the system of circle maps [63], this time with

$$
\eta_{3}\left(x_{1}, x_{2}, x_{3}\right)=-\left[\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)\right]^{-1}
$$

Instead of analyzing the three-dimensional case, it is feasible to first generalize the previous structures and then to provide a dimension-independent analysis. For this, and following Theorem 4.5.1.1, consider a domain $\Omega \subset \mathbb{R}^{n}$ in which the Poisson structure is to be defined, together with a generic function $\eta(x) \equiv \eta\left(x_{1}, \ldots, x_{n}\right)$ defined in $\Omega$ and complying to the requirements of such theorem. Moreover, we set $\varphi_{i}\left(x_{i}\right)=1$ and consistently $\psi_{i}\left(x_{i}\right)=x_{i}$ for every $i=1, \ldots, n$, as well as $\kappa_{i j}=0$ for every pair $(i, j)$. This leads to a natural $n$-dimensional generalization of the previous structure matrices (264):

$$
\begin{equation*}
J_{i j}(x)=\eta(x)\left(x_{i}-x_{j}\right), \quad i, j=1, \ldots, n \tag{265}
\end{equation*}
$$

In order to fully comply with the requirements of Theorem 4.5.1.1 (and necessarily for the application of Theorems 4.5.2.1 and 4.5.2.3) it must be also assumed that there exists at least one pair of indexes $(i, j)$ for which $\chi_{i j}\left(x_{i}, x_{j}\right)=x_{i}-x_{j} \neq 0$ everywhere in $\Omega$. Consistently with the previous style, in what follows this will be the case for $\chi_{12}$.

Therefore, according to (249) and Theorem 4.5.2.1 a complete set of functionally independent Casimir invariants associated in $\Omega$ to the structure matrices (265) is:

$$
\begin{equation*}
D_{k}(x)=\frac{x_{2}-x_{k}}{x_{1}-x_{2}}, \quad k=3, \ldots, n \tag{266}
\end{equation*}
$$

Then the reduction to Darboux form now makes use of the diffeomorphism (258), with the $D_{k}(x)$ given by (266). The inverse of this transformation is also a diffeomorphism, of equations:

$$
\left\{\begin{array}{l}
x_{1}=y_{1} \\
x_{2}=y_{2} \\
x_{k}=y_{2}-\left(y_{1}-y_{2}\right) y_{k}, \quad k=3, \ldots, n
\end{array}\right.
$$

After applying rule (257) the outcome is that $\mathcal{J}$ in (265) is transformed into the structure matrix (261), this time with

$$
J_{12}(x(y)) \equiv \tilde{J}_{12}(y)=\eta\left(y_{1}, y_{2}, y_{2}-\left(y_{1}-y_{2}\right) y_{3}, \ldots, y_{2}-\left(y_{1}-y_{2}\right) y_{n}\right)\left(y_{1}-y_{2}\right)
$$

The reduction is then completed by means of the time reparametrization $\mathrm{d} \tau=\tilde{J}_{12}(y) \mathrm{d} t$, which finally leads to the Darboux canonical form (263) with $\left(y_{3}, \ldots, y_{n}\right)$ being trivial decoupled Casimir functions and ( $y_{1}, y_{2}$ ) acting as classical canonical variables.

## Example 2. Generalization of the Poisson structure for the Euler top

We now consider the following cubic structure matrix appearing [63] in the analysis of the Euler equations for a triaxial top:

$$
\begin{equation*}
J_{i j}\left(x_{1}, x_{2}, x_{3}\right)=\left(\alpha_{j} x_{i}^{2}-\alpha_{i} x_{j}^{2}\right) \sum_{k=1}^{3}\left(\epsilon_{i j k}\right)^{2} x_{k}, \quad i, j=1,2,3 \tag{267}
\end{equation*}
$$

where $\epsilon_{i j k}$ is the Levi-Civita symbol, and for $i=1,2,3$, the $\alpha_{i}$ are real constants related to the principal moments of inertia $\mu_{i}$ of the top according to the expressions:

$$
\alpha_{1}=\frac{\mu_{2}-\mu_{3}}{\mu_{2} \mu_{3}}, \quad \alpha_{2}=\frac{\mu_{3}-\mu_{1}}{\mu_{1} \mu_{3}}, \quad \alpha_{3}=\frac{\mu_{1}-\mu_{2}}{\mu_{1} \mu_{2}}
$$

As in the previous example, an $n$-dimensional generalization will be developed and analyzed. Before doing that, it is interesting to first investigate how the structure matrix (267) fits in
the framework of Theorem 4.5.1.1. For this, and assuming that $\alpha_{1} \alpha_{2} \alpha_{3} \neq 0$, equations (267) can be equivalently written as:

$$
\left\{\begin{array}{l}
J_{12}\left(x_{1}, x_{2}, x_{3}\right)=\left(4 \alpha_{1} \alpha_{2} \alpha_{3} x_{1} x_{2} x_{3}\right) \frac{1}{\left(2 \alpha_{2} \alpha_{3} x_{1}\right)} \frac{1}{\left(2 \alpha_{1} \alpha_{3} x_{2}\right)}\left(\alpha_{2} \alpha_{3} x_{1}^{2}-\alpha_{1} \alpha_{3} x_{2}^{2}\right)  \tag{268}\\
J_{23}\left(x_{1}, x_{2}, x_{3}\right)=\left(4 \alpha_{1} \alpha_{2} \alpha_{3} x_{1} x_{2} x_{3}\right) \frac{1}{\left(2 \alpha_{1} \alpha_{3} x_{2}\right)} \frac{1}{\left(2 \alpha_{1} \alpha_{2} x_{3}\right)}\left(\alpha_{1} \alpha_{3} x_{2}^{2}-\alpha_{1} \alpha_{2} x_{3}^{2}\right) \\
J_{31}\left(x_{1}, x_{2}, x_{3}\right)=\left(4 \alpha_{1} \alpha_{2} \alpha_{3} x_{1} x_{2} x_{3}\right) \frac{1}{\left(2 \alpha_{1} \alpha_{2} x_{3}\right)} \frac{1}{\left(2 \alpha_{2} \alpha_{3} x_{1}\right)}\left(\alpha_{1} \alpha_{2} x_{3}^{2}-\alpha_{2} \alpha_{3} x_{1}^{2}\right)
\end{array}\right.
$$

Taking into account the alternative form (248) of the solution family, the structure matrix thus expressed in (268) can be recognized in terms of:

$$
\begin{equation*}
\eta\left(x_{1}, x_{2}, x_{3}\right)=4 \alpha_{1} \alpha_{2} \alpha_{3} x_{1} x_{2} x_{3} \quad, \quad \psi_{i}\left(x_{i}\right)=\alpha_{1} \alpha_{2} \alpha_{3} \frac{x_{i}^{2}}{\alpha_{i}}, \quad i=1,2,3 \tag{269}
\end{equation*}
$$

and $\kappa_{i j}=0$ for all $i, j=1,2,3$. Therefore, by means of the recasting (268-269), now it becomes possible to generalize this Poisson structure to arbitrary dimension $n$. For this, let $\Omega \subset \mathbb{R}^{n}$ be a domain such that $x_{i} \neq 0$ for every $x \in \Omega$ and for every $i=1, \ldots, n$, and let us also define:

- A set of $n$ nonvanishing real constants $\left(\alpha_{1}, \ldots, \alpha_{n}\right)$, their product being termed:

$$
\alpha=\prod_{k=1}^{n} \alpha_{k} \neq 0
$$

- An arbitrary function $\nu(x)$ which is $C^{\infty}(\Omega)$ and does not vanish in $\Omega$. Together with it, now function $\eta(x)$ is also defined as:

$$
\eta(x)=4 \alpha \nu(x) \prod_{k=1}^{n} x_{k}
$$

- Functions $\psi_{i}\left(x_{i}\right)=\alpha x_{i}^{2} / \alpha_{i}$ for every $i=1, \ldots, n$.
- Constants $\kappa_{i j}$, which are taken to vanish for all $i, j=1, \ldots, n$.
- To conclude, it is assumed for instance (and without loss of generality) that $\chi_{12}\left(x_{1}, x_{2}\right) \neq$ 0 in $\Omega$, where now it is:

$$
\chi_{12}\left(x_{1}, x_{2}\right)=\alpha\left(\frac{x_{1}^{2}}{\alpha_{1}}-\frac{x_{2}^{2}}{\alpha_{2}}\right)
$$

In this case, a generalized family of $n$-dimensional structure matrices is completely defined in $\Omega$ and complies to the requirements of Theorem 4.5.1.1, the result being of the form:

$$
\begin{equation*}
J_{i j}(x)=\nu(x)\left(\alpha_{j} x_{i}^{2}-\alpha_{i} x_{j}^{2}\right) \prod_{\substack{k=1 \\ k \neq i, j}}^{n} x_{k} \quad, \quad i, j=1, \ldots, n \tag{270}
\end{equation*}
$$

Notice that in the case $n=3$ and $\nu(x)=1$, the structure matrix (270) reduces to (267), as expected.

Therefore, Poisson systems with $n$-dimensional structure matrix of the generalized form (270) have the following set of $(n-2)$ functionally independent Casimir invariants in $\Omega$ :

$$
\begin{equation*}
D_{k}(x)=\frac{\alpha_{1} \alpha_{k} x_{2}^{2}-\alpha_{1} \alpha_{2} x_{k}^{2}}{\alpha_{2} \alpha_{k} x_{1}^{2}-\alpha_{1} \alpha_{k} x_{2}^{2}}, \quad k=3, \ldots, n \tag{271}
\end{equation*}
$$

Consequently, in this case the diffeomorphic coordinate transformation leading to the Darboux canonical form is defined by (258) and (271), and its inverse is also a diffeomorphism in $y(\Omega)$ given by:

$$
\left\{\begin{aligned}
x_{1} & =y_{1} \\
x_{2} & =y_{2} \\
x_{k} & =\varsigma_{k}\left[\frac{\alpha_{k}}{\alpha_{2}} y_{2}^{2}-\left(\frac{\alpha_{k}}{\alpha_{1}} y_{1}^{2}-\frac{\alpha_{k}}{\alpha_{2}} y_{2}^{2}\right) y_{k}\right]^{1 / 2}, \quad k=3, \ldots, n
\end{aligned}\right.
$$

where $\varsigma_{k} \equiv \operatorname{sign}\left(x_{k}\right)$ is a constant defined in terms of the usual sign function, namely $(+1)$ if $x_{k}>0$ and $(-1)$ if $x_{k}<0$ in $\Omega$ (recall that $x_{k} \neq 0$ in $\Omega$ for all $\left.k=1, \ldots, n\right)$. From this stage on, what remains of the Darboux reduction does not present any special feature not specified in the proof of Theorem 4.5.2.3, and therefore will be omitted for the sake of brevity.

## Example 3. Generalization of the cyclic Poisson structures of type I

Consider now the form of the hemiseparable structure matrices of type I, as provided in expression (248). Using this as starting point, let us now define the functions $\phi_{i}\left(x_{i}\right) \equiv \psi_{i}^{\prime}\left(x_{i}\right)$ for all $i=1, \ldots, n$. Accordingly, functions $\phi_{i}\left(x_{i}\right)$ are $C^{\infty}(\Omega)$ and do not vanish in $\Omega$. We are thus led to the following form for the structure matrices:

$$
\begin{equation*}
J_{i j}(x)=\frac{\eta(x)}{\phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)} \chi_{i j}\left(x_{i}, x_{j}\right)=\frac{\eta(x)}{\phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)}\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}\right], \quad i, j=1, \ldots, n \tag{272}
\end{equation*}
$$

The properties of the functions $\phi_{i}\left(x_{i}\right)$ just defined allow a simple redefinition of function $\eta(x)$ as

$$
\begin{equation*}
\hat{\eta}(x)=\eta(x) \prod_{k=1}^{n} \phi_{k}\left(x_{k}\right) \tag{273}
\end{equation*}
$$

Substituting (273) into (272) we finally obtain:
$J_{i j}(x)=\hat{\eta}(x) \chi_{i j}\left(x_{i}, x_{j}\right) \prod_{\substack{k=1 \\ k \neq i, j}}^{n} \phi_{k}\left(x_{k}\right)=\hat{\eta}(x)\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)+\kappa_{i j}\right] \prod_{\substack{k=1 \\ k \neq i, j}}^{n} \phi_{k}\left(x_{k}\right), \quad i, j=1, \ldots, n$

In the case $n=3$, it is therefore evident that the hemiseparable family of type I given in (274) becomes exactly the family of nontrivial cyclic structure matrices of type I, as it was developed in equation (71) and Theorem 2.3.1.1. Here the term "nontrivial" deserves a brief explanation. Notice that the cyclic Poisson structures of type I were defined in Theorem 2.3.1.1 in such a way that the rank could in principle vanish (something later excluded in their global analysis). The reason for doing this was the interest of analyzing and classifying the different possible reductions to the Darboux canonical form arising according to the vanishing of different functions $\omega_{i j}$. On the contrary, doing such kind of classification is not necessary in our present stage, and accordingly the possibility of a null rank was not present in Theorem 4.5.1.1, in which hemiseparable structure matrices of type I were directly defined in order to have nonzero rank. According to these considerations, we see that hemiseparable Poisson structures of type I are the natural generalization to arbitrary dimension of the 3-d family given by the cyclic structure matrices of type I. Actually, we see that the rank of the Poisson manifold has remained constant (and equal to 2 ) after such generalization to arbitrary dimension. This explains also the strong parallelism existing in the examples considered, in spite that in the present section the treatment has been different, mainly based on the idea of generalizing well-known 3-d Poisson structures to general dimension $n$.

### 4.6. HEMISEPARABLE SOLUTIONS OF TYPE II

A supplementary family of $n$-dimensional solutions [79] of the Jacobi equations (the last in the work) is considered in this section. The family is completely characterized and investigated globally. As it was the case in the previous section, this solution family has a functionally general form, defined in terms of functions of an arbitrary nonlinearity. In fact, the new family is formally reminiscent of the hemiseparable family of type I, and accordingly it is termed hemiseparable of type II. In spite of their parallelism, however, both families are different. In the new analysis it is also possible the explicit and global determination of the main features, such as the case-classification of the Casimir invariants and the global construction of the Darboux canonical form. Again, in this family it is found that the understanding of its global analysis implies a case classification such that different distinguished invariants are to be used in each case. Namely, only the knowledge of one complete set of Casimir invariants does not guarantee everywhere the reduction to the Darboux canonical form, in spite of the constancy of the rank. Thus, the analysis of this kind of Poisson structures requires the use of a case-dependent set of global Casimir invariants which determines the kind of subsequent global Darboux reduction to be constructed. Such classification is given in this section. Some applied examples of interest are also discussed.

The analysis starts in Subsection 4.6 .1 by providing the main results about the functional characterization of the solution family. In Subsection 4.6.2, the global analysis of the family is presented. We finish in Subsection 4.6.3 with some examples of applied significance.

### 4.6.1. Characterization of the family

The first result to be presented is the following one:

Theorem 4.6.1.1. Let $\eta(x)$ and $\varphi_{i}\left(x_{i}\right)$, for $i=1, \ldots, n$, be functions defined in a domain $\Omega \subset \mathbb{R}^{n}$, all of which are $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$. In addition let

$$
\begin{equation*}
\psi_{i}\left(x_{i}\right)=a_{i} \exp \left(\int \frac{\mathrm{~d} x_{i}}{\varphi_{i}\left(x_{i}\right)}\right), \quad i=1, \ldots, n \tag{275}
\end{equation*}
$$

where $a_{i} \neq 0, i=1, \ldots, n$, are arbitrary nonzero real constants, and every exponent in (275)

$$
\begin{equation*}
\int \frac{\mathrm{d} x_{i}}{\varphi_{i}\left(x_{i}\right)}, \quad i=1, \ldots, n \tag{276}
\end{equation*}
$$

denotes one primitive of $1 / \varphi_{i}\left(x_{i}\right)$. Finally, let the functions $\omega_{i j}\left(x_{i}, x_{j}\right)$ be defined as

$$
\omega_{i j}\left(x_{i}, x_{j}\right)=\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right), \quad i, j=1, \ldots, n
$$

and assume that $\omega_{i j}\left(x_{i}, x_{j}\right)$ is nonvanishing in $\Omega$ at least for one pair $(i, j)$. Then $\mathcal{J}=\left(J_{i j}\right)$ is a family of $n$-dimensional structure matrices globally defined in $\Omega$, where

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \varphi_{i}\left(x_{i}\right) \varphi_{j}\left(x_{j}\right) \omega_{i j}\left(x_{i}, x_{j}\right), \quad i, j=1, \ldots, n \tag{277}
\end{equation*}
$$

Proof. Skew-symmetry is clear from simple inspection in (277). We can then substitute $\mathcal{J}$ in (277) into the Jacobi identities (25) and obtain after some calculations:

$$
\sum_{l=1}^{n}\left(J_{l i} \partial_{l} J_{j k}+J_{l j} \partial_{l} J_{k i}+J_{l k} \partial_{l} J_{i j}\right)=\eta T_{1}+\eta^{2} T_{2}
$$

Here $T_{1}$ and $T_{2}$ are the following two terms, which we examine separately:

$$
\begin{gathered}
T_{1}=\sum_{l=1}^{n} \varphi_{i} \varphi_{j} \varphi_{k} \varphi_{l}\left(\partial_{l} \eta\right)\left(\omega_{i l} \omega_{j k}+\omega_{j l} \omega_{k i}+\omega_{k l} \omega_{i j}\right) \\
T_{2}=\sum_{l=1}^{n}\left\{\varphi_{i} \varphi_{l} \omega_{i l}\left[\delta_{l j} \varphi_{j}^{\prime} \varphi_{k} \omega_{j k}+\delta_{l k} \varphi_{j} \varphi_{k}^{\prime} \omega_{j k}+\varphi_{j} \varphi_{k}\left(\delta_{l j} \frac{\psi_{j}}{\varphi_{j}}-\delta_{l k} \frac{\psi_{k}}{\varphi_{k}}\right)\right]+\right. \\
\varphi_{j} \varphi_{l} \omega_{j l}\left[\delta_{l k} \varphi_{k}^{\prime} \varphi_{i} \omega_{k i}+\delta_{l i} \varphi_{k} \varphi_{i}^{\prime} \omega_{k i}+\varphi_{k} \varphi_{i}\left(\delta_{l k} \frac{\psi_{k}}{\varphi_{k}}-\delta_{l i} \frac{\psi_{i}}{\varphi_{i}}\right)\right]+ \\
\left.\varphi_{k} \varphi_{l} \omega_{k l}\left[\delta_{l i} \varphi_{i}^{\prime} \varphi_{j} \omega_{i j}+\delta_{l j} \varphi_{i} \varphi_{j}^{\prime} \omega_{i j}+\varphi_{i} \varphi_{j}\left(\delta_{l i} \frac{\psi_{i}}{\varphi_{i}}-\delta_{l j} \frac{\psi_{j}}{\varphi_{j}}\right)\right]\right\}
\end{gathered}
$$

With regard to $T_{1}$, if every $\omega_{i j}$ is substituted by its expression $\omega_{i j}=\psi_{i}-\psi_{j}$ and the result is simplified, it is found that:

$$
\omega_{i l} \omega_{j k}+\omega_{j l} \omega_{k i}+\omega_{k l} \omega_{i j}=0
$$

Then we have proved that $T_{1}=0$. We proceed now with $T_{2}$ by expanding its expression and cancelling out similar terms. After a rearrangement we are led to:

$$
T_{2}=\varphi_{i} \varphi_{j} \varphi_{k}\left[\omega_{i j} \psi_{j}-\omega_{i k} \psi_{k}+\omega_{j k} \psi_{k}-\omega_{j i} \psi_{i}+\omega_{k i} \psi_{i}-\omega_{k j} \psi_{j}\right]=0
$$

Thus we also have $T_{2}=0$ and the proof is finished.
Q.E.D.

Some brief but important remarks must be provided at this point. First of all recall that, as indicated in Theorem 4.6.1.1, for every $i$ the primitive (276) obtained from $\varphi_{i}\left(x_{i}\right)$ must be chosen to be one and the same for all the entries of $\mathcal{J}$. However, the specific choice is actually arbitrary. In such sense, notice that if different integration constants are selected, then the outcome is also a member of the solution family, this time with rescaled parameters $a_{i}$. In second term, notice that by construction the functions $\psi_{i}\left(x_{i}\right)$ and $\omega_{i j}\left(x_{i}, x_{j}\right)$ are $C^{\infty}(\Omega)$. In
third place, it can be observed that the definition (275) allows an alternative writing of the family just characterized, namely $\mathcal{J}=\left(J_{i j}\right)$ can also be expressed as

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \frac{\psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)}{\psi_{i}^{\prime}\left(x_{i}\right) \psi_{j}^{\prime}\left(x_{j}\right)} \omega_{i j}\left(x_{i}, x_{j}\right)=\eta(x) \frac{\psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)}{\psi_{i}^{\prime}\left(x_{i}\right) \psi_{j}^{\prime}\left(x_{j}\right)}\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right], \quad i, j=1, \ldots, n \tag{278}
\end{equation*}
$$

where functions $\psi_{i}\left(x_{i}\right)$ must be $C^{\infty}(\Omega)$ and nonvanishing in $\Omega$, and such that functions $\psi_{i}^{\prime}\left(x_{i}\right)$ are also nonvanishing in $\Omega$, while the rest of defining properties were already presented in Theorem 4.6.1.1. Under these assumptions, (278) can be taken as an alternative definition of the solution family of structure matrices. Both ways of expressing the family will be employed in the forthcoming analysis.

### 4.6.2. Casimir invariants and global Darboux analysis

The main global properties of the family identified in Theorem 4.6.1.1 are characterized in what follows:

Theorem 4.6.2.1. Let $\mathcal{J}$ be a structure matrix of the form (277) characterized in Theorem 4.6.1.1, which is defined in a domain $\Omega \subset \mathbb{R}^{n}$ and such that the pair $(i, j)$ verifies that function $\omega_{i j}\left(x_{i}, x_{j}\right)$ is nonvanishing in $\Omega$. Then $\operatorname{Rank}(\mathcal{J})=2$ everywhere in $\Omega$ and a complete set of independent Casimir invariants for $\mathcal{J}$ is given by:

$$
\begin{equation*}
D_{k}(x)=\frac{\psi_{i}\left(x_{i}\right)\left[\psi_{j}\left(x_{j}\right)-\psi_{k}\left(x_{k}\right)\right]}{\psi_{k}\left(x_{k}\right)\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right]}=\frac{\psi_{i}\left(x_{i}\right) \omega_{j k}\left(x_{j}, x_{k}\right)}{\psi_{k}\left(x_{k}\right) \omega_{i j}\left(x_{i}, x_{j}\right)} \quad, \quad k=1, \ldots, n ; \quad k \neq i, j \tag{279}
\end{equation*}
$$

Moreover, every Casimir invariant in (279) is globally defined in $\Omega$.

Proof. Since functions $\eta(x)$ and $\varphi_{i}\left(x_{i}\right)$ are nonvanishing in $\Omega$, the use of rank-preserving matrix transformations shows that $\operatorname{Rank}(\mathcal{J})=\operatorname{Rank}(W)$ in $\Omega$, where $W \equiv\left(\omega_{i j}\left(x_{i}, x_{j}\right)\right)$. Given that at least one of the entries of $W$ is also nonvanishing in $\Omega$, this implies that $\operatorname{Rank}(\mathcal{J}) \geq 2$ everywhere in $\Omega$. Matrix $W$ can be now submitted to additional rank-preserving transformations: notice that $\operatorname{Rank}(W)$ is also unaltered if we subtract the first row to the rest of rows, and then if on the resulting matrix we subtract the first column to every one of the remaining columns. The result is a new matrix $W^{*}$ of the form:

$$
W^{*}=\left(\begin{array}{cccc}
0 & \omega_{12} & \ldots & \omega_{1 n}  \tag{280}\\
-\omega_{12} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
-\omega_{1 n} & 0 & \ldots & 0
\end{array}\right)
$$

Thus, it is evident from (280) that $\operatorname{Rank}(\mathcal{J})=\operatorname{Rank}\left(W^{*}\right) \leq 2$ in every point of $\Omega$. Therefore we conclude that $\operatorname{Rank}(\mathcal{J})=2$ in $\Omega$. This proves the first part of the statement. For the second part, notice first that every function $D_{k}(x)$ in (279) always depends on $x_{i}, x_{j}$ and $x_{k}$ (since functions $\psi_{k}\left(x_{k}\right)$ cannot be constant for any $k$, according to the conditions established) and in addition $D_{k}(x)$ does not depend on the rest of variables. This implies immediately the functional independence of the set $\left\{D_{k}(x) \mid k=1, \ldots, n ; k \neq i, j\right\}$. Moreover, since all functions composing $D_{k}(x)$ are $C^{\infty}(\Omega)$ and $\psi_{k}\left(x_{k}\right) \omega_{i j}\left(x_{i}, x_{j}\right) \neq 0$ everywhere in $\Omega$, function $D_{k}(x)$ is necessarily $C^{\infty}(\Omega)$. Therefore, to complete the proof it is only required to prove that functions $D_{k}(x)$ are Casimir invariants for every $k$. The easiest way to see this is to verify that $\mathcal{J} \cdot \nabla D_{k}=0$ for every $k=1, \ldots, n$, with $k \neq i, j$ (notice as well that for both values $k=i, j$, function $D_{k}(x)$ is a constant, and then also a Casimir invariant, but trivial). We have then:

$$
\partial_{i} D_{k}(x)=\frac{\psi_{i}^{\prime} \psi_{j} \psi_{k} \omega_{k j}}{\left(\psi_{k} \omega_{i j}\right)^{2}}, \quad \partial_{j} D_{k}(x)=\frac{\psi_{i} \psi_{j}^{\prime} \psi_{k} \omega_{i k}}{\left(\psi_{k} \omega_{i j}\right)^{2}}, \quad \partial_{k} D_{k}(x)=\frac{\psi_{i} \psi_{j} \psi_{k}^{\prime} \omega_{j i}}{\left(\psi_{k} \omega_{i j}\right)^{2}}
$$

for $k=1, \ldots n, k \neq i, j$. Thus for every $l=1, \ldots, n$ it can be seen that:

$$
\begin{gather*}
\sum_{s=1}^{n} J_{l s} \partial_{s} D_{k}=J_{l i} \partial_{i} D_{k}+J_{l j} \partial_{j} D_{k}+J_{l k} \partial_{k} D_{k}= \\
\frac{\eta \varphi_{l} \psi_{i} \psi_{j}}{\psi_{k}\left(\omega_{i j}\right)^{2}}\left(\omega_{l i} \omega_{k j}+\omega_{l j} \omega_{i k}+\omega_{l k} \omega_{j i}\right) \tag{281}
\end{gather*}
$$

In equation (281) the last term vanishes for every choice of $i, j, k, l$,

$$
\omega_{l i} \omega_{k j}+\omega_{l j} \omega_{i k}+\omega_{l k} \omega_{j i}=0
$$

as it was already shown in the proof of Theorem 4.6.1.1. Therefore, $\mathcal{J} \cdot \nabla D_{k}=0$ for every $k$. Now the proof is complete.
Q.E.D.

Accordingly, every Poisson system of this kind has $(n-2)$ independent Casimir invariants, additional to the Hamiltonian function. This implies that:

Corollary 4.6.2.2. Consider that the assumptions of Remark 1.4.4.4 hold. Every n-dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (277) characterized in Theorem 4.6.1.1, is an algebraically integrable system in $\Omega$.

For the sake of completeness, we recall again the concept of time reparametrization for Poisson systems (see Definition 4.1.1) which are transformations of the form

$$
\begin{equation*}
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t \tag{282}
\end{equation*}
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \rightarrow \mathbb{R}$ is a $C^{\infty}(\Omega)$ function which does not vanish in $\Omega$. Therefore, if

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J} \cdot \nabla H \tag{283}
\end{equation*}
$$

is an arbitrary Poisson system defined in $\Omega$, then every time reparametrization (282) leads from (283) to the system (not necessarily of Poisson type):

$$
\begin{equation*}
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H \tag{284}
\end{equation*}
$$

With this issue in mind, another consequence of the previous results is that they allow the constructive and global determination of the Darboux canonical form for this family of Poisson structures. This statement is provided in the next:

Theorem 4.6.2.3. Let $\Omega \subset \mathbb{R}^{n}$ be a domain where a Poisson system

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x)
$$

is defined everywhere, for which $\mathcal{J}(x)$ is a structure matrix of the form (277) characterized in Theorem 4.6.1.1, and such that the pair $(i, j)$ verifies that function $\omega_{i j}\left(x_{i}, x_{j}\right)$ is nonvanishing in $\Omega$. Then such Poisson system can be globally reduced in $\Omega$ to an one degree of freedom Hamiltonian system and the Darboux canonical form is accomplished globally in $\Omega$ in the new coordinate system $\left(y_{1}, \ldots, y_{n}\right)$ and the new time $\tau$, where $\left(y_{1}, \ldots, y_{n}\right)$ are given by the diffeomorphism globally defined in $\Omega$

$$
\left\{\begin{array}{l}
y_{i}(x)=x_{i}  \tag{285}\\
y_{j}(x)=x_{j} \\
y_{k}(x)=D_{k}(x), \quad k=1, \ldots, n ; \quad k \neq i, j
\end{array}\right.
$$

in which the $D_{k}(x)$ are the Casimir invariants (279); and the new time $\tau$ is defined by the time reparametrization:

$$
\begin{equation*}
\mathrm{d} \tau=J_{i j}(x(y)) \mathrm{d} t \tag{286}
\end{equation*}
$$

Proof. It follows from Theorem 4.6.2.1 that Darboux' theorem is applicable because $\mathcal{J}$ has constant rank 2 in $\Omega$. Without loss of generality and for the sake of clarity, assume that it is $\omega_{12} \neq 0$ everywhere in $\Omega$. Recall also that, after a general smooth change $y \equiv y(x)$, an
arbitrary structure matrix $\mathcal{J}(x)$ is transformed into another one $\mathcal{J}^{*}(y)$ as:

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{287}
\end{equation*}
$$

For the present solution family, the reduction is carried out in two steps. We first perform the change of variables (285), which in this case is

$$
\left\{\begin{align*}
y_{1}(x) & =x_{1}  \tag{288}\\
y_{2}(x) & =x_{2} \\
y_{k}(x) & =D_{k}(x), \quad k=3, \ldots, n
\end{align*}\right.
$$

where the $D_{k}(x)$ are given by (279), namely:

$$
\begin{equation*}
D_{k}(x)=\frac{\psi_{1}\left(x_{1}\right) \omega_{2 k}\left(x_{2}, x_{k}\right)}{\psi_{k}\left(x_{k}\right) \omega_{12}\left(x_{1}, x_{2}\right)}=\frac{\psi_{1}\left(x_{1}\right)\left(\psi_{2}\left(x_{2}\right)-\psi_{k}\left(x_{k}\right)\right)}{\psi_{k}\left(x_{k}\right)\left(\psi_{1}\left(x_{1}\right)-\psi_{2}\left(x_{2}\right)\right)}, \quad k=3, \ldots, n \tag{289}
\end{equation*}
$$

Note that this change of variables is invertible everywhere in $\Omega$, and its inverse is

$$
\left\{\begin{array}{l}
x_{1}(y)=y_{1}  \tag{290}\\
x_{2}(y)=y_{2} \\
x_{k}(y)=\zeta_{k}\left(\frac{\psi_{1}\left(y_{1}\right) \psi_{2}\left(y_{2}\right)}{\psi_{1}\left(y_{1}\right)+y_{k} \omega_{12}\left(y_{1}, y_{2}\right)}\right), \quad k=3, \ldots, n
\end{array}\right.
$$

where function $\zeta_{k}$ is the inverse function of $\psi_{k}$ for every $k$. The examination of (288-290) shows that the variable transformation (288) to be performed exists and is a diffeomorphism everywhere in $\Omega$ as a consequence that by hypothesis we have $\omega_{12}\left(x_{1}, x_{2}\right) \neq 0$ in $\Omega$, as well as $\psi_{k}\left(x_{k}\right) \neq 0$ and $\psi_{k}^{\prime}\left(x_{k}\right) \neq 0$ for every $k$ in $\Omega$. Then, according to (288) and (289), and taking (287) into account, after some calculations we arrive at

$$
\mathcal{J}^{*}(y)=J_{12}(x(y))\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0  \tag{291}\\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right)
$$

where from equations (277) and (290) we have

$$
\begin{equation*}
J_{12}(x(y))=\eta\left(y_{1}, y_{2}, x_{3}(y), \ldots, x_{n}(y)\right) \varphi_{1}\left(y_{1}\right) \varphi_{2}\left(y_{2}\right) \omega_{12}\left(y_{1}, y_{2}\right) \tag{292}
\end{equation*}
$$

The explicit dependences of $\left(x_{3}(y), \ldots, x_{n}(y)\right)$ are obviously the ones given in (290) and were not displayed in (292) for the sake of clarity. Note that $J_{12}(x(y))$ is nonvanishing in $\Omega^{*}=y(\Omega)$ and $C^{\infty}\left(\Omega^{*}\right)$. These properties allow the accomplishment of the second step of the reduction
which is a reparametrization of time, which in this case does not suppress the Poisson structure of the vector field. Thus, making use of (292) in equation (286), the transformation $\mathrm{d} \tau=$ $J_{12}(x(y)) \mathrm{d} t$ is performed. According to (282-284) this leads from the structure matrix (291) to the Darboux canonical form:

$$
\mathcal{J}_{D[n, 2]}(y)=\left(\begin{array}{ccccc}
0 & 1 & 0 & \ldots & 0  \tag{293}\\
-1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right)=\mathbb{S}_{2} \oplus \mathbb{O}_{(n-2)}
$$

Thus the reduction is globally accomplished and the proof is complete.
Q.E.D.

As a consequence, we now have:

Corollary 4.6.2.4. Consider that the assumptions of Remark 1.4.4.4 hold. Every n-dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in which the structure matrix $\mathcal{J}(x)$ is of the kind (277) characterized in Theorem 4.6.1.1, can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

The analysis of the family of Poisson structures is thus completed, since at this stage the reduction directly relates the initial Poisson systems with their classical Hamiltonian counterparts. In the next subsection, the results developed are exemplified by means of some instances of applied interest. This provides several cases of systems embraced and generalized by the previous results.

### 4.6.3. Examples

The first example to be considered is related to a certain kind of Poisson structures appearing in different domains, including Lotka-Volterra equations and the system of circle maps. The motivation of the second example is to prove that the cyclic Poisson structures of type II considered in Section 2.4 are in fact a particular 3-d case of the solution family currently investigated, which thus provides the natural generalization of the former.

## Example 1. Generalization of the Poisson structures for the Lotka-Volterra equations and the system of circle maps

We shall first consider the family of three-dimensional structure matrices of the form:

$$
\begin{equation*}
J_{i j}\left(x_{1}, x_{2}, x_{3}\right)=\eta_{3}\left(x_{1}, x_{2}, x_{3}\right) x_{i} x_{j}\left(x_{i}-x_{j}\right), \quad i, j=1,2,3 \tag{294}
\end{equation*}
$$

This kind of Poisson structures have received some attention both for the study of conservative Lotka-Volterra systems [63], in which

$$
\eta_{3}\left(x_{1}, x_{2}, x_{3}\right)=1
$$

as well as for the Poisson formulation of the system of circle maps [63], this time with

$$
\eta_{3}\left(x_{1}, x_{2}, x_{3}\right)=-\left[\left(x_{1}-x_{2}\right)\left(x_{2}-x_{3}\right)\left(x_{3}-x_{1}\right)\right]^{-1}
$$

It is feasible to generalize the previous structure matrices in order to provide a dimensionindependent analysis, instead of directly analyzing the three-dimensional case. For this, and following Theorem 4.6.1.1, consider a domain $\Omega \subset \mathbb{R}^{n}$ in which the Poisson structure is to be defined, together with a generic function $\eta(x) \equiv \eta\left(x_{1}, \ldots, x_{n}\right)$ defined in $\Omega$ and complying to the requirements of such theorem. Moreover, we set $\varphi_{i}\left(x_{i}\right)=x_{i}$ and consistently $\psi_{i}\left(x_{i}\right)=x_{i}$ for every $i=1, \ldots, n$. This leads to a natural $n$-dimensional generalization of the structure matrices (294) previously introduced:

$$
\begin{equation*}
J_{i j}(x)=\eta(x) x_{i} x_{j}\left(x_{i}-x_{j}\right), \quad i, j=1, \ldots, n \tag{295}
\end{equation*}
$$

If we wish to fully comply with the requirements of Theorem 4.6.1.1 (and necessarily for the application of Theorems 4.6 .2 . 1 and 4.6.2.3) it must be assumed that $x_{i} \neq 0$ in $\Omega$ for every $i=1, \ldots, n$, and also that there exists at least one pair of indexes $(i, j)$ for which $\omega_{i j}\left(x_{i}, x_{j}\right)=x_{i}-x_{j} \neq 0$ everywhere in $\Omega$. In agreement with the previous style, in what follows this will be the case for $\omega_{12}$.

Then, according to (279) and Theorem 4.6.2.1 a complete set of functionally independent Casimir invariants associated in $\Omega$ to the structure matrices (295) is:

$$
\begin{equation*}
D_{k}(x)=\frac{x_{1}\left(x_{2}-x_{k}\right)}{x_{k}\left(x_{1}-x_{2}\right)}, \quad k=3, \ldots, n \tag{296}
\end{equation*}
$$

Therefore the reduction to the Darboux canonical form now makes use of the diffeomorphism (288), with the $D_{k}(x)$ given by (296). The inverse of this transformation is also a diffeomor-
phism, of equations:

$$
\left\{\begin{aligned}
x_{1} & =y_{1} \\
x_{2} & =y_{2} \\
x_{k} & =\frac{y_{1} y_{2}}{y_{1}+\left(y_{1}-y_{2}\right) y_{k}}, \quad k=3, \ldots, n
\end{aligned}\right.
$$

The outcome after applying rule (287) is that $\mathcal{J}$ in (295) is transformed into the structure matrix (291), this time with:

$$
J_{12}(x(y)) \equiv \tilde{J}_{12}(y)=\eta\left(y_{1}, y_{2}, \frac{y_{1} y_{2}}{y_{1}+\left(y_{1}-y_{2}\right) y_{3}}, \ldots, \frac{y_{1} y_{2}}{y_{1}+\left(y_{1}-y_{2}\right) y_{n}}\right) y_{1} y_{2}\left(y_{1}-y_{2}\right)
$$

Then the reduction is completed by means of the time reparametrization $\mathrm{d} \tau=\tilde{J}_{12}(y) \mathrm{d} t$, which finally leads to the Darboux canonical form (293) with ( $y_{1}, y_{2}$ ) now being Hamiltonian canonical variables, while $\left(y_{3}, \ldots, y_{n}\right)$ are trivially decoupled distinguished invariants.

## Example 2. Generalization of the cyclic Poisson structures of type II

Let us now turn back to the general form of the hemiseparable structure matrices of type II, as expressed in equations (278). Recall that this expression is found after making use of equation (275) and writing $\psi_{i}\left(x_{i}\right)=\varphi_{i}\left(x_{i}\right) \psi_{i}^{\prime}\left(x_{i}\right)$ for all $i=1, \ldots, n$. According to this, we have that functions $\psi_{i}\left(x_{i}\right)$ are $C^{\infty}(\Omega)$ and do not vanish in $\Omega$. Consistently, functions $\psi_{i}^{\prime}\left(x_{i}\right)$ are also $C^{\infty}(\Omega)$ and do not vanish in $\Omega$. Let us now define functions $\phi_{i}\left(x_{i}\right) \equiv \psi_{i}^{\prime}\left(x_{i}\right)$ for all $i=1, \ldots, n$. In this way, the following equivalent form for the type II hemiseparable structure matrices is found:

$$
\begin{equation*}
J_{i j}(x)=\eta(x) \frac{\psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)}{\phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)} \omega_{i j}\left(x_{i}, x_{j}\right)=\eta(x) \frac{\psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)}{\phi_{i}\left(x_{i}\right) \phi_{j}\left(x_{j}\right)}\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right], \quad i, j=1, \ldots, n \tag{297}
\end{equation*}
$$

Now notice that the nonvanishing properties of the functions $\phi_{i}\left(x_{i}\right)$ just defined allow a redefinition of function $\eta(x)$ as

$$
\begin{equation*}
\hat{\eta}(x)=\eta(x) \prod_{k=1}^{n} \phi_{k}\left(x_{k}\right) \tag{298}
\end{equation*}
$$

After substitution of (298) into (297) we finally arrive at:

$$
\begin{gather*}
J_{i j}(x)=\hat{\eta}(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right) \omega_{i j}\left(x_{i}, x_{j}\right) \prod_{\substack{k=1 \\
k \neq i, j}}^{n} \phi_{k}\left(x_{k}\right)= \\
\hat{\eta}(x) \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)\left[\psi_{i}\left(x_{i}\right)-\psi_{j}\left(x_{j}\right)\right] \prod_{\substack{k=1 \\
k \neq i, j}}^{n} \phi_{k}\left(x_{k}\right), \quad i, j=1, \ldots, n \tag{299}
\end{gather*}
$$

We then conclude easily from (299) that if $n=3$, then the hemiseparable family of type II reduces exactly to the family of nontrivial cyclic structure matrices of type II, as it was displayed in equation (91) and Theorem 2.4.1.1. And, as it was the case when considering type I Poisson structures in the previous section, now "nontrivial" requires a brief comment. This is based on the fact that the cyclic structure matrices of type II were defined in Theorem 2.4.1.1 in such a way that the rank could in principle vanish (of course, this feature was later excluded in the corresponding global analysis). On the contrary, such possibility is not present in Theorem 4.6.1.1, in which hemiseparable structures of type II were defined as having nonzero rank. In both cases, the reasons are entirely similar to those presented in Example 3 of Subsection 4.5.3 in connection with type I hemiseparable and cyclic structure matrices. Therefore, hemiseparable Poisson structures of type II are a natural generalization to arbitrary dimension of the 3 -d family given by the cyclic Poisson structures of type II. In fact, we have seen that the rank of the Poisson manifold has remained constant and equal to 2 after the generalization. Again, this accounts for the parallelism existing in the examples considered, in spite that in the present subsection the treatment has been essentially different, mainly based now on the goal of generalizing to an arbitrary dimension $n$ some 3-d Poisson structures already known in the literature.

### 4.7. THE ROLE OF TIME REPARAMETRIZATIONS

We now start the concluding section of this thesis, in which the use of time reparametrizations will provide a unifying perspective of many previously seen results. In this sense, it is of central importance Definition 4.1.1 of time reparametrization in the general $n$-dimensional case. The most relevant outcomes of the forthcoming analysis will be the following: (a) the characterization of some wide families of time reparametrizations that preserve the Poisson structure for general $n$-dimensional Poisson systems; (b) the generalization of some of the $n$ dimensional solution families already characterized in this chapter; (c) the explicit and global construction of the Darboux canonical form for $n$-dimensional Poisson structures of rank 2 .

### 4.7.1. Introduction and formulation of the problem

Provided $\mathcal{J}(x)$ is an $n$-dimensional structure matrix of constant rank in a domain $\Omega \subset \mathbb{R}^{n}$, we shall now consider the following problem: given an arbitrary function $\eta(x): \Omega \rightarrow \mathbb{R}$ which is smooth in $\Omega$ and does not vanish in $\Omega$, we would like to investigate the conditions such that the product $\eta(x) \mathcal{J}(x)$ is also a structure matrix.

The naturalness of this question should be clear at this stage: notice the close relationship of this issue with the problem of determining whether or not a specific time reparametrization preserves a given Poisson structure, as mentioned in Subsection 1.2.4 and in Section 4.1. According to Definition 4.1.1, recall that a time reparametrization (or NTT) is a transformation (164) of the form

$$
\mathrm{d} \tau=\frac{1}{\eta(x)} \mathrm{d} t
$$

where $t$ is the initial time variable, $\tau$ is the new time and $\eta(x): \Omega \subset \mathbb{R}^{n} \rightarrow \mathbb{R}$ is a smooth function in $\Omega$ which does not vanish in $\Omega$. Thus, given a Poisson system (165) defined in $\Omega$, then the time reparametrization (164) leads from (165) to the system (not necessarily of Poisson type) of the form (166):

$$
\frac{\mathrm{d} x}{\mathrm{~d} \tau}=\eta \mathcal{J} \cdot \nabla H
$$

As it was indicated in Subsection 1.2.4 and in Section 4.1, the new vector field obtained from (165) after a general time reparametrization is always a Poisson system in the cases of dimensions $n=2$ and $n=3$. On the contrary, this is not necessarily the situation for dimensions $n \geq 4$, which are then our subject in what follows. Therefore, apart from being a natural problem in this framework, the study of time reparametrizations is interesting in the present context because we have seen that sometimes they are required in order to achieve the

Darboux canonical form for $n$-dimensional solution families. In particular, notice that time reparametrizations are not required for such task in the case of separable and multiseparable solutions (both having arbitrary dimension and rank), while they are employed in the Darboux reduction of hemiseparable solutions of types I and II (which have arbitrary dimension and rank 2). As we shall see, the investigation of time reparametrizations provides important clues on which many of the preceding developments converge, including their role in the generalization of some of the previous solution families, as well as a tool for the determination of the Darboux canonical form.

In this context, it is also convenient to give a brief explanation about the condition $\eta(x) \neq 0$ in $\Omega$ just introduced. Two fundamental reasons account for this requirement. The first one is that by definition, every time reparametrization must be defined in terms of a smooth function $\eta(x)$ which does not vanish anywhere on its domain of definition (see Subsections 1.2.4 and 1.4.3, as well as Section 4.1, for additional details). Of course, it would be mathematically acceptable to investigate the conditions such that $\eta(x) \mathcal{J}(x)$ is a structure matrix provided $\mathcal{J}(x)$ is, with the only requirement of a smooth $\eta(x)$. However such problem could not be assimilated to the use of time reparametrizations, which is of central interest here. Additionally, the second key reason for choosing a nonvanishing function $\eta(x)$ is derived from the fact that if $\operatorname{Rank}(\mathcal{J}(x))$ is constant in $\Omega$, then the rank of $\eta(x) \mathcal{J}(x)$ will be also constant in $\Omega$. The interest in this constancy is of course the applicability of Darboux' theorem (recall that constancy of the rank is a requirement of such theorem). Therefore, the nonvanishing character of $\eta(x)$ preserves the applicability of Darboux' theorem, also of paramount importance in this work.

The following definition is natural for the problem considered:

Definition 4.7.1.1. Let $\mathcal{J}(x)$ be an $n \times n$ structure matrix defined everywhere in a domain $\Omega \subset \mathbb{R}^{n}$ and of constant rank in $\Omega$, and let $\eta(x): \Omega \rightarrow \mathbb{R}$ be a smooth function which does not vanish in $\Omega$ and such that $\eta(x) \mathcal{J}(x)$ is also a structure matrix defined everywhere in $\Omega$. Then, the function $\eta(x)$ will be called a reparametrization factor for $\mathcal{J}(x)$ in $\Omega$.

In connection with the previous definition, it will be of future use the following direct, but useful result:

Proposition 4.7.1.2. Let $\mathcal{J}(x)$ be an $n \times n$ structure matrix defined everywhere in a domain
$\Omega \subset \mathbb{R}^{n}$ and of constant rank in $\Omega$, and let $\eta(x)$ be a reparametrization factor for $\mathcal{J}(x)$ in $\Omega$. Then:
(a) Function $D(x)$ is a Casimir invariant of $\eta(x) \mathcal{J}(x)$ in $\Omega$ if and only if it is a Casimir invariant of $\mathcal{J}(x)$ in $\Omega$.
(b) If a Poisson system having the structure matrix $\mathcal{J}(x)$ can be reduced globally and diffeomorphically in $\Omega$ to the Darboux canonical form, then every Poisson system having the structure matrix $\eta(x) \mathcal{J}(x)$ can also be reduced globally and diffeomorphically in $\Omega$ to the Darboux canonical form.

Proof. The proof of (a) is clear since Casimir invariants are the solution set of the system of PDEs given by $\mathcal{J} \cdot \nabla D=0$. Regarding (b), for the reduction of $\eta(x) \mathcal{J}(x)$ it suffices to perform a preliminary time reparametrization $\mathrm{d} \tau=\eta(x) \mathrm{d} t$, where as usual $t$ is the initial time variable, and $\tau$ is the new time. The outcome is thus a Poisson system with structure matrix $\mathcal{J}(x)$ and time variable $\tau$. The rest of the global reduction then follows the diffeomorphic steps known by hypothesis for $\mathcal{J}(x)$.
Q.E.D.

The previous proposition thus implies that the identification of a reparametrization factor for a family of structure matrices immediately generalizes such family, while the operational framework provided by the knowledge of the Casimir invariants and the global Darboux reduction for the initial solution family is preserved in the generalization. This is an additional justification of the interest of reparametrization factors.

In addition, the investigation of reparametrization factors is physically relevant as far as it aims at characterizing those Poisson structures that are not destroyed by (certain, at least) time reparametrizations. In this sense, it is worth recalling also the general mathematical interest of time reparametrizations, which lie at the basis of many important aspects of dynamical systems theory, specially in connection with integrability issues [60]. In this context, the role of time reparametrizations should be clear in the framework of this thesis, since their use for the determination of the Darboux canonical form has led frequently to integrability results throughout the dissertation.

With regard to the problem formulation, let us first recall that the Jacobi PDEs

$$
\begin{equation*}
\sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{k l} \partial_{l} J_{i j}+J_{j l} \partial_{l} J_{k i}\right)=0, \quad i, j, k=1, \ldots, n \tag{300}
\end{equation*}
$$

vanish identically if $i, j, k$ are not all different, as it can be easily verified (see Subsection 1.3.1). Thus, for convenience, in what follows we shall sometimes make use of equations (300) with $i, j, k=1, \ldots, n$ together with the additional conditions $i \neq j, i \neq k$ and $j \neq k$. As indicated at the beginning of the present subsection, assume that an $n$-dimensional structure matrix $\mathcal{J}(x)$ is defined in a domain $\Omega \subset \mathbb{R}^{n}$, together with a function $\eta(x)$ which is $C^{\infty}(\Omega)$ and does not vanish in $\Omega$. If we substitute the product $\eta(x) \mathcal{J}(x)$ in equations (300) we arrive at the conditions:
$\eta \sum_{l=1}^{n}\left(J_{i l} \partial_{l} J_{j k}+J_{k l} \partial_{l} J_{i j}+J_{j l} \partial_{l} J_{k i}\right)+\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=0, \quad i, j, k=1, \ldots, n$
Since $\mathcal{J}$ is by hypothesis a structure matrix, this leads to:

$$
\begin{equation*}
\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=0, \quad i, j, k=1, \ldots, n \tag{301}
\end{equation*}
$$

Consistently, we see that equations (301) vanish if two or three of the indexes $i, j$ and $k$ take the same value. Moreover, identities (301) also vanish identically if one of such indexes coincides with $l$, even in the case in which $i, j$ and $k$ are all different. Accordingly, equations (301) can be equivalently expressed as:

$$
\sum_{\substack{l=1  \tag{302}\\
l \neq i, j, k}}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=0, \quad\left\{\begin{array}{l}
i, j, k=1, \ldots, n \\
i \neq j ; \quad i \neq k ; \quad j \neq k
\end{array}\right.
$$

In the future developments, either form (301) or (302) will be preferred according to convenience. Notice that the outcome of the ansatz $\eta(x) \mathcal{J}(x)$ is a new problem in which now only one unknown function $\eta(x)$ exists. Moreover, equations (301) or (302) constitute a set of linear PDEs. Both features imply a significant simplification of the problem. Note also that $\eta(x)=c \neq 0$, with $c \in \mathbb{R}$ being an arbitrary constant, is always a solution. This trivial result will become a particular case of the first solution family of reparametrization factors to be determined in brief. As an additional comment, it is worth saying that the linearity of equations (301) or (302) does not imply that a traditional technique such as the use of the characteristics method should be useful in this case, mainly for two reasons: in first place, because we are dealing with a system of simultaneous equations, which is quite difficult to handle in terms of such method; and secondly, because equations (301) or (302) are very generic in form, and in particular we are not making any assumption on the functional form of the structure functions $J_{i j}(x)$, something often necessary in order to perform the integrations required by the aforementioned method. Before proceeding further, it is also important to cite an additional comment of interest: as it can be seen in equations (301) or (302), we are again
reducing the full complexity of the Jacobi equations to the study of a system of linear PDEs for a single function $\eta(x)$. This is of course a significant improvement, very similar to the one first introduced in Section 2.5 in the three-dimensional situation. As it was the case there, the procedure aims at determining new solutions from a given one which is used as starting point, thus allowing a significant simplification of the problem. It can thus be appreciated how such a philosophy can be employed again now, and actually it will lead to developments that allow the conceptual unification of many of the results previously seen.

In what follows we shall provide two wide solution families relative to problem (301) or (302). This is the purpose of the next two subsections.

### 4.7.2. First family of reparametrization factor solutions

The result corresponding to a first family of solutions of equations (301) is described in the following:

Theorem 4.7.2.1. Let $\mathcal{J}(x)$ be an $n \times n$ structure matrix of constant rank everywhere in a domain $\Omega \subset \mathbb{R}^{n}$, and let $D(x)$ be a Casimir invariant of $\mathcal{J}(x)$ globally defined in $\Omega$. Then $D(x) \mathcal{J}(x)$ is a structure matrix everywhere in $\Omega$.

Proof. Let us consider the problem equations in the form (301). Such identities can be written in the following form:

$$
\begin{equation*}
J_{j k} \sum_{l=1}^{n} J_{i l} \partial_{l} \eta+J_{i j} \sum_{l=1}^{n} J_{k l} \partial_{l} \eta+J_{k i} \sum_{l=1}^{n} J_{j l} \partial_{l} \eta=0, \quad i, j, k=1, \ldots, n \tag{303}
\end{equation*}
$$

Thus, equations (303) can be expressed as:

$$
\begin{equation*}
J_{j k}(\mathcal{J} \cdot \nabla \eta)_{i}+J_{i j}(\mathcal{J} \cdot \nabla \eta)_{k}+J_{k i}(\mathcal{J} \cdot \nabla \eta)_{j}=0, \quad i, j, k=1, \ldots, n \tag{304}
\end{equation*}
$$

Consequently, if $\eta(x)$ is a Casimir invariant, equations (304) are identically satisfied, as far as Casimir functions constitute the solution set of the system $\mathcal{J}(x) \cdot \nabla D(x)=0$.
Q.E.D.

The previous proof is somehow reminiscent of the kind of manipulations employed in the study of distinguished solutions. Theorem 4.7.2.1 has a direct consequence:

Corollary 4.7.2.2. Let $\mathcal{J}(x)$ be an $n \times n$ structure matrix of constant rank $r$ in a domain $\Omega \subset \mathbb{R}^{n}$, having $(n-r)$ functionally independent Casimir invariants globally defined in $\Omega$.

Then there are $(n-r)$ functionally independent reparametrization factors for $\mathcal{J}(x)$ globally defined in $\Omega$, and every nonvanishing $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ function of them is also a reparametrization factor for $\mathcal{J}(x)$ everywhere in $\Omega$.

Proof. It is sufficient to make use of the following two remarks: in first place, every $C^{\infty}$ function of one or more Casimir invariants is also a Casimir invariant; and secondly, as a consequence of the previous statement, a Casimir invariant $D(x)$ which is vanishing somewhere in a given domain $\Omega$ can be trivially replaced by a nonvanishing one functionally dependent on it (for instance $D^{2}(x)+1$ ).
Q.E.D.

The previous results also allow regarding as a particular case the fact (already mentioned) that constants are always solutions of equations (301), just as a consequence that constants are (trivial) Casimir invariants of every structure matrix. Constant reparametrization factors are thus always present, even in the symplectic case $(r=n)$. On the other hand, if the rank is lower than the dimension $(r<n)$ then the number of nonconstant reparametrization factors is infinity. We shall turn back to these issues in the next subsection.

There is an alternative perspective that shows the naturalness of the result in Theorem 4.7.2.1. For this, consider a Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$. If we rescale the Hamiltonian as $H^{*}(x)=D(x) H(x)$ with $D(x)$ being a Casimir invariant associated with the structure matrix $\mathcal{J}(x)$, then the new system remains as a Poisson one, namely $\dot{x}=\mathcal{J}(x) \cdot \nabla[D(x) H(x)]$. However, this implies that:

$$
\dot{x}=\mathcal{J}(x) \cdot \nabla[D(x) H(x)]=\mathcal{J}(x) \cdot[D(x) \nabla H(x)+H(x) \nabla D(x)]=D(x) \mathcal{J}(x) \cdot \nabla H(x)
$$

And therefore it is clear that such rescaling of the Hamiltonian (which is thus equivalent to a rescaling of the structure matrix) must preserve the existence of a Poisson structure.

The family of reparametrization factors just characterized corresponds to a sufficient (but not necessary) condition for the verification of equations (301). An obvious question is if additional solutions might exist. The answer is positive, as the next subsection describes.

### 4.7.3. Second family of reparametrization factor solutions

Let us turn back to the problem of searching reparametrization factors, this time making use of the equations in the form (302). Obviously, a sufficient condition (different from the
one previously considered in Theorem 4.7.2.1) for the verification of (302) is that:

$$
J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}=0, \quad\left\{\begin{array}{l}
i, j, k, l=1, \ldots, n  \tag{305}\\
i \neq j, k, l ; \quad j \neq k, l ; \quad k \neq l
\end{array}\right.
$$

The interest of the conditions (305) is that they are merely algebraic, which is a remarkable simplification of the initial PDE problem. If (305) is verified, then every $C^{\infty}$ and nonvanishing function $\eta(x)$ will be a valid reparametrization factor. The investigation of this possibility is the subject of the next theorem, which is the main result of this subsection:

Theorem 4.7.3.1. Let $\mathcal{J}(x)$ be an $n \times n$ structure matrix defined in a domain $\Omega \subset \mathbb{R}^{n}$ and of constant rank $r$ everywhere in $\Omega$. Then the product $\eta(x) \mathcal{J}(x)$ is a structure matrix in $\Omega$ for every $C^{\infty}(\Omega)$ function $\eta(x)$ if and only if $r \leq 2$.

Proof. Every implication will be shown separately.
In one sense, let us first prove that if $\operatorname{Rank}(\mathcal{J}) \leq 2$, then the product by every $C^{\infty}$ function $\eta$ preserves the property of being a structure matrix. For this, consider the following submatrix of $\mathcal{J}$, which is obtained after deleting all its rows and columns different from those at the positions $i, j, k$ and $l$ (with $i, j, k, l$ all different):

$$
\mathcal{J}^{[i j k l]}=\left(\begin{array}{cccc}
0 & J_{i j} & J_{i k} & J_{i l}  \tag{306}\\
-J_{i j} & 0 & J_{j k} & J_{j l} \\
-J_{i k} & -J_{j k} & 0 & J_{k l} \\
-J_{i l} & -J_{j l} & -J_{k l} & 0
\end{array}\right)
$$

If $\operatorname{Rank}(\mathcal{J}) \leq 2$, then it must be $\left|\mathcal{J}^{[i j k l]}\right|=0$ in (306) for all possible values of the four indexes $i, j, k, l$. But notice that in fact it is:

$$
\left|\mathcal{J}^{[i j k l]}\right|=\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right)^{2}
$$

Consequently, identities (305) are verified and the proof in this sense is already accomplished.
Conversely, let us prove that if the product by every $C^{\infty}$ function $\eta$ preserves the character of structure matrix, then $\operatorname{Rank}(\mathcal{J}) \leq 2$. For convenience, we shall equivalently prove that if $\operatorname{Rank}(\mathcal{J}) \geq 4$, then the product by every possible $C^{\infty}$ function $\eta$ does not always preserve the property of being a structure matrix. For this, we shall consider a given point $x_{0} \in \Omega$, and let $J_{i j}\left(x_{0}\right) \equiv a_{i j}$ for all $i, j=1, \ldots, n$. The first part of the following reasoning is very similar to the one employed for the construction of the normal form for skew-symmetric matrices [14].

In first place, let us assume without loss of generality that $a_{12} \neq 0$. If this is not the case, it is always possible to place another nonzero element in the position $(1,2)$ : let $a_{i j} \neq 0$, then we can permute the first and second rows with the $i$-th and $j$-th rows, respectively, and later the first and second columns of the resulting matrix can also be permuted with the $i$-th and $j$-th columns, respectively. Therefore, independently of the value of $a_{12}$ the outcome is the following skew-symmetric matrix

$$
S_{x_{0}}^{*}=\left(\begin{array}{ccccc}
0 & a_{\pi_{1} \pi_{2}} & a_{\pi_{1} \pi_{3}} & \ldots & a_{\pi_{1} \pi_{n}}  \tag{307}\\
a_{\pi_{2} \pi_{1}} & 0 & a_{\pi_{2} \pi_{3}} & \ldots & a_{\pi_{2} \pi_{n}} \\
\vdots & \vdots & \vdots & & \vdots \\
a_{\pi_{n} \pi_{1}} & a_{\pi_{n} \pi_{2}} & a_{\pi_{n} \pi_{3}} & \ldots & 0
\end{array}\right) \equiv\left(\begin{array}{cc|c}
0 & a_{\pi_{1} \pi_{2}} & E_{2 \times(n-2)} \\
-a_{\pi_{1} \pi_{2}} & 0 & \\
\hline E_{(n-2) \times 2} & E_{(n-2) \times(n-2)}
\end{array}\right)
$$

where $\left(\pi_{1}, \ldots, \pi_{n}\right)$ is a permutation of $(1, \ldots, n)$ : if $a_{12} \neq 0$, then such permutation is the identical one; and if $a_{12}=0$, the permutation is given by $\pi_{1}=i, \pi_{i}=1, \pi_{2}=j, \pi_{j}=2$, and $\pi_{k}=k$ for every $k$ different from $1,2, i$ and $j$. Thus matrix (307) is our starting point in either case, with $a_{\pi_{1} \pi_{2}} \neq 0$. In the right-hand side of (307), the letter $E$ denotes three submatrices of the sizes indicated by their respective subindexes. Since row and column elementary operations do not alter the rank of a matrix, we can make use of them in order to transform (307) into the skew-symmetric matrix:

$$
S_{x_{0}}^{* *}=\left(\begin{array}{cc|c}
0 & a_{\pi_{1} \pi_{2}} & \mathbb{O}_{2 \times(n-2)}  \tag{308}\\
-a_{\pi_{1} \pi_{2}} & 0 & \\
\hline \mathbb{O}_{(n-2) \times 2} & & \tilde{E}_{(n-2) \times(n-2)}
\end{array}\right)
$$

In submatrix $\tilde{E}_{(n-2) \times(n-2)}$ of (308) we now have the entries,

$$
\tilde{E}_{(n-2) \times(n-2)}=\left(\begin{array}{cccc}
0 & \tilde{a}_{\pi_{3} \pi_{4}} & \ldots & \tilde{a}_{\pi_{3} \pi_{n}} \\
\tilde{a}_{\pi_{4} \pi_{3}} & 0 & \ldots & \tilde{a}_{\pi_{4} \pi_{n}} \\
\vdots & \vdots & & \vdots \\
\tilde{a}_{\pi_{n} \pi_{3}} & \tilde{a}_{\pi_{n} \pi_{4}} & \ldots & 0
\end{array}\right)
$$

where it is:

$$
\begin{equation*}
\tilde{a}_{\pi_{k} \pi_{l}}=a_{\pi_{k} \pi_{l}}+\frac{1}{a_{\pi_{1} \pi_{2}}}\left(a_{\pi_{1} \pi_{l}} a_{\pi_{2} \pi_{k}}-a_{\pi_{1} \pi_{k}} a_{\pi_{2} \pi_{l}}\right), \quad k, l=3, \ldots, n \tag{309}
\end{equation*}
$$

At this stage, since $\operatorname{Rank}\left(S_{x_{0}}^{* *}\right) \geq 4$, there must be a nonzero element in $\tilde{E}_{(n-2) \times(n-2)}$ : if $\tilde{a}_{\pi_{3} \pi_{4}} \neq 0$, then we do not need to perform any changes for what is to follow. On the contrary, if $\tilde{a}_{\pi_{3} \pi_{4}}=0$ we can again permute rows and columns in such a way that the position $(3,4)$
is occupied by a nonzero entry $\tilde{a}_{\pi_{k} \pi_{l}}$ from $\tilde{E}_{(n-2) \times(n-2)}$ (with both $\pi_{k}$ and $\pi_{l}$ different from $\pi_{1}$ and $\pi_{2}$ ), the resulting matrix being also skew-symmetric. Consequently, we can assume without loss of generality that it is $\tilde{a}_{\pi_{3} \pi_{4}} \neq 0$. Then, from matrix $S_{x_{0}}^{* *}$ in (308) we can pick out the following submatrix composed by the intersection of the first four rows and columns:

$$
\tilde{S}_{x_{0}}^{[1234]}=\left(\begin{array}{cc|cc}
0 & a_{\pi_{1} \pi_{2}} & 0 & 0  \tag{310}\\
-a_{\pi_{1} \pi_{2}} & 0 & 0 & 0 \\
\hline 0 & 0 & 0 & \tilde{a}_{\pi_{3} \pi_{4}} \\
0 & 0 & -\tilde{a}_{\pi_{3} \pi_{4}} & 0
\end{array}\right)
$$

with both $a_{\pi_{1} \pi_{2}} \neq 0$ and $\tilde{a}_{\pi_{3} \pi_{4}} \neq 0$, as indicated. The determinant of $\tilde{S}_{x_{0}}^{[1234]}$ in (310) is $\left|\tilde{S}_{x_{0}}^{[1234]}\right|=\left(a_{\pi_{1} \pi_{2}} \tilde{a}_{\pi_{3} \pi_{4}}\right)^{2} \neq 0$. Now without loss of generality and for the sake of clarity, let us assume $\pi_{i}=i$ for all $i=1, \ldots, 4$. From (309) we thus have that:

$$
\begin{equation*}
\left(a_{12} \tilde{a}_{34}\right)^{2}=\left[a_{12}\left(a_{34}+\frac{1}{a_{12}}\left(a_{14} a_{23}-a_{13} a_{24}\right)\right)\right]^{2} \neq 0 \tag{311}
\end{equation*}
$$

Equation (311) immediately implies that:

$$
\begin{equation*}
J_{12}\left(x_{0}\right) J_{34}\left(x_{0}\right)+J_{14}\left(x_{0}\right) J_{23}\left(x_{0}\right)+J_{31}\left(x_{0}\right) J_{24}\left(x_{0}\right) \neq 0 \tag{312}
\end{equation*}
$$

Let us investigate the implications of (312) in equations (302). For this we may consider, for instance, the equation in (302) corresponding to the choice $i=1, j=2$ and $k=3$. Such equation takes the form:

$$
\begin{equation*}
\left(J_{14} J_{23}+J_{34} J_{12}+J_{24} J_{31}\right) \partial_{4} \eta+\sum_{l=5}^{n}\left(J_{1 l} J_{23}+J_{3 l} J_{12}+J_{2 l} J_{31}\right) \partial_{l} \eta=0 \tag{313}
\end{equation*}
$$

Now two cases must be distinguished, namely $n=4$ and $n \geq 5$ :

Case I: $n=4$. We proceed by means of two auxiliary lemmas:

Lemma 4.7.3.2. Let $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ be an $n \times n$ skew-symmetric matrix defined in a domain $\Omega \subset \mathbb{R}^{n}$. Then, for every $x \in \Omega$ the quantities

$$
\Xi_{i j k l}(x) \equiv J_{i l}(x) J_{j k}(x)+J_{k l}(x) J_{i j}(x)+J_{j l}(x) J_{k i}(x), \quad i, j, k, l=1, \ldots, n
$$

are completely skew-symmetric in all the subindexes $(i, j, k, l)$.

Proof of Lemma 4.7.3.2. The result can be verified by direct evaluation of the index skew-symmetry properties.
Q.E.D.

The second lemma required now is:

Lemma 4.7.3.3. Let $\mathcal{J}(x)$ be a $4 \times 4$ structure matrix defined in a domain $\Omega \subset \mathbb{R}^{4}$ and such that $\operatorname{Rank}(\mathcal{J})=4$ everywhere in $\Omega$. Then the only possible reparametrization factors allowed for $\mathcal{J}(x)$ in $\Omega$ are the constant ones.

Proof of Lemma 4.7.3.3. Now equations (302) amount to:

$$
\begin{equation*}
\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=0 \tag{314}
\end{equation*}
$$

where in (314) the indexes $(i, j, k, l)$ may be every possible permutation of $(1,2,3,4)$. Due to the skew-symmetry property shown in Lemma 4.7.3.2, the number of independent equations in (314) is actually four:

$$
\left\{\begin{align*}
(i, j, k, l)=(1,2,3,4) & \Rightarrow\left(J_{14} J_{23}+J_{34} J_{12}+J_{24} J_{31}\right) \partial_{4} \eta=0  \tag{315}\\
(i, j, k, l)=(1,2,4,3) & \Rightarrow\left(J_{13} J_{24}+J_{43} J_{12}+J_{23} J_{41}\right) \partial_{3} \eta=0 \\
(i, j, k, l)=(1,3,4,2) & \Rightarrow\left(J_{12} J_{34}+J_{42} J_{13}+J_{32} J_{41}\right) \partial_{2} \eta=0 \\
(i, j, k, l)=(2,3,4,1) & \Rightarrow\left(J_{21} J_{34}+J_{41} J_{23}+J_{31} J_{42}\right) \partial_{1} \eta=0
\end{align*}\right.
$$

In addition, if $\mathcal{J}$ is a regular $4 \times 4$ skew-symmetric matrix, its determinant is:

$$
\begin{equation*}
|\mathcal{J}|=\left(J_{12} J_{34}+J_{31} J_{24}+J_{14} J_{23}\right)^{2} \neq 0 \tag{316}
\end{equation*}
$$

Hypothesis (316) implies that equations (315) are actually simplified to $\partial_{l} \eta=0$ for all $l=1, \ldots, 4$, namely $\eta$ is a constant. Lemma 4.7.3.3 is thus proven.
Q.E.D.

Therefore $\eta(x)$ cannot be an arbitrary function when $n=4$ and Case I is proved. Let us now turn to the second possibility considered:

Case II: $n \geq 5$. Notice now that equation (313) is valid, in particular, at $x_{0} \in \Omega$. Assume, for instance, that a function $\eta(x)$ is chosen in such a way that $\partial_{4} \eta \neq 0$ at $x_{0}$. Then, equation (312) implies that it is not possible at the same time to make the choice $\partial_{l} \eta=0$ at $x_{0}$ for all $l \geq 5$. Consequently, function $\eta(x)$ cannot be arbitrary in the complementary case $n \geq 5$. This proves Case II.

The results provided in the framework of this second family of reparametrization factors now investigated, can be complemented by means of an additional result, which actually generalizes Lemma 4.7.3.3:

Theorem 4.7.3.4. Let $\mathcal{J}(x)$ be an $n \times n$ structure matrix ( $n \geq 4$ ) defined in a domain $\Omega \subset \mathbb{R}^{n}$ and such that $\operatorname{Rank}(\mathcal{J})=n$ everywhere in $\Omega$. Then the only possible reparametrization factors allowed for $\mathcal{J}(x)$ in $\Omega$ are the constant ones.

Proof. We begin with an auxiliary result:

Lemma 4.7.3.5. Consider the matrix $\mathcal{J}_{D[n, n]}$, with $n \geq 4$ an even integer. Let $\Omega \subset \mathbb{R}^{n}$ be a domain. Then the only possible reparametrization factors allowed for $\mathcal{J}_{D[n, n]}$ in $\Omega$ are the constant ones.

Proof of Lemma 4.7.3.5. For the sake of clarity, recall that it is:

$$
\mathcal{J}_{D[n, n]}=\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right) \overbrace{\oplus \ldots \oplus}^{n / 2}\left(\begin{array}{cc}
0 & 1 \\
-1 & 0
\end{array}\right)
$$

Let us consider four different cases for the entries of $\mathcal{J}_{D[n, n]}$ :
Case I. Let $i$ be odd, with $1 \leq i \leq(n-3)$. Now we choose indexes $(i, j, k)=(i, i+1, i+2)$.
Then from equations (301) we obtain:

$$
\begin{equation*}
\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=\sum_{l=1}^{n} J_{i+2, l} \partial_{l} \eta=J_{i+2, i+3} \partial_{i+3} \eta=\partial_{i+3} \eta=0 \tag{317}
\end{equation*}
$$

and consequently (317) implies $\partial_{l} \eta=0$ for $l=i+3=4,6, \ldots n$.
Case II. Now let $(i, j, k)=(1,3,4)$. Again from (301) we are led to:

$$
\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=\sum_{l=1}^{n} J_{1 l} \partial_{l} \eta=J_{12} \partial_{2} \eta=\partial_{2} \eta=0
$$

Case III. This time we choose even values of $i$, with $2 \leq i \leq(n-2)$. Then, with indexes $(i, j, k)=(i, i+1, i+2)$ from equations (301) we now have:

$$
\begin{equation*}
\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=\sum_{l=1}^{n} J_{i l} \partial_{l} \eta=J_{i, i-1} \partial_{i-1} \eta=-\partial_{i-1} \eta=0 \tag{318}
\end{equation*}
$$

and thus (318) leads to $\partial_{l} \eta=0$ for $l=i-1=1,3, \ldots,(n-3)$.

Case IV. Finally, let $(i, j, k)=(n-3, n-2, n)$. Therefore (301) implies:

$$
\sum_{l=1}^{n}\left(J_{i l} J_{j k}+J_{k l} J_{i j}+J_{j l} J_{k i}\right) \partial_{l} \eta=\sum_{l=1}^{n} J_{n l} \partial_{l} \eta=J_{n, n-1} \partial_{n-1} \eta=-\partial_{n-1} \eta=0
$$

Together, Cases I-IV provide the result stated in Lemma 4.7.3.5.
Q.E.D.

Let us now continue the main proof. For this, it is worth noticing that after a general smooth change of variables $y \equiv y(x)$ transforming a structure matrix $\mathcal{J}(x)$ into a new one $\mathcal{J}^{*}(y)$, every reparametrization factor $\eta(x)$ is converted into $\eta^{*}(y)=\eta(x(y))$. To see this, it suffices to recall the general transformation rule for structure matrices subjected to smooth coordinate changes $y \equiv y(x)$ :

$$
\begin{equation*}
J_{i j}^{*}(y)=\sum_{k, l=1}^{n} \frac{\partial y_{i}}{\partial x_{k}} J_{k l}(x) \frac{\partial y_{j}}{\partial x_{l}}, \quad i, j=1, \ldots, n \tag{319}
\end{equation*}
$$

Clearly, according to (319) the transformation of $\eta(x) \mathcal{J}(x)$ leads to $\eta^{*}(y) \mathcal{J}^{*}(y)$, with $\eta^{*}(y)=$ $\eta(x(y))$, as indicated. Now let $x_{0} \in \Omega$ be a point, and consider the value of the matrix at that point, namely $\mathcal{J}\left(x_{0}\right)$. As it was already mentioned, it is well-known [14] that there exists a regular matrix $E_{x_{0}}$ such that $E_{x_{0}} \cdot \mathcal{J}\left(x_{0}\right) \cdot E_{x_{0}}^{T}=\mathcal{J}_{D[n, n]}$. On the basis of this relationship, we perform on $\mathcal{J}(x)$ the change of variables, diffeomorphic in $\mathbb{R}^{n}$, given by $y=E_{x_{0}} \cdot x$. According to (319), the outcome is evidently $\mathcal{J}^{*}(y)=E_{x_{0}} \cdot \mathcal{J}(x(y)) \cdot E_{x_{0}}^{T}$. Let $y_{0} \equiv E_{x_{0}} \cdot x_{0}$. Thus, in particular we have that $\mathcal{J}^{*}\left(y_{0}\right)=\mathcal{J}_{D[n, n]}$. Consider then equation (301) for the reparametrization factor in the new variables $y$ :

$$
\begin{equation*}
\sum_{l=1}^{n}\left[J_{i l}^{*}(y) J_{j k}^{*}(y)+J_{k l}^{*}(y) J_{i j}^{*}(y)+J_{j l}^{*}(y) J_{k i}^{*}(y)\right] \partial_{y_{l}} \eta^{*}(y)=0, \quad i, j, k=1, \ldots, n \tag{320}
\end{equation*}
$$

Since equations (320) are valid everywhere in $\Omega^{*}=y(\Omega)$, they are valid in particular in $y_{0} \in \Omega^{*}$, namely:

$$
\begin{equation*}
\sum_{l=1}^{n}\left[J_{i l}^{*}\left(y_{0}\right) J_{j k}^{*}\left(y_{0}\right)+J_{k l}^{*}\left(y_{0}\right) J_{i j}^{*}\left(y_{0}\right)+J_{j l}^{*}\left(y_{0}\right) J_{k i}^{*}\left(y_{0}\right)\right]\left(\left.\partial_{y_{l}} \eta^{*}(y)\right|_{y_{0}}\right)=0, \quad i, j, k=1, \ldots, n \tag{321}
\end{equation*}
$$

Given that $\mathcal{J}^{*}\left(y_{0}\right)=\mathcal{J}_{D[n, n]}$, as indicated, the analysis provided in Lemma 4.7.3.5 is immediately applicable to equations (321). Consequently we find that:

$$
\begin{equation*}
\left.\frac{\partial \eta^{*}(y)}{\partial y_{i}}\right|_{y_{0}}=0, \quad i=1, \ldots, n \tag{322}
\end{equation*}
$$

Taking into account that it is $\eta^{*}(y)=\eta(x(y))$, or equivalently that $\eta(x)=\eta^{*}(y(x))$, an application of the chain rule combined with (322) shows that:

$$
\begin{equation*}
\left.\frac{\partial \eta(x)}{\partial x_{i}}\right|_{x_{0}}=\left.\frac{\partial \eta^{*}(y(x))}{\partial x_{i}}\right|_{x_{0}}=\sum_{j=1}^{n}\left(\left.\frac{\partial \eta^{*}(y)}{\partial y_{j}}\right|_{y_{0}}\right)\left(\left.\frac{\partial y_{j}}{\partial x_{i}}\right|_{x_{0}}\right)=0, \quad i=1, \ldots, n \tag{323}
\end{equation*}
$$

Since the analysis leading to (323) can be carried out for every point $x_{0} \in \Omega$, we conclude that actually it is $\partial_{x_{i}} \eta(x)=0$ everywhere in $\Omega$ for all $i=1, \ldots, n$, namely $\eta(x)$ is in fact a constant. The proof of Theorem 4.7.3.4 is complete.
Q.E.D.

Of course, in Theorem 4.7.3.4 the maximal rank condition $\operatorname{Rank}(\mathcal{J})=n$ implies that we are dealing with even values of the dimension $n$. In spite of being a somehow exclusive result, such theorem complements the previous contributions for the characterization of reparametrization factors.

In the rest of the section the goal will be to provide different applications and illustrations of the results just considered. Such applications will be relevant to the understanding of important aspects of what has been developed along the entire work.

### 4.7.4. Miscellaneous examples

It is interesting to make use of this subsection in order to briefly recall the Poisson structures and systems for which the reduction to the Darboux canonical form was considered in Chapters 2 and 3. Now such Darboux reductions can be regarded with a wider perspective, and consequently all the systems analyzed in those chapters become valid instances of the results developed in the present section.

Specifically, Chapter 2 was devoted to the establishment of global reductions for a variety of 3-d Poisson systems. Obviously, a 3-d Poisson structure which is of constant rank in its domain of definition must be of rank 2 provided it is nontrivial. This was of course the case in Chapter 2 for the threefold family (Section 2.2 ), as well as for the cyclic families of types I and II (Sections 2.3 and 2.4, respectively). In all cases, NTTs of arbitrary form were employed in the different global reductions to the Darboux canonical form. That this could be done is understandable now, since we know at this stage that every time reparametrization preserves the existence of the Poisson structure if it has constant rank of value 2 .

In addition, the subject in Chapter 3 was a 4-d family, for which a case classification was developed in order to completely characterize the reduction to the Darboux canonical form in all possible situations. Without exception, use of arbitrary NTTs was again necessary. In connection with this, it was shown in the proof of Theorem 3.2.1.1 that in such family the Poisson structure is not lost after the multiplication by an arbitrary smooth and nonvanishing function, which is of course equivalent to the application of an arbitrary time reparametrization. Actually, the demonstration of this issue was a significant part of the proof of such theorem.

However, now all these features appear as natural since the rank of such family of structure matrices is also 2. With this knowledge in mind, the proof of Theorem 3.2.1.1 becomes simpler and we are able to recognize in advance that free use of time reparametrizations can be made for the reduction to the Darboux canonical form, at every stage of the reduction procedure.

### 4.7.5. Application to the generalization of separable solutions

The previous results of this section allow a direct generalization of separable structure matrices, as it is summarized in the next:

Corollary 4.7.5.1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain in which an $n \times n$ separable structure matrix $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ is defined as $J_{i j}(x)=s_{i j} \psi_{i}\left(x_{i}\right) \psi_{j}\left(x_{j}\right)$ in terms of the skew-symmetric constant matrix $S \equiv\left(s_{i j}\right)$ and the functions $\psi_{i}\left(x_{i}\right)$, for all $i, j=1, \ldots, n$. In addition, let $\operatorname{Rank}(S)=r$, and let $\left(k^{[r+1]}, \ldots, k^{[n]}\right)$ be a basis of $\operatorname{Ker}(S)$, such that

$$
D_{i}(x)=\sum_{j=1}^{n} k_{j}^{[i]} \int \frac{\mathrm{d} x_{j}}{\psi_{j}\left(x_{j}\right)}, \quad i=r+1, \ldots, n
$$

is a complete set of independent Casimir invariants of $\mathcal{J}(x)$ in $\Omega$. Then:
(a) If $\operatorname{Rank}(S)=2$, then $\tilde{\mathcal{J}}(x)=\eta(x) \mathcal{J}(x)$ is also a structure matrix in $\Omega$ for every function $\eta(x)$ which is $C^{\infty}(\Omega)$ and does not vanish in $\Omega$.
(b) If $n>\operatorname{Rank}(S) \geq 4$, then $\tilde{\mathcal{J}}(x)=\eta\left(D_{r+1}(x), \ldots, D_{n}(x)\right) \mathcal{J}(x)$ is also a structure matrix in $\Omega$ for every function $\eta\left(y_{1}, \ldots, y_{n-r}\right)$ which is $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ and does not vanish in $\mathbb{R}^{n-r}$.
(c) If $\operatorname{Rank}(S)=n \geq 4$, then $\tilde{\mathcal{J}}(x)=\eta(x) \mathcal{J}(x)$ is also a nontrivial structure matrix in $\Omega$ if and only if $\eta(x)$ is constant in $\Omega$, with $\eta \neq 0$.
(d) In Cases (a)-(c) the Darboux canonical form for $\tilde{\mathcal{J}}(x)$ can be constructed globally and diffeomorphically in $\Omega$ by means of the procedure indicated in Proposition 4.7.1.2.
(e) Consider in addition that the assumptions of Remark 1.4.4.4 hold. Every Poisson system $\dot{x}=\tilde{\mathcal{J}}(x) \cdot \nabla H(x)$ defined in $\Omega$ for which the structure matrix $\tilde{\mathcal{J}}(x)$ is of the kind specified in Case (a) is an algebraically integrable system in $\Omega$, and can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

We then see that the generalization is established in terms of a very simple property, such as the rank of the constant matrix $S$. Notice also that in Case (c) we do not obtain a generalization of separable structure matrices, contrarily to what is achieved in Cases (a) and (b). The reason accounting for such remark regarding Case (c) is that the product by a constant reparametrization factor amounts to a rescaling of matrix $S$, namely $\tilde{S}=\eta S$, thus preserving the separable nature of the structure matrix. On the contrary, the structure matrices obtained after Cases (a) and (b) are clearly not separable in general. At the same time, in all situations considered the reduction to the Darboux canonical form can still be carried out on the basis of the same algorithm employed for separable matrices (see Subsection 4.2.2), with the only addition of a previous time reparametrization, in agreement with the general procedure presented in Proposition 4.7.1.2. Accordingly, the separable family is generalized while maintaining the scope of the global results developed, both for the family properties (such as the Casimir invariants) and for the global reduction to the Darboux canonical form. Let us mention a simple example of how the generalization can be identified in practice.

## Example. Poisson structure for the Kermack-McKendrick model

Let us consider again the structure matrix $[63,135]$ arising in the study of the KermackMcKendrick model for epidemics:

$$
\mathcal{J}(x)=b x_{1} x_{2}\left(\begin{array}{ccc}
0 & 1 & -1  \tag{324}\\
-1 & 0 & 1 \\
1 & -1 & 0
\end{array}\right) \equiv\left(b x_{1} x_{2}\right) \hat{\mathcal{J}}(x)
$$

where $b$ is a positive real constant. Clearly, matrix (324) is not separable, but of course $\hat{\mathcal{J}}$ is. In addition, matrix $\hat{\mathcal{J}}$ is embraced by the generalization provided by Corollary 4.7.5.1, specifically in terms of Case (a): since it is $\operatorname{Rank}(\hat{\mathcal{J}})=2$, the reparametrization factor $\eta(x)=b x_{1} x_{2}$ is admissible, thus producing the structure matrix (324).

Many other separable Poisson brackets well-known in the literature can be generalized after Corollary 4.7.5.1. For instance, a sample can be found among the separable examples presented in Subsection 4.2.3. More precisely, we can mention the following instances:
(a) Separable Poisson structures with $n=r=2$ : we can include in this category different formulations of the LV system [29,136,143] as well as game-theoretic models [88].
(b) Separable Poisson structures with $n=3$ and $r=2$ : we also have LV models $[63,136]$, as well as the Poisson structures for the circle maps system [63] and the KermackMcKendrick model [63,135].
(c) Separable Poisson structures with arbitrary $n$ :

- With arbitrary $r$ : we can mention Poisson systems of the LV [143,146] and the QP [ 83,86$]$ forms.
- With $r=(n-1)$ : in the literature we can find some examples of the LV form in the domain of plasma physics [142], and specially among the very diverse Poisson structures associated with Toda lattices [4,36,37,38,39,40].

Needless to say, this enumeration is by no means exhaustive. Its only purpose is illustrative, the goal being to display the fact that the number of already known separable Poisson structures generalized after Corollary 4.7.5.1 is quite significant.

### 4.7.6. Application to the generalization of multiseparable solutions

As it was the case with the separable family, it is relevant at this stage to briefly consider the application of the results on time reparametrizations to the generalization of multiseparable structure matrices. As in the previous subsection, such results are summarized as follows:

Corollary 4.7.6.1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain in which an $n \times n$ multiseparable structure matrix $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ is defined according to Definition 4.3.1.1, namely as

$$
J_{i j}(x)=\sum_{k=1}^{r / 2} \Lambda_{i j}^{2 k-1,2 k} \psi_{2 k-1}(x) \psi_{2 k}(x), \quad i, j=1, \ldots, n
$$

in terms of the even number $r$ (with $n \geq r \geq 2$ ), the constant $n \times n$ matrices $A \equiv\left(a_{i j}\right)$ and $B=A^{-1} \equiv\left(b_{i j}\right)$, and the functions $\left(\psi_{1}(x), \ldots, \psi_{r}(x)\right)$. In addition, consider the functions

$$
D_{i}(x)=B_{i} \cdot x=\sum_{j=1}^{n} b_{i j} x_{j}, \quad i=r+1, \ldots, n
$$

which constitute a complete set of independent Casimir invariants of $\mathcal{J}(x)$ in $\Omega$. Then:
(a) If $r=2$, then $\tilde{\mathcal{J}}(x)=\eta(x) \mathcal{J}(x)$ is also a structure matrix in $\Omega$ for every function $\eta(x)$ which is $C^{\infty}(\Omega)$ and does not vanish in $\Omega$.
(b) If $n>r \geq 4$, then $\tilde{\mathcal{J}}(x)=\eta\left(D_{r+1}(x), \ldots, D_{n}(x)\right) \mathcal{J}(x)$ is also a structure matrix in $\Omega$ for every function $\eta\left(y_{1}, \ldots, y_{n-r}\right)$ which is $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ and does not vanish in $\mathbb{R}^{n-r}$.
(c) If $r=n \geq 4$, then $\tilde{\mathcal{J}}(x)=\eta(x) \mathcal{J}(x)$ is also a nontrivial structure matrix in $\Omega$ if and only if $\eta(x)$ is constant in $\Omega$, with $\eta \neq 0$.
(d) In Cases (a)-(c) the Darboux canonical form for $\tilde{\mathcal{J}}(x)$ can be constructed globally and diffeomorphically in $\Omega$ by means of the procedure indicated in Proposition 4.7.1.2.
(e) Consider in addition that the assumptions of Remark 1.4.4.4 hold. Every Poisson system $\dot{x}=\tilde{\mathcal{J}}(x) \cdot \nabla H(x)$ defined in $\Omega$ for which the structure matrix $\tilde{\mathcal{J}}(x)$ is of the kind specified in Case (a) is an algebraically integrable system in $\Omega$, and can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

We see that, again, the kind of generalization achieved becomes very natural, as far as it is entirely controlled by a simple solution feature such as the even number $r$. In addition, in Cases (a) and (b) the multiseparable family is clearly generalized, in the sense that the outcome is, in general, not multiseparable. In Case (c) such a generalization is not present, since a constant reparametrization factor can be used in order to redefine, for instance, the even (or, alternatively, the odd) functions $\psi_{k}$ as $\tilde{\psi}_{k}=\eta \psi_{k}$, thus remaining in the framework of multiseparable matrices. And, as usual, the results regarding the existence and form of the Casimir invariants and the global reduction to the Darboux canonical form remain valid after the generalization and can still be applied, with minor modifications, as in the original (non-generalized) family. Let us illustrate the previous ideas by means of the same instance just employed in the separable case.

## Example. Poisson structure for the Kermack-McKendrick model revisited

Let us consider again the structure matrix (324) for the Kermack-McKendrick equations. Such structure matrix is actually multiseparable, as it was shown thoroughly in Example 3 of Subsection 4.3.3 (see it for the details, which are not recalled here for the sake of conciseness). Since it is $\operatorname{Rank}(\mathcal{J})=2$, from Case (a) of Corollary 4.7.6.1 it is immediate that such multiseparable structure matrix can actually be generalized to:

$$
\tilde{\mathcal{J}}(x)=\eta\left(x_{1}, x_{2}, x_{3}\right)\left(\begin{array}{ccc}
0 & 1 & -1  \tag{325}\\
-1 & 0 & 1 \\
1 & -1 & 0
\end{array}\right)
$$

where $\eta\left(x_{1}, x_{2}, x_{3}\right)$ is a $C^{\infty}$ and nonvanishing function. This ensures that the reduction to the Darboux canonical form provided in Subsection 4.3.3 can be immediately extended to (325) by means of Proposition 4.7.1.2, as indicated in item (d) of Corollary 4.7.6.1.

### 4.7.7. Application to the generalization of distinguished solutions

To complete the family generalizations reported in the previous two subsections, it is also interesting to provide a brief description of the consequences that the results on time reparametrizations have, when applied to D-solutions.

Corollary 4.7.7.1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain in which an $n \times n$ distinguished structure matrix $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$ is defined and has constant rank $r$ everywhere. Let $\left(D_{r+1}(x), \ldots, D_{n}(x)\right)$ be a complete set of independent Casimir invariants of $\mathcal{J}(x)$ in $\Omega$. Then:
(a) If $r=2$, then $\tilde{\mathcal{J}}(x)=\eta(x) \mathcal{J}(x)$ is also a structure matrix in $\Omega$ for every function $\eta(x)$ which is $C^{\infty}(\Omega)$ and does not vanish in $\Omega$.
(b) If $n>r \geq 4$, then $\tilde{\mathcal{J}}(x)=\eta\left(D_{r+1}(x), \ldots, D_{n}(x)\right) \mathcal{J}(x)$ is also a structure matrix in $\Omega$ for every function $\eta\left(y_{1}, \ldots, y_{n-r}\right)$ which is $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ and does not vanish in $\mathbb{R}^{n-r}$.
(c) If $r=n \geq 4$, then $\tilde{\mathcal{J}}(x)=\eta(x) \mathcal{J}(x)$ is also a nontrivial structure matrix in $\Omega$ if and only if $\eta(x)$ is constant in $\Omega$, with $\eta \neq 0$.
(d) Consider in addition that the assumptions of Remark 1.4.4.4 hold. Every Poisson system $\dot{x}=\tilde{\mathcal{J}}(x) \cdot \nabla H(x)$ defined in $\Omega$ for which the structure matrix $\tilde{\mathcal{J}}(x)$ is of the kind specified in Case (a) is an algebraically integrable system in $\Omega$.

This corollary has nice implications when compared to the already known properties of the distinguished solutions. In spite of not being primarily concerned with the construction of the Darboux canonical form in the case of D-solutions, notice that the constancy of the rank is necessary in order to preserve unaltered the complete set of Casimir invariants, precisely because these are the entries of the D-solution structure matrix $\mathcal{J}(x)$.

In Case (a) of the previous corollary, it is worth a comparison with Case (d) of Theorem 4.4.1.4. (Recall that, according to the latter, if $\eta\left(y_{1}, \ldots, y_{n-r}\right): \mathbb{R}^{n-r} \rightarrow \mathbb{R}$ is an arbitrary $C^{\infty}\left(\mathbb{R}^{n-r}\right)$ real function, then the product $\eta\left(D_{r+1}(x), \ldots, D_{n}(x)\right) \mathcal{J}(x)$ is a D-solution in $\left.\Omega\right)$. Thus, Case (a) of Corollary 4.7.7.1 provides a direct generalization for D-solutions of rank two,
since now no limits are imposed to the functional form of $\eta(x)$. On the other hand, property (d) of Theorem 4.4.1.4 is interesting as far as it is family-preserving, namely the outcome is always a D-solution, something not valid in general in the present generalization.

Regarding Case (b) of Corollary 4.7.7.1, we now have that it coincides with statement (d) of Theorem 4.4.1.4. Thus now no generalization is provided by the corollary. However, it is interesting to recall that after the study of time reparametrizations, we know that such kind of property is actually valid for every structure matrix, and thus item (d) of Theorem 4.4.1.4 can now be seen as a particular case. In other words, property (d) of Theorem 4.4.1.4 can be regarded as an illustration for D-solutions of Theorem 4.7.2.1 characterizing the first family of reparametrization factor solutions.

Let us finally turn to Case (c) of Corollary 4.7.7.1. According to Example 1 in Subsection 4.4.3, we know that if $r=n \geq 4$ and $\mathcal{J}$ is a D-solution, then $\mathcal{J}$ must be a constant skewsymmetric matrix. When multiplying such a matrix by a constant reparametrization factor, the outcome is again a constant skew-symmetric real matrix, thus providing no generalization from the point of view of the functional form of the structure functions.

### 4.7.8. Global Darboux reduction for Poisson structures of rank two

An interesting consequence of the study of time reparametrizations for Poisson systems (and in particular of the second family of reparametrization factors, characterized in Subsection 4.7.3) is the possibility of constructing the global Darboux reduction for structure matrices of rank two and arbitrary dimension, something possible under very general assumptions, as we shall see now. This ameliorates significantly the scope of Darboux' theorem for such systems. The result is given in the next theorem:

Theorem 4.7.8.1. Let $\Omega \subset \mathbb{R}^{n}$ be a domain ( $n \geq 2$ ) where is defined a Poisson system

$$
\frac{\mathrm{d} x}{\mathrm{~d} t}=\mathcal{J}(x) \cdot \nabla H(x)
$$

having an $n \times n$ structure matrix $\mathcal{J}(x) \equiv\left(J_{i j}(x)\right)$, and such that $\operatorname{Rank}(\mathcal{J})=2$ everywhere in $\Omega$. Let $\left(D_{3}(x), \ldots, D_{n}(x)\right)$ be a complete set of independent Casimir invariants of $\mathcal{J}(x)$ in $\Omega$. In addition, let $\left(d_{1}(x), d_{2}(x)\right)$ be two arbitrary $C^{\infty}(\Omega)$ functions such that the transformation

$$
\begin{cases}y_{i}=d_{i}(x) \quad, \quad i=1,2  \tag{326}\\ y_{j}=D_{j}(x), & j=3, \ldots, n\end{cases}
$$

is one-to-one everywhere in $\Omega$ and its Jacobian matrix $M$ verifies:

$$
\begin{equation*}
|M|=\left|\frac{\partial\left(d_{1}(x), d_{2}(x), D_{3}(x), \ldots, D_{n}(x)\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}\right| \neq 0, \quad \text { for all } x \in \Omega \tag{327}
\end{equation*}
$$

Then such Poisson system can be reduced globally in $\Omega$ to an one degree of freedom Hamiltonian system and the Darboux canonical form is accomplished globally and diffeomorphically in $\Omega$ in the new coordinate system $\left(y_{1}, \ldots, y_{n}\right)$ and the new time $\tau$, where $\left(y_{1}, \ldots, y_{n}\right)$ are given by transformation (326) which is a diffeomorphism globally defined in $\Omega$; while the new time $\tau$ is defined by the time reparametrization:

$$
\begin{equation*}
\mathrm{d} \tau=\left.\left(\left\{d_{1}(x), d_{2}(x)\right\}_{\mathcal{J}}\right)\right|_{x(y)} \mathrm{d} t=\left.\left[\left(\nabla_{x} d_{1}(x)\right)^{T} \cdot \mathcal{J}(x) \cdot\left(\nabla_{x} d_{2}(x)\right)\right]\right|_{x(y)} \mathrm{d} t \equiv \eta(y) \mathrm{d} t \tag{328}
\end{equation*}
$$

Proof. The constancy of $\operatorname{Rank}(\mathcal{J})$ implies that Darboux' theorem is applicable. In column matrix notation for the gradients, the Jacobian matrix of (326) can be written as:

$$
M \equiv \frac{\partial\left(d_{1}(x), d_{2}(x), D_{3}(x), \ldots, D_{n}(x)\right)}{\partial\left(x_{1}, \ldots, x_{n}\right)}=\left(\begin{array}{llll}
\nabla_{x} d_{1}(x) & \nabla_{x} d_{2}(x) & \nabla_{x} D_{3}(x) & \ldots \tag{329}
\end{array} \nabla_{x} D_{n}(x)\right)^{T}
$$

Note in particular that functions $d_{1}(x)$ and $d_{2}(x)$ cannot be Casimir invariants because they are functionally independent of a complete set of independent Casimir functions. The effect of (326) is to transform $\mathcal{J}(x)$ into a new structure matrix $\mathcal{J}^{*}(y)=M \cdot \mathcal{J} \cdot M^{T}$ according to the rule (30). With the help of (329) we find, still in column matrix notation:

$$
\mathcal{J}^{*}=M \cdot\left(\left[\mathcal{J} \cdot \nabla_{x} d_{1}(x)\right] \quad\left[\begin{array}{llll}
\left.\mathcal{J} \cdot \nabla_{x} d_{2}(x)\right] & \mathbb{O}_{n \times 1} & \ldots & \mathbb{O}_{n \times 1} \tag{330}
\end{array}\right)\right.
$$

Using in (330) the fact that for any pair of matrices $A$ and $B$ that can be multiplied, we can write $A \cdot B=\left(B^{T} \cdot A^{T}\right)^{T}$, we immediately find that (330) becomes:

$$
\mathcal{J}^{*}=\left(\begin{array}{cc|ccc}
-\left(\nabla_{x} d_{1}\right)^{T} \cdot \mathcal{J} \cdot \nabla_{x} d_{1} & -\left(\nabla_{x} d_{2}\right)^{T} \cdot \mathcal{J} \cdot \nabla_{x} d_{1} & 0 & \ldots & 0  \tag{331}\\
-\left(\nabla_{x} d_{1}\right)^{T} \cdot \mathcal{J} \cdot \nabla_{x} d_{2} & -\left(\nabla_{x} d_{2}\right)^{T} \cdot \mathcal{J} \cdot \nabla_{x} d_{2} & 0 & \ldots & 0 \\
\hline 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & & \vdots \\
0 & 0 & 0 & \ldots & 0
\end{array}\right)
$$

But now recall that, as it was indicated in Chapter 1, for any two functions $f(x)$ and $g(x)$ it is, in agreement with (18) and (19):

$$
\begin{equation*}
\{f(x), g(x)\}_{\mathcal{J}(x)}=\left(\nabla_{x} f\right)^{T} \cdot \mathcal{J} \cdot \nabla_{x} g=\sum_{i, j=1}^{n}\left(\partial_{x_{i}} f\right) J_{i j}\left(\partial_{x_{j}} g\right) \tag{332}
\end{equation*}
$$

According to (331) and (332) we arrive at:

$$
J_{11}^{*}=\left\{d_{1}(x), d_{1}(x)\right\}_{\mathcal{J}}=0, \quad J_{22}^{*}=\left\{d_{2}(x), d_{2}(x)\right\}_{\mathcal{J}}=0
$$

and $J_{12}^{*}=\left\{d_{1}(x), d_{2}(x)\right\}_{\mathcal{J}}=-J_{21}^{*}$. Therefore we obtain by construction that matrix $\mathcal{J}^{*}$ is skew-symmetric, consistently with what is to be expected. Moreover, since $\operatorname{Rank}(\mathcal{J})=2$ and $\operatorname{Rank}(M)=n$ everywhere in $\Omega$ by hypothesis, matrix $\mathcal{J}^{*}$ is congruent on $\mathbb{R}$ with $\mathcal{J}$ and then it is also $\operatorname{Rank}\left(\mathcal{J}^{*}\right)=2$ everywhere in $\Omega^{*}=y(\Omega)$. Accordingly it is $J_{12}^{*}(y) \neq 0$ everywhere in $\Omega^{*}$. This implies that in order to fulfill the Darboux reduction we only need to perform the time reparametrization $\mathrm{d} \tau=\eta(y) \mathrm{d} t$ as detailed in (328), which is well defined everywhere because now

$$
\eta(y)=\left.\left(\left\{d_{1}(x), d_{2}(x)\right\}_{\mathcal{J}}\right)\right|_{x(y)}=\left.\left[\left(\nabla_{x} d_{1}(x)\right)^{T} \cdot \mathcal{J}(x) \cdot\left(\nabla_{x} d_{2}(x)\right)\right]\right|_{x(y)}=J_{12}^{*}(y)
$$

is $C^{\infty}\left(\Omega^{*}\right)$ and does not vanish in $\Omega^{*}$. Evidently, this time reparametrization transforms the structure matrix $\mathcal{J}^{*}$ into the Darboux canonical one $\mathcal{J}_{D[n, 2]}$, thus completing the global reduction. In order to conclude the proof, it is only required to show that transformation (326) is a global diffeomorphism in $\Omega$. This is actually a consequence of several facts: the change of coordinates (326) is a function globally onto (since $\Omega^{*}=y(\Omega)$ by definition) and by hypothesis one-to-one in $\Omega$. Consequently, (326) is a global bijection and the inverse function of (326) exists everywhere and is unique. Moreover, both the transformation (326) and its inverse are globally differentiable (and therefore continuous) since the functions $\left(d_{1}(x), d_{2}(x), D_{3}(x), \ldots D_{n}(x)\right)$ are $C^{\infty}(\Omega)$ and $|M| \neq 0$ in all points of $\Omega$, as indicated in (327). The proof is complete. Q.E.D.

We have the following direct consequence of the previous result:

Corollary 4.7.8.2. Suppose that the assumptions of Remark 1.4.4.4 hold. Consider an $n$-dimensional Poisson system $\dot{x}=\mathcal{J}(x) \cdot \nabla H(x)$ defined in a domain $\Omega \subset \mathbb{R}^{n}$ in such a way that the same hypotheses of Theorem 4.7.8.1 are verified. Then such Poisson system is algebraically integrable in $\Omega$, and it can be reduced globally and diffeomorphically in $\Omega$ to a Liouville integrable Hamiltonian system.

Now some remarks are in order. In principle, the previous results might appear as intuitively reasonable, since according to Darboux' theorem a Poisson system with a rank-two structure matrix essentially looks like a one degree of freedom classical Hamiltonian system embedded in an $n$-dimensional phase space by means of $(n-2)$ Casimir invariants that produce
the symplectic foliation. Thus, the two results in this subsection basically seem to show how to decouple the Casimir invariants in order to reverse such construction. Accordingly, since a Poisson system with a rank-two structure matrix appears essentially as a two-dimensional (embedded) system, it also seems to be natural that every time reparametrization is admissible, according to what was already said in Subsection 1.2 .4 , and thus we can always make use of such a reparametrization in order to complete the Darboux reduction. This picture of the situation looks reasonable, but it is somehow misleading. The reason is that such point of view is to some extent rough, precisely because it does not take into account the essential limitation of Darboux' theorem, namely that it is a merely local result. Therefore there is no guarantee at all in the sense that such embedding can be globally reversed -actually such a global embedding might not exist at all. In spite that the constant rank condition ensures that Darboux' theorem is applicable, the obtainment of a global result is precisely the improvement with respect to the local picture provided by Darboux' theorem, which in principle constructs the reduction in a neighborhood of each point.

A second remark concerns the hypotheses of Theorem 4.7.8.1. In particular, two key assumptions are that: (i) transformation (326) is one-to-one globally in $\Omega$; and (ii) $|M| \neq 0$ everywhere in $\Omega$ from condition (327). Clearly, such requirements are needed in order to imply the globally diffeomorphic character of the reduction, and are interesting as far as they can display to what extent such a reduction is feasible in practice. The naturalness of such conditions is very clear (as they become trivial) in the case $n=2$. Moreover, those conditions have been successfully tested for all $n \geq 3$ in Chapters 2 and 3 as well as in earlier sections of this chapter. Therefore most of the solution families previously considered in this work (among those being of rank two or admitting in particular this rank value, and for which the Darboux canonical form has been globally and diffeomorphically constructed) can be now regarded in a unified framework as applied instances of the results developed in this subsection.

### 4.8. A PERSPECTIVE OF THE RESULTS FOR ARBITRARY DIMENSION

At this stage, the relative roles played by the different contributions presented in this chapter should be understandable. In first place, we have considered some solution families of arbitrary rank (separable and multiseparable) for which the global Darboux reduction was completely general and direct by means of a pure diffeomorphism, namely without the use of time reparametrizations. On the contrary, when dealing with Poisson structures for which the rank is 2 (such as in the case of hemiseparable solutions of both types, as well as in Theorem 4.7.8.1) the use of arbitrary NTTs becomes possible, thus providing us with an additional degree of freedom that can be applied to the global construction of the Darboux canonical form (but obviously with no influence on the degree of difficulty associated with the determination of a complete set of independent Casimir invariants). Accordingly, in the case in which it is $\operatorname{Rank}(\mathcal{J})=2$ the global reduction procedure (if feasible) becomes to some extent algorithmic, as far as it suffices the direct dynamical decoupling of the distinguished functions, and the remaining two nonzero entries of the structure matrix are then reduced to the Darboux canonical form by means of an NTT, independently of how functionally involved such entries could be. This is the method implemented in Theorem 4.7.8.1. Clearly, this approach is possible in the case of rank 2, but arbitrary time reparametrizations are not generally valid for ranks equal or larger than 4. In such case this kind of NTT-based strategy is no longer valid, and the typical procedure must rely, in principle, on a direct reduction based on the only use of an adequate global coordinate transformation. These results clearly elucidate much of the previous story, providing a knowledge (about how and when a global Darboux reduction is feasible) that goes beyond the mere identification of solution families.

After Section 4.7 some additional aspects of the significance of time reparametrizations in the framework of Poisson structures have been uncovered. Specifically, this is valid regarding the conditions for the preservation of the Poisson structure itself. In particular, it is remarkable the rank dependence of the existence of reparametrization factors. As indicated, after this now it is a posteriori clear that NTTs were usable in all the Poisson systems considered in Chapters 2 and 3, since we were always dealing with Poisson structures of constant rank of value 2. Moreover, in the context of the validity of NTTs it is also interesting the importance of the distinguished functions, which induce time reparametrizations which always preserve the Poisson structure, irrespectively of its rank. This is also significant in the context of this work, in view of the fact that the existence of nontrivial Casimir invariants is a purely
noncanonical feature, not present in classical Hamiltonian systems (actually, not even present in noncanonical but symplectic Poisson systems).

The degrees of freedom provided by the distinguished functions have been fundamental also for the definition of the family of D-solutions. In this case the global analysis seems to be nontrivial, if possible, but this drawback is compensated by the remarkable (and uncommon) properties of D-solutions. Such properties lead to very general procedures for the generation of an infinity of new D-solutions obtainable from a given one. Precisely, this kind of strategy was also the one leading to the investigation of NTTs, already mentioned, since the characterization of a family of reparametrization factors automatically generalizes a known solution. Eventually, this point of view has allowed the generalization of the separable, multiseparable and D-solution families in Section 4.7.

To conclude, it is worth recalling the significance of the global analyses performed for very different $n$-d families of structure matrices, often allowing arbitrary ranks, and always defined in terms of functions of arbitrary nonlinearity. In spite that this is quite uncommon in the literature, it seems however a unavoidable issue of interest for the establishment of a bridge (dynamically global, and thus usable from the point of view of many applications) with the fruitful realm of classical Hamiltonian theory.

## CHAPTER 5.

## CONCLUSIONS

In this dissertation very diverse results regarding the study of the Jacobi PDEs for finitedimensional Poisson systems as well as the global analysis of their skew-symmetric solutions have been presented. The investigation of families of Poisson structures has comprised two different and complementary points of view. In first place, a search of solutions amenable to global analysis has involved 3 -d, 4 -d and $n$-d solution families. In second term, the characterization of novel Poisson structures has been focused on the most interesting $n$-d scenario. Of course, both categories overlap in some relevant $n$-d cases, such as in connection with separable and multiseparable structure matrices, but also has interesting exceptions such as the hemiseparable families of types I and II, in one sense, and the distinguished solutions, in the opposite. The results thus obtained cover different aspects of the theory and applications of finite-dimensional Poisson systems. In one hand, it is evident the contribution provided by the new families characterized, which are typically of a remarkable applied generality, as the number of Poisson structures well-known in the literature that become unified by them shows. In the other, it is clear the interest of the global results established since these provide us not only with new invariants (of the distinguished type) but also with reductions to the Darboux canonical form (a task only reported in the literature for a very limited sample of systems) which in addition improve the local scope of Darboux' theorem. Apart from the new solution families and global results developed, the analysis followed in the present thesis has led to the determination of new integrability results (both in the algebraic and in the Liouville senses) and also to a better understanding of the unexpected but important role of time reparametrizations in the context of finite-dimensional Poisson structures and systems.

A brief consideration of problems still to be investigated can also be helpful in order to properly establish the framework and relevance of the present contributions. From a purely mathematical perspective, the obvious open problem is the determination of the general solution of the skew-symmetric Jacobi equations in arbitrary dimension $n$. However, this does not
exhaust the list of issues deserving attention in this context. The problem of recasting (if possible) a given vector field for which a first integral is known, in terms of a finite-dimensional Poisson system, would be the second landmark in the route to a general understanding of the theory of dynamical systems in the framework of Poisson structures. A third domain of theoretical and applied progress would be the achievement of sufficiently general global results (in the sense of the determination of the Darboux canonical form) allowing a widespread transfer of information and methods between the classical Hamiltonian representation and the finite-dimensional Poisson format. The contributions displayed in this dissertation have been directed to an improvement in the context of the first and third previous issues, and also to the second in an indirect way (as far a the knowledge of novel solutions should also imply a better state of the art for such problem). But, needless to say, the answer of the three questions just enumerated constitutes an ambitious program of research that probably will require years, if not decades, of investigation.

Consequently, it can be said that every new contribution to the study of skew-symmetric solutions of the Jacobi equations tends to provide a more general perspective of the field of finite-dimensional Poisson structures. Typical features of this fact can be appreciated in the previous analysis. As already mentioned, not only the identification of new finite-dimensional Poisson structures constitutes in itself a relevant problem from the points of view of applied mathematics and mathematical physics but, as indicated, this knowledge provides a richer framework for the fundamental problem of recasting a given differential flow into a Poisson system, whenever possible. And, in addition, it is worth noting that the characterization of a sufficiently general family of solutions often allows the conceptual and operational unification of diverse Poisson structures and systems previously well-known but unrelated, which can hereafter be regarded from a more general and economic standpoint. Very diverse examples of this have been given throughout the work. In particular, in such sense it is physically interesting to identify the Casimir invariants and to develop the reduction procedure to the Darboux canonical form for the new solution families. These are features of special relevance when they can be globally achieved, thus providing an additional instance of a result that goes beyond the a priori scope of Darboux' theorem and has been reported in the literature only in a limited number of cases. This kind of results suggests that the direct investigation of the Jacobi equations constitutes a fruitful line of research not only for classification purposes but also for the detailed analysis of Poisson structures, not to mention its mathematical interest as an example of nonlinear system of coupled PDEs. Additionally to these considerations,
it is worth recalling that dimension three is the simplest nontrivial case for the analysis of the Jacobi equations and has consequently been studied in much more detail than higher dimensions, as discussed in Chapter 1. On the other hand, the Jacobi partial differential equations (25) become increasingly complex as the dimension of the Poisson manifold grows. This explains the relative scarcity of results for dimensions four and higher. Certainly, a complete knowledge of the skew-symmetric solutions of the Jacobi equations is probably still far, but nevertheless the investigation of the problem seems to be a unavoidable issue for a better understanding of finite-dimensional Poisson structures, and therefore of the scope of Hamiltonian dynamics.

## APPENDIX 1.

## THE PFAFFIAN METHOD

## A1.1. DESCRIPTION OF THE METHOD

Since it is in the basis of many of the calculations developed throughout this work, it seems convenient to provide the description of the Pfaffian method for the determination of Casimir invariants [84,175]. This approach is very advantageous when compared to traditional procedures such as the characteristics method $[8,67,170,176]$, as it will be seen in what follows.

Let us consider an $n$-d Poisson system (21-22), and a domain of the $n$-d phase space in which the rank of $\mathcal{J}(x)$ is constant and equal to $r=2 s<n$. Without loss of generality, if the $2 s$ first rows of $\mathcal{J}(x)$ are the linearly independent, then there exists a set of $2 s \times(n-2 s)$ smooth functions $\gamma_{i k}(x)$, where $i=2 s+1, \ldots, n$, and $k=1, \ldots, 2 s$, such that:

$$
\begin{equation*}
J_{i j}(x)=\sum_{k=1}^{2 s} \gamma_{i k}(x) J_{k j}(x), \quad j=1, \ldots, n \tag{333}
\end{equation*}
$$

The importance of the proportionality functions $\gamma_{i k}(x)$ was already noticed by Littlejohn [114]. Let us assume for the moment that they are known (their calculation is just a technical step for which we shall give a procedure later in this subsection). Then, the substitution of (333) into the Poisson system equations,

$$
\frac{\mathrm{d} x_{i}}{\mathrm{~d} t}=\sum_{j=1}^{n} J_{i j}(x) \partial_{j} H(x)=\left\{x_{i}, H\right\}, \quad i=1, \ldots, n
$$

gives immediately the following relations:

$$
\begin{equation*}
\dot{x}_{i}=\sum_{k=1}^{2 s} \gamma_{i k}(x) \dot{x}_{k}, \quad i=2 s+1, \ldots, n \tag{334}
\end{equation*}
$$

These equations reveal the degeneracy which is present in the Poisson system due to the fact that the rank of matrix $\mathcal{J}(x)$ is not maximum, i.e., they express all interdependences among the system variables induced by the existence of the Casimir functions, which arise from such
rank degeneracy according to equation (20), or $\mathcal{J}(x) \cdot \nabla D(x)=0$, as explained in Chapter 1. Then, from equations (334) we immediately obtain a system of $(n-2 s)$ Pfaffian DEs for the Casimir invariants:

$$
\begin{equation*}
\mathrm{d} x_{i}=\sum_{k=1}^{2 s} \gamma_{i k}(x) \mathrm{d} x_{k}, \quad i=2 s+1, \ldots, n \tag{335}
\end{equation*}
$$

Note that the system of Pfaffian equations (335) to be integrated, is equivalent to the traditional set of PDEs given by equation (20) namely $\mathcal{J}(x) \cdot \nabla D(x)=0$, as it can be demonstrated [175]. It is not difficult to see [84] that (335) leads to the Casimir functions: let $D^{[i]}(x)$ be a solution of the $i$-th equation, where $2 s+1 \leq i \leq n$. Then there exists a function $\vartheta(x)$ such that:

$$
\begin{equation*}
\mathrm{d} D^{[i]}=\vartheta(x)\left(\sum_{k=1}^{2 s} \gamma_{i k} \mathrm{~d} x_{k}-\mathrm{d} x_{i}\right) \tag{336}
\end{equation*}
$$

The $j$-th component of the vector $\mathcal{J} \cdot \nabla D^{[i]}$ will be:

$$
\begin{gathered}
\left(\mathcal{J} \cdot \nabla D^{[i]}\right)_{j}=\sum_{k=1}^{n} J_{j k} \partial_{k} D^{[i]}=\vartheta(x)\left(\sum_{k=1}^{2 s} J_{j k} \gamma_{i k}-J_{j i}\right)= \\
\vartheta(x)\left(J_{i j}-\sum_{k=1}^{2 s} \gamma_{i k} J_{k j}\right)=0, \quad j=1, \ldots, n
\end{gathered}
$$

Here we have applied the original degeneracy relations (333). This shows that the result of integrating each of the $(n-2 s)$ Pfaffian DEs (335) leads to the Casimir functions of matrix $\mathcal{J}(x)$. We know, on the other hand, that there are $(n-2 s)$ functionally independent Casimir invariants. From (336) it can be easily shown that the solutions of two different equations of the set (335) are always functionally independent. Consequently, the integration of equations (335) produces all the Casimir functions of the system [84,175].

We end this section by indicating how functions $\gamma_{i k}$ can be calculated. To do so we proceed to write (333) in matrix form as:

$$
\begin{equation*}
\left(\tilde{\mathcal{J}}^{[2 s]}\right)^{T} \cdot G=\left(\tilde{\mathcal{J}}^{[n-2 s]}\right)^{T} \tag{337}
\end{equation*}
$$

where $\tilde{\mathcal{J}}^{[2 s]}$ is the $2 s \times n$ submatrix composed by the first $2 s$ rows of $\mathcal{J}$, while $\tilde{\mathcal{J}}^{[n-2 s]}$ is the $(n-2 s) \times n$ submatrix composed by the last $(n-2 s)$ rows of $\mathcal{J}$, and

$$
G=\left(\begin{array}{ccc}
\gamma_{2 s+1,1} & \ldots & \gamma_{n 1} \\
\vdots & & \vdots \\
\gamma_{2 s+1,2 s} & \ldots & \gamma_{n, 2 s}
\end{array}\right)
$$

A rank analysis of the matrix equation (337) shows immediately that there always exists a unique matrix $G$ which is the solution. In fact, since $\tilde{\mathcal{J}}^{[2 s]}$ is a $2 s \times n$ matrix, there are $(n-2 s)^{2}$
redundant equations in (337). If we assume again that these redundant equations are those corresponding to the last $(n-2 s)$ rows of $\left(\tilde{\mathcal{J}}^{[2 s]}\right)^{T}$, we can write (337) in the nonredundant matrix form:

$$
\left(\mathcal{J}^{[2 s]}\right)^{T} \cdot G=\left(\mathcal{J}^{[n-2 s]}\right)^{T}
$$

where

$$
\mathcal{J}^{[2 s]}=\left(\begin{array}{ccc}
J_{11} & \ldots & J_{1,2 s} \\
\vdots & & \vdots \\
J_{2 s, 1} & \ldots & J_{2 s, 2 s}
\end{array}\right), \quad \mathcal{J}^{[n-2 s]}=\left(\begin{array}{ccc}
J_{2 s+1,1} & \ldots & J_{2 s+1,2 s} \\
\vdots & & \vdots \\
J_{n 1} & \ldots & J_{n, 2 s}
\end{array}\right)
$$

Since now $\mathcal{J}^{[2 s]}$ is an invertible matrix, the solution is:

$$
\begin{equation*}
G=\left(\mathcal{J}^{[n-2 s]} \cdot\left(\mathcal{J}^{[2 s]}\right)^{-1}\right)^{T} \tag{338}
\end{equation*}
$$

To summarize, the Pfaffian method for the determination of the Casimir functions proceeds in two steps:
(i) Calculation of matrix $G$ through relation (338).
(ii) Integration of the system of Pfaffian equations (335).

We shall now illustrate the procedure by means of some applied examples.

## A1.2. EXAMPLES

## Example 1. 3-d Lotka-Volterra system

Let us turn back to system (35) already considered in the Example 2 of Subsection 1.2.5. Nutku has shown [136] that the 3-d LV equations

$$
\left\{\begin{array}{l}
\dot{x}_{1}=x_{1}\left(\lambda_{1}+c x_{2}+x_{3}\right)  \tag{339}\\
\dot{x}_{2}=x_{2}\left(\lambda_{2}+x_{1}+a x_{3}\right) \\
\dot{x}_{3}=x_{3}\left(\lambda_{3}+b x_{1}+x_{2}\right)
\end{array}\right.
$$

are biHamiltonian when $a b c=-1$ and $\lambda_{3}=\lambda_{2} b-\lambda_{1} a b$. In this case, the vector field (339) can be written as a Poisson system in two different ways:

$$
\dot{x}=\mathcal{J}_{1} \cdot \nabla H_{1}=\mathcal{J}_{2} \cdot \nabla H_{2}
$$

Here, we have

$$
\begin{gathered}
\mathcal{J}_{1}(x)=\left(\begin{array}{ccc}
0 & c x_{1} x_{2} & b c x_{1} x_{3} \\
-c x_{1} x_{2} & 0 & -x_{2} x_{3} \\
-b c x_{1} x_{3} & x_{2} x_{3} & 0
\end{array}\right) \\
\mathcal{J}_{2}(x)=\left(\begin{array}{ccc}
c x_{1} x_{2}\left(a x_{3}+\lambda_{2}\right) & c x_{1} x_{3}\left(x_{2}+\lambda_{3}\right) \\
-c x_{1} x_{2}\left(a x_{3}+\lambda_{2}\right) & 0 & x_{1} x_{2} x_{3} \\
-c x_{1} x_{3}\left(x_{2}+\lambda_{3}\right) & -x_{1} x_{2} x_{3} & 0
\end{array}\right)
\end{gathered}
$$

and, in addition, the respective Hamiltonian functions are:

$$
\left\{\begin{array}{l}
H_{1}(x)=a b x_{1}+x_{2}-a x_{3}+\lambda_{3} \ln x_{2}-\lambda_{2} \ln x_{3} \\
H_{2}(x)=a b \ln x_{1}-b \ln x_{2}+\ln x_{3}
\end{array}\right.
$$

Since the rank of both $\mathcal{J}_{1}$ and $\mathcal{J}_{2}$ is 2 everywhere in the interior of the positive orthant, there is one independent Casimir invariant. We shall apply the Pfaffian method to both structure matrices.

For $\mathcal{J}_{1}$ we have, after simple inspection:

$$
(\text { row } 3)=\frac{x_{3}}{c x_{1}}(\text { row } 1)+\frac{b x_{3}}{x_{2}}(\text { row } 2)
$$

In other words, $\gamma_{31}=x_{3} / c x_{1}$ and $\gamma_{32}=b x_{3} / x_{2}$. The Pfaffian equation we must solve is then:

$$
\mathrm{d} x_{3}=\frac{x_{3}}{c x_{1}} \mathrm{~d} x_{1}+\frac{b x_{3}}{x_{2}} \mathrm{~d} x_{2}
$$

The integration of this equation is immediate and gives

$$
a b \ln x_{1}-b \ln x_{2}+\ln x_{3}=\mathrm{constant}
$$

which is Nutku's result. Since any function of a Casimir invariant is also a Casimir, the general solution will be

$$
D(x)=\Psi\left(a b \ln x_{1}-b \ln x_{2}+\ln x_{3}\right)
$$

with $\Psi$ a smooth one-variable function.
Similarly, for $\mathcal{J}_{2}$ we see that:

$$
(\text { row } 3)=-\frac{x_{3}}{c\left(a x_{3}+\lambda_{2}\right)}(\text { row1 })+\frac{x_{3}\left(x_{2}+\lambda_{3}\right)}{x_{2}\left(a x_{3}+\lambda_{2}\right)}(\text { row } 2)
$$

Consequently, $\gamma_{31}=-x_{3} /\left(c\left(a x_{3}+\lambda_{2}\right)\right)$ and $\gamma_{32}=x_{3}\left(x_{2}+\lambda_{3}\right) /\left(x_{2}\left(a x_{3}+\lambda_{2}\right)\right)$. This implies that:

$$
\mathrm{d} x_{3}=-\frac{x_{3}}{c\left(a x_{3}+\lambda_{2}\right)} \mathrm{d} x_{1}+\frac{x_{3}\left(x_{2}+\lambda_{3}\right)}{x_{2}\left(a x_{3}+\lambda_{2}\right)} \mathrm{d} x_{2}
$$

After integration we arrive easily at

$$
a b x_{1}+x_{2}-a x_{3}+\lambda_{3} \ln x_{2}-\lambda_{2} \ln x_{3}=\mathrm{constant}
$$

which is the solution. In general

$$
\begin{equation*}
D(x)=\Psi\left(a b x_{1}+x_{2}-a x_{3}+\lambda_{3} \ln x_{2}-\lambda_{2} \ln x_{3}\right) \tag{340}
\end{equation*}
$$

It is interesting to compare this procedure with the standard method of characteristics. We shall do it for $\mathcal{J}_{2}$. Since $\operatorname{Rank}\left(\mathcal{J}_{2}\right)=2$ in the domain of interest, the third equation of the system $\mathcal{J}_{2} \cdot \nabla D=0$ is a linear combination of the first and second ones, and can therefore be suppressed. The system of PDEs we have to solve in order to determine $D$ is then:

$$
\begin{gather*}
c x_{1} x_{2}\left(a x_{3}+\lambda_{2}\right) \frac{\partial D}{\partial x_{2}}+c x_{1} x_{3}\left(x_{2}+\lambda_{3}\right) \frac{\partial D}{\partial x_{3}}=0  \tag{341}\\
-c x_{1} x_{2}\left(a x_{3}+\lambda_{2}\right) \frac{\partial D}{\partial x_{1}}+x_{1} x_{2} x_{3} \frac{\partial D}{\partial x_{3}}=0 \tag{342}
\end{gather*}
$$

The characteristic equations for (341) are:

$$
\frac{\mathrm{d} x_{2}}{c x_{1} x_{2}\left(a x_{3}+\lambda_{2}\right)}=\frac{\mathrm{d} x_{3}}{c x_{1} x_{3}\left(x_{2}+\lambda_{3}\right)} \quad, \quad \mathrm{d} x_{1}=0
$$

Since $D$ is a function of three variables, we have to perform two integrations from the characteristic equations. It can be found easily that $x_{1}=k_{1}$ and $x_{2}-a x_{3}+\lambda_{3} \ln x_{2}-\lambda_{2} \ln x_{3}=k_{2}$, where $k_{1}$ and $k_{2}$ are constants of integration. Then, the general solution of equation (341) is of the form:

$$
\begin{equation*}
D^{[1]}=\Psi^{[1]}\left(x_{1}, x_{2}-a x_{3}+\lambda_{3} \ln x_{2}-\lambda_{2} \ln x_{3}\right) \tag{343}
\end{equation*}
$$

Similarly, for the second PDE (342), the system of characteristic equations is:

$$
-\frac{\mathrm{d} x_{1}}{c x_{1} x_{2}\left(a x_{3}+\lambda_{2}\right)}=\frac{\mathrm{d} x_{3}}{x_{1} x_{2} x_{3}}, \quad \mathrm{~d} x_{2}=0
$$

We can obtain without difficulty that $x_{2}=k_{1}$ and $a b x_{1}-a x_{3}-\lambda_{2} \ln x_{3}=k_{2}$, and then the general solution of (342) is:

$$
\begin{equation*}
D^{[2]}=\Psi^{[2]}\left(x_{2}, a b x_{1}-a x_{3}-\lambda_{2} \ln x_{3}\right) \tag{344}
\end{equation*}
$$

Now we must take into account that the Casimir invariants of the system are simultaneous solutions of (341) and (342). This means that they are functions of the $x_{i}$ complying to both formats (343) and (344). After inspection, one arrives directly to the solution (340). We shall further comment in Section A1.3 on the differences between both methods.

## Example 2. A higher-dimensional system: the light top

We shall now analyze in detail a six-dimensional example due to Weinstein [171]: the equations of motion of a rigid body anchored at one point, which moves in a gravitational field. The system variables are the components of the angular momentum in body coordinates, $L=\left(L_{1}, L_{2}, L_{3}\right)$, as well as those of the gravitational force, also in body coordinates, $F=$ $\left(F_{1}, F_{2}, F_{3}\right)$. From now on, we shall take the six variables according to the following order: $\left(L_{1}, L_{2}, L_{3}, F_{1}, F_{2}, F_{3}\right)$. Then, the structure matrix and the Hamiltonian are, respectively:

$$
\mathcal{J}=\left(\begin{array}{cccccc}
0 & L_{3} & -L_{2} & 0 & F_{3} & -F_{2} \\
-L_{3} & 0 & L_{1} & -F_{3} & 0 & F_{1} \\
L_{2} & -L_{1} & 0 & F_{2} & -F_{1} & 0 \\
0 & F_{3} & -F_{2} & 0 & 0 & 0 \\
-F_{3} & 0 & F_{1} & 0 & 0 & 0 \\
F_{2} & -F_{1} & 0 & 0 & 0 & 0
\end{array}\right),
$$

and

$$
H=\sum_{i=1}^{3}\left(\frac{L_{i}^{2}}{2 \mu_{i}}+x_{i} F_{i}\right)
$$

In $H$, the $\mu_{i}$ are the principal moments of inertia, and the $x_{i}$ are the coordinates of the body's center of mass measured from the anchor point (see [171] and references therein for further details).

We shall first apply the Pfaffian procedure for the determination of the Casimir functions of this system. For the sake of comparison, we shall later solve the same problem through the traditional method of characteristics
(a) Solution of the problem by the Pfaffian method.

Clearly, $\operatorname{Rank}(\mathcal{J})=4$, the third and the sixth rows being linear combinations of the rest. Then there are two independent Casimir functions. We can find the $\gamma_{i k}$ by means of (338):

$$
G=\left(\begin{array}{llll}
\gamma_{31} & \gamma_{32} & \gamma_{34} & \gamma_{35} \\
\gamma_{61} & \gamma_{62} & \gamma_{64} & \gamma_{65}
\end{array}\right)^{T}=\left(\mathcal{J}^{[2]} \cdot\left(\mathcal{J}^{[4]}\right)^{-1}\right)^{T}
$$

where

$$
\mathcal{J}^{[4]}=\left(\begin{array}{cccc}
0 & L_{3} & 0 & F_{3} \\
-L_{3} & 0 & -F_{3} & 0 \\
0 & F_{3} & 0 & 0 \\
-F_{3} & 0 & 0 & 0
\end{array}\right), \quad \mathcal{J}^{[2]}=\left(\begin{array}{cccc}
L_{2} & -L_{1} & F_{2} & -F_{1} \\
F_{2} & -F_{1} & 0 & 0
\end{array}\right)
$$

The solution for $G$ is:

$$
G=\left(\begin{array}{cc}
-F_{1} / F_{3} & 0 \\
-F_{2} / F_{3} & 0 \\
\left(F_{1} L_{3}-L_{1} F_{3}\right) / F_{3}^{2} & -F_{1} / F_{3} \\
\left(F_{2} L_{3}-L_{2} F_{3}\right) / F_{3}^{2} & -F_{2} / F_{3}
\end{array}\right)
$$

We then have to solve the following system of two Pfaffian DEs:

$$
\begin{gather*}
\mathrm{d} L_{3}=-\frac{F_{1}}{F_{3}} \mathrm{~d} L_{1}-\frac{F_{2}}{F_{3}} \mathrm{~d} L_{2}+\left(\frac{F_{1} L_{3}}{F_{3}^{2}}-\frac{L_{1}}{F_{3}}\right) \mathrm{d} F_{1}+\left(\frac{F_{2} L_{3}}{F_{3}^{2}}-\frac{L_{2}}{F_{3}}\right) \mathrm{d} F_{2}  \tag{345}\\
\mathrm{~d} F_{3}=-\frac{F_{1}}{F_{3}} \mathrm{~d} F_{1}-\frac{F_{2}}{F_{3}} \mathrm{~d} F_{2} \tag{346}
\end{gather*}
$$

The last one is straightforward and gives a first Casimir: $D_{1}=F_{1}^{2}+F_{2}^{2}+F_{3}^{2}=\|F\|^{2}$. Now, if we expand (345) and regroup terms we have:

$$
\begin{equation*}
F_{1} \mathrm{~d} L_{1}+F_{2} \mathrm{~d} L_{2}+F_{3} \mathrm{~d} L_{3}+L_{1} \mathrm{~d} F_{1}+L_{2} \mathrm{~d} F_{2}=L_{3}\left(\frac{F_{1}}{F_{3}} \mathrm{~d} F_{1}+\frac{F_{2}}{F_{3}} \mathrm{~d} F_{2}\right) \tag{347}
\end{equation*}
$$

Making use of equation (346) in the right-hand side of (347) leads immediately to $\mathrm{d}\left(L_{1} F_{1}+\right.$ $\left.L_{2} F_{2}+L_{3} F_{3}\right)=0$. Thus, the second independent Casimir is $D_{2}=L_{1} F_{1}+L_{2} F_{2}+L_{3} F_{3}=$ $L \cdot F$. We can write, as usual, the most general form of a Casimir invariant as

$$
D=\Psi\left(\left[F_{1}^{2}+F_{2}^{2}+F_{3}^{2}\right]^{1 / 2}, L_{1} F_{1}+L_{2} F_{2}+L_{3} F_{3}\right)=\Psi(\|F\|, L \cdot F)
$$

where $\Psi$ is a smooth two-variable function.
(b) Solution of the problem by the method of characteristics.

We can now compare the previous procedure with the direct solution of the system of PDEs given by $(20)$, namely $\mathcal{J} \cdot \nabla D=0$. For this, we should begin by recalling the same observation
than before: since $\operatorname{Rank}(\mathcal{J})=4$, two of the equations of the system will be redundant, which can be taken to be those corresponding to the third and sixth rows of $\mathcal{J}$. Therefore, the system of PDEs we have to solve is:

$$
\begin{gather*}
L_{3} \frac{\partial D}{\partial L_{2}}-L_{2} \frac{\partial D}{\partial L_{3}}+F_{3} \frac{\partial D}{\partial F_{2}}-F_{2} \frac{\partial D}{\partial F_{3}}=0  \tag{348}\\
-L_{3} \frac{\partial D}{\partial L_{1}}+L_{1} \frac{\partial D}{\partial L_{3}}-F_{3} \frac{\partial D}{\partial F_{1}}+F_{1} \frac{\partial D}{\partial F_{3}}=0  \tag{349}\\
F_{3} \frac{\partial D}{\partial L_{2}}-F_{2} \frac{\partial D}{\partial L_{3}}=0  \tag{350}\\
-F_{3} \frac{\partial D}{\partial L_{1}}+F_{1} \frac{\partial D}{\partial L_{3}}=0 \tag{351}
\end{gather*}
$$

The characteristic equations of (348) are:

$$
\begin{equation*}
\frac{\mathrm{d} L_{2}}{L_{3}}=-\frac{\mathrm{d} L_{3}}{L_{2}}=\frac{\mathrm{d} F_{2}}{F_{3}}=-\frac{\mathrm{d} F_{3}}{F_{2}}, \mathrm{~d} L_{1}=\mathrm{d} F_{1}=0 \tag{352}
\end{equation*}
$$

Since the unknown $D$ is a function of six variables, we have to find five constants from the characteristic equations (352) in order to construct the general solution of the PDE (348). We immediately find from (352) four of them:

$$
L_{1}=k_{1}, F_{1}=k_{2}, L_{2}^{2}+L_{3}^{2}=k_{3}, F_{2}^{2}+F_{3}^{2}=k_{4}
$$

We can derive a fifth one as follows:

$$
\begin{aligned}
0 \equiv & L_{3} \mathrm{~d} F_{3}-L_{3} \mathrm{~d} F_{3}+F_{3} \mathrm{~d} L_{3}-F_{3} \mathrm{~d} L_{3}= \\
& L_{3} \mathrm{~d} F_{3}+F_{3} \mathrm{~d} L_{3}+L_{2} \mathrm{~d} F_{2}+F_{2} \mathrm{~d} L_{2}= \\
& \mathrm{d}\left(L_{2} F_{2}+L_{3} F_{3}\right)
\end{aligned}
$$

Here we have made use of the characteristic equations (352). The fifth constant is thus $k_{5}=L_{2} F_{2}+L_{3} F_{3}$. The general solution of the $\operatorname{PDE}(348)$ is then:

$$
D^{[1]}=\Psi^{[1]}\left(L_{1}, F_{1}, L_{2}^{2}+L_{3}^{2}, F_{2}^{2}+F_{3}^{2}, L_{2} F_{2}+L_{3} F_{3}\right)
$$

The second PDE (349) can be obtained from the first one (348) if we exchange the subindexes 1 and 2. Then we can directly write:

$$
D^{[2]}=\Psi^{[2]}\left(L_{2}, F_{2}, L_{1}^{2}+L_{3}^{2}, F_{1}^{2}+F_{3}^{2}, L_{1} F_{1}+L_{3} F_{3}\right)
$$

For the third equation (350) we now have:

$$
\frac{\mathrm{d} L_{2}}{F_{3}}=-\frac{\mathrm{d} L_{3}}{F_{2}}, \mathrm{~d} L_{1}=\mathrm{d} F_{1}=\mathrm{d} F_{2}=\mathrm{d} F_{3}=0
$$

This leads to:

$$
L_{1}=k_{1}, F_{1}=k_{2}, F_{2}=k_{3}, F_{3}=k_{4}
$$

Since $F_{2}$ and $F_{3}$ are constants, we also arrive at $k_{5}=L_{2} F_{2}+L_{3} F_{3}$. Consequently, the general solution of the $\operatorname{PDE}$ (350) is:

$$
D^{[3]}=\Psi^{[3]}\left(L_{1}, F_{1}, F_{2}, F_{3}, L_{2} F_{2}+L_{3} F_{3}\right)
$$

And finally, we again obtain the fourth PDE (351) from the third one (350) by permutation of the subindexes 1 and 2 . Therefore:

$$
D^{[4]}=\Psi^{[4]}\left(L_{2}, F_{1}, F_{2}, F_{3}, L_{1} F_{1}+L_{3} F_{3}\right)
$$

Now, the Casimir functions are simultaneous solutions of all the PDEs (348-351). Then, we now have to compare the four solutions $D^{[i]}$, for $i=1, \ldots, 4$, and look for those functions of $L$ and $F$ compatible with all of them. After inspection, it is not difficult to arrive at the two most obvious possibilities: $\|F\|^{2}$ and $L \cdot F$, which are the two independent Casimir invariants already known.

## A1.3. CONCLUDING REMARKS

We have seen how the Pfaffian approach allows the calculation of the Casimir functions in a quite natural and rapid way. In addition, this procedure provides some insight on how a symplectic foliation arises from the degeneracy present in a singular Poisson structure.

A comparison with the traditional method relying on the system of PDEs (20) seems to be convenient. If we wish to solve equations (20), the two simplest strategies are separation of variables and the method of characteristics.

Separation of variables, which is rather lengthy even for simple PDEs and usually requires an eigenvalue analysis of the resulting ODEs, is clearly much less efficient than the Pfaffian method in the present situation.

On the other hand, we have already given in the examples a comparative solution of the problems by both the Pfaffian approach and the method of characteristics. Before entering in more quantitative and general arguments, two observations can be drawn from the examples. The first one is that the Pfaffian method is clearly less computationally consuming than the one of characteristics. Notice that the former technique reduces the problem to the solution of one Pfaffian DE per Casimir. The number of ODEs which has been necessary to handle and the number of quadratures which must be integrated in the method of characteristics is certainly higher, in both examples. The second important remark is that both techniques do not lead to the same set of equations, i.e., the Pfaffian method is not a shortcut for the obtainment of the characteristic equations, as it can be easily checked.

Let us compare in a quantitative way the complexity of both methods. We shall give as a measure of such complexity the number of quadratures which have to be calculated in every case to determine the solution. This number is $N_{p}=(n-2 s)$ for the Pfaffian method, namely the corank of the structure matrix, as we already know.

In the method of characteristics, on the other hand, we have to solve system (20), which consists of $2 s$ nonredundant PDEs (the remaining $(n-2 s)$ equations are redundant due to the degeneracy in rank of the structure matrix, and can therefore be suppressed, as we have seen in the examples). In order to compute the total number of quadratures in the method of characteristics, let us consider the $i$-th PDE of system (20). Its characteristic equations are of the form:

$$
\frac{\mathrm{d} x_{1}}{J_{i 1}}=\ldots=\frac{\mathrm{d} x_{i-1}}{J_{i, i-1}}=\frac{\mathrm{d} x_{i+1}}{J_{i, i+1}}=\ldots=\frac{\mathrm{d} x_{n}}{J_{i n}}, \quad \mathrm{~d} x_{i}=0
$$

Since $D$ is a function of $n$ variables, we need $(n-1)$ quadratures. However, we always have a trivial one, which is $x_{i}=$ constant. Therefore, we only have to carry out $(n-2)$ quadratures
per PDE, in general. Consequently, the total number of quadratures is $N_{c}=2 s(n-2)$ for the method of characteristics. It is then straightforward to verify the bound

$$
\begin{equation*}
\frac{N_{p}}{N_{c}}<1 \tag{353}
\end{equation*}
$$

in all nontrivial cases (the only situation in which (353) is not satisfied for a singular structure matrix, is the immaterial case corresponding to a trivial Poisson structure, namely to a null structure matrix). When the number of Casimir functions is large, for example if $2 s=2$, we obtain $N_{p} / N_{c}=1 / 2$. When such a number is medium, namely for $2 s \simeq n / 2$, we have that $N_{p} / N_{c} \simeq 1 /(n-2)$, thus decreasing with increasing size of the structure matrix. Finally, when the number of Casimir invariants is small, say $2 s \simeq(n-1)$, we arrive at $N_{p} / N_{c} \simeq$ $1 /[(n-1)(n-2)]$. In this case the ratio decreases as $n^{-2}$ as $n$ grows, and the Pfaffian method is now much more economic for a large structure matrix.

## APPENDIX 2.

## MATHEMATICAL NOTATION

The following symbols and expressions are of common use throughout the text:

| $a, b, \ldots$ | Real constants |
| :--- | :--- |
| $A, B, \ldots$ | Constant real matrices |
| $\\|a\\|$ | Absolute value of the real number $a$ |
| $\\|x\\|$ | Euclidean norm of vector $x \in \mathbb{R}^{m}$ |
| $C^{k}(\Omega)$ | $k$ times continously differentiable function in the set $\Omega$ |
| $C^{\infty}(\Omega)$ | Smooth function in the set $\Omega$ |
| $\partial_{i} f, \partial_{x_{i}} f$ | Euclidean scalar product in $\mathbb{R}^{m}$, or matrix product |
| $D, D(x)$ | Partial derivative of $f$ with respect to $x_{i}$ |
| $\delta$ | Distinguished function (Casimir invariant) |
| $\|A\|$ | Kronecker's delta or generalized Kronecer's delta |
| $\operatorname{diag}()$ | Determinant of the square matrix $A$ |
| $\dot{x}$ | Matrix formed by a direct sum of submatrices |
| $\epsilon$ | Time derivative of $x(t)$ |
| $\equiv$ | Levi-Civita symbol |
| $\eta, \eta(x)$ | Equal by definition, equivalent to |
| $H, H(x, t)$ | Time reparametrization factor |
| $I(x)$ | Hamiltonian function, either in the classical or in the Poisson sense |
| $I(x, t)$ | Time-independent first integral |
| $\mathbb{I}_{m}$ | Time-dependent first integral |
| $\infty$ | $m \times m$ identity matrix or submatrix |
| Int $(U)$ | Infinity |
| $\cap$ | Interior of the set $U \subset \mathbb{R}^{m}$ |
|  | Intersection of sets |


| $A^{-1}$ | Inverse of the square matrix $A$ |
| :---: | :---: |
| $\mathcal{J}, \mathcal{J}(x)$ | Structure matrix |
| $\mathcal{J}_{D[n, r]}$ | $n \times n$ Darboux canonical form structure matrix of rank $r$ |
| $\mathcal{J}^{[k]}, \mathcal{J}^{[k]}(x)$ | $k$-th submatrix of the structure matrix $\mathcal{J}(x)$ |
| $J_{i j}, J_{i j}(x)$ | Structure functions |
| $J_{i j}^{[k]}, J_{i j}^{[k]}(x)$ | Structure functions in the $k$-th submatrix $\mathcal{J}^{[k]}(x)$ of the structure matrix $\mathcal{J}(x)$ |
| $\frac{\partial\left(f_{1}, \ldots, f_{k}\right)}{\partial\left(x_{1}, \ldots, x_{m}\right)}$ | Jacobian matrix of functions $\left(f_{1}, \ldots, f_{k}\right)$ with respect to $\left(x_{1}, \ldots, x_{m}\right)$ |
| $\operatorname{Ker}(A)$ | Kernel (nullspace) of matrix $A$ |
| $\ln (a)$ | Natural (or Napierian) logarithm of $a$ (with $a$ real and positive) |
| M | Jacobian matrix of a differentiable change of coordinates |
| $\mathbb{M}$ | Smooth manifold |
| $M_{P}$ | Poisson manifold |
| $\mathbb{M}_{S}$ | Symplectic manifold |
| $a \bmod b$ | Remainder of the integer division $a / b$ (with $a, b$ positive integers) |
| $n$ | Dimension of the Poisson manifold |
| $\nabla$ | Nabla operator |
| $\nabla_{x} f$ | Column gradient vector of function $f$ with respect to coordinates $x$ |
| $0^{(1)}$ | $m \times m$ null matrix or submatrix |
| $\mathbb{0}_{m_{1} \times m_{2}}$ | $m_{1} \times m_{2}$ null matrix or submatrix |
| $\Omega$ | Open subset of $\mathbb{R}^{m}$, or domain (open and connected subset of $\mathbb{R}^{m}$ ) |
| $\oplus$ | Direct sum |
| $A_{1} \overbrace{\oplus \ldots \oplus}^{k} A_{k}$ | Direct sum of $k$ matrices $A_{1}, \ldots, A_{k}$ |
| $p \equiv\left(p_{i}\right)$ | Set of canonical generalized momenta $p_{i}$ |
| $\{f, g\}$ | Poisson bracket of functions $f$ and $g$ |
| $\{f, g\}_{c}$ | Canonical Poisson bracket of functions $f$ and $g$ |
| $\{f, g\}_{\mathcal{J}}$ | Poisson bracket with structure matrix $\mathcal{J}$ of functions $f$ and $g$ |
| $f^{\prime}$ | Derivative of the one-variable function $f$ |
| $q \equiv\left(q_{i}\right)$ | Set of canonical generalized coordinates $q_{i}$ |
| $r$ | Rank of a structure matrix |
| $\operatorname{Rank}(A)$ | Rank of matrix $A$ |
| $\mathbb{R}^{m}$ | $m$-dimensional Euclidean space |
| $\mathbb{R}_{+}^{m}$ | Interior of the positive orthant of $\mathbb{R}^{m}$ |

$s$
$\epsilon$
$\operatorname{sign}(a)$
$\simeq \quad$ Similar or equal
$\subset$
$S$
$\mathbb{S}_{n}$
1
$t$
$\tau$
$U \times V$
$m_{1} \times m_{2}$
$A^{T}$
$\cup$
$\Upsilon_{x}$
$u, u(x)$
$v, \quad v(x)$
$w, w(x)$
$x, y, z$
Set member and $(-1)$ if $a<0)$

Subset

Such that
Time variable

Transpose of matrix $A$
Union of sets
Neighborhood of $x \in \mathbb{R}^{m}$
$r / 2$, with $r$ being the rank of a structure matrix

Sign function of the real number $a \neq 0$ (it takes value $(+1)$ if $a>0$

Skew-symmetric constant real matrix
$n \times n$ symplectic matrix ( $n$ even)

New time variable (reparametrized time variable)
Cartesian product of the sets $U$ and $V$
Size of a matrix of $m_{1}$ rows and $m_{2}$ columns

Structure function $J_{12}(x)$ in three-dimensional structure matrices
Structure function $J_{31}(x)$ in three-dimensional structure matrices
Structure function $J_{23}(x)$ in three-dimensional structure matrices
Column vectors of the sets of local coordinates $\left(x_{1}, \ldots, x_{n}\right)$, $\left(y_{1}, \ldots, y_{n}\right)$ and $\left(z_{1}, \ldots, z_{n}\right)$, respectively

## APPENDIX 3.

## ABBREVIATIONS

In addition to the symbols indicated in Appendix 2, the abbreviations now enumerated are of common use in the present dissertation:

2-d Two-dimensional
3-d Three-dimensional
4-d Four-dimensional
$n$-d $n$-dimensional
DE Differential equation
D-solution Distinguished structure matrix
$\mathrm{D}_{\psi}$-solution $\quad$ D-solution of the kind characterized in Subsection 4.4.2
KAM Kolmogorov-Arnol'd-Moser (theorem)
KG Kernel-gradient (functions) -see Definition 4.4.1.3
LV Lotka-Volterra (systems of ordinary differential equations)
NTT New-time transformation (time reparametrization)
ODE Ordinary differential equation
PDE Partial differential equation
Q.E.D. End of a proof ("Quod erat demonstrandum")

QP Quasi-Polynomial (systems of ordinary differential equations)
SOE Smooth orbital equivalence
TOE Topological orbital equivalence
VV.AA. Various authors

## APPENDIX 4.

## PUBLICATIONS WITH IMPACT FACTOR

The following is a list of the articles (already published in journals with impact factor) which have been developed as a part of the research carried out in this thesis work:

1. B. Hernández-Bermejo: "New solutions of the Jacobi equations for three-dimensional Poisson structures", Journal of Mathematical Physics 42 (2001) 4984-4996.
2. B. Hernández-Bermejo: "One solution of the 3D Jacobi identities allows determining an infinity of them", Physics Letters A 287 (2001) 371-378.
3. B. Hernández-Bermejo: "Characterization and global analysis of a family of Poisson structures", Physics Letters A 355 (2006) 98-103.
4. B. Hernández-Bermejo: "New four-dimensional solutions of the Jacobi equations for Poisson structures", Journal of Mathematical Physics 47 (2006) 022901 1-13.
5. B. Hernández-Bermejo: "New solution family of the Jacobi equations: Characterization, invariants, and global Darboux analysis", Journal of Mathematical Physics 48 (2007) 022903 1-11.
6. B. Hernández-Bermejo: "Characterization, global analysis and integrability of a family of Poisson structures", Physics Letters A $\mathbf{3 7 2}$ (2008) 1009-1017.
7. B. Hernández-Bermejo and V. Fairén: "Simple evaluation of Casimir invariants in finitedimensional Poisson systems", Physics Letters A 241 (1998) 148-154.
8. B. Hernández-Bermejo and V. Fairén:"Separation of variables in the Jacobi identities", Physics Letters A 271 (2000) 258-263.
9. T. W. Yudichak, B. Hernández-Bermejo and P. J. Morrison: "Computing Casimir invariants from Pfaffian systems", Physics Letters A 260 (1999) 475-483.

In addition, the following manuscripts have been accepted for publication (again, in journals with impact factor) prior to the dissertation defense (thesis viva voce):
10. B. Hernández-Bermejo:"Generalization of solutions of the Jacobi PDEs associated to time reparametrizations of Poisson systems", Journal of Mathematical Analysis and Applications, in press.
11. B. Hernández-Bermejo: "An integrable family of Poisson systems: characterization and global analysis", Applied Mathematics Letters, in press.

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